Extension of Hermite-Hadamard Inequality

by

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Dedicated to My Parents

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Abstract

The concept of convex functions has indeed found an important place in modern mathematics as can be seen in a large number of research articles and books devoted to the field these days. The Hermite-Hadamard inequality, which, we can say, is the first fundamental result for convex functions with a natural geometrical interpretation and many applications, has attracted and continues to attract much interest in elementary mathematics.

In this thesis some new generalization related to the Hermite-Hadamard inequalities for superquadratic functions has been presented and Hermite-Hadamard inequalities via fractional integrals is also discussed. Some fundamental results like mean value theorems, Cauchy type means and exponential convexity have been developed for both cases of Hermite-Hadamard inequalities.

Contents

Chapter 1

Introduction and Preliminaries

Convex functions are very important in the theory of inequalities and play an important role in many areas of mathematics. They are especially important in the study of optimization problems where they are distinguished by a number of convenient properties. For instance, a (strictly) convex function on an open set has no more than one minimum. Even in infinite-dimensional spaces, under suitable additional hypotheses, convex functions continue to satisfy such properties and, as a result, they are the most well-understood functionals in the calculus of variations.

1.1 Convex Function of One Variable

A set C is *convex* if the line segment between any two points in C lies in C, that is, for any x, $y \in C$ and any λ with $0 \leq \lambda \leq 1$, we have

$$\lambda x + (1 - \lambda)y \in C. \tag{1.1.1}$$

Definition 1.1.1. Let I be an interval in $\mathbb{R}$. Then $f : \mathrm{I} \to \mathbb{R}$ is said to be *convex function* if for all x, $y \in I$ and all $0 \leq \lambda \leq 1$, the following inequality holds

$$f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y). \tag{1.1.2}$$

If above inequality is strict for all $x \neq y$ and $0 < \lambda < 1$ then, f is said to be strictly *convex*. lf the inequality is reversed, then f is said to be *concave* and for an *affine function* we have always equality in (1.1.2).

Definition 1.1.2. A very convenient equivalent definition of a *convex function* is in terms of its *epigraph*. Given a real-valued function $f : \mathbb{R} \to \mathbb{R}$, we define its *epigraph* as the set

$$epi f = \{(x,t) \in \mathbb{R}^2 | x \in \mathbb{R}, f(x) \leq t\}.$$

A function f is convex if and only if $epi f$ is convex.

1.1.1 Continuity and Differentiabillty of Convex Function

A function $f : I \to \mathbb{R}$ is said to be *midpoint convex function* if it satisfies the inequality

$$f\left(\frac{x+y}{2}\right) \leq \frac{f(x)+f(y)}{2}, \quad \text{for all } x,\ y \in I.$$

A convex function defined on some open interval is continuous and Lipschitz continuous on any closed subinterval. A continuous function that is midpoint convex will be convex.

Preposition 1.1.3 [8]. *Let a function $f : I \to \mathbb{R}$ is continuous. Then f is convex if and only if f is midpoint convex.*

Definition 1.1.4. Let $I \subset \mathbb{R}$ be an interval. A function $f : I \to \mathbb{R}$ is said to be is *absolutely continuous* on an interval I if for every positive number ϵ, there exists a positive number δ such that

$$\sum_i |f(\xi_i - \eta_i)| < \epsilon,$$

whenever a finite sequence of pairwise disjoint subintervals $(\xi_i,\ \eta_i)$ of I satisfies

$$\sum_i (\xi_i - \eta_i) < \delta.$$

Theorem 1.1.5 [5]. *If $f : I \to \mathbb{R}$ is convex then f satisfies the Lipschitz condition on any closed interval $[a, b]$ contained in the interior of I, that is there is a constant K so that for any two points $x,\ y \in [a, b]$,*

$$|\ f(x) - f(y)\ | \leq K\ |\ x - y\ |\ .$$

Consequently, f is absolutely continuous on [a, b] and continuous on interior of I.
The derivative of a convex function is best studied in terms of the left and right derivatives defined as

$$f'_-(x) = \lim_{y \uparrow x} \frac{f(y) - f(x)}{y - x}, \quad f'_+(x) = \lim_{y \downarrow x} \frac{f(y) - f(x)}{y - x},$$

and these are monotonically non-decreasing.

Theorem 1.1.6 [8]. *Let $I \subseteq \mathbb{R}$ be a convex set. A differentiable function $f : I \to \mathbb{R}$ is convex if and only if*

$$f^{'}(y) \leq \frac{f(x) - f(y)}{x - y}, \quad \text{for all } x,\ y\ \in\ I.$$

A function which is differentiable is convex on an interval if and only if its derivative is monotonically non-decreasing on that interval. If a differentiable function is convex then it is also continuously differentiable. A twice differentiable function f of one variable defined on the interval I is convex if its second derivative $f^{''} \geq 0$ for all $x \in I$.

1.2 Convex Function of Several Variables

In this section, the definition of a convex function of two or more variables are presented. We can extend the inequality given in the definition of convex function to the convex combination of finitely many points in a convex set C. This extension is known as discrete Jensen's inequality.

Definition 1.2.1. Consider a convex function $h : M \to \mathbb{R}$, where the domain $M \subseteq \mathbb{R}^n$ is a convex set in the n-dimensional euclidean space, then

$$h\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i h(x_i), \quad \text{for all } x_i \in M,$$

is called *Jensens Inequality*, where $\lambda_i \geq 0$ for all $i = 1, ..., n$ and $\sum_{i=1}^{n} \lambda_i = 1$.

Definition 1.2.2. The *epigraph* of a function h over $\mathbb{R}^n$ is the following set in $\mathbb{R}^{n+1}$

$$epih = \left\{(x, t) \in \mathbb{R}^{n+1} | x \in \mathbb{R}^n, h(x) \leq t\right\}.$$

Proposition 1.2.3 [8]. *A function h defined on a subset of $\mathbb{R}^n$ is convex if and only if its epigraph is a nonempty convex set in $\mathbb{R}^{n+1}$.*

Definition 1.2.4. The *gradient* of a function $h(x_1,\ .\ .\ .\ ,\ x_n)$ for each $x \in \mathbb{R}^n$ is given by

$$\bigtriangledown h(x_1,\ .\ .\ .\ ,\ x_n) = \left[\frac{\delta h}{\delta x_1},\ \frac{\delta h}{\delta x_2},\ .\ .\ .\ ,\ \frac{\delta h}{\delta x_n}\right].$$

Theorem 1.2.5 [13]. *Let $M \subseteq \mathbb{R}^n$ be a convex set. A differentiable function $h : M \to \mathbb{R}$ is convex if and only if*

$$h(y) \geq h(x) + \bigtriangledown h(x)^T (y - x), \quad for\ all\ x,\ y\ \in\ M.$$

Definition 1.2.6. The *Hessian* matrix of a function $h(x_1, \ldots, x_n)$ is a matrix given by

$$H_h(x_1, \ldots, x_n) = \left[\frac{\delta^2 h}{\delta x_i \delta x_j}\right] = \bigtriangledown^2 h, \quad \text{for } i,\ j = 1, \ldots, n.$$

Theorem 1.2.7 [13]. *Let $h : \mathbb{R}^n \to \mathbb{R}$ be twice continuously differentiable and $M \subseteq \mathbb{R}^n$ convex. Then h is convex on M if and only if the Hessian matrix $H_h(x)$ is positive semi-definite for all $x \in M$.*

Some examples of convex functions are given below:
- e^{ax} is convex on $\mathbb{R}$, for any $a \in \mathbb{R}$.
- $|x|^p$ is convex on $\mathbb{R}$ for $p \geq 1$.
- Every norm on $\mathbb{R}^n$ is convex.

1.3 Hermite-Hadamard Inequality

Hermite-Hadamard integral inequality is considered to be one of the most well-known inequality in mathematics for convex functions.

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)dt \leq \frac{f(a)+f(b)}{2}, \tag{1.3.1}$$

provided that for an interval $[a, b] \subseteq \mathbb{R}$, $f : [a, b] \to \mathbb{R}$ is a convex function. If the function f is concave, then the above inequality holds in the reverse direction. These inequalities for convex functions play an vital role in nonlinear analysis. In recent years there have been many extensions, generalizations and similar type results of the inequalities (1.3.1) can be found in [9].
These classical inequalities have been improved and generalized in many ways and applied for special means including Stolarsky-type means, logarithmic and p-logarithmic means. Also, many interesting applications of Hermite-Hadamard inequality can be found in [13].

1.4 Exponential Convexity

A function $h : (a, b) \to \mathbb{R}$ is *exponentially convex* if it is continuous and

$$\sum_{i,j=1}^{n} u_i u_j h(x_i + x_j) \geq 0,$$

for all $n \in \mathbb{N}$ and all choices $u_i \in \mathbb{R}$, $i = 1, 2, ..., n$ and $x_i \in (a, b)$, such that $x_i + x_j \in (a, b)$, $1 \leq i, j \leq n$.

Proposition 1.3.1. [3] *Let $h : (a, b) \to \mathbb{R}$. The following are equivalent:*
(i) h is exponential convex,
(ii) h is continuous and

$$\sum_{i,j=1}^{n} u_i u_j h\left(\frac{x_i + x_j}{2}\right) \geq 0,$$

for every $u_i \in \mathbb{R}$ and every x_i, $x_j \in (a, b)$, $1 \leq i, j \leq n$,
(iii) h is continuous and

$$\det\left[h\left(\frac{x_i + x_j}{2}\right)\right]_{i,j=1}^{m} \geq 0, \quad 1 \leq m \leq n,$$

for every $x_i \in (a, b)$, $i = 1, 2, ..., n$.

Corollary. [3] *If $h : (a, b) \to (0, \infty)$ is exponentially convex function, then h is a log-convex function:*

$$h\left(\frac{x + y}{2}\right) \leq \sqrt{h(x)h(y)}\ , \tag{1.4.1}$$

for all x, $y \in$ (a,b).

1.5 Cauchy Means

Mean-value theorems are of great importance in mathematical analysis. In particular, the Lagrange type and the Cauchy type mean-value theorems are most frequently used. The usual approach is to prove first the Lagrange type mean value theorems and then deduce from them the Cauchy type mean value theorems. We use Mean value theorem and its other generalized version to define new Cauchy means [12].

It states if two function $f(x)$ and $g(x)$ are continuous on closed interval $[a, b]$ and differentiable on (a, b) further that $g^{'}(x) \neq 0$, then there exists at least one c with $a < c < b$ satisfying

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f^{'}(c)}{g^{'}(c)}.$$

If the function $\frac{f'}{g'}$ is invertible, then the existence of c is unique and

$$c = \left(\frac{f'}{g'}\right)^{-1} \left(\frac{f(b) - f(a)}{g(b) - g(a)}\right). \tag{1.5.1}$$

This number c is called Cauchy mean value of the numbers a, b.

Chapter 2

Hermite-Hadamard Inequalities for Superquadratic Functions and Cauchy Type Means

In this chapter, some new generalizations are considered related to the Hermite-Hadamard inequality for superquadratic functions. Also defined mean value theorem, Cauchy means, and positive semi-definiteness, exponential convexity, log-convexity, that are associated with Hermite-Hadamard inequalities for superquadratic functions.

2.1 Superquadratic Functions

In this section, superquadratic functions are defined. Some results and examples related to superquadratic functions are also discussed.

Definition 2.1.1. A function $\phi : [0, \infty) \to \mathbb{R}$ is *superquadratic* provided that for all $x \geq 0$ there exists a constant $C(x) \in \mathbb{R}$ such that

$$\phi(y) - \phi(x) - \phi(|y - x|) \geq C(x)(y - x) \tag{2.1.1}$$

for all $y \geq 0$. We says that ϕ is subquadratic if -ϕ is a superquadratic function.

Lemma 2.1.2 [7]. *Let ϕ be a superquadratic function with C(x) as in above definition. Then*
(i) $\phi(0) \leq 0$,
(ii) If $\phi(0) = \phi^{'}(0) = 0$ then $C(x) = \phi^{'}(x)$ whenever ϕ is differentiable at $x > 0$.
(iii) If $\phi \geq 0$, then ϕ is convex and $\phi(0) = \phi^{'}(0) = 0$.

Lemma 2.1.3 [7]. *Suppose ϕ is differentiable and $\phi(0) = \phi^{'}(0) = 0$. If ϕ is superquadratic then $\phi(x)/x^2$ is non decreasing on $(0, \infty)$.*

A function $\phi : [0, \infty) \to \mathbb{R}$ is superadditive provided $\phi(p+q) \geq \phi(p) + \phi(q)$ for all $p,\ q \geq 0$.

Lemma 2.1.4. [1] *Suppose $\phi : [0, \infty) \to \mathbb{R}$ is continuously differentiable and $\phi(0) \leq 0$. If $\phi^{'}$ is superadditive or $\phi'(p)/p$ is non decreasing then ϕ is superquadratic.*

Example 2.1.5. The function $\phi(x) = x^q$ is superquadratic for $q \geq 2$ and subquadratic for $q \in (0,\ 2]$.

Proposition 2.1.6 [1] *Let $u : (0,\ \infty) \to \mathbb{R}$ be a continuously differentiable and a non decreasing function with*

$$\lim_{t \to 0+} tu(t) = 0 \quad \textit{and} \quad \lim_{t \to 0+} t^2 u(t) = 0, \tag{2.1.2}$$

such that the function $t \mapsto tu^{'}(t)$ is non decreasing.
Then the function $f : [0,\ \infty) \to \mathbb{R}$ defined by

$$f(t) = t^2 u(t), \quad t > 0,$$

is a differentiable and superquadratic function with $f(0) = f'(0) = 0$.

Example 2.1.7. The function $u(t) = \text{ln}t,\ t \in (0, \infty)$, is differentiable and non decreasing function which satisfies (2.1.2). Also $tu'(t) = 1$ is non decreasing. Therefore

$$f(t) = t^2 u(t) = t^2 \text{ln} t, \quad t > 0,$$

is a differentiable and superquadratic function with $f(0) = f'(0) = 0$ but f is not a convex function.

Proposition 2.1.8 [1]. *Let $v : (0,\ \infty) \to \mathbb{R}$ be a continuous and non decreasing function with $\lim\limits_{t \to 0+} tv(t) = 0$. Then the function $h : [0,\ \infty) \to \mathbb{R}$ defined by*

$$h(t) = \int_0^t xv(x)dx, \quad t > 0,$$

is a differentiable and superquadratic function with $h(0) = h'(0) = 0$.

Proof. We have $h'(t) = tv(t)$ for $t > 0$, so that $\lim\limits_{t \to 0+} h'(t) = 0 = h'(0)$. Hence, h is continuously differentiable on $[0, \infty)$. By our assumption $h'(t)/t = v(t)$ is non decreasing, so that h is a superquadratic function by Lemma (2.1.3). □

Example 2.1.9. The function $v(t) = (t-2)/\sqrt{t^2+1}$, $t > 0$ is non decreasing on $(0,\infty)$ and $\lim\limits_{t\to 0+} tv(t) = 0$. Therefore the function

$$h(t) = \int_0^t \frac{x(x-2)}{\sqrt{x^2+1}}dx = \frac{1}{2}t\sqrt{t^2+1} - 2\sqrt{t^2+1} - \frac{1}{2}ln\left(t+\sqrt{t^2+1}\right) + 2, \quad t \geq 0,$$

is superquadratic function. This function h is not convex function.

Example 2.1.10. The function $v(t) = \sinh t$ is non decreasing. Therefore the function

$$h(t) = \int_0^t x\sinh x dx = t\cosh t - \sinh t, \quad t \geq 0,$$

is superquadratic function. Moreover it is convex by Lemma 2.1.2.

2.2 Hermite-Hadamard Inequalities for Superquadratic Functions

In this section, by using some characterizations of superquadratic functions we obtain new inequalities and also discuss some special means.

Theorem 2.2.1 [3]. *Let* $\varphi : [0\ ,\ \infty\ ) \to \mathbb{R}$ *be an integrable superqudratic function; then for* $0 \leq a < b$ *one has*

$$\varphi\left(\frac{a+b}{2}\right) + \frac{1}{b-a}\int_a^b \varphi\left(\left|x - \frac{a+b}{2}\right|\right)dx \leq \frac{1}{b-a}\int_a^b \varphi(x)dx, \tag{2.2.1}$$

$$\frac{1}{b-a}\int_a^b \varphi(x)dx \leq \frac{\varphi(a)+\varphi(b)}{2} - \frac{1}{(b-a)^2}\int_a^b((b-x)\varphi(x-a)+(x-a)\varphi(b-x))dx. \tag{2.2.2}$$

2.2.1 Mean Value Theorems

In this section, mean value theorems are developed and calculate different cases of limit for new means $M_{r,t}$ and $\tilde{M}_{r,t}$ at $t = 2$, $t = r = 2$ and $t = r$.

Definition 2.2.2. Let $\varphi : [0,\infty) \to \mathbb{R}$ be an integrable function; for $0 \leq a < b$ one defines a linear functional Λ_φ as

$$\Lambda_\varphi = \int_a^b \varphi(x)dx - (b-a)\varphi\left(\frac{a+b}{2}\right) - \int_a^b \varphi\left(\left|x - \frac{a+b}{2}\right|\right)dx. \tag{2.2.3}$$

From above inequality (2.2.1) it is clear if φ is superquadratic function, then $\Lambda_\varphi \geq 0$.
We stated following Lemma.

Lemma 2.2.3 [3]. *Suppose that* $\varphi : [0,\infty) \to \mathbb{R}$ *is continuously differentiable and* $\varphi(0) \leq 0$. *If* φ' *is superadditive or* φ'/x *is increasing, then* φ *is superquadratic.*

Lemma 2.2.4 [4]. *Let* $\varphi \in C^2([0,\infty))$, $-\infty < m \leq M < \infty$ *such that*

$$m \leq \left(\frac{\varphi'(\xi)}{\xi}\right)' = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2} \leq M, \quad \text{for all } \xi \in [0,\infty). \tag{2.2.4}$$

Consider the function φ_1 and φ_2 defined as

$$\varphi_1(x) = \frac{Mx^3}{3} - \varphi(x), \quad \varphi_2(x) = \varphi(x) - \frac{mx^3}{3}.$$

Then ${\varphi'}_1/x$ *and* ${\varphi'}_2/x$ *are increasing functions. Also they are superquadratic functions if* $\varphi_i(0) = 0$, $i = 1, 2$.

Theorem 2.2.5 [3]. *If* $\varphi'/x \in C^1(I)$ *and* $\varphi(0) = 0$, *then the following equality holds:*

$$\Lambda_\varphi = \frac{1}{96}\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2}(b-a)\left(a^2(5a-7b) + b^2(3b-a)\right), \quad \xi \in I. \tag{2.2.5}$$

Proof. Suppose that φ'/x is bounded, where $\min \varphi/x = m$ and $\max \varphi/x = M$. Now by using φ_1 in place of φ in (2.2.1) we obtain

$$\frac{M}{3}(b-a)\left(\frac{a+b}{2}\right)^3 - \varphi\left(\frac{a+b}{2}\right) - \frac{M}{3}\left(\int_a^{\frac{a+b}{2}}\left(\frac{a+b-2x}{2}\right)^3 dx + \int_{\frac{a+b}{2}}^{2}\left(\frac{2x-a-b}{2}\right)^3 dx\right)$$
$$- \int_a^b \varphi\left(\left|x - \frac{a+b}{2}\right|\right) dx \leq \frac{M}{3}\int_a^b x^3 dx - \int_a^b \varphi(x)dx.$$

$$\int_a^b \varphi(x)dx - (b-a)\varphi\left(\frac{a+b}{2}\right) - \int_a^b \varphi\left(\left|x - \frac{a+b}{2}\right|\right) dx \leq \frac{M}{96}\left(a^2(5a-7b) + b^2(3b-a)\right).$$

Similarly, by using φ_2 in place of φ in (2.1.2) we obtain

$$\int_a^b \varphi(x)dx - (b-a)\varphi\left(\frac{a+b}{2}\right) - \int_a^b \varphi\left(\left|x - \frac{a+b}{2}\right|\right) dx \geq \frac{m}{96}\left(a^2(5a-7b) + b^2(3b-a)\right).$$

When we combine above inequalities we get that there exists $0 < \xi < \infty$ such that (2.2.5) holds. $\square$

Theorem 2.2.6 [3] *If φ'/x, $\psi'/x \in C^1(I)$, $\varphi(0) = \psi(0) = 0$, and $a^2(5a-7b) + b^2(3b-a) \neq 0$, then one has*

$$\frac{\Lambda_\varphi}{\Lambda_\psi} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = K(\xi), \quad \xi \in I, \tag{2.2.6}$$

provided the denominators are not equal to zero. If K is invertible then

$$\xi = K^{-1}\left(\frac{\Lambda_\varphi}{\Lambda_\psi}\right), \quad \Lambda_\psi \neq 0, \tag{2.2.7}$$

is a new mean.

We easily checked that the set of functions $\varphi(x) = x^r/(r(r-2))$, $r > 0$, $r \neq 2$, $x \geq 0$, satisfies Lemma 2.2.3. Therefore if we substitute $\varphi(x) = x^r/(r(r-2))$ and $\psi(x) = x^t/(t(t-2))$ in (2.2.3), we get

$$\Lambda_\varphi = \frac{1}{r(r-2)}\left(\int_a^b x^r dx - (b-a)\left(\frac{a+b}{2}\right)^r - \int_a^{\frac{a+b}{2}} (\frac{a+b-2x}{2})^r dx - \int_{\frac{a+b}{2}}^2 \left(\frac{2x-a-b}{2}\right)^r dx\right)$$

$$= \left(\frac{2^r(b^{r+1} - a^{r+1}) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1}}{2^r r(r+1)(r-2)}\right).$$

Similarly

$$\Lambda_\psi = \frac{1}{t(t-2)}\left(\int_a^b x^t dx - (b-a)\left(\frac{a+b}{2}\right)^t - \int_a^{\frac{a+b}{2}} \left(\frac{a+b-2x}{2}\right)^t dx - \int_{\frac{a+b}{2}}^2 \left(\frac{2x-a-b}{2}\right)^t dx\right)$$

$$= \left(\frac{2^r(b^{r+1} - a^{r+1}) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1}}{2^r r(r+1)(r-2)}\right).$$

Now using Λ_φ and Λ_ψ in (2.2.6) we obtain

$$\frac{\Lambda_\varphi}{\Lambda_\psi} = \frac{2^t t(t+1)(t-2)(2^r(b^{r+1} - a^{r+1}) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1})}{2^r r(r+1)(r-2)(2^t(b^{t+1} - a^{t+1}) - (b-a)(t+1)(a+b)^t - (b-a)^{t+1})},$$

and

$$\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = \xi^{r-t}.$$

Using this in Equation (2.2.7) then we have a new mean $M_{r,t}$ defined as follows, where $r,\ t > 0$, $r \neq t$ and $a, b > 0$, $a \neq b$.

$$M_{r,t} = \left(\frac{2^t t(t+1)(t-2)(2^r(b^{r+1}-a^{r+1}) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1})}{2^r r(r+1)(r-2)(2^t(b^{t+1}-a^{t+1}) - (b-a)(t+1)(a+b)^t - (b-a)^{t+1})} \right)^{1/(r-t)}, \quad r, t \neq 2. \tag{2.2.8}$$

To compute $M_{r,2} = M_{2,r}$, consider

$$\lim_{t\to 2} M_{r,t} = \lim_{t\to 2} \left(\frac{2^t t(t+1)(t-2)(2^r(b^{r+1}-a^{r+1}) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1})}{2^r r(r+1)(r-2)(2^t(b^{t+1}-a^{t+1}) - (b-a)(t+1)(a+b)^t - (b-a)^{t+1})} \right)^{1/(r-t)}.$$

Applying L'Hospital rule, we get

$$\lim_{t\to 2} \left(\frac{2^t(\ln 2(t^3-t^2-2t) + 3t^2 - 2t - 2)(2^r(b^{r+1}-a^{r+1}) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1})}{2^r r(r+1)(r-2)A^*} \right)^{1/(r-t)},$$

where

$A^* = 2^t \ln 2(b^{t+1} - a^{t+1}) + 2^t(b^{t+1}\ln b - a^{t+1}\ln a) - (b-a)(a+b)^t(1 + (t+1)\ln(a+b)) - (b-a)^{t+1}\ln(b-a)$.

Applying limit, we get

$$M_{r,2} = M_{2,r} = \left(\frac{24(2^r(b^{r+1}-a^{r+1}) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1})}{2^r r(r+1)(r-2)A} \right)^{1/(r-2)}, \ r \neq 2, \tag{2.2.9}$$

where

$$A = 4\ln 2(b^3 - a^3) + 4(b^3\ln b - a^3\ln a) - (b-a)(a+b)^2(1 + 3\ln(a+b)) - (b-a)^3\ln(b-a).$$

To compute $M_{2,2}$, consider

$$\lim_{r\to 2} M_{r,2} = \lim_{r\to 2} \left(\frac{24(2^r(b^{r+1}-a^{r+1}) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1})}{2^r r(r+1)(r-2)A} \right)^{1/(r-2)}.$$

Applying limit, we get

$$M_{2,2} = \exp\left(\frac{3B - (6\ln 2 + 5)A}{6A} \right), \tag{2.2.10}$$

where P is defined above and

$B = 2(\ln 2)^2(b^3 - a^3) + 8\ln 2(b^3\ln b - a^3\ln a) + 4(b^3(\ln b)^2 - a^3(\ln a)^2) - (b-a)(a+b)^2(\ln(a+$

$b)(2+3\ln(a+b))) - (b-a)^3(\ln(b-a))^2.$

To compute $M_{r,r}$, consider

$$\lim_{t\to r} M_{r,t} = \lim_{t\to r}\left(\frac{2^t t(t+1)(t-2)(2^r(b^{r+1}-a^{r+1})-(b-a)(r+1)(a+b)^r-(b-a)^{r+1})}{2^r r(r+1)(r-2)(2^t(b^{t+1}-a^{t+1})-(b-a)(t+1)(a+b)^t-(b-a)^{t+1})}\right)^{1/(r-t)}.$$

Taking log of both sides, we get

$$\begin{aligned}&\lim_{t\to r}\log M_{r,t}\\ &\quad = \lim_{t\to r}\ \frac{1}{r-t}\ (\log\ (2^t t(t+1)(t-2)(2^r(b^{r+1}-a^{r+1})-(b-a)(r+1)(a+b)^r-(b-a)^{r+1}))\\ &\qquad\qquad -\log\ (2^r r(r+1)(r-2)(2^t(b^{t+1}-a^{t+1})-(b-a)(t+1)(a+b)^t-(b-a)^{t+1}))).\end{aligned}$$

Applying L'Hospital rule, we get

$$= \lim_{t\to r}\left(\frac{d^{*'}}{d^*} - \frac{2^t\ln 2(t^3-t^2-2t)+2^t(3t^2-2t-2)}{2^t t(t+1)(t-2)}\right),$$

where

$d^* = 2^t(b^{t+1}-a^{t+1}) - (b-a)(t+1)(a+b)^t - (b-a)^{t+1}.$

Applying limit,we get

$$M_{r,r} = \exp\left(\frac{C}{D} - \frac{\ln 2(r^3-r^2-2r)+(3r^2-2r-2)}{r(r+1)(r-2)}\right),\quad r\neq 2, \tag{2.2.11}$$

where

$C = 2^r\ln 2(b^{r+1}-a^{r+1}) + 2^t(b^{r+1}\ln b - a^{r+1}\ln a) - (b-a)(a+b)^r(1+(r+1)\ln(a+b)) - (b-a)^{r+1}\ln(b-a).$
$D = 2^r(b^{r+1}-a^{r+1}) - (b-a)(r+1)(a+b)^r - (b-a)^{r+1}.$

Definition 2.2.7. Let $\varphi : [0,\infty) \to \mathbb{R}$ be an integrable function; for $0 \le a < b$ one defines a linear functional $\tilde{\Lambda}_\varphi$ as

$$\tilde{\Lambda}_\varphi = \frac{\varphi(a)+\varphi(b)}{2} - \frac{1}{b-a}\int_a^b \varphi(x)dx - \frac{1}{(b-a)^2}\int_a^b ((b-x)\varphi(x-a)+(x-a)\varphi(b-x))dx. \tag{2.2.12}$$

From above inequality (2.2.2) it is clear if φ is superquadratic function, then $\tilde{\Lambda}_\varphi \ge 0$.

Theorem 2.2.8 [3]. *If $\varphi'/x \in C^1(I)$ and $\varphi(0) = 0$, then the following equality holds:*

$$\tilde{\Lambda}_\varphi = \frac{1}{60}\frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi^2}\left(a^2(7a - 11b) + b^2(a + 3b)\right), \quad \xi \in I. \tag{2.2.13}$$

Proof. Suppose that φ'/x is bounded, where $min\ \varphi/x = m$ and $max\ \varphi/x = M$. Now by using φ_1 in place of φ in (2.2.2) we obtain

$$\begin{aligned}\frac{M}{3(b-a)}\int_a^b x^3 dx - \int_a^b \varphi(x)dx \leq{}& \frac{M}{3}\left(\frac{a^3+b^3}{2}\right) - \left(\frac{\varphi(a)+\varphi(b)}{2}\right)\\ &-\frac{M}{3(b-a)^2}\int_a^b \left((b-x)(x-a)^3 + (x-a)(b-x)^3\right)dx\\ &+\frac{1}{(b-a)^2}\int_a^b \left((b-x)\varphi(x-a) + (x-a)\varphi(b-x)\right)dx.\end{aligned}$$

After that

$$\begin{aligned}&\frac{\varphi(a)+\varphi(b)}{2} - \frac{1}{b-a}\int_a^b \varphi(x)dx - \frac{1}{(b-a)^2}\int_a^b ((b-x)\varphi(x-a) + (x-a)\varphi(b-x))dx\\ &\leq \frac{M}{60}\left(a^2(7a-11b) + b^2(a+3b)\right).\end{aligned}$$

Similarly, by using φ_2 in place of φ in (2.2.2) we obtain

$$\begin{aligned}&\frac{\varphi(a)+\varphi(b)}{2} - \frac{1}{b-a}\int_a^b \varphi(x)dx - \frac{1}{(b-a)^2}\int_a^b ((b-x)\varphi(x-a) + (x-a)\varphi(b-x))dx\\ &\geq \frac{m}{60}\left(a^2(7a-11b) + b^2(a+3b)\right).\end{aligned}$$

When we combine above inequalities we get that there exists $0 < \xi < \infty$ such that (2.2.13) holds. □

Theorem 2.2.9 [3]. *If φ'/x, $\psi'/x \in C^1(I)$, $\varphi(0) = \psi(0) = 0$, and $(a^2(7a - 11b) + b^2(a + 3b)) \neq 0$, then one has*

$$\frac{\tilde{\Lambda}_\varphi}{\tilde{\Lambda}_\psi} = \frac{\xi\varphi''(\xi) - \varphi'(\xi)}{\xi\psi''(\xi) - \psi'(\xi)} = T(\xi), \quad \xi \in I, \tag{2.2.14}$$

provided the denominators are not equal to zero. If T is invertible then

$$\xi = T^{-1}\left(\frac{\Lambda_\varphi}{\Lambda_\psi}\right), \quad \Lambda_\psi \neq 0, \tag{2.2.15}$$

is a new mean.

We easily checked that the set of functions $\varphi(x)= x^r/(r(r-2))$, $r>0$, $r\neq 2$, $x\geq 0$, satisfies Lemma 2.2.3. Therefore if we substitute $\varphi(x)=x^r/(r(r-2))$ and $\psi(x)=x^t/(t(t-2))$ in (2.2.12), we get

$$\tilde{\Lambda}_\varphi = \frac{1}{2r(r-2)}(a^r+b^r) - \frac{1}{(b-a)r(r-2)}\int_a^b x^r dx - \frac{1}{(b-a)^2 r(r-2)}$$
$$\int_a^b \left((b-x)(x-a)^r + (x-a)(b-x)^r\right) dx.$$

After simplification, it give us

$$\tilde{\Lambda}_\varphi = \frac{\left((b-a)(r+1)(r+2)(a^r+b^r) - 2(r+2)(b^{r+1}-a^{r+1}) - 4(b-a)^{r+1}\right)}{2r(r-2)(r+1)(r+2)(b-a)}.$$

Similarly

$$\tilde{\Lambda}_\psi = \frac{\left((b-a)(t+1)(t+2)(a^t+b^t) - 2(t+2)(b^{t+1}-a^{t+1}) - 4(b-a)^{t+1}\right)}{2t(t-2)(t+1)(t+2)(b-a)}.$$

Now using $\tilde{\Lambda}_\varphi$ and $\tilde{\Lambda}_\psi$ in (2.2.14) we obtain

$$\frac{\tilde{\Lambda}_\varphi}{\tilde{\Lambda}_\psi} = \frac{t(t+1)(t+2)(t-2)\left((b-a)(r+1)(r+2)(a^r+b^r) - 2(r+2)(b^{r+1}-a^{r+1}) - 4(b-a)^{r+1}\right)}{r(r+1)(r+2)(r-2)\left((b-a)(t+1)(t+2)(a^t+b^t) - 2(t+2)(b^{t+1}-a^{t+1}) - 4(b-a)^{t+1}\right)},$$

using this result in Equation(2.2.15)then we have a new mean $\tilde{M}_{r,t}$ defined as follows, where $r,\ t>0$, $r\neq t$ and $a,b>0$, $a\neq b$.

$$\tilde{M}_{r,t} = \left(\frac{t(t+1)(t+2)(t-2)\left((b-a)(r+1)(r+2)(a^r+b^r) - X\right)}{r(r+1)(r+2)(r-2)\left((b-a)(t+1)(t+2)(a^t+b^t) - Y\right)}\right)^{1/(r-t)}, \quad r,t\neq 2, \tag{2.2.16}$$

where X denotes $2(r+2)(b^{r+1}-a^{r+1})+4(b-a)^{r+1}$ and Y denotes $2(t+2)(b^{t+1}-a^{t+1})+4(b-a)^{t+1}$.

To compute $\tilde{M}_{r,2} = \tilde{M}_{2,r}$, consider

$$\lim_{t\to 2}\tilde{M}_{r,t} = \lim_{t\to 2}\left(\frac{t(t+1)(t+2)(t-2)\left((b-a)(r+1)(r+2)(a^r+b^r) - X\right)}{r(r+1)(r+2)(r-2)\left((b-a)(t+1)(t+2)(a^t+b^t) - Y\right)}\right)^{1/(r-t)}, \quad r\neq 2.$$

Applying L'Hospital rule, we get

$$= \lim_{t\to 2}\left(\frac{(4t^3+3t^2-8t-4)\left((b-a)(r+1)(r+2)(a^r+b^r) - X\right)}{r(r+1)(r+2)(r-2)E^*}\right)^{1/(r-t)},$$

where

$E^* = (b-a)\left((2t+3)(a^t+b^t)+(t+1)(t+2)(a^t\ln a+b^t\ln b)\right)-2(b^{t+1}+a^{t+1})-2(t+1)(b^{t+1}\ln b+a^{t+1}\ln a)-4(b-a)^{t+1}\ln(b-a)$.

$$\tilde{M}_{r,2}=\tilde{M}_{2,r}=\left(\frac{24\left((b-a)(r+1)(r+2)(a^r+b^r)-X\right)}{r(r+1)(r+2)(r-2)E}\right)^{1/(r-2)}, \quad r\neq 2, \tag{2.2.17}$$

where

$E=(b-a)\left(7(a^2+b^2)+12(a^2\ln a+b^2\ln b)\right)-2(b^3-a^3+4(b^3\ln b-a^3\ln a))-4(b-a)^3\ln(b-a)$.

To compute $\tilde{M}_{2,2}$, consider

$$\lim_{r\to 2}\tilde{M}_{r,2}=\lim_{r\to 2}\left(\frac{24\left((b-a)(r+1)(r+2)(a^r+b^r)-X\right)}{r(r+1)(r+2)(r-2)E}\right)^{1/(r-2)}.$$

Applying limit, we get

$$\tilde{M}_{2,2}=\exp\left(\frac{12F-13E}{12E}\right), \tag{2.2.18}$$

where E is defined above and

$F=(b-a)\left(a^2+b^2+7(a^2\ln a+b^2\ln b)+6(a^2(\ln a)^2+b^2(\ln b)^2)-2(b-a)^2(\ln(b-a))^2\right)-2(b^3\ln b-a^3\ln a)-4(b^3(\ln b)^2-a^3(\ln a)^2)$.

To compute $\tilde{M}_{r,r}$, consider

$$\lim_{t\to r}\tilde{M}_{r,t}=\lim_{t\to r}\left(\frac{t(t+1)(t+2)(t-2)\left((b-a)(r+1)(r+2)(a^r+b^r)-X\right)}{r(r+1)(r+2)(r-2)\left((b-a)(t+1)(t+2)(a^t+b^t)-Y\right)}\right)^{1/(r-t)}.$$

Taking log of both sides, we get

$$\begin{aligned}\lim_{t\to r}\log\ \tilde{M}_{r,t}=\lim_{t\to r}\ \frac{1}{r-t}\ &\left(\log\ \left(t(t+1)(t+2)(t-2)\left((b-a)(r+1)(r+2)(a^r+b^r)-X\right)\right)\right.\\ &\left.-\log\ \left(r(r+1)(r+2)(r-2)\left((b-a)(t+1)(t+2)(a^t+b^t)-Y\right)\right)\right).\end{aligned}$$

Applying L'Hospital rule, we get

$$=\lim_{t\to r}\left(\frac{g^{*\prime}}{g^*}-\frac{8t^3+3t^2-8t-4}{t(t+1)(t+2)(t-2)}\right),$$

where

$g^* = (b-a)(t+1)(t+2)(a^t+b^t) - Y.$

Applying limit,we get

$$\tilde{M}_{r,r} = \exp\left(\frac{G}{H} - \frac{8r^3+3r^2-8r-4}{r(r+1)(r+2)(r-2)}\right), \quad r \neq 2, \tag{2.2.19}$$

where

$G = (b-a)\left((2r+3)(a^r+b^r) + (r+1)(r+2)(a^r\ln a + b^r \ln b)\right) - 2(b^{r+1}+a^{r+1}) - 2(r+1)(b^{r+1}\ln b + a^{r+1}\ln a) - 4(b-a)^{r+1}\ln(b-a),$
$H = (b-a)(r+1)(r+2)(a^r+b^r) - 2(r+2)(b^{r+1}-a^{r+1}) - 4(b-a)^{r+1}.$

2.2.2 Cauchy Means

In this section, Cauchy type means are developed and calculate different cases of limit for Cauchy means $M^{[s]}_{r,t}$ and $\tilde{M}^{[s]}_{r,t}$ at $t = 2s$, $t = r = 2s$ and $t = r$. If we substitute $\varphi(x) = x^{r/s}/((r/s)(r/s-2))$ and $\psi(x) = x^{t/s}/((t/s)(t/s-2))$ in (2.2.3), then by substitution, $a = a^s$, $b = b^s$, we obtain

$$\Lambda_\varphi = \frac{1}{(r/s)(r/s-2)}\int_a^b x^{r/s}dx - \frac{b-a}{(r/s)(r/s-2)}\left(\frac{a+b}{2}\right)^{r/s} - \frac{1}{(r/s)(r/s-2)} \left(\int_a^{\frac{a+b}{2}}\left(\frac{a+b-2x}{2}\right)^{r/s}dx + \int_{\frac{a+b}{2}}^{2}\left(\frac{2x-a-b}{2}\right)^{r/s}dx\right).$$

$$= \left(\frac{s^2(s(b^{r+s}-a^{r+s}) - (r+s)(b^s-a^s)((a^s+b^s)/2)^{r/s} - 2s((b^s-a^s)/2)^{(r+s)/s})}{r(r+s)(r-2s)}\right).$$

Similarly

$$\Lambda_\varphi = \frac{1}{(t/s)(t/s-2)}\int_a^b x^{t/s}dx - \frac{b-a}{(t/s)(t/s-2)}\left(\frac{a+b}{2}\right)^{t/s} - \frac{1}{(t/s)(t/s-2)} \left(\int_a^{\frac{a+b}{2}}\left(\frac{a+b-2x}{2}\right)^{t/s}dx + \int_{\frac{a+b}{2}}^{2}\left(\frac{2x-a-b}{2}\right)^{t/s}dx\right).$$

$$= \left(\frac{s^2(s(b^{t+s}-a^{t+s}) - (t+s)(b^s-a^s)((a^s+b^s)/2)^{t/s} - 2s((b^s-a^s)/2)^{(t+s)/s})}{t(t+s)(t-2s)}\right).$$

Now using Λ_φ and Λ_ψ in (2.2.6) we obtain

$$\frac{\Lambda_\varphi}{\Lambda_\psi} = \frac{t(t+s)(t-2s)(s(b^{r+s}-a^{r+s}) - (r+s)(b^s-a^s)((a^s+b^s)/2)^{r/s} - 2s((b^s-a^s)/2)^{(r+s)/s})}{r(r+s)(r-2s)(s(b^{t+s}-a^{t+s}) - (t+s)(b^s-a^s)((a^s+b^s)/2)^{t/s} - 2s((b^s-a^s)/2)^{(t+s)/s})},$$

and

$$\frac{\xi\varphi''(\xi)-\varphi'(\xi)}{\xi\psi''(\xi)-\psi'(\xi)}=\xi^{r-t}.$$

Using these results in Equation (2.2.7) then we have a Cauchy mean $M^{[s]}_{r,t}$ defined as follows, where $r,\ t\ s\in\mathbb{R}$, $r\neq t$ and $a,b>0$, $a\neq b$.

$$M^{[s]}_{r,t}=\left(\frac{t(t+s)(t-2s)(s(b^{r+s}-a^{r+s})-(r+s)(b^s-a^s)((a^s+b^s)/2)^{r/s}-\Re}{r(r+s)(r-2s)(s(b^{t+s}-a^{t+s})-(t+s)(b^s-a^s)((a^s+b^s)/2)^{t/s}-\hat{\Re}}\right)^{1/(r-t)}, \quad r,t\neq 2s. \tag{2.2.20}$$

where $\Re=2s((b^s-a^s)/2)^{(r+s)/s})$ and $\hat{\Re}=2s((b^s-a^s)/2)^{(t+s)/s})$.

We calculate a limit when t goes to 2s, consider

$$\lim_{t\to 2s}M^{[s]}_{r,t}=\lim_{t\to 2s}\left(\frac{t(t+s)(t-2s)(s(b^{r+s}-a^{r+s})-(r+s)(b^s-a^s)((a^s+b^s)/2)^{r/s}-\Re}{r(r+s)(r-2s)(s(b^{t+s}-a^{t+s})-(t+s)(b^s-a^s)((a^s+b^s)/2)^{t/s}-\hat{\Re}}\right)^{1/(r-t)}.$$

Applying L'Hospital rule, we get

$$=\lim_{t\to 2s}\left(\frac{(3t^2-2ts-2s^2)(s(b^{r+s}-a^{r+s})-(r+s)(b^s-a^s)((a^s+b^s)/2)^{r/s}-\Re}{r(r+s)(r-2s)I^*}\right)^{1/(r-t)},$$

where

$I^*=s(b^{t+s}\ln b-a^{t+s}\ln a)-(b^s-a^s)((a^s+b^s)/2)^{t/s}(1+((t+s)/s)\ln((a^s+b^s)/2))-2s((b^s-a^s)/2)^{(t+s)/s}\ln((b^s-a^s)/2)$.

Applying limit, it give us

$$M^{[s]}_{r,2s}=\left(\frac{6s^2(s(b^{r+s}-a^{r+s})-(r+s)(b^s-a^s)((a^s+b^s)/2)^{r/s}-2((b^s-a^s)/2)^{(r+s)/s})}{r(r+s)(r-2s)I}\right)^{1/(r-2s)}, \tag{2.2.21}$$

where

$$I=s(b^{3s}\ln b-a^{3s}\ln a)-(b^s-a^s)((a^s+b^s)/2)^2(1+3\ln((a^s+b^s)/2))-2((b^s-a^s)/2)^3\ln((b^s-a^s)/2).$$

To compute $M^{[s]}_{2s,2s}$, consider

$$\lim_{r\to 2s}M^{[s]}_{r,2s}=\lim_{r\to 2s}\left(\frac{6s^2(s(b^{r+s}-a^{r+s})-(r+s)(b^s-a^s)((a^s+b^s)/2)^{r/s}-2s((b^s-a^s)/2)^{(r+s)/s})}{r(r+s)(r-2s)I}\right)^{1/(r-2s)}.$$

Applying limit, we get

$$M^{[s]}_{2s,2s} = \exp\left(\frac{J}{2sK} - \frac{5}{6s}\right), \tag{2.2.22}$$

where

$J = 4s^2\left(b^{3s}(\ln b)^2 - a^{3s}(\ln a)^2 - 2(b^s - a^s)(a^s + b^s)^2 \ln\left(\frac{a^s+b^s}{2}\right)\left(2 - 3\ln\left(\frac{a^s+b^s}{2}\right)\right)\right) - 2(b^s - a^s)^3 \left(\frac{\ln(b^s-a^s)}{2}\right)^2.$
$K = 4s(b^{3s}\ln b - a^{3s}\ln a) - (b^s - a^s)(a^s + b^s)^2\left(1 + 3\ln\left(\frac{a^s+b^s}{2}\right)\right) - (b^s - a^s)^3 \ln\left(\frac{b^s-a^s}{2}\right).$

To compute $M^{[s]}_{r,r}$, consider

$$\lim_{t\to r} M^{[s]}_{r,t} = \lim_{t\to r}\left(\frac{t(t+s)(t-2s)(s(b^{r+s} - a^{r+s}) - (r+s)(b^s - a^s)((a^s+b^s)/2)^{r/s} - \Re}{r(r+s)(r-2s)(s(b^{t+s} - a^{t+s}) - (t+s)(b^s - a^s)((a^s+b^s)/2)^{t/s} - \hat{\Re}}\right)^{1/(r-t)}.$$

Taking log of both sides, we have

$$\begin{aligned}&\lim_{t\to r}\log M^{[s]}_{r,t}\\ &\quad= \lim_{t\to r}\frac{1}{r-t}(\log(t(t+s)(t-2s)(s(b^{r+s} - a^{r+s}) - (r+s)(b^s - a^s)((a^s+b^s)/2)^{r/s} - \Re)\\ &\qquad\qquad \log(r(r+s)(r-2s)(s(b^{t+s} - a^{t+s}) - (t+s)(b^s - a^s)((a^s+b^s)/2)^{t/s} - \hat{\Re})).\end{aligned}$$

Applying L'Hospital rule, we have

$$= \lim_{t\to r}\left(\frac{\vartheta'}{\vartheta} - \frac{3t^2 - 2ts - 2s^2}{t(t+s)(t-2s)}\right),$$

where

$$\vartheta = s(b^{t+s} - a^{t+s}) - (t+s)(b^s - a^s)((a^s+b^s)/2)^{t/s} - \hat{\Re}.$$

Applying limit, it give

$$M^{[s]}_{r,r} = \exp\left(\frac{L}{M} - \frac{3r^2 - 2rs - 2s^2}{r(r+s)(r-2s)}\right), \tag{2.2.23}$$

where

$L = s(b^{r+s}\ln b - a^{r+s}\ln a) - (b^s - a^s)((a^s+b^s)/2)^{r/s}(1 + ((r+s)/s)\ln((a^s+b^s)/2)) - 2((b^s -$

$a^s)/2)^{(r+s)/s}\ln((b^s-a^s)/2)$.
$M = s(b^{r+s}-a^{r+s})-(r+s)(b^s-a^s)((a^s+b^s)/2)^{r/s}-2s((b^s-a^s)/2)^{(r+s)/s})$.

If we substitute $\varphi(x)=x^{r/s}/((r/s)(r/s-2))$ and $\psi(x)=x^{t/s}/((t/s)(t/s-2))$ in (2.2.12), then by substitution, $a=a^s$, $b=b^s$, we obtain

$$\tilde{\Lambda}_\varphi = \frac{s^2}{2r(r-2s)}(a^{r/s}+b^{r/s}) - \frac{s^2}{(b-a)r(r-2s)}\int_a^b x^{r/s}dx - \frac{s^2}{(b-a)^2 r(r-2s)} \int_a^b \left((b-x)(x-a)^{r/s}+(x-a)(b-x)^{r/s}\right)dx.$$

After simplification, it give us

$$\tilde{\Lambda}_\varphi = \left(\frac{s^2((r+s)(r+2s)(a^r+b^r)(b^s-a^s)-2s(r+2s)(b^{r+s}-a^{r+s})-4s^2(b-a)^{(r+s)/s}}{2(b-a)r(r+s)(r+2s)(r-2s)}\right).$$

Similarly

$$\tilde{\Lambda}_\psi = \left(\frac{s^2((t+s)(t+2s)(a^t+b^t)(b^s-a^s)-2s(t+2s)(b^{t+s}-a^{t+s})-4s^2(b-a)^{(t+s)/s}}{2(b-a)t(t+s)(t+2s)(t-2s)}\right).$$

Now using $\tilde{\Lambda}_\varphi$ and $\tilde{\Lambda}_\psi$ in (2.2.14) we obtain

$$\frac{\tilde{\Lambda}_\varphi}{\tilde{\Lambda}_\psi} = \frac{t(t+s)(t+2s)(t-2s)\left((r+s)(r+2s)(a^r+b^r)(b^s-a^s)-\omega\right)}{r(r+s)(r+2s)(r-2s)\left((t+s)(t+2s)(a^t+b^t)(b^s-a^s)-\hat{\omega}\right)},$$

where ω denotes $2s(r+2s)(b^{r+s}-a^{r+s})+4s^2(b-a)^{(r+s)/s}$, $\hat{\omega}$ denotes $2s(t+2s)(b^{t+s}-a^{t+s})+4s^2(b-a)^{(t+s)/s}$ and

$$\frac{\xi\varphi''(\xi)-\varphi'(\xi)}{\xi\psi''(\xi)-\psi'(\xi)} = \xi^{r-t}.$$

Using these results in (2.2.15)then we have a Cauchy mean $\tilde{M}_{r,t}^{[s]}$ defined as follows, where r, t $s\in\mathbb{R}$, $r\neq t$ and $a,b>0$, $a\neq b$.

$$\tilde{M}_{r,t}^{[s]} = \left(\frac{t(t+s)(t+2s)(t-2s)\left((r+s)(r+2s)(a^r+b^r)(b^s-a^s)-\omega\right)}{r(r+s)(r+2s)(r-2s)\left((t+s)(t+2s)(a^t+b^t)(b^s-a^s)-\hat{\omega}\right)}\right)^{1/(r-t)}, \quad r,t\neq 2s. \tag{2.2.24}$$

We calculate a limit when t goes to 2s, consider

$$\lim_{t\to 2s}\tilde{M}^{[s]}_{r,t}=\lim_{t\to 2s}\left(\frac{t(t+s)(t+2s)(t-2s)\left((r+s)(r+2s)(a^r+b^r)(b^s-a^s)-\omega\right)}{r(r+s)(r+2s)(r-2s)\left((t+s)(t+2s)(a^t+b^t)(b^s-a^s)-\hat{\omega}\right)}\right)^{1/(r-t)}.$$

Applying L'Hospital rule, we get

$$=\lim_{t\to 2s}\left(\frac{4t^3+3t^2s-8ts^2-4s^3\left((r+s)(r+2s)(a^r+b^r)(b^s-a^s)-\omega\right)}{r(r+s)(r+2s)(r-2s)N^*}\right)^{1/(r-t)},$$

where

$N^*=(b^s-a^s)\left((2t+3s)(a^t+b^t)+(t+s)(t+2s)(a^t\ln a+b^t\ln b)\right)-2s(b^{t+s}-a^{t+s})-2s(t+2s)(b^{t+s}\ln b-a^{t+s}\ln a)-4s(b^s-a^s)^{(t+s)/s}\ln(b^s-a^s).$

Applying limit, it give us

$$\tilde{M}^{[s]}_{r,2s}=\left(\frac{24s^3\left((r+s)(r+2s)(a^r+b^r)(b^s-a^s)-\omega\right)}{r(r+s)(r+2s)(r-2s)N}\right)^{1/(r-2s)}, \tag{2.2.25}$$

where

$N=(b^s-a^s)(7(a^{2s}+b^{2s})+12s^2(a^{2s}\ln a+b^{2s}\ln b))-2s(b^{3s}-a^{3s})-8s^2(b^{3s}\ln b-a^{3s}\ln a)-4s(b^s-a^s)^3\ln(b^s-a^s).$

To compute $\tilde{M}^{[s]}_{2s,2s}$, consider

$$\lim_{r\to 2s}\tilde{M}^{[s]}_{r,2s}=\lim_{r\to 2s}\left(\frac{24s^3\left((r+s)(r+2s)(a^r+b^r)(b^s-a^s)-\omega\right)}{r(r+s)(r+2s)(r-2s)N}\right)^{1/(r-2s)}.$$

Applying limit, we get

$$\tilde{M}^{[s]}_{2s,2s}=\exp\left(\frac{12O-13N}{12N}\right), \tag{2.2.26}$$

where N is defined above and

$O=(b^s-a^s)(2(a^{2s}+b^{2s}+7s(a^{2s}\ln a+b^{2s}\ln b)+6s^2(a^{2s}(\ln a)^2+b^{2s}(\ln b)^2)))-2(b^s-a^s)^3(\ln(b^s-a^s)^2)-2s(b^{3s}\ln b-a^{3s}\ln a)-4s(b^{3s}(\ln b)^2-a^{3s}(\ln a)^2).$

To compute $\tilde{M}_{r,r}^{[s]}$, consider

$$\lim_{t\to r}\tilde{M}_{r,t}^{[s]}=\lim_{t\to r}\left(\frac{t(t+s)(t+2s)(t-2s)\left((r+s)(r+2s)(a^r+b^r)(b^s-a^s)-\omega\right)}{r(r+s)(r+2s)(r-2s)\left((t+s)(t+2s)(a^t+b^t)(b^s-a^s)-\hat{\omega}\right)}\right)^{1/(r-t)},$$

$$r,t\neq 2s.$$

Taking log of both sides, we have

$$\lim_{t\to r}\log\tilde{M}_{r,t}^{[s]}=\lim_{t\to r}\frac{1}{r-t}(\log(t(t+s)(t+2s)(t-2s)\left((r+s)(r+2s)(a^r+b^r)(b^s-a^s)-\omega\right))$$

$$\log(r(r+s)(r+2s)(r-2s)\big((t+s)(t+2s)(a^t+b^t)(b^s-a^s)-\hat{\omega}\big)).$$

Applying L'Hospital rule, we have

$$=\lim_{t\to r}\left(\frac{u'}{u}-\frac{4t^3+3t^2s-8ts^2-4s^3}{t(t+s)(t+2s)(t-2s)}\right),$$

where

$u=(t+s)(t+2s)(a^t+b^t)(b^s-a^s)-\hat{\omega}$.

Applying limit, it give

$$\tilde{M}_{r,r}^{[s]}=\ \exp\left(\frac{P}{Q}-\frac{4r^3+3r^2s-8rs^2-4s^3}{r(r+s)(r+2s)(r-2s)}\right),\tag{2.2.27}$$

where

$P=(b^s-a^s)\left((2r+3s)(a^r+b^r)+(r+s)(r+2s)(a^r\ln a+b^r\ln b)\right)-2s(b^{r+s}-a^{r+s})-2s(r+2s)(b^{r+s}\ln b-a^{r+s}\ln a)-4s^2(b^s-a^s)^{(r+s)/s}\ln(b^s-a^s)$,
$Q=(r+s)(r+2s)(a^r+b^r)(b^s-a^s)-2s(r+2s)(b^{r+s}-a^{r+s})-4s^2(b-a)^{(r+s)/s}$.

2.3 Positive Semi-Definiteness, Exponential Convexity and Log-Convexity

In this section, the concept of positive semi-definiteness, exponential convexity and log-Convexity are introduced for Hermite-Hadamard inequalities.

Lemma 2.3.1 [2]. Consider the function φ_s for $s > 0$ defined as

$$\varphi_s(x) = \begin{cases} \frac{x^s}{s(s-2)}, & s \neq 2, \\ \frac{x^2}{2} logx, & s = 2. \end{cases}$$

Then, with the convention $0 \log 0 = 0$, $\varphi_s(x)$ is superquadratic.

Theorem 2.3.2 [3]. *For Λ_{φ_s} defined in* (2.2.3) *one has following.*
(a) The matrix $A = [\Lambda_{\varphi_{(p_i+p_j)/2}}]$, $1 \leq i,\ j \leq n$, is a positive semi-definite matrix, that is,

$$det\left(\left[\Lambda_{\varphi_{\frac{p_i+p_j}{2}}}\right]_{i,j=1}^{k}\right) \geq 0, \quad k = 1, 2, ..., n.$$

(b) One has

$$\Lambda^2_{\varphi_{(s+t)/2}} \leq \Lambda_{\varphi_s}\Lambda_{\varphi_t},$$

that is, Λ_{φ_s} is log convex in the Jensen sense.
(c) The function $s \mapsto \Lambda_{\varphi_s}$ is exponentially convex.
(d) Λ_{φ_s} is log convex, that is, for $r < s < t$ where $r,\ s,\ t \in \mathbb{R}_+$ one has

$$(\Lambda_{\varphi_s})^{t-r} \leq (\Lambda_{\varphi_r})^{t-s}(\Lambda_{\varphi_t})^{s-r}.$$

Corollary 2.3.3 [3]. *One has the following*
(i) For $s > 4$,

$$\Lambda_{\varphi_s} \geq \frac{(b-a)(3b^3 - ab^2 - 7a^2b + 5a^3)}{96}\left(\frac{3(a^2-b^2)^2}{2(3b^3 - ab^2 - 7a^2b + 5a^3)}\right)^{s-3}.$$

(ii) For $1 < s < 2$,

$$\Lambda_{\varphi_s} \leq (a-b)^{4-2s}(\Lambda_{\varphi_2})^{s-1}.$$

(iii) For $2 < s < 3$,

$$\Lambda_{\varphi_s} \leq \left(\frac{(b-a)(3b^3 - ab^2 - 7a^2b + 5a^3)}{96\Lambda_{\varphi_2}}\right)^{s-2}\Lambda_{\varphi_s}.$$

(iv) For $3 < s < 4$,

$$\Lambda_{\varphi_s} \leq \frac{(b-a)(3b^3 - ab^2 - 7a^2b + 5a^3)}{96}\left(\frac{3(a^2-b^2)^2}{2(3b^3 - ab^2 - 7a^2b + 5a^3)}\right)^{s-3}.$$

Theorem 2.3.4 [3]. *For* $\tilde{\Lambda}_{\varphi_s}$ *defined in* (2.2.12) *one has following.*
(a) The matrix $A = [\tilde{\Lambda}_{\varphi_{(p_i+p_j)/2}}]$, $1 \leq i,\ j \leq n$, *is a positive semi-definite matrix, that is,*

$$det\left(\left[\tilde{\Lambda}_{\varphi_{\frac{p_i+p_j}{2}}}\right]_{i,j=1}^{k}\right) \geq 0, \quad k = 1, 2, ..., n.$$

(b) One has

$$\tilde{\Lambda}^2_{\varphi_{(s+t)/2}} \leq \tilde{\Lambda}_{\varphi_s}\tilde{\Lambda}_{\varphi_t},$$

that is, $\tilde{\Lambda}_{\varphi_s}$ *is log convex in the Jensen sense.*
(c) The function $s \mapsto \tilde{\Lambda}_{\varphi_s}$ *is exponentially convex.*
(d) $\tilde{\Lambda}_{\varphi_s}$ *is log convex, that is, for* $r < s < t$ *where* $r,\ s,\ t \in \mathbb{R}_+$ *one has*

$$(\tilde{\Lambda}_{\varphi_s})^{t-r} \leq (\tilde{\Lambda}_{\varphi_r})^{t-s}(\tilde{\Lambda}_{\varphi_t})^{s-r}.$$

Lemma 2.3.5 [3]. *Let* g *be a log convex function, and if* $u_1 \leq v_1$, $u_2 \leq v_2$, $u_1 \neq u_2$, $v_1 \neq v_2$, *then the following inequality holds,*

$$\left(\frac{g(u_2)}{g(u_1)}\right)^{\frac{1}{u_2-u_1}} \leq \left(\frac{g(v_2)}{g(v_1)}\right)^{\frac{1}{v_2-v_1}}.$$

Chapter 3

Extensions of the Hermite-Hadamard Inequality for Convex Functions via Fractional Integrals

In this chapter, the extension and refinement Hermite-Hadamard for convex function via Riemann-Liouville fractional integrals is presented. How to relax the convexity property of the function f is also shown. After that, we obtain some results which involve a larger class of functions.

3.1 $\mathbf{L}_1$ Space

The space of functions which have integrable their absolute value in Lebesgue sense. A non negative measurable function f is called Lebesgue integrable if its Lebesgue integral $\int f d\mu$ is finite. An arbitrary measurable function is integrable if f^+ and f^- are each Lebesgue integrable, where f^+ and f^- represent the positive and negative parts of f, respectively.

3.2 Fractional Integrals

The main objects of classical calculus are derivatives and integrals of functions. In 1695 L'Hospital inquired of Leibniz what meaning could be ascribed to $D^n f$ if n were a fraction. Since that time the fractional calculus has drawn the attention of many famous mathematicians, such as Euler, Laplace, Fourier, Abel, Liouville, Riemann, and Laurent [11].
Here we introduce the notion of fractional integral as a generalization of the standard, integer-order integration and differentiation. Fractional integrals and derivatives arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium.

3.2.1 Left and Right Riemann-Liouville Fractional Integrals

The symbols $J_{a^+}^{\alpha} f$ and $J_{b^-}^{\alpha} f$ represent the left-sided and right-sided Riemann-Liouville fractional integrals of the order $\alpha > 0$, where $f \in L_1[a,b]$, which are defined by

$$J_{a^+}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \qquad x > a,$$

and

$$J_{b^-}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \qquad x < b,$$

respectively. Where, $\Gamma(\alpha)$ is the Gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t} t^{\alpha-1} dt.$$

The Gamma function is an generalization of the factorial function and satisfies the relation $\Gamma(n) = (n-1)!$.

3.3 Hermite-Hadamard Inequalities with Fractional Integrals

Due to the wide application of fractional integrals and importance of Hermite-Hadamard type inequalities, many researchers extended their studies to fractional Hermite-Hadamard type inequalities according to the Hermite-Hadamard type inequalities for functions of different classes. For example, see for convex functions [14, 20] and non decreasing functions [18], for m-convex functions [15, 19] and (s,m) convex functions [17], for functions satisfying $s-e$-condition [16] and the references therein.

Theorem 3.3.1 [10]. *Let $f:$ [a,b] $\to \mathbb{R}$ be a positive function with $a < b$ and $f \in L_1[a,b]$. If f is a convex function on [a,b], then the following inequalities for fractional integrals hold.*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}} [J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)] \leq \frac{f(a)+f(b)}{2}, \tag{3.3.1}$$

with $\alpha > 0$.

Proof. Since f is convex function on $[a,b]$, then we have

$$f\left(\frac{g+h}{2}\right) \leq \frac{f(g)+f(h)}{2}, \quad for\ g,h \in [a,b]\ and\ \lambda = 1/2,$$

for $g=\lambda a+(1-\lambda)b,\ h=(1-\lambda)a+\lambda b$, we get

$$f\left(\frac{a+b}{2}\right)\leq\frac{f(\lambda a+(1-\lambda)b)+f((1-\lambda)a+\lambda b)}{2}.$$

Multiplying both of sides by $\lambda^{\alpha-1}$, then integrating it with respect to λ on $[0,1]$, we get

$$f\left(\frac{a+b}{2}\right)\int_0^1\lambda^{\alpha-1}d\lambda\leq\frac{1}{2}\left(\int_0^1\lambda^{\alpha-1}f(\lambda a+(1-\lambda)b)d\lambda+\int_0^1\lambda^{\alpha-1}f((1-\lambda)a+\lambda b)d\lambda\right).$$

using $g=\lambda a+(1-\lambda)b,\ h=(1-\lambda)a+\lambda b$, we obtain

$$\frac{1}{\alpha}f\left(\frac{a+b}{2}\right)\leq\frac{1}{2}\left(\int_a^b\left(\frac{g-a}{b-a}\right)^{\alpha-1}f(g)\ dg+\int_b^a\left(\frac{h-b}{a-b}\right)^{\alpha-1}f(h)\ dh\right).$$

$$\frac{1}{\alpha}f\left(\frac{a+b}{2}\right)\leq\frac{1}{2(b-a)^\alpha}\left(\int_a^b(g-a)^{\alpha-1}f(g)\ dg+\int_a^b(b-h)^{\alpha-1}f(h)\ dh\right).$$

$$f\left(\frac{a+b}{2}\right)\leq\frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha}[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)],\tag{3.3.2}$$

the left side of inequality is proved. Now to show right side of inequality we know that f is convex function, then for $\lambda\in[0,1]$, it yields

$$f(\lambda a+(1-\lambda)b)\leq\lambda f(a)+(1-\lambda)f(b)\ \ and\ \ f((1-\lambda)a+\lambda b)\leq(1-\lambda)f(a)+\lambda f(b).$$

By adding both inequalities, we get

$$f(\lambda a+(1-\lambda)b)+f((1-\lambda)a+\lambda b)\leq\lambda f(a)+(1-\lambda)f(b)+(1-\lambda)f(a)+\lambda f(b).$$

Multiplying both of sides by $\lambda^{\alpha-1}$, then integrating it with respect to λ on $[0,1]$, we obtain

$$\int_0^1\lambda^{\alpha-1}f(\lambda a+(1-\lambda)b)d\lambda+\int_0^1\lambda^{\alpha-1}f((1-\lambda)a+\lambda b)d\lambda\leq(f(a)+f(b))\int_0^1\lambda^{\alpha-1}d\lambda.$$

using $g=\lambda a+(1-\lambda)b,\ h=(1-\lambda)a+\lambda b$, we obtain

$$\int_a^b\left(\frac{g-a}{b-a}\right)^{\alpha-1}f(g)\ dg+\int_b^a\left(\frac{h-b}{a-b}\right)^{\alpha-1}f(h)\ dh\leq\frac{f(a)+f(b)}{\alpha}.$$

$$\frac{\alpha\Gamma(\alpha)}{2(b-a)^\alpha}[J_{a^+}^{\alpha}f(b)+J_{b^-}^{\alpha}f(a)]\leq\frac{f(a)+f(b)}{2},\tag{3.3.3}$$

After combining inequality (3.3.2) and (3.3.3), we have the required result

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)] \leq \frac{f(a)+f(b)}{2}. \qquad \square$$

If we substitute $\alpha = 1$ in inequality (3.3.1) it reduced to Hermite-Hadamard Inequality. The above inequality is new refinement of Hermite-Hadamard Inequality. Clearly, inequality (3.1.1) can be rewritten as

$$f\left(\frac{a+b}{2}\right) - \frac{f(a)+f(b)}{2} \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)] - \frac{f(a)+f(b)}{2} \leq 0,$$

and

$$0 \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)] - f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2} - f\left(\frac{a+b}{2}\right).$$

This implies

$$\frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)] - \frac{f(a)+f(b)}{2} \leq 0 \leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)] - f\left(\frac{a+b}{2}\right).$$

We examine that the inequality (3.3.1) require function f to be convex. Appropriately, it is natural to consider that f is twice differentiable function. Therefore, $f^{''} \geq 0$. The first theorem concerns the case when $f^{''}$ is bounded in $[a, b]$. In other words, we do not require $f^{''} \geq 0$. Therefore, we can prove the following result.

Theorem 3.3.2 [10]. *Let f : $[a,b] \to \mathbb{R}$ be a positive, twice differentiable function with $a < b$ and $f \in L_1[a, b]$. If $f^{''}$ is bounded in $[a, b]$. Then we have*

$$\begin{aligned} &\frac{m\alpha}{2(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx \\ &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)] - f\left(\frac{a+b}{2}\right) \\ &\leq \frac{M\alpha}{2(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} \left(\frac{a+b}{2} - x\right)^2 [(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx, \end{aligned} \tag{3.3.4}$$

and

$$\begin{aligned} &\frac{-M\alpha}{2(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} (x-a)(b-x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx \\ &\leq \frac{\Gamma(\alpha+1)}{2(b-a)^{\alpha}}[J_{a^+}^{\alpha} f(b) + J_{b^-}^{\alpha} f(a)] - \frac{f(a)+f(b)}{2} \\ &\leq \frac{-m\alpha}{2(b-a)^{\alpha}} \int_a^{\frac{a+b}{2}} (x-a)(b-x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx, \end{aligned} \tag{3.3.5}$$

with $\alpha > 0$, *where* $m = inf_{t\in[a,b]} f''(t)$, $M = sup_{t\in[a,b]} f''(t)$.

Proof. We have

$$\begin{aligned}\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] &= \frac{\alpha}{2(b-a)^\alpha}\left[\int_a^b (b-x)^{\alpha-1}f(x)dx + \int_a^b (x-a)^{\alpha-1}f(x)dx\right]\\ &= \frac{\alpha}{2(b-a)^\alpha}\int_a^b f(x)[(b-x)^{\alpha-1} + (x-a)^{\alpha-1}]dx\\ &= \frac{\alpha}{2(b-a)^\alpha}\int_a^b f(a+b-x)[(b-x)^{\alpha-1} + (x-a)^{\alpha-1}]dx.\end{aligned}$$

Therefore

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)]\\ &= \frac{\alpha}{4(b-a)^\alpha}\int_a^b [f(x) + f(a+b-x)][(b-x)^{\alpha-1} + (x-a)^{\alpha-1}]dx.\end{aligned}$$

After that

$$\begin{aligned}&\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right)\\ &= \frac{\alpha}{4(b-a)^\alpha}\int_a^b \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right)\right][(b-x)^{\alpha-1} + (x-a)^{\alpha-1}]dx.\end{aligned}$$

As

$$\left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right)\right][(b-x)^{\alpha-1} + (x-a)^{\alpha-1}],$$

is symmetric about $x = \frac{a+b}{2}$, we have

$$\frac{\alpha}{4(b-a)^\alpha}\int_a^b \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right)\right][(b-x)^{\alpha-1} + (x-a)^{\alpha-1}]dx$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int_a^{\frac{a+b}{2}} \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right)\right][(b-x)^{\alpha-1} + (x-a)^{\alpha-1}]dx,$$

this implies that

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] - f\left(\frac{a+b}{2}\right)$$

$$= \frac{\alpha}{2(b-a)^{\alpha}} \int_{a}^{\frac{a+b}{2}} \left[f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) \right] [(b-x)^{\alpha-1} + (x-a)^{\alpha-1}] dx. \quad (3.3.6)$$

As

$$f(a+b-x) - f\left(\frac{a+b}{2}\right) = \int_{\frac{a+b}{2}}^{a+b-x} f^{'}(t) dt,$$

and

$$f\left(\frac{a+b}{2}\right) - f(x) = \int_{x}^{\frac{a+b}{2}} f^{'}(t) dt,$$

then we have

$$\begin{aligned} f(x) + f(a+b-x) - 2f\left(\frac{a+b}{2}\right) &= \int_{\frac{a+b}{2}}^{a+b-x} f^{'}(t) dt - \int_{x}^{\frac{a+b}{2}} f^{'}(t) dt \\ &= \int_{x}^{\frac{a+b}{2}} f^{'}(a+b-t) dt - \int_{x}^{\frac{a+b}{2}} f^{'}(t) dt \\ &= \int_{x}^{\frac{a+b}{2}} [f^{'}(a+b-t) - f^{'}(t)] dt. \end{aligned}$$

As

$$f^{'}(a+b-t) - f^{'}(t) = \int_{t}^{a+b-t} f^{''}(y) dy,$$

then for $t \in \left[a, \frac{a+b}{2}\right]$, we obtain

$$m(a+b-2t) \leq f^{'}(a+b-t) - f^{'}(t) \leq M(a+b-2t).$$

Thus,

$$\int_{x}^{\frac{a+b}{2}} m(a+b-2t) dt \leq f(a+b-t) - f(t) - 2f\left(\frac{a+b}{2}\right) \leq \int_{x}^{\frac{a+b}{2}} M(a+b-2t) dt.$$

That is

$$m\left(\frac{a+b}{2} - x\right)^{2} \leq f(a+b-t) - f(t) - 2f\left(\frac{a+b}{2}\right) \leq M\left(\frac{a+b}{2} - x\right)^{2},$$

using this in (3.3.6), we obtain required result (3.3.4)

$$\frac{m\alpha}{2(b-a)^\alpha}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx$$
$$\leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)]-f\left(\frac{a+b}{2}\right)$$
$$\leq \frac{M\alpha}{2(b-a)^\alpha}\int_a^{\frac{a+b}{2}}\left(\frac{a+b}{2}-x\right)^2[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx.$$

Similarly we use this

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)]$$
$$=\frac{\alpha}{4(b-a)^\alpha}\int_a^b[f(x)+f(a+b-x)][(b-x)^{\alpha-1}+(x-a)^{\alpha-1}]dx,$$

to show (3.3.5). Thus

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)]-\frac{f(a)+f(b)}{2}$$
$$=\frac{\alpha}{4(b-a)^\alpha}\int_a^b[f(x)+f(a+b-x)-(f(a)+f(b))]\,[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}]dx.$$

As

$$[f(x)+f(a+b-x)-(f(a)+f(b))]\,[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}]$$

is symmetric about $x=\frac{a+b}{2}$, we have

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b)+J_{b^-}^\alpha f(a)]-\frac{f(a)+f(b)}{2}$$

$$=\frac{\alpha}{2(b-a)^\alpha}\int_a^{\frac{a+b}{2}}[f(x)+f(a+b-x)-(f(a)+f(b))]\,[(b-x)^{\alpha-1}+(x-a)^{\alpha-1}]dx. \quad (3.3.7)$$

As

$$f(b)-f(a+b-x)=\int_{a+b-x}^b f'(t)dt,$$

and

$$f(x)-f(a)=\int_a^x f'(t)dt,$$

then we have

$$f(x) + f(a+b-x) - (f(a) + f(b)) = \int_a^x f'(t)dt - \int_{a+b-x}^b f'(t)dt$$
$$= \int_a^x f'(t)dt - \int_a^x f'(t)dt$$
$$= \int_a^x [f'(a+b-t) - f'(t)]dt.$$

As

$$f'(a+b-t) - f'(t) = \int_t^{a+b-t} f''(y)dy,$$

then for $t \in \left[a, \frac{a+b}{2}\right]$, we obtain

$$m(a+b-2t) \leq f'(a+b-t) - f'(t) \leq M(a+b-2t).$$

Thus,

$$-\int_a^x M(a+b-2t)dt \leq f(x) - f(a+b-x) - (f(a) + f(b)) \leq -\int_a^x m(a+b-2t)dt.$$

That is

$$-M(x-a)(b-x) \leq f(x) - f(a+b-x) - (f(a) + f(b)) \leq M\left(\frac{a+b}{2} - x\right)^2,$$

using this in (3.3.7), we obtain our required result (3.3.5). □

Remark 3.3.3. By applying above theorem (3.3.2), with function f such that $f'' \geq 0$ and we get refinement of Hermite-Hadamard Inequality (3.3.1).

Also, it is clear that $f'' \geq 0$ implies that f' is non decreasing. Consequently, in the following result, we consider that

$$f'(a+b-x) \geq f'(x),$$

for all $x \in [a, \frac{a+b}{2}]$. Obviously, if f' is non decreasing, then above inequality holds but it is easy to see that the reverse inequality is not true.

Theorem 3.3.4 [10]. *Let $f : [a,b] \to \mathbb{R}$ be a positive, differentiable function with $a < b$ and $f \in L_1[a,b]$. If $f'(a+b-x) \geq f'(x)$ for all $x \in \left[a, \frac{a+b}{2}\right]$. Then the following inequalities for fractional integrals hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J_{a^+}^\alpha f(b) + J_{b^-}^\alpha f(a)] \leq \frac{f(a)+f(b)}{2}, \tag{3.3.8}$$

with $\alpha > 0$.

Proof. From above theorem (3.3.2), we have

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)]-f\left(\frac{a+b}{2}\right)$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int_a^{\frac{a+b}{2}}\left[f(x)+f(a+b-x)-2f\left(\frac{a+b}{2}\right)\right][(b-x)^{\alpha-1}+(x-a)^{\alpha-1}]dx.$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int_a^{\frac{a+b}{2}}\left[\int_x^{\frac{a+b}{2}}[f^{'}(a+b-t)-f^{'}(t)]dt\right][(b-x)^{\alpha-1}+(x-a)^{\alpha-1}]dx \geq 0$$

Similarly

$$\frac{\Gamma(\alpha+1)}{2(b-a)^\alpha}[J^\alpha_{a^+}f(b)+J^\alpha_{b^-}f(a)]-\frac{f(a)+f(b)}{2}$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int_a^{\frac{a+b}{2}}[f(x)+f(a+b-x)-(f(a)+f(b))][(b-x)^{\alpha-1}+(x-a)^{\alpha-1}]dx.$$

$$= \frac{\alpha}{2(b-a)^\alpha}\int_a^{\frac{a+b}{2}}\left[-\int_a^{x}[f^{'}(a+b-t)-f^{'}(t)]dt\right][(b-x)^{\alpha-1}+(x-a)^{\alpha-1}]dx \leq 0.$$

This completes the proof. □

Chapter 4

Cauchy Type Means for Hermite-Hadamard Inequalities via Fractional Integrals

4.1 Introduction

In this chapter we develop Mean value theorems and Cauchy Means for Hermite-Hadamard inequalities involving Riemann-Liouville fractional integrals.
The inequality (3.3.1) defined in Chapter 3 can be rewritten as.

Theorem 4.1.1 *Let $f : [a, b] \to \mathbb{R}$ be a positive function with $f \in L_1[a, b]$ and $a < b$ then following inequalities hold*

$$f\left(\frac{a+b}{2}\right) \leq \frac{\alpha}{2(b-a)^{\alpha}} \int_a^b f(x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx. \tag{4.1.1}$$

$$\frac{\alpha}{2(b-a)^{\alpha}} \int_a^b f(x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx \leq \frac{f(a)+f(b)}{2}. \tag{4.1.2}$$

4.2 Mean Value Theorems

In this section, we develop men value theorem for Hermite-Hadamard inequalities involving fractional integrals.

Definition 4.2.1. Let $\varphi : [a, b] \to \mathbb{R}$ be an integrable function with $\varphi \in L_1$ and for $0 \leq a < b$ one defines a linear functional Ω_φ as

$$\Omega_\varphi = \frac{\alpha}{2(b-a)^{\alpha}} \int_a^b \varphi(x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx - \varphi\left(\frac{a+b}{2}\right). \tag{4.2.1}$$

From above inequality (4.1.1) it is clear $\Omega_\varphi \geq 0$.

Lemma 4.2.2 [6]. *Let $\varphi : I \to \mathbb{R}$ where $I \subseteq \mathbb{R}$ be a function such that $\varphi \in C^2(I)$. $\varphi''(x)$ is bounded and $m = \inf_{t\in I}\ \varphi''(t)$, $M = \sup_{t\in I}\ \varphi''(t)$, then the function $\varphi_1,\ \varphi_2 : I \to \mathbb{R}$ defined by*

$$\varphi_1 = \frac{M}{2}t^2 - \varphi(t), \quad \varphi_2 = \varphi(t) - \frac{m}{2}t^2, \tag{4.2.2}$$

are convex for $x > 0$.

Theorem 4.2.3. *Let $\varphi \in C^2(I)$ where $I = [a, b]$, then there exists $\xi \in I$ such that the following equality holds*

$$\Omega_\varphi = \frac{1}{8(\alpha+1)(\alpha+2)}\varphi''(\xi)\left(\alpha^2 - \alpha + 2\right)(b-a)^2, \tag{4.2.3}$$

$\alpha > 0$.

Proof. Suppose that φ'' is bounded, where inf $\varphi''(x) = m$ and sup $\varphi''(x) = M$. Now by using φ_1 in place of φ in (4.1.1) we obtain

$$\frac{M}{2}\left(\frac{a+b}{2}\right)^2 - \varphi\left(\frac{a+b}{2}\right) \leq \frac{\alpha M}{4(b-a)^\alpha}\int_a^b x^2[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx$$
$$- \frac{\alpha}{2(b-a)^\alpha}\int_a^b \varphi(x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx.$$

$$\frac{\alpha}{2(b-a)^\alpha}\int_a^b \varphi(x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx - \varphi\left(\frac{a+b}{2}\right)$$
$$\leq \frac{M(\alpha^2 - \alpha + 2)(b-a)^2}{8(\alpha+1)(\alpha+2)}.$$

Similarly, by using φ_2 in place of φ in (4.1.1) we obtain

$$\frac{\alpha}{2(b-a)^\alpha}\int_a^b \varphi(x)[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx - \varphi\left(\frac{a+b}{2}\right)$$
$$\geq \frac{m(\alpha^2 - \alpha + 2)(b-a)^2}{8(\alpha+1)(\alpha+2)}.$$

When we combine above inequalities we get that there exists $0 < \xi < \infty$ such that (4.2.3) holds. □

Theorem 4.2.4 *If $\varphi,\ \psi \in C^2(I)$, and $\frac{(\alpha^2-\alpha+2)(b-a)^2}{(\alpha+1)(\alpha+2)} \neq 0$, then one has*

$$\frac{\Omega_\varphi}{\Omega_\psi} = \frac{\varphi''(\xi)}{\psi''(\xi)} = K(\xi), \quad \xi \in I, \tag{4.2.4}$$

provided the denominators are not equal to zero. If K is invertible then

$$\xi = K^{-1}\left(\frac{\Omega_\varphi}{\Omega_\psi}\right), \quad \Omega_\psi \neq 0, \tag{4.2.5}$$

is a new mean.

Therefore if we substitute $\varphi(x) = x^r/(r(r-2))$ and $\psi(x) = x^t/(t(t-2))$ in (4.2.1), we get

$$\Omega_\varphi = \frac{\alpha}{2(b-a)^\alpha r(r-1)} \int_a^b x^r[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx - \frac{1}{r(r-1)}\left(\frac{a+b}{2}\right)^r.$$

$$= \frac{2^r\alpha \int_a^b x^r[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx - 2(a+b)^r(b-a)^\alpha}{2^r(b-a)^\alpha 2r(r-1)}.$$

Similarly

$$\Omega_\psi = \frac{\alpha}{2(b-a)^\alpha t(t-1)} \int_a^b x^t[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx - \frac{1}{t(t-1)}\left(\frac{a+b}{2}\right)^t.$$

$$= \frac{2^t\alpha \int_a^b x^t[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx - 2(a+b)^t(b-a)^\alpha}{2^t(b-a)^\alpha 2t(t-1)}.$$

Now using Ω_φ and Ω_ψ in (4.2.4) we obtain

$$\frac{\Omega_\varphi}{\Omega_\psi} = \frac{2^t t(t-1)(2^r\alpha \int_a^b x^r[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx - 2(a+b)^r(b-a)^\alpha)}{2^r r(r-1)(2^t\alpha \int_a^b x^t[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx - 2(a+b)^t(b-a)^\alpha)}$$

and

$$\frac{\varphi''(\xi)}{\psi''(\xi)} = \xi^{r-t}.$$

Using this in Equation (4.2.5) then we have a new mean $N_{r,t}$ defined as follows, where $r,\ t, > 0$, $r \neq t$ and $a,\ b,\ \alpha > 0$, $a \neq b$.

$$N_{r,t} = \left(\frac{2^t t(t-1)(2^r\alpha \int_a^b x^r[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx - 2(a+b)^r(b-a)^\alpha)}{2^r r(r-1)(2^t\alpha \int_a^b x^t[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx - 2(a+b)^t(b-a)^\alpha)}\right)^{1/(r-t)}, \quad r,t \neq 1. \tag{4.2.6}$$

Definition 4.2.5. Let $\varphi : [a,b] \to \mathbb{R}$ be an integrable function with $\varphi \in L_1$ and for $0 \le a < b$ one defines a linear functional Ω_φ as

$$\tilde{\Omega}_\varphi = \frac{\varphi(a)+\varphi(b)}{2} - \frac{\alpha}{2(b-a)^\alpha}\int_a^b \varphi(x)[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx. \tag{4.2.7}$$

From inequality (4.1.1) it is clear $\tilde{\Omega}_\varphi \ge 0$.

Theorem 4.2.6. *Let $\varphi \in C^2(I)$ where $I = [a,b]$, then there exists $\xi \in I$ such that the following equality holds*

$$\tilde{\Omega}_\varphi = \frac{1}{2}\varphi''(\xi)\frac{\alpha(b-a)^2}{(\alpha+1)(\alpha+2)}, \tag{4.2.8}$$

$\alpha > 0$.

Proof. Suppose that φ'' is bounded, where $\inf \varphi''(x) = m$ and $\sup \varphi''(x) = M$. Now by using φ_1 in place of φ in (4.1.2) we obtain

$$\frac{\alpha M}{4(b-a)^\alpha}\int_a^b x^2[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx - \frac{\alpha}{2(b-a)^\alpha}\int_a^b \varphi(x)[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx$$

$$\le \frac{M}{4}(a^2+b^2) - \frac{\varphi(a)+\varphi(b)}{2}.$$

$$\frac{\varphi(a)+\varphi(b)}{2} - \frac{\alpha}{2(b-a)^\alpha}\int_a^b \varphi(x)[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx \le \frac{M\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}.$$

Similarly, by using φ_2 in place of φ in (4.1.2) we obtain

$$\frac{\varphi(a)+\varphi(b)}{2} - \frac{\alpha}{2(b-a)^\alpha}\int_a^b \varphi(x)[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx \ge \frac{m\alpha(b-a)^2}{2(\alpha+1)(\alpha+2)}.$$

When we combine above inequalities we get that there exists $0 < \xi < \infty$ such that (4.2.8) holds. □

Theorem 4.2.7 *If φ, $\psi \in C^2(I)$, and $\frac{\alpha(b-a)^2}{(\alpha+1)(\alpha+2)} \neq 0$, then one has*

$$\frac{\tilde{\Omega}_\varphi}{\tilde{\Omega}_\psi} = \frac{\varphi''(\xi)}{\psi''(\xi)} = T(\xi), \quad \xi \in I, \tag{4.2.9}$$

provided the denominators are not equal to zero. If T is invertible then

$$\xi = T^{-1}\left(\frac{\tilde{\Omega}_\varphi}{\tilde{\Omega}_\psi}\right), \quad \tilde{\Omega}_\psi \neq 0, \tag{4.2.10}$$

is a new mean.

Therefore if we substitute $\varphi(x) = x^r/(r(r-2))$ and $\psi(x) = x^t/(t(t-2))$ in (4.2.7), we get

$$\tilde{\Omega}_\varphi = \frac{1}{2r(r-1)}(a^r + b^r) - \frac{\alpha}{2(b-a)^\alpha r(r-1)} \int_a^b x^r[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx.$$

$$= \frac{(a^r + b^r)(b-a)^\alpha - \alpha \int_a^b x^r[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx}{(b-a)^\alpha 2r(r-1)}.$$

Similarly

$$\tilde{\Omega}_\psi = \frac{1}{2t(t-1)}(a^t + b^t) - \frac{\alpha}{2(b-a)^\alpha t(t-1)} \int_a^b x^t[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx.$$

$$= \frac{(a^t + b^t)(b-a)^\alpha - \alpha \int_a^b x^t[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx}{(b-a)^\alpha 2t(t-1)}.$$

Now using $\tilde{\Omega}_\varphi$ and $\tilde{\Omega}_\psi$ in (4.2.9) we obtain

$$\frac{\tilde{\Omega}_\varphi}{\tilde{\Omega}_\psi} = \frac{t(t-1)((a^r + b^r)(b-a)^\alpha - \alpha \int_a^b x^r[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx}{r(r-1)((a^t + b^t)(b-a)^\alpha - \alpha \int_a^b x^t[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx},$$

and

$$\frac{\varphi''(\xi)}{\psi''(\xi)} = \xi^{r-t}.$$

Using this in Equation (4.2.10) then we have a new mean $\widehat{N}_{r,t}$ defined as follows, where $r,\ t, > 0$, $r \neq t$ and $a,\ b,\ \alpha\ >\ 0$, $a \neq b$.

$$\widehat{N}_{r,t} = \left(\frac{t(t-1)\left((b-a)^\alpha(a^r + b^r) - \alpha \int_a^b x^r[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx\right)}{r(r-1)\left((b-a)^\alpha(a^t + b^t) - \alpha \int_a^b x^t[(x-a)^{\alpha-1} + (b-x)^{\alpha-1}]dx\right)} \right)^{1/(r-t)}, \quad r, t \neq 1. \tag{4.2.11}$$

4.2.1 Different cases of Limit for $N_{r,t}$ and $\widehat{N}_{r,t}$

In this section, we calculate limit for $N_{r,t}$ and $\widehat{N}_{r,t}$ at $t=1$, $t=r=1$ and $t=r$. Let we have

$$N_{r,t}=\left(\frac{2^t t(t-1)(2^r\alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx-2(a+b)^r(b-a)^\alpha)}{2^r r(r-1)(2^t\alpha\int_a^b x^t[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx-2(a+b)^t(b-a)^\alpha)}\right)^{1/(r-t)}.$$

To compute $N_{r,1}=N_{1,r}$, consider

$$\lim_{t\to 1}N_{r,t}=\lim_{t\to 1}\left(\frac{2^t t(t-1)(2^r\alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx-2(a+b)^r(b-a)^\alpha)}{2^r r(r-1)(2^t\alpha\int_a^b x^t[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx-2(a+b)^t(b-a)^\alpha)}\right)^{1/(r-t)}.$$

Applying L'Hospital rule, we get

$$=\lim_{t\to 1}\left(\frac{2^t(\ln 2(t^2-t)+2t-1)(2^r\alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx-2(a+b)^r(b-a)^\alpha)}{2^r r(r-1)R^*}\right)^{1/(r-t)},$$

where

$R^* = 2^t\ln 2\alpha\int_a^b x^t[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx+2^t\alpha\int_a^b x^t\ln x[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx-2(a+b)^t\ln(a+b)(b-a)^\alpha$.

$$N_{r,1}=N_{1,r}=\left(\frac{2(2^r\alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx-2(a+b)^r(b-a)^\alpha)}{2^r r(r-1)R}\right)^{1/(r-1)},\ r\neq 1, \tag{4.2.12}$$

where

$$R=2(b-a)^\alpha[\ln 2(a+b)+b\ln b+a\ln a-(a+b)\ln(a+b)]-2\int_a^b\ln x[(x-a)^\alpha-(b-x)^\alpha]dx.$$

To compute $N_{1,1}$, consider

$$\lim_{r\to 1}N_{r,1}=\lim_{r\to 1}\left(\frac{2(2^r\alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx-2(a+b)^r(b-a)^\alpha)}{2^r r(r-1)R}\right)^{1/(r-1)}.$$

Applying limit, we get

$$N_{1,1}=\exp\left(\frac{S}{R}-(\ln 4+2)\right), \tag{4.2.13}$$

where R is defined above and

$S = 2(b-a)^\alpha[(\ln 2)^2(a+b) + 2\ln 2(b\ln b + a\ln a) + b(\ln b)^2 + a(\ln a)^2 - (a+b)(\ln(a+b))^2] - 2\int_a^b(\ln 4\ln x + \ln x^2 + (\ln x)^2)[(x-a)^\alpha - (b-x)^\alpha]dx.$

To compute $N_{r,r}$, consider

$$\lim_{t\to r} N_{r,t} = \lim_{t\to r}\left(\frac{2^t t(t-1)(2^r\alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx - 2(a+b)^r(b-a)^\alpha)}{2^r r(r-1)(2^t\alpha\int_a^b x^t[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx - 2(a+b)^t(b-a)^\alpha)}\right)^{1/(r-t)}, \quad r,t\neq 1.$$

Taking log of both sides, we get

$$\lim_{t\to r}\log N_{r,t} = \lim_{t\to r}\frac{1}{r-t}\ (\log\ (2^t t(t-1)(2^r\alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx - 2(a+b)^r(b-a)^\alpha)) - \log\ (2^r r(r-1)(2^t\alpha\int_a^b x^t[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx - 2(a+b)^t(b-a)^\alpha))).$$

Applying L'Hospital rule, we get

$$= \lim_{t\to r}\left(\frac{R^*}{2^t\alpha\int_a^b x^t[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx - 2(a+b)^t(b-a)^\alpha} - \frac{2^t ln2(t^2-t) + 2^t(2t-1)}{2^t t(t-1)}\right),$$

where R^* is defined above.
Applying limit,we get

$$N_{r,r} = \exp\left(\frac{U}{T} - \frac{2^r\ln 2(r^2-r) + (2r-1)}{2^r r(r-1)}\right), \quad r\neq 1, \tag{4.2.14}$$

where

$U = 2^r\ln 2\alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx + 2^r\alpha\int_a^b x^r\ln x[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx - 2(a+b)^r\ln(a+b)(b-a)^\alpha.$
$T = 2^r\alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx - 2(a+b)^r(b-a)^\alpha.$

Now we consider second function

$$\widehat{N}_{r,t} = \left(\frac{t(t-1)\left((b-a)^\alpha(a^r+b^r) - \alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx\right)}{r(r-1)\left((b-a)^\alpha(a^t+b^t) - \alpha\int_a^b x^t[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx\right)}\right)^{1/(r-t)}, \quad r,t\neq 1 \quad .$$

To compute $\widehat{N}_{r,1} = \widehat{N}_{1,r}$, consider

$$\lim_{t\to 1} \widehat{N}_{r,t} = \lim_{t\to 1} \left(\frac{t(t-1)\left((b-a)^\alpha(a^r+b^r) - \alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx\right)}{r(r-1)\left((b-a)^\alpha(a^t+b^t) - \alpha\int_a^b x^t[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx\right)} \right)^{1/(r-t)}, \quad r \neq 1 \quad .$$

Applying L'Hospital rule, we get

$$= \lim_{t\to 1} \left(\frac{(2t-1)\left((b-a)^\alpha(a^r+b^r) - \alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx\right)}{r(r-1)\left((b-a)^\alpha(a^t\ln a+b^t\ln b) - \alpha\int_a^b x^t\ln x[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx\right)} \right)^{1/(r-t)}.$$

$$\widehat{N}_{r,1} = \widehat{N}_{1,r} = \left(\frac{(b-a)^\alpha(a^r+b^r) - \alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx}{r(r-1)\left(\int_a^b \ln x[(x-a)^\alpha-(b-x)^\alpha]dx\right)} \right)^{1/(r-1)}, \quad r \neq 1. \tag{4.2.15}$$

To compute $\widehat{N}_{1,1}$, consider

$$\lim_{r\to 1} \widehat{N}_{r,1} = \lim_{r\to 1} \left(\frac{(b-a)^\alpha(a^r+b^r) - \alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx}{r(r-1)\left(\int_a^b \ln x[(x-a)^\alpha-(b-x)^\alpha]dx\right)} \right)^{1/(r-1)}.$$

Applying limit, we get

$$\widehat{N}_{1,1} = \exp\left(\frac{\int_a^b(2\ln x+(\ln x)^2)[(x-a)^\alpha-(b-x^\alpha)]dx}{\int_a^b \ln x[(x-a)^\alpha-(b-x^\alpha)]dx} - 2 \right). \tag{4.2.16}$$

To compute $\widehat{N}_{r,r}$, consider

$$\lim_{t\to r} \widehat{N}_{r,t} = \lim_{t\to r} \left(\frac{t(t-1)\left((b-a)^\alpha(a^r+b^r) - \alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx\right)}{r(r-1)\left((b-a)^\alpha(a^t+b^t) - \alpha\int_a^b x^t[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx\right)} \right)^{1/(r-t)}.$$

Taking log of both sides, we get

$$\lim_{t\to r} \log\ \widehat{N}_{r,t} = \lim_{t\to r} \frac{1}{r-t} \Big(\log\ (t(t-1)((b-a)^\alpha(a^r+b^r) - \alpha\int_a^b x^r[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx))$$
$$- \log\ (r(r-1)((b-a)^\alpha(a^t+b^t) - \alpha\int_a^b x^t[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx))\Big).$$

Applying L'Hospital rule, we get

$$= \lim_{t\to r}\left(\frac{(b-a)^{\alpha}(a^{t}\ln a+b^{t}\ln b)-\alpha\int_{a}^{b}x^{t}\ln x[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx}{(b-a)^{\alpha}(a^{t}+b^{t})-\alpha\int_{a}^{b}x^{t}[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx}-\frac{2t-1}{t(t-1)}\right).$$

Applying limit,we get

$$\widehat{N}_{r,r} = \exp\left(\frac{(b-a)^{\alpha}(a^{r}\ln a+b^{r}\ln b)-\alpha\int_{a}^{b}x^{r}\ln x[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx}{(b-a)^{\alpha}(a^{r}+b^{r})-\alpha\int_{a}^{b}x^{r}[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx}-\frac{2r-1}{r(r-1)}\right), \quad r\neq 1. \tag{4.2.17}$$

4.3 Cauchy Type Means

In this section, we define Cauchy type means for Hermite-Hadamard inequalities via fractional integrals.
If we substitute $\varphi(x) = x^{r/s}/((r/s)(r/s-1))$ and $\psi(x) = x^{t/s}/((t/s)(t/s-1))$ in (4.2.1), then by substitution, $a = a^s$, $b = b^s$, we obtain

$$\Omega_{\varphi} = \frac{\alpha}{2(b^{s}-a^{s})^{\alpha}(r/s)(r/s-1)}\int_{a^{s}}^{b^{s}}x^{r/s}[(x-a^{s})^{\alpha-1}+(b^{s}-x)^{\alpha-1}]dx-\frac{1}{(r/s)(r/s-1)}\left(\frac{a+b}{2}\right)^{r/s}.$$

$$= \frac{\alpha\int_{a^{s}}^{b^{s}}x^{r/s}[(x-a^{s})^{\alpha-1}+(b^{s}-x)^{\alpha-1}]dx-2\left(\frac{a+b}{2}\right)^{r/s}(b-a)^{\alpha}}{2(b^{s}-a^{s})^{\alpha}(r/s)(r/s-1)}.$$

Similarly

$$\Omega_{\psi} = \frac{\alpha}{2(b^{s}-a^{s})^{\alpha}(t/s)(t/s-1)}\int_{a^{s}}^{b^{s}}x^{t/s}[(x-a^{s})^{\alpha-1}+(b^{s}-x)^{\alpha-1}]dx-\frac{1}{(t/s)(t/s-1)}\left(\frac{a+b}{2}\right)^{t/s}.$$

$$= \frac{\alpha\int_{a^{s}}^{b^{s}}x^{t/s}[(x-a^{s})^{\alpha-1}+(b^{s}-x)^{\alpha-1}]dx-2\left(\frac{a+b}{2}\right)^{t/s}(b-a)^{\alpha}}{2(b^{s}-a^{s})^{\alpha}(t/s)(t/s-1)}.$$

Now using Ω_{φ} and Ω_{ψ} in (4.2.4) we obtain

$$\frac{\Omega_{\varphi}}{\Omega_{\psi}} = \frac{t(t-s)(\alpha\int_{a^{s}}^{b^{s}}x^{r/s}[(x-a^{s})^{\alpha-1}+(b^{s}-x)^{\alpha-1}]dx-2(b^{s}-a^{s})^{\alpha}\left(\frac{a^{s}+b^{s}}{2}\right)^{r/s}}{r(r-s)(\alpha\int_{a^{s}}^{b^{s}}x^{t/s}[(x-a^{s})^{\alpha-1}+(b^{s}-x)^{\alpha-1}]dx-2(b^{s}-a^{s})^{\alpha}\left(\frac{a^{s}+b^{s}}{2}\right)^{t/s}}.$$

and

$$\frac{\varphi''(\xi)}{\psi''(\xi)} = \xi^{r-t}.$$

Using these results in Equation (4.2.5) then we have a Cauchy mean $N_{r,t}^{[s]}$ defined as follows, where $r,\ t\ s \in \mathbb{R}$, $r \neq t$ and $a, b > 0$, $a \neq b$.

$$N_{r,t}^{[s]} = \left(\frac{t(t-s)(\alpha \int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^{\alpha} \left(\frac{a^s+b^s}{2}\right)^{r/s}}{r(r-s)(\alpha \int_{a^s}^{b^s} x^{t/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^{\alpha} \left(\frac{a^s+b^s}{2}\right)^{t/s}} \right)^{1/(r-t)}, \quad r, t \neq s. \tag{4.3.1}$$

If we substitute $\varphi(x) = x^{r/s}/((r/s)(r/s-1))$ and $\psi(x) = x^{t/s}/((t/s)(t/s-1))$ in (4.2.7), then by substitution, $a = a^s$, $b = b^s$, we obtain

$$\tilde{\Omega}_{\varphi} = \frac{1}{2(r/s)(r/s)}(a^r + b^r) - \frac{\alpha}{2(b^s-a^s)^{\alpha}(r/s)(r/s-1)} \int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx.$$

After simplification, it give us

$$\tilde{\Omega}_{\varphi} = \frac{(a^r + b^r)(b^s-a^s)^{\alpha} - \alpha \int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx}{2(b^s-a^s)^{\alpha}(r/s)(r/s-1)}.$$

Similarly

$$\tilde{\Omega}_{\psi} = \frac{(a^t + b^t)(b^s-a^s)^{\alpha} - \alpha \int_{a^s}^{b^s} x^{t/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx}{2(b^s-a^s)^{\alpha}(t/s)(t/s-1)}.$$

Now using $\tilde{\Omega}_{\varphi}$ and $\tilde{\Omega}_{\psi}$ in (4.2.9) we obtain

$$\frac{\tilde{\Omega}_{\varphi}}{\tilde{\Omega}_{\psi}} = \frac{t(t-s)((b^s-a^s)^{\alpha}(a^r+b^r) - \alpha \int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx)}{r(r-s)((b^s-a^s)^{\alpha}(a^t+b^t) - \alpha \int_{a^s}^{b^s} x^{t/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx)}$$

$$\frac{\varphi''(\xi)}{\psi''(\xi)} = \xi^{r-t}.$$

Using these results in Equation(4.2.10)then we have a Cauchy mean $\widehat{N}_{r,t}^{[s]}$ defined as follows, where $r,\ t\ s \in \mathbb{R}$, $r \neq t$ and $a,\ b,\ \alpha\ >\ 0$, $a \neq b$.

$$\widehat{N}_{r,t}^{[s]} = \left(\frac{t(t-s)((b^s-a^s)^{\alpha}(a^r+b^r) - \alpha \int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx)}{r(r-s)((b^s-a^s)^{\alpha}(a^t+b^t) - \alpha \int_{a^s}^{b^s} x^{t/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx)} \right)^{1/(r-t)}, \quad r, t \neq s. \tag{4.3.2}$$

4.3.1 Different cases of Limit for $N^{[s]}_{r,t}$ and $\widehat{N}^{[s]}_{r,t}$

In this section, we calculate limit for $N^{[s]}_{r,t}$ and $\widehat{N}^{[s]}_{r,t}$ at $t = s$, $t = r = s$ and $t = r$. Let we have

$$N^{[s]}_{r,t} = \left(\frac{t(t-s)(\alpha \int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^{\alpha} \left(\frac{a^s+b^s}{2}\right)^{r/s})}{r(r-s)(\alpha \int_{a^s}^{b^s} x^{t/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^{\alpha} \left(\frac{a^s+b^s}{2}\right)^{t/s})} \right)^{1/(r-t)}.$$

We calculate a limit when t goes to s, consider

$$\lim_{t\to s} N^{[s]}_{r,t} = \lim_{t\to s} \left(\frac{t(t-s)(\alpha \int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^{\alpha} \left(\frac{a^s+b^s}{2}\right)^{r/s})}{r(r-s)(\alpha \int_{a^s}^{b^s} x^{t/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^{\alpha} \left(\frac{a^s+b^s}{2}\right)^{t/s})} \right)^{1/(r-t)}.$$

Applying L'Hospital rule, we get

$$= \lim_{t\to s} \left(\frac{(2t-s)(\alpha \int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^{\alpha} \left(\frac{a^s+b^s}{2}\right)^{r/s})}{r(r-s)\left(\frac{\alpha}{s} \int_{a^s}^{b^s} x^{t/s}\ln x[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx - \frac{2}{s}(b^s-a^s)^{\alpha} \left(\frac{a^s+b^s}{2}\right)^{t/s} \ln\left(\frac{a^s+b^s}{2}\right)\right)} \right)^{1/(r-t)}.$$

Applying limit, it give us

$$N^{[s]}_{r,s} = \left(\frac{s(\alpha \int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^{\alpha} \left(\frac{a^s+b^s}{2}\right)^{r/s})}{r(r-s)V} \right)^{1/(r-s)}, \qquad (4.3.3)$$

where

$$V = \frac{(b^s-a^s)^{\alpha}}{s}[b^s \ln b^s + a^s \ln a^s - (a^s+b^s)\ln(\frac{a^s+b^s}{2})] - \frac{1}{s}\int_a^b \ln x[(x-a^s)^{\alpha} - (b^s-x)^{\alpha}]dx.$$

To compute $N^{[s]}_{s,s}$, consider

$$\lim_{r\to s} N^{[s]}_{r,s} = \lim_{r\to s} \left(\frac{s(\alpha \int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1} + (b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^{\alpha} \left(\frac{a^s+b^s}{2}\right)^{r/s})}{r(r-s)V} \right)^{1/(r-s)}.$$

Applying limit, we get

$$N^{[s]}_{s,s} = \exp\left(\frac{W}{V} - \frac{2}{s}\right), \qquad (4.3.4)$$

where V is defined above and

$W = \frac{(b^s-a^s)^\alpha}{s^2}[b^s(\ln b^s)^2 + a^s(\ln a^s)^2 - (a^s+b^s)(\ln(a^s+b^s)/2)^2] - \frac{1}{s^2}\int_{a^s}^{b^s}(2\ln x + (\ln x)^2)[(x-a^s)^\alpha - (b^s-x)^\alpha]dx$,

To compute $N_{r,r}^{[s]}$, consider

$$\lim_{t\to r} N_{r,t}^{[s]} = \lim_{t\to r}\left(\frac{t(t-s)(\alpha\int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^\alpha\left(\frac{a^s+b^s}{2}\right)^{r/s}}{r(r-s)(\alpha\int_{a^s}^{b^s} x^{t/s}[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx - 2(b^s-a^s)^\alpha\left(\frac{a^s+b^s}{2}\right)^{t/s}}\right)^{1/(r-t)}.$$

Taking log of both sides, we have

$$\lim_{t\to r}\log N_{r,t}^{[s]} = \lim_{t\to r}\frac{1}{r-t}(\log(t(t-s)(\alpha\int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^\alpha(\frac{a^s+b^s}{2})^{r/s}))$$

$$-\log(r(r-s)(\alpha\int_{a^s}^{b^s} x^{t/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^\alpha(\frac{a^s+b^s}{2})^{t/s}))).$$

Applying L'Hospital rule, we have

$$= \lim_{t\to r}\left(\frac{g'}{g} - \frac{2t-s}{t(t-s)}\right),$$

where

$g = \alpha\int_{a^s}^{b^s} x^{t/s}[(x-a)^{\alpha-1}+(b-x)^{\alpha-1}]dx - 2(b^s-a^s)^\alpha\left(\frac{a^s+b^s}{2}\right)^{t/s}$.

Applying limit, it give

$$N_{r,r}^{[s]} = \exp\left(\frac{X}{Y} - \frac{2r-s}{r(r-s)}\right), \tag{4.3.5}$$

where

$X = \frac{\alpha}{s}\int_{a^s}^{b^s} x^{r/s}\ln x[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx - \frac{2}{s}(b^s-a^s)^\alpha\left(\frac{a^s+b^s}{2}\right)^{r/s}\ln\left(\frac{a^s+b^s}{2}\right)$.
$Y = \alpha\int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx - 2(b^s-a^s)^\alpha\left(\frac{a^s+b^s}{2}\right)^{r/s}$.

Now we consider second case

$$\widehat{N}_{r,t}^{[s]} = \left(\frac{t(t-s)((b^s-a^s)^\alpha(a^r+b^r) - \alpha\int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)}{r(r-s)((b^s-a^s)^\alpha(a^t+b^t) - \alpha\int_{a^s}^{b^s} x^{t/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)}\right)^{1/(r-t)}.$$

We calculate a limit when t goes to s, consider

$$\lim_{t\to s}\widehat{N}_{r,t}^{[s]} = \lim_{t\to s}\left(\frac{t(t-s)((b^s-a^s)^\alpha(a^r+b^r)-\alpha\int_{a^s}^{b^s}x^{r/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)}{r(r-s)((b^s-a^s)^\alpha(a^t+b^t)-\alpha\int_{a^s}^{b^s}x^{t/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)}\right)^{1/(r-t)}.$$

Applying L'Hospital rule, we get

$$=\lim_{t\to s}\left(\frac{(2t-s)((b^s-a^s)^\alpha(a^r+b^r)-\alpha\int_{a^s}^{b^s}x^{r/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)}{r(r-s)\left((b^s-a^s)^\alpha(a^t\ln a+b^t\ln b)-\frac{\alpha}{s}\int_{a^s}^{b^s}x^{t/s}\ln x[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)\right)}\right)^{1/(r-t)}.$$

Applying limit, it give us

$$\widehat{N}_{r,s}^{[s]} = \left(\frac{s((b^s-a^s)^\alpha(a^r+b^r)-\alpha\int_{a^s}^{b^s}x^{r/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)}{r(r-s)J}\right)^{1/(r-s)}, \qquad (4.3.6)$$

where

$J=(b^s-a^s)^\alpha(a^s\ln a+b^s\ln b-(b^s\ln b^s+a^s\ln a^s)/s)+\frac{1}{s}\int_{a^s}^{b^s}\ln x[(x-a^s)^\alpha-(b^s-x)^\alpha]dx.$

To compute $\widehat{N}_{s,s}^{[s]}$, consider

$$\lim_{r\to s}\widehat{N}_{r,s}^{[s]} = \lim_{r\to s}\left(\frac{s((b^s-a^s)^\alpha(a^r+b^r)-\alpha\int_{a^s}^{b^s}x^{r/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)}{r(r-s)J}\right)^{1/(r-s)}.$$

Applying limit, we get

$$\widehat{N}_{s,s}^{[s]} = \exp\left(\frac{K}{J}-\frac{2}{s}\right), \qquad (4.3.7)$$

where J is defined above and

$K=(b^s-a^s)^\alpha(a^s(\ln a)^2+b^s(\ln b)^2-(b^s(\ln b^s)^2+a^s(\ln a^s)^2)/s^2)+\frac{1}{s^2}\int_{a^s}^{b^s}(2\ln x+(\ln x)^2)[(x-a^s)^\alpha-(b^s-x)^\alpha]dx.$

To compute $\widehat{N}_{r,r}^{[s]}$, consider

$$\lim_{t\to r}\widehat{N}_{r,t}^{[s]} = \lim_{t\to r}\left(\frac{t(t-s)((b^s-a^s)^\alpha(a^r+b^r)-\alpha\int_{a^s}^{b^s}x^{r/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)}{r(r-s)((b^s-a^s)^\alpha(a^t+b^t)-\alpha\int_{a^s}^{b^s}x^{t/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)}\right)^{1/(r-t)}, \ t\neq 2s.$$

Taking log of both sides, we have

$$\lim_{t\to r}\log \widehat{N}^{[s]}_{r,t} = \lim_{t\to r}\frac{1}{r-t}(\log(t(t-s)((b^s-a^s)^{\alpha}(a^r+b^r)-\alpha\int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx))$$

$$-\log(r(r-s)((b^s-a^s)^{\alpha}(a^t+b^t)-\alpha\int_{a^s}^{b^s} x^{t/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx))).$$

Applying L'Hospital rule, we have

$$=\lim_{t\to r}\left(\frac{(b^s-a^s)^{\alpha}(a^t\ln a+b^t\ln b)-\frac{\alpha}{s}\int_{a^s}^{b^s} x^{t/s}\ln x[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)}{(b^s-a^s)^{\alpha}(a^t+b^t)-\alpha\int_{a^s}^{b^s} x^{t/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx)}-\frac{2t-s}{t(t-s)}\right).$$

Applying limit, it give

$$\widehat{N}^{[s]}_{r,r} = \exp\left(\frac{M}{N}-\frac{2r-s}{r(r-s)}\right), \tag{4.3.8}$$

where

$M=(b^s-a^s)^{\alpha}(a^r\ln a+b^r\ln b)-\frac{\alpha}{s}\int_{a^s}^{b^s} x^{r/s}\ln x[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx.$
$N=(b^s-a^s)^{\alpha}(a^r+b^r)-\alpha\int_{a^s}^{b^s} x^{r/s}[(x-a^s)^{\alpha-1}+(b^s-x)^{\alpha-1}]dx.$

Conclusion

In the area of inequalities there are many results related to convex functions, but one of those is the classical Hermite-Hadamard inequality. The Hermite-Hadamard double inequality defined on a interval of real numbers is the first fundamental result for convex functions. In many area of analysis applications we have found many useful results, generalizations and extensions associated with Hermite-Hadamard inequality for different classes of functions.

In this thesis, we have discussed superquadratic functions and fractional integrals. Also extension of Hermite-Hadamard Inequalities for superquadratic functions and fractional integrals are presented. Using mean value theorem and Cauchy mean value theorem, we derived mean value theorems and Cauchy type means for Hermite-Hadamard inequalities via fractional integrals.

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