Existence and Stability Results for Generalized Fractional Differential Equations

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by

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MS THESIS WORK

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بِسْمِ اللّٰهِ الرَّحْمٰنِ الرَّحِیْمِ

Dedicated to my beloved parents.

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Too erroneus is human, In spite of best efforts to prepare the script of this thesis, it is quite possible that there may be some mistakes left. The author would appreciate the valuable feedback.

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Abstract

Most of the phenomenon in nature are generally nonlinear. Therefore, many mathematicians are always in pursuit of finding methods to solve nonlinear real world problems. One of the most elegant ways to solve a nonlinear fractional differential equation is to formulate it into the corresponding Volterra integral then solve it. But it is very applicative to comprehend whether there is a solution to a given differential equation beforehand, otherwise all the attempts to find a numerical or analytic solution will become valueless. We proved some equivalence results for the nonlinear BVP involving generalized Katugampola derivatives and coupled system of fractional differential equations involving generalized derivative operator. We proved the uniqueness of solutions using suitable fixed point theorems and discussed the stability of solutions by showing continuous dependence onto given parameters with suitable examples. The thesis presents many useful results and inequalities for generalized fractional operators and used them to calculate the estimated difference of solutions of two fractional differential equations. The generalization of Riesz fractional operators is introduced. Many properties for the generalized fractional operators are formulated and existence of solutions of generalized of fractional differential equations with Riesz derivative in certain spaces are discussed. The results presented in the dissertation can be viewed as a refinement and generalization of existence theory for fractional differential equations with R-L, Caputo , Katugampola and classical Riez derivative. The corresponding existence results for fractional differential equations involving said operators can be derived taking into account the special cases.

Contents

Chapter 1

Introduction

The development of differential equations is specifically a eloquent part in development of mathematics. The differential equations appear while modeling of many physical phenomenon. For example rate of change involves while discussing the motion of fluids, Flow of current in electrical circuits, Dissipation of heat in solid objects, Population dynamics models, Motion of mechanical systems, Seismic wave theory etc. Meanwhile the fractional differential equations are considered as an prolongation of the concept of derivative operator from integer order to any real or complex order. The analysis of fractional differential equations has been carried out by various Scientists and Mathematicians. For instance, see [1, 2, 3, 4, 6, 8, 9, 10]. In the recent past the research on different properties of solutions to numerous fractional differential and integral equations is the key topic of applied mathematics research because modeling of many systems and processes in physics, chemistry, optimal control theory, population dynamics, fluid dynamics, fiber optics, electro dynamics, electromegnetic theory etc. all involve fractional differential and integral operators. The purpose of this dissertation is to contribute in the field of Fractional Calculus particularly in the field of existence theory. The dissertation is organized as follows: Chapter 2 presents some basic definitions and lemmas from literature. We derived many useful properties and inequalities involving the generalized Katugampola fractional operators. In chapter 3 we proved equivalence results for generalized fractional differential equations and system of fractional differential equations involving the generalized Katugampola derivatives. We discussed existence and uniqueness of solutions with suitable examples. In chapter 4 we introduced the generalized Riesz fractional operators and derived some useful results. We established some equivalence results for the boundary value problem involving the generalized Riesz derivatives and discussed the existence and uniqueness of solutions. We proved the stability of solutions for the generalized fractional differential equations by means of continuous dependence on parameters.

Chapter 2

Generalized Fractional Integrals and Derivatives

2.1 Brief Introduction

Many Physical phenomenon in nature when expressed in mathematical language result ordinary or partial differential equations, while fractional differential equations are considered as an prolongation of the concept of derivative operator from integer order to any real or complex order. Fractional differential equations usually describes the non-local effects. From the last decade there is a blistering growth in the field of fractional calculus. Owing to the vast amount of applications, many mathematicians focused their engrossment to fractional calculus.

There exists several definitions for fractional derivatives and fractional integrals, like Riemann-Liouville, Caputo, Hadamard, Riesz, Grunwald-Letnikov, Marchaud, etc. Meanwhile the well-developed theory and many more applications of the said operators, are still spotlight area of research in applied sciences. Recently, U. Katugampola generalized the above mentioned integrals and differential operators.

In this chapter we discuss the different properties of the Katugampola integral and derivative operators in detail. Inasmuch as we shall discuss the results related to existence and uniqueness of solutions related to generalized fractional operators as well. We obtain the existence and uniqueness of solutions of linear and nonlinear fractional differential equations with initial and boundary conditions using the generalized Katungapola derivative [37]. In order to enhance the flow of this work, we provide here results required for later chapters, including useful definitions, examples and theorems related to fractional derivatives and integrals. This chapter gives a brief introduction of some structural properties which are explained with suitable examples and definitions. It also gives content of this thesis along with the inspiration for writing this thesis.

2.2 A Brief Historical Exposition on Different Approaches to Fractional Derivatives and Integrals

Fractional differential calculus has been an intriguing topic over the years. Initially the development in the theory of "Fractional Calculus" started during the 19th century through discernment and intellect of great mathematicians of that era. The beginning of "Fractional Calculus" is motivated by L'Hospital. In a letter to Leibniz in 1695, L'Hospital raised the question; "What does we mean by $\frac{d^n f}{dx^n}$, if n = 1/2 ?". Later on many mathematicians tried to answer this question and various kinds of differential and integrals operators were introduced like Riemann-Liouville, Caputo, Hadamard, Erdelyi-Kober, Katungampola, Grunwal-Letnikov, Marchand, and Riesz operators etc.[1, 2, 15, 37, 39, 38, 40]. We discuss some of these operators briefly, but throughout this research work our main focus will be Katungampola fractional operators. In 1819, the French mathematicians, S.F Lacroix [41] intimates a straightforward approach to define a derivative of arbitrary order, by considering the power function, $y = x^p$ with p being a positive integer, He found that nth derivative of x^p is

$$\begin{aligned} D_x^n y = \frac{d^n y}{dx^n} &= \frac{p!}{(p-n)!} x^{p-n} \\ &= \frac{\Gamma(p+1)}{\Gamma(p-n+1)} x^{p-n}, \qquad p \geq n \end{aligned} \tag{2.1}$$

where "Γ" is Gamma function (generalization of factorial function). For example, if we take $y = x$ and $n = 1/2$ then we have,

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} = \frac{2\sqrt{x}}{\sqrt{\pi}}.$$

Lacroix suggests that; since analytic functions can be expressed in terms of convergent power series, so we can use the above definition to determine the fractional order derivatives of all such functions. Likewise applying this formula with negative values of n gives the integral of a polynomial. For example to compute the whole integral of $y = x^3$ we set $p = 3$ and $n = -1$ i.e.

$$\frac{d^{-1}(x^3)}{dx^{-1}} = \frac{\Gamma(4)}{\Gamma(5)}x^4 = \frac{1}{4}x^4.$$

Abel was the first mathematician who presented the first application of fractional derivative in 1823 [3]. Using the fractional operator he determined the solution of the integral equation related to tautochronous problem [42],

$$\sqrt{2g}T = \int_{t=0}^{x} (x-t)^{-1/2} f(t)dt,$$

where g is gravitational acceleration. In terms of fractional differential operator Abel wrote the right hand side of integral as

$$T\sqrt{2g} = \sqrt{\pi}\frac{d^{-1/2}}{dx^{-1/2}}(f(x)).$$

By applying fractional derivative operator $\frac{d^{1/2}}{dx^{1/2}}$ on both sides of above equation he computed the result as follow

$$f(x) = \frac{T}{\pi}\sqrt{\frac{2g}{x}}.$$

In 1822 J.B.J. Fourier [1] derived the following integral representation of the fractional derivative,

$$D_x^{\alpha} f = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(\xi)d\xi \int_{-\infty}^{+\infty} p^{\alpha}\cos\{p(x-\xi) + \frac{\alpha\pi}{2}\}dp,$$

where α be any positive or negative real number.

Later on Joseph Liouville gave a better way of writing Fourier's formula. In 1832 [43] Liouville starting with the famous possible exponential approach to fractional order derivatives, in the way that for non-integer values of α the form of definition would be

$$\frac{d^{\alpha}e^{kx}}{dx^{\alpha}} = k^{\alpha}e^{kx}, \qquad \alpha \in \mathbb{R} \tag{2.2}$$

Liouville supposed the series representation of $f(x)$ as

$$f(x) = \sum_{k=0}^{\infty} b_k e^{c_k x}.$$

then α-derivative of $f(x)$ would be

$$D_x^{\alpha} f(x) = \sum_{k=0}^{\infty} b_k c_k^{\alpha} x e^{c_k x}.$$

where α is any real or complex number and negative values of α represents the integral of $f(x)$. However, this formula cannot be considered as a general definition of fractional derivative because this definition works for the functions which can be written in the form of exponential series or whose exponential Fourier representation exists. For example α-derivative of trigonometric functions using above definition becomes,

$$\frac{d^{\alpha}}{dx^{\alpha}}(e^{ikx}) = k^{\alpha}\{\cos(kx) + i\sin(kx)\}\{\cos(\frac{\alpha\pi}{2}) + i\sin(\frac{\alpha\pi}{2})\}.$$

In particularly if $f(x) = \cos(x)$ then,

$$\frac{d^{\alpha}}{dx^{\alpha}}\{\cos(x)\} = \frac{e^{i(x+\frac{\alpha\pi}{2})} - e^{-i(x+\frac{\alpha\pi}{2})}}{2} = \cos(x + \frac{\alpha\pi}{2}).$$

This shows that the fractional differential operator simply shifts the phase of cosine function and in the same way sine function as well. Liouville derived another method to calculate the fractional derivatives of the explicit functions. His method was applied to the explicit function x^{-n} [1]. He used the following indefinite integral to develop the definition of α-derivative for explicit functions.

$$I = \int_0^{\infty} t^{n-1} e^{-xt} dt.$$

Substitute $xt = u$ in the above integral gives,

$$I = x^{-n} \int_0^{\infty} u^{n-1} e^{-u} du = x^{-n}\Gamma(n), \qquad \Re\mathrm{e}\ n > 0$$

or

$$x^{-n} = \frac{I}{\Gamma(k)}.$$

Applying fractional differential operator D^α with respect to x on both sides gives,

$$D^\alpha(x^{-n}) = \frac{1}{\Gamma(n)} D^\alpha\{\int_0^\infty (t^{n-1}e^{-xt})\}dt.$$

or

$$= \frac{1}{\Gamma(n)} \int_0^\infty \{\frac{\partial^\alpha}{\partial x^\alpha}(t^{n-1}e^{-xt})\}dt.$$

Using equation 2.1 we have,

$$D^\alpha(x^{-n}) = \frac{(-1)^\alpha}{\Gamma(n)} \int_0^\infty (t^{(\alpha+n)-1}e^{-xt})dt.$$

Again Substituting $xt = u$ gives,

$$\begin{aligned} D^\alpha(x^{-n}) =& \frac{(-1)^\alpha}{\Gamma(n)} x^{-(\alpha+n)} \int_0^\infty (u^{(\alpha+n)-1}e^{-u})du \\ =& (-1)^\alpha \frac{\Gamma(\alpha+n)}{\Gamma(n)} x^{-n-\alpha}. \end{aligned}$$

This definition is a lot more reliable than that of Lacroix's definition [41] in equation (2.1), because the fractional derivative of a constant function $f(x) = Cx^0$ is zero due the fact that $\Gamma(0)=\infty$. But this definition is only applicable to the functions of the type x^{-n} where n is a non-negative real number. Liouville used the above result to explore the potential theory. He suggests, since the n-th order differential equation $\frac{d^n y}{dx^n} = 0$ has the complementary solution, $y = a_o + a_1 x + a_2 x^2 + a_3 x^3 + ... + a_{n-1}x^{n-1}$. So the fractional order differential equation $\frac{d^\alpha y}{dx^\alpha} = 0, \quad \alpha \in \mathbb{R}_+$ must have a appropriate complementary solution as well. For detail see [45].

2.2.1 Local fractional derivative and integral

Mostly fractional derivatives are computed using the fractional integrals, due to this they inherit non-local effects in terms of left and the right derivative. Due to this non locality most of the fractional derivative operators do not satisfy some familiar properties of classical derivative operator. For example fractional derivative operators do not satisfy the Chain Rule, Roll's Theorem, Mean Value Theorem, and Semi-group property. Except Caputo type derivative operator most of the fractional derivative operators violates the fact that derivative of a constant function is zero. Also Caputo type operators assume the differentiability condition as well, though there are many functions which are not differentiable everywhere but they have their fractional order derivative everywhere. For example the function $f(x) = 3x^{\frac{1}{3}}$ is not differentiable at $x = 0$ but it has a fractional order derivative everywhere. So in order to overcome these deficiencies many mathematicians defined local fractional derivative operators with similar properties of ordinary differentiation. These local derivatives also named as conformable derivatives. For detail see [59, 7, 60, 61].

Definition 1. *[7] Let $\phi : [0, \infty) \to \mathbb{R}$ then for $\mu > 0$ and $\beta \in (0, 1]$, The generalized local fractional derivative is defined by,*

$$D_n^{\beta}(\phi(\mu)) = \lim_{\varepsilon \to 0} \frac{\phi(\mu e_n^{\varepsilon\mu^{-\beta}}) - \phi(\mu)}{\varepsilon},$$

where e_n^{ε} is a truncated exponential function defined as $e_n^{\varepsilon} = \sum_{j=0}^{n} \frac{\mu^j}{j!}$. If ϕ ia β-differentiable on $(0, a)$, $a > 0$, and the $\lim_{\mu \to a} D_{\infty}^{\beta}(\phi(\mu))$ exists, then derivative at point a is

$$D_{\infty}^{\beta}(\phi)(a) = \lim_{\mu \to a^+} D_{\infty}^{\beta}(\phi(\mu))$$

In particular if $n = 1$ then above definition reduces to following conformable derivative [59, 60].

$$D_1^{\beta}(\phi(\mu)) = \lim_{\varepsilon \to 0} \frac{\phi(\mu + \varepsilon\mu^{1-\beta}) - \phi(\mu)}{\varepsilon}$$

For upcoming results we use the above definition for $n = \infty$ i.e,

$$D_{\infty}^{\beta}(\phi(\mu)) = \lim_{\varepsilon \to 0} \frac{\phi(\mu e^{\varepsilon\mu^{-\beta}}) - \phi(\mu)}{\varepsilon} \tag{2.3}$$

Theorem 2.2.1. *[7] Let $\beta \in (0, 1]$ and g, h are β-differentiable functions at a point $\mu > 0$, then following properties holds true:*

(i) $D_{\infty}^{\beta}(ag + bh) = aD_{\infty}^{\beta}(g) + aD_{\infty}^{\beta}(h),$ *for all $a, b \in \mathbb{R}$.*

(ii) $D_{\infty}^{\beta}(gh) = gD_{\infty}^{\beta}(h) + hD_{\infty}^{\beta}(g).$

(iii) $D^{\beta}_{\infty}\left(\frac{g}{h}\right) = \frac{hD^{\beta}_{\infty}(g)-gD^{\beta}_{\infty}(h)}{h^2}$

(iv) $D^{\beta}_{\infty}\left(goh\right)(\mu) = g'\left(h(\mu)\right) D^{\beta}_{\infty}g(\mu),$ *provided g is differentiable at $h(\mu)$.*

(v) $D^{\beta}_{\infty}(g)(\mu) = \mu^{1-\alpha}\frac{dg(\mu)}{d\mu},$ *if g is differentiable.*

Lemma 2.2.1. *[7] Let $\beta \in (0,1]$ and $a, n\mathbb{R}$ Then following results holds:*

(i) $D^{\beta}_{\infty}(c) = 0,$ *where c is any constant.*

(ii) $D^{\beta}_{\infty}(\mu^n) = n\mu^{n-\beta}.$

(iii) $D^{\beta}_{\infty}(e^{a\mu}) = a\mu^{1-\beta}e^{a\mu}$

(iv) $D^{\beta}_{\infty}(\sin a\mu) = a\mu^{1-\beta}\cos a\mu.$

(v) $D^{\beta}_{\infty}(\cos a\mu) = -a\mu^{1-\beta}\sin a\mu.$

Theorem 2.2.2. *[7](Fractional Roll's theorem) Let $a > 0$ and $\phi : [a,b] \to \mathbb{R}$ be a continuous function on a, b and α-differentiable on (a,b) fore $\mu > 0$ and $\beta \in (0,1]$, furthermore $\phi(a) = \phi(b)$. Then there exists a point $c \in (a,b)$ such that*

$$D^{\beta}_{\infty}(\phi)(c) = 0.$$

Theorem 2.2.3. *[7](Fractional Mean value theorem) Let $a > 0$ and $\phi : [a,b] \to \mathbb{R}$ be a continuous function on a, b and α-differentiable on (a,b) fore $\mu > 0$ and $\beta \in (0,1]$, then there exists a point $c \in (a,b)$ such that*

$$D^{\beta}_{\infty}(\phi)(c) = \frac{\phi(b) = \phi(a)}{\frac{1}{\beta}b^{\beta} - \frac{1}{\beta}a^{\beta}}.$$

Due to the fact that the fractional derivative (2.3) is not defined using the fractional integral as in the case of nonlocal derivative operators. So for the operator (2.3) there may be more than one fractional integrals that are considered as inverse of (2.3). One of such fractional integrals is defined by the UN. Katugampola as follows,

Definition 2.2.2. *[7] Let $\beta \in \mathbb{R}$ and $\mu \geq a$ with $a \in \mathbb{R}_+$, if $\phi : (a, \mu] \to \mathbb{R}$ then β-fractional integral of ϕ is defined by,*

$$I_a^\beta(\phi)(\mu) = \int_a^\mu \mu^{\beta-1}\phi(\mu)d\mu.$$

The above fractional integral is a special case of the generalized fractional integral defined by Prakash Agarwal, with $r = 1, s = 0$ and $K_\alpha = \mu^{\alpha-1}$, see definition 4.2.1.

Lemma 2.2.3. *[7] Let $\phi : (a, \mu] \to \mathbb{R}$ be a continuous function with such that $I_a^\beta(\phi)$ exists, then for $\beta \in (0, 1)$, $a \in \mathbb{R}_+$ following result holds true,*

$$D_\infty^\beta \left(I_a^\beta(\phi)\right)(\mu) = \phi(\mu).$$

Applications and different properties of the conformable derivative (2.3) has been discussed by various mathematicians for detail see [60, 59, 61, 62, 63] and the references therein.

Theorem 2.2.4. *Let $\beta \in (0, 1]$ and $t \in \mathbb{R}$ then the following fractional Chebyshev differential equation with conformable derivative has at-least one solution,*

$$\begin{aligned}
&(\cos^{-1}(t))^{2(1-\beta)}(1 - t^2)\phi^{2(\beta)}(t) \\
&+ \left\{\pm(\beta - 1)\sqrt{1 - t^2}\left((\cos^{-1}(t))^{1-2\beta} - (\cos^{-1}(t))^{2(1-\beta)}\right) - 3t(\cos^{-1}(t))^{2(1-\beta)}\right\}\phi^{(\beta)}(t) \\
&+ \frac{n(n + 2)}{(\cos^{-1}(t))^{2(\beta-1)}}\phi(t) = 0.
\end{aligned}$$

Proof. Since the conformable derivative is not defined by means of fractional integral unlike the many other fractional derivative operators. So this may not have unique integral representation. Here we use informal way to show the existence of solutions for the above fractional differential equation, knowing the fact that Chebyshev polynomials when are normalized presents the solution for corresponding Chebyshev differential equation. Since the Chebyshev polynomials of second kind [64] satisfies the following recurrence relation

$$S_n(\cos\vartheta)\sin\vartheta = \sin(n + 1)\vartheta \tag{2.4}$$

let $t = \cos\vartheta$, then from the above relation we can write

$$\phi(\vartheta) = \frac{\sin(n+1)\vartheta}{\sin\vartheta} \tag{2.5}$$

Taking D_∞^β on both sides, we have

$$\phi^{(\beta)}(\vartheta) = D_\infty^\beta(\phi(\vartheta)) = D_\infty^\beta\left(\frac{\sin(n+1)\vartheta}{\sin\vartheta}\right)$$

$$\begin{aligned}
\phi^{(\beta)}(\vartheta) =& \\
&= \frac{(n+1)\vartheta^{1-\beta}\sin\vartheta\cos(n+1)\vartheta - \vartheta^{1-\beta}\sin(n+1)\vartheta\cos\vartheta}{(\sin\vartheta)^2} \\
&= \frac{\vartheta^{1-\beta}\left\{(n+1)\sin\vartheta\cos(n+1)\vartheta - \sin(n+1)\vartheta\cos\vartheta\right\}}{\sin^2\vartheta} \\
&= \frac{\vartheta^{1-\beta}\left[\frac{(n+1)}{2}\left\{\sin(n+2)\vartheta - \sin n\vartheta\right\} - \frac{1}{2}\left\{\sin(n+2)\vartheta + \sin n\vartheta\right\}\right]}{\sin^2\vartheta} \\
&= \frac{(n+1)\sin(n+2)\vartheta - (n+1)\sin n\vartheta - \sin(n+2)\vartheta - \sin n\vartheta}{2\vartheta^{\beta-1}\sin^2\vartheta} \\
&= \frac{n\sin(n+2)\vartheta - (n+2)\sin n\vartheta}{2\vartheta^{\beta-1}\sin^2\vartheta}
\end{aligned}$$

or

$$2\phi^{(\beta)}(\vartheta) = \frac{n\sin(n+2)\vartheta - (n+2)\sin n\vartheta}{\vartheta^{\beta-1}\sin^2\vartheta} \tag{2.6}$$

Again applying D_∞^β on both sides, we have

$$\begin{aligned}
2\phi^{2(\beta)}(\vartheta) = 2D_\infty^\beta(\phi^{(\beta)}(\vartheta)) \\
= \frac{n(n+2)\sin^2\vartheta\left\{\cos(n+2)\vartheta - \cos n\vartheta\right\}}{\left(\vartheta^{\beta-1}\sin^2\vartheta\right)^2} \\
- \frac{(\vartheta-1)\sin^2\vartheta\left\{n\sin(n+2)\vartheta - (n+2)\sin n\vartheta\right\}}{\vartheta\left(\vartheta^{\beta-1}\sin^2\vartheta\right)^2} \\
- \frac{2\sin\vartheta\cos\vartheta\left\{n\sin(n+2)\vartheta - (n+2)\sin n\vartheta\right\}}{\left(\vartheta^{\beta-1}\sin^2\vartheta\right)^2}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n(n+2)}{\vartheta^{2(\beta-1)}\sin^2\vartheta}\left\{-2\sin(\frac{n+2+n}{2})\vartheta\sin(\frac{n+2-n}{2})\vartheta\right\} \\
&\quad - \frac{(\beta-1)\{n\sin(n+2)\vartheta-(n+2)\sin n\vartheta\}}{\vartheta^{2(\beta-1)}\sin^2\vartheta} \\
&\quad - \frac{2\cos\vartheta}{\vartheta^{2(\beta-1)}\sin\vartheta}\left\{\frac{n\sin(n+2)\vartheta-(n+2)\sin n\vartheta}{\sin^2\vartheta}\right\} \\
&= \frac{-2n(n+2)\sin(n+1)\vartheta}{\vartheta^{2(\beta-1)}\sin\vartheta} - \frac{(\beta-1)\{n\sin(n+2)\vartheta-(n+2)\sin n\vartheta\}}{\vartheta^{2(\beta-1)}\sin^2\vartheta} \\
&\quad - \frac{2\cot\vartheta}{\vartheta^{2(\beta-1)}}\left\{\frac{n\sin(n+2)\vartheta-(n+2)\sin n\vartheta}{\sin^2\vartheta}\right\}
\end{aligned}
$$

Using the equations (2.5) and (2.8) into the above equation, we have

$$
2\phi^{2(\beta)}(\vartheta) = \frac{-2n(n+2)}{\vartheta^{2(\beta-1)}}\phi(\vartheta) - \frac{2(\beta-1)}{\vartheta^{\beta-1}}\phi^{(\beta)}(\vartheta) - \frac{4\cot\vartheta}{\vartheta^{\beta-1}}\phi^{(\beta)}(\vartheta)
$$

or

$$
\phi^{2(\beta)}(\vartheta) + \frac{2\cot\vartheta}{\vartheta^{\beta-1}}\phi^{(\beta)}(\vartheta) + \frac{(\beta-1)}{\vartheta^{\beta-1}}\phi^{(\beta)}(\vartheta) + \frac{n(n+2)}{\vartheta^{2(\beta-1)}}\phi(\vartheta) = 0 \tag{2.7}
$$

Since, $S_n(t) = \frac{\sin(n+1)\vartheta}{\sin\vartheta} = \phi(\vartheta)$ with $t = \cos\vartheta$. Then by chain rule

$$
\phi^{(\beta)}(\vartheta) = S_n^{(\beta)}(t)t^{(\beta)}
$$

That is

$$
\phi^{(\beta)}(\vartheta) = -S_n^{(\beta)}(t)\vartheta^{1-\beta}\sin\vartheta
$$

Again using the chain rule, we have

$$
\begin{aligned}
\phi^{2(\beta)}(\vartheta) &= D_\infty^\beta(-\vartheta^{1-\beta}\sin\vartheta S_n^{(\beta)}(t)) \\
&= -(1-\beta)\vartheta^{1-2\beta}\sin\vartheta S_n^{(\beta)}(t) - \vartheta^{1-\beta}\vartheta^{1-\beta}\cos\vartheta S_n^{(\beta)}(t) - \vartheta^{1-\beta}\sin\vartheta S_n^{2(\beta)}(t)t^{(\beta)} \\
&= (\beta-1)\vartheta^{1-2\beta}\sin\vartheta S_n^{(\beta)}(t) - \vartheta^{2(1-\beta)}\cos\vartheta S_n^{(\beta)}(t) + \vartheta^{1-\beta}\vartheta^{1-\beta}\sin\vartheta\sin\vartheta S_n^{2(\beta)}(t) \\
&= (\beta-1)\vartheta^{1-2\beta}\sin\vartheta S_n^{(\beta)}(t) - \vartheta^{2(1-\beta)}\cos\vartheta S_n^{(\beta)}(t) + \vartheta^{2(1-\beta)}\sin^2\vartheta S_n^{2(\beta)}(t)
\end{aligned}
$$

Now using all values in equation (2.7), we get

$$
\begin{aligned}
&\vartheta^{2(1-\beta)}\sin^2\vartheta S_n^{2(\beta)}(t) + \left((\beta-1)\vartheta^{1-2\beta}\sin\vartheta - \vartheta^{2(1-\beta)}\cos\vartheta\right)S_n^{(\beta)}(t) \\
&\quad + \frac{2\cot\vartheta}{\vartheta^{\beta-1}}\left(-S_n^{(\beta)}(t)\vartheta^{1-\beta}\sin\vartheta\right) + \frac{(\beta-1)}{\vartheta^{\beta-1}}\left(-S_n^{(\beta)}(t)\vartheta^{1-\beta}\sin\vartheta\right) + \frac{n(n+2)}{\vartheta^{2(\beta-1)}}S_n(t) = 0
\end{aligned}
$$

or

$$\vartheta^{2(1-\beta)}\sin^2\vartheta S_n^{2(\beta)}(t) + \left\{\left(\vartheta^{1-2\beta} - \vartheta^{2(1-\beta)}\right)(\beta-1)\sin\vartheta - 3\vartheta^{2(1-\beta)}\cos\vartheta\right\} S_n^{(\beta)}(t)$$
$$+ \frac{n(n+2)}{\vartheta^{2(\beta-1)}} S_n(t) = 0$$

Since $\cos^{-1}(t) = \vartheta$ and $\sin^2\vartheta = 1 - \cos^2(\vartheta) = 1 - t^2$. Therefore, the above equations takes the form

$$(\cos^{-1}(t))^{2(1-\beta)}(1-t^2)S_n^{2(\beta)}(t)$$
$$+\left\{\pm(\beta-1)\sqrt{1-t^2}\left((\cos^{-1}(t))^{1-2\beta} - (\cos^{-1}(t))^{2(1-\beta)}\right) - 3t(\cos^{-1}(t))^{2(1-\beta)}\right\} S_n^{(\beta)}(t)$$
$$+ \frac{n(n+2)}{(\cos^{-1}(t))^{2(\beta-1)}} S_n(t) = 0$$

Whinch implies Chebyshev polynomials satisfy the given differential equation, and hence normalizing these polynomials gives the solution of given fractional differential equation. likewise, conversely one can easily verify the given equation by substituting the Chebyshev polynomials into the given equation. This completes the proof. □

For $\beta = 1$ the above equation coincides with the classical Chebyshev differential equation.

2.2.2 Riemann-Liouville and Hermann Weyl Approaches

Most of the fractional differential operators are defined using fractional integrals. Riemann-Liouville's contribution made the Fractional Calculus more interesting and gave it a new direction. He got motivation from Cauchy integral formula for repeated integration and defined the fractional integral and fractional order derivative as follows:

Definition 2.2.4. *[1] If $g(\xi) \in L_1[a,b]$, and $\alpha \in \mathbb{C}$ with $\Re\mathrm{e}(\alpha) > 0$, Then $J_{a+}^{\alpha} g$, $J_{b-}^{\alpha} g$ corresponding left and right Riemann-Liouville integral of order α are defined as,*

$$(J_{a+}^{\alpha} g)(\xi) = \frac{1}{\Gamma(\alpha)} \int_a^{\xi} (\xi - \mu)^{\alpha-1} f(\mu) d\mu. \qquad -\infty \leq a < \xi < \infty \tag{2.8}$$

and

$$(J_{b-}^{\alpha} g)(\xi) = \frac{1}{\Gamma(\alpha)} \int_{\xi}^{b} (\mu - \xi)^{\alpha-1} g(\mu) d\mu, \qquad -\infty < \xi < b \leq \infty, \tag{2.9}$$

respectively. When $\alpha = 0$, the integral J_a^o is called the identity operator.

While for $a = -\infty$ and $b = \infty$ one can refer equations (2.8) and (2.9) as corresponding Weyl's left and right hand integrals. In fact for different values of a and b there is a class of fractional integrals. For $a = 0$ equation (2.8) is known as Riemann-Liouville fractional integral without complementary function. For $a = n$ equation (2.8) is the unique solution of the following initial value problem.

$$y^{(n)}(\xi) = f(\xi), \qquad y(a) = y^1(a) = y^2(a) = ... = y^{(n-1)}(a) = 0, \qquad n \in \mathbb{N}$$

Theorem 2.2.5. *[2] Let $g(\xi) \in L_1[a,b]$ and $\alpha_1, \alpha_2 \in \mathbb{R}^+ U\{0\}$. Then,*

$$J_a^{\alpha_1} J_a^{\alpha_2} g(\xi) = J_a^{\alpha_1+\alpha_2} g(\xi) = J_a^{\alpha_2} J_a^{\alpha_1} g(\xi),$$

holds almost everywhere on $[a,b]$. Moreover if $g(\xi) \in C[a,b]$, or $\alpha_1 + \alpha_2 \geq 1$ then semi group property holds everywhere on $[a,b]$.

Since differentiation is the left inverse of integration, using the said property Riemann-Liouville defined the fractional differentiation operator as follows:

Definition 2.2.5. *If $g(\xi) \in L_1[a,b]$ and let $\alpha > 0$ and $c = \lceil \alpha \rceil$, then the Riemann Liouville fractional differential operator D_a^α, is defined by,*

$$D_a^\alpha g = D^c I_a^{c-\alpha} g.$$

For positive integer value of α this definition coincides with ordinary differentiation operator, and for $\alpha = 0$, D_a^0 is called the identity operator. There is an ambiguity in the Riemann Liouville's definition for fractional derivative that fractional order derivative of constant function is not zero. Later in 1967 by using the definition of Riemann Liouville fractional integral Caputo defined the fractional differentiation operator as follows:

Definition 2.2.6. *[2]**(Caputo differential operator)***
Let $g(\xi) \in L_1[a,b]$ and $\Re\mathrm{e}(\alpha) \geq 0$. and $c = \lceil \alpha \rceil$, Then

$$\begin{aligned} {}^*D_a^\alpha g(\xi) &= I_a^{c-\alpha} D^c g(\xi) \\ &= \int_a^\xi \frac{(\xi-\tau)^{c-\alpha-1} D^c g(\tau)}{\Gamma(c-\alpha)} d\tau, \qquad a < \xi < b, \end{aligned}$$

is called the Caputo fractional differentiation operator of order α. For positive integer values of α this definition gives the ordinary derivative of $g(\xi)$.

2.2.3 Hadamard and Erdelyi-Kober Operators

Since fractional integrals and derivatives of a function can be calculated in innumerable ways, Therefore many mathematicians are in a hunt to generalize the things further. In this context in 1892, Hadamard [15] used the similar approach that was used by Riemann-Liouville and he was able to generalize the Riemann-Lioville's definition of fractional integration and derivative to some extent. He considered the following n-tuple integral to define the fractional order integration.

$$\int_a^x \frac{1}{\eta_1} d\eta_1 \int_a^{\eta_1} \frac{1}{\eta 2} d\eta_2 \int_a^{\eta 2} \frac{1}{\eta 3} d\eta_3 \ldots \int_a^{\eta_{n-1}} \frac{1}{\eta_n} f(\eta_n) d\eta_n = \frac{1}{(n-1)!} \int_a^x (\log \frac{x}{\eta})^{n-1} \frac{f(\eta)}{\eta} d\eta.$$

Where n is positive integer.

Definition 2.2.7. *[15, 46] (**Hadamard fractional integral operator**) Let $g(\mu) : [a, b] \to \Re$ be an integrable function and $\Re(\alpha) > 0$, then the corresponding left and the right integral are defined by,*

$$(J^{\alpha}_{a+} g)(\mu) = \frac{1}{\Gamma(\alpha)} \int_a^{\mu} (\log \frac{\mu}{\eta})^{\alpha-1} \frac{g(\eta)}{\eta} d\eta, \qquad \mu > a,\ 1 \leq \eta < \infty, \tag{2.10}$$

and

$$(J^{\alpha}_{b-} g)(\mu) = \frac{1}{\Gamma(\alpha)} \int_{\mu}^{b} (\log \frac{\eta}{\mu})^{\alpha-1} \frac{g(\eta)}{\eta} d\eta, \qquad \mu < b,\ 1 \leq \eta < \infty, \tag{2.11}$$

Definition 2.2.8. *[15](**Hadamard derivative operators**) If $g(\mu) \in L_1[a, b]$, let $\Re(\alpha) > 0$ and $c = \lceil \alpha \rceil$, then for $\mu \in (0, \infty)$, Hadamard left and right derivative are defined by,*

$$(D^{\alpha}_{a+} g)(\mu) = \frac{1}{\Gamma(c-\alpha)} (\mu \frac{d}{d\mu})^c \int_a^{\mu} \frac{1}{\eta} (\log \frac{\mu}{\eta})^{c-\alpha+1} g(\eta) d\eta, \qquad \mu > \eta, \tag{2.12}$$

and

$$(D^{\alpha}_{b-} g)(\mu) = \frac{1}{\Gamma(c-\alpha)} (-\mu \frac{d}{d\mu})^c \int_{\mu}^{b} \frac{1}{\eta} (\log \frac{\eta}{\mu})^{c-\alpha+1} g(\eta) d\eta, \qquad \eta < \mu, \tag{2.13}$$

In 1940 Erdelyi-Kober [15, 46] defined the fractional integral operator as follows:

Definition 2.2.9. *[15] Let $g(\xi) \in L^p[a,b]$, $\rho \in \mathbb{R}_+$, and $\Re\text{e}(\alpha) > 0$ Then the corresponding left and the right fractional integral operators are defined as,*

$$(J^{\alpha}_{a+;\rho,\tau} g)(\xi) = \frac{\rho \xi^{-\rho(\alpha+\tau)}}{\Gamma(\alpha)} \int_a^{\xi} \eta^{\rho\tau+\rho-1} (\xi^{\rho} - \eta^{\rho})^{\alpha-1} g(\eta) d\eta, \quad -\infty \leq a < \xi < \infty \tag{2.14}$$

and

$$(J^{\alpha}_{b-;\rho,\tau} g)(\xi) = \frac{\rho \xi^{\rho\tau}}{\Gamma(\alpha)} \int_{\xi}^{b} \eta^{\rho(1-\alpha-\tau)-1} (\eta^{\rho} - \xi^{\rho})^{\alpha-1} g(\eta) d\eta, \quad -\infty < \xi < b \leq \infty \tag{2.15}$$

respectively.

Correspondingly Edelyi-Kober fractional derivative operators are defined as;

Definition 2.2.10. *[15] Let $g(\xi) \in L^p[a,b]$ and $\Re\text{e}(\alpha) > 0$, also $c = \lceil \alpha \rceil$, $\rho > 0$, and $\tau \in \mathbb{C}$ then the left and right Erdelyi-Kober fractional derivative operators are defined by,*

$$(D^{\alpha}_{a+;\rho,\tau} g)(\xi) = \xi^{-\rho\tau} (\frac{1}{\rho \xi^{\rho-1}} \frac{d}{d\xi})^c \xi^{\rho(c+\tau)} (J^{c-\alpha}_{a+;\rho,\tau+\alpha} g)(\xi), \tag{2.16}$$

and

$$(D^{\alpha}_{b-;\rho,\tau} g)(\xi) = \xi^{\rho(\tau+\alpha)} (-\frac{1}{\rho \xi^{\rho-1}} \frac{d}{d\xi})^c \xi^{\rho(c-\tau-\alpha)} (J^{c-\alpha}_{b-;\rho,\tau+\alpha-c} g)(\xi). \tag{2.17}$$

Although, like Riemann-Liouville's fractional integrals and derivatives operators Erdelyi-Kober's integrals and derivative operators satisfy the useful properties likewise, which we are not discussed here but we can not name the Erdelyi-Kober operators as generalized operators because from here we couldn't reduce the exact form of Riemann-Liouville's integral. When $\tau = 0$ and $\rho = 1$ Erdelyi-Kober fractional integral looks like Riemann-Liouville's integral but differ by the factor ξ^{α} called power weight, and thereby derivative operator as well. For detail see [15, 46, 47]. Before we discuss generalized fractional integral and and derivative operators by Katugampula, we prove the following case of Fubini's theorem first, which will be used frequently in our later results. In 1965 Whittaker [14] gave the following formula for double integrals.

Lemma 2.2.11. ***The Special Case of Fubini's Theorem (Dirichlet Formula)*** *If $F(u,v)$ be the jointly continuous map and let $\alpha, \beta \in \Re_+$ Then,*

$$\int_0^x (x-u)^{\alpha-1} du \int_0^u (u-v)^{\beta-1} F(u,v) dv = \int_0^x dv \int_v^x (x-u)^{\alpha-1} (u-v)^{\beta-1} F(u,v) du. \tag{2.18}$$

Proof. Consider the double integral, $\int_0^x \int_0^u (x-u)^{\alpha-1} (u-v)^{\beta-1} F(u,v) dv du$. This iterated integral can be written in another notation i.e. $\int_0^x (x-u)^{\alpha-1} du \int_0^u (u-v)^{\beta-1} F(u,v) dv$. Now the second integral in the above iterated integral is to be computed first with respect to v treating u as constant. That is first the strip with breadth du is integrated across the v-direction. By adding the infinite number of rectangles with breadth du along v-direction forms a 3-dimensional small slender fragments along the u-direction from $v = 0$ to $v = x$. To carry out the result when we interchange the order of integration by taking the strip of breadth dv along v-direction is integrated across the u-direction. Similarly, by considering the infinite number of such small strips we need to integrate from $u = v$ to $u = x$. Hence follows the required result. □

Corollary 2.2.12. *If $F(u,v) = g(u)h(v)$, with $g(u) = 1$ then the Dirichlet formula (2.18) reduces to,*

$$\int_0^x (x-u)^{\alpha-1} du \int_0^u (u-v)^{\beta-1} h(v) dv = B(\alpha,\beta) \int_0^x (x-v)^{\alpha+\beta-1} h(v) dv. \tag{2.19}$$

Proof. From equation (2.18) we have,

$$\int_0^x (x-u)^{\alpha-1} dv \int_0^u (u-v)^{\beta-1} h(v) du = \int_0^x h(v) dv \int_v^x (x-u)^{\alpha-1} (u-v)^{\beta-1} du. \tag{2.20}$$

Now we will show that,

$$\int_v^x (x-u)^{\alpha-1} (u-v)^{\beta-1} du = B(\alpha,\beta) (x-v)^{\alpha+\beta-1}.$$

For this, let's start from definition of beta function,

$$B(\alpha,\beta) = \int_0^1 (\xi)^{\alpha-1}(1-\xi)^{\beta-1}dt.$$

Let $\xi = \frac{u-v}{x-v} \quad \Rightarrow \quad d\xi = \frac{du}{x-v}$. Then substituting in the above integral we have,

$$\begin{aligned} B(\alpha,\beta) &= \int_v^x \left(\frac{u-v}{x-v}\right)^{\alpha-1}\left(\frac{x-u}{x-v}\right)^{\beta-1}\frac{du}{x-v} \\ &= \int_v^x \frac{(u-v)^{\alpha-1}(x-u)^{\beta-1}}{d} u(x-v)^{\alpha+\beta-1} \\ &= \frac{1}{(x-v)^{\alpha+\beta-1}}\int_v^x (u-v)^{\alpha-1}(x-u)^{\beta-1}du \\ \Rightarrow (x-v)^{\alpha+\beta-1}B(\alpha,\beta) &= \int_v^x (u-v)^{\alpha-1}(x-u)^{\beta-1}du. \end{aligned}$$

Using the above result in the equation (2.20) we have the required result.

$$\int_0^x (x-u)^{\alpha-1}du\int_0^u (u-v)^{\beta-1}h(v)dv = B(\alpha,\beta)\int_0^x (x-v)^{\alpha+\beta-1}h(v)dv.$$

□

2.3 Katugampola Fractional Integrals

So far in this chapter we have discussed briefly different approaches to define fractional integrals and derivatives. There are still a lot more definitions of fractional integrals and derivative operators which we are not going to discuss in thesis work; like, Marchaud, Gateaux, Grunwald; Riesz, Dzherbashyan's and Guy Jumarie's approaches etc. Our objective is to explore the useful properties of Katungampola's operators which generalizes Riemann-Liouville, Hadamard and somehow Erdelyi-Kober's operators as well. In 2011 Katugampola [48] come out with the generalized integral formula. He used the following n-fold integral to develop new definition for fractional order integral.

$$\int_a^x \eta_1^{\rho-1} d\eta_1 \int_a^{\eta_1} \eta_2^{\rho-1} d\eta_2 \int_a^{\eta 2} \eta_3^{\rho-1} d\eta_3 \ldots \int_a^{\eta_{n-1}} \eta_n^{\rho-1} \phi(\eta_n) d\eta_n.$$

For n being a natural number and $\rho \in \Re$ the above repeated integral can easily be computed using Fubini's theorem. For example consider the double integral first i.e,

$${}^{\rho}I_a^2 \phi(\mu) = \int_a^{\mu} \int_a^{\eta_1} \eta_1^{\rho-1} \eta_2^{\rho-1} \phi(\eta_2) d\eta_2 d\eta_1.$$

Using Fubini's theorem change in the order of integration brought in,

$$\int_a^{\mu} \eta_2^{\rho-1} \phi(\eta_2) \int_{\eta_2}^{\mu} \eta_1^{\rho-1} d\eta_1 d\eta_2 = \int_a^{\mu} \eta_2^{\rho-1} \phi(\eta_2) (\frac{\mu^{\rho}}{\rho} - \frac{{\eta_2}^{\rho}}{\rho}) d\mu_2.$$

$$= \frac{1}{\rho} \int_a^{\mu} \eta_2^{\rho-1} (\mu^{\rho} - {\eta_2}^{\rho}) \phi(\eta_2) d\eta_2. \tag{2.21}$$

Now

$$\begin{aligned} {}^{\rho}I_a^3 \phi(\mu) &= \int_a^{\mu} \int_a^{\eta_1} \int_a^{\eta_2} \eta_1^{\rho-1} \eta_2^{\rho-1} \eta_3^{\rho-1} \phi(\eta_3) d\eta_3 d\eta_2 d\eta_1 \\ &= \int_a^{\mu} \eta_1^{\rho-1} \{ \int_a^{\eta_1} \int_a^{\eta_2} \eta_2^{\rho-1} \eta_3^{\rho-1} \phi(\eta_3) d\eta_3 d\eta_2 \} d\eta_1. \end{aligned}$$

Using the result for double integral from equation (2.21) we have,

$${}^{\rho}I_a^3 \phi(\mu) = \frac{1}{\rho} \int_a^{\mu} \eta_1^{\rho-1} \int_a^{\eta_1} \eta_3^{\rho-1} (\eta_1^{\rho} - {\eta_3}^{\rho}) \phi(\eta_3) d\eta_3 d\eta_1.$$

Again by using Fubini's theorem and change in the order of integration yields;

$$
\begin{aligned}
{}^{\rho}I_a^3\phi(\mu) =& \frac{1}{\rho}\int\limits_a^{\mu} \eta_3^{\rho-1}\phi(\eta_3)\int\limits_{\eta 3}^{\mu} \eta_1^{\rho-1}(\mu_1^{\rho}-\eta_3{}^{\rho})d\eta_1 d\eta_3 \\
=& \frac{1}{\rho^2}\int\limits_a^{\mu} \eta_3^{\rho-1}\phi(\eta_3)\{\int\limits_{\eta_3}^{\mu} \rho\eta_1^{\rho-1}(\mu_1^{\rho}-\eta_3{}^{\rho})d\eta_1\}d\eta_3 \\
=& \frac{1}{2\rho^2}\int\limits_a^{\mu} \eta_3^{\rho-1}(\mu^{\rho}-\eta_3^{\rho})^2\phi(\eta_3)d\eta_3.
\end{aligned}
$$

Similarly after repeating the same steps up to $n-1$ iterations yields,

$$
\int\limits_a^{\mu}\eta_1^{\rho-1}d\eta_1\int\limits_a^{\eta_1}\eta_2^{\rho-1}d\eta_2\ldots\int\limits_a^{\eta_{n-1}}\eta_n^{\rho-1}\phi(\eta_n)d\eta_n = \frac{1}{(n-1)!\,\rho^{n-1}}\int\limits_a^{\mu}\eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{n-1}\phi(\eta)d\eta.
$$

Definition 2.3.1. *[48](**Katungampola fractional integral**)*
Let $\alpha \in \mathbb{R}_+, c \in \mathbb{R}$ and $g \in X_c^p(a,b)$ where, $X_c^p(a,b)$ is the space of Lebesgue measurable functions. Then corresponding generalized left and right-sided fractional integrals $({}^{\rho}I_{a+}^{\alpha}g)(\mu)$ and $({}^{\rho}I_{b-}^{\alpha}g)(\mu)$ of order $\alpha \in \mathbb{C}(Re(\alpha)) > 0$ are defined by

$$
\begin{aligned}
({}^{\rho}I_{a+}^{\alpha}g)(\mu) =& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\mu}\frac{\eta^{\rho-1}g(\eta)}{(\mu^{\rho}-\eta^{\rho})^{1-\alpha}}d\eta, \qquad \mu > a, \rho > 0, \\
({}^{\rho}I_{b-}^{\alpha}g)(\mu) =& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu}^{b}\frac{\eta^{\rho-1}g(\eta)}{(\eta^{\rho}-\mu^{\rho})^{1-\alpha}}d\eta, \qquad \mu < b, \rho > 0,
\end{aligned}
$$

respectively. Where $\Gamma(.)$ is a Euler's gamma function.

2.3.1 Some Properties of Katungampola Fractional Integral Operators

Theorem 2.3.1. *[48] If $\alpha \in \mathbb{R}_+ \cup \{0\}$, $1 \le p \le \infty$, and $\rho > 0, c \in \mathbb{R}$. Then, for $\rho \ge c$ with $g \in X_c^p(a,b)$ the integral operator ${}^{\rho}I_{a+}^{\alpha}g$ is bounded in $X_c^{\rho}(a,b)$*

$$
\left\|{}^{\rho}I_{a+}^{\alpha}g\right\|_{X_c^p} \le L\|g\|_{X_c^p},
$$

where

$$
L = \frac{(\rho+1)^{1-\alpha}}{b^{1-\alpha(\rho+1)}\Gamma(\alpha)}\int\limits_1^{\frac{b}{a}}\frac{(\xi^{\rho+1}-1)^{\alpha-1}}{\xi^{1-c+\alpha(\rho+1)}}d\xi, \qquad 0 < a < b < \infty.
$$

Katugampola integral operator is linear operator and different values of the parameter ρ yields different operators. In fact this operator represents a class of linear operators depending on the values of ρ. In a limiting case Katugampola integral operator reduces to Riemann-Lioville and Hadamard integral operators. The following theorem describes this relationship with Hadamard Riemann-Liouville and fractional integrals.

Theorem 2.3.2. *[5] Let* $\Re(\alpha) > 0$, *and* $\rho > 0$, *Then, for* $g \in X_c^\rho(a,b)$ *following relation holds:*

$$(1) \qquad \lim_{\rho\to 0^+} ({}^\rho I_{a+}^\alpha g)(\mu) = \frac{1}{\Gamma(\alpha)} \int\limits_a^\mu (\log \frac{\mu}{\eta})^{\alpha-1} \frac{g(\eta)}{\eta} d\eta,$$

$$(2) \qquad \lim_{\rho\to 1^+} ({}^\rho I_{a+}^\alpha g)(\mu) = \frac{1}{\Gamma(\alpha)} \int_a^\mu (\mu-\eta)^{\alpha-1} g(\eta) d\eta.$$

Theorem 2.3.3. *Let* $g(\mu) = (\mu - a)^p$, *and* $\alpha > 0$. *Then for some* $p \geq 0$, *the Katugampola fractional integral of* $g(\mu)$ *is*

$$({}^\rho I_{a+}^\alpha g)(\mu) = \frac{\Gamma(\frac{p}{\rho}+1)\rho^{-\alpha}}{\Gamma(\frac{p}{\rho}+\alpha+1)} \mu^{\alpha(\rho-1)} (\mu-a)^{p+\alpha}.$$

Before we prove this theorem, let's first discuss the following property of Beta integral which we will use in the proof.

Lemma 2.3.2. *Let* α, β, *and* ρ *be the positive real numbers. Then the Beta function satisfies the following equality.*

$$B(\alpha,\beta) = \frac{\rho}{(1-a_0{}^\rho)^{\alpha+\beta-1}} \int\limits_{a_0}^1 \{(\mu^\rho - a_0{}^\rho)^\rho\}^{\frac{\alpha-1}{\rho}} (1-\mu^\rho)^{\beta-1} \mu^{\rho-1} d\mu. \qquad (2.22)$$

Proof. By definition,

$$B(\alpha,\beta) = \int\limits_0^1 \{(\xi)^\rho\}^{\alpha-1} (1-\xi^\rho)^{\beta-1} \rho \xi^{\rho-1} d\xi.$$

The above form of the Beta function can easily be obtained by using the substitution $x = \xi^\rho$ in the standard definition of Beta integral. Now put $\xi^\rho = \frac{u^\rho - {a_0}^\rho}{1-a_0^\rho} \quad \Rightarrow \quad \rho\xi^{\rho-1}d\xi = \frac{\rho u^{\rho-1}du}{1-{a_0}^\rho}$. Then

$$B(\alpha,\beta) = \int_{a_0}^{1} (\frac{u^\rho - {a_0}^\rho}{1-{a_0}^\rho})^{\alpha-1} (1 - \frac{u^\rho - {a_0}^\rho}{1-{a_0}^\rho})^{\beta-1} \frac{\rho u^{\rho-1}du}{1-{a_0}^\rho} d\xi$$

$$= \frac{\rho}{(1-{a_0}^\rho)^{\alpha+\beta-1}} \int_{a_0}^{1} \{(u^\rho - {a_0}^\rho)^\rho\}^{\frac{\alpha-1}{\rho}} (1-u^\rho)^{\beta-1} u^{\rho-1} du.$$

or

$$\frac{(1-{a_0}^\rho)^{\alpha+\beta-1} B(\alpha,\beta)}{\rho} = \int_{a_0}^{1} \{(u^\rho - {a_0}^\rho)\}^{\alpha-1} (1-u^\rho)^{\beta-1} u^{\rho-1} du. \tag{2.23}$$

□

Now we prove the theorem (2.3.3).

Proof. Let $g(\mu) = (\mu - a)^p$, then by definition (2.3.1) we have,

$$ {}^\rho I^\alpha_{a+}(\mu - a)^p = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^\mu \frac{\eta^{\rho-1}(\eta - a)^p}{(\mu^\rho - \eta^\rho)^{1-\alpha}} d\eta$$

$$= \frac{\rho^{1-\alpha}\mu^{\alpha\rho-\rho}}{\Gamma(\alpha)} \int_a^\mu \eta^{\rho-1}(\eta - a)^p((1 - (\frac{\eta}{\mu})^\rho)^{\alpha-1} d\eta.$$

Let $u = \frac{\eta}{\mu} \quad \Rightarrow \quad d\eta = \mu du$. Then above integral becomes,

$$ {}^\rho I^\alpha_{a+}(\mu - a)^p = \frac{\rho^{1-\alpha}\mu^{\alpha\rho-\rho}}{\Gamma(\alpha)} \int_{\frac{a}{\mu}}^{1} (\mu u)^{\rho-1}(\mu u - a)^p (1-u^\rho)^{\alpha-1} \mu du$$

$$= \frac{\rho^{1-\alpha}\mu^{\alpha\rho+p}}{\Gamma(\alpha)} \int_{\frac{a}{\mu}}^{1} u^{\rho-1}(u - \frac{a}{\mu})^p (1-u^\rho)^{\alpha-1} du$$

$$= \frac{\rho^{-\alpha}\mu^{\alpha\rho+p}}{\Gamma(\alpha)} \int_{\frac{a}{\mu}}^{1} \{(u - \frac{a}{\mu})^\rho\}^{\frac{p}{\rho}+1-1} (1-u^\rho)^{\alpha-1} \rho u^{\rho-1} du.$$

Using the Lemma 2.3.2 we have,

$$\begin{aligned}
{}^{\rho}I^{\alpha}_{a+}(\mu-a)^p =& \frac{\rho^{-\alpha}\mu^{\alpha\rho+p}}{\Gamma(\alpha)}[(1-\frac{a}{\mu})^{p+\alpha}B(\frac{p}{\rho}+1,\alpha)] \\
=& \frac{\rho^{-\alpha}\mu^{\alpha\rho+p}}{\Gamma(\alpha)}\frac{(\mu-a)^{p+\alpha}}{\mu^{p+\alpha}}\frac{\Gamma(\frac{p}{\rho}+1)\Gamma(\alpha)}{\Gamma(\frac{p}{\rho}+\alpha+1)} \\
=& \frac{\Gamma(\frac{p}{\rho}+1)}{\Gamma(\frac{p}{\rho}+\alpha+1)}\rho^{-\alpha}\mu^{\alpha(\rho-1)}(\mu-a)^{p+\alpha} \qquad (2.24)
\end{aligned}$$

□

Corollary 2.3.3. *When $a=0$ then Theorem 2.3.3 yields,*

$$ {}^{\rho}I^{\alpha}_{0+}\mu^p = \frac{\Gamma(\frac{p}{\rho}+1)}{\Gamma(\frac{p}{\rho}+\alpha+1)}\rho^{-\alpha}\mu^{p+\alpha\rho}.$$

Lemma 2.3.4. *[48](**Semigroup Property**) Let $\alpha,\beta>0, 1\leq p\leq\infty$, and $\rho>0, c\in\mathbb{R}$. Then, for $\rho\geq c$ with $g\in X^{\rho}_{c}(a,b)$ where $0<a<b<\infty$. following relation holds:*

$$({}^{\rho}I^{\alpha}_{a+}\ {}^{\rho}I^{\beta}_{a+}g)(\mu) = ({}^{\rho}I^{\alpha+\beta}_{a+}g)(\mu)$$

Proof. Consider,

$$\begin{aligned}
&({}^{\rho}I^{\alpha}_{a+}\ {}^{\rho}I^{\beta}_{a+}g)(\mu) \\
&= {}^{\rho}I^{\alpha}_{a+}\ {}^{\rho}\left\{\frac{\rho^{1-\beta}}{\Gamma(\beta)}\int_a^{\mu_1}\frac{{\eta_1}^{\rho-1}g(\eta_1)}{({\mu_1}^{\rho}-{\eta_1}^{\rho})^{1-\beta}}d\eta_1\right\}(\mu) \\
&= \frac{\rho^{1-\alpha}\rho^{1-\beta}}{\Gamma(\alpha)\Gamma(\beta)}\int_a^{\mu}\frac{{\eta_2}^{\rho-1}}{(\mu^{\rho}-{\eta_2}^{\rho})^{1-\alpha}}\int_a^{\mu_1}\frac{{\eta_1}^{\rho-1}g(\eta_1)}{({\mu_1}^{\rho}-{\eta_1}^{\rho})^{1-\beta}}d\eta_1 d\eta_2 \\
&= \frac{1}{\rho^{\alpha+\beta-2}\Gamma(\alpha)\Gamma(\beta)}\int_a^{\mu}{\eta_2}^{\rho-1}(\mu^{\rho}-{\eta_2}^{\rho})^{\alpha-1}\int_a^{\mu_1}{\eta_1}^{\rho-1}({\mu_1}^{\rho}-{\eta_1}^{\rho})^{\beta-1}g(\eta_1)d\eta_1 d\eta_2.
\end{aligned}$$

Using Fubini's theorem changing the order of above double integral gives,

$$\begin{aligned}
&({}^{\rho}I^{\alpha}_{a+}\ {}^{\rho}I^{\beta}_{a+}g)(\mu) \\
&= \frac{1}{\rho^{\alpha+\beta-2}\Gamma(\alpha)\Gamma(\beta)}\int_a^{\mu}{\eta_1}^{\rho-1}g(\eta_1)\int_{\mu_1}^{\mu}{\eta_2}^{\rho-1}({\mu_1}^{\rho}-{\eta_1}^{\rho})^{\beta-1}(\mu^{\rho}-{\eta_2}^{\rho})^{\alpha-1}d\eta_2 d\eta_1.
\end{aligned}$$

Using the Lemma 2.3.2 substitute the value of second integral we get,

$$\begin{aligned}
({}^{\rho}I^{\alpha}_{a+}\ {}^{\rho}I^{\beta}_{a+}g)(\mu) =& \frac{1}{\rho^{\alpha+\beta-2}\Gamma(\alpha)\Gamma(\beta)}\int_a^{\mu}{\eta_1}^{\rho-1}g(\eta_1)\left\{\frac{(\mu^{\rho}-{\eta_1}^{\rho})^{\alpha+\beta-1}}{\rho}B(\alpha,\beta)\right\}d\eta_1 \\
=& \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)}\int_a^{\mu}{\eta_1}^{\rho-1}(\mu^{\rho}-{\eta_1}^{\rho})^{(\alpha+\beta)-1}g(\eta_1)d\eta_1 = ({}^{\rho}I^{\alpha+\beta}_{a+}g)(\mu)
\end{aligned}$$

□

Theorem 2.3.4. *Let $\alpha > 0$, and $\{\phi_j\}_{j=1}^{\infty}$ be a uniformly convergent sequence of continuous functions on $[a, b]$. Then we can interchange the Katungampola fractional integral operator and the limit. i.e,*

$$({}^{\rho}I_{a+}^{\alpha} \lim_{j\to 0} \phi_j)(\mu) = (\lim_{j\to\infty} {}^{\rho}I_{a+}^{\alpha}\phi_j)(\mu)$$

Proof. Let $\phi(\mu)$ be the limit of the sequence $\{\phi_j\}$. Since $\{\phi_j\}$ is the convergent sequence of continuous functions so ϕ is also continuous. To prove that under the given conditions we can interchange fractional integral and limit, it is enough to show that the sequence $\{{}^{\rho}I_{a+}^{\alpha}\phi_j\}_{j=1}^{\infty}$ is also uniformly convergent. i.e. $\left|{}^{\rho}I_{a+}^{\alpha}\phi_j(\mu) - {}^{\rho}I_{a+}^{\alpha}\phi(\mu)\right| \to 0$ as $j \to \infty$. For this consider,

$$\begin{aligned}
\left|{}^{\rho}I_{a+}^{\alpha}\phi_j(\mu) - {}^{\rho}I_{a+}^{\alpha}\phi(\mu)\right| &= \left|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\mu} \frac{\phi_j(\eta)\eta^{\rho-1}}{(\mu^{\rho}-\eta^{\rho})^{1-\alpha}}d\eta - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\mu} \frac{\phi_j(\eta)\eta^{\rho-1}}{(\mu^{\rho}-\eta^{\rho})^{1-\alpha}}d\eta\right| \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\mu} \left|\frac{(\phi_j(\eta)-\phi(\eta))\eta^{\rho-1}}{(\mu^{\rho}-\eta^{\rho})^{1-\alpha}}\right| d\eta \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\|\phi_j - \phi\|_{\infty} \int_a^{\eta} \eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{\alpha-1}d\eta. \qquad (2.25)
\end{aligned}$$

Now first we will evaluate the integral,

$$\int_a^{\eta} \eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{\alpha-1}d\eta = \mu^{\alpha\rho-\rho}\int_a^{\eta} \eta^{\rho-1}(1-\frac{\eta^{\rho}}{\mu^{\rho}})^{\alpha-1}d\eta.$$

Substituting $\frac{\eta^{\rho}}{\mu^{\rho}} = u$, we have

$$\int_a^{\eta} \eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{\alpha-1}d\eta = \frac{\mu^{\alpha\rho}}{\rho}\int_{\frac{\eta^{\rho}}{\mu^{\rho}}}^{1} u^{0}(1-u)^{\alpha-1}du.$$

or

$$\int_a^{\eta} \eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{\alpha-1}d\eta = \frac{\mu^{\alpha\rho}}{\rho}\int_{\frac{\eta^{\rho}}{\mu^{\rho}}}^{1} (u-\frac{\eta^{\rho}}{\mu^{\rho}})^{1-1}(1-u)^{\alpha-1}du.$$

By comparing with the following result,

$$\int_{\xi_1}^{\xi_2} (u-\xi_1)^{\alpha-1}(\xi_2-u)^{\beta-1}du = (\xi_2-\xi_1)^{\alpha+\beta-1}B(\alpha,\beta).$$

we have,

$$\begin{aligned}\int_a^{\eta} \eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1}d\eta =& \frac{\mu^{\alpha\rho}}{\rho}\left\{\frac{(\mu^\rho-a^\rho)^\alpha B(1,\alpha)}{\mu^{\alpha\rho}}\right\}\\ =& \frac{(\mu^\rho-a^\rho)^\alpha B(1,\alpha)}{\rho}\\ =& \frac{(\mu^\rho-a^\rho)^\alpha}{\alpha\rho}.\end{aligned}$$

Using this result in equation (2.25) we get,

$$\left|{}^\rho I^\alpha_{a+}\phi_j(\mu) - {}^\rho I^\alpha_{a+}\phi(\mu)\right| \le \frac{(\mu^\rho-a^\rho)^\alpha}{\rho\Gamma(\alpha+1)}\|\phi_j-\phi\|_\infty.$$

Since $\{\phi_j\}$ is uniformly convergence sequence, Thus

$$\left|{}^\rho I^\alpha_{a+}\phi_j(\mu) - {}^\rho I^\alpha_{a+}\phi(\mu)\right| \to 0. \quad as \quad j\to\infty.$$

Therefore, the sequence $({}^\rho I^\alpha_{a+}\phi_j)_{j=1}^\infty$ is also uniformly convergence and hence the result follows. □

Lemma 2.3.5. *If $\phi(\mu)$ is an analytic function in $(a_0-\xi, a_0+\xi)$, where $t>0$, and let $\alpha\in\mathbb{R}_+$. Then,*

$$({}^\rho I^\alpha_{a_0}\phi)(\xi) = \sum_{j=0}^{\infty}\frac{\Gamma(\frac{j}{\rho}+1)\xi^{\alpha(\rho-1)}(\xi-a_0)^{j+\alpha}}{j!\,\Gamma(\frac{j}{\rho}+\alpha+1)\rho^\alpha}\phi^j(a_0).$$

In particular, ${}^\rho I^\alpha_{a_0}\phi$ is also analytic.

Proof. Since $\phi(\xi)$ is an analytic function, Thus it can be written in the form of convergent power series, i.e.

$$\phi(\xi) = \sum_{j=0}^{\infty}\frac{(\xi-a_0)^j}{j!}\phi^j(a_0).$$

Using definition (2.3.1) we get,

$$({}^{\rho}I^{\alpha}_{a_0}\phi)(\xi) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int\limits_{a_0}^{\xi} \eta^{\rho-1}(\xi^{\rho} - \eta^{\rho})^{\alpha-1} \sum_{j=0}^{\infty} \frac{(\eta - a_0)^j}{j!} \phi^j(a_0) d\eta.$$

Using Theorem 2.3.4, interchange summation and integral sign we have,

$$({}^{\rho}I^{\alpha}_{a_0}\phi)(\xi) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{j=0}^{\infty} \frac{\phi^j(a_0)}{j!} \int\limits_{a_0}^{\xi} \eta^{\rho-1}(\eta - a_0)^j (\xi^{\rho} - \eta^{\rho})^{\alpha-1} d\eta.$$

Now by using Theorem 2.3.3 substitute the value of the integral on right gives,

$$\begin{aligned}({}^{\rho}I^{\alpha}_{a_0}\phi)(\xi) =& \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sum_{j=0}^{\infty} \frac{\phi^j(a_0)\xi^{\alpha(\rho-1)}(\xi - a_0)^{j+\alpha}}{\rho j!} B(\frac{j}{\rho} + 1, \alpha) \\ =& \sum_{j=0}^{\infty} \frac{\Gamma(\frac{j}{\rho} + 1)\xi^{\alpha(\rho-1)}(\xi - a_0)^{j+\alpha}}{j!\,\Gamma(\frac{j}{\rho} + \alpha + 1)\rho^{\alpha}} \phi^j(a_0).\end{aligned}$$

□

Theorem 2.3.5. *Let $\phi(\mu) \in X^p_c(a, b)$ and if $\{\lambda_j\}_{j=1}^{\infty}$ be a convergent sequence of non-negative real numbers with limit λ. Then,*

$$\lim_{j\to 0}(\ {}^{\rho}I^{\lambda_j}_{a+}\phi)(\mu) = ({}^{\rho}I^{\lambda}_{a+}\phi)(\mu).$$

Where convergence of the sequence $\{{}^{\rho}I^{\lambda_j}_{a+}\phi\}_{j=1}^{\infty}$ is signified in terms of $X^p_c a, b$ norm, with $1 \le p \le \infty$. $p, c \in \mathbb{R}$, $\rho > 0$ and $c \le \rho + 1$.

Proof. Let the sequence $\{\lambda_j\}_{j=1}^{\infty}$ converges to the limit λ. Then by definition,

$$({}^{\rho}I^{\lambda_j}_{a+}\phi)(\mu) = \frac{\rho^{1-\lambda_j}}{\Gamma(\lambda_j)} \int_a^{\mu} \frac{\eta^{\rho-1}\phi(\eta)}{(\mu^{\rho} - \eta^{\rho})^{1-\lambda_j}} d\eta, \quad \mu > a, \rho > 0,$$

Taking limit on both sides and by using the Theorem 2.3.4 we have,

$$\begin{aligned}\lim_{j\to 0}({}^{\rho}I^{\lambda_j}_{a+}\phi)(\mu) =& \frac{\rho^{\lim\limits_{j\to 0}(1-\lambda_j)}}{\lim\limits_{j\to 0}\{(\lambda_j - 1)!\}} \int_a^{\mu} \eta^{\rho-1}\phi(\eta) \left\{\lim_{j\to 0}(\mu^{\rho} - \eta^{\rho})^{\lambda_j - 1}\right\} d\eta \\ =& \frac{\rho^{(1-\lambda)}}{(\lambda - 1)!} \int_a^{\mu} \eta^{\rho-1}\phi(\eta)(\mu^{\rho} - \eta^{\rho})^{\lambda-1} d\eta \\ =& ({}^{\rho}I^{\lambda}_{a+}\phi)(\mu).\end{aligned}$$

□

For analytic functions the relation between Katugampola and the Riemann-Liouville's fractional integral operator is given in the following theorem.

Theorem 2.3.6. *Let $\phi(z)$ is an analytic function in a simply connected region R, which contains the origin as well, and let $\alpha, \beta, \rho \in \mathbb{C}$ with $[\Re\mathrm{e}(\alpha), \Re\mathrm{e}(\beta), \Re\mathrm{e}(\rho)] > 0$. Then the following relation holds:*

$$^{\rho}I^{\alpha}_{0+}(z^{\beta}\phi(z)) = \frac{\rho^{1-\alpha}\Gamma(2-\alpha)}{\Gamma(\alpha)} \sum_{j=0}^{\infty} \frac{(-1)^{j\rho} z^{\rho(\alpha-j-1)} (p)_{jq}}{j!} J^{\beta+\rho(j+1)}_{0^+} \phi(u-z)|_{u=z}.$$

and

$$^{\rho}I^{\alpha}_{0+}(z^{\beta}\phi(z)) = \frac{\rho^{1-\alpha}\Gamma(2-\alpha)}{\Gamma(\alpha)} \sum_{j=0}^{\infty} \frac{(-1)^{j\rho} z^{\rho(\alpha-j-1)} (p)_{jq}}{j!} J^{1-\alpha}_{0^+} z^{\rho(j+1)+\beta-1}\phi(z).$$

where $(p)_{jq}$ is the generalized Pochhammer symbol with $p = 2-\alpha-j$ and $q = \frac{\beta+\rho(j+1)-p}{j}$, and $J^{\beta+\rho(j+1)}_{0^+}$ is the Riemann-Liouville integral.

Proof. By definition,

$$\begin{aligned} ^{\rho}I^{\alpha}_{0+}(z^{\beta}\phi(z)) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \eta^{\rho-1}\eta^{\beta}(z^{\rho}-\eta^{\rho})^{\alpha-1}\phi(\eta)d\eta, \\ &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^z \eta^{\beta+\rho-1}(z^{\rho}-\eta^{\rho})^{\alpha-1}\phi(\eta)d\eta, \end{aligned}$$

Using the substitution $\eta = z - \mu$ we have,

$$\begin{aligned} ^{\rho}I^{\alpha}_{0+}(z^{\beta}\phi(z)) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^z (z-\mu)^{\beta+\rho-1}(z^{\rho}-(z-\mu)^{\rho})^{\alpha-1}\phi(z-\mu)d\mu, \\ &= \frac{\rho^{1-\alpha}z^{\alpha\rho-\rho}}{\Gamma(\alpha)} \int_0^z \left(1-\left(\frac{z-\mu}{z}\right)^{\rho}\right)^{\alpha-1} (z-\mu)^{\beta+\rho-1}\phi(z-\mu)d\mu, \end{aligned}$$

Now substituting the generalized binomial series [52] of $(1-(\frac{z-\mu}{z})^{\rho})^{\alpha-1}$ into the above expression we get,

$$
\begin{aligned}
{}^{\rho}I_{0+}^{\alpha}(z^{\beta}\phi(z)) =& \frac{\rho^{1-\alpha}z^{\alpha\rho-\rho}}{\Gamma(\alpha)}\int\limits_0^z\sum_{j=0}^{\infty}\binom{1-\alpha}{1-\alpha-j}\left(-(\frac{z-\mu}{z})^{\rho}\right)^{j}(z-\mu)^{\beta+\rho-1}\phi(z-\mu)d\mu,\\
=& \frac{\rho^{1-\alpha}z^{\alpha\rho-\rho}}{\Gamma(\alpha)}\int\limits_0^z\sum_{j=0}^{\infty}\frac{(-1)^{j\rho}\Gamma(2-\alpha)}{\Gamma(j+1)\Gamma(2-\alpha-j)}(\frac{z-\mu}{z})^{j\rho}(z-\mu)^{\beta+\rho-1}\phi(z-\mu)d\mu,
\end{aligned}
$$

Using Theorem 2.3.4 interchanging the integral and summation sign we have,

$$
{}^{\rho}I_{0+}^{\alpha}(z^{\beta}\phi(z)) = \frac{\rho^{1-\alpha}\Gamma(2-\alpha)}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{(-1)^{j\rho}z^{\rho(\alpha-j-1)}}{j!\,\Gamma(2-\alpha-j)}\int\limits_0^z(z-\mu)^{\beta+\rho+j\rho-1}\phi(z-\mu)d\mu, \tag{2.26}
$$

Since $\int\limits_0^z (z-\mu)^{\beta+\rho+j\rho-1}\phi(z-\mu)d\mu = \Gamma(\beta+\rho+j\rho)J_0^{\beta+\rho+j\rho}\phi(u-z)|_{u=z}$,
where $J_0^{\beta+\rho+j\rho}\phi(u-z)$ being a Riemann-Liouville integral operator 2.2.2. Hence the equation (2.26) becomes,

$$
\begin{aligned}
{}^{\rho}I_{0+}^{\alpha}(z^{\beta}\phi(z)) =& \frac{\rho^{1-\alpha}\Gamma(2-\alpha)}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{(-1)^{j\rho}z^{\rho(\alpha-j-1)}\Gamma(\beta+\rho+j\rho)}{j!\,\Gamma(2-\alpha-j)}J_0^{\beta+\rho+j\rho}\phi(u-z)|_{u=z},\\
=& \frac{\rho^{1-\alpha}\Gamma(2-\alpha)}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{(-1)^{j\rho}z^{\rho(\alpha-j-1)}(p)_{jq}}{j!}J_{0+}^{\beta+\rho(j+1)}\phi(u-z)|_{u=z}.
\end{aligned} \tag{2.27}
$$

Where $(p)_{jq}$ is the Pochhammer's symbol [54] defined by $\left\{(p)_{jq} = \frac{\Gamma(p+qj)}{\Gamma(p)},\ (p)_0 = 1\right\}$, with $p = 2-\alpha-j$ and $q = \frac{\beta+\rho(j+1)-p}{j}$.
For second part let us consider,

$$
{}^{\rho}I_{0+}^{\alpha}(z^{\beta}\phi(z)) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^z \eta^{\beta+\rho-1}(z^{\rho}-\eta^{\rho})^{\alpha-1}\phi(\eta)d\eta
$$

Taking limit as $\rho\to 1$ on both sides gives,

$$
\lim_{\rho\to 1}{}^{\rho}I_{0+}^{\alpha}(z^{\beta}\phi(z)) = \lim_{\rho\to 1}\left\{\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^z \eta^{\beta+\rho-1}(z^{\rho}-\eta^{\rho})^{\alpha-1}\phi(\eta)d\eta\right\}
$$

Using theorem (2.3.2) and (2.3.4) we have,

$$J_{0^+}^{\alpha}(z^{\beta}\phi(z)) = \frac{1}{\Gamma(\alpha)}\int_{0}^{z}\eta^{\beta}(z-\eta)^{\alpha-1}\phi(\eta)d\eta$$

Now substituting $\eta = z-\mu \ \Rightarrow \ d\eta = -d\mu$ gives,

$$\begin{aligned} J_{0^+}^{\alpha}(z^{\beta}\phi(z)) =& \frac{1}{\Gamma(\alpha)}\int_{0}^{z}(z-\mu)^{\beta}\mu^{\alpha-1}\phi(z-\mu)d\mu, \\ =& J_{0^+}^{\beta+1}z^{\alpha-1}\phi(u-z)|_{u=z}. \end{aligned}$$

Now by replacing α with $1-\alpha$ and β by $\beta+\rho+j\rho-1$ we get,

$$J_{0^+}^{\beta+\rho(j+1)}\phi(u-z)|_{u=z} = J_{0^+}^{1-\alpha}z^{\rho(j+1)+\beta-1}\phi(z).$$

Using the above value in equation (2.27) completes the proof. i.e,

$${}^{\rho}I_{0^+}^{\alpha}(z^{\beta}\phi(z)) = \frac{\rho^{1-\alpha}\Gamma(2-\alpha)}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{(-1)^{j\rho}z^{\rho(\alpha-j-1)}(p)_{jq}}{j!}J_{0^+}^{1-\alpha}z^{\rho(j+1)+\beta-1}\phi(z).$$

□

Corollary 2.3.6. *For $\rho = 0$ theorem 2.3.6 yields,*

$${}^{\rho}I_{0^+}^{\alpha}(z^{\beta}\phi(z)) = \frac{\Gamma(2-\alpha)}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{(-1)^{j}z^{(\alpha-j-1)}(p)_{jq}}{j!}J_{0^+}^{\beta+(j+1)}\phi(u-z)|_{u=z},$$

with $p = 2-\alpha-j$ and $q = \frac{\beta+(j+1)-p}{j}$,

Lemma 2.3.7. *[51] Let $\phi(z)$ is an analytic function in a simply connected region R, which contains the origin also, and let $\alpha,\rho \in \mathbb{C}$ with $\Re\mathrm{e}(\alpha), \Re\mathrm{e}(\rho) > 0$. Then the following relation holds:*

$${}^{\rho}I_{0^+}^{\alpha}\phi(z) = \frac{\rho^{1-\alpha}\Gamma(2-\alpha)}{\Gamma(\alpha)}\sum_{j=0}^{\infty}\frac{(-1)^{j\rho}z^{\rho(\alpha-j-1)}(p)_{jq}}{j!}J_{0^+}^{\rho(j+1)}\phi(u-z)|_{u=z}.$$

Proof. Proof simply follows from the Theorem 2.3.6 by substituting $\beta = 0$. □

Example 2.3.8. *Let $\phi(\mu) = e^{k\mu}$, where $k,\mu > 0$. Then Katugampola fractional integral ${}^{\rho}I_{0^+}^{\alpha}\phi$ is computed as follows.*

Proof. First we consider the Maclaurin series of $e^{k\mu}$ i.e. $\phi(\mu) = e^{k\mu} = \sum_{n=0}^{\infty} \frac{(k\mu)^n}{n!}$, and using the definition (2.3.1), we have

$$({}^{\rho}I_{0^+}^{\alpha}\phi)(\mu) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mu} \frac{\eta^{\rho-1} \sum_{n=0}^{\infty} \frac{(k\eta)^n}{n!}}{(\mu^{\rho} - \eta^{\rho})^{1-\alpha}} d\eta.$$

By using the Theorem 2.3.4 interchanging summation and the integral sign yields,

$$\begin{aligned}({}^{\rho}I_{0^+}^{\alpha}\phi)(\mu) =& \frac{\rho^{1-\alpha}\mu^{\alpha\rho-\rho}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\mu} \eta^{\rho-1}(k\eta)^n (1 - (\frac{\eta}{\mu})^{\rho})^{\alpha-1}, \\ =& \frac{\rho^{1-\alpha}\mu^{\alpha\rho-\rho}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^{\mu} \frac{(k\eta)^{n+\rho-1}}{k^{\rho-1}} (1 - (\frac{\eta}{\mu})^{\rho})^{\alpha-1},\end{aligned}$$

Using the substitution $u = \frac{\eta^{\rho}}{\mu^{\rho}}$ we get,

$$\begin{aligned}({}^{\rho}I_{0^+}^{\alpha}\phi)(\mu) =& \frac{\rho^{-\alpha}}{\Gamma(\alpha)} \sum_{n=0}^{\infty} \frac{(k\mu)^{\alpha\rho+n}}{n!\,k^{\alpha\rho}} \int_0^1 u^{(\frac{n+\rho}{\rho})-1}(1-u)^{\alpha-1} du, \\ =& \frac{\rho^{-\alpha}(k\mu)^{\alpha\rho}}{\Gamma(\alpha)k^{\alpha\rho}} \sum_{n=0}^{\infty} \frac{(k\mu)^n}{n!} B(\frac{n+\rho}{\rho}, \alpha), \\ =& \frac{\rho^{-\alpha}(k\mu)^{\alpha\rho}}{k^{\alpha\rho}} \sum_{n=0}^{\infty} \frac{(k\mu)^n}{\Gamma(\frac{n+\rho}{\rho} + \alpha)} \frac{\Gamma(\frac{n+\rho}{\rho})}{n!}, \\ =& \frac{\rho^{-\alpha}(\xi)^{\alpha\rho}}{k^{\alpha\rho}} \sum_{n=0}^{\infty} \frac{(\xi)^n}{\Gamma(\frac{1}{\rho}n + \alpha + 1)} \frac{\Gamma(1 + \frac{n}{\rho})}{\Gamma(1)n!}, \\ =& \frac{\rho^{-\alpha}(\xi)^{\alpha\rho}}{k^{\alpha\rho}} E_{\frac{1}{\rho},\alpha+1}^{1,\frac{1}{\rho}}(\xi).\end{aligned}$$

where "$E_{\frac{1}{\rho},\alpha+1}^{1,\frac{1}{\rho}}(\xi)$" is the generalized Mittag Leffler function [49] with $\xi = k\mu$.

Remark 1. *When $\rho = 1$ then fractional integral of $e^{k\mu}$ reduces to $\frac{\rho^{-\alpha}(\xi)^{\alpha\rho}}{k^{\alpha\rho}} E_{1,\alpha+1}^{1,1}(\xi)$ and this coincides with integer order integration of $e^{k\mu}$ when α is an integer.*

□

Lemma 2.3.9. *The Katugampola fractional integral of sin(μ) and cos(μ) are given as follows.*

$$\begin{aligned}
{}^{\rho}I^{\alpha}_{0+}\sin(\mu) =&\frac{\rho^{-\alpha}(-\mu)^{\alpha\rho}}{(-1)^{\alpha\rho+n-1}}\sum_{n=0}^{\infty}\frac{(-\mu)^{2n+1}}{\Gamma(\frac{2n+1}{\rho}+\alpha+1)}\frac{\Gamma(1+\frac{2n+1}{\rho})}{(2n+1)!},\\
=&\frac{\rho^{-\alpha}(-\mu)^{\alpha\rho}}{(-1)^{\alpha\rho+n-1}}\{E^{1,\frac{1}{\rho}}_{\frac{1}{\rho},\alpha+1}(-\mu)\}.
\end{aligned}$$

and

$$\begin{aligned}
{}^{\rho}I^{\alpha}_{0+}\cos(\mu) =&\frac{\rho^{-\alpha}(-\mu)^{\alpha\rho}}{(-1)^{\alpha\rho+n}}\sum_{n=0}^{\infty}\frac{(-\mu)^{2n}}{\Gamma(\frac{2n}{\rho}+\alpha+1)}\frac{\Gamma(\frac{2n}{\rho}+1)}{(2n)!},\\
=&\frac{\rho^{-\alpha}(-\mu)^{\alpha\rho}}{(-1)^{\alpha\rho+n}}E^{1,\frac{1}{\rho}}_{\frac{1}{\rho},\alpha+1}(-\mu).
\end{aligned}$$

respectively.

Proof. The proof is similar to that of Example 2.3.8, considering the Maclaurin series of *sine* and *cosine* one can easily prove the above results. One may prove the above results either by considering the exponential form of *sine* and *cosine* functions or by taking the Katugampola fractional integral of $e^{i\mu}$ and then compute the real and imaginary parts. But this is rather complicated because fractional order integral of *cosine* and *sine* may have imaginary coefficients too depending on value of α. When $\rho = 1$ and α is integer the results coincides with classical integrals. Applications of above operators are discussed in next chapters. □

Now we are quite familiar with fractional integration, so let's go forward to generalized Katugampola fractional order differentiation.

2.4 Katugampola Fractional Derivatives

Fractional derivatives are usually defined by means of fractional integrals, so fractional derivatives are as many fractional integrals. Katugampola [5] used the notation ${}^{\rho}D^{\alpha}_{a+}$ for fractional order differentiation. Using the left inverse property of derivatives for integer order i.e, $D^n f = D^m I^{m-n} f, m > n$, He defined the generalized derivative operator as follows.

Definition 2.4.1. *[48](**Type-**1 K**-derivative**)*
Let $\alpha \in \mathbb{C}$, with $\Re(\alpha)>0$, and $g \in X_c^p(a,b)$. Then the left derivative operator ${}^{\rho}D_{a^+}^{\alpha}$, is defined as,

$$({}^{\rho}D_{a^+}^{\alpha}g)(\mu) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\frac{d^n}{d\mu^n}\int_a^{\mu} \eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{n-\alpha-1}g(\eta)d\eta, \quad \mu > a, \rho > 0, \tag{2.28}$$

where $n = \lceil \alpha \rceil$.

When $\rho = 1$ this definition reduces to Riemann-Lioville differential operator. The Caputo modification of above operator is given as follows;

Definition 2.4.2. *[48] Let $\alpha \in \mathbb{C}$, with $\Re(\alpha)>0$, and $g \in X_c^p(a,b)$, also $n = \lceil \alpha \rceil$. Then the Caputo type left derivative operator ${}_{*}^{\rho}D_{a^+}^{\alpha}$, is defined as,*

$$({}_{*}^{\rho}D_{a^+}^{\alpha}g)(\mu) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\int_a^{\mu} \frac{\eta^{\rho-1}g^n(\eta)}{(\mu^{\rho}-\eta^{\rho})^{\alpha-n+1}}d\eta, \quad \mu > a, \rho > 0,$$

When $\rho = 1$ and α is an integer then this definition coincides with ordinary derivatives.

Theorem 2.4.1. *Let $\phi(\mu) = (\mu - a_0)^p$. Then for some $p > -1$, the type-1 generalized K-derivative of $\phi(\mu)$ is given as,*

$$({}^{\rho}D_{a_0^+}^{\alpha}\phi)(\mu) = \frac{\rho^{\alpha-n}\Gamma(\frac{p}{\rho}+1)}{\Gamma(\frac{p}{\rho}+n-\alpha+1)}\sum_{i=0}^{n}\binom{n}{i}\phi_{1,n}^{(n-i)}(\mu)\phi_{2,n}^{(i)}(\mu).$$

where $\phi_{1,n}(\mu) = \mu^{(\rho-1)(n-\alpha)}$ and $\phi_{2,n}(\mu) = (\mu - a_0)^{p+n-\alpha}$, $\rho, \alpha \in \mathbb{R}_+$ and $n = \lceil \alpha \rceil$.

Proof. From equation (2.28) we have,

$$({}^{\rho}D_{a_0^+}^{\alpha}\phi)(\mu) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}\frac{d^n}{d\mu^n}\int_{a_0}^{\mu} \eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{n-\alpha-1}(\eta-a_0)^p d\eta. \tag{2.29}$$

First we will evaluate the integral,

$$\int_{a_0}^{\mu}(\eta-a_0)^p(\mu^{\rho}-\eta^{\rho})^{n-\alpha-1}\eta^{\rho-1}d\eta = \mu^{\rho(n-\alpha-1)}\int_{a_0}^{\mu}(\eta-a_0)^p\left(1-\left(\frac{\eta}{\mu}\right)^{\rho}\right)^{n-\alpha-1}\eta^{\rho-1}d\eta.$$

Using the substitution $u = \frac{\eta}{\mu}$ we have,

$$\begin{aligned}\int_{a_0}^{\mu} (\eta - a_0)^p (\mu^\rho - \eta^\rho)^{n-\alpha-1} \eta^{\rho-1} d\eta =& \mu^{\rho(n-\alpha-1)} \int_{\frac{a_0}{\mu}}^{1} (\mu u - a_0)^p (1 - u^\rho)^{n-\alpha-1} (\mu u)^{\rho-1} \mu du, \\ =& \mu^{\rho n - \alpha\rho + p)} \int_{\frac{a_0}{\mu}}^{1} (u - \frac{a_0}{\mu})^p (1 - u^\rho)^{n-\alpha-1} (u)^{\rho-1} du.\end{aligned} \tag{2.30}$$

Using the Lemma 2.3.2 computing the integral on the right hand side we have,

$$\int_{\frac{a_0}{\mu}}^{1} (u - \frac{a_0}{\mu})^p (1 - u^\rho)^{n-\alpha-1} (u)^{\rho-1} du = \frac{(1 - \frac{a_0}{\mu})^{p+n-\alpha}}{\rho} B(\frac{p}{\rho} + 1, n - \alpha).$$

Substituting this value in equation (2.30) yields,

$$\begin{aligned}\int_{a_0}^{\mu} (\eta - a_0)^p (\mu^\rho - \eta^\rho)^{n-\alpha-1} \eta^{\rho-1} d\eta =& \mu^{\rho n - \alpha\rho + p)} \left\{ \frac{(1 - \frac{a_0}{\mu})^{p+n-\alpha}}{\rho} B(\frac{p}{\rho} + 1, n - \alpha) \right\}, \\ =& \frac{\mu^{(\rho-1)(n-\alpha)}}{\rho} \left\{ \frac{\Gamma(\frac{p}{\rho} + 1)\Gamma(n - \alpha)}{\Gamma(\frac{p}{\rho} + n - \alpha + 1)} (\mu - a_0)^{p+n-\alpha} \right\}.\end{aligned}$$

Now substituting the value of above integral into equation (2.29) gives,

$$\begin{aligned}({}^\rho D_{a_0+}^\alpha \phi)(\mu) =& \frac{\rho^{\alpha-n}\Gamma(\frac{p}{\rho} + 1)}{\Gamma(\frac{p}{\rho} + n - \alpha + 1)} \frac{d^n}{d\mu^n} \left\{ \mu^{(\rho-1)(n-\alpha)} (\mu - a_0)^{p+n-\alpha} \right\}, \\ =& \frac{\rho^{\alpha-n}\Gamma(\frac{p}{\rho} + 1)}{\Gamma(\frac{p}{\rho} + n - \alpha + 1)} \frac{d^n}{d\mu^n} \left\{ \phi_{1,n}(\mu)\phi_{2,n}(\mu) \right\},\end{aligned}$$

By applying the Leibniz rule [50] and hence completes the proof.

$$({}^\rho D_{a_0+}^\alpha \phi)(\mu) = \frac{\rho^{\alpha-n}\Gamma(\frac{p}{\rho} + 1)}{\Gamma(\frac{p}{\rho} + n - \alpha + 1)} \sum_{i=0}^{n} \binom{n}{i} \phi_{1,n}^{(n-i)}(\mu) \phi_{2,n}^{(i)}(\mu).$$

□

Corollary 2.4.3. *If $a_0 = 0$ then ${}^\rho D_0^\alpha \mu^p$ takes the form,*

$${}^\rho D_0^\alpha \mu^p = \frac{\rho^{\alpha-n}\Gamma(\frac{p}{\rho} + 1)\Gamma(p + \rho(n - \alpha) + 1)}{\Gamma(\frac{p}{\rho} + n - \alpha + 1)\Gamma(p + \rho(n - \alpha) - n + 1)} \mu^{p-n+\rho(n-\alpha)}.$$

Proof. The proof follows from the Theorem 2.4.1 by substituting $a_0 = 0$. □

Corollary 2.4.4. *For $\alpha \in (0,1)$, $p > -1$, The K-derivative of $\phi(\mu) = \mu^p$ takes the form,*

$$({}^\rho D^\alpha_{a+}\phi)(\mu) = \frac{\rho^\alpha \Gamma(\frac{p}{\rho}+1)}{\Gamma(1+\frac{p}{\rho}-\alpha)} \left(\mu^{p+\rho-\alpha\rho-1}\right).$$

Proof. The proof simply follows from the Theorem 2.4.1 by substituting $a_0 = 0$ and $n = 1$. □

U.N Katugampola [5] gave another beautiful definition of fractional derivative operator, which also generalizes the two familiar fractional derivative operators, namely the "Riemann-Liouville" and the "Hadamard" frational derivatives [2, 15] to a single form.

Definition 2.4.5. *[5]* ***Generalized Derivative Operator (Type-2 K-derivative)*** *Let $\alpha \in \mathbb{C}$, with $\Re(\alpha)>0$, and $g \in X^p_c(a,b)$. Then left and right-sided Katugampola derivative operators ${}^\rho D^\alpha_{a+}$ and ${}^\rho D^\alpha_{b-}$ are defined as,*

$$({}^\rho D^\alpha_{a+}g)(\mu) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}(\mu^{1-\rho}\frac{d}{d\mu})^n \int_a^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{n-\alpha-1} g(\eta)d\eta, \quad \mu > a, \rho > 0, \tag{2.31}$$

$$({}^\rho D^\alpha_{b-}g)(\mu) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)}(-\mu^{1-\rho}\frac{d}{d\mu})^n \int_\mu^b \eta^{\rho-1}(\eta^\rho - \mu^\rho)^{n-\alpha-1} g(\eta)d\eta, \quad \mu < b, \rho > 0, \tag{2.32}$$

The association of this generalized operator with "Riemann-Liouville" and "Hadamard" derivatives is postulated in theorem 2.4.4.

2.4.1 Some Properties of Generalized Katungampola Fractional Derivative Operators

Theorem 2.4.2. *[5]**(Linearity)*** *Let $\alpha \in \mathbb{C}$, and ρ, $\Re(\alpha) > 0$, and $\phi_1, \phi_2 \in X^p_c(a,b)$. Then,*

$${}^\rho D^\alpha_{a+}(\phi_1+\phi_2)(\mu) = {}^\rho D^\alpha_{a+}\phi(\mu) + {}^\rho D^\alpha_{a+}\phi(\mu).$$

Proof. The proof is straight forward, just follows from linearity of K-integral. $\square$

Theorem 2.4.3. *Let $\phi(\mu) = (\mu - a_0)^p$ with $p > -1$ and $\alpha \in (0,1)$, then the generalized K-derivative of $\phi(\mu)$ is,*

$$({}^\rho D^\alpha_{a_0+}\phi)(\mu)$$
$$= \frac{\rho^{\alpha-1}\Gamma(\frac{p}{\rho}+1)}{\Gamma(\frac{p}{\rho}-\alpha+2)}\left\{(\rho-\alpha\rho+\alpha-1)\mu^{\alpha-\alpha\rho-1}(\mu-a_0)^{p-\alpha+1} + (p-\alpha+1)\mu^{\alpha-\alpha\rho}(\mu-a_0)^{p-\alpha}\right\}.$$

Proof. From equation (2.31) we have,

$$({}^\rho D^\alpha_{a_0+}\phi)(\mu) = \frac{\rho^\alpha}{\Gamma(1-\alpha)}(\mu^{1-\rho}\frac{d}{d\mu})\int_{a_0}^{\mu}\eta^{\rho-1}(\mu^\rho-\eta^\rho)^{-\alpha}(\eta-a_0)^p d\eta. \tag{2.33}$$

First we will evaluate the integral,

$$\int_{a_0}^{\mu}(\eta-a_0)^p(\mu^\rho-\eta^\rho)^{-\alpha}\eta^{\rho-1}d\eta = \mu^{-\rho\alpha}\int_{a_0}^{\mu}(\eta-a_0)^p\left(1-(\frac{\eta}{\mu})^\rho\right)^{-\alpha}\eta^{\rho-1}d\eta,$$

Using the substitution $u = \frac{\eta}{\mu}$ we have,

$$\int_{a_0}^{\mu}(\eta-a_0)^p(\mu^\rho-\eta^\rho)^{-\alpha}\eta^{\rho-1}d\eta = \mu^{-\rho\alpha}\int_{\frac{a_0}{\mu}}^{1}(\mu u-a_0)^p(1-u^\rho)^{-\alpha}(\mu u)^{\rho-1}\mu du,$$
$$= \mu^{\rho-\alpha\rho+p)}\int_{\frac{a_0}{\mu}}^{1}(u-\frac{a_0}{\mu})^p(1-u^\rho)^{-\alpha}(u)^{\rho-1}du. \tag{2.34}$$

Using the lemma (2.3.2) computing the integral on the right hand side we have,

$$\int_{\frac{a_0}{\mu}}^{1}(u-\frac{a_0}{\mu})^p(1-u^\rho)^{-\alpha}(u)^{\rho-1}du = \frac{(1-\frac{a_0}{\mu})^{p-\alpha+1}}{\rho}B(\frac{p}{\rho}+1, 1-\alpha).$$

Substituting this value in equation (2.34) yields,

$$\int_{a_0}^{\mu}(\eta-a_0)^p(\mu^\rho-\eta^\rho)^{-\alpha}\eta^{\rho-1}d\eta = \mu^{\rho-\alpha\rho+p}\left\{\frac{(1-\frac{a_0}{\mu})^{p-\alpha+1}}{\rho}B(\frac{p}{\rho}+1,1-\alpha)\right\},$$
$$= \frac{\mu^{(\rho-1)(1-\alpha)}}{\rho}\left\{\frac{\Gamma(\frac{p}{\rho}+1)\Gamma(1-\alpha)}{\Gamma(\frac{p}{\rho}-\alpha+2)}(\mu-a_0)^{p-\alpha+1}\right\}.$$

Now substituting the value of above integral into equation (2.33) gives,

$$({}^\rho D^\alpha_{a_0+}\phi)(\mu) = \frac{\rho^{\alpha-1}\mu^{1-\rho}\Gamma(\frac{p}{\rho}+1)}{\Gamma(\frac{p}{\rho}-\alpha+2)}\frac{d}{d\mu}\left\{\mu^{(\rho-1)(1-\alpha)}(\mu-a_0)^{p-\alpha+1}\right\},$$
$$= \frac{\rho^{\alpha-1}\Gamma(\frac{p}{\rho}+1)}{\Gamma(\frac{p}{\rho}-\alpha+2)}\left\{(\rho-\alpha\rho+\alpha-1)\mu^{\alpha-\alpha\rho-1}(\mu-a_0)^{p-\alpha+1} + (p-\alpha+1)\mu^{\alpha-\alpha\rho}(\mu-a_0)^{p-\alpha}\right\}.$$

$\square$

Theorem 2.4.4. *[5] Let $\Re(\alpha) > 0$, and $\rho > 0$, Then, for $g \in X_c^\rho(a,b)$ following relation holds:*

$$(1).\quad \lim_{\rho\to 0^+}({}^\rho D_{a^+}^\alpha g)(\mu) = \frac{1}{\Gamma(n-\alpha)}(\mu\frac{d}{d\mu})^n \int_a^\mu (\log\frac{\mu}{\eta})^{n-\alpha-1}\frac{g(\eta)}{\eta}d\eta,$$

$$(2).\quad \lim_{\rho\to 1^+}({}^\rho D_{a^+}^\alpha g)(\mu) = \frac{1}{\Gamma(n-\alpha)}(\frac{d}{d\mu})^n \int_a^\mu (\mu-\eta)^{n-\alpha-1}g(\eta)d\eta,$$

Theorem 2.4.5. *Let $\phi(z)$ is an analytic function in a simply connected region R, which contains the origin as well, and let $\alpha, \beta, \rho \in \mathbb{C}$ with $[\Re(\alpha), \Re(\beta), \Re(\rho)] > 0$. Then the following relation holds:*

$$\begin{aligned}&{}^\rho D_{0^+}^\alpha(z^\beta\phi(z))\\ &= \frac{\rho^{1-\alpha}\Gamma(2+\alpha-n)}{\Gamma(n-\alpha)}(z^{1-\rho}\frac{d}{dz})^n\sum_{j=0}^{\infty}\frac{(-1)^{j\rho}z^{\rho(n-\alpha-j-1)}(p)_{jq}}{j!}J_{0^+}^{\beta+\rho(j+1)}\phi(u-z)|_{u=z}.\end{aligned}$$

and

$$\begin{aligned}&{}^\rho D_{0^+}^\alpha(z^\beta\phi(z))\\ &= \frac{\rho^{1-\alpha}\Gamma(2+\alpha-n)}{\Gamma(n-\alpha)}(z^{1-\rho}\frac{d}{dz})^n\sum_{j=0}^{\infty}\frac{(-1)^{j\rho}z^{\rho(n-\alpha-j-1)}(p)_{jq}}{j!}J_{0^+}^{1+\alpha-n}z^{\rho(j+1)+\beta-1}\phi(z).\end{aligned}$$

where $n = \lceil\alpha\rceil$ and $(p)_{jq}$ is the generalized Pochhamer symbol with $p = 2+\alpha-(n+j)$ and $q = \frac{\beta+\rho(j+1)-p}{j}$. $J_{0^+}^{\beta+\rho(j+1)}$, and $J_{0^+}^{1+\alpha-n}$ denotes the Riemann - Liouville integrals of order $\beta+\rho(j+1)$ and $1+\alpha-n$ respectively.

Proof. Since, ${}^\rho D_{0^+}^\alpha(z^\beta\phi(z))$ is defined as,

$${}^\rho D_{0^+}^\alpha(z^\beta\phi(z)) = (z^{1-\rho}\frac{d}{dz})^n({}^\rho I_{0^+}^{n-\alpha}z^\beta\phi(z)).$$

Now using Theorem 2.3.6 substituting the value of ${}^\rho I_{0^+}^{n-\alpha}z^\beta\phi(z)$ completes the proof. □

Lemma 2.4.6. *[40] Let $\phi(z)$ is an analytic function in a simply connected region R, which contains the origin as well, and let $\alpha, \beta, \rho \in \mathbb{C}$ with $0 < \Re(\alpha) < 1$, and*

$[\Re e(\beta), \Re e(\rho)] > 0$. *Then* ${}^{\rho}D^{\alpha}_{0+}\phi(z)$ *is defined in terms of Riemann-Liouville integral operator as follows.*

$$ {}^{\rho}D^{\alpha}_{0+}\phi(z) = \frac{\rho^{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1-\alpha)}(z^{1-\rho}\frac{d}{dz})\sum_{j=0}^{\infty}\frac{(-1)^{j\rho}z^{-\rho(\alpha+j)}(p)_{jq}}{j!}J^{\rho(j+1)}_{0+}\phi(u-z)|_{u=z}. $$

and

$$ {}^{\rho}D^{\alpha}_{0+}\phi(z) = \frac{\rho^{1-\alpha}\Gamma(1+\alpha)}{\Gamma(1-\alpha)}(z^{1-\rho}\frac{d}{dz})^{n}\sum_{j=0}^{\infty}\frac{(-1)^{j\rho}z^{-\rho(\alpha+j)}(p)_{jq}}{j!}J^{\alpha}_{0+}z^{\rho j}\phi(z). $$

where $(p)_{jq}$ *is the generalized Pochhamer symbol with* $p = 1+\alpha-j$ *and* $q = \frac{\rho(j+1)-p}{j}$, *and* J^{α}_{0+} *denotes the Riemann-Liouville integral.*

Proof. The proof simply follows from theorem 2.4.5 by putting $\beta = 0$ and $n = 1$. □

2.5 Composition of Generalized K-Derivatives and Integrals

In this section we will present some useful relations between generalized Katugampola derivatives and integral operators.

Theorem 2.5.1. *[5](**Inverse property**) Let* $\alpha \in (0,1)$, *such that and* $\rho, a > 0$ *Then for* $\phi \in X^{p}_{c}(a,b)$ *following relation holds:*

$$ ({}^{\rho}D^{\alpha}_{a^+}{}^{\rho}I^{\alpha}_{a^+})\phi(\mu) = \phi(\mu). $$

Proof. Consider,

$$ \begin{aligned} &({}^{\rho}D^{\alpha}_{a^+}{}^{\rho}I^{\alpha}_{a^+})\phi(\mu) \\ &= \frac{\rho^{\alpha}}{\Gamma(1-\alpha)}(\mu^{1-\rho}\frac{d}{d\mu})\int_a^{\mu}\eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{-\alpha}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_a^{\eta}\tau^{\rho-1}(\eta^{\rho}-\tau^{\rho})^{\alpha-1}\phi(\tau)d\tau d\eta, \\ &= \frac{\rho}{\Gamma(\alpha)\Gamma(1-\alpha)}(\mu^{1-\rho}\frac{d}{d\mu})\int_a^{\mu}\eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{-\alpha}\int_a^{\eta}\tau^{\rho-1}(\eta^{\rho}-\tau^{\rho})^{\alpha-1}\phi(\tau)d\tau d\eta. \end{aligned} \tag{2.35} $$

Using the Lemma 2.2.11 we have,

$$
\begin{aligned}
&({}^\rho D^\alpha_{a+} {}^\rho I^\rho_{a+})\phi(\mu)\\
&= \frac{\rho}{\Gamma(\alpha)\Gamma(1-\alpha)}(\mu^{1-\rho}\frac{d}{d\mu})\int_a^\mu \phi(\tau)\tau^{\rho-1}\int_\tau^\mu (\mu^\rho-\eta^\rho)^{-\alpha}(\eta^\rho-\tau^\rho)^{\alpha-1}\eta^{\rho-1}d\eta d\tau.
\end{aligned} \tag{2.36}
$$

Now consider the inner integral and using lemma (2.3.2) we get,

$$
\begin{aligned}
\int_\tau^\mu (\mu^\rho-\eta^\rho)^{-\alpha}(\eta^\rho-\tau^\rho)^{\alpha-1}\eta^{\rho-1}d\eta =& \frac{(\mu^\rho-\tau^\rho)^{-\alpha+1-\alpha-1}}{\rho}B(1-\alpha,\alpha),\\
=&\frac{B(1-\alpha,\alpha)}{\rho}.
\end{aligned}
$$

Using the value of the above integral into the equation (2.36) we have,

$$
\begin{aligned}
({}^\rho D^\alpha_{a+} {}^\rho I^\alpha_{a+})\phi(\mu) =& \frac{\rho}{\Gamma(\alpha)\Gamma(1-\alpha)}(\mu^{1-\rho}\frac{d}{d\mu})\int_a^\mu \phi(\tau)\tau^{\rho-1}\frac{B(1-\alpha,\alpha)}{\rho}d\tau,\\
=&(\mu^{1-\rho}\frac{d}{d\mu})\int_a^\mu \phi(\tau)\tau^{\rho-1}d\tau.
\end{aligned}
$$

Using the fundamental theorem of calculus we have the required result, i.e.

$$({}^\rho D^\alpha_{a+} {}^\rho I^\alpha_{a+})\phi(\mu) = \phi(\mu).$$

□

Theorem 2.5.2. *[5] Let $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha), \Re(\beta) \in (0,1)$, and $\rho, a > 0$ Then for $\phi \in X^p_c(a,b)$ following relation holds:*

$${}^\rho D^\alpha_{a+} {}^\rho I^\beta_{a+}\phi(\mu) = {}^\rho I^{\beta-\alpha}_{a+}\phi(\mu).$$

Proof. The proof is similar to the proof of theorem 2.5.1. □

Likewise the above composition rule holds for right sided K-integrals and derivative operators as well.

Theorem 2.5.3. *[30] Let $\alpha \in \mathbb{N}$, and $\rho > 0$. Then for $g(\mu) \in X^\rho_c(a,b)$, following relation holds,*

$$({}^\rho I^\alpha_{a+}\delta^\alpha_\rho g)(\mu) = g(\mu) - \sum_{i=0}^{\alpha-1}\frac{\delta^i_\rho g(a)}{i!}\left(\frac{\mu^\rho - a^\rho}{\rho}\right)^i.$$

Where $\delta^\alpha_\rho = \left(\eta^{1-\rho}\frac{d}{d\eta}\right)^\alpha$.

2.5.1 Caputo Type Modification of Generalized K-Derivative

Ricardo Almeida et.al [29] demonstrated a Caputo version of generalized Katugampola generalized derivative.

Definition 2.5.1. *[29] Let $\alpha, \rho \in \mathbb{C}$ with $\Re\text{e}(\alpha), \Re\text{e}(\rho) > 0$, and if $g(\mu) \in X_c^\rho(a,b)$, Then the left and the right sided generalized Caputo type derivative operators ${}^\rho_* D^\alpha_{a+} g$ and ${}^\rho_* D^\alpha_{b-} g$ are defined as,*

$$\begin{aligned}{}^\rho_* D^\alpha_{a+} g =& \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^\mu \frac{\eta^{\rho-1}}{(\mu^\rho - \eta^\rho)^{\alpha-n+1}} (\eta^{1-\rho}\frac{d}{d\eta})^n g(\eta) d\eta, \quad \mu > \eta, \rho > 0,\\ =& ({}^\rho I^{n-\alpha}_{a^+} \delta^\alpha_\rho g)(\mu).\end{aligned}$$

and

$$\begin{aligned}{}^\rho_* D^\alpha_{b-} g =& \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_\mu^b \frac{\eta^{\rho-1}}{(\eta^\rho - \mu^\rho)^{\alpha-n+1}} (-\mu^{1-\rho}\frac{d}{d\eta})^n g(\eta) d\eta, \quad \mu < \eta, \rho > 0,\\ =& (-1)^n ({}^\rho I^{n-\alpha}_{b^-} \delta^\alpha_\rho g)(\mu).\end{aligned}$$

where $\delta^\alpha_\rho = \left(\eta^{1-\rho}\frac{d}{d\eta}\right)^\alpha$ and $n = \lceil \alpha \rceil$.

Theorem 2.5.4. *[30] Let $\alpha, \rho \in \mathbb{R}$, such that and $\rho, a > 0$ Then for $\phi \in X_c^p(a,b)$ following relation holds:*

$$({}^\rho_* D^\alpha_{a^+} {}^\rho I^\alpha_{a^+} \phi)(\mu) = \phi(\mu).$$

Similarly the inverse property holds for right-sided integral and derivative operator as well.

Lemma 2.5.2. *Let $\phi(\mu) = \mu^\beta$, then for some $\beta > -1$, $\rho \in \mathbb{R}_+$ and $1 < \alpha < 2$, the generalized Caputo derivative of $\phi(\mu)$ is computed as follows,*

$${}^\rho_* D^\alpha_{0+} \mu^\beta = \frac{\rho^{\alpha-2}\beta(\beta-\rho)\Gamma(\frac{\beta-\rho}{\rho})}{\Gamma(\beta-\alpha\rho+\rho)} \mu^{\beta-\alpha\rho}.$$

Proof. Using the definition 2.5.1 we have,

$$\begin{aligned}{}^\rho_* D^\alpha_{0+} \mu^\beta &= \frac{\rho^{\alpha-1}}{\Gamma(2-\alpha)} \int_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{1-\alpha} (\eta^{1-\rho}\frac{d}{d\eta})^2 \eta^\beta d\eta\\ &= \frac{\rho^{\alpha-1}\beta(\beta-\rho)}{\Gamma(2-\alpha)} \int_0^\mu \eta^{\beta-\rho-1}(\mu^\rho - \eta^\rho)^{1-\alpha} d\eta\\ &= \frac{\rho^{\alpha-1}\beta(\beta-\rho)\mu^{\rho(1-\alpha)}}{\Gamma(2-\alpha)} \int_0^\mu \eta^{\beta-\rho-1}(1 - (\frac{\eta}{\mu})^\rho)^{1-\alpha} d\eta,\end{aligned}$$

Using the substitution $u = \frac{\eta}{\mu}$ we have,

$$
\begin{aligned}
{}_{*}^{\rho}D_{0+}^{\alpha}\mu^{\beta} &= \frac{\rho^{\alpha-1}\beta(\beta-\rho)\mu^{\beta-\alpha\rho}}{\Gamma(2-\alpha)}\int_0^1 u^{\beta-\rho-1}(1-(u)^{\rho})^{1-\alpha}du \\
&= \frac{\rho^{\alpha-2}\beta(\beta-\rho)\mu^{\beta-\alpha\rho}}{\Gamma(2-\alpha)}B(\frac{\beta-\rho}{\rho}, 2-\alpha) \\
&= \frac{\rho^{\alpha-2}\beta(\beta-\rho)\Gamma(\frac{\beta-\rho}{\rho})}{\Gamma(\beta-\alpha\rho+\rho)}\mu^{\beta-\alpha\rho}.
\end{aligned}
$$

□

Corollary 2.5.3. *For $0 < \alpha < 1$ the above result becomes,*

$$
{}_{*}^{\rho}D_{0+}^{\alpha}\mu^{\beta} = \frac{\rho^{\alpha-1}\Gamma(\beta/\rho+1)}{\Gamma(\beta/\rho-\alpha+1)}\mu^{\beta-\alpha\rho}.
$$

Theorem 2.5.5. *[30] Let $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha), \Re(\beta) \in (0,1)$, and $\rho, a > 0$ Then for $\phi \in X_c^p(a,b)$ following relation holds:*

$$
{}_{*}^{\rho}D_{a^+}^{\alpha}\, {}^{\rho}I_{a^+}^{\beta}\phi(\mu) = {}^{\rho}I_{a^+}^{\beta-\alpha}\phi(\mu).
$$

Theorem 2.5.6. *[30] Let $\alpha, \rho \in \mathbb{R}_+$, Then for $g(\mu) \in X_c^{\rho}(a,b)$, following relation between Generalized Katugampola derivative and Caputo type generalized derivative operators holds:*

$$
({}_{*}^{\rho}D_{a^+}^{\alpha}g)(\mu) = ({}^{\rho}D_{a^+}^{\alpha}g)(\mu) - \sum_{j=0}^{n-1}\frac{\delta_{\rho}^{j}g(a)}{\Gamma(j-\alpha+1)}\left(\frac{\mu^{\rho}-a^{\rho}}{\rho}\right)^{j-\alpha}.
$$

Where $\delta_{\rho}^{j} = \left(\eta^{1-\rho}\frac{d}{d\eta}\right)^{j}$ and $n = \lceil\alpha\rceil$.

Corollary 2.5.4. *When $0 < \alpha < 1$ the above relation takes the following form,*

$$
({}_{*}^{\rho}D_{a^+}^{\alpha}g)(\mu) = ({}^{\rho}D_{a^+}^{\alpha}g)(\mu) - \frac{\rho^{\alpha}g(a)}{\Gamma(1-\alpha)(\mu^{\rho}-a^{\rho})^{\alpha}}.
$$

Theorem 2.5.7. *[30] Let $\alpha, \rho \in \mathbb{R}_+$, and $\phi(\mu) \in AC_{\delta}^{n}[a,b]$, Then following relation holds:*

$$
\left({}^{\rho}I_{a^+}^{\alpha}\, {}_{*}^{\rho}D_{a^+}^{\alpha}g\right)(\mu) = g(\mu) - \sum_{j=0}^{n-1}\frac{\delta_{\rho}^{j}g(a)}{j!}\left(\frac{\mu^{\rho}-a^{\rho}}{\rho}\right)^{j}.
$$

Where $n = \lceil\alpha\rceil$ and $\delta_{\rho}^{j} = \left(\eta^{1-\rho}\frac{d}{d\eta}\right)^{j}$

Remark 2. *[45] Let $\alpha > 0$ and $\phi(\mu) \in C(0,1) \cap L^1(0,1)$ then the following fractional differential equation*

$$D^{\alpha}_{0+}\phi(\mu) = 0.$$

has solution of the form,

$$\phi(\mu) = c_1\mu^{\alpha-1} + c_2\mu^{\alpha-1} + c_3\mu^{\alpha-1} + \ldots + c_n\mu^{\alpha-n}.$$

Where $c_n \in \mathbb{R}$ and $n \in \mathbb{N}$.

Remark 3. *[56] Let $\alpha > 0$ and $\phi(\mu), D^{\alpha}_{0+}\phi(\mu) \in C(0,1) \cap L^1(0,1)$ then the following relation holds true.*

$$I^{\alpha}_{0+}\, D^{\alpha}_{0+}\phi(\mu) = \phi(\mu) + c_1\mu^{\alpha} + c_2\mu^{\alpha-1} + c_3\mu^{\alpha-1} + \ldots + c_n\mu^{\alpha-n}.$$

Where $c_n \in \mathbb{R}$ and $n = \lceil \alpha \rceil$, and I^{α}_{0+}, I^{α}_{0+} are Riemann Liouville's integral and derivative operator respectively.

In the same way one can easily deduce the same composition rule for Katugampola integral and derivative operator.

Lemma 2.5.5. *Let $\alpha > 0$ and $\phi(\mu), {}^{\rho}D^{\alpha}_{0+}\phi(\mu) \in C(0,1) \cap L^1(0,1)$ then the following relation holds true.*

$$\begin{aligned} {}^{\rho}I^{\alpha}_{0+}\, {}^{\rho}_{*}D^{\alpha}_{0+}\phi(\mu) =& \phi(\mu) + c_0 + c_1\frac{\mu^{\rho}}{\rho} + c_2(\frac{\mu^{\rho}}{\rho})^2 + \ldots + c_{n-1}(\frac{\mu^{\rho}}{\rho})^{n-1}, \\ =& \phi(\mu) + \sum_{j=0}^{n-1}\frac{c_j}{j!}(\frac{\mu^{\rho}}{\rho})^{n-1}. \end{aligned}$$

Where $c_j \in \mathbb{R}$ and $n = \lceil \alpha \rceil$.

Proof. The proof is straight forward. One can easily verify the above result by taking into account the Remark 2, theorem 2.5.4 and using the Theorem 2.5.7. We will frequently use this lemma in our later results. □

Now its look quite adequate to start the study of fractional differential equations which involves generalized K-derivative operator because we have discussed the necessary properties of Katugampola integro-differential operators which we shall use in the adjacent chapters. Interested reader may find the some other properties of above operators in [48, 51, 37, 30]. In the next chapters we will also discuss local Katugampola derivative operator and some of its applications.

Chapter 3

Analysis of Fractional differential equations involving Katugampola Derivative Operators

3.1 Introdution

As we know many natural Phenomenons when modeled in mathematical language, involves rates at which particular, activity or things happen and are non-linear in nature. So its useful to discuss the generalized behavior. In this chapter we consider the problems which involves, generalized Katugampola fractional derivative operators. Before we begin a solemn study on existence and uniqueness of solutions of particular fractional differential equations which involves K-derivative operators, we try to discuss basic perspectives on existence and uniqueness of solutions of fractional differential equations. Existence theory is meat-and-potatoes in every field of Science. As it is very applicative to comprehend whether there is a solution to a given differential equation beforehand, otherwise all the attempts to find a numerical or analytic solution will become valueless. The analysis of fractional differential equations has been carried out by various authors. For details, see [1, 2]. In this chapter we will study existence and uniqueness of solutions for fractional boundary value problems. There are various techniques to show the existence and uniqueness of solutions. The commonly used technique is to transform the given differential equation into intego-differential operator and then using the contractive mapping principle, Lipschitz conditions and some suitable fixed point theorems we can check the existence and uniqueness of solutions. In this chapter wee also use the same approach because many boundary value problems can be transfigured into the comparable voltera-integral operator through some so-called green functions. Zhanbing Bai et.al [57] considered the following fractional boundary value problem with Riemann-Liouville differential operator $D^{\alpha}_{0^+}$.

$$\begin{aligned} & D^{\alpha}_{0+}\phi(\mu) = g(\mu, \phi(\mu)) \qquad 0 < \mu < 1 \\ & \phi(0) = 0 = \phi(1) \end{aligned}$$

where $\alpha \in (1, 2)$.

We will discuss the same problem but with the generalized Katugampola derivative operator i.e.

$$\begin{aligned} & {}^{\rho}_{*}D^{\alpha}_{0+}\phi(\mu) = g(\mu, \phi(\mu)), \\ & \phi(0) = \phi_0, \quad \phi(1) = \phi_1 \end{aligned} \tag{3.1}$$

(H_1) $0 < \mu < 1$, $1 < \alpha \leq 2$, $\phi : [0, 1] \to \mathbb{R}$ and $g : [0, 1] \times \mathbb{R} \to \mathbb{R}$ are continuous and $|g(\mu, \phi(\mu))| \leq c + d\,|\phi(\mu)|$ where $c, d \in \mathbb{R}_+$.

And define the space $B = \{\phi \in C\,[0, 1] : \|\phi\| \leq r,\ \mu \in [0, 1]\}$, clearly B is the Banach space. The existence and uniqueness of solutions of similar two point boundary value problem is also discussed by many mathematicians with different approaches like using Laplace Transform, Mellin Transform methods and using different fixed point theorems. In this section our objective is to overview the different criterions to check the existence and uniqueness of solutions for fractional order boundary value problems which involves Caputo type generalized K-derivative operator. Numerous methods have been developed to the check the existence and uniqueness of solutions for FDE's. But all the methods and approaches are girdled by fixed point theorems. We will follow the analogous approaches used in [37, 2, 57, 55, 56] to check the existence and uniqueness results for FBVP 3.1.

By means of Schauder fixed point theorem, Xinwei Su [58] demonstrated the existence result for the following coupled system of fractional boundary value problem.

$$\begin{aligned} & D^{\alpha}_{0+}\phi(\mu) = g_1(\mu, \tilde{\phi}(\mu), D^{\alpha^*}\tilde{\phi}(\mu)) \qquad 0 < \mu < 1, \\ & D^{\beta}_{0+}\tilde{\phi}(\mu) = g_2(\mu, \phi(\mu), D^{\beta^*}\phi(\mu)) \qquad 0 < \mu < 1, \\ & \phi(0) = \phi(1) = \tilde{\phi}(0) = \tilde{\phi}(1) = 0. \end{aligned} \tag{3.2}$$

Where $\alpha, \beta \in (1,2)$, $\alpha^*, \beta^* > 0$, $\alpha-\beta^* \geq 1$, $\beta-\alpha^* \geq 1$, and $g_1, g_2 : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ are known functions and D is the Riemann-Liouville differentiation operator. The papers [57, 58, 55, 56] by various authors demonstrates the existence and uniqueness results for solutions of fractional boundary value problems. Motivated by [58] and previous results, we present the analysis on existence of solutions for the following non-linear system of fractional differential equations involving generalized K-derivative operator with general boundary conditions.

$$\begin{cases} {}^{\rho}_{*}D^{\alpha}_{0+}\phi(\mu) = g_1(\mu, \tilde{\phi}(\mu), {}^{\rho}D^{\alpha^*}\tilde{\phi}(\mu)) & 0 < \mu < 1 \\ {}^{\rho}_{*}D^{\beta}_{0+}\tilde{\phi}(\mu) = g_2(\mu, \phi(\mu), {}^{\rho}D^{\beta^*}\phi(\mu)) & 0 < \mu < 1 \\ \phi(0) = \tilde{\phi}(0) = A_0,\ \phi(1) = \tilde{\phi}(1) = A_1. \end{cases} \tag{3.3}$$

Where $\alpha, \beta \in (1,2)$, $\alpha^*, \beta^* > 0$, $\alpha - \beta^* \geq 1$, $\beta - \alpha^* \geq 1$, and $g_1, g_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous known functions and ${}^{\rho}_{*}D^{\cdot}_{0+}$ is the Caputo type generalized Katugampola differentiation operator.

3.2 Main Results

3.2.1 Existence Results for Generalized Fractional Differential Equations

Lemma 3.2.1. *Let condition (H_1) holds then the problem (3.1) is equivalent to the following Voltera integral equation of second kind. i.e fixed points of the following voltera integral operator are solutions of boundary value problem 3.1.*

$$\phi(\mu) = \rho(\phi_1 - \phi_0)\mu + \phi_0 + \int_0^1 \{G(\mu,\eta)g(\eta,\phi(\eta))\}d\eta. \tag{3.4}$$

where $G(\mu,\eta)$ is the Green's function for problem 3.1 and defined as,

$$G(\mu,\eta) = \begin{cases} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{\alpha-1} - \mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\alpha-1}\right), & if\ \ 0 \leq \eta \leq \mu. \\ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\alpha-1}\right), & if\ \ \mu \leq \eta \leq 1. \end{cases} \tag{3.5}$$

Lemma 3.2.2. *Let $\phi(\mu) \in [0,1]$ be the given function such that $M = \sup_{(\mu,\phi)\in B} |g(\mu,\phi(\mu))|$, then for $\mu, \eta \in (0,1)$, the Green's function $G(\mu,\eta)$ defined by equation (3.5) satisfies the following properties:*

1. $G(\mu,\eta) \in C([0,1]\times[0,1))$ *and* $G(\mu,\eta) \geq 0.$

2. $\int\limits_0^1 \{G(\mu,\eta)g(\eta,\phi(\eta))\}d\eta \in B.$

Proof. It is easily perceptible from the expression of $G(\mu,\eta)$ that $G(\mu,\eta) \in C([0,1]\times[0,1))$. Also since $\mu \in [0,1],\ 1 < \alpha < 2,\ \rho > 0$, therefore $(\mu^\rho - \eta^\rho)^{\alpha-1} \geq 0$ for $\eta \leq \mu$ and $(1-\eta^\rho)^{\alpha-1} \geq 0$ for $\eta \geq \mu$.

Therefore $G_1(\mu,\eta) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left\{\eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1} - \mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}\right\} \geq 0$. Also $G_2(\mu,\eta) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left\{\mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}\right\} \geq 0$. As $G(\mu,\eta) = G_1(\mu,\eta) + G_2(\mu,\eta)$, and both $G_1(\mu,\eta)$ and $G_2(\mu,\eta)$ are positive. Hence $G(\mu,\eta) \geq 0$ for $\mu,\eta \in (0,1)$.

Now consider,

$$\begin{aligned}
&\int\limits_0^1 \{G(\mu,\eta)g(\eta,(\eta))\}d\eta \\
&\qquad = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^\mu \left\{\eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1} - \mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}\right\} g(\eta,\phi(\eta))d\eta \\
&\qquad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\mu^1 \mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}g(\eta,\phi(\eta))d\eta.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\left|\int\limits_0^1 \{G(\mu,\eta)g(\eta,(\eta))\}d\eta\right| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^\mu \left|\eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1}g(\eta,\phi(\eta))\right|d\eta \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^\mu \left|\mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}g(\eta,\phi(\eta))\right|d\eta \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\mu^1 \left|\mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}g(\eta,\phi(\eta))\right|d\eta.
\end{aligned}$$

Since $M = \sup\limits_{(\mu,\phi)\in B}|g(\mu,\phi(\mu))|$. So above inequality becomes,

$$\left|\int\limits_0^1 \{G(\mu,\eta)g(\eta,(\eta))\}d\eta\right| \tag{3.6}$$

$$\leq \frac{\rho^{1-\alpha}M}{\Gamma(\alpha)}\left[\int\limits_0^\mu \left|\eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1}\right|d\eta + \int\limits_0^\mu \left|\mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}\right|d\eta + \int_\mu^1 \left|\mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}\right|d\eta\right].$$

Now consider,

$$\int_0^\mu \eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1}d\eta = \mu^{\rho(\alpha-1)}\int_0^\mu \eta^{\rho-1}\left(1-(\frac{\eta}{\mu})^\rho\right)^{\alpha-1}d\eta.$$

Using the substitution $u=\frac{\eta^\rho}{\mu^\rho}$. $\Rightarrow$ $d\eta=\frac{\mu}{\rho}u^{\frac{1}{\rho}-1}du$. we get,

$$\begin{aligned}\int_0^\mu \eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1}d\eta &= \mu^{\alpha\rho}\int_0^1 u^0(1-u^\rho)^{\alpha-1}d\eta\\ &= \mu^{\alpha\rho}\int_0^1 u^{1-1}(1-u^\rho)^{\alpha-1}u.\end{aligned}$$

By comparing with beta function [] we have,

$$\begin{aligned}\int_0^\mu \eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1}d\eta &= \frac{\mu^{\alpha\rho}}{\rho}\frac{\Gamma(1)\Gamma(\alpha)}{\Gamma(1+\alpha)}\\ &= \frac{\mu^{\alpha\rho}}{\rho\alpha}.\end{aligned}$$

Similarly,

$$\begin{aligned}\mu\int_0^\mu \eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1}d\eta &= \frac{\mu^{\alpha\rho+1}}{\rho}\int_0^1 u^{1-1}(1-u\mu^\rho)^{\alpha-1}du,\\ &= \frac{\mu^{\alpha\rho+1}}{\rho}\int_0^1 (u\mu^\rho)^{1-1}(1-u\mu^\rho)^{\alpha-1}du,\\ &= \frac{\mu^{\alpha\rho+1}}{\rho}B(1,\alpha),\\ &= \frac{\mu^{\alpha\rho+1}}{\rho\alpha}.\end{aligned}$$

Therefore, from inequality (3.6) we have,

$$\begin{aligned}\left|\int_0^1 \{G(\mu,\eta)g(\eta,(\eta))\}d\eta\right| &\le \frac{\rho^{1-\alpha}M}{\Gamma(\alpha)}\left\{\frac{\mu^{\alpha\rho}}{\rho\alpha}+\frac{\mu^{\alpha\rho+1}}{\rho\alpha}+\frac{\mu^\rho(1-\mu^\rho)^\alpha}{\rho\alpha}\right\},\\ &= \frac{M}{\rho^\alpha\Gamma(\alpha+1)}\left\{\mu^{\alpha\rho}+\mu^{\alpha\rho+1}+\mu^\rho(1-\mu^\rho)^\alpha\right\}.\end{aligned}$$

That is,

$$\left| \int_0^1 \{G(\mu,\eta)g(\eta,(\eta))\}d\eta \right| \leq \frac{MK_\mu}{\rho^\alpha \Gamma(\alpha+1)},$$

where $K_\mu = \mu^{\alpha\rho} + \mu^{\alpha\rho+1} + \mu^\rho(1-\mu^\rho)^\alpha$. Thereupon, $\int_0^1 \{G(\mu,\eta)g(\eta,(\eta))\}d\eta \in B$. □

Proof.(Lemma 3.2.1). Let $\phi \in B$ is the solution of problem (3.1), then by applying the generalized integral operator on both sides of equation (3.1) and using definition (2.3.1) and the lemma 2.5.5 yields,

$$\phi(\mu) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \frac{\eta^{\rho-1} g(\eta,\phi(\eta))}{(\mu^\rho - \eta^\rho)^{1-\alpha}} d\eta + c_0 + c_1 \frac{\mu^\rho}{\rho}. \tag{3.7}$$

Using the boundary conditions $\phi(0) = \phi_0$ and $\phi(1) = \phi_1$ into above equation we get $c_0 = \phi_0$, and $c_1 = \rho\phi_1 - \rho c_0 - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^1 \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g(\eta,\phi(\eta))d\eta$. Now substituting these values of constants into equation (3.7) we get,

$$\begin{aligned}
\phi(\mu) = & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} g(\eta,\phi(\eta))d\eta \\
& - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g(\eta,\phi(\eta))d\eta + \phi_0 + (\phi_1 - \phi_0)\mu^\rho, \\
= & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} g(\eta,\phi(\eta))d\eta \\
& - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left\{ \int_0^\mu \mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g(\eta,\phi(\eta))d\eta + \int_\mu^1 \mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g(\eta,\phi(\eta)) \right\} \\
& + \phi_0 + (\phi_1 - \phi_0)\mu^\rho, \\
= & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \left\{ \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} - \mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} \right\} g(\eta,\phi(\eta))d\eta \\
& + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^1 \mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g(\eta,\phi(\eta)) + \phi_0 + (\phi_1 - \phi_0)\mu^\rho.
\end{aligned}$$

or

$$\phi(\mu) = \rho(\phi_1 - \phi_0)\mu + \phi_0 + \int_0^1 \{G(\mu,\eta)g(\eta,\phi(\eta))\}d\eta.$$

Where,

$$G(\mu,\eta)=\begin{cases}\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1}-\mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}\right), & if\, 0\leq\eta\leq\mu.\\ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}\right), & if\, \mu\leq\eta\leq 1.\end{cases}$$

is called Green function. Therefore from lemma (3.2.2) we find that the voltera integral (3.4) is bounded and hence belongs to B. i.e, Every solution that satisfies problem (3.1) also satisfies integral operator (3.4). Conversely, let $\phi(\mu)\in B$ is the solution of the integral operator (3.4), then using the definition of the Green's function we can write the right hand side of integral operator (3.4) as follows, and we denote it by $\psi(\mu)$.

$$\begin{aligned}\psi(\mu)=&\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\mu\eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-1}g(\eta,\phi(\eta))d\eta\\&-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^1\mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}g(\eta,\phi(\eta))d\eta+\phi_0+(\phi_1-\phi_0)\mu^\rho.\end{aligned}$$

Applying the generalized Caputo type derivative operator on both sides and using the theorem 2.5.4 and the Lemma 2.5.2 we get,

$${}^\rho D^\alpha_{o^+}\psi(\mu)=g(\mu,\phi(\mu)).$$

Now, on the other side we perceive that,

$$\phi(0)=\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^0\eta^{\rho-1}(0-\eta^\rho)^{\alpha-1}g(\eta,\phi(\eta))d\eta-0+\phi_0+0=\phi_0$$

and

$$\begin{aligned}\phi(1)=&\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^1\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}g(\eta,\phi(\eta))d\eta\\&-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^1\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}g(\eta,\phi(\eta))d\eta+\phi_0+\phi_1-\phi_0\\=&\,\phi_1\end{aligned}$$

Thus $\phi(\mu)\in B$ is also a solution of problem (3.1). This completes the proof. □

Now define an operator $T:B\to B$ by,

$$T(\phi(\mu))=\rho(\phi_1-\phi_0)\mu+\phi_0+\int_0^1\{G_1(\mu,\eta)+G_2(\mu,\eta)\}\,g(\eta,\phi(\eta))d\eta. \tag{3.8}$$

Considering the expression of $G(\mu,\eta)$ and lemma 3.2.2 we can see-through that $T(\phi(\mu))=\phi(\mu)$, $\mu\in[0,1]$. Lemma 3.2.1 signifies that solutions of boundary value problem (3.1) coincides with fixed points of $T(\phi(\mu))$.

Remark 4. *The operator defined by equation (3.8) is continuous.*

Proof. Let $\phi_1, \phi_2 \in B$, and we consider,

$$|T(\phi_1(\mu)) - T(\phi_2(\mu))|$$
$$= \left|\rho(\phi_1 - \phi_0)\mu + \phi_0 - \rho(\phi_1 - \phi_0)\mu - \phi_0 + \int_0^1 \{G(\mu,\eta)(g(\eta,\phi_1(\eta)) - g(\eta,\phi_2(\eta)))\}d\eta\right|.$$
$$= \left|\int_0^1 \{G(\mu,\eta)\left(g(\eta,\phi_1(\eta))\right)) - g(\eta,\phi_2(\eta))\}d\eta\right|$$
$$\leq \int_0^1 G(\mu,\eta)\left|(g(\eta,\phi_1(\eta))) - g(\eta,\phi_2(\eta))\right|d\eta.$$

Since g is uniform continuous function i.e, $|(g(\mu,\phi_1(\mu))) - g(\mu,\phi_1(\mu))| < \varepsilon$ whenever $\|\phi_1(\mu) - \phi_2(\mu)\| < \delta(\varepsilon)$ for all $\mu \in [0,1]$. Also $G(\mu,\eta)$ is bounded therefore from lemma 3.2.2 and the continuity of g implies, $|T(\phi_1(\mu)) - T(\phi_2(\mu))| < \varepsilon$ provided that $\|\phi_1(\mu) - \phi_2(\mu)\| < \delta(\varepsilon)$ for $\phi_1, \phi_2 \in B$ and $\mu \in [0,1]$. Hence the operator (3.8) is continuous. □

Lemma 3.2.3. *Let the condition (H_1) holds then, the operator (3.8) is relatively compact subset of $J = C[0,1]$.*

Proof. First we show that the operator $T(\phi(\mu))$ is uniformly bounded. For this, by using the definition of Green's function into above operator we get,

$$\begin{aligned} T(\phi(\mu)) = {} & \phi_0 + (\phi_1 - \phi_0)\mu^\rho \\ & + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\mu \left\{\eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} - \mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}\right\} g(\eta,\phi(\eta))d\eta \\ & + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\mu^1 \mu^\rho\eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g(\eta,\phi(\eta))d\eta. \end{aligned}$$

Now

$$\begin{aligned}
&|T(\phi(\mu))| \\
&\quad \leq \|\phi_0 + (\phi_1 - \phi_0)\mu^\rho\| \\
&\qquad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left\{ \int_0^\mu \left|\eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} g(\eta, \phi(\eta))\right| d\eta + \int_0^\mu \left|\mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g(\eta, \phi(\eta))\right| d\eta \right\} \\
&\qquad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^1 \left|\mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g(\eta, \phi(\eta))\right| d\eta.
\end{aligned}$$

Using lemma 3.2.2 we get,

$$\begin{aligned}
|T(\phi(\mu))| &\leq \|\phi_0 + (\phi_1 - \phi_0)\mu^\rho\| + \frac{(c + d\,|\phi(\mu)|)\,K_\mu}{\rho^\alpha \Gamma(\alpha+1)}, \\
&\leq L + \frac{MK_\mu}{\rho^\alpha \Gamma(\alpha+1)}.
\end{aligned}$$

or

$$|T(\phi(\mu))| \leq R.$$

where $R = L + \frac{MK_\mu}{\rho^\alpha \Gamma(\alpha+1)}$. and $L = sup\,\|\phi_0 + (\phi_1 - \phi_0)\mu^\rho\|$. That is $T(\phi(\mu))$ is uniformly bounded. Now we show that $T(\phi(\mu))$ is an equicontinuous operator. For this consider,

$$\begin{aligned}
|T^r(\phi(\mu_1)) - T^r(\phi(\mu_2))| \leq\ & \|(\phi_1 - \phi_0)({\mu_1}^\rho - {\mu_2}^\rho)\| \\
& + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mu_1} \left|{\eta_r}^{\rho-1}({\mu_1}^\rho - {\eta_r}^\rho)^{\alpha-1} g(\eta_r, \phi(\eta_r))\right| d\eta_r \\
& + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mu_2} \left|{\eta_r}^{\rho-1}({\mu_2}^\rho - {\eta_r}^\rho)^{\alpha-1} g(\eta_r, \phi(\eta_r))\right| d\eta_r \\
& + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mu_1} \left|{\mu_1}^\rho {\eta_r}^{\rho-1}(1 - {\eta_r}^\rho)^{\alpha-1} g(\eta_r, \phi(\eta_r))\right| d\eta_r \\
& + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\mu_2} \left|{\mu_2}^\rho {\eta_r}^{\rho-1}(1 - {\eta_r}^\rho)^{\alpha-1} g(\eta_r, \phi(\eta_r))\right| d\eta_r \\
& + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\mu_1}^1 \left|{\mu_1}^\rho {\eta_j}^{\rho-1}(1 - {\eta_r}^\rho)^{\alpha-1} g(\eta_r, \phi(\eta_r))\right| d\eta_r \\
& + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_{\mu_2}^1 \left|{\mu_2}^\rho {\eta_j}^{\rho-1}(1 - {\eta_r}^\rho)^{\alpha-1} g(\eta_r, \phi(\eta_r))\right| d\eta_r.
\end{aligned}$$

Since $|g(\mu,\phi(\mu))| \leq c + d\,|\phi(\mu)|$, therefore from the above expression we can write,

$$
\begin{aligned}
&|T^r(\phi(\mu_1)) - T^r(\phi(\mu_2))| \leq \; |(\phi_1-\phi_0)|\,\|(\mu_1{}^{\rho}-\mu_2{}^{\rho})\| \\
&\quad + \frac{\rho^{1-\alpha}\,(c+d\,|\phi(\mu)|)}{\Gamma(\alpha)}\left\{\int_0^{\mu_1}\left|\eta_r{}^{\rho-1}(\mu_1{}^{\rho}-\eta_r{}^{\rho})^{\alpha-1}\right|d\eta_r + \int_0^{\mu_2}\left|\eta_r{}^{\rho-1}(\mu_2{}^{\rho}-\eta_r{}^{\rho})^{\alpha-1}\right|d\eta_r\right\} \\
&\quad + \frac{\rho^{1-\alpha}\,(c+d\,|\phi(\mu)|)}{\Gamma(\alpha)}\left\{\int_0^{\mu_1}\left|\mu_1{}^{\rho}\eta_r{}^{\rho-1}(1-\eta_r{}^{\rho})^{\alpha-1}\right|d\eta_r + \int_0^{\mu_2}\left|\mu_2{}^{\rho}\eta_r{}^{\rho-1}(1-\eta_r{}^{\rho})^{\alpha-1}\right|d\eta_r\right\} \\
&\quad + \frac{\rho^{1-\alpha}\,(c+d\,|\phi(\mu)|)}{\Gamma(\alpha)}\left\{\int_{\mu_1}^{1}\left|\mu_1{}^{\rho}\eta_j{}^{\rho-1}(1-\eta_r{}^{\rho})^{\alpha-1}\right|d\eta_r + \int_{\mu_2}^{1}\left|\mu_2{}^{\rho}\eta_j{}^{\rho-1}(1-\eta_r{}^{\rho})^{\alpha-1}\right|d\eta_r\right\}.
\end{aligned}
$$

Using the substitution $u = \frac{\eta_r{}^{\rho}}{\mu^{\rho}}$ one can easily verify that, $\int_0^{\mu}\frac{\eta_r{}^{\rho-1}g(\eta_r,\phi(\eta_r))}{(\mu^{\rho}-\eta_r{}^{\rho})^{1-\alpha}}d\eta_r = \frac{\mu^{\rho\alpha}}{\alpha\rho}$ and $\mu\int_0^{\mu}\frac{\eta_r{}^{\rho-1}g(\eta_r,\phi(\eta_r))}{(\mu^{\rho}-\eta_r{}^{\rho})^{\alpha-1}}d\eta_r = \frac{\mu^{\rho\alpha+1}}{\alpha\rho}$. Also since $\int_{\mu}^{1}\eta_r{}^{\rho-1}(1-\eta_r{}^{\rho})^{\alpha-1}g(\eta_r,\phi(\eta_r))d\eta_r = \frac{\mu(1-\mu^{\rho})^{\alpha}}{\alpha\rho}$. Hence the above inequality becomes,

$$
\begin{aligned}
&|T^r(\phi(\mu_1)) - T^r(\phi(\mu_2))| \\
&\leq \frac{(c+d\,|\phi(\mu)|)\,\rho^{1-\alpha}}{\Gamma(\alpha)}\left\{\frac{\mu_1{}^{\rho\alpha}}{\alpha\rho}+\frac{\mu_2{}^{\rho\alpha}}{\alpha\rho}+\frac{\mu_1{}^{\rho\alpha+1}}{\alpha\rho}+\frac{\mu_2{}^{\rho\alpha+1}}{\alpha\rho}+\frac{\mu_1(1-\mu_1{}^{\rho})^{\alpha}}{\alpha\rho}+\frac{\mu_2(1-\mu_2{}^{\rho})^{\alpha}}{\alpha\rho}\right\} \\
&\quad + |(\phi_1-\phi_0)|\,\|(\mu_1{}^{\rho}-\mu_2{}^{\rho})\|\,, \\
&= \frac{(c+d\,|\phi(\mu)|)}{\rho^{\alpha}\Gamma(\alpha+1)}\left\{\mu_1{}^{\rho\alpha}(1+\mu_1{}^{\rho\alpha})+\mu_2{}^{\rho\alpha}(1+\mu_2{}^{\rho\alpha})+\mu_1(1-\mu_1{}^{\rho})^{\alpha}+\mu_2(1-\mu_2{}^{\rho})^{\alpha}\right\} \\
&\quad + |(\phi_1-\phi_0)|\,\|(\mu_1{}^{\rho}-\mu_2{}^{\rho})\|\,.
\end{aligned}
$$

Since $\mu \in [0,1]$, $\alpha \in (1,2)$ and $\rho > 0$. Also $(1-\mu^{\rho})^{\alpha}, \mu(1-\mu^{\rho})^{\alpha} \in [0,1]$. Therefore,

$$
|T^r(\phi(\mu_1)) - T^r(\phi(\mu_2))| \leq \frac{(c+d\,|\phi(\mu)|)\,K_{\mu_1,\mu_2}}{\rho^{\alpha}\Gamma(\alpha+1)} + \delta_1 = \varepsilon.
$$

Provided that,

$$
\|(\mu_1{}^{\rho}-\mu_2{}^{\rho})\| \leq \frac{\delta_1}{|(\phi_1-\phi_0)|} = \delta.
$$

Where $K_{\mu_1,\mu_2} = \mu_1{}^{\rho\alpha}(1+\mu_1{}^{\rho\alpha})+\mu_2{}^{\rho\alpha}(1+\mu_2{}^{\rho\alpha})+\mu_1(1-\mu_1{}^{\rho})^{\alpha}+\mu_2(1-\mu_2{}^{\rho})^{\alpha}$. Since $\mu^{\rho\alpha}, \mu(1-\mu^{\rho})^{\alpha}$ and $\mu^{\rho\alpha}(1+\mu^{\rho\alpha})$ are uniformly continuous, hence the operator defined by equation (3.8) is equicontinuous. Therefore by Arzela Ascoli [] the operator (3.8) is relatively compact. □

Lemma 3.2.4. *Assume that condition H_1 holds then the boundary value problem (3.1) has at-least one solution in B.*

Proof. The remark (4) asserts that the operator (3.8) is continuous and from Lemma 3.2.3 we see through that $|T(\phi(\mu))| \leq R$. Therefore T is self mapped and lemma 3.2.3 shows that the operator (3.8) is compact subset of $C(J)$. Hence the lemma 3.2.1 and Schauder's fixed point theorem [] indicates the existence of atleast one solution of boundary value problem (3.1). □

For the existence results we use the following assumption.

(H_2) $0 < \mu < 1$, $1 < \alpha \leq 2$, $\phi : [0,1] \to \mathbb{R}$ and $g : [0,1] \times \mathbb{R} \to \mathbb{R}$ are continuous and $|g(\mu, \phi_1(\mu)) - g(\mu, \phi_2(\mu))| \leq L_g \|\phi_1(\mu) - \phi_2(\mu)\|$ where L_g is such that $0 < L_g K_\mu < \rho^\alpha \Gamma(\alpha + 1)$.

Lemma 3.2.5. *Let the condition H_2 holds then the equation (3.4) and (3.5) describe the unique solution of boundary value problem (3.1).*

Proof. To prove this lemma we use the Banach fixed point theorem []. For this we necessitate to confirm that T is self mapped and a contraction mapping. Lemma 3.2.3 shows $T(\phi(\mu)) \in B$ and taking into account the definition of the operator T, lemma 3.2.2 and the remark (4) shows that T is self mapped. Now we need to check contraction principle. For this let $\phi, \tilde{\phi} \in B$, we consider

$$\begin{aligned}
&\left|T(\phi(\mu)) - T(\tilde{\phi}(\mu))\right| \\
&= \left|\rho(\phi_1 - \phi_0)\mu + \phi_0 - \rho(\phi_1 - \phi_0)\mu - \phi_0 + \int_0^1 \left\{G(\mu,\eta)(g(\eta,\phi(\eta)) - g(\eta,\tilde{\phi}(\eta)))\right\} d\eta\right|. \\
&= \left|\int_0^1 \{G(\mu,\eta)\,(g(\eta,\phi(\eta))) - g(\eta,\phi(\eta))\} d\eta\right| \\
&\leq \int_0^1 G(\mu,\eta) \left|(g(\eta,\phi(\eta))) - g(\eta,\tilde{\phi}(\eta))\right| d\eta.
\end{aligned}$$

Since the function g satisfies the Lipchitz condition i.e, $|g(\mu,\phi_1(\mu))-g(\mu,\phi_2(\mu))| \leq L_g \|\phi_1(\mu)-\phi_2(\mu)\|$. Therefore we have,

$$\begin{aligned}\left|T(\phi(\mu))-T(\tilde{\phi}(\mu))\right| &\leq L_g \|\phi(\mu)-\phi(\mu)\| \int_0^1 G(\mu,\eta)d\eta \\ &= \frac{L_g \|\phi(\mu)-\phi(\mu)\|}{\rho^\alpha\Gamma(\alpha+1)}\left\{\mu^{\alpha\rho}+\mu^{\alpha\rho+1}+\mu^\rho(1-\mu^\rho)^\alpha\right\}, \\ &= \frac{L_g K_\mu}{\rho^\alpha\Gamma(\alpha+1)}\|\phi(\mu)-\phi(\mu)\|.\end{aligned}$$

The Banach fixed point theorem [] asserts that the operator T has a unique fixed point. Hence from lemma 3.2.1 shows that, (3.4) is the unique solution of the problem (3.1). □

Now for the existence results and discussion for the boundary value system (3.3) we use the following conditions. Let $J=[0,1]$ and $C(J)$ be the space of all continuous functions defined on J. We define the space $X=\left\{\phi(\mu)|\phi(\mu)\in C(J) \text{ and } {}^\rho D^{\beta^*}\phi(\mu)\in C(J)\right\}$ characterized by the norm $\|\phi(\mu)\|_X=\max_{\mu\in J}|\phi(\mu)|+\max_{\mu\in J}\left|{}^\rho D^{\beta^*}\phi(\mu)\right|$.

Lemma 3.2.6. *[58] $(X,\|.\|_X)$ ia a Banach space.*

Proof. Let $(\phi_j)_{j=0}^\infty$ be a cauchy sequence in $(X,\|.\|_X)$, then clearly $\left({}^\rho D^{\beta^*}\phi_j\right)_{j=0}^\infty$ is also a cauchy sequence in the space $C(J)$. Therefore both $(\phi_j(\mu))_{j=0}^\infty$ and $\left({}^\rho D^{\beta^*}\phi_j(\mu)\right)_{j=0}^\infty$ converges uniformly, say $u(\mu)$ and $v(\mu)$ respectively in the space $C(J)$. We just have to show that $v={}^\rho D^{\beta^*}u$. For this consider.

$$\begin{aligned}\left|{}^\rho I_{0+}^{\beta^*}\,{}^\rho D_{0+}^{\beta^*}\phi_j(\mu)-{}^\rho I_{0+}^{\beta^*}v(\mu)\right| &= \left|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\mu \frac{{}^\rho D_{0+}^{\beta^*}\phi_j(\eta)\eta^{\rho-1}}{(\mu^\rho-\eta^\rho)^{1-\alpha}}d\eta-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\mu \frac{v(\eta)\eta^{\rho-1}}{(\mu^\rho-\eta^\rho)^{1-\alpha}}d\eta\right| \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\mu\left|\frac{({}^\rho D_{0+}^{\beta^*}\phi_j(\eta)-v(\eta))\eta^{\rho-1}}{(\mu^\rho-\eta^\rho)^{1-\alpha}}\right|d\eta \\ &\leq \frac{\mu^{\rho\alpha}}{\rho\Gamma(\alpha+1)}\max_{\mu\in J}\left|{}^\rho D_{0+}^{\beta^*}\phi_j(\mu)-v(\mu)\right|.\end{aligned}$$

Since $\left({}^\rho D^{\beta^*}\phi_j(\mu)\right)_{j=0}^\infty$ converges uniformly to $v(\mu)$ for $\mu\in J$.
Hence $\left|{}^\rho I_{0+}^{\beta^*}\,{}^\rho D_{0+}^{\beta^*}\phi_j(\mu)-{}^\rho I_{0+}^{\beta^*}v(\mu)\right|\to 0$ as $j\to\infty$. i.e, $\lim_{j\to\infty} I_{0+}^{\beta^*}\,{}^\rho D_{0+}^{\beta^*}\phi_j(\mu)\cong {}^\rho I_{0+}^{\beta^*}v(\mu)$.

Now considering ${}^{\rho}D^{\beta^*}_{0^+}(\lim_{j\to\infty} I^{\beta^*}_{0^+}{}^{\rho}D^{\beta^*}_{0^+}\phi_j(\mu)) = {}^{\rho}D^{\beta^*}_{0^+}{}^{\rho}I^{\beta^*}_{0^+}v(\mu)$ and taking into account theorem 2.3.4 and theorem 2.5.1 we get that, $v(\mu)={}^{\rho}D^{\beta^*}u(\mu)$. This completes the proof. □

For upcoming results we consider the following Banach space, $Y = \left\{\tilde{\phi}(\mu) \mid \tilde{\phi}(\mu) \in C(J)\, and\, {}^{\rho}D^{\alpha^*}\tilde{\phi}(\mu) \in C(J)\right\}$ characterized by the norm $\left\|\tilde{\phi}(\mu)\right\|_Y = \max_{\mu\in J}\left|\tilde{\phi}(\mu)\right| + \max_{\mu\in J}\left|{}^{\rho}D^{\alpha^*}\tilde{\phi}(\mu)\right|$. Then for $\left(\phi, \tilde{\phi}\right) \in X \times Y$, let $\left\|(\phi, \tilde{\phi})\right\|_{X\times Y} = \max\left\{\|\phi\|_X, \left\|\tilde{\phi}\right\|_Y\right\}$. Then certainly $\left(X \times Y, \|.\|_{X\times Y}\right)$ is a Banach space. Furthermore for our convenience let,

$$R_1^* \geq \max\left\{4M_21a_1k_2, 4M_1a_2k_2 \max\left|\tilde{\phi}(\mu)\right|, 4M_1k_2b_1 \max\left|{}^{\rho}_{*}D^{\alpha^*}\tilde{\phi}(\mu)\right|\right\}.$$

and

$$R_2^* \geq \max\left\{4M_1a_3k_2, 4M_1a_4k_2 \max|\phi(\mu)|, 4M_1k_2b_2 \max\left|{}^{\rho}_{*}D^{\beta^*}\phi(\mu)\right|\right\}.$$

Here $M_1 = \max_{\mu\in J,\rho\in\mathbb{R}_+}|\mu^{\alpha\rho} + \mu^{\alpha\rho+1} + \mu^{\rho}(1-\mu^{\rho})^{\alpha}|$, $k_1 = \max|\rho(A_1 - A_0)\mu^{\rho}|$, and $k_2 = \frac{\rho^{-\alpha}}{\Gamma(\alpha+1)}$.

Lemma 3.2.7. *Let* $\alpha \in (1, 2), \alpha^* \in (0, 1)$ *and* $g_1 \in C(J)$*, then the problem*

$$\begin{cases} {}^{\rho}_{*}D^{\alpha}_{0^+}\phi(\mu) = g_1(\mu, \phi(\mu), {}^{\rho}D^{\alpha^*}\phi(\mu)) & 0 < \mu < 1 \\ \phi(0) = A_0,\ \phi(1) = A_1. \end{cases}$$

is equivalent to the following Volterra integral.

$$\phi(\mu) = \rho(A_1 - A_0)\mu + A_0 + \int_0^1 \left\{G_1(\mu, \eta)g_1(\eta, \phi(\eta), {}^{\rho}D^{\alpha^*}\phi(\eta))\right\}d\eta,$$

where

$$G_1(\mu, \eta) = \begin{cases} \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\eta^{\rho-1}(\mu^{\rho} - \eta^{\rho})^{\alpha-1} - \mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\alpha-1}\right), & if\ \ 0 \leq \eta \leq \mu. \\ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\left(\mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\alpha-1}\right), & if\ \ \mu \leq \eta \leq 1. \end{cases} \quad (3.9)$$

Proof. The proof is similar to the proof of the lemma 3.2.1. □

Let

$$G_2(\mu,\eta)=\begin{cases}\frac{\rho^{1-\beta}}{\Gamma(\beta)}\left(\eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{\beta-1}-\mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\beta-1}\right), & if\, 0\leq\eta\leq\mu.\\ \frac{\rho^{1-\beta}}{\Gamma(\beta)}\left(\mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\beta-1}\right), & if\ \mu\leq\eta\leq 1.\end{cases}\tag{3.10}$$

We bring out (G_1, G_2) to be the Green's function of coupled system (3.3). Let's ensue the following coupled system of volterra integral equations.

$$\begin{cases}\phi(\mu)=\rho(A_1-A_0)\mu+A_0+\int\limits_0^1\left\{G_1(\mu,\eta)g_1(\eta,\tilde{\phi}(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}(\eta))\right\}d\eta,\\ \tilde{\phi}(\mu)=\rho(A_1-A_0)\mu+A_0+\int\limits_0^1\left\{G_2(\mu,\eta)g_2(\eta,\phi(\eta),{}^{\rho}D^{\beta^*}\phi(\eta))\right\}d\eta.\end{cases}\tag{3.11}$$

Before we demonstrate main results, for our convenience let's define the following discussion first. we define

$$B^*=\left\{(\phi(\mu),\tilde{\phi}(\mu))\,|\,(\phi(\mu),\tilde{\phi}(\mu))\in X\times Y,\left\|(\phi(\mu),\tilde{\phi}(\mu))\right\|_{X\times Y}\leq R,\,\mu\in J\right\}.$$

Where,

$$R\geq\max\left\{4k_1+A_0,\ R_1^*,\ R_2^*,\ 2R_3^*\mu^{\rho(1-\beta^*)},\ 2R_4^*\mu^{\rho(1-\alpha^*)},\ 2R_5^*,\ 2R_6^*\right\}.$$

Ahead of the existence results we discuss the important property of Green's function $G_1(\mu,\eta)$ and $G_2(\mu,\eta)$ that we use in our later results. We use the following conditions,

$(\tilde{H}_1)$ $1<\alpha<2$, $0<\alpha^*<1$ and $g_1:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous function, furthermore $\left|g_1(\mu,\tilde{\phi}(\mu),{}^{\rho}_{*}D^{\alpha^*}\tilde{\phi}(\mu))\right|\leq a_1+a_2\max\left|\tilde{\phi}(\mu)\right|+b_1\max\left|{}^{\rho}_{*}D^{\alpha^*}\tilde{\phi}(\mu)\right|$, where $a_1,a_2,b_1\in\mathbb{R}_+$.

$(\tilde{H}_2)$ $1<\alpha<2$, $0<\beta^*<1$ and $g_2:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ is continuous function, furthermore $\left|g_2(\mu,\phi(\mu),{}^{\rho}_{*}D^{\beta^*}\phi(\mu))\right|\leq a_3+a_4\max|\phi(\mu)|+b_2\max\left|{}^{\rho}_{*}D^{\beta^*}\phi(\mu)\right|$, where $a_3,a_4,b_2\in\mathbb{R}_+$.

Lemma 3.2.8. *Assume that functions g_1 and g_2 satisfies condition $(\tilde{H}_1)$ and $(\tilde{H}_2)$ respectively, then Green's function (3.9) and (3.10) satisfies the following properties:*

1. $\int\limits_0^1\left\{G_1(\mu,\eta)g_1(\eta,\tilde{\phi}(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}(\eta))\right\}d\eta\in B^*$.

2. $\int\limits_0^1 \left\{G_2(\mu,\eta) g_2(\eta, \phi(\eta), {}^\rho D^{\beta^*}\phi(\eta))\right\} d\eta \in B^*$.

Proof. Consider,

$$\int\limits_0^1 \{G_1(\mu,\eta) g_1(\eta, \phi(\eta), {}^\rho D^{\alpha^*}\phi(\eta))\} d\eta$$

$$= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int\limits_0^\mu \left\{\eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} - \mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1}\right\} g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*}\tilde{\phi}(\eta)) d\eta$$
$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^1 \mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*}\phi(\eta)) d\eta.$$

Therefore,

$$\left| \int\limits_0^1 \left\{G_1(\mu,\eta) g_1(\mu, \tilde{\phi}(\mu), {}^\rho D^{\alpha^*}\tilde{\phi}(\mu))\right\} d\eta \right|$$
$$\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int\limits_0^\mu \left|\eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*}\tilde{\phi}(\eta))\right| d\eta$$
$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int\limits_0^\mu \left|\mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*}\tilde{\phi}(\eta))\right| d\eta$$
$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^1 \left|\mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*}\tilde{\phi}(\eta))\right| d\eta.$$

Since $\left|g_1(\mu, \tilde{\phi}(\mu), D^\beta \tilde{\phi}(\mu))\right| \leq a_1 + a_2 \max\left|\tilde{\phi}(\mu)\right| + b_1 \max\left|D^{\alpha^*}\tilde{\phi}(\mu)\right|$. Therefore,

$$\left| \int\limits_0^1 \left\{G_1(\mu,\eta) g_1(\mu, \tilde{\phi}(\mu), {}^\rho D^{\alpha^*}\tilde{\phi}(\mu))\right\} d\eta \right|$$
$$\leq \frac{\rho^{1-\alpha}(a_1 + a_2 \max\left|\tilde{\phi}(\mu)\right| + b_1 \max\left|D^{\alpha^*}\tilde{\phi}(\mu)\right|)}{\Gamma(\alpha)} \left\{\frac{\mu^{\alpha\rho}}{\rho\alpha} + \frac{\mu^{\alpha\rho+1}}{\rho\alpha} + \frac{\mu^\rho(1-\mu^\rho)^\alpha}{\rho\alpha}\right\}$$
$$= \frac{M_1 M_2}{\rho^\alpha \Gamma(\alpha+1)} \leq R.$$

Here $M_2 = (a_1 + a_2 \max\left|\tilde{\phi}(\mu)\right| + b_1 \max\left|D^{\alpha^*}\tilde{\phi}(\mu)\right|)$.

Thus $\int\limits_0^1 \left\{G_1(\mu, \tilde{\phi}(\mu), {}^\rho D^{\alpha^*}\tilde{\phi}(\mu)\right\} d\eta \in B^*$. One can easily prove the second part likewise.

□

3.2.2 Existence Results for Coupled System of Generalized Nonlinear Fractional Differential Equations

Lemma 3.2.9. *Assume that $g_1, g_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Then $(\phi(\mu), \tilde{\phi}(\mu)) \in B^*$ is a solution of nonlinear coupled system (3.3) if and only if $(\phi(\mu), \tilde{\phi}(\mu))$ is a solution of the system (3.11).*

Proof. Let $(\phi(\mu), \tilde{\phi}(\mu)) \in B^*$ is a solution of (3.3), now applying the generalized Katugampola integral operator on both sides of the first equation in (3.3) and making use of the definition (2.3.1) and the lemma 2.5.5 yields,

$$\phi(\mu) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \frac{\eta^{\rho-1} g_1(\eta, \phi(\eta), {}^\rho D^{\alpha^*}\phi(\eta))}{(\mu^\rho - \eta^\rho)^{1-\alpha}} d\eta + c_0 + c_1 \frac{\mu^\rho}{\rho}.$$

Using the boundary conditions $\phi(0) = A_0$ and $\phi(1) = A_1$ into the above equation we get,

$$\begin{aligned}
\phi(\mu) = & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} g_1(\eta, \phi(\eta), {}^\rho D^{\alpha^*}\phi(\eta)) d\eta \\
& - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \mu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-1} g_1(\eta, \phi(\eta), {}^\rho D^{\alpha^*}\phi(\eta)) d\eta + A_0 + (A_1 - A_0)\mu^\rho, \\
= & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} g_1(\eta, \phi(\eta), {}^\rho D^{\alpha^*}\phi(\eta)) d\eta \\
& - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \mu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-1} g_1(\eta, \phi(\eta), {}^\rho D^{\alpha^*}\phi(\eta)) d\eta \\
& - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^1 \mu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-1} g_1(\eta, \phi(\eta), {}^\rho D^{\alpha^*}\phi(\eta)) + A_0 + (A_1 - A_0)\mu^\rho, \\
= & \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \left\{ \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} - \mu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-1} \right\} g_1(\eta, \phi(\eta), {}^\rho D^{\alpha^*}\phi(\eta)) d\eta \\
& + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^1 \mu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-1} g_1(\eta, \phi(\eta), {}^\rho D^{\alpha^*}\phi(\eta)) + A_0 + (A_1 - A_0)\mu^\rho.
\end{aligned}$$

or

$$\phi(\mu) = \rho(A_1 - A_0)\mu^\rho + A_0 + \int_0^1 \left\{ G_1(\mu, \eta) g_1(\eta, \phi(\eta), {}^\rho D^{\alpha^*}\phi(\eta)) \right\} d\eta.$$

Where $G_1(\mu,\eta)$ is the Green's function defined by (3.9). Similarly by applying the same procedure on second equation of the non-linear system (3.3) we get,

$$\tilde{\phi}(\mu) = \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int\limits_0^{\mu} \left\{ \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\beta-1} - \mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\beta-1} \right\} g_2(\eta, \tilde{\phi}(\eta), {}^\rho D^{\beta^*}\tilde{\phi}(\eta))d\eta$$
$$+ \frac{\rho^{1-\beta}}{\Gamma(\beta)} \int_\mu^1 \mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\beta-1} g_2(\eta, \tilde{\phi}(\eta), {}^\rho D^{\beta^*}\tilde{\phi}(\eta)) + A_0 + (A_1 - A_0)\mu^\rho.$$

or

$$\tilde{\phi}(\mu) = \rho(A_1 - A_0)\mu + A_0 + \int\limits_0^1 \left\{ G_2(\mu,\eta) g_2(\eta, \phi(\eta), {}^\rho D^{\beta^*}\phi(\eta)) \right\} d\eta.$$

where $G_2(\mu,\eta)$ is the Green's function defined by (3.10). The lemma 3.2.8 shows that the system (3.11) of volterra integral equations is bounded. i.e, every solution that satisfies system (3.3) also satisfies the non-linear system (3.11) of velterra integral equations. Conversely, let $(\phi(\mu), \tilde{\phi}(\mu)) \in B^*$ is the solution of the system (3.11), Now we denote the right hand side of the first equation of the system (3.11) by $\psi(\mu)$ then by using the definition of Green's function $G_1(\mu,\eta)$ we can write,

$$\psi(\mu) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*}\tilde{\phi}(\eta))d\eta$$
$$- \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \mu^\rho \eta^{\rho-1}(1-\eta^\rho)^{\alpha-1} g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*}\tilde{\phi}(\eta))d\eta + A_0 + (A_1 - A_0)\mu^\rho.$$

Now by applying the generalized Caputo derivative operator on both sides and using the Theorem 2.5.4 and Lemma 2.5.2 we get,

$${}^\rho D^\alpha_{o^+}\psi(\mu) = g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*}\tilde{\phi}(\eta)).$$

Cognitively, by applying the same procedure on the second equation of the system (3.11) yields,

$${}^\rho_* D^\alpha_{o^+}\tilde{\phi}(\mu) = g_2(\eta, \phi(\eta), {}^\rho D^{\alpha^*}\phi(\eta)).$$

Now we observe that the system (3.11) of volterra integrals satisfy the boundary conditions. i.e. $\phi(0) = \tilde{\phi}(0) = A_0$ and $\phi(1) = \tilde{\phi}(1) = A_1$. Hence the problem (3.3) is equivalent to (3.11). □

Now we present the existence and uniqueness results for the non-linear boundary value problem (3.3). Before we present the detailed investigation, let us have the following consideration first. We define an operator $\tilde{T} : B^* \to B^*$ by:

$$\tilde{T}(\phi,\tilde{\phi})(\mu) = \begin{pmatrix} f(\mu) + \int\limits_0^1 \left(G_1(\mu,\eta)g_1(\eta,\tilde{\phi}(\eta),{}^\rho D^{\alpha^*}\tilde{\phi}(\eta))\right)d\eta, \\ f(\mu) + \int\limits_0^1 \left(G_2(\mu,\eta)g_2(\eta,\phi(\eta),{}^\rho D^{\beta^*}\phi(\eta))\right)d\eta \end{pmatrix} =: \left(\tilde{T}_1\tilde{\phi}(\mu), \tilde{T}_2\phi(\mu)\right). \tag{3.12}$$

Where $f(\mu) = \rho(A_1 - A_0)\mu^\rho + A_0$. As long as we descry the Lemma 3.2.9, then clearly we see through that the fixed points of the integral operator (3.12) are the solutions of the boundary value problem (3.3). For our convenience, we define the following confabulation.

$$R_3^* = \frac{(A_1 - A_0)\rho^{\beta^* - \lceil \beta^* \rceil}}{\Gamma(2-\beta^*)}.$$

$$R_4^* = \frac{(A_1 - A_0)\rho^{\alpha^* - \lceil \alpha^* \rceil}\Gamma(2 - \lceil \alpha^* \rceil)}{\Gamma(2-\alpha^*)}.$$

$$R_5^* = \frac{M_1^* a_1 \rho^{\beta^*-\alpha}}{\Gamma(\alpha-\beta^*+1)} + \frac{M_1^*\rho^{\beta^*-\alpha}a_2 \max\left|\tilde{\phi}(\mu)\right|}{\Gamma(\alpha-\beta^*+1)} + \frac{M_1^*\rho^{\beta^*-\alpha}b_1 \max\left|D^{\alpha^*}\tilde{\phi}(\mu)\right|}{\Gamma(\alpha-\beta^*+1)}.$$

$$R_6^* = \frac{M_2^* a_3 \rho^{\alpha^*-\alpha}}{\Gamma(\alpha-\alpha^*+1)} + \frac{M_2^*\rho^{\alpha^*-\alpha}a_4 \max\left|\phi(\mu)\right|}{\Gamma(\alpha-\alpha^*+1)} + \frac{M_2^*\rho^{\alpha^*-\alpha}b_2 \max\left|D^{\alpha^*}\phi(\mu)\right|}{\Gamma(\alpha-\alpha^*+1)}.$$

Where $M_1^* = \max\limits_{\mu\in J,\rho\in\mathbb{R}_+}\left|\mu^{\rho(\alpha-\beta^*)} + \mu^{(\alpha-\beta^*)(\rho+1)} + \mu^\rho(1-\mu^\rho)^{\alpha-\beta^*}\right|$, and
$M_2^* = \max\limits_{\mu\in J,\rho\in_+}\left|\mu^{\rho(\alpha-\alpha^*)} + \mu^{(\alpha-\alpha^*)(\rho+1)} + \mu^\rho(1-\mu^\rho)^{\alpha-\alpha^*}\right|$.

$(\tilde{H}_3)$ $1 < \alpha < 2$, $0 < \alpha^* < 1$ and $g_1 : [0,1]\times\mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous function, furthermore g satisfies the Lipschitz condition i.e,

$$\left|g_1(\mu,\phi_1(\mu),{}^\rho D^{\alpha^*}\phi_1(\eta)) - g_1(\mu,\phi_2(\mu),{}^\rho D^{\alpha^*}\phi_2(\eta))\right| \le L_1\left(|\phi_1(\mu)-\phi_2(\mu)| + \left|{}^\rho D^{\alpha^*}\phi_2(\eta) - {}^\rho D^{\alpha^*}\phi_1(\eta)\right|\right).$$

($\tilde{H}_4$) $1 < \beta < 2$, $0 < \beta^* < 1$ and $g_2 : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function, furthermore g satisfies the Lipschitz condition i.e,

$$\left|g_2(\mu, \phi_3(\mu), {}^\rho D^{\beta^*} \phi_3(\eta)) - g_2(\mu, \phi_4(\mu), {}^\rho D^{\beta^*} \phi_4(\eta))\right| \le L_2 \left(|\phi_3(\mu) - \phi_4(\mu)| + \left|{}^\rho D^{\beta^*} \phi_4(\eta) - {}^\rho D^{\beta^*} \phi_3(\eta)\right|\right).$$

Where L_1 and L_2 are Lipschitz constants such that $0 < L_1 < K_1^*$ and $0 < L_2 < K_2^*$.

Remark 5. *Let g_1 and g_2 satisfy the condition ($\tilde{H}_1$) and ($\tilde{H}_2$) then the operator defined by (3.12) is continuous.*

Proof. The result follows from the boundedness of $G_1(\mu, \eta)$, $G_2(\mu, \eta)$ and the continuity of $g_1(\mu)$ and $g_2(\mu)$. □

Theorem 3.2.1. *Assume that the conditions ($\tilde{H}_1$) and ($\tilde{H}_2$) hold then the problem (3.3) has at-least one solution in B^*.*

Proof. We prove this result using the Schauder fixed point theorem. First we show that the operator $\tilde{T} : B^* \to B^*$ is self mapped. For this let us consider,

$$\begin{aligned}
\left\|\tilde{T}_1 \tilde{\phi}(\mu)\right\| &\le |\rho(A_1 - A_0)\mu^\rho + A_0| + \left| \int_0^1 \left\{ G_1(\mu, \eta) g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*} \tilde{\phi}(\eta)) \right\} d\eta \right| \\
&\le \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \left| \eta^{\rho-1} (\mu^\rho - \eta^\rho)^{\alpha-1} g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*} \tilde{\phi}(\eta)) \right| d\eta \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \left| \mu^\rho \eta^{\rho-1} (1 - \eta^\rho)^{\alpha-1} g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*} \tilde{\phi}(\eta)) \right| d\eta \\
&\quad + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^1 \left| \mu^\rho \eta^{\rho-1} (1 - \eta^\rho)^{\alpha-1} g_1(\eta, \tilde{\phi}(\eta), {}^\rho D^{\alpha^*} \tilde{\phi}(\eta)) \right| d\eta, \\
&\le |A_0| + |\rho(A_1 - A_0)\mu^\rho| \\
&\qquad + \frac{(a_1 + a_2 \max\left|\tilde{\phi}(\mu)\right| + b_1 \max\left|D^{\alpha^*} \tilde{\phi}(\mu)\right|)}{\rho^\alpha \Gamma(\alpha+1)} \left(\mu^{\alpha\rho} + \mu^{\alpha\rho+1} + \mu^\rho (1 - \mu^\rho)^\alpha\right) \\
&\le k_1 + M_2(a_1 k_2 + a_2 \max\left|\tilde{\phi}(\mu)\right| k_2 + b_1 k_2 \max\left|D^{\alpha^*} \tilde{\phi}(\mu)\right|).
\end{aligned}$$

and hence,

$$\left\|\tilde{T}_1 \tilde{\phi}(\mu)\right\|_X \le \frac{R}{4} + \frac{R}{4} + \frac{R}{4} + \frac{R}{4} = R.$$

Also,

$$\left\|{}^{\rho}D^{\beta^*}\tilde{T}_1\tilde{\phi}(\mu)\right\| \leq \left|{}^{\rho}D^{\beta^*}f(\mu)\right| + \left|{}^{\rho}D^{\beta^*}\int\limits_0^1 \Big(G_1(\mu,\eta)g_1(\eta,\tilde{\phi}(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}(\eta))\Big)d\eta\right|.$$

Where $f(\mu)=\rho(A_1-A_0)\mu^{\rho}+A_0$ and since by the Lemma 2.5.2 ${}^{\rho}_{*}D^{\beta^*}_{0^+}f(\mu) = R_3^*\mu^{\rho(1-\beta^*)}$. Therefore using the definition of Green's function (3.9) above inequality becomes,

$$\begin{aligned}\left\|{}^{\rho}D^{\beta^*}\tilde{T}_1\tilde{\phi}(\mu)\right\| \leq\ & R_3^*\mu^{\rho(1-\beta^*)}\\ &+\left|{}^{\rho}_{*}D^{\beta^*}\left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^{\mu}\left\{\eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{\alpha-1}\right\}g_1(\eta,\tilde{\phi}(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}(\eta))d\eta\right]\right|\\ &+\left|{}^{\rho}_{*}D^{\beta^*}\left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^{\mu}\left\{\mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\alpha-1}\right\}g_1(\eta,\tilde{\phi}(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}(\eta))d\eta\right]\right|\\ &+\left|{}^{\rho}_{*}D^{\beta^*}\left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu}^{1}\mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\alpha-1}g_1(\eta,\tilde{\phi}(\eta),{}^{\rho}D^{\alpha^*}\phi(\eta))d\eta\right]\right|.\end{aligned}$$

Using the Theorem 2.5.5 we get,

$$\begin{aligned}\left\|{}^{\rho}D^{\beta^*}\tilde{T}_1\tilde{\phi}(\mu)\right\| \leq\ & R_3^*\mu^{\rho(1-\beta^*)}\\ &+\frac{\rho^{1-(\alpha-\beta^*)}}{\Gamma(\alpha-\beta^*)}{}^{\rho}\int\limits_0^{\mu}\left\{\eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{\alpha-\beta^*-1}\right\}\left|g_1(\eta,\tilde{\phi}(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}(\eta))\right|d\eta\\ &+\frac{\rho^{1-(\alpha-\beta^*)}}{\Gamma(\alpha-\beta^*)}{}^{\rho}\int\limits_0^{\mu}\left\{\mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\alpha-\beta^*-1}\right\}\left|g_1(\eta,\tilde{\phi}(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}(\eta))\right|d\eta\\ &+\frac{\rho^{1-(\alpha-\beta^*)}}{\Gamma(\alpha-\beta^*)}\int_{\mu}^{1}\left\{\mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\alpha-\beta^*-1}\right\}\left|g_1(\eta,\tilde{\phi}(\eta),{}^{\rho}D^{\alpha^*}\phi(\eta))\right|d\eta.\end{aligned}$$

Using the condition $(\tilde{H}_1)$ and the Theorem 2.3.3 we get,

$$\begin{aligned}\left\|{}^{\rho}D^{\beta^*}\tilde{T}_1\tilde{\phi}(\mu)\right\| \leq\ & R_3^*\mu^{\rho(1-\beta^*)}\\ &+\frac{\rho^{1-(\alpha-\beta^*)}(a_1+a_2\max\left|\tilde{\phi}(\mu)\right|+b_1\max\left|D^{\alpha^*}\tilde{\phi}(\mu)\right|)}{\Gamma(\alpha-\beta^*)}\Big(\frac{\mu^{\rho(\alpha-\beta^*)}}{\rho(\alpha-\beta^*)}\\ &+\frac{\mu^{(\alpha-\beta^*)(\rho+1)}}{\rho(\alpha-\beta^*)}+\frac{\mu^{\rho}(1-\mu^{\rho})^{\alpha-\beta^*}}{\rho(\alpha-\beta^*)}\Big),\end{aligned}$$

That is

$$\left\|{}^{\rho}D^{\beta^*}\tilde{T}_1\tilde{\phi}(\mu)\right\| \leq R_1^*\mu^{\rho(1-\beta^*)} + R_5^* \leq \frac{R}{2} + \frac{R}{2} = R.$$

Executing the same arguments for the operator $\tilde{T}_2\phi(\mu)$ we observed that, $\left\|\tilde{T}_2\phi(\mu)\right\|_Y \leq R$, and $\left\|{}^{\rho}D^{\alpha^*}\tilde{T}_2\phi(\mu)\right\| \leq R$. Thus $\left\|\tilde{T}(\phi,\tilde{\phi})\right\| \leq R$ for any $(\phi,\tilde{\phi}) \in B^*$. Therefore, the operator $\tilde{T}: B^* \to B^*$ is self mapped. Next we show that the operator $\tilde{T}$ is completely continuously. For this let $(\phi,\tilde{\phi}) \in B^*$ and $\mu,\nu \in J$, then we have

$$\begin{aligned}
&\left|\tilde{T}_1\tilde{\phi}(\mu) - \tilde{T}_1\tilde{\phi}(\nu)\right| \\
&= \left|\rho(A_1 - A_0)(\mu^{\rho} - \nu^{\rho}) + \int_0^1 \left\{(G_1(\mu,\eta) - G_1(\nu,\eta))g_1(\eta,\phi(\eta),{}^{\rho}D^{\alpha^*}\phi(\eta))\right\}d\eta\right|, \\
&\leq \left|\rho(A_1 - A_0)(\mu^{\rho} - \nu^{\rho}) + \left|g_1(\eta,\phi(\eta),{}^{\rho}D^{\alpha^*}\phi(\eta))\right|\right| \int_0^1 |(G_1(\mu,\eta) - G_1(\nu,\eta))|d\eta, \\
&\leq L + \frac{M_2}{\rho^{\alpha}\Gamma(\alpha+1)}[(\mu^{\alpha\rho} - \nu^{\alpha\rho}) + (\mu^{\alpha(\rho+1)} - \nu^{\alpha\rho+1}) + (\mu^{\rho}(1-\nu^{\rho})^{\alpha} - \nu^{\rho}(1-\nu^{\rho})^{\alpha})].
\end{aligned}$$

Where $L = \sup_{\mu,\nu\in J} |\rho(A_1 - A_0)(\mu^{\rho} - \nu^{\rho})|$. Next we consider,

$$\begin{aligned}
&\left|{}^{\rho}_{*}D^{\beta^*}\tilde{T}_1\tilde{\phi}(\mu) - {}^{\rho}_{*}D^{\beta^*}\tilde{T}_1\tilde{\phi}(\nu)\right| \\
&= \left|{}^{\rho}_{*}D^{\beta^*}f(\mu) - {}^{\rho}_{*}D^{\beta^*}f(\nu) + {}^{\rho}_{*}D^{\beta^*}\int_0^1 \left\{(G_1(\mu,\eta) - G_1(\nu,\eta))g_1(\eta,\phi(\eta),{}^{\rho}D^{\alpha^*}\phi(\eta))\right\}d\eta\right|, \\
&\leq \left|R_3^*\mu^{\rho(1-\beta^*)} + R_3^*\nu^{\rho(1-\beta^*)}\right| \\
&\quad + \left|g_1(\mu,\phi(\mu),{}^{\rho}D^{\alpha^*}\phi(\mu))\right| {}^{\rho}_{*}D^{\beta^*}(\int_0^1 |(G_1(\mu,\eta) - G_1(\nu,\eta))|d\eta).
\end{aligned}$$

Using the Lemma 2.5.2, definition of the Green's function (3.9) and the given condition

$(\tilde{H}_1)$ we have,

$$
\begin{aligned}
&\leq R_3^* \left(\mu^{\rho(1-\beta^*)} + \nu^{\rho(1-\beta^*)}\right) \\
&\quad + \left| M_2 {}_*^\rho D^{\beta^*} \left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \left\{ \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} - \eta^{\rho-1}(\nu^\rho - \eta^\rho)^{\alpha-1} \right\} d\eta \right] \right| \\
&\quad + \left| M_2 {}_*^\rho D^{\beta^*} \left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\mu \left\{ \mu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-1} - \nu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-1} \right\} d\eta \right] \right| \\
&\quad + \left| M_2 {}_*^\rho D^{\beta^*} \left[\frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_\mu^1 \left\{ \mu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-1} - \nu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-1} \right\} d\eta \right] \right|.
\end{aligned}
$$

Using the Theorem 2.5.5 yields,

$$
\begin{aligned}
&\left| {}_*^\rho D^{\beta^*} \tilde{T}_1 \tilde{\phi}(\mu) - {}_*^\rho D^{\beta^*} \tilde{T}_1 \tilde{\phi}(\nu) \right| \\
&\leq R_3^* \left(\mu^{\rho(1-\beta^*)} + \nu^{\rho(1-\beta^*)}\right) \\
&\quad + \frac{M_2 \rho^{1-(\alpha-\beta^*)}}{\Gamma(\alpha-\beta^*)} \int_0^\mu \left\{ \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-\beta^*-1} - \eta^{\rho-1}(\nu^\rho - \eta^\rho)^{\alpha-\beta^*-1} \right\} d\eta \\
&\quad + \frac{M_2 \rho^{1-(\alpha-\beta^*)}}{\Gamma(\alpha-\beta^*)} \int_0^\mu \left\{ \mu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-\beta^*-1} - \nu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-\beta^*-1} \right\} d\eta \\
&\quad + \frac{M_2 \rho^{1-(\alpha-\beta^*)}}{\Gamma(\alpha-\beta^*)} \int_\mu^1 \left\{ \mu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-\beta^*-1} - \nu^\rho \eta^{\rho-1}(1 - \eta^\rho)^{\alpha-\beta^*-1} \right\} d\eta, \\
&= R_3^* \left(\mu^{\rho(1-\beta^*)} + \nu^{\rho(1-\beta^*)}\right) + \frac{M_2 \rho^{\beta^*-\alpha}}{\Gamma(\alpha-\beta^*+1)} \{ \mu^{\rho(\alpha-\beta^*)} - \nu^{\rho(\alpha-\beta^*)} + \mu^{(\alpha-\beta^*)(\rho+1)} \\
&\qquad - \nu^{(\alpha-\beta^*)(\rho+1)} - \mu^\rho(1-\mu^\rho)^{\alpha-\beta^*} - \nu^\rho(1-\nu^\rho)^{\alpha-\beta^*} \}.
\end{aligned}
$$

Here we omit the calculations for $T_2\phi$, similarly one can easily make out that,

$$
\begin{aligned}
\left| \tilde{T}_2\phi(\mu) - \tilde{T}_2\phi(\nu) \right| \leq & L + \frac{M_3}{\rho^\alpha \Gamma(\alpha+1)} [(\mu^{\alpha\rho} - \nu^{\alpha\rho}) + (\mu^{\alpha(\rho+1)} - \nu^{\alpha\rho+1}) \\
& + (\mu^\rho(1-\nu^\rho)^\alpha - \nu^\rho(1-\nu^\rho)^\alpha)].
\end{aligned}
$$

and

$$
\begin{aligned}
\left| {}_*^\rho D^{\alpha^*} \tilde{T}_2\phi(\mu) - {}_*^\rho D^{\alpha^*} \tilde{T}_2\phi(\nu) \right| = & R_4^* \left(\mu^{\rho(1-\alpha^*)} + \nu^{\rho(1-\alpha^*)}\right) \\
& + \frac{M_3 \rho^{\alpha^*-\alpha}}{\Gamma(\alpha-\alpha^*+1)} \{ \mu^{\rho(\alpha-\alpha^*)} - \nu^{\rho(\alpha-\alpha^*)} + \mu^{(\alpha-\alpha^*)(\rho+1)} \\
& - \nu^{(\alpha-\alpha^*)(\rho+1)} - \mu^\rho(1-\mu^\rho)^{\alpha-\alpha^*} - \nu^\rho(1-\nu^\rho)^{\alpha-\alpha^*} \}.
\end{aligned}
$$

Where $M_3 = a_3 + a_4 \max|\phi(\mu)| + b_2 \max\left|{}_{*}^{\rho}D^{\beta^*}\phi(\mu)\right|$. Since the functions $\mu^{\alpha\rho}, \nu^{\alpha\rho}$, $\mu^{\rho}(1-\nu^{\rho})^{\alpha}$, $\mu^{(\alpha-\beta^*)(\rho+1)}$, $\nu^{\rho}(1-\nu^{\rho})^{\alpha-\alpha^*}, \mu^{\rho}(1-\mu^{\rho})^{\alpha-\beta^*}, \mu^{(\alpha-\beta^*)(\rho+1)}$, and $\nu^{(\alpha-\beta^*)(\rho+1)}$ are uniform continuous for $\mu, \nu \in J$, so we see-through that the operator $\tilde{T}B^*$ is equicontinuous. Also since $\tilde{T}B^* \subseteq B^*$, implies $\tilde{T}B^*$ is uniformly bounded as well. Henceforth, $\tilde{T}$ is completely continuous and thus Schauder fixed point theorem [] assures the existence of atleast one fixed point of the operator (3.12). Hence taking into account the Lemma 3.2.9 completes the proof. □

Theorem 3.2.2. *Assume that the conditions* ($\tilde{H}_3$) *and* ($\tilde{H}_4$) *are satisfied then the problem (3.3) has a unique solution.*

Proof. To prove this theorem we use the Banach fixed point theorem []. For this first we necessitate to confirm that $\tilde{T}$ is a self mapped and then we show that the operator (3.12) satisfy the contraction mapping priciple as well. Since the conditions ($\tilde{H}_3$) and ($\tilde{H}_4$) are stronger than we used in the Theorem 3.2.1, so obviously the operator $\tilde{T}$ satisfies the self mappedness condition under these conditions as well. The only stipulation that we need to verify here is contraction. For this consider,

$$\begin{aligned}
&\left|\tilde{T}_1\tilde{\phi}_1(\mu) - \tilde{T}_1\tilde{\phi}_2(\mu)\right| \\
&= \left|\int_0^1 \left\{(G_1(\mu,\eta)g_1(\eta,\tilde{\phi}_1(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}_1(\eta)) - G_1(\mu,\eta))g_1(\eta,\tilde{\phi}_2(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}_2(\eta))\right\}d\eta\right|, \\
&= \left|\int_0^1 \left[(G_1(\mu,\eta)\left\{g_1(\eta,\tilde{\phi}_1(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}_1(\eta)) - g_1(\eta,\tilde{\phi}_2(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}_2(\eta))\right\}\right]d\eta\right|, \\
&\le \int_0^1 (G_1(\mu,\eta)\left|g_1(\eta,\tilde{\phi}_1(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}_1(\eta)) - g_1(\eta,\tilde{\phi}_2(\eta),{}^{\rho}D^{\alpha^*}\tilde{\phi}_2(\eta))\right|d\eta, \\
&\le L_1\left(\left|\tilde{\phi}_1(\mu) - \tilde{\phi}_2(\mu)\right| + \left|{}^{\rho}D^{\alpha^*}\tilde{\phi}_2(\mu) - {}^{\rho}D^{\alpha^*}\tilde{\phi}_1(\mu)\right|\right)\int_0^1 (G_1(\mu,\eta)d\eta, \\
&= L_1\left(\left|\tilde{\phi}_1(\mu) - \tilde{\phi}_2(\mu)\right| + \left|{}^{\rho}D^{\alpha^*}\tilde{\phi}_2(\mu) - {}^{\rho}D^{\alpha^*}\tilde{\phi}_1(\mu)\right|\right)\left\{\frac{\mu^{\alpha\rho} + \mu^{\alpha\rho+1} + \mu^{\rho}(1-\mu^{\rho})^{\alpha}}{\rho^{\alpha}\Gamma(\alpha+1)}\right\}, \\
&\le \frac{L_1K_{\mu}}{\rho^{\alpha}\Gamma(\alpha+1)}\left\|\tilde{\phi}_1(\mu) - \tilde{\phi}_2(\mu)\right\|.
\end{aligned}$$

Also,

$$\left|{}^{\rho}_{*}D^{\beta^*}\tilde{T}_1\tilde{\phi}_1(\mu) - {}^{\rho}_{*}D^{\beta^*}\tilde{T}_1\tilde{\phi}_1(\mu)\right|$$

$$= \left|{}^{\rho}_{*}D^{\beta^*}\int_0^1 \left\{(G_1(\mu,\eta)\left(g_1(\eta,\phi_1(\eta),{}^{\rho}_{*}D^{\alpha^*}\phi_1(\eta) - g_1(\eta,\phi_2(\eta),{}^{\rho}_{*}D^{\alpha^*}\phi_2(\eta)\right)\right\}d\eta\right|,$$

$$\leq {}^{\rho}_{*}D^{\beta^*}\int_0^1 \left\{(G_1(\mu,\eta)\left|g_1(\eta,\phi_1(\eta),{}^{\rho}_{*}D^{\alpha^*}\phi_1(\eta) - g_1(\eta,\phi_2(\eta),{}^{\rho}_{*}D^{\alpha^*}\phi_2(\eta)\right|\right\}d\eta.$$

Using the definition of Green's function (3.9) and the Theorem 2.5.6 we get,

$$\left|{}^{\rho}_{*}D^{\beta^*}\tilde{T}_1\tilde{\phi}_1(\mu) -{}^{\rho}_{*}D^{\beta^*}\tilde{T}_1\tilde{\phi}_2(\mu)\right| \leq \frac{\rho^{1-(\alpha-\beta^*)}M_4}{\Gamma(\alpha-\beta^*)}\int_0^{\mu}\left\{\eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{\alpha-\beta^*-1}\right\}d\eta$$

$$+ \frac{\rho^{1-(\alpha-\beta^*)}M_4}{\Gamma(\alpha-\beta^*)}\int_0^{\mu}\left\{\mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\alpha-\beta^*-1}\right\}d\eta$$

$$+ \frac{\rho^{1-(\alpha-\beta^*)}M_4}{\Gamma(\alpha-\beta^*)}\int_{\mu}^{1}\left\{\mu^{\rho}\eta^{\rho-1}(1-\eta^{\rho})^{\alpha-\beta^*-1}\right\}d\eta.$$

Where $M_4 = \left|g_1(\eta,\phi_1(\eta),{}^{\rho}_{*}D^{\alpha^*}\phi_1(\eta) - g_1(\eta,\phi_2(\eta),{}^{\rho}_{*}D^{\alpha^*}\phi_2(\eta)\right|$. Now after some simple calculations and using the condition $(\tilde{H})$ gives,

$$\left|{}^{\rho}_{*}D^{\beta^*}\tilde{T}_1\tilde{\phi}_1(\mu) - {}^{\rho}_{*}D^{\beta^*}\tilde{T}_1\tilde{\phi}_2(\mu)\right|$$

$$\leq \frac{\rho^{1-(\alpha-\beta^*)}L_1\left(\left|\tilde{\phi}_1(\mu)-\tilde{\phi}_2(\mu)\right| + \left|{}^{\rho}D^{\alpha^*}\tilde{\phi}_2(\mu) - {}^{\rho}D^{\alpha^*}\tilde{\phi}_1(\mu)\right|\right)}{\Gamma(\alpha-\beta^*)}\{\frac{\mu^{\rho(\alpha-\beta^*)}}{\rho(\alpha-\beta^*)}$$

$$+ \frac{\mu^{(\alpha-\beta^*)(\rho+1)}}{\rho(\alpha-\beta^*)} + \frac{\mu^{\rho}(1-\mu^{\rho})^{\alpha-\beta^*}}{\rho(\alpha-\beta^*)}\},$$

$$\leq \frac{L_1\left\|\tilde{\phi}_1(\mu)-\tilde{\phi}_2(\mu)\right\|}{\rho^{\alpha-\beta^*}\Gamma(\alpha-\beta^*)}\left(\mu^{\rho(\alpha-\beta^*)} + \mu^{(\alpha-\beta^*)(\rho+1)} + \mu^{\rho}(1-\mu^{\rho})^{\alpha-\beta^*}\right);$$

$$= \frac{L_1K_{\mu}^{\beta^*}}{\rho^{\alpha-\beta^*}\Gamma(\alpha-\beta^*+1)}\left\|\tilde{\phi}_1(\mu)-\tilde{\phi}_2(\mu)\right\|.$$

Where $K_{\mu}^{\beta^*} = \mu^{\rho(\alpha-\beta^*)} + \mu^{(\alpha-\beta^*)(\rho+1)} + \mu^{\rho}(1-\mu^{\rho})^{\alpha-\beta^*}$. Therefore,

$$\left|\tilde{T}_1\tilde{\phi}_1(\mu) - \tilde{T}_1\tilde{\phi}_2(\mu)\right| \leq \left(\frac{L_1K_{\mu}}{\rho^{\alpha}\Gamma(\alpha+1)} + \frac{L_1K_{\mu}^{\beta^*}}{\rho^{\alpha-\beta^*}\Gamma(\alpha-\beta^*)}\right)\left\|\tilde{\phi}_1(\mu)-\tilde{\phi}_2(\mu)\right\|,$$

$$= \frac{L_1}{K_1^*}\left\|\tilde{\phi}_1(\mu)-\tilde{\phi}_2(\mu)\right\|.$$

where $K_1^* = \frac{\rho^{\alpha-\beta^*}\Gamma(\alpha+1)\Gamma(\alpha-\beta^*)}{\rho^{-\beta^*}\Gamma(\alpha-\beta^*)K_\mu+\Gamma(\alpha+1)K_\mu^{\beta^*}}$. Likewise, one can easily see-through that,

$$\left|\tilde{T}_2\phi_1(\mu) - \tilde{T}_2\phi_2(\mu)\right| \leq \frac{L_2}{K_2^*} \left\|\phi_1(\mu) - \phi_2(\mu)\right\|.$$

With $K_\mu^{\alpha^*} = \mu^{\rho(\alpha-\alpha^*)} + \mu^{(\alpha-\alpha^*)(\rho+1)} + \mu^\rho(1-\mu^\rho)^{\alpha-\alpha^*}$ and $K_2^* = \frac{\rho^{\alpha-\alpha^*}\Gamma(\alpha+1)\Gamma(\alpha-\alpha^*)}{\rho^{-\beta^*}\Gamma(\alpha-\alpha^*)K_\mu+\Gamma(\alpha+1)K_\mu^{\alpha^*}}$. Therefore,

$$\left|\tilde{T}(\phi_1, \tilde{\phi}_1) - \tilde{T}(\phi_2, \tilde{\phi}_2)\right| \leq \frac{L_1}{K}\left(\left\|\phi_1(\mu) - \phi_2(\mu)\right\| + \left\|\tilde{\phi}_1(\mu) - \tilde{\phi}_2(\mu)\right\|\right).$$

Where, $K = \max(K_1^*, K_2^*)$. Thus Banach fixed point theorem [] assures the existence of a unique fixed point of the operator (3.12). Therefore taking into account the Lemma 3.2.9 we concluded that the boundary problem (3.3) has a unique solution. □

Chapter 4

Existence and stability results for fractional differential equations involving generalized Riesz-Caputo Katugampola derivative

4.1 Introduction

The analysis of fractional differential equations has been carried out by various authors. For details, see [1, 2, 3, 4, 6]. Mostly fractional derivatives are computed using the fractional integrals, due to this they describe the non-local effects in terms of left and the right derivative. In 1892, Riesz [15] demonstrated the two-sided fractional operators using the both left and the right Riemann-Liouville's fractional differential and integral operators. In the resent past the research on different properties of solutions to numerous fractional differential and integral equations is the key topic of applied mathematics research. While modeling of many systems and processes in physics, chemistry, optimal control theory, population dynamics, fluid dynamics, fiber optics, electro dynamics, electromegnetic theory etc. all involve fractional differential and integral operators. Due to the two-sided nature of Riesz's differential operator, the interesting differential is specifically used for fractional modeling on finite domain. Some optimality conditions are discussed by Almeida [16] for fractional variational problems with Riesz-Caputo derivative. Frederico et.al derived a Noether's theorem for variational problems having

Riesz-Caputo derivatives. Mandelbrot [17] demonstrated that there ia a close connection between Brownian motion and fractional calculus. Sami,I Muslih and P Agrawal [18] solved the fractional poisson's equation having Riesz derivative using Fourier transform. Due the validity of Riesz derivative operator on the whole domain it appears in the fractional turbulent diffusion model. For example the advection-diffusion process relies on the whole space at any position. Ding, Hengfei et.al [19] numerically solved the advection-diffusion equation having Riesz derivative. For further applications of Riesz derivative on the anomalous diffusion see [20, 21, 22, 23].

In this chapter we define the generalized Riesz-Caputo type derivative operator using the generalized Katugampola integral and differential operators. We present basic perspectives on existence and uniqueness of solutions of fractional differential equations. Motivated by [24, 25], we present the analysis on existence of solutions for the following non-linear fractional differential equation involving generalized Riesz -Caputo type derivative operator with general boundary conditions.

$$\begin{cases} {}_0^{RC}D_\varsigma^{\alpha,\rho}\phi(\mu) = g(\mu, \phi(\mu), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\mu)), & \mu \in [0,\varsigma], \\ \phi(0) = \phi_0, \quad \phi(\varsigma) = \phi_\varsigma, \end{cases} \tag{4.1}$$

where $g : [0,\varsigma] \times \mathbb{R} \to \mathbb{R}$ is continuous with $1 < \alpha \leq 2$ and $0 < \alpha^* \leq 1$.

The rest of the chapter is organized as follows: The Section 4.2 presents some basic definitions and lemmas from literature. In Section 4.3 we introduce the generalized Riesz's fractional operators and derived some useful results. While, in the section 4.4 we established some equivalence results for the boundary value problem (4.1) and establish the results for the existence and uniqueness of solutions for BVP (4.1). The last section of this chapter presents the stability of solutions for BVP (4.1) by means of continuous dependence on parameters.

4.2 Preliminaries

In this section we demonstrate some useful results including definitions and lemmas related to Riesz-Caputo derivatives and integrals that will help us in our later discussions. Following the same traditional definitions of Riesz-Caputo derivative and integral [15, 24, 26] we can generalize these definitions using generalized Caputo type Katugampola derivative operator. Some preliminaries structural properties are also introduced in this section, which we will frequently use in our later discussion. In 2010, Prakash Agrawal defined the Generalized fractional in the following way.

Definition 4.2.1. *[27] Let $\alpha > 0$, then the generalized fractional integral operator $A^{\alpha}_{(a,\varsigma;r,s)}$ is defined as,*

$$A^{\alpha}_{(a,\varsigma;r,s)}\phi(\mu) := \frac{1}{\Gamma(\alpha)}\left[r\int_{a}^{\mu} K_{\alpha}(\mu,\eta)\phi(\eta)d\eta + s\int_{\mu}^{\varsigma} K_{\alpha}(\mu,\eta)\phi(\eta)d\eta \right],$$

where the kernel function $K_{\alpha}(\mu,\eta)$ may depend on α furthermore, $a < \mu < \varsigma$ and $r, s \in \mathbb{R}$.

This is the generalized fractional integral operator, using the specific kernel function leads to the specific operator for example if $K_{\alpha}(\mu,\eta) = \frac{(\mu-\eta)^{\alpha-1}}{\Gamma(\alpha)}$ and taking $\varsigma = 0$ will leads to the left sided R-L integral operator and taking $K_{\alpha}(\mu,\eta) = \frac{\eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{\alpha-1}}{\rho^{\alpha-1}\Gamma(\alpha)}$ with $\varsigma = 0$ gives the left Katugampola integral defined below. Furthermore, the limits of integration a and ς can be extended to $-\infty$ and ∞ respectively.

Lemma 4.2.2. *[30] Let $\alpha, \rho \in \mathbb{R}_{+}$, and $g(\mu) \in AC^{n}_{\delta}[0,\varsigma]$, Then for $0 \leq \mu \leq \varsigma$ following relations holds:*

(i) $$\left({}^{\rho}_{0}I^{\alpha}_{\mu}{}^{\rho}_{*}D^{\alpha}_{0,\mu}g\right)(\mu) = g(\mu) - \sum_{j=0}^{n-1} \frac{\delta^{j}_{\rho}g(0)}{j!}\left(\frac{\mu^{\rho}-0}{\rho}\right)^{j},$$

(ii) $$\left({}^{\rho}_{\mu}I^{\alpha}_{\varsigma}{}^{\rho}_{*}D^{\alpha}_{\mu,\varsigma}g\right)(\varsigma) = (-1)^{n}\left\{ g(\mu) - \sum_{j=0}^{n-1} \frac{\delta^{j}_{\rho}g(\varsigma)}{j!}\left(\frac{\varsigma^{\rho}-\mu^{\rho}}{\rho}\right)^{j} \right\},$$

where $n = \lceil \alpha \rceil$ and $\delta^{j}_{\rho} = \left(\eta^{1-\rho}\frac{d}{d\eta}\right)^{j}$.

Lemma 4.2.3. *[31] Let $\alpha > 0$, $g(\mu)$ and $u_1(\mu)$ are locally integrable, nonnegative and nondecreasing functions with $\mu \in [0,\varsigma]$, also assume that $v_1(\mu)$ be nondecreasing continuous function, such that $0 \leq v_1(\mu) < L$ where L is a constant. Furthermore, if*

$$g(\mu) \leq u_1(\mu) + \rho^{1-\alpha}v_1(\mu)\int_{0}^{\mu} \eta^{\rho-1}(\mu^{\rho}-\eta^{\rho})^{\alpha-1}g(\eta)d\eta, \qquad 0 \leq \mu \leq \varsigma.$$

then the following inequality holds true,

$$g(\mu) \leq u_1(\mu) + \int\limits_0^\mu \left[\sum_{j=1}^{\infty} \frac{\rho^{1-n\alpha}(v_1(\varsigma)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \eta^{\rho-1} u_1(\eta)(\mu^\rho - \eta^\rho)^{n\alpha-1} \right] d\eta.$$

Corollary 4.2.4. *[31] Let $\alpha > 0$ and assume that $g(\mu)$, $u_1(\mu)$, and $v_1(\mu)$ are defined in the same way as in the Lemma 4.2.3. Furthermore if g satisfies,*

$$g(\mu) \leq u_1(\mu) + \rho^{1-\alpha} v_1(\mu) \int\limits_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} g(\eta) d\eta, \qquad 0 \leq \eta \leq \mu.$$

on $\mu \in [0, \varsigma]$, then

$$g(\mu) \leq u_1(\mu) E_{\alpha,1}\left(\rho^{-\alpha} v_1(\mu)\Gamma(\alpha)\mu^{\alpha\rho}\right).$$

Where $E_{\alpha,1}(.)$ is a Mittag-Leffler function [49].

Likewise, the Gronwall inequality for generalized right sided Katugampola fractional operator is defined as;

Lemma 4.2.5. *[31] Let $\alpha > 0$, $\mu \in [0, \varsigma]$ and assume that $g(\mu)$, $u_2(\mu)$, and $v_2(\mu)$ are defined in the same way as in the Lemma 4.2.3. Furthermore if,*

$$g(\mu) \leq u_2(\mu) + \rho^{1-\alpha} v_2(\mu) \int_\mu^\varsigma \eta^{\rho-1}(\eta^\rho - \mu^\rho)^{\alpha-1} g(\eta) d\eta, \qquad 0 \leq \mu \leq \eta.$$

then the following inequality holds true,

$$g(\mu) \leq u_2(\mu) + \int_\mu^\varsigma \left[\sum_{j=1}^{\infty} \frac{\rho^{1-n\alpha}(v_2(\varsigma)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \eta^{\rho-1} u_2(\eta)(\eta^\rho - \mu^\rho)^{n\alpha-1} \right] d\eta.$$

Proof. To prove this lemma we define an operator,

$$\tilde{T}g(\mu) = \rho^{1-\alpha} u_2(\mu) \int_\mu^\varsigma \eta^{\rho-1}(\eta^\rho - \mu^\rho)^{\alpha-1} g(\eta) d\eta.$$

and the sequence $\tilde{T}^j (j \in \mathbb{N})$ as $\tilde{T} = \tilde{T}$, $\tilde{T}^j = \tilde{T}\tilde{T}^{j-1}$ $(j \in \mathbb{N} - \{1\})$. Therefore,

$$g(\mu) \leq u_2(\mu) + \tilde{T}g(\mu).$$

Which implies,

$$g(\mu) \leq \sum_{j=1}^{n-1} \tilde{T}^j u_2(\mu) + \tilde{T}^n g(\mu).$$

Next we claim,

$$\tilde{T}^n g(\mu) \leq \rho^{1-n\alpha} \int_{\mu}^{\varsigma} \frac{(v_2(\varsigma)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \eta^{\rho-1} (\eta^\rho - \mu^\rho)^{n\alpha-1} g(\eta) d\eta. \tag{4.2}$$

and $\tilde{T}^n g(\mu) \to 0$ as $n \to \infty$ for $\mu \in [0, \varsigma]$. We prove the above inequality by induction. Clearly the inequality 4.2 holds true for $n = 1$. Let us assume that it true for $n = k$ as well. i.e,

$$\tilde{T}^k g(\mu) \leq \rho^{1-k\alpha} \int_{\mu}^{\varsigma} \frac{(v_2(\varsigma)\Gamma(\alpha))^k}{\Gamma(k\alpha)} \eta^{\rho-1} (\eta^\rho - \mu^\rho)^{k\alpha-1} g(\eta) d\eta.$$

Now if $n = k + 1$ and using the fact that $\tilde{T}^{k+1} = \tilde{T}\tilde{T}^k$ we have,

$$\begin{aligned} &\tilde{T}^{k+1} g(\mu) \\ &\leq \rho^{2-(k+1)\alpha} (v_2(\varsigma))^{k+1} \int_{\mu}^{\varsigma} \xi^{\rho-1} (\xi^\rho - \mu^\rho)^{\alpha-1} \int_{\xi}^{\varsigma} \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} \eta^{\rho-1} (\eta^\rho - \xi^\rho)^{k\alpha-1} g(\eta) d\eta d\xi, \\ &= \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} (v_2(\varsigma))^{k+1} \int_{\mu}^{\varsigma} \xi^{\rho-1} \left(\frac{\xi^\rho - \mu^\rho}{\rho}\right)^{\alpha-1} \left[\int_{\xi}^{\varsigma} \eta^{\rho-1} \left(\frac{\eta^\rho - \xi^\rho}{\rho}\right)^{k\alpha-1} g(\eta) d\eta\right] d\xi. \end{aligned}$$

Now by changing the order of integration using the special Case of Fubini's Theorem (Dirichlet Formula) introduced by Whittaker [14] in 1965, i.e

$$\int_0^x (x-u)^{\alpha-1} du \int_0^u (u-v)^{\beta-1} F(u,v) dv = \int_0^x dv \int_v^x (x-u)^{\alpha-1} (u-v)^{\beta-1} F(u,v) du.$$

We get,

$$\tilde{T}^{k+1} g(\mu) \leq \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)} (v_2(\varsigma))^{k+1} \int_{\mu}^{\varsigma} \eta^{\rho-1} \left[\int_{\mu}^{\eta} \xi^{\rho-1} \left(\frac{\xi^\rho - \mu^\rho}{\rho}\right)^{\alpha-1} \left(\frac{\eta^\rho - \xi^\rho}{\rho}\right)^{k\alpha-1} d\xi\right] d\eta.$$

Using the substitution $u = \frac{\xi^\rho - \mu^\rho}{\eta^\rho - \mu^\rho}$ into the inner integral and evaluating it yields,

$$\tilde{T}^{k+1}g(\mu) \leq \frac{(\Gamma(\alpha))^k}{\Gamma(k\alpha)}(v_2(\varsigma))^{k+1} \int_\mu^\varsigma \eta^{\rho-1}\left\{B(k\alpha,\alpha)(\eta^\rho - \mu^\rho)^{(k+1)\alpha}\right\}d\eta,$$

$$= \frac{(\Gamma(\alpha))^{k+1}}{\Gamma((k+1)\alpha)}(v_2(\varsigma))^{k+1} \int_\mu^\varsigma \eta^{\rho-1}(\eta^\rho - \mu^\rho)^{(k+1)\alpha}d\eta.$$

Therefore by induction method the relation 4.2 is true. Since $v_2(\varsigma)$ is bounded function and $\lim_{n\to\infty} \frac{\Gamma(\alpha))^n}{\Gamma(n\alpha)} = 0$ as denominator goes to infinity faster than numerator. Therefore,

$$\tilde{T}^n g(\mu) \leq \rho^{1-n\alpha}\int_\mu^\varsigma \frac{(v_2(\varsigma)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\eta^{\rho-1}(\eta^\rho - \mu^\rho)^{n\alpha-1}g(\eta)d\eta \to 0 \ \ as \ \ k \to \infty$$

for $\mu \in [0,\varsigma]$. Hence,

$$g(\mu) \leq u_2(\mu) + \sum_{j=1}^{n-1}\tilde{T}^j u_2(\mu) + \tilde{T}^n g(\mu),$$

$$= u_2(\mu) + \sum_{j=1}^{n-1}\tilde{T}^j u_2(\mu) = u_2(\mu)$$

$$+ \int_\mu^\varsigma \left[\sum_{j=1}^{\infty}\frac{\rho^{1-n\alpha}(v_2(\varsigma)\Gamma(\alpha))^n}{\Gamma(n\alpha)}\eta^{\rho-1}u_2(\eta)(\eta^\rho - \mu^\rho)^{n\alpha-1}\right]d\eta.$$

□

4.3 Generalized Riesz-Caputo fractional operators.

In this section we introduce the generalized Riesz fractional integrals and derivative operators.

Definition 4.3.1. *[15] For $g(\mu) \in C(0,\varsigma)$ the classical Riesz-Caputo derivative is defined by,*

$$ {}^R_* D^\alpha_{0,\varsigma} g(\mu) = \frac{1}{\Gamma(n-\alpha)}\int_0^\varsigma |\mu - \eta|^{n-\alpha-1}g^{(n)}(\eta)d\eta,$$

$$= \frac{1}{2}({}_*D^\alpha_{0,\mu} + (-1)^n {}_*D^\alpha_{\mu,\varsigma})g(\mu).$$

Where ${}_*D^\alpha_{0,\mu}$ and ${}_*D^\alpha_{\mu,\varsigma}$ are left and the right Caputo derivative operators respectively.

Following the same mechanism we generalize the Riesz fractional integral by means of the definition 2.3.1 as follows,

Definition 4.3.2. *Let $g(\mu) \in X_c^p(a,b)$ and $\alpha, \rho > 0$ then, for $0 \leq \mu \leq \varsigma$ the generalized Riesz type integral is defined as,*

$$\begin{aligned}({}_0^{\rho}I_{\varsigma}^{\alpha} g)(\mu) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\varsigma} \eta^{\rho-1} |(\eta^{\rho} - \mu^{\rho})|^{\alpha-1} g(\eta) d\eta \\ &= {}_0^{\rho}I_{\mu}^{\alpha} g(\mu) + {}_{\mu}^{\rho}I_{\varsigma}^{\alpha} g(\mu).\end{aligned}$$

Accordingly, the Riesz-Caputo derivative [15] can be generalized by means of generalized Katugampola Caputo type derivative operators [5] as follows.

Definition 4.3.3. *Let $\alpha, \rho \in \mathbb{C}$ with $\Re\mathrm{e}(\alpha), \Re\mathrm{e}(\rho) > 0$ and if $g(\mu) \in X_c^{\rho}(a,b)$ then for $0 \leq \mu \leq \varsigma$ then the generalized Riesz-Caputo type derivative operator is defined as,*

$$\begin{aligned}{}_0^{RC}D_{\varsigma}^{\alpha,\rho} g(\mu) &= \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_0^{\varsigma} \frac{\eta^{\rho-1}}{|(\mu^{\rho} - \eta^{\rho})|^{\alpha-n+1}} \left(\eta^{1-\rho} \frac{d}{d\eta}\right)^n g(\eta) d\eta \\ &= \frac{1}{2} \left({}_*^{\rho}D_{0,\mu}^{\alpha} + (-1)^n {}_*^{\rho}D_{\mu,\varsigma}^{\alpha} \right) g(\mu),\end{aligned}$$

where ${}_*^{\rho}D_{0,\mu}^{\alpha}$ and ${}_*^{\rho}D_{\mu,\varsigma}^{\alpha}$ are the left and the right generalized Caputo type Katugampola derivatives [29] respectively.

$${}_*^{\rho}D_{0,\mu}^{\alpha} = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_0^{\mu} \frac{\eta^{\rho-1}}{(\mu^{\rho} - \eta^{\rho})^{\alpha-n+1}} \left(\eta^{1-\rho} \frac{d}{d\eta}\right)^n g(\eta) d\eta,$$

and

$${}_*^{\rho}D_{\mu,\varsigma}^{\alpha} = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_{\mu}^{\varsigma} \frac{\eta^{\rho-1}}{(\eta^{\rho} - \mu^{\rho})^{\alpha-n+1}} \left(-\eta^{1-\rho} \frac{d}{d\eta}\right)^n g(\eta) d\eta,$$

where $n = \lceil \alpha \rceil$.

Since, for $\alpha = 1$ the right Katugampola derivative is the negative of the left Katugampola derivative, so for integer values of α, the generalized Riesz type derivative defined above comes to term with conventional definitions of derivative.

Lemma 4.3.4. *Let $g \in AC_{\delta}^n[0,\varsigma]$ with $0 \leq \mu \leq \varsigma$, then following relation holds true:*

$${}_0^{\rho}I_{\varsigma}^{\alpha} {}_0^{RC}D_{\varsigma}^{\alpha,\rho} g(\mu) = \frac{1}{2} \left({}_0^{\rho}I_{\mu}^{\alpha} {}_*^{\rho}D_{0,\mu}^{\alpha} + (-1)^n {}_{\mu}^{\rho}I_{\varsigma}^{\alpha} {}_*^{\rho}D_{\mu,\varsigma}^{\alpha} \right) g(\mu). \qquad (*)$$

Proof. Using the above definitions, we can write

$$\begin{aligned}
{}_{0}^{\rho}I_{\varsigma}^{\alpha}{}_{0}^{RC}D_{\varsigma}^{\alpha,\rho}g(\mu) &= \frac{1}{2}{}_{0}^{\rho}I_{\varsigma}^{\alpha}\left({}_{*}^{\rho}D_{0,\mu}^{\alpha} + (-1)^{n}{}_{*}^{\rho}D_{\mu,\varsigma}^{\alpha}\right)g(\mu) \\
&= \frac{1}{2}{}_{0}^{\rho}I_{\varsigma}^{\alpha}{}_{*}^{\rho}D_{0,\mu}^{\alpha}g(\mu) + \frac{(-1)^{n}}{2}{}_{0}^{\rho}I_{\varsigma}^{\alpha}{}_{*}^{\rho}D_{\mu,\varsigma}^{\alpha}g(\mu) \\
&= \frac{1}{2}\left({}_{0}^{\rho}I_{\mu}^{\alpha}{}_{*}^{\rho}D_{0,\mu}^{\alpha} +_{\mu}^{\rho} I_{\varsigma}^{\alpha}{}_{*}^{\rho}D_{0,\mu}^{\alpha}\right)g(\mu) + \frac{(-1)^{n}}{2}\left({}_{0}^{\rho}I_{\mu}^{\alpha}{}_{*}^{\rho}D_{\mu,\varsigma}^{\alpha} +_{\mu}^{\rho} I_{\varsigma}^{\alpha}{}_{*}^{\rho}D_{\mu,\varsigma}^{\alpha}\right)g(\mu) \\
&= \frac{1}{2}\left({}_{0}^{\rho}I_{\mu}^{\alpha}{}_{*}^{\rho}D_{0,\mu}^{\alpha} + (-1)^{n}{}_{\mu}^{\rho}I_{\varsigma}^{\alpha}{}_{*}^{\rho}D_{\mu,\varsigma}^{\alpha}\right)g(\mu).
\end{aligned}$$

□

Remark 6. *If* $0 < \alpha \leq 1$, *and for* $g(\mu) \in C[0, \varsigma]$. *Then relation in* $(*)$ *becomes,*

$$ {}_{0}^{\rho}I_{\varsigma}^{\alpha}{}_{0}^{RC}D_{\varsigma}^{\alpha,\rho}g(\mu) = g(0) - \frac{1}{2}\left(g(0) + g(\varsigma)\right). $$

Proof. The proof simply follows by using $n = 1$ in the Lemma 4.3.4 and using the Lemma 4.2.2, yields the required result. □

Theorem 4.3.1. *Let* $\alpha > 0$, *and* $\{\phi_j\}_{j=1}^{\infty}$ *be a uniformly convergent sequence of continuous functions on* $[a, b]$. *Then we can interchange the generalized Riesz fractional integral operator and the limit. i.e,*

$$ ({}_{0}^{\rho}I_{\varsigma}^{\alpha}\lim_{j\to 0}\phi_j)(\mu) = (\lim_{j\to\infty}{}_{0}^{\rho}I_{\varsigma}^{\alpha}\phi_j)(\mu). $$

Proof. The result follows taking into account the definition 4.3.2, the Theorem 2.3.4 and the fact that sum of two convergent sequence is convergent. □

Lemma 4.3.5. *Let* $\alpha > 0$ *and assume that* $g(\mu)$, $u_1(\mu)$, *and* $v_1(\mu)$ *are defined in the same way as in the Lemma 4.2.3. Furthermore, if*

$$ g(\mu) \leq u_2(\mu) + \rho^{1-\alpha}v_2(\mu)\int_{\mu}^{\varsigma}\eta^{\rho-1}(\eta^{\rho} - \mu^{\rho})^{\alpha-1}g(\eta)d\eta, \qquad 0 \leq \mu \leq \eta. $$

on $\mu \in [0, \varsigma]$, *then*

$$ g(\mu) \leq u_2(\mu)E_{\alpha,1}\left(\rho^{-\alpha}v_2(\mu)\Gamma(\alpha)(\varsigma^{\rho} - \mu^{\rho})^{\alpha}\right). $$

Proof. From Lemma 4.2.5,

$$g(\mu) \leq u_2(\mu) + \int_{\mu}^{\varsigma} \left[\sum_{j=1}^{\infty} \frac{\rho^{1-n\alpha}(v_2(\varsigma)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \eta^{\rho-1} u_2(\eta)(\eta^{\rho} - \mu^{\rho})^{n\alpha-1} \right] d\eta.$$

Since $u_2(\eta)$ is nondecreasing function, therefore $u_2(\eta) \leq u_2(\mu)$ for all $\eta \in [0, \varsigma]$, and hence

$$\begin{aligned}
g(\mu) &\leq +u_2(\mu) \left\{ 1 + \int_{\mu}^{\varsigma} \sum_{j=1}^{\infty} \frac{\rho^{1-n\alpha}(v_2(\varsigma)\Gamma(\alpha))^n}{\Gamma(n\alpha)} \eta^{\rho-1} u_2(\eta)(\eta^{\rho} - \mu^{\rho})^{n\alpha-1} d\eta \right\} \\
&= u_2(\mu) \left\{ 1 + \sum_{j=1}^{\infty} \frac{\rho^{-n\alpha}(v_2(\varsigma)\Gamma(\alpha))^n}{\Gamma(n\alpha+1)} (\varsigma^{\rho} - \mu^{\rho})^{n\alpha} \right\} \\
&= u_2(\mu) \left\{ \sum_{j=0}^{\infty} \frac{(\rho^{-\alpha} v_2(\varsigma)\Gamma(\alpha)(\varsigma^{\rho} - \mu^{\rho})^{\alpha})^n}{\Gamma(n\alpha+1)} \right\} \\
&= u_2(\mu) E_{\alpha,1} \left(\rho^{-\alpha} v_2(\mu)\Gamma(\alpha)(\varsigma^{\rho} - \mu^{\rho})^{\alpha} \right).
\end{aligned}$$

□

Lemma 4.3.6. *Let $\alpha > 0$, $0 < \mu < \varsigma$ and assume that $g(\mu)$, $u_1(\mu)$, $u_2(\mu)$, $v_1(\mu)$ $v_2(\mu)$ are defined in the same way as in the Lemma 4.2.3 and the Lemma 4.2.5. Furthermore if $g(\mu)$ satisfies the inequality,*

$$\begin{aligned}
g(\mu) \leq u_1(\mu) + \rho^{1-\alpha} v_1(\mu) \int_{0}^{\mu} \eta^{\rho-1}(\mu^{\rho} - \eta^{\rho})^{\alpha-1} g(\eta) d\eta \\
+ u_2(\mu) + \rho^{1-\alpha} v_2(\mu) \int_{\mu}^{\varsigma} \eta^{\rho-1}(\eta^{\rho} - \mu^{\rho})^{\alpha-1} g(\eta) d\eta,
\end{aligned}$$

then the following inequality holds true,

$$g(\mu) \leq (u_1(\mu) + u_2(\mu)) E_{\alpha,1} \left(\rho^{-\alpha} v_2(\mu)\Gamma(\alpha)(\varsigma^{\rho} - \mu^{\rho})^{\alpha} \right) E_{\alpha,1} \left(\rho^{-\alpha} v_1(\mu)\Gamma(\alpha)\mu^{\alpha\rho} \right).$$

where $E_{\alpha,1}(.)$ is a Mittag-Leffler function.

Proof. Conflating the Lemma 4.2.3 and the Lemma 4.3.5 gives,

$$\begin{aligned} g(\mu) &\leq \left(u_1(\mu) + u_2(\mu) + \rho^{1-\alpha} v_2(\mu) \int_\mu^\varsigma \eta^{\rho-1}(\eta^\rho - \mu^\rho)^{\alpha-1} d\eta\right) E_{\alpha,1}\left(\rho^{-\alpha} v_1(\mu)\Gamma(\alpha)\mu^{\alpha\rho}\right), \\ &\leq (u_1(\mu) + u_2(\mu)) E_{\alpha,1}\left(\rho^{-\alpha} v_2(\mu)\Gamma(\alpha)(\varsigma^\rho - \mu^\rho)^\alpha\right) E_{\alpha,1}\left(\rho^{-\alpha} v_1(\mu)\Gamma(\alpha)\mu^{\alpha\rho}\right). \end{aligned}$$

□

These Gronwall inequalities are helpful to compute the estimated difference of solutions of two differential equations. Now we define the similar type of inequality for generalized Caputo type Riesz-Katugampola fractional operators.

4.4 Main Results

For the upcoming existence results and discussion for the boundary value (4.1) we use the following conditions. Let $J = [0, \varsigma]$ and $C(J)$ be the space of all continuous functions defined on J. We define the space $X = \left\{\phi(\mu) | \phi(\mu) \in C(J) \ \ and \ \ {}^\rho_* D^{\alpha^*} \phi(\mu) \in C(J)\right\}$ characterized by the norm $\|\phi(\mu)\|_X = \max_{\mu \in J} |\phi(\mu)| + \max_{\mu \in J} \left|{}^\rho_* D^{\alpha^*} \phi(\mu)\right|$.

Lemma 4.4.1. $(X, \|.\|_X)$ *ia a Banach space.*

Proof. Let $\{\phi_j\}_{j=0}^\infty$ be a cauchy sequence in $(X, \|.\|_X)$, then clearly $\left\{{}^\rho_* D^{\alpha^*} \phi_j\right\}_{j=0}^\infty$ is also a cauchy sequence in the space $C(J)$. Therefore both $\{\phi_j(\mu)\}_{j=0}^\infty$ and $\left\{{}^\rho_* D^{\alpha^*} \phi_j(\mu)\right\}_{j=0}^\infty$ converges uniformly, say $u(\mu)$ and $v(\mu)$ respectively in the space $C(J)$. We just have to show that $v = {}^\rho_* D^{\alpha^*} u$. For this consider.

$$\begin{aligned} &\left|{}^\rho I_{0+}^{\beta^*} {}^\rho_* D_{0+}^{\alpha^*} \phi_j(\mu) - {}^\rho I_{0+}^{\alpha^*} v(\mu)\right| \\ &= \left|\frac{\rho^{1-\alpha^*}}{\Gamma(\alpha^*)} \int_0^\mu \frac{{}^\rho_* D_{0+}^{\alpha^*} \phi_j(\eta)\eta^{\rho-1}}{(\mu^\rho - \eta^\rho)^{1-\alpha^*}} d\eta - \frac{\rho^{1-\alpha^*}}{\Gamma(\alpha^*)} \int_0^\mu \frac{v(\eta)\eta^{\rho-1}}{(\mu^\rho - \eta^\rho)^{1-\alpha^*}} d\eta\right|, \\ &\leq \frac{\rho^{1-\alpha^*}}{\Gamma(\alpha^*)} \int_0^\mu \left|\frac{({}^\rho_* D_{0+}^{\alpha^*} \phi_j(\eta) - v(\eta))\eta^{\rho-1}}{(\mu^\rho - \eta^\rho)^{1-\alpha^*}}\right| d\eta, \\ &\leq \frac{\mu^{\rho\alpha^*}}{\rho\Gamma(\alpha^* + 1)} \max_{\mu \in J} \left|{}^\rho_* D_{0+}^{\alpha^*} \phi_j(\mu) - v(\mu)\right|. \end{aligned}$$

Since $\left\{{}^{\rho}_{*}D^{\alpha^*}\phi_j(\mu)\right\}_{j=0}^{\infty}$ converges uniformly to $v(\mu)$ for $\mu \in J$.
Hence $\left|{}^{\rho}I_{0+}^{\alpha^*}{}^{\rho}_{*}D_{0+}^{\alpha^*}\phi_j(\mu) - {}^{\rho}I_{0+}^{\beta^*}v(\mu)\right| \to 0.$ as $j \to \infty$. i.e, $\lim_{j\to\infty} {}^{\rho}I_{0+}^{\alpha^*}{}^{\rho}_{*}D_{0+}^{\alpha^*}\phi_j(\mu) \cong {}^{\rho}I_{0+}^{\alpha^*}v(\mu)$. Now considering ${}^{\rho}_{*}D_{0+}^{\alpha^*}(\lim_{j\to\infty} {}^{\rho}I_{0+}^{\alpha^*}{}^{\rho}_{*}D_{0+}^{\beta^*}\phi_j(\mu)) = {}^{\rho}_{*}D_{0+}^{\alpha^*}{}^{\rho}I_{0+}^{\alpha^*}v(\mu)$ and taking into account the Theorem 2.3.4 and the Theorem 2.5.4 we get, $v(\mu) = {}^{\rho}_{*}D^{\alpha^*}u(\mu)$. This completes the proof. □

Lemma 4.4.2. *Let $\alpha \in (1, 2), \alpha^* \in (0, 1)$ and $g \in C(J)$, then the problem (4.1) is equivalent to the following integral equation,*

$$\begin{aligned}
\phi(\mu) = &\frac{1}{2}(\phi_0 + \phi_\varsigma) + \left(\frac{\phi_\varsigma - \phi_0}{2\varsigma^\rho}\right)(2\mu^\rho - \varsigma^\rho) \\
&+ \frac{\rho^{1-\alpha}(\mu^\rho - \varsigma^\rho)}{\varsigma^\rho\Gamma(\alpha)} \int_0^{\varsigma} \eta^{\alpha\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))d\eta \\
&- \frac{\mu^\rho\rho^{1-\alpha}}{\varsigma^\rho\Gamma(\alpha)} \int_0^{\varsigma} \eta^{\rho-1}(\varsigma^\rho - \eta^\rho)^{\alpha-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))d\eta \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\varsigma} \eta^{\rho-1}|\eta^\rho - \mu^\rho|^{\alpha-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))d\eta. \\
= &\frac{1}{2}(\phi_0 + \phi_s) + \psi(\mu).
\end{aligned} \tag{4.3}$$

where,

$$\psi(\mu) = \begin{cases} \frac{(\phi_s-\phi_0)\mu^\rho}{2\varsigma^\rho} - \frac{\mu^\rho\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int_0^{\varsigma} \frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\varsigma^\rho-\eta^\rho)^{1-\alpha}}d\eta + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mu} \frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\mu^\rho-\eta^\rho)^{1-\alpha}}d\eta, & \mu > \eta, \\ \frac{(\phi_s-\phi_0)(\mu^\rho-\varsigma^\rho)}{2\varsigma^\rho} + \frac{(\mu^\rho-\varsigma^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int_0^{\varsigma} \frac{g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}d\eta + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\mu^{\varsigma} \frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho-\mu^\rho)^{1-\alpha}}d\eta, & \eta > \mu, \end{cases}$$

Proof. Let $\phi(\mu) \in X$ be a solution of the boundary value problem (4.1), then by applying the generalized Riesz-type integral operator on both sides of equation (4.1) and using the definition 4.3.2, the Lemma 4.2.2 and the Lemma 4.3.4 yields,

$$\begin{aligned}
&\frac{1}{2}\phi(\mu) - \frac{1}{2}\phi(0) - c_0\frac{\mu^\rho}{2\rho} + \frac{1}{2}\phi(\mu) - \frac{1}{2}\phi(\varsigma) - \frac{1}{2}c_1(\frac{\mu^\rho - \varsigma^\rho}{\rho}) \\
&= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{\varsigma} \eta^{\rho-1}|(\eta^\rho - \mu^\rho)|^{\alpha-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*}\phi(\eta))d\eta.
\end{aligned}$$

or

$$\phi(\mu) = \frac{1}{2}(\phi_0 + \phi_\varsigma) + \frac{c_0\mu^\rho}{2\rho} + \frac{1}{2}c_1(\frac{\mu^\rho - \varsigma^\rho}{\rho})$$
$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*}\phi(\eta))d\eta$$
$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\mu^\varsigma \eta^{\rho-1}(\eta^\rho - \mu^\rho)^{\alpha-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*}\phi(\eta))d\eta.$$

Using the boundary conditions $\phi(0) = \phi_0$ and $\phi(\varsigma) = \phi_\varsigma$ into above equation we get $c_0 = \frac{\rho(\phi_\varsigma - \phi_0)}{\varsigma^\rho} - \frac{2\rho^{2-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int_0^\varsigma \frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\varsigma^\rho-\eta^\rho)^{1-\alpha}}d\eta.$ and $c_1 = \frac{\rho(\phi_s - \phi_0)}{\varsigma^\rho} + \frac{2\rho^{2-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int_0^\varsigma \frac{g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}d\eta.$
Now again substituting these values of constants into the above equation we get,

$$\phi(\mu) = \frac{1}{2}(\phi_0 + \phi_\varsigma)\frac{(\phi_s - \phi_0)\mu^\rho}{2\varsigma^\rho} - \frac{\mu^\rho\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int_0^\varsigma \frac{\eta^{\rho-1}g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\varsigma^\rho - \eta^\rho)^{1-\alpha}}d\eta$$
$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\mu \frac{\eta^{\rho-1}g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\mu^\rho - \eta^\rho)^{1-\alpha}}d\eta + \frac{(\phi_s - \phi_0)(\mu^\rho - \varsigma^\rho)}{2\varsigma^\rho}$$
$$+ \frac{(\mu^\rho - \varsigma^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int_0^\varsigma \frac{g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}d\eta$$
$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\mu^\varsigma \frac{\eta^{\rho-1}g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho - \mu^\rho)^{1-\alpha}}d\eta.$$

$$\phi(\mu) = \frac{1}{2}(\phi_0 + \phi_\varsigma) + \left(\frac{\phi_\varsigma - \phi_0}{2\varsigma^\rho}\right)(2\mu^\rho - \varsigma^\rho)$$
$$+ \frac{\rho^{1-\alpha}(\mu^\rho - \varsigma^\rho)}{\varsigma^\rho\Gamma(\alpha)}\int_0^\varsigma \eta^{\alpha\rho-1}g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))d\eta$$
$$- \frac{\mu^\rho\rho^{1-\alpha}}{\varsigma^\rho\Gamma(\alpha)}\int_0^\varsigma \eta^{\rho-1}(\varsigma^\rho - \eta^\rho)^{\alpha-1}g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))d\eta$$
$$+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\varsigma \eta^{\rho-1}|\eta^\rho - \mu^\rho|^{\alpha-1}g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))d\eta.$$
$$= \frac{1}{2}(\phi_0 + \phi_\varsigma) + \psi(\mu).$$

Where,

$$\psi(\mu) = \begin{cases} \frac{(\phi_\varsigma - \phi_0)\mu^\rho}{2\varsigma^\rho} - \frac{\mu^\rho \rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho} \int\limits_0^\varsigma \frac{\eta^{\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\varsigma^\rho - \eta^\rho)^{1-\alpha}} d\eta + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int\limits_0^\mu \frac{\eta^{\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\mu^\rho - \eta^\rho)^{1-\alpha}} d\eta, & \mu > \eta, \\ \frac{(\phi_\varsigma - \phi_0)(\mu^\rho - \varsigma^\rho)}{2\varsigma^\rho} + \frac{(\mu^\rho - \varsigma^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho} \int\limits_0^\varsigma \frac{g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}} d\eta + \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int\limits_\mu^\varsigma \frac{\eta^{\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho - \mu^\rho)^{1-\alpha}} d\eta, & \eta > \mu, \end{cases}$$

Conversely, let $\phi(\mu) \in X$ be a solution of the fractional integral operator (4.3) and we denote the right hand side of the equation (4.3) by $\Phi(\mu)$. i.e,

$$\Phi(\mu) = \frac{1}{2}(\phi_0 + \phi_\varsigma) + \psi(\mu).$$

Now taking left and the right Caputo-Katugampola derivative on both sides of above equation we get,

$$\begin{aligned} {}_*^\rho D_{0,\mu}^\alpha \Phi(\mu) =& {}_*^\rho D_{0,\mu}^\alpha (\frac{1}{2}(\phi_0 + \phi_s)) + \frac{(\phi_\varsigma - \phi_0)}{2\varsigma^\rho} {}_*^\rho D_{0,\mu}^\alpha (\mu^\rho) \\ & - \frac{{}_0^\rho I_\varsigma^\alpha g(\varsigma, \phi(\varsigma), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\varsigma))}{\varsigma^\rho} {}_*^\rho D_{0,\mu}^\alpha (\mu^\rho) \\ & + {}_*^\rho D_{0,\mu}^\alpha {}_0^\rho I_\mu^\alpha g(\mu, \phi(\mu), {}_0^{RC}D_\varsigma^{\alpha^*}\phi(\mu)) \\ =& g(\mu, \phi(\mu), {}_0^{RC}D_\varsigma^{\alpha^*}\phi(\mu)). \end{aligned} \tag{4.4}$$

$$\begin{aligned} {}_*^\rho D_{\mu,\varsigma}^\alpha \Phi(\mu) =& {}_*^\rho D_{\mu,\varsigma}^\alpha (\frac{1}{2}(\phi_0 + \phi_s)) + \frac{(\phi_\varsigma - \phi_0)}{2\varsigma^\rho} {}_*^\rho D_{\mu,\varsigma}^\alpha (\mu^\rho - \varsigma^\rho) \\ & + \frac{{}_0^\rho I_\varsigma^\alpha g(\varsigma, \phi(\varsigma), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\varsigma))}{\varsigma^\rho} {}_*^\rho D_{\mu,\varsigma}^\alpha (\mu^\rho - \varsigma^\rho) \\ & + {}_*^\rho D_{\mu,\varsigma}^\alpha {}_\mu^\rho I_\varsigma^\alpha g(\mu, \phi(\mu), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\mu)) \\ =& g(\mu, \phi(\mu), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\mu)). \end{aligned} \tag{4.5}$$

Here we have used the Theorem 2.5.4 and some simple calculation leads to the facts that ${}_*^\rho D_{0,\mu}^\alpha (\mu^\rho) = 0$ and ${}_*^\rho D_{\mu,\varsigma}^\alpha (\mu^\rho - \varsigma^\rho) = 0$. Consequently from equations (4.4), (4.5) and the definition 4.3.3 the required result follows, i.e.

$$\frac{1}{2}\left({}_*^\rho D_{0,\mu}^\alpha \Phi(\mu) + {}_*^\rho D_{\mu,\varsigma}^\alpha \Phi(\mu)\right) = {}_0^{RC}D_\varsigma^{\alpha,\rho}\Phi(\mu) = g(\mu, \phi(\mu), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\mu)).$$

□

Now we present the existence and uniqueness results for the non-linear boundary value problem (4.1). We define an operator $\tilde{T} : X \to X$ by,

$$\begin{aligned}\tilde{T}(\phi(\mu)) =& \frac{1}{2}\left(\phi_0 + \phi_\varsigma\right) + \left(\frac{\phi_\varsigma - \phi_0}{2\varsigma^\rho}\right)(2\mu^\rho - \varsigma^\rho) \\ &+ \frac{\rho^{1-\alpha}\left(\mu^\rho - \varsigma^\rho\right)}{\varsigma^\rho\Gamma(\alpha)} \int_0^\varsigma \eta^{\alpha\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))d\eta \\ &- \frac{\mu^\rho\rho^{1-\alpha}}{\varsigma^\rho\Gamma(\alpha)} \int_0^\varsigma \eta^{\rho-1}(\varsigma^\rho - \eta^\rho)^{\alpha-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))d\eta \\ &+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^\varsigma \eta^{\rho-1}|\eta^\rho - \mu^\rho|^{\alpha-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))d\eta. \end{aligned} \tag{4.6}$$

The Lemma 4.4.2 signifies that solutions of the problem 4.1 coincides with the fixed points of the operator $T(\phi(\mu))$. Ahead of the the detailed existence results, let us have the following consideration first.

(H_1^*) $1 < \alpha < 2$, $0 < \alpha^* < 1$ and $g : [0, \varsigma] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function and $U(\mu) \in L^1[J, \mathbb{R}_+]$, be a nonnegative function such that $U(\mu) \leq \phi(\mu)$. Furthermore g satisfies,

$$\left|g(\mu, \phi(\mu), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\mu))\right| \leq \rho^\alpha \left(a_1 \left|\phi(\mu)\right| + a_2 \left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\mu)\right|\right) + \frac{b\rho^\alpha}{\varsigma^\rho}U(\mu).$$

where $a_1, a_2, b \in \mathbb{R}_+$.

(H_2^*) $1 < \alpha < 2$, $0 < \alpha^* < 1$ and $g : [0, \varsigma] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function and g satisfies, the Lipschitz condition, i.e.

$$\begin{aligned}&\left|g(\eta, \phi_1(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta)) - g(\eta, \phi_2(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right| \\ &\quad \leq \lambda_1 \left(\left|\phi_1(\mu) - \phi_2(\mu)\right| + \left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta)) - {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|\right).\end{aligned}$$

where $0 < \lambda_1 < \frac{1}{2}\max\{K_1, K_2\}$.

Let $M_1 = \max_{\mu \in J} \{h_1(\mu) : |h_1(\mu)| \leq d_2\}$ and $M^* = \max_{\mu \in J} \{f(\mu) : |f(\mu)| \leq d_1\}$, where,

$$\begin{aligned} h_1(\mu) = & \frac{\rho^{\alpha^*} |\phi_\varsigma - \phi_0| \mu^{\rho(1-\alpha^*)}}{2\varsigma^\rho \Gamma(2-\alpha^*)} + \frac{2a_3 K \mu^{\rho(\alpha-\alpha^*-1)}}{\varsigma^\rho \rho^{1-\alpha^*} \Gamma(2-\alpha^*)} \\ & + \frac{2a_3 \rho^{\alpha-1} K \mu^{\rho(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*+1)} + \frac{b\rho^{1+\alpha^*} K^* \mu^{\rho(1-\alpha^*)}}{\Gamma(2-\alpha^*)\varsigma^{2\rho}}. \end{aligned}$$

and

$$f(\mu) = \varsigma^{\rho(\alpha-1)} + \mu^{\alpha\rho} + (\varsigma^\rho - \mu^\rho)^\alpha.$$

Furthermore, let

$$K^* := \frac{\rho}{\Gamma(\alpha)} \max \left\{ \int_0^\varsigma \eta^{\rho-1} (\varsigma^\rho - \eta^\rho)^{\alpha-1} U(\eta) d\eta, \int_0^\varsigma \eta^{\alpha\rho-1} U(\eta) d\eta \right\}.$$

$$L_1 = \sup \left[\max_{\mu \in J} \left(\varsigma^{\alpha\rho} + \mu^{\alpha\rho} + (\varsigma^\rho - \mu^\rho)^\alpha \right) \right].$$

$$\begin{aligned} L_2 = \sup[\max_{\mu \in J} \{ & \frac{\varsigma^{\rho(\alpha-1)} \mu^{\rho(1-\alpha^*)}}{\Gamma(\alpha+1)\Gamma(2-\alpha^*)} + \frac{\mu^{(\alpha-\alpha^*)}}{\Gamma(2-\alpha^*+1)} + \frac{\varsigma^{\rho(\alpha-1)} (\varsigma^\rho - \mu^\rho)^{1-\alpha^*}}{\Gamma(\alpha+1)\Gamma(2-\alpha^*)} \\ & + \frac{(\varsigma^\rho - \mu^\rho)^{\alpha-\alpha^*}}{\Gamma(\alpha-\alpha^*+1)} \}]. \end{aligned}$$

By means of local integrability of $U(\mu)$, K^* exists certainly. Define a set $A_r = \{\phi \in C(J) : \|\phi\| < r\}$, where $r = \left\{ 4 \max \left(|\phi_s|, |\phi_0|, \frac{2bK^*}{\varsigma^\rho}, \frac{2Ka_3 M^*}{\Gamma(\alpha+1)} \right) \right\} E^2_{\alpha,1}(b)$. Then manifestly the set A_r is a closed, bounded and convex subset of the above defined Banach space $(X, \|.\|_X)$.

Theorem 4.4.1. *Assume that the condition* (H_1^*) *holds then the problem 4.1 has atleast one solution in* A_r.

Proof. We prove this result using the Schauder fixed point theorem. First we show that the operator $\tilde{T} : A_r \to A_r$ is self mapped. Suppose $\phi \in A_r$ and for $L \in (0,1)$, the operator (4.6) satisfies $\phi(\mu) = L\tilde{T}(\phi(\mu))$, then from (4.6) and using the condition (H_1^*) we have that,

$$
\begin{aligned}
|\phi(\mu)| \leq \left|\tilde{T}\phi(\mu)\right| \leq & \frac{1}{2}\left|(\phi_0 + \phi_\varsigma)\right| + \frac{\left|\phi_s - \phi_0\right| \mu^\rho}{2\varsigma^\rho} \\
& + \frac{\left|\phi_0 - \phi_\varsigma\right| (\varsigma^\rho - \mu^\rho)}{2\varsigma^\rho} \\
& + \frac{a_1 \rho \mu^\rho}{\varsigma^\rho \Gamma(\alpha)} \int_0^\varsigma \eta^{\rho-1} (\varsigma^\rho - \eta^\rho)^{\alpha-1} \left|\phi(\eta)\right| d\eta \\
& + \frac{a_2 \rho \mu^\rho}{\varsigma^\rho \Gamma(\alpha)} \int_0^\varsigma \eta^{\rho-1} (\varsigma^\rho - \eta^\rho)^{\alpha-1} \left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta)\right| d\eta \\
& + \frac{b\rho\mu^\rho}{\varsigma^{2\rho}\Gamma(\alpha)} \int_0^\varsigma \eta^{\rho-1} (\varsigma^\rho - \eta^\rho)^{\alpha-1} U(\eta) d\eta + \frac{a_1\rho}{\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1} (\mu^\rho - \eta^\rho)^{\alpha-1} \left|\phi(\eta)\right| d\eta \\
& + \frac{a_2\rho}{\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1} (\mu^\rho - \eta^\rho)^{\alpha-1} \left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta)\right| d\eta \\
& + \frac{b\rho}{\varsigma^\rho\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1} (\mu^\rho - \eta^\rho)^{\alpha-1} U(\eta) d\eta \\
& + \frac{a_1\rho(\varsigma^\rho - \mu^\rho)}{\Gamma(\alpha)\varsigma^\rho} \int_0^\varsigma \eta^{\alpha\rho-1} \left|\phi(\eta)\right| d\eta + \frac{a_2\rho(\varsigma^\rho - \mu^\rho)}{\Gamma(\alpha)\varsigma^\rho} \int_0^\varsigma \eta^{\alpha\rho-1} \left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta)\right| d\eta \\
& + \frac{b\rho(\varsigma^\rho - \mu^\rho)}{\Gamma(\alpha)\varsigma^{2\rho}} \int_0^\varsigma \eta^{\alpha\rho-1} U(\eta) d\eta + \frac{a_1\rho}{\Gamma(\alpha)} \int_\mu^\varsigma \eta^{\rho-1} (\eta^\rho - \mu^\rho)^{\alpha-1} \left|\phi(\eta)\right| d\eta \\
& + \frac{a_2\rho}{\Gamma(\alpha)} \int_\mu^\varsigma \eta^{\rho-1} (\eta^\rho - \mu^\rho)^{\alpha-1} \left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta)\right| d\eta \\
& + \frac{b\rho}{\varsigma^\rho\Gamma(\alpha)} \int_\mu^\varsigma \eta^{\rho-1} (\eta^\rho - \mu^\rho)^{\alpha-1} U(\eta) d\eta.
\end{aligned}
$$

$$\leq |\phi_s| + |\phi_0| + \frac{2a_3K\rho\mu^\rho}{\varsigma^\rho\Gamma(\alpha)} \int_0^\varsigma \eta^{\rho-1}(\varsigma^\rho - \eta^\rho)^{\alpha-1}d\eta + \frac{b\rho\mu^\rho}{\varsigma^{2\rho}\Gamma(\alpha)} \int_0^\varsigma \eta^{\rho-1}(\varsigma^\rho - \eta^\rho)^{\alpha-1}U(\eta)d\eta$$
$$+ \frac{2a_3K\rho}{\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1}d\eta + \frac{b\rho}{\varsigma^\rho\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1}U(\eta)d\eta$$
$$+ \frac{2a_3K\rho(\varsigma^\rho - \mu^\rho)}{\Gamma(\alpha)\varsigma^\rho} \int_0^\varsigma \eta^{\alpha\rho-1}d\eta + \frac{b\rho(\varsigma^\rho - \mu^\rho)}{\Gamma(\alpha)\varsigma^{2\rho}} \int_0^\varsigma \eta^{\alpha\rho-1}U(\eta)d\eta$$
$$+ \frac{2a_3K\rho}{\Gamma(\alpha)} \int_\mu^\varsigma \eta^{\rho-1}(\eta^\rho - \mu^\rho)^{\alpha-1}d\eta + \frac{b\rho}{\varsigma^\rho\Gamma(\alpha)} \int_\mu^\varsigma \eta^{\rho-1}(\eta^\rho - \mu^\rho)^{\alpha-1}U(\eta)d\eta,$$

where $K = \max_{\mu\in[J} \left(|\phi(\mu)|, \left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\mu)\right|\right)$.

$$\begin{aligned}
\left|\tilde{T}\phi(\mu)\right| &\leq |\phi_s| + |\phi_0| + \frac{2a_3K\mu^\rho}{\varsigma^{\rho(2-\alpha)}\Gamma(\alpha+1)} + \frac{bK^*}{\varsigma^\rho} + \frac{2a_3K\mu^{\alpha\rho}}{\Gamma(\alpha+1)} \\
&\quad + \frac{2a_3K(\varsigma^\rho - \mu^\rho)}{\varsigma^{\rho(2-\alpha)}\Gamma(\alpha+1)} + \frac{2a_3K(\varsigma^\rho - \mu^\rho)^\alpha}{\Gamma(\alpha+1)} \\
&\quad + \frac{b\rho}{\varsigma^\rho\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1}U(\eta)d\eta + \frac{b\rho}{\varsigma^\rho\Gamma(\alpha)} \int_\mu^\varsigma \eta^{\rho-1}(\eta^\rho - \mu^\rho)^{\alpha-1}U(\eta)d\eta \\
&= |\phi_s| + |\phi_0| + \frac{bK^*}{\varsigma^\rho} + \frac{2Ka_3}{\Gamma(\alpha+1)} \left\{\varsigma^{\rho(\alpha-1)} + \mu^{\alpha\rho} + (\varsigma^\rho - \mu^\rho)^\alpha\right\} \\
&\quad + \frac{b\rho}{\varsigma^\rho\Gamma(\alpha)} \int_0^\mu \eta^{\rho-1}(\mu^\rho - \eta^\rho)^{\alpha-1}U(\eta)d\eta + \frac{b\rho}{\varsigma^\rho\Gamma(\alpha)} \int_\mu^\varsigma \eta^{\rho-1}(\eta^\rho - \mu^\rho)^{\alpha-1}U(\eta)d\eta.
\end{aligned}$$

Since the functions $\mu^{\alpha\rho}$ and $(\varsigma^\rho - \mu^\rho)^\alpha$ are integrable, uniformly continuous and non-negative for $\mu \in [0, \varsigma]$, also $U(\mu) \leq \phi(\mu)$, hence by applying the Lemma 4.3.6 gives,

$$\begin{aligned}
\left|\tilde{T}\phi(\mu)\right| &\leq \left\{|\phi_s| + |\phi_0| + \frac{2bK^*}{\varsigma^\rho} + \frac{2Ka_3}{\Gamma(\alpha+1)} \left\{\varsigma^{\rho(\alpha-1)} + \mu^{\alpha\rho} + (\varsigma^\rho - \mu^\rho)^\alpha\right\}\right\} E_{\alpha,1}\left(\frac{b(\varsigma^\rho - \mu^\rho)^\alpha}{\varsigma^\rho}\right) E_{\alpha,1}\left(\frac{b\mu^{\alpha\rho}}{\varsigma^\rho}\right) \\
&\leq \left\{|\phi_s| + |\phi_0| + \frac{bK^*}{\varsigma^\rho} + \frac{2Ka_3M^*}{\Gamma(\alpha+1)}\right\} E_{\alpha,1}\left(\frac{b(\varsigma^\rho - \mu^\rho)^\alpha}{\varsigma^\rho}\right) E_{\alpha,1}\left(\frac{b\mu^{\alpha\rho}}{\varsigma^\rho}\right)
\end{aligned}$$

$$\left|\tilde{T}\phi(\mu)\right| \leq \left\{|\phi_s| + |\phi_0| + \frac{2bK^*}{\varsigma^\rho} + \frac{2Ka_3M^*}{\Gamma(\alpha+1)}\right\} E_{\alpha,1}^2(b) < r.$$

Also,

$$\left|{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}(\tilde{T}\phi(\mu))\right| \leq {}^{\rho}_{*}D^{\alpha^*}_{0,\mu}(\frac{1}{2}(\phi_0+\phi_s)) + \frac{(\phi_\varsigma-\phi_0)}{2\varsigma^\rho}\left({}^{\rho}_{*}D^{\alpha^*}_{0,\mu}(\mu^\rho)\right) + \left(\frac{2a_3\rho^{\alpha}{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}(\mu^\rho)}{\varsigma^\rho}\right){}^{\rho}_{0}I^{\alpha}_{\varsigma}K$$
$$+\left(\frac{b\rho^\alpha}{\varsigma^{2\rho}}{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}(\mu^\rho)\right){}^{\rho}_{0}I^{\alpha}_{\varsigma}U(\mu) + 2a_3\rho^{\alpha}{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}{}^{\rho}_{0}I^{\alpha}_{\mu}K + \frac{b\rho^\alpha}{\varsigma^\rho}{}^{\rho}_{*}D^{\alpha}_{0,\mu}{}^{\rho}_{0}I^{\alpha}_{\mu}U(\eta).$$

Using the Theorem 2.5.4 and the Lemma 4.2.2 we get,

$$\left|{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}(\tilde{T}\phi(\mu))\right| \leq \frac{\rho^{\alpha^*}\left|\phi_\varsigma-\phi_0\right|\mu^{\rho(1-\alpha^*)}}{2\varsigma^\rho\Gamma(2-\alpha^*)} + \frac{2a_3K\mu^{\rho(\alpha-\alpha^*-1)}}{\varsigma^\rho\rho^{1-\alpha^*}\Gamma(2-\alpha^*)}$$
$$+\frac{b\rho^{\alpha+\alpha^*}\mu^{\rho(1-\alpha^*)}{}^{\rho}_{0}I^{\alpha}_{\varsigma}U(\eta)}{\Gamma(2-\alpha^*)\varsigma^{2\rho}} + 2a_3\rho^{\alpha}{}^{\rho}_{0}I^{\alpha-\alpha^*}_{\mu}K + \frac{b\rho^\alpha}{\varsigma^\rho}{}^{\rho}_{0}I^{\alpha-\alpha^*}_{\mu}U(\eta).$$

$$= \frac{\rho^{\alpha^*}\left|\phi_\varsigma-\phi_0\right|\mu^{\rho(1-\alpha^*)}}{2\varsigma^\rho\Gamma(2-\alpha^*)} + \frac{2a_3K\mu^{\rho(\alpha-\alpha^*-1)}}{\varsigma^\rho\rho^{1-\alpha^*}\Gamma(2-\alpha^*)} + \frac{2a_3\rho^{\alpha-1}K\mu^{\rho(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*+1)}$$
$$+\frac{b\rho^{1+\alpha^*}K^*\mu^{\rho(1-\alpha^*)}}{\Gamma(2-\alpha^*)\varsigma^{2\rho}} + \frac{b\rho^{1+\alpha^*}}{\varsigma^\rho\Gamma(\alpha-\alpha^*)}\int_0^\mu \eta^{\rho-1}(\mu^\rho-\eta^\rho)^{\alpha-\alpha^*-1}U(\eta)d\eta.$$

Since $U(\mu)\leq\phi(\mu)$, and $\mu^{\rho(1-\alpha^*)}$, $\mu^{\rho(\alpha-\alpha^*-1)}$ are measurable and continuous functions for $\mu\in[0,\varsigma]$, therefore by using the assumption $\phi(\mu)=L\tilde{T}(\phi(\mu))$ and the Corollary 4.2.4 we get,

$$\left|{}^{\rho}_{*}D^{\alpha^*}_{0,\mu}(\tilde{T}\phi(\mu))\right| \leq M_1E_{(\alpha-\alpha^*),1}\left(\frac{\rho^{\alpha^*}b}{\varsigma^\rho}\mu^{(\alpha-\alpha^*)\rho}\right) < r_1. \tag{4.7}$$

Moreover,

$$\left|{}^{\rho}_{*}D^{\alpha^*}_{\mu,\varsigma}(\tilde{T}\phi(\mu))\right| \leq {}^{\rho}_{*}D^{\alpha^*}_{\mu,\varsigma}(\frac{1}{2}(\phi_0+\phi_s)) + \frac{\left|\phi_\varsigma-\phi_0\right|}{2\varsigma^\rho}{}^{\rho}_{*}D^{\alpha^*}_{\mu,\varsigma}(\varsigma^\rho-\mu^\rho) + \frac{2a_3K}{\Gamma(\alpha)\varsigma^{\rho(1-\alpha)}}{}^{\rho}_{*}D^{\alpha^*}_{\mu,\varsigma}(\varsigma^\rho-\mu^\rho)$$
$$+\frac{bK^*}{\varsigma^{2\rho}}{}^{\rho}_{*}D^{\alpha^*}_{\mu,\varsigma}(\varsigma^\rho-\mu^\rho) + 2a_3\rho^{\alpha}{}^{\rho}_{*}D^{\alpha^*}_{\mu,\varsigma}{}^{\rho}_{\mu}I^{\alpha}_{\varsigma}K + \frac{b\rho^\alpha}{\varsigma^\rho}{}^{\rho}_{*}D^{\alpha^*}_{\mu,\varsigma}I^{\alpha}_{\varsigma}U(\eta).$$

Using the Theorem 2.5.4 and the Lemma 4.2.2 we get,

$$\begin{aligned}\left|{}_{*}^{\rho}D_{\mu,\varsigma}^{\alpha^*}(\tilde{T}\phi(\mu))\right| &\leq \frac{\rho^{\alpha^*}\left|\phi_\varsigma-\phi_0\right|\left(\varsigma^\rho-\mu^\rho\right)^{1-\alpha^*}}{2\varsigma^\rho\Gamma(2-\alpha^*)}+\frac{2a_3K\rho^{\alpha^*}\left(\varsigma^\rho-\mu^\rho\right)^{1-\alpha^*}}{\Gamma(\alpha)\Gamma(2-\alpha^*)}\\ &+\frac{bK^*\rho^{\alpha^*}\left(\varsigma^\rho-\mu^\rho\right)^{1-\alpha^*}}{\Gamma(2-\alpha^*)\varsigma^{2\rho}}+2a_3\rho^{\alpha}{}_{\mu}^{\rho}I_{\varsigma}^{\alpha-\alpha^*}K+\frac{b\rho^\alpha}{\varsigma^\rho}{}_{\mu}^{\rho}I_{\varsigma}^{\alpha-\alpha^*}U(\eta),\\ &=\frac{\rho^{\alpha^*}\left|\phi_\varsigma-\phi_0\right|\left(\varsigma^\rho-\mu^\rho\right)^{1-\alpha^*}}{2\varsigma^\rho\Gamma(2-\alpha^*)}+\frac{2a_3K\rho^{\alpha^*}\left(\varsigma^\rho-\mu^\rho\right)^{1-\alpha^*}}{\Gamma(\alpha)\Gamma(2-\alpha^*)}\\ &+\frac{bK^*\rho^{\alpha^*}\left(\varsigma^\rho-\mu^\rho\right)^{1-\alpha^*}}{\Gamma(2-\alpha^*)\varsigma^{2\rho}}+\frac{2a_3K\rho^{\alpha^*}\left(\varsigma^\rho-\mu^\rho\right)^{\alpha-\alpha^*}}{\Gamma(\alpha-\alpha^*+1)}\\ &+\frac{b\rho^{1+\alpha^*}}{\varsigma^\rho\Gamma(\alpha-\alpha^*)}\int_{\mu}^{\varsigma}\eta^{\rho-1}(\eta^\rho-\mu^\rho)^{\alpha-\alpha^*-1}U(\eta)d\eta.\end{aligned}$$

Since $U(\mu)\leq\phi(\mu)$, and $(\varsigma^\rho-\mu^\rho)^{\alpha-\alpha^*}\in L^1[J,\mathbb{R}_+]$ for $\mu\in[0,\varsigma]$, therefore taking into account the assumption that $\phi(\mu)=L\tilde{T}(\phi(\mu))$ with $L\in(0,1)$ and the Corollary 4.2.4 yields,

$$\left|{}_{*}^{\rho}D_{\mu,\varsigma}^{\alpha^*}(\tilde{T}\phi(\mu))\right|\leq M_1E_{(\alpha-\alpha^*),1}\left(\frac{\rho^{\alpha^*}b}{\varsigma^\rho}(\varsigma^\rho-\mu^\rho)^{\alpha-\alpha^*}\right)<r_2 \tag{4.8}$$

From the definition 4.3.3 the inequalities (4.7) and (4.8) we have that,

$$\left|{}_{0}^{RC}D_{\varsigma}^{\alpha^*,\rho}(\tilde{T}\phi(\mu))\right|=\frac{1}{2}\left|{}_{*}^{\rho}D_{0,\mu}^{\alpha^*}(\tilde{T}\phi(\mu))-{}_{*}^{\rho}D_{\mu,\varsigma}^{\alpha^*}(\tilde{T}\phi(\mu))\right|<r.$$

Which implies $\tilde{T}\phi\in A_r$, that is the operator $\tilde{T}:A_r\to A_r$ is self-mapped. Next we show that the operator (4.6) is continuous. For this let $\phi_1(\mu),\phi_2(\mu)\in A_r$, then we have

$$\begin{aligned}&\left|\tilde{T}\phi_1(\mu)-\tilde{T}\phi_2(\mu)\right|\\ &\leq\frac{\mu^\rho\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int_0^\varsigma\frac{\eta^{\rho-1}\left|g(\eta,\phi_1(\eta),{}_{0}^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_{0}^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_2(\eta))\right|}{\left(\varsigma^\rho-\eta^\rho\right)^{1-\alpha}}d\eta\\ &+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^\mu\frac{\eta^{\rho-1}\left|g(\eta,\phi_1(\eta),{}_{0}^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_{0}^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_2(\eta))\right|}{\left(\mu^\rho-\eta^\rho\right)^{1-\alpha}}d\eta\end{aligned}$$

$$+\frac{(\varsigma^\rho-\mu^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int_0^\varsigma \frac{\left|g(\eta,\phi_1(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|}{\eta^{1-\alpha\rho}}d\eta$$

$$+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_\mu^\varsigma \frac{\eta^{\rho-1}\left|g(\eta,\phi_1(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|}{(\eta^\rho-\mu^\rho)^{1-\alpha}}d\eta,$$

$$=\frac{\{\varsigma^{\alpha\rho}+(\varsigma^\rho-\mu^\rho)^\alpha\}}{\rho^\alpha\Gamma(\alpha+1)}\left|g(\eta,\phi_1(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|.$$

Since g is continuous on A_r, hence for all $\mu\in[0,\varsigma]$ there exists $\delta>0$ such that $\|\phi_1(\eta)-\phi_2(\eta)\|<\delta$, and for any $\epsilon>0$, $\left|g(\eta,\phi_1(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|<\frac{\rho^\alpha\Gamma(\alpha+1)}{\varsigma^{\alpha\rho}}\varepsilon$. Therefore,

$$\begin{aligned}&\left|\tilde{T}\phi_1(\mu)-\tilde{T}\phi_2(\mu)\right|\\&\le\frac{\{\varsigma^{\alpha\rho}+(\varsigma^\rho-\mu^\rho)^\alpha\}}{\rho^\alpha\Gamma(\alpha+1)}\left|g(\eta,\phi_1(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|,\\&\le\frac{\varepsilon}{2}+\frac{(\varsigma^\rho-\mu^\rho)^\alpha}{2\varsigma^{\alpha\rho}}\varepsilon\\&<\varepsilon.\end{aligned}$$

Likewise one can prove ${}_0^{RC}D_\varsigma^{\alpha^*,\rho}(\tilde{T}\phi(\mu))$ is continuous on A_r. Moreover, we show that the operator (4.6) is completely continuous. For this, let $\eta_1,\eta_2\in J$, with $\eta_1<\eta_2$, and $\phi\in A_r$. Then

$$\begin{aligned}&\left|\tilde{T}\phi(\mu_1)-\tilde{T}\phi(\mu_2)\right|\\&\le\frac{|\phi_\varsigma-\phi_0|(\mu_1^\rho-\mu_2^\rho)}{\varsigma^\rho}+\left|\frac{(\mu_1^\rho-\mu_2^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int_0^\varsigma\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\varsigma^\rho-\eta^\rho)^{1-\alpha}}d\eta\right|\\&\quad+\left|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mu_1}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\mu_1^\rho-\eta^\rho)^{1-\alpha}}d\eta-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_0^{\mu_2}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\mu_2^\rho-\eta^\rho)^{1-\alpha}}d\eta\right|\\&\quad+\left|\frac{(\mu_1^\rho-\mu_2^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int_0^\varsigma\frac{g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}d\eta\right|\\&\quad+\left|\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu_1}^\varsigma\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho-\mu_1^\rho)^{1-\alpha}}d\eta-\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu_2}^\varsigma\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho-\mu_2^\rho)^{1-\alpha}}d\eta\right|,\end{aligned}$$

$$
\leq \frac{\left|\phi_{\varsigma}-\phi_{0}\right|(\mu_1^{\rho}-\mu_2^{\rho})}{\varsigma^{\rho}}+\frac{2a_3K\rho(\mu_1^{\rho}-\mu_2^{\rho})}{\varsigma^{\rho}\Gamma(\alpha)}\int_{0}^{\varsigma}\eta^{\rho-1}(\varsigma^{\rho}-\eta^{\rho})^{\alpha-1}d\eta
$$

$$
+\frac{b\rho(\mu_1^{\rho}-\mu_2^{\rho})}{\varsigma^{2\rho}\Gamma(\alpha)}\int_{0}^{\varsigma}\eta^{\rho-1}(\varsigma^{\rho}-\eta^{\rho})^{\alpha-1}U(\eta)d\eta
$$

$$
+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{0}^{\mu_1}\left\{\eta^{\rho-1}(\mu_1^{\rho}-\eta^{\rho})^{\alpha-1}-\eta^{\rho-1}(\mu_2^{\rho}-\eta^{\rho})^{\alpha-1}\right\}\left|g(\eta,\phi(\eta),{}_{0}^{RC}D_{\varsigma}^{\alpha^{*},\rho}\phi(\eta))\right|d\eta
$$

$$
+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}\eta^{\rho-1}(\mu_2^{\rho}-\eta^{\rho})^{\alpha-1}\left|g(\eta,\phi(\eta),{}_{0}^{RC}D_{\varsigma}^{\alpha^{*},\rho}\phi(\eta))\right|d\eta
$$

$$
+\frac{(\mu_1^{\rho}-\mu_2^{\rho})\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^{\rho}}\int_{0}^{\varsigma}\frac{\left|g(\eta,\phi(\eta),{}_{0}^{RC}D_{\varsigma}^{\alpha^{*},\rho}\phi(\eta))\right|}{\eta^{1-\alpha\rho}}d\eta
$$

$$
+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu_2}^{\varsigma}\left\{\frac{\eta^{\rho-1}}{(\eta^{\rho}-\mu_1^{\rho})^{1-\alpha}}-\frac{\eta^{\rho-1}}{(\eta^{\rho}-\mu_2^{\rho})^{1-\alpha}}\right\}\left|g(\eta,\phi(\eta),{}_{0}^{RC}D_{\varsigma}^{\alpha^{*},\rho}\phi(\eta))\right|d\eta
$$

$$
+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}\frac{\eta^{\rho-1}\left|g(\eta,\phi(\eta),{}_{0}^{RC}D_{\varsigma}^{\alpha^{*},\rho}\phi(\eta))\right|}{(\eta^{\rho}-\mu_1^{\rho})^{1-\alpha}}d\eta,
$$

$$
\leq \frac{\left|\phi_{\varsigma}-\phi_{0}\right|(\mu_1^{\rho}-\mu_2^{\rho})}{\varsigma^{\rho}}+\frac{2a_3K\varsigma^{\alpha\rho}(\mu_1^{\rho}-\mu_2^{\rho})}{\Gamma(\alpha+1)}+\frac{b\rho K^{*}(\mu_1^{\rho}-\mu_2^{\rho})}{\varsigma^{2\rho}\Gamma(\alpha)}
$$

$$
+\frac{2Ka_3\rho}{\Gamma(\alpha)}\int_{0}^{\mu_1}\left\{\eta^{\rho-1}(\mu_1^{\rho}-\eta^{\rho})^{\alpha-1}-\eta^{\rho-1}(\mu_2^{\rho}-\eta^{\rho})^{\alpha-1}\right\}d\eta
$$

$$
+\frac{b\rho}{\varsigma^{\rho}\Gamma(\alpha)}\int_{0}^{\mu_1}\left\{\eta^{\rho-1}(\mu_1^{\rho}-\eta^{\rho})^{\alpha-1}-\eta^{\rho-1}(\mu_2^{\rho}-\eta^{\rho})^{\alpha-1}\right\}U(\eta)d\eta
$$

$$
+\frac{2Ka_3\rho}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}\eta^{\rho-1}(\mu_2^{\rho}-\eta^{\rho})^{\alpha-1}d\eta+\frac{b\rho}{\varsigma^{\rho}\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}\eta^{\rho-1}(\mu_2^{\rho}-\eta^{\rho})^{\alpha-1}U(\eta)d\eta
$$

$$
+\frac{2Ka_3(\mu_1^{\rho}-\mu_2^{\rho})\varsigma^{\rho(\alpha-1)}}{\Gamma(\alpha+1)}+\frac{(\mu_1^{\rho}-\mu_2^{\rho})K^{*}}{\varsigma^{\rho}}
$$

$$
+\frac{2Ka_3\rho}{\Gamma(\alpha)}\int_{\mu_2}^{\varsigma}\left\{\eta^{\rho-1}(\eta^{\rho}-\mu_1^{\rho})^{1-\alpha}-\eta^{\rho-1}(\eta^{\rho}-\mu_2^{\rho})^{1-\alpha}\right\}d\eta
$$

$$
+\frac{b\rho}{\varsigma^{\rho}\Gamma(\alpha)}\int_{\mu_2}^{\varsigma}\left\{\eta^{\rho-1}(\eta^{\rho}-\mu_1^{\rho})^{1-\alpha}-\eta^{\rho-1}(\eta^{\rho}-\mu_2^{\rho})^{1-\alpha}\right\}U(\eta)d\eta
$$

$$
+\frac{2Ka_3\rho}{\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}\frac{\eta^{\rho-1}}{(\eta^{\rho}-\mu_1^{\rho})^{1-\alpha}}d\eta+\frac{b\rho}{\varsigma^{\rho}\Gamma(\alpha)}\int_{\mu_1}^{\mu_2}\frac{\eta^{\rho-1}U(\eta)}{(\eta^{\rho}-\mu_1^{\rho})^{1-\alpha}}d\eta.
$$

Since $U(\mu) \in L^1[J, \mathbb{R}_+]$, therefore the functions $\left(\eta^{\rho-1}(\mu_1^\rho - \eta^\rho)^{\alpha-1} - \eta^{\rho-1}(\mu_2^\rho - \eta^\rho)^{\alpha-1}\right) U(\eta)$, $\left(\eta^{\rho-1}(\mu_2^\rho - \eta^\rho)^{\alpha-1} U(\eta)\right)$ and $\left(\eta^{\rho-1}(\eta^\rho - \mu_1^\rho)^{1-\alpha} - \eta^{\rho-1}(\eta^\rho - \mu_2^\rho)^{1-\alpha}\right) U(\eta)$ are Lebesgue integrable in η, also $(\mu_1^\rho - \mu_2^\rho)\varsigma^{\rho(\alpha-1)}$, $\varsigma^{\alpha\rho}(\mu_1^\rho - \mu_2^\rho)$ are uniformly continuous for $\mu_1, \mu_2 \in J$. So we see-through that right hand side of the above inequality tends to zero as $\mu_1 \to \mu_2$. Furthermore, we prove that $\left| {}_0^{RC}D_\varsigma^{\alpha^*,\rho}(\tilde{T}\phi(\mu_1)) - {}_0^{RC}D_\varsigma^{\alpha^*,\rho}(\tilde{T}\phi(\mu_2)) \right| \to 0$ as $\mu_1 \to \mu_2$ for all $\mu_1, \mu_2 \in [0, \varsigma]$ with $\mu_1 < \mu_2$. For this let us compute first the left and the right Caputo-Katugampola derivative of the operator (4.6).

$$
\begin{aligned}
{}_*^\rho D_{0,\mu}^{\alpha^*}(\tilde{T}\phi(\mu)) =& \frac{\rho^{\alpha^*}(\phi_\varsigma - \phi_0)\mu_1^{\rho(1-\alpha^*)}}{2\varsigma^\rho \Gamma(2-\alpha^*)} - \frac{\rho^{1-(\alpha-\alpha^*)}\mu_1^{\rho(1-\alpha^*)}}{\varsigma^\rho \Gamma(\alpha)\Gamma(2-\alpha^*)} \int_0^\varsigma \frac{\eta^{\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\varsigma^\rho - \eta^\rho)^{1-\alpha}} d\eta \\
&+ \frac{\rho^{1-(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*)} \int_0^{\mu_1} \frac{\eta^{\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\mu_1^\rho - \eta^\rho)^{1-(\alpha-\alpha^*)}} d\eta, \qquad (4.9)
\end{aligned}
$$

and

$$
\begin{aligned}
{}_*^\rho D_{\mu,\varsigma}^{\alpha^*}(\tilde{T}\phi(\mu)) = & \frac{\rho^{\alpha^*}(\phi_0 - \phi_\varsigma)(\varsigma^\rho - \mu_1^\rho)^{1-\alpha^*}}{2\varsigma^\rho \Gamma(2-\alpha^*)} \\
& - \frac{\rho^{1-(\alpha-\alpha^*)}(\varsigma^\rho - \mu_1^\rho)^{1-\alpha^*}}{\varsigma^\rho \Gamma(\alpha)\Gamma(2-\alpha^*)} \int_0^\varsigma \frac{g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}} d\eta \qquad (4.10) \\
& + \frac{\rho^{1-(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*)} \int_{\mu_1}^\varsigma \frac{\eta^{\rho-1} g(\eta, \phi(\eta), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho - \mu_1^\rho)^{1-(\alpha-\alpha^*)}} d\eta.
\end{aligned}
$$

From the Definition 4.3.3 the equations (4.9) and (4.10) we have,

$$
{}_0^{RC}D_\varsigma^{\alpha^*,\rho}(\tilde{T}\phi(\mu_1)) = \frac{1}{2}\left({}_*^\rho D_{0,\mu_1}^{\alpha^*}(\tilde{T}\phi(\mu_1)) - {}_*^\rho D_{\mu_1,\varsigma}^{\alpha^*}(\tilde{T}\phi(\mu_1)) \right)
$$

$$
\begin{aligned}
{}_{0}^{RC}D_{\varsigma}^{\alpha^*,\rho}(\tilde{T}\phi(\mu_1)) = \frac{1}{2}\Bigg[& \frac{\rho^{\alpha^*}(\phi_\varsigma - \phi_0)\mu_1^{\rho(1-\alpha^*)}}{2\varsigma^\rho\Gamma(2-\alpha^*)} \\
& - \frac{\rho^{1-(\alpha-\alpha^*)}\mu_1^{\rho(1-\alpha^*)}}{\varsigma^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_0^{\varsigma}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi(\eta))}{(\varsigma^\rho-\eta^\rho)^{1-\alpha}}d\eta + \frac{\rho^{1-(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*)}\int_0^{\mu_1} \\
& \frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi(\eta))}{(\mu_1^\rho-\eta^\rho)^{1-(\alpha-\alpha^*)}}d\eta + \frac{\rho^{\alpha^*}(\phi_\varsigma-\phi_0)(\varsigma^\rho-\mu_1^\rho)^{1-\alpha^*}}{2\varsigma^\rho\Gamma(2-\alpha^*)} + \\
& \frac{\rho^{1-(\alpha-\alpha^*)}(\varsigma^\rho-\mu_1^\rho)^{1-\alpha^*}}{\varsigma^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_0^{\varsigma}\frac{g(\eta,\phi(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}d\eta \\
& - \frac{\rho^{1-(\alpha-\alpha^*)}}{\Gamma(\alpha-\alpha^*)}\int_{\mu_1}^{\varsigma}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho-\mu_1^\rho)^{1-(\alpha-\alpha^*)}}d\eta\Bigg].
\end{aligned}
$$

Therefore by using the above equation we established that,

$$
\begin{aligned}
\left|{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}(\tilde{T}\phi(\mu_1)) - {}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}(\tilde{T}\phi(\mu_2))\right| \leq & \left|\frac{\rho^{\alpha^*}(\phi_\varsigma-\phi_0)(\mu_1^{\rho(1-\alpha^*)}-\mu_1^{\rho(1-\alpha^*)})}{4\varsigma^\rho\Gamma(2-\alpha^*)}\right| \\
& + \left|\frac{\rho^{1-(\alpha-\alpha^*)}(\mu_2^{\rho(1-\alpha^*)}-\mu_1^{\rho(1-\alpha^*)})}{2\varsigma^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_0^{\varsigma}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi(\eta))}{(\varsigma^\rho-\eta^\rho)^{1-\alpha}}d\eta\right| \\
& + \left|\frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)}\int_0^{\mu_1}\left\{\frac{\eta^{\rho-1}}{(\mu_1^\rho-\eta^\rho)^{1-(\alpha-\alpha^*)}}-\frac{\eta^{\rho-1}}{(\mu_2^\rho-\eta^\rho)^{1-(\alpha-\alpha^*)}}\right\}g(\eta,\phi(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi(\eta))d\eta\right| \\
& + \left|\frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)}\int_{\mu_1}^{\mu_2}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi(\eta))}{(\mu_2^\rho-\eta^\rho)^{1-(\alpha-\alpha^*)}}d\eta\right| \\
& + \left|\frac{\rho^{\alpha^*}(\phi_\varsigma-\phi_0)\left\{(\varsigma^\rho-\mu_1^\rho)^{1-\alpha^*}-(\varsigma^\rho-\mu_2^\rho)^{1-\alpha^*}\right\}}{4\varsigma^\rho\Gamma(2-\alpha^*)}\right| \\
& + \left|\frac{\rho^{1-(\alpha-\alpha^*)}\left\{(\varsigma^\rho-\mu_1^\rho)^{1-\alpha^*}-(\varsigma^\rho-\mu_2^\rho)^{1-\alpha^*}\right\}}{2\varsigma^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int_0^{\varsigma}\frac{g(\eta,\phi(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi(\eta))}{\eta^{1-\alpha\rho}}d\eta\right|
\end{aligned}
$$

$$+\left|\frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)}\int\limits_{\mu_1}^{\mu_2}\frac{\eta^{\rho-1}g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))}{(\eta^\rho-\mu_1^\rho)^{1-(\alpha-\alpha^*)}}d\eta\right|$$

$$+\left|\frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)}\left\{\int\limits_{\mu_2}^{\varsigma}\frac{\eta^{\rho-1}}{(\eta^\rho-\mu_2^\rho)^{1-(\alpha-\alpha^*)}}-\int\limits_{\mu_2}^{\varsigma}\frac{\eta^{\rho-1}}{(\eta^\rho-\mu_1^\rho)^{1-(\alpha-\alpha^*)}}\right\}g(\eta,\phi(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\eta))d\eta\right|.$$

Using the condition (H_1^*)

$$\left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}(\tilde{T}\phi(\mu_1))-{}_0^{RC}D_\varsigma^{\alpha^*,\rho}(\tilde{T}\phi(\mu_2))\right|$$

$$\leq\frac{\rho^{\alpha^*}|\phi_\varsigma-\phi_0|\,(\mu_1^{\rho(1-\alpha^*)}-\mu_2^{\rho(1-\alpha^*)})}{4\varsigma^\rho\Gamma(2-\alpha^*)}$$

$$+\frac{\rho^{1+\alpha^*}a_3K(\mu_1^{\rho(1-\alpha^*)}-\mu_2^{\rho(1-\alpha^*)})}{\varsigma^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int\limits_0^\varsigma\eta^{\rho-1}(\varsigma^\rho-\eta^\rho)^{\alpha-1}d\eta$$

$$+\frac{b\rho^{1+\alpha^*}(\mu_1^{\rho(1-\alpha^*)}-\mu_2^{\rho(1-\alpha^*)})}{2\varsigma^{2\rho}\Gamma(\alpha)}\int\limits_0^\varsigma\eta^{\rho-1}(\varsigma^\rho-\eta^\rho)^{\alpha-1}U(\eta)d\eta$$

$$+\frac{\rho^{1+\alpha^*}a_3K}{\Gamma(\alpha-\alpha^*)}\int\limits_0^{\mu_1}\left\{(\mu_1^\rho-\eta^\rho)^{(\alpha-\alpha^*)-1}\eta^{\rho-1}-\eta^{\rho-1}(\mu_2^\rho-\eta^\rho)^{(\alpha-\alpha^*)-1}\right\}d\eta$$

$$+\frac{b\rho^{1+\alpha^*}}{2\varsigma^\rho\Gamma(\alpha-\alpha^*)}\int\limits_0^{\mu_1}\left\{(\mu_1^\rho-\eta^\rho)^{(\alpha-\alpha^*)-1}\eta^{\rho-1}-\eta^{\rho-1}(\mu_2^\rho-\eta^\rho)^{(\alpha-\alpha^*)-1}\right\}U(\eta)d\eta$$

$$+\frac{\rho^{1+\alpha^*}a_3K}{\Gamma(\alpha-\alpha^*)}\int\limits_{\mu_1}^{\mu_2}\eta^{\rho-1}(\mu_2^\rho-\eta^\rho)^{(\alpha-\alpha^*)-1}d\eta+\frac{b\rho^{1+\alpha^*}}{2\varsigma^\rho\Gamma(\alpha-\alpha^*)}\int\limits_{\mu_1}^{\mu_2}\eta^{\rho-1}(\mu_2^\rho-\eta^\rho)^{(\alpha-\alpha^*)-1}U(\eta)d\eta$$

$$+\frac{\rho^{\alpha^*}|\phi_\varsigma-\phi_0|\left\{(\varsigma^\rho-\mu_1^\rho)^{1-\alpha^*}-(\varsigma^\rho-\mu_2^\rho)^{1-\alpha^*}\right\}}{4\varsigma^\rho\Gamma(2-\alpha^*)}+\frac{a_3K\rho^{\alpha^*}\left\{(\varsigma^\rho-\mu_1^\rho)^{1-\alpha^*}-(\varsigma^\rho-\mu_2^\rho)^{1-\alpha^*}\right\}}{\Gamma(\alpha)(\alpha-\alpha^*)\Gamma(2-\alpha^*)\varsigma^{\rho(1-\alpha+\alpha^*)}}$$

$$+\frac{b\rho^{1+\alpha^*}\left\{(\varsigma^\rho-\mu_1^\rho)^{1-\alpha^*}-(\varsigma^\rho-\mu_2^\rho)^{1-\alpha^*}\right\}}{2\Gamma(\alpha)\Gamma(2-\alpha^*)\varsigma^{2\rho}}\int\limits_0^\varsigma\eta^{\rho(\alpha-\alpha^*)-1}U(\eta)d\eta$$

$$+\frac{a_3K\rho^{\alpha^*}\left\{(\varsigma^\rho-\mu_1^\rho)^{(\alpha-\alpha^*)}-(\varsigma^\rho-\mu_2^\rho)^{(\alpha-\alpha^*)}\right\}}{\Gamma(\alpha)(\alpha-\alpha^*)\Gamma(2-\alpha^*)}$$

$$+\frac{b\rho^{1+\alpha^*}}{2\varsigma^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int\limits_{\mu_2}^{\varsigma}\Big\{\eta^{\rho-1}(\eta^\rho-\mu_1^\rho)^{(\alpha-\alpha^*)-1}-\eta^{\rho-1}(\eta^\rho-\mu_2^\rho)^{(\alpha-\alpha^*)-1}\Big\}U(\eta)d\eta$$

$$+\frac{b\rho^{1+\alpha^*}}{2\varsigma^\rho\Gamma(\alpha)\Gamma(2-\alpha^*)}\int\limits_{\mu_1}^{\mu_2}\eta^{\rho-1}(\eta^\rho-\mu_1^\rho)^{(\alpha-\alpha^*)-1}U(\eta)d\eta.$$

Since $U(\mu)\in L^1[J,\mathbb{R}_+]$ and the functions $\eta^{\rho-1}(\mu_2^\rho-\eta^\rho)^{(\alpha-\alpha^*)-1}$, $(\eta^{\rho(\alpha-\alpha^*)-1}U(\eta)$, $((\mu_1^\rho-\eta^\rho)^{(\alpha-\alpha^*)-1}\eta^{\rho-1}-\eta^{\rho-1}(\mu_2^\rho-\eta^\rho)^{(\alpha-\alpha^*)-1}U(\eta))$ are Lebesgue integrable on $[0,\varsigma]$, so the right hand side of the above inequality tends to zero as $\mu_1\to\mu_2$. Hence the set of operators $\tilde{T}A_r$ is equicontinuous. Also $\tilde{T}A_r\subseteq A_r$, implies $\tilde{T}A_r$ is uniformly bounded. Henceforth, $\tilde{T}$ is completely continuous and thus Schauder fixed point theorem assures the existence of atleast one fixed point of the operator (4.6). Hence taking into account the Lemma 4.4.2 completes the proof. □

Theorem 4.4.2. *Assume that the conditions (H_1^*) and (H_2^*) hold then the equation (4.3) comports as a unique solution of the Problem (4.1).*

Proof. To prove this theorem we use the Banach fixed point theorem. For this first we necessitate to confirm that (4.6) is a self mapped operator and afterwards we show that $\tilde{T}$ satisfy the contraction mapping principle. Since, we have shown in the Theorem 4.4.1 that, $\tilde{T}\phi(\mu),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}(\tilde{T}\phi(\mu))\in A_r$, so the operator $\tilde{T}$ satisfies the self mappedness property under these conditions. Hence, the only stipulation that we need to verify here is contraction. For this consider,

$$\Big|\tilde{T}\phi_1(\mu)-\tilde{T}\phi_2(\mu)\Big|$$

$$\le\left|g(\eta,\phi_1(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|\{\frac{\mu^\rho\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int\limits_0^\varsigma\frac{\eta^{\rho-1}}{(\varsigma^\rho-\eta^\rho)^{1-\alpha}}d\eta$$

$$+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_0^\mu\frac{\eta^{\rho-1}}{(\mu^\rho-\eta^\rho)^{1-\alpha}}d\eta+\frac{(\varsigma^\rho-\mu^\rho)\rho^{1-\alpha}}{\Gamma(\alpha)\varsigma^\rho}\int\limits_0^\varsigma\frac{1}{\eta^{1-\alpha\rho}}d\eta+\frac{\rho^{1-\alpha}}{\Gamma(\alpha)}\int\limits_\mu^\varsigma\frac{\eta^{\rho-1}}{(\eta^\rho-\mu^\rho)^{1-\alpha}}d\eta\}$$

$$\le\frac{\lambda_1\left(|\phi_1(\mu)-\phi_2(\mu)|+\left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta))-{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|\right)(\varsigma^{\alpha\rho}+\mu^{\alpha\rho}+(\varsigma^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)},$$

$$\le\frac{\lambda_1}{K_1}\left\|\phi_1(\mu)-\phi_2(\mu)\right\|,$$

where $K_1 = \frac{\rho^{\alpha}\Gamma(\alpha+1)}{L_1}$. Moreover,

$$\begin{aligned}
&\left|{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}(\tilde{T}\phi_1(\mu)) - {}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}(\tilde{T}\phi_2(\mu))\right| \\
&\leq \frac{\rho^{1-(\alpha-\alpha^*)}\mu^{\rho(1-\alpha^*)}}{2\varsigma^{\rho}\Gamma(\alpha)\Gamma(2-\alpha^*)} \int_0^{\varsigma} \frac{\eta^{\rho-1}\left|g(\eta,\phi_1(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_1(\eta)) - g(\eta,\phi_2(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_2(\eta))\right|}{(\varsigma^{\rho}-\eta^{\rho})^{1-\alpha}}d\eta \\
&\quad + \frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)} \int_0^{\mu} \frac{\eta^{\rho-1}\left|g(\eta,\phi_1(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_1(\eta)) - g(\eta,\phi_2(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_2(\eta))\right|}{(\mu^{\rho}-\eta^{\rho})^{1-(\alpha-\alpha^*)}}d\eta \\
&\quad + \frac{\rho^{1-(\alpha-\alpha^*)}(\varsigma^{\rho}-\mu^{\rho})^{1-\alpha^*}}{2\varsigma^{\rho}\Gamma(\alpha)\Gamma(2-\alpha^*)} \int_0^{\varsigma} \frac{\left|g(\eta,\phi_1(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_1(\eta)) - g(\eta,\phi_2(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_2(\eta))\right|}{\eta^{1-\alpha\rho}}d\eta \\
&\quad + \frac{\rho^{1-(\alpha-\alpha^*)}}{2\Gamma(\alpha-\alpha^*)} \int_{\mu}^{\varsigma} \frac{\eta^{\rho-1}\left|g(\eta,\phi_1(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_1(\eta)) - g(\eta,\phi_2(\eta),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_2(\eta))\right|}{(\eta^{\rho}-\mu^{\rho})^{1-(\alpha-\alpha^*)}}d\eta, \\
&\leq \lambda_1 \frac{\rho^{(\alpha^*-\alpha)}\varsigma^{\rho(\alpha-1)}\mu^{\rho(1-\alpha^*)}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)}\left(|\phi_1(\mu)-\phi_2(\mu)| + \left|{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_1(\eta)) - {}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_2(\eta))\right|\right) \\
&\quad + \lambda_1 \frac{\rho^{(\alpha^*-\alpha)}\mu^{(\alpha-\alpha^*)}}{2\Gamma(2-\alpha^*+1)}\left(|\phi_1(\mu)-\phi_2(\mu)| + \left|{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_1(\eta)) - {}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_2(\eta))\right|\right) \\
&\quad + \lambda_1 \frac{\rho^{(\alpha^*-\alpha)}\varsigma^{\rho(\alpha-1)}(\varsigma^{\rho}-\mu^{\rho})^{1-\alpha^*}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)}\left(|\phi_1(\mu)-\phi_2(\mu)| + \left|{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_1(\eta)) - {}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_2(\eta))\right|\right) \\
&\quad + \lambda_1 \frac{\rho^{(\alpha^*-\alpha)}(\varsigma^{\rho}-\mu^{\rho})^{\alpha-\alpha^*}}{2\Gamma(\alpha-\alpha^*+1)}\left(|\phi_1(\mu)-\phi_2(\mu)| + \left|{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_1(\eta)) - {}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_2(\eta))\right|\right) \\
&= \frac{\lambda_1}{K_2}\left\|\phi_1(\mu)-\phi_2(\mu)\right\|,
\end{aligned}$$

where $K_2 = \frac{2\rho^{(\alpha-\alpha^*)}}{L_2}$. Therefore,

$$\left|\tilde{T}\phi_1(\mu) - \tilde{T}\phi_2(\mu)\right| \leq \frac{2\lambda_1}{M}\left\|\phi_1(\mu)-\phi_2(\mu)\right\|.$$

where, $M = \max(K_1, K_2)$. Thus Banach fixed point theorem assures the existence of a unique fixed point of the operator (4.6). So in consequence of the Lemma 4.4.2 we concluded that (4.3) is the unique solution of the boundary value problem (4.1). □

Lemma 4.4.3. *Assume that* $1 < \alpha < 2$*,* $0 < \beta^* < 1$ *and* $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous function, furthermore* g *satisfies,*

$$\left|g(\mu,\phi(\mu),{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi(\mu))\right| \leq a_3 + a_4 \max|\phi(\mu)| + b_2 \max\left|{}_0^{RC}D_{\varsigma}^{\alpha^*,\rho}\phi_2(\mu)\right|.$$

where $a_3, a_4, b_2 \in \mathbb{R}_+$. Then the solution $\phi(\mu)$ of (4.1) exists in A_r.

Proof. The result follows from the Theorem 4.4.1. □

Lemma 4.4.4. *Assume that $1 < \alpha < 2$, $0 < \beta^* < 1$ and $g : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous function, furthermore g satisfies the following condition,*

$$\left|g(\mu, \phi(\mu), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\mu))\right| \leq \frac{\rho^\alpha}{\varsigma^\rho}\left|\phi(\mu)\right|.$$

Then the problem (4.1) has atleast one solution in A_r.

Proof. Let $a_1 = a_2 = 0$ and $U(\mu) = |\phi|$, then taking into account the Theorem 4.4.1 the result holds. □

Example 4.4.5. *Consider the following fractional differential equation*

$$\begin{cases} {}_o^{RC}D_\pi^{\frac{7}{4},2}u(\mu) = \frac{|u|}{(\mu+4)^2(1+|u|)}, & \mu \in [0,\pi]. \\ u(0) = 0, \quad u(\pi) = 1. \end{cases}$$

Where $g(\mu, u) = \frac{|u|}{(\mu+4)^2(1+|u|)}$, $\alpha = \frac{7}{4}$ and $\varsigma = \pi$. Also since, $\|g(\mu, u) - g(\mu, v)\| \leq \lambda_1 \|u - v\|$ with $\lambda_1 = \frac{1}{16}$, Therefore the Theorem 4.4.2 assures that the boundary value problem has a unique solution on $[0.\pi]$.

4.4.1 Dependence of Solutions on the Parameters

The stability analysis of fractional differential equations has been carried out by many mathematicians. For detail one can see [2, 33, 34, 35] and the references therein. The solutions satisfy various type of stability, continuous dependence on initial data is one of them. This section demonstrates that solution of the problem (4.1) depends on the parameters α, ϕ_0, ϕ_ς and g provided that the function g satisfy the conditions (H_1^*) and (H_2^*). Continuous dependence of solutions on the parameters indicates the stability of solutions.

Theorem 4.4.3. *Assume that $\phi_1(\eta)$ be the solution of the BVP (4.1) and $\phi_2(\eta)$ be the solution of the following problem,*

$$\begin{cases} {}_0^{RC}D_\varsigma^{\alpha-\epsilon,\rho}\phi(\mu) = g(\mu, \phi(\mu), {}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\mu)), & \mu \in [0,\varsigma], \\ \phi(0) = \phi_0, \quad \phi(\varsigma) = \phi_\varsigma. \end{cases}$$

where, $1 < \alpha - \epsilon < \alpha \leq 2$, $0 < \alpha^* \leq 1$, and g is continuous. Then $\|\phi_1 - \phi_2\| = O(\varepsilon)$.

Proof. Using the equation (4.3) we have that,

$$\begin{aligned}
&|\phi_1(\mu)-\phi_2(\mu)|\\
&\leq \left|g(\eta,\phi_1(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|\{\frac{(\varsigma^{\alpha\rho}+\mu^{\alpha\rho}+(\varsigma^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)}\\
&\qquad+\frac{(\varsigma^{\rho(\alpha-\varepsilon)}+\mu^{\rho(\alpha-\varepsilon)}+(\varsigma^\rho-\mu^\rho)^{\alpha-\varepsilon})}{\rho^{\alpha-\varepsilon}\Gamma(\alpha-\varepsilon+1)}\},\\
&\leq \lambda_1\left\{\frac{(\varsigma^{\alpha\rho}+\mu^{\alpha\rho}+(\varsigma^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)}+\frac{(\varsigma^{\rho(\alpha-\varepsilon)}+\mu^{\rho(\alpha-\varepsilon)}+(\varsigma^\rho-\mu^\rho)^{\alpha-\varepsilon})}{\rho^{\alpha-\varepsilon}\Gamma(\alpha-\varepsilon+1)}\right\}\|\phi_1-\phi_2\|\\
&=\mathrm{O}(\varepsilon).
\end{aligned}$$

Also,

$$\left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}(\phi_1(\mu))-{}_0^{RC}D_\varsigma^{\alpha^*,\rho}(\phi_2(\mu))\right|\leq\lambda_1\|\phi_1-\phi_2\|(H(\mu)+H(\mu,\varepsilon))=\mathrm{O}(\varepsilon).$$

where,

$$\begin{aligned}
H(\mu)=&\frac{\rho^{(\alpha^*-\alpha)}\varsigma^{\rho(\alpha-1)}\mu^{\rho(1-\alpha^*)}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)}+\frac{\rho^{(\alpha^*-\alpha)}\mu^{(\alpha-\alpha^*)}}{2\Gamma(2-\alpha^*+1)}\\
&+\frac{\rho^{(\alpha^*-\alpha)}\varsigma^{\rho(\alpha-1)}(\varsigma^\rho-\mu^\rho)^{1-\alpha^*}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)}+\frac{\rho^{(\alpha^*-\alpha)}(\varsigma^\rho-\mu^\rho)^{\alpha-\alpha^*}}{2\Gamma(\alpha-\alpha^*+1)}.
\end{aligned}$$

and

$$\begin{aligned}
H(\mu,\varepsilon)=&\frac{\rho^{(\alpha^*-\alpha-\varepsilon)}\varsigma^{\rho(\alpha-\varepsilon-1)}\mu^{\rho(1-\alpha^*)}}{2\Gamma(\alpha-\varepsilon+1)\Gamma(2-\alpha^*)}+\frac{\rho^{(\alpha^*-\alpha-\varepsilon)}\mu^{(\alpha-\varepsilon-\alpha^*)}}{2\Gamma(2-\alpha^*+1)}\\
&+\frac{\rho^{(\alpha^*-\alpha-\varepsilon)}\varsigma^{\rho(\alpha-\varepsilon-1)}(\varsigma^\rho-\mu^\rho)^{1-\alpha^*}}{2\Gamma(\alpha-\varepsilon+1)\Gamma(2-\alpha^*)}+\frac{\rho^{(\alpha^*-\alpha-\varepsilon)}(\varsigma^\rho-\mu^\rho)^{\alpha-\varepsilon-\alpha^*}}{2\Gamma(\alpha-\alpha^*+1)}.
\end{aligned}$$

This completes the proof. □

Theorem 4.4.4. *Assume that the conditions of the Theorem 4.4.2 hold and if $\phi_1(\eta)$ be the solution of the BVP (4.1) and $\phi_2(\eta)$ be the solution of the following problem,*

$$\begin{cases}{}_0^{RC}D_\varsigma^{\alpha,\rho}\phi(\mu)=g(\mu,\phi(\mu),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\mu)), & \mu\in[0,\varsigma].\\ \phi(0)=\phi_0+\varepsilon_1, \qquad \phi(\varsigma)=\phi_\varsigma+\varepsilon_2.\end{cases}$$

Then, $\|\phi_1-\phi_2\|=\mathrm{O}(\max\{\varepsilon_1,\varepsilon_2\})$.

Proof.

$$
\begin{aligned}
&|\phi_1(\mu)-\phi_2(\mu)| \\
&\quad\le \frac{(\varepsilon_1+\varepsilon_2)\mu^\rho}{2}+\frac{(\varepsilon_1+\varepsilon_2)}{2\varsigma^\rho} \\
&\qquad+\frac{\left|g(\eta,\phi_1(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|(\varsigma^{\alpha\rho}+\mu^{\alpha\rho}+(\varsigma^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)} \\
&\quad\le \frac{(\varepsilon_1+\varepsilon_2)\mu^\rho}{2}+\frac{(\varepsilon_1+\varepsilon_2)}{2\varsigma^\rho}+\frac{\lambda_1(\varsigma^{\alpha\rho}+\mu^{\alpha\rho}+(\varsigma^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)}\|\phi_1(\mu)-\phi_2(\mu)\| \\
&\quad= \mathrm{O}(\{\varepsilon_1,\varepsilon_2).
\end{aligned}
$$

This follows the desired result. □

Theorem 4.4.5. *Assume that $\phi_1(\eta)$ be the solution of the BVP (4.1) and $\phi_2(\eta)$ be the solution of the following problem,*

$$
\begin{cases}
{}_0^{RC}D_\varsigma^{\alpha,\rho}\phi(\mu)=g(\mu,\phi(\mu),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi(\mu))+\epsilon, & \mu\in[0,\varsigma]. \\
\phi(0)=\phi_0, \qquad \phi(\varsigma)=\phi_\varsigma. &
\end{cases}
$$

where, $1<\alpha-\epsilon<\alpha\le 2$ and $0<\alpha^*\le 1$, and g is continuous. Then $\|\phi_1-\phi_2\|=O(\varepsilon)$.

Proof. From the Lemma 4.4.2 we have that,

$$
\begin{aligned}
&|\phi_1(\mu)-\phi_2(\mu)| \\
&\le \left|g(\eta,\phi_1(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta))-g(\eta,\phi_2(\eta),{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|\{\frac{(\varsigma^{\alpha\rho}+\mu^{\alpha\rho}+(\varsigma^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)}\} \\
&\quad+\frac{\varepsilon(\varsigma^{\alpha\rho}+\mu^{\alpha\rho}+(\varsigma^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)} \\
&\le \left\{\lambda_1\left(|\phi_1(\mu)-\phi_2(\mu)|+\left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_1(\eta))-{}_0^{RC}D_\varsigma^{\alpha^*,\rho}\phi_2(\eta))\right|+\varepsilon\right)\right\}\frac{(\varsigma^{\alpha\rho}+\mu^{\alpha\rho}+(\varsigma^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)} \\
&\le \frac{(\varsigma^{\alpha\rho}+\mu^{\alpha\rho}+(\varsigma^\rho-\mu^\rho)^\alpha)}{\rho^\alpha\Gamma(\alpha+1)}\{\lambda_1\|\phi_1-\phi_2\|+\varepsilon\} \\
&= \mathrm{O}(\varepsilon).
\end{aligned}
$$

Moreover,

$$
\left|{}_0^{RC}D_\varsigma^{\alpha^*,\rho}(\tilde{T}\phi_1(\mu))-{}_0^{RC}D_\varsigma^{\alpha^*,\rho}(\tilde{T}\phi_2(\mu))\right|\le H(\mu)\{\lambda_1\|\phi_1-\phi_2\|+\varepsilon\}=\mathrm{O}(\varepsilon),
$$

where

$$H(\mu) = \frac{\rho^{(\alpha^*-\alpha)}\varsigma^{\rho(\alpha-1)}\mu^{\rho(1-\alpha^*)}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)} + \frac{\rho^{(\alpha^*-\alpha)}\mu^{(\alpha-\alpha^*)}}{2\Gamma(2-\alpha^*+1)} + \frac{\rho^{(\alpha^*-\alpha)}\varsigma^{\rho(\alpha-1)}(\varsigma^{\rho}-\mu^{\rho})^{1-\alpha^*}}{2\Gamma(\alpha+1)\Gamma(2-\alpha^*)} + \frac{\rho^{(\alpha^*-\alpha)}(\varsigma^{\rho}-\mu^{\rho})^{\alpha-\alpha^*}}{2\Gamma(\alpha-\alpha^*+1)}.$$

This completes the proof. □

Chapter 5

Conclusions

This thesis contributes in the field of fractional calculus. Many mathematical properties of fractional operators are discussed in this thesis. A generalization of Riesz's fractional operators is presented. Some useful results and inequalities for new generalized Riesz's fractional operators are studied. Some generalized Gronwall inequalities are derived that are helpful to compute the estimated difference of solutions of two fractional differential equations. We proved some equivalence results for the nonlinear BVP involving generalized Katugampola derivatives and coupled system of fractional differential equations involving generalized derivative operator. We proved the uniqueness of solutions using suitable fixed point theorems and several mathematical techniques, and discussed the stability of solutions by showing continuous dependence onto given parameters. An instructive comparison with literature shows this thesis presents the generalization of various results in the field of fractional calculus. Moreover, the results presented in this thesis can be used in several directions, like diffusion process where the diffusion rate at any position depends on the whole space so Riesz fractional derivative can be useful in a sense that α-differentiable function needs not to be differentiable on whole domain. For example the function $f(x) = 3x^{\frac{1}{3}}$ is not differentiable at $x = 0$ but it has a fractional order derivative everywhere. Although the physical and geometrical interpretation of fractional derivative is the area that needs a lot more attention. The author would like to describe the geometrical interpretation of fractional derivative hereafter.

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