Integral Inequalities of Gronwall Type

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by

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Dedicated to

my grand father and mother

for their Love, Support and Encouragement.

Abstract

N. H. Abel introduced the theory of Integral Equations in 1812. Since then, it has gained prominence and many other great mathematicians have contributed to the development of Integral Equations. Integral inequality that gives an explicit bound to the unknown function provides a handy tool to investigate qualitative properties of solutions of differential and integral equations. One of the best known and widely used inequalities in the study of non-linear differential equations is Gronwall-Bellman inequality.

Gronwall-Bellman inequality play an important role in the area of Integral and Differential Equations, and is used as a technical tool to prove existence, uniqueness and stability of a solution and to obtain various estimates for the solutions.

In this thesis we deals with the Gronwall-Bellman type inequalities involving functions of two independent variables. We study how to obtain optimal bounds of the unknown functions that satisfy a certain differential or integral inequality. Also how to generalizes the result of Gronwall-Bellman inequalities to a new type of retarded inequalities which includes both a nonconstant term outside the integrals and more than one distinct non-linear integrals. Finally, we extended the work by choosing suitable function for w, as $w(s) = s^r$ in Chapter 3.

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Mehwish Akbar

Contents

Chapter 1

Introduction to Integral Inequalities of Gronwall Type

1.1 Introduction

An equation in which an unknown function appears under an integral sign is called an *integral equation*, and an inequality which involves integrals is called *integral inequality*.

The pioneer of the theory of Integral Equations was N. H. Abel. In 1812, he formulated the first integral equation, while studing a problem in mechanics. Since then, several mathematicians have contributed to the development of Integral Equations. The major work was done in the late eighteenth and early nineteenth century by J. Liouville, J. Hadamard, V. Volterra, I. Fredholm, E. Goursat, D. Hilbert, E. Picard, and H. Poincaré. In the year 1911, the first dissertation on Integral Equations was written by T. Lalescu (see [11]). The integral equations have played an important role in Pure and Applied Mathematics, with applications in differential equations, integral equations, partial differential equations, functional analysis, numerical computations and others (see for example [17] and [19]).

Gronwall inequality plays a fundamental role in the area of Differential and Integral Equations. It is used as a handy tool to prove existence, stability, uniqueness and other estimates of a solution. In the past few years, due to the importance of such inequalities in the theory of differential and integral equations various investigators have discovered many useful inequalities in order to reach the diversity of desired goal. For example, the bounds given by Gronwall-Bellman inequality [3, 5] and its nonlinear generalization due to Bihari [4] are used to a noticeable extent in the literature, (see for instance [1, 2, 3, 4, 5, 8, 14, 15, 16] and the references cited therein).

1.2 The Inequalities of Gronwall and Bellman

Some integral inequalities play an important role in the study of the qualitative behaviour of solutions of differential and integral equations. This section presents the basic inequality due to Gronwall (1919) and Bellman (1943) which gives numerous applications in the study of different classes of differential and integral equations.

1.2.1 Gronwall Inequality

Gronwall's Lemma has two main classes: one is the integral inequalities and second is the differential inequalities. Both allow one to bound a function on $\mathbb{R}_+$ that satisfies an integral or differential inequality. Thomas Hakon Gronwall $(1877-1932)$ introduced the Gronwall Lemma in 1919 [5]. It is stated as follows:

Lemma 1.2.1. [5] *Let $z : [a, a+h] \longrightarrow \mathbb{R}$ be a continuous function that satisfies the inequalities*

$$0 \leq z(t) \leq \int_a^x A + Mz(s)ds, \tag{1.2.1}$$

for all $a \leq x \leq a+h$, where $A, M \geq 0$ are constants. Then

$$0 \leq z(t) \leq Ahe^{Mh}, \tag{1.2.2}$$

for all $a \leq t \leq a+h$. In particular, one has the estimate

$$z(t) \leq A(x-a)e^{M(t-a)}, \tag{1.2.3}$$

for all $a \leq t \leq a+h$.

In terms of differential inequality, it is stated as:

Lemma 1.2.2. [5] *Let I denote an interval of the real line of the form $[a,\infty)$ or $[a,b]$ or $[a,b)$ with $a<b$. Let g and u be real-valued continuous functions defined on I. If u is differentiable in the interior I^o of I and satisfies the differential inequality*

$$u'(t) \leq g(t)u(t), \qquad t \in I^o, \tag{1.2.4}$$

then u is bounded by the solution:

$$u(t) \leq u(a) \exp\left(\int_a^t g(s)ds\right), \tag{1.2.5}$$

for all $t \in I$.

1.2.2 Bellman Inequality

Later, in 1943, Richard Bellman proved the integral form of the Gronwall inequality [3]. But before that, we quote another form of Gronwall-Bellman inequality which is widely used in the study of nonlinear differential equations:

Lemma 1.2.3. [1] *Let $u(t)$ and $g(t)$ be non-negative continuous functions on an interval $I=[0,\infty)$ satisfying*

$$u(t) \leq c + \int_a^t g(s)u(s)ds, \qquad t \in I, \tag{1.2.6}$$

for some constant $c \geq 0$, then

$$u(t) \leq c \exp\left(\int_a^t g(s)ds\right), \qquad t \in I. \tag{1.2.7}$$

Now, the Bellman's inequality is stated as follows:

Theorem 1.2.4. [3] *Let I denote an interval of the real line of the form $[a,\infty)$ or $[a,b]$ or $[a,b)$ with $a<b$. Let f, g and u be real-valued functions defined on I. Assume that g and u are continuous and that f is integrable on every closed and bounded subinterval of I.*

(a) If g is non-negative and if u satisfies the integral inequality

$$u(t) \leq f(t) + \int_a^t g(s)u(s)ds, \qquad \forall t \in I, \tag{1.2.8}$$

then

$$u(t) \le f(t) + \int_a^t f(s)g(s)\exp\left(\int_s^t g(r)dr\right)ds, \qquad t \in I. \tag{1.2.9}$$

(b) If, in addition, the function f is non-decreasing, then

$$u(t) \le f(t)\exp\left(\int_a^t g(s)ds\right), \qquad t \in I. \tag{1.2.10}$$

Remark 1.2.1. The above theorem concludes that in Lemma 1.2.3, we may omit the requirements $u(t)$ and c to be non-negative.

Gronwall inequality also known as Gronwall's lemma or Gronwall-Bellman inequality, provides an explicit bound to the unknown functions and is an important tool to obtain various estimates in the theory of ordinary and partial differential equations. The above results are the most influential results in the theory of inequalities and a great number of monographs were written on the generalizations and analogous results of these inequalities (see [1, 6, 8, 14]). The applications of the Gronwall-Bellman inequality were developed in a remarkable way in the discussion of the existence, the uniqueness, the stability, boundedness and continuation and other qualitative properties of the solutions of differential and integral equations.

1.3 The Integral Inequalities of Gronwall's Type

Gronwall-Bellman inequality has many generalizations, one of them is the Bihari's inequality (see [4]), which is the non-linear generalization of Gronwall inequality. Bihari's inequality is proved in the year 1956 by Hungarian mathematician Imre Bihari $(1915-1998)$. The inequality is stated as:

Theorem 1.3.1. [4] *Let $u(t)$ and $g(t)$ be non-negative continuous functions defined on $[0,\infty)$ and let $w(t)$ be a continuous non-decreasing function defined on $[0,\infty)$ and $w(u) > 0$ on $[0,\infty)$. If u satisfies the following integral inequality*

$$u(t) \le c + \int_0^t g(s)w\left(u(s)\right)ds, \qquad t \in [0,\infty), \tag{1.3.1}$$

for some non-negative constant c, then

$$u(t) \le G^{-1}\left(G(c) + \int_0^t g(s)ds\right), \qquad t \in [0,T]. \tag{1.3.2}$$

Where the function G is defined by

$$G(x) = \int_{x_0}^{x} \frac{1}{w(y)} dy, \qquad x \ge 0,\ x_0 > 0, \tag{1.3.3}$$

and G^{-1} is the inverse function of G and T is chosen so that

$$G(c) + \int_0^t g(s)ds \in Dom(G^{-1}), \qquad \forall \quad t \in [0,T]. \tag{1.3.4}$$

In 1930, a mathematician Reid established the following result:

Theorem 1.3.2. [1] *Let $u(t)$ and $g(t)$ be non-negative continuous functions on an interval $[a,b]$ and suppose*

$$u(t) \le c + \int_{t_0}^{t} g(s)u(s)|ds|, \qquad t \in [a,b], \tag{1.3.5}$$

for some constant c and $t_0 \in [a,b]$, then

$$u(t) \le c \exp\left(\int_{t_0}^{t} g(s)|ds|\right), \qquad t \in [a,b]. \tag{1.3.6}$$

Chandirov, a mathematician in 1958 established a corollary and then in 1970, he proved a theorem which gives the best possible estimate for a function $u(t)$ satisfying equation (1.3.7). Both of them are given below:

Corollary 1.3.3. [1] *Let $u(t)$ and $f(t)$, $g(t)$ and $h(t)$ be continuous functions on an interval $[a,b]$ and let $g(t)$ and $h(t)$ be non-negative in $[a,b]$, and suppose*

$$u(t) \le f(t) + \int_a^t [g(s)u(s) + h(s)]\, ds, \qquad t \in [a,b], \tag{1.3.7}$$

then

$$u(t) \le \left[\sup_{s\in[a,t]} f(s) + \int_a^t h(s)ds\right] \exp\left(\int_a^t g(s)ds\right), \qquad t \in [a,b]. \tag{1.3.8}$$

Theorem 1.3.4. [1] *Let $u(t)$ and $f(t)$, $g(t)$ and $h(t)$ be continuous functions on an interval $[a,b]$ and let $g(t)$ be non-negative in $[a,b]$, and suppose*

$$u(t) \leq f(t) + \int_a^t [g(s)u(s) + h(s)]\, ds, \qquad t \in [a,b],$$

then

$$u(t) \leq f(t) + \int_a^t [f(s)g(s) + h(s)] \exp\left(\int_s^t g(r)dr\right) ds, \qquad t \in [a,b]. \tag{1.3.9}$$

A useful general version of the Lemma 1.2.3 stated in Section 1.2 was given by B.G. Pachpatte in 1973:

Theorem 1.3.5. [15] *Let $u(t)$, $f(t)$ and $g(t)$ be real valued non-negative continuous functions in a real interval $I = [0,\infty)$, for which the inequality*

$$u(t) \leq c + \int_0^t f(s)u(s)ds + \int_0^t f(s)\left(\int_0^s g(\tau)u(\tau)d\tau\right) ds, \qquad t \in I, \tag{1.3.10}$$

holds, where c is a non-negative constant, then

$$u(t) \leq c\left[1 + \int_0^t f(s)\exp\left(\int_0^s (f(\tau) + g(\tau))\, d\tau\right) ds\right], \qquad t \in I. \tag{1.3.11}$$

J. A. Oguntuase obtained a bound on the following integral inequality:

Theorem 1.3.6. [13] *Let $u(t)$, $g(t)$ be non-negative continuous functions in a real interval $I = [a,b]$. Suppose that $k(t,s)$ and its partial derivatives $k_t(t,s)$ exist and are non-negative continuous functions for almost every $t, s \in I$. If the inequality*

$$u(t) \leq c + \int_a^t g(s)u(s)ds + \int_a^t g(s)\left(\int_a^s k(s,\tau)u(\tau)d\tau\right) ds, \qquad a \leq \tau \leq s \leq t \leq b, \tag{1.3.12}$$

holds, where c is a non-negative constant, then

$$u(t) \leq c\left[1 + \int_a^t g(s)\exp\left(\int_a^s (g(\tau) + k(\tau,\tau))\, d\tau\right) ds\right]. \tag{1.3.13}$$

Many new inequalities have been established so far. The author motivated by the work of Zareen A. Khan and B. G. Pachpatte establishes some new retarded integral inequalities involving functions of two independent variables. He generalizes their work, the details of which are given in Chapter 3.

Chapter 2

On Certain New Gronwall-Bellman Type Integral Inequalities of Two Independent Variables

Due to various motivations, many generalizations and applications of the Lemma 1.2.3, have been established and used extensively. In this chapter, the detailed proofs of integral inequalities involving functions of two independent variables have been given. These results can be used as tools in the qualitative theory of certain partial differential equations.

2.1 Integral Inequalities for Non-decreasing Continuous Functions

In this section, we proved some lemma's for non-decreasing continuous functions, that are useful in our main results (see [6]).

Lemma 2.1.1. *Let $z(x, y)$, $A(x, y)$ and $B(x, y)$ be real valued non-negative, non-*

decreasing continuous functions defined for $x, y \in \mathbb{R}_+$ and suppose

$$z(x,y) \le 1 + \int_0^x A(s,y)z(s,y)ds + \int_0^y B(x,t)z(x,t)dt, \tag{2.1.1}$$

for $x, y \in \mathbb{R}_+$. Then

$$z(x,y) \le Q(x,y)E(x,y),$$

where

$$Q(x,y) = \exp\left[\int_0^y B(x,t)E(x,t)dt\right], \tag{2.1.2}$$

and

$$E(x,y) = \exp\left[\int_0^x A(s,y)ds\right]. \tag{2.1.3}$$

Proof. Define

$$b(x,y) = 1 + \int_0^y B(x,t)z(x,t)dt, \tag{2.1.4}$$

$$b(x,0) = 1. \tag{2.1.5}$$

By substituting (2.1.4) in (2.1.1), we get

$$z(x,y) \le b(x,y) + \int_0^x A(s,y)z(s,y)ds. \tag{2.1.6}$$

Since $b(x,y)$ is positive, monotonic non-decreasing continuous function, therefore

$$\frac{z(x,y)}{b(x,y)} \le 1 + \int_0^x A(s,y)\frac{z(s,y)}{b(s,y)}ds. \tag{2.1.7}$$

Let

$$v(x,y) = 1 + \int_0^x A(s,y)\frac{z(s,y)}{b(s,y)}ds, \tag{2.1.8}$$

$$v(0,y) = 1. \tag{2.1.9}$$

From (2.1.7) and (2.1.8), we have

$$\frac{z(x,y)}{b(x,y)} \le v(x,y). \tag{2.1.10}$$

Differentiating (2.1.8) w.r.t. x implies

$$v_x(x,y) = A(x,y)\frac{z(x,y)}{b(x,y)}.$$

Now using (2.1.10), we have

$$v_x(x,y) \le A(x,y)v(x,y).$$

This implies

$$\frac{v_x(x,y)}{v(x,y)} \le A(x,y). \tag{2.1.11}$$

Keeping y fixed, set $x = s$ and integrate from 0 to x, we get

$$\ln v(x,y) \le \int_0^x A(s,y)ds.$$

This implies

$$v(x,y) \le \exp\left[\int_0^x A(s,y)ds\right] = E(x,y). \tag{2.1.12}$$

From (2.1.10) and (2.1.12) we have

$$z(x,y) \le b(x,y)E(x,y). \tag{2.1.13}$$

Differentiating (2.1.4) w.r.t. y implies

$$b_y(x,y) = B(x,y)z(x,y). \tag{2.1.14}$$

Now substituting (2.1.13) in (2.1.14)

$$b_y(x,y) \le B(x,y)b(x,y)E(x,y).$$

This implies

$$\frac{b_y(x,y)}{b(x,y)} \le B(x,y)E(x,y).$$

Now keeping x fixed, set $y = t$ and integrate from 0 to y, we get

$$\ln b(x,y) \le \int_0^y B(x,t)E(x,t)dt,$$

$$b(x,y) \le \exp\left[\int_0^y B(x,t)E(x,t)dt\right] = Q(x,y). \tag{2.1.15}$$

From (2.1.13) and (2.1.15), we have

$$z(x,y) \le E(x,y)Q(x,y). \tag{2.1.16}$$

□

Lemma 2.1.2. *Let $z(x,y)$, $A(x,y)$, $B(x,y)$ and $m(x,y)$ be real valued non-negative, non-decreasing continuous functions defined for $x, y \in \mathbb{R}_+$, $0 < p < 1$ and suppose*

$$z(x,y) \leq 1 + \int_0^x A(s,y)z(s,y)ds + \int_0^y B(x,t)m^{p-1}(x,t)z^p(x,t)dt, \qquad (2.1.17)$$

for $x, y \in \mathbb{R}_+$. Then

$$z(x,y) \leq Q_1(x,y)E(x,y),$$

where

$$Q_1(x,y) = \left[1 + (1-p)\int_0^y B(x,t)m^{p-1}(x,t)E^p(x,t)dt\right]^{\frac{1}{1-p}}, \qquad (2.1.18)$$

and $E(x,y)$ is defined as (2.1.3).

Proof. Define

$$b(x,y) = 1 + \int_0^y B(x,t)m^{p-1}(x,t)z^p(x,t)dt, \qquad (2.1.19)$$

$$b(x,0) = 1. \qquad (2.1.20)$$

By substituting (2.1.19) in (2.1.17), we get

$$z(x,y) \leq b(x,y) + \int_0^x A(s,y)z(s,y)ds. \qquad (2.1.21)$$

Now following the same steps from (2.1.6) to (2.1.13) as in Lemma 2.1.1, we have

$$z(x,y) \leq b(x,y)E(x,y). \qquad (2.1.22)$$

Differentiating (2.1.19) w.r.t. y implies

$$b_y(x,y) = B(x,y)m^{p-1}(x,y)z^p(x,y). \qquad (2.1.23)$$

Now substituting (2.1.22) in (2.1.23)

$$b_y(x,y) \leq B(x,y)m^{p-1}(x,y)b^p(x,y)E^p(x,y). \qquad (2.1.24)$$

This implies

$$\frac{b_y(x,y)}{b^p(x,y)} \leq B(x,y)m^{p-1}(x,y)E^p(x,y).$$

Now by keeping x fixed, set $y = t$ and integrate from 0 to y, and using (2.1.20), we get

$$\frac{1}{1-p}\left[b^{1-p}(x,y) - 1\right] \le \int_0^y B(x,t)m^{p-1}(x,t)E^p(x,t)dt,$$

$$b(x,y) \le \left[1 + (1-p)\int_0^y B(x,t)m^{p-1}(x,t)E^p(x,t)dt\right]^{\frac{1}{1-p}} = Q_1(x,y). \qquad (2.1.25)$$

From (2.1.22) and (2.1.25), we get

$$z(x,y) \le E(x,y)Q_1(x,y). \qquad (2.1.26)$$

□

Lemma 2.1.3. *Let $z(x,y)$, $A(x,y)$, $B(x,y)$ and $m(x,y)$ be real valued non-negative, non-decreasing continuous functions defined for $x, y \in \mathbb{R}_+$, $0 < p < 1$ and suppose*

$$z(x,y) \le 1 + \int_0^x A(s,y)m^{p-1}(s,y)z^p(s,y)ds + \int_0^y B(x,t)m^{p-1}(x,t)z^p(x,t)dt, \qquad (2.1.27)$$

for $x, y \in \mathbb{R}_+$. Then

$$z(x,y) \le Q_3(x,y)E_2(x,y),$$

where

$$E_2(x,y) = \left[1 + (1-p)\int_0^x A(s,y)m^{p-1}(s,y)Q_3^p(s,y)ds\right]^{\frac{1}{1-p}}, \qquad (2.1.28)$$

and

$$Q_3(x,y) = \left[1 + (1-p)\int_0^y B(x,t)m^{p-1}(x,t)b^{p-1}(x,t)dt\right]^{\frac{1}{1-p}}. \qquad (2.1.29)$$

Proof. Define

$$b(x,y) = 1 + \int_0^x A(s,y)m^{p-1}(s,y)z^p(s,y)ds, \qquad (2.1.30)$$

$$b(0,y) = 1. \qquad (2.1.31)$$

By substituting (2.1.30) in (2.1.27), we get

$$z(x,y) \le b(x,y) + \int_0^y B(x,t)m^{p-1}(x,t)z^p(x,t)dt. \qquad (2.1.32)$$

Since $b(x,y)$ is positive, monotonic non-decreasing continuous function, therefore

$$\frac{z(x,y)}{b(x,y)} \leq 1 + \int_0^y B(x,t)m^{p-1}(x,t)\frac{z^p(x,t)}{b(x,t)}dt. \tag{2.1.33}$$

Let

$$v(x,y) = 1 + \int_0^y B(x,t)m^{p-1}(x,t)\frac{z^p(x,t)}{b(x,t)}dt, \tag{2.1.34}$$

$$v(x,0) = 1. \tag{2.1.35}$$

From (2.1.33) and (2.1.34), we have

$$\frac{z(x,y)}{b(x,y)} \leq v(x,y). \tag{2.1.36}$$

Differentiating (2.1.34) w.r.t. y implies

$$v_y(x,y) = B(x,y)m^{p-1}(x,y)\frac{z^p(x,y)}{b(x,y)}. \tag{2.1.37}$$

Now using (2.1.36), we have

$$v_y(x,y) \leq B(x,y)m^{p-1}(x,y)b^{p-1}(x,y)v^p(x,y).$$

This implies

$$\frac{v_y(x,y)}{v^p(x,y)} \leq B(x,y)m^{p-1}(x,y)b^{p-1}(x,y). \tag{2.1.38}$$

Keeping x fixed, set $y = t$ and integrate from 0 to y, we get

$$\frac{1}{1-p}\left[v^{1-p}(x,y) - 1\right] \leq \int_0^y B(x,t)m^{p-1}(x,t)b^{p-1}(x,t)dt,$$

$$v(x,y) \leq \left[1 + (1-p)\int_0^y B(x,t)m^{p-1}(x,t)b^{p-1}(x,t)dt\right]^{\frac{1}{1-p}} = Q_3(x,y). \tag{2.1.39}$$

From (2.1.36) and (2.1.39), we have

$$z(x,y) \leq b(x,y)Q_3(x,y). \tag{2.1.40}$$

Differentiating (2.1.30) w.r.t. x implies

$$b_x(x,y) = A(x,y)m^{p-1}(x,y)z^p(x,y). \tag{2.1.41}$$

Now substituting (2.1.40) in (2.1.41)

$$b_x(x,y) \le A(x,y)m^{p-1}(x,y)b^p(x,y){Q_3}^p(x,y).$$

This implies

$$\frac{b_y(x,y)}{b^p(x,y)} \le A(x,y)m^{p-1}(x,y){Q_3}^p(x,y).$$

Now by keeping y fixed, set $x = s$ and integrate from 0 to x, and using (2.1.31), we get

$$\frac{1}{1-p}\left[b^{1-p}(x,y) - 1\right] \le \int_0^x A(s,y)m^{p-1}(s,y){Q_3}^p(s,y)ds,$$

$$b(x,y) \le \left[1 + (1-p)\int_0^x A(s,y)m^{p-1}(s,y){Q_3}^p(s,y)ds\right]^{\frac{1}{1-p}} = E_2(x,y). \quad (2.1.42)$$

From (2.1.40) and (2.1.42), we have

$$z(x,y) \le E_2(x,y)Q_3(x,y). \quad (2.1.43)$$

□

2.2 Gronwall-Bellman Type Integral Inequalities of Two Independent Variables

Theorem 2.2.1. *Let $\Phi(x,y)$, $A(x,y)$, $B(x,y)$ and $H(x,y)$ be real valued non-negative, non-decreasing continuous functions defined for $x,y \in \mathbb{R}_+$, $c > 0$ and suppose*

$$\Phi(x,y) \le c + \int_0^x A(s,y)\Phi(s,y)ds + \int_0^y B(x,t)\Phi(x,t)dt + \int_0^x\int_0^y H(s,t)\Phi(s,t)ds\,dt, \quad (2.2.1)$$

for $x,y \in \mathbb{R}_+$. Then

$$\Phi(x,y) \le cQ(x,y)E(x,y)\exp\left[\int_0^x\int_0^y H(s,t)Q(s,t)E(s,t)ds\,dt\right].$$

Where $Q(x,y)$ and $E(x,y)$ are defined as in (2.1.2) and (2.1.3) respectively.

Proof. Define

$$m(x,y) = c + \int_0^x \int_0^y H(s,t)\Phi(s,t)ds\,dt, \tag{2.2.2}$$

$$m(x,0) = m(0,y) = c, \qquad m_x(x,0) = m_y(0,y) = 0. \tag{2.2.3}$$

By substituting (2.2.2) in (2.2.1), we get

$$\Phi(x,y) \le m(x,y) + \int_0^x A(s,y)\Phi(s,y)ds + \int_0^y B(x,t)\Phi(x,t)dt. \tag{2.2.4}$$

Since $m(x,y)$ is positive, non-decreasing continuous function, therefore

$$\frac{\Phi(x,y)}{m(x,y)} \le 1 + \int_0^x A(s,y)\frac{\Phi(s,y)}{m(s,y)}ds + \int_0^y B(x,t)\frac{\Phi(x,t)}{m(x,t)}dt. \tag{2.2.5}$$

Let

$$z(x,y) = 1 + \int_0^x A(s,y)\frac{\Phi(s,y)}{m(s,y)}ds + \int_0^y B(x,t)\frac{\Phi(x,t)}{m(x,t)}dt, \tag{2.2.6}$$

then,

$$z(0,0) = 1. \tag{2.2.7}$$

Substituting (2.2.6) in (2.2.5), implies

$$\frac{\Phi(x,y)}{m(x,y)} \le z(x,y). \tag{2.2.8}$$

Also from (2.2.6) and (2.2.8), we have

$$z(x,y) \le 1 + \int_0^x A(s,y)z(s,y)ds + \int_0^y B(x,t)z(x,t)dt. \tag{2.2.9}$$

Using Lemma 2.1.1, we have

$$z(x,y) \le E(x,y)Q(x,y). \tag{2.2.10}$$

Substituting (2.2.10) in (2.2.8), we get

$$\frac{\Phi(x,y)}{m(x,y)} \le E(x,y)Q(x,y),$$

$$\Phi(x,y) \le E(x,y)Q(x,y)m(x,y). \tag{2.2.11}$$

Differentiating (2.2.2) w.r.t x and y, we have

$$m_{xy}(x,y) = H(x,y)\Phi(x,y),$$

using (2.2.11), we have

$$m_{xy}(x,y) \le H(x,y)E(x,y)Q(x,y)m(x,y).$$

This implies

$$\frac{m_{xy}(x,y)}{m(x,y)} \le H(x,y)E(x,y)Q(x,y),$$

$$\frac{m_{xy}(x,y)m(x,y)}{m^2(x,y)} - \frac{m_x(x,y)m_y(x,y)}{m^2(x,y)} \le H(x,y)E(x,y)Q(x,y),$$

$$\frac{\partial}{\partial y}\left[\frac{m_x(x,y)}{m(x,y)}\right] \le H(x,y)E(x,y)Q(x,y). \tag{2.2.12}$$

By keeping y fixed, set $x = s$, in (2.2.12) and integrate from 0 to x, using

$$\frac{d}{dy}\int_a^b f(x,y)dx = \int_a^b \frac{\partial}{\partial y} f(x,y)dx, \tag{2.2.13}$$

and (2.2.3), we have

$$\frac{d}{dy}\left[\ln m(x,y)\right] \le \int_0^x H(s,y)E(s,y)Q(s,y)ds.$$

Again keeping x fixed, set $y = t$, in the above inequality and integrate from 0 to y, using (2.2.3), we obtain

$$\ln\left[\frac{m(x,y)}{c}\right] \le \int_0^x \int_0^y H(s,t)E(s,t)Q(s,t)ds\,dt.$$

This implies

$$m(x,y) \le c\exp\left[\int_0^x \int_0^y H(s,t)E(s,t)Q(s,t)ds\,dt\right].$$

Substituting the above bound in (2.2.11), we have

$$\Phi(x,y) \le cE(x,y)Q(x,y)\exp\left[\int_0^x \int_0^y H(s,t)E(s,t)Q(s,t)ds\,dt\right].$$

□

Theorem 2.2.2. *Let $\Phi(x,y)$, $A(x,y)$, $B(x,y)$ and $H(x,y)$ be real valued non-negative, non-decreasing continuous functions defined for $x, y \in \mathbb{R}_+$, $c > 0$ and $0 < p < 1$ are constants and suppose*

$$\Phi(x,y) \le c + \int_0^x A(s,y)\Phi(s,y)ds + \int_0^y B(x,t)\Phi(x,t)dt + \int_0^x \int_0^y H(s,t)\Phi^p(s,t)ds\,dt, \tag{2.2.14}$$

for $x, y \in \mathbb{R}_+$. *Then*

$$\Phi(x,y) \le Q(x,y)E(x,y)\left[c^{1-p} + (1-p)\int_0^x\int_0^y H(s,t)Q^p(s,t)E^p(s,t)ds\,dt\right]^{\frac{1}{1-p}}. \tag{2.2.15}$$

Where $Q(x,y)$ *and* $E(x,y)$ *are defined in* (2.1.2) *and* (2.1.3).

Proof. Define

$$m(x,y) = c + \int_0^x\int_0^y H(s,t)\Phi^p(s,t)ds\,dt, \tag{2.2.16}$$

$$m(x,0) = m(0,y) = c, \qquad m_x(x,0) = m_y(0,y) = 0. \tag{2.2.17}$$

By substituting (2.2.16) in (2.2.14), we get

$$\Phi(x,y) \le m(x,y) + \int_0^x A(s,y)\Phi(s,y)ds + \int_0^y B(x,t)\Phi(x,t)dt.$$

Now following the same steps from (2.2.4) to (2.2.11) as in Theorem 2.2.1, we have

$$\Phi(x,y) \le E(x,y)Q(x,y)m(x,y). \tag{2.2.18}$$

Differentiating (2.2.16) w.r.t x and y, we have

$$m_{xy}(x,y) = H(x,y)\Phi^p(x,y).$$

Using (2.2.18), we have

$$m_{xy}(x,y) \le H(x,y)E^p(x,y)Q^p(x,y)m^p(x,y).$$

This implies

$$\frac{m_{xy}(x,y)}{m^p(x,y)} \le H(x,y)E^p(x,y)Q^p(x,y),$$

$$\frac{m_{xy}(x,y)m(x,y)}{m^{p+1}(x,y)} - \frac{m_x(x,y)m_y(x,y)}{m^{p+1}(x,y)} \le H(x,y)E^p(x,y)Q^p(x,y),$$

$$\frac{\partial}{\partial y}\left[\frac{m_x(x,y)}{m^p(x,y)}\right] \le H(x,y)E^p(x,y)Q^p(x,y). \tag{2.2.19}$$

By keeping y fixed, set $x = s$, in (2.2.19) and integrate from 0 to x, using (2.2.13) and (2.2.17), we have

$$\frac{d}{dy}\left[\left[m^{1-p}(x,y) - cm^{-p}(x,y)\right] + p\int_0^x m^{-p}(s,y) - cm^{-1-p}(s,y)ds\right] \le C,$$

$$\frac{d}{dy}\left[m^{1-p}(x,y)+\frac{p}{1-p}m^{1-p}(x,y)-\frac{p}{1-p}c^{1-p}-c^{1-p}\right]\le C,$$

$$\frac{d}{dy}\left[\frac{1}{1-p}\left[m^{1-p}(x,y)-c^{1-p}\right]\right]\le C.$$

Where

$$C=\int_0^x H(s,y)E^p(s,y)Q^p(s,y)ds.$$

Again keeping x fixed, set $y=t$, in the last inequality and integrate from 0 to y, using (2.2.17), we obtain

$$\frac{1}{1-p}\left[m^{1-p}(x,y)-c^{1-p}\right]\le\int_0^x\int_0^y H(s,t)E^p(s,t)Q^p(s,t)ds\,dt,$$

this implies

$$m(x,y)\le\left[c^{1-p}+(1-p)\int_0^x\int_0^y H(s,t)E^p(s,t)Q^p(s,t)ds\,dt\right]^{\frac{1}{1-p}},$$

substituting the above bound in (2.2.18), we have

$$\Phi(x,y)\le E(x,y)Q(x,y)\left[c^{1-p}+(1-p)\int_0^x\int_0^y H(s,t)E^p(s,t)Q^p(s,t)ds\,dt\right].$$

□

2.3 Generalization of Gronwall-Bellman Type Integral Inequalities of Two Independent Variables

Theorem 2.3.1. *Let $\Phi(x,y)$, $A(x,y)$, $B(x,y)$ and $H(x,y)$ be real valued non-negative, non-decreasing continuous functions defined for $x,y\in\mathbb{R}_+$, $c>0$ and $0<p<1$ are constants and suppose*

$$\Phi(x,y)\le c+\int_0^x A(s,y)\Phi(s,y)ds+\int_0^y B(x,t)\Phi^p(x,t)dt+\int_0^x\int_0^y H(s,t)\Phi^p(s,t)ds\,dt, \tag{2.3.1}$$

for $x, y \in \mathbb{R}_+$. *Then*

$$\Phi(x,y) \le Q_1(x,y)E(x,y)\left[c^{1-p} + (1-p)\int_0^x\int_0^y H(s,t){Q_1}^p(s,t)E^p(s,t)ds\,dt\right]^{\frac{1}{1-p}}. \tag{2.3.2}$$

Where $E(x,y)$ *is defined as in* (2.1.3) *of Lemma 2.1.1 and* $Q_1(x,y)$ *is defined as in* (2.1.18) *of Lemma 2.1.2.*

Proof. Define

$$m(x,y) = c + \int_0^x\int_0^y H(s,t)\Phi^p(s,t)ds\,dt, \tag{2.3.3}$$

$$m(x,0) = m(0,y) = c, \qquad m_x(x,0) = m_y(0,y) = 0. \tag{2.3.4}$$

By substituting (2.3.3) in (2.3.1), we get

$$\Phi(x,y) \le m(x,y) + \int_0^x A(s,y)\Phi(s,y)ds + \int_0^y B(x,t)\Phi^p(x,t)dt. \tag{2.3.5}$$

Since $m(x,y)$ is positive, non-decreasing continuous function, therefore

$$\frac{\Phi(x,y)}{m(x,y)} \le 1 + \int_0^x A(s,y)\frac{\Phi(s,y)}{m(s,y)}ds + \int_0^y B(x,t)\frac{\Phi^p(x,t)}{m(x,t)}dt. \tag{2.3.6}$$

Let

$$z(x,y) = 1 + \int_0^x A(s,y)\frac{\Phi(s,y)}{m(s,y)}ds + \int_0^y B(x,t)\frac{\Phi^p(x,t)}{m(x,t)}dt, \tag{2.3.7}$$

then,

$$z(0,0) = 1. \tag{2.3.8}$$

Substituting (2.3.7) in (2.3.6), implies

$$\frac{\Phi(x,y)}{m(x,y)} \le z(x,y). \tag{2.3.9}$$

Also from (2.3.7) and (2.3.9), we have

$$z(x,y) \le 1 + \int_0^x A(s,y)z(s,y)ds + \int_0^y B(x,t)m^{p-1}(x,t)z^p(x,t)dt, \tag{2.3.10}$$

where

$$\frac{\Phi^p(x,y)}{m(x,y)} \le m^{p-1}(x,y)z^p(x,y). \tag{2.3.11}$$

Using Lemma 2.1.2, implies

$$z(x,y) \leq E(x,y)Q_1(x,y). \tag{2.3.12}$$

By substituting from (2.3.12) in (2.3.9), we get

$$\frac{\Phi(x,y)}{m(x,y)} \leq E(x,y)Q_1(x,y), \tag{2.3.13}$$

this implies

$$\Phi(x,y) \leq E(x,y)Q_1(x,y)m(x,y). \tag{2.3.14}$$

Differentiating (2.3.3) w.r.t x and y, we have

$$m_{xy}(x,y) = H(x,y)\Phi^p(x,y),$$

using (2.3.14), we have

$$m_{xy}(x,y) \leq H(x,y)E^p(x,y){Q_1}^p(x,y)m^p(x,y).$$

This implies

$$\frac{m_{xy}(x,y)}{m^p(x,y)} \leq H(x,y)E^p(x,y){Q_1}^p(x,y),$$

$$\frac{m_{xy}(x,y)m(x,y)}{m^{p+1}(x,y)} - \frac{m_x(x,y)m_y(x,y)}{m^{p+1}(x,y)} \leq H(x,y)E^p(x,y){Q_1}^p(x,y),$$

$$\frac{\partial}{\partial y}\left[\frac{m_x(x,y)}{m^p(x,y)}\right] \leq H(x,y)E^p(x,y){Q_1}^p(x,y). \tag{2.3.15}$$

By keeping y fixed, set $x = s$, in (2.3.15) and integrate from 0 to x, using (2.2.13) and (2.3.4), we have

$$\frac{d}{dy}\left[\left[m^{1-p}(x,y) - cm^{-p}(x,y)\right] + p\int_0^x m^{-p}(s,y) - cm^{-1-p}(s,y)ds\right] \leq D,$$

$$\frac{d}{dy}\left[m^{1-p}(x,y) + \frac{p}{1-p}m^{1-p}(x,y) - \frac{p}{1-p}c^{1-p} - c^{1-p}\right] \leq D,$$

$$\frac{d}{dy}\left[\frac{1}{1-p}\left[m^{1-p}(x,y) - c^{1-p}\right]\right] \leq D.$$

Where

$$D = \int_0^x H(s,y)E^p(s,y){Q_1}^p(s,y)ds.$$

Again keeping x fixed, set $y = t$, in the last inequality and integrate from 0 to y, using (2.3.4), we obtain

$$\frac{1}{1-p}\left[m^{1-p}(x,y) - c^{1-p}\right] \leq \int_0^x \int_0^y H(s,t)E^p(s,t){Q_1}^p(s,t)ds\,dt,$$

this implies

$$m(x,y) \leq \left[c^{1-p} + (1-p)\int_0^x \int_0^y H(s,t)E^p(s,t){Q_1}^p(s,t)ds\,dt\right]^{\frac{1}{1-p}},$$

substituting the above bound in (2.3.14), we have

$$\Phi(x,y) \leq E(x,y)Q_1(x,y)\left[c^{1-p} + (1-p)\int_0^x \int_0^y H(s,t)E^p(s,t){Q_1}^p(s,t)ds\,dt\right]^{\frac{1}{1-p}}.$$

□

Theorem 2.3.2. *Let $\Phi(x,y)$, $A(x,y)$, $B(x,y)$ and $H(x,y)$ be real valued non-negative, non-decreasing continuous functions defined for $x,y \in \mathbb{R}_+$, $c > 0$ and $0 < p < 1$ are constants and suppose*

$$\Phi(x,y) \leq c + \int_0^x A(s,y)\Phi^p(s,y)ds + \int_0^y B(x,t)\Phi(x,t)dt + \int_0^x \int_0^y H(s,t)\Phi^p(s,t)ds\,dt, \tag{2.3.16}$$

for $x,y \in \mathbb{R}_+$. Then

$$\Phi(x,y) \leq Q_2(x,y)E_1(x,y)\left[c^{1-p} + (1-p)\int_0^x \int_0^y H(s,t){Q_2}^p(s,t){E_1}^p(s,t)ds\,dt\right]^{\frac{1}{1-p}},$$

where

$$Q_2(x,y) = \exp\left[\int_0^y B(x,t)dt\right], \tag{2.3.17}$$

$$E_1(x,y) = \left[1 + (1-p)\int_0^x A(s,y){Q_2}^p(s,y)m^{p-1}(s,y)ds\right]^{\frac{1}{1-p}}. \tag{2.3.18}$$

Proof. Define

$$m(x,y) = c + \int_0^x \int_0^y H(s,t)\Phi^p(s,t)ds\,dt, \tag{2.3.19}$$

$$m(x,0) = m(0,y) = c, \qquad m_x(x,0) = m_y(0,y) = 0. \tag{2.3.20}$$

By substituting (2.3.19) in (2.3.16), we get

$$\Phi(x,y) \le m(x,y) + \int_0^x A(s,y)\Phi^p(s,y)ds + \int_0^y B(x,t)\Phi(x,t)dt. \tag{2.3.21}$$

Since $m(x,y)$ is positive, non-decreasing continuous function, therefore

$$\frac{\Phi(x,y)}{m(x,y)} \le 1 + \int_0^x A(s,y)\frac{\Phi^p(s,y)}{m(s,y)}ds + \int_0^y B(x,t)\frac{\Phi(x,t)}{m(x,t)}dt. \tag{2.3.22}$$

Let

$$z(x,y) = 1 + \int_0^x A(s,y)\frac{\Phi^p(s,y)}{m(s,y)}ds + \int_0^y B(x,t)\frac{\Phi(x,t)}{m(x,t)}dt, \tag{2.3.23}$$

then,

$$z(0,0) = 1. \tag{2.3.24}$$

Substituting (2.3.23) in (2.3.22), implies

$$\frac{\Phi(x,y)}{m(x,y)} \le z(x,y). \tag{2.3.25}$$

Also from (2.3.23) and (2.3.25), we have

$$z(x,y) \le 1 + \int_0^x A(s,y)m^{p-1}(s,y)z^p(s,y)ds + \int_0^y B(x,t)z(x,t)dt. \tag{2.3.26}$$

Where we have used equation (2.3.11). Now by interchanging $A(x,y)$ by $B(x,y)$ and the limits of integration in Lemma 2.1.2, and following the same steps as before we get the result:

$$z(x,y) \le E_1(x,y)Q_2(x,y). \tag{2.3.27}$$

Where $E_1(x,y)$ and $Q_2(x,y)$ are defined as (2.3.18) and (2.3.17) respectively. Now by substituting (2.3.27) in (2.3.25), we get

$$\frac{\Phi(x,y)}{m(x,y)} \le E_1(x,y)Q_2(x,y),$$

this implies

$$\Phi(x,y) \le E_1(x,y)Q_2(x,y)m(x,y). \tag{2.3.28}$$

Differentiating (2.3.19) w.r.t x and y, we have

$$m_{xy}(x,y) = H(x,y)\Phi^p(x,y),$$

using (2.3.28), we have

$$m_{xy}(x,y) \le H(x,y){E_1}^p(x,y){Q_2}^p(x,y)m^p(x,y).$$

This implies

$$\frac{m_{xy}(x,y)}{m^p(x,y)} \le H(x,y){E_1}^p(x,y){Q_2}^p(x,y),$$

$$\frac{m_{xy}(x,y)m(x,y)}{m^{p+1}(x,y)} - \frac{m_x(x,y)m_y(x,y)}{m^{p+1}(x,y)} \le H(x,y){E_1}^p(x,y){Q_2}^p(x,y),$$

$$\frac{\partial}{\partial y}\left[\frac{m_x(x,y)}{m^p(x,y)}\right] \le H(x,y){E_1}^p(x,y){Q_2}^p(x,y). \tag{2.3.29}$$

By keeping y fixed, set $x = s$, in (2.3.29) and integrate from 0 to x, using (2.2.13) and (2.3.20), we have

$$\frac{d}{dy}\left[\frac{1}{1-p}\left[m^{1-p}(x,y) - c^{1-p}\right]\right] \le \int_0^x H(s,y){E_1}^p(s,y){Q_2}^p(s,y)ds.$$

Again keeping x fixed, set $y = t$, in the above inequality and integrate from 0 to y, using (2.3.20), we obtain

$$\frac{1}{1-p}\left[m^{1-p}(x,y) - c^{1-p}\right] \le \int_0^x\int_0^y H(s,t){E_1}^p(s,t){Q_2}^p(s,t)ds\,dt.$$

This implies

$$m(x,y) \le \left[c^{1-p} + (1-p)\int_0^x\int_0^y H(s,t){E_1}^p(s,t){Q_2}^p(s,t)ds\,dt\right]^{\frac{1}{1-p}},$$

substituting the above bound in (2.3.28), we have

$$\Phi(x,y) \le E_1(x,y)Q_2(x,y)\left[c^{1-p} + (1-p)\int_0^x\int_0^y H(s,t){E_1}^p(s,t){Q_2}^p(s,t)ds\,dt\right]^{\frac{1}{1-p}}.$$

□

Theorem 2.3.3. *Let $\Phi(x,y)$, $A(x,y)$, $B(x,y)$ and $H(x,y)$ be real valued non-negative, non-decreasing continuous functions defined for $x, y \in \mathbb{R}_+$, $c > 0$ and $0 < p < 1$ are constants and suppose*

$$\Phi(x,y) \le c + \int_0^x A(s,y)\Phi^p(s,y)ds + \int_0^y B(x,t)\Phi^p(x,t)dt + \int_0^x\int_0^y H(s,t)\Phi^p(s,t)ds\,dt, \tag{2.3.30}$$

for $x, y \in \mathbb{R}_+$. *Then*

$$\Phi(x,y) \leq Q_3(x,y)E_2(x,y)\left[c^{1-p} + (1-p)\int_0^x\int_0^y H(s,t){Q_3}^p(s,t){E_2}^p(s,t)ds\,dt\right]^{\frac{1}{1-p}}. \tag{2.3.31}$$

Where $Q_3(x,y)$, $E_2(x,y)$ *are defined as in* (2.1.29) *and* (2.1.28) *respectively.*

Proof. Define

$$m(x,y) = c + \int_0^x\int_0^y H(s,t)\Phi^p(s,t)ds\,dt, \tag{2.3.32}$$

$$m(x,0) = m(0,y) = c, \qquad m_x(x,0) = m_y(0,y) = 0. \tag{2.3.33}$$

By substituting (2.3.32) in (2.3.30), we get

$$\Phi(x,y) \leq m(x,y) + \int_0^x A(s,y)\Phi^p(s,y)ds + \int_0^y B(x,t)\Phi^p(x,t)dt. \tag{2.3.34}$$

Since $m(x,y)$ is positive, non-decreasing continuous function, therefore

$$\frac{\Phi(x,y)}{m(x,y)} \leq 1 + \int_0^x A(s,y)\frac{\Phi^p(s,y)}{m(s,y)}ds + \int_0^y B(x,t)\frac{\Phi^p(x,t)}{m(x,t)}dt. \tag{2.3.35}$$

Let

$$z(x,y) = 1 + \int_0^x A(s,y)\frac{\Phi^p(s,y)}{m(s,y)}ds + \int_0^y B(x,t)\frac{\Phi^p(x,t)}{m(x,t)}dt, \tag{2.3.36}$$

then,

$$z(0,0) = 1. \tag{2.3.37}$$

Substituting (2.3.36) in (2.3.35), implies

$$\frac{\Phi(x,y)}{m(x,y)} \leq z(x,y). \tag{2.3.38}$$

Also from (2.3.36) and (2.3.38), we have

$$z(x,y) \leq 1 + \int_0^x A(s,y)m^{p-1}(s,y)z^p(s,y)ds + \int_0^y B(x,t)m^{p-1}(x,t)z^p(x,t)dt. \tag{2.3.39}$$

Where we have used the equation (2.3.11). Now by using Lemma 2.1.3, we get

$$z(x,y) \leq E_2(x,y)Q_3(x,y). \tag{2.3.40}$$

Substituting (2.3.40) in (2.3.38), we get

$$\frac{\Phi(x,y)}{m(x,y)} \leq E_2(x,y)Q_3(x,y).$$

This implies

$$\Phi(x,y) \leq E_2(x,y)Q_3(x,y)m(x,y). \tag{2.3.41}$$

Differentiating (2.3.32) w.r.t x and y, we have

$$m_{xy}(x,y) = H(x,y)\Phi^p(x,y),$$

using (2.3.41), we have

$$m_{xy}(x,y) \leq H(x,y){E_2}^p(x,y){Q_3}^p(x,y)m^p(x,y),$$

this implies

$$\frac{m_{xy}(x,y)}{m^p(x,y)} \leq H(x,y){E_2}^p(x,y){Q_3}^p(x,y),$$

$$\frac{m_{xy}(x,y)m(x,y)}{m^{p+1}(x,y)} - \frac{m_x(x,y)m_y(x,y)}{m^{p+1}(x,y)} \leq H(x,y){E_2}^p(x,y){Q_3}^p(x,y),$$

$$\frac{\partial}{\partial y}\left[\frac{m_x(x,y)}{m^p(x,y)}\right] \leq H(x,y){E_2}^p(x,y){Q_3}^p(x,y). \tag{2.3.42}$$

By keeping y fixed, set $x = s$, in (2.3.42) and integrate from 0 to x, using (2.2.13) and (2.3.33), we have

$$\frac{d}{dy}\left[\frac{1}{1-p}\left[m^{1-p}(x,y) - c^{1-p}\right]\right] \leq \int_0^x H(s,y){E_2}^p(s,y){Q_3}^p(s,y)ds.$$

Again keeping x fixed, set $y = t$, in the above inequality and integrate from 0 to y, using (2.3.33), we obtain

$$\frac{1}{1-p}\left[m^{1-p}(x,y) - c^{1-p}\right] \leq \int_0^x \int_0^y H(s,t){E_2}^p(s,t){Q_3}^p(s,t)ds\,dt,$$

which implies

$$m(x,y) \leq \left[c^{1-p} + (1-p)\int_0^x \int_0^y H(s,t){E_2}^p(s,t){Q_3}^p(s,t)ds\,dt\right]^{\frac{1}{1-p}},$$

substituting the above bound in (2.3.41), we have

$$\Phi(x,y) \leq E_2(x,y)Q_3(x,y)\left[c^{1-p} + (1-p)\int_0^x \int_0^y H(s,t){E_2}^p(s,t){Q_3}^p(s,t)ds\,dt\right]^{\frac{1}{1-p}}.$$

□

Chapter 3

On Some New Retarded Integral Inequalities of Gronwall-Bihari Type in Two Independent Variables

Various linear and non-linear generalizations of Gronwall inequality have been established, and vast numbers of monographs have been dedicated on these inequalities and their applications (see for instance [1, 7, 10, 12, 15] and many more). In the past few years, some new inequalities have been discovered and many authors have generalized these inequalities to more than one variable such as [8, 9, 18, 20, 21]. This chapter is concerned with some new generalized retarded non-linear integral inequalities arising from well-known Gronwall-Bellman inequality and Bihari integral inequality. The established integral inequalities involve functions of two independent variables and are a generalized version of [6] and [16].

3.1 Retarded Integral Inequalities of Gronwall-Bihari Type in Two Independent Variables

The Lemma 3.1.1 proved below, is a generalized form of Gronwall-Bellman type inequality with retardation and is useful in our main results. Before stating the lemma, we will define some notations as follows: X and Y are any two subsets of $\mathbb{R}_+$. Intervals I, J of $\mathbb{R}_+$ are defined as: $I = [x_0, X)$, $J = [y_0, Y)$ and $\mathbb{I} = I \times J$.

Lemma 3.1.1. *Let c, Φ and $a \in C(I, \mathbb{R}_+)$ be non-negative continuous functions with $c(x)$ is non-decreasing function for $x \in I$ and assume that $\alpha \in C^1(I, I)$ be non-decreasing with $\alpha(x) \leq x$ on I. Suppose that $q \geq p > 0$ are constants. If $\Phi(x)$ satisfies the inequality:*

$$\Phi^q(x) \leq c(x) + \int_{\alpha(x_0)}^{\alpha(x)} a(s)\Phi^p(s)ds, \tag{3.1.1}$$

for $x_0 \leq s \leq x$, then the following inequalities are true

(1) If $p = q$

$$\Phi(x) \leq c^{1/p}(x) \exp\left(\frac{1}{p}\int_{\alpha(x_0)}^{\alpha(x)} a(s)ds\right). \tag{3.1.2}$$

(2) If $p < q$

$$\Phi(x) \leq c^{1/q}(x)\left[1 + \frac{q-p}{q}\int_{\alpha(x_0)}^{\alpha(x)} c^{\frac{(p-q)}{q}}(s)a(s)ds\right]^{\frac{1}{q-p}}, \tag{3.1.3}$$

for $x \in I$.

Proof. For $c(x) > 0$

(1) If $p = q$ holds, letting

$$z(x) = \left[\frac{\phi(x)}{c^{1/p}(x)}\right]^p. \tag{3.1.4}$$

Substituting (3.1.4) in (3.1.1), gives

$$z(x) \leq 1 + \int_{\alpha(x_0)}^{\alpha(x)} a(s)z(s)ds, \tag{3.1.5}$$

for $x \in I$. Let

$$v(x) = 1 + \int_{\alpha(x_0)}^{\alpha(x)} a(s)z(s)ds. \tag{3.1.6}$$

Where $v(x)$ is a positive, continuous and non-decreasing function and also

$$v(x_0) = 1. \tag{3.1.7}$$

Putting (3.1.6) in (3.1.5), we get

$$z(x) \leq v(x). \tag{3.1.8}$$

Now differentiating (3.1.6) w.r.t. x, using fundamental theorem of calculus, we obtain

$$v'(x) = \alpha'(x)a(\alpha(x))z(\alpha(x)), \tag{3.1.9}$$

since $\alpha(x) \leq x$ on I, from (3.1.8), we have

$$\begin{aligned} v'(x) &\leq \alpha'(x)a(\alpha(x))z(x), \\ &\leq \alpha'(x)a(\alpha(x))v(x). \end{aligned}$$

This implies

$$\frac{v'(x)}{v(x)} \leq \alpha'(x)a(\alpha(x)), \qquad x \in I. \tag{3.1.10}$$

By integration of (3.1.10) from x_0 to x, then we have

$$\ln v(x) \leq \int_{\alpha(x_0)}^{\alpha(x)} a(s)ds.$$

Using (3.1.8), we get

$$\ln z(x) \leq \int_{\alpha(x_0)}^{\alpha(x)} a(s)ds.$$

This implies

$$\ln \left[\frac{\phi(x)}{c^{1/p}(x)}\right]^p = \ln z(x) \leq \int_{\alpha(x_0)}^{\alpha(x)} a(s)ds.$$

Hence, we obtain

$$p \ln \left[\frac{\phi(x)}{c^{1/p}(x)}\right] \leq \int_{\alpha(x_0)}^{\alpha(x)} a(s)ds.$$

This inequality implies the desired inequality (3.1.2).

$$\Phi(x) \leq c^{1/p}(x) \exp\left(1/p \int_{\alpha(x_0)}^{\alpha(x)} a(s)ds\right).$$

(2) If $p < q$, letting

$$y(x) = \frac{\phi(x)}{c^{1/q}(x)}. \tag{3.1.11}$$

Substituting (3.1.11) in (3.1.1), gives

$$y^q(x) \leq 1 + \int_{\alpha(x_0)}^{\alpha(x)} a(s) y^p(s) c^{\frac{(p-q)}{q}}(s) ds. \tag{3.1.12}$$

Let

$$h(x) = 1 + \int_{\alpha(x_0)}^{\alpha(x)} a(s) y^p(s) c^{\frac{(p-q)}{q}}(s) ds. \tag{3.1.13}$$

Where $h(x)$ is a positive, continuous and non-decreasing function and also

$$h(x_0) = 1. \tag{3.1.14}$$

Putting (3.1.13) in (3.1.12), we get

$$y^q(x) \leq h(x),$$

$$y(x) \leq h^{1/q}(x). \tag{3.1.15}$$

Now differentiating (3.1.13) w.r.t. x, using fundamental theorem of calculus, $\alpha(x) \leq x$ on I and (3.1.15), we have obtain

$$\begin{aligned} h'(x) &= \alpha'(x) a(\alpha(x)) c^{\frac{(p-q)}{q}}(\alpha(x)) y^p(\alpha(x)), \\ &\leq \alpha'(x) a(\alpha(x)) c^{\frac{(p-q)}{q}}(\alpha(x)) y^p(x), \\ &\leq \alpha'(x) a(\alpha(x)) c^{\frac{(p-q)}{q}}(\alpha(x)) h^{p/q}(x). \end{aligned} \tag{3.1.16}$$

This implies

$$\frac{h'(x)}{h^{p/q}(x)} \leq \alpha'(x) a(\alpha(x)) c^{\frac{(p-q)}{q}}(\alpha(x)), \qquad x \in I. \tag{3.1.17}$$

By integration of (3.1.17) from x_0 to x, we have

$$\frac{q}{q-p}\left[h^{\frac{(q-p)}{q}}(x) - 1\right] \leq \int_{\alpha(x_0)}^{\alpha(x)} a(s) c^{\frac{(p-q)}{q}}(\alpha(x)) ds,$$

$$h^{\frac{(q-p)}{q}}(x) \leq 1 + \frac{q-p}{q} \int_{\alpha(x_0)}^{\alpha(x)} a(s) c^{\frac{(p-q)}{q}}(\alpha(x)) ds.$$

Using (3.1.15), we get

$$y^{(q-p)}(x) \leq 1 + \frac{q-p}{q} \int_{\alpha(x_0)}^{\alpha(x)} a(s) c^{\frac{(p-q)}{q}}(\alpha(x)) ds.$$

This implies

$$\left[\frac{\phi(x)}{c^{1/q}(x)}\right]^{(q-p)} \leq 1 + \frac{q-p}{q} \int_{\alpha(x_0)}^{\alpha(x)} a(s) c^{\frac{(p-q)}{q}}(\alpha(x)) ds.$$

Hence we obtain

$$\Phi(x) \leq c^{1/q}(x) \left[1 + \frac{q-p}{q} \int_{\alpha(x_0)}^{\alpha(x)} a(s) c^{\frac{(p-q)}{q}}(\alpha(x)) ds\right]^{\frac{1}{q-p}}.$$

For $c(x) \geq 0$, we take $c(x) + \epsilon$ instead of $c(x)$, in the above proof, where $\epsilon > 0$ is an arbitrary small constant, and $\epsilon \to 0$ to obtain (3.1.2) and (3.1.3). This completes the proof. □

Theorem 3.1.2. *Let c, Φ, A and $B \in C(\mathbb{I}, \mathbb{R}_+)$ be non-decreasing, continuous functions in each variables and assume that $\alpha \in C^1(I, I)$, $\beta \in C^1(J, J)$ be non-decreasing with $\alpha(x) \leq x$ on I, $\beta(y) \leq y$ on J. Suppose that $1 > p > 0$ is constant.*

(A1) If $\Phi(x, y)$ satisfies

$$\Phi(x,y) \leq c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi(s,y) ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t) dt, \quad (3.1.18)$$

for all $(x, y) \in \mathbb{I}$, then

$$\Phi(x,y) \leq c(x,y) E_1(x,y) Q_1(x,y), \quad (3.1.19)$$

for all $(x, y) \in \mathbb{I}$. Where

$$E_1(x,y) = \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} A(s,y) ds\right), \quad (3.1.20)$$

$$Q_1(x,y) = \left[1 + (1-p) \int_{\beta(y_0)}^{\beta(y)} B(x,t) c^{(p-1)}(x,t) E_1^p(x,t) dt\right]^{\frac{1}{1-p}}. \quad (3.1.21)$$

(A2) If $\Phi(x,y)$ satisfies

$$\Phi(x,y) \leq c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi^p(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi(x,t)dt, \quad (3.1.22)$$

for all $(x,y) \in \mathbb{I}$, then

$$\Phi(x,y) \leq c(x,y)E_2(x,y)Q_2(x,y), \quad (3.1.23)$$

for all $(x,y) \in \mathbb{I}$. Where

$$E_2(x,y) = \exp\left(\int_{\beta(y_0)}^{\beta(y)} B(x,t)dt\right), \quad (3.1.24)$$

$$Q_2(x,y) = \left[1 + (1-p)\int_{\alpha(x_0)}^{\alpha(x)} A(s,y)c^{(p-1)}(s,y)E_2^p(s,y)ds\right]^{\frac{1}{1-p}}. \quad (3.1.25)$$

Proof. (A1) We define a function $z(x,y)$ by

$$z(x,y) = c(x,y) + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t)dt, \quad (3.1.26)$$

by substituting (3.1.26) in (3.1.18), we get

$$\Phi(x,y) \leq z(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi(s,y)ds, \qquad (x,y) \in \mathbb{I}. \quad (3.1.27)$$

Clearly $z(x,y)$ is a non-negative, continuous and non-decreasing function in x. Keeping $y \in J$ fixed in (3.1.27), a suitable application of Lemma 3.1.1 to (3.1.27) implies

$$\phi(x,y) \leq z(x,y)\exp\left(\int_{\alpha(x_0)}^{\alpha(x)} A(s,y)ds\right),$$

for $(x,y) \in \mathbb{I}$, where

$$\exp\left(\int_{\alpha(x_0)}^{\alpha(x)} A(s,y)ds\right) = E_1(x,y),$$

as defined in (3.1.20). Thus

$$\phi(x,y) \leq z(x,y)E_1(x,y). \quad (3.1.28)$$

By (3.1.26) and (3.1.28), we obtain

$$z(x,y) \leq c(x,y) + \int_{\beta(y_0)}^{\beta(y)} B(x,t){E_1}^p(x,t)z^p(x,t)dt. \tag{3.1.29}$$

Keeping x fixed in (3.1.29), an estimation of $z(x,y)$ can be obtained by a suitable application of Lemma 3.1.1 to (3.1.29), after that, we obtained

$$z(x,y) \leq c(x,y)\left[1 + (1-p)\int_{\beta(y_0)}^{\beta(y)} c^{(p-1)}(x,t)B(x,t){E_1}^p(x,t)dt\right]^{\frac{1}{1-p}},$$

for $(x,y) \in \mathbb{I}$, where

$$\left[1 + (1-p)\int_{\beta(y_0)}^{\beta(y)} c^{(p-1)}(x,t)B(x,t){E_1}^p(x,t)dt\right]^{\frac{1}{1-p}} = Q_1(x,y),$$

as defined in (3.1.21). This implies

$$z(x,y) \leq c(x,y)Q_1(x,y). \tag{3.1.30}$$

Finally substituting the last inequality into (3.1.28), we have

$$\Phi(x,y) \leq c(x,y)E_1(x,y)Q_1(x,y).$$

(A2) We define a function $z(x,y)$ by

$$z(x,y) = c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi^p(s,y)ds, \tag{3.1.31}$$

by substituting (3.1.31) in (3.1.22), we get

$$\Phi(x,y) \leq z(x,y) + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi(x,t)dt, \qquad (x,y) \in \mathbb{I}. \tag{3.1.32}$$

Clearly $z(x,y)$ is a non-negative, continuous and non-decreasing function in y. Keeping $x \in I$ fixed in (3.1.32), a suitable application of Lemma 3.1.1 to (3.1.32) implies

$$\phi(x,y) \leq z(x,y)\exp\left(\int_{\beta(y_0)}^{\beta(y)} B(x,t)dt\right),$$

for $(x,y) \in \mathbb{I}$, where

$$\exp\left(\int_{\beta(y_0)}^{\beta(y)} B(x,t)dt\right) = E_2(x,y),$$

as defined in (3.1.24). Thus

$$\phi(x,y) \leq z(x,y)E_2(x,y). \tag{3.1.33}$$

By (3.1.31) and (3.1.33), we obtain

$$z(x,y) \leq c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y){E_2}^p(s,y)z^p(s,y)ds. \tag{3.1.34}$$

Keeping y fixed in (3.1.34), an estimation of $z(x,y)$ can be obtained by a suitable application of Lemma 3.1.1 to (3.1.34), after that, we obtained

$$z(x,y) \leq c(x,y)\left[1 + (1-p)\int_{\alpha(x_0)}^{\alpha(x)} c^{(p-1)}(s,y)A(s,y){E_2}^p(s,y)ds\right]^{\frac{1}{1-p}},$$

for $(x,y) \in \mathbb{I}$, where

$$\left[1 + (1-p)\int_{\alpha(x_0)}^{\alpha(x)} c^{(p-1)}(s,y)A(s,y){E_2}^p(s,y)ds\right]^{\frac{1}{1-p}} = Q_2(x,y),$$

as defined in (3.1.25). This implies

$$z(x,y) \leq c(x,y)Q_2(x,y). \tag{3.1.35}$$

Finally substituting the last inequality into (3.1.33), we have

$$\Phi(x,y) \leq c(x,y)E_2(x,y)Q_2(x,y).$$

□

Theorem 3.1.3. *Let c, ϕ, A, B, α and β be defined as in Theorem 3.1.2. Suppose that $q \geq p > 0$ are constants. If $\phi(x,y)$ satisfies the inequality:*

$$\Phi^q(x,y) \leq c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi^p(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t)dt, \tag{3.1.36}$$

for all $(x,y) \in \mathbb{I}$*, then we have:*

$$\Phi(x,y) \leq \left\{ \begin{array}{ll} [c(x,y)E_1(x,y)Q_3(x,y)]^{\frac{1}{p}}, & \text{if } p = q \\ c^{1/q}(x,y)E_4(x,y)Q_4(x,y), & \text{if } p < q \end{array} \right\} \tag{3.1.37}$$

for all $(x,y) \in \mathbb{I}$*, where*

$$Q_3(x,y) = \exp\left(\int_{\beta(y_0)}^{\beta(y)} B(x,t)E_1(x,t)dt\right), \tag{3.1.38}$$

and $E_1(x,y)$ *is defined as* (3.1.20)*, and*

$$Q_4(x,y) = \left[1 + \frac{q-p}{q}\int_{\beta(y_0)}^{\beta(y)} z^{\frac{(p-q)}{q}}(x,t)B(x,t)dt\right]^{\frac{1}{q-p}}, \tag{3.1.39}$$

$$E_4(x,y) = \left[1 + \frac{q-p}{q}\int_{\alpha(x_0)}^{\alpha(x)} c^{\frac{(p-q)}{q}}(s,y)A(s,y){Q_4}^p(s,y)ds\right]^{\frac{1}{q-p}}, \tag{3.1.40}$$

where $z(x,y) \leq c(x,y){E_4}^q(x,y)$*, for all* $(x,y) \in \mathbb{I}$*.*

Proof. (1) If $p < q$ holds, we define a function $z(x,y)$ by

$$z(x,y) = c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi^p(s,y)ds, \tag{3.1.41}$$

by substituting (3.1.41) in (3.1.36), we get

$$\Phi^q(x,y) \leq z(x,y) + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t)dt. \tag{3.1.42}$$

Clearly $z(x,y)$ is a non-negative, continuous and non-decreasing function in y. Treating x fixed in (3.1.42), and a suitable application of Lemma 3.1.1 to (3.1.42) gives

$$\phi(x,y) \leq z(x,y)^{\frac{1}{q}}\left[1 + \frac{q-p}{q}\int_{\beta(y_0)}^{\beta(y)} z^{\frac{(p-q)}{q}}(x,t)B(x,t)dt\right]^{\frac{1}{q-p}},$$

for $(x,y) \in \mathbb{I}$, where

$$\left[1 + \frac{q-p}{q}\int_{\beta(y_0)}^{\beta(y)} z^{\frac{(p-q)}{q}}(x,t)B(x,t)dt\right]^{\frac{1}{q-p}} = Q_4(x,y),$$

as defined in (3.1.39). This implies

$$\phi(x,y) \leq z(x,y)^{\frac{1}{q}} Q_4(x,y). \tag{3.1.43}$$

By (3.1.43) and (3.1.41), we obtain

$$z(x,y) \leq c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y) {Q_4}^p(s,y) z^{p/q}(s,y) ds. \tag{3.1.44}$$

Keeping y fixed in (3.1.44), an estimation of $z(x,y)$ can be obtained by a suitable application of Lemma 3.1.1 to (3.1.44), after that, we obtain

$$z(x,y) \leq c(x,y) \left[1 + \frac{q-p}{q} \int_{\alpha(x_0)}^{\alpha(x)} c^{\frac{(p-q)}{q}} A(s,y) {Q_4}^p(s,y) \right]^{\frac{q}{q-p}},$$

for $(x,y) \in \mathbb{I}$, where

$$\left[1 + \frac{q-p}{q} \int_{\alpha(x_0)}^{\alpha(x)} c^{\frac{(p-q)}{q}} A(s,y) {Q_4}^p(s,y) \right]^{\frac{1}{q-p}} = E_4(x,y),$$

as defined in (3.1.40). This implies

$$z(x,y) \leq c(x,y) {E_4}^q(x,y). \tag{3.1.45}$$

Finally, substituting the last inequality into (3.1.43), the desired inequality (3.1.37) follows i.e

$$\phi(x,y) \leq c^{1/q}(x,y) E_4(x,y) Q_4(x,y).$$

(2) If $p = q$, we define a function $z(x,y)$ by

$$z(x,y) = c(x,y) + \int_{\beta(y_0)}^{\beta(y)} B(x,t) \Phi^p(x,t) dt, \tag{3.1.46}$$

by substituting (3.1.46) in (3.1.36), we get

$$\Phi^q(x,y) \leq z(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y) \Phi^p(s,y) ds. \tag{3.1.47}$$

Clearly $z(x,y)$ is a non-negative, continuous and non-decreasing function in x. Treating y fixed in (3.1.47), and a suitable application of Lemma 3.1.1 to (3.1.47) gives

$$\begin{aligned}\phi(x,y) &\leq z(x,y)^{\frac{1}{p}} \exp\left[\frac{1}{p}\int_{\alpha(x_0)}^{\alpha(x)} A(s,y)ds\right], \\ &= z(x,y)^{\frac{1}{p}} \left[\exp\left(\int_{\alpha(x_0)}^{\alpha(x)} A(s,y)ds\right)\right]^{\frac{1}{p}},\end{aligned} \tag{3.1.48}$$

for $(x,y) \in \mathbb{I}$, where

$$\exp\left[\int_{\alpha(x_0)}^{\alpha(x)} A(s,y)ds\right] = E_1(x,y),$$

as defined in (3.1.20). This implies

$$\phi(x,y) \leq z(x,y)^{\frac{1}{p}} E_1^{\frac{1}{p}}(x,y). \tag{3.1.49}$$

By (3.1.46) and (3.1.49), we obtain

$$z(x,y) \leq c(x,y) + \int_{\beta(y_0)}^{\beta(y)} B(x,t)E_1(x,t)z(x,t)dt. \tag{3.1.50}$$

Keeping x fixed in (3.1.50), an estimation of $z(x,y)$ can be obtained by a suitable application of Lemma 3.1.1 to (3.1.50), after that, we obtain

$$z(x,y) \leq c(x,y) \exp\left[\int_{\beta(y_0)}^{\beta(y)} B(x,t)E_1(x,t)dt\right],$$

for $(x,y) \in \mathbb{I}$, where

$$\exp\left[\int_{\beta(y_0)}^{\beta(y)} B(x,t)E_1(x,t)dt\right] = Q_3(x,y),$$

as defined in (3.1.38). This implies

$$z(x,y) \leq c(x,y)Q_3(x,y). \tag{3.1.51}$$

Finally, substituting the last inequality into (3.1.49), the desired inequality (3.1.37) follows i.e

$$\phi(x,y) \le \left[c(x,y)E_1(x,y)Q_3(x,y)\right]^{\frac{1}{p}}.$$

□

Remark 3.1.1. If we take $B(x,y) = 0$ and keeping y fixed, then Theorem 3.1.3 reduce exactly to Lemma 3.1.1.

3.2 Generalization of Retarded Integral Inequalities of Gronwall-Bihari Type in Two Independent Variables

Before stating the theorem, we will define the set S as follows:

$$S = \{(x,y,s,t) \in \mathbb{I}^2 : x_0 \le s \le x \le X; y_0 \le t \le y \le Y\}$$

Theorem 3.2.1. *Let c, ϕ, A, B, α and β be defined as in Theorem 3.1.2. Let $H(x,y,s,t) \in C(S,\mathbb{R}_+)$ be a non-decreasing and continuous functions in x and y for each $(s,t) \in \mathbb{I}$. Let $w \in C(\mathbb{R}_+,\mathbb{R}_+)$ be non-decreasing and sub-multiplicative function with $w(\phi) > 0$ for $\phi > 0$.*

(B1) If $\phi(x,y)$ satisfies

$$\Phi(x,y) \le c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t)dt + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)w(\Phi(s,t))ds\,dt, \tag{3.2.1}$$

for all $(x,y) \in \mathbb{I}$. Then

$$\Phi(x,y) \le M_1(x,y)E_1(x,y)\tilde{Q}_1(x,y), \tag{3.2.2}$$

for all $x_0 \le x \le x_1$, $y_0 \le y \le y_1$. Where

$$M_1(x,y) \le G^{-1}\left[G(c(x,y)) + C_1\right], \tag{3.2.3}$$

for all $x_0 \leq x \leq x_1$*,* $y_0 \leq y \leq y_1$*. Where*

$$C_1 = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) w(E_1(s, t)) w(\tilde{Q}_1(s, t)) ds\, dt, \tag{3.2.4}$$

and

$$G(\phi) = \int_{\phi_0}^{\phi} \frac{\delta t}{w(t)}, \qquad \phi \geq \phi_0 > 0. \tag{3.2.5}$$

Where $E_1(x, y)$ *is defined in* (3.1.20) *and*

$$\tilde{Q}_1(x, y) = \left[1 + (1 - p) \int_{\beta(y_0)}^{\beta(y)} B(x, t) {M_1}^{(p-1)}(x, t) {E_1}^p(x, t) dt \right]^{\frac{1}{1-p}}, \tag{3.2.6}$$

where G^{-1} *is the inverse function of* G *and the real numbers* $x_1, y_1 \in \mathbb{R}_+$ *are chosen so that* $G(c(x, y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) w(E_1(s, t)) w(\tilde{Q}_1(s, t)) ds\, dt \in Dom(G^{-1})$*.*

(B2) If $\phi(x, y)$ *satisfies*

$$\Phi(x, y) \leq c(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s, y) \Phi^p(s, y) ds + \int_{\beta(y_0)}^{\beta(y)} B(x, t) \Phi(x, t) dt + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) w(\Phi(s, t)) ds\, dt, \tag{3.2.7}$$

for all $(x, y) \in \mathbb{I}$*. Then*

$$\Phi(x, y) \leq M_2(x, y) E_2(x, y) \tilde{Q}_2(x, y), \tag{3.2.8}$$

for all $x_0 \leq x \leq x_2$*,* $y_0 \leq y \leq y_2$*. Where*

$$M_2(x, y) \leq G^{-1} \left[G(c(x, y)) + C_2 \right], \tag{3.2.9}$$

for all $x_0 \leq x \leq x_2$*,* $y_0 \leq y \leq y_2$*. Where*

$$C_2 = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) w(E_1(s, t)) w(\tilde{Q}_1(s, t)) ds\, dt, \tag{3.2.10}$$

and G *and* E_2 *are defined in* (3.2.5) *and* (3.1.24) *with*

$$\tilde{Q}_2(x, y) = \left[1 + (1 - p) \int_{\alpha(x_0)}^{\alpha(x)} A(s, y) {M_2}^{(p-1)}(s, y) {E_2}^p(s, y) ds \right]^{\frac{1}{1-p}}, \tag{3.2.11}$$

where G^{-1} is the inverse function of G and the real numbers $x_2, y_2 \in \mathbb{R}_+$ are chosen so that $G(c(x,y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t) w(E_2(s,t)) w(\tilde{Q}_2(s,t)) ds\, dt \in Dom(G^{-1})$.

Proof. (B1) If $c(x,y) > 0$. Setting

$$M_1(x,y) = c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t) w(\phi(s,t)) ds\, dt, \tag{3.2.12}$$

the inequality (3.2.1) can be restated as

$$\Phi(x,y) \leq M_1(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi(s,y) ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t) dt. \tag{3.2.13}$$

Clearly $M_1(x,y)$ is non-negative and non-decreasing function in each x and y. Now a suitable application of the inequality (3.1.18) in Theorem 3.1.2 to (3.2.13), yields

$$\Phi(x,y) \leq M_1(x,y) E_1(x,y) \tilde{Q}_1(x,y), \tag{3.2.14}$$

where $E_1(x,y)$ and $\tilde{Q}_1(x,y)$ are defined in (3.1.20) and (3.2.6). From (3.2.12) and (3.2.14) and by using the fact that w is a sub-multiplicative, we have

$$M_1(x,y) \leq c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t) w(E_1(s,t)\tilde{Q}_1(s,t)) w(M_1(s,t)) ds\, dt, \tag{3.2.15}$$

for $(x,y) \in \mathbb{I}$.

Fixing any numbers $\tilde{x_1}$ and $\tilde{y_1}$ with $0 < \tilde{x_1} \leq x_1$ and $0 < \tilde{y_1} \leq y_1$, from (3.2.15) we have

$$M_1(x,y) \leq c(\tilde{x_1}, \tilde{y_1}) + \tilde{D}_1,$$

for $x_0 \leq x \leq \tilde{x_1}$, $y_0 \leq y \leq \tilde{y_1}$. Defining

$$r_1(x,y) = c(\tilde{x_1}, \tilde{y_1}) + \tilde{D}_1,$$

where

$$\tilde{D}_1 = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t) w(E_1(s,t)\tilde{Q}_1(s,t)) w(M_1(s,t)) ds\, dt.$$

Then $r_1(x_0, y) = r_1(x, y_0) = c(\tilde{x_1}, \tilde{y_1})$ and

$$M_1(x, y) \le r_1(x, y), \tag{3.2.16}$$

with $r_1(x, y)$ is positive and non-decreasing in $y \in [y_0, \tilde{y_0}]$, and

$$\begin{aligned} D_1 r_1(x, y) &= u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, u, t) w(E_1(u, t)\tilde{Q_1}(u, t)) w(M_1(u, t)) dt, \\ &\le u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, u, t) w(E_1(u, t)\tilde{Q_1}(u, t)) w(r_1(u, t)) dt, \\ &\le w(r_1(x, y)) u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, u, t) w(E_1(u, t)\tilde{Q_1}(u, t)) dt. \end{aligned} \tag{3.2.17}$$

Where $u = \alpha(x)$ implying $u' = \alpha'(x)$. Dividing both sides of (3.2.17) by $w(r_1(x, y))$, we obtain

$$\frac{D_1 r_1(x, y)}{w(r_1(x, y))} \le \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, \alpha(x), t) w(E_1(\alpha(x), t)\tilde{Q_1}(\alpha(x), t)) dt, \tag{3.2.18}$$

from (3.2.5) and (3.2.18) we have

$$D_1 G(r_1(x, y)) \le \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, \alpha(x), t) w(E_1(\alpha(x), t)\tilde{Q_1}(\alpha(x), t)) dt. \tag{3.2.19}$$

Now setting $x = s$ in (3.2.19) and then integrating with respect to s from x_0 to x, we obtain

$$G(r_1(x, y)) \le G(r_1(x_0, y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t) w(E_1(s, t)\tilde{Q_1}(s, t)) ds\, dt.$$

Noting $G(r_1(x_0, y)) = G(c(\tilde{x_1}, \tilde{y_1}))$, we have

$$G(r_1(x, y)) \le G(c(\tilde{x_1}, \tilde{y_1})) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t) w(E_1(s, t)\tilde{Q_1}(s, t)) ds\, dt.$$

Taking $x = \tilde{x_1}$, $y = \tilde{y_1}$ in (3.2.16) and the last inequality, we obtain

$$M_1(\tilde{x_1}, \tilde{y_1}) \le r_1(\tilde{x_1}, \tilde{y_1}), \tag{3.2.20}$$

and

$$G(r_1(\tilde{x_1},\tilde{y_1})) \leq G(c(\tilde{x_1},\tilde{y_1}))+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t)w(E_1(s,t)\tilde{Q}_1(s,t))ds\,dt. \tag{3.2.21}$$

Since $0<\tilde{x_1}\leq x_1$ and $0<\tilde{y_1}\leq y_1$ are arbitrary, from (3.2.20) and (3.2.21) we have

$$M_1(x,y)\leq r_1(x,y), \tag{3.2.22}$$

and

$$r_1(x,y)\leq G^{-1}\left[G(c(x,y))+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)w(E_1(s,t)\tilde{Q}_1(s,t))ds\,dt\right], \tag{3.2.23}$$

for all $x_0<x\leq x_1$, $y_0<y\leq y_1$. Hence by (3.2.22) and (3.2.23), we obtain

$$M_1(x,y)\leq G^{-1}\left[G(c(x,y))+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)w(E_1(s,t)\tilde{Q}_1(s,t))ds\,dt\right], \tag{3.2.24}$$

for all $x_0<x\leq x_1$, $y_0<y\leq y_1$. By (3.2.1), (3.2.24) holds also when $x=x_0$ and $y=y_0$.

Finally substituting the last inequality into (3.2.14), the desired inequality (3.2.2) follows immediately.

(B2) If $c(x,y)>0$. Setting

$$M_2(x,y)=c(x,y)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)w(\phi(s,t))ds\,dt, \tag{3.2.25}$$

the inequality (3.2.7) can be restated as

$$\Phi(x,y)\leq M_2(x,y)+\int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi^p(s,y)ds+\int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi(x,t)dt. \tag{3.2.26}$$

Clearly $M_2(x,y)$ is non-negative and non-decreasing function in each x and y. Now a suitable application of the inequality (3.1.22) in Theorem 3.1.2 to (3.2.26), yields

$$\Phi(x,y)\leq M_2(x,y)E_2(x,y)\tilde{Q}_2(x,y), \tag{3.2.27}$$

where $E_2(x,y)$ and $\tilde{Q}_2(x,y)$ are defined in (3.1.24) and (3.2.11).

From (3.2.25) and (3.2.27) and by using the fact that w is a sub-multiplicative, we have

$$M_2(x,y) \le c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t) w(E_2(s,t)\tilde{Q}_2(s,t)) w(M_2(s,t)) ds\, dt, \tag{3.2.28}$$

for $(x,y) \in \mathbb{I}$.

Fixing any numbers $\tilde{x_2}$ and $\tilde{y_2}$ with $0 < \tilde{x_2} \le x_2$ and $0 < \tilde{y_2} \le y_2$, from (3.2.28) we have

$$M_2(x,y) \le c(\tilde{x_2}, \tilde{y_2}) + \tilde{D_2},$$

for $x_0 \le x \le \tilde{x_2}$, $y_0 \le y \le \tilde{y_2}$. Defining

$$r_2(x,y) = c(\tilde{x_2}, \tilde{y_2}) + \tilde{D_2},$$

where

$$\tilde{D_2} = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_2}, \tilde{y_2}, s, t) w(E_2(s,t)\tilde{Q}_2(s,t)) w(M_2(s,t)) ds\, dt.$$

Then $r_2(x_0,y) = r_2(x,y_0) = c(\tilde{x_2}, \tilde{y_2})$ and

$$M_2(x,y) \le r_2(x,y), \tag{3.2.29}$$

with $r_2(x,y)$ is positive and non-decreasing in $x \in [x_0, \tilde{x_0}]$, and

$$\begin{aligned} D_2 r_2(x,y) &= v' \int_{\alpha(x_0)}^{\alpha(x)} H(\tilde{x_2}, \tilde{y_2}, s, v) w(E_2(s,v)\tilde{Q}_2(s,v)) w(M_2(s,v)) ds, \\ &\le v' \int_{\alpha(x_0)}^{\alpha(x)} H(\tilde{x_2}, \tilde{y_2}, s, v) w(E_2(s,v)\tilde{Q}_2(s,v)) w(r_2(s,v)) ds, \\ &\le w(r_2(x,y)) v' \int_{\alpha(x_0)}^{\alpha(x)} H(\tilde{x_2}, \tilde{y_2}, s, v) w(E_2(s,v)\tilde{Q}_2(s,v)) ds. \end{aligned} \tag{3.2.30}$$

Where $v = \beta(y)$ implying $v' = \beta'(y)$. Dividing both sides of (3.2.30) by $w(r_2(x,y))$, we obtain

$$\frac{D_2 r_2(x,y)}{w(r_2(x,y))} \le \beta'(y) \int_{\alpha(x_0)}^{\alpha(x)} H(\tilde{x_2}, \tilde{y_2}, s, \beta(y)) w(E_2(s,\beta(y))\tilde{Q}_2(s,\beta(y))) ds, \tag{3.2.31}$$

from (3.2.5) and (3.2.31) we have

$$D_2G(r_2(x,y)) \le \beta'(y)\int_{\alpha(x_0)}^{\alpha(x)} H(\tilde{x_2},\tilde{y_2},s,\beta(y))w(E_2(s,\beta(y))\tilde{Q}_2(s,\beta(y)))ds. \tag{3.2.32}$$

Now setting $y = t$ in (3.2.32) and then integrating with respect to t from y_0 to y, we obtain

$$G(r_2(x,y)) \le G(r_2(x,y_0)) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_2},\tilde{y_2},s,t)w(E_2(s,t)\tilde{Q}_2(s,t))ds\,dt.$$

Noting $G(r_2(x,y_0)) = G(c(\tilde{x_2},\tilde{y_2}))$, we have

$$G(r_2(x,y)) \le G(c(\tilde{x_2},\tilde{y_2})) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_2},\tilde{y_2},s,t)w(E_2(s,t)\tilde{Q}_2(s,t))ds\,dt.$$

Taking $x = \tilde{x_2}$, $y = \tilde{y_2}$ in (3.2.29) and the last inequality, we obtain

$$M_2(\tilde{x_2},\tilde{y_2}) \le r_2(\tilde{x_2},\tilde{y_2}), \tag{3.2.33}$$

and

$$G(r_2(\tilde{x_2},\tilde{y_2})) \le G(c(\tilde{x_2},\tilde{y_2})) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_2},\tilde{y_2},s,t)w(E_2(s,t)\tilde{Q}_2(s,t))ds\,dt. \tag{3.2.34}$$

Since $0 < \tilde{x_2} \le x_2$ and $0 < \tilde{y_2} \le y_2$ are arbitrary, from (3.2.33) and (3.2.34) we have

$$M_2(x,y) \le r_2(x,y), \tag{3.2.35}$$

and

$$r_2(x,y) \le G^{-1}\left[G(c(x,y)) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)w(E_2(s,t)\tilde{Q}_2(s,t))ds\,dt\right], \tag{3.2.36}$$

for all $x_0 < x \le x_2$, $y_0 < y \le y_2$. Hence by (3.2.35) and (3.2.36), we obtain

$$M_2(x,y) \le G^{-1}\left[G(c(x,y)) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)w(E_2(s,t)\tilde{Q}_2(s,t))ds\,dt\right], \tag{3.2.37}$$

for all $x_0 < x \le x_2$, $y_0 < y \le y_2$. By (3.2.7), (3.2.37) holds also when $x = x_0$ and $y = y_0$.

Finally substituting the last inequality into (3.2.27), the desired inequality (3.2.8) follows immediately.

If $c(x, y) \ge 0$ is non-negative, we carry out the above procedure in $(B1)$ and $(B2)$ with $c(x, y) + \epsilon$ instead of $c(x, y)$ where $\epsilon > 0$ is an arbitrary small constant and subsequently pass to the limit as $\epsilon \to 0$ to obtain (3.2.2) and (3.2.8). This completes the proof.

□

Remark 3.2.1. If we take $H(x, y, s, t) = 0$ in the last theorem then Theorem 3.2.1 reduce to Theorem 3.1.2.

Remark 3.2.2. Theorem 3.2.1 (**B2**) reduces to Theorem 2.3.3 in Chapter 2, if $p = 1$, $w(s) = s$, $c(x, y) = c$, $\alpha(x) = x$, $\beta(y) = y$, $H(x, y, s, t) = H(s, t)$ $x_0 = y_0 = 0$.

Using Theorem 3.1.3, we can get some more generalized results as follows:

Theorem 3.2.2. *Let c, ϕ, A, B, H, α, β and w be defined as in Theorem 3.2.1. Suppose that $q \ge p > 0$ are constants. If $\phi(x, y)$ satisfies*

$$\Phi^q(x, y) \le c(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s, y)\Phi^p(s, y)ds + \int_{\beta(y_0)}^{\beta(y)} B(x, t)\Phi^p(x, t)dt + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t)w(\Phi(s, t))ds\,dt, \tag{3.2.38}$$

for all $(x, y) \in \mathbb{I}$, then the following conclusions are true:

(C1) If $p = q$, then

$$\Phi(x, y) \le \left[N_1(x, y)E_1(x, y)Q_3(x, y)\right]^{1/p}, \tag{3.2.39}$$

for all $x_0 \le x \le x_3$, $y_0 \le y \le y_3$. Where

$$N_1(x, y) \le K^{-1}\left[K(c(x, y)) + C_3\right], \tag{3.2.40}$$

for all $x_0 \le x \le x_3$, $y_0 \le y \le y_3$. Where

$$C_3 = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t)w(E_1{}^{\frac{1}{p}}(s, t))w(Q_3{}^{\frac{1}{p}}(s, t))ds\,dt. \tag{3.2.41}$$

and

$$K(\phi) = \int_{\phi_0}^{\phi} \frac{\delta t}{w(t^{1/q})}, \qquad \phi \geq \phi_0 > 0. \tag{3.2.42}$$

Where $E_1(x,y)$ *and* $Q_3(x,y)$ *are defined in* (3.1.20) *and* (3.1.38). *And* K^{-1} *is the inverse function of* K *and the real numbers* x_3, y_3 *are chosen so that* $K(c(x,y))+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)w(E_1{}^{\frac{1}{p}}(s,t))w(Q_3{}^{\frac{1}{p}}(s,t))ds\, dt \in Dom(K^{-1})$.

(C2) If $p < q$, *then*

$$\Phi(x,y) \leq N_2{}^{1/q}(x,y)\tilde{E}_4(x,y)\tilde{Q}_4(x,y), \tag{3.2.43}$$

for all $x_0 \leq x \leq x_4$, $y_0 \leq y \leq y_4$. *Where*

$$N_2(x,y) \leq K^{-1}\left[K(c(x,y)) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)w(\tilde{E}_4(s,t)\tilde{Q}_4(s,t))ds\, dt\right], \tag{3.2.44}$$

for all $x_0 \leq x \leq x_4$, $y_0 \leq y \leq y_4$, K *is defined in* (3.2.42), *with*

$$\tilde{Q}_4(x,y) = \left[1 + \frac{(q-p)}{q}\int_{\beta(y_0)}^{\beta(y)} B(x,t)\tilde{z}^{\frac{(p-q)}{q}}(x,t)dt\right]^{\frac{1}{q-p}}, \tag{3.2.45}$$

$$\tilde{E}_4(x,y) = \left[1 + \frac{(q-p)}{q}\int_{\alpha(x_0)}^{\alpha(x)} A(s,y)N_2{}^{\frac{(p-q)}{q}}(s,y)\tilde{Q}_4{}^{p}(s,y)ds\right]^{\frac{1}{q-p}}, \tag{3.2.46}$$

for all $x_0 \leq x \leq x_4$, $y_0 \leq y \leq y_4$, *where* $\tilde{z}(x,y) \leq N_2(x,y)\tilde{E}_4{}^{p}(x,y)$.
Where K^{-1} *is the inverse function of* K *and the real numbers* x_4, y_4 *are chosen so that* $K(c(x,y)) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)w\tilde{E}_4(s,t))w(\tilde{Q}_4(s,t))ds\, dt \in Dom(K^{-1})$.

Proof. (C1) If $p = q$, we define a function $N_1(x,y)$ by

$$N_1(x,y) = c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)w(\Phi(s,t))ds\, dt, \tag{3.2.47}$$

by substituting (3.2.47) in (3.2.38), we get

$$\Phi^q(x,y) \leq N_1(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi^p(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t)dt. \tag{3.2.48}$$

Clearly $N_1(x,y)$ is a non-negative and non-decreasing function in each variable x and y. A suitable application of the inequality (3.1.36) in Theorem 3.1.3 to (3.2.48) gives

$$\phi(x,y) \leq \left[N_1(x,y)E_1(x,y)Q_3(x,y)\right]^{1/p}, \tag{3.2.49}$$

where $E_1(x,y)$, $Q_3(x,y)$ are defined in (3.1.20) and (3.1.38).
From (3.2.47) and (3.2.49) and by using the fact that w is sub-multiplicative, we obtain

$$N_1(x,y) \leq c(x,y) + D_3, \tag{3.2.50}$$

for $(x,y) \in \mathbb{I}$. Where

$$D_3 = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t) w(N_1{}^{\frac{1}{p}}(s,t)) w(E_1{}^{\frac{1}{p}}(s,t) Q_3{}^{\frac{1}{p}}(s,t)) ds\, dt.$$

Fixing any numbers $\tilde{x_3}$ and $\tilde{y_3}$ with $0 < \tilde{x_3} \leq x_3$ and $0 < \tilde{y_3} \leq y_3$, from (3.2.50) we have

$$N_1(x,y) \leq c(\tilde{x_3},\tilde{y_3}) + \tilde{D_3},$$

for $x_0 \leq x \leq \tilde{x_3}$, $y_0 \leq y \leq \tilde{y_3}$. Defining

$$r_3(x,y) = c(\tilde{x_3},\tilde{y_3}) + \tilde{D_3}.$$

Where

$$\tilde{D_3} = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_3},\tilde{y_3},s,t) w(E_1{}^{\frac{1}{p}}(s,t) Q_3{}^{\frac{1}{p}}(s,t)) w(N_1{}^{\frac{1}{p}}(s,t)) ds\, dt.$$

Then $r_3(x_0,y) = r_3(x,y_0) = c(\tilde{x_3},\tilde{y_3})$ and

$$N_1(x,y) \leq r_3(x,y), \tag{3.2.51}$$

with $r_3(x,y)$ is positive and non-decreasing in $y \in [y_0, \tilde{y_0}]$, and

$$\begin{aligned} D_1 r_3(x,y) &= u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_3},\tilde{y_3},u,t) w(E_1{}^{\frac{1}{p}}(u,t) Q_3{}^{\frac{1}{p}}(u,t)) w(N_1{}^{\frac{1}{p}}(u,t)) dt, \\ &\leq u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_3},\tilde{y_3},u,t) w(E_1{}^{\frac{1}{p}}(u,t) Q_3{}^{\frac{1}{p}}(u,t)) w(r_3{}^{\frac{1}{p}}(u,t)) dt, \\ &\leq w(r_3{}^{\frac{1}{p}}(x,y)) u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_3},\tilde{y_3},u,t) w(E_1{}^{\frac{1}{p}}(u,t) Q_3{}^{\frac{1}{p}}(u,t)) dt. \end{aligned} \tag{3.2.52}$$

Where $u = \alpha(x)$ implying $u' = \alpha'(x)$. Dividing both sides of (3.2.52) by $w({r_3}^{1/p}(x,y))$, we obtain

$$\frac{D_1 r_3(x,y)}{w({r_3}^{\frac{1}{p}}(x,y))} \le \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_3}, \tilde{y_3}, \alpha(x), t) w({E_1}^{\frac{1}{p}}(\alpha(x), t){Q_4}^{\frac{1}{p}}(\alpha(x), t))dt, \tag{3.2.53}$$

from (3.2.42) and (3.2.53) we have

$$D_1 K(r_3(x,y)) \le \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_3}, \tilde{y_3}, \alpha(x), t) w({E_1}^{\frac{1}{p}}(\alpha(x), t){Q_3}^{\frac{1}{p}}(\alpha(x), t))dt. \tag{3.2.54}$$

Now setting $x = s$ in (3.2.54) and then integrating with respect to s from x_0 to x, we obtain

$$K(r_3(x,y)) \le K(r_3(x_0,y)) + \tilde{C_3}.$$

Noting $K(r_3(x_0,y)) = K(c(\tilde{x_3}, \tilde{y_3}))$, we have

$$K(r_3(x,y)) \le K(c(\tilde{x_3}, \tilde{y_3})) + \tilde{C_3}.$$

Taking $x = \tilde{x_3}$, $y = \tilde{y_3}$ in (3.2.51) and the last inequality, we obtain

$$N_1(\tilde{x_3}, \tilde{y_3}) \le r_3(\tilde{x_3}, \tilde{y_3}), \tag{3.2.55}$$

and

$$K(r_3(\tilde{x_3}, \tilde{y_3})) \le K(c(\tilde{x_3}, \tilde{y_3})) + \tilde{C_3}. \tag{3.2.56}$$

Where

$$\tilde{C_3} = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_3}, \tilde{y_3}, s, t) w({E_1}^{\frac{1}{p}}(s,t){Q_3}^{\frac{1}{p}}(s,t)) ds\, dt.$$

Since $0 < \tilde{x_3} \le x_3$ and $0 < \tilde{y_3} \le y_3$ are arbitrary, from (3.2.55) and (3.2.56), and using the fact that w is sub-multiplicative, we have

$$N_1(x,y) \le r_3(x,y), \tag{3.2.57}$$

and

$$r_3(x,y) \le K^{-1}\left[K(c(x,y)) + C_3\right], \tag{3.2.58}$$

for all $x_0 < x \leq x_3$, $y_0 < y \leq y_3$. Hence by (3.2.57) and (3.2.58), we obtain

$$N_1(x, y) \leq K^{-1}\left[K(c(x, y)) + C_3\right], \tag{3.2.59}$$

for all $x_0 < x \leq x_3$, $y_0 < y \leq y_3$. Where C_3 is defined in (3.2.41). By (3.2.38), (3.2.59) holds also when $x = x_0$ and $y = y_0$.

Finally substituting the last inequality into (3.2.49), the desired inequality (3.2.39) follows immediately.

(C2) If $p < q$ holds, we define a function $N_2(x, y)$ by

$$N_2(x, y) = c(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) w(\Phi(s, t)) ds\, dt, \tag{3.2.60}$$

by substituting (3.2.60) in (3.2.38), we get

$$\Phi^q(x, y) \leq N_2(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s, y)\Phi^p(s, y) ds + \int_{\beta(y_0)}^{\beta(y)} B(x, t)\Phi^p(x, t) dt. \tag{3.2.61}$$

Clearly $N_2(x, y)$ is a non-negative and non-decreasing function in each variable x and y. A suitable application of the inequality (3.1.36) in Theorem 3.1.3 to (3.2.61) gives

$$\phi(x, y) \leq {N_2}^{1/q}(x, y)\tilde{E}_4(x, y)\tilde{Q}_4(x, y), \tag{3.2.62}$$

where $\tilde{E}_4(x, y)$, $\tilde{Q}_4(x, y)$ are defined in (3.2.46) and (3.2.45).

From (3.2.60) and (3.2.62) and by using the fact that w is sub-multiplicative, we obtain

$$N_2(x, y) \leq c(x, y) + D_4, \tag{3.2.63}$$

for $(x, y) \in \mathbb{I}$. Where

$$D_4 = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) w({N_2}^{\frac{1}{q}}(s, t)) w(\tilde{E}_4(s, t)\tilde{Q}_4(s, t)) ds\, dt.$$

Fixing any numbers $\tilde{x_4}$ and $\tilde{y_4}$ with $0 < \tilde{x_4} \leq x_4$ and $0 < \tilde{y_4} \leq y_4$, from (3.2.63) we have

$$N_2(x, y) \leq c(\tilde{x_4}, \tilde{y_4}) + \tilde{D}_4,$$

for $x_0 \leq x \leq \tilde{x_4}$, $y_0 \leq y \leq \tilde{y_4}$. Defining

$$r_4(x,y) = c(\tilde{x_4}, \tilde{y_4}) + \tilde{D}_4,$$

Where

$$\tilde{D_4} = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_4}, \tilde{y_4}, s, t) w(\tilde{E_4}(s,t)\tilde{Q_4}(s,t)) w({N_2}^{\frac{1}{q}}(s,t)) ds\, dt.$$

Then $r_4(x_0, y) = r_4(x, y_0) = c(\tilde{x_4}, \tilde{y_4})$ and

$$N_2(x,y) \leq r_4(x,y), \tag{3.2.64}$$

with $r_4(x,y)$ is positive and non-decreasing in $y \in [y_0, \tilde{y_0}]$, and

$$\begin{aligned} D_1 r_4(x,y) &= u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_4}, \tilde{y_4}, u, t) w(\tilde{E_4}(u,t)\tilde{Q_4}(u,t)) w({N_2}^{\frac{1}{q}}(u,t)) dt, \\ &\leq u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_4}, \tilde{y_4}, u, t) w(\tilde{E_4}(u,t)\tilde{Q_4}(u,t)) w({r_4}^{\frac{1}{q}}(u,t)) dt, \\ &\leq w({r_4}^{\frac{1}{q}}(x,y)) u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_4}, \tilde{y_4}, u, t) w(\tilde{E_4}(u,t)\tilde{Q_4}(u,t)) dt. \end{aligned} \tag{3.2.65}$$

Where $u = \alpha(x)$ implying $u' = \alpha'(x)$. Dividing both sides of (3.2.65) by $w({r_4}^{1/q}(x,y))$, we obtain

$$\frac{D_1 r_4(x,y)}{w({r_4}^{1/q}(x,y))} \leq \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_4}, \tilde{y_4}, \alpha(x), t) w(\tilde{E_4}(\alpha(x),t)\tilde{Q_4}(\alpha(x),t)) dt, \tag{3.2.66}$$

from (3.2.42) and (3.2.66) we have

$$D_1 K(r_4(x,y)) \leq \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_4}, \tilde{y_4}, \alpha(x), t) w(\tilde{E_4}(\alpha(x),t)\tilde{Q_4}(\alpha(x),t)) dt. \tag{3.2.67}$$

Now setting $x = s$ in (3.2.67) and then integrating with respect to s from x_0 to x, we obtain

$$K(r_4(x,y)) \leq K(r_4(x_0,y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_4}, \tilde{y_4}, s, t) w(\tilde{E_4}(s,t)\tilde{Q_4}(s,t)) ds\, dt.$$

Noting $K(r_4(x_0, y)) = K(c(\tilde{x_4}, \tilde{y_4}))$, we have

$$K(r_4(x, y)) \leq K(c(\tilde{x_4}, \tilde{y_4})) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_4}, \tilde{y_4}, s, t) w(\tilde{E_4}(s, t) \tilde{Q_4}(s, t)) ds\, dt.$$

Taking $x = \tilde{x_4}$, $y = \tilde{y_4}$ in (3.2.64) and the last inequality, we obtain

$$N_2(\tilde{x_4}, \tilde{y_4}) \leq r_4(\tilde{x_4}, \tilde{y_4}), \tag{3.2.68}$$

and

$$K(r_4(\tilde{x_4}, \tilde{y_4})) \leq K(c(\tilde{x_4}, \tilde{y_4})) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_4}, \tilde{y_4}, s, t) w(\tilde{E_4}(s, t) \tilde{Q_4}(s, t)) ds\, dt. \tag{3.2.69}$$

Since $0 < \tilde{x_4} \leq x_4$ and $0 < \tilde{y_4} \leq y_4$ are arbitrary, from (3.2.68) and (3.2.69) we have

$$N_2(x, y) \leq r_4(x, y), \tag{3.2.70}$$

and

$$r_4(x, y) \leq K^{-1} \left[K(c(x, y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) w(\tilde{E_4}(s, t) \tilde{Q_4}(s, t)) ds\, dt \right], \tag{3.2.71}$$

for all $x_0 < x \leq x_4$, $y_0 < y \leq y_4$. Hence by (3.2.70) and (3.2.71), we obtain

$$N_2(x, y) \leq K^{-1} \left[K(c(x, y)) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) w(\tilde{E_4}(s, t) \tilde{Q_4}(s, t)) ds\, dt \right], \tag{3.2.72}$$

for all $x_0 < x \leq x_4$, $y_0 < y \leq y_4$. By (3.2.38), (3.2.72) holds also when $x = x_0$ and $y = y_0$.

Finally substituting the last inequality into (3.2.62), the desired inequality (3.2.43) follows immediately.

□

Remark 3.2.3. Theorem 3.2.2 (**C2**) reduces to Theorem 2.2.1 in Chapter (2), if $q = 1$, $w(s) = s$, $c(x, y) = c$, $\alpha(x) = x$, $\beta(y) = y$, $H(x, y, s, t) = H(s, t)$ $x_0 = y_0 = 0$.

3.3 Applications of the Generalized Retarded Integral Inequalities of Gronwall-Bihari Type in Two Independent Variables

Some exciting new inequalities can be obtained from Theorem 3.2.1 and Theorem 3.2.2, by choosing suitable functions for w. For example, if we take $w(s) = s^r$, then we have the following results.

Corollary 3.3.1. *Let p, c, ϕ, A, H, B, α and β be defined as in Theorem 3.2.1. Let $0 < r < 1$ is a constant and if $\phi(x,y)$ satisfies*

$$\Phi(x,y) \le c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t)dt + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)\Phi^r(s,t)ds\,dt, \tag{3.3.1}$$

for all $(x,y) \in \mathbb{I}$. Then

$$\Phi(x,y) \le m_1(x,y)e_1(x,y)\tilde{q_1}(x,y), \tag{3.3.2}$$

for all $(x,y) \in \mathbb{I}$. Where

$$m_1(x,y) \le \left[c^{1-r}(x,y) + (1-r)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t){e_1}^r(s,t){\tilde{q_1}}^r(s,t)ds\,dt\right]^{\frac{1}{1-r}}, \tag{3.3.3}$$

for all $(x,y) \in \mathbb{I}$, and

$$\tilde{q_1}(x,y) = \left[1 + (1-p)\int_{\beta(y_0)}^{\beta(y)} B(x,t){m_1}^{p-1}(x,t){e_1}^p(x,t)dt\right]^{\frac{1}{1-p}}, \tag{3.3.4}$$

$$e_1(x,y) = \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} A(s,y)ds\right). \tag{3.3.5}$$

Proof. Setting

$$m_1(x,y) = c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)\phi^r(s,t)ds\,dt, \tag{3.3.6}$$

the inequality (3.3.1) can be restated as

$$\Phi(x,y) \le m_1(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t)dt. \tag{3.3.7}$$

Clearly $m_1(x,y)$ is non-negative and non-decreasing function in each x and y. Now a suitable application of the inequality (3.1.18) in Theorem 3.1.2 to (3.3.7), yields

$$\Phi(x,y) \le m_1(x,y)e_1(x,y)\tilde{q_1}(x,y), \tag{3.3.8}$$

where $e_1(x,y)$ and $\tilde{q_1}(x,y)$ are defined in (3.3.5) and (3.3.4). From (3.3.6) and (3.3.8), we have

$$m_1(x,y) \le c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t){e_1}^r(s,t)\tilde{q_1}^r(s,t){m_1}^r(s,t)ds\,dt, \tag{3.3.9}$$

for $(x,y) \in \mathbb{I}$. Fixing any numbers $\tilde{x_1}$, and $\tilde{y_1}$ with $0 < \tilde{x_1} \le x_1$ and $0 < \tilde{y_1} \le y_1$, we have

$$m_1(x,y) \le c(\tilde{x_1},\tilde{y_1}) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_1}^r(s,t)\tilde{q_1}^r(s,t){m_1}^r(s,t)ds\,dt,$$

for $x_0 \le x \le \tilde{x_1}$, $y_0 \le y \le \tilde{y_1}$. Defining

$$z_1(x,y) = c(\tilde{x_1},\tilde{y_1}) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_1}^r(s,t)\tilde{q_1}^r(s,t){m_1}^r(s,t)ds\,dt, \tag{3.3.10}$$

then $z_1(x_0,y) = z_1(x,y_0) = c(\tilde{x_1},\tilde{y_1})$ and

$$m_1(x,y) \le z_1(x,y), \tag{3.3.11}$$

with $z_1(x,y)$ is positive and non-decreasing in y, and

$$\begin{aligned} D_1 z_1(x,y) &= \alpha'(x)\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},\alpha(x),t){e_1}^r(\alpha(x),t)\tilde{q_1}^r(\alpha(x),t){m_1}^r(\alpha(x),t)dt, \\ &\le \alpha'(x)\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},\alpha(x),t){e_1}^r(\alpha(x),t)\tilde{q_1}^r(\alpha(x),t){z_1}^r(\alpha(x),t)dt, \\ &\le {z_1}^r(x,y)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},\alpha(x),t){e_1}^r(\alpha(x),t)\tilde{q_1}^r(\alpha(x),t)dt. \end{aligned} \tag{3.3.12}$$

Dividing both sides of (3.3.12) by ${z_1}^r(x,y)$, we obtain

$$\frac{D_1 z_1(x,y)}{{z_1}^r(x,y)} \leq \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, \alpha(x), t) {e_1}^r(\alpha(x), t) \tilde{q_1}^r(\alpha(x), t)) dt. \qquad (3.3.13)$$

Now setting $x = s$ in (3.3.13) and then integrating with respect to s from x_0 to x, we obtain

$$\frac{1}{1-r}\left({z_1}^{1-r}(x,y) - c^{1-r}(\tilde{x_1}, \tilde{y_1})\right) \leq \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t) {e_1}^r(s,t) \tilde{q_1}^r(s,t) ds\, dt,$$

$${z_1}^{1-r}(x,y) - c^{1-r}(\tilde{x_1}, \tilde{y_1}) \leq (1-r) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t) {e_1}^r(s,t) \tilde{q_1}^r(s,t) ds\, dt,$$

and

$$z_1(x,y) \leq \left[c^{1-r}(\tilde{x_1}, \tilde{y_1}) + (1-r) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t) {e_1}^r(s,t) \tilde{q_1}^r(s,t) ds\, dt\right]^{\frac{1}{1-r}}.$$

Taking $x = \tilde{x_1}$, $y = \tilde{y_1}$ in (3.3.11) and the last inequality, gives:

$$m_1(\tilde{x_1}, \tilde{y_1}) \leq z_1(\tilde{x_1}, \tilde{y_1}),$$

and

$$z_1(\tilde{x_1}, \tilde{y_1}) \leq \left[c^{1-r}(\tilde{x_1}, \tilde{y_1}) + (1-r) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t) {e_1}^r(s,t) \tilde{q_1}^r(s,t) ds\, dt\right]^{\frac{1}{1-r}}.$$

Since $0 < \tilde{x_1} \leq x_1$ and $0 < \tilde{y_1} \leq y_1$ are arbitrary, from the previous two equations we have

$$m_1(x,y) \leq z_1(x,y),$$

and

$$z_1(x,y) \leq \left[c^{1-r}(x,y) + (1-r) \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) {e_1}^r(s,t) \tilde{q_1}^r(s,t) ds\, dt\right]^{\frac{1}{1-r}}, \qquad (3.3.14)$$

for all $(x,y) \in \mathbb{I}$. Finally substituting the last inequality into (3.3.11), the desired inequality (3.3.3) follows immediately, and this completes the proof. □

Remark 3.3.1. (1) If $r = 0$, an estimation of the inequality (3.3.1) can be easily obtained.

(2) If $r = 1$, an estimation of the inequality (3.3.1) can be easily obtained.

Remark 3.3.2. Corollary 3.3.1 reduces to Theorem 2.3.1, when $c(x,y) = c$, $\alpha(x) = x$, $\beta(y) = y$, $H(x,y,s,t) = H(s,t)$ $x_0 = y_0 = 0$ and $r = p$

Corollary 3.3.2. *Let p, c, ϕ, A, H, B, α and β be defined as in Theorem 3.2.1. Let $0 < r < 1$ is a constant and if $\phi(x,y)$ satisfies*

$$\Phi(x,y) \leq c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi^p(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi(x,t)dt + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)\Phi^r(s,t)ds\,dt, \tag{3.3.15}$$

for all $(x,y) \in \mathbb{I}$. Then

$$\Phi(x,y) \leq m_2(x,y)e_2(x,y)\tilde{q}_2(x,y), \tag{3.3.16}$$

for all $(x,y) \in \mathbb{I}$. Where

$$m_2(x,y) \leq \left[c^{1-r}(x,y) + (1-r)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t){e_2}^r(s,t){\tilde{q}_2}^r(s,t)ds\,dt\right]^{\frac{1}{1-r}}, \tag{3.3.17}$$

for all $(x,y) \in \mathbb{I}$, and

$$\tilde{q}_2(x,y) = \left[1 + (1-p)\int_{\alpha(x_0)}^{\alpha(x)} A(s,y){m_2}^{p-1}(s,y){e_2}^p(s,y)ds\right]^{\frac{1}{1-p}}, \tag{3.3.18}$$

$$e_2(x,y) = \exp\left(\int_{\beta(y_0)}^{\beta(y)} B(x,t)dt\right). \tag{3.3.19}$$

Proof. Setting

$$m_2(x,y) = c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)\phi^r(s,t)ds\,dt, \tag{3.3.20}$$

the inequality (3.3.15) can be restated as

$$\Phi(x,y) \leq m_2(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi^p(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi(x,t)dt. \tag{3.3.21}$$

Clearly $m_2(x, y)$ is non-negative and non-decreasing function in each x and y. Now a suitable application of the inequality (3.1.18) in Theorem 3.1.2 to (3.3.21), yields

$$\Phi(x, y) \leq m_2(x, y)e_2(x, y)\tilde{q_2}(x, y), \tag{3.3.22}$$

where $e_2(x, y)$ and $\tilde{q_2}(x, y)$ are defined in (3.3.19) and (3.3.18). From (3.3.20) and (3.3.22), we have

$$m_2(x, y) \leq c(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t){e_2}^r(s, t)\tilde{q_2}^r(s, t){m_2}^r(s, t)ds\, dt, \tag{3.3.23}$$

for $(x, y) \in \mathbb{I}$. Fixing any numbers $\tilde{x_1}$, and $\tilde{y_1}$ with $0 < \tilde{x_1} \leq x_1$ and $0 < \tilde{y_1} \leq y_1$, we have

$$m_2(x, y) \leq c(\tilde{x_1}, \tilde{y_1}) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t){e_2}^r(s, t)\tilde{q_2}^r(s, t){m_2}^r(s, t)ds\, dt,$$

for $x_0 \leq x \leq \tilde{x_1}$, $y_0 \leq y \leq \tilde{y_1}$. Defining

$$z_2(x, y) = c(\tilde{x_1}, \tilde{y_1}) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t){e_2}^r(s, t)\tilde{q_2}^r(s, t){m_2}^r(s, t)ds\, dt, \tag{3.3.24}$$

then $z_2(x_0, y) = z_2(x, y_0) = c(\tilde{x_1}, \tilde{y_1})$ and

$$m_2(x, y) \leq z_2(x, y), \tag{3.3.25}$$

with $z_2(x, y)$ is positive and non-decreasing in y, and

$$\begin{aligned} D_1 z_2(x, y) &= \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, \alpha(x), t){e_2}^r(\alpha(x), t)\tilde{q_2}^r(\alpha(x), t){m_2}^r(\alpha(x), t)dt, \\ &\leq \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, \alpha(x), t){e_2}^r(\alpha(x), t)\tilde{q_2}^r(\alpha(x), t){z_2}^r(\alpha(x), t)dt, \\ &\leq {z_2}^r(x, y)\alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, \alpha(x), t){e_2}^r(\alpha(x), t)\tilde{q_2}^r(\alpha(x), t)dt. \end{aligned} \tag{3.3.26}$$

Dividing both sides of (3.3.26) by ${z_2}^r(x, y)$, we obtain

$$\frac{D_1 z_2(x, y)}{{z_2}^r(x, y)} \leq \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, \alpha(x), t){e_2}^r(\alpha(x), t)\tilde{q_2}^r(\alpha(x), t))dt. \tag{3.3.27}$$

Now setting $x = s$ in (3.3.27) and then integrating with respect to s from x_0 to x, we obtain

$$\frac{1}{1-r}\left({z_2}^{1-r}(x,y) - c^{1-r}(\tilde{x_1},\tilde{y_1})\right) \leq \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_2}^r(s,t)\tilde{q_2}^r(s,t)ds\,dt,$$

$${z_2}^{1-r}(x,y) - c^{1-r}(\tilde{x_1},\tilde{y_1}) \leq (1-r)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_2}^r(s,t)\tilde{q_2}^r(s,t)ds\,dt,$$

and

$$z_2(x,y) \leq \left[c^{1-r}(\tilde{x_1},\tilde{y_1}) + (1-r)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_2}^r(s,t)\tilde{q_2}^r(s,t)ds\,dt\right]^{\frac{1}{1-r}}.$$

Taking $x = \tilde{x_1}$, $y = \tilde{y_1}$ in (3.3.25) and the last inequality, gives:

$$m_2(\tilde{x_1},\tilde{y_1}) \leq z_2(\tilde{x_1},\tilde{y_1}),$$

and

$$z_2(\tilde{x_1},\tilde{y_1}) \leq \left[c^{1-r}(\tilde{x_1},\tilde{y_1}) + (1-r)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_2}^r(s,t)\tilde{q_2}^r(s,t)ds\,dt\right]^{\frac{1}{1-r}}.$$

Since $0 < \tilde{x_1} \leq x_1$ and $0 < \tilde{y_1} \leq y_1$ are arbitrary, from the previous two equations we have

$$m_2(x,y) \leq z_2(x,y),$$

and

$$z_2(x,y) \leq \left[c^{1-r}(x,y) + (1-r)\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t){e_2}^r(s,t)\tilde{q_2}^r(s,t)ds\,dt\right]^{\frac{1}{1-r}}, \tag{3.3.28}$$

for all $(x,y) \in \mathbb{I}$. Finally substituting the last inequality into (3.3.25), the desired inequality (3.3.17) follows immediately, and this completes the proof. □

Remark 3.3.3. Corollary 3.3.2 reduces to Theorem 2.3.2, when $c(x,y) = c$, $\alpha(x) = x$, $\beta(y) = y$, $H(x,y,s,t) = H(s,t)$ $x_0 = y_0 = 0$ and $r = p$

Corollary 3.3.3. *Let p, q, c, ϕ, A, H, B, α and β be defined as in Theorem 3.2.2. Suppose that $q > r > 0$ are constants and if $\phi(x, y)$ satisfies*

$$\Phi^q(x,y) \leq c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi^p(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t)dt + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)\Phi^r(s,t)ds\,dt, \tag{3.3.29}$$

for all $(x, y) \in \mathbb{I}$. Then the following conclusions are true

(D1) If $p = q$, then

$$\Phi(x,y) \leq \left[n_1(x,y)e_1(x,y)q_3(x,y)\right]^{1/p}, \tag{3.3.30}$$

for all $(x, y) \in \mathbb{I}$. Where

$$n_1(x,y) \leq \left[c^{\frac{p-r}{p}}(x,y) + (\frac{p-r}{p})C_4\right]^{\frac{p}{p-r}}, \tag{3.3.31}$$

for all $(x, y) \in \mathbb{I}$. Where

$$C_4 = \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t){e_1}^{\frac{r}{p}}(s,t){q_3}^{\frac{r}{p}}(s,t)ds\,dt, \tag{3.3.32}$$

and

$$q_3(x,y) = \exp\left(\int_{\beta(y_0)}^{\beta(y)} B(x,t)e_1(x,t)dt\right), \tag{3.3.33}$$

and $e_1(x, y)$ is defined in (3.3.5).

(D2) If $p < q$

$$\Phi(x,y) \leq {n_2}^{1/q}(x,y)e_4(x,y)q_4(x,y), \tag{3.3.34}$$

for all $(x, y) \in \mathbb{I}$. Where

$$n_2(x,y) \leq \left[c^{\frac{q-r}{q}}(x,y) + (\frac{q-r}{q})C_5\right]^{\frac{q}{q-r}}, \tag{3.3.35}$$

for all $(x, y) \in \mathbb{I}$. Where

$$C_5 = \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t){e_4}^{r}(s,t){q_4}^{r}(s,t)ds\,dt, \tag{3.3.36}$$

and

$$q_4(x,y)=\left[1+\frac{q-p}{q}\int_{\beta(y_0)}^{\beta(y)} z^{\frac{p-q}{q}}(x,t)B(x,t)dt\right]^{\frac{1}{q-p}}, \tag{3.3.37}$$

and

$$e_4(x,y)=\left[1+\frac{q-p}{q}\int_{\alpha(x_0)}^{\alpha(x)} {n_2}^{\frac{p-q}{q}}(s,y)A(s,y){q_4}^{p}(s,y)ds\right]^{\frac{1}{q-p}}. \tag{3.3.38}$$

Proof. (D1) If $p=q$. Setting

$$n_1(x,y)=c(x,y)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)\phi^r(s,t)ds\,dt, \tag{3.3.39}$$

the inequality (3.3.29) can be restated as

$$\Phi^q(x,y)\le n_1(x,y)+\int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi^p(s,y)ds+\int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t)dt. \tag{3.3.40}$$

Clearly $n_1(x,y)$ is non-negative and non-decreasing function in each x and y. Now a suitable application of the inequality (3.1.36) in Theorem 3.1.3 to (3.3.40), yields

$$\Phi(x,y)\le\left[n_1(x,y)e_1(x,y)q_3(x,y)\right]^{1/p}, \tag{3.3.41}$$

where $e_1(x,y)$ and $q_3(x,y)$ are defined in (3.3.5) and (3.3.33). From (3.3.41) and (3.3.39), we have

$$n_1(x,y)\le c(x,y)+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t){e_1}^{\frac{r}{p}}(s,t){q_3}^{\frac{r}{p}}(s,t){n_1}^{\frac{r}{p}}(s,t)ds\,dt, \tag{3.3.42}$$

for $(x,y)\in\mathbb{I}$. Fixing any numbers $\tilde{x_1}$, and $\tilde{y_1}$ with $0<\tilde{x_1}\le x_1$ and $0<\tilde{y_1}\le y_1$, we have

$$n_1(x,y)\le c(\tilde{x_1},\tilde{y_1})+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_1}^{\frac{r}{p}}(s,t){q_3}^{\frac{r}{p}}(s,t){n_1}^{\frac{r}{p}}(s,t)ds\,dt,$$

for $x_0\le x\le\tilde{x_1}$, $y_0\le y\le\tilde{y_1}$. Defining

$$z_1(x,y)=c(\tilde{x_1},\tilde{y_1})+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_1}^{r/p}(s,t){q_3}^{r/p}(s,t){n_1}^{r/p}(s,t)ds\,dt, \tag{3.3.43}$$

then $z_1(x_0, y) = z_1(x, y_0) = c(\tilde{x_1}, \tilde{y_1})$ and

$$n_1(x, y) \leq z_1(x, y), \tag{3.3.44}$$

with $z_1(x, y)$ is positive and non-decreasing in y, and

$$\begin{aligned} D_1 z_1(x, y) &= u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, u, t) {e_1}^{\frac{r}{p}}(u, t) {q_3}^{\frac{r}{p}}(u, t) {n_1}^{\frac{r}{p}}(u, t) dt, \\ &\leq u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, u, t) {e_1}^{\frac{r}{p}}(u, t) {q_3}^{\frac{r}{p}}(u, t) {z_1}^{\frac{r}{p}}(u, t) dt, \\ &\leq {z_1}^{\frac{r}{p}}(x, y) u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, u, t) {e_1}^{\frac{r}{p}}(u, t) {q_3}^{\frac{r}{p}}(u, t) dt. \end{aligned} \tag{3.3.45}$$

Where $u = \alpha(x)$ implying $u' = \alpha'(x)$. Dividing both sides of (3.3.45) by ${z_1}^{r/p}(x, y)$, we obtain

$$\frac{D_1 z_1(x, y)}{{z_1}^{r/p}(x, y)} \leq \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, \alpha(x), t) {e_1}^{r/p}(\alpha(x), t) {q_3}^{r/p}(\alpha(x), t)) dt. \tag{3.3.46}$$

Now setting $x = s$ in (3.3.46) and then integrating with respect to s from x_0 to x, we obtain

$$\frac{1}{1 - \frac{r}{p}} \left({z_1}^{1-\frac{r}{p}}(x, y) - c^{1-\frac{r}{p}}(\tilde{x_1}, \tilde{y_1}) \right) \leq \tilde{C_4},$$

$${z_1}^{\frac{p-r}{p}}(x, y) - c^{\frac{p-r}{p}}(\tilde{x_1}, \tilde{y_1}) \leq \left(1 - \frac{r}{p} \right) \tilde{C_4},$$

and

$$z_1(x, y) \leq \left[c^{\frac{p-r}{p}}(\tilde{x_1}, \tilde{y_1}) + (\frac{p-r}{p}) \tilde{C_4} \right]^{\frac{p}{p-r}}.$$

Taking $x = \tilde{x_1}$, $y = \tilde{y_1}$ in (3.3.44) and the last inequality, gives:

$$n_1(\tilde{x_1}, \tilde{y_1}) \leq z_1(\tilde{x_1}, \tilde{y_1}),$$

and

$$z_1(\tilde{x_1}, \tilde{y_1}) \leq \left[c^{\frac{p-r}{p}}(\tilde{x_1}, \tilde{y_1}) + (\frac{p-r}{p}) \tilde{C_4} \right]^{\frac{p}{p-r}}.$$

Where

$$\tilde{C_4} = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t) {e_1}^{\frac{r}{p}}(s, t) {q_3}^{\frac{r}{p}}(s, t) ds\, dt.$$

Since $0 < \tilde{x_1} \le x_1$ and $0 < \tilde{y_1} \le y_1$ are arbitrary, from the previous two equations we have

$$n_1(x,y) \le z_1(x,y),$$

and

$$z_1(x,y) \le \left[c^{\frac{p-r}{p}}(x,y) + (\frac{p-r}{p})C_4\right]^{\frac{p}{p-r}}, \tag{3.3.47}$$

for all $(x,y) \in \mathbb{I}$. Where C_4 is defined in (3.3.32). Finally substituting the last inequality into (3.3.44), the desired inequality (3.3.31) follows immediately, and this completes the proof.

(D2) If $p < q$. Setting

$$n_2(x,y) = c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t)\phi^r(s,t)ds\,dt, \tag{3.3.48}$$

the inequality (3.3.29) can be restated as

$$\Phi^q(x,y) \le n_2(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s,y)\Phi^p(s,y)ds + \int_{\beta(y_0)}^{\beta(y)} B(x,t)\Phi^p(x,t)dt. \tag{3.3.49}$$

Clearly $n_2(x,y)$ is non-negative and non-decreasing function in each x and y. Now a suitable application of the inequality (3.1.36) in Theorem 3.1.3 to (3.3.49), yields

$$\Phi(x,y) \le {n_2}^{1/q}(x,y)e_4(x,y)q_4(x,y), \tag{3.3.50}$$

where $e_4(x,y)$ and $q_4(x,y)$ are defined in (3.3.38) and (3.3.37). From (3.3.50) and (3.3.48), we have

$$n_2(x,y) \le c(x,y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t){e_4}^r(s,t){q_4}^r(s,t){n_2}^{r/q}(s,t)ds\,dt, \tag{3.3.51}$$

for $(x,y) \in \mathbb{I}$. Fixing any numbers $\tilde{x_1}$, and $\tilde{y_1}$ with $0 < \tilde{x_1} \le x_1$ and $0 < \tilde{y_1} \le y_1$, we have

$$n_2(x,y) \le c(\tilde{x_1},\tilde{y_1}) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_4}^r(s,t){q_4}^r(s,t){n_2}^{r/q}(s,t)ds\,dt,$$

for $x_0 \leq x \leq \tilde{x_1}$, $y_0 \leq y \leq \tilde{y_1}$. Defining

$$z_2(x,y) = c(\tilde{x_1}, \tilde{y_1}) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t) {e_4}^r(s,t) {q_4}^r(s,t) {n_2}^{r/q}(s,t) ds\, dt, \tag{3.3.52}$$

then $z_2(x_0, y) = z_2(x, y_0) = c(\tilde{x_1}, \tilde{y_1})$ and

$$n_2(x,y) \leq z_2(x,y), \tag{3.3.53}$$

with $z_2(x,y)$ is positive and non-decreasing in y, and

$$\begin{aligned} D_1 z_2(x,y) &= u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, u, t) {e_4}^r(u,t) {q_4}^r(u,t) {n_2}^{\frac{r}{q}}(u,t) dt, \\ &\leq u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, u, t) {e_4}^r(u,t) {q_4}^r(u,t) {z_2}^{\frac{r}{q}}(u,t) dt, \\ &\leq {z_2}^{\frac{r}{q}}(x,y) u' \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, u, t) {e_4}^r(u,t) {q_4}^r(u,t) dt. \end{aligned} \tag{3.3.54}$$

Where $u = \alpha(x)$ implying $u' = \alpha'(x)$. Dividing both sides of (3.3.54) by ${z_2}^{r/q}(x,y)$, we obtain

$$\frac{D_1 z_2(x,y)}{{z_2}^{\frac{r}{q}}(x,y)} \leq \alpha'(x) \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, \alpha(x), t) {e_4}^r(\alpha(x), t) {q_4}^r(\alpha(x), t) dt. \tag{3.3.55}$$

Now setting $x = s$ in (3.3.55) and then integrating with respect to s from x_0 to x, we obtain

$$\left(1 + \frac{r/q}{1 - \frac{r}{q}}\right) \left({z_2}^{1-\frac{r}{q}}(x,y) - c^{1-\frac{r}{q}}(\tilde{x_1}, \tilde{y_1})\right) \leq \tilde{C_5}.$$

This implies

$$\frac{1}{1 - \frac{r}{q}} \left({z_2}^{1-\frac{r}{q}}(x,y) - c^{1-\frac{r}{q}}(\tilde{x_1}, \tilde{y_1})\right) \leq \tilde{C_5},$$

$${z_2}^{\frac{q-r}{q}}(x,y) - c^{\frac{q-r}{q}}(\tilde{x_1}, \tilde{y_1}) \leq \left(1 - \frac{r}{q}\right) \tilde{C_5},$$

and

$$z_2(x,y) \leq \left[c^{\frac{q-r}{q}}(\tilde{x_1}, \tilde{y_1}) + (\frac{q-r}{q}) \tilde{C_5}\right]^{\frac{q}{q-r}}.$$

Taking $x = \tilde{x_1}$, $y = \tilde{y_1}$ in (3.3.53) and the last inequality gives:

$$n_2(\tilde{x_1}, \tilde{y_1}) \leq z_2(\tilde{x_1}, \tilde{y_1}),$$

and

$$z_2(\tilde{x_1}, \tilde{y_1}) \leq \left[c^{\frac{q-r}{q}}(\tilde{x_1}, \tilde{y_1}) + (\frac{q-r}{q})\tilde{C_5}\right]^{\frac{q}{q-r}}.$$

Where

$$\tilde{C_5} = \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t) {e_4}^r(s,t) {q_4}^r(s,t) ds\, dt.$$

Since $0 < \tilde{x_1} \leq x_1$ and $0 < \tilde{y_1} \leq y_1$ are arbitrary, from the previous two equations we have

$$n_2(x, y) \leq z_2(x, y),$$

and

$$z_2(x, y) \leq \left[c^{\frac{q-r}{q}}(x, y) + (\frac{q-r}{q})C_5\right]^{\frac{q}{q-r}}, \tag{3.3.56}$$

for all $(x, y) \in \mathbb{I}$. Where C_5 is defined in (3.3.36). Finally substituting the last inequality into (3.3.53), the desired inequality (3.3.34) follows immediately, and this completes the proof.

□

Corollary 3.3.4. *Let p, q, c, ϕ, A, H, B, α and β be defined as in Theorem 3.2.2. Suppose that $q = r > 0$ are constants and if $\phi(x, y)$ satisfies*

$$\Phi^q(x, y) \leq c(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s, y)\Phi^p(s, y) ds + \int_{\beta(y_0)}^{\beta(y)} B(x, t)\Phi^p(x, t) dt$$
$$+ \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t)\Phi^r(s, t) ds\, dt, \tag{3.3.57}$$

for all $(x, y) \in \mathbb{I}$. Then the following conclusions are true

(E1) If $p = q$, then

$$\Phi(x, y) \leq [n_1(x, y) e_1(x, y) q_3(x, y)]^{1/p}, \tag{3.3.58}$$

for all $(x, y) \in \mathbb{I}$. Where

$$n_1(x, y) \leq c(x, y) \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) e_1(s, t) q_3(s, t) ds\, dt\right), \tag{3.3.59}$$

for all $(x, y) \in \mathbb{I}$*, and* $q_3(x, y)$ *and* $e_1(x, y)$ *are defined in* (3.3.33) *and* (3.3.5) *respectively.*

(E2) If $p < q$

$$\Phi(x, y) \leq {n_2}^{1/q}(x, y)e_4(x, y)q_4(x, y), \tag{3.3.60}$$

for all $(x, y) \in \mathbb{I}$*. Where*

$$n_2(x, y) \leq c(x, y) \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t){e_4}^r(s, t){q_4}^r(s, t) ds\, dt\right), \tag{3.3.61}$$

for all $(x, y) \in \mathbb{I}$*, and* $e_4(x, y)$ *and* $q_4(x, y)$ *are defined in* (3.3.38) *and* (3.3.37) *respectively.*

Proof. (E1) If $p = q$. Setting

$$n_1(x, y) = c(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t)\phi^r(s, t) ds\, dt, \tag{3.3.62}$$

the inequality (3.3.57) can be restated as

$$\Phi^q(x, y) \leq n_1(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s, y)\Phi^p(s, y) ds + \int_{\beta(y_0)}^{\beta(y)} B(x, t)\Phi^p(x, t) dt. \tag{3.3.63}$$

Clearly $n_1(x, y)$ is non-negative and non-decreasing function in each x and y. Now a suitable application of the inequality (3.1.36) in Theorem 3.1.3 to (3.3.63), yields

$$\Phi(x, y) \leq [n_1(x, y)e_1(x, y)q_3(x, y)]^{1/p}, \tag{3.3.64}$$

where $e_1(x, y)$ and $q_3(x, y)$ are defined in (3.3.5) and (3.3.33). From (3.3.64) and (3.3.62), we have

$$n_1(x, y) \leq c(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t)e_1(s, t)q_3(s, t)n_1(s, t) ds\, dt, \tag{3.3.65}$$

for $(x, y) \in \mathbb{I}$. Fixing any numbers $\tilde{x_1}$, and $\tilde{y_1}$ with $0 < \tilde{x_1} \leq x_1$ and $0 < \tilde{y_1} \leq y_1$, we have

$$n_1(x, y) \leq c(\tilde{x_1}, \tilde{y_1}) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t)e_1(s, t)q_3(s, t)n_1(s, t) ds\, dt,$$

for $x_0 \le x \le \tilde{x_1}$, $y_0 \le y \le \tilde{y_1}$. Defining

$$z_1(x,y) = c(\tilde{x_1},\tilde{y_1}) + \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t)e_1(s,t)q_3(s,t)n_1(s,t)ds\,dt, \tag{3.3.66}$$

then $z_1(x_0,y) = z_1(x,y_0) = c(\tilde{x_1},\tilde{y_1})$ and

$$n_1(x,y) \le z_1(x,y), \tag{3.3.67}$$

with $z_1(x,y)$ is positive and non-decreasing in y, and

$$\begin{aligned} D_1 z_1(x,y) &= \alpha'(x)\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},\alpha(x),t)e_1(\alpha(x),t)q_3(\alpha(x),t)n_1(\alpha(x),t)dt, \\ &\le \alpha'(x)\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},\alpha(x),t)e_1(\alpha(x),t)q_3(\alpha(x),t)z_1(\alpha(x),t)dt, \\ &\le z_1(x,y)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},\alpha(x),t)e_1(\alpha(x),t)q_3(\alpha(x),t)dt. \end{aligned} \tag{3.3.68}$$

Dividing both sides of (3.3.68) by $z_1(x,y)$, we obtain

$$\frac{D_1 z_1(x,y)}{z_1(x,y)} \le \alpha'(x)\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},\alpha(x),t)e_1(\alpha(x),t)q_3(\alpha(x),t)dt. \tag{3.3.69}$$

Now setting $x = s$ in (3.3.69) and then integrating with respect to s from x_0 to x, we obtain

$$\ln z_1(x,y) - \ln c(\tilde{x_1},\tilde{y_1}) \le \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t)e_1(s,t)q_3(s,t)ds\,dt.$$

This implies

$$\ln\left(\frac{z_1(x,y)}{c(\tilde{x_1},\tilde{y_1})}\right) \le \int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t)e_1(s,t)q_3(s,t)ds\,dt,$$

$$\frac{z_1(x,y)}{c(\tilde{x_1},\tilde{y_1})} \le \exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t)e_1(s,t)q_3(s,t)ds\,dt\right),$$

and

$$z_1(x,y) \le c(\tilde{x_1},\tilde{y_1})\exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t)e_1(s,t)q_3(s,t)ds\,dt\right).$$

Taking $x = \tilde{x_1}$, $y = \tilde{y_1}$ in (3.3.67) and the last inequality, gives:

$$n_1(\tilde{x_1}, \tilde{y_1}) \leq z_1(\tilde{x_1}, \tilde{y_1}),$$

and

$$z_1(\tilde{x_1}, \tilde{y_1}) \leq c(\tilde{x_1}, \tilde{y_1}) \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1}, \tilde{y_1}, s, t) e_1(s,t) q_3(s,t) ds\, dt\right).$$

Since $0 < \tilde{x_1} \leq x_1$ and $0 < \tilde{y_1} \leq y_1$ are arbitrary, from the previous two equations we have

$$n_1(x, y) \leq z_1(x, y),$$

and

$$z_1(x, y) \leq c(x, y) \exp\left(\int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) e_1(s,t) q_3(s,t) ds\, dt\right), \quad (3.3.70)$$

for all $(x, y) \in \mathbb{I}$. Finally substituting the last inequality into (3.3.67), the desired inequality (3.3.59) follows immediately, and this completes the proof.

(E2) If $p < q$. Setting

$$n_2(x, y) = c(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) \phi^r(s,t) ds\, dt, \quad (3.3.71)$$

the inequality (3.3.29) can be restated as

$$\Phi^q(x, y) \leq n_2(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} A(s, y) \Phi^p(s, y) ds + \int_{\beta(y_0)}^{\beta(y)} B(x, t) \Phi^p(x, t) dt. \quad (3.3.72)$$

Clearly $n_2(x, y)$ is non-negative and non-decreasing function in each x and y. Now a suitable application of the inequality (3.1.36) in Theorem 3.1.3 to (3.3.72), yields

$$\Phi(x, y) \leq {n_2}^{1/q}(x, y) e_4(x, y) q_4(x, y), \quad (3.3.73)$$

where $e_4(x, y)$ and $q_4(x, y)$ are defined in (3.3.38) and (3.3.37). From (3.3.73) and (3.3.71), we have

$$n_2(x, y) \leq c(x, y) + \int_{\alpha(x_0)}^{\alpha(x)} \int_{\beta(y_0)}^{\beta(y)} H(x, y, s, t) {e_4}^r(s,t) {q_4}^r(s,t) n_2(s,t) ds\, dt, \quad (3.3.74)$$

for $(x,y)\in\mathbb{I}$. Fixing any numbers $\tilde{x_1}$, and $\tilde{y_1}$ with $0<\tilde{x_1}\le x_1$ and $0<\tilde{y_1}\le y_1$, we have

$$n_2(x,y)\le c(\tilde{x_1},\tilde{y_1})+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}H(\tilde{x_1},\tilde{y_1},s,t){e_4}^r(s,t){q_4}^r(s,t)n_2(s,t)ds\,dt,$$

for $x_0\le x\le\tilde{x_1}$, $y_0\le y\le\tilde{y_1}$. Defining

$$z_2(x,y)=c(\tilde{x_1},\tilde{y_1})+\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}H(\tilde{x_1},\tilde{y_1},s,t){e_4}^r(s,t){q_4}^r(s,t)n_2(s,t)ds\,dt, \tag{3.3.75}$$

then $z_2(x_0,y)=z_2(x,y_0)=c(\tilde{x_1},\tilde{y_1})$ and

$$n_2(x,y)\le z_2(x,y), \tag{3.3.76}$$

with $z_2(x,y)$ is positive and non-decreasing in y, and

$$\begin{aligned}D_1z_2(x,y)&=\alpha'(x)\int_{\beta(y_0)}^{\beta(y)}H(\tilde{x_1},\tilde{y_1},\alpha(x),t){e_4}^r(\alpha(x),t){q_4}^r(\alpha(x),t)n_2(\alpha(x),t)dt,\\&\le\alpha'(x)\int_{\beta(y_0)}^{\beta(y)}H(\tilde{x_1},\tilde{y_1},\alpha(x),t){e_4}^r(\alpha(x),t){q_4}^r(\alpha(x),t)z_2(\alpha(x),t)dt,\\&\le z_2(x,y)\alpha'(x)\int_{\beta(y_0)}^{\beta(y)}H(\tilde{x_1},\tilde{y_1},\alpha(x),t){e_4}^r(\alpha(x),t){q_4}^r(\alpha(x),t)dt.\end{aligned} \tag{3.3.77}$$

Dividing both sides of (3.3.77) by $z_2(x,y)$, we obtain

$$\frac{D_1z_2(x,y)}{z_2(x,y)}\le\alpha'(x)\int_{\beta(y_0)}^{\beta(y)}H(\tilde{x_1},\tilde{y_1},\alpha(x),t){e_4}^r(\alpha(x),t){q_4}^r(\alpha(x),t)dt. \tag{3.3.78}$$

Now setting $x=s$ in (3.3.78) and then integrating with respect to s from x_0 to x, we obtain

$$\ln z_2(x,y)-\ln c(\tilde{x_1},\tilde{y_1})\le\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}H(\tilde{x_1},\tilde{y_1},s,t){e_4}^r(s,t){q_4}^r(s,t)ds\,dt.$$

This implies

$$\ln\frac{z_2(x,y)}{c(\tilde{x_1},\tilde{y_1})}\le\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)}H(\tilde{x_1},\tilde{y_1},s,t){e_4}^r(s,t){q_4}^r(s,t)ds\,dt,$$

$$\frac{z_2(x,y)}{c(\tilde{x_1},\tilde{y_1})} \le \exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_4}^r(s,t){q_4}^r(s,t)ds\,dt\right),$$

and

$$z_2(x,y) \le c(\tilde{x_1},\tilde{y_1})\exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_4}^r(s,t){q_4}^r(s,t)ds\,dt\right).$$

Taking $x=\tilde{x_1}$, $y=\tilde{y_1}$ in (3.3.76) and the last inequality, gives:

$$n_2(\tilde{x_1},\tilde{y_1}) \le z_2(\tilde{x_1},\tilde{y_1}),$$

and

$$z_2(\tilde{x_1},\tilde{y_1}) \le c(\tilde{x_1},\tilde{y_1})\exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(\tilde{x_1},\tilde{y_1},s,t){e_4}^r(s,t){q_4}^r(s,t)ds\,dt\right).$$

Since $0 < \tilde{x_1} \le x_1$ and $0 < \tilde{y_1} \le y_1$ are arbitrary, from the previous two equations we have

$$n_2(x,y) \le z_2(x,y),$$

and

$$z_2(x,y) \le c(x,y)\exp\left(\int_{\alpha(x_0)}^{\alpha(x)}\int_{\beta(y_0)}^{\beta(y)} H(x,y,s,t){e_4}^r(s,t){q_4}^r(s,t)ds\,dt\right), \tag{3.3.79}$$

for all $(x,y) \in \mathbb{I}$. Finally substituting the last inequality into (3.3.76), the desired inequality (3.3.60) follows immediately, and this completes the proof.

□

Chapter 4

Conclusion

This thesis is concerned with Gronwall-Bellman type integral inequalities. The main objective of our study was to establish some inequalities of Gronwall type involving functions of two independent variables which provide bounds on unknown functions. Thomas Hakon Gronwall and Richard Bellman has given the concept of Gronwall type inequalities. We have generalized the Gronwall-Bellman and Gronwall-Bihari type inequalities to two independent variables. The main purpose was to establish explicit bounds on retarded Gronwall Bellman and Bihari-like inequalities which can be used to study the qualitative behavior of the solutions of certain classes of retarded differential equations. The two independent variable generalizations of the main results and some applications of one of our results are also given. The results obtained originated from the celebrated Gronwall-Bellman-Bihari inequality has been of vital importance in the study of existence, uniqueness, continuous dependence, comparison, boundedness and stability of solutions of integral and differential equations. In the last three decades, more than one variable generalizations of these inequalities have been obtained and these results have generated a lot of research interests due to its usefulness in the theory of differential and integral equations. In future, these non-linear inequalities can be further generalized to the functions of n independent variables, which will compliment the existing results in the literature on Gronwall- Bellman- Bihari type inequalities in several variables.

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