Exact Solutions of Monge-Ampere and Zabolotskaya-Khokhlov Equations Using Optimal System

by

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A thesis submitted in partial fulfillment of the requirements
for the degree of Master of Philosophy in Mathematics

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FORM TH-4

National University of Sciences & Technology

M.Phil THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Mohsin Umair, Regn No. NUST201361935MSNS78013F Titled: Exact Solutions of Monge-Ampere & ZabolotsKaya - Khokholov Equations Using Optimal System be accepted in partial fulfillment of the requirements for the award of **M.Phil** degree.

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Abstract

In this thesis, using the Lie symmetry method, optimal systems, group invariants and exact solutions of Monge-Ampere and $(2+1)-$dimensional Zabolotskaya-Khokholov equations are obtained.

Acknowledgement

First of all, I am thankful to Almighty ALLAH, who gives power to weak, help the helpless and without help of whom, nothing can be possible, the most Gracious and the most Merciful.

I am very grateful to my honorable and respected supervisor Dr. Tooba Feroze for her affectionate guidance, continuous encouragement and help throughout my work. I am really short of words of gratitude to her valuable suggestions and support. I would like to extend my thanks to honorable and respected GEC members, Prof. Asghar Qadir and Dr. Sajid Ali for helpful discussions and suggestions.

School of Natural Sciences (SNS) has an inspiring research friendly environment for which, I am grateful to respected and honorable Prof. Azad Akhter Siddiqui, the Principal SNS, National University of Sciences and Technology (NUST). I am also grateful to all the respected faculty members at SNS.

Most importantly, I am most grateful to my parents, brother, sisters and my respected teacher Dr. Rabia Kamal of University of Wah for their patience, love, guidance and endless support.

Finally, I am grateful to all of my friends and colleagues, I cannot name all of them but some are Afaq, Ali, Bilal, Bilal Malik, Danish, Ehtisham, Faizan, Hamza, Ijaz, Kamran, Mehtab, Muaz, Nouman, Noveel, Rizwan, Saad, Shahid, Shahnwaz, Wajahat, waqas and all others for their encouragement and support (Friend names are arranged alphabetic order).

Mohsin Umair

Dedicated to

My Parents

Contents

Chapter 1

Fundamentals of symmetry analysis of differential equations

1.1 Introduction

There is an old Armenian saying, "He who lacks a sense of the past is condemned to live in the narrow darkness of his own generation." Mathematics without history is mathematics stripped of its greatness [25]. The history of mathematics is full of the stories of people who spent almost all of their lives in solving equations. At first they solved algebraic equations and later, they done differential equations. Since the last three centuries mathematical analysis has been the most dominant branch of mathematics. Differential equation is at the heart of mathematical analysis. It is a fact that nothing is permanent except change. The basic purpose of differential equations are to serve as a tool for the study of change in different phenomena of the physical world. Differential equations are the natural goal of elementary calculus. It was independently invented by the English physicist Isaac Newton $(1643-1716)$ and the German mathematician Gottfried Leibniz $(1646-1716)$. Also differential equations are the most important part of mathematics for understanding the physical sciences. Isaac Newton wrote laws of nature in terms of differential equations. The first book on the subject of differential equations was written between 1701 and 1704, by the Italian mathematician Gabriel Manfredi's. The book was on first degree differential equations and was published in Latin. Differential equations have diverse uses in physics, engineering, chemistry, biology, astrophysics, economics etc. Mostly differential equations represent different physical phenomena in respective applied sciences. From the history of mathematics we came to know that mathematicians are interested in finding the solutions of equations. In comparison of solution of ordinary differential equations with the solution of partial differential equations, it is simpler to find the solution of ordinary differential equation. It is because of the fact that the solution space of partial differential equations is infinite dimensional. Actually a problem arises when we have to deal with non-linear differential equations. The reason is that, up to now many techniques have been developed for finding the analytic solution of non-linear ordinary as well as partial differential equations. But we do not have any generalized way except the Lie technique.

In 1867, the Norwegian mathematician Sophus Lie, introduced a powerful technique for the solutions of differential equations [4, 9, 15, 19]. The best part of the technique is that it is applicable to all types of differential equations, whether they are homogeneous, non homogeneous, linear or nonlinear of any order. The idea of this technique came into his mind when he was attending lectures of Ludwing Sylow on the work of Galios on solubility of algebraic equations. After that he thought that the same approach can be used for solution of differential equations. Galois theory associates permutation groups with the solution of algebraic equation. Lie used the same idea for the solution of differential equations. He assumed that there must be group of transformations (Lie groups) which will be associated to the solution of differential equations. This idea construct a new branch of mathematics that is Lie group analysis (Lie symmetry analysis). This analysis combine three branches of mathematics namely: algebra, analysis and geometry. The main idea of Lie group of transformations is that it employs transformations which form a vector space closed under Lie algebra. It also replace the global object that is group, with its local infinitesimal group which is known as its Lie algebra. In this chapter we briefly explain fundamentals of symmetry analysis, comprising of manifolds, vector space, tangent space, Lie brackets, Lie group and Lie algebra, Lie point transformations, adjoint representations and optimal system.

In second chapter, we review the optimal algebra of Korteweg de Vries equations of fifth order. In third chapter, we find the optimal algebra of non linear non homogeneous Monge-Ampere equation for two particular cases. In fourth chapter, we find exact solutions of $(2+1)$ dimensional non linear Zabolotskaya Khokholov equation by using its optimal algebra.

1.2 Manifold

An $n-$dimensional manifold is a set M, together with a countable collection of subsets $m_i \subset M$, called coordinate charts, and one$-$one functions $f_i : m_i \to v_i$ onto connected open subsets $v_i \subset \mathbb{R}^n$, called local coordinate maps [19], which satisfy the following properties:

(a) The coordinate chart covers M, i.e.

$$\bigcup_i m_i = M.$$

(b) On the overlap of any pair of coordinate charts $m_i \cap m_j$ the composite map

$$f_i \circ f_j^{-1} : f_i(m_i \cap m_j) \to f_j(m_i \cap m_j),$$

is smooth (infinitely differentiable) function.

(c) For distinct points $p \in m_i$, $q \in m_j$ in M, there exists open sets $u \subset v_i$, $v \subset v_j$, with the property $f_i(p) \in u$, $f_i(q) \in v$, satisfying

$$f_i^{-1}(u) \cap f_j^{-1}(v) = \Phi.$$

An $n-$dimensional manifold is actually a topological space which is similar to Euclidean space at the neighborhood of each point, where topological space (X, T) is the set X together with the topology T on the set X.

Euclidean space $\mathbb{R}^n$ itself is the n-dimensional manifold. There is a single coordinate chart $U = \mathbb{R}^n$, with local coordinate map given by the identity : $x = I : \mathbb{R}^n \to \mathbb{R}^n$. More generally, any open subset $U \subset \mathbb{R}^n$ is an $n-$dimensional manifold with a single coordinate chart given by U itself, with local coordinate map identity. Conversely, if M is any manifold with a single global coordinate chart $x : M \to V \subset \mathbb{R}^n$, we can identify M with its image V, an open subset of $\mathbb{R}^n$.

Another example of manifold is unit circle i.e. a circle of unit radius

$$S^1 = \{(x, y) : x^2 + y^2 = 1\},$$

which is seen to be a one dimensional manifold with two coordinate charts.

The unit sphere

$$S^2 = \{(x, y) : x^2 + y^2 + z^2 = 1\},$$

is also an example of non-trivial two dimensional manifold realized as surface in $\mathbb{R}^n$ when $n = 3$.

1.2.1 Vector field

A vector field $\mathbf{V}$ on a manifold M is a function which assigns a tangent vector $\mathbf{V}|_{\mathbf{p}}$ at each point $\mathbf{P} \in M$, where $\mathbf{V}|_{\mathbf{P}}$ varies smoothly while moving from point to point on the manifold M [19]. For local coordinates $x = (\overset{1}{x}, \overset{2}{x}, \overset{3}{x}, \ldots, \overset{n}{x})$, then the vector field takes the form as

$$\mathbf{V}|_{\mathbf{P}} = \xi^1(x)\frac{\partial}{\partial x^1} + \xi^2(x)\frac{\partial}{\partial x^2} + \xi^3(x)\frac{\partial}{\partial x^3} + \cdots + \xi^n(x)\frac{\partial}{\partial x^n}, \tag{1.1}$$

where every $\xi^{i,s}(x)$ are smooth functions of x. A simple example of vector field is the velocity field of steady fluid flow at some open subset $M \subset \mathbb{R}^3$. At every point $(x, y, z) \in \mathbf{V}|_{(x,y,z)}$ will give the velocity of fluid particles which are passing through the point (x, y, z).

Suppose $N \subset M$ is a submanifold of M i.e. N is a subset of M and satisfying all the properties of manifolds parameterized by the immersion $\Psi : \tilde{N} \to M$, $\tilde{N} \subset N$. The tangent space to N at $x \in N$, is the image of the tangent space to $\tilde{N}$ at the corresponding point $\tilde{x}$:

$$TN|_x = d\Psi(T\tilde{N}|_{\tilde{x}}), \qquad \tilde{x} = \Psi(x).$$

$TN|_x$ is a subspace of $TM|_x$ of the same dimension as that of N [19]. So, simply we can say that the collection of all tangent vectors at point $P \in N$ to all possible curves those passing through the point P is called the *tangent space* to N at point P to which we denote as $TN|_P$, also an $n-$dimensional tangent space is generated by basis vectors

$$\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \cdots, \frac{\partial}{\partial x^n}.$$

The collection of all possible tangent spaces passing through point $P \in N$ over the manifold N is called the *tangent bundle* of N, that is

$$TN = \bigcup_{P \in N} TN|_P.$$

1.3 Lie group and Lie algebra

A Lie group is somewhat a relation between the algebraic concept of groups and geometric notion of manifold. This combination of algebra and calculus results the powerful techniques for the study of symmetry. We start discussing Lie group by first defining the abstract group.

A *group* $\mathbb{G}$ is an ordered pair $(\mathbb{G}, *)$ with its binary operation $*$ satisfying the following axioms

(i) $\mathbb{G}$ is closed under the binary operation $*$.

(ii) $a * (b * c) = (a * b) * c \quad \forall a, b, c \in \mathbb{G}$ i.e. $\mathbb{G}$ is associative under binary operation $*$.

(iii) There must exist an element e such that $a * e = e * a = a$ i.e. $\mathbb{G}$ has an identity element.

(iv) For each element $a \in \mathbb{G}$ there is an element $a^{-1} \in \mathbb{G}$, such that $a^{-1} * a = a * a^{-1} = e$, i.e. inverse of every $a \in \mathbb{G}$ [6]. Some examples of group are given below.

(i) $\mathbb{G} = \mathbb{R}\backslash\{0\}$ is a group under binary operations of multiplication($\times$), under the binary operation of multiplication $e = 1$ is the identity and inverse of every element a of $\mathbb{G}$ is reciprocal of a i.e. $a = \dfrac{1}{a}$.

(ii) $\mathbb{G} = \mathbb{R}$ is also a group under the binary operation of addition $e = 0$ is the identity and inverse of every element a of $\mathbb{G}$ is negative of a i.e. $a = -a$.

(iii) In a similar way, $\mathbb{GL}(n, \mathbb{R})$, the set of all invertible matrices of order $n \times n$ with real entries is a group under the binary operation of matrix multiplication, with identity element be identity matrix and inverse is the ordinary inverse of a matrix which normally have rational entries. Usually we denote general linear matrix by just $\mathbb{GL}(n)$.

Now moving towards Lie group, the remarkable trait of Lie group is that it also have the structure of smooth manifolds, so the elements of group can be varied continuously. From above examples the manifold structure is clear for $\mathbb{R}$, for general linear group, it can be identified with open subset

$$\mathbb{GL}(n) = V : detV \neq 0,$$

of the space $M_{n\times n}$ of all $n \times n$ matrices. But $M_{n\times n}$ is isomorphic to $\mathbb{R}^{n^2}$, thus $\mathbb{GL}(n)$ is also an n^2-dimensional manifold. In both cases analytically the group operation is smooth. This exemplifies to the general definition of Lie group, now moving towards the definition of Lie group.

An $n-$dimensional *Lie group* is a group $\mathbb{G}$ which has the structure of an $n-$dimensional manifold such that the following composition function σ and inversion function κ are smooth for all elements of group $\mathbb{G}$ [19].

$$\sigma : \mathbb{G} \times \mathbb{G} \to \mathbb{G}\ , \qquad \sigma(a, b) = a \cdot b, \qquad a, b \in \mathbb{G},$$
$$\kappa : \mathbb{G} \to \mathbb{G}\ , \qquad \kappa(a) = a^{-1}, \qquad a \in \mathbb{G}.$$

$\mathbb{G} = \mathbb{R}^n$, with the obvious manifold structure be the group under the action of vector addition $(\mathbf{a,b}) \to \mathbf{a+b}$. The identity element is the null vector and the inverse element of vector $\mathbf{a}$ be negative vector $-\mathbf{a}$. So $\mathbb{R}^n$ is an example of $n-$parameter abelian Lie group.
Also consider the group of orthogonal $n \times n$ matrices $SO(n)$

$$SO(n) = V \in \mathbb{GL}(n) : V^\intercal V = I : detV = I,$$

thus $SO(n)$ is the subset of $\mathbb{R}^{n^2}$ defined by n^2 equations

$$V^\intercal V - I = 0,$$

involving the matrix entries v_{ij} of V.

A *Lie subgroup* $\mathbb{H}$ of a Lie group $\mathbb{G}$ is given by a submanifold $\phi : \tilde{\mathbb{H}} \to \mathbb{G}$, here $\tilde{\mathbb{H}}$ is itself a Lie group such that, $\mathbb{H} = \phi(\tilde{\mathbb{H}})$ is the image of ϕ, and ϕ is a Lie group homomorphism[19].
Usually, Lie group occurs as subgroups of some larger Lie groups, like orthogonal groups are subgroup of general linear groups of every invertible matrices. Generally we will be interested in Lie subgroups during our work which are considered as Lie group.

If $\mathbb{G}$ is a Lie group then there are certain distinguished vector fields on $\mathbb{G}$ characterized by their invariance under the group multiplication. These invariant vector fields form finite dimensional vector space called the *Lie algebra* of Lie group

$\mathbb{G}$. Precisely, Lie algebra is the infinitesimal generator of Lie group $\mathbb{G}$. Also an $n-$parameter Lie group has an $n-$dimensional Lie algebra which form a vector space and is denoted by $\boldsymbol{g}$ it contains all the generators of $n-$dimensioanl Lie group, satisfying the conditions of

(i) **Bilenearity:**

For vector fields $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ and $\mathbf{V}_4$ on any manifold and a, b, c and d be constants then the bilenearity condition is

$$[a\mathbf{V}_1 + b\mathbf{V}_2, c\mathbf{V}_3 + d\mathbf{V}_4] = ac[\mathbf{V}_1, \mathbf{V}_3] + ad[\mathbf{V}_1, \mathbf{V}_4] + bc[\mathbf{V}_2, \mathbf{V}_3] + bd[\mathbf{V}_2, \mathbf{V}_4].$$

(ii) **Skew-Symmetry:**

For vector fields $\mathbf{V}_1$ and $\mathbf{V}_2$ on the manifold M are called skew-symmetric if following condition holds

$$[\mathbf{V}_1, \mathbf{V}_2] = -[\mathbf{V}_2, \mathbf{V}_1].$$

(iii) **Jacobi identity:**

Let $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ be vector fields on the manifold M then they satisfy the following condition

$$[\mathbf{V}_1, [\mathbf{V}_2, \mathbf{V}_3]] + [\mathbf{V}_2, [\mathbf{V}_3, \mathbf{V}_1]] + [\mathbf{V}_3, [\mathbf{V}_1, \mathbf{V}_2]] = 0,$$

called the Jacobi identity. Lie algebra is called *abelian* if

$$[\mathbf{V}_i, \mathbf{V}_j] = 0 \qquad \forall \mathbf{V}_i, \mathbf{V}_j \in \boldsymbol{g}.$$

Commutator table (commutator relation table) are used to display the structure of given Lie algebra. Also to display structure of Lie algebra, it is convenient to write it in the tabular form. For an $n-$dimensional Lie algebra $\boldsymbol{g}$, with basis vectors $\mathbf{V}_1, \mathbf{V}_2 \cdots, \mathbf{V}_n$ commutator relation table will be the $n \times n$ table, whose $(i, j)-$th entry represent the lie bracket $[\mathbf{V}_i, \mathbf{V}_j]$. Its all diagonal entries are zero and the table is always skew-symmetric.

For example, consider Benjamin Bona Mahony (BBM) equation

$$u_t + u_x + uu_x - u_{xxt} = 0.$$

Symmetry generators of BBM equation are as follow

$$\mathbf{V}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{V}_2 = \frac{\partial}{\partial x}, \qquad \mathbf{V}_3 = t\frac{\partial}{\partial t} - (u+1)\frac{\partial}{\partial u}.$$

One can write its Lie algebra as

$$\begin{array}{lll} [\mathbf{V}_1, \mathbf{V}_1] = 0, & [\mathbf{V}_1, \mathbf{V}_2] = 0, & [\mathbf{V}_1, \mathbf{V}_1] = \mathbf{V}_1, \\ [\mathbf{V}_2, \mathbf{V}_1] = 0, & [\mathbf{V}_2, \mathbf{V}_2] = 0, & [\mathbf{V}_2, \mathbf{V}_3] = 0, \\ [\mathbf{V}_3, \mathbf{V}_1] = 0, & [\mathbf{V}_3, \mathbf{V}_2] = -\mathbf{V}_2, & [\mathbf{V}_3, \mathbf{V}_3] = 0. \end{array}$$

Using these results of commutator relations one can construct commutator relation table as:

,	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$
$\mathbf{V}_1$	0	0	$\mathbf{V}_1$
$\mathbf{V}_2$	0	0	0
$\mathbf{V}_3$	$-\mathbf{V}_1$	0	0

Commutator relation table

1.4 Lie point transformations

Here our main goal is to use symmetry generators to make given differential equation that can be ordinary differential equation or partial differential equation, such that if we are given with ordinary differential equation, then to reduce its order and if we are given with partial differential equation, then to reduce its dimension or make it an ordinary differential equation. Here we are dealing with non linear partial differential equations, so our focus will be on reducing the dimension of given non linear partial differential equation. For this we need a proper definition of symmetry generators and a proper technique that describe the way to find them. This in turn requires few knowledge about transformations and their symmetry generators. Starting from somehow a simpler case of ordinary differential equation. For coordinates (x, y) x is independent variable while y is dependent variable. To make ordinary differential equations of lower order we often need to find appropriate change of variables i.e. by a transformation of independent variable x and dependent variable y

$$\tilde{x} = \tilde{x}(x, y; \varepsilon) = x + \varepsilon\xi(x, y),$$

$$\tilde{y} = \tilde{y}(x, y; \varepsilon) = y + \varepsilon\eta(x, y),$$

are known as Lie point transformations, also one parameter Lie point transformations as they depend on only one parameter ε. These one parameter point transformations form a group with identity transformation when $\varepsilon = 0$. The group of one parameter point transformation defines a family of curves in manifold M, These parametric curves can be viewed as the integral curves of a differentiable vector field $\mathbf{V} \in M$. These curves are parameterized by the parameter ε and also known as the orbits of the group of transformations.

As $\tilde{x}(x, y; \varepsilon)$, $\tilde{y}(x, y; \varepsilon)$ are parametric equations of group of transformation through the point p. The tangent vector $\mathbf{V}|_p$ i.e. tangent vector $\mathbf{V}$ at the point $p = p(x, y; 0)$ at $\varepsilon = 0$ is given by

$$\mathbf{V}|_p = \frac{\partial\tilde{x}}{\partial\varepsilon}|_{\varepsilon\to 0}\frac{\partial}{\partial x}|_p + \frac{\partial\tilde{y}}{\partial\varepsilon}|_{\varepsilon\to 0}\frac{\partial}{\partial y}|_p, \tag{1.2}$$

where,

$$\begin{aligned} \tilde{x} &= x + \varepsilon\xi; \qquad & \xi = \frac{\partial \tilde{x}}{\partial \varepsilon}|_{\varepsilon=0}, \\ \tilde{y} &= y + \varepsilon\eta; \qquad & \eta = \frac{\partial \tilde{y}}{\partial \varepsilon}|_{\varepsilon=0}. \end{aligned} \tag{1.3}$$

Then equation (1.2) can be written as

$$\mathbf{V} = \xi(x,y)\frac{\partial}{\partial x} + \eta(x,y)\frac{\partial}{\partial y}, \tag{1.4}$$

is the *infinitesimal generator*.

The transformation (1.3) is a one parameter point transformation which is also known as *infinitesimal point transformation.*

If we have a differential equation of some arbitrary order say n, of the form

$$f(x, y, y', y'', \cdots, y^n) = 0,$$

here y^n denotes the arbitrary nth order derivative of dependent variable y with respect to independent variable x, then differentiating the Lie point transformations successively, which takes the form

$$\begin{aligned} \tilde{x} &= \tilde{x}(x, y; \varepsilon) = x + \varepsilon\xi(x, y), \\ \tilde{y} &= \tilde{y}(x, y; \varepsilon) = y + \varepsilon\eta(x, y), \\ \tilde{y}' &= \tilde{y}'(x, y, y'; \varepsilon) = y'(x, y, y') + \varepsilon\eta'(x, y, y'), \end{aligned}$$

continuing in this manner,m we have

$$\tilde{y}^{(n)} = \tilde{y}^{(n)}(x, y, y' \cdots, y^{(n)}; \varepsilon) = y^{(n)}(x, y, y', \cdots, y^{(n)}) + \varepsilon\eta^{(n)}(x, y, y', \cdots, y^{(n)}).$$

Now moving towards the prolongation of generator for ordinary differential equation, for the first prolongation function $\eta^{(1)}$, we have

$$\begin{aligned} \eta^{(1)} &= \frac{d\eta}{dx} - y^{(1)}\frac{d\xi}{dx}, \\ \eta^{(2)} &= \frac{d\eta^{(1)}}{dx} - y^{(2)}\frac{d\xi}{dx}, \\ &\vdots \\ \eta^{(n)} &= \frac{d\eta^{(n-1)}}{dx} - y^{(n)}\frac{d\xi}{dx}. \end{aligned}$$

Now the expressions for the infinitesimal generators be as

$$
\begin{aligned}
\mathbf{V}^{(1)} &= \mathbf{V} + \eta^{(1)}\frac{\partial}{\partial y^1}, \\
\mathbf{V}^{(2)} &= \mathbf{V} + \eta^{(1)}\frac{\partial}{\partial y^1} + \eta^{(2)}\frac{\partial}{\partial y^2}, \\
&\vdots \\
\mathbf{V}^{(n)} &= \mathbf{V} + \eta^{(1)}\frac{\partial}{\partial y^1} + \eta^{(2)}\frac{\partial}{\partial y^2} + \cdots + \eta^{(n)}\frac{\partial}{\partial y^n}.
\end{aligned}
$$

For ordinary differential equations relations for $\eta^{(1)}$ and $\eta^{(2)}$ are

$$
\begin{aligned}
\eta^{(1)} &= \eta_{,x} + (\eta_{,y} - \xi_{,x})y' - \xi_{,y}{y'}^2, \\
\eta^{(2)} &= \eta_{,xx} + (2\eta_{,xy} - \xi_{,xx})y' + (\eta_{,yy} - 2\xi_{,xy}){y'}^2 - \xi_{,yy}{y'}^3 + (\eta_{,y} - 2\xi_{,x} - 3\xi_{,y}y')y''.
\end{aligned}
$$

Now extending this concept for system of partial differential equations. If we have more than one independent variable say k such that $\mathbf{x} = (x^1, x^2, x^3, \cdots, x^k)$ here x^1 be the first independent variable, x^2 be the second independent variable and similarly x^k be the kth independent variable. For more than one dependent variable say m, such that $u = (u^1, u^2, u^3, \cdots, u^m)$ here u^1 be the first dependent variable, u^2 be the second dependent variable similarly u^m be the mth dependent variable. If we have a partial differential equation of kth order as

$$
f(x, u, u_{i_1}, u_{i_1 i_2}, \ldots, u_{i_1 i_2 \ldots i_k}) = 0,
$$

then its infinitesimal transformations will be of the form

$$
\begin{aligned}
\tilde{x}^i &= \tilde{x}^i(x^i, u^j; \varepsilon) = x^i + \varepsilon\xi^i(x^i, u^j), \\
\tilde{y}^j &= \tilde{y}^j(x^i, u^j; \varepsilon) = y^j + \varepsilon\eta^j(x^i, u^j), \\
\tilde{u^j}_{i_1} &= \tilde{u^j}_{i_1}(x^i, u^j; \varepsilon) = {u^j}_{i_1}(x^i, u^j, u_{i_1}) + \varepsilon{\eta^j}_{i_1}(x^i, u^j, {u^j}_{i_1}), \\
&\vdots \\
\tilde{u^j}_{i_1,i_2,\cdots,i_k} &= \tilde{u^j}_{i_1,i_2,\cdots,i_k}(x^i, u^j, {u^j}_{i_1}, \cdots, {u^j}_{i_1,i_2,\cdots,i_k}; \varepsilon) \\
&= {u^j}_{i_1,i_2,\cdots,i_k}(x^i, u^j, {u^j}_{i_1}, \cdots, {u^j}_{i_1,i_2,\cdots,i_k}) + \varepsilon{\eta^j}_{i_1,i_2,\cdots,i_k}(x^i, u^j, {u^j}_{i_1,i_2,\cdots,i_k}).
\end{aligned}
$$

Here the subscripts represent derivative while superscripts represent coordinates.

Now extending this concept to prolongation of generator for partial differential equation. The generator for partial differential equation is

$$
\mathbf{V} = \xi^i\frac{\partial}{\partial x^i} + \eta^j\frac{\partial}{\partial u^j}, \tag{1.5}
$$

$$\mathbf{V}^{(1)} = \mathbf{V} + \eta^j{}_{i_1}\frac{\partial}{\partial u^j{}_{i_1}},$$

$$\mathbf{V}^{(2)} = \mathbf{V}^{(1)} + \eta^j{}_{i_1 i_2}\frac{\partial}{\partial u^j{}_{i_1 i_2}},$$

$$\vdots$$

$$\mathbf{V}^{(n)} = \xi^i\frac{\partial}{\partial x^i} + \eta^j\frac{\partial}{\partial u^j} + \eta^j{}_{i_1}\frac{\partial}{\partial u^j{}_{i_1}} + \eta^j{}_{i_1 i_2}\frac{\partial}{\partial u^j{}_{i_1 i_2}} + \cdots + \eta^j{}_{i_1 i_2\cdots i_n}\frac{\partial}{\partial u^j{}_{i_1 i_2\cdots i_n}}, \quad (1.6)$$

here (1.6) is nth order prolongation of (1.5), we will extend the generator according to the order of partial differential differential equation. Also the expressions for general first and second order prolongation of $\eta^{,s}$ are as follows

$$\begin{aligned} \eta^\alpha{}_{,n} &= \frac{D\eta^\alpha}{Dx^n} - u^\alpha{}_{,i}\frac{D\xi^i}{Dx^n}, \\ \eta^\alpha{}_{,nm} &= \frac{D\eta^\alpha_m}{Dx^n} - u^\alpha{}_{,im}\frac{D\xi^i}{Dx^n}, \\ \frac{D}{Dx^n} &= \frac{\partial}{\partial x^n} + u^\alpha{}_{,n}\frac{\partial}{\partial u^\alpha} + u^\alpha{}_{,nm}\frac{\partial}{\partial u^\alpha{}_{,m}}, \end{aligned}$$

here superscript of $\eta^{,s}$ is for number of depending variable subscript is for order of derivative, superscript of ξ is for independent variable and similarly superscripts of u representing number of dependent variable while its subscripts representing independent variable.

1.5 Optimal system

Lie group analysis is a powerful tool for obtaining exact similarity solutions of nonlinear differential equations. To calculate the group invariant solutions, one first need to find the full Lie point symmetry group admitted by the given differential equations and to determine all the subgroups of this Lie group. An effective systematic way to classify the similarity solutions is in terms of the set of representatives of all the conjugacy classes of Lie group, which is called representative system. Now an *optimal* is defined as the best, most efficient or of the greatest value, sometimes under certain parameters or constraints. Here the *optimal system* is the one with the minimal representatives of each class of similar vectors. That is a list of group invariant solutions from which every other solution can be derived. The problem to find an optimal system of similarity solutions leads to construct an optimal system of sub-algebras for the Lie algebra of the known Lie point symmetry group.

Let us suppose we have a differential equation which admits a Lie algebra $\mathbf{L}_n$ of dimension $n > 1$. Then in principle one can consider invariant solutions of differential equations based on one, two or more sub-algebras of $\mathbf{L}_n$. But we also knew that there are an infinite number of sub algebras for one dimensional Lie algebra. We can make this problem manageable by recognizing that if two sub-algebras are similar that is if they can be connected with each other by means of some transformation from symmetry group which also have Lie algebra $\mathbf{L}_n$, then their corresponding invariant solutions can also be connected by means of transformation which connected sub-algebras. Therefore, it will be enough to put all the sub-algebras of same dimension, say s, and then select one representative from each class. The set of all these representatives of all these is an optimal system of order s. If we have to find all the invariant solutions with respect to $s-$dimensional sub-algebras, then it will be enough to construct invariant solutions for the optimal solution of order s. The set of invariant solution obtained, as a result of these optimal algebras, is called an optimal system of invariant solutions. To find optimal system of given differential equation we follow the following steps.

(i) First we calculate the commutator table for symmetry generators of given differential equation.

(ii) Next we construct adjoint representation table, by conjunction of adjoint map with already calculated commutator relation table.

(iii) After that, we consider the construction of optimal system of given differential equation. Where the method requires simplification of general vector by using judicious application of adjoint map.

Let $\mathbf{V}$ be a vector field on a manifold $\mathbf{M}$ and a function f which maps from manifold $\mathbf{M}$ to the real numbers $\mathbb{R}$ is smooth [19],

$$f : \mathbf{M} \to \mathbb{R}.$$

Here we are interested in seeing how the function f changes $f(exp(\varepsilon\mathbf{V}))$ as ε varies.

In local coordinates $\mathbf{V} = \sum \xi^i \frac{\partial}{\partial x^i}$, then

$$\begin{aligned}
\frac{d}{d\varepsilon} f(exp(\varepsilon\mathbf{V}x)) &= \sum_{i=1}^{m} \xi^i(exp(\varepsilon\mathbf{V}x)) \frac{\partial f}{\partial x^i}(exp(\varepsilon\mathbf{V}x)), \\
\frac{d}{d\varepsilon} f(exp(\varepsilon\mathbf{V}x)) &= \mathbf{V}(f)[exp(\varepsilon\mathbf{V}x)],
\end{aligned}$$

in particular at $\varepsilon = 0$,

$$\frac{d}{d\varepsilon} f(exp(\varepsilon\mathbf{V}x))|_{\varepsilon=0} = \mathbf{V}(f)(x),$$

the vector field $\mathbf{V}$ is the first order partial differential operator on real valued functions $f(x)$ on $\mathbf{M}$. Furthermore, by Taylor's theorem

$$f(exp(\varepsilon \mathbf{V}x)) = f(x) + \varepsilon \mathbf{V}f(x) + o(\varepsilon^2).$$

So, $\mathbf{V}(f)$ gives the infinitesimal change in the function f under the flow generated by $\mathbf{V}$. We can continue the process of differentiation and substitution into the Taylor series, obtaining

$$f(exp(\varepsilon \mathbf{V}x)) = f(x) + \varepsilon \mathbf{V}f(x) + \frac{\varepsilon^2}{2}\mathbf{V}^2(f)(x) + \cdots,$$

where $\mathbf{V}^2(f) = \mathbf{V}(\mathbf{V}(f))$.

1.6 Adjoint representation

To define the adjoint representation first we define a group representation and then the adjoint representation and its tabular form.

1.6.1 Group representation

Let $\mathbf{G}$ be a group. A representation of $\mathbf{G}$ is a homomorphism

$$\rho : \mathbf{G} \longrightarrow \mathbf{G}_n(\mathbb{C}) \tag{1.7}$$

for some number n, which is called the *dimension* or *degree* of the representation.

Let $\mathbf{V}$ be an $n-$dimensional vector space. The set of all invertible linear maps from $\mathbf{V}$ to $\mathbf{V}$ form a group which we call general linear matrix over the vector space $\mathbf{V}$ as $GL(\mathbf{V})$. A representation of a group $\mathbf{G}$ is a choice of vector space and a homomorphism

$$\rho : \mathbf{G} \longrightarrow \mathbf{G}(\mathbf{V}). \tag{1.8}$$

If we pick a basis of $\mathbf{V}$, we get a representation in the previous sense. Informally a representation of a group is a way of writing it down as a group of matrices. Representation of a group is a map between any group element $'\mathbf{g}'$ of a group $\mathbf{G}$ and a linear transformation $\rho(\mathbf{g})$ of some vector space in such a way that the identity element of the group transforms the identity. Every inverse element is mapped to the corresponding inverse transformation. The combination of transformations corresponding to any elements of group g ,h is the same as the transformation corresponding to the point gh.
(i) $\rho(e) = I$,
(ii) $\rho(g^{-1}) = (\rho())$,

(iii) $\rho(g)\rho(h) = \rho(gh)$.

Linear actions of Lie groups play a vital role in development and applications of mathematics. For theoretical aspects one is interested in considering an abstract group with its group action operation and manifold structure. While considering physical applications one is more interested in what the group actually does that is the group action. Most of time in physics one is more concerned with linear transformations in a vector space. Therefore the concept of a representation is more relevant to physics. A representation identifies with each point of the group manifold a linear transformation of a vector space.

1.6.2 Representation of a Lie algebra

The representation of a Lie algebra $\boldsymbol{g}$ is defined by a Lie algebra homomorphism

$$\rho : \boldsymbol{g} \longrightarrow gl(\mathbf{V}),$$

the space of linear maps on vector space $\mathbf{V}$. Also we can say that representation of Lie algebra is a linear map which preserves the Lie bracket operation:

$$\rho([\mathbf{v}, \mathbf{w}]) = \rho(\mathbf{w})\rho(\mathbf{v}) - \rho(\mathbf{v})\rho(\mathbf{w}),$$

while

$$\rho(\mathbf{v})(\mathbf{x}) = \frac{d}{dt}\rho(exp(t\mathbf{v}))\mathbf{x} \mid_{t=0} \qquad ; \mathbf{x} \in \mathbf{V}, \mathbf{v} \in g,$$

defines the infinitesimal version of the Lie algebra $\mathbf{g}$. If the representation

$$\rho : \mathbf{G} \longrightarrow \mathbf{G}(n)$$

acts on $\mathbb{R}^n$, with coordinates $x = (x^1, x^2, \ldots, x^n)$ its infinitesimal generators

$$\mathbf{V}_A = \sum_{i,j=1}^{n} a^i_j \partial x_i$$

are linear vector fields on $\mathbb{R}^n$.

Consider a Lie algebra $\boldsymbol{g}$ over the real field, $gl(\boldsymbol{g})$ the Lie algebra of linear operators on $\boldsymbol{g}$ and the map

$$ad : \boldsymbol{g} \longrightarrow gl(\boldsymbol{g}), \quad ad(x)y = [x, y], \quad \forall x, y \in \boldsymbol{g}.$$

The map

$$ad(x) : \boldsymbol{g} \longrightarrow \boldsymbol{g}$$

is known as the *adjoint action* of the element x on $\boldsymbol{g}$.

1.6.3 Adjoint representation

Let G be a Lie group and $\boldsymbol{g}$ be its Lie algebra, then the differential of conjugation action

$$K_{\mathbf{g}} : h \longrightarrow \mathbf{g}h\mathbf{g}^{-1}$$

of G defines a representation of G on its Lie algebra $\boldsymbol{g}$, called the *adjoint representation* of $\mathbf{G}$:

$$K_{\mathbf{g}}(h) = \mathbf{g}h\mathbf{g}^{-1},$$

$$Ad_{\boldsymbol{g}}(\mathbf{v}) = dK_{\boldsymbol{g}}(\mathbf{v}) \quad ; \quad \mathbf{v} \in \boldsymbol{g}.$$

Also, if $\mathbf{v} \in \boldsymbol{g}$ generates the one parameter subgroup $H = \{exp(\varepsilon\mathbf{v}) : \varepsilon \in R\}$, then $Ad_{\boldsymbol{g}}(\mathbf{v})$ generates the conjugate one parameter subgroup $K_{\boldsymbol{g}}(H) = \boldsymbol{g}H\boldsymbol{g}^{-1}$.

The best part of Lie group analysis is that a curved object Lie group $\mathbf{G}$ can be almost completely captured by a flat one that is the tangent space $T_e\mathbf{G}$ of $\mathbf{G}$ at identity. Lie algebra of Lie group is also the tangent space at identity. The adjoint representation is a homeomorphism from group G to the space of linear operators on the tangent space at identity $T_e\mathbf{G}$. Denoted by $K_{\boldsymbol{g}}(h) = \boldsymbol{g}h\boldsymbol{g}^{-1}$ and is known as the *adjoint action.*

$$K_{\boldsymbol{g}}(hj) = \boldsymbol{g}hj\boldsymbol{g}^{-1},$$

$$K_{\boldsymbol{g}}(hj) = \boldsymbol{g}hej\boldsymbol{g}^{-1},$$

$$K_{\boldsymbol{g}}(hj) = \boldsymbol{g}h(\boldsymbol{g}^{-1}\boldsymbol{g})j\boldsymbol{g}^{-1} \quad ; \quad \therefore e = \boldsymbol{g}\boldsymbol{g}^{-1}$$

$$K_{\boldsymbol{g}}(hj) = (\boldsymbol{g}h\boldsymbol{g}^{-1})(\boldsymbol{g}j\boldsymbol{g}^{-1}),$$

$$K_{\boldsymbol{g}}(hj) = K_{\boldsymbol{g}}(h)K_{\boldsymbol{g}}(j) \quad where \quad h, j \in \mathbf{v}.$$

Also this homomorphism maps the identity to identity for every group element $\boldsymbol{g}$:

$$k_{\boldsymbol{g}}(e) = \boldsymbol{g}e\boldsymbol{g}^{-1}$$

$$k_{\boldsymbol{g}}(e) = e,$$

which shows that any curve through identity e on the manifold G is mapped by this homeomorphism to another curve through e. Therefore, the adjoint representation maps any tangent vector of the curve on G in tangent space of G at identity to another vector in $T_e\mathbf{G}$. Now we are going to show that adjoint action is given by commutator relation.
Consider a curve $\alpha(t)$ on the manifold G with $\alpha(0) = e \in \mathbf{G}$ and tangent vector $\alpha'(0) = \mathbf{X} \in T_e\mathbf{G}$. Also, assume that the curve goes through some arbitrary element $\boldsymbol{g} \in \mathbf{G}$. Then the adjoint action

$$Ad_{\boldsymbol{g}}(\mathbf{X}) = \boldsymbol{g}\mathbf{X}\boldsymbol{g}^{-1}$$

as

$$Ad_{\boldsymbol{g}}(\mathbf{Y}) = Ad_{\alpha(t)}(\mathbf{Y}) = \alpha(t)\mathbf{Y}\alpha(t)^{-1}.$$

Differentiating this map at the identity, $t = 0$, gives the Lie algebra homomorphism.

$$\begin{aligned}
\frac{d}{dt}Ad_{\alpha(t)}(\mathbf{Y}) \mid_{t=0} &= \frac{d}{dt}(\alpha(t)\mathbf{Y}\alpha(t)^{-1}) \mid_{t=0}, \\
\frac{d}{dt}Ad_{\alpha(t)}(\mathbf{Y}) \mid_{t=0} &= \alpha'(0)\mathbf{Y}\alpha(0)^{-1} + \alpha(0)\mathbf{Y}\frac{d}{dt}\alpha(t)^{-1} \mid_{t=0} .
\end{aligned}$$

Here, using the definition of a matrix Lie algebra to calculate $\frac{d}{dt}\alpha(t)^{-1}$, we get

$$\begin{aligned}
\frac{d}{dt}Ad_{\alpha(t)}(\mathbf{Y}) \mid_{t=0} &= \alpha'(0)\mathbf{Y}\alpha(0)^{-1} + \alpha(0)\mathbf{Y}(-\alpha(0)^{-1}\alpha'(0)\alpha(0)^{-1}), \\
\frac{d}{dt}Ad_{\alpha(t)}(\mathbf{Y}) \mid_{t=0} &= \alpha'(0)\mathbf{Y}\alpha(0)^{-1} - \alpha(0)\mathbf{Y}\alpha(0)^{-1}\alpha'(0)\alpha(0)^{-1}. \\
\frac{d}{dt}Ad_{\alpha(t)}(\mathbf{Y}) \mid_{t=0} &= \mathbf{XY}e - e\mathbf{Y}e\mathbf{X}e, \\
\frac{d}{dt}Ad_{\alpha(t)}(\mathbf{Y}) \mid_{t=0} &= \mathbf{XY} - \mathbf{YX}, \\
\frac{d}{dt}Ad_{\alpha(t)}(\mathbf{Y}) \mid_{t=0} &= [\mathbf{X}, \mathbf{Y}].
\end{aligned}$$

The adjoint action of the Lie algebra on itself is given by commutator relation. Thus the natural product of tangent space at identity is given by Lie bracket. The representation of Lie algebra on itself is given by a Lie algebra. Now to get more understanding physically, we consider an example of group of rotations in three dimensions i.e. $SO(3)$.

The orthogonal group in three dimensions is comprised of transformations that leaves $x^2 + y^2 + z^2$ invariant. If we restrict ourselves to transformations with unit determinant, we obtain the group of proper rotations on three dimensions $SO(3)$.

Consider first rotation about x-axis by an angle ϕ,

$$R^x_\phi = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi & cos\phi \end{bmatrix}$$

its corresponding infinitesimal generator is

$$A^x = \mathbf{V}_1 = \frac{d}{d\phi}R^x_\phi \mid_{\phi=0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

For rotation about y-axis with an angle ϕ, we have

$$R^y_\phi = \begin{bmatrix} cos\phi & 0 & sin\phi \\ 0 & 1 & 0 \\ -sin\phi & 0 & cos\phi \end{bmatrix}$$

and its generator is

$$A^y = \mathbf{V}_2 = \frac{d}{d\phi} R^y_\phi \mid_{\phi=0} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$

Finally, for rotation about z-axis by an angle ϕ

$$R^z_\phi = \begin{bmatrix} cos\phi & -sin\phi & 0 \\ sin\phi & cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the corresponding generator is

$$A^z = \mathbf{V}_3 = \frac{d}{d\phi} R^z_\phi \mid_{\phi=0} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The adjoint action of R^x_ϕ on the generator A^y can be found by differentiating the product $R^x_\phi R^y_\varepsilon R^x_{-\phi}$ with respect to ε and setting $\varepsilon = 0$. We get

$$AdR^x_\phi A^{(y)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & -sin\phi \\ 0 & sin\phi & cos\phi \end{bmatrix} \begin{bmatrix} cos\varepsilon & 0 & sin\varepsilon \\ 0 & 1 & 0 \\ -sin\varepsilon & 0 & cos\varepsilon \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & cos\phi & sin\phi \\ 0 & -sin\phi & cos\phi \end{bmatrix}$$

$$AdR^x_\phi A^{(y)} = \begin{bmatrix} cos\varepsilon & -sin\phi cos\varepsilon & cos\varepsilon cos\phi \\ sin\phi cos\varepsilon & -sin^2\phi sin\varepsilon & sin\phi cos\phi sin\varepsilon \\ cos\phi cos\varepsilon & sin\phi cos\phi sin\varepsilon & -cos^2\phi sin\varepsilon \end{bmatrix}$$

Now differentiating with respect to ε and setting $\varepsilon = 0$, we obtain

$$AdR^x_\phi A^{(y)} = \begin{bmatrix} 0 & -sin\phi & cos\phi \\ sin\phi & 0 & 0 \\ cos\phi & 0 & 0 \end{bmatrix}$$

$$AdR^x_\phi A^{(y)} = cos\phi A^y + sin\phi A^z.$$

Similarly, we obtain

$$AdR^x_\phi A^{(x)} = A^x,$$

and

$$AdR_\phi^x A^{(z)} = -sin\phi A^y + cos\phi A^z.$$

Thus the adjoint action of the subgroup R_ϕ^x of rotations around the x-axis in physical space is the same as the group rotations around the A^x-axis in the Lie algebra space $SO(3)$. Finally the infinitesimal generators are found by differentiation

$$adA^x \mid_{A^y} = \frac{d}{d\phi} AdR_\phi^x A^{(y)} \mid_{\phi=0} = A^z$$

which can also be found by

$$R_\phi^x A^y R_{-\phi}^x = A^z,$$

which agrees with the commutator relation

$$[A^y, A^x] = A^x A^y - A^y A^x = A^z$$

If we know the infinitesimal adjoint action $'ad\boldsymbol{g}'$ of a Lie algebra, we can construct the $Ad\mathbf{G}$ of the underlying Lie group by integrationg the system of linear ordinary differential equations.

$$ad\mathbf{V}|_{\mathbf{W}} = \frac{d}{d\varepsilon} Ad(exp(\varepsilon\mathbf{V}))\mathbf{W}|_{\varepsilon=0}; \qquad \mathbf{W}\in\boldsymbol{g}.$$

A fundamental fact is that the infinitesimal adjoint action agrees with the Lie bracket on $\boldsymbol{g}$.

$$ad\mathbf{V}|_{\mathbf{W}} = \frac{d\mathbf{W}}{d\varepsilon}; \mathbf{W}(0) = \mathbf{W}_o,$$

with solution

$$\mathbf{W}(\varepsilon) = Ad(exp(\varepsilon\mathbf{V}))\mathbf{W}_o.$$

By using Lie series

$$Ad(exp(\varepsilon\mathbf{V}))\mathbf{W}_o = \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!}(ad\mathbf{V})^n(\mathbf{W}_o),$$

$$Ad(exp(\varepsilon\mathbf{V}))\mathbf{W}_o = \mathbf{W}_o - \varepsilon[\mathbf{V}, \mathbf{W}_o] + \frac{\varepsilon^2}{2!}[\mathbf{V}, [\mathbf{X}, \mathbf{W}_o] - \cdots.$$

Adjoint representation table are used to display the structure of conjugacy maps of given Lie algebra. Also it is convenient to display conjugacy relations of each sub algebra with every other sub algebra in the tabular form. For an $n-$dimensional Lie algebra $\boldsymbol{g}$, adjoint representation table will be the $n \times n$ table, whose $(i, j)-$th entry represent the adjoint action of $\mathbf{V}_i$ on $\mathbf{V}_j$ as $Ad(exp(\varepsilon\mathbf{V}_i))\mathbf{V}_j$. Adjoint action is defined by using Lie series in conjunction with commutator relation table as:

$$Ad(exp(\varepsilon \mathbf{V}_i))\mathbf{V}_j = \mathbf{V}_j - \varepsilon[\mathbf{V}_i, \mathbf{V}_j] + \frac{\varepsilon^2}{2!}[\mathbf{V}_i, [\mathbf{V}_i, \mathbf{V}_j]] - \cdots,$$

where $[\mathbf{V}_i, \mathbf{V}_j]$ is the Lie bracket for the generators $\mathbf{V}_i$ and $\mathbf{V}_j$. Using this definition of adjoint action one can easily construct adjoint representation table.

Consider Benjamin Bona Mahony (BBM) equation

$$u_t + u_x + uu_x - u_{xxt}.$$

Symmetry generators of BBM equation are as follow

$$\mathbf{V}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{V}_2 = \frac{\partial}{\partial x}, \qquad \mathbf{V}_3 = t\frac{\partial}{\partial t} - (u+1)\frac{\partial}{\partial u}.$$

By using Lie series definition of adjoint action one can easily calculate adjoint actions of each of its sub-algebras.

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_1, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_1]] - ....,$$

as $[\mathbf{V}_1, \mathbf{V}_1] = 0$, adjoint action of $\mathbf{V}_1$ on itself be

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_1, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_2]] - ....,$$

we have $[\mathbf{V}_1, \mathbf{V}_2] = 0$, hence

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_1, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_3]] - ....,$$

since $[\mathbf{V}_1, \mathbf{V}_3] = \mathbf{V}_1$ and

$$[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_3]] = [\mathbf{V}_1, \mathbf{V}_1] = 0.$$

Therefore, the adjoint action of $\mathbf{V}_1$ on $\mathbf{V}_3$ be

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_2, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_1]] - ....,$$

from commutator relation table of BBM equation, we have $[\mathbf{V}_2, \mathbf{V}_1] = 0$,
by using this we get

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_2, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_2]] - ....,$$

as $[\mathbf{V}_2, \mathbf{V}_2] = 0$. So, the adjoint action of $\mathbf{V}_2$ on itself be

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_2, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_3]] - ....,$$

from commutator relation table of BBM equation, we have $[\mathbf{V}_2, \mathbf{V}_3] = 0$. Therefore,

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_3 = \mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_3, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_1]] - ....,$$

we know that $[\mathbf{V}_3, \mathbf{V}_1] = -\mathbf{V}_1$,

$$[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_1]] = -[\mathbf{V}_3, \mathbf{V}_1] = (-1)^2\mathbf{V}_1.$$

Therefore,

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_1 = \mathbf{V}_1 e^{\epsilon}.$$

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_3, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_2]] - ....,$$

since $[\mathbf{V}_3, \mathbf{V}_2] = 0$, by using this we get

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_3, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_3]] - ....,$$

from commutator relation table of BBM equation, we have $[\mathbf{V}_3, \mathbf{V}_3] = 0$. So, the adjoint action of $\mathbf{V}_3$ on itself be

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_3 = \mathbf{V}_3.$$

From these results of adjoint maps one can construct adjoint representation table as:

Ad	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$
$\mathbf{V}_1$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3 + \epsilon\mathbf{V}_1$
$\mathbf{V}_2$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$
$\mathbf{V}_3$	$\mathbf{V}_1 e^{\epsilon}$	$\mathbf{V}_2$	$\mathbf{V}_3$

Adjoint representation table

Chapter 2

Optimal algebra of Korteweg de Vries type equation of order five

2.1 Introduction

The Korteweg de Vries (KdV) type equations are very important because of their physical applications. The generalized fifth order KdV type equations are given by

$$u_t + \alpha u u_x + \beta u_x u_{xx} + \gamma u u_{xxx} + u_{xxxxx} = 0,$$

where $u(t, x)$ is a function of temporal variable t and spatial variable x. The coefficients α, β and γ are arbitrary real constants. Usually, KdV type equations arise while studying shallow water waves. Particularly, KdV type equations describe the traveling of long water waves. It is formally proved that KdV type equations have solitary wave solutions. KdV type equations have wide range of applications in quantum mechanics and nonlinear optics. Because of such wide range of applications KdV type equations got a lot of attention and have been studied extensively. For different values of α, β and γ, they represent different fifth order KdV type evolutionary equations.

In this chapter, optimal system of fifth order KdV type equations which usually arise during modeling of many physical phenomena such as gravity-capillary waves on shallow layer and magneto-sound propagation in plasma are discussed. After that ordinary differential equations are obtained by using transformations obtained from optimal algebra of each equation.

In this chapter we review the formation of optimal algebra of KdV type equations of order five [16].

2.2 Simplified Kawahara equation

In this section optimal algebra of simplified Kawahara equation is discussed

$$u_t + \lambda u u_x + \mu u_{xxxxx} = 0, \tag{2.1}$$

where λ and μ are arbitrary constants. To find the optimal system of simplified Kawahara equation (2.1), one can follow the following procedure,

Step 1: Find vector fields.

Step 2: Use vector fields to form commutator relation table.

Step 3: Construct adjoint representation table by conjunction of adjoint map with commutator relation table.

Step 4: Use adjoint representation table to find the set of spanning sub algebras.

From sub algebras of optimal system one can find transformations which reduces simplified Kawahara equation (2.1) to the ordinary differential equation.

2.2.1 Lie symmetries and commutator relation table

One can easily get the Lie point symmetries of simplified Kawahara equation (2.1) by using Lie symmetry method [4, 9, 15, 19]. For equation (2.1) one parameter group of transformations are

$$\begin{aligned}
\tilde{t} &= t + \epsilon\xi^1(t,x,u) + O(\epsilon^2),\\
\tilde{x} &= x + \epsilon\xi^2(t,x,u) + O(\epsilon^2),\\
\tilde{u} &= u + \epsilon\eta(t,x,u) + O(\epsilon^2),
\end{aligned}$$

where ϵ is the parameter of group of transformations. For general case of two independent and one dependent variables the infinitesimal generator is

$$\mathbf{V} = \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u}. \tag{2.2}$$

It is needed to prolongate infinitesimal generator (2.2) upto fifth order as the simplified Kawahara equation involved first and fifth order derivatives w.r.t. x, while first order derivative w.r.t. t. Therefore, the form of infinitesimal generator becomes

$$\begin{aligned}
\mathbf{V} &= \xi^1(t,x,u)\frac{\partial}{\partial t} + \xi^2(t,x,u)\frac{\partial}{\partial x} + \eta(t,x,u)\frac{\partial}{\partial u} + \\
&\quad \eta_t(t,x,u)\frac{\partial}{\partial u_{,t}} + \eta_x(t,x,u)\frac{\partial}{\partial u_{,x}} + \eta_{xxxxx}(t,x,u)\frac{\partial}{\partial u_{,xxxxx}}.
\end{aligned} \tag{2.3}$$

Apply infinitesimal generator (2.3) to simplified Kawahra equation (2.1). It yields the system of over determined linear partial differential equations in ξ^1, ξ^2 and η. By solving those we get symmetry generators of simplified Kawahara equation (2.1) as:

$$\mathbf{V}_1 = \frac{\partial}{\partial t}, \quad \mathbf{V}_2 = \frac{\partial}{\partial x}, \quad \mathbf{V}_3 = \lambda t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}, \quad \mathbf{V}_4 = x\frac{\partial}{\partial x} + 5t\frac{\partial}{\partial t} - 4u\frac{\partial}{\partial u},$$

which are closed under Lie bracket operation. One can write their commutator relation table as:

,	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$
$\mathbf{V_1}$	0	0	$\lambda\mathbf{V}_2$	$5\mathbf{V}_1$
$\mathbf{V_2}$	0	0	0	$\mathbf{V}_2$
$\mathbf{V_3}$	$-\lambda\mathbf{V}_3$	0	0	$-4\mathbf{V}_3$
$\mathbf{V_4}$	$-5\mathbf{V}_1$	$-\mathbf{V}_2$	$-\mathbf{V}_3$	0

Table 1 (a)

2.2.2 Construction of adjoint representation table

Construct adjoint representation table, by using Lie series in conjunction with commutator relation Table 1(a).

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_1, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_1]] - ....,$$

as $[\mathbf{V}_1, \mathbf{V}_1] = 0$, it gives

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_1, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_2]] - ....,$$

since $[\mathbf{V}_1, \mathbf{V}_2] = 0$, it yields

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_1, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_3]] - ....,$$

from commutator relation Table 1(a), $[\mathbf{V}_1, \mathbf{V}_3] = \lambda\mathbf{V}_2$, and

$$[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_3]] = \lambda[\mathbf{V}_1, \mathbf{V}_2] = 0,$$

by using these, it gives

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon\lambda\mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_1, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_4]] - ....,$$

as $[\mathbf{V}_1, \mathbf{V}_4] = 5\mathbf{V}_5$, and

$$[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_4]] = 5[\mathbf{V}_1, \mathbf{V}_1] = 0,$$

then adjoint map of $\mathbf{V}_1$ on $\mathbf{V}_4$ is

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_4 = \mathbf{V}_4 - 5\epsilon \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_2, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_1]] - ....,$$

from Table 1(a), $[\mathbf{V}_2, \mathbf{V}_1] = 0$, it yields

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_2, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_2]] - ....,$$

since $[\mathbf{V}_2, \mathbf{V}_2] = 0$, it implies

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_2, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_3]] - ....,$$

from commutator relation Table 1(a), $[\mathbf{V}_2, \mathbf{V}_3] = 0$, using this, it yields

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_3 = \mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_2, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_4, \mathbf{V}_4]] - ....,$$

as $[\mathbf{V}_2, \mathbf{V}_4] = \mathbf{V}_2$, and

$$[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_4]] = [\mathbf{V}_2, \mathbf{V}_2] = 0,$$

therefore

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_3, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_1]] - ....,$$

since $[\mathbf{V}_3, \mathbf{V}_1] = -\lambda \mathbf{V}_2$, and

$$[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_1]] = -\lambda[\mathbf{V}_3, \mathbf{V}_2] = 0,$$

so, adjoint action of $\mathbf{V}_3$ on $\mathbf{V}_1$ is

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_1 = \mathbf{V}_1 + \epsilon\lambda \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_3, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_2]] - ....,$$

from Table $3(a)$, $[\mathbf{V}_3, \mathbf{V}_2] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_3, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_3]] - ....,$$

as $[\mathbf{V}_3, \mathbf{V}_3] = 0$, it gives

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_3 = \mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_3, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_4]] - ....,$$

from commutator relation Table $1(a)$, $[\mathbf{V}_3, \mathbf{V}_4] = -4\mathbf{V}_3$, and

$$[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_4]] = -4[\mathbf{V}_3, \mathbf{V}_3] = 0,$$

using these one can find adjoint action of $\mathbf{V}_3$ on $\mathbf{V}_4$ as

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_4 = \mathbf{V}_4 + \epsilon \mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_4, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_1]] - ....,$$

since $[\mathbf{V}_4, \mathbf{V}_1] = -5\mathbf{V}_1$, and

$$[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_1]] = -5[\mathbf{V}_4, \mathbf{V}_1] = (-5)^2\mathbf{V}_1,$$

hence

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_1 = e^{5\varepsilon}\mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_4, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_2]] - ....,$$

as $[\mathbf{V}_4, \mathbf{V}_2] = -\mathbf{V}_2$, then the adjoint action of $\mathbf{V}_4$ on $\mathbf{V}_2$ is

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_2 = e^{\varepsilon}\mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_3 = \mathbf{V}_4 - \epsilon[\mathbf{V}_4, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_3]] - ....,$$

from Table $3(a)$, $[\mathbf{V}_4, \mathbf{V}_3] = 4\mathbf{V}_3$, therefore

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_3 = e^{4\varepsilon}\mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_3, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_4]] - ....,$$

as $[\mathbf{V}_4, \mathbf{V}_4] = 0$, hence

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_4 = \mathbf{V}_4.$$

Using above calculated results of adjoint maps to write the adjoint representation Table $2(a)$.

Ad	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$
$\mathbf{V}_1$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3 - \epsilon\lambda\mathbf{V}_2$	$\mathbf{V}_4 - 5\varepsilon\mathbf{V}_1$
$\mathbf{V}_2$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4 - \varepsilon\mathbf{V}_2$
$\mathbf{V}_3$	$\mathbf{V}_1 + \epsilon\lambda\mathbf{V}_2$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4 + 4\varepsilon\mathbf{V}_3$
$\mathbf{V}_4$	$e^{5\varepsilon}\mathbf{V}_1$	$e^{\varepsilon}\mathbf{V}_2$	$e^{-\varepsilon}\mathbf{V}_3$	$\mathbf{V}_4$

Table 2 (a)

2.2.3 Formation of optimal system

Using adjoint representation Table 2(a) to find the optimal system of simplified Kawahara equation (2.3). Consider a general non-zero vector

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4. \tag{2.4}$$

One-dimensional sub algebras are spanned by vector of the form (2.4). Suppose first that $a_4 \neq 0$ (for sake of convenience assume $a_4 = 1$), then the general non zero vector (2.4) becomes

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + \mathbf{V}_4.$$

From adjoint representation Table 2(a), it can be seen that, if one act on $\mathbf{V}$ by $Ad(exp(\frac{-a_3}{4})\mathbf{V}_3)$, then coefficient of $\mathbf{V}_3$ vanishes. For distinction one can name the resultant vector as $\mathbf{V}'$

$$\mathbf{V}' = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + \mathbf{V}_4.$$

Also by acting on $\mathbf{V}'$ by $Ad(exp(a_2)\mathbf{V}_2)$, then coefficient of $\mathbf{V}_2$ vanishes, which is represented by $\mathbf{V}''$

$$\mathbf{V}'' = a_1\mathbf{V}_1 + \mathbf{V}_4.$$

Similarly, by acting on $\mathbf{V}''$ by $Ad(exp(\frac{a_1}{5})\mathbf{V}_1)$, coefficient of $\mathbf{V}_1$ vanish. Here one can see that the coefficients of all those symmetry generators which are written in linear combination with $\mathbf{V}_4$ are vanished. That is, the vector $\mathbf{V}$ is equivalent to $\mathbf{V}_4$ under adjoint representation. In other words, every one dimensional sub algebra spanned by the vector $\mathbf{V}$ with $a_4 \neq 0$ is equivalent to sub algebra spanned by $\mathbf{V}_4$.

The remaining sub algebras are obtained by vector (2.4) with $a_4 = 0$. If $a_3 \neq 0$, then for convenience one can scale it to make $a_3 = 1$. Then the general non zero vector (2.4) get the form

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + \mathbf{V}_3.$$

By observing adjoint representation Table 2(a), one can see that the symmetry generator $\mathbf{V}_3$ written only with $\mathbf{V}_2$. So, by acting on $\mathbf{V}$ with $Ad(exp(\frac{a_2}{\lambda})\mathbf{V}_2)$, then one can make coefficient of $\mathbf{V}_2$ vanish. Name the resulting vector as $\mathbf{V}'$

$$\mathbf{V}' = a_1\mathbf{V}_1 + \mathbf{V}_3.$$

Now it can be seen that coefficients of all those symmetry generators which are written in linear combination with $\mathbf{V}_3$ are vanished. Therefore, symmetry generators which are included in optimal system of (2.1) are:

(i) $a\mathbf{V}_1 + \mathbf{V}_3$,

(ii) $\mathbf{V}_3$.

Proceeding the same method for symmetry generators $\mathbf{V}_1$ and $\mathbf{V}_2$. One can find the optimal system of simplified Kawahara equation (2.1) as

$$\mathbf{V}_1, \qquad \mathbf{V}_2 + a\mathbf{V}_1 \qquad \mathbf{V}_3, \qquad \mathbf{V}_4, \qquad a\mathbf{V}_1 + \mathbf{V}_3,$$

with a be any constant. Which are also same as the classification of real three and four dimensional Lie algebras done by J. Patera and P. Winternitz in [20].

2.2.4 Reduction

In this section reduction of simplified Kawahara equation into ordinary differential equations is discussed.

(i) For the symmetry generator $\mathbf{V}_1 = \dfrac{\partial}{\partial t}$, which can be written as

$$\frac{dt}{1} = \frac{dx}{0} = \frac{adu}{0},$$

using method of characteristics, one can calculate

$$\begin{aligned} x &= \xi, \\ u &= f(\xi). \end{aligned}$$

Substituting these to simplified Kawahara equation (2.1), they reduces it to the following ordinary differential equation

$$\mu f^5 + \lambda f f' = 0.$$

(ii) For the symmetry generator $\mathbf{V}_2 = \dfrac{\partial}{\partial x}$ one can write it as

$$\frac{dt}{0} = \frac{dx}{1} = \frac{adu}{0},$$

from here trivial solution $u(t,x) = c$ obtain, where c is any constant.

(iii) For the symmetry generator $\mathbf{V}_3 = \lambda t\frac{\partial}{\partial t} + \frac{\partial}{\partial u}$,
by using method of characteristics one can find that

$$\begin{aligned} t &= \xi, \\ u &= \frac{1}{\lambda}xt^{-1} + f(\xi). \end{aligned}$$

Substitute t and u in simplified Kawahara equation (2.1), then it becomes an ordinary differential equation

$$\xi f f' + f = 0.$$

(iv) For the symmetry generator $\mathbf{V}_4 = \lambda t\frac{\partial}{\partial x} + \frac{\partial}{\partial t} - 4u\frac{\partial}{\partial u}$,
one can find

$$\begin{aligned} xt^{-5} &= \xi, \\ u &= t^{\frac{-4}{5}} f(\xi). \end{aligned}$$

By using u and ξ in simplified Kawahara equation (2.1), then one get reduced ordinary differential equation

$$\mu f^5 + \lambda f f' - \frac{\xi f'}{5} + \frac{4f}{5} = 0.$$

(v) For the combination of symmetry generators $a\mathbf{V}_1 + \mathbf{V}_3 = a\frac{\partial}{\partial t} + \lambda t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}$,
from method of characteristics one can calculate

$$\begin{aligned} x - \frac{\lambda t^2}{a} &= \xi, \\ u &= \frac{t}{a} + f(\xi). \end{aligned}$$

By using the values of u and ξ in simplified Kawahara equation (2.1), then it will be reduced to the following ordinary differential equation,

$$\mu f^5 + \lambda f f' + \frac{1}{a} = 0.$$

2.3 General Kawahara equation

In this section optimal algebra of another type of fifth order KdV type equation

$$u_t + \alpha u u_x + \beta u_{xxx} + \gamma u_{xxxxx} = 0, \tag{2.5}$$

known as general Kawahara equation is discussed. Here α, β and γ are arbitrary constants.

2.3.1 Lie symmetries and commutator relation table

Following the method that adopted in finding symmetry generators of simplified Kawahara equation (2.1). One can get the symmetry generators of general Kawahara equation (2.5) as:

$$\mathbf{V}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{V}_2 = \frac{\partial}{\partial x}, \qquad \mathbf{V}_3 = \alpha t\frac{\partial}{\partial x} + \frac{\partial}{\partial u},$$

which are closed under the Lie bracket operation. The Lie algebra of these symmetry generators is

$$[\mathbf{V}_1, \mathbf{V}_1] = 0, \qquad [\mathbf{V}_1, \mathbf{V}_2] = 0, \qquad [\mathbf{V}_1, \mathbf{V}_3] = \alpha\mathbf{V}_2,$$

$$[\mathbf{V}_2, \mathbf{V}_1] = 0, \qquad [\mathbf{V}_2, \mathbf{V}_2] = 0, \qquad [\mathbf{V}_2, \mathbf{V}_3] = 0,$$

$$[\mathbf{V}_3, \mathbf{V}_1] = -\alpha\mathbf{V}_2, \qquad [\mathbf{V}_3, \mathbf{V}_2] = 0, \qquad [\mathbf{V}_3, \mathbf{V}_1] = 0.$$

One can write commutator relation table as:

,	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$
$\mathbf{V}_1$	0	0	$\alpha\mathbf{V}_2$
$\mathbf{V}_2$	0	0	0
$\mathbf{V}_3$	-$\alpha\mathbf{V}_2$	0	0

Table 3 (a)

2.3.2 Construction of adjoint representation table

To compute adjoint representation, one can use Lie series in conjunction with commutator relation Table 1(c). The adjoint action is given by the Lie series as

$$Ad(exp(\varepsilon\mathbf{V}_i))\mathbf{V}_j = \mathbf{V}_j - \varepsilon[\mathbf{V}_i, \mathbf{V}_j] + \frac{\varepsilon^2}{2!}[\mathbf{V}_i, [\mathbf{V}_i, \mathbf{V}_j]] - \cdots,$$

where $[\mathbf{V}_i, \mathbf{V}_j]$ is the Lie bracket for the generators $\mathbf{V}_i$ and $\mathbf{V}_j$. Using this definition of adjoint action.

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_1, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_1]] - ....,$$

as $[\mathbf{V}_1, \mathbf{V}_1] = 0$,
therefore

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_1 = \mathbf{V}_1.$$

In this manner, one can construct the table with (i, j)-th the entry representing $Ad(exp(\varepsilon\mathbf{V}_i))\mathbf{V}_j$.

Ad	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$
$\mathbf{V}_1$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3+\epsilon\alpha\mathbf{V}_2$
$\mathbf{V}_2$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$
$\mathbf{V}_3$	$\mathbf{V}_1+\epsilon\alpha\mathbf{V}_2$	$\mathbf{V}_2$	$\mathbf{V}_3$

Table 4 (a)

2.3.3 Formation of optimal algebra

From the adjoint representation Table 4(a), one can form the optimal algebra of general Kawahara equation (2.5). For this consider a general non-zero vector

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3. \tag{2.6}$$

First consider $a_3 \neq 0$ and then for sake of convenience consider $a_3 = 1$, general non zero vector (2.6) get the form

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + \mathbf{V}_3,$$

$$\mathbf{V} = a_1\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial x} + \alpha t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}.$$

Referring adjoint representation Table 4(a). It can be seen that by acting on $\mathbf{V}$ by $Ad(exp(\frac{1}{\alpha}a_2\mathbf{V}_1))$, then the coefficient of $\mathbf{V}_2$ vanish. The resultant vector is represented by $\mathbf{V}'$

$$\mathbf{V}' = Ad(exp(\frac{1}{\alpha}a_2\mathbf{V}_1))\mathbf{V} = \mathbf{V} - \frac{a_2}{\alpha}[\mathbf{V}_1,\mathbf{V}] + (\frac{a_2}{\alpha})^2\frac{1}{2!}[\mathbf{V}_1,[\mathbf{V}_1,-\mathbf{V}],$$

as

$$[\mathbf{V}_1,\mathbf{V}] = \Big(\frac{\partial}{\partial t}, a_1\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial x} + \alpha t\frac{\partial}{\partial x} + \frac{\partial}{\partial u}\Big),$$

$[\mathbf{V}_1,\mathbf{V}] = \alpha\frac{\partial}{\partial x}$,
and
$[\mathbf{V}_1,[\mathbf{V}_1,\mathbf{V}]] = 0$,
therefore

$$\begin{aligned}
\mathbf{V}' &= Ad(exp(\frac{1}{\alpha}a_2\mathbf{V}_1))\mathbf{V} = \mathbf{V} - \frac{a_2}{\alpha}(\alpha\frac{\partial}{\partial x}) \\
\mathbf{V}' &= a_1\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial x} + \alpha t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} - \frac{a_2}{\alpha}\alpha\frac{\partial}{\partial x}, \\
\mathbf{V}' &= a_1\frac{\partial}{\partial t} + \alpha t\frac{\partial}{\partial x} + \frac{\partial}{\partial u} \\
\mathbf{V}' &= a_1\mathbf{V}_1 + \mathbf{V}_3.
\end{aligned}$$

From Table 4(a), it can be seen that coefficients of all those symmetry generators which are written in linear combination with $\mathbf{V}_3$. So, from here the sub algebras which are included in optional algebra of general Kawahara equation (2.5) are

(i) $\mathbf{V}_3$,

(ii) $\mathbf{V}_3 + a\mathbf{V}_1$,
where a be any constant except zero.

The remaining one-dimensional sub algebras are spanned by vector of above form (2.6) with $a_3 = 0$. If $a_2 \neq 0$, then scale to make $a_2 = 1$ for convenience, then the general vector (2.6) become

$$\mathbf{V} = a_1\mathbf{V}_1 + \mathbf{V}_2.$$

From adjoint representation Table 2(a), it is clear that there is no linear combination of $\mathbf{V}_2$ with any other symmetry generator in second column. But in first column of adjoint representation Table 2(a) there is a combination of $\mathbf{V}_2$ and $\mathbf{V}_1$. Therefore, by acting on $\mathbf{V}$ by $Ad(exp(\alpha a_1)\mathbf{V}_1)$, then coefficient of $\mathbf{V}_1$ vanishes. Call the resulting vector as $\mathbf{V}'$

$$\mathbf{V}' = \mathbf{V}_2.$$

Therefore, from here the symmetry generators included in optimal algebra of general Kawahara equation (2.5) is $\mathbf{V}_2$. The remaining one dimensional sub algebras are spanned by vector of above form (2.6) with $a_3 = a_2 = 0$, $a_1 \neq 0$ and for convenience scale to make $a_1 = 1$, then the general vector $\mathbf{V}$ get the form

$$\mathbf{V} = \mathbf{V}_1.$$

As there is no linear combination of $\mathbf{V}_1$ with any other symmetry generator. Therefore, the only symmetry generator which is included in optimal algebra from here is $\mathbf{V}_1$. The set of optimal algebra (optimal system) of general Kawahara equation are

$$\mathbf{V}_1, \quad \mathbf{V}_2, \quad \mathbf{V}_3, \quad a\mathbf{V}_1 + \mathbf{V}_3,$$

where a is an arbitrary constant. Which are also same as the classification of real three and four dimensional Lie algebras done by J. Patera and P. Winternitz in [20].

2.3.4 Reduction

Here symmetry generators of optimal system of general Kawahara equation (2.5) are use to reduce it to ordinary differential equation.

(i) For the generator $\mathbf{V}_1 = \frac{\partial}{\partial t}$,
one can write it as

$$\frac{dx}{0} = \frac{dt}{1} = \frac{du}{0},$$

by method of characteristics, it yields

$$\begin{aligned} \xi &= x, \\ u &= f(\xi). \end{aligned}$$

Substituting these in general Kawahara equation (2.5), they reduces it to the following ordinary differential equation

$$\gamma f^{(5)} + \beta f''' + \alpha f f' = 0,$$

where $f' = \frac{df}{d\xi}$.

(ii) For the generator, $\mathbf{V}_2 = \frac{\partial}{\partial x}$,

trivial solution $u(t, x) = c$ exists, where c is an arbitrary constant.

(iii) For the generator, $\mathbf{V}_3 = \alpha t \frac{\partial}{\partial x} + \frac{\partial}{\partial u}$,

using method of characteristics one can obtain

$$\begin{aligned} t &= \xi, \\ u &= \frac{1}{\alpha} t^{-1} x + f(\xi). \end{aligned}$$

By substitution of t and u in general Kawahara (2.5), which results the reduction of (2.5) to the following ordinary differential equation

$$\xi f' + f = 0.$$

(iv) For the linear combination, $a\mathbf{V}_1 + \mathbf{V}_3 = a\dfrac{\partial}{\partial t} + \alpha t \dfrac{\partial}{\partial x} + \dfrac{\partial}{\partial u}$,
by the use of methods of characteristics one can find

$$\begin{aligned} x - \frac{\alpha t^2}{2} &= \xi, \\ u &= \frac{t}{a} + f(\xi). \end{aligned}$$

Using these in general Kawahara (2.5), then (2.5) reduce to the following ordinary differential equation,

$$\mu f^5 + \lambda f f' + \frac{1}{a} = 0.$$

2.4 General modified Kawahara equation

In this section optima system of one more type of fifth order KdV equation

$$u_t + \alpha u^2 u_x + \beta u_{xxx} + \gamma u_{xxxxx} = 0, \tag{2.7}$$

known as general modified Kawahara equation is dicussed. With α, β, and γ are arbitrary constants. To get optimal algebra first vector fields of general modified Kawahara equation (2.7) are to find. Then use these to construct commutator relation table and adjoint representation table. These tables use in finding optimal algebra of general modified Kawahara equation (2.7). From optimal system one can get transformations which reduces it to the ordinary differential equation.

2.4.1 Lie Symmetries and Commutator Relation Table

Following the method that adopted in finding symmetry generators of general Kawahara equation (2.1), one can find the symmetry generators of general modified Kawahara equation (2.7) as:

$$\mathbf{V}_1 = \frac{\partial}{\partial t}, \qquad \mathbf{V}_2 = \frac{\partial}{\partial x},$$

their commutator relation table be

,	$\mathbf{V}_1$	$\mathbf{V}_2$
$\mathbf{V}_1$	0	0
$\mathbf{V}_2$	0	0

Table 5 (a)

Construction of adjoint representation table

To compute adjoint representation, one can use Lie series in conjunction with commutator relation Table $1(c)$. The adjoint action is given by the Lie series as

$$Ad(exp(\varepsilon \mathbf{V}_i))\mathbf{V}_j = \mathbf{V}_j - \varepsilon[\mathbf{V}_i, \mathbf{V}_j] + \frac{\varepsilon^2}{2!}[\mathbf{V}_i, [\mathbf{V}_i, \mathbf{V}_j]] - \cdots,$$

where $[\mathbf{V}_i, \mathbf{V}_j]$ is the Lie bracket for the generators $\mathbf{V}_i$ and $\mathbf{V}_j$. Using this definition of adjoint action.

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_1, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_1]] - ....,$$

as $[\mathbf{V}_1, \mathbf{V}_1] = 0$,
therefore

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_1 = \mathbf{V}_1.$$

In this manner, one can construct the Table 6(a) with (i, j)-th the entry representing $Ad(exp(\varepsilon\mathbf{V}_i))\mathbf{V}_j$.

Ad	$\mathbf{V}_1$	$\mathbf{V}_2$
$\mathbf{V}_1$	$\mathbf{V}_1$	$\mathbf{V}_2$
$\mathbf{V}_2$	$\mathbf{V}_1$	$\mathbf{V}_2$

Table 6 (a)

2.4.2 Formation of optimal system

Use adjoint representation Table 6(a), to find the optimal system of general modified Kawahara equation (2.7). Consider a general non-zero vector

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2. \tag{2.8}$$

First assume that $a_2 \neq 0$ (for sake of convenience consider that $a_2 = 1$). Then the general non zero vector (2.6) get the form

$$\mathbf{V} = a_1\mathbf{V}_1 + \mathbf{V}_2.$$

Observing adjoint representation Table 6(a), one can see that it is not possible to vanish symmetry generator $\mathbf{V}_1$ from above vector $\mathbf{V}$. Because in adjoint representation Table 6(a) there is no linear combination of $\mathbf{V}_1$ and $\mathbf{V}_2$ is written. Therefore, from here possible symmetry generators included in optimal algebra of (2.7) are

(i) $a\mathbf{V}_1 + \mathbf{V}_2$,

(ii) $\mathbf{V}_1$.

2.4.3 Reduction

(i) For the combination of symmetry generators, $a\mathbf{V}_1 + \mathbf{V}_2 = a\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial x}$, one can write it as

$$\frac{dt}{a} = \frac{dx}{1} = \frac{adu}{0},$$

using method of characteristics one can find

$$\begin{aligned}\xi &= t - ax,\\ u &= \frac{t}{a} + f(\xi).\end{aligned}$$

Substituting these to general modified Kawahara equation (2.7), then they reduces it to the following ordinary differential equation

$$a^5\gamma f^{(5)} + a^3\beta f''' + a\alpha f^2 f' - f' = 0.$$

(ii) For symmetry generator, $\mathbf{V}_1 = \dfrac{\partial}{\partial t}$,
from here trivial solution $u(t,x) = c$ obtained, where c is an arbitrary constant.

Chapter 3

Optimal system of Monge-Ampere equation

3.1 Introduction

In this chapter we find the optimal system of the semi-linear non-homogeneous Monge-Ampere equation

$$u_{xx}u_{yy} - u_{xy}^2 + a(x, y) = 0. \tag{3.1}$$

Here $a(x, y)$ is a non-homogeneous part and $u(x, y)$ is the dependent variable with x, y as independent variables. The name "Monge-Ampere equation"have been derived from its early formulation in two different directions. One by the French mathematician and civil engineer Gaspard Monge $(1746 - 1818)$ while the second by the French physicist Andre Marie Ampere $(1775 - 1836)$. Gaspard Monge is the inventor of descriptive geometry. In 1781, Gaspard Monge originally formulated and analyzed the problem of optimal transportation, initiating a profound mathematical theory, which connects the different areas of differential geometry, nonlinear partial differential equations, linear programming and probability theory. It was later studied by Minkowski $(1864 - 1909)$, Schauder $(1899 - 1943)$, Lewy $(1904 - 1988)$, Bernstein $(1918 - 1990)$ and mnay others. During the last century the development of Monge-Ampere equation was closely related to geometric problems. It also arise in meteorology and fluid mechanics. In fluid mechanics it is coupled with transport equation, like semi-geostrophic equation. Due to these abundance of applications and beautiful theory, equations of Monge Ampere type are important and got lot of attention and studied extensively [18].

In this chapter we find an optimal system of the semi-linear non-homogeneous Monge-Ampere equation. On optimal system of nonlinear partial differential equations a lot of excellent work has been done by experts [19, 15, 12, 8, 11, 7]. Many techniques have been developed for obtaining optimal systems. Here we use Peter. J. Olver's technique [19] to derive optimal system for different cases of Monge-Ampere equation by assuming different particular values of the non-homogeneous part $a(x,y)$. Then we use transformations from sub algebras of an optimal system to reduce the semi-linear non-homogeneous Monge-Ampere equation (3.1) to an ordinary differential equation.

3.1.1 Lie symmetries and commutator relation table

In equation (3.1) we have two independent variables x and y while u the dependent variable. For one parameter ε the one parameter group of transformations for equation (3.1) are

$$\tilde{x} = x + \varepsilon\xi^1(x,y,u) + o(\varepsilon^2),$$
$$\tilde{y} = y + \varepsilon\xi^2(x,y,u) + o(\varepsilon^2),$$
$$\tilde{u} = u + \varepsilon\eta(x,y,u) + o(\varepsilon^2).$$

Equation (3.1) verifies the above set of transformations. For general case of two independent variables and one dependent variable the symmetry generator is

$$\mathbf{V} = \xi^1(x,y,u)\frac{\partial}{\partial x} + \xi^2(x,y,u)\frac{\partial}{\partial y} + \eta(x,y,u)\frac{\partial}{\partial u}. \tag{3.2}$$

Second order prolonged generator for non-homogeneous semi-linear Monge-Ampere equation (3.1) is

$$\mathbf{V} = \xi_1\frac{\partial}{\partial x} + \xi_2\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \eta_x\frac{\partial}{\partial u_{,x}} + \eta_y\frac{\partial}{\partial u_{,y}} + \eta_{xx}\frac{\partial}{\partial u_{,xx}} + \eta_{xy}\frac{\partial}{\partial u_{,xy}} + \eta_{yy}\frac{\partial}{\partial u_{,yy}}. \tag{3.3}$$

By applying generator (3.3) to the equation (3.1), we obtain a system of over determined linear partial differential equations in ξ^1, ξ^2 and η. Solving these to get symmetry generators of equation (3.1).

3.2 Case I: $a(x,y) = e^x$

First consider particular value for non-homogeneous part of non-homogeneous Monge-Ampere equation (3.1) to be e^x. For this case symmetry generators are: $\mathbf{V}_1 = \frac{\partial}{\partial u}$, $\mathbf{V}_2 = \frac{\partial}{\partial y}$, $\mathbf{V}_3 = x\frac{\partial}{\partial y}$, $\mathbf{V}_4 = x\frac{\partial}{\partial u}$, $\mathbf{V}_5 = y\frac{\partial}{\partial u}$, $\mathbf{V}_6 = \frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$, $\mathbf{V}_7 = y\frac{\partial}{\partial y} + u\frac{\partial}{\partial u}$.

One can write its commutator relation table as:

,	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$	$\mathbf{V_5}$	$\mathbf{V_6}$	$\mathbf{V_7}$
$\mathbf{V_1}$	0	0	0	0	0	$\mathbf{V}_1$	$\mathbf{V}_7$
$\mathbf{V_2}$	0	0	0	0	$\mathbf{V}_1$	0	$\mathbf{V}_2$
$\mathbf{V_3}$	0	0	0	0	$\mathbf{V}_4$	$-\mathbf{V}_2$	$\mathbf{V}_3$
$\mathbf{V_4}$	0	0	0	0	0	$\mathbf{V}_4 - \mathbf{V}_1$	$\mathbf{V}_4$
$\mathbf{V_5}$	0	0	$-\mathbf{V}_4$	0	0	$\mathbf{V}_5$	0
$\mathbf{V_6}$	$-\mathbf{V}_4$	0	$\mathbf{V}_2$	$\mathbf{V}_1 - \mathbf{V}_4$	$-\mathbf{V}_5$	0	0
$\mathbf{V_7}$	$-\mathbf{V}_1$	$-\mathbf{V}_2$	$-\mathbf{V}_3$	$-\mathbf{V}_4$	0	0	0

Table 1 (b)

3.2.1 Construction of adjoint representation table

To compute adjoint representation, we use the Lie series in conjunction with commutator relation Table 1(b). The adjoint action is given by the Lie series as

$$Ad(exp(\varepsilon \mathbf{V}_i))\mathbf{V}_j = \mathbf{V}_j - \varepsilon[\mathbf{V}_i, \mathbf{V}_j] + \frac{\varepsilon^2}{2!}[\mathbf{V}_i, [\mathbf{V}_i, \mathbf{V}_j]] - \cdots,$$

where $[\mathbf{V}_i, \mathbf{V}_j]$ is the Lie bracket for the generators $\mathbf{V}_i$ and $\mathbf{V}_j$. Using this definition of adjoint action.

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_1, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_1]] - ....,$$

as $[\mathbf{V}_1, \mathbf{V}_1] = 0$, adjoint action of $\mathbf{V}_1$ on itself be

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_1, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_2]] - ....,$$

we have $[\mathbf{V}_1, \mathbf{V}_2] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_1, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_3]] - ....,$$

since $[\mathbf{V}_1, \mathbf{V}_3] = 0$, using this we get

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_3 = \mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_1, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_4]] - ....,$$

as $[\mathbf{V}_1, \mathbf{V}_4] = 0$, hence

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_4 = \mathbf{V}_4.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_1, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_5]] - ....,$$

since $[\mathbf{V}_1, \mathbf{V}_5] = 0$, adjoint action of $\mathbf{V}_1$ on $\mathbf{V}_4$ be

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_4 = \mathbf{V}_5.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_1, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_6]] - ....,$$

we have $[\mathbf{V}_1, \mathbf{V}_6] = \mathbf{V}_1$, using this we obtain

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_6 = \mathbf{V}_6 - \varepsilon \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_1, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_7]] - ....,$$

from commutator Table $1(b)$, $[\mathbf{V}_1, \mathbf{V}_7] = \mathbf{V}_1$, hence

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_7 = \mathbf{V}_7 - \varepsilon \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_2, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_1]] - ....,$$

we have $[\mathbf{V}_2, \mathbf{V}_1] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_2, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_2]] - ....,$$

we know that $[\mathbf{V}_2, \mathbf{V}_2] = 0$, using this we get

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_2, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_3]] - ....,$$

as $[\mathbf{V}_2, \mathbf{V}_3] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_3 = \mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_2, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_4, \mathbf{V}_4]] - ....,$$

we know that $[\mathbf{V}_2, \mathbf{V}_4] = 0$, it yields

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_4 = \mathbf{V}_4.$$

$$Ad(exp(\epsilon\mathbf{V}_2))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_2, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_5]] - ....,$$

from commutator relation Table 1(b), $[\mathbf{V}_2, \mathbf{V}_5] = \mathbf{V}_1$, hence

$$Ad(exp(\epsilon\mathbf{V}_2))\mathbf{V}_5 = \mathbf{V}_5 - \varepsilon\mathbf{V}_1.$$

$$Ad(exp(\epsilon\mathbf{V}_2))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_2, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_6]] - ....,$$

as $[\mathbf{V}_2, \mathbf{V}_6] = 0$, adjoint action of $\mathbf{V}_2$ on $\mathbf{V}_6$ be

$$Ad(exp(\epsilon\mathbf{V}_2))\mathbf{V}_6 = \mathbf{V}_6.$$

$$Ad(exp(\epsilon\mathbf{V}_2))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_2, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_7]] - ....,$$

we have $[\mathbf{V}_2, \mathbf{V}_7] = \mathbf{V}_7$, therefore

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_7 = \mathbf{V}_7 - \varepsilon\mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_3, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_1]] - ....,$$

since $[\mathbf{V}_3, \mathbf{V}_1] = 0$, using this we get

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_3, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_2]] - ....,$$

as $[\mathbf{V}_3, \mathbf{V}_2] = 0$, hence

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_3, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_3]] - ....,$$

we have $[\mathbf{V}_3, \mathbf{V}_3] = 0$, using this, it yields

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_3 = \mathbf{V}_3.$$

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_3, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_4]] - ....,$$

we know that $[\mathbf{V}_3, \mathbf{V}_4] = 0$, therefore

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_3 = \mathbf{V}_4.$$

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_3, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_5]] - ....,$$

from commutator relation Table 1(b), $[\mathbf{V}_3, \mathbf{V}_5] = \mathbf{V}_4$, using this we get

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_5 = \mathbf{V}_5 - \varepsilon \mathbf{V}_4.$$

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_3, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_6]] - ....,$$

as $[\mathbf{V}_3, \mathbf{V}_6] = -\mathbf{V}_2$, hence

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_6 = \mathbf{V}_6 + \varepsilon \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_3, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_7]] - ....,$$

we have $[\mathbf{V}_3, \mathbf{V}_7] = \mathbf{V}_3$, adjoint action of $\mathbf{V}_3$ on $\mathbf{V}_7$ be

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_7 = \mathbf{V}_7 - \varepsilon \mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_4, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_1]] - ....,$$

since $[\mathbf{V}_4, \mathbf{V}_1] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_4, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_2]] - ....,$$

we have $[\mathbf{V}_4, \mathbf{V}_2] = 0$, by using this we obtain

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_4, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_3]] - ....,$$

we know that $[\mathbf{V}_4, \mathbf{V}_3] = 0$, it yields

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_3 = \mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_4, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_4]] - ....,$$

as $[\mathbf{V}_4, \mathbf{V}_4] = 0$, hence

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_4 = \mathbf{V}_4.$$

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_4, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_5]] - ....,$$

since $[\mathbf{V}_4, \mathbf{V}_5] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_4))\mathbf{V}_5 = \mathbf{V}_5.$$

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_6=\mathbf{V}_6-\epsilon[\mathbf{V}_4,\mathbf{V}_6]+\frac{\epsilon^2}{2!}[\mathbf{V}_4,[\mathbf{V}_4,\mathbf{V}_6]]-....,$$

we know that $[\mathbf{V}_4,\mathbf{V}_6]=\mathbf{V}_4-\mathbf{V}_1$, adjoint action of $\mathbf{V}_4$ on $\mathbf{V}_6$ be

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_6=\mathbf{V}_6+\varepsilon\mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_7=\mathbf{V}_7-\epsilon[\mathbf{V}_4,\mathbf{V}_7]+\frac{\epsilon^2}{2!}[\mathbf{V}_4,[\mathbf{V}_4,\mathbf{V}_7]]-....,$$

since $[\mathbf{V}_4,\mathbf{V}_7]=\mathbf{V}_4$, adjoint action of $\mathbf{V}_4$ on $\mathbf{V}_7$ be

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_7=\mathbf{V}_7-\varepsilon\mathbf{V}_4.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_1=\mathbf{V}_1-\epsilon[\mathbf{V}_5,\mathbf{V}_1]+\frac{\epsilon^2}{2!}[\mathbf{V}_5,[\mathbf{V}_5,\mathbf{V}_1]]-....,$$

we have $[\mathbf{V}_5,\mathbf{V}_1]=0$, using this we obtain

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_1=\mathbf{V}_1.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_2=\mathbf{V}_2-\epsilon[\mathbf{V}_5,\mathbf{V}_2]+\frac{\epsilon^2}{2!}[\mathbf{V}_5,[\mathbf{V}_5,\mathbf{V}_2]]-....,$$

as $[\mathbf{V}_5,\mathbf{V}_2]=0$, therefore

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_2=\mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_3=\mathbf{V}_3-\epsilon[\mathbf{V}_5,\mathbf{V}_3]+\frac{\epsilon^2}{2!}[\mathbf{V}_5,[\mathbf{V}_5,\mathbf{V}_3]]-....,$$

from commutator relation Table 1(b), $[\mathbf{V}_5,\mathbf{V}_3]=-\mathbf{V}_4$, it yields

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_3=\mathbf{V}_3+\varepsilon\mathbf{V}_4.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_4=\mathbf{V}_4-\epsilon[\mathbf{V}_5,\mathbf{V}_4]+\frac{\epsilon^2}{2!}[\mathbf{V}_5,[\mathbf{V}_5,\mathbf{V}_4]]-....,$$

we have $[\mathbf{V}_5,\mathbf{V}_4]=0$, hence

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_4=\mathbf{V}_4.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_5=\mathbf{V}_5-\epsilon[\mathbf{V}_5,\mathbf{V}_5]+\frac{\epsilon^2}{2!}[\mathbf{V}_5,[\mathbf{V}_5,\mathbf{V}_5]]-....,$$

we know that, $[\mathbf{V}_5,\mathbf{V}_5]=0$, adjoint action of $\mathbf{V}_5$ on itself be

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_5=\mathbf{V}_5.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_6=\mathbf{V}_6-\epsilon[\mathbf{V}_5,\mathbf{V}_6]+\frac{\epsilon^2}{2!}[\mathbf{V}_5,[\mathbf{V}_5,\mathbf{V}_6]]-....,$$

from commutator relation Table 1(b), $[\mathbf{V}_5, \mathbf{V}_6] = \mathbf{V}_5$, using this we get

$$Ad(exp(\epsilon \mathbf{V}_5))\mathbf{V}_6 = \mathbf{V}_6 - \varepsilon \mathbf{V}_5.$$

$$Ad(exp(\epsilon \mathbf{V}_5))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_5, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_5, [\mathbf{V}_5, \mathbf{V}_7]] - ....,$$

as $[\mathbf{V}_5, \mathbf{V}_7] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_5))\mathbf{V}_7 = \mathbf{V}_7.$$

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_6, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_1]] - ....,$$

we have $[\mathbf{V}_6, \mathbf{V}_1] = -\mathbf{V}_1$, adjoint action of $\mathbf{V}_6$ on $\mathbf{V}_1$ be

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_1 = e^{\varepsilon} \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_6, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_2]] - ....,$$

as $[\mathbf{V}_6, \mathbf{V}_2] = 0$, using this we get

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_6, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_3]] - ....,$$

since $[\mathbf{V}_6, \mathbf{V}_3] = \mathbf{V}_2$, therefore

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_3 = \mathbf{V}_3 - \varepsilon \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_6, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_4]] - ....,$$

we have $[\mathbf{V}_6, \mathbf{V}_4] = \mathbf{V}_1 - \mathbf{V}_4$, hence

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_4 = e^{\varepsilon}(\mathbf{V}_4 - \varepsilon \mathbf{V}_1).$$

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_6, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_5]] - ....,$$

we know that $[\mathbf{V}_6, \mathbf{V}_5] = -\mathbf{V}_5$, adjoint action of $\mathbf{V}_6$ on $\mathbf{V}_5$ be

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_5 = e^{\varepsilon} \mathbf{V}_5.$$

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_6, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_6]] - ....,$$

as $[\mathbf{V}_6, \mathbf{V}_6] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_6 = \mathbf{V}_6.$$

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_6, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_7]] - ....,$$

we know that $[\mathbf{V}_6, \mathbf{V}_7] = 0$, hence

$$Ad(exp(\epsilon \mathbf{V}_6))\mathbf{V}_7 = \mathbf{V}_7.$$

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_7, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_1]] - ....,$$

we have $[\mathbf{V}_7, \mathbf{V}_1] = -\mathbf{V}_1$, using this, it yields

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_1 = e^{\varepsilon}\mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_7, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_2]] - ....,$$

as $[\mathbf{V}_7, \mathbf{V}_2] = -\mathbf{V}_2$, therefore

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_2 = e^{\varepsilon}\mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_7, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_3]] - ....,$$

since $[\mathbf{V}_7, \mathbf{V}_3] = -\mathbf{V}_3$, hence

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_3 = e^{\varepsilon}\mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_7, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_4]] - ....,$$

we have $[\mathbf{V}_7, \mathbf{V}_4] = -\mathbf{V}_4$, using this we obtain

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_4 = e^{\varepsilon}\mathbf{V}_4.$$

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_7, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_5]] - ....,$$

we know that $[\mathbf{V}_7, \mathbf{V}_5] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_5 = \mathbf{V}_5.$$

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_7, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_6]] - ....,$$

from commutator relation Table 1(b), $[\mathbf{V}_7, \mathbf{V}_6] = 0$, adjoint action of $\mathbf{V}_7$ on $\mathbf{V}_6$ be

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_6 = \mathbf{V}_7.$$

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_7, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_7]] - ....,$$

since $[\mathbf{V}_7, \mathbf{V}_7] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_7))\mathbf{V}_7 = \mathbf{V}_7.$$

Now use these results of adjoint actions to construct an adjoint representation table as:

Ad	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$	$\mathbf{V_5}$	$\mathbf{V_6}$	$\mathbf{V_7}$
$\mathbf{V_1}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6 - \varepsilon\mathbf{V}_1$	$\mathbf{V}_7 - \varepsilon\mathbf{V}_1$
$\mathbf{V_2}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5 - \varepsilon\mathbf{V}_1$	$\mathbf{V}_6$	$\mathbf{V}_7 - \varepsilon\mathbf{V}_2$
$\mathbf{V_3}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5 - \varepsilon\mathbf{V}_4$	$\mathbf{V}_6 + \varepsilon\mathbf{V}_2$	$\mathbf{V}_7 - \varepsilon\mathbf{V}_3$
$\mathbf{V_4}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6 - \varepsilon(\mathbf{V}_4 - \mathbf{V}_1)$	$\mathbf{V}_7 - \varepsilon\mathbf{V}_4$
$\mathbf{V_5}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$-\mathbf{V}_4$	$\mathbf{V}_3 + \varepsilon\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6 - \varepsilon\mathbf{V}_5$	$\mathbf{V}_7$
$\mathbf{V_6}$	$e^{\varepsilon}\mathbf{V}_1$	$e^{-\varepsilon}\mathbf{V}_2$	$\mathbf{V}_3 - \varepsilon\mathbf{V}_2$	$e^{\varepsilon}(\mathbf{V}_4 - \varepsilon\mathbf{V}_1)$	$e^{\varepsilon}\mathbf{V}_5$	$\mathbf{V}_6$	$\mathbf{V}_7$
$\mathbf{V_7}$	$e^{\varepsilon}\mathbf{V}_1$	$e^{\varepsilon}\mathbf{V}_2$	$e^{\varepsilon}\mathbf{V}_3$	$e^{\varepsilon}\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6$	$\mathbf{V}_7$

Table 2 (b)

3.2.2 Formation of optimal system

Optimal system defined earlier in (1.5), it constitutes the set of conjugacy classes of group of transformations. As we know that adjoint action gives the conjugacy classes of group of transformations which are written in columns of adjoint representation Table 2(b). Our aim is to find the set of one dimensional sub algebras, that cover all conjugacy classes. Following Olver's technique [7, 8, 11, 16, 19] we assume a general vector $\mathbf{V}$ as the combination of all symmetry generators. Then by observing columns of adjoint representation Table 2(b), we try to vanish coefficients of as much symmetry generators as possible by using appropriate adjoint action on general vector $\mathbf{V}$. In this case there are seven symmetry generators. So, non zero general vector is

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4 + a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + a_7\mathbf{V}_7. \tag{3.4}$$

Use judicious application of adjoint map to make as many as constants $a's$ vanish. Assume that $a_7 \neq 0$ and also for convenience $a_7 = 1$. Then the general non zero vector (3.4) become

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4 + a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + \mathbf{V}_7.$$

Referring adjoint representation Table 2(b), if we act on $\mathbf{V}$ by $Ad(exp(a_4\mathbf{V}_4))$, then coefficient of $\mathbf{V}_4$ vanish. Name the resulting vector as $\mathbf{V}'$

$$\mathbf{V}' = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + \mathbf{V}_7.$$

Similarly, if we act on $\mathbf{V}'$ by adjoint map $Ad(exp(a_3\mathbf{V}_3))$, then the coefficient of $\mathbf{V}_3$ vanish. Call the resulting vector as $\mathbf{V}''$

$$\mathbf{V}'' = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + \mathbf{V}_7.$$

Working on same lines we find that, if we act on $\mathbf{V}''$ by $Ad(exp(a_2\mathbf{V}_2))$, then the coefficient of $\mathbf{V}_2$ vanish from the general vector $\mathbf{V}''$, which is represented in $\mathbf{V}'''$

$$\mathbf{V}''' = a_1\mathbf{V}_1 + a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + \mathbf{V}_7.$$

Also if we act on $\mathbf{V}'''$ by $Ad(exp(a_1\mathbf{V}_1))$, then coefficient of $\mathbf{V}_1$ vanish and we got the vector free from the coefficients a_4, a_3, a_2 and a_1, that is

$$\mathbf{V}^{iv} = a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + \mathbf{V}_7.$$

Referring adjoint representation Table 2(b), coefficient of symmetry generator $\mathbf{V}_5$ can be vanish from above vector if we act on above general non-zero vector by $Ad(exp(a_5)\mathbf{V}_5)$. After that, we cannot vanish any other coefficient of symmetry generators. Therefore, from here we have two symmetry generators which are include in optimal algebra of non homogeneous Monge-Ampere equation.

(i) $a_6\mathbf{V}_6 + \mathbf{V}_7$.

(ii) $\mathbf{V}_7$.

Remaining sub algebras are spanned by the vector (3.4), when $a_7 = 0$. If $a_6 \neq 0$ then for convenience also assume that $a_6 = 1$, then general nonzero vector get the form

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4 + a_5\mathbf{V}_5 + \mathbf{V}_6.$$

Referring adjoint representation Table 2(b), if we act on $\mathbf{V}$ by $Ad(exp(a_5\mathbf{V}_5))$, then coefficient of $\mathbf{V}_5$ cancel from above vector and we call it $\mathbf{V}'$

$$\mathbf{V}' = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4 + \mathbf{V}_6.$$

In the same column of adjoint representation Table 2(b), there is a relation of $\mathbf{V}_6$ with $\mathbf{V}_4$ and $\mathbf{V}_1$. Here choice is our that to whom vanish first. Now we act on $\mathbf{V}'$ by $Ad(exp(a_4\mathbf{V}_4))$, so, that coefficient of $\mathbf{V}_4$ vanish and we call the resulting vector as $\mathbf{V}''$

$$\mathbf{V}'' = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + \mathbf{V}_6.$$

Continuing in the same way, we observe from adjoint representation Table 2(b), that by acting on $\mathbf{V}''$ by $Ad(exp(a_2\mathbf{V}_2))$ the coefficient of $\mathbf{V}_2$ vanish and we call thee resultant vector as $\mathbf{V}'''$

$$\mathbf{V}''' = a_1\mathbf{V}_1 + a_3\mathbf{V}_3 + \mathbf{V}_6.$$

To vanish the coefficient of $\mathbf{V}_1$, we act on $\mathbf{V}'''$ by $Ad(exp(a_1\mathbf{V}_1))$. Name the resulting vector as $\mathbf{V}^{iv}$

$$\mathbf{V}^{iv} = a_3\mathbf{V}_3 + \mathbf{V}_6.$$

Here we are succeeded in vanishing all the coefficients $a's$, whose symmetry generators are written in linear combination of $\mathbf{V}_6$. Therefore, from here the symmetry generators which are included in optimal algebra of non homogeneous Monge-Ampere equation (3.1) with e^x as non homogeneous part are

(i) $a_3\mathbf{V}_3 + \mathbf{V}_6$,

(ii) $\mathbf{V}_6$.

Remaining sub algebras are spanned by the vector (3.4), if $a7 = a_6 = 0, a_{5\neq 0}$. For sake of convenience we further assume that $a_5 = 1$, then the vector (3.4) become

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4 + \mathbf{V}_5.$$

From adjoint representation Table 2(b) we come to know that, if we act on $\mathbf{V}$ by $Ad(exp(a_4\mathbf{V}_4))$, then the coefficient of $\mathbf{V}_4$ vanish and call the new resulting vector as $\mathbf{V}'$

$$\mathbf{V}' = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + \mathbf{V}_5.$$

Similarly, if we act on $\mathbf{V}'$ by $Ad(exp(a_1\mathbf{V}_1))$, then coefficient of $\mathbf{V}_1$ vanish and name the resulting vector as $\mathbf{V}''$

$$\mathbf{V}'' = +a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + +\mathbf{V}_5.$$

Now in whole adjoint representation table there is not any symmetry generator which is written in linear combination with $\mathbf{V}_5$. From here, the set of symmetry generators included in optimal algebra of non homogeneous Monge-Ampere equation (3.1) with non homogeneous part e^x are

(i) $a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + \mathbf{V}_5$,

(ii) $\mathbf{V}_5$.

For remaining sub algebras of optimal system we assume that $a_7 = a_6 = a_5 =$

$0, a_4 \neq 0$ and further for sake of convenience assume that $a_4 = 1$. Then the general non zero vector (3.4) become

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + \mathbf{V}_4.$$

Referring adjoint representation Table $2(b)$, if we act on above vector $\mathbf{V}$ by $Ad(exp(-a_3\mathbf{V}_3))$, then coefficient of $\mathbf{V}_3$ vanish and we call the resulting vector as $\mathbf{V}'$

$$\mathbf{V}' = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + \mathbf{V}_4.$$

Also the coefficient of $\mathbf{V}_1$ cancel, if we act on $\mathbf{V}'$ by $Ad(exp(a_1\mathbf{V}_1))$ and name the resulting generator to $\mathbf{V}''$

$$\mathbf{V}'' = a_2\mathbf{V}_2 + \mathbf{V}_4.$$

Now there is no any other linear combination of $\mathbf{V}_4$ exist whose coefficient has not been vanish till yet. Therefore, from here the symmetry generators included in optimal algebra of non homogeneous Monge-Ampere equation (3.1) are

(i) $a_2\mathbf{V}_2 + \mathbf{V}_4$,

(ii) $\mathbf{V}_4$.

Remaining sub algebras are spanned by vector (3.4), if $a_7 = a_6 = a_5 = a_4 = 0$. If $a_3 \neq 0$ then for sake of convenience assume that $a_3 = 1$. Then the general non zero vector (3.4) become

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + \mathbf{V}_3.$$

From adjoint representation Table $2(b)$, we come to know that, if we act on above vector $\mathbf{V}$ by $Ad(exp(a_2\mathbf{V}_2))$. Then coefficient of $\mathbf{V}_2$ vanish and we call the resulting vector as $\mathbf{V}'$

$$\mathbf{V}' = a_1\mathbf{V}_1 + \mathbf{V}_3.$$

Now there is no any other linear combination of $\mathbf{V}_3$ exist, whose coefficient has not been vanish till yet. Therefore, from here the symmetry generators included in optimal algebra of non homogeneous Monge-Ampere equation (3.1) are

(i) $a_1\mathbf{V}_1 + \mathbf{V}_3$,

(ii) $\mathbf{V}_3$. Remaining sub algebras are spanned by the vector (3.4), if $a_7 = a_6 = a_5 = a_4 = a_3 = 0, a_{2 \neq 0}$ and further for sake of convenience assume that $a_2 = 1$. Then the general non zero vector (3.4) become

$$\mathbf{V} = a_1\mathbf{V}_1 + \mathbf{V}_2.$$

From adjoint representation Table 2(b), we find that $\mathbf{V}_2$ is not written with any other symmetry generator. So, from here the only symmetry generator will belong to optimal system is

(i) $a_1\mathbf{V}_1 + \mathbf{V}_2$.

Similarly, by assuming $a_7 = a_6 = a_5 = a_4 = a_3 = 0, a_2 = 0$, for general non zero vector (3.11). We get symmetry generator $\mathbf{V}_1$, which is included in optimal algebra of non homogeneous Monge-Ampere equation (3.1).

Therefore, the optimal system of one dimensional sub algebras of Non homogeneous Monge-Ampere equation(3.1) with e^x as the non homogeneous part is
$\mathbf{V}_7+a_6\mathbf{V}_6$, $\mathbf{V}_5+a_3\mathbf{V}_3+a_2\mathbf{V}_2$, $\mathbf{V}_4+a_2\mathbf{V}_2$, $\mathbf{V}_6+a_3\mathbf{V}_3$, $\mathbf{V}_3+a_1\mathbf{V}_1$, $\mathbf{V}_2+a_1\mathbf{V}_1$, $\mathbf{V}_7$, $\mathbf{V}_6$, $\mathbf{V}_5$, $\mathbf{V}_4$, $\mathbf{V}_3$, $\mathbf{V}_1$.

3.2.3 Reduction

As non homogeneous Monge-Ampere equation (3.1) is semi linear partial differential equation. So, by using any transformation and reducing it to ordinary differential equation is not an easy task. Here we are going to show reduction of those optimal sub algebras because of which semi-linear non-homogeneous Monge-Ampere equation either reduces to an ordinary differential equation or gives solution. Remaining sub algebras yields either the trivial solution or reduces the order of semi-linear non-homogeneous Monge-Ampere equation.

(i) For symmetry generator $\mathbf{V}_7$, $\mathbf{V}_7 = y\dfrac{\partial}{\partial y} + u\dfrac{\partial}{\partial u}$,
it can be written as

$$\frac{dx}{0} = \frac{dy}{y} = \frac{du}{u},$$

from here we have

$$x = \xi, y = e^{\xi}, u = U(\xi)e^{\xi}.$$

Substituting these in non homogeneous Monge-Ampere equation (3.1) with non homogeneous part as e^x, it reduces to the following ordinary differential equation

$$U'^2 - e^{\xi} = 0, \tag{3.5}$$

where $U' = \frac{dU}{d\xi}$.
While

$$u(x, y) = \pm\sqrt{-2C_1x - 8e^{(x+y)} + 2C_2} \tag{3.6}$$

be the solution of semi-linear non-homogeneous Monge-Ampere equation (3.1) with e^x the non-homogeneous part.

(ii) For combination of $\mathbf{V}_3$ and $\mathbf{V}_6$, one can write their combination as

$$\mathbf{V} = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} + u\frac{\partial}{\partial u},$$

it yields

$x = \xi$, $y = \dfrac{\eta^2}{2} + \xi$, $u = Ue^{\xi}$.

Using these we obtain an ordinary differential equation as

$$U''(U - U')e^{\xi} - U'^2 e^{\xi} = 1, \tag{3.7}$$

where $U' = \dfrac{dU}{d\xi}$.

(iii) For combination of $\mathbf{V}_6$ and $\mathbf{V}_3$, one can write their combination as

$$\mathbf{V} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{3u}{2}\frac{\partial}{\partial u},$$

it can also be written as

$$\frac{dx}{1} = \frac{dy}{y} = \frac{2du}{3u},$$

it yields

$x = \xi$, $y = \xi e^{\eta}$, $u = Ue^{\frac{3\eta}{2}}$. Using these we get an ordinary differential equation as

$$4\xi U'U'' + U'^2 - 9UU'' + 4 = 0, \tag{3.8}$$

where $U' = \dfrac{dU}{d\xi}$.

(iv) For combination of $\mathbf{V}_5$, $\mathbf{V}_6$ and $\mathbf{V}_7$, one can write their combination as

$$\mathbf{V} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + (y + \frac{3u}{2})\frac{\partial}{\partial u}.$$

It can also be written as

$$\frac{dx}{1} = \frac{dy}{y} = \frac{2du}{2y + 3u}.$$

From here we get solution of semi-linear non-homogeneous Monge-Ampere equation (3.1) as

$$u(x, y) = \frac{2y(e^{\frac{-3x}{2}}) - 1}{1 - e^{\frac{-3x}{2}}}. \tag{3.9}$$

(v) For symmetry generator $\mathbf{V}_6 = \frac{\partial}{\partial x} + u\frac{\partial}{\partial u}$, one can also write it as

$$\frac{dx}{1} = \frac{dy}{0} = \frac{du}{u}.$$

It yields
$x = \xi$, $y = \eta$, $u = Ue^{\xi}$. Using these we obtain

$$u(x, y) = \frac{\sqrt{C_1 e^x}[(e^{\frac{-\sqrt{C_1 e^x}(y+C_2)}{e^x}})^2 + e^x]}{2(e^{\frac{-\sqrt{C_1 e^x}(y+C_2)}{e^x}})} \tag{3.10}$$

the solution of semi-liner non-homogeneous Monge-Ampere equation (3.1).

3.3 Case II: $a(x, y) = e^x\phi(y)$

Now consider another case of family of non homogeneous Monge-Ampere equation (3.1) with particular value of non homogeneous part as $e^x\phi(y)$.

3.3.1 Lie symmetries and commutator relation table

Adopting the method that we adopted for finding symmetry generators for the first case of non homogeneous Monge-Ampere equation. We get symmetry generators of equation (3.1) with $e^x\phi(y)$ as non homogeneous part, as: $\mathbf{V}_1 = \frac{\partial}{\partial u}$,

$\mathbf{V}_2 = x\frac{\partial}{\partial u}$, $\mathbf{V}_3 = y\frac{\partial}{\partial u}$, $\mathbf{V}_4 = \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u}$,

which are closed under Lie bracket operation. Symmetry generator $\mathbf{V}_1 = \frac{\partial}{\partial u}$ representing translation, $\mathbf{V}_2 = x\frac{\partial}{\partial u}$, $\mathbf{V}_3 = y\frac{\partial}{\partial u}$ representing Galilean transformation while $\mathbf{V}_4 = \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u}$ representing translation in x direction and scaling in u direction.

One can write their commutator relation table as:

3.3.2 Construction of adjoint representation table

To construct the Adjoint representation table we use results of commutator relation Table 3(b). The adjoint action is given by the Lie series as,
$Ad(exp(\varepsilon\mathbf{V}_i))\mathbf{V}_j = \mathbf{V}_j - \varepsilon[\mathbf{V}_i, \mathbf{V}_j] + \frac{\varepsilon^2}{2!}[\mathbf{V}_i, [\mathbf{V}_i, \mathbf{V}_j]] - \cdots$,

,	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$
$\mathbf{V_1}$	0	0	0	$\frac{1}{2}\mathbf{V}_1$
$\mathbf{V_2}$	0	0	0	$\frac{1}{2}\mathbf{V}_2 - \mathbf{V}_1$
$\mathbf{V_3}$	0	0	0	$\frac{1}{2}\mathbf{V}_3$
$\mathbf{V_4}$	$-\frac{1}{2}\mathbf{V_1}$	$\mathbf{V}_1 - \frac{1}{2}\mathbf{V}_2$	$-\frac{1}{2}\mathbf{V}_3$	0

Table 3 (b)

where $[\mathbf{V}_i, \mathbf{V}_j]$ is the Lie bracket for the generators $\mathbf{V}_i$ and $\mathbf{V}_j$. Using this definition of adjoint action.

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_1, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_4]] - ....,$$

from commutator relation Table $1(b)$, $[\mathbf{V}_1, \mathbf{V}_4] = \frac{1}{2}\mathbf{V}_1$,
and

$$[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_4]] = \frac{1}{2}[\mathbf{V}_1, \mathbf{V}_1] = 0,$$

adjojnt action of $\mathbf{V}_1$ on $\mathbf{V}_4$ be

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_4 = \mathbf{V}_4 - \frac{\varepsilon}{2}\mathbf{V}_1.$$

In this manner we the construct adjoint representation table.

Ad	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$
$\mathbf{V_1}$	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4} - \frac{\varepsilon}{\mathbf{2}}\mathbf{V_1}$
$\mathbf{V_2}$	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4} - \frac{\varepsilon}{2}\mathbf{V_2} + \varepsilon\mathbf{V_1}$
$\mathbf{V_3}$	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4} - \frac{\varepsilon}{2}\mathbf{V_3}$
$\mathbf{V_4}$	$\mathbf{V_1}\mathbf{e}^{\frac{\varepsilon}{\mathbf{2}}}$	$\mathbf{V_2}\mathbf{e}^{\frac{\varepsilon}{\mathbf{2}}} - \varepsilon\mathbf{V_1}\mathbf{e}^{\frac{\varepsilon}{\mathbf{2}}}$	$\mathbf{V_3}\mathbf{e}^{\frac{\varepsilon}{\mathbf{2}}}$	$\mathbf{V_4}$

Table 4 (b)

3.3.3 Formation of optimal system

Following Olver's technique [7, 8, 11, 16, 19] we assume a general vector $\mathbf{V}$ as the combination of all symmetry generators. Then by observing columns of adjoint representation Table 4(b), we try to vanish coefficients of as much symmetry generators as possible by using appropriate adjoint action on general vector $\mathbf{V}$. For this now consider a general non zero vector

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4. \tag{3.11}$$

Suppose first that $a_4 \neq 0$, further assume for convenience that $a_4 = 1$, we have

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + \mathbf{V}_4,$$

$$\mathbf{V} = a_1\frac{\partial}{\partial u} + a_2 x\frac{\partial}{\partial u} + a_3 y\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial}u,$$

observing the adjoint representation Table 4(b), if we act on $\mathbf{V}$ by $Ad(exp(2a_3\mathbf{V}_3))$, then coefficient of $\mathbf{V}_3$ vanishes, we call the resultant vector as $\mathbf{V}'$

$$\mathbf{V}' = Ad(exp(2a_3\mathbf{V}_3))\mathbf{V} = \mathbf{V} - 2a_3[\mathbf{V}_3, \mathbf{V}] + \frac{(2a_3)^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}]] - \cdots,$$

$$[\mathbf{V}_3, \mathbf{V}] = [y\frac{\partial}{\partial u}, a_1\frac{\partial}{\partial u} + a_2 x\frac{\partial}{\partial u} + a_3 y\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial}u],$$

$$[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}]] = 0,$$

therefore

$$\mathbf{V}' = Ad(exp(2a_3\mathbf{V}_3))\mathbf{V} = a_1\frac{\partial}{\partial u} + a_2 x\frac{\partial}{\partial u} + a_3 y\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u} - 2a_3\frac{y}{2}\frac{\partial}{\partial u},$$

here a_3 vanishes. We have

$$\mathbf{V}' = a_1\frac{\partial}{\partial u} + a_2 x\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u},$$

that is

$$\mathbf{V}' = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + \mathbf{V}_4.$$

Again referring adjoint representation Table 4(b), if we act on $\mathbf{V}'$ by $Ad(exp(2a_2\mathbf{V}_2))$, then coefficient of $\mathbf{V}_2$ vanish. Represented it in $\mathbf{V}''$

$$\mathbf{V}'' = Ad(exp(2a_2\mathbf{V}_2))\mathbf{V}' = \mathbf{V}' - 2a_2[\mathbf{V}_2, \mathbf{V}'] + \frac{(2a_2)^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}']] - \cdots,$$

$$[\mathbf{V}_2, \mathbf{V}'] = [x\frac{\partial}{\partial u}, a_1\frac{\partial}{\partial u} + a_2 x\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u}],$$

$$[\mathbf{V}_2, \mathbf{V}'] = \frac{x}{2}\frac{\partial}{\partial u} - \frac{\partial}{\partial u},$$

and

$$[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}']] = 0,$$

therefore

$$\mathbf{V}'' = Ad(exp(2a_2\mathbf{V}_2))\mathbf{V}' = a_1\frac{\partial}{\partial u} + a_2 x\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u} - 2a_2(\frac{x}{2}\frac{\partial}{\partial u} - \frac{\partial}{\partial u}),$$

$$\mathbf{V}'' = (a_1 + 2a_2)\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u},$$

taking $\quad {a_1}' = a_1 + 2a_2,$

we get, $\quad \mathbf{V}'' = {a_1}'\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u},$

$$\mathbf{V}'' = {a_1}'\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u},$$

that is,

$$\mathbf{V}'' = {a_1}'\mathbf{V}_1 + \mathbf{V}_4.$$

Continuing in the same way, if we act on $\mathbf{V}''$ by $Ad(exp(2{a_1}'\mathbf{V}_1))$, then coefficient of $\mathbf{V}_1$ vanish, we call the resultant vector as $\mathbf{V}'''$

$$\mathbf{V}''' = Ad(exp(2{a_1}'\mathbf{V}_1))\mathbf{V}'' = \mathbf{V}'' - 2{a_1}'[\mathbf{V}_1, \mathbf{V}''] + \frac{(2{a_1}')^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}'']] - \cdots,$$

$$[\mathbf{V}_1, \mathbf{V}''] = [\frac{\partial}{\partial u}, {a_1}'\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u}],$$

$$[\mathbf{V}_1, \mathbf{V}''] = \frac{1}{2}\frac{\partial}{\partial u},$$

and

$$[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}'']] = 0,$$

therefore

$$\mathbf{V}''' = Ad(exp(2{a_1}'\mathbf{V}_1))\mathbf{V}'' = {a_1}'\frac{\partial}{\partial u} + \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u} - 2{a_1}'(\frac{1}{2}\frac{\partial}{\partial u}),$$

$$\mathbf{V}''' = \frac{\partial}{\partial x} + \frac{u}{2}\frac{\partial}{\partial u},$$

that is

$$\mathbf{V}''' = \mathbf{V}_4.$$

Here we are succeeded in vanishing all the coefficients from general vector $\mathbf{V}$ and finally we have a relation $\mathbf{V}''' = \mathbf{V}_4$. Therefore $\mathbf{V}_4$ is included in optimal system of one dimensional sub algebras of non homogeneous non linear Monge-Ampere equation with non homogeneous part $e^x\phi(y)$.

The remaining one dimensional sub algebras are spanned by vector of above form with $a_4 = 0$. If $a_3 \neq 0$ and for convenience we assume that $a_3 = 1$. Then the general vector $\mathbf{V}$ given by equation (3.5) get the form

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + \mathbf{V}_3.$$

In adjoint representation Table $4(b)$, the symmetry generator $\mathbf{V}_3$ is not written in linear combination of any other symmetry generators. So, it is impossible for us to vanish any coefficient $a's$ from above general vector. Therefore the vector

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + \mathbf{V}_3,$$

is included in optimal system of one dimensional sub algebras of non homogeneous non linear Monge-Ampere equation (3.1) with non homogeneous part $e^x\phi(y)$.
Beside these sub algebras are spanned by above vector (3.5) of the form $a_4 = a_3 = 0$, and $a_2 \neq 0$. For convenience we scale it to make $a_2 = 1$. Then the general vector $\mathbf{V}$ (3.5) get the form

$$\mathbf{V} = a_1\mathbf{V}_1 + \mathbf{V}_2.$$

From adjoint representation Table $4(b)$, we come to know that the symmetry generator $\mathbf{V}_2$ is not written in the linear combination of any other symmetry generators. So, it is impossible for us to vanish coefficient of symmetry generator $\mathbf{V}_1$. Therefore, the symmetry generator

$$\mathbf{V} = a_1\mathbf{V}_1 + \mathbf{V}_2,$$

is included in optimal system. Also here $e^{\frac{\varepsilon}{2}}$ term is involved so, for simplicity we are taking $a_1 = +1, -1, 0$. Then we have, $\mathbf{V}_1 + \mathbf{V}_2$, $\mathbf{V}_2 - \mathbf{V}_1$, $\mathbf{V}_2$. Also from first column we get $\mathbf{V}_1$ as the one dimensional sub algebra which is included in optimal system.

Therefore, the optimal system of one dimensional sub algebras of (3.1) be

$\mathbf{V}_4,\quad a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + \mathbf{V}_3,\quad a_1\mathbf{V}_1 + \mathbf{V}_2\quad \mathbf{V}_2 - a_1\mathbf{V}_1,\quad \mathbf{V}_2,\quad \mathbf{V}_1.$ Which are also same as the classification of real three and four dimensional Lie algebras done by J. Patera and P. Winternitz in [20].

3.3.4 Reduction

We have non homogeneous Monge-Ampere equation which is semi linear partial differential equation, for this we found set of optimal algebras. As equation (3.1) is semi linear partial differential equation so by using any transformation and reducing it to ordinary differential equation is not an easy task. But here if we try to reduce it with optimal algebra $\mathbf{V}_4$ we succeed to get its reduced ordinary differential equation.

$$\mathbf{V}_4 = \frac{\partial}{\partial x} + \frac{1}{2}u\frac{\partial}{\partial u},$$

it can be written as

$$\frac{dx}{1} = \frac{dy}{0} = \frac{2du}{u},$$

we have

$z = y$, $w = ue^{-\frac{x}{2}}$. Using these in (3.1), we get the ordinary differential equation

$$\frac{ww''}{4} + \frac{w'^2}{4} + \phi(z) = 0. \tag{3.12}$$

Its solution be,

$$w^2 = -8\int\int \phi(z)dz + 4\ C_1\int dz + C_2, \tag{3.13}$$

for convenience considering particular value of $\phi(z) = z$, also using the substitutions $z = y$, $w = ue^{-\frac{x}{2}}$, we get solution

$$u^2 = \frac{4}{3}e^x y^3 + C\ e^x,$$

which also satisfies the semi-linear non-homogeneous Monge-Ampere equation (3.1) with $e^x\phi(y)$.

By using any other optimal algebra we did not get any transformation, so, that we can reduce equation (3.1) to ordinary differential equation. But we are able to just reduce the order of semi linear non homogeneous Monge-Ampere equation or we get its trivial solution.

3.4 Conclusion

In this chapter we found solutions of semi-linear non-homogeneous Monge-Ampere equation (3.1) from its optimal systems. We consider two particular cases by considering e^x and $e^x\phi(y)$ as non-homogeneous parts in Case I and Case II respectively. All solutions satisfies original partial differential equation (semi-linear non-homogeneous Monge-Ampere equation (3.1)) with respective conditions. Since we know that equation (3.1) is semi-linear. It also involves three basic symmetries (symmetries depending on homogeneous part only) as $\frac{\partial}{\partial u}$, $x\frac{\partial}{\partial u}$, and $y\frac{\partial}{\partial u}$. Symmetries of this form basically defines translation and Galelian translation. These type of symmetry generators does not follow translation geometrically. That is why reduction by using such symmetries is sometimes very difficult. Because of this we are unable to find solutions from all sub algebras of optimal system. Here, because of optimal algebra we are able to find such sub algebra which reduces equation (3.1) to an ordinary differential equation with its respective conditions. Therefore, either by using any sub algebra of equation (3.1) or linear combination of its sub algebras will only give these solutions. Otherwise they reduce the order of equation or give trivial solution. We can further extends this work to find all solutions of those differential equations whose symmetry generators are of the type as given here.

Chapter 4

One dimensional symmetry reduction of $(2+1)$ dimensional nonlinear Zabolotskaya Khokhlov equation

4.1 Introduction

As we have already discuss in (1.1) about the role of differential equations especially the role of nonlinear partial differential equations in applied sciences and engineering. With the help of non linear partial differential equations many problems are modeled in plasma physics, geometry, fluid dynamics, biology and nonlinear acoustics. In this chapter we find optimal system of $(2+1)$ dimensional nonlinear Zabolotskaya-Khokhlov equation

$$u_{tx} - (uu_x)_x - u_{yy} = 0. \tag{4.1}$$

This is one of the basic equation in nonlinear acoustic and nonlinear wave theory. It is named after two Russian mathematicians R. V. Khokhlov and E. A. Zabolotskaya. They derived it for the first time and presented an approximate solution which describes some important features of nonlinear waves [23]. After development of new medical devices for nonlinear diagnostic ultrasound imaging, acoustic surgery for noninvasive destruction of tumors and stone communication for kidney, extensive research have been made in nonlinear acoustics. Propagation of confined wave beam or sound beam on nonlinear medium without dispersion or with dispersion is describe by Zabolotskaya-Khokhlov equation. It also investigates the deformation of beam which is associated with the properties of nonlinear medium [14, 24]. Zabolotskaya-Khokhlov equation has its applications in many fields of life such as it is used to simulate the estimation of fish stock abundance, discrimination between fish species, the effect of excess attenuation which occurs due to nonlinear sound propagation in water absorption and diffraction in the focused sound beams [5, 13, 22, 27].

4.2 Lie symmetries and commutator relation table

Zabolotskaya-Khokhlov equation (4.1) have three independent variables the temporal variable t while x, y the spatial variable, whereas u the dependent variable. For one parameter ε, one parameter group of transformations for Zabolotskaya-Khokhlov equation (4.1) be

$$\begin{aligned}
\tilde{t} &= t + \varepsilon\xi^1(x, y, u) + O(\varepsilon^2),\\
\tilde{x} &= x + \varepsilon\xi^2(x, y, u) + O(\varepsilon^2),\\
\tilde{y} &= y + \varepsilon\xi^3(x, y, u) + O(\varepsilon^2),\\
\tilde{u} &= u + \varepsilon\eta(x, y, u) + O(\varepsilon^2).
\end{aligned}$$

Also Zabolotskaya-Khokhlov equation (4.1) verifies the above set of transformations. For general case of three independent variables and one dependent variable the symmetry generator is

$$\mathbf{V} = \xi^1(x, y, u)\frac{\partial}{\partial t} + \xi^2(x, y, u)\frac{\partial}{\partial x} + \xi^3(x, y, u)\frac{\partial}{\partial y} + \eta(x, y, u)\frac{\partial}{\partial u}. \tag{4.2}$$

We need prolongation of generator (4.2) according to order of derivatives involved in equation (4.1). As equation (4.1) have one derivative of first order with respect to x, double derivatives with respect to x and y, while one mixed double derivative of t and x. So, generator (4.2) get the form

$$\mathbf{V} = \xi_1\frac{\partial}{\partial t} + \xi_2\frac{\partial}{\partial x} + \xi_3\frac{\partial}{\partial y} + \eta\frac{\partial}{\partial u} + \eta_x\frac{\partial}{\partial u_{,x}} + \eta_{xx}\frac{\partial}{\partial u_{,xx}} + \eta_{tx}\frac{\partial}{\partial u_{,tx}} + \eta_{yy}\frac{\partial}{\partial u_{,yy}}. \tag{4.3}$$

By applying generator (4.3) to equation (4.1), we get a system of over determined linear partial differential equations in ξ^1, ξ^2, ξ^3 and η. Solving these we get symmetry generators of equation (4.1) as:

$$\mathbf{V}_1 = \frac{\partial}{\partial t}, \quad \mathbf{V}_2 = \frac{\partial}{\partial x}, \quad \mathbf{V}_3 = \frac{\partial}{\partial y}, \quad \mathbf{V}_4 = y\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}, \quad \mathbf{V}_5 = t\frac{\partial}{\partial x} - \frac{\partial}{\partial u},$$
$$\mathbf{V}_6 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, \qquad \mathbf{V}_7 = 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}.$$

One can find Lie algebra of these symmetry generators as:

$[\mathbf{V}_1,\mathbf{V}_1]=0,$ $[\mathbf{V}_1,\mathbf{V}_2]=0,$ $[\mathbf{V}_1,\mathbf{V}_3]=0,$ $[\mathbf{V}_1,\mathbf{V}_4]=0,$ $[\mathbf{V}_1,\mathbf{V}_5]=0,$
$[\mathbf{V}_1,\mathbf{V}_6]=\mathbf{V}_1,$ $[\mathbf{V}_1,\mathbf{V}_7]=4\mathbf{V}_1,$
$[\mathbf{V}_2,\mathbf{V}_1]=0,$ $[\mathbf{V}_2,\mathbf{V}_2]=0,$ $[\mathbf{V}_2,\mathbf{V}_3]=0,$ $[\mathbf{V}_2,\mathbf{V}_4]=0,$ $[\mathbf{V}_2,\mathbf{V}_5]=0,$
$[\mathbf{V}_2,\mathbf{V}_6]=\mathbf{V}_2,$ $[\mathbf{V}_2,\mathbf{V}_7]=4\mathbf{V}_1,$
$[\mathbf{V}_3,\mathbf{V}_1]=0,$ $[\mathbf{V}_3,\mathbf{V}_2]=0,$ $[\mathbf{V}_3,\mathbf{V}_3]=0,$ $[\mathbf{V}_3,\mathbf{V}_4]=\mathbf{V}_2,$ $[\mathbf{V}_3,\mathbf{V}_5]=0,$
$[\mathbf{V}_3,\mathbf{V}_6]=\mathbf{V}_3,$ $[\mathbf{V}_3,\mathbf{V}_7]=3\mathbf{V}_3,$
$[\mathbf{V}_4,\mathbf{V}_1]=-2\mathbf{V}_3,$ $[\mathbf{V}_4,\mathbf{V}_2]=0,$ $[\mathbf{V}_4,\mathbf{V}_3]=-\mathbf{V}_2,$ $[\mathbf{V}_4,\mathbf{V}_4]=0,$ $[\mathbf{V}_4,\mathbf{V}_5]=0,$
$[\mathbf{V}_4,\mathbf{V}_6]=0,$ $[\mathbf{V}_4,\mathbf{V}_7]=-\mathbf{V}_4,$
$[\mathbf{V}_5,\mathbf{V}_1]=0,$ $[\mathbf{V}_5,\mathbf{V}_2]=0,$ $[\mathbf{V}_5,\mathbf{V}_3]=0,$ $[\mathbf{V}_5,\mathbf{V}_4]=0,$ $[\mathbf{V}_5,\mathbf{V}_5]=0,$
$[\mathbf{V}_5,\mathbf{V}_6]=0,$ $[\mathbf{V}_5,\mathbf{V}_7]=2\mathbf{V}_7,$
$[\mathbf{V}_6,\mathbf{V}_1]=-\mathbf{V}_1,$ $[\mathbf{V}_6,\mathbf{V}_2]=-\mathbf{V}_2,$ $[\mathbf{V}_6,\mathbf{V}_3]=-\mathbf{V}_3,$ $[\mathbf{V}_6,\mathbf{V}_4]=0,$ $[\mathbf{V}_6,\mathbf{V}_5]=0,$
$[\mathbf{V}_6,\mathbf{V}_6]=0,$ $[\mathbf{V}_6,\mathbf{V}_7]=0,$
$[\mathbf{V}_7,\mathbf{V}_1]=-4\mathbf{V}_1,$ $[\mathbf{V}_7,\mathbf{V}_2]=-2\mathbf{V}_2,$ $[\mathbf{V}_7,\mathbf{V}_3]=-3\mathbf{V}_3,$ $[\mathbf{V}_7,\mathbf{V}_4]=\mathbf{V}_4,$
$[\mathbf{V}_7,\mathbf{V}_5]=-2\mathbf{V}_7,$ $[\mathbf{V}_7,\mathbf{V}_6]=0,$ $[\mathbf{V}_7,\mathbf{V}_7]=0.$

Also one can construct commutator relation table from these as:

,	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$	$\mathbf{V_5}$	$\mathbf{V_6}$	$\mathbf{V_7}$
$\mathbf{V_1}$	0	0	0	$2\mathbf{V}_3$	0	$\mathbf{V}_1$	$4\mathbf{V}_1$
$\mathbf{V_2}$	0	0	0	0	0	$\mathbf{V}_2$	$2\mathbf{V}_2$
$\mathbf{V_3}$	0	0	0	$\mathbf{V}_2$	0	$\mathbf{V}_3$	$3\mathbf{V}_3$
$\mathbf{V_4}$	$-2\mathbf{V}_3$	0	$-\mathbf{V}_2$	0	0	0	$-\mathbf{V}_4$
$\mathbf{V_5}$	0	0	0	0	0	0	$2\mathbf{V}_7$
$\mathbf{V_6}$	$-\mathbf{V}_1$	$-\mathbf{V}_2$	$-\mathbf{V}_3$	0	0	0	0
$\mathbf{V_7}$	$-4\mathbf{V}_1$	$-2\mathbf{V}_2$	$-3\mathbf{V}_3$	$\mathbf{V}_4$	$-2\mathbf{V}_7$	0	0

Table 1 (c)

4.3 Construction of adjoint representation table

To compute adjoint representation, we use the Lie series in conjunction with commutator relation Table 1(c). The adjoint action is given by the Lie series as
$Ad(exp(\varepsilon\mathbf{V}_i))\mathbf{V}_j=\mathbf{V}_j-\varepsilon[\mathbf{V}_i,\mathbf{V}_j]+\dfrac{\varepsilon^2}{2!}[\mathbf{V}_i,[\mathbf{V}_i,\mathbf{V}_j]]-\cdots,$
where $[\mathbf{V}_i,\mathbf{V}_j]$ is the Lie bracket for the generators $\mathbf{V}_i$ and $\mathbf{V}_j$. Using this definition of adjoint action.

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_1=\mathbf{V}_1-\epsilon[\mathbf{V}_1,\mathbf{V}_1]+\frac{\epsilon^2}{2!}[\mathbf{V}_1,[\mathbf{V}_1,\mathbf{V}_1]]-....$$

as $[\mathbf{V}_1, \mathbf{V}_1] = 0$, adjoint action of $\mathbf{V}_1$ on itself be

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_1, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_2]] - ....,$$

as $[\mathbf{V}_1, \mathbf{V}_2] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_1, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_3]] - ....,$$

since $[\mathbf{V}_1, \mathbf{V}_3] = 0$, we have

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_3 = \mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_1, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_4]] - ....,$$

we have $[\mathbf{V}_1, \mathbf{V}_4] = 2\mathbf{V}_3$, therefore

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_4 = \mathbf{V}_4 - 2\varepsilon \mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_1, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_5]] - ....,$$

we know that $[\mathbf{V}_1, \mathbf{V}_5] = 0$, hence

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_4 = \mathbf{V}_5.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_1, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_6]] - ....,$$

from commutator relation Table 1(c), $[\mathbf{V}_1, \mathbf{V}_6] = \mathbf{V}_1$, using this we get

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_6 = \mathbf{V}_6 - \varepsilon \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_1, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_1, [\mathbf{V}_1, \mathbf{V}_7]] - ....,$$

as $[\mathbf{V}_1, \mathbf{V}_7] = 4\mathbf{V}_1$, adjoint action of $\mathbf{V}_1$ on $\mathbf{V}_7$ be

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_7 = \mathbf{V}_7 - 4\varepsilon \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_2, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_1]] - ....,$$

as $[\mathbf{V}_2, \mathbf{V}_1] = 0$, hence

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_2, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_2]] - ....,$$

we have $[\mathbf{V}_2, \mathbf{V}_2] = 0$, adjoint action of $\mathbf{V}_2$ on itself be

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_2, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_3]] - ....,$$

we know that $[\mathbf{V}_2, \mathbf{V}_3] = 0$, therefore

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_3 = \mathbf{V}_3.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_2, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_4, \mathbf{V}_4]] - ....,$$

from commutator relation Table 1(c) $[\mathbf{V}_2, \mathbf{V}_4] = 0$, using this we get

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_4 = \mathbf{V}_4.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_2, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_5]] - ....,$$

since $[\mathbf{V}_2, \mathbf{V}_5] = 0$, hence

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_5 = \mathbf{V}_5.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_2, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_6]] - ....,$$

we have $[\mathbf{V}_2, \mathbf{V}_6] = \mathbf{V}_2$, therefore

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_6 = \mathbf{V}_6 - \varepsilon \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_2))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_2, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_2, [\mathbf{V}_2, \mathbf{V}_7]] - ....,$$

from commutator relation Table 1(c), $[\mathbf{V}_2, \mathbf{V}_7] = 2\mathbf{V}_2$, using this we obtain

$$Ad(exp(\epsilon \mathbf{V}_1))\mathbf{V}_7 = \mathbf{V}_7 - 2\varepsilon \mathbf{V}_2.$$

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_3, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_1]] - ....,$$

we know that $[\mathbf{V}_3, \mathbf{V}_1] = 0$, adjoint action of $\mathbf{V}_3$ on $\mathbf{V}_1$ be

$$Ad(exp(\epsilon \mathbf{V}_3))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_3, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_2]] - ....,$$

as $[\mathbf{V}_3, \mathbf{V}_2] = 0$, therfore

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_3, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_3]] - ....,$$

since we know that $[\mathbf{V}_3, \mathbf{V}_3] = 0$, so, adjoint action of $\mathbf{V}_3$ on itself be

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_3 = \mathbf{V}_3.$$

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_3, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_4]] - ....,$$

we have $[\mathbf{V}_3, \mathbf{V}_4] = \mathbf{V}_2$, hence

$$Ad(exp(\epsilon\mathbf{V}_1))\mathbf{V}_3 = \mathbf{V}_4 - \varepsilon\mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_3, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_5]] - ....,$$

from commutator relation Table 1(c), $[\mathbf{V}_3, \mathbf{V}_5] = 0$, using this it yields

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_5 = \mathbf{V}_5.$$

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_3, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_6]] - ....,$$

as $[\mathbf{V}_3, \mathbf{V}_6] = \mathbf{V}_3$, therefore

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_6 = \mathbf{V}_6 - \varepsilon\mathbf{V}_3.$$

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_3, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_3, [\mathbf{V}_3, \mathbf{V}_7]] - ....,$$

since $[\mathbf{V}_3, \mathbf{V}_7] = 3\mathbf{V}_3$, adjoint action of $\mathbf{V}_3$ on $\mathbf{V}_7$ be

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_7 = \mathbf{V}_7 - 3\varepsilon\mathbf{V}_3.$$

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_4, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_1]] - ....,$$

as $[\mathbf{V}_4, \mathbf{V}_1] = -2\mathbf{V}_3$, from here we obtain

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_1 = \mathbf{V}_1 + 2\varepsilon\mathbf{V}_3 + \varepsilon^2\mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_4, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_2]] - ....,$$

since $[\mathbf{V}_4, \mathbf{V}_2] = 0$, using this we get

$$Ad(exp(\epsilon\mathbf{V}_3))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_4, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_3]] - ....,$$

we have $[\mathbf{V}_4, \mathbf{V}_3] = -\mathbf{V}_2$, hence

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_3 = \mathbf{V}_3 + \varepsilon\mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_4, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_4]] - ....,$$

from commutator relation table $1(c)$, $[\mathbf{V}_4, \mathbf{V}_4] = 0$, therefore

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_4 = \mathbf{V}_4.$$

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_4, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_5]] - ....,$$

we know that $[\mathbf{V}_4, \mathbf{V}_5] = 0$, using this we obtain

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_5 = \mathbf{V}_5.$$

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_4, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_6]] - ....,$$

since $[\mathbf{V}_4, \mathbf{V}_6] = 0$, adjoint action of $\mathbf{V}_4$ on $\mathbf{V}_6$ be

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_6 = \mathbf{V}_6.$$

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_4, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_4, [\mathbf{V}_4, \mathbf{V}_7]] - ....,$$

from commutator relation Table $1(c)$, $[\mathbf{V}_4, \mathbf{V}_7] = -\mathbf{V}_4$, therefore

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_7 = \mathbf{V}_7 + \varepsilon\mathbf{V}_4.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_5, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_5, [\mathbf{V}_5, \mathbf{V}_1]] - ....,$$

as $[\mathbf{V}_5, \mathbf{V}_1] = 0$, using this we get adjoint action of $\mathbf{V}_5$ on $\mathbf{V}_1$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_1 = \mathbf{V}_1.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_5, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_5, [\mathbf{V}_5, \mathbf{V}_2]] - ....,$$

since $[\mathbf{V}_5, \mathbf{V}_2] = 0$, using this we get

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_2 = \mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_5, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_5, [\mathbf{V}_5, \mathbf{V}_3]] - ....,$$

we have $[\mathbf{V}_5, \mathbf{V}_3] = 0$, therefore

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_3 = \mathbf{V}_3.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_5, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_5, [\mathbf{V}_5, \mathbf{V}_4]] - ....,$$

from commutator relation Table $1(c)$, $[\mathbf{V}_5, \mathbf{V}_4] = 0$, hence

$$Ad(exp(\epsilon\mathbf{V}_4))\mathbf{V}_4 = \mathbf{V}_4.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_5, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_5, [\mathbf{V}_5, \mathbf{V}_5]] - ....,$$

we know that $[\mathbf{V}_5, \mathbf{V}_5] = 0$, adjoint action of $\mathbf{V}_5$ on itself be

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_5 = \mathbf{V}_5.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_5, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_5, [\mathbf{V}_5, \mathbf{V}_6]] - ....,$$

as $[\mathbf{V}_5, \mathbf{V}_6] = 0$, hence

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_6 = \mathbf{V}_6.$$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_5, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_5, [\mathbf{V}_5, \mathbf{V}_7]] - ....,$$

we have $[\mathbf{V}_5, \mathbf{V}_7] = 2\mathbf{V}_7$, using this we get adjoint action of $\mathbf{V}_5$ on $\mathbf{V}_7$

$$Ad(exp(\epsilon\mathbf{V}_5))\mathbf{V}_7 = \mathbf{V}_7 e^{-2\varepsilon}.$$

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_6, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_1]] - ....,$$

we have $[\mathbf{V}_6, \mathbf{V}_1] = -\mathbf{V}_1$, therefore

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_1 = e^{\varepsilon}\mathbf{V}_1.$$

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_6, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_2]] - ....,$$

as $[\mathbf{V}_6, \mathbf{V}_2] = -\mathbf{V}_2$, using this we get

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_2 = e^{\varepsilon}\mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_6, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_3]] - ....,$$

we have $[\mathbf{V}_6, \mathbf{V}_3] = -\mathbf{V}_3$, therefore

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_3 = e^{\varepsilon}\mathbf{V}_3.$$

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_6, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_4]] - ....,$$

we know that $[\mathbf{V}_6, \mathbf{V}_4] = 0$, adjoint action of $\mathbf{V}_6$ on $\mathbf{V}_4$ is

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_4 = \mathbf{V}_4.$$

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_6, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_5]] - ....,$$

from commutator relation Table $1(c)$, $[\mathbf{V}_6, \mathbf{V}_5] = 0$, adjoint action of $\mathbf{V}_6$ on $\mathbf{V} - 5$ be

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_5 = \mathbf{V}_5.$$

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_6, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_6]] - ....,$$

since $[\mathbf{V}_6, \mathbf{V}_6] = 0$, adjoint action of $\mathbf{V}_6$ on itself be

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_6 = \mathbf{V}_6.$$

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_6, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_6, [\mathbf{V}_6, \mathbf{V}_7]] - ....,$$

we know that $[\mathbf{V}_6, \mathbf{V}_7] = 0$, using this we get

$$Ad(exp(\epsilon\mathbf{V}_6))\mathbf{V}_7 = \mathbf{V}_7.$$

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_1 = \mathbf{V}_1 - \epsilon[\mathbf{V}_7, \mathbf{V}_1] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_1]] - ....,$$

since $[\mathbf{V}_7, \mathbf{V}_1] = -4\mathbf{V}_1$, adjoint action of $\mathbf{V}_7$ on $\mathbf{V}_1$ be

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_1 = e^{4\varepsilon}\mathbf{V}_1.$$

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_2 = \mathbf{V}_2 - \epsilon[\mathbf{V}_7, \mathbf{V}_2] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_2]] - ....,$$

we know that $[\mathbf{V}_7, \mathbf{V}_2] = -2\mathbf{V}_2$, therefore

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_2 = e^{2\varepsilon}\mathbf{V}_2.$$

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_3 = \mathbf{V}_3 - \epsilon[\mathbf{V}_7, \mathbf{V}_3] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_3]] - ....,$$

from commutator relation Table $1(c)$, $[\mathbf{V}_7, \mathbf{V}_3] = -3\mathbf{V}_3$, hence

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_3 = e^{3\varepsilon}\mathbf{V}_3.$$

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_4 = \mathbf{V}_4 - \epsilon[\mathbf{V}_7, \mathbf{V}_4] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_4]] - ....,$$

we have $[\mathbf{V}_7, \mathbf{V}_4] = \mathbf{V}_4$, using this we get

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_4 = e^{-\varepsilon}\mathbf{V}_4.$$

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_5 = \mathbf{V}_5 - \epsilon[\mathbf{V}_7, \mathbf{V}_5] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_5]] - ....,$$

since $[\mathbf{V}_7, \mathbf{V}_5] = -2\mathbf{V}_7$, adjoint action of $\mathbf{V}_7$ on $\mathbf{V}_5$ be

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_5 = \mathbf{V}_5 + 2\varepsilon\mathbf{V}_7.$$

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_6 = \mathbf{V}_6 - \epsilon[\mathbf{V}_7, \mathbf{V}_6] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_6]] - ....,$$

we have $[\mathbf{V}_7, \mathbf{V}_6] = 0$, using this we obtain

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_6 = \mathbf{V}_7.$$

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_7 = \mathbf{V}_7 - \epsilon[\mathbf{V}_7, \mathbf{V}_7] + \frac{\epsilon^2}{2!}[\mathbf{V}_7, [\mathbf{V}_7, \mathbf{V}_7]] - ....,$$

from commutator relation Table $1(c)$, $[\mathbf{V}_7, \mathbf{V}_7] = 0$, using this we get adjoint action of $\mathbf{V}_7$ on itself

$$Ad(exp(\epsilon\mathbf{V}_7))\mathbf{V}_7 = \mathbf{V}_7.$$

Using these results one can construct an adjoint representation table.

Ad	$\mathbf{V_1}$	$\mathbf{V_2}$	$\mathbf{V_3}$	$\mathbf{V_4}$	$\mathbf{V_5}$	$\mathbf{V_6}$	$\mathbf{V_7}$
$\mathbf{V_1}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4 - \varepsilon\mathbf{V}_3$	$\mathbf{V}_5$	$\mathbf{V}_6 - \varepsilon\mathbf{V}_1$	$\mathbf{V}_7 - 4\varepsilon\mathbf{V}_1$
$\mathbf{V_2}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6 - \varepsilon\mathbf{V}_2$	$\mathbf{V}_7 - 2\varepsilon\mathbf{V}_2$
$\mathbf{V_3}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4 - \varepsilon\mathbf{V}_2$	$\mathbf{V}_5$	$\mathbf{V}_6 - \varepsilon\mathbf{V}_3$	$\mathbf{V}_7 - 3\varepsilon\mathbf{V}_3$
$\mathbf{V_4}$	$\mathbf{V}_1 + 2\varepsilon\mathbf{V}_3$	$\mathbf{V}_2$	$\mathbf{V}_3 + \varepsilon\mathbf{V}_2$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6$	$\mathbf{V}_7 + \varepsilon\mathbf{V}_4$
$\mathbf{V_5}$	$\mathbf{V}_1$	$\mathbf{V}_2$	$\mathbf{V}_3$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6$	$e^{-2\varepsilon}\mathbf{V}_7$
$\mathbf{V_6}$	$e^{\varepsilon}\mathbf{V}_1$	$e^{\varepsilon}\mathbf{V}_2$	$\mathbf{V}_3 e^{\varepsilon}$	$\mathbf{V}_4$	$\mathbf{V}_5$	$\mathbf{V}_6$	$\mathbf{V}_7$
$\mathbf{V_7}$	$e^{4\varepsilon}\mathbf{V}_1$	$e^{2\varepsilon}\mathbf{V}_2$	$e^{3\varepsilon}\mathbf{V}_3$	$e^{-\varepsilon}\mathbf{V}_4$	$\mathbf{V}_5 + 2\varepsilon\mathbf{V}_7$	$\mathbf{V}_6$	$\mathbf{V}_7$

Table 2 (c)

4.4 Formation of optimal system

We have defined optimal system in 1.5, it constitutes the set of conjugacy classes of group of transformations. Also we know that adjoint action gives the conjugacy classes of group of transformations which are written in columns of adjoint representation table. Our aim is to find the set of one dimensional sub algebras which cover all conjugacy classes. Following Olver's technique [12, 15, 16, 17, 19], assume a general vector $\mathbf{V}$ as the combination of all symmetry generators and then by observing columns of adjoint representation table try to vanish as much symmetry generators as possible by using appropriate adjoint action on general vector $\mathbf{V}$. Consider a general non zero vector of the form

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4 + a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + a_7\mathbf{V}_7. \tag{4.4}$$

Our task is to vanish as many as of the coefficients $a's$ as possible by the judicious application of adjoint map to $\mathbf{V}$. Suppose first that $a_7 \neq 0$ and also further assume for convenience that $a_7 = 1$

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4 + a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + \mathbf{V}_7$$

$$\begin{aligned} \mathbf{V} \;=\;& a_1\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial x} + a_3\frac{\partial}{\partial y} + a_4 y\frac{\partial}{\partial x} + 2a_4 t\frac{\partial}{\partial y} + a_5 t\frac{\partial}{\partial x} - a_5\frac{\partial}{\partial u} + \\ & a_6 t\frac{\partial}{\partial t} + a_6 x\frac{\partial}{\partial x} + a_6 y\frac{\partial}{\partial y} + 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}. \end{aligned}$$

Referring to Table 2(c), if we act on $\mathbf{V}$ by $Ad(exp(-a_4)\mathbf{V}_4)$ then we can make the coefficient of $\mathbf{V}_4$ vanish. We call the resulting vector as $\mathbf{V}'$

$$\mathbf{V}' = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + \mathbf{V}_7,$$

$$\begin{aligned} \mathbf{V}' \;=\;& a_1\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial x} + a_3\frac{\partial}{\partial y} + a_5 t\frac{\partial}{\partial x} - a_5\frac{\partial}{\partial u} + a_6 t\frac{\partial}{\partial t} + \\ & a_6 x\frac{\partial}{\partial x} + a_6 y\frac{\partial}{\partial y} + 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}. \end{aligned}$$

Next we act on $\mathbf{V}'$ by $Ad(exp(\frac{1}{3}a_3)\mathbf{V}_3)$ to cancel the coefficient of $\mathbf{V}_3$. Which is represented by

$$\mathbf{V}'' = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + \mathbf{V}_7,$$

$$\begin{aligned} \mathbf{V}'' \;=\;& a_1\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial x} + a_5 t\frac{\partial}{\partial x} - a_5\frac{\partial}{\partial u} + a_6 t\frac{\partial}{\partial t} + a_6 x \\ & \frac{\partial}{\partial x} + a_6 y\frac{\partial}{\partial y} + 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}. \end{aligned}$$

Similarly, if we act on $\mathbf{V}''$ by $Ad(exp(\frac{1}{2}a_2)\mathbf{V}_2)$, then coefficient of $\mathbf{V}_2$ vanish. We call the resulting vector as $\mathbf{V}'''$

$$\mathbf{V}''' = a_1\mathbf{V}_1 + a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + \mathbf{V}_7,$$

$$\begin{aligned}\mathbf{V}''' &= a_1\frac{\partial}{\partial t} + a_5 t\frac{\partial}{\partial x} - a_5\frac{\partial}{\partial u} + a_6 t\frac{\partial}{\partial t} + a_6 x\frac{\partial}{\partial x} + \\ & \quad a_6 y\frac{\partial}{\partial y} + 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}\end{aligned}$$

In adjoint representation Table 2(c), we observe that $\mathbf{V}_7$ and $\mathbf{V}_1$ are written in a combination. So we can vanish coefficient of $\mathbf{V}_1$, if we act on $\mathbf{V}'''$ by $Ad(exp(\frac{1}{4}a_1)\mathbf{V}_1)$. We call the resulting vector as $\mathbf{V}^{iv}$

$$\mathbf{V}^{iv} = a_5\mathbf{V}_5 + a_6\mathbf{V}_6 + \mathbf{V}_7$$

$$\mathbf{V}^{iv} = a_5 t\frac{\partial}{\partial x} - a_5\frac{\partial}{\partial u} + a_6 t\frac{\partial}{\partial t} + a_6 x\frac{\partial}{\partial x} + a_6 y\frac{\partial}{\partial y} + 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}.$$

Referring to adjoint representation Table 2(c), if we act on $\mathbf{V}^{iv}$ by $Ad(exp(2a_5)\mathbf{V}_5)$, then coefficient of $\mathbf{V}_5$ cancel from $\mathbf{V}^{iv}$. We name the resulting vector as $\mathbf{V}^{v}$

$$\mathbf{V}^{iv} = a_6\mathbf{V}_6 + \mathbf{V}_7,$$

$$\mathbf{V}^{iv} = a_6 t\frac{\partial}{\partial t} + a_6 x\frac{\partial}{\partial x} + a_6 y\frac{\partial}{\partial y} + 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}.$$

Up to now we are succeed in vanishing all those symmetry generators written with $\mathbf{V}_7$ in its linear combination in adjoint representation Table 2(c). So, from here the symmetry generators which are included in optimal system be:

(i) $\mathbf{V}_7$,

(ii) $a\mathbf{V}_6 + \mathbf{V}_7$, where a be an arbitrary constant. The remaining one-dimensional sub algebras are spanned by vector (4.4) of the above form with $a_7 = 0$. If $a_6 \neq 0$, we scale to make $a_6 = 1$. Then the general non zero vector (4.4) get the form

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4 + a_5\mathbf{V}_5 + \mathbf{V}_6.$$

Referring adjoint representation Table 2(c), symmetry generator $\mathbf{V}_6$ is written in linear combination of $\mathbf{V}_3, \mathbf{V}_2$ and $\mathbf{V}_1$. We can make their coefficients vanish by acting on $\mathbf{V}$ by $Ad(exp(a_3)\mathbf{V}_3), Ad(exp(a_2)\mathbf{V}_2)$ and $Ad(exp(a_1)\mathbf{V}_1)$ respectively. After acting $Ad(exp(a_3)\mathbf{V}_3)$ on $\mathbf{V}$, we get vector $\mathbf{V}'$ free of coefficient of $\mathbf{V}_3$

$$\mathbf{V}' = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_4\mathbf{V}_4 + a_5\mathbf{V}_5 + \mathbf{V}_6,$$

$$\mathbf{V}' = a_1\frac{\partial}{\partial t} + a_2\frac{\partial}{\partial x} + a_4 y\frac{\partial}{\partial x} + 2a_4 t\frac{\partial}{\partial y} + a_5 t\frac{\partial}{\partial x} - a_5\frac{\partial}{\partial u} + a_6 t\frac{\partial}{\partial t} + a_6 x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 4t\frac{\partial}{\partial t}.$$

After acting $Ad(exp(a_2)\mathbf{V}_2)$ on $\mathbf{V}'$, we get the vector free of coefficient of $\mathbf{V}_2$, we call resulting vector as $\mathbf{V}''$

$$\mathbf{V}'' = a_1\mathbf{V}_1 + a_4\mathbf{V}_4 + a_5\mathbf{V}_5 + \mathbf{V}_6,$$

$$\mathbf{V}'' = a_1\frac{\partial}{\partial t} + a_4 y\frac{\partial}{\partial x} + 2a_4 t\frac{\partial}{\partial y} + a_5 t\frac{\partial}{\partial x} - a_5\frac{\partial}{\partial u} + a_6 t\frac{\partial}{\partial t} + a_6 x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 4t\frac{\partial}{\partial t}.$$

Similarly, by applying $Ad(exp(a_1)\mathbf{V}_1)$ on $\mathbf{V}''$, the coefficient of $\mathbf{V}_1$ vanish, which is represented in the vector $\mathbf{V}'''$

$$\mathbf{V}''' = a_4\mathbf{V}_4 + a_5\mathbf{V}_5 + \mathbf{V}_6,$$

$$\mathbf{V}''' = a_4 y\frac{\partial}{\partial x} + 2a_4 t\frac{\partial}{\partial y} + a_5 t\frac{\partial}{\partial x} - a_5\frac{\partial}{\partial u} + a_6 t\frac{\partial}{\partial t} + a_6 x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 4t\frac{\partial}{\partial t}.$$

Therefore, from here the sub algebras which are included in optimal system are:

(i) $\mathbf{V}_6$,

(ii) $a\mathbf{V}_4 + b\mathbf{V}_5 + \mathbf{V}_6$,

where a ,b be arbitrary constants, which we assume to be one during reduction. The remaining one-dimensional sub algebras are spanned by vector (4.4) of the above form with $a_7 = a_6 = 0$. If $a_5 \neq 0$, for convenience we assume that $a_5 = 1$. Then general non zero vector (4.4) be

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4 + \mathbf{V}_5.$$

From adjoint representation Table 2(c), we came to know that $\mathbf{V}_5$ written only with $\mathbf{V}_7$, but we have already utilized this relation while we were interested in finding the representative for the class involving $\mathbf{V}_7$. Therefore, from here the only only sub algebra which included in optimal system is the combination of symmetry generators:

(i) $a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + a_4\mathbf{V}_4 + \mathbf{V}_5$,

where these $a's$ are coefficients and are arbitrary constants, which we assumed for simplicity to be one during reduction.

Beside these one-dimensional sub algebras are spanned by vector (4.4) of the above form with $a_7 = a_6 = a_5 = 0$. For $a_4 \neq 0$ also for our our convenience we further assume that $a_4 = 1$ then the general non zero vector (4.4) be

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + a_3\mathbf{V}_3 + \mathbf{V}_4.$$

Concerning adjoint representation Table 2(c), if we act on $\mathbf{V}$ by $Ad(exp(a_2)\mathbf{V}_4)$, then we can make the coefficient of $\mathbf{V}_4$ vanish, which is represented in $\mathbf{V}'$

$$\mathbf{V}' = a_1\mathbf{V}_1 + a_3\mathbf{V}_3 + \mathbf{V}_4.$$

On similar lines if we act on $\mathbf{V}'$ by $Ad(exp(a_3)\mathbf{V}_3)$, then we vanish coefficient of $\mathbf{V}_3$, we call that resulting vector as $\mathbf{V}''$

$$\mathbf{V}'' = a_1\mathbf{V}_1 + \mathbf{V}_4.$$

Till now we are succeed in vanishing coefficients of all those symmetry generators written with the combination of $\mathbf{V}_4$. So, from here the symmetry generators which are included in the optimal system are:

(i) $\mathbf{V}_4$,

(ii) $a\mathbf{V}_1 + \mathbf{V}_4$.

Now for remaining one dimensional sub algebras we assume in general non zero vector (4.4) that $a_7 = a_6 = a_5 = 0 = a_4 = 0$. If $a_3 \neq 0$ then for convenience we scale it to $a_3 = 1$. General non zero vector get the form

$$\mathbf{V} = a_1\mathbf{V}_1 + a_2\mathbf{V}_2 + \mathbf{V}_3.$$

Referring adjoint representation Table 2(c), we came to know that the symmetry generator $\mathbf{V}_3$ written in linear combination with $\mathbf{V}_2$ in second column and with $\mathbf{V}_1$ in first column. If we act on $\mathbf{V}$ by $Ad(exp(-a_2)\mathbf{V}_2)$ then coefficient of $\mathbf{V}_2$ vanish and we call the resulting generator as $\mathbf{V}'$

$$\mathbf{V}' = a_1\mathbf{V}_1 + \mathbf{V}_3.$$

Similarly, if we act on $\mathbf{V}'$ by $Ad(exp(-2a_1)\mathbf{V}_1)$ then we can make coefficient of $\mathbf{V}_1$ vanish and we name the resulting vector as $\mathbf{V}''$

$$\mathbf{V}'' = \mathbf{V}_3.$$

So far we are succeed in vanishing coefficients of all those symmetry generators written in the linear combination with $\mathbf{V}_3$. So, from here the symmetry generator which included in the optimal system is $\mathbf{V}_3$ only.

Working on same lines for considering all coefficients to be zero except a_2 and a_1 respectively. we find out that the symmetry generator which are included to the optimal system be

(i) $a\mathbf{V}_1 + \mathbf{V}_2$,

(ii) $\mathbf{V}_1$.

Recapitulating, we have found an optimal system of one dimensional sub algebras to be those spanned by

$\mathbf{V}_5+a_4\mathbf{V}_4+a_3\mathbf{V}_3+a_2\mathbf{V}_2+a_1\mathbf{V}_1$, $\mathbf{V}_6+a_5\mathbf{V}_5+a_4\mathbf{V}_4$, $\mathbf{V}_7+a_6\mathbf{V}_6$, $\mathbf{V}_4+a_1\mathbf{V}_1$, $\mathbf{V}_2+a_1\mathbf{V}_1$, $\mathbf{V}_7$, $\mathbf{V}_6$, $\mathbf{V}_4$, $\mathbf{V}_3$, $\mathbf{V}_1$.

4.5 Reduction

(i) For symmetry generator $\mathbf{V}_7 = 4t\frac{\partial}{\partial t} + 2x\frac{\partial}{\partial x} + 3y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u}$,

one can write it as
$$\frac{dt}{4t} = \frac{dx}{2x} = \frac{dy}{3y} = \frac{du}{-2u},$$

from here we get

$$t = e^{4\rho}, \quad x = \xi\sqrt{e^{4\rho}}, \quad y = \eta e^{3\rho}, \quad u = \frac{U(\xi,\eta)}{\sqrt{e^{4\rho}}}.$$

Substituting in equation (4.1), we get

$$U_{\xi\xi}(\frac{1}{2}\xi + U) + U_{\eta\eta} + \frac{3}{4}\eta U_{\xi\eta} + {U_\xi}^2 + U_\xi = 0, \tag{4.5}$$

which is one dimension less than the equation (4.1).

Symmetry generators of equation (4.5) are

$$\mathbf{V}_1 = -2\frac{\partial}{\partial \xi} + \frac{\partial}{\partial U}, \qquad \mathbf{V}_2 = 4\eta\frac{\partial}{\partial \xi} + \frac{32}{3}\frac{\partial}{\partial \eta} + y\frac{\partial}{\partial U}, \qquad \mathbf{V}_3 = \xi\frac{\partial}{\partial \xi} + \frac{\eta}{2}\frac{\partial}{\partial \eta} + U\frac{\partial}{\partial U}.$$

Using these symmetries we get solutions of Zabolotskaya-Khokhlov equation (4.1) as:

$$u(t,x,y) = \frac{y^2}{8t^2} - \frac{x}{t} + C_1\frac{y}{t^{\frac{3}{4}}} + C_2, \tag{4.6}$$

which is defined for all values of x, y and t except when $t = 0$. From these symmetry generators we obtain another solution as:

$$\begin{aligned} u(t,x,y) \;\; = \;\; & \frac{1}{t^{\frac{1}{2}}}[\frac{39y^2}{512t^6} - \frac{3x}{64t^{\frac{1}{2}}} + \frac{e^{C_1}}{288}(192C_1e^{C_1} + 27e^{C_1}(\frac{5y^2}{86} - \frac{x}{t^{\frac{1}{2}}}) \\ & -\sqrt{(384C_1e^{C_1} - 27e^{C_1}(\frac{5y^2}{86} - \frac{x}{t^{\frac{1}{2}}}) + 4)} + 2)], \end{aligned}$$

which is also undefined for $t = 0$.

(ii) For symmetry generator $\mathbf{V}_3 = \frac{\partial}{\partial y}$,

it can be written as

$$\frac{dt}{0} = \frac{dx}{0} = \frac{dy}{1} = \frac{du}{0},$$

from here we get

$$t = \xi, \quad x = \eta, \quad y = \rho, \quad u = U(\xi, \eta).$$

By substituting these in equation (4.1), we get

$$U_{\xi\eta} - UU_{\eta\eta} - {U_\eta}^2 = 0,$$

which is one dimension less than the equation (4.1). It yields trivial solution of Zabolotskaya-Khokhlov equation (4.1).

(iii) For symmetry generator $\mathbf{V}_4 = y\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial y}$,

one can write it as

$$\frac{dt}{0} = \frac{dx}{y} = \frac{dy}{2t} = \frac{du}{0},$$

from here we get

$$t = \xi, \quad x = \frac{4\rho^2\xi^2 - \eta}{4\xi}, \quad y = 2\rho\xi, \quad u = U(\xi, \eta).$$

Substituting in equation (4.1), we get

$$U_{\eta\eta}(4\eta + 16\xi^2 U) + 16\xi^2 {U_\eta}^2 + 4\xi U_{\xi\eta} + 6U_\eta = 0,$$

which is one dimension less than the equation (4.1). It also gives trivial solution of Zabolotskaya-Khokhlov equation (4.1).

(iv) For symmetry generator $\mathbf{V}_6 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$,

we can write it as

$$\mathbf{V}_6 = t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + 0\frac{\partial}{\partial u},$$

$$\frac{dt}{t} = \frac{dx}{x} = \frac{dy}{y} = \frac{du}{0},$$

from here we get

$t = e^{\rho}, \quad x = \xi e^{\rho}, \quad y = \eta e^{\rho}, \quad u = U(\xi, \eta).$

Using these in equation (4.1), we obtain

$$U_{\xi\xi}(\xi + U) + U_{\eta\eta} + \eta U_{\xi\eta} + {U_{\xi}}^2 + u_{\xi} = 0, \tag{4.7}$$

which is one dimension less than the equation (4.1). Here (4.7) have three dimensional Lie algebra as:

$$\mathbf{V}_1 = -\frac{\partial}{\partial \xi} + \frac{\partial}{\partial U}, \qquad \mathbf{V}_2 = \frac{\eta}{2}\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \qquad \mathbf{V}_3 = \xi\frac{\partial}{\partial \xi} + \frac{\eta}{2}\frac{\partial}{\partial \eta} + U\frac{\partial}{\partial U}.$$

Using these symmetry generators one can easily get the solution of Zabolotskaya-Khokhlov equation (4.1) as

$$u(t, x, y) = C_1\frac{y}{t} - \frac{x}{t} + C_2, \tag{4.8}$$

which is undefined when $t = 0$.

(v) For symmetry generator $\mathbf{V}_1 = \frac{\partial}{\partial t}$,

one can write it as

$$\frac{dt}{1} = \frac{dx}{0} = \frac{dy}{0} = \frac{du}{0},$$

from here we get

$t = \rho, \quad x = \xi, \quad y = \eta, \quad u = U(\xi, \eta).$

Using in equation (4.1), we get

$$UU_{\xi\xi} - {U_{\xi}}^2 - U_{\eta\eta} = 0,$$

which is one dimension less than the equation (4.1). Its solution can be easily calculated in terms of Wiestrass function.

(vi) For combination of symmetry generators $\mathbf{V}_1$ and $\mathbf{V}_2$, one can write their combination as

$$\mathbf{V} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x},$$

it can also be written as

$$\frac{dt}{1} = \frac{dx}{1} = \frac{dy}{0} = \frac{du}{0}$$

from here we get

$t = \rho, \quad x = \xi + \rho, \quad y = \eta, \quad u = U(\xi, \eta).$

Using these in equation (4.1), we obtain

$$U_{\xi\xi}(U+1)+U_{\eta\eta}+{U_{\xi}}^2=0 \tag{4.9}$$

which is one dimension less than the equation (4.1). Equation (4.9) has four dimensional Lie algebra as:

$$\mathbf{V}_1=\frac{\partial}{\partial\xi},\quad \mathbf{V}_2=\frac{\partial}{\partial\eta},\quad \mathbf{V}_3=\xi\frac{\partial}{\partial xi}+(2U-2)\frac{\partial}{\partial U},\quad \mathbf{V}_4=\eta\frac{\partial}{\partial\eta}+(2-2U)\frac{\partial}{\partial U}.$$

Using these we get the solution of Zabolotskaya-Khokhlov equation (4.1) as

$$u(t,x,y)=1+\frac{C_1}{2y^2}\frac{A(e^{\frac{x-t}{C_1}})^2+12AC_1(e^{\frac{x-t}{C_1}})+36{C_1}^2}{A(e^{\frac{x-t}{C_1}})}, \tag{4.10}$$

where $A=e^{\frac{C_2}{C_1}}$. Also solution is undefined when $y=o$

(vii) For combination of symmetry generators $\mathbf{V}_1$ and $\mathbf{V}_4$, one can write their combination as

$$\mathbf{V}=\frac{\partial}{\partial t}+y\frac{\partial}{\partial x}+2t\frac{\partial}{\partial y},$$

it can also be written as

$$\frac{dt}{1}=\frac{dx}{y}=\frac{dy}{2t}=\frac{du}{0},$$

it yields

$$t=\rho,\quad x=\frac{1}{3}\rho^3+\rho\xi+\eta,\quad y=\rho^2+\xi,\quad u=U(\xi,\eta).$$

Using these in equation (4.1), we get

$$U_{\eta\eta}(U+\xi)+U_{\xi\xi}+{U_{\eta}}^2=0, \tag{4.11}$$

which is one dimension less than the equation (4.1). Equation (4.11) has four dimensional Lie algebra as:

$$\mathbf{V}_1=\frac{\partial}{\partial\eta},\quad \mathbf{V}_2=\frac{\partial}{\partial\xi}-\frac{\partial}{\partial U},\quad \mathbf{V}_3=\eta\frac{\partial}{\partial\eta}+2(\xi+U)\frac{\partial}{\partial U},\quad \mathbf{V}_4=\xi\frac{\partial}{\partial\xi}-(2U+3\xi)\frac{\partial}{\partial U}.$$

Using these we get following solutions of Zabolotskaya-Khokhlov equation (4.1) as

$$u(t,x,y)=t^2-y+C_2e^{C_1(x+\frac{2t^3}{3}-ty)}, \tag{4.12}$$

and

$$u(t,x,y)=t^2-y+\frac{C_1}{2(y-t^2)}\frac{A^2(e^{\frac{x+\frac{2t^3}{3}-ty}{C_1}})^2+12AC_1(e^{\frac{x+\frac{2t^3}{3}-ty}{C_1}})+36{C_1}^2}{A(e^{\frac{x+\frac{2t^3}{3}-ty}{C_1}})}, \tag{4.13}$$

where $A = e^{\frac{C_2}{C_1}}$. Here first solution is defined every where for t, x and y while second solution becomes undefined when $y = \pm t$.

(viii) For combination of symmetry generators $\mathbf{V}_6$ and $\mathbf{V}_7$, one can write their combination as

$$\mathbf{V} = 5t\frac{\partial}{\partial t} + 3x\frac{\partial}{\partial x} + 4y\frac{\partial}{\partial y} - 2u\frac{\partial}{\partial u},$$

one can also write it as

$$\frac{dt}{5t} = \frac{dx}{3x} = \frac{dy}{4y} = \frac{du}{-2u},$$

it yields

$$t = e^{5\rho},\ \ x = \xi e^{3\rho},\ \ y = \eta e^{4\rho},\ \ u = U(\xi,\eta)e^{-2\rho}.$$

Using these in equation (4.1), it yields

$$U_{\xi\xi}(U + \frac{3}{5}\xi) + U_{\eta\eta} + \frac{4}{5}\eta U_{\xi\eta} + U_{\xi}(U_{\xi} + 1) = 0, \tag{4.14}$$

which is one dimension less than the equation (4.1). Equation (4.14) has two dimensional algebra as:

$$\mathbf{V}_1 = \eta\frac{\partial}{\partial U}, \qquad\qquad \mathbf{V}_2 = \frac{\partial}{\partial U}.$$

Using these one can easily obtain solution of Zabolotskaya-Khokhlov equation (4.1).

(ix) For combination of symmetry generators $\mathbf{V}_4, \mathbf{V}_5$ and $\mathbf{V}_6$, we can write their combination as

$$\mathbf{V} = \frac{\partial}{\partial t} + (x + y + t)\frac{\partial}{\partial x} + (2t + y)\frac{\partial}{\partial y} - \frac{\partial}{\partial u},$$

we can also write it as

$$\frac{dt}{1} = \frac{dx}{(x + y + t)} = \frac{dy}{(2t + y)} = \frac{du}{-1},$$

from here we get

$$t = e^{\rho}, \quad x = e^{\rho}(\rho^2 + \rho\xi + \rho + \eta), \quad y = e^{\rho}(2\rho + \xi), \quad u = U(\xi,\eta).$$

substituting in equation (4.1), we get

$$U_{\eta\eta}(\xi + \eta + U + 1) + U_{\xi\eta}(\xi + 2) + U_{\xi\xi} + U_{\eta}^2 + U_{\eta} = 0, \tag{4.15}$$

which is one dimension less than the equation (4.1). Equation (4.15) has three dimensional Lie algebra as:

$$\mathbf{V}_1 = -2\frac{\partial}{\partial\eta} + \frac{\partial}{\partial U}, \qquad \mathbf{V}_2 = 2\frac{\partial}{\partial\xi} + x\frac{\partial}{\partial\eta}, \qquad \mathbf{V}_3 = (\frac{\xi}{2}+1)\frac{\partial}{\partial\xi} + \eta\frac{\partial}{\partial\eta} + (\frac{\xi}{4}+U)\frac{\partial}{\partial U}.$$

Using these we obtain the following solution of Zabolotskaya-Khokhlov equation (4.1) as:

$$u(t,x,y) = \frac{-1}{2}[\frac{x}{t} - (lnt)^2 + lnt(1 - lnt^2 + \frac{y}{t})] + (\frac{y}{t} - lnt^2)(\frac{1}{8} + C_1) + C_2, \quad (4.16)$$

which is undefined when $t = 0$.

(x) For combination of symmetry generators $\mathbf{V}_1, \mathbf{V}_2, \mathbf{V}_3, \mathbf{V}_4$ and $\mathbf{V}_5$, their combination can be written as

$$\mathbf{V} = \frac{\partial}{\partial t} + (1 + y + t)\frac{\partial}{\partial x} + (2t + 1)\frac{\partial}{\partial y} - \frac{\partial}{\partial u},$$

one can write it as

$$\frac{dt}{1} = \frac{dx}{(x + y + t)} = \frac{dy}{(2t + y)} = \frac{du}{-1},$$

it yields

$$t = \rho, \quad x = \frac{1}{3}(\rho^3 + \rho^2 + \rho\xi + \rho + \eta), \quad y = \rho^2 + \rho + \xi, \quad u = U(\xi,\eta) - \rho.$$

Using these in equation (4.1), we get

$$U_{\eta\eta}(\xi + U + 1) + U_{\xi\eta} + U_{\xi\xi} + U_\eta^2 = 0 \quad (4.17)$$

which is one dimension less than the equation (4.1). Equation (4.17) has four dimensional Lie algebra as:

$$\mathbf{V}_1 = \frac{\partial}{\partial\eta}, \qquad \mathbf{V}_2 = -\frac{\partial}{\partial\xi} + \frac{\partial}{\partial U}, \qquad \mathbf{V}_3 = -\xi\frac{\partial}{\partial\xi} - \eta\frac{\partial}{\partial\eta} + \xi\frac{\partial}{\partial U},$$

$$\mathbf{V}_4 = (\xi + \frac{5}{4})\frac{\partial}{\partial\xi} + (-\frac{\xi}{4} + \frac{3\eta}{2})\frac{\partial}{\partial\eta} + U\frac{\partial}{\partial U}.$$

Using these symmetry generators we obtain the solution of Zabolotskaya-Khokhlov equation as

$$u(t,x,y) = y - t^2 - 1 + C_2 e^{C_1(x + \frac{2t^3}{3} + x^2 - t(1+y))}, \quad (4.18)$$

which is defined for every value of t, x and y.

4.6 Conclusion

In this chapter we found solutions of nonlinear Zabolotsaya-Khokhlov equation (4.1). Nonlinear Zabolotsaya-Khokholov equation (4.1) involves three independent variables, one temporal variable t and two spatial variables x and y, while one dependent variable u. We find ten sub algebras which are included in optimal system of equation (4.1). As a result of these optimal algebras we get reduced partial differential equations. These reduced partial differential equations are one dimension less than the original equation (Zabolotskaya-Khokhlov equation (4.1)). Then we use symmetry generators of each reduced equation to find their solutions. Then we use transformations of sub algebras, included in optimal systems to get the solutions of Zabolotskaya-Khokhlov equation. Since we know that the reduced partial differential equations are obtained from the transformations, which are calculated from optimal system. Therefore, either by using any Lie algebra of equation (4.1) or combination of these Lie algebras, one can reduce the one dimension of Zabolotskaya-Khokhlov equation (4.1). For future work we can extend this work of classification of seven dimensional Lie algebras according to the fundamental work done by J. Patera and P. Winternitz [20, 21, 26] in a generalized way for three and four dimensional Lie algebras.

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