Existence and uniqueness results of terminal value problems for right fractional calculus

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To my Parents

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Abstract

The theory of fractional calculus gained considerable attention due to its various practical uses in past three decades. Which provides the natural generalization of ordinary integrals and derivatives to arbitrary ones. This thesis is particularly devoted to right fractional calculus. We review basic definitions and important results for the right fractional integral and differential operators. Some new properties for right Riemann-Liouville fractional integral and differential operators and right fractional Caputo operator are discussed.

We discuss the generalized Taylor's formula for right fractional calculus with integral remainder. We also consider Mean Value theorem for right fractional order differential equations.

Some adequate conditions are developed for the existence and uniqueness of results for terminal value problems of non-linear right fractional order differential equations on bounded domain using Riemann-Liouville fractional integral and differential operators. Furthermore we generalized these results for a coupled system and establish existence and uniqueness results by employing Banach fixed point theorem and Schauder's fixed point theorem.

Finally, we develop some new conditions for terminal value problem of non-linear right fractional order differential equations on unbounded domain using right fractional Caputo derivative. We construct Green's function and develop some of its useful properties. We investigate existence and uniqueness of the solutions using Banach fixed point theorem and Schauder's fixed point theorem.

Contents

Chapter 1

Basic definitions and preliminaries

1.1 Introduction

Fractional calculus is now days a standout amongst the mathematical analysis. Fractional calculus is as old as integral order calculus established autonomously by Newton and Leibniz. First time Leibniz used the notation $\frac{d^n f}{dx^n}$ in his publications for the nth-derivative of the function $f(x)$. However it has grown particularly seriously since 1974 when first international conference in the field took place. Fractional calculus deals with the integrals and derivatives of arbitrary order. We can say that it deals with the natural generalization of ordinary integrals and derivatives to arbitrary ones. The role of this kind of calculus is to solve problems of complex systems that appears in various fields of science and technology. However many people remain unaware of fractional calculus because it is not being taught in schools and colleges. The mathematics involved appeared very different from that of integer order calculus. One of the significant preferences of fractional calculus is that it can be considered as a superset of integer order calculus. During the last decades fractional calculus has been applied to almost every field of science, engineering and mathematics [9, 12, 16, 22, 24, 28, 29]

Fourier, Euler, Laplace and many other mathematicians used their own notations, methodology and definitions that justify the concept of integral and derivatives of arbitrary order. The most famous definitions that have been used in fractional calculus are the Riemann-Liouville and Caputo definitions.

This chapter is review of [7]. In 2nd Section, we explain Gamma function and its properties. In 3rd Section, we discuss right Riemann-Liouville fractional integrals and derivatives, properties of right fractional integrals and derivatives. Taylor's formula with right fractional integral remainder is also discussed. In 4th Section, we discuss the right Caputo fractional derivative, properties of right Caputo fractional derivative and relation between Caputo and Riemann-Liouville fractional derivative. At the end some important results from analysis are briefly discussed.

1.2 Gamma function

The Euler's Gamma function $\Gamma(p)$ is one of the basic and most important functions for fractional calculus, which is generalization of factorial function $n!$. Factorial function is defined for integers, whereas gamma function also takes non integral values. Therefore we can say gamma function is continuous extension of factorial function to real number arguments.

Definition 1.2.1. [12] The Euler's Gamma function $\Gamma(p) : (0, \infty) \longrightarrow \mathbb{R}$ defined by Euler's integral of the second kind is given by

$$\Gamma(p) = \int_0^\infty \xi^{p-1} e^{-\xi} d\xi. \tag{1.2.1}$$

Since the integral on right side is uniformly convergent for all $p \in \mathbb{R}^+$, so the Γ is a continuous function for all $p \in \mathbb{R}^+$.

1.2.1 Properties of gamma function

The gamma function satisfies the recurrence relation

$$\Gamma(p) = (p-1)\Gamma(p-1), \quad p > 0 \tag{1.2.2}$$

which can be proved by integrating by parts equation (1.2.1)

$$\begin{aligned} \Gamma(p) &= \int_0^\infty \xi^{p-1} e^{-\xi} d\xi \\ &= \big[-e^{-\xi}\xi^{p-1} \big]_0^\infty + (p-1)\int_0^\infty \xi^{p-2} e^{-\xi} d\xi \\ &= (p-1)\Gamma(p-1). \end{aligned}$$

From equation (1.2.2) we can write for $p > -1$

$$\Gamma(p) = \frac{\Gamma(p+1)}{p}, \quad p \neq 0.$$

Similarly

$$\Gamma(p) = \frac{\Gamma(p+n)}{p(p+1)(p+2)\dots(p+n-1)}, \quad n \in \mathbb{N}.$$

Thus $\Gamma(p)$ is defined for all $p \in \mathbb{R}$ except $p = 0, -1, -2, \dots$
Obviously, $\Gamma(1) = 1$, similarly using equation (1.2.2) for $p = 2, 3, 4 \dots$

$$\Gamma(2) = 1.\Gamma(1) = 1 = 1!,$$

$$\Gamma(3) = 2.\Gamma(2) = 2.1 = 2!,$$

$$\Gamma(4) = 3.\Gamma(3) = 3.2.1 = 3!.$$

In general for any $n \in \mathbb{N}$

$$\Gamma(n) = (n-1)!.$$

1.3 Right Riemann-Liouville fractional integrals and derivatives

In this section we establish fundamental properties of right Riemann-Liouville fractional integral and differential operators on finite interval in suitable space of functions.

1.3.1 Right Riemann-Liouville fractional integrals

Definition and important properties of right Riemann-Liouville fractional integral are given here.

Definition 1.3.1. Suppose $f \in L_1[a,b]$ and $p \in \mathbb{R}^+$, then right Riemann-Liouville fractional integral of order p is given by

$$I_b^p f(x) = \frac{1}{\Gamma(p)} \int_x^b (\xi - x)^{p-1} f(\xi) d\xi, \tag{1.3.1}$$

for each $x \in [a,b]$ [7].

Lemma 1.3.2. *Let $p \in \mathbb{R}^+$ and $f \in L_1[a,b]$. Then*

$$I_b^p f(x) = \frac{1}{\Gamma(p)} \int_x^b (\xi - x)^{p-1} f(\xi) d\xi, \tag{1.3.2}$$

for each $x \in [a,b]$.

Proof. We could write

$$I_b^1 f(x) = \int_x^b f(\xi) d\xi.$$

The second iterate will then be

$$I_b^2 f(x) = \int_x^b \int_{\xi_2}^b (f(\xi_1) d\xi_1) d\xi_2.$$

Interchanging the order of integration using Fubinis theorem, we have

$$I_b^2 f(x) = \int_x^b \int_x^{\xi_1} f(\xi_1) d\xi_2 d\xi_1.$$

Since f is independent of ξ_2, so we can move it outside the integral

$$I_b^2 f(x) = \int_x^b f(\xi_1) \int_x^{\xi_1} d\xi_2 d\xi_1 = \int_x^b f(\xi_1)(\xi_1 - x) d\xi_1,$$

or

$$I_b^2 f(x) = \int_x^b f(\xi)(\xi - x) d\xi.$$

Similarly third iterate will be of form

$$I_b^3 f(x) = I_b^1(I_b^2 f(x)) = \int_x^b \left(\int_x^{\xi_1} (\xi_2 - x) f(\xi_1) d\xi_2 \right) d\xi_1 = \int_x^b \frac{(\xi_1 - x)^2}{2!} f(\xi_1) d\xi_1$$
$$= \int_x^b \frac{(\xi - x)^{3-1}}{(3-1)!} f(\xi) d\xi.$$

Similarly,

$$I_b^4 f(x) = \int_x^b \frac{(\xi - x)^{4-1}}{(4-1)!} f(\xi) d\xi.$$

So in general for nth order integral we have

$$I_b^n f(x) = \int_x^b \frac{(\xi - x)^{n-1}}{(n-1)!} f(\xi) d\xi.$$

Since $\Gamma(n) = (n-1)!$, and replacing n with $p > 0$, we get

$$I_b^p f(x) = \int_x^b \frac{(\xi - x)^{p-1}}{\Gamma(p)} f(\xi) d\xi.$$

□

Theorem 1.3.3. *Suppose $p \in \mathbb{R}^+$ and $f \in L_1[a,b]$. Then*

$$\lim_{p \longrightarrow 0} I_b^p f(x) = f(x). \tag{1.3.3}$$

Proof. Here arises two cases. If $f(x)$ has continuous derivative for $x \geq 0$, then using definition (1.3.1) and integrating by parts we get

$$\begin{aligned} I_b^p f(x) &= \frac{1}{\Gamma(p)}\left[\frac{(\xi - x)^p}{p} f(\xi)\right]_x^b - \frac{1}{\Gamma(p)}\int_x^b \frac{(\xi - x)^p}{p} f^{'}(\xi)d\xi \\ &= \frac{1}{\Gamma(p+1)}\left[(b-x)^p f(b) - \int_x^b (\xi - x)^p f^{'}(\xi)d\xi\right], \end{aligned}$$

so we obtain by taking limit

$$\lim_{p \longrightarrow 0} I_b^p f(x) = f(b) - \int_x^b f^{'}(\xi)d\xi = f(b) - [f(b) - f(x)] = f(x).$$

Now we consider the case if $f(x)$ has continuous derivative for $x \leq b$. Here we can write right fractional integral operator in the form

$$I_b^p f(x) = \frac{1}{\Gamma(p)}\int_x^b (\xi - x)^{p-1}[f(\xi) - f(x)]d\xi + \frac{1}{\Gamma(p)}\int_x^b (\xi - x)^{p-1} f(x) d\xi \tag{1.3.4}$$

$$= \frac{1}{\Gamma(p)}\int_x^{x+\delta} (\xi - x)^{p-1}[f(\xi) - f(x)] \tag{1.3.5}$$

$$+ \frac{1}{\Gamma(p)}\int_{x+\delta}^b (\xi - x)^{p-1}[f(\xi) - f(x)] \tag{1.3.6}$$

$$+ \frac{f(x)}{\Gamma(p+1)}(b-x)^p. \tag{1.3.7}$$

Let us say integral (1.3.5) is I_1. Since $f(x)$ is continuous for every $\delta > 0$ there exist $\epsilon > 0$ such that

$$|f(\xi) - f(x)| < \epsilon.$$

Then we can estimate integral (1.3.5) as follow

$$I_1 < \frac{\epsilon}{\Gamma(p)}\int_x^{x+\delta} (\xi - x)^{p-1} d\xi = \frac{\epsilon \delta^p}{\Gamma(p)},$$

for an arbitrary $\epsilon > 0$, choose $\delta > 0$ such that

$$|I_1| < \epsilon, \quad \text{for all } p \geq 0.$$

Now let us take $\epsilon \longrightarrow 0$ as $\delta \longrightarrow 0$ for all $p \geq 0$

$$\lim_{\delta \longrightarrow 0} |I_1| = 0. \tag{1.3.8}$$

Let for a fixed $\delta > 0$, $M = \max|f(\xi) - f(x)|$. Then we can estimate the integral (1.3.6) as

$$I_2 \leq \frac{M}{\Gamma(p)}\int_{x+\delta}^b (\xi - x)^{p-1} d\xi = \frac{M}{\Gamma(p)}[(b-x)^p - \delta^p],$$

obviously for $\delta > 0$ we have

$$\lim_{p \longrightarrow 0} |I_2| = 0 \tag{1.3.9}$$

Now consider

$$|I_b^p f(x) - f(x)| \leq |I_1| + |I_2| + |f(x)| \left| \frac{(b-x)^p}{\Gamma(p+1)} - 1 \right|,$$

and using limits (1.3.8) and (1.3.9) we can write

$$\lim_{p \longrightarrow 0} |I_b^p f(x) - f(x)| \leq \epsilon.$$

Thus expression (1.3.3) holds for $x \leq b$. $\square$

Theorem 1.3.4. [7] *Suppose $p \in \mathbb{R}^+$ and $g \in L_1[a,b]$. Then, the right Riemann-Liouville fractional integral $I_b^p g(x)$ exists almost everywhere on the interval $[a,b]$. Moreover, the function $I_b^p g$ itself is also an element of $L_1[a,b]$.*

Proof. Let $\zeta := [a,b] \times [a,b]$, consider a function $\chi : \zeta \longrightarrow \mathbb{R}$ defined as $\chi(\tau, x) = (\tau - x)^{p-1}$, that is

$$\chi(\tau, x) = \begin{cases} (\tau - x)^{p-1}, & if \quad a \leq x \leq \tau \leq b, \\ 0, & if \quad a \leq \tau \leq x \leq b. \end{cases}$$

Then χ is measureable. We have to prove that $\int_a^b \chi(\tau, x) g(\tau) d\tau$ is integrable, i.e

$$\int_a^b \chi(\tau, x) |g(\tau)| d\tau < \infty.$$

We can define Lesbesgue integral for measureable function as

$$\int_a^b \chi(\tau, x) dx = \int_a^\tau \chi(\tau, x) dx + \int_\tau^b \chi(\tau, x) dx = \int_a^\tau (\tau - x)^{p-1} dx = \frac{(\tau - x)^p}{p}$$

Now

$$\begin{aligned} \int_a^b \left(\int_a^b \chi(\tau, x) |g(\tau)| dx \right) d\tau = \int_a^b |g(\tau)| \left(\int_a^b \chi(\tau, x) dx \right) d\tau = \int_a^b |g(\tau)| \frac{(\tau - x)^p}{p} d\tau \\ \leq \frac{(b-a)^p}{p} \int_a^b |g(\tau)| d\tau = \frac{(b-a)^p}{p} \|g(\tau)\|_{L_1[a,b]} < \infty. \end{aligned}$$

Therefore $\chi(\tau, x) g(\tau)$ is integrable on ζ by Tonelli's theorem. By Fubini's theorem $\int_a^b \chi(\tau, x) g(\tau) d\xi$ is integrable on [a,b]. So

$$I_b^p g(x) = \frac{1}{\Gamma(p)} \int_x^b (\tau - x)^{p-1} g(\tau) d\tau$$

exist on [a,b]. $\square$

Theorem 1.3.5. *Suppose $g \in L_1[a,b]$, then right Riemann-Liouville fractional integral operator $I_b^p g$ is continuous for $p \geq 1$.*

Proof. Let $\zeta := [a,b] \times [a,b]$, and $\chi(\tau, x) = (\tau - x)^{p-1}$ be continuous on ζ. Then define

$$I_b^p g(x) = \frac{1}{\Gamma(p)} \int_x^b \chi(\tau, x) g(\tau) d\tau.$$

We have to show that $|I_b^p g(x)| \leq M \|g(x)\|$ for all $g \in C[a,b]$.

Let there exist $h > 0$, then

$$\begin{aligned}
|I_b^p g(x+h) - I_b^p g(x)| &= \frac{1}{\Gamma(p)} \left| \int_{x+h}^b \chi(\tau, x+h) g(\tau) d\tau - \int_x^b \chi(\tau, x) g(\tau) d\tau \right| \\
&= \frac{1}{\Gamma(p)} \left| \int_{x+h}^b \chi(\tau, x+h) g(\tau) d\tau - \int_x^{x+h} \chi(\tau, x) g(\tau) d\tau - \int_{x+h}^b \chi(\tau, x) g(\tau) d\tau \right| \\
&\leq \frac{1}{\Gamma(p)} \left[\int_{x+h}^b |\chi(\tau, x+h) - \chi(\tau, x)| \, |g(\tau)| \, d\tau + \int_x^{x+h} \chi(\tau, x) \, |g(\tau)| \, d\tau \right] \\
&\leq \frac{1}{\Gamma(p)} \left[\int_{x+h}^b |\chi(\tau, x+h) - \chi(\tau, x)| \, |g(\tau)| \, d\tau + h^{p-1} \|g(\tau)\|_{L_1[a,b]} \right].
\end{aligned}$$

As $h \longrightarrow 0$ we get $\chi(\tau, x+h) \longrightarrow \chi(\tau, x)$, thus

$$|\chi(\tau, x+h) - \chi(\tau, x)| \longrightarrow 0,$$

also

$$|\chi(\tau, x+h) - \chi(\tau, x)| \leq |2| \|\chi\|.$$

Thus

$$|\chi(\tau, x+h) - \chi(\tau, x)||g(\tau)| \leq |2| \|\chi\| \|g(\tau)\| \in L_1[a,b],$$

and also $|\chi(\tau, x+h) - \chi(\tau, x)||g(\tau)| \longrightarrow 0$ as $h \longrightarrow 0$, almost for all $\tau \in [a,b]$.

We conclude that as $h \longrightarrow 0$, then

$$\int_x^b |\chi(\tau, x+h) - \chi(\tau, x)||g(\tau)| d\xi \longrightarrow 0,$$

by Dominated Convergence Theorem. So, $|I_b^p g(x+h) - I_b^p g(x)| \longrightarrow 0$ as $h \longrightarrow 0$, also $|I_b^p g(x)| \leq \|\chi\| \|g(x)\|$.

Therefor $I_b^p g$ is a continuous function. □

Now we have semigroup property of right fractional integral operator.

Theorem 1.3.6. [7] *Suppose $f \in L_1[a,b]$ and $p, q \in \mathbb{R}^+$. Then for interval $J = [a,b]$*

$$I_b^p I_b^q f = I_b^{p+q} f = I_b^q I_b^p f, \tag{1.3.10}$$

valid almost everywhere on J. Moreover the identity (1.3.10) *is true everywhere on J if $f \in C[a,b]$ or $p + q \geq 1$.*

Proof. If $p, q = 0$, then statement is trivial. Consider the case $p, q > 0$.

We have

$$I_b^p I_b^q f(x) = \frac{1}{\Gamma(p)\Gamma(q)} \int_x^b (\tau - x)^{p-1} \left(\int_\tau^b (\xi - \tau)^{q-1} f(\xi) d\xi \right) d\tau.$$

The integral exists from Theorem 1.3.4, and we may interchange order of integration by Fubini's theorem to obtain

$$\begin{aligned} I_b^p I_b^q f(x) &= \frac{1}{\Gamma(p)\Gamma(q)} \int_x^b \left(\int_x^\xi (\tau - x)^{p-1} (\xi - \tau)^{q-1} f(\xi) d\tau \right) d\xi \\ &= \frac{1}{\Gamma(p)\Gamma(q)} \int_x^b f(\xi) \left(\int_x^\xi (\tau - x)^{p-1} (\xi - \tau)^{q-1} d\tau \right) d\xi. \end{aligned}$$

Substitute $\tau = s + x$ to get

$$\begin{aligned} I_b^p I_b^q f(x) &= \frac{1}{\Gamma(p)\Gamma(q)} \int_x^b f(\xi) \left(\int_0^{\xi - x} (s)^{p-1} ((\xi - x) - s)^{q-1} ds \right) d\xi \\ &= \frac{1}{\Gamma(p)\Gamma(q)} \int_x^b f(\xi) \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} (\xi - x)^{p+q-1} d\xi \\ &= \frac{1}{\Gamma(p+q)} \int_x^b f(\xi)(\xi - x)^{p+q-1} d\xi = I_b^{p+q} f(x). \end{aligned}$$

So

$$I_b^p I_b^q f(x) = I_b^{p+q} f(x), \tag{1.3.11}$$

almost everywhere on [a,b]. Moreover if f is continuous, then $I_b^p f$ is also continuous and we have that $I_b^{p+q} f \in C[a, b]$. Since the functions on either side of equation (1.3.11) are continuous almost everywhere, they must be equal everywhere.

As we have assumed $f \in L_1[a, b]$, now if additionally $p + q \geq 1$, then from equation (1.3.11) we have

$$I_b^p I_b^q f(x) = I_b^{p+q} f(x) = I_b^{p+q-1} I_b^1.$$

Since I_b^1 is a continuous function, therefore $I_b^{p+q} f(x) = I_b^{p+q-1} I_b^1$ is also continuous. And once again we may conclude that either side of equality coincides almost everywhere, hence they must be identical everywhere. □

Theorem 1.3.7. [12] *The operators $\{I_b^p : L_1[a, b] \longrightarrow L_1[a, b]; p \geq 0\}$ form a commutative semigroup with respect to concatenation. The identity operator I_b^0 is the neutral element of this semigroup.*

Example 1.3.8. Consider $f(x) = (b - x)^\gamma$ and $\gamma > -1$. Then for $p \in \mathbb{R}^+$ we have

$$I_b^p f(x) = \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + p + 1)} (b - x)^{\gamma + p}.$$

From the definition of right fractional integral we have

$$I_b^p f(x) = \frac{1}{\Gamma(p)} \int_x^b (\xi - x)^{p-1} (b - \xi)^\gamma d\xi.$$

Substitute $\xi = b - s(b - x)$

$$\begin{aligned} I_b^p f(x) &= \frac{1}{\Gamma(p)} \int_0^1 ((b - x)(1 - s))^{p-1} (s(b - x))^\gamma (b - x) ds \\ &= \frac{(b - x)^{\gamma + p}}{\Gamma(p)} \int_0^1 (1 - s)^{p-1} (s)^\gamma ds \\ &= \frac{(b - x)^{\gamma + p}}{\Gamma(p)} \times \frac{\Gamma(p)\Gamma(\gamma + 1)}{\Gamma(\gamma + p + 1)} \\ &= \frac{\Gamma(\gamma + 1)}{\Gamma(\gamma + p + 1)} (b - x)^{\gamma + p}. \end{aligned}$$

Which is required result.

1.3.2 Right Riemann-Liouville fractional derivative

After giving a brief introduction of right Riemann-Liouville fractional integral operator we establish integro-differential operator known as fractional derivative. Here definition and important properties of right Riemann-Liouville fractional derivative are given.

Definition 1.3.9. Let $p \in \mathbb{R}^+$ and $m = \lceil p \rceil$. Then the right Riemann-Liouville fractional differential operator D_b^p of order p is defined by

$$D_b^p f(x) = (-1)^m D^m I_b^{m-p} f(x), \tag{1.3.12}$$

for $m - 1 < p \leq m$. It can also be written as

$$D_b^p f(x) = \frac{(-1)^m}{\Gamma(m-p)} \frac{d^m}{dx^m} \int_x^b (\xi - x)^{m-p-1} f(\xi) d\xi. \tag{1.3.13}$$

For $p = 0$, $D_b^0 = I$,is the identity operator.

Lemma 1.3.10. *Suppose $f \in L_1[a, b]$ and $n \in \mathbb{N}$. Then*

$$D^n I_b^n f(x) = (-1)^n f(x), \tag{1.3.14}$$

for each $x \in [a, b]$.

Proof. Assume Riemann-Liouville integral

$$I_b^n f(x) = \frac{1}{\Gamma(n)} \int_x^b (\xi - x)^{n-1} f(\xi) d\xi.$$

Apply D^n on both sides,

$$D^n I_b^n f(x) = \frac{1}{\Gamma(n)} \frac{d^n}{dx^n} \int_x^b (\xi - x)^{n-1} f(\xi) d\xi.$$

Applying Lebniz rule we get

$$\begin{aligned} D^n I_b^n f(x) &= \frac{1}{\Gamma(n)} \frac{d}{dx} \left[(-1)^{n-1} \Gamma(n) \int_x^b f(\xi) d\xi \right] \\ &= (-1)^{n-1} \frac{d}{dx} \int_x^b f(\xi) d\xi \\ &= (-1)^n f(x). \end{aligned}$$

Hence proved. □

Lemma 1.3.11. *Let $p \in \mathbb{R}^+$ and $m \in \mathbb{N}$ such that $m > p$. Then*

$$D_b^p = (-1)^m D^m I_b^{m-p}.$$

Proof. Since $m \geq \lceil p \rceil$ from our assumption. Thus from Theorem 1.3.6

$$D^m I_b^{m-p} = D^{m+\lceil p \rceil - \lceil p \rceil} I_b^{m-p+\lceil p \rceil - \lceil p \rceil} = D^{\lceil p \rceil} D_b^{m-\lceil p \rceil} I^{m-\lceil p \rceil} I_b^{\lceil p \rceil - p}$$

From Lemma 1.3.10 we can write $D_b^{m-\lceil p \rceil} I^{m-\lceil p \rceil} = (-1)^{m-\lceil p \rceil} I,$. Thus

$$D^m I_b^{m-p} = (-1)^{m-\lceil p \rceil} D^{\lceil p \rceil} I_b^{\lceil p \rceil - p} = (-1)^{m-\lceil p \rceil} D^{\lceil p \rceil} I_b^{\lceil p \rceil} I_b^{-p}$$

Again we have $D^{\lceil p \rceil} I_b^{\lceil p \rceil} = (-1)^{\lceil p \rceil} I$, and also $I_b^{-p} = D_b^p$, so

$$D^m I_b^{m-p} = (-1)^{m-\lceil p \rceil} (-1)^{\lceil p \rceil} D_b^p = (-1)^m D_b^p.$$

Which is required result.

Definition 1.3.12. Let us consider an interval $\Lambda \subset \mathbb{R}$ such that $b \in \Lambda$, $x \leq b$ for all $x \in \Lambda$. Then for $p \in \mathbb{R}^+$ following sets of functions are defined

$$_bI_p := \{f \in C(\Lambda); I_b^p \text{ exists and is finite in } \Lambda\},$$

$$_bD_p := \{f \in C(\Lambda); D_b^p \text{ exists and is finite in } \Lambda\}.$$

□

Proposition 1.3.13. [10] Suppose $p, q \in \mathbb{R}^+$ and $f \in {}_bI_q([a,b])$ such that $I_b^q f \in {}_bD_q([a,b])$ and $f \in C[a,b]$. Then for all $x \in [a,b]$

(a) if $p \geq q$

$$D_b^p I_b^q f(x) = D_b^{p-q} f(x),$$

(b) if $p < q$

$$D_b^p I_b^q f(x) = I_b^{q-p} f(x).$$

Proof. (a) Let $m = \lceil p \rceil$, then from the Definition 1.3.9

$$\begin{aligned} D_b^p I_b^q f(x) &= (-1)^m D^m I_b^{m-p} I_b^q f(x) \\ &= (-1)^m D^m I_b^{m-(p-q)} f(x) \\ &= D_b^{p-q} f(x). \end{aligned}$$

(b) Let $m = \lceil p \rceil$. then from the Definition 1.3.9 and Lemma 1.3.10

$$\begin{aligned} D_b^p I_b^q f(x) &= (-1)^m D^m I_b^{m-p} I_b^q f(x) \\ &= (-1)^m D^m I_b^m I_b^{q-p} f(x) \\ &= I_b^{q-p} f(x). \end{aligned}$$

□

Remark 1.3.14. If $p = q$, then from proposition 1.3.13 we can write

$$D_b^p I_b^p f(x) = f(x).$$

Example 1.3.15. Consider $f(x) = (b - x)^\gamma$ and $\gamma > -1$. Then for $p \in \mathbb{R}^+$ we have

$$D_b^p f(x) = \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-p+1)}(b-x)^{\gamma-p}.$$

From the definition of right fractional integral we have

$$\begin{aligned} D_b^p f(x) &= \frac{(-1)^m}{\Gamma(m-p)}\left(\frac{d}{dx}\right)^m \int_x^b (\xi - x)^{m-p-1} f(\xi) d\xi \\ &= \frac{(-1)^m}{\Gamma(m-p)}\left(\frac{d}{dx}\right)^m \int_x^b (\xi - x)^{m-p-1} (b-\xi)^\gamma d\xi \\ &= \frac{(-1)^m}{\Gamma(m-p)}\left(\frac{d}{dx}\right)^m \frac{\Gamma(\gamma+1)}{\Gamma(\gamma+m-p+1)}(b-x)^{\gamma+m-p} \\ &= \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-p+1)}(b-x)^{\gamma-p}. \end{aligned}$$

Which is the required result.

Next we explain Taylor's expansion with Riemann-Liouville integral remainder.

Theorem 1.3.16. *Let $g \in AC^m[a,b]$. Then for every $x \in [a,b]$*

$$g(x) = \sum_{k=0}^{m-1} \frac{(-1)^k g^{(k)}(b)}{\Gamma(k+1)}(b-x)^k + \frac{(-1)^m}{\Gamma(m)} \int_x^b (\xi - x)^{m-1} g^{(m)}(\xi) d\xi. \tag{1.3.15}$$

Proof. From given conditions we can write

$$\int_x^b g'(\xi) d\xi = g(b) - g(x), \tag{1.3.16}$$

$$g(x) = g(b) - \int_x^b g'(\xi) d\xi.$$

Now integrating by parts and using equation (1.3.16) we have

$$\begin{aligned} \int_x^b (\xi - x) g''(\xi) d\xi &= \left[(\xi - x) g'(\xi)\right]_x^b - \int_x^b g'(\xi) d\xi \\ &= (b-x) g'(b) - g(b) + g(x), \end{aligned} \tag{1.3.17}$$

.

$$g(x) = g(b) - (b-x) g'(b) + \int_x^b (\xi - x) g''(\xi) d\xi.$$

Repeat this process of integration by parts to get

$$\begin{aligned} \int_x^b (\xi - x)^2 g'''(\xi) d\xi &= \frac{\left[(\xi - x)^2 g'(\xi)\right]_x^b}{2} - \int_x^b (\xi - x) g''(\xi) d\xi \\ &= \frac{(b-x)^2}{2} g'(b) - (b-x) g'(b) + g(b) - g(x), \end{aligned} \tag{1.3.18}$$

$$g(x) = g(b) - (b-x)g'(b) + \frac{(b-x)^2}{2}g''(b) - \int_x^b \frac{(\xi - x)^2}{2} g'''(\xi)d\xi.$$

Similarly

$$g(x) = g(b) - (b-x)g'(b) + \frac{(b-x)^2}{2!}g''(b) - \frac{(b-x)^3}{3!}g'''(b) + \int_x^b \frac{(b-x)^3}{3!} g^{iv}(\xi)d\xi.$$

Reapeating this process for m-times we get required result. ☐

To present next theorem, we define the space $I_b^p(L_{\widetilde{p}})$ for $p \in \mathbb{R}^+$ and $1 \leq \widetilde{p} < \infty$ [16]

$$I_b^p(L_{\widetilde{p}}) := \{g(x) | g(x) = I_b^p \phi(x), \phi \in L_{\widetilde{p}}(a,b)\}.$$

Theorem 1.3.17. [16] *Consider a function $g(x)$ such that $g(x) \in AC^m[a,b]$. Let $p \in \mathbb{R}^+$ and $m = \lceil p \rceil$. Then for every $x \in [a,b]$:*

(a) if $g(x) \in I_b^p(L_p)$, then we have

$$I_b^p D_b^p g(x) = g(x), \tag{1.3.19}$$

(b) if $I_b^{m-p} g(x) \in AC^m[a,b]$, then we have

$$I_b^p D_b^p g(x) = g(x) - \sum_{k=0}^{m-1} \frac{(-1)^{m-k-1}(b-x)^{p-k-1}}{\Gamma(p-1)} \lim_{z \longrightarrow b^-} D^{m-k-1} I_b^{m-p} g(z). \tag{1.3.20}$$

Specificaly, for $0 < p < 1$ we can write

$$I_b^p D_b^p g(x) = g(x) - \frac{(b-x)^{p-1}}{\Gamma(p)} \lim_{z \longrightarrow b^-} I_b^{1-p} g(z).$$

Proof. (a) From our assumption if $g(x) \in I_b^p(L_p)$ then we have $g(x) = I_b^p \phi(x)$. Using Definition (1.3.9) and Remark (1.3.14) we can write

$$\begin{aligned} I_b^p D_b^p g(x) &= I_b^p D_b^p I_b^p \phi(x) \\ &= I_b^p \phi(x) \\ &= g(x). \end{aligned}$$

(b) Consider the function $g(x) \in AC^m[a,b]$, from Theorem 1.3.16 we have

$$g(x) = \sum_{k=0}^{m-1} \frac{(-1)^k g^{(k)}(b)}{\Gamma(k+1)}(b-x)^k + \frac{(-1)^m}{\Gamma(m)} \int_x^b (t-x)^{m-1} g^{(m)}(t)dt. \tag{1.3.21}$$

Which gives

$$(-1)^m I_b^m D^m g(x) = g(x) - \sum_{k=0}^{m-1} \frac{(-1)^k g^{(k)}(b)}{\Gamma(k+1)}(b-x)^k. \tag{1.3.22}$$

Since $I_b^{m-p} g \in AC^m[a,b]$, by definition we can write

$$D^{(m-1)} I_b^{(m-p)} g(x) = D^{(m-1)} I_b^{(m-p)} g(b) - I_b^1 \phi(x) = -I_b^1 \phi(x), \tag{1.3.23}$$

where $\phi \in L_1$ space, and $\phi(x) = g^n(x)$.

Apply $(-1)^m I_b^{(m-1)}$ on both sides of equation (1.3.23), we get

$$(-1)^m I_b^{(m-1)} D^{(m-1)} I_b^{(m-p)} g(x) = -(-1)^m I_b^m \phi(x),$$

and using (1.3.22) we have

$$I_b^{m-p} g(x) = \sum_{k=0}^{m-1} \frac{(-1)^k (b-x)^k}{\Gamma(k+1)} \lim_{z \longrightarrow b^-} D^k I_b^{m-p} g(z) + (-1)^m I_b^m \phi(x). \tag{1.3.24}$$

Applying differential operator D_b^{m-p} on both sides of equation (1.3.24) we get

$$g(x) = \sum_{k=0}^{m-1} \frac{(-1)^k D_b^{m-p} (b-x)^k}{\Gamma(k+1)} \lim_{z \longrightarrow b^-} D^k I_b^{m-p} g(z) + (-1)^m D_b^{m-p} I_b^m \phi(x).$$

Where $D_b^{m-p}(b-x)^k = \frac{\Gamma(k+1)(b-x)^{k+p-m}}{\Gamma(k+p-m+1)}$. Using semigroup property $D_b^{m-p} I_b^m \phi(x) = (-1)^m I_b^p \phi(x)$ we will get

$$g(x) = \sum_{k=0}^{m-1} \frac{(-1)^k (b-x)^{k+p-m}}{\Gamma(k+p-m+1)} \lim_{z \longrightarrow b^-} D^k I_b^{m-n} g(z) + I_b^p \phi(x).$$

Which can be written as

$$I_b^p \phi(x) = g(x) - \sum_{k=0}^{m-1} \frac{(-1)^k (b-x)^{k+p-m}}{\Gamma(k+p-m+1)} \lim_{z \longrightarrow b^-} D^k I_b^{m-p} g(z).$$

Now consider left side of equation (1.3.20), using equation (1.3.24) and definition of differential operator D_b^p, we get

$$\begin{aligned} I_b^p D_b^p g(x) &= (-1)^m I_b^p D^m I_b^{m-p} g(x) \\ &= (-1)^m I_b^p D^m [\sum_{k=0}^{m-1} \frac{(-1)^k (b-x)^k}{\Gamma(k+1)} \lim_{z \longrightarrow b^-} D^k I_b^{m-p} g(z) + (-1)^m I_b^m \phi(x)] \\ &= \sum_{k=0}^{m-1} \frac{(-1)^k (-1)^m I_b^p D^m (b-x)^k}{\Gamma(k+1)} \lim_{z \longrightarrow b^-} D^k I_b^{m-p} g(z) + (-1)^{2m} I_b^p D^m I_b^m \phi(x) \\ &= I_b^p \phi(x). \end{aligned}$$

Which gives

$$I_b^p D_b^p g(x) = g(x) - \sum_{k=0}^{m-1} \frac{(-1)^k (b-x)^{k+p-m}}{\Gamma(k+p-m+1)} \lim_{z \longrightarrow b^-} D^k I_b^{m-p} g(z).$$

Replacing k by $m-k-1$, we get

$$I_b^p D_b^p g(x) = g(x) - \sum_{k=0}^{m-1} \frac{(-1)^{m-k-1} (b-x)^{p-k-1}}{\Gamma(p-1)} \lim_{z \longrightarrow b^-} D^{m-k-1} I_b^{m-p} g(z).$$

Which is required result.

□

1.4 Right Caputo fractional derivative

Riemann-Liouville fractional derivative played significant role in fractional calculus, however it has certain disadvantages while dealing with real-world situations. The Riemann-Liouville derivative of a constant function is not zero. Moreover, if a function is constant at the origin, its Riemann-Liouville fractional derivation has a singularity at the origin. Therefore the field of application of the Riemann-Liouville fractional derivative has reduced to some extent. Caputo's fractional derivative demands stronger conditions to compute the fractional derivative of a function. It is defined only for differentiable functions. On contrary to Riemann-Liouville fractional derivative for Caputo's fractional derivative, we must calculate its derivative first. Therefore we can say Caputo fractional derivative is most important modified form of a fractional derivative.

Definition 1.4.1. Suppose $f^{(m)}(x) \in L_1[a,b]$, $p \in \mathbb{R}^+$ and $m = \lceil p \rceil$. Then

$$ {}^cD_b^p f(x) = (-1)^m I_b^{m-p} f^{(m)}(x), \tag{1.4.1}$$

is right Caputo fractional derivative ${}^cD_b^p$.
It can also be written as

$$ {}^cD_b^p = \frac{(-1)^m}{\Gamma(m-p)} \int_x^b (\xi - x)^{m-p-1} f^{(m)}(\xi) d\xi. \tag{1.4.2}$$

Example 1.4.2. Consider $f(x) = (b-x)^\gamma$ and $\gamma \in \mathbb{R}^+$. Then,

$$ {}^cD_b^p f(x) = \begin{cases} 0, \quad \gamma \in \{0, 1, 2, \ldots, m-1\}, \\ \frac{\Gamma(\gamma+1)}{\Gamma(\gamma-p+1)}(b-x)^{\gamma-p}, \quad \gamma \in \mathbb{N}, \gamma \geq m, \\ \qquad \qquad \textit{or} \quad \gamma \notin \mathbb{N} \quad \textit{and} \quad \gamma > m-1. \end{cases}$$

Proposition 1.4.3. *Let $p \in \mathbb{R}^+$, $m = \lceil p \rceil$ and $f(x)$ is such that ${}^cD_b^p f(x)$ exists. Then*

$$ \lim_{p \longrightarrow m} {}^cD_b^p f(x) = (-1)^m f^{(m)}(x).$$

Proof. Considering equation (1.4.2) and performing the integration by parts, we get

$$\begin{aligned} {}^cD_b^p f(x) &= (-1)^m \left(\frac{\left[(\xi - x)^{m-p} f^{(m)}(\xi)\right]_x^b}{\Gamma(m-p+1)} - \frac{1}{\Gamma(m-p+1)} \int_x^b (\xi - x)^{m-p} f^{(m+1)}(\xi) d\xi \right) \\ &= \frac{(-1)^m}{\Gamma(m-p+1)} \left((b-x)^{m-p} f^{(m)}(b) - \int_x^b (\xi - x)^{m-p} f^{(m+1)}(\xi) d\xi \right). \end{aligned}$$

Now, by taking limit $p \longrightarrow m$, we get

$$\begin{aligned} \lim_{p \longrightarrow m} {}^cD_b^p f(x) &= \lim_{p \longrightarrow m} \frac{(-1)^m}{\Gamma(m-p+1)} \left((b-x)^{m-p} f^{(m)}(b) - \int_x^b (\xi - x)^{m-p} f^{(m+1)}(\xi) d\xi \right) \\ &= (-1)^m \left(f^{(m)}(b) - \left[f^{(m)}(\xi)\right]_x^b \right) \\ &= (-1)^m f^{(m)}(x). \end{aligned}$$

□

Theorem 1.4.4. [7] *Let $p \in \mathbb{R}^+$ and $m = \lceil p \rceil$. Then the following relation between right Riemann-Liouville and right Caputo derivative holds*

$$^cD_b^p f(x) = D_b^p f(x) - \sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i-p+1)}(b-x)^{i-p}.$$

Proof. Let us consider Taylor series expansion given in Theorem 1.3.16

$$f(x) = \sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i+1)}(b-x)^i + \frac{(-1)^m}{\Gamma(m)} \int_x^b (\xi - x)^{m-1} f^{(m)}(\xi) d\xi,$$

where

$$\frac{(-1)^m}{\Gamma(m)} \int_x^b (\xi - x)^{m-1} f^{(m)}(\xi) d\xi = (-1)^m I_b^m f^{(m)}(x)$$

Now, applying right Riemann-Liouville fractional derivative on both sides and using example (1.3.15)

$$\begin{aligned} D_b^p f(x) &= D_b^p \Big(\sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i+1)}(b-x)^i + (-1)^m I_b^m f^{(m)}(x) \Big) \\ &= \sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i-p+1)}(b-x)^{i-p} + (-1)^m D_b^p I_b^m f^{(m)}(x) \\ &= \sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i-p+1)}(b-x)^{i-p} + (-1)^m I_b^{m-p} f^{(m)}(x) \\ &= \sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i-p+1)}(b-x)^{i-p} +^c D_b^p f(x). \end{aligned}$$

After rearranging we get

$$^cD_b^p f(x) = D_b^p f(x) - \sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i-p+1)}(b-x)^{i-p}.$$

□

Remark 1.4.5. If $f^{(i)}(b) = 0, k = 0, 1, 2, ...m-1$, then right Riemann-Liouville and right Caputo fractional derivatives are equivalent. i.e

$$^cD_b^p f(x) = D_b^p f(x).$$

Proposition 1.4.6. [7] *Let $f \in C[a, b]$, then for every $x \in [a, b]$:*

(a) if $p, q \in \mathbb{R}^+$ such that $\lceil p \rceil < q$, then

$$^cD_b^p I_b^q f(x) = I_b^{q-p} f(x),$$

(b) if $f \in AC^{m-n}[a, b], p \in \mathbb{R}^+$ and $n \in \mathbb{N}$ such that $n \leq m - 1 < p \leq m$, then

$$^cD_b^p I_b^n f(x) = {}^cD_b^{p-n} f(x).$$

Proof. (a) Let $m = \lceil p \rceil$. Then from definition of Caputo fractional derivative and Theorem 1.3.6, we have

$$\begin{aligned} {}^cD_b^p I_b^q f(x) &= (-1)^m I_b^{m-p} D^m I_b^q f(x) \\ &= (-1)^{2m} I_b^{m-p} I_b^{q-m} f(x) \\ &= I_b^{q-p} f(x). \end{aligned}$$

(b) Let $m = \lceil p \rceil$. Then from definition of Caputo fractional derivative and Lemma 1.3.10, we have

$$\begin{aligned} {}^cD_b^p I_b^n f(x) &= (-1)^m I_b^{m-p} D^m I_b^n f(x) \\ &= (-1)^{m+n} I_b^{m-p} I_b^{m-n} f(x) \\ &= (-1)^{m+n} I_b^{(m-n)-(p-n)} f(x)^{(m-n)} \\ &= (-1)^{m+n}(-1)^{m-n}\ {}^cD_b^{p-n} f(x) = {}^cD_b^{p-n} f(x). \end{aligned}$$

□

Remark 1.4.7. Let $f \in C[a,b]$ and $p = q$, then from Proposition 1.4.6-(a) we have

$${}^cD_b^p I_b^p f(x) = f(x).$$

Theorem 1.4.8. *Suppose $p \in \mathbb{R}^+$ and $f(y) \in AC^m[a,b]$. Then for $m = \lceil p \rceil$*

$$I_b^p\ {}^cD_b^p f(y) = f(y) - \sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i+1)}(b-y)^i. \tag{1.4.3}$$

Particularly, for $0 < p \leq 1$ we have

$$I_b^p\ {}^cD_b^p f(y) = f(y) - f(b).$$

Proof. From definition of Caputo fractional derivative and semigroup property

$$\begin{aligned} I_b^p\ {}^cD_b^p f(y) &= (-1)^m I_b^p I_b^{m-p} f^{(m)}(y) \\ &= (-1)^m I_b^{m-p+p} f^{(m)}(y) \\ &= (-1)^m I_b^m f^{(m)}(y). \end{aligned}$$

Now, from Theorem 1.3.16 we have

$$f(y) = \sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i+1)}(b-y)^i + (-1)^m I_b^m f^{(m)}(y).$$

So

$$\begin{aligned} I_b^p\ {}^cD_b^p f(y) &= (-1)^{2m}\left(f(y) - \sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i+1)}(b-y)^i\right) \\ &= f(y) - \sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i+1)}(b-y)^i. \end{aligned}$$

□

Corollary 1.4.9. *Under the assumptions of Theorem 1.4.8 we can write*

$$f(y) = \sum_{i=0}^{m-1} \frac{(-1)^i f^{(i)}(b)}{\Gamma(i+1)}(b-y)^i + I_b^p \, {}^cD_b^p f(y),$$

which is Taylor expansion for the right Caputo fractional derivative.

1.5 Results from analysis

Some important results and definitions are given in this section. Which are used to construct further results.

Theorem 1.5.1. ***(Fubini's theorem for integrable functions)***
Let X and Y are complete measure spaces and $R = X \times Y$. If $f(x,y)$ is measureable and

$$\int_R |f(x,y)| d(x,y) < \infty,$$

then

$$\int_X \left(\int_Y f(x,y) dy \right) dx = \int_Y \left(\int_X f(x,y) dx \right) dy = \int_R f(x,y) d(x,y).$$

The two iterated integrals may actually have different values when the absolute value of function does not have finite integral.

Theorem 1.5.2. ***(Tonelli's theorem for non-negative functions)***
Suppose $f(x,y)$ is Lebesgue measurable on a rectangle $R = X \times Y$. If $f \geq 0$, then

$$\int_X \left(\int_Y f(x,y) dy \right) dx = \int_Y \left(\int_X f(x,y) dx \right) dy = \int_R f(x,y) d(x,y).$$

Outcome of Tonelli's theorem is identic to Fubini's theorem, but the assumption that $|f|$ has a finite integral is replaced by the assumption that f is non-negative.

Theorem 1.5.3. ***(Dominated convergence theorem)***
[15] Suppose $f_n : \mathbb{R} \longrightarrow \mathbb{R}$ are Lebesgue measurable functions and $f_n \longrightarrow f$ pointwise almost everywhere as $n \longrightarrow \infty$. If $g \geq 0$ is integrable function such that $|f_n| \leq g$ for all $n \in \mathbb{R}^+$. Then f is integrable and

$$\int_{\mathbb{R}} f d\mu = \lim_{n \longrightarrow \infty} \int_{\mathbb{R}} f_n d\mu.$$

Theorem 1.5.4. ***(Leibniz integral rule)***
Suppose the function $f(x,\xi)$ and $\frac{\partial f(x,\xi)}{\partial x}$ are continuous in some region of the (x,ξ)-plane, for all $u(x) \leq \xi \leq v(x)$. Also $u(x)$, $v(x)$ and their derivatives are continuous for $x_0 \leq x \leq x_1$. Then

$$\frac{d}{dx} \int_{u(x)}^{v(x)} f(x,\xi) d\xi = \int_{u(x)}^{v(x)} \frac{\partial f(x,\xi)}{\partial x} d\xi + f(x,v(x)) \frac{d}{dx} v(x) - f(x,u(x)) \frac{d}{dx} u(x).$$

Definition 1.5.5. A function $f : [a,b] \longrightarrow \mathbb{R}$ is absolutely continuous iff

$$f^{(n-1)}(x) = f^{(n-1)}(b) + \int_x^b f^{(n)}(t) dt,$$

for $f^{(n)} \in L_1[a,b]$. The collection of all absolutely continuous functions on given interval is denoted by $AC^n[a,b]$.

Definition 1.5.6. Let $p, q > 0$. The function $E_{p,q}$ defined by the following convergent series

$$E_{p,q}(x) := \sum_{i=0}^{\infty} \frac{x^i}{\Gamma(ip+q)},$$

depending upon two parameters p and q is called the two-parameter Mittag-Leffler function.

Definition 1.5.7. [15] Suppose $\mathcal{X}$ is measure space and $f : \mathcal{X} \longrightarrow \mathbb{R}$ be a measurable function. Then the $L_{\widetilde{p}}(\mathcal{X})$ space consists of equivalence classes of measurable functions such that for $1 \leq \widetilde{p} < \infty$

$$\int |f|^{\widetilde{p}} d\mu < \infty.$$

The $L_{\widetilde{p}}$ norm of $f \in L_{\widetilde{p}}(\mathcal{X})$ is defined by

$$\|f\|_{\widetilde{p}} = \left(\int |f|^{\widetilde{p}} d\mu \right)^{1/\widetilde{p}}.$$

Definition 1.5.8. [18] An element $x \in \mathcal{X}$ is a fixed point of map $T : \mathcal{X} \longrightarrow \mathcal{X}$, which is kept fixed by T, such that

$$Tx = x.$$

The image Tx coincides with x.

Example 1.5.9. The fixed points of mapping $T : \mathbb{R} \longrightarrow \mathbb{R}^+$ defined by $Tx = x^2$ are $x = 0$ and $x = 1$.

Definition 1.5.10. Suppose B is Banach space and $\mathcal{X} \subseteq B$ is closed. Then $Q : \mathcal{X} \longrightarrow \mathcal{X}$ is a contraction mapping on $\mathcal{X}$ if there exist $0 < k < 1$ such that

$$\|Qy_1 - Qy_2\| \leq k\|y_1 - y_2\|, \quad y_1, y_2 \in \mathcal{X}.$$

Example 1.5.11. Define the mapping $g : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$g(x) = \frac{x}{a} + b, \quad \text{for all } a > 1 \text{ and } b \in \mathbb{R}.$$

Then

$$|g(x) - g(y)| = \frac{1}{a}|x - y|.$$

Since $0 < \frac{1}{a} < 1$, thus g is contraction mapping.

Proposition 1.5.12. *Let $J \subseteq \mathbb{R}$ and $f : J \longrightarrow J$ be a differentiable function such that*

$$|f'(x)| \leq k \quad forall \quad x \in J,$$

for $k < 1$. Then f is contraction.

Definition 1.5.13. The family of functions $\mathcal{S} \subseteq C[a, b]$ is equicontinuous iff for every $\epsilon > 0$ there exist $\delta > 0$ such that for all $y_1, y_2 \in [a, b]$

$$|f(y_1) - f(y_2)| < \epsilon \quad \text{whenever } |y_1 - y_2| < \delta,$$

for all $f \in \mathcal{S}$.

Theorem 1.5.14. ***(Banach-Contraction principal)***

[18] Consider a Banach space B. Let $\mathcal{X} \subseteq B$ is closed, then there is a unique fixed point for each contraction mapping $T : \mathcal{X} \longrightarrow \mathcal{X}$ in $\mathcal{X}$.

Theorem 1.5.15. ***(Arzelà-Ascoli)***

[15] Let B be a Banach space and $S \subseteq B$ is bounded. Then $Q \subseteq C(S)$ is relatively compact iff Q is equicontinuous and bounded.

Theorem 1.5.16. ***(Schauder's fixed point theorem)***

[15] Consider a Banach space B. Let $Q \subseteq B$, that is non-empty closed bounded convex, and suppose the mapping $T : Q \longrightarrow Q$ is compact in B. Then T has at least one fixed point.

Chapter 2

Generalized Taylor's formula for right fractional calculus

2.1 Introduction

Now days the power series expansion has got extensive importance in mathematics to obtain an uncomplicated approximation of complicated functions. No one can deny the importance of Taylor series in the history of all sciences. Taylor series also linearized the complex problems which secure the easy analysis and allowed the scientist to approximate many complexed systems, neglecting higher order terms around the equilibrium point.

Many mathematicians has discussed generalized Taylor's formula for left fractional calculus. G. Hardy [14] wrote a formal version of the generalized Taylor's formula using Riemann-Liouville fractional derivative and integral:

$$f(x+h) = \sum_{m=-\infty}^{\infty} \frac{h^{m+r}}{\Gamma(m+r+1)} (j_a^{m+r} f)(x),$$

where j_a^{m+r} is Riemann-Liouville fractional integral of order $m+r$.

Trujillo, Rivero and Bonilla [35] also gave the generalized Taylor's formula using Riemann-Liouville fractional derivative:

$$f(x) = \sum_{i=0}^{n} \frac{c_i (x-a)^{(i+1)p-1}}{\Gamma((i+1)p)} + R_n(x,a),$$

with

$$R_n(x,a) = D_a^{(n+1)p} f(\xi) \frac{(x-a)^{(n+1)p}}{\Gamma((n+1)p+1)}, \qquad x \leq \xi \leq b$$

and for each $i \in \mathbb{N}, 0 \leq i \leq n$,

$$c_i = \Gamma(p) \left[(x-a)^{1-p} D_a^{ip} f(x) \right] (a^+).$$

D. Usero [36] and Z.M. Odibat, N.T. Shawagfeh [23] provided generalized version of Taylor's formula using Caputo fractional derivative.

In this chapter we established generalized Taylor's formula for right fractional calculus

$$f(x) = \sum_{k=0}^{n} \frac{a_k (b-x)^{(k+1)p-1}}{\Gamma((k+1)p)} + R_n^p(x,b),$$

with

$$R_n^p(x,b) = D_b^{(n+1)p} f(\xi) \frac{(b-x)^{(n+1)p}}{\Gamma((n+1)p+1)},$$

and

$$a_k = I_b^{1-p} D_b^{kp} f(b).$$

where $x \leq \xi \leq b,\ k \in \mathbb{N}$ and D_b^p is right Riemann-Liouville fractional derivative.

2.2 Preliminaries

Here we establish necessary results which are used in our main results. First we state and prove fractional version of Mean Value Theorem for right Riemann-Liouville fractional derivative.

Proposition 2.2.1. *Let $f(x) \in {}_bD_p$ and $0 < p \leq 1$. Then for all $x \in [a,b)$*

$$I_b^p D_b^p f(x) = f(x) - f(b).$$

This can be explain from Theorem 1.3.17.

Proposition 2.2.2. *Let $f(x) \in {}_bD_p$ and $0 < p \leq 1$. Then*

$$f(x) = f(b) - D_b^p f(\xi) \frac{(b-x)^p}{\Gamma(p+1)},$$

where $x \leq \xi \leq b$ for all $x \in [a,b)$.

Proof. From Proposition 2.2.1 we can write

$$f(x) = f(b) + I_b^p D_b^p f(x)$$

Using definition of right integral operator and integral mean value theorem we can write

$$\begin{aligned} I_b^p D_b^p f(x) &= \frac{1}{\Gamma(p)} \int_x^b (t-x)^{p-1} D_b^p f(t) dt \\ &= -D_b^p f(\xi) \frac{(b-x)^p}{\Gamma(p+1)}. \end{aligned} \tag{2.2.1}$$

Thus required result is obtained. □

Next proposition would be initiative of higher order approximations.

Proposition 2.2.3. [36] Let f be an analytic function, $0 < p \leq 1$ and $m \in \mathbb{N}$ such that

1. $D_b^{mp} f, D_b^{(m+1)p} f \in C[a,b)$,
2. $D_b^{(m+1)p} f \in {}_bI_p[a,b]$.

Then

$$I_b^{mp} D_b^{mp} f(x) - I_b^{(m+1)p} D_b^{(m+1)p} f(x) = \frac{(b-x)^{(m+1)p-1}}{\Gamma((m+1)p)} I_b^{1-p} D_b^{mp} f(b).$$

Proof. The proof is obvious for $m = 0$. Now for the case $m > 0$, using Theorems 1.3.6 and 1.3.17 we have

$$\begin{aligned} I_b^{mp} D_b^{mp} f(x) - I_b^{(m+1)p} D_b^{(m+1)p} f(x) &= I_b^{mp}\left[D_b^{mp} f(x) - I_b^p D_b^{(m+1)p} f(x)\right] \\ &= I_b^{mp}\left[D_b^{mp} f(x) - \left(I_b^p D_b^p\right) D_b^{mp} f(x)\right] \\ &= I_b^{mp}\left[\frac{(b-x)^{p-1}}{\Gamma(p)} I_b^{1-p} D_b^{mp} f(b)\right] \\ &= \frac{(b-x)^{(m+1)p-1}}{\Gamma((m+1)p)} I_b^{1-p} D_b^{mp} f(b), \end{aligned}$$

where $I_b^{mp}\left(\frac{(b-x)^{p-1}}{\Gamma(p)}\right) = \frac{(b-x)^{(m+1)p-1}}{\Gamma((m+1)p)}$. □

Proposition 2.2.4. [36] Assume the conditions of Proposition 2.2.3 hold and $m, l \in \mathbb{N}$. Then

$$I_b^{lp} D_b^{mp} f(x) = -D_b^{mp} f(\xi) \frac{(b-x)^{lp}}{\Gamma(lp+1)},$$

where $x \leq \xi \leq b$ for all $x \in [a, b)$.

2.3 Generalized Taylor's formula

Now we are on the stage to establish our main focused result generalized Taylor's formula with right Riemann-Liouville fractional derivative.

Theorem 2.3.1. *Let $0 < p \leq 1$ and $n \in \mathbb{N}$ satisfying the following conditions:*

1. *for all $k = 1, ..., n$, $D_b^{kp} f \in C[a, b)$ and $D_b^{kp} f \in {}_bI_p[a, b]$,*
2. *$D_b^{(n+1)p} f$ is continuous in $[a, b]$.*

Then for all $x \in [a, b)$,

$$f(x) = \sum_{k=0}^{n} \frac{a_k (b-x)^{(k+1)p-1}}{\Gamma((k+1)p)} + R_n^p(x, b), \tag{2.3.1}$$

with

$$R_n^p(x, b) = D_b^{(n+1)p} f(\xi) \frac{(b-x)^{(n+1)p}}{\Gamma((n+1)p+1)}, \qquad x \leq \xi \leq b \tag{2.3.2}$$

and for each $k \in \mathbb{N} \cup \{0\}$,

$$a_k = I_b^{1-p} D_b^{kp} f(b). \tag{2.3.3}$$

Proof. The given below results can be achieved by induction method using Proposition 2.2.3.

For $m = 0$

$$I_b^0 D_b^0 f(x) - I_b^p D_b^p f(x) = \frac{(b-x)^{p-1}}{\Gamma(p)} I_b^{1-p} D_b^0 f(b)$$

$$f(x) = \frac{(b-x)^{p-1}}{\Gamma(p)} I_b^{1-p} f(b) + I_b^p D_b^p f(x).$$

For $m = 1$

$$I_b^p D_b^p f(x) - I_b^{2p} D_b^{2p} f(x) = \frac{(b-x)^{2p-1}}{\Gamma(2p)} I_b^{1-p} D_b^p f(b)$$

$$I_b^p D_b^p f(x) = \frac{(b-x)^{2p-1}}{\Gamma(2p)} I_b^{1-p} D_b^p f(b) + I_b^{2p} D_b^{2p} f(x).$$

For $m = 2$

$$I_b^{2p} D_b^{2p} f(x) - I_b^{3p} D_b^{3p} f(x) = \frac{(b-x)^{3p-1}}{\Gamma(3p)} I_b^{1-p} D_b^{2p} f(b)$$

$$I_b^{2p} D_b^{2p} f(x) = \frac{(b-x)^{3p-1}}{\Gamma(3p)} I_b^{1-p} D_b^{2p} f(b) + I_b^{3p} D_b^{3p} f(x).$$

In general we can write

$$I_b^{mp} D_b^{mp} f(x) = \frac{(b-x)^{(m+1)p-1}}{\Gamma((m+1)p)} I_b^{1-p} D_b^{mp} f(b) + I_b^{(m+1)p} D_b^{(m+1)p} f(x).$$

Summing all up to n term we obtain the series

$$f(x) = \sum_{k=0}^{n} \frac{(b-x)^{(k+1)p-1}}{\Gamma((k+1)p)} I_b^{1-p} D_b^{kp} f(b) + I_b^{(n+1)p} D_b^{(n+1)p} f(x).$$

Where last term is remainder term, using equation (2.2.1) we can write

$$R_n^p(x,b) = I_b^{(n+1)p} D_b^{(n+1)p} f(x) = -D_b^{(n+1)p} f(\xi) \frac{(b-x)^{(n+1)p}}{\Gamma((n+1)p+1)}, \qquad x \leq \xi \leq b,$$

from expression (2.3.3) we have

$$f(x) = \sum_{k=0}^{n} \frac{a_k (b-x)^{(k+1)p-1}}{\Gamma((k+1)p)} + R_n^p(x,b).$$

□

Corollary 2.3.2. *Set $0 < p \leq 1$, $n \in \mathbb{N}$ and g is a continuous function, such that*

$$f(x) = (b-x)^{p-1} g(x)$$

satisfies the conditions of the above theorem. Then, for all $x \in [a,b)$,

$$g(x) = \sum_{k=0}^{n} \frac{a_k (b-x)^{kp}}{\Gamma((k+1)p)} + \widetilde{R}_n^p(x,b) \tag{2.3.4}$$

with

$$\widetilde{R}_n^p(x,b) = \frac{-[D_b^{(n+1)p}(b-x)^{p-1} g(x)](\xi)}{\Gamma((n+1)p+1)} (b-x)^{np+1}, \qquad x \leq \xi \leq b$$

and for each $k \in \mathbb{N} \cup \{0\}$,

$$a_k = [I_b^{1-p} D_b^{kp} (x-b)^{p-1} g(x)](b).$$

Proof. It follows from the above theorem. □

2.4 Application

Example 2.4.1. Consider the problem:

$$D_b^p u(x) = \omega u(x), \qquad (D_b^{p-1} u)(b) = c, \ (c \in \mathbb{R}). \tag{2.4.1}$$

where $0 < p \leq 1$, $\omega \in \mathbb{R}$ and $x > 0$.

Since $u(x)$ is b-singular of order p. Using the generalized Taylor's formula, solution of $u(x)$ can be written as

$$u(x) = \sum_{k=0}^{\infty} \frac{a_k (b-x)^{(k+1)p-1}}{\Gamma((k+1)p)} + R_n^p(b, x).$$

Since

$$\lim_{n \longrightarrow \infty} R_n^p(b, x) = 0.$$

So

$$u(x) = \sum_{k=0}^{\infty} \frac{a_k (b-x)^{(k+1)p-1}}{\Gamma((k+1)p)}. \tag{2.4.2}$$

Using Example 1.3.15, we obtain

$$D_b^p u(x) = \sum_{k=0}^{\infty} \frac{a_k (b-x)^{kp-1}}{\Gamma(kp)}. \tag{2.4.3}$$

Substituting (2.4.2) and (2.4.3) into (2.4.1) yields

$$\sum_{k=0}^{\infty} \frac{a_{k+1} (b-x)^{(k+1)p-1}}{\Gamma((k+1)p)} - \omega \sum_{k=0}^{\infty} \frac{a_k (b-x)^{(k+1)p-1}}{\Gamma((k+1)p)} = 0. \tag{2.4.4}$$

Equating the coefficient of $(b-x)^{(k+1)p-1}$, we get

$$a_{k+1} = \omega a_k.$$

$$a_k = \omega^k a_0. \tag{2.4.5}$$

Now substituting (2.4.5) in (2.4.2)

$$u(x) = a_0 \sum_{k=0}^{\infty} \frac{\omega^k (b-x)^{(k+1)p-1}}{\Gamma((k+1)p)}, \tag{2.4.6}$$

from initial condition we get

$$\begin{aligned} u(x) &= c \sum_{k=0}^{\infty} \frac{\omega^k (b-x)^{kp+p-1}}{\Gamma(kp+p)} \\ &= c(b-x)^{p-1} E_{p,p}(\omega (b-x)^p), \end{aligned} \tag{2.4.7}$$

where $E_{p,p}(x)$ is the Mittag-Leffler function.

In particular, the solution of problem (2.4.1) for $p = 3/2$:

$$D_b^{3/2} u(x) = \omega u(x), \qquad (D_b^{1/2} u)(b) = c, \ (c \in \mathbb{R}),$$

is given by

$$u(x) = c(b-x)^{3/2} E_{3/2,3/2}(\omega (b-x)^{3/2}).$$

Example 2.4.2. Let us consider problem

$$D_1^{2p}u(x) + D_1^p u(x) - u(x) = 0, \tag{2.4.8}$$

with $0 < p \leq 1$, $x > 0$ and $D_1^{2p} = D_1^p.D_1^p$.
Since $u(x)$ is 1-singular of order p. Using the generalized Taylor's formula, solution of $u(x)$ can be written as

$$u(x) = \sum_{k=0}^{\infty} \frac{a_k(1-x)^{(k+1)p-1}}{\Gamma((k+1)p)}. \tag{2.4.9}$$

Applying right Riemann-Liouville fractional derivative, we obtain

$$D_1^p u(x) = \sum_{k=0}^{\infty} \frac{a_k(1-x)^{kp-1}}{\Gamma(kp)}. \tag{2.4.10}$$

$$D_1^{2p}u(x) = \sum_{k=0}^{\infty} \frac{a_k(1-x)^{(k-1)p-1}}{\Gamma((k-1)p)}. \tag{2.4.11}$$

Substituting (2.4.9), (2.4.10) and (2.4.11) into (2.4.8) yields

$$\sum_{k=0}^{\infty} \frac{a_{k+2}(1-x)^{(k+1)p-1}}{\Gamma((k+1)p)} + \sum_{k=0}^{\infty} \frac{a_{k+1}(1-x)^{(k+1)p-1}}{\Gamma((k+1)p)} - \sum_{k=0}^{\infty} \frac{a_k(1-x)^{(k+1)p-1}}{\Gamma((k+1)p)} = 0. \tag{2.4.12}$$

Equating the coefficient of $(1-x)^{(k+1)p-1}$ in (2.4.12), we obtain recursive relation

$$a_{k+2} = a_k - a_{k+1}. \tag{2.4.13}$$

This gives

$$a_2 = a_0 - a_1,$$

$$a_3 = -a_0 + 2a_1,$$

$$a_4 = 2a_0 - 3a_1,$$

$$a_5 = -3a_0 + 5a_1,$$

$$a_6 = 5a_0 - 8a_1.$$

We obtain following solution:

$$\begin{aligned} u_1(x) =& a_0\bigg(\frac{1}{\Gamma(p)}(1-x)^{p-1} + \frac{1}{\Gamma(3p)}(1-x)^{3p-1} - \frac{1}{\Gamma(4p)}(1-x)^{4p-1} + \frac{2}{5\Gamma(p)}(1-x)^{5p-1} \\ & - \frac{3}{6\Gamma(p)}(1-x)^{6p-1} + \frac{5}{7\Gamma(p)}(1-x)^{7p-1} + ...\bigg), \end{aligned} \tag{2.4.14}$$

$$\begin{aligned} u_2(x) =& a_1\bigg(\frac{1}{\Gamma(2p)}(1-x)^{2p-1} - \frac{1}{\Gamma(3p)}(1-x)^{3p-1} + \frac{2}{\Gamma(4p)}(1-x)^{4p-1} - \frac{3}{5\Gamma(p)}(1-x)^{5p-1} \\ & + \frac{5}{6\Gamma(p)}(1-x)^{6p-1} - \frac{8}{7\Gamma(p)}(1-x)^{7p-1} + ...\bigg). \end{aligned} \tag{2.4.15}$$

Chapter 3

Terminal value problems for fractional order nonlinear differential equations on bounded domain

3.1 Introduction

There is a great work particularly dealing with the solvability of nonlinear fractional differential equations including initial and boundary value problems. Lots of the papers and books are dedicated for this purpose. Although, terminal value problem for nonlinear fractional differential equations is open to discussion until now and many aspect of terminal value problem may take into account in detail. Recently it has received quit attention. Here we develop the existence and uniqueness results for fractional terminal value problems on bounded domain. Indeed, terminal value problems have numerous applications such as chemical engineering, thermo elasticity, underground water flow, viscoelasticity, cellular systems, electromagnetic heat transmission and so fourth. Several researchers investigated boundary value problems in fractional calculus, R.P. Agarwal, M. Benchohra, S. Hamani [3], Kilbas [16], K. Karthikeyan, J.J. Trujillo [17], Podlubny [24], M. Rehman, R. Khan [26], G.Wanga, A. Cabadab, L. Zhanga [37], Zhang [41], Samko [30] and many more. The theory of terminal value problems has been considered by various research workers, A.R. Aftabizadeh, V. Lakshmikantham [2], K. Diethelm [13], M. Rehman, S. A. Hussain [25]and W. E. Shreve [32]. The fixed point theorems have been applied by some authors to investigate the existence of solutions.

K. Diethelm [13] determined the existence and stability results for the unique solution of terminal value problems for fractional order with Caputo derivative taking finite interval.

X. Su and S. Zhang [33] discussed the nonlinear boundary value problem on half-line

$$D_0^p y(t) - f(t, y(t), D_0^{p-1} y(t)) = 0, \quad 1 < p \leq 2, \quad t \in [0, \infty),$$
$$y(0) = 0, \; D_0^{p-1} y_(\infty) = y_\infty, \quad y_\infty \in \mathbb{R},$$

where f is a continuous function, $D_0^{p-1} y_(\infty) := \lim_{t \longrightarrow \infty} D_0^{p-1} y_(t)$. D_0^p is standard Riemann-Liouville fractional derivative.

Here we develop new conditions to establish existence and uniqueness results. We consider the problem

$$D_b^p y(t) = f(t, y(t)), \quad 1 < p \leq 2, \tag{3.1.1}$$

$$y(b) = 0, \quad D_b^{p-1} y(b) = \xi y(\tau), \tag{3.1.2}$$

for all $t \in [a, b]$. Furthermore, we generalized the problem (3.1.1)

$$D_b^p y(t) = f(t, y(t), D_b^q y(t)), \quad 1 < p \leq 2, \tag{3.1.3}$$

$$y(b) = 0, \quad D_b^{p-1} y(b) = \xi y(\tau). \tag{3.1.4}$$

for all $t \in [a, b]$. Where f is a continuous function, $p - q \geq 1$, $\xi > 0$, $\tau \in (a, b)$ and $\Gamma(p) > \xi(b-\tau)^{p-1}$. D_b^p is right Riemann-Liouville fractional derivative.
Examples are also comprised to present the application of our results.

Lemma 3.1.1. *Let $p > 0$ and if $y \in C(0,1) \cap L(0,1)$. Then the unique solution of fractional differential equation*

$$D_b^p y(t) = 0$$

is given by

$$y(t) = c_1(b-t)^{p-1} + c_2(b-t)^{p-2} + c_3(b-t)^{p-3} + ... + c_n(b-t)^{p-n},$$

for some $c_i \in \mathbb{R},\ i = 1, 2, ..., n$.

Lemma 3.1.2. *If $y(t) \in C(0,1) \cap L(0,1)$ and $D_b^p y(t) \in C(0,1) \cap L(0,1)$. Then for $p > 0$*

$$I_b^p D_b^p y(t) = y(t) - c_1(b-t)^{p-1} - c_2(b-t)^{p-2} - c_3(b-t)^{p-3} - ... - c_n(b-t)^{p-n},$$

where $c_i \in \mathbb{R},\ i = 1, 2, ..., n$.

3.2 Terminal value problem-I

Here we develop the existence and uniqueness results for (3.1.1) and (3.1.2).
For the convenience of our results, we have following hypothesis for all real valued functions x and y on $[a, b]$:

(H_1) $f : [a, b] \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2) $|f(t, y)| \leq \phi(t) + \psi|y|^\gamma$, $0 < \gamma \leq 1$, $\psi > 0$, where $\phi \in L_1[a, b]$ is non-negative function.

(H_3) $|f(t, x) - f(t, y)| \leq \rho(t)|x - y|$, where $\rho \in C([a, b])$ is non-negative function.

We use following notations for convenience:

$\Omega = \Gamma(p) - \xi(b-\tau)^{p-1}$,

$\lambda_1 = \max\limits_{t \in [a,b]} \big(\frac{1}{\Gamma(p)} \int_t^b (s-t)^{p-1} \phi(s) ds \big)$,

$\lambda_2 = \max\limits_{t\in[a,b]}\big(\frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b(s-\tau)^{p-1}\phi(s)ds\big),$

$\mu = \frac{(b-a)^p}{\Gamma(p+1)} + \frac{\xi(b-a)^{p-1}}{\Gamma(p+1)\Omega}(b-\tau)^p,$

$\zeta = \frac{1}{\Gamma(p)} + \frac{\xi(b-a)^{p-1}}{\Gamma(p)\Omega}.$

Lemma 3.2.1. *Suppose that κ is continuous at $[a,b]$. Then y is solution of terminal value problem*

$$D_b^p y(t) = \kappa(t), \quad 1 < p \leq 2, \tag{3.2.1}$$

$$y(b) = 0, \quad D_b^{p-1}y(b) = \xi y(\tau), \tag{3.2.2}$$

for all $t\in[a,b]$. iff y satisfies

$$y(t) = \frac{1}{\Gamma(p)}\int_t^b(s-t)^{p-1}\kappa(s)ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b(s-\tau)^{p-1}\kappa(s)ds.$$

Proof. Suppose y is solution of (3.2.1) and (3.2.2), using Lemma 3.1.1 we have

$$y(t) = \frac{1}{\Gamma(p)}\int_t^b(s-t)^{p-1}\kappa(s)ds + c_1(b-t)^{p-1} + c_2(b-t)^{p-2}, \tag{3.2.3}$$

with $c_1, c_2 \in \mathbb{R}$. Using condition $y(b)=0$ we get $c_2 = 0$.
Now

$$D_b^{p-1}y(t) = \int_t^b \kappa(s)ds + c_1\Gamma(p),$$

$$D_b^{p-1}y(b) = c_1\Gamma(p),$$

and

$$\xi y(\tau) = \frac{\xi}{\Gamma(p)}\int_\tau^b(s-\tau)^{p-1}\kappa(s)ds + c_1\xi(b-\tau)^{p-1}.$$

Using condition $D_b^{p-1}y(b) = \xi y(\tau)$, we get

$$c_1 = \frac{\xi}{\Gamma(p)\Omega}\int_\tau^b(s-\tau)^{p-1}\kappa(s)ds.$$

Thus

$$y(t) = \frac{1}{\Gamma(p)}\int_t^b(s-t)^{p-1}\kappa(s)ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b(s-\tau)^{p-1}\kappa(s)ds.$$

□

3.2.1 Existence and uniqueness of solution

Now, we establish our main result for terminal value problem (3.1.1) and (3.1.2).
Let us define the space $\mathcal{X} := \{x \in C[a,\infty) : \|x\| < \infty\}$. The space $\mathcal{X}$ with norm $\|x\| = \sup\limits_{t\in[a,b]}|x(t)|$ is a Banach space.
Define the operator $\mathcal{T}$, by

$$\mathcal{T}y(t) = \frac{1}{\Gamma(p)}\int_t^b(s-t)^{p-1}f(s,y(s))ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b(s-\tau)^{p-1}f(s,y(s))ds.$$

Solutions of problem (3.1.1) and (3.1.2) are fixed points of $\mathcal{T}$.

Theorem 3.2.2. *Suppose* (H_1) *and* (H_2) *hold. Then at least one solution of* (3.1.1) *and* (3.1.2) *exists.*

Proof. Choose $R \geq \max\{(3\mu\psi)^{\frac{1}{1-\gamma}}, 3\lambda_1, 3\lambda_2\}$. Define the set $\mathcal{Q} := \{y \in \mathcal{X} : \|y\| \leq R\}$. The set $\mathcal{Q} \subseteq \mathcal{X}$, that is closed and convex. Since $(b-t)^p \leq (b-a)^p$ and y be an arbitrary element in $\mathcal{Q}$, then

$$
\begin{aligned}
|\mathcal{T}y(t)| \leq& \left|\frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1} f(s,y(s))ds\right| + \left|\frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1} f(s,y(s))ds\right| \\
=& \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}|f(s,y(s))|ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}|f(s,y(s))|ds \\
\leq& \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}(\phi(s)+\psi|R|^\gamma)ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}(\phi(s)+\psi|R|^\gamma)ds \\
=& \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}\phi(s)ds + \frac{\psi|R|^\gamma}{\Gamma(p)}\int_t^\infty (s-t)^{p-1}ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}\times \\
& \phi(s)ds + \frac{\xi(b-t)^{p-1}\psi|R|^\gamma}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}ds \\
=& \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}\phi(t)ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}\phi(s)ds \\
&+ \left(\frac{\psi|y|^\gamma}{\Gamma(p+1)}(b-t)^p + \frac{\xi(b-t)^{p-1}\psi|R|^\gamma}{\Gamma(p+1)\Omega}(b-\tau)^p\right) \\
\leq& \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}\phi(t)ds + \frac{\xi(b-a)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}\phi(s)ds \\
&+ \left(\frac{\psi|y|^\gamma}{\Gamma(p+1)}(b-a)^p + \frac{\xi(b-a)^{p-1}\psi|R|^\gamma}{\Gamma(p+1)\Omega}(b-\tau)^p\right)
\end{aligned}
$$

From H_2 we have $|\mathcal{T}y(t)| \leq \lambda_1 + \lambda_2 + \psi|R|^\gamma\mu \leq \frac{R}{3} + \frac{R}{3} + \frac{R}{3} = R$. Thus $\|\mathcal{T}y\| \leq R$.

Let $a \leq t \leq \widetilde{t} < \infty$, then we have

$$
\begin{aligned}
|\mathcal{T}y(t) - \mathcal{T}y(\widetilde{t})| \leq& \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}|f(s,y(s))|ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}|f(s,y(s))|ds \\
&- \frac{1}{\Gamma(p)}\int_{\widetilde{t}}^b (s-\widetilde{t})^{p-1}|f(s,y(s))|ds - \frac{\xi(b-\widetilde{t})^{p-1}}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}|f(s,y(s))|ds
\end{aligned}
$$

$$
\begin{aligned}
=& \frac{1}{\Gamma(p)}\int_t^{\widetilde{t}} (s-t)^{p-1}|f(s,y(s))|ds + \frac{1}{\Gamma(p)}\int_{\widetilde{t}}^b [(s-t)^{p-1} - (s-\widetilde{t})^{p-1}]|f(s,y(s))|ds \\
&+ \frac{\xi[(b-t)^{p-1} - (b-\widetilde{t})^{p-1}]}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}|f(s,y(s))|ds \\
\leq& \frac{1}{\Gamma(p)}\int_t^{\widetilde{t}} (s-t)^{p-1}(\phi(s)+\psi|R|^\gamma)ds + \frac{1}{\Gamma(p)}\int_{\widetilde{t}}^b [(s-t)^{p-1} - (s-\widetilde{t})^{p-1}](\phi(s)+\psi|R|^\gamma)ds \\
&+ \frac{\xi[(b-t)^{p-1} - (b-\widetilde{t})^{p-1}]}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}(\phi(s)+\psi|R|^\gamma)ds \\
=& \frac{1}{\Gamma(p)}\int_t^{\widetilde{t}} (s-t)^{p-1}\phi(s)ds + \frac{1}{\Gamma(p)}\int_{\widetilde{t}}^b [(s-t)^{p-1} - (s-\widetilde{t})^{p-1}]\phi(s)ds \\
&+ \frac{\xi[(b-t)^{p-1} - (b-\widetilde{t})^{p-1}]}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}\phi(s)ds + \frac{\psi|R|^\gamma}{\Gamma(p+1)}\Bigg[(b-t)^p - (b-\widetilde{t})^p \\
&+ \frac{\xi[(b-t)^{p-1} - (b-\widetilde{t})^{p-1}]}{\Omega}(b-\tau)^p\Bigg].
\end{aligned}
$$

It can be easily observe that $\int_t^{\widetilde{t}}(s-t)^{p-1}\phi(s)ds \longrightarrow 0$, as $t \longrightarrow \widetilde{t}$.
Let $f(t)=(s-t)^{p-1}$ is a differentiable function. Then using Mean Value theorem for $\eta \in (t,\widetilde{t})$, we can write

$$f(t)-f(\widetilde{t}) = f^{'}(\eta)(t-\widetilde{t}),$$
$$(s-t)^{p-1}-(s-\widetilde{t})^{p-1} = -(p-1)(s-\eta)^{p-2}(t-\widetilde{t}).$$

Thus

$$\int_{\widetilde{t}}^{b}[(s-t)^{p-1}-(s-\widetilde{t})^{p-1}]\phi(s)ds = -(p-1)(t-\widetilde{t})\int_{\widetilde{t}}^{b}(s-\eta)^{p-2}\phi(s)ds.$$

So we have $\int_{\widetilde{t}}^{b}[(s-t)^{p-1}-(s-\widetilde{t})^{p-1}]\phi(s)ds \longrightarrow 0$, as $t \longrightarrow \widetilde{t}$.
More over $\frac{\xi[(b-t)^{p-1}-(b-\widetilde{t})^{p-1}]}{\Gamma(p)\Omega}\int_\tau^b(s-\tau)^{p-1}\phi(s)ds \longrightarrow 0$, as $t \longrightarrow \widetilde{t}$, and $\frac{\psi|R|^\gamma}{\Gamma(p+1)}\Bigg[(b-t)^p-(b-\widetilde{t})^p+\frac{\xi[(b-t)^{p-1}-(b-\widetilde{t})^{p-1}]}{\Omega}(b-\tau)^p\Bigg] \longrightarrow 0$, as $t \longrightarrow \widetilde{t}$.
Thus $|\mathcal{T}y(t)-\mathcal{T}y(\widetilde{t})| \longrightarrow 0$ as $t \longrightarrow \widetilde{t}$. Thus $\|\mathcal{T}y(t)-\mathcal{T}y(\widetilde{t})\| \longrightarrow 0$ as $t \longrightarrow \widetilde{t}$. Hence $\mathcal{T}y(t)$ is equicontinuous and compact operator by Arzela-Ascoli theorem. Thus by Schauder's fixed point theorem, there is at least one fixed point of $\mathcal{T}$, that is the solution of the problem (3.1.1) and (3.1.2). □

Theorem 3.2.3. *Assume (H_1), (H_3) hold and $\zeta < 1$. Then fractional order terminal value problem* (3.1.1) *and* (3.1.2) *has unique solution.*

Proof. From Theorem 3.2.2 it follows that $\mathcal{T}$ has a fixed point. Now we only show that operator $\mathcal{T}$ is contraction. Since $(b-t)^{p-1} < (b-a)^{p-1}$, then from our assumption $\zeta < 1$ we have for $x, y \in \mathcal{X}$

$$\begin{aligned}
|\mathcal{T}x(t)-\mathcal{T}y(t)| =& \frac{1}{\Gamma(p)}\int_t^b(s-t)^{p-1}|f(s,x(s))-f(s,y(s))|ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b(s-\tau)^{p-1}\times \\
& |f(s,x(s))-f(s,y(s))|ds \\
\leq& \frac{1}{\Gamma(p)}\int_t^b(s-t)^{p-1}\rho(s)|x(s)-y(s)|ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b(s-\tau)^{p-1}\times \\
& \rho(s)|x(s)-y(s)|ds \\
\leq& \frac{1}{\Gamma(p)}\|x-y\|\int_t^b(s-t)^{p-1}\rho(s)ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\|x-y\|\int_\tau^b(s-\tau)^{p-1}\rho(s)ds \\
\leq& \frac{1}{\Gamma(p)}\|x-y\| + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\|x-y\| \\
\leq& \frac{1}{\Gamma(p)}\|x-y\| + \frac{\xi(b-a)^{p-1}}{\Gamma(p)\Omega}\|x-y\| \\
=& \zeta\|x-y\|.
\end{aligned}$$

Hence $\mathcal{T}$ is contraction. Thus by Banach contraction theorem, unique solution of the problem (3.1.1) and (3.1.2) exists. □

Example 3.2.4. Let us consider the problem on $I=[a,\infty)$:

$$D_b^p y(t) = \frac{ln(1+y(t))}{e^t+t} + \sqrt{|y(t)|}, \quad 1 < p \leq 2, \quad t \in [a,b], \tag{3.2.4}$$

$$y(b)=0, \quad D_b^{p-1}y(b) = \xi y(\tau). \tag{3.2.5}$$

Here $f(t,y(t)) = \frac{ln(1+y(t))}{e^t+t} + \sqrt{|y(t)|}$. First we verify the conditions of Theorem 3.2.2.

$$\begin{aligned}|f(t,y(t))| &= \left|\frac{ln(1+y(t))}{e^t+t} + \sqrt{|y(t)|}\right| \\ &\leq \left|\frac{y(t)}{e^t+t} + \sqrt{|y(t)|}\right| \\ &\leq \left|\frac{y(t)}{e^t+t}\right| + \left|\sqrt{|y(t)|}\right|,\end{aligned}$$

where $\phi(t) = \frac{y(t)}{e^t+t}$, $\psi = 1$ and $\gamma = 1/2$. Now, we verify the condition of Theorem 3.2.3.

$$\begin{aligned}|f(t,x(t)) - f(t,y(t))| &= \left|\frac{ln(1+x(t))}{e^t+t} + \sqrt{|x(t)|} - \frac{ln(1+y(t))}{e^t+t} - \sqrt{|y(t)|}\right| \\ &\leq \left|\frac{x(t)}{e^t+t} + \sqrt{|x(t)|} - \frac{y(t)}{e^t+t} - \sqrt{|y(t)|}\right| \\ &\leq \frac{1}{e^t+t}|x(t) - y(t)| \\ &\leq \frac{1}{e^t+t}\|x(t) - y(t)\|,\end{aligned}$$

where $\zeta(t) = \frac{1}{e^t+t}$. Hence, all conditions are satisfied. Thus by Theorem 3.2.2 and Theorem 3.2.3 unique solution exists.

3.3 Termina value problem-II

Here we establish existence and uniqueness results for problem (3.1.3) and (3.1.4).
For the convenience of our results, we have following hypothesis for all real valued functions $x, \widetilde{x}, y$ and $\widetilde{y}$ on $[a,b]$:

(H_1) $f : [a,b] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous.

(H_2)**:** $|f(t,x,y)| \leq \phi(t) + \psi|x|^{\gamma_1} + \sigma|y|^{\gamma_2}$, $0 < \gamma_1, \gamma_2 \leq 1$, $\psi, \sigma > 0$, where $\phi \in L_1[a,b]$ is non-negative function.

(H_3)**:** $|f(t,x,y) - f(t,\widetilde{x},\widetilde{y})| \leq \varrho(t)|x - \widetilde{x}| + \varsigma(t)|y - \widetilde{y}|$, where $\varrho, \varsigma \in C([a,b]$ are non-negative functions.

We use following notations for convenience:

$$\mathcal{G}_1 = \max_{t\in[a,b]}\left(\frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}\phi(t)ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\varOmega}\int_\tau^b (s-\tau)^{p-1}\phi(t)ds + \frac{1}{\Gamma(p-q)}\int_t^b (s-t)^{p-q-1}\phi(s)ds + \frac{\xi(b-t)^{p-q-1}}{\Gamma(p-q)\varOmega}\int_\tau^b (s-\tau)^{p-1}\phi(s)ds\right),$$

$$\mathcal{G}_2 = \max_{t\in[a,b]}\left(\frac{(b-t)^p}{\Gamma(p+1)} + \frac{\xi(b-t)^{p-1}}{\Gamma(p+1)\varOmega}(b-\tau)^p + \frac{(b-t)^{p-q}}{\Gamma(p-q+1)} + \frac{\xi(b-t)^{p-q-1}(b-\tau)^p}{p\Gamma(p-q)\varOmega}\right)$$

$$k_1 = \frac{1}{\Gamma(p)} + \frac{\xi(b-a)^{p-1}}{\Gamma(p)\varOmega},$$

$$k_2 = \frac{1}{\Gamma(p-q)} + \frac{\xi(b-a)^{p-q-1}}{\Gamma(p-q)\varOmega}.$$

3.3.1 Existence and uniqueness of solution

Let us define the Banach space $\mathcal{Y} = \{y \in C[a,b] : D_b^q y \in C[a,b]\}$, with norm $\|y\| = \max_{t\in[a,b]}|y(t)| + \max_{t\in[a,b]}|D_b^q y(t)|$.

Define the operator $\mathcal{K}$, by

$$\mathcal{K}y(t) = \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1} f(s, y(s), D_b^q y(s))ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\mathit{\Omega}}\int_\tau^b (s-\tau)^{p-1} f(s, y(s), D_b^q y(s))ds.$$

Solution of terminal value problem (3.1.3) and (3.1.4) are fixed points of $\mathcal{K}$.

Choose $R \geq \{(3\psi\mathcal{G}_2)^{\frac{1}{1-\gamma_1}}, (3\sigma\mathcal{G}_2)^{\frac{1}{1-\gamma_2}}, 3\mathcal{G}_1\}$.

Theorem 3.3.1. *Assume* (H_1) *and* (H_2) *hold. Then at least one solution of* (3.1.3) *and* (3.1.4) *exists.*

Proof. Let $E \subseteq \mathcal{Y}$, that is closed and convex, defined as $E := \{y \in \mathcal{Y} : \|y\| \leq R\}$. Let y be an arbitrary element in E, then we have

$$\begin{aligned}
|\mathcal{K}y(t)| \leq & \left|\frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1} f(s, y(s), D_b^q y(s))ds\right| + \left|\frac{\xi(b-t)^{p-1}}{\Gamma(p)\mathit{\Omega}}\int_\tau^b (s-\tau)^{p-1} \times \right. \\
& \left. f(s, y(s), D_b^q y(s))ds\right| \\
= & \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}|f(s, y(s), D_b^q y(s))|ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\mathit{\Omega}}\int_\tau^b (s-\tau)^{p-1} \times \\
& |f(s, y(s), D_b^q y(s))|ds \\
\leq & \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}(\phi(t) + \psi|R|^{\gamma_1} + \sigma|R|^{\gamma_2})ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\mathit{\Omega}}\int_\tau^b (s-\tau)^{p-1} \times \\
& (\phi(t) + \psi|R|^{\gamma} + \sigma|R|^{\gamma_2})ds \\
= & \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}\phi(t)ds + \frac{(\psi|R|^{\gamma} + \sigma|R|^{\gamma_2})}{\Gamma(p)}\int_t^\infty (s-t)^{p-1}ds \\
& + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\mathit{\Omega}}\int_\tau^b (s-\tau)^{p-1}\phi(t)ds + \frac{\xi(b-t)^{p-1}(\psi|R|^{\gamma} + \sigma|R|^{\gamma_2})}{\Gamma(p)\mathit{\Omega}}\int_\tau^b (s-\tau)^{p-1}ds \\
= & \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}\phi(t)ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\mathit{\Omega}}\int_\tau^b (s-\tau)^{p-1}\phi(t)ds \\
& + \left(\frac{(\psi|R|^{\gamma} + \sigma|R|^{\gamma_2})}{\Gamma(p+1)}(b-t)^p + \frac{\xi(b-t)^{p-1}(\psi|R|^{\gamma} + \sigma|R|^{\gamma_2})}{\Gamma(p+1)\mathit{\Omega}}(b-\tau)^p\right),
\end{aligned}$$

and

$$\begin{aligned}
|D_b^q \mathcal{K}y(t)| = & \left|D_b^q I_b^p f(t, y(t), D_b^q y(t)) + \frac{\xi D_b^q (b-t)^{p-1}}{\mathit{\Omega}} I_b^p f(\tau, y(\tau), D_b^q y(\tau))\right| \\
= & \left|I_b^{p-q} f(t, y(t), D_b^q y(t)) + \frac{\xi\Gamma(p)(b-t)^{p-q-1}}{\Gamma(p-q)\mathit{\Omega}} I_b^p f(\tau, y(\tau), D_b^q y(\tau))\right|
\end{aligned}$$

$$
\begin{aligned}
\leq & \frac{1}{\Gamma(p-q)} \int_t^b (s-t)^{p-q-1}(\phi(s)+\psi|R|^{\gamma_1}+\sigma|R|^{\gamma_2})ds \\
& + \frac{\xi(b-t)^{p-q-1}}{\Gamma(p-q)\Omega} \int_\tau^b (s-\tau)^{p-1}(\phi(s)+\psi|R|^{\gamma_1}+\sigma|R|^{\gamma_2})ds \\
= & \frac{1}{\Gamma(p-q)} \int_t^b (s-t)^{p-q-1}\phi(s)ds + \frac{\xi(b-t)^{p-q-1}}{\Gamma(p-q)\Omega} \int_\tau^b (s-\tau)^{p-1}\phi(s)ds \\
& + \left(\frac{(\psi|R|^{\gamma}+\sigma|R|^{\gamma_2})}{\Gamma(p-q+1)}(b-t)^{p-q} + \frac{\xi(b-t)^{p-q-1}(\psi|R|^{\gamma}+\sigma|R|^{\gamma_2})}{p\Gamma(p-q)\Omega}(b-\tau)^p \right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\mathcal{K}y(t)\| &= \max_{t\in[a,b]} |\mathcal{K}y(t)| + \max_{t\in[a,b]} |D^q\mathcal{K}y(t)| \\
&\leq \mathcal{G}_1 + (\psi|R|^{\gamma_1}+\sigma|R|^{\gamma_2})\mathcal{G}_2 \leq \frac{R}{3}+\frac{R}{3}+\frac{R}{3} = R.
\end{aligned}
$$

Now we have to show that $\mathcal{K}y(t)$ is a completely continuous. Let $a \leq t \leq \widetilde{t} < \infty$, then

$$
\begin{aligned}
|\mathcal{K}y(t) - \mathcal{K}y(\widetilde{t})| \leq & \frac{1}{\Gamma(p)} \int_t^b (s-t)^{p-1}|f(s,y(s),D_b^q y(s))|ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega} \int_\tau^b (s-\tau)^{p-1}\times \\
& |f(s,y(s),D_b^q y(s))|ds - \frac{1}{\Gamma(p)} \int_{\widetilde{t}}^b (s-\widetilde{t})^{p-1}|f(s,y(s),D_b^q y(s))|ds \\
& - \frac{\xi(b-\widetilde{t})^{p-1}}{\Gamma(p)\Omega} \int_\tau^b (s-\tau)^{p-1}|f(s,y(s),D_b^q y(s))|ds \\
= & \frac{1}{\Gamma(p)} \int_t^{\widetilde{t}} (s-t)^{p-1}|f(s,y(s),D_b^q y(s))|ds + \frac{1}{\Gamma(p)} \int_{\widetilde{t}}^b [(s-t)^{p-1}-(s-\widetilde{t})^{p-1}]\times \\
& |f(s,y(s),D_b^q y(s))|ds + \frac{\xi[(b-t)^{p-1}-(b-\widetilde{t})^{p-1}]}{\Gamma(p)\Omega} \int_\tau^b (s-\tau)^{p-1}|f(s,y(s),D_b^q y(s))|ds \\
\leq & \frac{1}{\Gamma(p)} \int_t^{\widetilde{t}} (s-t)^{p-1}(\phi(s)+\psi|R|^{\gamma_1}+\sigma|R|^{\gamma_2})ds + \frac{1}{\Gamma(p)} \int_{\widetilde{t}}^b [(s-t)^{p-1}-(s-\widetilde{t})^{p-1}]\times \\
& (\phi(s)+\psi|R|^{\gamma_1}+\sigma|R|^{\gamma_2})ds + \frac{\xi[(b-t)^{p-1}-(b-\widetilde{t})^{p-1}]}{\Gamma(p)\Omega} \int_\tau^b (s-\tau)^{p-1}(\phi(s)+\psi|R|^{\gamma_1} \\
& + \sigma|R|^{\gamma_2})ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|\mathcal{K}y(t) - \mathcal{K}y(\widetilde{t})| \leq & \frac{1}{\Gamma(p)} \int_t^{\widetilde{t}} (s-t)^{p-1}\phi(s)ds + \frac{1}{\Gamma(p)} \int_{\widetilde{t}}^b [(s-t)^{p-1}-(s-\widetilde{t})^{p-1}]\phi(s)ds \\
& + \frac{\xi[(b-t)^{p-1}-(b-\widetilde{t})^{p-1}]}{\Gamma(p)\Omega} \int_\tau^b (s-\tau)^{p-1}\phi(s)ds + \frac{(\psi|R|^{\gamma_1}+\sigma|R|^{\gamma_2})}{\Gamma(p+1)} \Big[(b-t)^p - (b-\widetilde{t})^p \\
& + \frac{\xi[(b-t)^{p-1}-(b-\widetilde{t})^{p-1}]}{\Omega}(b-\tau)^p \Big].
\end{aligned} \tag{3.3.1}
$$

And

$$
\begin{aligned}
|D_b^q\mathcal{K}y(t)-D_b^q\mathcal{K}y(\widetilde{t})| =& \Big|D_b^q I_b^p f(t,y(t),D_b^q y(t)) + \frac{\xi D_b^q(b-t)^{p-1}}{\Omega} I_b^p f(\tau,y(\tau),D_b^q y(\tau)) \\
&- D_b^q I_b^p f(\widetilde{t},y(\widetilde{t}),D_b^q y(\widetilde{t})) - \frac{\xi D_b^q(b-\widetilde{t})^{p-1}}{\Omega} I_b^p f(\tau,y(\tau),D_b^q y(\tau))\Big| \\
=& \Big|I_b^{p-q} f(t,y(t),D_b^q y(t)) + \frac{\xi\Gamma(p)(b-t)^{p-q-1}}{\Gamma(p-q)\Omega} I_b^p f(\tau,y(\tau),D_b^q y(\tau)) \\
&- I_b^{p-q} f(\widetilde{t},y(\widetilde{t}),D_b^q y(\widetilde{t})) - \frac{\xi\Gamma(p)(b-\widetilde{t})^{p-q-1}}{\Gamma(p-q)\Omega} I_b^p f(\tau,y(\tau),D_b^q y(\tau))\Big| \\
\le& \frac{1}{\Gamma(p-q)}\int_t^{\widetilde{t}} (s-t)^{p-q-1}(\phi(s)+\psi|R|^{\gamma_1}+\sigma|R|^{\gamma_2})ds \\
&+ \frac{1}{\Gamma(p-q)}\int_{\widetilde{t}}^b [(s-t)^{p-q-1}-(s-\widetilde{t})^{p-q-1}](\phi(s)+\psi|R|^{\gamma_1}+\sigma|R|^{\gamma_2})ds \\
&+ \frac{\xi[(b-t)^{p-q-1}-(b-\widetilde{t})^{p-q-1}]}{\Gamma(p-q)\Omega}\int_\tau^b (s-\tau)^{p-1}(\phi(s)+\psi|R|^{\gamma_1}+\sigma|R|^{\gamma_2})ds.
\end{aligned}
$$

Thus

$$
\begin{aligned}
|D_b^q\mathcal{K}y(t)-D_b^q\mathcal{K}y(\widetilde{t})| \le& \frac{1}{\Gamma(p-q)}\int_t^{\widetilde{t}} (s-t)^{p-q-1}\phi(s)ds + + \frac{1}{\Gamma(p-q)}\int_{\widetilde{t}}^b [(s-t)^{p-q-1}-(s-\widetilde{t})^{p-q-1}]\times \\
&\phi(s)ds + \frac{\xi[(b-t)^{p-q-1}-(b-\widetilde{t})^{p-q-1}]}{\Gamma(p-q)\Omega}\int_\tau^b (s-\tau)^{p-1}\phi(s)ds \\
&+ \Bigg(\frac{(\psi|R|^{\gamma}+\sigma|R|^{\gamma_2})}{\Gamma(p-q+1)}[(b-t)^{p-q}-(b-\widetilde{t})^{p-q}] \\
&+ \frac{\xi(\psi|R|^{\gamma}+\sigma|R|^{\gamma_2})}{p\Gamma(p-q)\Omega}(b-\tau)^p\Bigg).
\end{aligned} \tag{3.3.2}
$$

Note that all terms in equations (3.3.1) and (3.3.2) are uniformly continuous and bounded, we conclude that $\mathcal{K}y(t)$ and $D_b^q\mathcal{K}y(t)$ are equicontinuous for all real valued functions y. Thus Schauder fixed point theorem applies that there exist at least one solution of fractional terminal value problem (3.1.3) and (3.1.4). □

Theorem 3.3.2. *Assume* (H_1), (H_3) *hold and* $k=\max(k_1+k_2)<1$. *Then there the fractional terminal value problem* (3.1.3) *and* (3.1.4) *has unique solution.*

Proof. We have proved in Theorem 3.2.3 the existence of fractional terminal value problem (3.1.3) and (3.1.4). Now we establish the contraction of $\mathcal{K}$. Let x,y be arbitrary elements of the Banach space $\mathcal{Y}$.

Also we know that $(b-t)^{p-1} < (b-a)^{p-1}$, then we have

$$
\begin{aligned}
|\mathcal{K}x(t)-\mathcal{K}y(t)| \leq & \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}|f(s,x(s),D_b^q x(s))-f(s,y(s),D_b^q y(s))|ds \\
& + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}|f(s,x(s),D_b^q x(s))-f(s,y(s),D_b^q y(s))|ds \\
\leq & \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}(\varrho(t)|x(s)-y(s)|+\varsigma(t)|D_b^q x(s)-D_b^q y(s)|)ds \\
& + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\int_\tau^b (s-\tau)^{p-1}(\varrho(t)|x(s)-y(s)|+\varsigma(t)|D_b^q x(s)-D_b^q y(s)|)ds \\
\leq & \frac{1}{\Gamma(p)}\max_{t\in[a,b]}|x-y|\int_t^b (s-t)^{p-1}\varrho(s)ds + \frac{1}{\Gamma(p)}\max_{t\in[a,b]}|D_b^q x(t)-D_b^q y(t)|\int_t^b (s-t)^{p-1}\varsigma(s)ds \\
& + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\max_{t\in[a,b]}|x-y|\int_\tau^b (s-\tau)^{p-1}\varrho(s)ds \\
& + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\max_{t\in[a,b]}|D_b^q x(t)-D_b^q y(t)|\int_\tau^b (s-\tau)^{p-1}\varsigma(s)ds \\
\leq & \frac{1}{\Gamma(p)}\max_{t\in[a,b]}|x-y| + \frac{1}{\Gamma(p)}\max_{t\in[a,b]}|D_b^q x(t)-D_b^q y(t)| + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\max_{t\in[a,b]}|x-y| \\
& + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega}\max_{t\in[a,b]}|D_b^q x(t)-D_b^q y(t)|
\end{aligned}
$$

$$
\begin{aligned}
\leq & \left(\frac{1}{\Gamma(p)}+\frac{\xi(b-a)^{p-1}}{\Gamma(p)\Omega}\max_{t\in[a,b]}|x-y|\right) \\
& + \left(\frac{1}{\Gamma(p)}+\frac{\xi(b-a)^{p-1}}{\Gamma(p)\Omega}\right)\max_{t\in(a,b)}|D_b^q x(t)-D_b^q y(t)| \\
= & k_1\left(\max_{t\in[a,b]}|x-y| + \max_{t\in[a,b]}|D_b^q x(t)-D_b^q y(t)|\right),
\end{aligned}
$$

and

$$
\begin{aligned}
|D_b^q\mathcal{K}x(t)-D_b^q\mathcal{K}y(t)| = & \Bigg| D_b^q I_b^p\big[f(t,x(t),D_b^q x(t))-f(t,y(t),D_b^q y(t))\big] + \frac{\xi D_b^q(b-t)^{p-1}}{\Omega}I_b^p\times \\
& \big[f(\tau,x(\tau),D_b^q x(\tau))-f(\tau,y(\tau),D_b^q y(\tau))\big]\Bigg| \\
= & \Bigg| I_b^{p-q}\big[f(t,x(t),D_b^q x(t))-f(t,y(t),D_b^q y(t))\big] + \frac{\xi\Gamma(p)(b-t)^{p-q-1}}{\Gamma(p-q)\Omega}I_b^p\times \\
& \big[f(\tau,x(\tau),D_b^q x(\tau))-f(\tau,y(\tau),D_b^q y(\tau))\big]\Bigg| \\
\leq & \frac{1}{\Gamma(p-q)}\int_t^b (s-t)^{p-q-1}(\varrho(t)|x(s)-y(s)|+\varsigma(t)|D_b^q x(s)-D_b^q y(s)|)ds \\
& + \frac{\xi(b-t)^{p-q-1}}{\Gamma(p-q)\Omega}\int_\tau^b (s-\tau)^{p-1}(\varrho(t)|x(s)-y(s)| \\
& + \varsigma(t)|D_b^q x(s)-D_b^q y(s)|)ds
\end{aligned}
$$

$$\leq \frac{\max_{t\in[a,b]}|x-y|}{\Gamma(p-q)} \int_t^b (s-t)^{p-q-1}\varrho(s)ds + \frac{\max_{t\in[a,b]}|D_b^q x(t) - D_b^q y(t)|}{\Gamma(p-q)} \int_t^b (s-t)^{p-q-1} \times$$
$$\varsigma(s)ds + \frac{\xi(b-t)^{p-q-1}\max_{t\in[a,b]}|x-y|}{\Gamma(p-q)\Omega} \int_\tau^b (s-\tau)^{p-1}\varrho(s)ds$$
$$+ \frac{\xi(b-t)^{p-q-1}\max_{t\in[a,b]}|D_b^q x(t) - D_b^q y(t)|}{\Gamma(p-q)\Omega} \int_\tau^b (s-\tau)^{p-1}\varrho(s)ds$$
$$\leq \frac{1}{\Gamma(p-q)} \max_{t\in[a,b]}|x-y| + \frac{1}{\Gamma(p-q)} \max_{t\in(a,b)}|D_b^q x(t) - D_b^q y(t)|$$
$$+ \frac{\xi(b-a)^{p-q-1}}{\Gamma(p-q)\Omega} \max_{t\in[a,b]}|x-y|$$
$$+ \frac{\xi(b-a)^{p-q-1}}{\Gamma(p-q)\Omega} \max_{t\in[a,b]}|D_b^q x(t) - D_b^q y(t)|$$
$$= k_2\left(\max_{t\in[a,b]}|x-y| + \max_{t\in[a,b]}|D_b^q x(t) - D_b^q y(t)|\right).$$

Thus

$$\begin{aligned}\|\mathcal{K}x(t) - \mathcal{K}y(t)\| \leq & \max_{t\in[a,b]}|\mathcal{K}x(t) - \mathcal{K}y(t)| + \max_{t\in[a,b]}|D_b^q\mathcal{K}x(t) - D_b^q\mathcal{K}y(t)| \\ =& k_1\left(\max_{t\in[a,b]}|x-y| + \max_{t\in[a,b]}|D_b^q x(t) - D_b^q y(t)|\right) + k_2\left(\max_{t\in[a,b]}|x-y| \right. \\ & \left. + \max_{t\in[a,b]}|D_b^q x(t) - D_b^q y(t)|\right) \\ =& (k_1+k_2)\left(\max_{t\in[a,b]}|x-y| + \max_{t\in[a,b]}|D_b^q x(t) - D_b^q y(t)|\right) \\ \leq & k\left(\|x-y\| + \|D_b^q x(t) - D_b^q y(t)\|\right).\end{aligned}$$

Hence $\mathcal{K}$ is contraction. Thus by Banach contraction theorem, fractional terminal value problem (3.1.3) and (3.1.4) has unique solution. □

Example 3.3.3. Consider the problem on $I = [a, \infty)$.

$$D_b^{3/2}x(t) = ln(1 + D_b^{7/5}x(t) + \sqrt{|x(t)|}\sin^2(x(t)), \quad t \in [a,b], \tag{3.3.3}$$
$$x(b) = 0, \quad D_b^{1/2}x(b) = \xi x(\tau). \tag{3.3.4}$$

Here $f(t,x,y) = ln(1+y) + \sqrt{|x|}\sin^2(x)$. We attempt to verify the conditions of Theorem 3.3.1

$$\begin{aligned}|f(t,x,y)| &= |ln(1+y) + \sqrt{|x|}\sin^2(x)| \\ &\leq |y + \sqrt{|x|}\sin^2(x)| \\ &\leq |y| + \sqrt{|x|}|\sin^2(x)| \\ &\leq |y| + \sqrt{|x|},\end{aligned}$$

where $\phi(t) = 0, \psi = |x|, \sigma = |y|, \gamma_1 = 1/2$ and $\gamma_2 = 0.$. Now, we will verify the condition of Theorem 3.3.2.

$$\begin{aligned}|f(t,x,y) - f(t,\widetilde{x},\widetilde{y})| &= |ln(1+y) + \sqrt{|x|}\sin^2(x) - ln(1+\widetilde{y}) - \sqrt{|\widetilde{x}|}\sin^2(\widetilde{x})| \\ &\le |ln(1+y) - ln(1+\widetilde{y})| + |\sqrt{|x|}\sin^2(x) - \sqrt{|\widetilde{x}|}\sin^2(\widetilde{x})| \\ &\le |x-\widetilde{x}| + |y-\widetilde{y}|,\end{aligned}$$

where $\varrho(t) = \varsigma(t) = 1$. Hence, all conditions are satisfied. Thus unique solution of (3.3.3) exists.

Chapter 4

A coupled system of terminal value problems for fractional order nonlinear differential equations on bounded domain

4.1 Introduction

Recently the debate on coupled systems of differential equations in fractional calculus has gained very significant importance. Z.Z.E. Abidine [1], Y. Li, Y. Sang, H. Zhang [19], K. Zhang, J. Xu, D. O'Regan [43], Y. Liu, B. Ahmad, R.P. Agrawal [20], Y. Chen, D. Chen, Z. Lv [11] and L. Zhang, B. Ahmad, G. Wang [38] investigated the existence of coupled systems involving fractional differential equations.

X. Su [34] discussed a coupled system for two-point boundary value problem

$$\begin{cases} D^p x(t) = f(t, y(t), D^u y(t)), \quad 1 < p < 2, \\ D^q y(t) = g(t, x(t), D^v x(t)), \quad 1 < p < 2 \\ x(0) = x(1) = y(0) = y(1) = 0, \end{cases}$$

for all $t \in [0,1]$, where D^p is right Riemann-Liouville derivative.

B. Ahmad, J.J. Nieto [5] discussed existence of three-point boundary conditions for a coupled system

$$\begin{cases} D^p x(t) = f(t, y(t), D^u y(t)), \quad 1 < p < 2, \\ D^q y(t) = g(t, x(t), D^v x(t)), \quad 1 < q < 2, \\ x(0) = 0, \quad x(1) = \gamma x(\eta), \quad y(0) = 0, \quad y(1) = \gamma y(\eta), \end{cases}$$

for all $t \in [0,1]$ and $p, q \in (1,2)$, where p, q, u, v, γ, η satisfy certain conditions.

Motivated by work coted above, here we generalized the results of previous chapter to a coupled system

$$\begin{cases} D_b^p x(t) = f(t, y(t), D_b^u y(t)), \quad 1 < p \leq 2, \\ D_b^q y(t) = g(t, x(t), D_b^v x(t)), \quad 1 < q \leq 2, \\ x(b) = 0, \quad D_b^{p-1} x(b) = \xi x(\tau), \quad y(b) = 0, \quad D_b^{q-1} y(b) = \xi y(\tau), \end{cases} \tag{4.1.1}$$

for all $t \in [a,b]$, with right Riemann-Liouville fractional derivative D_b^p. Where $u, v > 0, p - v \geq 1, q - u \geq 1, \xi > 0, \tau \in (a,b), \Gamma(p) > \xi(b-\tau)^{p-1}$ and $\Gamma(q) > \xi(b-\tau)^{q-1}$.

For the convenience of our results, we have following hypothesis for all real valued functions $x, \widetilde{x}, y$ and $\widetilde{y}$ on $[a,b]$:

(H_1) $f, g : [a,b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions

(H_2) $|f(t,x,y)| \leq \phi(t) + \sigma_1|x|^{\gamma_1} + \sigma_2|y|^{\gamma_2}$, $0 < \gamma_1, \gamma_2 \leq 1$, $\sigma_1, \sigma_2 > 0$, where $\phi \in L_1[a,b]$ is non-negative function.

(H_3)**:** $|g(t,x,y)| \leq \psi(t) + \varrho_1|x|^{\rho_1} + \varrho_2|y|^{\rho_2}$, $0 < \rho_1, \rho_2 \leq 1$, $\varrho_1, \varrho_2 > 0$, where $\psi \in L_1[a,b]$ is non-negative function.

(H_4)**:** $|f(t,x,y) - f(t,\widetilde{x},\widetilde{y})| \leq \delta(t)|x - \widetilde{x}| + \varsigma(t)|y - \widetilde{y}|$, where $\delta, \varsigma \in C([a,b])$ are non-negative functions.

(H_5)**:** $|g(t,x,y) - g(t,\widetilde{x},\widetilde{y})| \leq \zeta(t)|x - \widetilde{x}| + \kappa(t)|y - \widetilde{y}|$, where $\zeta, \kappa \in C([a,b])$ are non-negative functions.

We use following notations for convenience:

$$\Omega_1 = \Gamma(p) - \xi(b-\tau)^{p-1},$$

$$\Omega_2 = \Gamma(q) - \xi(b-\tau)^{q-1},$$

$$\mathcal{A}_1 = \max_{t\in[a,b]}\left(\frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}\phi(t)ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1}\int_\tau^b (s-\tau)^{p-1}\phi(t)ds + \frac{1}{\Gamma(p-v)}\int_t^b (s-t)^{p-v-1}\phi(s)ds + \frac{\xi(b-t)^{p-v-1}}{\Gamma(p-v)\Omega_1}\int_\tau^b (s-\tau)^{p-1}\phi(s)ds\right),$$

$$\mathcal{Q}_1 = \max_{t\in[a,b]}\left(\frac{(b-t)^p}{\Gamma(p+1)} + \frac{\xi(b-t)^{p-1}}{\Gamma(p+1)\Omega_1}(b-\tau)^p + \frac{(b-t)^{p-v}}{\Gamma(p-v+1)} + \frac{\xi(b-\tau)^p}{p\Gamma(p-v)\Omega_1}\right)$$

$$\mathcal{A}_2 = \max_{t\in[a,b]}\left(\frac{1}{\Gamma(q)}\int_t^b (s-t)^{q-1}\psi(t)ds + \frac{\xi(b-t)^{q-1}}{\Gamma(q)\Omega_2}\int_\tau^b (s-\tau)^{q-1}\psi(t)ds + \frac{1}{\Gamma(q-u)}\int_t^b (s-t)^{q-u-1}\phi(s)ds + \frac{\xi(b-t)^{p-u-1}}{\Gamma(q-u)\Omega_2}\int_\tau^b (s-\tau)^{q-1}\phi(s)ds\right),$$

$$\mathcal{Q}_2 = \max_{t\in[a,b]}\left(\frac{(b-t)^q}{\Gamma(q+1)} + \frac{\xi(b-t)^{q-1}}{\Gamma(q+1)\Omega_2}(b-\tau)^q + \frac{(b-t)^{q-u}}{\Gamma(q-u+1)} + \frac{\xi(b-\tau)^q}{q\Gamma(q-u)\Omega_2}\right)$$

$$l_1 = \frac{1}{\Gamma(p)} + \frac{\xi(b-a)^{p-1}}{\Gamma(p)\Omega_1},$$

$$l_2 = \frac{1}{\Gamma(p-v)} + \frac{\xi(b-a)^{p-v-1}}{\Gamma(p-v)\Omega_1},$$

$$k_1 = \frac{1}{\Gamma(q)} + \frac{\xi(b-a)^{q-1}}{\Gamma(q)\Omega_2},$$

$$k_2 = \frac{1}{\Gamma(q-u)} + \frac{\xi(b-a)^{q-u-1}}{\Gamma(q-u)\Omega_2}.$$

4.2 Main results

Lemma 4.2.1. *Let $h \in C[a,b]$ be a function. Then x is the solution of terminal value problem*

$$D_b^p x(t) = h(t), \quad 1 < p \leq 2, \quad t \in [a,b],$$
$$x(b)) = 0, \quad D_b^{p-1} x(b) = \xi x(\tau),$$

if and only if x satisfies

$$x(t) = \frac{1}{\Gamma(p)} \int_t^b (s-t)^{p-1} h(s) ds + \frac{\xi (b-t)^{p-1}}{\Gamma(p) \Omega_1} \int_\tau^b (s-\tau)^{p-1} h(s) ds.$$

Proof. Proof is same as that of Lemma (3.2.1), so we omit it. □

Similarly, y is the solution of terminal value problem

$$D_b^q y(t) = h(t), \quad 1 < q \leq 2, \quad t \in [a,b],$$
$$y(b)) = 0, \quad D_b^{q-1} y(b) = \xi y(\tau),$$

if and only if y satisfies

$$y(t) = \frac{1}{\Gamma(q)} \int_t^b (s-t)^{q-1} h(s) ds + \frac{\xi (b-t)^{q-1}}{\Gamma(q) \Omega_2} \int_\tau^b (s-\tau)^{q-1} h(s) ds.$$

Let us define the Banach space

$$\mathcal{X} = \{x | x \in C[a,b] : D^v x \in C[a,b]\}$$

with norm

$$\|x\| = \max_{t \in [a,b]} |x(t)| + \max_{t \in [a,b]} |D_b^v x(t)|$$

and Banach space

$$\mathcal{Y} = \{y | y \in C[a,b] : D^u y \in C[a,b]\}$$

with norm

$$\|y\| = \max_{t \in [a,b]} |y(t)| + \max_{t \in [a,b]} |D_b^u y(t)|.$$

Clearly $\mathcal{X} \times \mathcal{Y}$ is a Banach space with norm

$$\|(x,y)\| = \max\{\|x\|, \|y\|\},$$

for all $(x,y) \in \mathcal{X} \times \mathcal{Y}$.

Now let us have the system of integral equations:

$$\begin{cases} x(t) = \frac{1}{\Gamma(p)} \int_t^b (s-t)^{p-1} f(s, y(s), D_b^u y(s)) ds + \frac{\xi (b-t)^{p-1}}{\Gamma(p) \Omega_1} \int_\tau^b (s-\tau)^{p-1} f(s, y(s), D_b^u y(s)) ds, \\ y(t) = \frac{1}{\Gamma(q)} \int_t^b (s-t)^{q-1} g(s, x(s), D_b^v x(s)) ds + \frac{\xi (b-t)^{q-1}}{\Gamma(q) \Omega_2} \int_\tau^b (s-\tau)^{q-1} g(s, x(s), D_b^v x(s)) ds. \end{cases} \tag{4.2.1}$$

Lemma 4.2.2. *Let H_1 holds. Then $(x,y) \in \mathcal{X} \times \mathcal{Y}$ satisfy* (4.1.1) *if and only if $(x,y) \in \mathcal{X} \times \mathcal{Y}$ satisfy the integral equations* (4.2.1).

Proof. Proof is obvious from Lemma (4.2.1). □

4.2.1 Existence and uniqueness of solutions

Let us define operator $\mathcal{K} : \mathcal{X} \times \mathcal{Y} \longrightarrow \mathcal{X} \times \mathcal{Y}$ as

$$\mathcal{K}(x, y)(t) = (\mathcal{K}_1 y(t), \mathcal{K}_2 x(t)).$$

Solutions of coupled system (4.1.1) are fixed points of operator $\mathcal{K}$. Where

$$\begin{aligned}\mathcal{K}_1 y(t) =& \frac{1}{\Gamma(p)} \int_t^b (s-t)^{p-1} f(s, y(s), D_b^u y(s)) ds \\ &+ \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1} \int_\tau^b (s-\tau)^{p-1} f(s, y(s), D_b^u y(s)) ds,\end{aligned}$$

and

$$\begin{aligned}\mathcal{K}_2 x(t) =& \frac{1}{\Gamma(q)} \int_t^b (s-t)^{q-1} g(s, x(s), D_b^v x(s)) ds \\ &+ \frac{\xi(b-t)^{q-1}}{\Gamma(q)\Omega_2} \int_\tau^b (s-\tau)^{q-1} g(s, x(s), D_b^v x(s)) ds.\end{aligned}$$

Define

$$\mathcal{S} = \{(x(t), y(t)) | (x(t), y(t)) \in \mathcal{X} \times \mathcal{Y}, \|(x(t), y(t))\| \leq R\},$$

where $R \geq \max_{t \in [a,b]} \{(3\sigma_1 \mathcal{Q}_1)^{\frac{1}{1-\gamma_1}}, (3\sigma_2 \mathcal{Q}_1)^{\frac{1}{1-\gamma_2}}, (3\varrho_1 \mathcal{Q}_2)^{\frac{1}{1-\rho_1}}, (3\varrho_2 \mathcal{Q}_2)^{\frac{1}{1-\rho_1}}, 3\mathcal{A}_1, 3\mathcal{A}_2\}$.
Note that S is the ball in the Banach space $\mathcal{X} \times \mathcal{Y}$. Now we establish existence and uniqueness results.

Theorem 4.2.3. *Suppose that* (H_1), (H_2) *and* (H_3) *hold. Then coupled system* (4.1.1) *has at least one solution.*

Proof. Let x be an arbitrary element in $\mathcal{S}$, then we have

$$\begin{aligned}|\mathcal{K}_1 y(t)| \leq& \left| \frac{1}{\Gamma(p)} \int_t^b (s-t)^{p-1} f(s, y(s), D_b^u y(s)) ds \right| \\ &+ \left| \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1} \int_\tau^b (s-\tau)^{p-1} f(s, y(s), D_b^u y(s)) ds \right| \\ =& \frac{1}{\Gamma(p)} \int_t^b (s-t)^{p-1} |f(s, y(s), D_b^u y(s))| ds \\ &+ \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1} \int_\tau^b (s-\tau)^{p-1} |f(s, y(s), D_b^u y(s))| ds \\ \leq& \frac{1}{\Gamma(p)} \int_t^b (s-t)^{p-1} (\phi(t) + \sigma_1 |R|^{\gamma_1} + \sigma_2 |R|^{\gamma_2}) ds \\ &+ \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1} \int_\tau^b (s-\tau)^{p-1} (\phi(t) + \sigma_1 |R|^{\gamma_1} + \sigma_2 |R|^{\gamma_2}) ds\end{aligned}$$

$$
\begin{aligned}
=&\frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}\phi(t)ds + \frac{(\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})}{\Gamma(p)}\int_t^\infty (s-t)^{p-1}ds\\
&+\frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1}\int_\tau^b (s-\tau)^{p-1}\phi(t)ds + \frac{\xi(b-t)^{p-1}(\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})}{\Gamma(p)\Omega_1}\int_\tau^b (s-\tau)^{p-1}ds\\
=&\frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}\phi(t)ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1}\int_\tau^b (s-\tau)^{p-1}\phi(t)ds\\
&+\left(\frac{(\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})}{\Gamma(p+1)}(b-t)^p + \frac{\xi(b-t)^{p-1}(\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})}{\Gamma(p+1)\Omega_1}(b-\tau)^p\right),
\end{aligned}
$$

and

$$
\begin{aligned}
|D_b^v\mathcal{K}_1y(t)| =&\left|D_b^vI_b^pf(t,y(t),D_b^uy(t)) + \frac{\xi D_b^v(b-t)^{p-1}}{\Omega_1}I_b^pf(\tau,y(\tau),D_b^uy(\tau))\right|\\
=&\left|I_b^{p-v}f(t,y(t),D_b^uy(t)) + \frac{\xi\Gamma(p)(b-t)^{p-v-1}}{\Gamma(p-v)\Omega_1}I_b^pf(\tau,y(\tau),D_b^uy(\tau))\right|\\
\le&\frac{1}{\Gamma(p-v)}\int_t^b (s-t)^{p-v-1}(\phi(s)+\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})ds\\
&+\frac{\xi(b-t)^{p-v-1}}{\Gamma(p-v)\Omega_1}\int_\tau^b (s-\tau)^{p-1}(\phi(s)+\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})ds\\
=&\frac{1}{\Gamma(p-v)}\int_t^b (s-t)^{p-v-1}\phi(s)ds + \frac{\xi(b-t)^{p-v-1}}{\Gamma(p-v)\Omega_1}\int_\tau^b (s-\tau)^{p-1}\phi(s)ds\\
&+\left(\frac{(\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})}{\Gamma(p-v+1)}(b-t)^{p-v} + \frac{\xi(\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})}{p\Gamma(p-v)\Omega_1}(b-\tau)^p\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\mathcal{K}_1y(t)\| &= \max_{t\in[a,b]}|\mathcal{K}_1y(t)| + \max_{t\in[a,b]}|D^v\mathcal{K}_1y(t)|\\
&\le \mathcal{A}_1 + (\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})\mathcal{Q}_1 \le \frac{R}{3}+\frac{R}{3}+\frac{R}{3} = R.
\end{aligned}
$$

Similarly, $\|\mathcal{K}_2x(t)\| \le \mathcal{A}_2 + (\varrho_1|R|^{\rho_1}+\varrho_2|R|^{\rho_2})\mathcal{Q}_2 \le \frac{R}{3}+\frac{R}{3}+\frac{R}{3} = R$. Thus $\|\mathcal{K}(x,y)\| \le R$. Since $\mathcal{K}_1y(t), \mathcal{K}_2x(t), D^v\mathcal{K}_1y(t)$ and $D^u\mathcal{K}_2x(t)$ are continuous on $[a,b]$, thus $\mathcal{K}:\mathcal{S}\longrightarrow\mathcal{S}$.
Now we have to show that $\mathcal{K}$ is equicontinuous. Let $a\le t\le\widetilde{t}<b$, then we can write

$$
\begin{aligned}
|\mathcal{K}_1y(t)-\mathcal{K}_1y(\widetilde{t})| \le&\frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}|f(s,y(s),D_b^uy(s))|ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1}\int_\tau^b (s-\tau)^{p-1}\times\\
&|f(s,y(s),D_b^uy(s))|ds - \frac{1}{\Gamma(p)}\int_{\widetilde{t}}^b (s-\widetilde{t})^{p-1}|f(s,y(s),D_b^uy(s))|ds\\
&-\frac{\xi(b-\widetilde{t})^{p-1}}{\Gamma(p)\Omega_1}\int_\tau^b (s-\tau)^{p-1}|f(s,y(s),D_b^uy(s))|ds\\
=&\frac{1}{\Gamma(p)}\int_t^{\widetilde{t}} (s-t)^{p-1}|f(s,y(s),D_b^uy(s))|ds + \frac{1}{\Gamma(p)}\int_{\widetilde{t}}^b [(s-t)^{p-1}-(s-\widetilde{t})^{p-1}]\times\\
&|f(s,y(s),D_b^uy(s))|ds + \frac{\xi[(b-t)^{p-1}-(b-\widetilde{t})^{p-1}]}{\Gamma(p)\Omega_1}\int_\tau^b (s-\tau)^{p-1}|f(s,y(s),D_b^uy(s))|ds
\end{aligned}
$$

$$\leq \frac{1}{\Gamma(p)}\int_t^{\widetilde{t}}(s-t)^{p-1}(\phi(s)+\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})ds+\frac{1}{\Gamma(p)}\int_{\widetilde{t}}^b[(s-t)^{p-1}-(s-\widetilde{t})^{p-1}]\times$$
$$(\phi(s)+\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})ds+\frac{\xi[(b-t)^{p-1}-(b-\widetilde{t})^{p-1}]}{\Gamma(p)\Omega_1}\int_\tau^b(s-\tau)^{p-1}(\phi(s)$$
$$+\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})ds.$$

Thus

$$\begin{aligned}|\mathcal{K}_1y(t)-\mathcal{K}_1y(\widetilde{t})|\leq&\frac{1}{\Gamma(p)}\int_t^{\widetilde{t}}(s-t)^{p-1}\phi(s)ds+\frac{1}{\Gamma(p)}\int_{\widetilde{t}}^b[(s-t)^{p-1}-(s-\widetilde{t})^{p-1}]\phi(s)ds\\&+\frac{\xi[(b-t)^{p-1}-(b-\widetilde{t})^{p-1}]}{\Gamma(p)\Omega_1}\int_\tau^b(s-\tau)^{p-1}\phi(s)ds+\frac{(\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})}{\Gamma(p+1)}\Big[(b-t)^p-(b-\widetilde{t})^p\\&+\frac{\xi[(b-t)^{p-1}-(b-\widetilde{t})^{p-1}]}{\Omega_1}(b-\tau)^p\Big].\end{aligned}\tag{4.2.2}$$

And

$$\begin{aligned}|D_b^v\mathcal{K}_1y(t)-D_b^v\mathcal{K}_1y(\widetilde{t})|=&\Big|D_b^vI_b^pf(t,y(t),D_b^uy(t))+\frac{\xi D_b^v(b-t)^{p-1}}{\Omega_1}I_b^pf(\tau,y(\tau),D_b^uy(\tau))\\&-D_b^vI_b^pf(\widetilde{t},y(\widetilde{t}),D_b^uy(\widetilde{t}))-\frac{\xi D_b^v(b-\widetilde{t})^{p-1}}{\Omega_1}I_b^pf(\tau,y(\tau),D_b^uy(\tau))\Big|\\=&\Big|I_b^{p-v}f(t,y(t),D_b^uy(t))+\frac{\xi\Gamma(p)(b-t)^{p-v-1}}{\Gamma(p-v)\Omega_1}I_b^pf(\tau,y(\tau),D_b^uy(\tau))\\&-I_b^{p-v}f(\widetilde{t},y(\widetilde{t}),D_b^uy(\widetilde{t}))-\frac{\xi\Gamma(p)(b-\widetilde{t})^{p-v-1}}{\Gamma(p-v)\Omega_1}I_b^pf(\tau,y(\tau),D_b^uy(\tau))\Big|\\\leq&\frac{1}{\Gamma(p-v)}\int_t^{\widetilde{t}}(s-t)^{p-v-1}(\phi(s)+\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})ds\\&+\frac{1}{\Gamma(p-v)}\int_{\widetilde{t}}^b[(s-t)^{p-v-1}-(s-\widetilde{t})^{p-v-1}](\phi(s)+\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})ds\\&+\frac{\xi[(b-t)^{p-v-1}-(b-\widetilde{t})^{p-v-1}]}{\Gamma(p-v)\Omega_1}\int_\tau^b(s-\tau)^{p-1}(\phi(s)+\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})ds.\end{aligned}$$

Thus

$$\begin{aligned}|D_b^v\mathcal{K}_1y(t)-D_b^v\mathcal{K}_1y(\widetilde{t})|\leq&\frac{1}{\Gamma(p-v)}\int_t^{\widetilde{t}}(s-t)^{p-v-1}\phi(s)ds+\frac{1}{\Gamma(p-v)}\int_{\widetilde{t}}^b[(s-t)^{p-v-1}-(s-\widetilde{t})^{p-v-1}]\phi(s)ds\\&+\frac{\xi[(b-t)^{p-v-1}-(b-\widetilde{t})^{p-v-1}]}{\Gamma(p-v)\Omega_1}\int_\tau^b(s-\tau)^{p-1}\phi(s)ds+\Bigg(\frac{(\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})}{\Gamma(p-v+1)}\times\\&[(b-t)^{p-v}-(b-\widetilde{t})^{p-v}]+\frac{\xi(\sigma_1|R|^{\gamma_1}+\sigma_2|R|^{\gamma_2})}{p\Gamma(p-v)\Omega_1}(b-\tau)^p\Bigg).\end{aligned}\tag{4.2.3}$$

On similar steps it can be prove that

$$|\mathcal{K}_2x(t)-\mathcal{K}_2x(\widetilde{t})| \leq \frac{1}{\Gamma(q)}\int_t^{\widetilde{t}}(s-t)^{q-1}\psi(s)ds+\frac{1}{\Gamma(p)}\int_{\widetilde{t}}^b[(s-t)^{q-1}-(s-\widetilde{t})^{q-1}]\phi(s)ds$$
$$+\frac{\xi[(b-t)^{q-1}-(b-\widetilde{t})^{q-1}]}{\Gamma(q)\Omega_2}\int_\tau^b(s-\tau)^{q-1}\psi(s)ds+\frac{(\varrho_1|R|^{\rho_1}+\varrho_2|R|^{\rho_2})}{\Gamma(q+1)}\Big[(b-t)^q-(b-\widetilde{t})^q$$
$$+\frac{\xi[(b-t)^{p-1}-(b-\widetilde{t})^{p-1}]}{\Omega_2}(b-\tau)^q\Big]. \tag{4.2.4}$$

and

$$|D_b^u\mathcal{K}_2x(t)-D_b^u\mathcal{K}_2x(\widetilde{t})| \leq \frac{1}{\Gamma(q-u)}\int_t^{\widetilde{t}}(s-t)^{q-u-1}\psi(s)ds+\frac{1}{\Gamma(q-u)}\int_{\widetilde{t}}^b[(s-t)^{q-u-1}-(s-\widetilde{t})^{q-u-1}]\psi(s)ds$$
$$+\frac{\xi[(b-t)^{q-u-1}-(b-\widetilde{t})^{q-u-1}]}{\Gamma(q-u)\Omega_2}\int_\tau^b(s-\tau)^{p-1}\psi(s)ds+\Big(\frac{(\varrho_1|R|^{\rho_1}+\varrho_2|R|^{\rho_2})}{\Gamma(q-u+1)}\times$$
$$[(b-t)^{q-u}-(b-\widetilde{t})^{q-u}]+\frac{\xi(\varrho_1|R|^{\rho_1}+\varrho_2|R|^{\rho_2})}{p\Gamma(q-u)\Omega_2}(b-\tau)^q\Big). \tag{4.2.5}$$

Since all terms in equations (4.2.2), (4.2.3), (4.2.4) and (4.2.5) are uniformly continuous and bounded, we conclude that $\mathcal{K}S$ is equicontinuous for all real valued functions x and y. Thus Schauder fixed point theorem applies that there exists a solution of (4.1.1). □

Example 4.2.4. Consider the problem

$$\begin{cases} D_b^{3/2}x(t)=(e^t+1)+(\ln(1+D_b^{1/6}y(t))^{1/3})+\sin^2(y(t)), & t\in[a,b],\\ D_b^{5/3}y(t)=(t^2+1)+(\ln(1+D_b^{1/5}x(t)))^{1/3}+\sin^2(x(t)), & t\in[a,b],\\ x(b)=0,\quad D_b^{1/2}x(b)=\xi x(\tau),\quad y(b)=0,\quad D_b^{2/3}y(b)=\xi y(\tau). \end{cases} \tag{4.2.6}$$

Here $p=3/2$ and $q=5/3$.

$$f(t,x,y)=(e^t+1)+(\ln(1+y))^{1/3}+\sin^2(x(t)),$$

and

$$g(t,x,y)=(t^2+1)+(\ln(1+x))^{1/3}+\sin^2(y(t)).$$

So

$$\begin{aligned}|f(t,x,y)| &\leq |(e^t+1)|+|(\ln(1+y))^{1/3}|+|\sin^2(x(t))|\\ &\leq (e^t+1)+|y|^{1/3}+|x|,\end{aligned}$$

and

$$\begin{aligned}|g(t,x,y)| &\leq |(t^2+1)|+|(\ln(1+x))^{1/3}|+|\sin^2(y(t))|\\ &\leq (t^2+1)+|x|^{1/3}+|y|,\end{aligned}$$

where $\phi(t)=e^t+1\in L_1[a,b], \psi(t)=(t^2+1)\in L_1[a,b], \gamma_1=1/3, \gamma_2=1, \rho_1=1/3, \rho_2=1$ and $\sigma_1=\sigma_2=\varrho_1=\varrho_2=1$. Thus Theorem 4.2.3 verified. Hence there exist a solution of (4.2.6).

Theorem 4.2.5. *Suppose that (H_1), (H_4) and (H_5) hold. Let $l = \max(l_1+l_2) < 1$ and $k = \max(k_1+k_2) < 1$. Then there exists unique solution for coupled system of the fractional order terminal value problem (4.1.1).*

Proof. From Theorem 4.2.3 it follows that (4.1.1) has at least one solution. Now we establish the contraction of $\mathcal{K}$. Since $(b-t)^{p-1} < (b-a)^{p-1}$, then for all $x, \widetilde{x}, y, \widetilde{y}$

$$
\begin{aligned}
|\mathcal{K}_1 y(t) - \mathcal{K}_1\widetilde{y}(t)| \leq & \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}|f(s,y(s),D_b^u y(s)) - f(s,\widetilde{y}(s),D_b^u\widetilde{y}(s))|ds \\
& + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1}\int_\tau^b (s-\tau)^{p-1}|f(s,y(s),D_b^u y(s)) - f(s,\widetilde{y}(s),D_b^u\widetilde{y}(s))|ds \\
\leq & \frac{1}{\Gamma(p)}\int_t^b (s-t)^{p-1}(\delta(t)|y(s)-\widetilde{y}(s)| + \varsigma(t)|D_b^u y(s) - D_b^u\widetilde{y}(s)|)ds \\
& + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1}\int_\tau^b (s-\tau)^{p-1}(\delta(t)|y(s)-\widetilde{y}(s)| \\
& + \varsigma(t)|D_b^u y(s) - D_b^u\widetilde{y}(s)|)ds \\
\leq & \frac{1}{\Gamma(p)}\max_{t\in[a,b]}|y-\widetilde{y}|\int_t^b (s-t)^{p-1}\delta(s)ds + \frac{1}{\Gamma(p)}\max_{t\in[a,b]}|D_b^u y(t) - D_b^u\widetilde{y}(t)| \times \\
& \int_t^b (s-t)^{p-1}\varsigma(s)ds + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1}\max_{t\in[a,b]}|y-\widetilde{y}|\int_\tau^b (s-\tau)^{p-1}\delta(s)ds \\
& + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1}\max_{t\in[a,b]}|D_b^u y(t) - D_b^u\widetilde{y}(t)|\int_\tau^b (s-\tau)^{p-1}\varsigma(s)ds \\
\leq & \frac{1}{\Gamma(p)}\max_{t\in[a,b]}|y-\widetilde{y}| + \frac{1}{\Gamma(p)}\max_{t\in[a,b]}|D_b^u y(t) - D_b^u\widetilde{y}(t)| + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1}\times \\
& \max_{t\in[a,b]}|y-\widetilde{y}| + \frac{\xi(b-t)^{p-1}}{\Gamma(p)\Omega_1}\max_{t\in[a,b]}|D_b^u y(t) - D_b^u\widetilde{y}(t)| \\
\leq & \left(\frac{1}{\Gamma(p)} + \frac{\xi(b-a)^{p-1}}{\Gamma(p)\Omega_1}\max_{t\in[a,b]}|x-y|\right) \\
& + \left(\frac{1}{\Gamma(p)} + \frac{\xi(b-a)^{p-1}}{\Gamma(p)\Omega_1}\right)\max_{t\in[a,b]}|D_b^u y(t) - D_b^u\widetilde{y}(t)| \\
= & l_1\left(\max_{t\in[a,b]}|y-\widetilde{y}| + \max_{t\in[a,b]}|D_b^u y(t) - D_b^u\widetilde{y}(t)|\right),
\end{aligned}
$$

and

$$
\begin{aligned}
|D_b^v\mathcal{K}_1 y(t) - D_b^v\mathcal{K}_1\widetilde{y}(t)| = & \Big| D_b^v I_b^p\big[f(t,y(t),D_b^u y(t)) - f(t,\widetilde{y}(t),D_b^u\widetilde{y}(t))\big] + \frac{\xi D_b^v(b-t)^{p-1}}{\Omega_1} I_b^p \times \\
& \big[f(\tau,y(\tau),D_b^u y(\tau)) - f(\tau,\widetilde{y}(\tau),D_b^u\widetilde{y}(\tau))\big]\Big| \\
= & \Big| I_b^{p-v}\big[f(t,y(t),D_b^u y(t)) - f(t,\widetilde{y}(t),D_b^u\widetilde{y}(t))\big] + \frac{\xi\Gamma(p)(b-t)^{p-v-1}}{\Gamma(p-v)\Omega_1} I_b^p \times \\
& \big[f(\tau,y(\tau),D_b^u y(\tau)) - f(\tau,\widetilde{y}(\tau),D_b^u\widetilde{y}(\tau))\big]\Big|
\end{aligned}
$$

$$
\begin{aligned}
\leq & \frac{1}{\Gamma(p-v)}\int_t^b (s-t)^{p-v-1}(\delta(t)|y(s)-\widetilde{y}(s)|+\varsigma(t)|D_b^u y(s)-D_b^u\widetilde{y}(s)|)ds \\
& + \frac{\xi(b-t)^{p-v-1}}{\Gamma(p-v)\Omega_1}\int_\tau^b (s-\tau)^{p-1}(\delta(t)|y(s)-\widetilde{y}(s)| \\
& + \varsigma(t)|D_b^u y(s)-D_b^u\widetilde{y}(s)|)ds \\
\leq & \frac{\max\limits_{t\in[a,b]}|y-\widetilde{y}|}{\Gamma(p-v)}\int_t^b (s-t)^{p-v-1}\delta(s)ds + \frac{\max\limits_{t\in[a,b]}|D_b^u y(t)-D_b^u\widetilde{y}(t)|}{\Gamma(p-v)}\int_t^b (s-t)^{p-v-1}\times \\
& \varsigma(s)ds + \frac{\xi(b-t)^{p-v-1}\max\limits_{t\in[a,b]}|y-\widetilde{y}|}{\Gamma(p-v)\Omega_1}\int_\tau^b (s-\tau)^{p-1}\delta(s)ds \\
& + \frac{\xi(b-t)^{p-v-1}\max\limits_{t\in[a,b]}|D_b^u y(t)-D_b^u\widetilde{y}(t)|}{\Gamma(p-v)\Omega_1}\int_\tau^b (s-\tau)^{p-1}\varsigma(s)ds \\
\leq & \frac{1}{\Gamma(p-v)}\max_{t\in[a,b]}|y-\widetilde{y}| + \frac{1}{\Gamma(p-v)}\max_{t\in[a,b]}|D_b^u y(t)-D_b^u\widetilde{y}(t)| \\
& + \frac{\xi(b-a)^{p-v-1}}{\Gamma(p-v)\Omega_1}\max_{t\in[a,b]}|y-\widetilde{y}| \\
& + \frac{\xi(b-a)^{p-v-1}}{\Gamma(p-v)\Omega_1}\max_{t\in[a,b]}|D_b^u y(t)-D_b^u\widetilde{y}(t)| \\
= & l_2\left(\max_{t\in[a,b]}|y-\widetilde{y}| + \max_{t\in[a,b]}|D_b^u y(t)-D_b^u\widetilde{y}(t)|\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\mathcal{K}_1 y(t)-\mathcal{K}_1\widetilde{y}(t)\| \leq & \max_{t\in[a,b]}|\mathcal{K}_1 y(t)-\mathcal{K}_1\widetilde{y}(t)| + \max_{t\in[a,b]}|D_b^u\mathcal{K}_1 y(t)-D_b^u\mathcal{K}_1\widetilde{y}(t)| \\
= & l_1\left(\max_{t\in[a,b]}|y-\widetilde{y}| + \max_{t\in[a,b]}|D_b^u y(t)-D_b^u\widetilde{y}(t)|\right) + l_2\left(\max_{t\in[a,b]}|y-\widetilde{y}|\right. \\
& \left. + \max_{t\in[a,b]}|D_b^u y(t)-D_b^u\widetilde{y}(t)|\right) \\
= & (l_1+l_2)\left(\max_{t\in[a,b]}|y-\widetilde{y}| + \max_{t\in[a,b]}|D_b^u y(t)-D_b^u\widetilde{y}(t)|\right) \\
\leq & l\left(\|y-\widetilde{y}\| + \|D_b^u y(t)-D_b^u\widetilde{y}(t)\|\right)
\end{aligned}
$$

Similarly

$$
|\mathcal{K}_2 x(t)-\mathcal{K}_2\widetilde{x}(t)| \leq k_1\left(\max_{t\in[a,b]}|x-\widetilde{x}| + \max_{t\in[a,b]}|D_b^v x(t)-D_b^v\widetilde{x}(t)|\right),
$$

and

$$
|D_b^u\mathcal{K}_2 x(t)-D_b^u\mathcal{K}_2\widetilde{x}(t)| \leq k_2\left(\max_{t\in[a,b]}|x-\widetilde{x}| + \max_{t\in[a,b]}|D_b^v x(t)-D_b^v\widetilde{x}(t)|\right).
$$

Thus

$$\begin{aligned}
\|\mathcal{K}_2x(t)-\mathcal{K}_2\widetilde{x}(t)\| \leq & \max_{t\in[a,b]}|\mathcal{K}_2x(t)-\mathcal{K}_2\widetilde{x}(t)|+\max_{t\in[a,b]}|D_b^v\mathcal{K}_2x(t)-D_b^v\mathcal{K}_2\widetilde{x}(t)|\\
=&k_2\bigg(\max_{t\in[a,b]}|x-\widetilde{x}|+\max_{t\in[a,b]}|D_b^vx(t)-D_b^v\widetilde{x}(t)|\bigg)+k_2\bigg(\max_{t\in[a,b]}|x-\widetilde{x}|\\
&+\max_{t\in[a,b]}|D_b^vx(t)-D_b^v\widetilde{x}(t)|\bigg)\\
=&(k_1+k_2)\bigg(\max_{t\in[a,b]}|x-\widetilde{x}|+\max_{t\in[a,b]}|D_b^vx(t)-D_b^v\widetilde{x}(t)|\bigg)\\
\leq&k\bigg(\|x-\widetilde{x}\|+\|D_b^vx(t)-D_b^v\widetilde{x}(t)\|\bigg).
\end{aligned}$$

Hence $\mathcal{K}$ is contraction. Thus by Banach contraction theorem, unique solution of (4.1.1) exists. □

Example 4.2.6. Consider the problem

$$\begin{cases}
D_b^{3/2}x(t)=(e^t+t)+\sqrt{\ln(1+D_b^{1/5}y(t))}+\cos^2(y(t)), \quad t\in[a,b],\\
D_b^{5/3}y(t)=(t+1)^2+\sqrt{\ln(1+D_b^{1/6}x(t))}+\cos^2(x(t)), \quad t\in[a,b],\\
x(b)=0, \quad D_b^{1/2}x(b)=\xi x(\tau), \quad y(b)=0, \quad D_b^{2/3}y(b)=\xi y(\tau).
\end{cases} \tag{4.2.7}$$

Here $p=3/2$ and $q=5/3$.

$$f(t,x,y)=(e^t+t)+\sqrt{\ln(1+y)}+\cos^2(x(t)),$$

and

$$g(t,x,y)=(t+1)^2+\sqrt{\ln(1+x)}+\cos^2(y(t)).$$

Now

$$\begin{aligned}
|f(t,x,y)-f(t,\widetilde{x},\widetilde{y})| &= |(e^t+t)+\sqrt{\ln(1+y)}+\cos^2(x(t))-(e^t+t)-\sqrt{\ln(1+\widetilde{y})}-\cos^2(\widetilde{x}(t))|\\
&\leq |\sqrt{\ln(1+y)}-\sqrt{\ln(1+\widetilde{y})}|+|\cos^2(x(t))-\cos^2(\widetilde{x}(t))|\\
&\leq |y-\widetilde{y}|+|x-\widetilde{x}|,
\end{aligned}$$

where $\varsigma(t)=\delta(t)=1$. Similarly we can prove it for $g(t,x,y)$. Thus Theorem 4.2.5 verified. Hence (4.2.7) has unique solution.

Chapter 5

Terminal value problem for fractional order nonlinear differential equations on an unbounded domain

5.1 Introduction

In chapter 3 we have discussed existence and uniqueness results on bounded domain. We have also discussed coupled system of fractional terminal value problem in chapter 4. In this chapter we discuss fractional terminal value problem on unbounded domain. However many authors established existence results for boundary value problems on infinite intervals. B. Ahmad, J.J. Nieto, D. Garout and A. Alsaedi [6], A. Arara, M. Benchohra, N. Hamidi, [8], S. Liang, J. Zhang [21], X. Su, S. Zhang [33], G. Wang [39], B. Ahmad, R. P. Agarwal, L. Zhang, G.Wang [40], X. Zhao, W. Ge [42], G.Wanga, A. Cabadab, L. Zhanga [37] and Zhang [41] studied existence results for boundary value problem with fractional order on infinite interval.

M. Rehman, S.A. Hussain [25] developed adequate conditions for fractional order terminal value problem to establish existence and uniqueness results on infinite interval with right Caputo fractional derivative

$$\begin{aligned} {}^{c}D^{p}_{\infty}x(t) &= f(t, x(t), x'(t)), \quad 1 < p \leq 2, \quad t \in [a, \infty), \\ x(\infty) &= \mu, \ x'(\infty) = 0, \end{aligned}$$

where x' is derivative of x and f is continuous. G. Wanga, A. Cabadab and L. Zhanga [37] discussed the existence of the solutions for boundary conditions on an unbounded domain

$$\begin{aligned} D^{p}_{0}x(t) + f(t, x(t), x'(t)) &= 0, \quad 1 < p \leq 2, \quad t \in [0, \infty), \\ x(0) = 0, \ D^{p-1}_{0}x(\infty) &= \lambda \int_0^{\tau} x(t)dt, \end{aligned}$$

where f is continuous, $\lambda, \tau \in [0, \infty)$ and D^{p}_{0} is Riemann-Liouville derivative.

Here we establish the existence and uniqueness results for a class of terminal value problem of nonlinear fractional differential equations [27]

$$
\begin{aligned}
{}^{c}D^{p}_{\infty}x(t) &= f(t,x(t)), \quad 1<p\le 2, \quad t\in[a,\infty),\\
x(\infty) &= \lambda x(\tau), \qquad x^{'}(\infty)=0, \qquad for\ \lambda\neq 1,
\end{aligned}
\tag{5.1.1}
$$

where $0<\lambda<1$ and $\tau\in[a,\infty)$. We discuss terminal vale problem instead of initial boundary value problem on semi-infinite domain with right Caputo fractional differential operator ${}^{c}D^{p}_{\infty}$.
We establish Green's function and its properties for terminal value problem (5.1.1) on infinite interval.
For convenience of our results we have following hypothesis for all real valued functions x and y on $[a,b]$:

(H_1) $f:[a,\infty)\times\mathbb{R}\to\mathbb{R}$ is continuous.

(H_2)**:** $|f(t,x)|\le\rho(t)+\sigma(t)|x|^{\mu}$, $\quad 0<\mu\le 1$ where $\rho,\sigma\in C([a,\infty)$ are non-negative functions such that $\int_a^\infty(s-a)^{p-1}\rho(s)ds=\mathcal{G}_1<\infty$, $\int_a^\infty(s-a)^{p-1}\sigma(s)ds=\mathcal{G}_2<\infty$.

(H_3)**:** Assume there exists non-negative function $\eta\in C([a,\infty))$ such that $\int_a^\infty(s-a)^{p-1}\eta(s)ds=\mathcal{G}_3<\infty$ and

$$
|f(t,x)-f(t,y)|\le\eta(t)|x-y|.
$$

5.2 Main results

First, we determine an expression for the Green's function of fractional differential terminal value problem.

Lemma 5.2.1. *Assume that $x(t)x'(t)\le 0$ and ζ is continuous on $[a,\infty)$. Then x is solution of terminal value problem*

$$
\begin{aligned}
{}^{c}D^{p}x(t) &= \zeta(t), \quad 1<p\le 2, \quad t\in[0,\infty),\\
x(\infty) &= \lambda x(\tau), \qquad x^{'}(\infty)=0, \qquad for\ \lambda\neq 1.
\end{aligned}
\tag{5.2.1}
$$

iff x satisfies

$$
x(t)=\begin{cases}\int_t^\infty G_1(s,t)\zeta(s)ds, & if\ t\le\tau,\\ \int_\tau^\infty G_2(s,t)\zeta(s)ds, & if\ \tau\le t,\end{cases}
\tag{5.2.2}
$$

where

$$
G_1(s,t)=\begin{cases}\frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}+\frac{(s-t)^{p-1}}{\Gamma(p)}, & t\le\tau\le s,\\ \frac{(s-t)^{p-1}}{\Gamma(p)}, & t\le s\le\tau,\end{cases}
\tag{5.2.3}
$$

and

$$
G_2(s,t)=\begin{cases}\frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}, & \tau\le s\le t,\\ \frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}+\frac{(s-t)^{p-1}}{\Gamma(p)}, & \tau\le t\le s.\end{cases}
\tag{5.2.4}
$$

Proof. Applying right fractional integral operator I_b^p on both sides of equation (5.2.1) and using Theorem 1.4.8, we get

$$x(t)=x(b)+(t-b)x^{'}(b)+\int_t^b\frac{(s-t)^{p-1}}{\Gamma(p)}\zeta(s)ds.$$

From our assumption $x(t)x'(t)\le 0$, we can write $\lim_{t\longrightarrow\infty}tx'(t)=0$ (see [32]). Taking limit $b\longrightarrow\infty$, and using $x(t)x'(t)\le 0$ and from our assumed conditions we can write

$$\begin{aligned}x(t)&=\lim_{b\longrightarrow\infty}x(b)+\lim_{b\longrightarrow\infty}(t-b)x^{'}(b)+\lim_{b\longrightarrow\infty}\int_t^b\frac{(s-t)^{p-1}}{\Gamma(p)}\zeta(s)ds\\&=\lambda x(\tau)+\lim_{b\longrightarrow\infty}tx^{'}(b)-\lim_{b\longrightarrow\infty}bx^{'}(b)+\int_t^\infty\frac{(s-t)^{p-1}}{\Gamma(p)}\zeta(s)ds\\&=\lambda x(\tau)+0-0+\int_t^\infty\frac{(s-t)^{p-1}}{\Gamma(p)}\zeta(s)ds.\end{aligned}$$

Consequently

$$x(t)=\lambda x(\tau)+\int_t^\infty\frac{(s-t)^{p-1}}{\Gamma(p)}\zeta(s)ds.\tag{5.2.5}$$

Now, replacing t with τ on both sides of equation (5.2.5)

$$\begin{aligned}x(\tau)&=\lambda x(\tau)+\int_\tau^\infty\frac{(s-\tau)^{p-1}}{\Gamma(p)}\zeta(s)ds\\x(\tau)(1-\lambda)&=\int_\tau^\infty\frac{(s-\tau)^{p-1}}{\Gamma(p)}\zeta(s)ds\\x(\tau)&=\frac{1}{(1-\lambda)}\int_\tau^\infty\frac{(s-\tau)^{p-1}}{\Gamma(p)}\zeta(s)ds.\end{aligned}\tag{5.2.6}$$

Substituting equation (5.2.6) in equation (5.2.5), we get

$$x(t)=\frac{\lambda}{(1-\lambda)}\int_\tau^\infty\frac{(s-\tau)^{p-1}}{\Gamma(p)}\zeta(s)ds+\int_t^\infty\frac{(s-t)^{p-1}}{\Gamma(p)}\zeta(s)ds.\tag{5.2.7}$$

Here arises two cases, $t\le\tau$ and $\tau\le t$.

For $t\le\tau$, equation (5.2.7) gives

$$\begin{aligned}x(t)&=\frac{\lambda}{(1-\lambda)}\int_\tau^\infty\frac{(s-\tau)^{p-1}}{\Gamma(p)}\zeta(s)ds+\int_t^\tau\frac{(s-t)^{p-1}}{\Gamma(p)}\zeta(s)ds+\int_\tau^\infty\frac{(s-t)^{p-1}}{\Gamma(p)}\zeta(s)ds\\&=\int_\tau^\infty\left(\frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}+\frac{(s-t)^{p-1}}{\Gamma(p)}\right)\zeta(s)ds+\int_t^\tau\frac{(s-t)^{p-1}}{\Gamma(p)}\zeta(s)ds\\&=\int_t^\infty G_1(s,t)\zeta(s)ds\end{aligned}$$

where

$$G_1(s,t)=\begin{cases}\frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}+\frac{(s-t)^{p-1}}{\Gamma(p)}, & t\le\tau\le s,\\ \frac{(s-t)^{p-1}}{\Gamma(p)}, & t\le s\le\tau,\end{cases}\tag{5.2.8}$$

Now, for $\tau\le t$, we get from equation (5.2.7)

$$\begin{aligned}x(t)&=\frac{\lambda}{(1-\lambda)}\int_\tau^t\frac{(s-\tau)^{p-1}}{\Gamma(p)}\zeta(s)ds+\frac{\xi}{(1-\xi)}\int_t^\infty\frac{(s-\tau)^{p-1}}{\Gamma(p)}\zeta(s)ds+\int_t^\infty\frac{(s-t)^{p-1}}{\Gamma(p)}\zeta(s)ds\\&=\frac{\lambda}{(1-\lambda)}\int_\tau^t\frac{(s-\tau)^{p-1}}{\Gamma(p)}\zeta(s)ds+\int_t^\infty\left(\frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}+\frac{(s-t)^{p-1}}{\Gamma(p)}\right)\zeta(s)ds\\&=\int_\tau^\infty G_2(s,t)\zeta(s)ds\end{aligned}$$

where

$$G_2(s,t) = \begin{cases} \frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}, & \tau \le s \le t, \\ \frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)} + \frac{(s-t)^{p-1}}{\Gamma(p)}, & \tau \le t \le s. \end{cases} \tag{5.2.9}$$

Combining both cases, we can write

$$x(t) = \begin{cases} \int_t^\infty G_1(s,t)\zeta(s)ds, & if\ t \le \tau, \\ \int_\tau^\infty G_2(s,t)\zeta(s)ds, & if\ \tau \le t. \end{cases}$$

□

5.2.1 Properties of The Green's function

Here we discuss important properties of Green's function.

(a) $G_1(s,t) \ge 0,\ G_2(s,t) \ge 0$.

(b) $G_1(s,t) \le \frac{(1+\lambda)(s-a)^{p-1}}{(1-\lambda)\Gamma(p)}$.

(c) $\frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)} \le G_2(s,t) \le \frac{(1+\lambda)(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}$.

Proof. (a) The conclusion is obvious.

(b) If $a \le t$, then for $t \le \tau \le s$, we have $(s-\tau)^{p-1} \le (s-a)^{p-1}$ and $(s-t)^{p-1} \le (s-a)^{p-1}$. Thus

$$\begin{aligned} G_1(s,t) &= \frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)} + \frac{(s-t)^{p-1}}{\Gamma(p)} \\ &\le \frac{\lambda(s-a)^{p-1}}{(1-\lambda)\Gamma(p)} + \frac{(s-a)^{p-1}}{\Gamma(p)} \\ &= \frac{(1+\lambda)(s-a)^{p-1}}{(1-\lambda)\Gamma(p)}. \end{aligned}$$

Similarly for $t \le s \le \tau$, we have

$$G_1(s,t) \le \frac{(s-a)^{p-1}}{\Gamma(p)} \le \frac{(1+\lambda)(s-a)^{p-1}}{(1-\lambda)\Gamma(p)}.$$

as $\frac{1+\lambda}{1-\lambda} \ge 1$. Thus for all s, t and τ, $G_1(s,t) \le \frac{(1+\lambda)(s-a)^{p-1}}{(1-\lambda)\Gamma(p)}$.

(c) Obviously $G_2(s,t) \ge \frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}$, for all s and τ. For $\tau \le s \le t$, we have $G_2(s,t) \le \frac{(1+\lambda)(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}$. Now for $\tau \le t \le s$, we have

$$\begin{aligned} G_2(s,t) &= \frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)} + \frac{(s-t)^{p-1}}{\Gamma(p)} \\ &\le \frac{\lambda(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)} + \frac{(s-\tau)^{p-1}}{\Gamma(p)} \\ &= \frac{(1+\lambda)(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}. \end{aligned}$$

□

5.3 Existence and uniqueness of solutions

Let us define the space

$$\mathcal{X} = \{x(t) \in C(0,\infty) : \|x\| < \infty\},$$

with norm

$$\|x\| = \sup_{t\in[0,\infty)} |x(t)|.$$

Let $\mathcal{T}$ is operator define by

$$\mathcal{T}x(t) = \begin{cases} \int_t^\infty G_1(s,t)f(s,x(s))ds, & if\ t \le \tau, \\ \int_\tau^\infty G_2(s,t)f(s,x(s))ds, & if\ \tau \le t. \end{cases} \tag{5.3.1}$$

Solutions of problem (5.1.1) are fixed points of $\mathcal{T}$.

Theorem 5.3.1. *Assume* (H_1) *and* (H_2) *hold. Then at least one solution of* (5.1.1) *exists.*

Proof. Choose $\mathcal{R} \ge \frac{\widetilde{a}\mathcal{G}_1}{1+\widetilde{a}\mathcal{G}_2}$, where $\widetilde{a} := \frac{1+\lambda}{(1-\lambda)\Gamma(p)}$. Define the set $\mathcal{A} := \{x \in \mathcal{B} : \|x\| \le \mathcal{R}\}$, where $\mathcal{B}$ is a banach space and $\mathcal{A} \subseteq \mathcal{B}$, that is closed and convex. Let $u \in \mathcal{A}$ be an arbitrary element, then we have following estimate for $\mathcal{T}x$:
For $t \le \tau$, using property (b) of G_1

$$\begin{aligned} |\mathcal{T}x(t)| &\le \int_t^\infty G_1(t,s)|f(s,x(s))|ds \\ &\le \int_t^\infty \frac{(1+\lambda)(s-a)^{p-1}}{(1-\lambda)\Gamma(p)}|f(s,x(s))|ds \\ &\le \widetilde{a}\int_a^\infty (s-a)^{p-1}(\rho(s)+\sigma(s)|x|^\mu ds. \end{aligned}$$

From hypothesis (H_2) we have $|\mathcal{T}x(t)| \le \widetilde{a}(\mathcal{G}_1 + \mathcal{G}_2\mathcal{R}^\mu) \le \mathcal{R}$.
For $t \ge \tau$, the property (c) of G_2 and the inequality $(s-\tau)^{p-1} \le (s-a)^{p-1}$ lead us to the estimate

$$\begin{aligned} |\mathcal{T}x(t)| &\le \int_\eta^\infty G_2(t,s)|f(s,x(s))|ds \\ &\le \int_\tau^\infty \frac{(1+\lambda)(s-\tau)^{p-1}}{(1-\lambda)\Gamma(p)}|f(s,x(s))|ds \\ &\le \widetilde{a}\int_a^\infty (s-a)^{p-1}(\phi(s)+\psi(s)|x|^\mu ds. \end{aligned}$$

From hypothesis (H_2) we have $|\mathcal{T}x(t)| \le \widetilde{a}(\mathcal{G}_1 + \mathcal{G}_2\mathcal{R}^\mu) \le \mathcal{R}$. Thus $\|\mathcal{T}x(t)\| \le \mathcal{R}$ for all $x \in \mathcal{A}$. Hence $\mathcal{T}(\mathcal{A}) \subset \mathcal{A}$.
Now we show that $\mathcal{T}(\mathcal{A})$ is equicontinuous. Let $a \le t \le \widetilde{t} < \infty$, then for $t \le \tau$

$$\begin{aligned} |\mathcal{T}x(t) - \mathcal{T}x(\widetilde{t})| &= \left| \int_t^\infty G_1(t,s)f(s,x(s))ds - \int_{\widetilde{t}}^\infty G_1(\widetilde{t},s)f(s,x(s))ds \right| \\ &\le \int_t^{\widetilde{t}} G_1(t,s)|f(s,x(s))|ds + \int_{\widetilde{t}}^\infty |G_1(t,s) - G_1(\widetilde{t},s)|f(s,x(s))|ds. \end{aligned}$$

We observe that $|G_1(t,s)-G_1(\widetilde{t},s)| = |\frac{1}{\Gamma(p)}((s-t)^{p-1}-(s-\widetilde{t})^{p-1})| \le \frac{2}{\Gamma(p)}(s-t)^{p-1}$. $|G_1(t,s)-G_1(\widetilde{t},s)| \to 0$ as $t \to \widetilde{t}$. From hypothesis (H_2), we have

$$\begin{aligned}
|G_1(t,s) - G_1(\widetilde{t},s)||f(t,x(t))| &\le \frac{2}{\Gamma(p)}(s-t)^{p-1}|f(t,x(t))| \\
&\le \frac{2}{\Gamma(p)}(s-a)^{p-1}(\rho(t)+\sigma(x)|x|^{\mu}) \\
&\le \frac{2}{\Gamma(p)}(s-a)^{p-1}(\rho(t)+R^{\mu}\sigma(t)) \in L_1[a,\infty).
\end{aligned}$$

Also $|G_1(t,s) - G_1(\widetilde{t},s)||f(t,x(t))| \to 0$ as $t \to \widetilde{t}$ for all $s \in [0,\infty)$. Thus by Lebesgue Dominated Convergence Theorem we have $\int_{\widetilde{t}}^{\infty} |G_1(t,s) - G_1(\widetilde{t},s)||f(t,t(t))|ds \to 0$ as $t \to \widetilde{t}$.
Furthermore

$$\begin{aligned}
\int_t^{\widetilde{t}} G_1(t,s)|f(s,x(s))|ds &\le \int_t^{\widetilde{t}} \frac{(1+\lambda)(s-a)^{p-1}}{(1-\lambda)\Gamma(p)}|f(s,x(s))|ds \\
&\le \widetilde{a}\widetilde{A}\int_t^{\widetilde{t}}(s-a)^{p-1}ds \\
&= \frac{1}{p}\widetilde{a}\widetilde{A}((\widetilde{t}-a)^p - (\widetilde{t}-a)^p) \\
&= \frac{p-1}{p}\widetilde{a}\widetilde{A}(\varrho-a)^{p-1}(\widetilde{t}-t) \quad \text{for some} \quad \varrho \in (t,\widetilde{t}),
\end{aligned}$$

where $\widetilde{A} \le \max\limits_{\substack{x\in\mathcal{A} \\ s\in[t,\widetilde{t}]}}|f(s,x(s))|$. Therefore $\int_t^{\widetilde{t}} G_1(t,s)|f(s,x(s))|ds \to 0$ as $t \to \widetilde{t}$. Consequently, for the case, $t \le \tau$, we conclude $\|\mathcal{T}x(t) - \mathcal{T}x(\widetilde{t})\| \to 0$ as $t \to \widetilde{t}$.
When $t \ge \tau$,

$$|\mathcal{T}x(t) - \mathcal{T}x(\widetilde{t})| \le \int_{\tau}^{\infty} |G_2(t,s) - G_2(\widetilde{t},s)|f(s,x(s))|ds.$$

Repeating the same arguments as for the case $t \le \tau$, we have

$$|G_2(t,s) - G_2(\widetilde{t},s)||f(t,x(t))| \le \frac{2}{\Gamma(p)}(s-a)^{p-1}(\rho(t)+\mathcal{R}^{\mu}\sigma(t)) \in L_1[a,\infty).$$

Again, by Lebesgue Dominated Convergence Theorem we have $\int_{\eta}^{\infty} |G_1(t,s)-G_1(\widetilde{t},s)||f(t,x(t))|ds \to 0$ as $t \to \widetilde{t}$. Thus we conclude $\|\mathcal{T}x(t) - \mathcal{T}x(\widetilde{t})\| \to 0$ as $t \to \widetilde{t}$ for all $t \in [0,\infty)$. Hence $\mathcal{T}(\mathcal{A})$ is equicontinuous. Furthermore $\mathcal{T} : \mathcal{A} \to \mathcal{A}$ is compact operator by Arzela-Ascoli theorem. Thus there exists a fixed point of $\mathcal{T}$, by Schauder's fixed point theorem, which is the solution of (5.1.1). □

Theorem 5.3.2. *Assume* (H_1), (H_3) *hold and* $\mathcal{Q} := \frac{(1+\lambda)\mathcal{G}_3}{(1-\lambda)\Gamma(p)} < 1$. *Then* (5.1.1) *has unique solution.*

Proof. From Theorem 5.3.1, it follows that $\mathcal{T}$ maps closed bounded subset of Banach space into itself. Here we shall only show that under the hypothesis in the statement the operator $\mathcal{T}$ defined in (5.3.1) is contraction. For this, let x, y be arbitrary elements of the Banach space $\mathcal{B}$.

For $t \leq \tau$, assumption (H_3) and property of G_1 gives

$$\begin{aligned}|\mathcal{T}x(t) - \mathcal{T}y(t)| &\leq \int_t^\infty G_1(t,s)|f(s,x(s)) - f(s,y(s))|ds \\ &\leq \frac{1+\lambda}{(1-\lambda)\Gamma(p)} \int_t^\infty (s-a)^{p-1}\chi(s)|x(s) - y(s)|ds \\ &\leq \frac{1+\lambda}{(1-\lambda)\Gamma(p)} \|x-y\| \int_a^\infty (s-a)^{p-1}\chi(s)ds \\ &\leq \mathcal{Q}\|x-y\|.\end{aligned}$$

Similar computations for the case $t \geq \tau$ lead to inequality

$$|\mathcal{T}x(t) - \mathcal{T}y(t)| \leq \mathcal{Q}\|x-y\|.$$

Therefore, $\mathcal{T}$ is contraction. Hence by Banach fixed point theorem (5.1.1) has unique solution. □

Example 5.3.3. Consider the following problem on $I = [0,\infty)$:

$$\begin{aligned}{}^cD^p_\infty x(t) =& \frac{e^{-t^2}}{(t+1)^{p-1}} \sin^2(\sqrt{x(t)}), \quad \text{for all } t \in I, \quad 1 < p \leq 2, \\ x(\tau) =& \lambda x(\infty), \quad x'(\infty) = 0, \quad \tau \in I.\end{aligned} \tag{5.3.2}$$

Where $f(t,x(t)) = \frac{e^{-t^2}}{(t+1)^{p-1}} \sin^2(\sqrt{x(t)})$.
First, we will verify the conditions of Theorem 5.3.1

$$\begin{aligned}|f(t,x(t))| &= \left|\frac{e^{-t^2}}{(t+1)^{p-1}} \sin^2(\sqrt{x(t)})\right| \\ &= \left|\frac{e^{-t^2}}{(t+1)^{p-1}} - \frac{e^{-t^2}}{(t+1)^{p-1}} \cos^2(\sqrt{x(t)})\right| \\ &\leq \left|\frac{e^{-t^2}}{(t+1)^{p-1}}\right| + \left|\frac{e^{-t^2}}{(t+1)^{p-1}} \cos^2(\sqrt{x(t)})\right| \\ &\leq \frac{e^{-t^2}}{(t+1)^{p-1}} + \frac{e^{-t^2}}{(t+1)^{p-1}}|x(t)|^{1/2} \\ &= \rho(t) + \sigma(t)|x(t)|^{1/2},\end{aligned}$$

where $\rho(t) = \sigma(t) = \frac{e^{-t^2}}{(t+1)^{p-1}}$ and $\mu = 1/2$.
So,

$$\begin{aligned}\mathcal{G}_1 = \mathcal{G}_2 &= \int_0^\infty s^{p-1} \frac{e^{-s^2}}{(s+1)^{p-1}} ds \\ &\leq \int_0^\infty e^{-s^2} ds = \frac{\sqrt{\pi}}{2} < \infty.\end{aligned}$$

Now, we will verify the conditions of Theorem 5.3.2. For $x, y \in \mathbb{R}$ we observe that

$$\begin{aligned}|f(t,x(t)) - f(t,y(t))| &= \left|\frac{e^{-t^2}}{(t+1)^{p-1}}\sin^2(\sqrt{x(t)}) - \frac{e^{-t^2}}{(t+1)^{p-1}}\sin^2(\sqrt{y(t)})\right| \\ &= \frac{e^{-t^2}}{(t+1)^{p-1}}\left|\sin^2(\sqrt{x(t)}) - \sin^2(\sqrt{y(t)})\right| \\ &\leq \frac{e^{-t^2}}{(t+1)^{p-1}}|x-y| = \zeta(t)|x-y|,\end{aligned}$$

where $\zeta(t) = \frac{e^{-t^2}}{(t+1)^{p-1}}$.

So

$$\mathcal{G}_3 = \int_0^\infty s^{p-1}\frac{e^{-s^2}}{(s+1)^{p-1}} = \frac{\sqrt{\pi}}{2} < \infty.$$

Hence Theorem 5.3.2 verified. Therefore there is a unique solution of (5.3.2).

Example 5.3.4. Consider the following problem on $I = [1,\infty)$:

$$\begin{aligned}{}^cD^p_\infty x(t) =& \frac{e^{-t}}{t(t^2+1)^p}\ln(1+|x(t)|), \quad \text{for all } t \in I, \quad 1 < p \leq 2, \\ x(\tau) =& \lambda x(\infty), \quad x'(\infty) = 0, \quad \tau \in I.\end{aligned} \tag{5.3.3}$$

Where $f(t,x(t)) = \frac{e^{-t}}{t(t^2+1)^p}\ln(1+|x(t)|)$.

First, we will verify the conditions of Theorem 5.3.1

$$\begin{aligned}|f(t,x(t))| &= \left|\frac{e^{-t}}{t(t^2+1)^p}\ln(1+|x(t)|)\right| \\ &\leq \left|\frac{e^{-t}}{t(t^2+1)^p}|x(t)|\right| \\ &\leq \frac{e^{-t}}{t(t^2+1)^p}|x(t)| \\ &= \sigma(t)|x(t)|,\end{aligned}$$

where $\rho(t) = 0$, $\sigma(t) = \frac{e^{-t}}{t(t^2+1)^p}$ and $\mu = 1$.

So,

$$\mathcal{G}_1 = 0,$$

and

$$\begin{aligned}\mathcal{G}_2 &= \int_1^\infty (s-1)^{p-1}\frac{e^{-s}}{s(s^2+1)^p}ds \\ &\leq \int_1^\infty e^{-s}ds = \frac{1}{e} < \infty.\end{aligned}$$

Now, to verify the conditions of Theorem 5.3.2. let for $x, y \in \mathbb{R}$ we have

$$\begin{aligned}|f(t,x(t)) - f(t,y(t))| &= \left|\frac{e^{-t}}{t(t^2+1)^p}\ln(x(t)) - \frac{e^{-t}}{t(t^2+1)^p}\ln(y(t))\right| \\ &= \frac{e^{-t}}{t(t^2+1)^p}|\ln(x(t)) - \ln(y(t))| \\ &\leq \frac{e^{-t}}{t(t^2+1)^p}|x-y| = \eta(t)|x-y|.\end{aligned}$$

where $\eta(t) = \frac{e^{-t}}{t(t^2+1)^p}$.
So

$$\begin{aligned} \mathcal{G}_3 &= \int_1^\infty (s-1)^{p-1} \frac{e^{-s}}{s(s^2+1)^p} \\ &\leq \int_1^\infty e^{-s} ds = \frac{1}{e} < \infty. \end{aligned}$$

Hence, all conditions are satisfied. Therefore, Theorem 5.3.1 and Theorem 5.3.2 implies that (5.3.3) has a unique solution.

Chapter 6

Conclusion

We discussed basic definitions and some essential properties for the right fractional calculus and established some new properties and results. Particularly we discussed Riemann-Liouville and Caputo operators. We developed a generalized Taylor's formula for right fractional calculus.

We established sufficient conditions for existence and uniqueness results for terminal value problems on bounded domain. Also we developed the existence and uniqueness results of coupled system for non-linear right fractional differential equations on bounded domain. Right Riemann-Liouville fractional derivative has been used in these results.

Finally, we set up three point terminal value conditions for the existence and uniqueness results on unbounded domain with right fractional Caputo derivative.

We constructed all existence and uniqueness results by employing Banach contraction theorem and Schauder's fixed point theorem.

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