Construction of Some New Super Edge Magic Total Graphs Using Old Ones

by

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Abstract

A bijection that assigns non-negative integers to vertices and/or edges of a graph is called a *labeling*. In a labeled graph, we calculate weights of vertices or edges. In this thesis we study the edge weights of different graphs under a labeling. If the edge weights for a graph are constant, that is, all the edge weights are same, then the labeling is called *magic*. Similarly, if all weights are different, then the labeling is called *antimagic*. Sedláček used the idea of *magic* for the first time in graphs in 1963, since then many variations of magic and antimagic labelings are introduced.

Chen et al. [11] introduced *banana trees* and conjectured that banana trees are *graceful*. Javaid et al. [22, 23] introduced *w-graphs*, *w-trees*, *extended w-trees* and constructed their *super edge magic total labelings*. In this thesis, after discussing different types of magic and antimagic labelings, we construct *extended umbrella graph* using *umbrella graph*, and study its *super edge magic total labeling*. We construct another family of graphs using w-graphs and banana trees, referred as *reflexive w-trees*. We also construct some generalizations of these graphs, and study their *super edge magic total labelings*.

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(Friend names are alphabetically arranged;)

Mehar Ali Malik

To My Parents

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Chapter 1

Fundamentals of Graph Theory

1.1 Introduction

In Euclidean geometry, the objects we use are points, lines and a plane. Consider the plane as a chessboard, points as chesspieces and lines as the moves of the chess-pieces. The rules of chess tell us how different pieces can be moved. The opening arrangement is the positions of the pieces at the start of the game, so this can be considered as the list of axioms accepted without proofs. So there is an underlying abstract structure which is infact the essence of 'chess'. Names, shapes and colors of squares and pieces doesn't matter. Even the physical existence of the chessboard and pieces is irrelevant. The interpretation of chess in such abstract way is analogous of chess as a battle. Here what is relevant is the set of rules under which a piece can move, the number of geometric arrangements of different squares and the number of different pieces and number of pieces of each type.

Euclid's words "point", "line"and "plane"suggest that geometry deals with flat surfaces, tiny points and stretched lines. But studying this geometry in abstract way redefines these terms. Now the points (also called nodes or vertices) are just a set of objects, and the lines are connections (also called links or edges) between these points.

This configuration of points and lines occur in great diversity of applications. They can represent physical networks, for example electrical circuits, road networks, organic molecules structures, or structures in operations research. They can also be used in representing less tangible interactions as might occur in sociological systems like friends and family relations, office colleges, etc, or the flow of control in computer programs or the interactions in ecosystems. Formally, we model such configurations by means of special combinatorial structures called *Graphs*.

There are many other situations in which we need such abstract geometrical structures such as, when we need to find out how many layers does a printed circuit board (pcb) need so that the conductive paths do not cross, is actually a *graph planarity* problem. If we want to color the regions of a map in such a way that all the countries that share border have different colors, then it is a *graph coloring* problem. There are many other practical problems which are solved using various *graph theoretic techniques*. In this chapter we give a brief and formal introduction to this theory.

The origin of *graph theory* probably goes back to 1763 when the great Leonhard Euler was proposed a problem to travel all the seven bridges of the city of Königsberg in a single round trip when every bridge is travelled exactly once. All the graphs satisfying this property are named *Eulerian* after Leonhard Euler. He modeled this problem in the form of a graph and gave the solution that; *"a graph is Eulerian if and only if every vertex has even degree"*. The graph-theoretic concepts used here are elaborated in the upcoming sections.

1.2 What are graphs?

A *graph* $G = (V, E)$ is a structure consisting of a set V of objects, called *vertex set*, and a set E of adjacency relations of these objects, called *edge set*. Two vertices are said to be *adjacent* whenever they are directly joined by an edge. If two vertices u and v of a graph G are joined by an edge e then we say that e is incident on u and v. Clearly, each edge has two vertices as its endpoints, so usually we denote an edge by its end vertices, that is, e will be denoted as uv. If an edge e joins a vertex v with itself, then the edge e is called a *loop*. When two edges have same endpoints, they are called *multiple edges*. If there is no edge incident on a vertex v, it is called an *isolated vertex*. A graph is called *simple* when it has no loops or multiple edges.

The *neighbourhood* of a vertex v in G is the set of all vertices of G which are adjacent to v, denoted as $N_G(v)$ or $N(v)$. The number of vertices and edges in a graph is called the *order* (denoted as $|V(G)|$ or ν or p) and *size* (denoted as $|E(G)|$ or ϵ or q) of the graph, respectively. When $V(G)$ is empty, the graph is called *null* graph. Similarly a graph is said to be *finite* if its vertex and edge sets are finite.

Consider a non-simple graph G_1 in figure 1.1. The vertex set of the graph is $V(G_1) = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, and the edge set is $E(G_1) = \{v_1v_2, v_2v_3, v_1v_3, v_1v_3, v_4v_5, v_6v_6\}$. There is a multiple edge v_1v_3 in G_1 so it is written twice in the edge set. The edge v_6 is a loop. There is no edge incident on v_7, so it is an isolated vertex. The neighbourhood of v_2 in this graph is $N_(v_2) = \{v_1, v_3\}$.

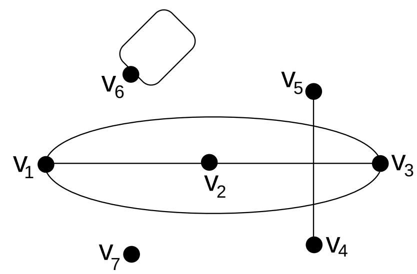


Figure 1.1: G_1

All the edges in figure 1.1 have certain lengths, different shapes and cross each other at different points. The vertices are at specific positions with respect to each other. But according to the abstract definition of a graph, G_1 is merely a graph with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and edge set $\{v_1v_2, v_2v_3, v_1v_3, v_1v_3, v_4v_5, v_6v_6\}$. These abstract properties of the graph G_1 in figure 1.1 are the same as that of another graph G_2 in figure 1.2. So we say that the graph in figure 1.1 is the same as that in figure 1.2 or that G_1 and G_2 are the redrawings of each other.

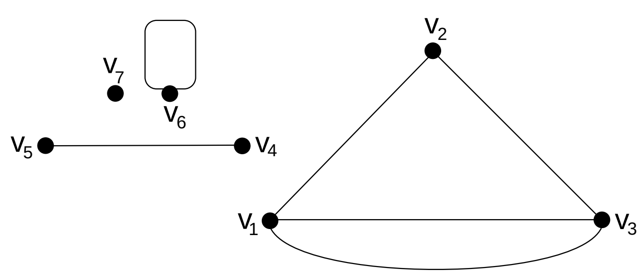


Figure 1.2: G_2

The vertices in a graph can be considered as places and the edges can be considered as links between these vertices. We travel on these edges to reach different vertices. Suppose we travel from a vertex u to a vertex v through an edge e, then if we can travel back to the vertex u through the same edge e then the edge e is unordered pair of vertices u and v, that is, $uv = vu = e$, but if we can go through e in only one direction (say from u to v) then the edge e is an ordered par of vertices, in this case $e = uv \neq vu$. The edges discussed in the second case are called *directed* edges, and the graphs all of whose edges have directions are called *directed graphs* or simply *digraphs*. To avoid ambiguity, the undirected edge or simple edge uv is written as $\{uv\}$ and directed edge uv is written as $[uv]$. A graph having both directed and undirected edges is called *mixed graph*.

A graph G can also be represented by its *adjacency matrix*, denoted as $A(G)$. The matrix $A(G)$ is a square matrix whose rows and columns are indexed by identical ordering of $V(G)$, such that the entry a_{ij} is the number of edges from v_i to v_j. A loop contributes two to the degree of any vertex. The adjacency matrix of the graph G_1 in figure 1.1 is given below.

$$A(G_1) = \begin{pmatrix} 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

The entries on the principal diagonal represent loops on the respective vertices. If a_{ii} is 4, it means that there are two loops on vertex v_i. The sum of any row or column is equal to the degree of that vertex. In case of a directed graph, the entry a_{ij} of the adjacency matrix is the number of edges having directions from v_i to v_j.

In a graph G, the degree of a vertex v, denoted as $deg(v)$, is the number of edges incident on v. The *maximum degree* of a graph G is the maximum vertex degree in that graph, denoted as $\triangle(G)$, and the *minimum degree* is the minimum vertex degree in the graph, denoted as $\delta(G)$. The *degree sequence* or *graphic sequence* is the non-increasing sequence formed by vertex degrees of the graph. The *degree* of a graph is the sum of degrees of all the vertices. A graph is said to be *even or odd* if all of its vertex degrees are even or odd, respectively. An *isolated vertex* is a vertex of degree 0, and a vertex of degree 1 is called a *leaf*. A vertex is *even* (odd) if its degree is even (odd). In a simple graph of order p, the maximum degree of a vertex is $p-1$, since any vertex can at most be adjacent to all other $p-1$ vertices. Hence for any vertex v in a graph, $deg(v)$ lies between 0 and $p-1$. A graph is called *regular* or *k-regular* if all the vertex degrees are same, that is, $\triangle(G) = \delta(G) = k$. The 3-regular graphs are known as *cubic* graphs. All graphs in figure 1.3 are cubic.

Now we state the *fundamental theorem of graph theory* also known as the *Euler degree-sum theorem*.

Theorem 1.2.1. [19] *The sum of degrees of the vertices of a graph is twice the number of edges. That is,*

$$\sum_{v \in V(G)} d(v) = 2|E(G)|.$$

Since each edge contributes 'one' to the degree of each of its end-vertex, it contributes 'two' to the degree of the graph. Alternatively, summing up all the vertex degrees includes every edge twice, so degree of the graph equals twice the number of edges. We can draw some trivial results from this theorem, such as
1) The degree of a graph is always even.
2) In any graph, the number of vertices of odd degree is even.
3) No graph of odd order is regular with odd degree.

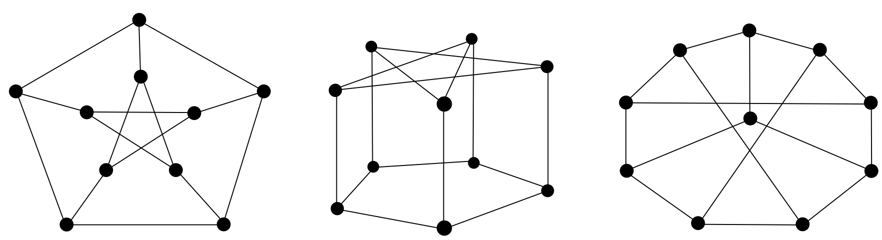


Figure 1.3: Some 3-regular graphs

A graph H is *subgraph* of a graph G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and the edges of H have same endpoints as that in G. The subgraph H is said to be *spanning* if $V(H) = V(G)$. If $S \subset V(G)$, a subgraph H induced by S, denoted as $H[S]$, is a graph with $V(H) = S$ and all the edges of G which have both ends in S.

A *walk* in a graph G is an alternating sequence of vertices and their incident edges. A *trail* is a walk in which no edge is repeated. A *path* is a trail in which no vertex is repeated. A *circuit* is a closed trail, that is, a trail in which first and last vertex coincides. Similarly, a closed path is called a *cycle*. A cycle of order n is denoted as C_n. A cycle with a *chord* is a cycle in which any two non-adjacent vertices are joined with an edge. The *girth* of a graph is the length of smallest cycle in the graph. If there is no cycle in the graph then girth is undefined. In figure 1.4, a circuit $\{abcdefghdia\}$, a path $\{jklm\}$ and a cycle $\{nopqrn\}$ are highlighted. We can observe that a circuit is an edge disjoint union of cycles. When vertices are written in the way {abc}, it means that there is an edge between vertices a and b, and another edge between b and c.

Any graph with $\delta(G) \geq 2$ contains a cycle. This can be proved by a simple algorithm. Suppose there is a maximal path $P = \{v_1, v_2, \ldots, v_n\}$ in the graph. Then the first and the last vertex of P have at least one neighbour which is not on the path, say v_{n+1} is such a vertex. Extend P to $P' = \{v_1, v_2, \ldots, v_n, v_{n+1}\}$, which contradicts the maximality of P. Hence the graph contains a cycle.

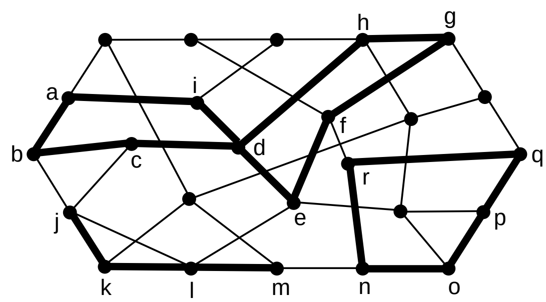


Figure 1.4: Circuit, path and cycle in a graph

An *isomorphism* between simple graphs G and H is a bijection $f : V(G) \to V(H)$ such that $uv \in E(G)$ if and only if $f(u)f(v) \in E(H)$. If such mapping exist we say that G is *isomorphic* to H ($G \cong H$). In an isomorphism, the adjacency relations of one graph are preserved in the other. While discussing a graph G in this theory we actually are talking about an isomorphism class of graphs containing all the graphs isomorphic to that particular graph G. Our structural comments and conclusions will be true for all the graphs isomorphic to G. For the isomorphic graphs G and H, the image $f(v) \in V(H)$ of a vertex $v \in V(G)$ preserves all the adjacency properties of v in H. For example, the degree of v remains same, if v was adjacent to two vertices of degree 4 and 1 in G then $f(v)$ has the neighbours of same degrees in H, if v was on a cycle of length 5 in G then its image is on a cycle of length 5 in H.

Isomorphic graphs satisfy many other properties, for example, both graphs must have the same degree sequence, if the graph G can be drawn in such a way that no two edges cross each other, then H can also be drawn in this way. We can use adjacency matrices to check graph isomorphism as well. If we order the vertices of one graph in such a way that its adjacency matrix becomes identical with the adjacency matrix of some other graph then both graphs are isomorphic. All graphs in figure 1.3 are isomorphic to each other.

A *clique* in a graph G is the set of pairwise adjacent vertices. The number of vertices in maximum such set is called *clique number*, denoted as $\omega(G)$. An *independent set* is the set of pairwise non-adjacent vertices in the graph, and the number of vertices in maximal such set is called *independence number*, denoted as $\beta(G)$.

The length of a path or walk is the number of edges in it. The *distance* between two vertices u and v in a graph is the length of shortest path between these vertices, denoted as $d(u, v)$. If no such path exist then $d(a, b) = \infty$. A graph is *connected* if for every pair of vertices there is at least one path between them, otherwise graph is *disconnected*. The graph in figure 1.4 is connected but that in figure 1.1 is disconnected.

An *Eulerian circuit* in a graph G is a circuit (closed trail) which contains all the edges of G. A graph is *Eulerian* if it contains an Eulerian circuit. In the following theorem a characterization of Eulerian graphs is presented.

Theorem 1.2.2. [19] *The following statements are equivalent for a connected graph* G.
(1) G *is Eulerian.*
(2) *The degree of every vertex in* G *is even.*
(3) $E(G)$ *is the union of edge-disjoint cycles in* G.

When a graph is Eulerian then by definition we have a trail passing from every edge and ending at the point from where we started it. The trail visits a particular vertex through an edge and leaves through some other edge, so every time the trail visits the vertex it uses two from the degree of the vertex. Since the trail used all the edges so it visited all the vertices even number of times, which implies that all vertex degrees are even. When degree of all vertices are even we can partition the edge set in cycles. It can be proved with the help of an algorithm. Consider a maximal path in a graph and extend any of its leaves using the given condition that all vertex degrees are even. It will result in a cycle (or a contradiction). Now, delete the edges of the cycle we obtain and apply the algorithm again on the resultant graph.

A cycle is said to be *hamiltonian* if it passes through every vertex of the graph. A graph containing a *hamiltonian cycle* is called a *hamiltonian graph*. Clearly, a hamiltonian cycle in a graph is a spanning subgraph of the graph. There is no characterization known for hamiltonian graph, but we have some results sufficient for a graph to be hamiltonian. Some of the simplest conditions are stated below.

Theorem 1.2.3. [41] *If* G *is a simple graph with at least three vertices and* $\delta(G) \geq \frac{|G|}{2}$, *then* G *is hamiltonian.*

Theorem 1.2.4. [19] *Let* G *be a simple* n*-vertex graph, where* $n \geq 3$, *such that* $deg(x) + deg(y) \geq n$ *for each pair of non-adjacent vertices* x *and* y, *then* G *is hamiltonian.*

A *bipartite* graph is a graph whose vertex set can be partitioned in two sets, called partite sets, in such a way such that the vertices in each partite set are pairwise non-adjacent. The two partite sets are independent of each other and are called *bipartition* of the graph. Similarly, a graph is *s-partite* if its vertex set is the union of S independent sets. A very useful characterization of bipartite graphs is given in the following theorem. A partite graph is called *balanced* if all the partite sets have equal cardinality.

Theorem 1.2.5. [24] *A graph G is bipartite if and only if it contains no odd cycle.*

In a bipartite graph with bipartition $[X, Y]$, let us construct a cycle with initial vertex in any partite set, say X. To create a cycle we need to visit both partite sets alternatively, since vertices in one partite set can not be adjacent. So every time we return to the partite set from where we started, we travel even number of edges.

A graph is *complete* if all of its vertices are pairwise adjacent. A complete graph of order n is denoted as K_n. Clearly, a complete graph K_n is a regular graph of degree $n-1$. A complete bipartite graph, denoted as $K_{m,n}$, is a complete graph with m vertices in one partite set and n vertices in the other. In figure 1.5, we present the complete graph on 10 vertices (K_{10}) and complete bipartite graph $K_{4,3}$.

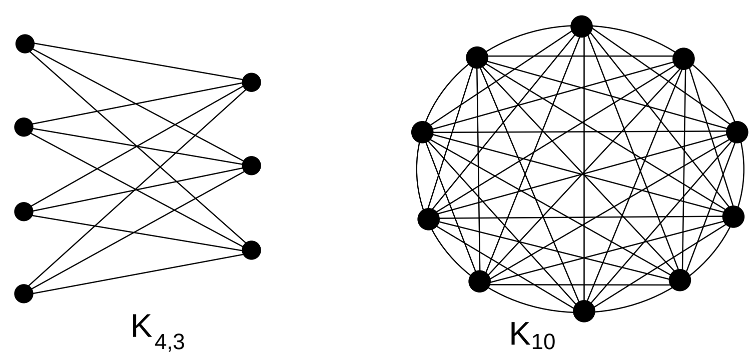


Figure 1.5: Complete graphs

The maximal connected subgraph of a graph is called a *component*. A disconnected graph obviously has atleast two components. The number of components of a graph G is denoted as $c(G)$. The graph in figure 1.2 has 4 components. A *separating set* (also called *vertex-cut*) of a connected graph G is a set $S \subset V(G)$ such that $G - S$ is disconnected. Similarly an *edge-cut* is a set $H \subset E(G)$ such that $G - H$ is disconnected. A $\{u, v\}$-separating set in a graph is the set of edges whose deletion disconnect the vertices u and v.

A single vertex whose removal from a graph increases its number of components, is called a *cut-vertex*. Similarly a single edge whose removal from a connected graph leaves it disconnected, is called a *bridge*. We can observe that a bridge can never be a part of a cycle. It is interesting to know that the deletion of one vertex from a connected graph G can create many components in G, but deletion of one edge can increase the number of components by at most 1.

The *vertex connectivity* $\kappa(G)$ of a connected graph G is the minimum number of vertices whose removal can disconnect the graph G. When $\kappa(G) \geq k$, graph is called k-connected. In the same way, the *edge connectivity* $\lambda(G)$ of a connected graph is minimum number of edges whose removal can disconnect the graph. The graph G is called *k-edge connected* when $\lambda(G) \geq k$. When a graph is k-edge connected, every two vertices are joined by k internally disjoint paths. The converse of this statement also hold.

The vertex connectivity of a complete graph K_n is not defined since it has no separating set, so we adopt a convention that a graph with one isolated vertex is disconnected, then $\kappa(K_n) = n - 1$. Similarly, the vertex connectivity of a complete bipartite graph $K_{m,n}$ is $min\{m, n\}$, because vertices in one partite set are only adjacent to the vertices of other partite set, so deleting the vertices of one partite set leaves the other partite set isolated. The graph (a) in figure 1.6 has no cut-vertex, the minimal vertex-cut is $\{x, y\}$ and the minimum $\{u, v\}$-separating set consists of edges $\{e_1, e_2, e_3, e_4\}$. In graph (b) of the same figure, there is a bridge e, and the end-vertices of e are both cut-vertices. So, the vertex and edge connectivity of the graph (b) is 1.

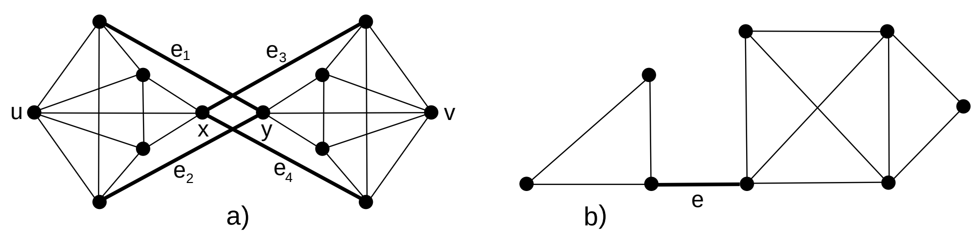


Figure 1.6: vertex/edge-cut in a graph

The connectivity of a graph can also be regarded as the strength of connections between its vertices. After removing some vertices or edges, if the graph is still connected means the connections in the graph are strong. While trying to disconnect a graph by removing edges, it is direct observation that if we remove the edges incident on the vertex with minimum degree in the graph, the connection of all other vertices with that vertex will be destroyed, and the graph becomes disconnected. So, we can say that for a connected graph, $\lambda(G) \leq \delta(G)$. The vertex connectivity and the edge connectivity can be compared with the following theorem.

Theorem 1.2.6. [12] *If G is a simple graph then vertex connectivity never exceeds edge connectivity, i.e., $\kappa(G) \leq \lambda(G)$.*

For a k-edge connected graph G, there will be an edge cut say S with k edges in G. The edges of S partition $V(G)$ in two sets say V_1 and V_2, each edge of S has on end in V_1 and other end in V_2. If we remove at most k vertices from any of V_1 or V_2, on which the edges in S are incident, we can disconnect the graph G. Hence the edge connectivity exceeds the vertex connectivity.

A graph G is said to be *planar* if it can be drawn in a plane in such a way that no two edges cross each other. Such a drawing, if it exist, is called the *planar embedding* of the graph. If a graph is not planar, the minimum number of edges-crossing is called the *crossing number* of the graph. A planar embedding of a graph divides the plane in different regions. Each maximal region which is not partitioned by an edge or path in further subregions is called a *face*. The planar embedding of a finite graph G always has an *unbounded face*. The *boundary* of a face is the set of vertices and edges surrounding the face, and the *length* of the i-th face (denoted as $l(f_i)$) is the length of the closed walk around the face in G. In planar embedding of a graph, a bridge can be included in only one face, and it contributes 2 to the length of that face. We can observe that the sum of lengths of all the faces of a planar graphs equals twice the size of the graph. Since every edge can be at most at the boundary of two neighbouring faces, they are counted twice when adding all face lengths. The edges which lie in a unique face are counted twice because they are the bridges, so we get a degree sum and face relationship $\sum_i l(f_i) = 2|E(G)|$.

Theorem 1.2.7. [10] *If a connected graph G has exactly n vertices, e edges and f faces, then $n - e + f = 2$.*

The formula in the above theorem is known as *Euler's Formula*, and it is satisfied by every connected planar graph. For a graph with k components, *Euler's formula* becomes $n - e + f = k + 1$, because we only need to add $k - 1$ edges to make the graph connected. We can find an upper bound for the size of a graph using this theorem. Using the fact that every face should be of length 3 or more, from the degree sum and face relation we get $3f \leq 2|E(G)|$, where f is the number of faces. Now using this inequality in *Euler's Formula* we get $|E(G)| \leq 3|V(G)| - 6$. The planar graphs are characterized by the *Kuratowski's Theorem* stated below.

Theorem 1.2.8. [41] *A graph is planar if and only if it does not contain a subdivision of K_5 or $K_{3.3}$.*

A subdivision of a graph means the subdivision of its edges by replacing the edges with paths internally disjoint from each other. The graphs K_5 and $K_{3,3}$ are the smallest graphs which can not be drawn on a plane without edge-crossing. These graphs are known as *Kuratowski's graphs.*

The *eccentricity* of a vertex v in a graph G, denoted as $ecc(v)$, is the distance from v to the farthest vertex in G, that is, $max\{d(v,x)\}$ for every vertex x in G. The *diameter* of a graph, denoted as $diam(G)$, is the maximum distance between any two vertices of G. So, in terms of eccentricity, $diam(G) = max\{ecc(v)\}$, for every vertex v in G. Similarly, the *radius* of a graph G is the minimum of all the eccentricities of the graph, that is, $rad(G) = min\{ecc(v)\}$, for every vertex v in G. All the vertices having minimum eccentricities are called *central* vertices. The graph (a) in figure 1.7 has diameter 9 and radius 5, while the graph (b) has constant eccentricity, so $diam(b) = rad(b) = 5$.

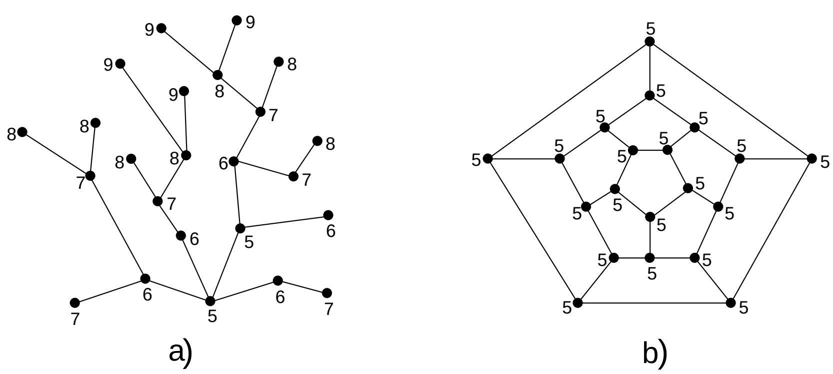


Figure 1.7: Two graphs labeled with vertex eccentricities

1.3 Trees and forests

There is a special type of graphs having no cycles, called *acyclic* graphs. An acyclic graph is called a *forest* and a connected forest is called a *tree.* So it is clear that every component of a forest is a tree. This family of graphs is important to the structural understanding of graphs and to the algorithms of the information processing, and they play central role in design of connected networks. Some special tree structures are used in information management to store data in space efficient ways that allow their retrieval and modification to be time efficient [19].

Now we discuss some properties of trees. A very basic property is that every tree of size more than one has at least two leaves. No graph can has one vertex of degree 1, since the number of odd vertices must be even, this can be followed from theorem 1.2.1. So if a tree does not has 2 vertices of degree 1, then its minimum degree is at least two, then there must be a cycle in the graph. Hence a tree contains at least two leaves. A *spanning* tree of a graph G is a spanning subgraph of G which is a tree.

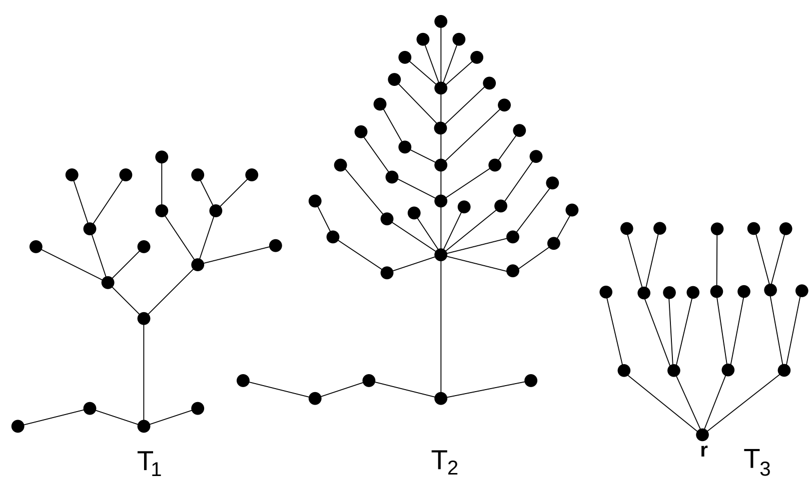
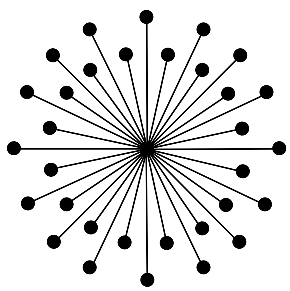


Figure 1.8: Trees in a forest

A tree is a *minimal connected* graph. That is, the removal of one edge leaves the tree disconnected. A tree of order n has exactly $n-1$ edges. We can apply induction on the number of vertices to prove this statement. If there is one vertex then the statement is trivial. Let there are k vertices in the tree, then the size of the tree is $k-1$. Now, consider any tree T on $k+1$ vertices. Let x is a leaf in T, then $T-x$ is also acyclic and connected. So, $T-x$ is a tree on k vertices and $k-1$ edges. Since $T-x$ has one vertex less than T, so, T has exactly k edges. Similarly, a tree on n vertices and k components has exactly $n-k$ edges.

Earlier in this chapter a *path* was defined as a trail with no repeated vertex. We can also define a *path* as a tree in which every i-th vertex is adjacent with $(i+1)$st vertex. A path of order n is denoted as P_n. A star is a tree in which a single fixed vertex is adjacent to all other vertices. A star with $n+1$ vertices is denoted as S_n. The forest of stars is known as a *galaxy*. A graph in which all edges are incident on the vertices of a fixed path is called a *caterpillar*.



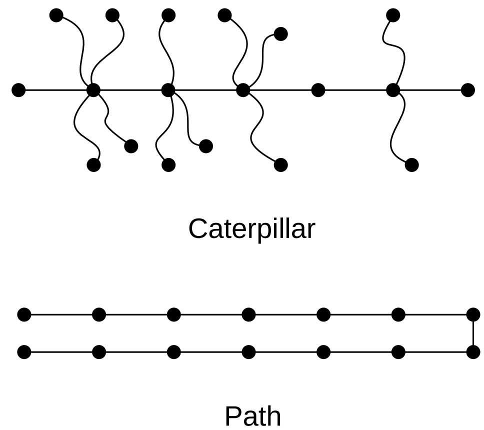


Figure 1.9: Star, path and caterpillar

A *rooted tree* is a tree in which a fixed vertex r (say) is designated as the *root* of the tree. For every vertex v the unique paths from the root r to that vertex v is directed away form the root. In a rooted tree if vertex u and v are adjacent and u lies in the path from the root r to v, then u is called the *parent* of v, and v is called the *child* of u, and all other vertices in the path are called the *ancestors* of v. Every tree can be drawn in the form of a rooted tree. The tree T_3 in figure 1.8 is a rooted tree. The next theorem provides some useful characterizations of trees.

Theorem 1.3.1. [19] *Let T be a connected graph with n vertices. Then the following statements are equivalent.*
(i) T is a tree.
(ii) T contains no cycles and has $n-1$ edges.
(iii) T is connected and has $n-1$ edges.
(iv) T is connected, and every edge is a cut-edge.
(v) Any two vertices of T are connected by exactly one path.
(vi) T has no cycles, and for each new edge e, the graph $T+e$ has exactly one cycle.

These statements can be derived easily in the light of previously discussed properties of trees. Since a tree is a minimal connected graph, so deletion of any edge leads to disconnection, so every edge is cut-edge. Only a cycle provides alternative paths between vertices of a graph. Since a tree has no cycle so there is a unique path between any two vertices. Now, a tree is connected so there is a path between every pair of vertices say x and y. If we add an edge xy in the tree, it provides an alternative path to travel between x and y, hence produces a cycle.

1.4 Operations on graphs

One very common operation that we perform on graphs is *vertex/edge deletion* from the graph. Once a vertex is deleted from a graph, all the edges incident on that vertex are also removed, and when an edge is deleted from a graph, no difference occur other than the size of the graph is reduced by 1. For $e \in E(G)$, the edge deleted graph G is denoted as $G - e$. Another operation that we perform only on edges of a graph is *contraction* of an edge. An edge uv is contracted by coinciding both of its end-vertices into a single vertex x and joining all edges which were incident on u and v to the new vertex x. The graph we obtain after contracting the edge uv is denoted as $G|uv$ or $G|e$. Studying simple graphs, any loops or multiple edges that occur after edge contraction are removed. The deletion and contraction of an edge e in P_5 is figured in 1.10.

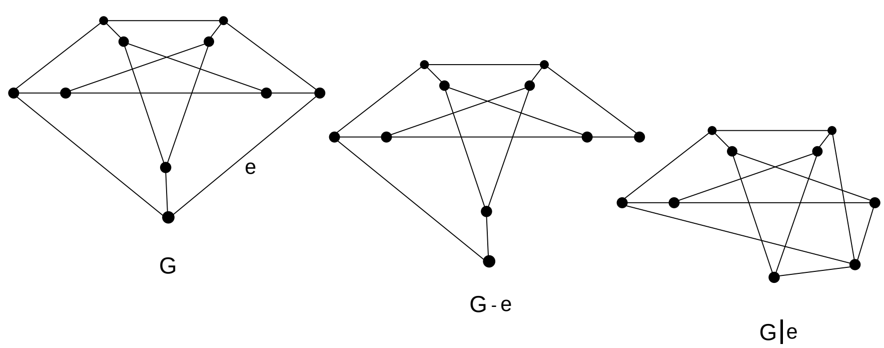


Figure 1.10: Edge contraction in a graph

The *disjoint union* of graphs $H_1, H_2, \ldots, H_n$ is the graph $G = H_1, H_2, \ldots, H_n$ with vertex set $\bigcup_{i=1}^{n} V(G_i)$ and edge set $\bigcup_{i=1}^{n} E(G_i)$. The disjoint union of k copies of G is denoted as kG. The *join* of two graphs G and H, denoted as $G + H$, is a graph obtained by adding every edge between the vertices of G and H. That is, $E(G + H) = E(G) + E(H) + \{uv : u \in V(G), v \in V(H)\}$.

The *cartesian product* of two graphs G and H, denoted as $G \square H$, is a graph with vertex set $V(G \square H) = V(G) \times V(H)$ and edge set $E(G \square H) = V(G) \times E(H) \cup E(G) \times V(H)$. The join and cartesian product of C_4 and P_5 are shown in figure 1.11.

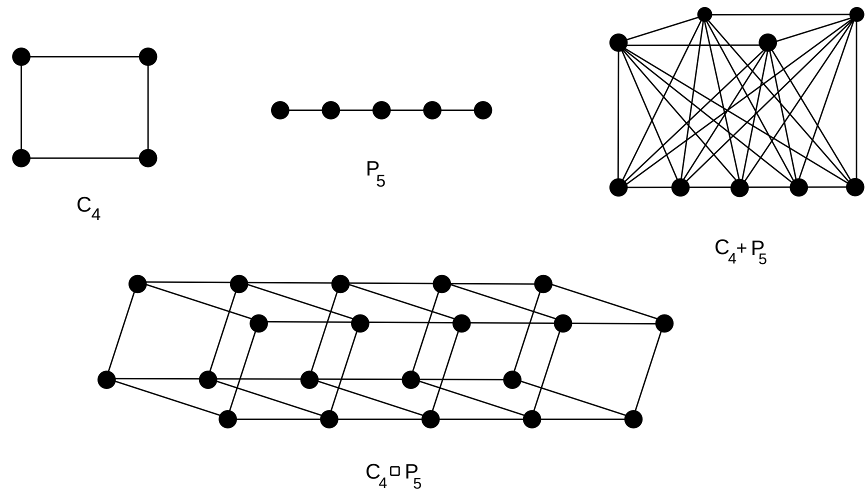


Figure 1.11: Join and cartesian product of C_4 and P_5

1.5 Some special classes of graphs

A *wheel* is a graph obtained by joining all the vertices of a cycle to a new vertex. A wheel of n vertices adjacent with one other vertex is denoted as W_n. If we remove one edge from the outer cycle of a wheel, the resulting graph is called *fan*, denoted by f_n. If we remove the edges from the outer cycle of the wheel alternatively, we get a *friendship* graph, denoted by F_n. The wheel, fan and friendship graphs are elaborated in figure 1.12.

A fan graph f_n can also be obtained by $P_n \Box K_1$, where K_1 represents complete graph on a single vertex (an isolated vertex). Similarly a wheel graph can be described as $C_n \Box K_1$, and a friendship graph is $(2n)P_2 \Box K_1$, where $(2n)P_2$ represents even number of copies of P_2.

A *circular ladder* is a graph obtained by $C_n \Box P_2$, denoted as D_n. A circular ladder is also called a *prism.* An *antiprism* (denoted as A_n) is a graph formed by combining two n sided polygons by a band of $2n$ triangles. The graph C_4 is known as 2-*hypercube* (Q_2). The graph $C_4 \Box P_2$ is called 3-*hypercube* (Q_3). Similarly, the n-*hypercube* (Q_n) is obtained by $Q_{n-1} \Box P_2$.

The *petersen* graph, commonly known as *generalized petersen graph* $P(n, k)$, for $n \geq 3$ and $k < \frac{n}{2}$ is the graph obtained by joining n vertices to corresponding vertices of an n-cycle and joining each vertex to the k-th vertex in the cyclic order. The prism D_8, antiprism A_8 and the generalized petersen graph $P(8,3)$ (also known as Möbius-Cantor graph) are shown in figure 1.12.

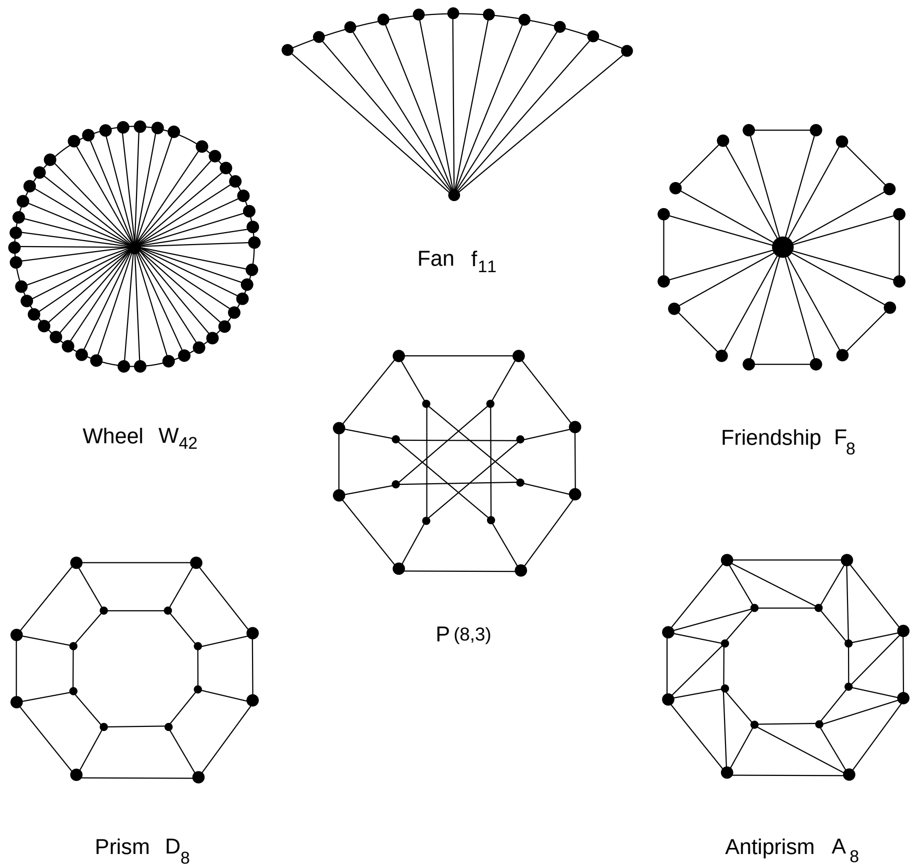
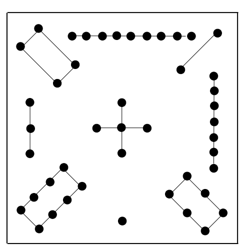


Figure 1.12: Wheel, Fan, Friendship, Prism, Antiprism and Petersen graphs

Chapter 2

Introduction to Graph Labeling

A *magic square* of order n is an array of n^2 distinct integers in the form of a square, such that the sum of all n numbers in a row, column, and both diagonals is equal to a constant (say) k. This constant is called the *magic constant*. A *normal* magic square contains 1 to n^2 consecutive integers. A normal magic square of order n has the magic constant $k = \frac{n(n^2+1)}{2}$ (see [1]). Magic squares are amongst the most popular mathematical recreations. The first magic square known to be recorded is known as *Lo Shu* magic square discovered around 2200 BC. According to the legend about the Chinese Emperor *Yu*, from the book *Yih King*, the diagram was found on the shell of a divine turtle [27]. The following figure depicts the *Lo Shu* magic square and the corresponding normal magic square.



4	9	2
3	5	7
8	1	6

Figure 2.1: Lo Shu magic square

In 1963, Sedláček used the concept of magic squares for the first time in graphs [35]. A *labeled* graph is a graph in which labels (usually non-negative integers) are assigned to the elements (vertices, edges, faces or any combination of these) of the graph. These labels are assigned with respect to some specified conditions. The graphs having no specific names or labels for their elements are called *unlabeled* graphs. The labels are used to identify the elements of a graph. The process of assigning labels to the elements of a graph is called *graph labeling*. Sedláček defined the notion of a *magic graph* to be a graph whose edges are labeled with real numbers such that the sum of the labels around any vertex is constant. Afterwards, the concept of *super magic* was introduced by Stewart in 1966 (see [37]). He defined a *magic* labeling to be *super magic* if all the labels are consecutive integers.

The *weight* of a vertex $x \in V(G)$, denoted as $\omega(x)$, under some labeling is the sum of the labels of the vertex x and all the edges which are incident on it. Similarly, the *weight* of an edge is the sum of the labels of the edge and its endpoints. In case when the edges of the graph are not labeled, the edge weight is just the sum of the labels of its endpoints.

In 1970, Kotzig and Rosa introduced the term *total labeling* in such a way that a *magic* labeling is *total* if all the vertices and edges of the graph are labeled, and the sum of the labels of any edge and its endpoints is constant [25]. In 1996, Ringel and Llado redefined the terms introduced by Kotzig and called this labeling *edge magic* labeling [33]. Again in 1998, Enomoto et al. redefined the term *magic*, introduced by Stewart, by adding the property that a labeling is *super* if the smallest possible labels are assigned to the vertices of the graphs [13]. In [28], MacDougall, Miller, Slamin and Wallis introduced the concept of *vertex magic total labeling*.

The popularity which the subject of graph labeling has gained in the area of graph theory can be realised by putting a glance on a survey of graph labeling by Gallian [16]. More than a thousand papers are appeared on different kinds of graph labelings. This popularity is due to the range of applications of the graph labelings in other branches of science. Most of these applications are found in x-ray crystallography, coding theory, cryptography, astronomy, radar, circuit design and communication network design [6, 7].

To avoid ambiguities in the definitions of different kind of labelings just discussed, we define all of them in a sequence. First we list the types of labelings with respect to the weights calculated for different elements of the graphs.

- Harmonious labeling - The vertices are labeled with distinct integers and the edge weights are calculated which are distinct.
- Graceful labeling - The vertices are labeled with distinct integers such that the edge weights form consecutive integers.
- Magic labeling - A labeling in which all the calculated weights of the elements of the graph are same.
- Antimagic labeling - A labeling in which all the weights of the elements of the graph are different.

In the following we present the second listing of graph labelings depending upon the elements of the graph which are labeled.

- Edge labeling - All the edges of the graph are labeled.
- Vertex labeling - All the vertices of the graph are labeled.
- Face labeling - All the faces of the graph are labeled.
- Super labeling - All the vertices are labeled with smallest possible labels.
- Total labeling - All the vertices and edges of the graph are labeled.
- Supertotal labelings - All the vertices, edges and faces are labeled.

In the light of above mentioned types of a graph labeling, one can guess that a *super edge magic total labeling* will be a labeling in which all the vertices and edges are labeled (because of *total*), with the smallest labels assigned to the vertices (because of *super*), and the edge weights are calculated under the labeling which are all same (because of *edge magicness*). All these labelings are discussed with examples in the next sections of this chapter.

2.1 Graceful labeling

A *graceful labeling* is a bijection $\lambda : V(G) \to \{0, 1, 2, \ldots, q\}$, where q is the number of edges in G, such that each edge $xy \in E(G)$ is assigned a unique label $|\lambda(x) - \lambda(y)|$, where all the vertex labels are distinct as well, and the absolute value of the difference of $\lambda(x)$ and $\lambda(y)$ is called the weight of the edge xy.

At first, Rosa [34] called this labeling β-valuation but afterwards Golomb searched out the same kind of labeling independently, and called it *graceful labeling*. A graph G is called *graceful* if it admits a *graceful* labeling. The most popular conjecture on graceful labelings which is still open was proposed by Ringel and Kotzig, it states that all trees are graceful [25].

The graceful labeling of a 4-regular graph is shown in the figure 2.2. The labels in bold-italic shows the edge weights which are the absolute differences of the labels of the incident vertices.

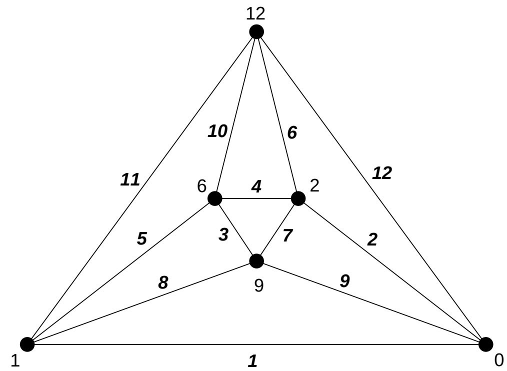


Figure 2.2: Graceful Graph

2.2 Harmonious labeling

This labeling was first introduced in 1980 by Graham and Sloane [17]. A *harmonious* labeling of a graph G is a vertex labeling defined as a bijection $\lambda : V(G) \to \mathbb{Z}_{|E|}$, such that the mapping λ' from the edge set $E(G)$ to Z_q defined by $\lambda'(uv) = \lambda(u) + \lambda(v)$ for every $uv \in E(G)$, assigns different labels to the edges of G. If the graph G admits a harmonious labeling then it is called a *harmonious* graph.

In Erdöes unpublished results, it is proved that no graph is neither graceful nor harmonious [17]. A harmonious graph is shown in figure 2.3. The labels in bold-italic shows the edge weights. Graham and Sloane showed that this is a maximal sized harmonious graph on 7 vertices [18].

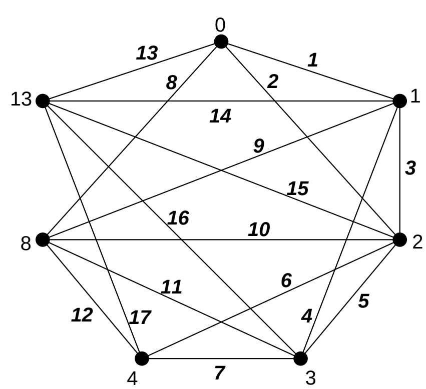


Figure 2.3: Harmonious Graph

2.3 Antimagic labelings

An *antimagic square* of order n is an arrangement of numbers from 1 to n^2 in the form of a square such that the sum of all the n elements of any row, column and both diagonals form a sequence of $2n + 2$ consecutive integers. The smallest antimagic square is of order 4. Two antimagic squares of order 4 are shown below [39].

4	13	12	1
11	6	2	14
5	15	10	8
16	3	7	9

1	13	3	12
15	9	4	10
7	2	16	8
14	6	11	5

In each of these two antimagic squares; the rows, columns and the diagonals sum up to ten different numbers between 29 and 38. As the order increases, the construction becomes easier. It is still an open problem to find a method of constructing an antimagic square of every order.

An (a, d)-*antimagic labeling* is a labeling in which all calculated weights of the elements of a graph form an arithmetic progression, starting from a constant a (> 0) with common difference d (≥ 0). A graph having an antimagic labeling is called an *antimagic graph*. The magic labelings are a spacial case of (a, d)-antimagic labelings when common difference of the arithmetic progression is zero $(d = 0)$.

The concept of antimagic squares was applied in graphs and in the result *antimagic graphs* were introduced. The notion of an antimagic graph was first introduced by Hartsfield and Ringel in 1989, afterwards Nicholas et al. Bodendiek and Walther [8, 9] in 1996 were the first to introduce the concept of an (a, d)-vertex antimagic edge labeling. They called that labeling (a, d)-*antimagic labeling*. Bača et al. [2] defined the concept of an (a, d)-vertex antimagic total labeling, and (a, d)-edge antimagic total labeling was introduced by Simanjuntank et al. in [36].

Hartsfield and Ringel [20] pointed out that all paths P_n for $n \geq 3$, cycles C_n, wheels W_n, and complete graphs K_n for $n \geq 3$, are antimagic. The Antimagic labelings are further divided in two cases.

- Edge antimagic - When edge weights form an arithmetic progression.
- Vertex antimagic - When vertex weights form an arithmetic progression.

2.3.1 Edge antimagic labelings

One thing is clear by the name that in this labeling edge weights are calculated and they form an arithmetic progression. Since the set of weights W (say) forms an arithmetic progression, it has an initial term a and a common difference d. So in an antimagic labeling this set of weights is written as $W = \{a, a+d, a+2d, \ldots, a+(q-1)d\}$, where q is the number of edges in the graph. There are further two kinds in which an edge antimagic labeling can fall, depending upon the choice of labeling the elements of the graph.

- (a, d)-edge antimagic vertex labeling (EAV labeling)
- (a, d)-edge antimagic total labeling (EAT labeling)

Edge antimagic vertex labeling

A bijection $\lambda : V(G) \to \{1, 2, 3, \ldots, p\}$, where p is the number of vertices in the graph G, is called an (a, d)-edge antimagic vertex labeling if all the edge weights of the graph form an arithmetic progression with some starting term $a > 0$ and a common difference $d \geq 0$. The edge weight of an edge $uv \in E(G)$ under this labeling is calculated as $\omega(uv) = \lambda(u) + \lambda(v)$.

Simanjuntak et al. [36], proved that there is no (a, d)-EAV labeling of even cycles. They also showed that every path P_n has (3,2)-EAV labeling. Bača et al. [3], showed that for every symmetric complete bipartite graph $K_{n,n}$, there is no (a, d)-EAV labeling. A $(6, 1)$ and a $(3, 2)$-edge antimagic vertex labeling of two graphs are showed in figure 2.4. The labels in bold-italic represent the edge weights.

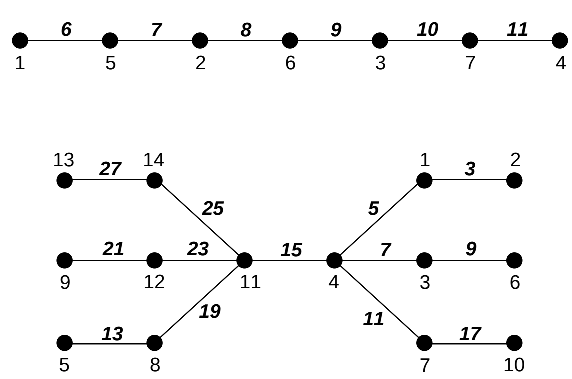


Figure 2.4: $(6, 1)$ and $(3, 2)$-EAV labelings (respectively)

Edge antimagic total labeling

An (a, d)-*edge antimagic total labeling* of a (p, q) graph G is a bijection $\lambda : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p+q\}$ such that the edge weights form an arithmetic progression with some starting term $a > 0$ and common difference $d \geq 0$. The weight of an edge $uv \in E(G)$ under this labeling is calculated as $\omega(uv) = \lambda(u)+\lambda(v)+\lambda(uv)$.

Simanjuntak et al. [36] proved that a graph with all vertices of odd degrees cannot have an (a, d)-EAT labeling with a and d both even. They also proved that if f is an (a, d)-EAT labeling of a graph G then f' is an $(3p+3q+3-a-(q-1)d, d)$-EAT labeling of G. In a (p, q) graph G if the smallest possible labels are assigned to the vertices then this labeling is called *super* (a, d)-*edge antimagic total labeling*. A super $(41, 1)$-edge antimagic total labeling of a caterpillar is shown in figure 2.5.

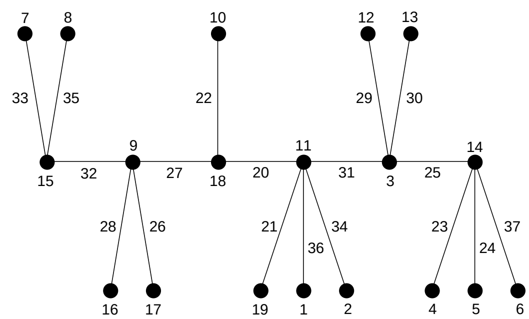


Figure 2.5: Super $(41, 1)$-edge antimagic total labeling of a caterpillar

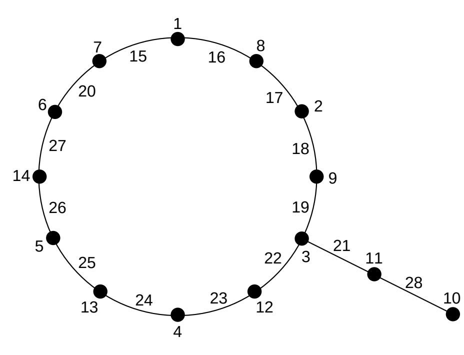


Figure 2.6: Super $(23, 2)$-edge antimagic total labeling of $(12, 3)$-kite

2.3.2 Vertex antimagic labelings

In vertex antimagic labelings we calculate all the vertex weights of the graph, which form an arithmetic progression $\{a, a+1, a+2, \ldots, a+(p-1)d\}$, starting from a with common difference d. This labeling is called an (a, d)-*vertex antimagic labeling*. As before, the vertex antimagic labelings are also of two types depending upon the choice of labeling different elements of the graph.

- Vertex antimagic edge labeling (VAE labeling)
- Vertex antimagic total labeling (VAT labeling)

Vertex antimagic edge labeling

As it is an edge labeling so we just label the edges of the graph with a bijection $\lambda : E(G) \to \{1, 2, 3, \ldots, q\}$, and the vertex weights are calculated which form an arithmetic progression $W = \{a, a+d, a+2d, \ldots, a+(p-1)d\}$. The weight of a vertex $x \in V(G)$ under this labeling is calculated as $\omega(x) = \sum\limits_{y \in N(x)} \lambda(xy)$. Such a labeling is called (a, d)-*vertex antimagic edge labeling.*

Hartsfield and Ringel [20] called this labeling *antimagic*, and they conjectured that all graphs except K_2 are antimagic. Miller et al. [31] proved that every antiprism A_n has a $(6n+3, 2)$-VAE labeling and a $(4n+4, 4)$-VAE labeling as well. They also showed that A_3 does not has a $(11, 5)$-VAE labeling. The $(13, 6)$-*VAE labeling* of the antiprism A_4 is shown in Figure 2.7 [4]. The labels in bold-italic represent vertex weights.

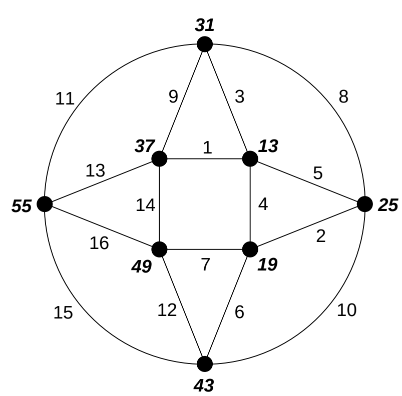


Figure 2.7: $(13, 6)$-vertex antimagic edge labeling of A_4

Vertex antimagic total labeling

For a (p,q) graph G, an (a,d)-*vertex antimagic total labeling* is a bijection $\lambda : V(G) \cup E(G) \to \{1,2,3,\ldots,p+q\}$ such that all the vertex weights form a sequence $W = \{a, a+d, a+2d, \ldots, a+(p-1)d\}$, with starting term $a > 0$ and common difference $d \geq 0$. The weight of a vertex $x \in V(G)$ under this labeling is calculated as

$$\omega(x) = \lambda(x) + \sum_{y \in N(x)} \lambda(xy).$$

Bača et al. in [2], introduced this kind of labeling for the first time, they also established a relationship between the super magic labeling and vertex antimagic total labeling stated below.

Lemma 2.3.1. [2] *Every super magic graph has an $(a,1)$-vertex antimagic total labeling.*

They proved another relationship between vertex antimagic edge labeling and vetex antimagic total labeling of a graph, which is stated below in the form of a lemma.

Lemma 2.3.2. [2] *(i) If $d > 1$, then every (a,d)-VAE graph has an $(a+p+q, d-1)$-VAT labeling.*
(ii) Every (a,d)-VAE graph has an $(a+q+1, d+1)$-VAT labeling.

If the labels $\{1,2,3,\ldots,p\}$ are assigned to the vertices of the graph then this labeling is called *super* and the graph possessing such a labeling is called *super* (a,d)-VAT.

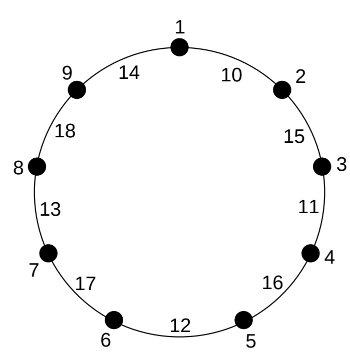


Figure 2.8: Super $(25,2)$-vertex antimagic total labeling of C_9

2.4 Magic labelings

Same like the antimagic labelings, the concept of magic labelings was adopted from the magic square which were discovered long ago. The magic labelings form a subclass of antimagic labelings with $d = 0$. So, if the labeling is *vertex magic* then the weight of every vertex is same, and if the labeling is *edge magic* then the weight of every edge is equal to a fixed constant, which is called the *magic constant*. A graph possessing a magic labeling is called a *magic graph*. The same concepts of *super* and *total* labeling in antimagic labelings are followed in magic labelings as well.

2.4.1 Magic labeling (VME)

As described in the beginning of this chapter, Sedláček [35] in 1963, introduced the notion of a *magic graph*, to be a graph with an edge labeling with real numbers in such a way that the vertex weights are constant. With the passage of time many more labelings are introduced so, to avoid any ambiguity of notations we call this labeling a *vertex magic edge labeling*. Stewart called this labeling *super magic* if the edge labels are consecutive integers.

If the edges incident on a vertex of an n-regular graph of order $2n$ are labeled with the elements of a row or column of a magic square, the vertex weights would obviously be constant. This can be understood by examining the magic labeling of $K_{4,4}$ using a magic square of order 4, in figure 2.9. The magic constant under this labeling will be the magic constant of the magic square, which is in this case 34.

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Magic square of order 4

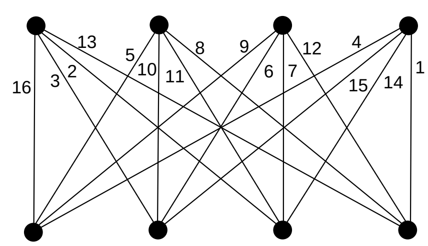


Figure 2.9: Magic labeling of $K_{4,4}$

Stewart in [37], proved that K_n is magic for $n = 2$ and for all $n \geq 5$. He also proved that $K_{n,n}$ is magic for all $n \geq 3$, and all fans f_n are magic for $n \geq 3$ iff n is odd. In [38], Stewart showed that K_n is super magic for $n \geq 5$ iff $n > 5$ and $n \not\equiv 0 \,(\mathrm{mod}\ 4)$. A magic graph with magic constant 27 is shown in the figure 2.10.

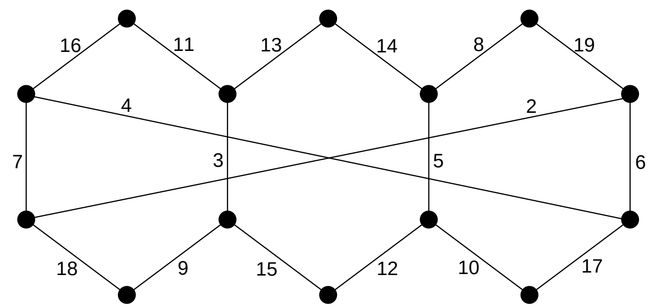


Figure 2.10: Super magic graph

2.4.2 Vertex magic total labeling

A bijection $\lambda : V(G) \cup E(G) \to \{1, 2, 3, \ldots, p+q\}$ is called a *vertex magic total labeling* if the weights of all the vertices of G are same. The weight of a vertex $v \in V(G)$ under this labeling is calculated as $\omega(v) = \lambda(v) + \sum\limits_{u \in N_v(G)} \lambda(uv)$. A graph possessing such a labeling is called a *vertex magic total graph* (VMT).

A VMT graph with magic constant 15 is shown in figure 2.11.

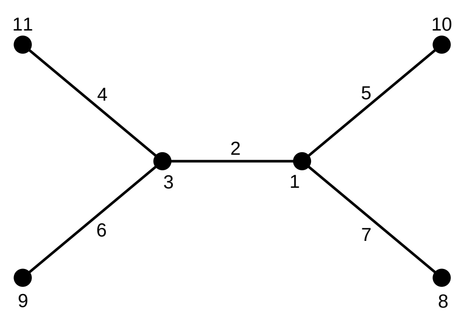


Figure 2.11: Vertex magic total graph

MacDougall et al. in [28], proved that all cycles C_n, paths P_n and balanced complete bipartite graphs $K_{n,n}$ are VMT. MacDougall, Miller and Wallis [29] proved that the wheel W_n has no VMT labeling for $n > 11$, the fan graph f_n has no VMT labeling for $n > 10$, and the friendship graph F_n has no VMT labeling for $n > 3$. Bača, Miller and Slamin in [5] proved that every generalized petersen graph $P_{n,k}$ has a VMT labeling for $n > 3$ and $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$. The vertex magic total labeling of products of cycles is discussed in [15].

Another VMT graph with magic constant 48 is shown in figure 2.12.

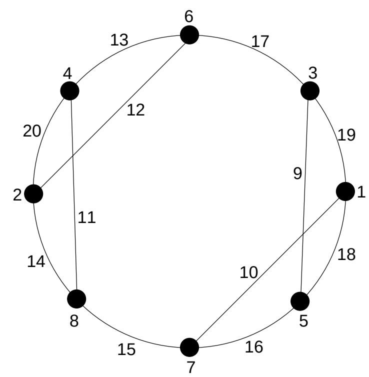


Figure 2.12: Vertex magic total graph

Chapter 3

Super Edge Magic Total Labeling

The edge magic total labelings were first discussed by Kotzig and Rosa [25] in 1970. In this chapter we define edge magic total and super edge magic total labelings, and a brief introduction to some super edge magic total graphs studied in [21, 22, 23] is also included.

3.1 Edge magic total labeling

An edge magic total labeling of a graph G can be defined as a bijection

$$\lambda : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p + q\}.$$

defined in such a way that the weights of all the edges are equal to a fixed constant k (say). The weight of an edge $uv \in E(G)$ under this labeling function is calculated as

$$\omega(uv) = \lambda(u) + \lambda(v) + \lambda(uv).$$

The constant k is called the magic constant of the graph G under the labeling λ. A graph with an edge magic total labeling is called *edge magic total graph* or just EMT.

Kotzig and Rosa [25] proved that the complete bipartite graphs $K_{m,n}$ for any m and n and cycle C_n for all $n \geq 3$ are EMT. Ringel and Lladó [33] proved that a (p, q) graph G is not EMT if all vertices are of odd degree and q is even and $p + q \equiv 2\,(mod\,4)$. Enomoto et al. [13] proposed a conjecture that all wheels except the ones described in [33] are EMT. Wallis, Baskoro, Miller and Slamin [40] constructed EMT labelings of the complete graph K_n for $n \in \{1, 2, 3, 4, 5, 6\}$ for all possible values of magic constant. They also showed that all paths, complete bipartite graphs and all cycles with a chord admit EMT labeling.

Kotzig and Rosa [26], conjectured that every tree is EMT. A tree with EMT labeling is shown in figure 3.1. The magic constant for this graph under given EMT labeling is 36.

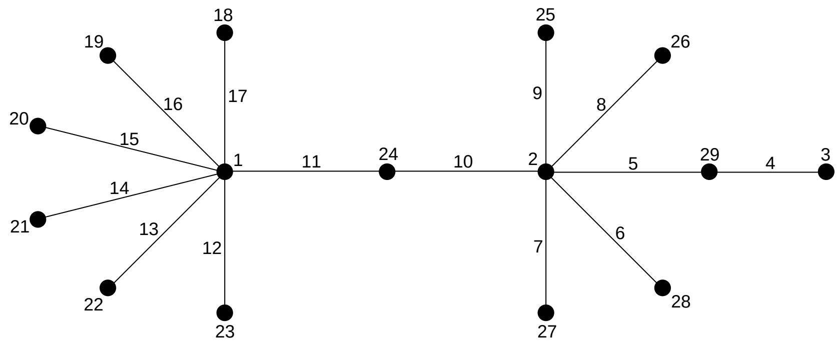


Figure 3.1: Edge magic total graph

3.2 Super edge magic total labeling

The concept of *super edge magic total labeling* (SEMT) was introduced by Enomoto, Lladó, Nakamigawa and Ringel. They defined the SEMT labeling of a graph G to be a bijection

$$\lambda : V(G) \cup E(G) \rightarrow \{1, 2, 3, \ldots, p + q\}.$$

such that the weight of every edge is equal to a fixed constant, and being a super labeling it satisfies another property that all vertices are labeled with the smallest available labels $\{1, 2, 3, \ldots, p\}$, and the rest of labels $\{p + 1, p + 2, p + 3, \ldots, p + q\}$ are assigned to the edges of the graph. The weight of an edge $uv \in E(G)$ under SEMT labeling is calculated in the same way as was calculated in EMT labeling.

The SEMT labeling of a disconnected graph $C_{12} \cup P_2$ with magic constant 36, is shown in the figure 3.2.

Enomoto, Lladó, Nakamigawa and Ringel [13] proved that the cycle C_n is super edge magic total if and only if n is odd. The complete bipartite graph $K_{m,n}$ is super edge magic total if and only if $m = 1$ or $n = 1$. If the cardinality of any partite set of a bipartite graph is one, it becomes a star S_n, so it means that every star is super edge magic total. The complete graph K_n is super edge magic total if and only if $n = 1, 2$ or 3. They also gave an upper bound for a graph to be SEMT in terms of

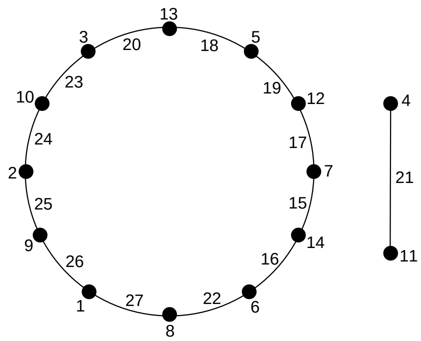


Figure 3.2: Super edge magic total graph $C_{12} \cup P_2$

the size q of a graph which is $q \leq 2p - 3$.

Enomoto, Lladó, Nakamigawa and Ringel proposed one of the most popular conjecture in graph labeling known as the *tree conjecture*.

Conjecture 3.2.1. [13] *Every tree is super edge magic total.*

This conjecture has been verified for the trees of up to 17 order with the help of computer. Kotzig and Rosa [26] proved that all caterpillars are super edge magic total.

The most useful lemma which provides a necessary and sufficient condition for a graph to be super edge magic total, is given by Figueroa et al. in [14].

Lemma 3.2.1. [14] *A graph $G(p, q)$ is super edge magic total if and only if there exists a bijective function $\lambda : V(G) \to \{1, 2, 3, \ldots, p\}$ such that the set of edge weights*

$$S = \{\lambda(u) + \lambda(v) : uv \in E(G)\}.$$

consists of q consecutive integers. In such a case, λ extends to a super edge magic total labeling of G with magic constant $k = p + q + s$, where $s = min(S)$ and

$$S = \{k - (p + 1), k - (p + 2), k - (p + 3), \ldots, k - (p + q)\}.$$

Another super edge magic total graph with magic constant 42 is presented in figure 3.3.

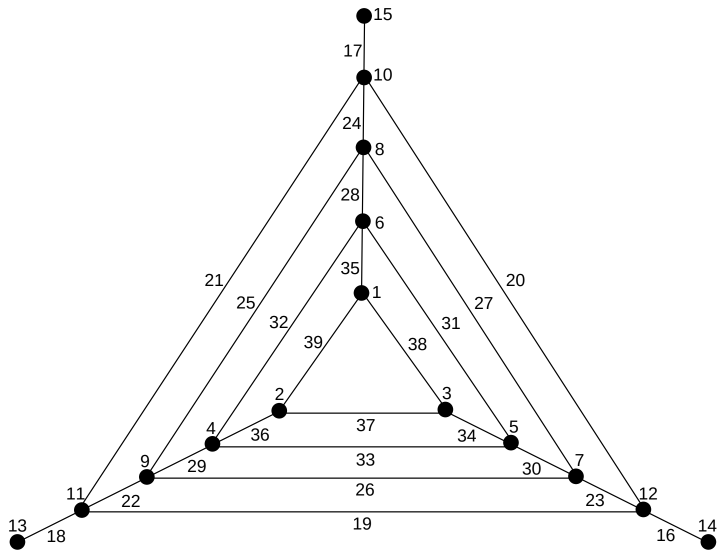


Figure 3.3: Super edge magic total graph

3.3 Some classes of SEMT graphs

In [22], Javaid et al. introduced the term *w-graphs* $W(n)$, and defined it to be a graph constructed from two stars by coinciding a vertex. They constructed a new family of graphs by joining one vertex each from k isomorphic copies of $W(n)$ to a new vertex a, and called this graph a *w-tree*, denoted as $WT(n,k)$. They studied the super edge magic total labeling of w-trees and disjoint union of *w-trees* under certain conditions. The vertex and edge sets of $W(n)$ are as follows:
$V(W(n)) = \{c_1, c_2, b, w, d\} \cup \{x^1, x^2, \ldots, x^n\} \cup \{y^1, y^2, \ldots, y^n\}$,
$E(W(n)) = \{c_1x^i, c_2y^i : 1 \leq i \leq n\} \cup \{c_1b, c_1w, c_2w, c_2d\}$.

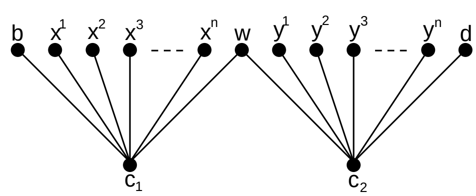


Figure 3.4: $W(n)$

The vertex and edge sets of $WT(n,k)$ are as follows:

$$\begin{aligned} V(WT(n,k)) &= \{a\} \cup \{b_i, w_i, d_i, c_{i1}, c_{i2} : 1 \le i \le k\} \cup \\ &\quad \{x_i^l : 1 \le i \le k, 1 \le l \le n\} \cup \\ &\quad \{y_i^l : 1 \le i \le k, 1 \le l \le n\}, \\ E(WT(n,k)) &= \{b_i c_{i1}, w_i c_{i1}, w_i c_{i2}, d_i c_{i2}, a d_i : 1 \le i \le k\} \cup \\ &\quad \{x_i^l c_{i1} : 1 \le i \le k, 1 \le l \le n\} \cup \\ &\quad \{y_i^l c_{i2} : 1 \le i \le k, 1 \le l \le n\}. \end{aligned}$$

The super edge magic total labeling of a w-tree $WT(4,2)$ is presented in figure 3.5 with magic constant 77.

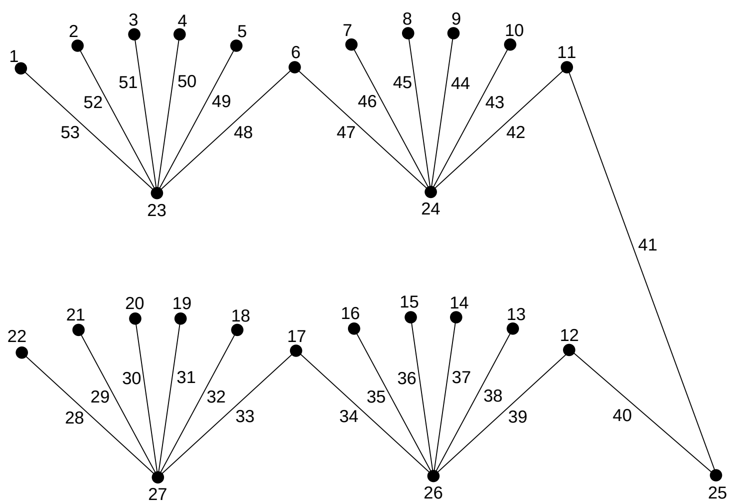


Figure 3.5: $WT(4,2)$

Javaid et al. [23] introduced *extended w-graphs* $Ew(n,r)$ constructed from a forest of r isomorphic stars $K_{1,n}$ with $V(rK_{1,n}) = \{c_m, x_i^m : 1 \le i \le n, 1 \le m \le r\}$. The extended w-graph is obtained by merging the vertices x_n^i with x_1^{i+1} for $1 \le i \le r-1$. Similarly, *extended w-tree* $Ewt(n,k,r)$ is constructed from k isomorphic copies of $Ew(n,r)$ by taking a new vertex a and joining it with the vertex d in each copy of $Ew(n,r)$. The vertex and edge sets of $Ewt(n,k,r)$ are as follows:

$$\begin{aligned} V(Ewt(n,k,r)) &= \{a\} \cup \{b_i, d_i : 1 \le i \le k\} \cup \\ &\quad \{c_i^s : 1 \le s \le r, 1 \le i \le k\} \cup \\ &\quad \{w_i^s, y_i^l, x_{is}^l : 1 \le i \le k, 1 \le l \le n, 1 \le s \le r-1\}, \\ E(Ewt(n,k,r)) &= \{a d_i, b_i c_i^1, y_i^l c_i^r : 1 \le i \le k, 1 \le l \le n\} \cup \\ &\quad \{x_{is}^l c_i^s, w_i^s c_i^s, w_i^s c_i^{s+1} : 1 \le i \le k, 1 \le l \le n, 1 \le s \le r-1\}. \end{aligned}$$

Chen et al. [11] defined a *banana tree* $BT(n_1, n_2, \ldots, n_k)$, to be a graph obtained from the disjoint union of stars K_{1,n_i} with $V(K_{1,n_i}) = \{c_i, a_{in_i}\}$ for $1 \le i \le k$, by adding a new vertex a and joining it with one vertex from every star. Hussain et al. [21] constructed the super edge magic total labeling of banana trees for different cases, the same labeling for disjoint union of two isomorphic copies of banana trees, and also for disjoint union of m isomorphic copies of banana trees containing two stars. The vertex and edge sets of $BT(n_1, n_2, \ldots, n_k)$ are as follows:
$V(BT(n_1, n_2, \ldots, n_k)) = \{a\} \cup \{c_i : 1 \le i \le k\} \cup \{a_{ij} : 1 \le i \le k, 1 \le j \le n_i\}$,
$E(BT(n_1, n_2, \ldots, n_k)) = \{aa_{i1} : 1 \le i \le k\} \cup \{c_i a_{ij} : 1 \le i \le k, 1 \le j \le n_i\}$.

The super edge magic total labeling of banana tree $BT(6, 7)$ with magic constant 46 in presented in the figure 3.6.

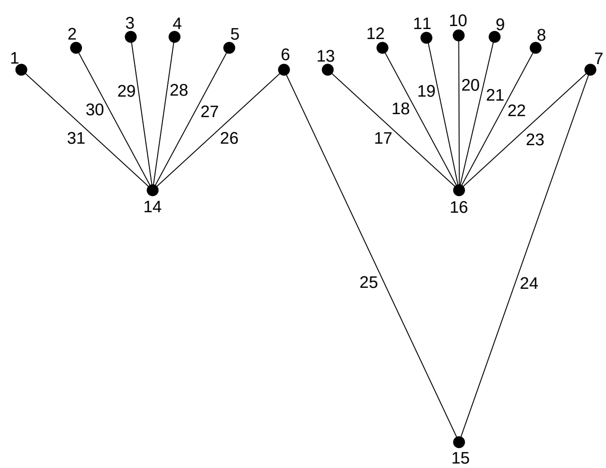


Figure 3.6: $BT(6, 7)$

Using the w-graph $W(n)$, Malik et al. [30] constructed a new family of graphs by taking the reflection of a w-graph and joining both with a path of order m, and called it a reflexive w-graph, denoted as $RW(m, n)$. They also gave different generalizations of $RW(m, n)$ and constructed their super edge magic total labelings. The vertex and edge sets of $RW(m, n)$ are as follows:

$$\begin{aligned} V(RW(m,n)) &= \{b_{i,j} : 1 \le i \le 2,\ 1 \le j \le 3\} \cup \{c_{i,j} : 1 \le i, j \le 2\} \cup \\ &\quad \{x^l_{i,t} : 1 \le i, l \le 2,\ 1 \le t \le n\} \cup \{y_i : 1 \le i \le m-2\}, \end{aligned}$$

$$\begin{aligned} E(RW(m,n)) &= \{b_{i,j}c_{i,j}, b_{i,j+1}c_{i,j} : 1 \le i, j \le 2\} \cup \{y_1 c_{1,1}, y_{m-2} c_{2,1}\} \cup \\ &\quad \{y_i y_{i+1} : 1 \le i \le m-3\} \cup \{x^l_{i,t} c_{l,i} : 1 \le i, l \le 2,\ 1 \le t \le n\}. \end{aligned}$$

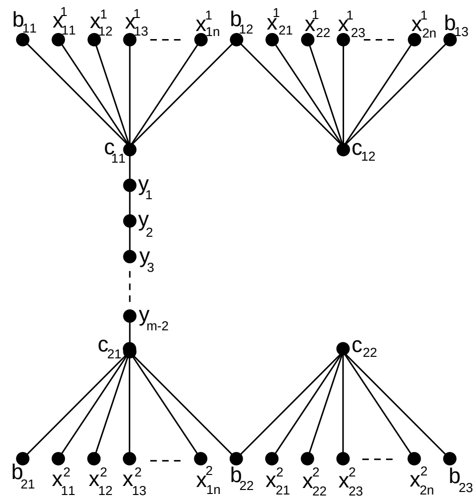


Figure 3.7: $RW(m, n)$

In the following we present a very comprehensive list of graphs labeled with SEMT labeling and some graphs which are not SEMT. The source of this data is a survey on graph labelings conducted by J.A. Gallian [16].

Graph	**Types**	**Notes**
C_n	SEM	iff n is odd
caterpillars	SEM	
$K_{m,n}$	SEM	iff $m = 1$ or $n = 1$
K_n	SEM	iff $n = 1, 2$ or 3
trees	SEM?	
nK_2	SEM	iff n odd
$K_{1,m} \cup K_{1,n}$	SEM	if m is a multiple of $n + 1$
$K_{1,m} \cup K_{1,n}$	SEM?	iff m is a multiple of $n + 1$
$K_{1,2} \cup K_{1,n}$	SEM	iff n is a multiple of 3
$K_{1,3} \cup K_{1,n}$	SEM	iff n is a multiple of 4
$P_m \cup K_{1,n}$	SEM	if $m \geq 4$ is even
$2P_n$	SEM	iff n is not 2 or 3
$2P_{4n}$	SEM	for all n
$K_{1,m} \cup 2nK_{1,2}$	SEM	for all m and n

Graph	Types	Notes
$C_3 \cup C_n$	SEM	iff $n \geq 6$ even
$C_4 \cup C_n$	SEM	iff $n \geq 5$ odd
$C_5 \cup C_n$	SEM	iff $n \geq 4$ even
$C_m \cup C_n$	SEM	if $m \geq 6$ even and n odd, $n \geq m/2+2$
$C_m \cup C_n$	SEM?	iff $m+n \geq 9$ and $m+n$ odd
$C_4 \cup P_n$	SEM	iff $n \in 3$
$C_5 \cup P_n$	SEM	if $n \in 4$
$C_m \cup P_n$	SEM	if $m \geq 6$ even and $n \geq m/2+2$
$P_m \cup P_n$	SEM	iff $(m,n) \in (2,2)$ or $(3,3)$
corona $C_n \odot \bar{K_m}$	SEM	$n \geq 3$
$G \odot \bar{K_n}$	SEM	if G is SEM 2-regular graph
$C_m \odot \bar{K_n}$	SEM	
$St(m,n)$	SEM	$n \equiv 0 \, mod(m+1)$
$St(1,k,n)$	SEM	$k = 1, 2$ or n
$St(2,k,n)$	SEM	$k = 2, 3$
$St(1,1,k,n)$	SEM	$k = 2, 3$
$St(k,2,2,n)$	SEM	$k = 1, 2$
$St(a_1, ..., a_n)$	SEM?	for $n > 1$ odd
books B_n	SEM	if n even
books B_n	SEM?	if n even or $n \equiv 5 \, mod(8)$
nP_3	SEM	if $n \geq 4$ even
$K_2 \times C_{2n+1}$	SEM	
$P_3 \cup kP_2$	SEM	for all k
kP_n	SEM	if k is odd
$k(P_2 \cup P_n)$	SEM	if k is odd and $n = 3, 4$
fans F_n	SEM	iff $n \leq 6$
friendship graph of n triangles	SEM	iff $n = 3, 4, 5,$ or 7
generalized Petersen graph $P(n,2)$	SEM	if $n \geq 3$ odd

Graph	Types	Notes
trees with -labelings	SEM	
$P_{2m+1} \times P_2$	SEM	
$C_{2m+1} \times P_m$	SEM	
join of K_1 with any subgraph of a star	SEM	
if G is k-regular SEM graph		then $k \leq 3$
G is connected (p, q)-graph	SEM	G exists iff $p - 1 \leq q \leq 2p - 3$
G is connected 3-regular graph on p vertices	SEM	iff $p \equiv 2\, mod(4)$
nG	SEM	if G is a bipartite or tripartite SEM graph and n odd
$nK_2 + nK_2$	not SEM	

Chapter 4

Some New Classes of Graphs and Their SEMT Labeling

In this chapter we construct new super edge magic total graphs by using some old classes of graphs. We construct a new family of graphs which we call *reflexive w-trees* (denoted as $RT(m,n,k)$). It is constructed using k copies of reflexive w-graph $RW(m,n)$, and joining the vertex $b_{2,3}$ (or $c_{2,2}$) of every i-th copy to the vertex $b_{1,3}$ of $(i+1)$-st copy. We also construct the super edge magic total labeling of this graph and its generalizations. We construct another graph by joining the tail of an umbrella $U(m,n)$ (introduced in [32]) with a star S_k. This graph is referred as *extended umbrella graph* $U(m,n,k)$. In the end of this chapter, we study the super edge magic total labeling of $U(m,n,k)$.

4.1 SEMT labeling of reflexive w-trees

Definition 4.1.1. The *reflexive w-tree* denoted by $RT(m,n,k)$ is obtained by joining k isomorphic copies of $RW(m,n)$ with new vertices a_i $(1 \leq i \leq k-1)$ by adding the edges $a_i b_{2i,3}$ (or $a_i c_{2i,2}$) and $a_i b_{2i+1,3}$ for $1 \leq i \leq k-1$. Here
$V(kRW(m,n)) = \{b_{i,j} : 1 \leq i \leq 2k,\ 1 \leq j \leq 3\} \cup \{c_{i,j} : 1 \leq i \leq 2k,\ 1 \leq j \leq 2\} \cup \{x^l_{i,t} : 1 \leq l \leq 2k,\ 1 \leq i \leq 2,\ 1 \leq t \leq n\} \cup \{y^l_i : 1 \leq i \leq m-2,\ 1 \leq l \leq k\}$.

Now, we present the SEMT labeling of different generalizations of reflexive w-trees in the form of theorems. In the following theorems, we several times encounter the term $\sum_{1}^{0} f(n,r)$, which apparently has no meaning. So we use a convention that $\sum_{1}^{0} f(n,r) = 0$. An unlabeled drawing of two reflexive w-trees is presented in figure 4.1, to explain the structure of these trees. The graph (a) in figure 4.1 is the case when m is odd and m is even in graph (b).

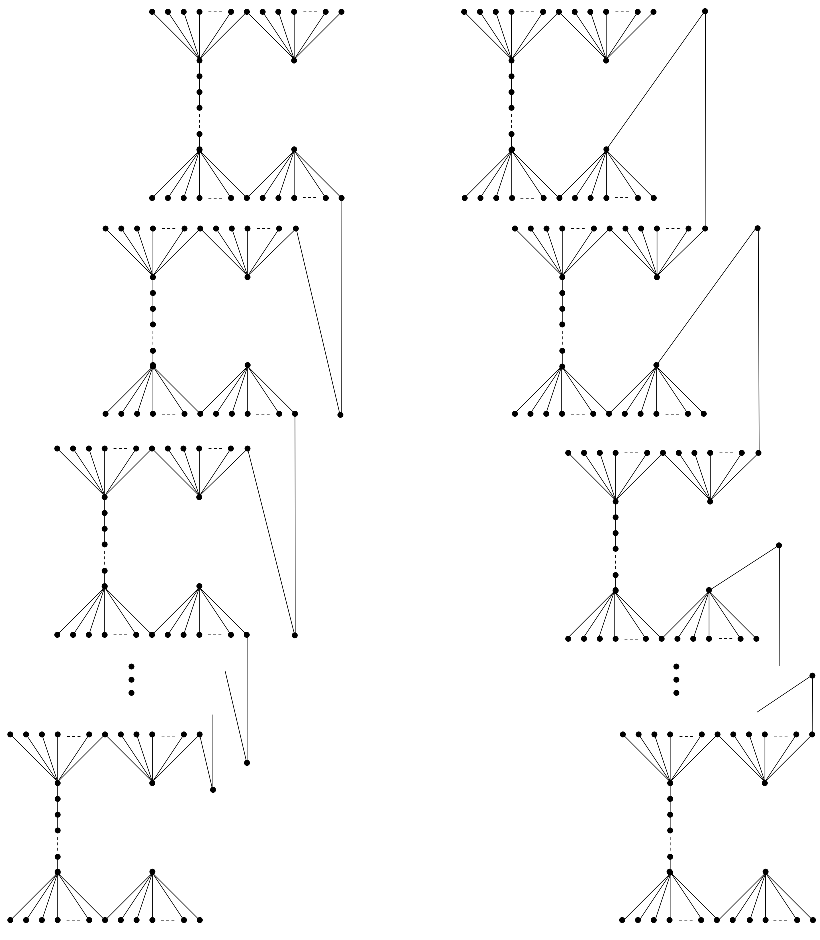


Figure 4.1: Reflexive w-trees

Theorem 4.1.1. *The graph $G \cong RT(m,n,k)$, for any $m,n,k \in \mathbb{Z}^+$, admits super edge-magic total labeling.*

Proof. The order and size of the graph $RT(m,n,k)$ is $k(4n+m+9)-1$ and $k(4n+m+9)-2$, respectively. The vertex set of $RT(m,n,k)$ is defined as follows:

$$\begin{aligned} V(G) &= \{b_{i,j} : 1 \leq i \leq 2k,\ 1 \leq j \leq 3\} \cup \{a_i : 1 \leq i \leq k-1\} \cup \\ &\quad \{c_{i,j} : 1 \leq i \leq 2k, 1 \leq j \leq 2\} \cup \{y_i^l : 1 \leq i \leq m-2, 1 \leq l \leq k\} \cup \\ &\quad \{x_{i,t}^l : 1 \leq l \leq 2k, 1 \leq i \leq 2,\ 1 \leq t \leq n\}. \end{aligned}$$

The edge set of $RT(m,n,k)$ is defined below as:

$$\begin{aligned}
E(G) &= \{b_{i,j}c_{i,j}, b_{i,j+1}c_{i,j} : 1 \leq i \leq 2k, 1 \leq j \leq 2\} \cup \\
&\quad \{x^l_{i,t}c_{l,i} : 1 \leq l \leq 2k, 1 \leq i \leq 2,\ 1 \leq t \leq n\} \cup \\
&\quad \{a_i b_{2i,3}, a_i b_{2i+1,3}, 1 \leq i \leq k-1,\ m = odd\} \cup \\
&\quad \{a_i c_{2i,2}, a_i b_{2i+1,3}, 1 \leq i \leq k-1,\ m = even\} \cup \\
&\quad \{y^l_i y^l_{i+1}, y^l_1 c_{2l-1,1}, y^l_{m-2}c_{2l,1} : 1 \leq i \leq m-3, 1 \leq l \leq k\}.
\end{aligned}$$

The labeling of $RT(m,n,k)$ is defined in two cases depending upon m, by the function

$$f : V(RT(m,n,k)) \to \big\{1, 2, \ldots, k(4n+m+9)-1\big\}.$$

Case 1: When the order m of the path y_m in $RT(m,n,k)$, which connect two copies of $W(n)$, is odd.

The vertices $b_{i,j}$, which appear on both sides of every star in all copies of $W(n)$, are labeled by the following formulas:

$$\begin{aligned}
f(b_{2i,j}) &= 2n + 3 + \lceil \frac{m-2}{2} \rceil + \big(\lceil \frac{m-2}{2} \rceil + 4n + 6\big)(i-1) - \\
&\quad n(1-j) + j, \\
f(b_{2i-1,j}) &= 2n + 4 + \big(\lceil \frac{m-2}{2} \rceil + 4n + 6\big)(i-1) + n(1-j) - j, \\
&\qquad\qquad 1 \leq i \leq k, 1 \leq j \leq 3.
\end{aligned}$$

The vertices $c_{i,j}$, which lie at the center of each star in all copies of $W(n)$, are labeled as follows:

$$\begin{aligned}
f(c_{2i,j}) &= 2k(2n+3) + k\lceil \frac{m-2}{2} \rceil + \big(\lfloor \frac{m-2}{2} \rfloor + 5\big)(i-1) + \\
&\quad \lfloor \frac{m-2}{2} \rfloor + j + 2, \\
f(c_{2i-1,j}) &= 2k(2n+3) + k\lceil \frac{m-2}{2} \rceil + \big(\lfloor \frac{m-2}{2} \rfloor + 5\big)(i-1) - j + 3, \\
&\qquad\qquad 1 \leq i \leq k, 1 \leq j \leq 2.
\end{aligned}$$

The vertices a_i, which connect a pair of $W(n)$ lying together in $RT(m,n,k)$, are labeled as follows:

$$f(a_i) = 2k(2n+3) + k\lceil \frac{m-2}{2} \rceil + i\lfloor \frac{m-2}{2} \rfloor + 5i, \quad 1 \leq i \leq k-1.$$

The vertices x_{it}^{l}, lying between the vertices $b_{i,j}$ in every copy of $W(n)$, have labels according to the following formulas:

$$\begin{aligned}
f(x_{it}^{2l}) &= 2n+4+\lceil\frac{m-2}{2}\rceil+\big(\lceil\frac{m-2}{2}\rceil+4n+6\big)(l-1)+\\
&\quad (n+1)(i-1)+t,\\
f(x_{it}^{2l-1}) &= 2n+3+\big(\lceil\frac{m-2}{2}\rceil+4n+6\big)(l-1)-(n+1)(i-1)-t,\\
&\qquad 1\le i\le 2, 1\le l\le k, 1\le t\le n.
\end{aligned}$$

The labeling of the vertices which lie on the path y_m, is defined by the following formulas:

$$\begin{aligned}
f(y_{2i-1}^{l}) &= 2n+3+\big(\lceil\frac{m-2}{2}\rceil+4n+6\big)(l-1)+i,\\
&\qquad 1\le l\le k,\ 1\le i\le\lceil\frac{m-2}{2}\rceil.\\
f(y_{2i}^{l}) &= 2k(2n+3)+k\lceil\frac{m-2}{2}\rceil+\big(\lfloor\frac{m-2}{2}\rfloor+5\big)(l-1)+i+2,\\
&\qquad 1\le l\le k,\ 1\le i\le\lfloor\frac{m-2}{2}\rfloor.
\end{aligned}$$

The labeling f defined by the these formulas label all the vertices of $RT(m,n,k)$. We calculate the weights of all the edges $uv\in E(RT(m,n,k))$ under the labeling f, by the formula $\omega(uv)=f(u)+f(v)$. All the edge weights calculated under this labeling, form a sequence of $|E(G)|$ consecutive integers:

$$\Big\{2k(2n+3)+k\lceil\frac{m-2}{2}\rceil+2,\quad 2\quad k(2n+3)+k\lceil\frac{m-2}{2}\rceil+3,$$
$$\ldots\quad ,\quad 8kn+k\lceil\frac{m-2}{2}\rceil+mk+15k-1\Big\}.$$

We can label the edges of $RT(m,n,k)$, by assigning the minimum possible edge label to the edge with maximum weight. Similarly, assign the second lowest edge label to the edge with second highest weight, and so on. Hence the labeling f of this graph extend to the *super edge magic total labeling*. By lemma 3.2.1, the magic constant under this labeling is $12kn+k\lceil\frac{m-2}{2}\rceil+2mk+24k-1$.

Case 2: When the order m of the path connecting two copies of $W(n)$, is even.

The leaves $b_{i,j}$, appearing on both sides of every star in all copies of $W(n)$, are labeled by the following formulas:

$$\begin{aligned} f(b_{2i,j}) &= k\big(\frac{m-2}{2}+2n+5\big)+\frac{m-2}{2}+\big(\frac{m-2}{2}+2n+6\big)(i-1)+\\ &\quad (n+1)(j-1)+3,\\ f(b_{2i-1,j}) &= 2n+\big(\frac{m-2}{2}+2n+5\big)(i-1)+(n+1)(1-j)+3,\\ &\qquad\qquad\qquad\qquad 1\le i\le k, 1\le j\le 3. \end{aligned}$$

The vertices $c_{i,j}$, which lie at the center of each star in all copies of $W(n)$, are labeled as follows:

$$\begin{aligned} f(c_{2i,j}) &= 2n+\frac{m-2}{2}+\big(\frac{m-2}{2}+2n+5\big)(i-1)+j+3,\\ f(c_{2i-1,j}) &= k\big(\frac{m-2}{2}+2n+5\big)+\big(\frac{m-2}{2}+2n+6\big)(i-1)-j+3,\\ &\qquad\qquad\qquad\qquad 1\le i\le k, 1\le j\le 2. \end{aligned}$$

The vertices a_i, connecting $2k$ copies of $W(n)$ in k pairs, are labeled by the following labeling function:

$$f(a_i) = k\big(\frac{m-2}{2}+2n+5\big)+i\big(\frac{m-2}{2}+2n+6\big),\ \ 1\le i\le k-1.$$

The labeling scheme of the vertices x^l_{it}, which lie inside every star in all copies of $W(n)$, are labeled as follows:

$$\begin{aligned} f(x^{2l}_{it}) &= k\big(\frac{m-2}{2}+2n+5\big)+\big(\frac{m-2}{2}+2n+6\big)(l-1)+\frac{m-2}{2}\\ &\quad +(n+1)(i-1)+t+3,\\ f(x^{2l-1}_{it}) &= 2n+\big(\frac{m-2}{2}+2n+5\big)(l-1)-(n+1)(i-1)-t+3,\\ &\qquad\qquad\qquad\qquad 1\le i\le 2, 1\le l\le k, 1\le t\le n. \end{aligned}$$

The vertices y^l_i, which lie on the path connecting the copies of $W(n)$, are labeled as follows:

$$\begin{aligned} f(y^l_{2i-1}) &= 2n+3+(\frac{m-2}{2}+2n+5)(l-1)+i,\\ f(y^l_{2i}) &= k\big(\frac{m-2}{2}+2n+5\big)+\big(\frac{m-2}{2}+2n+6\big)(l-1)+i+2,\\ &\qquad\qquad\qquad\qquad 1\le l\le k,\ 1\le i\le \frac{m-2}{2}. \end{aligned}$$

The labeling f, defined by a set of formulas in this case, labels all the vertices of $RT(m,n,k)$. All the edge weights calculated under this labeling function appear in a sequence of $|E(G)|$ consecutive integers, which are:

$$\Big\{k\big(\frac{m-2}{2}+2n+5\big)+2, k\big(\frac{m-2}{2} \quad + \quad 2n+5\big)+3,$$
$$\ldots \quad , \quad k\big(\frac{m-2}{2}\big)+6nk+14k+mk-1\Big\}.$$

If we extend the labeling f of $RT(m,n,k)$ to a total labeling using lemma 3.2.1, we get a *super edge magic total labeling* of $RT(m,n,k)$. The magic constant under this labeling is $k\big(\frac{m-2}{2}\big)+10nk+23k+2mk-1$. $\square$

Definition 4.1.2. The *extended reflexive w-tree* $RT_E(m,n,r,k)$ is a graph in which all copies of w-graphs are replaced by extended w-graphs. That is, the number of stars in each copy of $W(n)$ is increased upto any $k \geq 2$.

Theorem 4.1.2. *The graph $G \cong RT_E(m,n,r,k)$, for any $m,n,r,k \in \mathbb{Z}^+$, admits super edge-magic total labeling.*

Proof. The order and size of the graph $RT_E(m,n,r,k)$ is $2nrk+4rk+mk+k-1$ and $2nrk+4rk+mk+k-2$, respectively. The vertex set of $RT_E(m,n,r,k)$ is defined as:

$$\begin{aligned} V(G) \quad = \quad & \{b_{i,j} : 1 \leq i \leq 2k,\ 1 \leq j \leq r+1\} \cup \{c_{i,j} : 1 \leq i \leq 2k, 1 \leq j \leq r\} \cup \\ & \{y_i^l : 1 \leq i \leq m-2, 1 \leq l \leq k\} \cup \{a_i : 1 \leq i \leq k-1\} \cup \\ & \{x_{i,t}^l : 1 \leq l \leq 2k, 1 \leq i \leq r,\ 1 \leq t \leq n\}. \end{aligned}$$

The edge set of $RT_E(m,n,r,k)$ is defined below as:

$$\begin{aligned} E(G) \quad = \quad & \{b_{i,j}c_{i,j}, b_{i,j+1}c_{i,j} : 1 \leq i \leq 2k, 1 \leq j \leq r\} \cup \\ & \{x_{i,t}^l c_{l,i} : 1 \leq l \leq 2k, 1 \leq i \leq r,\ 1 \leq t \leq n\} \cup \\ & \{a_i b_{2i+1,r+1}, a_i b_{2i,r+1}, 1 \leq i \leq k-1,\ \textit{for} \quad m\textit{=odd}\ \} \cup \\ & \{a_i b_{2i+1,r+1}, a_i c_{2i,r}, 1 \leq i \leq k-1,\ \textit{for} \quad m\textit{=even}\ \} \cup \\ & \{y_i^l y_{i+1}^l, y_1^l c_{2l-1,1}, y_{m-2}^l c_{2l,1} : 1 \leq i \leq m-3, 1 \leq l \leq k\}. \end{aligned}$$

The labeling of $RT_E(m,n,r,k)$ is defined in two cases depending upon the order m of the path y_m, by the function

$$f : V(RT_E(m,n,r,k)) \rightarrow \big\{1,2,\ldots,2nrk+4rk+mk+k-1\big\}.$$

Case 1: When the path y_m, which connect the copies of $W(n)$ in $RT_E(m,n,r,k)$, is of odd order.

The vertices $c_{i,j}$, which lie at the center of each star in all copies of $W(n)$, are labeled by the following formulas:

$$\begin{aligned}
f(c_{2i,j}) &= k\big(\lceil\frac{m-2}{2}\rceil + 2nr + 2r + 2\big) + \big(\lfloor\frac{m-2}{2}\rfloor + 2r + 1\big)(i-1) + \\
&\quad \lfloor\frac{m-2}{2}\rfloor + r + j, \\
f(c_{2i-1,j}) &= k\big(\lceil\frac{m-2}{2}\rceil + 2nr + 2r + 2\big) + \big(\lfloor\frac{m-2}{2}\rfloor + 2r + 1\big)(i-1) + \\
&\quad r - j + 1, \qquad\qquad 1 \le i \le k, 1 \le j \le r.
\end{aligned}$$

The vertices $b_{i,j}$, which are adjacent to the vertices $c_{i,j}$ and lie on both sides of every star in all copies of $W(n)$, are labeled as follows:

$$\begin{aligned}
f(b_{2i,j}) &= nr + r + \lceil\frac{m-2}{2}\rceil + 2 + \big(\lceil\frac{m-2}{2}\rceil + 2nr + 2r + 2\big)(i-1) + \\
&\quad (n+1)(j-1), \\
f(b_{2i-1,j}) &= nr + r + \big(\lceil\frac{m-2}{2}\rceil + 2nr + 2r + 2\big)(i-1) - (n+1)(j-1) + 1, \\
&\qquad\qquad 1 \le i \le k, 1 \le j \le r+1.
\end{aligned}$$

The vertices a_i, connecting a pair of $W(n)$ lying together in $RW_E(m,n,r)$, are labeled as follows:

$$\begin{aligned}
f(a_i) &= k\big(\lceil\frac{m-2}{2}\rceil + 2nr + 2r + 2\big) + \big(\lfloor\frac{m-2}{2}\rfloor + 2r + 1\big)(i-1) + \\
&\quad 2r + \lfloor\frac{m-2}{2}\rfloor + 1, \qquad\qquad 1 \le i \le k-1.
\end{aligned}$$

The vertices x_{it}^{l}, which are adjacent to the vertices $c_{i,j}$ and lie between the vertices $b_{i,j}$, are labeled as follows:

$$\begin{aligned}
f(x_{it}^{2l}) &= nr + r + \big(\lceil\frac{m-2}{2}\rceil + 2nr + 2r + 2\big)(l-1) + \lceil\frac{m-2}{2}\rceil + \\
&\quad (n+1)(i-1) + t + 2, \\
f(x_{it}^{2l-1}) &= nr + r + \big(\lceil\frac{m-2}{2}\rceil + 2nr + 2r + 2\big)(l-1) - (n+1)(i-1) - \\
&\quad t + 1, \qquad\qquad 1 \le i \le r, 1 \le l \le k, 1 \le t \le n.
\end{aligned}$$

The vertices on the paths y_i^l receive labels according to the following formulas:

$$\begin{aligned}
f(y_{2i-1}^{l}) &= nr + r + \big(\lceil\frac{m-2}{2}\rceil + 2nr + 2r + 2\big)(l-1) + i + 1, \\
&\qquad\qquad 1 \le l \le k,\ 1 \le i \le \lceil\frac{m-2}{2}\rceil, \\
f(y_{2i}^{l}) &= k\big(\lceil\frac{m-2}{2}\rceil + 2nr + 2r + 2\big) + \big(\lfloor\frac{m-2}{2}\rfloor + 2r + 1\big)(l-1) + \\
&\quad r + i, \qquad\qquad 1 \le l \le k,\ 1 \le i \le \lfloor\frac{m-2}{2}\rfloor.
\end{aligned}$$

The formulas, stated above, constitute a vertex labeling of $RT_E(m,n,r,k)$ in such a way that under this labeling scheme, the edge weights of the graph form a sequence of $|E(G)|$ consecutive integers:

$$\Big\{k\big(\lceil\frac{m-2}{2}\rceil+2nr+2r+2\big) \quad + \quad 2, k\big(\lceil\frac{m-2}{2}\rceil+2nr+2r+2\big)+3,$$
$$\dots \quad , \quad k\lceil\frac{m-2}{2}\rceil+4nrk+6rk+3k+mk-1\Big\}.$$

We can extend the vertex labeling f of $RT_E(m,n,r,k)$ to a total labeling, such that all the edge weights under that labeling are same, by using lemma 3.2.1. Hence, the labeling f becomes the *super edge magic total labeling* of the graph $RT_E(m,n,r,k)$. The magic constant under this labeling is $k\lceil\frac{m-2}{2}\rceil+6nrk+10rk+4k+2mk-1$.

Case 2: When the order m of the path y_m, which connect two copies of $W(n)$, is even.

The vertices $c_{i,j}$, which lie at the center of every star in all copies of $W(n)$, are labeled as:

$$\begin{aligned}
f(c_{2i,j}) &= (nr+r+1)+\frac{m-2}{2}+\big(\frac{m-2}{2}+nr+2r+1\big)(i-1)+j,\\
f(c_{2i-1,j}) &= k\big(\frac{m-2}{2}+nr+2r+1\big)+\big(\frac{m-2}{2}+nr+2r+2\big)(i-1)+\\
&\quad r-j+1, \qquad\qquad 1\le i\le k, 1\le j\le r.
\end{aligned}$$

The vertices $b_{i,j}$, which are adjacent to the vertices $c_{i,j}$ and appear on both sides of each star in all copies of $W(n)$, are labeled as follows:

$$\begin{aligned}
f(b_{2i,j}) &= k\big(\frac{m-2}{2}+nr+2r+1\big)+\big(\frac{m-2}{2}+nr+2r+2\big)(i-1)+\\
&\quad \frac{m-2}{2}+(n+1)(j-1)+r+1,\\
f(b_{2i-1,j}) &= nr+r+1+\big(\frac{m-2}{2}+nr+2r+1\big)(i-1)-(n+1)(j-1),\\
&\qquad\qquad 1\le i\le k, 1\le j\le r+1.
\end{aligned}$$

The labels of the vertices a_i, which connect a pair of $W(n)$ lying together, are as follows:

$$\begin{aligned}
f(a_i) &= k\big(\frac{m-2}{2}+nr+2r+1\big)+\frac{m-2}{2}+\big(\frac{m-2}{2}+nr+2r+2\big)(i-1)+\\
&\quad nr+2r+2, \qquad\qquad 1\le i\le k-1.
\end{aligned}$$

The vertices x_{it}^{l}, which are adjacent to the vertices $c_{i,j}$ and lie between the vertices $b_{i,j}$, are labeled by the following formulas:

$$\begin{aligned} f(x_{it}^{2l}) &= \big(\frac{m-2}{2}+nr+2r+1\big)(k+l-1)+\frac{m-2}{2}+(n+1)(i-1)+l+r+t, \\ f(x_{it}^{2l-1}) &= (nr+r+1)+\big(\frac{m-2}{2}+nr+2r+1\big)(l-1)-(n+1)(i-1)-t, \\ & \qquad\qquad\qquad\qquad 1 \le i \le r, 1 \le l \le k, 1 \le t \le n. \end{aligned}$$

The vertices lying on the paths y_i^l, which connect two copies of $W(n)$, are labeled as follows:

$$\begin{aligned} f(y_{2i-1}^{l}) &= nr+r+1+\big(\frac{m-2}{2}+nr+2r+1\big)(l-1)+i, \\ f(y_{2i}^{l}) &= k\big(\frac{m-2}{2}+nr+2r+1\big)+\big(\frac{m-2}{2}+nr+2r+2\big)(l-1)+r+i, \\ & \qquad\qquad\qquad\qquad 1 \le l \le k,\ 1 \le i \le \frac{m-2}{2}. \end{aligned}$$

The edge weights of the graph $RT_E(m,n,r,k)$ under this labeling, forms a sequence of $|E(G)|$ consecutive integers, which are:

$$\Big\{k\big(\frac{m-2}{2}+nr+2r+1\big)+2 \quad , \quad k\big(\frac{m-2}{2}+nr+2r+1\big)+3, \\ \ldots \quad , \quad k\big(\frac{m-2}{2}\big)+3nrk+6rk+mk+2k-1\Big\}.$$

Hence the labeling f of the graph $RT_E(m,n,r,k)$ can be converted into the *super edge-magic total labeling*, by using lemma 3.2.1. The magic constant under this labeling is $k\big(\frac{m-2}{2}\big)+5rk(n+2)+2mk+3k-1$. □

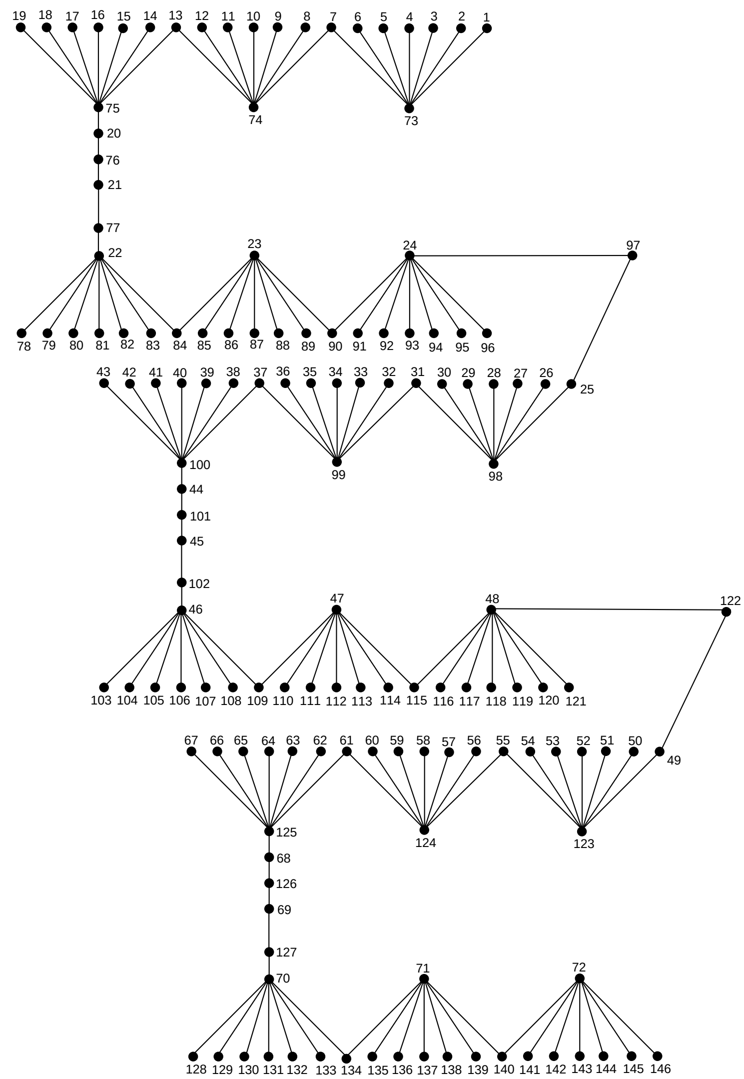


Figure 4.2: SEMT labeling of $RT_E(6,5,3,3)$ with magic constant 365

In the next theorem we generalize the graph $RT(m,n,k)$ for k copies of $RW(m,n)$ and any number of stars in it, such that every star contains any arbitrary number of vertices. This graph is referred as *generalized reflexive w-tree* and is denoted by $RT_G(m;n_{i,j};r_i;k)$.

Theorem 4.1.3. *For $1 \leq i \leq 2k$,$1 \leq j \leq r_i$ and for any $m, n_{i,j}, r_i, k \in \mathbb{Z}^+$, the graph $G \cong RT_G(m;n_{i,j};r_i;k)$ admits super edge-magic total labeling.*

Proof. The order of the graph $RT_G(m; n_{i,j}; r_i; k)$ is

$$\sum_{i=1}^{2k}\sum_{j=1}^{r_i} n_{i,j} + \sum_{i=1}^{2k}(2r_i+1) + k(m-1) - 1,$$

and size of this graph is

$$\sum_{i=1}^{2k}\sum_{j=1}^{r_i} n_{i,j} + \sum_{i=1}^{2k}(2r_i+1) + k(m-1) - 2.$$

The vertex set of $RT_G(m; n_{i,j}; r_i; k)$ is defined as follows:

$$\begin{aligned} V(G) &= \{b_{i,j} : 1 \le i \le 2k,\ 1 \le j \le r_i+1\} \cup \{c_{i,j} : 1 \le i \le 2k, 1 \le j \le r_i\} \cup \\ &\quad \{a_i : 1 \le i \le k-1\} \cup \{y_i^l : 1 \le i \le m-2, 1 \le l \le k\} \cup \\ &\quad \{x_{i,t}^l : 1 \le l \le 2k, 1 \le i \le r_l,\ 1 \le t \le n_{li}\}. \end{aligned}$$

The edge set of $RT_G(m; n_{i,j}; r_i; k)$ is defined below as:

$$\begin{aligned} E(G) &= \{b_{i,j}c_{i,j}, b_{i,j+1}c_{i,j} : 1 \le i \le 2k, 1 \le j \le r_i\} \cup \\ &\quad \{x_{i,t}^l c_{l,i} : 1 \le l \le 2k, 1 \le i \le r_l,\ 1 \le t \le n_{li}\} \cup \\ &\quad \{y_i^l y_{i+1}^l, y_1^l c_{2l-1,1}, y_{m-2}^l c_{21,1} : 1 \le i \le m-3, 1 \le l \le k\} \cup \\ &\quad \{a_i b_{2i+1,r_{2i+1}+1}, a_i b_{2i,r_{2i}+1},\ 1 \le i \le k-1,\ for \quad m{=}odd\ \} \cup \\ &\quad \{a_i b_{2i+1,r_{2i+1}+1}, a_i c_{2i,r_{2i}},\ 1 \le i \le k-1,\ for \quad m{=}even\ \}. \end{aligned}$$

The labeling of $RT_G(m; n_{i,j}; r_i; k)$ is defined in two cases depending upon m, by the function

$$f : V\big(RT_G(m; n_{i,j}; r_i; k)\big) \to \Big\{1, 2, \dots, \sum_{i=1}^{2k}\sum_{j=1}^{r_i} n_{ij} + \sum_{i=1}^{2k}(2r_i+1) + k(m-1) - 1\Big\}.$$

Case 1: When m is odd.

For $1 \le i \le k$, $1 \le j \le r_{2i-1}+1$,

$$\begin{aligned} f(b_{2i-1,j}) &= \sum_{t=1}^{1-j+r_{2i-1}} n_{2i-1,1-t+r_{2i-1}} + \sum_{s=1}^{2i-2}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2i-2}(r_t+1) \\ &\quad + r_{2i-1} + \lceil \frac{m-2}{2} \rceil (i-1) - j + 2. \end{aligned}$$

For $1 \le i \le k, 1 \le j \le r_{2i}+1$,

$$f(b_{2i,j}) = \sum_{s=1}^{2i-1}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2i-1}(r_t+1) + \sum_{t=1}^{j-1} n_{2i,t} + i\lceil \frac{m-2}{2} \rceil + j.$$

For $1 \le i \le k, 1 \le j \le r_{2i-1}$,

$$\begin{aligned} f(c_{2i-1,j}) &= \sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2k}(r_t+1) + \sum_{t=1}^{2i-2} r_t + k\lceil\frac{m-2}{2}\rceil \\ &\quad +(i-1)\lfloor\frac{m-2}{2}\rfloor + r_{2i-1} - j + i. \end{aligned}$$

For $1 \le i \le k, 1 \le j \le r_{2i}$,

$$f(c_{2i,j}) = \sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2k}(r_t+1) + \sum_{t=1}^{2i-1} r_t + k\lceil\frac{m-2}{2}\rceil + i\lfloor\frac{m-2}{2}\rfloor + j + i - 1.$$

For $1 \le l \le k, 1 \le i \le r_{2l-1}, 1 \le t \le n_{2l-1,i}$,

$$\begin{aligned} f(x_{it}^{2l-1}) &= \sum_{s=1}^{2l-2}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2l-2}(r_t+1) + \sum_{t=1}^{1-i+r_{2l-1}} n_{2l-1,1-t+r_{2l-1}} \\ &\quad +(l-1)\lceil\frac{m-2}{2}\rceil + r_{2l-1} - i - t + 1. \end{aligned}$$

For $1 \le l \le k, 1 \le i \le r_{2l}, 1 \le t \le n_{2l,i}$,

$$f(x_{it}^{2l}) = \sum_{s=1}^{2l-1}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2l-1}(r_t+1) + l\lceil\frac{m-2}{2}\rceil + \sum_{t=1}^{i-1} n_{2l,t} + i + t.$$

For $1 \le i \le k-1$,

$$f(a_i) = \sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + k\lceil\frac{m-2}{2}\rceil + \sum_{t=1}^{2k}(r_t+1) + \sum_{t=1}^{2i} r_t + i\lfloor\frac{m-2}{2}\rfloor + i.$$

For $1 \le l \le k$, $1 \le i \le \lceil\frac{m-2}{2}\rceil$,

$$f(y_{2i-1}^l) = \sum_{s=1}^{2l-1}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2l-1}(r_t+1) + (l-1)\lceil\frac{m-2}{2}\rceil + i.$$

For $1 \le l \le k$, $1 \le i \le \lfloor\frac{m-2}{2}\rfloor$,

$$f(y_{2i}^l) = \sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t} + \sum_{t=1}^{2k}(r_t+1) + \sum_{t=1}^{2l-1} r_t + k\lceil\frac{m-2}{2}\rceil + (l-1)\lfloor\frac{m-2}{2}\rfloor + l + i - 1.$$

The set of edge edge weights under this labeling function forms a sequence of $|E(G)|$ consecutive integers:

$$\Big\{\sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t}+\sum_{t=1}^{2k}(r_t+1)+k\lceil\frac{m-2}{2}\rceil+\ 2\ ,$$
$$\sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t}+\sum_{t=1}^{2k}(r_t+1)+k\lceil\frac{m-2}{2}\rceil+\ 3\ ,$$
$$\ldots,2\sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t}+3\sum_{t=1}^{2k}r_t\ +\ k\lceil\frac{m-2}{2}\rceil+km+3k-1\Big\}.$$

Hence the labeling f can be extended to the *super edge-magic total labeling* of the graph $RT_G(m;n_{i,j};r_i;k)$, by using lemma 3.2.1. The magic constant under this labeling is

$$3\sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t}+5\sum_{t=1}^{2k}r_t+k\lceil\frac{m-2}{2}\rceil+2km+4k-1.$$

Case 2: When m is even.

For $1\le i\le k$, $1\le j\le r_{2i-1}+1$,

$$\begin{aligned}f(b_{2i-1,j}) &= \sum_{t=1}^{1-j+r_{2i-1}} n_{2i-1,1-t+r_{2i-1}}+\sum_{s=1}^{i-1}\sum_{t=1}^{r_s} n_{2s-1,t}+\sum_{t=1}^{i-1}(r_{2t-1}+1)\\ &\quad+\sum_{t=1}^{i-1}r_{2t}+(i-1)(\frac{m-2}{2})+r_{2i-1}+2-j.\end{aligned}$$

For $1\le i\le k, 1\le j\le r_{2i}+1$,

$$\begin{aligned}f(b_{2i,j}) &= \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s,t}+\sum_{s=1}^{i-1}\sum_{t=1}^{r_s} n_{2s,t}+\sum_{t=1}^{k}(r_{2t-1}+1)+\sum_{t=1}^{k}r_{2t}+\sum_{t=1}^{i}r_{2t-1}\\ &\quad+\sum_{t=1}^{i-1}(r_{2t}+1)+\sum_{t=1}^{j-1}n_{2i,t}+i(\frac{m-2}{2})+i+j-1.\end{aligned}$$

For $1\le i\le k, 1\le j\le r_{2i-1}$,

$$\begin{aligned}f(c_{2i-1,j}) &= \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t}+\sum_{s=1}^{i-1}\sum_{t=1}^{r_s} n_{2s,t}+\sum_{t=1}^{k}r_{2t}+\sum_{t=1}^{i}r_{2t-1}\\ &\quad+\sum_{t=1}^{k}(r_{2t-1}+1)+\sum_{t=1}^{i-1}(r_{2t}+1)+(k+i-1)(\frac{m-2}{2})+i-j.\end{aligned}$$

For $1 \le i \le k, 1 \le j \le r_{2i}$,

$$f(c_{2i,j}) = \sum_{s=1}^{i}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{i}(r_{2t-1}+1) + (i-1)\Big(\frac{m-2}{2}\Big) + j.$$

For $1 \le l \le k, 1 \le i \le r_{2l-1}, 1 \le t \le n_{2l-1,i}$,

$$\begin{aligned} f(x_{it}^{2l-1}) &= \sum_{s=1}^{l-1}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{1-i+r_{2l-1}} n_{2l-1,1-k+r_{2l-1}} + \sum_{t=1}^{l-1}(r_{2t-1}+1) \\ &\quad + \sum_{t=1}^{l-1} r_{2t} + (l-1)\Big(\frac{m-2}{2}\Big) + r_{2l-1} - i - t + 2. \end{aligned}$$

For $1 \le l \le k, 1 \le i \le r_{2l-1}, 1 \le t \le n_{2l-1,i}$,

$$\begin{aligned} f(x_{it}^{2l}) &= \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{k}(r_{2t-1}+1) + \sum_{t=1}^{k} r_{2t} + \sum_{t=1}^{l} r_{2t-1} + \sum_{t=1}^{i-1} n_{2l,t} \\ &\quad + (i+k)\Big\lceil\frac{m-2}{2}\Big\rceil + i + t + l - 1. \end{aligned}$$

For $1 \le i \le k-1$,

$$\begin{aligned} f(a_i) &= \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{s=1}^{i}\sum_{t=1}^{r_s} n_{2s,t} + \sum_{t=1}^{k}(r_{2t-1}+1) + \sum_{t=1}^{i}(r_{2t}+1) \\ &\quad + \sum_{t=1}^{k} r_{2t} + \sum_{t=1}^{i} r_{2t-1} + (i+k)\Big(\frac{m-2}{2}\Big) + i. \end{aligned}$$

For $1 \le l \le k, 1 \le i \le \big(\frac{m-2}{2}\big)$,

$$\begin{aligned} f(y_{2i-1}^{l}) &= \sum_{s=1}^{l}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{t=1}^{l}(r_{2t-1}+1) + \sum_{t=1}^{l-1} r_{2t} \\ &\quad + (l-1)\Big(\frac{m-2}{2}\Big) + i. \end{aligned}$$

For $1 \le l \le k, 1 \le i \le \big(\frac{m-2}{2}\big)$,

$$\begin{aligned} f(y_{2i}^{l}) &= \sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} + \sum_{s=1}^{l-1}\sum_{t=1}^{r_s} n_{2s,t} + \sum_{t=1}^{k}(r_{2t-1}+1) + \sum_{t=1}^{l-1}(r_{2t}+1) \\ &\quad + \sum_{t=1}^{k} r_{2t} + \sum_{t=1}^{l} r_{2t-1} + (k+l-1)\Big(\frac{m-2}{2}\Big) + i + l - 1. \end{aligned}$$

All the edge weights under this labeling constitute a set of $|E(G)|$ consecutive integers, which are:

$$\Big\{\sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t}+\sum_{t=1}^{k}(r_{2t-1}+1) \quad + \quad \sum_{t=1}^{k} r_{2t}+k\big(\frac{m-2}{2}\big)+2,$$
$$\sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t}+\sum_{t=1}^{k}(r_{2t-1}+1) \quad + \quad \sum_{t=1}^{k} r_{2t}+k\big(\frac{m-2}{2}\big)+3,\dots,$$
$$\sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t}+\sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t} \quad + \quad \sum_{t=1}^{2k} r_t+\sum_{t=1}^{2k} r_t+\frac{km}{2}+km+k-1\Big\}.$$

Hence by using lemma 3.2.1, the labeling f of the graph $RT_G(m; n_{i,j}; r_i; k)$ extends to the *super edge magic total labeling.* The magic constant under this labeling is

$$2\sum_{s=1}^{2k}\sum_{t=1}^{r_s} n_{s,t}+3\sum_{t=1}^{2k} r_t+\sum_{s=1}^{k}\sum_{t=1}^{r_s} n_{2s-1,t}+\frac{km}{2}+2km+2k-1.$$

□

Example 4.1.1. *For $m = 6$ and $k = 4$, the super edge magic total labeling of the generalized reflexive w-tree $RT_G(6; n_{i,j}; r_i; 4)$ for $1 \le i \le 8$, $1 \le j \le r_i$, is presented in the figure 4.3.*

$$\begin{pmatrix} r_1\\ r_2\\ r_3\\ r_4\\ r_5\\ r_6\\ r_7\\ r_8 \end{pmatrix} = \begin{pmatrix} 3\\ 4\\ 5\\ 2\\ 4\\ 4\\ 3\\ 6 \end{pmatrix}, \quad \begin{pmatrix} n_{11}, n_{12}, n_{13}\\ n_{21}, n_{22}, n_{23}, n_{24}\\ n_{31}, n_{32}, n_{33}, n_{34}, n_{35}\\ n_{41}, n_{42}\\ n_{51}, n_{52}, n_{53}, n_{54}\\ n_{61}, n_{62}, n_{63}, n_{64}\\ n_{71}, n_{72}, n_{73}\\ n_{81}, n_{82}, n_{83}, n_{84}, n_{85}, n_{86} \end{pmatrix} = \begin{pmatrix} 5,4,2\\ 7,5,3,4\\ 3,5,3,2,6\\ 6,4\\ 3,5,2,4\\ 7,6,2,4\\ 3,5,4\\ 10,4,5,5,4,10 \end{pmatrix}$$

The edge weights form the sequence {101,102,103,...,330} and hence by lemma 3.2.1 *this labeling can be extended to the super edge magic total labeling with magic constant 562.*

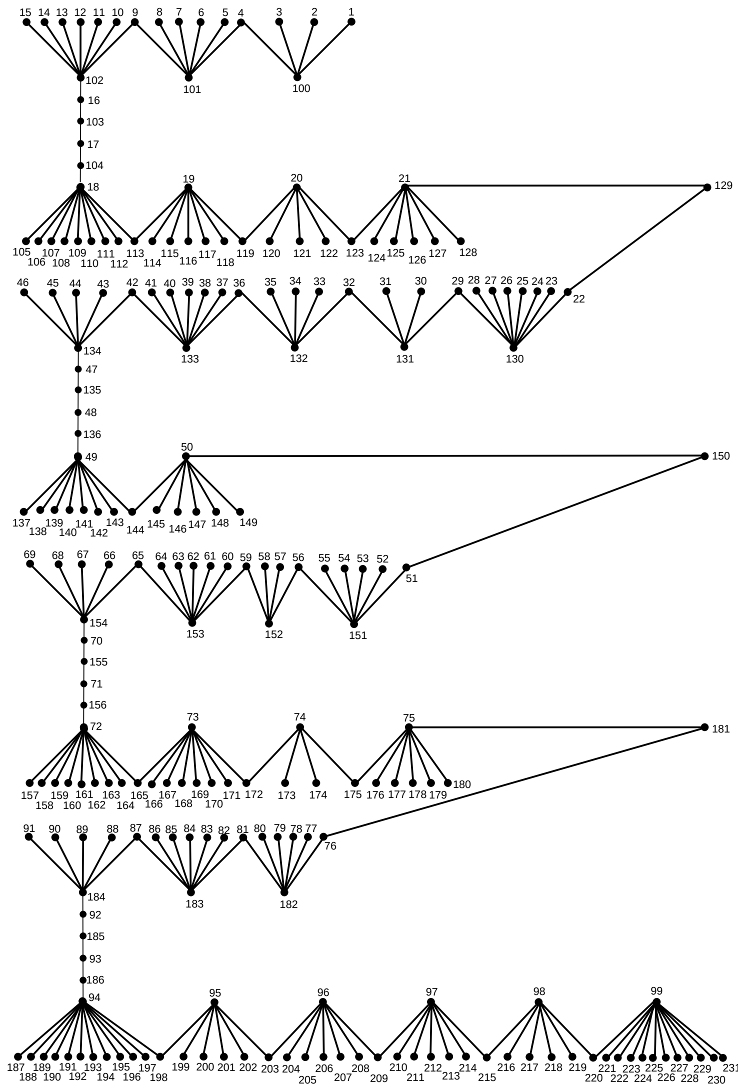


Figure 4.3: $RT_G(6; n_{i,j}; r_i; 4)$ For $1 \leq i \leq 8$, $1 \leq j \leq r_i$

4.2 SEMT labeling of extended umbrella graphs

Sin-Min Lee and Nien-Tsu Lee [32] defined an *umbrella graph* $U(m,n)$ to be a graph obtained by joining a path P_n with the central vertex of a fan f_m. The vertex and edge sets of $U(m,n)$ are as follows.

$$\begin{aligned} V(U(m,n)) &= \{x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_n\}, \\ E(U(m,n)) &= \{x_i x_{i+1} : 1 \le i \le m-1\} \cup \{y_i y_{i+1} : 1 \le i \le n-1\} \cup \\ &\quad \{x_i y_1 : 1 \le i \le m\}. \end{aligned}$$

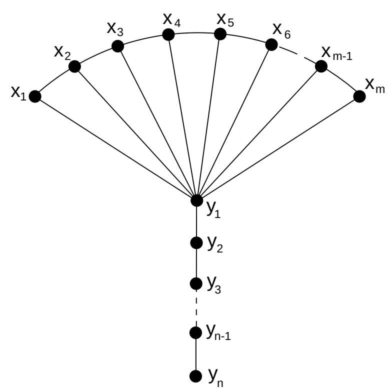


Figure 4.4: $U(m,n)$

Theorem 4.2.1. *For any $m \in \mathbb{Z}^+$ and $n = \begin{cases} m, m-1, & m \equiv 1 (mod\, 2); \\ m-1, m-2, & m \equiv 0 (mod\, 2). \end{cases}$ The umbrella graph $U(m,n)$ admits the super edge magic total labeling.*

Proof. The order and size of the graph $U(m,n)$ is $m+n$ and $2m+n-2$, respectively. We define the labeling of $U(m,n)$ in two cases depending upon m, by the function

$$f : V(U(m,n)) \to \{1, 2, \ldots, m+n\}.$$

Case 1: When m is odd.

$$f(x_{2i-1}) = i, \qquad i = 1, 2, 3, \ldots, \lfloor \frac{m}{2} \rfloor + 1.$$

$$f(x_{2i}) = \lfloor \frac{m}{2} \rfloor + i + 1, \qquad i = 1, 2, 3, \ldots, \lfloor \frac{m}{2} \rfloor.$$

$$f(y_{2j-1}) = m + \lfloor\frac{n}{2}\rfloor + j, \qquad \jmath = 1, 2, 3, \ldots, \lfloor\frac{n}{2}\rfloor + 1.$$

$$f(y_{2j}) = m + j, \qquad j = 1, 2, 3, \ldots, \lfloor\frac{n}{2}\rfloor.$$

All the edge weights of the graph $U(m, n)$ under this labeling function form a sequence of $|E(G)|$ consecutive integers:

$$\left\{\lceil\frac{m}{2}\rceil + 2, \lceil\frac{m}{2}\rceil + 3, \ldots, \lceil\frac{m}{2}\rceil + 2m + n - 1\right\}.$$

Hence this labeling can be extended to the *super edge magic total labeling*, by using lemma 3.2.1. The magic constant of the graph $U(m, n)$ under this labeling is $\lceil\frac{m}{2}\rceil + 3m + 2n$.

Case 2: When m is even.

$$f(x_{2i-1}) = i, \qquad i = 1, 2, 3, \ldots, \lfloor\frac{m}{2}\rfloor.$$

$$f(x_{2i}) = \lfloor\frac{m}{2}\rfloor + i, \qquad i = 1, 2, 3, \ldots, \lfloor\frac{m}{2}\rfloor.$$

$$f(y_{2j-1}) = m + \lfloor\frac{n}{2}\rfloor + j, \qquad \jmath = 1, 2, 3, \ldots, \lfloor\frac{n}{2}\rfloor.$$

$$f(y_{2j}) = m + j, \qquad j = 1, 2, 3, \ldots, \lfloor\frac{n}{2}\rfloor.$$

In this case, the edge weights of the graph $U(m, n)$ form a sequence of $|E(G)|$ consecutive integers, which are:

$$\left\{\frac{m}{2} + 2, \frac{m}{2} + 3, \ldots, \frac{m}{2} + 2m + n - 1\right\}.$$

Hence by using lemma 3.2.1, the labeling f of this graph can be converted into the *super edge magic total labeling* of $U(m, n)$. The magic constant under this labeling is $\frac{m}{2} + 3m + 2n$. □

Definition 4.2.1. We construct a graph by connecting the tail of an *umbrella* $U(m, n)$ with a *star* S_k, and refer this graph as *extended umbrella graph* and denote it as $U(m, n, k)$.

Theorem 4.2.2. *The graph $G \cong U(m, n, k)$ admits super edge magic total labeling.*

Proof. The order and size of the graph $G \cong U(m,n,k)$ is $m+n+k$ and $2m+n+k-2$, respectively. The vertex set of $U(m,n,k)$ is defined as follows:

$$V(U(m,n,k)) = \{x_i : 1 \le i \le m\} \cup \{y_i : 1 \le i \le n\} \cup \{z_i : 1 \le i \le k\}.$$

The edge set of $U(m,n,k)$ is defined as:

$$\begin{aligned} E(U(m,n,k)) \quad = \quad & \{x_i x_{i+1} : 1 \le i \le m-1\} \cup \{x_i y_1 : 1 \le i \le m\} \cup \{y_i y_{i+1} : 1 \le i \le n-1\} \\ & \{z_i y_n : 1 \le i \le k, \text{ when m is odd and n = m} - 1, \text{ or when m is even} \\ & \text{and n = m} - 2\} \cup \{z_i y_{n-1} : 1 \le i \le k, \text{ when m is odd and n = m, or} \\ & \text{when m is even and n = m} - 1\}. \end{aligned}$$

The labeling of $U(m,n,k)$ is defined in two cases depending upon m, by the bijection

$$f' : V(U(m,n,k)) \rightarrow \{1, 2, \ldots, m+n+k\}.$$

The labels of the vertices x_i and y_i of $U(m,n,k)$ under f' are the same as the labels of x_i and y_i under *super edge magic total labeling* (f) of $U(m,n)$ in theorem 4.2.1. The vertices of the star z_i are labeled as

$$f'(z_i) = m+n+i, \qquad 1 \le i \le k.$$

When z_i is adjacent to y_n, all the edge weights in this labeling function constitute a sequence of $|E(G)|$ consecutive integers:

$$\left\{ \lceil \frac{m}{2} \rceil + 2, \lceil \frac{m}{2} \rceil + 3, \ldots, \lceil \frac{m}{2} \rceil + 2m+n+k-1 \right\}.$$

Hence the graph $U(m,n,k)$ is SEMT, by using the lemma 3.2.1. The magic constant under this labeling is $\lceil \frac{m}{2} \rceil + 3m + 2n + 2k$.

When z_i is adjacent to y_{n-1}, the edge weights under the labeling function f' forms a sequence of $|E(G)|$ consecutive integers, which are:

$$\left\{ \frac{m}{2} + 2, \frac{m}{2} + 3, \ldots, \frac{m}{2} + 2m+n+k-1 \right\}.$$

And again by using the lemma 3.2.1, the graph $U(m,n,k)$ is SEMT. The magic constant under this labeling is $\frac{m}{2} + 3m + 2n + 2k$. □

Example 4.2.1. *The super edge magic total labeling of extended umbrella graph* $U(10,9,8)$ *is presented in figure 4.5. The magic constant of* $U(10,9,8)$ *under this labeling is* 69.

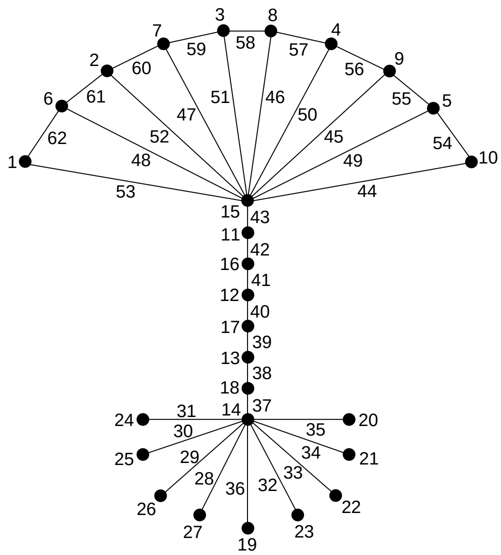


Figure 4.5: $U(10, 9, 8)$

4.3 Concluding remarks and open problems

The main objective of the work was an attempt to prove the *tree conjecture*; that all trees are super edge magic total. But in this thesis we just constructed a couple of graph classes and their super edge magic total labelings. We invite the readers to investigate:

- SEMT labeling of a forest of reflexive w-trees.
- SEMT labeling of disjoint union of umbrella and extended umbrella graphs.
- SEMT labeling of disjoint union of w-trees and umbrella graphs.

Appendix A – graph-theoretic symbols

$E(G)$	edge set of G
ε	size of G
$V(G)$	vertex set of G
n	order of G
$deg(v)$	degree of vertex v (in G)
$\delta(G)$	minimum degree of G
$N(S), N_G(S)$	neighborhood of set S in G
B_n	basket
C_n	cycle on n vertices
D_n	totally disconnected graph on n vertices
F_n	friendship graph with n triangles
f_n	fan with n blades
G^c	complement of G
K_n	complete graph on n vertices
$K_{m,n}$	complete bipartite graph with parts of cardinalities m and n
$K_{k[n]}$	regular complete k-partite graph of degree $(k-1)n$
M_n	Möbius ladder on n vertices
Q_n	n-dimensional cube
$L(G)$	line graph of G
P_n	path on n vertices
$P(n,k)$	generalized Petersen graph
$\mathcal{P}_{2n}$	prism
T	tree
W_n	wheel with n spokes
$\vec{G}$	digraph
mG	union of m disjoint copies of G
G^i	i-th power of G
$G \times H$	cartesian product of G and H
$G \circ H$	composition of G and H
$G \oplus H$	join of G and H
$G \cup H$	union of G and H
$G = F_1 \uplus F_2$	edge–disjoint factors of G
$G + e$	addition of e

Bibliography

[1] W.S. Andrews, *Magic squares and cubes*, Open Court Pub. Co., Chicago, 1917.

[2] M. Bača, F. Bertault, J.A. MacDougall, M. Miller, R.Simanjuntak and Slamin, *Vertex antimagic total labelings of graphs*, Discuss. Math. Graph Theory, **23** (2003), 67–83.

[3] M. Bača, Y. Lin, M. Miller, R. Simanjuntak, *New construction of magic and antimagic graph labelings*, Utilitas Math., **60** (2001), 229–239.

[4] M. Bača, M. Miller, *Super Edge Antimagic Graphs*, 2008.

[5] M. Bača, M. Miller, Slamin, *Vertex magic total labelings of generalized Petersen graphs*, Intern. J. Comput. Math., **79** (2002), 1259–1263.

[6] G.S. Bloom, S.W. Golomb, *Applications of numbered undirected graphs*, Proc. IEEE, **65** (1977), 562–570.

[7] G.S. Bloom, S.W. Golomb, *Numbered complete graphs, unusual rules, and assorted applications*, Theory and Applications of graphs, Lecture Notes in Math., **642**. Springer - Verlag (1978), 53–65.

[8] R. Bodendiek, G. Walther, *Aritmetisch antimagische Graphen*, K. Wagner and R. Bodendiek, ed., Graphentheorie III, BI-Wiss. Verl., Mannheim, 1993.

[9] R. Bodendiek, G. Walther, *On arithmetic antimagic edge labelings of graphs*, Mitt. Math. Ges. Hamburg, **17** (1998), 85–99.

[10] J.A. Bondy, U.S.R. Murty, *Graph theory with applications*, Department of Combinatorics and Optimization, University of Waterloo, 1976.

[11] W.C. Chen, H.I. Lü, and Y.N. Yeh, *Operations of interlaced trees and graceful trees*, Southeast Asian Bull. Math., **21** (1997), 337-348.

[12] N. Deo, *Graph Theory with applications to engineering and computer science*, Department of Computer Science, University of Central Florida, 2004.

[13] H. Enomoto, A.S. Llado, T. Nakamigawa, G. Ringel, *Super edge magic graphs*, SUTJ. Math., **2** (1998), 105–109.

[14] R. Figueroa-Centeno, R. Ichishima, F.A. Muntaner-Batle, *The place of super edge magic labelings among other classes of labelings*, Discrete Math., **231** (2001), 153–168.

[15] D. Fronček, P. Kovář, T. Kovářová, *Vertex magic total labeling of products of cycles*, Australas. J. Combin., in press.

[16] J.A. Gallian, *A dynamic survey of graph lebeling*, The Electronic Journal of Combinatorics, #05C78 (Nov. 2011 version).

[17] R.L. Graham, N.J.A. Sloan, *On additive bases and harmonious graphs*, SIAM J. Alg. Discrete Math., **1** (1980), 382–404.

[18] R.L. Graham, N.J.A. Sloan, *On constant weight codes and harmonious graphs*, Bell laboratories, Murray Hill.

[19] J.L. Gross, J. Yellen, *Graph Theory and its Applications*, Chapman and Hall/CRC, USA, 2006.

[20] N. Hartsfield, G. Ringel, *Pearls in Graph Theory*, Academic Press, Boston, 1990.

[21] M. Hussain, E.T. Baskoro, Slamin, *On super edge magic total labeling of banana trees*, Utilitus Math., (2009), in press.

[22] M. Javaid, M. Hussain, K. Ali, K.H. Dar, *Super edge-magic total labeling on w-trees*, Utilitas Math., **86** (2011), 183–191.

[23] M. Javaid, A. A. Bhatti, M. Hussain, K. Ali, *Super edge-magic total labeling on forest of extended w-trees*, Utilitas Math., in press.

[24] D. König, *Theorie der endlichen und unendlichen Graphen*. Akademische Verlagsgesellschaft, 1936 (reprinted Chelsea 1950).

[25] A. Kotzig, A. Rosa, *Magic valuations of finite graphs*, Canad. Math. Bull., **13** (1970), 451–461.

[26] A. Kotzig, A. Rosa, *Magic valuations of complete graphs*, Publ. CRM, **8** (1972), 175.

[27] K.W. Lih, B.Q. Shou and his Hun Yuan Polyhedra, *Philosophy and conceptual history of science in Taiwan*. Kluwer Academic Publishers, (1993), 209–220.

[28] J.A. MacDougall, M. Miller, Slamin, W.D. Wallis, *Vertex magic total labelings*, Utilitas Math., in press.

[29] J.A. MacDougall, M. Miller, W.D. Wallis, *Vertex magic total labelings of wheels and related graphs*, Utilitas Math., **62** (2002), 175–183.

[30] M.A. Malik, M.Y.H. Malik, M. Imran, On super edge magic total labeling of reflexive w-graphs, preprint.

[31] M. Miller, M. Bača, Y. Lin, *On two conjectures concerning (a, d)-antimagic labelings of antiprisms*, J. Combin. Math. Combin. Comput., **37** (2001), 251–254.

[32] S. Min-Lee, *On super edge-magic graphs with many odd cycles*, (with Alexander Nien-Tsu Lee), Congressus Numerantium, **163** (2003), 65–80.

[33] G. Ringel, A.S. Llado, *Another tree conjecture*, Bull. Inst. Combin. Appl., **18** (1996), 83–85.

[34] A. Rosa, *On certain valuations of the vertices of a graphs*, Theory of Graphs (International Symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris (1967), 349–355.

[35] J. Sedláček, Problem 27, *Theory of graphs and its applications*, (Smolenice, 1963), Publ. House Czechoslovak Acad. Sci., Prague, (1964), 163–164.

[36] R. Simanjuntak, F. Bertault, M. Miller, *Two new (a, d)-antimagic graph labelings*, Proc. of Eleventh Australian Workshop of Combinatorial Algorithms, (2000), 179–189.

[37] B.M. Stewart, *Magic Graphs*, Canad. J. Math., **18** (1966), 1031–1059.

[38] B.M. Stewart, *Supermagic complete graphs*, Canad. J. Math., **19** (1967), 427–438.

[39] W.D. Wallis, *Magic Graphs*, Brikhäuser, 2001.

[40] W.D. Wallis, E.T. Baskoro, M. Miller, Slamin, *Edge magic total labelings*, Austral. J. Combin., **22** (2000), 177–190.

[41] D.B. West, *An Introduction to Graph Theory*, Prentice - Hall, 1996.