Wave propagation in an isotropic incompressible plate

by

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Abstract

This dissertation deals with the study of elastic waves in compressible and incompressible isotropic materials. The propagation of elastic plane waves and the Rayleigh-Lamb dispersion relation for the elastic plate is discussed. Dispersion relation for Rayleigh-Lamb waves in a compressible isotropic plate are reintroduced using Helmholtz representation. Some earlier work of Ogden and Vinh [20] concerning the existence of a Rayleigh wave in an incompressible orthotropic half space is discussed. Also work of Hussain et al. [15] is reviewed in which Lamb modes for an incompressible isotropic plate is studied in which it is shown that plateau region does not exist in the spectrum and all the modes start off with negative slope and the slope retains its sign till the end. This fact is explained analytically.

In this dessertation, work of Ogden and Vinh [20] and Hussain et al. [15] is used and by using modified method dispersion relation is derived for Lamb modes in an isotropic plate under incompressibility constraint [15]. Shapes of dispersion curves are also plotted for an incompressible isotropic plate.

Dedicated to

My parents

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Contents

Chapter 1

Introduction

Waves which are contrived to follow a path, defined by the material walls of a structure, are known as guided waves and the structure is termed as a wave guide. A hollow metal pipe is a simple example of a wave guide. A wave whose amplitude decreases exponentially as we go down a half space is known as Rayleigh wave. For this reason such a wave is called a surface wave. When these waves are guided in layers they are referred to as Lamb waves, Rayleigh-Lamb waves, or generalized Rayleigh waves. Lamb waves propagate in solid plates. They are elastic waves whose particle motion lies in the plane that contains the direction of wave propagation and the plate normal (the direction perpendicular to the plate) [2].

Lamb modes in an infinite elastic isotropic plate were treated for the first time by Lord Rayleigh [10] and Lamb [11]. For an incompressible isotropic plate these waves behave in a manner different from those for compressible plate. Lamb modes for an isotropic compressible plate of thickness $2h$ can be divided into two systems of symmetric and anti-symmetric modes. Symmetric modes are symmetric with respect to the mid plane of the plate, whereas, Anti-symmetric modes are anti-symmetric with respect to the mid plane of the plate [12]. These modes are also known as flexural modes.

Wave propagation in solids is of interest in a number of engineering applications. Propagation of guided Rayleigh-Lamb waves in a plate is of interest in seismology, ultrasonic material characterization [13] and in electrical devices. Ultrasonic waves can also be used in medicine. For instance, a wave can propagate through human body, as it can play the role of a cylindrical wave guide, so fracture, thickness and other properties of the bone can be examined. Lamb waves are guided dispersive waves and these waves have great applications in Non-destructive testing of materials (NDT). Nondestructive testing (NDT) of materials is a branch of mechanical engineering, in which materials are being tested with out any damage, such as to detect cracks and fatigue. Lamb waves can be utilized for the Non-destructive testing (NDT) of the waveguide having plate like structure, the whole area of the plate can be examined because the stresses are produced all over the plate thickness. Thus, it is conceivable to find cracks inside the structure [14] [15].

The symmetry axes of an object are lines about which it can be rotated through some angle which brings the object to a new orientation which appears identical to its starting position. The symmetry planes of an object are imaginary mirrors in which it can be reflected while appearing unchanged. When a mirror is placed on a line of symmetry of a two dimensional shape and looked at from either side, the shapes look identical. In other words, each half of the shape is a mirror image of the other half. In a similar way, when a plane cuts a 3-D shape in two so that each half is a mirror image of the other half, the plane is called a plane of symmetry.

Elastic materials can be divided into eight classes on the basis of their symmetries. Let a, b, c denote the sides of a unit cell of a crystal and let α, β, γ denote the angles between these vectors so that γ is the angle between a and b etc. For an isotropic material every line is an axis of symmetry and every plane is a plane of symmetry.

The study of wave propagation in a plate has been the subject of great interest for long time [10] [16–19]. This thesis studies wave propagation in elastic solids, especially in plates. The thesis consists of five chapters, references, and appendixes.

In chapter 2, we studied basic definitions and concepts including waves in an isotropic compressible plate in which the dispersion relations for the symmetric and antisymmetric modes are obtained from the Rayleigh-Lamb frequency equations by using the Helmholtz displacement decomposition. These relations are illustrated as plots of frequency versus wave number. In chapter 3, Secular equation, existence and uniqueness of Rayleigh wave in an incompressible orthotropic elastic material is derived. Speed of Rayleigh wave is also calculated and it is found that at what point wave speed for orthotropic meet isotropic incompressible plate [20]. In chapter 4, we reviewed the work of Hussain et al. [15] in which the dispersion relation for an incompressible isotropic plate is derived. Zero-group velocity Lamb modes considered in chapter 4 are similar to the case of compressible materials [3], i-e,

1. Except the lowest S_0 mode, all modes asymptotically approach the line $C = C_T$.
2. The lowest mode asymptotically approaches the the line $C = C_R$.

However their behaviour is not similar in following points.

1. From the spectrum, the plateau region disappears.
2. Shapes of the curves is independent of the material.
3. No ZGV mode exist.

In chapter 5, we modify the work of Hussain et al. [15] i-e we used another method to find the dispersion relation of Lamb modes for an incompressible isotropic plate by taking a general solution of fourth order linear homogeneous equation and then draw the dispersion curves. Chapter 6 contains conclusion.

Most of the work presented in this thesis is a review of the following papers:

1. Ogden R.W. and Vinh P.C., On Rayleigh waves in incompressible orthotropic elastic solids, *J. Acoust. Soc. Am.* **115** (2004) 530-533.
2. T. Hussain, M.A. Awan, M. Shams and F. Ahmad, "Lamb modes for an isotropic incompressible plate", *Mathematical Problems in Engineering*, (2013).

Chapter 2

Preliminaries

This chapter contains some basic definitions and mathematical preliminaries. Waves and types of waves, Stress-strain relation, Fourth-order tensor, Isotropy and Rayleigh-Lamb dispersion relation are presented in this chapter. These definitions and concepts will be used throughout this dissertation.

2.1 Waves

A wave can be described as a disturbance(or oscillation) that passes through space and time accompanied by the transfer of energy.

2.1.1 Plane Wave

Plane wave is a constant-frequency wave which propagates along the direction of a vector $\mathbf{n}$ and a phase velocity $\mathbf{c}$. Conveniently, Plane harmonic wave is represented as

$$u = A\mathbf{p}e^{i\eta}, \tag{2.1.1}$$

where

$$\eta = k(\mathbf{x}.\mathbf{n} - \mathbf{c}t). \tag{2.1.2}$$

Equation (2.1.1) describes a plane wave propagating with phase velocity $\mathbf{c}$ in a direction of the unit propagation vector $\mathbf{n}$. There are two basic types of waves, i-e, longitudinal and transverse waves. In longitudinal waves, the particle is displaced in a direction parallel to the direction of wave propagation. While in transverse waves, the particle is displaced perpendicular to the direction of wave propagation.

2.1.2 Rayleigh and Lamb Waves

Rayleigh wave is a type of surface wave as it travel across surfaces. Close to the surface of mediums, Rayleigh waves are made of longitudinal and traverse waves that decreases exponentially in amplitude as distance from the surface increases. They waves can be used in non-destructive testing for detecting defects.

2.2 Phase and Group velocity

The velocity of a wave with which the phase of a wave propagate in a medium is called phase velocity. In general, the phase velocity is given by

$$v_p = \frac{\omega}{k},$$

where, ω and k are angular frequency and wave number given as

$$\omega = \frac{2\pi}{T}, \qquad k = \frac{2\pi}{\lambda}.$$

While, the velocity of a wave with which the overall structure of wave's amplitude propagates through medium is called group velocity. In general, it is given by

$$v_g = \frac{\partial \omega}{\partial k}.$$

2.3 Strain and Stress

Strain is a measure of change of shape or deformation of material after applying a force. In cartesian coordinates, the displacement of material points is denoted by $u(x_i, t)$, where $i = 1, 2$. The components of *strain tensor* can be expressed by the following relation

$$S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \qquad i, j = 1, 2 \tag{2.3.1}$$

where, $T_{ij} = T_{ji}$ is a symmetric tensor of order 2. In an orthonormal frame, *stress tensor* is

$$T_{ij} = \lim_{\Delta s_j \to 0}\left(\frac{\Delta F_i}{\Delta s_j}\right). \tag{2.3.2}$$

where, ΔF_i is the *i-th* component of the force $\mathbf{\Delta F}$ applied on the surface element Δs_j (perpendicular to the *j-axis*) by the medium. For the linearized theory of elasticity the stress tensor is symmetric, i.e, $T_{ij} = T_{ji}$ [2].

2.4 Generalized Hooke's Law

If a medium returns to its original state after the stress (e.g external forces) that deforms it, is removed then it is said to be elastic. The relative amount of deformation is called the strain. There is a *1-1* correspondence between stress and strain. Let us denote stress and strain tensors by T_{ij} and S_{ij}, respectively. Suppose that T_{ij} is a function of S_{ij}, i.e, $T_{ij}(S_{ij})$.The elastic behaviour of most of the substances is suitably described (for small deformations) by the Taylor series expansion:

$$T_{ij}(S_{kl}) = T_{ij}(0) + \frac{\partial T_{ij}}{\partial S_{kl}}|_{S_{kl}=0} S_{kl} + \frac{\partial^2 T_{ij}}{\partial S_{kl} \partial S_{mn}}|_{\substack{S_{kl}=0 \\ S_{mn}=0}} S_{kl} S_{mn} + ...$$

or, since $T_{ij}(0) = 0$, therefore

$$T_{ij} = c_{ijkl} S_{kl}, \tag{2.4.1}$$

where

$$c_{ijkl} = \frac{\partial T_{ij}}{\partial S_{kl}}|_{S_{kl}=0}. \tag{2.4.2}$$

Equation (2.4.1) is called generalized *Hook's Law*, where c_{ijkl} is an elasticity tensor. This relation between stress and strain was first stated by Hooke in the $17th$ century, for the case of a stretched elastic string.

2.5 Elasticity tensor

The generalized Hook's law is given by equation (2.4.1). The elasticity tensor c_{ijkl} has $3^4 = 81$ components which are called elastic constants or elastic parameters.

Since T_{ij} and S_{kl} are symmetric, it shows that elasticity tensor c_{ijkl} is symmetric with respect to first and second in two indices

$$c_{ijkl} = c_{jikl},$$

$$c_{ijkl} = c_{ijlk}.$$

In terms of displacement, *Hook's law* can be written as

$$T_{ij} = \frac{1}{2} c_{ijkl} [\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k}].$$

As two indices in $c_{ijkl} = c_{ijlk}$ are equal so,

$$T_{ij} = c_{ijkl} \frac{\partial u_l}{\partial x_k}.$$

Since c_{ijkl} has $3^4 = 81$ components but due to above symmetry relations it reduces to 36. The contracted notations are numbered from 1 to 6 which are as follows:

$$\begin{array}{ccc} (11) \longleftrightarrow 1 & (22) \longleftrightarrow 2 & (33) \longleftrightarrow 3 \\ (23) = (32) \longleftrightarrow 4 & (13) = (31) \longleftrightarrow 5 & (12) = (21) \longleftrightarrow 6 \end{array} \tag{2.5.1}$$

The transformation of contracted notations can be written in the following form [1]

$$\alpha = \begin{cases} i, & i = j \\ 9 - i - j, & i \neq j \end{cases}$$

$$\beta = \begin{cases} k, & k = l \\ 9 - k - l, & k \neq l \end{cases}$$

So the equation (2.4.1) becomes

$$T_{ij} = c_{\alpha\beta} S_{kl}. \tag{2.5.2}$$

By using this notation, the components of c_{ijkl} can be represented in the 6 order square matrix as

$$c_{\alpha\beta} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & c_{14} & c_{15} & c_{16} \\ c_{12} & c_{22} & c_{23} & c_{24} & c_{25} & c_{26} \\ c_{13} & c_{23} & c_{33} & c_{34} & c_{35} & c_{36} \\ c_{14} & c_{24} & c_{34} & c_{44} & c_{45} & c_{46} \\ c_{15} & c_{25} & c_{35} & c_{45} & c_{55} & c_{56} \\ c_{16} & c_{26} & c_{36} & c_{46} & c_{56} & c_{66} \end{pmatrix}. \tag{2.5.3}$$

2.6 Isotropy

Isotropic tensor is an elasticity tensor of order 4, i-e c_{ijkl}. A tensor which has the same components under all transformations of the reference frame is called an *isotropic* tensor. For instance, $\lambda\delta_{ij}$ is an *isotropic* tensor of order 2, where λ is scalar and $\delta_{ij}\delta_{kl}$, $\delta_{ik}\delta_{jl}$ and $\delta_{il}\delta_{jk}$ are all *isotropic* tensors of order 4. The elasticity tensor c_{ijkl} must be a linear combination of the above three forth-order tensors, which can be written as:

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu_1\delta_{ik}\delta_{jl} + \mu_2\delta_{il}\delta_{jk}, \tag{2.6.1}$$

where, λ and μ are Lamé constants. Interchanging i and j in the above equation and using $\delta_{ij} = \delta_{ji}$, we get

$$c_{jikl} = \lambda\delta_{ji}\delta_{kl} + \mu_1\delta_{jk}\delta_{il} + \mu_2\delta_{jl}\delta_{ik},$$

$$\Rightarrow \mu_1 = \mu_2 = \mu,$$

so,

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \tag{2.6.2}$$

where,

$$c_{ijkl} = c_{jikl}.$$

The matrix $[c_{\alpha\beta}]$ becomes symmetric i.e,

$$[c_{\alpha\beta}] = [c_{\beta\alpha}].$$

Putting equations (2.6.2) in (2.4.1), the Hook's Law takes the form

$$T_{ij} = \lambda S_{kk} + 2\mu S_{ij}. \tag{2.6.3}$$

From equation (2.6.2),

$$c_{11} = c_{22} = c_{33} = \lambda + 2\mu$$

$$c_{12} = c_{13} = c_{23} = \lambda$$

$$c_{44} = c_{55} = c_{66} = \mu$$

Similarly,

$$c_{14} = c_{15} = c_{16} = 0$$

$$c_{24} = c_{25} = c_{26} = 0$$

$$c_{34} = c_{35} = c_{36} = 0$$

$$c_{45} = c_{46} = c_{56} = 0$$

The matrix representation of elastic tensor is as follows

$$c_{\alpha\beta} = \begin{pmatrix} \lambda+2\mu & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & \lambda+2\mu & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & \lambda+2\mu & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{pmatrix}. \tag{2.6.4}$$

Where, λ and μ are *Lame's constants.* All 81 components can be expressed in terms of these two independent parameters.

2.7 Equation of motion for displacement $\overline{u}$

From the fundamental 2nd law of dynamics

$$F = ma, \tag{2.7.1}$$

where F is the forces, m is the mass and a is the acceleration. Eq. (2.7.1) can also be written as

$$\begin{aligned} &\sum F = ma, \\ &\text{body forces} + \text{surface forces} = m\frac{\partial^2 u_i}{\partial t^2}, \\ &\rho b_i + \frac{\partial T_{ij}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2}. \end{aligned}$$

where m is the mass density, b_i is the body force density.

By ignoring the body forces, above equation can be written as

$$\frac{\partial T_{ij}}{\partial x_j} = \rho\frac{\partial^2 u_i}{\partial t^2}. \tag{2.7.2}$$

By using the generalized Hooke's law in equation (2.7.2) we get the equation of motion as

$$C_{ijkl}\frac{\partial^2 u_l}{\partial x_j \partial x_k} = \rho\frac{\partial^2 u_i}{\partial t^2},$$

This implies

$$T_{ij} = \rho\frac{\partial^2 u_i}{\partial t^2}. \tag{2.7.3}$$

For isotropic material, *Hook's law* is given as

$$\begin{aligned}
T_{ij} &= \lambda S_{kk}\delta_{ij} + 2\mu S_{ij} \\
&= \lambda(\frac{\partial u_k}{\partial x_k})\delta_{ij} + \mu(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}) \\
&= \lambda(\frac{\partial^2 u_k}{\partial x_k \partial x_j})\delta_{ij} + \mu(\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_i \partial x_j}) \\
&= (\lambda + \mu)\frac{\partial}{\partial x_i}(\frac{\partial u_k}{\partial x_j})\delta_{ij} + \mu(\frac{\partial^2 u_i}{\partial x_j \partial x_j}) \\
T_{ij} &= (\lambda + \mu)\nabla(\nabla.\bar{u}) + \mu\nabla^2\bar{u}.
\end{aligned}$$

Putting value of $T_{ij,j}$ in equation (2.7.3),

$$(\lambda + \mu)\nabla(\nabla.\bar{u}) + \mu\nabla^2\bar{u} = \rho\ddot{u}. \tag{2.7.4}$$

This is called the equation of motion for displacement $\overline{u}$.

Assume a solution of the form

$$\bar{u} = \mathbf{p}f(\mathbf{x}.\mathbf{n} - vt), \tag{2.7.5}$$

Where, $\mathbf{n}$ is a wave vector and $\mathbf{p}$ is a polarization vector. Plugging value of $\bar{u}$ in equation (2.7.4) and solving, we get

$$(\lambda + \mu(p.n))\mathbf{n} + (\mu - v^2\rho)\mathbf{p} = 0.$$

Case (1) : When $\mathbf{p}$ and $\mathbf{n}$ are not parallel ($\mathbf{p}.\mathbf{n} = 0$)

$$\begin{aligned}
\lambda + \mu(p.n) &= 0 \\
\mu - v^2\rho &= 0
\end{aligned}$$

This implies,

$$v_T = \sqrt{\frac{\mu}{\rho}}. \tag{2.7.6}$$

The above equation shows the velocity of transverse wave in an isotropic medium.

Case (2) : When **p** and **n** are parallel ($\mathbf{p}.\mathbf{n} \neq 0$)

$$(\lambda + \mu(p.n) + \mu - v^2\rho)\mathbf{p} = 0$$
$$(2\mu + \lambda - v^2\rho) = 0$$

This implies,

$$v_L = \sqrt{\frac{\lambda + 2\mu}{\rho}}. \tag{2.7.7}$$

The above equation shows the velocity of longitudinal wave in an isotropic medium.

2.8 Displacement potentials

In the absence of body forces, the general displacement equation for an isotropic homogeneous material can be written as

$$(\lambda + \mu)\boldsymbol{\nabla}(\boldsymbol{\nabla}.\bar{u}) + \mu\boldsymbol{\nabla}^2\bar{u} = \rho\ddot{u},$$

where $\overline{u}$ is displacement.

According to Helmholtz theorem [2], a vector function **u** can be decomposed as the sum of the gradient of a scalar field and the curl of vector field

$$\mathbf{u} = \boldsymbol{\nabla}\phi + \boldsymbol{\nabla} \times \boldsymbol{\psi}, \tag{2.8.1}$$

Such that

$$\boldsymbol{\nabla}.\boldsymbol{\psi} = 0.$$

Where, ϕ is a scalar displacement potential and $\boldsymbol{\psi}$ is a vector displacement potential.

In Cartesian coordinates $\psi = \psi_x e_x + \psi_y e_y + \psi_z e_z$, and the Helmholtz decomposition will be of the form

$$u_1 = \frac{\partial\phi}{\partial x} + \frac{\partial\psi_z}{\partial y} - \frac{\partial\psi_y}{\partial z},$$
$$u_2 = \frac{\partial\phi}{\partial y} - \frac{\partial\psi_z}{\partial x} + \frac{\partial\psi_x}{\partial z},$$
$$u_3 = \frac{\partial\phi}{\partial z} + \frac{\partial\psi_y}{\partial x} - \frac{\partial\psi_x}{\partial y}.$$

Now, substituting Eqn. (2.8.1) in Eqn.(2.7.4) and using $\nabla.\nabla\phi = \nabla^2\phi$ and $\nabla.\nabla \times \psi = 0$, we get

$$(\lambda + \mu)\nabla(\nabla^2\varphi) + \mu\nabla^2(\nabla\phi + \nabla \times \psi) = \rho\frac{\partial^2}{\partial t^2}(\nabla\phi + \nabla \times \psi),$$

$$\nabla((\lambda + \mu + \mu)\nabla^2\phi - \rho\ddot{\phi}) + \nabla \times (\mu\nabla^2\psi - \rho\ddot{\psi}),$$

$$(\lambda + 2\mu)\nabla^2\phi - \rho\ddot{\phi} = 0, \tag{2.8.2}$$

$$\mu\nabla^2\psi - \rho\ddot{\psi} = 0. \tag{2.8.3}$$

Equations (2.8.2) and (2.8.3) gives

$$\nabla^2\phi = \frac{1}{c_L^2}\frac{\partial^2\phi}{\partial t^2}, \tag{2.8.4}$$

$$\nabla^2\psi = \frac{1}{c_T^2}\frac{\partial^2\psi}{\partial t^2}, \tag{2.8.5}$$

where,

$$c_L^2 = \frac{\lambda + 2\mu}{\rho},$$
$$c_T^2 = \frac{\mu}{\rho}.$$

Equations (2.8.4) and (2.8.5) are the equations for scalar and vector displacement potential. c_L and c_T are longitudinal and transverse wave velocities.
In cartesian coordinate system, equation for scalar displacement potential remains the same, while equation for vector displacement potential can be written as

$$\nabla^2\psi_1 = \frac{1}{c_T^2}\frac{\partial^2\psi_x}{\partial t^2}, \qquad \nabla^2\psi_2 = \frac{1}{c_T^2}\frac{\partial^2\psi_y}{\partial t^2}, \qquad \nabla^2\psi_3 = \frac{1}{c_T^2}\frac{\partial^2\psi_3}{\partial t^2}.$$

2.9 Guided waves in a compressible isotropic plate

A material is said to be in plane strain if there is no strain in z-direction (no change in thickness) but has stress and strain in xy-direction, i-e

$$u_1 = u_1(x_1, x_2, t),$$

$$u_2 = u_2(x_1, x_2, t),$$

$$u_3 = 0.$$

The scalar and vector potential relations are then written using equation (2.8.1)

$$u_1 = \frac{\partial \phi}{\partial x_1} + \frac{\partial \psi}{\partial x_2} \tag{2.9.1}$$

$$u_2 = \frac{\partial \phi}{\partial x_2} - \frac{\partial \psi}{\partial x_1} \tag{2.9.2}$$

For simplicity ψ_3 is taken as ψ in equations (2.9.1) and (2.9.2) provided that ϕ satisfies the equation (2.8.4)

$$\frac{\partial^2 \phi}{\partial x_1^2} + \frac{\partial^2 \phi}{\partial x_2^2} = \frac{1}{c_L^2}\frac{\partial^2 \phi}{\partial t^2}. \tag{2.9.3}$$

Similarly, ψ satisfy the equation (2.8.5)

$$\frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2} = \frac{1}{c_T^2}\frac{\partial^2 \psi}{\partial t^2}. \tag{2.9.4}$$

From *Hook's law*, the relevant components of stress tensor are

$$T_{21} = \mu(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}), \tag{2.9.5}$$

$$T_{22} = \lambda(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2}) + 2\mu(\frac{\partial u_2}{\partial x_2}). \tag{2.9.6}$$

For the wave motion in elastic layer, we consider solutions of equations (2.9.3) and (2.9.4) of the form

$$\phi = \Phi(x_2)e^{i(kx_1 - \omega t)}. \tag{2.9.7}$$

$$\psi = \Psi(x_2)exp[i(kx_1 - \omega t)]. \tag{2.9.8}$$

Putting equations (2.9.7) and (2.9.8) in (2.9.3) and (2.9.4), we get

$$\Phi'' + (\frac{c^2}{c_L^2} - 1)k^2\Phi = 0 \tag{2.9.9}$$

$$\Psi'' + (\frac{c^2}{c_L^2} - 1)k^2\Psi = 0 \tag{2.9.10}$$

Solving the above differential equations, the resulting solutions are given as

$$\Phi(x_1) = A_1 \sin(px_2) + A_2 \cos(px_2) \tag{2.9.11}$$

$$\Psi(x_2) = B_1 \sin(qx_2) + B_2 \cos(qx_2) \tag{2.9.12}$$

Since the exponential appears in all of expressions but it does not play any further role to determine the frequency equation, so it can be omitted, thus equations. (2.9.7) and (2.9.8) becomes (2.9.2)

$$\phi = [A_1 \sin(px_2) + A_2 \cos(px_2)]e^{i(kx_1 - \omega t)} \tag{2.9.13}$$

$$\psi = [B_1 \sin(qx_2) + B_2 \cos(qx_2)]e^{i(kx_1 - \omega t)} \tag{2.9.14}$$

Equations (2.9.1), (2.9.5) and (2.9.6) can be written as

$$u_1 = ik\Phi + \frac{d\Psi}{dx_2}, \tag{2.9.15}$$

$$u_2 = \frac{d\Phi}{dx_2} - ik\Psi, \tag{2.9.16}$$

$$T_{21} = \mu(2ik\frac{d\Phi}{dx_2} + k^2\Psi + \frac{d^2\Psi}{dx_2^2}), \tag{2.9.17}$$

$$T_{22} = \lambda(k^2\Phi + \frac{d^2\Phi}{dx_2^2}) + 2\mu(\frac{d^2\Phi}{dx_2^2} - ik\frac{d^2\Psi}{dx_2^2}). \tag{2.9.18}$$

From equations (2.9.18) and (2.9.19), it can be shown that the displacement components can be expressed in terms of elementary functions. For the displacement in x_1-direction the motion is symmetric (anti-symmetric) with respect to $x_2 = 0$, if u_1 holds cosines (sines). The displacement in the x_2-direction is symmetric (antisymmetric) if u_2 holds sines (cosines).The wave propagation modes in the elastic layer can be separated in two systems of symmetric and anti-symmetric modes, respectively.

Symmetric modes:

$$\Phi = A_2 \cos(px_2), \tag{2.9.19}$$

$$\Psi = B_1 \sin(qx_2), \tag{2.9.20}$$

$$u_1 = [A_2 \cos(px_2)ik + B_1 q \cos(qx_2)], \tag{2.9.21}$$

$$u_2 = [-A_2 p \sin(px_2) - B_1 ik \sin(qx_2)], \tag{2.9.22}$$

$$T_{21} = \mu[2ikp \sin(px_2) + (k^2 - q^2)B_1 \sin(qx_2)], \tag{2.9.23}$$

$$T_{22} = -\lambda(k^2 + p^2)A_2 \cos(px_2) - 2\mu[p^2 A_2 \cos(px_2) + ikqB_1 \cos(qx_2)]. \tag{2.9.24}$$

Anti-Symmetric modes:

$$\Phi = A_1 \sin(px_2), \tag{2.9.25}$$

$$\Psi = B_2 \cos(qx_2), \tag{2.9.26}$$

$$u_1 = [A_1 \sin(px_2) ik + B_2 q \sin(qx_2)], \tag{2.9.27}$$

$$u_2 = [A_1 p \cos(px_2) - B_2 ik \cos(qx_2)], \tag{2.9.28}$$

$$T_{21} = \mu[2ikpA_1 \cos(px_2) + (k^2 - q^2) B_2 \cos(qx_2)], \tag{2.9.29}$$

$$T_{22} = -\lambda(k^2 + p^2) A_1 \sin(px_2) - 2\mu[p^2 A_1 \sin(px_2) + ikqB_2 \sin(qx_2)]. \tag{2.9.30}$$

Now assume free boundaries at $x_2 = \pm h$, i.e.,

$$T_{21} = T_{22} = 0. \tag{2.9.31}$$

For the symmetric modes the boundary conditions gives a system of two homogeneous equations for the constants A_2 and B_1. Similarly, the two homogeneous equations for the constants A_1 and B_2. are obtained for anti-symmetric modes.Since, the systems are homogeneous, the determinant of the coefficient matrix must disappear, which gives the frequency equation. Therefore, for the symmetric modes, we have

$$\frac{-2\mu ikp \sin(ph)}{-\lambda(k^2 - (\lambda + 2\mu)p^2) \cos(ph)} = \frac{(k^2 - q^2) B_1 \sin(qh)}{-2\mu ikp \cos(qh)},$$

Above equation can be written as

$$\frac{\tan(qh)}{\tan(ph)} = \frac{4\mu k^2 pq}{[\lambda k^2 + (\lambda + 2\mu)p^2]}, \tag{2.9.32}$$

using

$$\frac{\mu}{\rho} = c_T^2, \qquad \frac{\lambda + 2\mu}{\rho} = c_L^2$$

we get,

$$\frac{\tan(qh)}{\tan(ph)} = \frac{4\mu k^2 pq}{(k^2 - q^2)[c_L^2 k^2 - 2k^2 c_T^2 + c_L^2 p^2]}, \tag{2.9.33}$$

Equation (2.9.33) simplifies to

$$\frac{\tan(qh)}{\tan(ph)} = \frac{-4k^2 pq}{(q^2 - k^2)^2}, \tag{2.9.34}$$

For the anti-symmetric modes the boundary conditions yield

$$\frac{(k^2 - q^2)\cos(qh)}{-2ikp\cos(qh)} = \frac{-2\mu ikq\sin(qh)}{(\lambda k^2 + \lambda p^2 + 2\mu p^2)\sin(ph)},$$

or

$$\frac{\tan(qh)}{\tan(ph)} = \frac{-(q^2 - k^2)^2}{4k^2pq}. \tag{2.9.35}$$

where,

$$p = \sqrt{\frac{\omega^2}{c_L^2} - k^2}$$

$$q = \sqrt{\frac{\omega^2}{c_T^2} - k^2}$$

Equations (2.9.34) and (2.9.35) are well known Rayleigh-Lamb frequency equations for free compressible isotropic plate. c_T and c_L, respectively denote the phase speed of transverse and longitudinal waves.

Equation for Symmetric modes explains the behaviour of all modes, anomalous or otherwise. By plotting the dispersion curves for phase velocity as a function of wave number or frequency,the normalized phase velocity with phase speed of transverse wave c_T is usually normalized. Some salient features of the spectrum are as follows [3].

1. No mode exist with $c < c_R$.

2. There is only one mode whose speed asymptotically approaches c_R.

3. A horizontal line above $c = c_T$ (including the line $c = c_L$) cannot be an asymptote to any of the modes.

4. Phase speed of all modes, except the lowest mode, approaches c_T as the wave number becomes very large.

In Fig 2.1, below, the spectrum of an isotropic steel plate has been plotted.

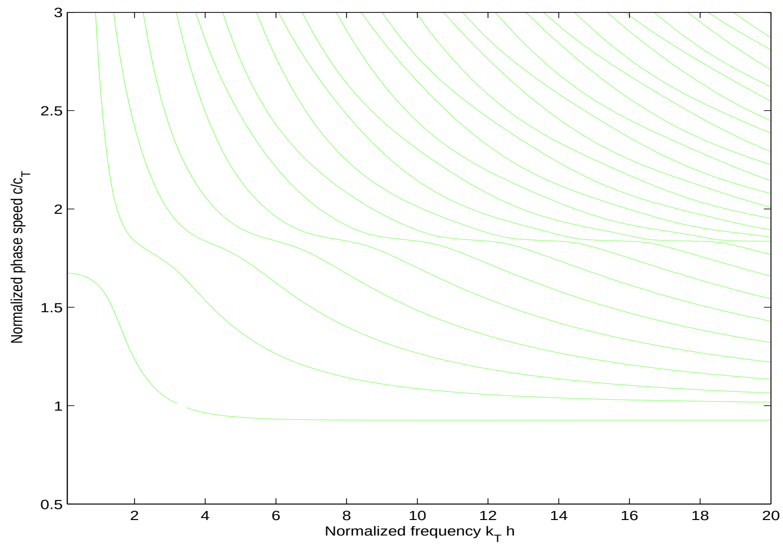

Figure 2.1: Symmetric Lamb modes on a steel plate ($k = 1.83$) showing phase velocity as a function of normalized frequency.

2.10 Anomalous Lamb mode Spectrum in an isotropic plate

A Lamb mode in the spectrum of a thin plate is said to be anomalous, in $w - c$ plane, if slope of the mode changes its sign. To study the anomalous behaviour of Lamb modes, we plot phase velocity, c, of a mode as a function of frequency, w, In the narrow range of frequency, in which change of slope occurs, phase velocity is double valued function and the point where the slope is undefined corresponds to zero-group velocity (ZGV) point of the Lamb mode. Due to the presence of ZGV point, there is a bulge in the mode of spectrum. Lamb modes with single ZGV point exist in the spectrum of an isotropic plate. Tolstoy and Usdin [5] were the first who predicted this anomalous behavior of modes in 1957. They studied this peculiar property of modes in an infinite plate having Poissons ratio, v, $\frac{1}{4}$. If the slopes at two points differ in sign, it will indicate that a zero group velocity point occurs there. And it is well known fact that the spectrum of the dispersion curves have zero-group velocity points (ZGV) corresponding to compressible isotropic plate.

Chapter 3

Rayleigh waves for an incompressible orthotropic plate

In this chapter, we reviewed [20] to describe Rayleigh waves for an incompressible isotropic and orthotropic elastic solids. The derivation of secular equation, existence, uniqueness and an explicit formula for the Rayleigh wave speed are discussed for orthotropic plate and hence specialized for an isotropic plate. Graph is plotted among wave speed $(\rho c^2/\gamma)$ and $\Delta(>0)$. It shows the dependence of the wave speed on the ratio of material constants i-e the wave speed is very small for small Δ and increases rapidly as Δ increases, reaching its isotropic value for $\Delta = 4$.

3.1 Orthotropic materials

Assume that we have three material axes of symmetry for an orthotropic incompressible medium, denoted by x_1, x_2 and x_3. Thus for an orthotropic material we have

$$\Delta c_{\alpha\beta} = \begin{pmatrix} c_{11} & c_{12} & c_{13} & 0 & 0 & 0 \\ c_{12} & c_{22} & c_{23} & 0 & 0 & 0 \\ c_{13} & c_{23} & c_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & c_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & c_{66} \end{pmatrix}. \tag{3.1.1}$$

Also the linear stress-strain relations for the material are

$$T_{11} = -p + c_{11}S_{11} + c_{12}S_{22} + c_{13}S_{33},$$

$$T_{22} = -p + c_{12}S_{11} + c_{22}S_{12} + c_{23}S_{33},$$

$$T_{33} = -p + c_{13}S_{11} + c_{23}S_{12} + c_{33}S_{33}, \qquad (3.1.2)$$

$$T_{23} = 2c_{44}S_{23},$$

$$T_{13} = 2c_{55}S_{13},$$

$$T_{12} = 2c_{66}S_{12}.$$

where $T's$, $S's$ and $c's$ denote the stress, strain and elasticity constant components respectively and p is the hydrostatic pressure.

Assume a plane wave motion in (x_1, x_2) plane and incompressibility which implies,

$$S_{13} = S_{23} = S_{33} = 0, \qquad (3.1.3)$$

and

$$S_{11} + S_{22} = 0. \qquad (3.1.4)$$

Orthotropic stress-strain relations (3.1.2) reduces due to above (3.1.4) conditions to

$$T_{11} = -p + (c_{11} - c_{12})S_{11},$$

$$T_{22} = -p + (c_{12} - c_{22})S_{11}, \qquad (3.1.5)$$

$$T_{12} = 2c_{66}S_{12}.$$

To get the positive strain energy function the following inequalities must be true

$$c_{66} \geq 0, \qquad c_{11} + c_{22} - 2c_{12} \geq 0 \qquad (3.1.6)$$

where c_{11}, c_{22}, c_{12} and c_{66} are material constants. From equation (3.1.4) we have

$$u_{1,1} + u_{2,2} = 0, \qquad (3.1.7)$$

which is satisfied by introducing a scalar function $\psi(x_1, x_2, t)$, such that

$$u_1 = \psi_{,2}, \qquad u_2 = -\psi_{,1}. \qquad (3.1.8)$$

The equation of motion by neglecting body forces is given as

$$T_{ij,j} = \rho \ddot{u}_i, \qquad i = 1, 2 \tag{3.1.9}$$

This implies,

$$T_{11,1} + T_{12,2} = \rho \ddot{u}_1, \tag{3.1.10}$$

$$T_{12,1} + T_{22,2} = \rho \ddot{u}_2 \tag{3.1.11}$$

using equations (3.1.5) and (3.1.8) in (3.1.10) yields

$$-p_{,1} + (c_{11} - c_{12} - c_{66}) + c_{66}\psi_{,222} = \rho\psi_{,2}, \tag{3.1.12}$$

similarly, equation (3.1.11) gives

$$p_{,2} + (c_{22} - c_{12} - c_{66}) + c_{66}\psi_{,111} = \rho\psi_{,1}, \tag{3.1.13}$$

In order to eliminate p, take partial derivative of (3.1.12) and (3.1.13) w.r.t x_1 and x_2 and then adding, we have

$$(c_{11} - 2c_{12} + c_{22})\psi_{,1122} - 2c_{66}\psi_{,1122} + c_{66}(\psi_{,1111} + \psi_{,2222}) = \rho\Delta\psi, \tag{3.1.14}$$

From equation (3.1.14), we have

$$\gamma\psi_{,1111} + 2\beta\psi_{,1122} + \gamma\psi_{,2222} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}). \tag{3.1.15}$$

where

$$2\beta = \delta - 2\gamma. \tag{3.1.16}$$

The traction free boundary conditions in terms of the stress components are written as

$$T_{21} = T_{22} = 0 \quad at \quad at \;\; x_2 = 0.$$

$$\gamma(\psi_{,22} - \psi_{,11}) = 0$$

$$\gamma(\psi_{,222} - \psi_{,112}) + \delta\psi_{,112} - \rho\psi_{,2} = 0 \quad at \quad x_2 = 0 \tag{3.1.17}$$

Now consider propagation of harmonic waves in the x_1 direction. We write ψ in the form

$$\psi = \phi(y)e^{ik(ct - x_1)} \tag{3.1.18}$$

Where c and is the velocity of a wave, k is a wave number, $y = kx_2$ and the function ϕ is to be determined.

Using ψ in equation (3.1.15) yields

$$k^4\gamma\phi - 2k^4\beta\phi'' + k^4\gamma\phi'''' = c^2\rho\left[\phi k^4 - k^4\phi''\right] \tag{3.1.19}$$

Hence we get

$$\gamma\phi^{iv} - (2\beta - \rho c^2)\phi'' + (\gamma - \rho c^2)\phi = 0 \tag{3.1.20}$$

and the boundary conditions given in (3.1.17) yields

$$\phi''(0) + \phi(0) = 0 \quad \text{since} \quad \gamma \neq 0, \tag{3.1.21}$$

$$\gamma\phi''(0) + (\gamma - \delta + \rho c^2)\phi'(0) = 0. \tag{3.1.22}$$

First we have to omit the factor γ on the assumption that $\gamma = 0$. Thus we have to solve equation (3.1.20) with the boundary conditions given in (3.1.21). Assume that the general solution for $\phi(y)$ that satisfies these boundary conditions is

$$\phi(y) = Pe^{(s_1 y)} + Qe^{(s_2 y)}, \tag{3.1.23}$$

where P and Q are constants, while s_1 and s_2 are the solutions of the equation

$$\gamma s^4 - (2\beta - \rho c^2)s^2 + (\gamma - \rho c^2) = 0, \tag{3.1.24}$$

From equation (3.1.24) it follows that

$$s_1^2 + s_2^2 = (2\beta - \rho c^2)/\gamma, \qquad s_1^2 s_2^2 = (\gamma - \rho c^2)/\gamma. \tag{3.1.25}$$

If the roots s_1^2 and s_2^2 are real of the quadratic equation (3.1.24), then they must be positive to guarantee that s_1 and s_2 can have a positive real part. If they are complex then they are conjugate. In that case the product $s_1^2 s_2^2$ must be positive and hence a real wave speed c which satisfies the inequalities

$$0 < \rho c^2 < \gamma.$$

Also $\rho c^2 = \gamma$ is a speed of shear body wave, not a surface wave.

Substituting equation (3.1.23) into the boundary conditions (3.1.21) and (3.1.22). we get the equations

$$(s_1^2 + 1)P + (s_2^2 + 1)Q = 0,$$

$$\gamma[(s_1^2+1)+\rho c^2-\delta]s_1 P+\gamma[(s_2^2+1)+\rho c^2-\delta]s_2 Q=0, \tag{3.1.26}$$

for P and Q. For non-trivial solution, the determinant of coefficients of the system (3.1.26) must be zero

$$\frac{1+s_1^2}{\gamma[(s_1^2+1)+\rho c^2-\delta]s_1}=\frac{1+s_2^2}{\gamma[(s_2^2+1)+\rho c^2-\delta]s_2},$$

After removing factor (s_1-s_2), this yields

$$\gamma(s_1^2+s_2^2+s_1^2s_1^2)+(\delta-\rho c^2)s_1s_2+\gamma-\delta+\rho c^2=0, \tag{3.1.27}$$

Use of equation (3.1.25) in equation (3.1.27) then leads to

$$(\delta-\rho c^2)\sqrt{1-\rho c^2/\gamma}-\rho c^2=0, \tag{3.1.28}$$

Equation (3.1.28) is the required secular equation for wave speed through ρc^2.

3.2 Existence and uniqueness of Rayleigh wave

Now we show that both inequalities $\gamma>0$ and $\delta>0$ guarantees the existence and uniqueness of a Rayleigh wave. For this purpose it is better to introduce the new variable $\eta=\sqrt{1-\rho c^2/\gamma}$ so that the secular equation (3.1.28) may be rewritten as

$$\begin{aligned} g(\eta) &= \frac{(\delta-\rho c^2)\eta}{\gamma}-\frac{\rho c^2}{\gamma}, \\ &= (\frac{\delta}{\gamma}-1+\eta^2)\eta-(1-\eta^2), \\ &= \frac{\delta\eta}{\gamma}-\eta+\eta^3-1+\eta^2, \end{aligned}$$

$$g(\eta)=\eta^3+\eta^2+(\delta/\gamma-1)\eta-1=0, \qquad 0<\eta<1. \tag{3.2.1}$$

Then

$$g(0)=-1<0, \qquad g(1)=\delta/\gamma>0, \tag{3.2.2}$$

which shows that equation (3.2.1) has at least one solution (by intermediate value theorem) in the interval (0,1).

Also

$$g'(\eta)=3\eta^2+2\eta+\delta/\gamma-1, \qquad g''(\eta)>0 \quad (\eta>0). \tag{3.2.3}$$

- If $\delta\geq\gamma$ then it follows that for $\eta>0$, $g'(\eta)>0$ and hence g is a monotonic increasing. In this case solution for η is unique.

- If $0 < \delta < \gamma$ then $g'(0) < 0$. Thus, g has a maximum for $\eta < 0$ and a minimum for $\eta > 0$.

By the inequality in equation (3.2.3) g therefore decreases to a minimum as η increases from 0, and therefore increases monotonically. So in this case the solution is also unique. Hence it is concluded that there exists a unique Rayleigh wave in an incompressible orthotropic elastic half-space in which the material constants satisfy the conditions (3.1.6), which guarantee positive definiteness of the strain-energy function for the considered plane strain restriction. Also it is noted that if $\delta \leq 0$ then the equation (3.1.28) has no real non-zero solution for c any sign of γ, although it is not physically meaningful to consider non positive values of these constants.

3.3 Wave speed formula

An explicit formula for wave speed is derived in this section, given that $\gamma > 0$ and $\delta > 0$, by finding the unique root, η_0 say, of equation (3.2.1) in the interval $(0, 1)$. The wave speed c is given by

$$\rho c^2 = \gamma(1 - \eta_0^2). \tag{3.3.1}$$

Now we show that the cubic equation (3.2.1) has only one real root, η_0, the other two are complex.
According to the theory of cubic equation, the nature of the three roots of the cubic i-e [21]

$$\eta^3 + b_2\eta^2 + b_1\eta + b_0 = 0, \tag{3.3.2}$$

is determined by the sign of the discriminant D defined by

$$D = R^2 + Q^3, \tag{3.3.3}$$

where R and Q are given in terms of the coefficients b_0, b_1, b_2 by

$$R = -\frac{1}{54}(9b_1b_2 - 27b_0 - 2b_2^3), \qquad Q = \frac{1}{9}(3b_1 - b_2^2), \tag{3.3.4}$$

Real Coefficients

1. If $D > 0$, one root of equation (3.3.2) real and two are complex conjugates.

2. If $D = 0$, the equation has three real roots , at least two of which are equal.

3. If $D < 0$, equation (3.3.2) has three distinct real roots.

In the first case, $(D > 0)$ the single root η_0 is given by Cardano's formula [21] in the form

$$\eta_0 = -\frac{1}{3}b_2 + (R + \sqrt{D})^{1/3} + (R - \sqrt{D})^{1/3}. \tag{3.3.5}$$

By comparing equations (3.2.1) and (3.3.2), we have

$$b_0 = -1, \qquad b_1 = \Delta - 1, \qquad b_2 = 1, \tag{3.3.6}$$

and hence

$$R = \frac{\Delta}{6} + \frac{8}{27}, \qquad Q = \frac{\Delta}{3} - \frac{4}{9}, \tag{3.3.7}$$

where $\Delta = \delta/\gamma$. Using equation (3.3.6) in (3.3.3) ,

$$\begin{aligned} D &= \left(\frac{\Delta}{6} + \frac{8}{27}\right)^2 + \left(\frac{\Delta}{3} - \frac{4}{9}\right)^3, \\ &= \frac{\Delta^2}{36} + \frac{64}{729} + \frac{\Delta}{3}.\frac{8}{27} + \left(\frac{\Delta^3}{27} - \frac{64}{729} - \frac{\Delta^2}{3}.\frac{4}{9} + \frac{16\Delta}{81}\right), \\ &= \frac{\Delta^2}{36} + \frac{8\Delta}{81} + \frac{16\Delta}{81} + \frac{\Delta^3}{27} - \frac{4\Delta^2}{27}, \end{aligned}$$

it gives

$$D = \frac{1}{108}\Delta(4\Delta^2 - 13\Delta + 32), \tag{3.3.8}$$

Since $\Delta > 0$, so it is clear from equation (3.3.8) that $D > 0$. So, equation (3.3.2) has only one real root, necessarily with in the range of values required.

Use of equations (3.3.6), (3.3.7) and (3.3.8) in (3.3.5) leads to

$$\begin{aligned} \eta_0 = \frac{1}{3}[-1 &+ \sqrt[3]{[9\Delta + 16 + 3\sqrt{3}\sqrt{\Delta(4\Delta^2 - 13\Delta + 32)}]/2} \\ &+ \sqrt[3]{[9\Delta + 16 - 3\sqrt{3}\sqrt{\Delta(4\Delta^2 - 13\Delta + 32)}]/2}], \end{aligned} \tag{3.3.9}$$

From equations (3.3.1) and (3.3.8) the speed c of Rayleigh wave is

$$\begin{aligned} \rho c^2/\gamma = 1 - \frac{1}{9}[-1 &+ \sqrt[3]{[9\Delta + 16 + 3\sqrt{3}\sqrt{\Delta(4\Delta^2 - 13\Delta + 32)}]/2} \\ &+ \sqrt[3]{[9\Delta + 16 - 3\sqrt{3}\sqrt{\Delta(4\Delta^2 - 13\Delta + 32)}]/2}]. \end{aligned} \tag{3.3.10}$$

From equation (3.3.10), plot of $\rho c^2/\gamma$ against $\Delta(> 0)$ is shown in Fig. 3.1

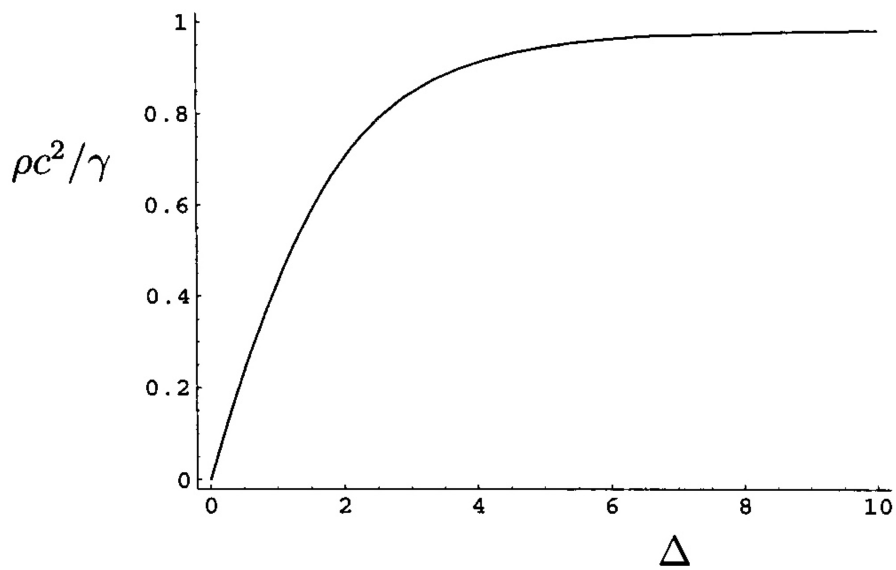

Figure 3.1: Plot of $\rho c^2/\gamma$ against $\Delta(>0)$

Isotropic value for Δ

For an (incompressible) isotropic material $c_{11} = c_{22}$, $c_{11} - c_{12} = 2\mu$, and $c_{66} = \mu$, where μ is the classical shear modulus, and hence, by equation (3.1.6), $\Delta = 4$. Substituting value of $\Delta(>0)$ in equation (3.3.10) yields

$$\rho c^2/\gamma = 1 - \frac{1}{9}\left[\sqrt[3]{6\sqrt{33}+26} - \sqrt[3]{6\sqrt{33}+26} - 1\right]^2. \qquad (3.3.11)$$

This is approximately 0.9126, which is the classical value for an incompressible elastic solid [22].

In order to illustrate the dependence of the wave speed on the ratio of material constants, a plot of $\rho c^2/\gamma$ against $\Delta(>0)$ based on equation (3.3.11) is shown in Fig. 3.1. For a small Δ wave speed is very small and as Δ increases it increases rapidly and reaches its isotropic value for $\Delta = 4$ and then approaching an asymptotic value for $\rho c^2/\gamma \to 1$ as Δ becomes very large. It is noted that Δ may be interpreted as a shear modulus of the material; indeed, in the isotropic case $\Delta = 2\mu$, where μ is the Lame' shear modulus. Thus, the limit $\Delta \to 0$, (that is not applicable for isotropic materials) corresponds to a material with one very small shear modulus. In a similar manner, γ is a shear modulus and, if $\delta \neq 0$ in the limit $\gamma \to 0$ we have $\Delta \to \infty$. Thus, we have interpretations for the two extreme values of Δ [20].

Chapter 4

Lamb Modes for an incompressible isotropic plate

This chapter is concerned with the behaviour of Lamb modes for an isotropic plate using incompressibility condition.

We review [15] in which Hussain et al. discussed Lamb modes for an isotropic incompressible plate. Here we have considered the dispersion relation for an isotropic plate under the constraint of incompressibility. As a result, a single parameter c_T, phase speed of transverse wave is needed to plot the dispersion curves, hence the plateau region disappears in the spectrum. Then the dispersion curves, corresponding to the given dispersion relation, are plotted.

4.1 Symmetric dispersion relation for an incompressible isotropic plate

If the volume of a material does not change, it is said to be incompressible, thus, it can hold only isochoric deformation.

Consider an isotropic plate having thickness $2h$. In a cartesian coordinate system, x_1 x_2 x_3, the propagation of wave is along x_1-direction, where, x_2-axis is normal to the plate surface. Consider (x_1, x_2) as plane of motion, the displacement components (u_1, u_2, u_3) are such that

$$u_i = u_i(x_1, x_2, t), \qquad i = 1, 2 \quad u_3 = 0 \tag{4.1.1}$$

The linearized incompressibility condition using $u_3 = 0$, is given as

$$u_{1,1} + u_{2,2} = 0, \tag{4.1.2}$$

From equation (4.1.2) we can have a scalar function $\psi(x_1, x_2, t)$, such that

$$u_1 = \psi_{,2}, \qquad u_2 = -\psi_{,1}. \tag{4.1.3}$$

Constitutive relation for an isotropic elastic material is given by

$$T_{ij} = -p\delta_{ij} + 2\mu S_{ij}. \tag{4.1.4}$$

where, T_{ij} is stress tensor and S_{ij} is a strain tensor. p is the arbitrary hydrostatic pressure associated with the incompressibility constraint.

Relevant components of stress tensor are given by

$$\begin{aligned} T_{11} &= -p + 2\mu u_{1,1}, \\ T_{22} &= -p + 2\mu u_{2,2}, \\ T_{12} &= \mu(u_{1,2} + u_{2,1}). \end{aligned} \tag{4.1.5}$$

Now, As we know that force at a point due to stresses in the continuum is given by

$$\frac{\partial T_{ij}}{\partial x_j} + \rho F_i$$

The above equation must be equal to acceleration

$$\frac{\partial T_{ij}}{\partial x_j} + \rho F_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

so, the equation of motion in the absence of body force is given as

$$\frac{\partial T_{ij}}{\partial x_j} = \rho \frac{\partial^2 u_i}{\partial t^2}, \qquad i = 1, 2$$

or

$$T_{ij,j} = \rho \ddot{u}_i, \qquad i = 1, 2 \tag{4.1.6}$$

This implies,

$$T_{11,1} + T_{12,2} = \rho \ddot{u}_1 \tag{4.1.7}$$

$$T_{12,1} + T_{22,2} = \rho \ddot{u}_2 \tag{4.1.8}$$

Putting equations (4.1.3) and (4.1.5) in (3.1.7), it gives

$$\mu(\psi_{,211} + \psi_{,222}) - p_{,1} = \rho\ddot{\psi}_{,2} \tag{4.1.9}$$

Similarly, equation (3.1.8) gives

$$-\mu(\psi_{,122} + \psi_{,111}) - p_{,2} = -\rho\ddot{\psi}_{,1} \tag{4.1.10}$$

In order to eliminate p, differentiate equation (4.1.9) w.r.t x_2 and equation (4.1.10) w.r.t x_1 and then subtracting, we get

$$\mu(\psi_{,1111} + \psi_{,2222}) + 2\mu\psi_{,1122} = \rho(\ddot{\psi}_{,11} + \ddot{\psi}_{,22}) \tag{4.1.11}$$

We consider the propagation of harmonic waves in x_1 direction, so we can write ψ in the form

$$\psi = \phi(y)e^{ik(ct-x_1)} \tag{4.1.12}$$

Where c is the velocity of a wave, k is a wave number, $y = kx_2$ and the ϕ function is to be determined.

Using ψ in equation (4.1.11) yields

$$\mu[\phi(y)(ik)^4 + \phi^{(iv)}(y)k^4] + 2\mu[-\phi''(y)k^4] = \rho[\phi(y)k^4c^2 - \phi''(y)k^4c^2],$$

$$\mu\phi(y)(k)^4 + \mu\phi^{(iv)}(y)k^4 - 2\mu\phi''(y)k^4 = \rho\phi(y)k^4c^2 - \rho\phi''(y)k^4c^2,$$

$$\mu\phi(y) - \rho c^2\phi(y) + \mu\phi^{(iv)}(y) - 2\mu\phi''(y) + \rho c^2\phi''(y) = 0,$$

and hence,

$$\mu\phi^{(iv)}(y) - (2\mu - \rho c^2)\phi''(y) + (\mu - \rho c^2)\phi(y) = 0. \tag{4.1.13}$$

Boundary conditions are given by

$$T_{12} = T_{22} = 0 \qquad on \quad x_2 = \pm h \tag{4.1.14}$$

These conditions, in terms of ψ are

$$\mu(\psi_{,22} - \psi_{,11}) = 0 \tag{4.1.15}$$

$$(3\mu - \rho c^2)k^2\psi_{,2} - \mu\psi_{,222} = 0 \tag{4.1.16}$$

We assume general solution for ϕ that satisfies boundary conditions are as follows

$$\phi(y) = P\sin(s_1 y) + Q\sin(s_2 y) \tag{4.1.17}$$

where P and Q are constants, while s_1^2 and s_2^2 are roots of the following quadratic equation

$$s^4 + (2 - \frac{\rho c^2}{\mu})s^2 + (1 - \frac{\rho c^2}{\mu}) = 0, \tag{4.1.18}$$

where,

$$s_1^2 = -1, \qquad s_2^2 = \frac{\rho c^2}{\mu} - 1 \tag{4.1.19}$$

Using ψ and equation (4.1.5) in equations (4.1.15) and (4.1.16), which gives

$$A\sin(s_1 kh)(1 - s_1^2) + B\sin(s_2 kh)(1 - s_2^2) = 0, \tag{4.1.20}$$

$$s_1 P(3\mu - \rho c^2 + \mu s_1^2)\cos(s_1 kh) + s_2 Q(3\mu - \rho c^2 + \mu s_2^2)\cos(s_2 kh) = 0, \tag{4.1.21}$$

For a non trivial solution the determinant of the above system of equations must be zero. So we have

$$\frac{\sin(s_1 kh)(1 - s_1^2)}{\cos(s_1 kh)s_1(3\mu - \rho c^2 + \mu s_1^2)} = \frac{\sin(s_2 kh)(1 - s_2^2)}{\cos(s_2 kh)s_2(3\mu - \rho c^2 + \mu s_2^2)},$$

By simplifying, we get

$$\frac{\tan(s_1 kh)}{\tan(s_2 kh)} = \frac{s_1(1 - s_2^2)^2}{s_2(1 - s_1^2)^2}. \tag{4.1.22}$$

This is a symmetric dispersion relation for an isotropic incompressible plate.

Using equation (4.1.19) in (4.1.22) yields

$$\frac{\tan(\sqrt{-1}kh)}{\tan(\sqrt{\frac{\rho c^2}{\mu} - 1}kh)} = \frac{\sqrt{-1}(2 - \frac{\rho c^2}{\mu})^2}{2^2(\sqrt{\frac{\rho c^2}{\mu} - 1})}$$

$$\frac{\tan(\sqrt{-1}x)}{\tan(\sqrt{y^2 - 1}x)} = \frac{\sqrt{-1}(2 - y^2)^2}{4\sqrt{y^2 - 1}}$$

$$\frac{\tan(ix)}{\tan(\sqrt{y^2-1}x)} = \frac{i(2-y^2)^2}{4\sqrt{y^2-1}}$$

and hence,

$$\frac{\tanh(x)}{\tan(\sqrt{y^2-1}x)} = \frac{(2-y^2)^2}{4\sqrt{y^2-1}}$$

or

$$\frac{\tan(\sqrt{y^2-1}x)}{\tanh(x)} = \frac{4\sqrt{y^2-1}}{(2-y^2)^2} \tag{4.1.23}$$

Where the dimensionless wave number hk is defined by x and normalized velocity $\frac{c}{c_T}$ ($c_T = \sqrt{\frac{\mu}{\rho}}$) by y.

Equation (4.1.23) depends only on one parameter which is independent of c_L, hence any analytic expression for slope at $c = c_L$ can not be obtained. This indication shows the absence of plateau region from the spectrum.

4.2 Dispersion curves

The dispersion curves for an isotropic plate under incompressibility, in the $k-c$ plane, are shown in Fig. 4.1 [24].

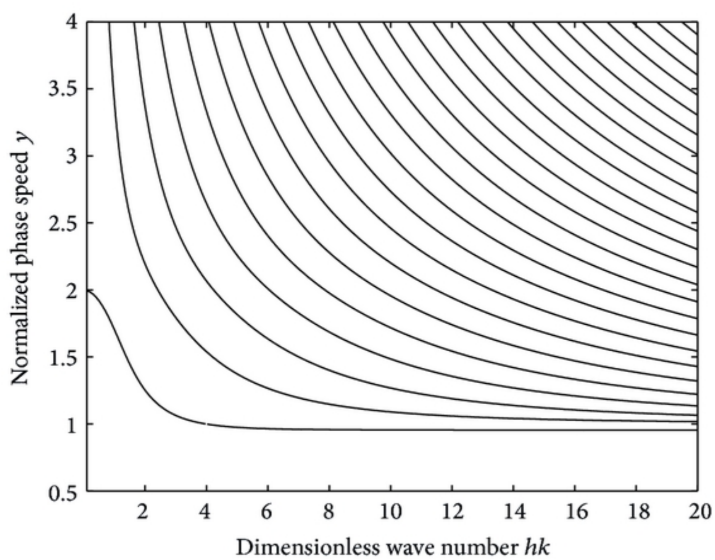

Figure 4.1: Symmetric Lamb modes for an isotropic plate, under incompressibility, in k-c plane.

Except the lowest S_0 mode, all curves approach the line $y = 1$ corresponding $c = c_T$. It can be concluded that Fig. 4.1 represent dispersion curves for all isotropic incompressible materials, it means that the shape of dispersion curves does not depend on material properties.

When $0 < c < c_T$, s_2^2 become negative, hence equation (4.1.23) can be written as follows

$$\frac{\tan(\sqrt{1-y^2}x)}{\tanh(x)} = \frac{4\sqrt{1-y^2}}{(2-y^2)^2},$$

Since $|\tanh(x)| \leq 1$, thus for large wave number, we can write

$$\frac{4\sqrt{1-y^2}}{(2-y^2)^2} = 1,$$

$$4\sqrt{1-y^2} - (2-y^2)^2 = 0,$$

This last expression is the same as equation (3.2.8) of Dowaikh [23], it has only one root in interval (0, 1), which is 0.9553125, hence the phase speed of S_0 mode approaches the speed, $c_R = 0.9553125c_T$, which is the speed of the Rayleigh wave.

4.3 Anomalous dispersion curves in case of Incompressibility

It is well known [4–8] that the Lamb modes for an isotropic plate under compressibility in $w - c$ plane exhibit a phenomenon known as "anomalous dispersion."

A question naturally arises whether the anomaly persists even when the incompressibility constraint is enforced. To verify this define

$$u = \frac{\omega h}{c_T} = \frac{c.kh}{c_T} = yx$$

Putting this value in equation (4.1.23), the equation, in terms of u and y becomes

$$\frac{\tan(\sqrt{y^2-1}u/y)}{\tanh(u/y)} = \frac{4\sqrt{y^2-1}}{(2-y^2)^2} \tag{4.3.1}$$

Where u is the normalized frequency and y is the normalized phase speed. Dispersion curve corresponds to equation (4.3.1) is shown in Fig. 4.2 [24].

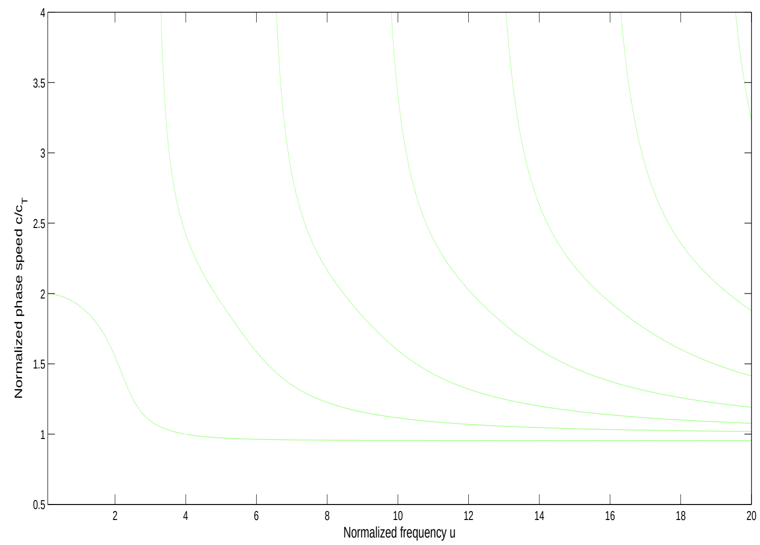

Figure 4.2: Symmetric Lamb modes for an isotropic plate, under incompressibility, in u-y plane.

From Figure 4.2, it becomes clear that anomalous behaviour of S_1 mode has disappeared from the spectrum. Analytically this phenomenon can be explained by examining the slopes of all modes, first for large phase velocity, $y >> 1$, secondly when $y \to 1^+$. We rewrite equation (4.3.1) in the form

$$h(u, y) = \tan(\sqrt{y^2 - 1}u/y)(2 - y^2)^2 - 4\tanh(u/y)\sqrt{y^2 - 1} = 0 \tag{4.3.2}$$

For $y >> 1$, we have

$$\sqrt{y^2 - 1} \simeq y,$$

$$(2 - y^2)^2 \simeq (y^2)^2,$$

$$y^4 \tan(u) - 4y \tanh(u/y) \simeq 0,$$

$$4\tanh(u/y) \simeq y^3 \tan(u),$$

$$\tan(u) = \frac{4\tanh(u/y)}{y^3},$$

Since for large y

$$\tanh(u/y) \simeq u/y,$$

hence

$$\tan(u) \simeq \frac{4u}{y^4}, \tag{4.3.3}$$

or

$$\tan(u) \simeq 0,$$

$$u_n = n\pi + \epsilon, \qquad n = 0, 1, 2, 3, \ldots..,$$

where ϵ is an infinitesimally small positive number. To find the slope, $\frac{\mathrm{d}y}{\mathrm{d}u}$, of all modes, following formula is used

$$\frac{\mathrm{d}y}{\mathrm{d}u} = -\frac{\partial h/\partial u}{\partial h/\partial y} \tag{4.3.4}$$

For large value of y, the partial derivatives $\partial h/\partial u$ and $\partial h/\partial y$ can be as follows,

$$\frac{\partial h}{\partial u} \simeq -iy^4 \cos(u) - iy^2 u \sin(u)$$

$$\frac{\partial h}{\partial y} \simeq -iuy \cos(u) - 4iy^3 u \sin(u)$$

Plugging the above relations in equation (4.3.4)

$$\frac{\mathrm{d}y}{\mathrm{d}u} \simeq -\frac{iy^4 \cos(u) + iy^2 u \sin(u)}{iuy \cos(u) + 4iy^3 u \sin(u)}$$

$$\simeq -\frac{y^3 + yu \tan(u)}{u + 4y^2 \tan(u)} \tag{4.3.5}$$

using value of $\tan(u)$, from equation (4.3.3), in above equation (4.3.5) we get,

$$\frac{\mathrm{d}y}{\mathrm{d}u}\Big|_{u=u_n} \simeq -\frac{y^3}{u_n}, \tag{4.3.6}$$

It shows that

$$\frac{\mathrm{d}y}{\mathrm{d}u} < 0, \tag{4.3.7}$$

for all modes.

Now we have to find the approximation when phase velocity, y, is small and we will show that $\frac{\mathrm{d}y}{\mathrm{d}u} < 0$ for small y.

Let

$$y^2 = 1 + \epsilon^2,$$

Plugging the above value in equation (4.3.1)

$$\frac{\tan(\epsilon u)}{\tanh(u)} \simeq \frac{4\epsilon}{(1 - \epsilon^2)^2},$$

$$\tan(\epsilon u) \simeq \frac{4\epsilon \tanh(u)}{(1-\epsilon^2)^2}, \tag{4.3.8}$$

By taking partial derivatives of $\frac{\partial h}{\partial u}$ and $\frac{\partial h}{\partial y}$, for $y^2 = 1+\epsilon^2$, it becomes

$$\frac{\partial h}{\partial u} \simeq 3\epsilon \cos(u\epsilon) \cos h(u) - \sin(u\epsilon) \sinh(u), \tag{4.3.9}$$

$$\frac{\partial h}{\partial y} \simeq \frac{u}{\epsilon} \cos(u\epsilon) \cosh(u) + 4 \sin(u\epsilon) \cosh(u) - 3u \sin(u\epsilon) \sinh(u) - 3u\epsilon \cos(u\epsilon) \cosh(u), \tag{4.3.10}$$

By inserting above three Equations and using $\tanh(u) \to 1$ in equation (4.3.4), it becomes

$$\frac{\mathrm{d}y}{\mathrm{d}u} \simeq \frac{\epsilon^2}{-u_n + 4}, \tag{4.3.11}$$

It clearly shows that

$$\frac{\mathrm{d}y}{\mathrm{d}u} < 0, \tag{4.3.12}$$

for all modes when $u_n > 4$, $n = 1, 2, 3, 4, ...$ an assumption which can be made.

Hence from equation (4.3.6) and (4.3.11), we can infer that there is no anomalous dispersion in the spectrum of an incompressible isotropic plate.

Chapter 5

Modified method for Lamb modes in an incompressible isotropic plate

In this chapter we reviewed the work of Hussain et al. [15] and used another method to derive the dispersion relation of Lamb modes for an isotropic plate under incompressibility. In this method the general solution of fourth order homogeneous linear ordinary differential equation (4.1.13) is obtained and then used its particular cases (solutions) in the derivation of Symmetric and anti-symmetric dispersion relations. Dispersion curves for both symmetric and anti-symmetric cases are also plotted among normalized frequency and phase velocity.

5.1 Frequency equations and Boundary conditions

We consider the fourth order homogeneous linear ordinary differential equation (4.1.13)

$$\mu\phi^{(iv)}(y) - (2\mu - \rho c^2)\phi''(y) + (\mu - \rho c^2)\phi(y) = 0. \tag{5.1.1}$$

It is well known that the solution of of above equation can be written in the form

$$\phi(y) = e^{my}. \tag{5.1.2}$$

Now by taking derivatives of equation (5.1.2)

$$\phi'(y) = me^{my}, \qquad \phi''(y) = m^2 e^{my},$$
$$\phi'''(y) = m^3 e^{my} \qquad \phi^{(iv)}(y) = m^4 e^{my}.$$

So equation (5.1.1) becomes

$$\mu m^4 e^{my} - (2\mu - \rho c^2)m^2 e^{my} + (\mu - \rho c^2)e^{my} = 0,$$

$$[\mu m^4 - (2\mu - \rho c^2)m^2 + (\mu - \rho c^2)m]e^{my} = 0,$$

since $e^{my} \neq 0$ so,

$$\mu m^4 - (2\mu - \rho c^2)m^2 + (\mu - \rho c^2) = 0,$$
$$\mu[m^4 + (\frac{\rho c^2}{\mu} - 2)m^2 + (1 - \frac{\rho c^2}{\mu})] = 0$$
$$m^4 + (\frac{\rho c^2}{\mu} - 2)m^2 + (1 - \frac{\rho c^2}{\mu}) = 0.$$
$$(m+1)(m-1)(m^2 + \frac{\rho c^2}{\mu} - 1) = 0, \tag{5.1.3}$$

Therefore the four roots of the auxiliary equation (5.1.3) are

$$m = \pm 1, \pm\sqrt{1 - \frac{\rho c^2}{\mu}}.$$

or

$$m = \pm 1, \pm q, \text{ where } q = \sqrt{1 - \frac{\rho c^2}{\mu}}.$$

Now we are able to write the general solution which satisfy the boundary conditions (4.1.15) and (4.1.16). Due to the nature of the last two roots of the auxiliary equation (5.1.3), the general solution can be written in three different ways and we will discuss each one separately.

CASE 1: $1 - \frac{\rho c^2}{\mu} > 0 \Rightarrow \ \mu > \rho c^2$

In this case the last two roots of the auxiliary equation (5.1.3) become real and distinct and hence the general solution $\phi(y)$ of the equation (5.1.1) or (4.1.13) can be written in the form

$$\phi(y) = C_1 e^{y} + C_2 e^{-y} + C_3 e^{qy} + C_4 e^{-qy}. \tag{5.1.4}$$

Now by using hyperbolic functions we can write the following relations

$$e^{y} = \cosh(y) + \sinh(y),$$

$$e^{-y} = \cosh(y) - \sinh(y),$$

$$e^{qy} = \cosh(qy) + \sin h(qy),$$

$$e^{-qy} = \cosh(qy) - \sin h(qy).$$

Hence the solution (5.1.4) becomes

$$\phi(y) = A\sinh(y) + B\cosh(y) + C\sinh(qy) + D\cosh(qy), \tag{5.1.5}$$

where

$$(C_1 - C_2) = A, (C_1 + C_2) = B, (C_3 - C_4) = C \text{ and } (C_3 + C_4) = D.$$

The general solution (5.1.5) of the linear homogeneous ordinary differential equation (5.1.1) or (4.1.13) is the the linear combination of four linearly independent functions $\cosh(y)$, $\sinh(y)$, $\cosh(qy)$ and $\sinh(qy)$) where A, B, C and D are arbitrary real numbers (constants).

In order to derive symmetric and anti-symmetric dispersion relations we split the general solution in two forms:

(a) Symmetric dispersion relation:

If we assume that $B = D = 0$, then the solution (5.1.5) reduces to the following only

$$\phi(y) = A\sinh(y) + C\sinh(qy). \tag{5.1.6}$$

With the aid of equations (4.1.12) and (5.1.6) the boundary condition given in equation (4.1.15), that is

$$\mu(\psi_{,22} - \psi, 11) = 0,$$

becomes as follows

$$[A\sinh(y)e^{ik(ct-x_1)} + C\sinh(qy)e^{ik(ct-x_1)}]_{,22} - [A\sinh(y)e^{ik(ct-x_1)} + C\sinh(qy)e^{ik(ct-x_1)}]_{,11} = 0,$$

$$[A\sinh(y)k^2 + C\sinh(qy)(qk)^2] - [A\sinh(y)(ik)^2 + C\sinh(qy)(ik)^2] = 0,$$

$$A\sinh(y) + C\sinh(qy)q^2 + A\sinh(y) + C\sinh(qy) = 0,$$

$$2A\sinh(y) + C(q^2+1)\sinh(qy) = 0. \tag{5.1.7}$$

Similarly, using equations (4.1.12) and (5.1.6) in the second boundary condition given by the equation (4.1.16)

$$(3\mu - \rho c^2)k^2\psi_{,2} - \mu\psi_{,222} = 0,$$

we have

$$(3\mu - \rho c^2)k^2[A\sinh(y) + C\sinh(qy)]_{,2} - \mu[A\sinh(y) + C\sinh(qy)]_{,222} = 0,$$

$$(3\mu - \rho c^2)k^2[A\cosh(y).k + C\cosh(qy)(qk)]_{,2} - \mu[A\cosh(y)k^3 + C\cosh(qy)(qk)^3] = 0,$$

$$A\cosh(y)(3\mu - \rho c^2 - \mu) + C\cosh(qy)(3\mu q - \rho c^2 q - \mu^3 q) = 0. \tag{5.1.8}$$

The equations (5.1.7) and (5.1.8) form a system of two homogeneous linear equations in unknowns A and C. The non-trivial solution of above mentioned system exists only if the determinant of its coefficient matrix is zero. This condition gives us

$$\frac{2\sinh(y)}{\cosh(y)(3\mu - \rho c^2 - \mu)} = \frac{\sinh(qy)(q^2+1)}{\cosh(qy)(3\mu q - \rho c^2 q - \mu q^3)},$$

$$\frac{\tanh(y)}{\tan h(qy)} = \frac{(1+q^2)(2\mu - \rho c^2)}{2\mu q(3 - \frac{\rho c^2}{\mu} - q^2)},$$

$$\frac{\tanh(y)}{\tan h(qy)} = \frac{(1+q^2)^2}{4q}. \tag{5.1.9}$$

This is the symmetric dispersion relation for an isotropic plate under incompressibility. Spectrum of the dispersion curves appears as in Figure 5.1 [24] corresponds to (5.1.6).

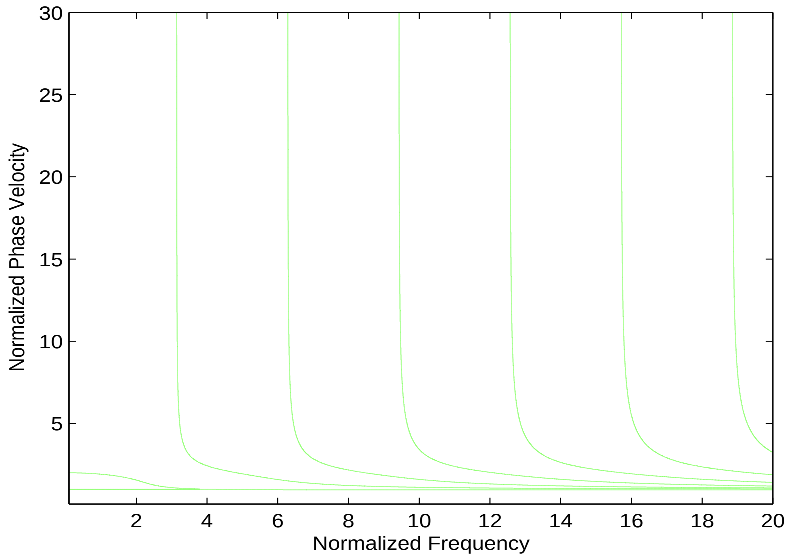

Figure 5.1: Symmetric Lamb modes for an isotropic plate, under incompressibility, in u-y plane

(b) Anti-symmetric dispersion relation:

If we assume that$A = C = 0$, then the solution (5.1.5) reduces to the following only

$$\phi(y) = B\cosh y + D\cosh(qy) \tag{5.1.10}$$

With the aid of equations (4.1.12) and (5.1.10) the boundary condition given in equation (4.1.15), that is

$$\mu(\psi_{,22} - \psi, 11) = 0,$$

becomes

$$[B\cosh(y)e^{ik(ct-x_1)} + D\cosh(qy)e^{ik(ct-x_1)}]_{,22} - [B\cosh(y)e^{ik(ct-x_1)} + D\cosh(qy)e^{ik(ct-x_1)}]_{,11} = 0,$$

$$[B\cosh(y)k^2 + D\cosh(qy)(qk)^2] - [B\cosh(y)(-ik)^2 + D\cosh(qy)(-ik)^2] = 0,$$

$$[B\cosh(y) + D\cosh(qy)q^2] - [-B\cosh(y) - D\cosh(qy)] = 0,$$

$$B\cosh(y) + D\cosh(qy)q^2 + B\cosh(y) + D\cosh(qy) = 0,$$

$$2B\cosh(y) + D(q^2+1)\cosh(qy) = 0, \tag{5.1.11}$$

Similarly, using equations (4.1.12) and (5.1.10) in the second boundary condition given by the equation (4.1.16)

$$(3\mu - \rho c^2)k^2\psi_{,2} - \mu\psi_{,222} = 0,$$

we have

$$(3\mu - \rho c^2)k^2[B\cosh(y)e^{ik(ct-x_1)} + D\cosh(qy)e^{ik(ct-x_1)}]_{,2} - \mu[B\cosh(y)e^{ik(ct-x_1)} + D\cosh(qy)e^{ik(ct-x_1)}]_{,222} = 0,$$

$$(3\mu - \rho c^2)k^2[B\sinh(y)ke^{ik(ct-x_1)} + D\sinh(qy)(qk)e^{ik(ct-x_1)}] - \mu[B\sinh(y)k^3e^{ik(ct-x_1)} + D\sinh(qy)(qk)^3e^{ik(ct-x_1)}] = 0,$$

$$(3\mu - \rho c^2)k^2[B\sinh(y)k + D\sinh(qy)(qk)] - \mu[B\sinh(y)k^3 + D\sinh(qy)(qk)^3] = 0,$$

$$3\mu B\sinh(y)k^3 + 3\mu D\sinh(qy)(qk)k^2 - \rho c^2 B\sinh(y)k^3 - \rho c^2 D\sinh(qy)(qk)k^2 - \mu B\sinh(y)k^3 - \mu D\sinh(qy)(qk)^3 = 0,$$

$$B\sinh(y)[3\mu - \rho c^2 - \mu] + D\sinh(qy)[3\mu q - \rho c^2 q - \mu q^3] = 0. \tag{5.1.12}$$

The equations (5.1.11) and (5.1.12) form a system of two homogeneous linear equations in unknowns B and D. The non-trivial solution of above mentioned system exists only if the determinant of its coefficient matrix is zero. This condition implies that

$$\frac{2\cosh(y)}{\sinh(y)(3\mu - \rho c^2 - \mu)} = \frac{(q^2+1)\cosh(qy)}{\sinh(qy)(3\mu q - \rho c^2 q - \mu q^3)},$$

$$\frac{\sinh(qy)(3\mu q - \rho c^2 q - \mu q^3)}{(q^2+1)\cosh(qy)} = \frac{\sinh(y)(3\mu - \rho c^2 - \mu)}{2\cosh(y)},$$

$$\frac{\tanh(qy)}{\tanh(y)} = \frac{(1+q^2)(3\mu - \rho c^2 - \mu)}{2q\mu(3 - \frac{\rho c^2}{\mu} - q^2)},$$

$$\frac{\tanh(qy)}{\tanh(y)} = \frac{(1+q^2)\mu(2 - \frac{\rho c^2}{\mu})}{2q\mu(3 - \frac{\rho c^2}{\mu} - q^2)},$$

$$\frac{\tanh(qy)}{\tanh(y)} = \frac{(1+q^2)^2}{4q}. \tag{5.1.13}$$

Equation (5.1.13) gives anti-symmetric dispersion relation and spectrum of the dispersion curves appears as in Figure 5.2 [24] corresponds to (5.1.10).

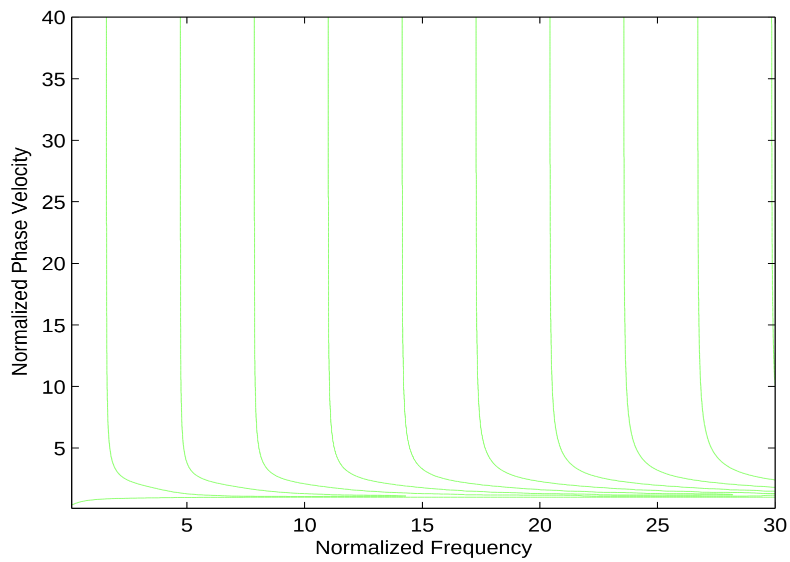

Figure 5.2: Symmetric Lamb modes for an isotropic plate, under incompressibility, in u-y plane

By inspecting Figure 5.1 and 5.2, it becomes clear that anomalous behaviour of all Lamb modes has disappeared from the spectrum. Hence there is no anomalous dispersion in the spectrum of an incompressible isotropic plate.

CASE 2: $1 - \frac{\rho c^2}{\mu} < 0 \Rightarrow \ \mu < \rho c^2$

In this case the last two roots of the auxiliary equation (5.1.3) become pure imaginary, that is the roots are now written as

$$m = \pm 1, \pm iq, \text{ where } q = \sqrt{1 - \frac{\rho c^2}{\mu}}.$$

Hence the general solution $\phi(y)$ of the equation (5.1.1) or (4.1.13) can be written as

$$\phi(y) = C_1 e^{y} + C_2 e^{-y} + C_3 e^{iqy} + C_4 e^{-iqy}.$$

or

$$\phi(y) = A\sinh(y) + B\cosh(y) + C\sin(qy) + D\cos(qy). \tag{5.1.14}$$

(a) Symmetric dispersion relation:

If we assume that $B = D = 0$, then the solution (5.1.14) reduces to the following only

$$\phi(y) = A\sinh(y) + C\sin(qy). \tag{5.1.15}$$

With the aid of equations (4.1.12) and (5.1.15) the boundary condition given in equation (4.1.15), that is

$$\mu(\psi_{,22} - \psi, 11) = 0,$$

becomes

$$A\sinh(y) - C\sin(qy)q^2 + A\sinh(y) + C\sin(qy) = 0,$$

$$2A\sinh(y) + C(1 - q^2)\sin(qy) = 0. \tag{5.1.16}$$

Similarly, using equations (4.1.12) and (5.1.15) in the second boundary condition given by the equation (4.1.16)

$$(3\mu - \rho c^2)k^2\psi_{,2} - \mu\psi_{,222} = 0,$$

we have

$$(3\mu - \rho c^2)k^2[A\sinh(y) + C\sin(qy)]_{,2} - \mu[A\sinh(y) + C\sin(qy)]_{,222} = 0,$$

$$A\cosh(y)(2\mu - \rho c^2) + C\cos(py)(3\mu q - \rho c^2 q + \mu q^2)q = 0. \tag{5.1.17}$$

For the existence of non-trivial solution of the system of equations (5.1.16) and (5.1.17) the determinant of the coefficient matrix must be zero, which implies

$$\frac{2\sinh(y)}{\cosh(y)(2\mu - \rho c^2)} = \frac{(1 - q^2)\sin(qy)}{\cos(qy)(3\mu q - \rho c^2 + \mu q^2)q},$$

$$\frac{\tanh(y)}{\tan(qy)} = \frac{(1 - q^2)^2}{4q}. \tag{5.1.18}$$

This is the symmetric dispersion relation for an isotropic plate under incompressibility.

(b)Anti-symmetric dispersion relation: If we assume that$A = C = 0$, then the solution (5.1.14) reduces to the following only

$$\phi(y) = B\cosh(y) + D\cos(qy) \tag{5.1.19}$$

With the aid of equations (4.1.12) and (5.1.19) the boundary condition given in equation (4.1.15), that is

$$\mu(\psi_{,22} - \psi, 11) = 0,$$

becomes

$$[B\cosh(y) + D\cos(py)q^2] + [B\cosh(y) - D\cos(qy)] = 0,$$

$$2B\cosh(y) + D(1-q^2)\cos(qy) = 0. \tag{5.1.20}$$

Similarly, using equations (4.1.12) and (5.1.19) in equation (4.1.16)

$$(3\mu - \rho c^2)k^2\psi_{,2} - \mu\psi_{,222} = 0,$$

we have

$$(3\mu - \rho c^2)k^2[B\sinh(y) + D\sin(qy)q] - \mu[B\sinh(y)k^3 + D\sin(qy)(qk)^3] = 0,$$

$$B\sinh(y)[2\mu - \rho c^2] - D\sin(qy)[3\mu - \rho c^2 q + \mu q^2]q = 0. \tag{5.1.21}$$

For the existence of non-trivial solution of the system of equations (5.1.20) and (5.1.21) the determinant of the coefficient matrix must be zero, which implies

$$\frac{2\cosh(y)}{\sinh(y)(2\mu - \rho c^2)} = \frac{(1-q^2)\cos(qy)}{-\sin(qy)(3\mu q - \rho c^2 q + \mu q^3)},$$

$$\frac{\tan(qy)}{\tanh(y)} = \frac{(1-q^2)^2}{4q}. \tag{5.1.22}$$

This is an anti-symmetric dispersion relation for an isotropic plate under incompressibility.

CASE 3: $1 - \frac{\rho c^2}{\mu} = 0 \Rightarrow \ \mu = \rho c^2$

In this case the last two roots of the auxiliary equation (5.1.3) become zero, that is the roots are now written as

$$m = \pm 1, 0, 0.$$

Hence the general solution $\phi(y)$ of the equation (5.1.1) or (4.1.13) can be written as

$$\phi(y) = A + By + C\cosh(y) + D\sinh(y). \tag{5.1.23}$$

But if we use the solution (5.1.23) of this case (similar to previous cases), we cannot derive any dispersion relation.

5.2 Analytical behaviour of Lamb modes

Since symmetric dispersion relation for an isotropic incompressible plate is given by equation (5.1.9), i-e

$$\frac{\tanh(y)}{\tanh(qy)} = \frac{(1+q^2)^2}{4q}.$$

If we insert $y = kx_2 = kh$, where k is the wave number and $2h$ is the length of isotropic plate, whereas $x_2 = \pm h$ on the boundaries. Hence equation (5.1.9) can also be written as

$$\frac{\tanh(qkh)}{\tanh(kh)} = \frac{4q}{(1+q^2)^2}. \tag{5.2.1}$$

If define the dimensionless wave number kh by x and normalized velocity $\frac{c}{c_T}$ $(c_T = \sqrt{\frac{\mu}{\rho}})$ by y, then equation (5.2.1) becomes

$$\frac{\tanh(\sqrt{1-y^2}x)}{\tanh(y)} = \frac{4\sqrt{1-y^2}}{(1+(1-y^2)^2},$$

$$\frac{i\tanh(\sqrt{1-y^2}x)}{\tanh(y)} = \frac{4i\sqrt{1-y^2}}{(1+(1-y^2)^2},$$

$$\frac{\tan i(\sqrt{1-y^2}x)}{\tanh(y)} = \frac{4i\sqrt{1-y^2}}{(2-y^2)^2},$$

$$\frac{\tan(\sqrt{y^2-1}x)}{\tanh(y)} = \frac{4\sqrt{y^2-1}}{(2-y^2)^2}. \tag{5.2.2}$$

Equation (5.2.2) is the same as equation (4.1.23) that depends only on one parameter and independent of c_L. In terms of u and y equation (5.2.2) becomes

$$\frac{\tan(\sqrt{y^2-1}u/y)}{\tan h(u/y)} = \frac{4\sqrt{y^2-1}}{(2-y^2)^2} \tag{5.2.3}$$

where $u = xy$, u denotes the normalized frequency and y is the normalized phase speed. Analytically the behaviour of modes in Figure 4.1 can be explained by examining the slopes of all modes, first for large phase velocity, $y >> 1$, secondly for small phase velocity y . We rewrite equation (5.2.2) in the form

$$h(u,y) = \tan(\sqrt{y^2-1}u/y)(2-y^2)^2 - 4\tanh(u/y)\sqrt{y^2-1} = 0$$

By approximation, it is clear from equations (4.3.6) and (4.3.11) [15] that all modes end up with negative slope and there is no anomalous dispersion in the spectrum of an isotropic incompressible plate.

Chapter 6

Conclusion

Elastic waves in compressible and incompressible isotropic materials has been studied. In [20] Ogden and Vinh have studied Rayleigh wave in a compressible orthotropic material in which an explicit formula is derived which shows speed of Rayleigh waves for orthotropic reaches its isotropic value for specific ratio of material constant. Then behaviour of Lamb modes for an isotropic incompressible plate are discussed. Also modified method is used to derive dispersion relation for Lamb modes for an isotropic incompressible plate. Shapes of dispersion curves are also plotted which agrees Hussain et al. [15] work.

From this dissertation it is concluded that Lamb modes in case of incompressibility is similar to the case of compressible materials [3], i-e,

1. Except the lowest S_0 mode, all modes asymptotically approach the line $C = C_T$.

2. The lowest mode asymptotically approaches the the line $C = C_R$.

However their behaviour is not similar in following points.

1. From the spectrum, the plateau region disappears.

2. Shapes of the curves is independent of the material.

3. No ZGV mode exist.

Appendix A

By using mathematica we have computed partial derivatives of $g(u,y)$ with respect to u and y and these expressions are given as follows

$$\begin{aligned}\frac{\partial g}{\partial u} =& \frac{4i\sqrt{y^2-1}\cos(\frac{u\sqrt{y^2-1}}{y})\cosh(u/y)}{y} - \frac{i(2-y^2)^2\sqrt{y^2-1}\cos(\frac{u\sqrt{y^2-1}}{y})\cosh(u/y)}{y} \\ &- \frac{i(2-y^2)^2\sin(\frac{u\sqrt{y^2-1}}{y})\sinh(u/y)}{y} - \frac{4i(y^2-1)\sin(\frac{u\sqrt{y^2-1}}{y})\sinh(u/y)}{y}, \qquad (A.1)\end{aligned}$$

$$\begin{aligned}\frac{\partial g}{\partial y} =& \frac{-4u\sqrt{y^2-1}\cos(\frac{u\sqrt{y^2-1}}{y})\cosh(u/y)}{y^2} \\ &- i(2-y^2)^2(\frac{u}{\sqrt{y^2-1}} - \frac{u\sqrt{y^2-1}}{y^2})\cos(\frac{u\sqrt{y^2-1}}{y})\cosh(u/y) \\ &+ 4iy(2-y^2)\cosh(u/y)\sin(\frac{u\sqrt{y^2-1}}{y}) + \frac{4iy\sinh(u/y)\cos(\frac{u\sqrt{y^2-1}}{y})}{\sqrt{y^2-1}} \\ &+ \frac{iu(2-y^2)^2\sinh(u/y)\sin(\frac{u\sqrt{y^2-1}}{y})}{y^2} \\ &- i\sqrt{y^2-1}(\frac{u}{\sqrt{y^2-1}} - \frac{u\sqrt{y^2-1}}{y^2})\sinh(u/y)\sin(\frac{u\sqrt{y^2-1}}{y}). \qquad (A.2)\end{aligned}$$

The partial derivative, $\frac{\partial g}{\partial y}$, can be approximated as following

$$\frac{4i\sqrt{y^2-1}\cos(\frac{u\sqrt{y^2-1}}{y})\cosh(u/y)}{y} \simeq 4i\cos(u)\cosh(u/y),$$

$$\frac{i(2-y^2)^2\sqrt{y^2-1}\cos(\frac{u\sqrt{y^2-1}}{y})\cosh(u/y)}{y} \simeq iy^3\cos u\cosh(u/y),$$

$$\frac{i(2-y^2)^2\sin(\frac{u\sqrt{y^2-1}}{y})\sinh(u/y)}{y} \simeq iy^3\sin u\sinh(u/y),$$

$$\frac{4i(y^2-1)\sin(\frac{u\sqrt{y^2-1}}{y})\sinh(u/y)}{y} \simeq 4iy\sin u\sinh(u/y).$$

For large y, we can have

$$\cosh(u/y) \simeq 1,$$

$$\sinh(u/y) \simeq u/y,$$

hence

$$\frac{4i\sqrt{y^2-1}\cos(\frac{u\sqrt{y^2-1}}{y})\cosh(u/y)}{y} \simeq 4i\cos(u),$$

$$\frac{i(2-y^2)^2\sqrt{y^2-1}\cos(\frac{u\sqrt{y^2-1}}{y})\cosh(u/y)}{y} \simeq iy^4\cos(u),$$

$$\frac{i(2-y^2)^2\sin(\frac{u\sqrt{y^2-1}}{y})\sinh(u/y)}{y} \simeq iuy^2\sin(u),$$

$$\frac{4i(y^2-1)\sin(\frac{u\sqrt{y^2-1}}{y})\sinh(u/y)}{y} \simeq 4iu\sin(u).$$

Then Eq. $(A.1)$ becomes

$$\frac{\partial g}{\partial u} \simeq 4i\cos(u) - iy^4\cos(u) - iuy^2\sin(u) - 4iu\sin(u),$$

$$\frac{\partial g}{\partial u} \simeq (4-y^4)i\cos(u) - (y^2+1)u\sin(u),$$

$$\frac{\partial g}{\partial u} \simeq -iy^4\cos(u) - iy^2u\sin(u). \qquad (A.3)$$

Now, for $\frac{\partial g}{\partial y}$ we shall proceed as follows

$$\frac{4iu\sqrt{y^2-1}\cos(\frac{u\sqrt{y^2-1}}{y})\cosh(u/y)}{y^2} \simeq \frac{4iu\cos(u)\cosh(u/y)}{y},$$

$$i(2-y^2)^2(\frac{u}{\sqrt{y^2-1}} - \frac{u\sqrt{y^2-1}}{y^2})\cos(\frac{u\sqrt{y^2-1}}{y})\cosh(u/y) \simeq iuy\cos(u)\cosh(u/y),$$

$$4iy(2-y^2)\cosh(u/y)\sin(\frac{u\sqrt{y^2-1}}{y}) \simeq -4iy^3\cosh(u/y)\sin(u),$$

$$\frac{4iy\sinh(u/y)\cos(\frac{u\sqrt{y^2-1}}{y})}{\sqrt{y^2-1}} \simeq 4i\sinh(u/y)\cos(u),$$

$$\frac{iu(2-y^2)^2\sinh(u/y)\sin(\frac{u\sqrt{y^2-1}}{y})}{y^2} \simeq iuy^2\sin(u)\sinh(u/y),$$

$$i\sqrt{y^2-1}(\frac{u}{\sqrt{y^2-1}} - \frac{u\sqrt{y^2-1}}{y^2})\sinh(u/y)\sin(\frac{u\sqrt{y^2-1}}{y}) \simeq \frac{-4iu\sin(u)\sinh(u/y)}{y^2}.$$

Using approximations

$$\cosh(u/y) \simeq 1,$$

$$\sinh(u/y) \simeq u/y,$$

for large y, we can write above expressions as follows

$$\frac{4iu\sqrt{y^2-1}\cos(\frac{u\sqrt{y^2-1}}{y})\cosh(u/y)}{y^2} \simeq \frac{4iu\cos(u)}{y},$$

$$i(2-y^2)^2\left(\frac{u}{\sqrt{y^2-1}}-\frac{u\sqrt{y^2-1}}{y^2}\right)\cos\left(\frac{u\sqrt{y^2-1}}{y}\right)\cosh(u/y) \simeq iuy\cos(u),$$

$$4iy(2-y^2)\cosh(u/y)\sin\left(\frac{u\sqrt{y^2-1}}{y}\right) \simeq -4iy^3\sin(u),$$

$$\frac{4iy\sinh(u/y)\cos(\frac{u\sqrt{y^2-1}}{y})}{\sqrt{y^2-1}} \simeq \frac{4iu\cos(u)}{y},$$

$$\frac{iu(2-y^2)^2\sinh(u/y)\sin(\frac{u\sqrt{y^2-1}}{y})}{y^2} \simeq iu^2y\sin(u),$$

$$i\sqrt{y^2-1}\left(\frac{u}{\sqrt{y^2-1}}-\frac{u\sqrt{y^2-1}}{y^2}\right)\sinh(u/y)\sin\left(\frac{u\sqrt{y^2-1}}{y}\right) \simeq \frac{-4iu^2\sin(u)}{y^3}.$$

Then Eq. (A.2) becomes

$$\frac{\partial g}{\partial y} \simeq -\frac{4iu\cos(u)}{y} - iuy\cos(u) - 4iy^3\sin(u) + \frac{4iu\cos(u)}{y} + iu^2y\sin(u) - \frac{4iu^2\sin(u)}{y^3},$$

$$\frac{\partial g}{\partial y} \simeq -iuy\cos(u) - 4iy^3\sin(u) + iu^2y\sin(u) - \frac{4iu^2\sin(u)}{y^3},$$

ignoring last term as it involves $\frac{1}{y^3}$, we have

$$\frac{\partial g}{\partial y} \simeq -iuy\cos(u) - 4iy^3\sin(u). \qquad (A.4)$$

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