Group Classification and Invariant Characterization of Systems of Second Order Partial Differential Equations

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Dedicated To My Beloved Parents

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Muhammad Huzaifa Yaseen

Abstract

Differential invariants for linear and nonlinear ordinary and partial differential equations have been derived using Lie infinitesimal method. These invariants help in reduction of differential equations to their simplest possible solvable forms through invertible transformations of the dependent and independent variables (point transformations). We employ Lie infinitesimal method here to derive differential invariants for systems of two nonlinear parabolic type partial differential equations. Canonical forms for the considered systems are derived using obtained invariants which lead to solutions of the systems of nonlinear parabolic type partial differential equations.

Contents

Chapter 1

Introduction

Differential equations (DEs) initially appeared in the theory of calculus, developed by Isaac Newton (1642-1727) and Gottfried Wilhelm Leibniz (1646-1716) in the seventeenth century. Newton formulated three kinds of DEs in 1671. Leibniz was the first, who introduced the term DEs in 1676, in his letter to Newton and then utilized it in his publication [1] in 1684. Since then, in many disciplines including physics, engineering, economics, cosmology, epidemiology etc., the term DEs is frequently used to express mathematical models.

An equation that attains derivatives is named as *differential equation*, which relates certain function with it derivatives. In DEs functions generally represent physical quantities while its derivatives express their rates of change. DEs must contain dependent and independent variables. If in DEs dependent variables contain only one independent variable, then the equations are named as *ordinary differential equations* (ODEs). Whereas, *partial differential equations* (PDEs) are referred to those equations in which dependent variables are functions of more than one independent variable.

There are numerous strategies to find the exact solutions of DEs, however these do not address and solve all classes of ODEs and PDEs. In the nineteenth century, a Norwegian mathematician, Marius Sophus Lie [2], developed a method named as *Lie symmetry methods for DEs* to get their solutions. Lie's method for integrating the DEs is based on the groups of continuous transformations, known as *Lie groups*. The significance of this method is that it is applicable to any class of DEs. Whereas, neither the nature of DE nor the number of variables involved in the equation effects application of this method [4, 6, 21, 23].

Equivalence transformations are those transformations which maintain the differential structure of the DEs i.e., it leaves the DEs form invariant. The set of all the equivalence transformations makes a continuous group. Equivalence transformations play a vital role in the course of the calculations of differential invariants. In 1770 two semi differential invariants acquired by Euler in his integral calculus [24] and then, in 1773, Laplace [25] presented semi differential invariants for the linear hyperbolic PDEs in his fundamental memoir on the integration of linear PDEs, recognized as the Laplace invariants. Laplace invariants are those invariants which remain unchanged under a subgroup of equivalence transformation corresponding to the dependent variable only, therefore these quantities are named as semi differential invariants. In 1900, Cotton [18] obtained semi differential invariants for the linear elliptic PDEs , known as the Cotton invariants. Laplace and Cotton invariants remain conserved under the linear changes of the dependent variable, which respectively, map the linear hyperbolic and elliptic PDEs into themselves. Hyperbolic and elliptic PDEs can be transformed into each other by the application of linear complex transformations of the independent variables, as do Laplace and Cotton invariants.

Modern group analysis declares that differential invariants provide a powerful tool for handling initial value problems, quantitative analysis of DEs etc. semi differential invariants for linear ODEs were discussed in the 1870-1880 by J. Cockle [27], E. Laguerre [28], J.C. Malet [30], G.H. Halphen [32], R Harley [34] and A.R. Forsyth [36]. The restriction to linear equation was necessary in their approach, as these calculations were extremely lengthy in case of nonlinear equations. Indeed, when Roger Liouville investigated invariants for nonlinear ODEs introduced by Lie, the direct method led to 70 pages of calculations. Lie pointed that all variational problems and invariant DEs can be expressed in terms of differential invariants [37, 39]. He also declared that the theory of differential invariants is based on the infinitesimal method. Later on, Ovsiannikov [13] and Ibragimov [7, 14, 16, 20, 33, 38, 43, 45, 46] systematically developed the infinitesimal method to calculate invariants of the algebraic and DEs, known as *Lie* *infinitesimal method*. This method is applicable to algebraic equations and DEs possessing finite or infinite equivalence group [5, 8, 10, 11, 15, 22, 26, 29, 31, 35, 40, 44, 47].

In this thesis first chapter provides some basic definitions, theorems of symmetry analysis and differential invariants. In second chapter, we review equivalence transformations and differential invariants of scalar PDEs by using Lie infinitesimal method. In the third chapter we find the set of equivalence transformations associated with systems of two nonlinear parabolic type PDEs. In forth chapter, joint and semi differential invariants for systems of two nonlinear parabolic type PDEs are obtained by using Lie infinitesimal method. Then these invariants are shown to reduce such systems into their canonical forms via transformations of the dependent, independent and only the dependent variables. Last chapter concludes this work.

1.1 Lie Symmetry Analysis for ODEs

A symmetry group of a system of DEs is the largest group of transformations acting on the space of dependent and independent variables that maps a solution of the system of DEs into another solution. In other words, the solution manifold of the system of DEs remains invariant under a symmetry transformation of that system of DEs.

1.1.1 One Parameter Group of Transformations

Consider m and w be independent and dependent variables respectively. A point transformation

$$\tilde{m} = \tilde{m}(m, w), \qquad \tilde{w} = \tilde{w}(m, w), \tag{1.1}$$

can be used to simplify system of DEs. A set of invertible transformations that depends on an arbitrary parameter δ

$$\tilde{m} = \tilde{m}(m, w; \delta), \qquad \tilde{w} = \tilde{w}(m, w; \delta), \tag{1.2}$$

such that it contains the identity i.e., for $\delta = 0, \quad \tilde{m}(m, w; 0) = m, \quad \tilde{w}(m, w; 0) = w,$ and composition also belongs to same family. For example

$$\tilde{\tilde{m}}(\tilde{m}, \tilde{w}; \tilde{\delta}) = \tilde{\tilde{m}}(m, w; \tilde{\tilde{\delta}}), \tag{1.3}$$

for some $\tilde{\tilde{\delta}} = \tilde{\tilde{\delta}}(\tilde{\delta}, \delta)$, then the set of transformations (1.2) forms a group named as the *one-parameter group of point transformations*. If the group of transformations (1.2) is such that δ is a continuous parameter, transformations are infinitely differentiable with respect to the independent and dependent variables and $\tilde{\tilde{\delta}}(\tilde{\delta}, \delta)$ is an analytic function of $\tilde{\delta}$ and δ then it form a *one-parameter Lie group of continuous transformations*.

The one-parameter transformations (1.2) map one point (m, w) to another point $(\tilde{m}, \tilde{w})$ in the mw-plane and when the parameter δ changes from some initial value, say δ_o to some other value then the point $(\tilde{m}, \tilde{w})$ moves along some curve. For different initial points, different curves are obtained which can be mapped into one another under the action of the group (1.2). The set of these curves, called the orbits of the groups and can be completely described by the field of its tangent vectors **X** and vice versa.

1.1.2 Infinitesimal Transformations and Their Generators

Consider one-parameter Lie group of transformations

$$\tilde{m} = \tilde{m}(m, w; \delta), \qquad \tilde{w} = \tilde{w}(m, w; \delta), \tag{1.4}$$

with

$$\tilde{m}(m, w; 0) = m, \qquad \tilde{w}(m, w; 0) = w. \tag{1.5}$$

If we consider that δ is small, then we expand Taylor series of (1.4) about $\delta = 0$. Then

$$\begin{aligned} \tilde{m} &= \tilde{m}(m, w; 0) + \delta \frac{\partial \tilde{m}}{\partial \delta}|_{\delta=0} + O(\delta^2), \\ \tilde{w} &= \tilde{w}(m, w; 0) + \delta \frac{\partial \tilde{w}}{\partial \delta}|_{\delta=0} + O(\delta^2). \end{aligned} \tag{1.6}$$

Assume that

$$\xi(m, w) = \frac{\partial \tilde{m}}{\partial \delta}|_{\delta=0}, \qquad \eta(m, w) = \frac{\partial \tilde{w}}{\partial \delta}|_{\delta=0}, \tag{1.7}$$

after using (1.5), (1.7) in (1.6), we get

$$\begin{aligned} \tilde{m} &= m + \delta \xi(m, w) + O(\delta^2), \\ \tilde{w} &= w + \delta \eta(m, w) + O(\delta^2). \end{aligned} \tag{1.8}$$

The above equation can also be written as

$$\begin{aligned}\tilde{m} &= m + \delta \mathbf{X} m + O(\delta^2), \\ \tilde{w} &= w + \delta \mathbf{X} w + O(\delta^2),\end{aligned} \tag{1.9}$$

where the operator $\mathbf{X}$ is given by

$$\mathbf{X} = \xi(m,w)\frac{\partial}{\partial m} + \eta(m,w)\frac{\partial}{\partial w}. \tag{1.10}$$

The operator $\mathbf{X}$ is called *infinitesimal generator* of (1.1). It is also known as symmetry operator having ξ and η as its components that are called infinitesimal coordinates. It indicates that by repeating the application of infinitesimal transformation one can get finite transformation which is an alternative way of expression that the integral curves of vector field $\mathbf{X}$ are the group of orbits.

Example 1.1.1. Corresponding to one-parameter group of rotation the infinitesimal transformations

$$\tilde{m} = m\cos\delta - w\sin\delta, \qquad \tilde{w} = m\sin\delta + w\cos\delta, \tag{1.11}$$

gives the associated generator

$$\mathbf{X} = -w\frac{\partial}{\partial m} + m\frac{\partial}{\partial w}. \tag{1.12}$$

Example 1.1.2. Infinitesimal generator for the group of translation is

$$\mathbf{X} = \frac{\partial}{\partial m}, \tag{1.13}$$

from (1.13), we have

$$\frac{\partial \tilde{m}}{\partial \delta}|_{\delta=0} = 1, \qquad \frac{\partial \tilde{w}}{\partial \delta}|_{\delta=0} = 0, \tag{1.14}$$

it gives the infinitesimal transformations of the form

$$\tilde{m} = m + \delta, \qquad \tilde{w} = w. \tag{1.15}$$

1.1.3 Extension of Point Transformations and Their Symmetry Generators

To apply a point transformation (1.1) or (1.2) on a differential equation we require an extension or prolongation to include all the derivatives. For instance consider the following n-th order ODE

$$H(m, w, w^{(1)}, w^{(2)}, ..., w^{(n)}) = 0, \qquad w^{(1)} \equiv \frac{dw}{dm} \tag{1.16}$$

the derivatives can be transformed as

$$d\tilde{w} = \left(\frac{\partial \tilde{w}}{\partial w}\right) dw + \left(\frac{\partial \tilde{w}}{\partial m}\right) dm,$$

$$d\tilde{m} = \left(\frac{\partial \tilde{m}}{\partial w}\right) dw + \left(\frac{\partial \tilde{m}}{\partial m}\right) dm,$$

$$\begin{aligned}
\tilde{w}^{(1)} &= \frac{d\tilde{w}(m, w; \delta)}{d\tilde{m}(m, w; \delta)} \\
&= \frac{(\partial \tilde{w}/\partial w)\, w^{(1)} + (\partial \tilde{w}/\partial m)}{(\partial \tilde{m}/\partial w) w^{(1)} + (\partial \tilde{m}/\partial m)} = \tilde{w}^{(1)}(m, w, w^{(1)}; \delta), \\
\tilde{w}^{(2)} &= \frac{d\tilde{w}^{(1)}(m, w, w^{(1)}; \delta)}{d\tilde{m}(m, w; \delta)} \\
&= \frac{(\partial \tilde{w}^{(1)}/\partial w^{(1)}) w^{(2)} + (\partial \tilde{w}^{(1)}/\partial w) w^{(1)} + (\partial \tilde{w}^{(1)}/\partial m)}{(\partial \tilde{m}/\partial w) w^{(1)} + (\partial \tilde{m}/\partial m)} = \tilde{w}^{(1)}(m, w, w^{(1)}, w^{(2)}; \delta), \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
\tilde{w}^{(n)} &= \frac{(\partial \tilde{w}^{(n-1)}/\partial w^{(n-1)}) w^{(n)} + ... + (\partial \tilde{w}^{(n-1)}/\partial w) w^{(1)} + (\partial \tilde{w}^{(n-1)}/\partial m)}{(\partial \tilde{m}/\partial w) w^{(1)} + (\partial \tilde{m}/\partial m)} \\
&= \tilde{w}^{(1)}(m, w, w^{(1)}, w^{(2)}, ..., w^{(n)}; \delta).
\end{aligned} \tag{1.17}$$

Now, the n-th order extension of the infinitesimal generator (1.10) is given as follows

$$\begin{aligned}
\tilde{m} &= m + \delta\xi(m, w) + O(\delta^2) = m + \delta \mathbf{X}_m + O(\delta^2), \\
\tilde{w} &= w + \delta\eta(m, w) + O(\delta^2) = m + \delta \mathbf{X}_w + O(\delta^2), \\
\tilde{w}^{(1)} &= w^{(1)} + \delta\eta^{(1)}(m, w, w^{(1)}) + O(\delta^2) = w^{(1)} + \delta \mathbf{X}_{w^{(1)}} + O(\delta^2), \\
&\vdots \qquad\qquad \vdots \qquad\qquad \vdots
\end{aligned}$$

$$\tilde{w}^{(n)} = w^{(n)} + \delta\eta^{(n)}(m, w, w^{(1)}, ..., w^{(n)}) + O(\delta^2) = w^{(n)} + \delta\mathbf{X}_{w^{(n)}} + O(\delta^2). \quad (1.18)$$

Where $\eta^{(1)}$,..., $\eta^{(n)}$ are defined as

$$\eta^{(1)} = \frac{\partial \tilde{w}^{(1)}}{\partial \delta}|_{\delta=0}, ..., \eta^{(n)} = \frac{\partial \tilde{w}^{(n)}}{\partial \delta}|_{\delta=0}. \quad (1.19)$$

Substituting the expression (1.18) in (1.17), we obtain

$$\begin{aligned} \tilde{w}^{(1)} &= w^{(1)} + \delta\eta^{(1)} + O(\delta^2) = \frac{d\tilde{w}}{d\tilde{m}} \\ &= \frac{dw + \delta d\eta + O(\delta^2)}{dm + \delta d\xi + O(\delta^2)} = \frac{w^{(1)} + \delta(d\eta/dm) + O(\delta^2)}{1 + \delta(d\xi/dm) + O(\delta^2)} \\ &= w^{(1)} + \delta\left[\left(\frac{d\eta}{dm}\right) - w^{(1)}\left(\frac{d\xi}{dm}\right)\right] + O(\delta^2). \end{aligned} \quad (1.20)$$

Similarly, for the $\eta^{(n)}(m, w, w^{(1)}, ..., w^{(n)})$ we have

$$\begin{aligned} \tilde{w}^{(n)} &= w^{(n)} + \delta\eta^{(n)} + O(\delta^2) = \frac{d\tilde{w}^{(n-1)}}{d\tilde{m}} \\ &= w^{(n)} + \delta\left[\left(\frac{d\eta^{(n-1)}}{dm}\right) - w^{(n)}\left(\frac{d\xi}{dm}\right)\right] + O(\delta^2). \end{aligned} \quad (1.21)$$

Here $\eta^{(n)}(m, w, w^{(1)}, ..., w^{(n)})$ is the n-th prolongation of $\eta(m, w)$ [9]. The results are summarized in the following theorem.

Theorem 1.1.1. *For one-parameter Lie group of point transformations (1.2) the infinitesimal generator can be extended as follows [9]*

$$\begin{aligned} \eta^{(1)} &= \frac{d\eta}{dm} - w^{(1)}\frac{d\xi}{dm}, \\ &\vdots \\ \eta^{(n)} &= \frac{d\eta^{(n-1)}}{dm} - w^{(n)}\frac{d\xi}{dm}, \end{aligned} \quad (1.22)$$

as the corresponding n-th order infinitesimal generator is expressed

$$\boldsymbol{X}^{(n)} = \xi\frac{\partial}{\partial m} + \eta\frac{\partial}{\partial w} + \eta^{(1)}\frac{\partial}{\partial w^{(1)}} + ... + \eta^{(n)}\frac{\partial}{\partial w^{(n)}}. \quad (1.23)$$

Definition 1.1.3. A point transformation

$$\tilde{m} = \tilde{m}(m, w; \delta), \qquad \tilde{w} = \tilde{w}(m, w; \delta), \quad (1.24)$$

is termed as a *point symmetry of the DE*

$$H(m, w, w^{(1)}, w^{(2)}, ..., w^{(n)}) = 0, \tag{1.25}$$

if and only if the DE remains same under the n-th prolongation of transformations (1.24), i.e., it remains invariant and preserves the form of DE [41] that can be expressed as

$$H(\tilde{m}, \tilde{w}, \tilde{w}^{(1)}, \tilde{w}^{(2)}, ..., \tilde{w}^{(n)}) = 0. \tag{1.26}$$

In simple words we say, any solution of (1.25) can be mapped into a solution of (1.26).

1.1.4 Multiple Parameter Lie Groups of Transformations and Their Generators

A transformation (1.2) can depend on more than one parameter such as

$$\tilde{m} = \tilde{m}(m, w; \delta_N), \quad \tilde{w} = \tilde{w}(m, w; \delta_N). \quad N = 1, 2, ..., r \tag{1.27}$$

The transformation (1.27) is said to be *r-parameter* lie group of transformation if it satisfies all properties of one-parameter Lie group transformations [9] with distinct δ_N. A symmetry generator $\mathbf{X}_N$ can be associated with each parameter δ_N by the following expression

$$\mathbf{X}_N = \xi_N \frac{\partial}{\partial m} + \eta_N \frac{\partial}{\partial w}, \tag{1.28}$$

where

$$\xi_N(m, w) = \frac{\partial \tilde{m}}{\partial \delta_N}|_{\delta_N=0}, \qquad \eta_N(m, w) = \frac{\partial \tilde{w}}{\partial \delta_N}|_{\delta_N=0}. \tag{1.29}$$

1.2 Lie Symmetry Analysis for PDEs

Consider $\mathbf{m} = (m_i)$ and $\mathbf{w} = (w_\alpha)$ be q independent and p dependent variables respectively. The derivatives of $\mathbf{w}$ with respect to $\mathbf{m}$ are denoted by

$$\partial \mathbf{w} = w_{\alpha,i} = D_i(w_\alpha), \tag{1.30}$$

$$\partial^2 \mathbf{w} = w_{\alpha,ij} = D_i D_j(w_\alpha), \tag{1.31}$$

and so on n-th term becomes

$$\partial^{(n)}\mathbf{w} = w_{\alpha,i_1 i_2 \ldots i_k} = D_{i_1} D_{i_2} \ldots D_{i_k}(w_\alpha), \tag{1.32}$$

where

$$D_i = \frac{\partial}{\partial m_i} + w_{\alpha,i}\frac{\partial}{\partial w_\alpha} + w_{\alpha,ij}\frac{\partial}{\partial w_{\alpha,j}} + \ldots, \tag{1.33}$$

is the total derivative operator. Then a system of PDEs can be reported as

$$H^\sigma(\mathbf{m}, \mathbf{w}, \partial\mathbf{w}, \partial^2\mathbf{w}, \ldots, \partial^{(n)}\mathbf{w}) = 0, \qquad \sigma = 1, 2, \ldots, s. \tag{1.34}$$

To deal with symmetries of system of PDEs (1.34), we form the group of invertible transformations that depend on the real parameter δ which leaves (1.34) invariant.

1.2.1 Point Transformations and Their Symmetry Generators

For p dependent $\mathbf{w} = (w_\alpha)$ and q independent $\mathbf{m} = (m_i)$ variables one-parameter Lie point transformations can be written as

$$\begin{aligned} \tilde{m}_i &= \tilde{m}_i(m_i, w_\alpha; \delta), \\ \tilde{w}_\alpha &= \tilde{w}_\alpha(m_i, w_\alpha; \delta), \end{aligned} \tag{1.35}$$

since δ is a small parameter therefore series expansion of transformation (1.35) can be written as

$$\begin{aligned} \tilde{m}_i &= m_i + \delta\xi_i(m_i, w_\alpha) + O(\delta^2), \qquad i = 1, 2, \ldots, q \\ \tilde{w}_\alpha &= w_\alpha + \delta\eta_\alpha(m_i, w_\alpha) + O(\delta^2), \qquad \alpha = 1, 2, \ldots, p \end{aligned} \tag{1.36}$$

where

$$\xi_i(m_i, w_\alpha) = \frac{\partial m_i}{\partial \delta}|_{\delta=0}, \qquad \eta_\alpha(m_i, w_\alpha) = \frac{\partial w_\alpha}{\partial \delta}|_{\delta=0}, \tag{1.37}$$

The infinitesimal transformations are generated by an operator of the form

$$\mathbf{X} = \xi_i\frac{\partial}{\partial m_i} + \eta_\alpha\frac{\partial}{\partial w_\alpha}. \tag{1.38}$$

Theorem 1.2.1. *For a one-parameter Lie group of point transformations (1.35) the nth extension of the corresponding infinitesimal generator (1.38) is given by*

$$\boldsymbol{X}^{(n)} = \xi_i \frac{\partial}{\partial m_i} + \eta_\alpha \frac{\partial}{\partial w_\alpha} + \eta^{(1)}_{\alpha,i} \frac{\partial}{\partial w_{\alpha,i}} + ... + \eta^{(n)}_{\alpha,i_1 i_2 ... i_k} \frac{\partial}{\partial w_{\alpha,i_1 i_2 ... i_k}}, \tag{1.39}$$

where

$$\begin{aligned} \eta^{(1)}_{\alpha,i} &= D_i \eta_\alpha - w_{\alpha,j} D_i \xi_j, \\ \eta^{(n)}_{\alpha,i_1 i_2 ... i_k} &= D_{i_n} \eta^{(n-1)}_{\alpha,i_1 i_2 ... i_{n-1}} - w_{\alpha,i_1 i_2 ... i_{n-1}} D_{i_k} \xi_j. \end{aligned} \tag{1.40}$$

1.2.2 Lie Point Symmetries

A one-parameter Lie group of point transformations (1.35) is called a Lie point symmetry of a system (1.34) if and only if the system remains invariant and can be elaborated as

$$H^\sigma(\tilde{\mathbf{m}}, \tilde{\mathbf{w}}, \partial\tilde{\mathbf{w}}, \partial^2\tilde{\mathbf{w}}, ..., \partial^{(n)}\tilde{\mathbf{w}}) = 0, \qquad \sigma = 1, 2, ..., s. \tag{1.41}$$

In other words, the solution manifold of the system (1.34) remains invariant under the transformations (1.35).

Theorem 1.2.2. *A one-parameter Lie group of point transformations (1.35) with the n-th order extended generator (1.39) is said to be point symmetry of the ststem (1.34) if and only if [9]*

$$\boldsymbol{X}^{(n)} H^\sigma(\boldsymbol{m}, \boldsymbol{w}, \partial\boldsymbol{w}, \partial^2\boldsymbol{w}, ..., \partial^{(n)}\boldsymbol{w}) = 0, \qquad \sigma = 1, 2, ..., s \tag{1.42}$$

whenever

$$H^\sigma(\boldsymbol{m}, \boldsymbol{w}, \partial\boldsymbol{w}, \partial^2\boldsymbol{w}, ..., \partial^{(n)}\boldsymbol{w}) = 0, \qquad \sigma = 1, 2, ..., s.$$

1.3 Equivalence Transformations

Equivalence transformations are invertible transformations that preserve the differential structure of the equations. One-parameter Lie group of point transformations

$$\tilde{m}_i = \tilde{m}_i(m_i, w_\alpha; \delta), \qquad i = 1, 2, ..., q$$

$$
\begin{aligned}
\tilde{w}_\alpha &= \tilde{w}_\alpha(m_i, w_\alpha; \delta), \qquad \alpha = 1, 2, ..., p \\
\mathbf{P}_l &= \mathbf{P}_l(m_i, w_\alpha, \mathbf{P}_l; \delta), \qquad l = 1, 2, ..., r.
\end{aligned} \tag{1.43}
$$

is called one-parameter Lie group of equivalence transformations if it maps a system of DEs into a system of the same family. According to Ovsiannikov [42], an equivalence transformation is represented by a generator of continuous equivalence group of transformations behaving in the expanded space of dependent variables, independent variables, functions and their derivatives (arbitrary coefficients of the DEs) which does not alter the form of the equation under investigation.

Equivalence transformations play very important role to classify the DEs , where the nature of transformations help to characterize the DEs. In the theory of invariants, equivalence transformations are also used. Derivation of equivalence transformations for the class of equations under consideration is the first step towards determination of differential invariants. The set of all equivalence transformations of a given family of DEs forms a group which is called the equivalence group. The method used here to derive equivalence transformations is called infinitesimal method. System of PDEs may involve arbitrary functions $\mathbf{P}_l$, thus the equivalence operator $\mathbf{X}$ is written in the following form

$$
\mathbf{X} = \xi_i \frac{\partial}{\partial m_i} + \eta_\alpha \frac{\partial}{\partial w_\alpha} + \sum_{l=1}^{r} \mu_l \frac{\partial}{\partial \mathbf{P}_l}, \tag{1.44}
$$

where functions ξ_i, η_α represents independent and dependent variables, while μ_l express arbitrary functions that appears in DEs.

Lie Infinitesimal Method

We consider an example here to illustrate Lie infinitesimal method, we take a well known Korteweg-de Vries equation [19] which is given as

$$
w_m + w_{sss} + w w_s = 0. \tag{1.45}
$$

The operator corresponding to this equation is of the form

$$
\mathbf{X} = \xi_1(m, s, w) \frac{\partial}{\partial m} + \xi_2(m, s, w) \frac{\partial}{\partial s} + \eta(m, s, w) \frac{\partial}{\partial w}. \tag{1.46}
$$

Equation (1.46) is a symmetry generator of (1.45) if

$$\mathbf{X}^{[3]}(w_m + w_{sss} + ww_s)|_{w_m=-w_{sss}-ww_s} = 0. \tag{1.47}$$

For this case the third prolongation of operator (1.46) is

$$\mathbf{X}^{[3]} = \xi_1\frac{\partial}{\partial m} + \xi_2\frac{\partial}{\partial s} + \eta\frac{\partial}{\partial w} + \eta^m\frac{\partial}{\partial w_m} + \eta^s\frac{\partial}{\partial w_s} + \eta^{ss}\frac{\partial}{\partial w_{ss}} + \eta^{sss}\frac{\partial}{\partial w_{sss}}, \tag{1.48}$$

where

$$\begin{aligned}
\eta^m &= D_m(\eta) - w_m D_m(\xi_1) - w_s D_m(\xi_2),\\
\eta^s &= D_s(\eta) - w_m D_s(\xi_1) - w_s D_s(\xi_2),\\
\eta^{ss} &= D_s(\eta^s) - w_m D_s(\xi_1) - w_s D_s(\xi_2),\\
\eta^{sss} &= D_s(\eta^{ss}) - w_m D_s(\xi_1) - w_s D_s(\xi_2),
\end{aligned}$$

and

$$\begin{aligned}
D_m &= \frac{\partial}{\partial m} + w_m\frac{\partial}{\partial w} + w_{mm}\frac{\partial}{\partial w_m} + w_{ms}\frac{\partial}{\partial w_s} + ...,\\
D_s &= \frac{\partial}{\partial m} + w_s\frac{\partial}{\partial w} + w_{ms}\frac{\partial}{\partial w_m} + w_{ss}\frac{\partial}{\partial w_s} + ...,
\end{aligned} \tag{1.49}$$

which provides

$$\begin{aligned}
\eta^m &= \eta_m + w_m(\eta_w - \xi_{1,m}) - w_m^2\xi_{1,w} - w_s\xi_{2,m} - w_m w_s\xi_{2,w},\\
\eta^s &= \eta_s + w_s(\eta_w - \xi_{2,s}) - w_s^2\xi_{2,w} - w_m\xi_{1,s} - w_m w_s\xi_{1,w},\\
\eta^{ss} &= \eta_{ss} + 2w_s\eta_{sw} - w_m\xi_{1,ss} - 2w_m w_s\xi_{1,sw} - w_s\xi_{2,ss} - 2w_s^2\xi_{2,sw} + w_s^2\eta_{ww}\\
&\quad -w_m w_s^2\xi_{1,ww} - w_s^3\xi_{2,ww} - 2w_{ms}\xi_{1,s} - 2w_{ms}w_s\xi_{1,w} + w_{ss}\eta_w - w_{ss}w_m\xi_{1,w}\\
&\quad -2w_{ss}\xi_{2,s} - 3w_{ss}w_s\xi_{2,w}\\
\eta^{sss} &= \eta_{sss} - 3w_m w_s^2\xi_{1,sww} - 3w_m w_s w_{ss}\xi_{1,ww} + 3w_{ss}\eta_{sw} - 3w_{ms}\xi_{1,ss}\\
&\quad +3w_s\eta_{1,ssw} - w_m\xi_{1,sss} - w_s\xi_{2,sss} - 3w_s^2\xi_{2,ssw} + 3w_s^2\eta_{sww} - 3w_s^3\xi_{2,sww}\\
&\quad -3w_m w_s\xi_{1,ssw} - 6w_{ms}w_s\xi_{1,sw} - 3w_m w_{ss}\xi_{1,sw} - 9w_s w_{ss}\xi_{2,sw} - w_m w_{sss}\xi_{1,w}\\
&\quad -4w_s w_{sss}\xi_{2,w} - w_m w_s^3\xi_{1,www} - 3w_{ms}w_s^2\xi_{1,ww} + 3w_s w_{ss}\eta_{ww} - 6w_{ss}w_s^2\xi_{2,ww}\\
&\quad -3w_{ms}w_{ss}\xi_{1,w} - 3w_s w_{mss}\xi_{1,w} - 3w_{ss}\xi_{2,ss} + w_s^3\eta_{www} - w_s^4\xi_{2,www} - 3w_{ss}^2\xi_{2,w}
\end{aligned}$$

$$-3w_{mss}\xi_{1,s} + w_{sss}\eta_w - 3w_{sss}\xi_{2,s}. \tag{1.50}$$

After applying third order prolongated generator, the determining equation (1.47) gives

$$\eta^m + \eta^{sss} + w_s\eta + w\eta^s|_{w_m=-w_{sss}-ww_s} = 0. \tag{1.51}$$

Substituting (1.50) in (1.51) and replacing w_m with $-(w_{sss}+ww_s)$. Afterwards, coefficients of $w_{mss}w_s$, w_{mss}, w_{ss}^2, $w_{ss}w_s$, w_{ss}, w_{sss}, w_s and constant terms gives the following system of linear homogenous PDEs

$$\begin{aligned}
&\xi_{1,w} = 0,\\
&\xi_{1,s} = 0,\\
&\xi_{2,w} = 0,\\
&\eta_{w,w} = 0,\\
&\eta_{sw} - \xi_{2,ss} = 0,\\
&-3\xi_{2,s} + \xi_{1,m} = 0,\\
&3\eta_{ssw} - \xi_{2,sss} - w\xi_{2,s} - \xi_{2,m} + w\xi_{1,m} + \eta = 0,\\
&\eta_{sss} + \eta_m + w\eta_s = 0,
\end{aligned} \tag{1.52}$$

that generates

$$\begin{aligned}
\xi_1 &= -\frac{3c_1}{2}m + c_2,\\
\xi_2 &= c_2 m - \frac{c_1}{2}s + c_3,\\
\eta &= c_1 w + c_4,
\end{aligned} \tag{1.53}$$

when solved, here c_i for $i = 1, 2, 3, 4$ are arbitrary constants. The expression (1.53) is termed as an equivalence transformation for equation (1.45). The corresponding generator becomes

$$\mathbf{X} = (-\frac{3c_1}{2}m + c_2)\frac{\partial}{\partial m} + (c_2 m - \frac{c_1}{2}s + c_3)\frac{\partial}{\partial s} + (c_1 w + c_4)\frac{\partial}{\partial w}, \tag{1.54}$$

which can separately be written as

$$\mathbf{X}_1 = \frac{\partial}{\partial m},$$

$$\begin{aligned}
\mathbf{X}_2 &= \frac{\partial}{\partial s},\\
\mathbf{X}_3 &= m\frac{\partial}{\partial s}+\frac{\partial}{\partial w},\\
\mathbf{X}_4 &= -\frac{3m}{2}\frac{\partial}{\partial m}-\frac{s}{2}\frac{\partial}{\partial s}+w\frac{\partial}{\partial w}.
\end{aligned}\tag{1.55}$$

1.4 Differential Invariants

Invariants of a DE are mathematical expressions written in terms of its coefficients, while differential invariants are those that also involve derivatives of the coefficients. Differential invariant of DEs remains invariant under the group of equivalence transformations and satisfies invariance test (Infinitesimal criteria of invariance). A differential invariant of order r can be expressed as

$$J(\mathbf{P}_l, \partial\mathbf{P}_l, \partial^2\mathbf{P}_l, ..., \partial^r\mathbf{P}_l),\tag{1.56}$$

where $\mathbf{P}_l$ represents coefficients (arbitrary) of the considered DEs and $\partial\mathbf{P}_l$ are their partial derivatives. Given mathematical form of the criteria for zeroth order invariants, we employ $\mathbf{X}$ that is given in (1.44) in the following equation

$$\mathbf{X}J(\mathbf{P}_l) = 0.\tag{1.57}$$

To get first order differential invariants we extend $\mathbf{X}$ once and apply

$$\mathbf{X}^{[1]}J(\mathbf{P}_l, \partial\mathbf{P}_l) = 0.\tag{1.58}$$

Likewise, r-th order differential invariants have the invariance condition

$$\mathbf{X}^{[r]}J(\mathbf{P}_l, \partial\mathbf{P}_l, \partial^2\mathbf{P}_l, ..., \partial^r\mathbf{P}_l) = 0.\tag{1.59}$$

For reader complete extension procedure is given in later sections e.g., (2.1.2).

Differential invariants play an essential role in converting DEs into canonical and integrable forms. In order to deduce differential invariants for a DE, one has to obtain the associated set of equivalence transformations. Once the differential invariants of a DE are obtained then one can attempt reduction of DE into its simpler forms. For instance these invariants can be employed to linearize nonlinear DEs. If differential invariants for any two DEs are same then it guarantees that they are mappable into each other through point transformations.

Differential invariants have two major categories. The first one is *joint differential invariants* which are derived under transformations of both the independent and dependent variables. The invariants which are obtained under the transformations of only the dependent or independent variables separately, are known as *semi differential invariants*.

Chapter 2

Equivalence Transformations and Differential Invariants for Scalar PDEs

A general second order scalar linear PDE with two independent variables m and s is of the form

$$aw_{mm} + bw_{ms} + cw_{ss} + dw_m + ew_s + fw = g, \tag{2.1}$$

where a, b, c, d, e, f, g are given differentiable functions of m and s. The discriminant is defined by $b^2 - 4ac$, and equation (2.1) can be classified with the sign of discriminant. If $b^2 - 4ac = 0$, then (2.1) is called parabolic equation, which describe heat flow and diffusion process. Hyperbolic equations satisfy the property $b^2 - 4ac > 0$, which for example describe vibrating system and wave motion, while elliptic equations describe processes in equilibrium which satisfy the property $b^2 - 4ac < 0$.

This chapter is a review of literature where we present equivalence transformations and differential invariants for scalar PDEs. The first section is on Laplace invariants for linear scalar parabolic PDEs, in which study equivalence transformations and then corresponding to those transformations we get semi differential invariants. Second section deals with differential invariants of scalar nonlinear hyperbolic type PDEs.

2.1 Differential Invariants of Scalar Linear PDEs

In 1773, Laplace discovered invariants for linear scalar hyperbolic DEs and applied in his theory of integration for hyperbolic equations. These invariants are termed as

Laplace invariants [17]. Cotton expanded Laplace invariants and obtained invariants for linear elliptic equations [18] in 1900, named as Cotton's invariants. Afterwards, N. H. Ibragimov find Laplace type invariants for linear scalar parabolic equations by Lie infinitesimal method in 2001 and fill the gap.

2.1.1 Equivalence Transformations

Consider the scalar parabolic equation with two independent variables m, s of the form

$$w_m + a(m,s)w_{ss} + b(m,s)w_s + c(m,s)w = 0, \tag{2.2}$$

where a, b, c are arbitrary coefficients while the subscripts represents the partial derivatives, i.e., $w_m = \partial w/\partial m$, etc. The equivalence transformation of (2.2) is an invertible transformation

$$\tilde{m} = \phi_1(m,s,w), \qquad \tilde{s} = \phi_2(m,s,w), \qquad \tilde{w} = \phi_3(m,s,w), \tag{2.3}$$

such that (2.2) remains the same, as for example in order, homogeneity and linearity etc under (2.3). The set of all the equivalence transformations of (2.2) makes an equivalence group and in order to find the continuous group by making use of infinitesimal method, we apply the following operator

$$\mathbf{X} = \xi_1\frac{\partial}{\partial m} + \xi_2\frac{\partial}{\partial s} + \eta\frac{\partial}{\partial w} + \mu_1\frac{\partial}{\partial a} + \mu_2\frac{\partial}{\partial b} + \mu_3\frac{\partial}{\partial c}. \tag{2.4}$$

Here, the coordinates ξ_1, ξ_2, η are functions of (m,s,w) while μ_1, μ_2, μ_3 are functions of (m,s,w,a,b,c). The Lie invariance criterion for (2.2) reads as

$$\mathbf{X}^{[2]}(w_m + aw_{ss} + bw_s + cw)|_{(2.2)} = 0, \tag{2.5}$$

where $\mathbf{X}^{[2]}$ is the second order prolongation of the operator (2.4) which is expressed as

$$\mathbf{X}^{[2]} = \mathbf{X} + \eta^m\frac{\partial}{\partial w_m} + \eta^s\frac{\partial}{\partial w_s} + \eta^{mm}\frac{\partial}{\partial w_{mm}} + \eta^{ms}\frac{\partial}{\partial w_{ms}} + \eta^{ss}\frac{\partial}{\partial w_{ss}}. \tag{2.6}$$

By applying (2.6) on (2.2), we obtain

$$\eta^m + a\eta^{ss} + b\eta^s + c\eta + \mu_1 w_{ss} + \mu_2 w_s + \mu_3 w|_{(2.2)} = 0, \tag{2.7}$$

where η^m, η^s, η^{ss} are obtainable from (1.49). Substituting these extension coefficients, replacing w_m with $-(aw_{ss}+bw_s+cw)$ and equating coefficients of $w_{ms}w_s$, w_{ms}, $w_{ss}w_s$, w_s^2 and other terms in (2.7), we get

$$\xi_{1,w}=0, \qquad \xi_{1,s}=0, \qquad \xi_{2,w}=0, \qquad \eta_{ww}=0, \tag{2.8}$$

it implies

$$\begin{aligned} \xi_1 &:= \xi_1(m), \\ \xi_2 &:= \xi_2(m,s), \\ \eta &:= \eta_1(m,s)w+\eta_2(m,s). \end{aligned} \tag{2.9}$$

Subsequently, coefficients of w_{ss}, w_s and remaining terms in (2.7) gives

$$\begin{aligned} \mu_1 &= 2a\xi_{2,s}-a\xi_{1,m}, \\ \mu_2 &= a\xi_{2,ss}+b\xi_{2,s}+\xi_{2,m}-2a\eta_{sw}-b\xi_{1,m} \\ \mu_3 &= cw\eta_w-a\eta_{ss}-\eta_m-b\eta_s-cw\xi_{1,m}. \end{aligned} \tag{2.10}$$

2.1.2 Semi Differential Invariants for Parabolic PDEs

Laplace type invariants (or semi differential invariants) of scalar parabolic PDEs corresponding to the dependent variable, are reviewed in this subsection by the use of the Lie infinitesimal method. In order to find these differential invariants for (2.2), we get the following generator

$$\mathbf{X}=\mu_1\frac{\partial}{\partial a}+\mu_2\frac{\partial}{\partial b}+\mu_3\frac{\partial}{\partial c}, \tag{2.11}$$

by considering ξ_1, ξ_2 and all their derivatives equal to zero in (2.9) and (2.10), which leads to

$$\mu_1=0, \qquad \mu_2=2a\eta_s, \qquad \mu_3=\eta_m+a\eta_{ss}+b\eta_s. \tag{2.12}$$

Using (2.12) in (2.11), gives

$$\mathbf{X}=2a\eta_s\frac{\partial}{\partial b}+(\eta_m+a\eta_{ss}+b\eta_s)\frac{\partial}{\partial c}. \tag{2.13}$$

The infinitesimal test $\mathbf{X}K = 0$, for zeroth order invariant $K(a,b,c)$ gets to the form

$$2a\eta_s\frac{\partial K}{\partial b} + (\eta_m + a\eta_{ss} + b\eta_s)\frac{\partial K}{\partial c} = 0. \tag{2.14}$$

Equating coefficients of η_s, η_m equal to zero, one obtains

$$\frac{\partial K}{\partial b} = 0, \qquad \frac{\partial K}{\partial c} = 0. \tag{2.15}$$

As a result, there is only independent invariant $K = a$. Now to find first order differential invariants of the form $K(a, a_m, a_s; b, b_m, b_s; c, c_m, c_s)$ for (2.2), one need to extend (2.13) which has the expression

$$\mathbf{X}^{[1]} = \mathbf{X} + \mu_{2,m}\frac{\partial}{\partial b_m} + \mu_{2,s}\frac{\partial}{\partial b_s} + \mu_{3,m}\frac{\partial}{\partial c_m} + \mu_{3,s}\frac{\partial}{\partial c_s}. \tag{2.16}$$

Where

$$\begin{aligned}
\mu_{2,m} &= \tilde{D}_m(\mu_2) - b_m\tilde{D}_m(\xi_1) - b_s\tilde{D}_m(\xi_2),\\
\mu_{2,s} &= \tilde{D}_s(\mu_2) - b_m\tilde{D}_s(\xi_1) - b_s\tilde{D}_s(\xi_2),\\
\mu_{3,m} &= \tilde{D}_m(\mu_3) - c_m\tilde{D}_m(\xi_1) - c_s\tilde{D}_m(\xi_2),\\
\mu_{3,s} &= \tilde{D}_s(\mu_3) - c_m\tilde{D}_s(\xi_1) - c_s\tilde{D}_s(\xi_2),
\end{aligned} \tag{2.17}$$

and

$$\begin{aligned}
\tilde{D}_m &= \frac{\partial}{\partial m} + a_m\frac{\partial}{\partial a} + a_{mm}\frac{\partial}{\partial a_m} + a_{ms}\frac{\partial}{\partial a_s} + ... + b_m\frac{\partial}{\partial b} + b_{mm}\frac{\partial}{\partial b_m} + b_{ms}\frac{\partial}{\partial b_s}\\
&\quad + ... + c_m\frac{\partial}{\partial c} + c_{mm}\frac{\partial}{\partial c_m} + c_{ms}\frac{\partial}{\partial c_s},\\
\tilde{D}_s &= \frac{\partial}{\partial s} + a_s\frac{\partial}{\partial a} + a_{ss}\frac{\partial}{\partial a_s} + a_{ms}\frac{\partial}{\partial a_m} + ... + b_s\frac{\partial}{\partial b} + b_{ss}\frac{\partial}{\partial b_s} + b_{ms}\frac{\partial}{\partial b_m}\\
&\quad + ... + c_s\frac{\partial}{\partial c} + c_{ss}\frac{\partial}{\partial c_s} + c_{ms}\frac{\partial}{\partial c_m}.
\end{aligned} \tag{2.18}$$

So the once extended generator reads as

$$\begin{aligned}
\mathbf{X}^{[1]} &= 2a\eta_s\frac{\partial}{\partial b} + (\eta_m + a\eta_{ss} + b\eta_s)\frac{\partial}{\partial c} + 2(a\eta_{ms} + a_m\eta_s)\frac{\partial}{\partial b_m} + 2(a\eta_{ss} + a_s\eta_x)\frac{\partial}{\partial b_s}\\
&\quad + (\eta_{mm} + a\eta_{mss} + a_m\eta_{ss} + b\eta_{ms} + b_m\eta_s)\frac{\partial}{\partial c_m}
\end{aligned}$$

$$+(\eta_{ms}+a\eta_{sss}+a_s\eta_{ss}+b\eta_{ss}+b_s\eta_s)\frac{\partial}{\partial c_s}. \tag{2.19}$$

From the invariance criteria

$$\mathbf{X}^{[1]}K=0, \tag{2.20}$$

equating coefficients of η_{sss}, η_s, η_{ms}, η_m, η_{ss}, η_s to zero yields

$$\begin{aligned} &\frac{\partial K}{\partial c_m}=0, \qquad \frac{\partial K}{\partial c_s}=0, \qquad \frac{\partial K}{\partial b_m}=0,\\ &\frac{\partial K}{\partial c}=0, \qquad \frac{\partial K}{\partial b_s}=0, \qquad \frac{\partial K}{\partial b}=0. \end{aligned} \tag{2.21}$$

Hence, there exist first order differential invariants only of the form $K(a,a_m,a_s)$. Accordingly, we consider derivation of second order differential invariants by using

$$\mathbf{X}^{[2]}K(a,a_m,a_s,a_{mm},a_{ms},a_{ss};b,b_m,b_s,b_{mm},b_{ms},b_{ss};c,c_m,c_s,c_{mm},c_{ms},c_{ss})=0, \tag{2.22}$$

where

$$\begin{aligned} \mathbf{X}^{[2]} \;=\; & \mathbf{X}^{[1]}+\mu_{2,mm}\frac{\partial}{\partial b_{mm}}+\mu_{2,ms}\frac{\partial}{\partial b_{ms}}+\mu_{2,ss}\frac{\partial}{\partial b_{ss}}+\mu_{3,mm}\frac{\partial}{\partial c_{mm}}\\ &+\mu_{3,ms}\frac{\partial}{\partial c_{ms}}+\mu_{3,ss}\frac{\partial}{\partial c_{ss}}. \end{aligned} \tag{2.23}$$

Applying the same procedure as above, one first finds the equations

$$\begin{aligned} &\frac{\partial K}{\partial c_{mm}}=0, \qquad \frac{\partial K}{\partial c_{ms}}=0, \qquad \frac{\partial K}{\partial c_{ss}}=0, \qquad \frac{\partial K}{\partial b_{mm}}=0,\\ &\frac{\partial K}{\partial b_{ms}}=0, \qquad \frac{\partial K}{\partial c_m}=0, \qquad \frac{\partial K}{\partial c}=0. \end{aligned} \tag{2.24}$$

It follows that $K=K(a,a_m,a_s,a_{mm},a_{ms},a_{ss};b,b_m,b_s,b_{ss};c_s)$. Afterwards, (2.22) gives following system of equations

$$\begin{aligned} &\frac{\partial K}{\partial c_s}+2a\frac{\partial K}{\partial b_m}=0, \qquad a\frac{\partial K}{\partial b_m}-\frac{\partial K}{\partial b_{ss}}=0,\\ &a\frac{\partial K}{\partial b_s}+(a_s-b)\frac{\partial K}{\partial b_{ss}}=0,\\ &a\frac{\partial K}{\partial b}+a_m\frac{\partial K}{\partial b_m}+a_s\frac{\partial K}{\partial b_s}+(a_{ss}-b_s)\frac{\partial K}{\partial b_{ss}}=0. \end{aligned} \tag{2.25}$$

One obtain following differential invariants after solving (2.25)

$$K=K(a,a_m,a_s,a_{mm},a_{ms},a_{ss};K_1), \tag{2.26}$$

with

$$K_1 = \frac{1}{2}b^2 a_s + (a_m + a a_{ss} - a_s^2)b + (a a_s - ab)b_s - a b_m - a^2 b_{ss} + 2a^2 c_s, \tag{2.27}$$

which is termed as Laplace type invariant for parabolic equation (2.2).

2.2 Differential Invariants of Scalar Nonlinear PDEs

In this section, we observe how Lie infinitesimal method works to calculate differential invariants for the class of scalar nonlinear DEs. Here first we derive equivalence transformations corresponding to second order scalar nonlinear hyperbolic type PDEs, then to determine joint differential invariants we get differential invariants under transformations of both independent and dependent variables.

2.2.1 Equivalence Transformations

Consider a second order scalar nonlinear equation[12]

$$w_{mm} = a(s, w_s)w_{ss} + b(s, w_s), \tag{2.28}$$

where a, b are differentiable functions which involve first order derivatives. An equivalence transformation of (2.28) is an invertible transformation of the variables m, s and w, of the type

$$\tilde{m} = \phi_1(m, s, w), \qquad \tilde{s} = \phi_2(m, s, w), \qquad \tilde{w} = \phi_3(m, s, w), \tag{2.29}$$

that map an equation of the form (2.28) into an equation of the same form. To obtain equivalence transformations of (2.28), we use an operator

$$\mathbf{X} = \xi_1 \frac{\partial}{\partial m} + \xi_2 \frac{\partial}{\partial s} + \eta \frac{\partial}{\partial w} + \mu_1 \frac{\partial}{\partial a} + \mu_2 \frac{\partial}{\partial b}. \tag{2.30}$$

The Lie invariance condition for (2.28) is

$$\mathbf{X}^{[2]}(w_{mm} - a w_{ss} - b)|_{(2.28)} = 0, \tag{2.31}$$

where, $\mathbf{X}^{[2]}$ is second prolongation of equivalence operator (2.30) which have the form

$$\mathbf{X}^{[2]} = \mathbf{X} + \eta^m \frac{\partial}{\partial w_m} + \eta^s \frac{\partial}{\partial w_s} + \eta^{mm} \frac{\partial}{\partial w_{mm}} + \eta^{ss} \frac{\partial}{\partial w_{ss}}. \tag{2.32}$$

The invariance condition (2.31), after operating generator (2.32) can be written as

$$\eta^{mm} - a\eta^{ss} - \mu_1 w_{ss} - \mu_2|_{(2.28)} = 0. \tag{2.33}$$

Inserting η^{mm}, η^{ss} and introducing the relation $w_{mm} = (aw_{ss} + b)$ to eliminate w_{ss}, one can easily find

$$\begin{aligned}
\xi_1 &= c_2 m + c_3, \\
\xi_2 &= \xi(s), \\
\eta &= c_1 w + c_4 m^2 + c_5 m + \eta(s), \\
\mu_1 &= 2a\xi_{2,s} - 2ac_2, \\
\mu_2 &= 2c_4 + a(w_s \xi_{2,ss} - \eta_{ss}) + b(c_1 - 2c_2).
\end{aligned} \tag{2.34}$$

Where $\xi(s)$, $\eta(s)$ are two arbitrary functions and c_i for $i = 1, 2, 3, 4, 5$ are constants. Hence, the class of equations (2.28) has infinite continuous group of equivalence transformations which is spanned by the following infinitesimal operators [3]

$$\begin{aligned}
\mathbf{X}_1 &= \frac{\partial}{\partial m}, \\
\mathbf{X}_2 &= \frac{\partial}{\partial w}, \\
\mathbf{X}_3 &= m\frac{\partial}{\partial w}, \\
\mathbf{X}_4 &= s\frac{\partial}{\partial w}, \\
\mathbf{X}_5 &= m\frac{\partial}{\partial m} + s\frac{\partial}{\partial s} + 2w\frac{\partial}{\partial w}, \\
\mathbf{X}_6 &= m\frac{\partial}{\partial m} - 2a\frac{\partial}{\partial a} - 2b\frac{\partial}{\partial b}, \\
\mathbf{X}_7 &= m^2\frac{\partial}{\partial w} + 2\frac{\partial}{\partial b}, \\
\mathbf{X}_\xi &= \xi\frac{\partial}{\partial s} + 2a\xi'\frac{\partial}{\partial a} + a\xi'' w_s\frac{\partial}{\partial b}, \\
\mathbf{X}_\eta &= \eta\frac{\partial}{\partial w} - a\eta''\frac{\partial}{\partial b}.
\end{aligned} \tag{2.35}$$

Here, the prime represents differentiation with respect to s.

2.2.2 Joint Differential Invariants for Hyperbolic Type PDEs

To acquire the differential invariants of order zero, i.e., invariants of the form $J = J(m, s, w, w_m, w_s, a, b)$, one employ the invariant test $\mathbf{X}J = 0$, using operators $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{X}_3$, $\mathbf{X}_4$, $\mathbf{X}_\xi$ and determine that the zeroth order invariants does not depend on m, s, w, w_m and w_s. Therefore, $J = J(a, b)$. Consequently, applying the invariant test to the operators $\mathbf{X}_6$ and $\mathbf{X}_7$, one gets

$$\frac{\partial J}{\partial a} = 0, \qquad \frac{\partial J}{\partial b} = 0. \tag{2.36}$$

So, equation (2.28) do not have differential invariants of order zero.

Differential Invariants of the First Order

In order to derive differential invariants of the first order, the following criteria is used

$$\mathbf{X}^{[1]} J(a, b, a_s, a_{w_s}, b_s, b_{w_s}) = 0, \tag{2.37}$$

where

$$\mathbf{X}^{[1]} = \mathbf{X} + \mu_{1,i}\frac{\partial}{\partial a_i} + \mu_{2,i}\frac{\partial}{\partial b_i}, \qquad i \in \{s, w_s\}. \tag{2.38}$$

Here, $\mu_{1,i}$, $\mu_{2,i}$ is expressed as

$$\begin{aligned} \mu_{1,i} &= \tilde{D}_i(\mu_1) - a_s\tilde{D}_i(\xi_2) - a_{w_s}\tilde{D}_i(\eta_{w_s}), \\ \mu_{2,i} &= \tilde{D}_i(\mu_2) - b_s\tilde{D}_i(\xi_2) - b_{w_s}\tilde{D}_i(\eta_{w_s}), \end{aligned}$$

with

$$\begin{aligned} \tilde{D}_s &= \frac{\partial}{\partial s} + a_s\frac{\partial}{\partial a} + a_{ss}\frac{\partial}{\partial a_s} + a_{sw_s}\frac{\partial}{\partial a_{w_s}} + ... + \frac{\partial}{\partial s} + b_s\frac{\partial}{\partial b} \\ &\quad + b_{ss}\frac{\partial}{\partial b_s} + b_{sw_s}\frac{\partial}{\partial b_{w_s}} ..., \\ \tilde{D}_{w_s} &= \frac{\partial}{\partial w_s} + a_{w_s}\frac{\partial}{\partial a} + a_{sw_s}\frac{\partial}{\partial a_s} + a_{w_s w_s}\frac{\partial}{\partial a_{w_s}} + ... + \frac{\partial}{\partial w_s} + b_{w_s}\frac{\partial}{\partial b} \\ &\quad + b_{sw_s}\frac{\partial}{\partial b_s} + b_{w_s w_s}\frac{\partial}{\partial b_{w_s}} + ... \end{aligned} \tag{2.39}$$

The invariance conditions $\mathbf{X}_1^{[1]}J = 0, ..., \mathbf{X}_4^{[1]}J = 0$ are verified identically. Furthermore, if one takes first prolongation $\mathbf{X}_7^{[1]}$ of the operator $\mathbf{X}_7$, then it can readily observe that this prolongation matches with $\mathbf{X}_7$ itself, and so

$$\mathbf{X}_7^{[1]}J = \frac{\partial J}{\partial b} = 0. \tag{2.40}$$

It implies that

$$J = J(a, a_s, a_{w_s}, b_s, b_{w_s}). \tag{2.41}$$

Similarly, holding in the first prolongation of the operator $\mathbf{X}_5$ with quantities, we get

$$\mathbf{X}_5^{[1]} = -a_s\frac{\partial}{\partial a_s} - b_s\frac{\partial}{\partial b_s} - a_{w_s}\frac{\partial}{\partial a_{w_s}} - b_{w_s}\frac{\partial}{\partial b_{w_s}}. \tag{2.42}$$

Now applying this operator on (2.41), we find

$$\mathbf{X}_5^{[1]} = -a_s\frac{\partial J}{\partial a_s} - b_s\frac{\partial J}{\partial b_s} - a_{w_s}\frac{\partial J}{\partial a_{w_s}} - b_{w_s}\frac{\partial J}{\partial b_{w_s}} = 0. \tag{2.43}$$

From the characteristics equations

$$\frac{da_s}{a_s} = \frac{db_s}{b_s} = \frac{da_{w_s}}{a_{w_s}} = \frac{db_{w_s}}{b_{w_s}}, \tag{2.44}$$

it follows that $J = J(a, J_1, J_2, J_3)$, where

$$J_1 = \frac{b_s}{a_s}, \qquad J_2 = \frac{a_{w_s}}{a_s}, \qquad J_3 = \frac{b_{w_s}}{a_s}, \tag{2.45}$$

provided that $a_s \neq 0$. The first extension of the operator $\mathbf{X}_6$, in the form which we require, is

$$\mathbf{X}_6^{[1]} = a\frac{\partial}{\partial a} + a_s\frac{\partial J}{\partial a_s} + b_s\frac{\partial J}{\partial b_s} + a_{w_s}\frac{\partial J}{\partial a_{w_s}} + b_{w_s}\frac{\partial J}{\partial b_{w_s}} = 0. \tag{2.46}$$

Employing this operator on the invariants (2.45), one finds that

$$\mathbf{X}_6^{[1]}J_1 = \mathbf{X}_6^{[1]}J_2 = \mathbf{X}_6^{[1]}J_3 = 0, \tag{2.47}$$

and hence

$$\mathbf{X}_6^{[1]}J \equiv a\frac{\partial J}{\partial a} = 0. \tag{2.48}$$

It acquires that the terms (2.45) present a basis of invariants (2.37) for $\mathbf{X}_1, ..., \mathbf{X}_7$, $J = J(J_1, J_2, J_3)$. Now we move on the first prolongation of the operator $\mathbf{X}_\eta$ and write it in the following form

$$\mathbf{X}_\eta^{[1]} = -\eta'' a_{w_s} \frac{\partial}{\partial a_s} - (a\eta''' + \eta'' a_s + \eta'' b_{w_s}) \frac{\partial}{\partial b_s} - \eta'' a_{w_s} \frac{\partial}{\partial b_{w_s}}. \tag{2.49}$$

The invariant test $\mathbf{X}_\eta^{[1]} J = 0$, implies

$$\eta'' a_{w_s} \frac{\partial J}{\partial a_s} + (a\eta''' + \eta'' a_s + \eta'' b_{w_s}) \frac{\partial J}{\partial b_s} + \eta'' a_{w_s} \frac{\partial}{\partial b_{w_s}} = 0. \tag{2.50}$$

As $\eta(s)$ is an arbitrary function and its derivatives η'', η''' are functionally independent, so (2.50) could be separated by equating coefficients of η and its derivatives to zero, which provides the following system

$$\begin{aligned} &a \frac{\partial J}{\partial b_s} = \frac{a}{a_s} \frac{\partial J}{\partial J_1} = 0, \\ &a_{w_s} \left(\frac{\partial J}{\partial a_s} + \frac{\partial J}{\partial b_{w_s}} \right) = -J_2 \left[J_2 \frac{\partial J}{\partial J_2} + (J_3 - 1) \frac{\partial J}{\partial J_3} \right] = 0. \end{aligned} \tag{2.51}$$

It follows that $J = J(\omega)$ with

$$\omega = \frac{J_3 - 1}{J_2} \equiv \frac{b_{w_s} - a_s}{a_{w_s}}, \tag{2.52}$$

with $a_{w_s} \neq 0$. Finally, we keep the first prolongation of the operator $\mathbf{X}_\xi$ taking only the essential terms

$$\mathbf{X}_\xi^{[1]} = (2a\xi'' + a_s\xi' + a_{w_s}\xi'' w_s) \frac{\partial}{\partial a_s} + 3a_{w_s}\xi' \frac{\partial}{\partial a_{w_s}} + (a\xi'' + a_{w_s}\xi'' w_s + b_{w_s}\xi') \frac{\partial}{\partial b_{w_s}}. \tag{2.53}$$

Considering (2.52), one has

$$\mathbf{X}_\xi^{[1]} J = -\frac{1}{a_{w_s}} \left[a\xi'' - 2\xi'(a_s - b_{w_s}) \right] \frac{\partial J}{\partial \omega} = 0. \tag{2.54}$$

Treating ξ' and ξ'' as independent functions and accepting that $a_{w_s} \neq 0$, one obtains

$$\frac{\partial J}{\partial \omega} = 0. \tag{2.55}$$

Differential Invariants of the Second Order: With the objective, to attain differential invariants occupying second order derivatives $J = J(a, b, a_i, b_i, a_{ij}, b_{ij})$ for $i, j \in \{s, w_s\}$. The second extension of the generator $\mathbf{X}$ is written as

$$\mathbf{X}^{[2]} = \mathbf{X}^{[1]} + \mu_{1,ij}\frac{\partial}{\partial a_{ij}} + \mu_{2,ij}\frac{\partial}{\partial b_{ij}}, \tag{2.56}$$

with

$$\begin{aligned} \mu_{1,ij} &= \tilde{D}_i(\mu_{1,i}) - a_{s,i}\tilde{D}_i(\xi_2) - a_{w_s,i}\tilde{D}_i(\eta_{w_s}), \\ \mu_{2,ij} &= \tilde{D}_i(\mu_{2,i}) - b_{s,i}\tilde{D}_i(\xi_2) - b_{w_s,i}\tilde{D}_i(\eta_{w_s}). \end{aligned} \tag{2.57}$$

Repeating the same procedure, the following second order differential invariants are derived

$$\begin{aligned} J_4 &= a\frac{a_{w_s w_s}}{(a_{w_s})^2}, \\ J_5 &= \frac{a a_{w_s w_s (2b_{w_s} - a_s) - a a_{w_s} a_{s w_s} - 3(a_{w_s})^2 (b_{w_s} - a_s)}}{a_{w_s}[a_{w_s}(b_{w_s} - a_s) + a(a_{s w_s} - b_{w_s w_s})]}, \\ J_6 &= \frac{a a_s(a_s a_{w_s w_s} + 2a_{w_s} b_{w_s w_s}) + 4b_{w_s}[a_{w_s w_s}(b_{w_s} - a_s) - a_{w_s} b_{w_s w_s}]}{[a_{w_s}(b_{w_s} - a_s) + a(a_{s w_s} - b_{w_s w_s})]^2} \\ &\quad - \frac{2(a_{w_s})^2[(a_s)^2 + (b_{w_s})^2] + a(a_{ss} - 2b_{s w_s}) + a_{w_s} b_s - 5a_s b_{w_s}}{[a_{w_s}(b_{w_s} - a_s) + a(a_{s w_s} - b_{w_s w_s})]^2}. \end{aligned} \tag{2.58}$$

Hence it is observed that, scalar nonlinear hyperbolic type PDEs (2.28) have no differential invariants of zero order. However, it contains three functionally independent differential invariants of first and second order expressed in (2.45) and (2.58).

Chapter 3

Equivalence Group Classification for Systems of Two Nonlinear Parabolic Type PDEs

Equivalence transformations are the basic source to find differential invariants and with the help of these differential invariants one can linearize nonlinear complicated forms of DEs to linear or simple solvable nonlinear forms of DEs.

In this chapter, we address a major class and few special cases of systems of two second order nonlinear parabolic type PDEs to investigate associated equivalence transformations. Moreover, we also discuss linearity and nonlinearity of equivalence transformations corresponding to our considered systems of nonlinear parabolic type PDEs. To characterize on the bases of equivalence transformations, here first we study the equivalence transformations for a system of nonlinear parabolic type PDEs which involves first order derivatives in its arbitrary coefficients and then we consider its various subclasses.

3.1 Equivalence Transformations

Consider a system of two second order nonlinear parabolic type PDEs with a, b, c, d as its arbitrary functions

$$\begin{aligned} w_m + a(m, s, w, v)w_{ss} + b(m, s, w, v, w_s, v_s) = 0, \\ v_m + c(m, s, w, v)v_{ss} + d(m, s, w, v, w_s, v_s) = 0, \end{aligned} \tag{3.1}$$

where, m and s in subscripts denotes partial derivatives i.e., $w_m = \frac{\partial w}{\partial m}$, $w_s = \frac{\partial w}{\partial s}$, $v_m = \frac{\partial v}{\partial m}$, $v_s = \frac{\partial v}{\partial s}$, $w_{ss} = \frac{\partial^2 w}{\partial s^2}$, $v_{ss} = \frac{\partial^2 v}{\partial s^2}$. An equivalence transformation of (3.1) is an invertible transformation of the dependent and independent variables which maps (3.1) into itself. Lie infinitesimal method engages the following operator

$$\mathbf{X} = \xi_1 \frac{\partial}{\partial m} + \xi_2 \frac{\partial}{\partial s} + \eta_1 \frac{\partial}{\partial w} + \eta_2 \frac{\partial}{\partial v} + \mu_1 \frac{\partial}{\partial a} + \mu_2 \frac{\partial}{\partial b} + \mu_3 \frac{\partial}{\partial c} + \mu_4 \frac{\partial}{\partial d}, \tag{3.2}$$

to provide the set of equivalence transformations for (3.1), where $\xi_k = \xi_k(m, s, w, v)$, $\eta_k = \eta_k(m, s, w, v)$, $\mu_l = \mu_l(m, s, w, v, w_s, v_s, a, b, c, d)$ for $k = 1, 2$, $l = 1, 2, 3, 4$. For system (3.1), second order prolongation of the above generator is needed that reads as

$$\mathbf{X}^{[2]} = \mathbf{X} + \eta_1^m \frac{\partial}{\partial w_m} + \eta_2^m \frac{\partial}{\partial v_m} + \eta_1^s \frac{\partial}{\partial w_s} + \eta_2^s \frac{\partial}{\partial v_s} + \eta_1^{ss} \frac{\partial}{\partial w_{ss}} + \eta_2^{ss} \frac{\partial}{\partial v_{ss}}, \tag{3.3}$$

where

$$\begin{aligned}
\eta_1^m &= D_m(\eta_1) - w_m D_m(\xi_1) - w_s D_m(\xi_2), \\
\eta_2^m &= D_m(\eta_2) - v_m D_m(\xi_1) - v_s D_m(\xi_2), \\
\eta_1^s &= D_s(\eta_1) - w_m D_s(\xi_1) - w_s D_s(\xi_2), \\
\eta_2^s &= D_s(\eta_2) - v_m D_s(\xi_1) - v_s D_s(\xi_2), \\
\eta_1^{ss} &= D_s(\eta_1^s) - w_{ms} D_s(\xi_1) - w_{ss} D_s(\xi_2), \\
\eta_2^{ss} &= D_s(\eta_2^s) - v_{ms} D_s(\xi_1) - v_{ss} D_s(\xi_2),
\end{aligned}$$

with

$$\begin{aligned}
D_m &= \frac{\partial}{\partial m} + w_m \frac{\partial}{\partial w} + v_m \frac{\partial}{\partial v} + w_{mm} \frac{\partial}{\partial w_m} + v_{mm} \frac{\partial}{\partial v_m} + \cdots, \\
D_s &= \frac{\partial}{\partial s} + w_s \frac{\partial}{\partial w} + v_s \frac{\partial}{\partial v} + w_{ss} \frac{\partial}{\partial w_s} + v_{ss} \frac{\partial}{\partial v_s} + \cdots.
\end{aligned} \tag{3.4}$$

These expressions finally leads us to

$$\begin{aligned}
\eta_1^m = {} & \eta_{1,m} + v_m \eta_{1,v} + w_m \eta_{1,w} - w_m \xi_{1,m} - w_m v_m \xi_{1,v} - w_m^2 \xi_{1,w} \\
& - w_s \xi_{2,m} - w_s v_m \xi_{2,v} - w_m w_s \xi_{2,w},
\end{aligned}$$

$$
\begin{aligned}
\eta_2^m &= \eta_{2,m} + v_m\eta_{2,v} + w_m\eta_{2,w} - v_m\xi_{1,m} - w_m v_m\xi_{1,w} - v_m^2\xi_{1,v} \\
&\quad -v_s\xi_{2,m} - v_m v_s\xi_{2,v} - w_m v_s\xi_{2,w}, \\
\eta_1^s &= \eta_{1,s} + v_s\eta_{1,v} + w_s\eta_{1,w} - w_m\xi_{1,s} - w_m v_s\xi_{1,v} - w_s^2\xi_{2,w} \\
&\quad -w_s\xi_{2,s} - w_m w_s\xi_{1,w} - w_s v_s\xi_{2,v}, \\
\eta_2^s &= \eta_{2,s} + v_s\eta_{2,v} + w_s\eta_{2,w} - v_m\xi_{1,s} - v_m v_s\xi_{1,v} - v_s^2\xi_{2,v} \\
&\quad -v_s\xi_{2,s} - v_m w_s\xi_{1,w} - w_s v_s\xi_{2,w}, \\
\eta_1^{ss} &= \eta_{1,ss} + 2v_s\eta_{1,sv} + 2w_s\xi_{1,sw} - w_m\xi_{1,ss} - w_s\xi_{2,ss} - w_s v_{ss}\xi_{2,v} + v_{ss}\eta_{1,v} \\
&\quad -2w_s^2\xi_{2,sw} + v_s^2\eta_{1,vv} + w_s^2\eta_{1,ww} - w_s^3\xi_{2,ww} + 2w_s v_s\eta_{1,wv} - 2w_m w_s v_s\xi_{1,wv} \\
&\quad -2w_{ms}\xi_{1,s} + w_{ss}\eta_{1,w} - 2w_{ss}\xi_{2,s} - 2w_{ms}v_s\xi_{1,v} - w_s v_s^2\xi_{2,vv} - w_m v_{ss}\xi_{1,v} \\
&\quad -2w_{ms}w_s\xi_{1,w} - w_m v_s^2\xi_{1,vv} - 3w_{ss}w_s\xi_{2,w} - w_{ss}w_m\xi_{1,w} - 2w_s^2 v_s\xi_{2,wv} \\
&\quad -2w_{ss}v_s\xi_{2,v} - w_m w_s^2\xi_{1,ww} - 2w_m v_s\xi_{1,sv} - 2w_m w_s\xi_{1,sw} - 2w_s v_s\xi_{2,sv}, \\
\eta_2^{ss} &= \eta_{2,ss} - 2w_s v_m v_s\xi_{1,wv} + 2v_s\eta_{2,sv} + 2w_s\eta_{2,sw} - v_m\xi_{1,ss} - v_s\xi_{2,ss} - 2v_s^2\xi_{2,sv} \\
&\quad +v_s^2\eta_{2,vv} - v_s^3\xi_{2,vv} - 2v_{ms}\xi_{1,s} + v_{ss}\eta_{2,v} - 2v_{ss}\xi_{2,s} + w_s^2\eta_{2,ww} + w_{ss}\eta_{2,w} \\
&\quad -2v_m v_s\xi_{1,sv} - 2w_s v_m\xi_{1,sw} - 2w_s v_s\xi_{2,sw} - w_{ss}v_m\xi_{1,w} - w_{ss}v_s\xi_{2,w} \\
&\quad +2w_s v_s\eta_{2,wv} - v_m v_s^2\xi_{1,vv} - 2w_s v_s^2\xi_{2,wv} - 2v_{ms}v_s\xi_{1,v} - 2v_{ms}w_s\xi_{1,w} \\
&\quad -v_{ss}v_m\xi_{1,v} - 3v_{ss}v_s\xi_{2,v} - 2w_s v_{ss}\xi_{2,w} - w_s^2 v_m\xi_{1,ww} - v_s w_s^2\xi_{2,ww}.
\end{aligned}
$$

Lie invariance condition for system (3.1) is

$$
\begin{aligned}
\mathbf{X}^{[2]}(w_m + aw_{ss} + b)|_{(3.1)} = 0, \\
\mathbf{X}^{[2]}(v_m + cv_{ss} + d)|_{(3.1)} = 0,
\end{aligned} \tag{3.5}
$$

which expands to the following equations

$$\eta_1^m + a\eta_1^{ss} + \mu_1 w_{ss} + \mu_2|_{(3.1)} = 0, \tag{3.6}$$

$$\eta_2^m + c\eta_2^{ss} + \mu_3 v_{ss} + \mu_4|_{(3.1)} = 0. \tag{3.7}$$

Substituting η_1^m, η_1^{ss} in (3.6), η_2^m, η_2^{ss} in (3.7) and replacing w_m with $-(aw_{ss}+b)$, v_m with $-(cv_{ss}+d)$, provides

$$\eta_{1,m} - w_s\xi_{2,m} - (aw_{ss}+b)\eta_{1,w} + (aw_{ss}+b)\xi_{1,m} - (aw_{ss}+b)(cv_{ss}+d)\xi_{1,v}$$

$$
\begin{aligned}
&-(aw_{ss}+b)^2\xi_{1,w}-(cv_{ss}+d)\eta_{1,v}+w_s(cv_{ss}+d)\xi_{2,v}+w_s(aw_{ss}+b)\xi_{2,w}\\
&+a[2w_sv_s\eta_{1,wv}+w_{ss}(aw_{ss}+b)\xi_{1,w}-2w_s^2v_s\xi_{2,wv}+2v_s(aw_{ss}+b)\xi_{1,sv}-w_sv_s^2\xi_{2,vv}\\
&+2w_sv_s(aw_{ss}+b)\xi_{1,wv}+2v_s\eta_{1,sv}+2w_s\eta_{1,sw}-w_s\xi_{2,ss}-2w_s^2\xi_{2,sw}+v_s^2\eta_{1,vv}+v_{ss}\eta_{1,v}\\
&+w_s^2\eta_{1,ww}-w_s^3\xi_{2,ww}-2w_{ms}\xi_{1,s}+w_{ss}\eta_{1,w}-2w_{ss}\xi_{2,s}+(aw_{ss}+b)\xi_{1,ss}+\eta_{1,ss}\\
&-w_sv_{ss}\xi_{2,v}-2w_{ms}v_s\xi_{1,v}-2w_{ms}w_s\xi_{1,w}-2w_{ss}v_s\xi_{2,v}-3w_{ss}w_s\xi_{2,w}+v_s^2(aw_{ss}+b)\xi_{1,vv}\\
&+w_s^2(aw_{ss}+b)\xi_{1,ww}+v_{ss}(aw_{ss}+b)\xi_{1,v}+2w_s(aw_{ss}+b)\xi_{1,sw}-2w_sv_s\xi_{2,sv}]\\
&+\mu_1w_{ss}+\mu_2=0, \qquad (3.8)
\end{aligned}
$$

$$
\begin{aligned}
&\eta_{2,m}-(cv_{ss}+d)\eta_{2,v}-(aw_{ss}+b)\eta_{2,w}+(cv_{ss}+d)\xi_{1,m}+(cv_{ss}+d)^2\xi_{1,v}\\
&-(cv_{ss}+d)(aw_{ss}+b)\xi_{1,w}-v_s\xi_{2,m}+v_s(cv_{ss}+d)\xi_{2,v}+v_s(aw_{ss}+b)\xi_{2,w}\\
&+c[2w_sv_s\eta_{2,wv}+2w_sv_s(cv_{ss}+d)\xi_{1,wv}+2v_s\eta_{2,sv}+2w_s\eta_{2,sw}-2w_sv_s\xi_{2,sw}\\
&-2v_{ms}v_s\xi_{1,v}+2v_s(cv_{ss}+d)\xi_{1,sv}-2w_sv_{ss}\xi_{2,w}+w_s^2(cv_{ss}+d)\xi_{1,ww}-w_{ss}v_s\xi_{2,w}\\
&-3v_sv_{ss}\xi_{2,v}+2w_s(cv_{ss}+d)\xi_{1,sw}-2w_sv_{ms}\xi_{1,w}+w_{ss}(cv_{ss}+d)\xi_{1,w}+v_s^2\eta_{2,vv}\\
&+v_s^2(cv_{ss}+d)\xi_{1,vv}-2w_sv_s^2\xi_{2,wv}-v_sw_s^2\xi_{2,ww}-2v_{ms}\xi_{1,s}+v_{ss}(cv_{ss}+d)\xi_{1,v}\\
&+(cv_{ss}+d)\xi_{1,ss}-2v_{ss}\xi_{2,s}-v_s^3\xi_{2,vv}+v_{ss}\eta_{2,v}-2v_s^2\xi_{2,sv}+w_s^2\eta_{2,ww}+w_{ss}\eta_{2,w}\\
&-v_s\xi_{2,ss}+\eta_{2,ss}]+\mu_3v_{ss}+\mu_4=0, \qquad (3.9)
\end{aligned}
$$

respectively. After simplification coefficients of $w_{ss}w_s, w_{ss}v_s, w_{ms}, w_{ms}w_s, w_{ms}v_s, v_{ss}$ in (3.8) and w_{ss} in (3.9), when equating to zero, yields

$$
\begin{aligned}
&2a^2\xi_{1,sw}-2a\xi_{2,w}=0,\\
&2a^2\xi_{1,sv}-2a\xi_{2,v}=0,\\
&\xi_{1,s}=0,\\
&\xi_{1,w}=0,\\
&\xi_{1,v}=0,\\
&(ab-bc)\xi_{1,v}+(a-c)\eta_{1,v}=0,\\
&(cd-ad)\xi_{1,w}+(c-a)\eta_{2,w}=0. \qquad (3.10)
\end{aligned}
$$

Solving (3.10), we obtain

$$\begin{aligned}&\xi_{1,s}=0, \qquad \xi_{1,w}=0, \qquad \xi_{1,v}=0, \qquad \xi_{2,w}=0,\\ &\xi_{2,v}=0, \qquad\qquad \eta_{1,v}=0, \qquad\qquad \eta_{2,w}=0,\end{aligned} \tag{3.11}$$

which implies

$$\begin{aligned}\xi_1 &:= \xi_1(m),\\ \xi_2 &:= \xi_2(m,s),\\ \eta_1 &:= \eta_1(m,s,w),\\ \eta_2 &:= \eta_2(m,s,v).\end{aligned} \tag{3.12}$$

Afterwards using (3.11) in (3.8) and (3.9), reduces these equations to

$$\begin{aligned}&aw_s^2\eta_{1,ww}+2aw_s\eta_{1,sw}+a\eta_{1,ss}-aw_s\xi_{2,ss}-b\eta_{1,w}-w_s\xi_{2,m}\\ &+\eta_{1,m}-2aw_{ss}\xi_{2,s}+aw_{ss}\xi_{1,m}+b\xi_{1,m}+\mu_1 w_{ss}+\mu_2=0,\end{aligned} \tag{3.13}$$

$$\begin{aligned}&cv_s^2\eta_{2,vv}+2cv_s\eta_{2,sv}+c\eta_{2,ss}-cv_s\xi_{2,ss}-d\eta_{2,v}-v_s\xi_{2,m}\\ &+\eta_{2,m}-2cv_{ss}\xi_{2,s}+cv_{ss}\xi_{1,m}+d\xi_{1,m}+\mu_3 v_{ss}+\mu_4=0.\end{aligned} \tag{3.14}$$

Now coefficients of w_{ss} and remaining terms in (3.13) as well as the coefficients of v_{ss} and other terms in (3.14) provides

$$\begin{aligned}\mu_1 &= 2a\xi_{2,s}-a\xi_{1,m},\\ \mu_2 &= aw_s\xi_{2,ss}+b\eta_{1,w}+w_s\xi_{2,m}-aw_s^2\eta_{1,ww}-\eta_{1,m}\\ &\quad -2aw_s\eta_{1,sw}-a\eta_{1,ss}-b\xi_{1,m}\\ \mu_3 &= 2c\xi_{2,s}-c\xi_{1,m},\\ \mu_4 &= cv_s\xi_{2,ss}+d\eta_{2,v}+v_s\xi_{2,m}-cv_s^2\eta_{2,vv}-\eta_{2,m}\\ &\quad -2cv_s\eta_{2,sv}-c\eta_{2,ss}-d\xi_{1,m}.\end{aligned} \tag{3.15}$$

Here, ξ_i, η_i, μ_j characterize the infinitesimal changes in the dependent, independent variables and arbitrary coefficients of considered system for $i=1,2,\ j=1,2,3,4$.

Now we consider a few special cases of (3.1) to investigate associated equivalence transformations.

Case-I

Consider a system of nonlinear parabolic type PDEs

$$\begin{aligned} w_m + a_1(m,s,w,v)w_{ss} + a_2(m,s,w,v)v_{ss} + a_3(m,s,w,v,w_s,v_s) = 0, \\ v_m + b_1(m,s,w,v)v_{ss} + b_2(m,s,w,v)w_{ss} + b_3(m,s,w,v,w_m,w_s) = 0, \end{aligned} \tag{3.16}$$

where a_i, b_i for $i = 1,2,3$ are arbitrary coefficients. Second order prolonged generator for (3.16) is

$$\begin{aligned} \mathbf{X}^{[2]} &= \xi_1\frac{\partial}{\partial m} + \xi_2\frac{\partial}{\partial s} + \eta_1\frac{\partial}{\partial w} + \eta_2\frac{\partial}{\partial v} + \eta_1^m\frac{\partial}{\partial w_m} + \eta_2^m\frac{\partial}{\partial v_m} + \eta_1^{ss}\frac{\partial}{\partial w_{ss}} \\ &\quad + \eta_2^{ss}\frac{\partial}{\partial v_{ss}} + \mu_1\frac{\partial}{\partial a_1} + \mu_2\frac{\partial}{\partial a_2} + \mu_3\frac{\partial}{\partial a_3} + \mu_4\frac{\partial}{\partial b_1} + \mu_5\frac{\partial}{\partial b_2} + \mu_6\frac{\partial}{\partial b_3}, \end{aligned} \tag{3.17}$$

where μ_1 to μ_6 are functions of $(m,s,w,v,w_s,v_s,a_i,b_i)$. Operating generator (3.17) on (3.16), we get

$$\eta_1^m + a_1\eta_1^{ss} + a_2\eta_2^{ss} + \mu_1 w_{ss} + \mu_2 v_{ss} + \mu_3|_{(3.16)} = 0, \tag{3.18}$$

$$\eta_2^m + b_1\eta_2^{ss} + b_2\eta_1^{ss} + \mu_4 v_{ss} + \mu_5 w_{ss} + \mu_6|_{(3.16)} = 0. \tag{3.19}$$

Coefficients of $w_{ss}v_{ss}$, $w_{ss}w_s$, w_{ms} in (3.18) gives the following expression after substituting η_1^m, η_1^{ss}, η_2^{ss}, and replacing w_m with $-(a_1w_{ss}+a_2v_{ss}+a_3)$, v_m with $-(b_1v_{ss}+b_2w_{ss}+b_3)$

$$\xi_{1,w} = 0, \qquad \xi_{1,v} = 0, \qquad \xi_{2,w} = 0, \qquad \xi_{2,v} = 0, \qquad \xi_{1,s} = 0. \tag{3.20}$$

It implies

$$\begin{aligned} \xi_1 &:= \xi_1(m), \\ \xi_2 &:= \xi_2(m,s), \\ \eta_1 &:= \eta_1(m,s,w,v), \\ \eta_2 &:= \eta_2(m,s,w,v). \end{aligned} \tag{3.21}$$

Afterwards coefficients of w_{ss}, v_{ss} and remaining terms in (3.18), (3.19) gives

$$\begin{aligned}
\mu_1 &= 2a_1\xi_{2,s} + b_2\eta_{1,v} - a_2\eta_{2,w} - a_1\xi_{1,m},\\
\mu_2 &= 2a_2\xi_{2,s} + a_2\eta_{1,w} + b_1\eta_{1,v} - a_1\eta_{1,v} - a_2\eta_{2,v} - a_2\xi_{1,m},\\
\mu_3 &= a_1w_s\xi_{2,ss} + w_s\xi_{2,m} + a_2v_s\xi_{2,ss} + b_3\eta_{1,v} + a_3\eta_{1,w} - a_1w_s^2\eta_{1,ww}\\
&\quad -a_2w_s^2\eta_{2,ww} - a_1v_s^2\eta_{1,vv} - a_2v_s^2\eta_{2,vv} - 2a_1w_sv_s\eta_{1,wv} - 2a_2w_sv_s\eta_{2,wv}\\
&\quad -2a_1w_s\eta_{1,sw} - 2a_2w_s\eta_{2,sw} - 2a_1v_s\eta_{1,sv} - 2a_2v_s\eta_{2,sv} - a_1\eta_{1,ss}\\
&\quad -a_2\eta_{2,ss} - \eta_{1,m} - a_3\xi_{1,m},\\
\mu_4 &= 2b_1\xi_{2,s} + a_2\eta_{2,w} - b_2\eta_{1,v} - b_1\xi_{1,m},\\
\mu_5 &= 2b_2\xi_{2,s} + b_2\eta_{2,v} + a_1\eta_{2,w} - b_1\eta_{2,w} - b_2\eta_{1,w} - b_2\xi_{1,m},\\
\mu_6 &= b_2w_s\xi_{2,ss} + v_s\xi_{2,m} + b_1v_s\xi_{2,ss} + a_3\eta_{2,w} + b_3\eta_{2,v} - b_2w_s^2\eta_{1,ww}\\
&\quad -b_1w_s^2\eta_{2,ww} - b_2v_s^2\eta_{1,vv} - b_1v_s^2\eta_{2,vv} - 2b_2w_sv_s\eta_{1,wv} - 2b_1w_sv_s\eta_{2,wv}\\
&\quad -2b_2w_s\eta_{1,sw} - 2b_1w_s\eta_{2,sw} - 2b_2v_s\eta_{1,sv} - 2b_1v_s\eta_{2,sv} - b_2\eta_{1,ss}\\
&\quad -b_1\eta_{2,ss} - \eta_{2,m} - b_3\xi_{1,m}.
\end{aligned} \tag{3.22}$$

Here, μ_1 to μ_6 represents the arbitrary coefficients.

Case-II

A system of two second order nonlinear parabolic type PDEs

$$\begin{aligned}
&w_m + a_1w_{ss} + a_2v_{ss} + a_3w_s^2 + a_4w_sv_s + a_5v_s^2 + a_6w_s + a_7v_s + a_8 = 0,\\
&v_m + b_1v_{ss} + b_2w_{ss} + b_3v_s^2 + b_4v_sw_s + b_5w_s^2 + b_6v_s + b_7w_s + b_8 = 0,
\end{aligned} \tag{3.23}$$

where a_1 to a_8 and b_1 to b_8 all are functions of (m,s,w,v). For (3.23) we have second-order prolonged generator of the form

$$\begin{aligned}
\mathbf{X}^{[2]} &= \xi_1\frac{\partial}{\partial m} + \xi_2\frac{\partial}{\partial s} + \eta_1\frac{\partial}{\partial w} + \eta_2\frac{\partial}{\partial v} + \eta_1^m\frac{\partial}{\partial w_m} + \eta_1^s\frac{\partial}{\partial w_s} + \eta_2^m\frac{\partial}{\partial v_m} + \eta_2^s\frac{\partial}{\partial v_s}\\
&\quad +\eta_1^{mm}\frac{\partial}{\partial w_{mm}} + \eta_1^{ss}\frac{\partial}{\partial w_{ss}} + \eta_2^{mm}\frac{\partial}{\partial v_{mm}} + \eta_2^{ss}\frac{\partial}{\partial v_{ss}} + \mu_1\frac{\partial}{\partial a_1} + \mu_2\frac{\partial}{\partial a_2}\\
&\quad +\mu_4\frac{\partial}{\partial a_4} + \mu_5\frac{\partial}{\partial a_5} + \mu_6\frac{\partial}{\partial a_6} + \mu_7\frac{\partial}{\partial a_7} + \mu_8\frac{\partial}{\partial a_8} + \mu_9\frac{\partial}{\partial b_1} + \mu_{10}\frac{\partial}{\partial b_2}\\
&\quad +\mu_{11}\frac{\partial}{\partial b_3} + \mu_{12}\frac{\partial}{\partial b_4} + \mu_{13}\frac{\partial}{\partial b_5} + \mu_{14}\frac{\partial}{\partial b_6} + \mu_{15}\frac{\partial}{\partial b_7} + \mu_{16}\frac{\partial}{\partial b_8},
\end{aligned} \tag{3.24}$$

where μ_1 to μ_{16} are functions of (m,s,w,v,a_i, b_i) for $i = 1, 2, ..., 8$. The Lie invariance condition is given by

$$\mathbf{X}^{[2]}(w_m + a_1 w_{ss} + a_2 v_{ss} + a_3 w_s^2 + a_4 w_s v_s + a_5 v_s^2 + a_6 w_s + a_7 v_s + a_8)|_{(3.23)} = 0,$$
$$\mathbf{X}^{[2]}(v_m + b_1 v_{ss} + b_2 w_{ss} + b_3 v_s^2 + b_4 v_s w_s + b_5 w_s^2 + b_6 v_s + b_7 w_s + b_8)|_{(3.23)} = 0,$$

which gives

$$\eta_1^m + a_1\eta_1^{ss} + a_2\eta_2^{ss} + 2a_3\eta_1^s w_s + 2a_5\eta_2^s v_s + a_6\eta_1^s + a_4\eta_1^s v_s + a_4\eta_2^s w_s + a_7\eta_2^s$$
$$+\mu_1 w_{ss} + \mu_2 v_{ss} + \mu_3 w_s^2 + \mu_4 w_s v_s + \mu_5 v_s^2 + \mu_6 w_s + \mu_7 v_s + \mu_8|_{(3.23)} = 0, \qquad (3.25)$$

$$\eta_2^m + b_1\eta_2^{ss} + b_2\eta_1^{ss} + 2b_3\eta_2^s v_s + 2b_5\eta_1^s w_s + b_6\eta_2^s + b_4\eta_1^s v_s + b_4\eta_2^s w_s + b_7\eta_1^s + \mu_9 v_{ss}$$
$$+\mu_{10} w_{ss} + \mu_{11} v_s^2 + \mu_{12} v_s w_s + \mu_{13} w_s^2 + \mu_{14} v_s + \mu_{15} w_s + \mu_{16}|_{(3.23)} = 0. \qquad (3.26)$$

Inserting η_1^m, η_2^m, η_1^s, η_2^s, η_1^{ss}, η_2^{ss} in above equations and replacing w_m with $-(a_1 w_{ss} + a_2 v_{ss} + a_3 w_s^2 + a_4 w_s v_s + a_5 v_s^2 + a_6 w_s + a_7 v_s + a_8)$, v_m with $-(b_1 v_{ss} + b_2 w_{ss} + b_3 v_s^2 + b_4 v_s w_s + b_5 w_s^2 + b_6 v_s + b_7 w_s + b_8)$, we obtain the determining equations. Coefficients of w_{ms}, $w_{ms} w_s$, $w_{ms} v_s$, w_s^3 in (3.25) provides

$$\xi_{1,s} = 0, \quad \xi_{1,w} = 0, \quad \xi_{1,v} = 0, \quad \xi_{2,w} = 0, \quad \xi_{2,v} = 0, \qquad (3.27)$$

it implies

$$\begin{aligned} \xi_1 &:= \xi_1(m), \\ \xi_2 &:= \xi_2(m, s), \\ \eta_1 &:= \eta_1(m, s, w, v), \\ \eta_2 &:= \eta_2(m, s, w, v). \end{aligned} \qquad (3.28)$$

Subsequently utilizing (3.27) in (3.25), (3.26) simplifies these equations. After the said insertions, comparing coefficients of $w_{ss}, v_{ss}, w_s^2, w_s v_s, v_s^2, w_s, v_s$ and remaining terms provide

$$\begin{aligned}
\mu_1 &= 2a_1\xi_{2,s} + b_2\eta_{1,v} - a_1\xi_{1,m} - a_2\eta_{2,w},\\
\mu_2 &= a_2\eta_{1,w} + 2a_2\xi_{2,s} + b_1\eta_{1,v} - a_1\eta_{1,v} - a_2\eta_{2,v} - a_2\xi_{1,m},\\
\mu_3 &= 2a_3\xi_{2,s} + b_5\eta_{1,v} - a_1\eta_{1,ww} - a_2\eta_{2,ww} - a_3\eta_{1,w} - a_4\eta_{2,w} - a_3\xi_{1,m},\\
\mu_4 &= 2a_4\xi_{2,s} + b_4\eta_{1,v} - 2a_1\eta_{1,wv} - 2a_2\eta_{2,wv} - 2a_3\eta_{1,v} - a_4\eta_{2,v}\\
&\quad -2a_5\eta_{2,w} - a_4\xi_{1,m},\\
\mu_5 &= a_5\eta_{1,w} + 2a_5\xi_{2,s} + b_3\eta_{1,v} - a_1\eta_{1,vv} - a_2\eta_{2,vv} - a_4\eta_{1,v}\\
&\quad -2a_4\eta_{2,v} - a_4\xi_{1,m},\\
\mu_6 &= a_1\xi_{2,ss} + \xi_{2,m} + b_7\eta_{1,v} - 2a_1\eta_{1,sw} - 2a_2\eta_{2,sw} - a_7\eta_{2,w}\\
&\quad -2a_3\eta_{1,s} - a_4\eta_{2,s} - a_6\xi_{1,m} - a_6\xi_{2,s},\\
\mu_7 &= a_2\xi_{2,ss} + a_7\eta_{1,w} + a_7\xi_{2,s} + b_6\eta_{1,v} - 2a_1\eta_{1,sv} - 2a_2\eta_{2,sv} - a_6\eta_{1,v}\\
&\quad -a_7\eta_{2,v} - a_4\eta_{1,s} - 2a_5\eta_{2,s} - a_7\xi_{1,m},\\
\mu_8 &= a_8\eta_{1,w} + b_8\eta_{1,v} - a_1\eta_{1,ss} - a_2\eta_{2,ss} - a_6\eta_{1,s} - a_7\eta_{2,s} - \eta_{1,m} - a_8\xi_{1,m},\\
\mu_9 &= a_2\eta_{2,w} + 2b_1\xi_{2,s} - b_2\eta_{1,v} - b_1\xi_{1,m},\\
\mu_{10} &= a_1\eta_{2,w} + b_2\eta_{2,v} + 2b_2\xi_{2,s} - b_1\eta_{2,w} - b_2\eta_{1,w} - b_2\xi_{1,m},\\
\mu_{11} &= a_5\eta_{2,w} + 2b_3\xi_{2,s} - b_2\eta_{1,vv} - b_1\eta_{2,vv} - b_3\eta_{2,v} - b_4\eta_{1,v} - b_3\xi_{1,m},\\
\mu_{12} &= a_4\eta_{2,w} + 2b_4\xi_{2,s} - 2b_2\eta_{1,wv} - 2b_1\eta_{2,wv} - 2b_3\eta_{2,w} - 2b_6\eta_{1,v}\\
&\quad -b_4\eta_{1,w} - b_4\xi_{1,m},\\
\mu_{13} &= a_3\eta_{2,w} + b_5\eta_{2,v} + 2b_5\xi_{2,s} - b_2\eta_{1,ww} - b_1\eta_{2,ww} - b_4\eta_{2,w}\\
&\quad -2b_5\eta_{1,w} - b_5\xi_{1,m},\\
\mu_{14} &= a_7\eta_{2,w} + b_1\xi_{2,ss} + b_6\xi_{2,s} + \xi_{2,m} - 2b_2\eta_{1,sv} - 2b_1\eta_{2,sv}\\
&\quad -b_7\eta_{1,v} - b_4\eta_{1,s} - 2b_3\eta_{2,s} - b_6\xi_{1,m},\\
\mu_{15} &= a_6\eta_{2,w} + b_7\eta_{2,v} + b_2\xi_{2,ss} + b_7\xi_{2,s} - 2b_2\eta_{1,sw} - 2b_1\eta_{2,sw} - b_6\eta_{2,w}\\
&\quad -b_7\eta_{1,w} - 2b_5\eta_{1,s} - b_4\eta_{2,s} - b_7\xi_{1,m},\\
\mu_{16} &= a_8\eta_{2,w} + b_8\eta_{2,v} - b_2\eta_{1,ss} - b_1\eta_{2,ss} - b_7\eta_{1,s} - b_6\eta_{2,s} - \eta_{2,m} - b_8\xi_{1,m}. \qquad (3.29)
\end{aligned}$$

Case-III

For a system of nonlinear parabolic type PDEs

$$\begin{aligned} &w_m + a_1 w_{ss} + a_2 w_s^2 + a_3 w_s + a_4 = 0, \\ &v_m + b_1 v_{ss} + b_2 v_s^2 + b_3 v_s + b_4 = 0, \end{aligned} \tag{3.30}$$

where all coefficients are functions of (m,s,w,v), second-order prolonged generator has the form

$$\begin{aligned} \mathbf{X}^{[2]} \;=\; & \xi_1 \frac{\partial}{\partial m} + \xi_2 \frac{\partial}{\partial s} + \eta_1 \frac{\partial}{\partial w} + \eta_2 \frac{\partial}{\partial v} + \eta_1^m \frac{\partial}{\partial w_m} + \eta_1^s \frac{\partial}{\partial w_s} + \eta_2^m \frac{\partial}{\partial v_m} + \eta_2^s \frac{\partial}{\partial v_s} \\ & + \eta_1^{ss} \frac{\partial}{\partial w_{ss}} + \eta_2^{ss} \frac{\partial}{\partial v_{ss}} + \mu_1 \frac{\partial}{\partial a_1} + \mu_2 \frac{\partial}{\partial a_2} + \mu_3 \frac{\partial}{\partial a_3} + \mu_4 \frac{\partial}{\partial a_4} + \mu_5 \frac{\partial}{\partial b_1} \\ & + \mu_6 \frac{\partial}{\partial b_2} + \mu_7 \frac{\partial}{\partial b_3} + \mu_8 \frac{\partial}{\partial b_4}. \end{aligned} \tag{3.31}$$

Where μ_1 to μ_8 are functions of (m,s,w,v,a_i, b_i). After applying generator (3.31) on (3.30), we get

$$\eta_1^m + a_1 \eta_1^{ss} + 2a_2 \eta_1^s w_s + a_3 \eta_1^s + \mu_1 w_{ss} + \mu_2 w_s^2 + \mu_3 w_s + \mu_4 |_{(3.30)} = 0, \tag{3.32}$$

$$\eta_2^m + b_1 \eta_2^{ss} + 2b_2 \eta_2^s v_s + b_3 \eta_2^s + \mu_5 v_{ss} + \mu_6 v_s^2 + \mu_7 v_s + \mu_8 |_{(3.30)} = 0. \tag{3.33}$$

Coefficients of w_s^4, $w_s^3 v_s$, w_s^3, $v_s^2 w_s$, $w_s v_s$ in (3.32) along with the coefficients of $w_s v_s$ in (3.33) respectively gives the following expression after substituting η_1^m, η_1^s, η_2^m, η_2^s, η_1^{ss}, η_2^{ss}, as a consequence replacing w_m with $-(a_1 w_{ss} + a_2 w_s^2 + a_3 w_s + a_4)$, v_m with $-(b_1 v_{ss} + b_2 v_s^2 + b_3 v_s + b_4)$, we get

$$\begin{aligned} &\xi_{1,w} = 0, \quad \xi_{1,v} = 0, \quad \xi_{2,w} = 0, \quad \xi_{1,s} = 0, \\ &\xi_{2,v} = 0, \quad \eta_{1,v} = 0, \quad \eta_{2,w} = 0. \end{aligned} \tag{3.34}$$

The above equations implies

$$\begin{aligned} \xi_1 &:= \xi_1(m), \\ \xi_2 &:= \xi_2(m, s), \\ \eta_1 &:= \eta_1(m, s, w), \end{aligned}$$

$$\eta_2 \quad := \quad \eta_2(m,s,v). \tag{3.35}$$

Further, coefficients of w_{ss}, w_s^2, w_s and constant terms in (3.32) as well as the coefficients of v_{ss}, v_s^2, v_s and constant terms in (3.33) gives

$$\begin{aligned}
\mu_1 &= 2a_1\xi_{2,s} - a_1\xi_{1,m},\\
\mu_2 &= 2a_2\xi_{2,s} - a_1\eta_{1,ww} - a_2\eta_{1,w} - a_2\xi_{1,m},\\
\mu_3 &= a_1\xi_{2,ss} + a_3\xi_{2,s} + \xi_{2,m} - 2a_1\eta_{1,sw} - 2a_2\eta_{1,s} - a_3\xi_{1,m},\\
\mu_4 &= a_4\eta_{1,w} - a_1\eta_{1,ss} - a_3\eta_{1,s} - \eta_{1,m} - a_4\xi_{1,m},\\
\mu_5 &= 2b_1\xi_{2,s} - b_1\xi_{1,m},\\
\mu_6 &= 2b_2\xi_{2,s} - b_1\eta_{2,vv} - b_2\eta_{2,v} - b_2\xi_{1,m},\\
\mu_7 &= b_1\xi_{2,ss} + b_3\xi_{2,s} + \xi_{2,m} - 2b_1\eta_{2,sv} - 2b_2\eta_{2,s} - b_3\xi_{1,m},\\
\mu_8 &= b_4\eta_{2,v} - b_1\eta_{2,ss} - b_3\eta_{2,s} - \eta_{2,m} - b_4\xi_{1,m}.
\end{aligned} \tag{3.36}$$

Case-IV

A nonlinear system of PDEs of the type

$$\begin{aligned}
w_m + a_1w_s^2 + a_2v_s^2 + a_3w_sv_s = 0,\\
v_m + b_1v_s^2 + b_2w_s^2 + b_3v_sw_s = 0,
\end{aligned} \tag{3.37}$$

where a_i, b_i, for $i = 1,2,3$, are functions of (m,s,w,v), the second-order prolonged generator is written as

$$\begin{aligned}
\mathbf{X}^{[2]} &= \xi_1\frac{\partial}{\partial m} + \xi_2\frac{\partial}{\partial s} + \eta_1\frac{\partial}{\partial w} + \eta_2\frac{\partial}{\partial v} + \eta_1^m\frac{\partial}{\partial w_m} + \eta_1^s\frac{\partial}{\partial w_s} + \eta_2^m\frac{\partial}{\partial v_m} + \eta_2^s\frac{\partial}{\partial v_s}\\
&+\mu_1\frac{\partial}{\partial a_1} + \mu_2\frac{\partial}{\partial a_2} + \mu_3\frac{\partial}{\partial a_3} + \mu_4\frac{\partial}{\partial b_1} + \mu_5\frac{\partial}{\partial b_2} + \mu_6\frac{\partial}{\partial b_3}.
\end{aligned} \tag{3.38}$$

Here μ_1 to μ_6 are functions of (m,s,w,v,a_i, b_i). Applying (3.38) on (3.37), we get

$$\begin{aligned}
&\eta_1^m + 2a_1\eta_1^sw_s + 2a_2\eta_2^sv_s + a_3\eta_1^sv_s + a_3\eta_2^sw_s\\
&+\mu_1w_s^2 + \mu_2v_s^2 + \mu_3w_sv_s|_{(3.37)} = 0,
\end{aligned} \tag{3.39}$$

$$\eta_2^m + 2b_1\eta_2^s v_s + 2b_2\eta_1^s w_s + b_3\eta_1^s v_s + b_3\eta_2^s w_s$$
$$+\mu_4 v_s^2 + \mu_5 w_s^2 + \mu_6 v_s w_s|_{(3.37)} = 0. \tag{3.40}$$

Using η_1^m, η_1^s, η_2^m, η_2^s in above equations and replacing w_m with $-(a_1w_s^2+a_2v_s^2+a_3w_sv_s)$, v_m with $-(b_1v_s^2+b_2w_s^2+b_3v_sw_s)$. After this coefficients of w_s^4, w_s^3, w_s, v_s and constant terms in (3.39) and also coefficients of constant terms in (3.40) provides

$$\begin{aligned} &\xi_{1,w} = 0, \quad \xi_{1,v} = 0, \quad \xi_{2,w} = 0, \quad \xi_{2,v} = 0, \quad \xi_{1,s} = 0, \\ &\xi_{2,m} = 0, \quad \eta_{1,s} = 0, \quad \eta_{2,s} = 0, \quad \eta_{1,m} = 0. \quad \eta_{2,m} = 0, \end{aligned} \tag{3.41}$$

it generates

$$\begin{aligned} \xi_1 &:= \xi_1(m), \\ \xi_2 &:= \xi_2(s), \\ \eta_1 &:= \eta_1(w,v), \\ \eta_2 &:= \eta_2(w,v). \end{aligned} \tag{3.42}$$

Later on, coefficients w_s^2, v_s^2, w_sv_s in (3.39), v_s^2, w_s^2, v_sw_s in (3.40) gives

$$\begin{aligned} \mu_1 &= 2a_1\xi_{2,s} + b_2\eta_{1,v} - a_1\eta_{1,w} - a_3\eta_{2,w} - a_1\xi_{1,m}, \\ \mu_2 &= 2a_2\xi_{2,s} + a_2\eta_{1,w} + b_1\eta_{1,v} - a_3\eta_{1,v} - 2a_2\eta_{2,v} - a_2\xi_{1,m}, \\ \mu_3 &= 2a_3\xi_{2,s} + b_3\eta_{1,v} - 2a_1\eta_{1,v} - a_3\eta_{2,v} - 2a_2\eta_{2,w} - a_3\xi_{1,m}, \\ \mu_4 &= 2b_1\xi_{2,s} + a_2\eta_{2,w} - b_3\eta_{1,v} - b_1\eta_{2,v} - b_1\xi_{1,m}, \\ \mu_5 &= 2b_2\xi_{2,s} + a_1\eta_{2,w} + b_2\eta_{2,v} - b_3\eta_{2,w} - 2b_2\eta_{1,w} - b_2\xi_{1,m}, \\ \mu_6 &= a_3\eta_{2,w} + 2b_3\xi_{2,s} - 2b_1\eta_{2,w} - 2b_2\eta_{1,v} - b_3\eta_{1,w} - b_3\xi_{1,m}. \end{aligned} \tag{3.43}$$

Case-V

For a second order system of nonlinear parabolic type PDEs

$$\begin{aligned} w_m + a_1w_{ss} + a_2v_{ss} + a_3 = 0, \\ v_m + b_1v_{ss} + b_2w_{ss} + b_3 = 0, \end{aligned} \tag{3.44}$$

where arbitrary coefficients are functions of (m,s,w,v), we have following generator

$$\begin{aligned}\mathbf{X}^{[2]} &= \xi_1\frac{\partial}{\partial m}+\xi_2\frac{\partial}{\partial s}+\eta_1\frac{\partial}{\partial w}+\eta_2\frac{\partial}{\partial v}+\eta_1^m\frac{\partial}{\partial w_m}+\eta_2^m\frac{\partial}{\partial v_m}+\eta_1^{ss}\frac{\partial}{\partial w_{ss}}\\ &+\eta_2^{ss}\frac{\partial}{\partial v_{ss}}+\mu_1\frac{\partial}{\partial a_1}+\mu_2\frac{\partial}{\partial a_2}+\mu_3\frac{\partial}{\partial a_3}+\mu_4\frac{\partial}{\partial b_1}+\mu_5\frac{\partial}{\partial b_2}+\mu_6\frac{\partial}{\partial b_3}. \quad (3.45)\end{aligned}$$

Employing (3.45) on (3.44), provides

$$\eta_1^m+a_1\eta_1^{ss}+a_2\eta_2^{ss}+\mu_1w_{ss}+\mu_2v_{ss}+\mu_3|_{(3.44)}=0, \quad (3.46)$$

$$\eta_2^m+b_1\eta_2^{ss}+b_2\eta_1^{ss}+\mu_4v_{ss}+\mu_5w_{ss}+\mu_6|_{(3.44)}=0. \quad (3.47)$$

Substituting η_1^m, η_2^m, η_1^{ss}, η_2^{ss} and w_m as $-(a_1w_{ss}+a_2v_{ss}+a_3)$, v_m as $-(b_1v_{ss}+b_2w_{ss}+b_3)$, and equating coefficients of $w_{ss}v_{ss}$, $w_{ss}w_s$, w_{ms}, w_s^2, v_s^2, w_sv_s, v_s, w_s in (3.46) to zero, we get

$$\begin{aligned}&\xi_{1,w}=0, \quad \xi_{1,v}=0, \quad \xi_{2,w}=0, \quad \xi_{2,v}=0, \quad \xi_{1,s}=0, \quad \eta_{1,ww}=0,\\ &\eta_{2,ww}=0, \quad \eta_{1,vv}=0, \quad \eta_{2,vv}=0, \quad \eta_{1,wv}=0, \quad \eta_{2,wv}=0, \quad \eta_{1,sv}=0,\\ &2\eta_{2,sv}=\xi_{2,ss}, \quad \eta_{1,sw}=0, \quad 2\eta_{2,sw}=\xi_{2,ss}, \quad \xi_{2,m}=0. \quad (3.48)\end{aligned}$$

Solving these equations, we find

$$\begin{aligned}\xi_1 &:= \xi_1(m),\\ \xi_2 &:= \xi_2(s),\\ \eta_1 &:= f_1(m)w+f_2(m)v+f_3(m,s),\\ \eta_2 &:= f_4(m)w+f_5(m)v+f_6(m,s)+\frac{1}{2}(w+v)\xi_{2,s}. \quad (3.49)\end{aligned}$$

Moreover, coefficients of w_{ss}, v_{ss} and remaining terms in (3.46), (3.47) gives

$$\begin{aligned}\mu_1 &= 2a_1\xi_{2,s}+b_2\eta_{1,v}-a_2\eta_{2,w}-a_1\xi_{1,m},\\ \mu_2 &= 2a_2\xi_{2,s}+a_2\eta_{1,w}+b_1\eta_{1,v}-a_1\eta_{1,v}-a_2\eta_{2,v}-a_2\xi_{1,m},\\ \mu_3 &= a_3\eta_{1,w}+b_3\eta_{1,v}-a_1\eta_{1,ss}-a_2\eta_{2,ss}-\eta_{1,m}-a_3\xi_{1,m},\\ \mu_4 &= 2b_1\xi_{2,s}+a_2\eta_{2,w}-b_2\eta_{1,v}-b_1\xi_{1,m},\\ \mu_5 &= 2b_2\xi_{2,s}+b_2\eta_{2,v}+a_1\eta_{2,w}-b_1\eta_{2,w}-b_2\eta_{1,w}-b_2\xi_{1,m},\end{aligned}$$

$$\mu_6 = a_3\eta_{2,w} + b_3\eta_{2,v} - b_2\eta_{1,ss} - b_1\eta_{2,ss} - \eta_{2,m} - b_3\xi_{1,m}. \tag{3.50}$$

Notice that, here η_1, η_2 are linear in w and v.

Case-VI

Consider a system

$$\begin{aligned} w_m + a_1 w_{ss} + a_2 v_{ss} = 0, \\ v_m + b_1 v_{ss} + b_2 w_{ss} = 0, \end{aligned} \tag{3.51}$$

where all the coefficients are functions of (m,s,w,v). Second-order extended generator for (3.51) has the form

$$\begin{aligned} \mathbf{X}^{[2]} &= \xi_1\frac{\partial}{\partial m} + \xi_2\frac{\partial}{\partial s} + \eta_1\frac{\partial}{\partial w} + \eta_2\frac{\partial}{\partial v} + \eta_1^m\frac{\partial}{\partial w_m} + \eta_2^m\frac{\partial}{\partial v_m} \\ &+ \eta_1^{ss}\frac{\partial}{\partial w_{ss}} + \eta_2^{ss}\frac{\partial}{\partial v_{ss}} + \mu_1\frac{\partial}{\partial a_1} + \mu_2\frac{\partial}{\partial a_2} + \mu_3\frac{\partial}{\partial b_1} + \mu_4\frac{\partial}{\partial b_2}, \end{aligned} \tag{3.52}$$

where μ_1 to μ_4 are functions of (m,s,w,v,a_1, a_2, b_1, b_2). The Lie invariance condition in this case is

$$\mathbf{X}^{[2]}(w_m + a_1 w_{ss} + a_2 v_{ss})|_{(3.51)} = 0, \tag{3.53}$$

$$\mathbf{X}^{[2]}(v_m + b_1 v_{ss} + b_2 w_{ss})|_{(3.51)} = 0. \tag{3.54}$$

Applying the generator and replacing w_m with $-(a_1 w_{ss} + a_2 v_{ss})$, v_m with $-(b_1 v_{ss} + b_2 w_{ss})$. Coefficients of w_{ss}^2, $w_{ss}w_s$, $v_{ss}v_s$, v_{ms}, v_s^2, w_s^2, $w_s v_s$, w_s and constants terms in (3.53) along with coefficients of v_s and remaining terms in (3.54) gives

$$\begin{aligned} &\xi_{1,w} = 0, \quad \xi_{1,v} = 0, \quad \xi_{2,w} = 0, \quad \xi_{2,v} = 0, \quad \xi_{1,s} = 0 \quad \eta_{1,vv} = 0, \\ &\eta_{2,vv} = 0, \quad \eta_{1,ww} = 0, \quad \eta_{2,ww} = 0, \quad \eta_{1,wv} = 0, \quad \eta_{2,wv} = 0, \\ &2\eta_{1,sw} = \xi_{2,ss}, \quad \eta_{2,sw} = 0, \quad \xi_{2,m} = 0, \quad \eta_{1,ss} = 0, \quad \eta_{2,ss} = 0, \\ &\eta_{1,m} = 0, \quad \eta_{1,sv} = 0, \quad 2\eta_{2,sv} = \xi_{2,ss}. \quad \eta_{2,m} = 0, \end{aligned} \tag{3.55}$$

Solving (3.55), we obtain

$$\xi_1 := \xi_1(m),$$

$$\begin{aligned}
\xi_2 &:= c_1 s^2 + c_2 s + c_3,\\
\eta_1 &:= (c_1 w + c_4)s + c_5 w + c_6 v + c_7,\\
\eta_2 &:= (c_1 v + c_8)s + c_9 w + c_{10} v + c_{11}.
\end{aligned} \tag{3.56}$$

Here c_i for $i = 1, 2, ..., 11$ are arbitrary constants. Thereupon, coefficients of w_{ss}, v_{ss}, in (3.53) and v_{ss}, w_{ss}, in (3.54) provides

$$\begin{aligned}
\mu_1 &= 2a_1\xi_{2,s} + b_2\eta_{1,v} - a_2\eta_{2,w} - a_1\xi_{1,m},\\
\mu_2 &= 2a_2\xi_{2,s} + a_2\eta_{1,w} + b_1\eta_{1,v} - a_1\eta_{1,v} - a_2\eta_{2,v} - a_2\xi_{1,m},\\
\mu_3 &= 2b_1\xi_{2,s} + a_2\eta_{2,w} - b_2\eta_{1,v} - b_1\xi_{1,m},\\
\mu_4 &= 2b_2\xi_{2,s} + b_2\eta_{2,v} + a_1\eta_{2,w} - b_1\eta_{2,w} - b_2\eta_{1,w} - b_2\xi_{1,m}.
\end{aligned} \tag{3.57}$$

As is evident from (3.56), both η_1, η_2 are linear in s, w and v.

Case-VII

A system of PDEs of the form

$$\begin{aligned}
w_m + a_1 w_{ss} + a_2 = 0,\\
v_m + b_1 v_{ss} + b_2 = 0,
\end{aligned} \tag{3.58}$$

where a_1, a_2, b_1, b_2, all are functions of (m,s,w,v), applying (3.52) on (3.58), provide

$$\eta_1^m + a_1\eta_1^{ss} + \mu_1 w_{ss} + \mu_2|_{(3.58)} = 0, \tag{3.59}$$

$$\eta_2^m + b_1\eta_2^{ss} + \mu_3 v_{ss} + \mu_4|_{(3.58)} = 0. \tag{3.60}$$

Utilizing η_1^m, η_1^{ss}, η_2^m, η_2^{ss} and changing w_m with $-(a_1 w_{ss} + a_2)$, v_m with $-(b_1 v_{ss} + b_2)$. Coefficients of $w_{ms}w_s$, $w_{ss}v_{ss}$, $w_{ss}w_s$, $v_{ss}w_s$, w_{ms}, v_{ss}, w_s^2, w_s in (3.59) w_{ss}, v_s^2, v_s in (3.60) gives

$$\begin{aligned}
&\xi_{1,w} = 0, \quad \xi_{1,v} = 0, \quad \xi_{2,w} = 0, \quad \xi_{2,v} = 0, \quad \xi_{1,s} = 0,\\
&\eta_{1,v} = 0, \quad \eta_{1,ww} = 0, \quad 2\eta_{1,sw} = \xi_{2,ss}, \quad \xi_{2,m} = 0,\\
&\eta_{2,w} = 0, \quad \eta_{2,vv} = 0, \quad 2\eta_{2,sv} = \xi_{2,ss},
\end{aligned} \tag{3.61}$$

which implies

$$
\begin{aligned}
\xi_1 &:= \xi_1(m), \\
\xi_2 &:= \xi_2(s), \\
\eta_1 &:= f_1(m)w + f_2(m,s) + \frac{w}{2}\xi_{2,s}, \\
\eta_2 &:= f_3(m)v + f_4(m,s) + \frac{v}{2}\xi_{2,s}.
\end{aligned} \tag{3.62}
$$

With these coordinates, coefficients of w_{ss} and constant terms in (3.59) along with coefficients of v_{ss} and remaining terms in (3.60) provides

$$
\begin{aligned}
\mu_1 &= 2a_1\xi_{2,s} - a_1\xi_{1,m}, \\
\mu_2 &= a_2\eta_{1,w} - a_1\eta_{1,ss} - \eta_{1,m} - a_2\xi_{1,m}, \\
\mu_3 &= 2b_1\xi_{2,s} - b_1\xi_{1,m}, \\
\mu_4 &= b_2\eta_{2,v} - b_1\eta_{2,ss} - \eta_{2,m} - b_2\xi_{1,m}.
\end{aligned} \tag{3.63}
$$

Again the noticeable factor is linearity of η_1 and η_2 in w and v, respectively.

It is identified that, for the systems (3.1), (3.16), (3.23), (3.30), (3.37) we get non-linearities in the infinitesimal coordinates η_1 and η_2 with respect to w and v. Therefore for such systems one can further pursue differential invariants which enable reduction in nonlinearities of these systems. On the other hand, systems (3.44), (3.51), (3.58) have linear form of equivalence transformations coordinates η_1 and η_2 in w and v, which can not be used to linearize them, by driving associated differential invariants.

Chapter 4

Differential Invariants for Systems of Two Nonlinear Parabolic Type PDEs

In this chapter, differential invariants of systems of two second order nonlinear parabolic type PDEs are studied by using Lie infinitesimal method. Here in first section we derive joint differential invariants. Semi differential invariants under the transformations of only the dependent variables are deduced in second section. Applications corresponding to semi and joint differential invariants are also discussed in their relevant sections. In the last section, different subclasses of these systems are investigated and characterized them by using both the semi and joint differential invariants.

4.1 Joint Differential Invariants

In this section the joint differential invariants of system of PDEs (3.1) are derived under the transformations of both the dependent and independent variables. For this system the equivalence transformations are given in (3.12) and (3.15). Corresponding to the equivalence transformations, we have an infinitesimal generator

$$\mathbf{X} = \xi_1 \frac{\partial}{\partial m} + \xi_2 \frac{\partial}{\partial s} + \eta_1 \frac{\partial}{\partial w} + \eta_2 \frac{\partial}{\partial v} + \mu_1 \frac{\partial}{\partial a} + \mu_2 \frac{\partial}{\partial b} + \mu_3 \frac{\partial}{\partial c} + \mu_4 \frac{\partial}{\partial d}. \tag{4.1}$$

To deduce a zeroth order invariant we apply the infinitesimal test

$$\mathbf{X} J(m, s, w, v, a, b, c, d) = 0, \tag{4.2}$$

which leads us to

$$\xi_1\frac{\partial J}{\partial m}+\xi_2\frac{\partial J}{\partial s}+\eta_1\frac{\partial J}{\partial w}+\eta_2\frac{\partial J}{\partial v}+(2a\xi_{2,s}-a\xi_{1,m})\frac{\partial J}{\partial a}+(aw_s\xi_{2,ss}+b\eta_{1,w}$$
$$+w_s\xi_{2,m}-aw_s^2\eta_{1,ww}-\eta_{1,m}-2aw_s\eta_{1,sw}-a\eta_{1,ss}-b\xi_{1,m})\frac{\partial J}{\partial b}+(2c\xi_{2,s}$$
$$-c\xi_{1,m})\frac{\partial J}{\partial c}+(cv_s\xi_{2,ss}+d\eta_{2,v}+v_s\xi_{2,m}-cv_s^2\eta_{2,vv}-\eta_{2,m}-2cv_s\eta_{2,sv}$$
$$-c\eta_{2,ss}-d\xi_{1,m})\frac{\partial J}{\partial d}=0, \tag{4.3}$$

This equation splits into the following equations

$$J_m=0,\quad J_s=0,\quad J_w=0,\quad J_v=0,\quad -J_b=0,\quad -J_d=0, \tag{4.4}$$

obtained by annulling the terms with ξ_1, ξ_2, η_1, η_2, $\eta_{1,m}$, $\eta_{2,m}$. Hence, $J=J(a,c)$. Now the terms with $\xi_{1,m}$, $\xi_{2,s}$ provide the following system of two equations

$$-aJ_a-cJ_c=0,\qquad 2aJ_a+2cJ_c=0, \tag{4.5}$$

after solving the above system of equations we get zeroth order invariant

$$J_1=\frac{c}{a}. \tag{4.6}$$

In order to find first order differential invariants, i.e., the invariants of the form $J(m,s,w,v,a,b,c,d,a_i,b_j,c_i,d_j)$, the once extended generator is

$$\mathbf{X}^{[1]}=\mathbf{X}+\mu_{1,i}\frac{\partial}{\partial a_i}+\mu_{2,j}\frac{\partial}{\partial b_j}+\mu_{3,i}\frac{\partial}{\partial c_i}+\mu_{4,j}\frac{\partial}{\partial d_j}, \tag{4.7}$$

where, $i\in\{m,s,w,v\}$ and $j\in\{m,s,w,v,w_s,v_s\}$. In above equation $\mu_{1,i}$, $\mu_{2,j}$, $\mu_{3,i}$, $\mu_{4,j}$ are expressed as

$$\begin{aligned}
\mu_{1,i} &= \tilde{D}_i(\mu_1)-a_m\tilde{D}_i(\xi_1)-a_s\tilde{D}_i(\xi_2)-a_w\tilde{D}_i(\eta_1)-a_v\tilde{D}_i(\eta_2),\\
\mu_{2,j} &= \tilde{D}_j(\mu_2)-b_m\tilde{D}_j(\xi_1)-b_s\tilde{D}_j(\xi_2)-b_w\tilde{D}_j(\eta_1)-b_v\tilde{D}_j(\eta_2)-b_{w_s}\tilde{D}_j(\eta_1^s)-b_{v_s}\tilde{D}_j(\eta_2^s),\\
\mu_{3,i} &= \tilde{D}_i(\mu_3)-c_m\tilde{D}_i(\xi_1)-c_s\tilde{D}_i(\xi_2)-c_w\tilde{D}_i(\eta_1)-c_v\tilde{D}_i(\eta_2),\\
\mu_{4,j} &= \tilde{D}_j(\mu_4)-d_m\tilde{D}_j(\xi_1)-d_s\tilde{D}_j(\xi_2)-d_w\tilde{D}_j(\eta_1)-d_v\tilde{D}_j(\eta_2)-d_{w_s}\tilde{D}_j(\eta_1^s)-d_{v_s}\tilde{D}_j(\eta_2^s).
\end{aligned}$$

Since $i \subset j$, therefore $\tilde{D}_i \subset \tilde{D}_j$ and generally total derivative operator is defined by

$$\begin{aligned}\tilde{D}_j &= \frac{\partial}{\partial j} + a_i \frac{\partial}{\partial a} + a_{i,i} \frac{\partial}{\partial a_i} + ... + \frac{\partial}{\partial j} + b_j \frac{\partial}{\partial b} + b_{j,j} \frac{\partial}{\partial b_j} + ... \\ & \frac{\partial}{\partial j} + c_j \frac{\partial}{\partial c} + c_{i,i} \frac{\partial}{\partial c_i} + ... + \frac{\partial}{\partial j} + d_j \frac{\partial}{\partial d} + d_{j,j} \frac{\partial}{\partial d_j} + ... \end{aligned} \tag{4.8}$$

Thus, after substituting concerned from the above in (4.7), one can find the following first order extended generator

$$\begin{aligned}\mathbf{X}^{[1]} &= \xi_1 \partial_m + \xi_2 \partial_s + \eta_1 \partial_w + \eta_2 \partial_v + (2a\xi_{2,s} - a\xi_{1,m})\partial_a + (aw_s\xi_{2,ss} + b\eta_{1,w} \\ &+ w_s\xi_{2,m} - aw_s^2\eta_{1,ww} - \eta_{1,m} - 2aw_s\eta_{1,sw} - a\eta_{1,ss} - b\xi_{1,m})\partial_b + (2c\xi_{2,s} \\ &- c\xi_{1,m})\partial_c + (cv_s\xi_{2,ss} + d\eta_{2,v} + v_s\xi_{2,m} - cv_s^2\eta_{2,vv} - \eta_{2,m} - 2cv_s\eta_{2,sv} - c\eta_{2,ss} \\ &- d\xi_{1,m})\partial_d + (2a\xi_{2,ms} - a\xi_{1,mm} + 2a_m\xi_{2,s} - 2a_m\xi_{1,m} - a_s\xi_{2,m} - a_w\eta_{1,m} \\ &- a_v\eta_{2,m})a_m + (2a\xi_{2,ss} + a_s\xi_{2,s} - a_s\xi_{1,m} - a_w\eta_{1,s} - a_v\eta_{2,s})a_s + (2a_w\xi_{2,s} \\ &- a_w\xi_{1,m} - a_w\eta_{1,w})a_w + (2a_v\xi_{2,s} - a_v\xi_{1,m} - a_v\eta_{2,v})a_v + (aw_s\xi_{2,mss} + w_s\xi_{2,mm} \\ &+ b\eta_{1,mw} - aw_s^2\eta_{1,mww} - 2aw_s\eta_{1,msw} - a\eta_{1,mss} - \eta_{1,mm} - b\xi_{1,mm} + a_m w_s\xi_{2,ss} \\ &- a_m w_s^2\eta_{1,ww} - 2a_m w_s\eta_{1,sw} - a_m\eta_{1,ss} + b_m\eta_{1,w} - 2b_m\xi_{1,m} - b_s\xi_{2,m} - b_w\eta_{1,m} \\ &- b_v\eta_{2,m} - b_{w_s}\eta_{1,ms} - b_{w_s}w_s\eta_{1,mw} + b_{w_s}w_s\xi_{2,ms} - b_{v_s}\eta_{2,ms} - b_{v_s}v_s\eta_{2,mv} \\ &+ b_{v_s}v_s\xi_{2,ms})b_m + (aw_s\xi_{2,sss} + w_s\xi_{2,ms} + b\eta_{1,sw} - aw_s^2\eta_{1,sww} - 2aw_s\eta_{1,ssw} \\ &- a\eta_{1,sss} - \eta_{1,ms} + a_s w_s\xi_{2,ss} - a_s w_s^2\eta_{1,ww} - 2a_s w_s\eta_{1,sw} - a_s\eta_{1,ss} + b_s\eta_{1,w} \\ &- b_s\xi_{1,m} - b_s\xi_{2,s} - b_w\eta_{1,s} - b_v\eta_{2,s} - b_{w_s}\eta_{1,ss} - b_{w_s}w_s\eta_{1,sw} + b_{w_s}w_s\xi_{2,ss} - b_{v_s}\eta_{2,ss} \\ &- b_{v_s}v_s\eta_{2,sv} + b_{v_s}v_s\xi_{2,ss})b_s + (b\eta_{1,ww} - aw_s^2\eta_{1,www} - 2aw_s\eta_{1,sww} - a\eta_{1,ssw} \\ &- \eta_{1,mw} + a_w w_s\xi_{2,ss} - a_w w_s^2\eta_{1,ww} - 2a_w w_s\eta_{1,sw} - a_w\eta_{1,ss} - b_w\xi_{1,m} - b_{w_s}\eta_{1,sw} \\ &- b_{w_s}w_s\eta_{1,ww})b_w + (a_v w_s\xi_{2,ss} - a_v w_s^2\eta_{1,ww} - 2a_v w_s\eta_{1,sw} - a_v\eta_{1,ss} + b_v\eta_{1,w} \\ &- b_v\xi_{1,m} - b_v\eta_{2,v} - b_{v_s}\eta_{2,sv} - b_{v_s}v_s\eta_{2,vv})b_v + (a\xi_{2,ss} + \xi_{2,m} - 2aw_s\eta_{1,ww} - 2a\eta_{1,sw} \\ &- b_{w_s}\xi_{1,m} + b_{w_s}\xi_{2,s})b_{w_s} + (b_{v_s}\eta_{1,w} - b_{v_s}\xi_{1,m} - b_{v_s}\eta_{2,v} + b_{v_s}\xi_{2,s})b_{v_s} + (2c\xi_{2,ms} \\ &- c\xi_{1,mm} + 2c_m\xi_{2,s} - 2c_m\xi_{1,m} - c_s\xi_{2,m} - c_w\eta_{1,m} - c_v\eta_{2,m})c_m + (2c\xi_{2,ss} + c_s\xi_{2,s} \\ &- c_s\xi_{1,m} - c_w\eta_{1,s} - c_v\eta_{2,s})c_s + (2c_w\xi_{2,s} - c_w\xi_{1,m} - c_w\eta_{1,w})c_w + (2c_v\xi_{2,s} - c_v\xi_{1,m} \\ &- c_v\eta_{2,v})c_v + (cv_s\xi_{2,mss} + v_s\xi_{2,mm} + d\eta_{2,mv} - cv_s^2\eta_{2,mvv} - 2cv_s\eta_{2,msv} - c\eta_{2,mss}\end{aligned}$$

$$-\eta_{2,mm} - d\xi_{1,mm} + c_m v_s \xi_{2,ss} - c_m v_s^2 \eta_{2,vv} - 2c_m v_s \eta_{2,sv} - c_m \eta_{2,ss} + d_m \eta_{2,v}$$
$$-2d_m \xi_{1,m} - d_s \xi_{2,m} - d_w \eta_{1,m} - d_v \eta_{2,m} - d_{w_s} \eta_{1,ms} - d_{w_s} w_s \eta_{1,mw} + d_{w_s} w_s \xi_{2,ms}$$
$$-d_{v_s} \eta_{2,ms} - d_{v_s} v_s \eta_{2,mv} + d_{v_s} v_s \xi_{2,ms}) d_m + (c v_s \xi_{2,sss} + v_s \xi_{2,ms} + d\eta_{2,sv} - c v_s^2 \eta_{2,svv}$$
$$-2c v_s \eta_{2,ssv} - c\eta_{2,sss} - \eta_{2,ms} + c_s v_s \xi_{2,ss} - c_s v_s^2 \eta_{2,vv} - 2c_s v_s \eta_{2,sv} - c_s \eta_{2,ss} + d_s \eta_{2,v}$$
$$-d_s \xi_{1,m} - d_s \xi_{2,s} - d_w \eta_{1,s} - d_v \eta_{2,s} - d_{w_s} \eta_{1,ss} - d_{w_s} w_s \eta_{1,sw} + d_{w_s} w_s \xi_{2,ss} - d_{v_s} \eta_{2,ss}$$
$$-d_{v_s} v_s \eta_{2,sv} + d_{v_s} v_s \xi_{2,ss}) d_s + (c_w v_s \xi_{2,ss} - c_w v_s^2 \eta_{2,vv} - 2c_w v_s \eta_{2,sv} - c_w \eta_{2,ss}$$
$$+d_w \eta_{2,v} - d_w \xi_{1,m} - d_w \eta_{1,w} - d_{w_s} \eta_{1,sw} - d_{w_s} w_s \eta_{1,ww}) d_w + (d\eta_{2,vv} - c v_s^2 \eta_{2,vvv}$$
$$-2c v_s \eta_{2,svv} - c\eta_{2,ssv} - \eta_{2,mv} + c_v v_s \xi_{2,ss} - c_v v_s^2 \eta_{2,vv} - 2c_v v_s \eta_{2,sv} - c_v \eta_{2,ss}$$
$$-d_v \xi_{1,m} - d_{v_s} \eta_{2,sv} - d_{v_s} v_s \eta_{2,vv}) d_v + (d_{w_s} \eta_{2,v} - d_{w_s} \xi_{1,m} - d_{w_s} \eta_{1,w} + d_{w_s} \xi_{2,s}) d_{w_s}$$
$$+(c\xi_{2,ss} + \xi_{2,m} - 2c v_s \eta_{2,vv} - 2c\eta_{2,sv} - d_{v_s} \xi_{1,m} + d_{v_s} \xi_{2,s}) d_{v_s}. \tag{4.9}$$

For derivation of first order joint differential invariants we consider the invariance criterion

$$\mathbf{X}^{[1]} J(m, s, w, v, a, b, c, d, a_i, b_j, c_i, d_j) = 0. \tag{4.10}$$

Upon equating to zero the terms with ξ_1, ξ_2, η_1, η_2, $\eta_{1,mm}$, $\eta_{1,sss}$, $\eta_{1,www}$, $\eta_{2,mm}$, $\eta_{2,sss}$, $\eta_{2,vvv}$ in (4.10) respectively gives the following results after simplification

$$\begin{aligned} &J_m = 0, \quad J_s = 0, \quad J_w = 0, \quad J_v = 0, \quad J_{b_m} = 0, \\ &J_{b_s} = 0, \quad J_{b_w} = 0, \quad J_{d_m} = 0, \quad J_{d_s} = 0, \quad J_{d_v} = 0, \end{aligned} \tag{4.11}$$

hence, $J = J(a, b, c, d, a_m, a_s, a_w, a_v, b_v, b_{w_s}, b_{v_s}, c_m, c_s, c_w, c_v, d_w, d_{w_s}, d_{v_s})$. Now the terms with $\xi_{1,m}$, $\xi_{1,mm}$, $\xi_{2,m}$, $\xi_{2,s}$, $\xi_{2,ms}$, $\xi_{2,ss}$, $\eta_{1,m}$, $\eta_{1,s}$, $\eta_{1,w}$, $\eta_{1,ss}$, $\eta_{1,sw}$, $\eta_{1,ww}$, $\eta_{2,m}$, $\eta_{2,s}$, $\eta_{2,v}$, $\eta_{2,ss}$, $\eta_{2,sv}$, $\eta_{2,vv}$ provide following system of equations

$$-b_{v_s} J_{b_{v_s}} - d_{w_s} J_{d_{w_s}} - b_{w_s} J_{b_{w_s}} - aJ_a - bJ_b - cJ_c - dJ_d - 2a_m J_{a_m} - 2c_m J_{c_m} - a_s J_{a_s}$$
$$-c_s J_{c_s} - a_w J_{a_w} - c_w J_{c_w} - d_w J_{d_w} - a_v J_{a_v} - b_v J_{b_v} - c_v J_{c_v} - d_{v_s} J_{d_{v_s}} = 0,$$
$$-aJ_{a_m} - cJ_{c_m} = 0,$$
$$J_{b_{w_s}} + w_s J_b + v_s J_d - a_s J_{a_m} - c_s J_{c_m} + J_{d_{v_s}} = 0,$$
$$b_{v_s} J_{b_{v_s}} + d_{w_s} J_{d_{w_s}} + b_{w_s} J_{b_{w_s}} + 2aJ_a + 2cJ_c + 2a_m J_{a_m} + 2c_m J_{c_m} + a_s J_{a_s}$$
$$+c_s J_{c_s} + 2a_w J_{a_w} + 2c_w J_{c_w} + 2a_v J_{a_v} + 2c_v J_{c_v} + d_{v_s} J_{d_{v_s}} = 0,$$

$$
\begin{aligned}
&2aJ_{a_m} + 2cJ_{c_m} = 0,\\
&aJ_{b_{w_s}} + aw_sJ_b + cv_sJ_d + 2aJ_{a_s} + 2cJ_{c_s} + c_wv_sJ_{d_w} + a_vw_sJ_{b_v} + cJ_{d_{v_s}} = 0,\\
&-J_b - a_wJ_{a_m} - c_wJ_{c_m} = 0,\\
&a_wJ_{a_s} - c_wJ_{c_s} = 0,\\
&b_{v_s}J_{b_{v_s}} - d_{w_s}J_{d_{w_s}} + bJ_b - a_wJ_{a_w} - c_wJ_{c_w} - d_wJ_{d_w} + b_vJ_{b_v} = 0,\\
&-aJ_b - a_vJ_{b_v} = 0,\\
&-2aJ_{b_{w_s}} - 2aw_sJ_b - d_{w_s}J_{d_w} - 2a_vw_sJ_{b_v} = 0,\\
&-2aw_sJ_{b_{w_s}} - aw_s^2J_b - d_{w_s}w_sJ_{d_w} - a_vw_s^2J_{b_v} = 0,\\
&-J_d - a_vJ_{a_m} - c_vJ_{c_m} = 0,\\
&-a_vJ_{a_s} - c_vJ_{c_s} = 0,\\
&-b_{v_s}J_{b_{v_s}} + d_{w_s}J_{d_{w_s}} + dJ_d + d_wJ_{d_w} - a_vJ_{a_v} - b_vJ_{b_v} - c_vJ_{c_v} = 0,\\
&-cJ_d - c_wJ_{d_w} = 0,\\
&-2cv_sJ_d - 2c_wv_sJ_{d_w} - b_{v_s}J_{b_v} - 2cJ_{d_{v_s}} = 0,\\
&-cv_s^2J_d - c_wv_s^2J_{d_w} - b_{v_s}v_sJ_{b_v} - 2cv_sJ_{d_{v_s}} = 0. \qquad (4.12)
\end{aligned}
$$

After solving (4.12) in MAPLE, we get following first order joint differential invariants along with J_1,

$$
J_2 = \frac{c_w}{a_w}, \qquad J_3 = \frac{c_v}{a_v}, \qquad J_4 = \frac{b_{v_s}a_w^2}{d_{w_s}a_v^2}. \qquad (4.13)
$$

4.1.1 Applications

In this subsection a few examples of systems of two second order nonlinear parabolic type PDEs are provided to illustrate the invariance criteria developed.

Example 1

A coupled parabolic type system of PDEs

$$
\begin{aligned}
&w_m + \left(\frac{sw^2 + sv}{m}\right)w_{ss} + \left(\frac{sw^2 + sv}{mw}\right)w_s^2 + \left(\frac{2w^2 - s + 2v}{m}\right)w_s = 0,\\
&v_m - \left(\frac{s}{m}\right)v_s + \left(\frac{2w}{m}\right)w_s + \frac{w^2}{ms} = 0, \qquad (4.14)
\end{aligned}
$$

having

$$a = (\frac{sw^2 + sv}{m}), \qquad b = (\frac{sw^2 + sv}{mw})w_s^2 + (\frac{2w^2 - s + 2v}{m})w_s,$$
$$c = 0, \qquad d = -(\frac{s}{m})v_s + (\frac{2w}{m})w_s + \frac{w^2}{ms},$$

can be transformed into

$$\begin{aligned} &u_t + (u + c)u_{xx} = 0, \\ &c_t + u_x = 0, \end{aligned} \tag{4.15}$$

by means of transformations

$$m = t, \qquad s = \frac{x}{t}, \qquad w = \sqrt{\frac{u}{x}}, \qquad v = \frac{c}{x}. \tag{4.16}$$

The joint differential invariants for (4.14) and (4.15) are same, which are

$$J_1 = 0, \quad J_2 = 0, \quad J_3 = 0, \quad J_4 = 0. \tag{4.17}$$

Example 2

Consider a system of nonlinear parabolic type PDEs

$$\begin{aligned} &w_m + (\frac{swv\sin(m)}{m^2})w_{ss} - (\frac{s}{m})w_s + (\frac{s}{m\sin(m)})v_s + w\cot(m) + \frac{v}{m\sin(m)} = 0, \\ &v_m + (\frac{1}{m^2})v_{ss} + (\frac{2 - s^2 m}{sm^2})v_s + (\frac{\sin(m)}{ms})w_s - \frac{v}{m} = 0. \end{aligned} \tag{4.18}$$

Joint differential invariants corresponding to both the dependent and independent variables for (4.18) can be calculated as

$$\begin{aligned} J_1 &= \frac{c}{a} = \frac{1}{swv\sin(m)}, \\ J_2 &= \frac{c_w}{a_w} = 0, \\ J_3 &= \frac{c_v}{a_v} = 0, \\ J_4 &= \frac{b_{v_s} a_w^2}{d_{w_s} a_v^2} = \frac{s^2 v^2}{\sin^2(m) w^2}, \end{aligned} \tag{4.19}$$

where

$$a = (\frac{swv\sin(m)}{m^2}), \qquad b = (\frac{-s}{m})w_s + (\frac{s}{m\sin(m)})v_s + w\cot(m) + \frac{v}{m\sin(m)},$$
$$c = (\frac{1}{m^2}), \qquad d = (\frac{2 - s^2 m}{sm^2})v_s + (\frac{\sin(m)}{ms})w_s - \frac{v}{m}.$$

Equation (4.18) can be mapped into

$$\begin{aligned} &u_t + ucu_{xx} + c_x = 0, \\ &c_t + c_{xx} + u_x = 0, \end{aligned} \tag{4.20}$$

by using the equivalence transformations

$$m = t, \qquad s = \frac{x}{t}, \qquad w = \frac{u}{\sin(t)}, \qquad v = \frac{tc}{x}. \tag{4.21}$$

Joint differential invariants for (4.20) are also same as for (4.18), under the transformations (4.21).

Example 3

Consider a coupled system of PDEs

$$\begin{aligned} &w_m + (v\sqrt{w})w_{ss} - (\frac{v}{2\sqrt{w}})w_s^2 + (\frac{2\sqrt{w}}{m})v_s - (\frac{s}{m})w_s + \frac{2w}{m} = 0, \\ &v_m + (\frac{v}{m})v_{ss} - (\frac{s}{m})v_s + (\frac{1}{2m\sqrt{w}})w_s + \frac{v}{m} = 0, \end{aligned} \tag{4.22}$$

with the following joint differential invariants

$$J_1 = J_3 = \frac{1}{m\sqrt{w}}, \qquad J_2 = 0, \qquad J_4 = \frac{v^2}{w}. \tag{4.23}$$

Equation (4.22) is transformable to

$$\begin{aligned} &u_t + ucu_{xx} + c_x = 0, \\ &c_t + cc_{xx} + u_x = 0, \end{aligned} \tag{4.24}$$

under invertible transformations

$$m = t, \qquad s = \frac{x}{t}, \qquad w = \frac{u^2}{t^2}, \qquad v = \frac{c}{t}. \tag{4.25}$$

Joint differential invariants for (4.24) are

$$J_1 = J_3 = \frac{1}{u}, \qquad J_2 = 0, \qquad J_4 = \frac{c^2}{u^2}. \tag{4.26}$$

The joint differential invariants for both (4.22) and (4.24) are same with transformations (4.25).

Example 4

Both the following systems

$$\begin{aligned}
&w_m + (m^2w^2 + \frac{m^2 s}{v})w_{ss} + (m^2 w + \frac{m^2 s}{wv})w_s^2 + (\frac{s}{m})w_s - (\frac{ms}{2wv^2})v_s \\
&+\frac{m}{2wv} = 0, \\
&v_m + (\frac{m^2 s w^2}{v})v_{ss} - (\frac{2m^2 s w^2}{v^2})v_s^2 + (\frac{sv + 2m^3 w^2}{mv})v_s - (\frac{2mwv^2}{s})w_s \\
&-\frac{v}{m} = 0,
\end{aligned} \tag{4.27}$$

and

$$\begin{aligned}
&u_t + (u+c)u_{xx} + c_x = 0, \\
&c_t + ucc_{xx} + u_x = 0,
\end{aligned} \tag{4.28}$$

are mappable into each other under invertible transformations

$$m = t, \qquad s = tx, \qquad w = \sqrt{u}, \qquad v = \frac{tx}{c}. \tag{4.29}$$

Joint differential invariants for both of them read as

$$J_1 = \frac{sw^2}{w^2 v + s}, \qquad J_2 = \frac{s}{v}, \qquad J_3 = w^2, \qquad J_4 = 1, \tag{4.30}$$

$$J_1 = \frac{uc}{u+c}, \qquad J_2 = c, \qquad J_3 = u, \qquad J_4 = 1, \tag{4.31}$$

that are equal under transformation (4.29).

4.2 Semi Differential Invariants

Semi differential invariants of system of two second order nonlinear parabolic type PDEs (3.1) are investigated in this section by Lie infinitesimal approach. To get these semi differential invariants corresponding to only the dependent variables we put ξ_1, ξ_2 and all their derivatives equal to zero in (3.12) and (3.15), which give

$$\begin{aligned} &\xi_1 := 0, \quad \xi_2 := 0, \quad \eta_1 := \eta_1(m,s,w), \quad \eta_2 := \eta_2(m,s,v) \\ &\mu_1 = 0, \quad \mu_2 = b\eta_{1,w} - aw_s^2\eta_{1,ww} - \eta_{1,m} - 2aw_s\eta_{1,sw} - a\eta_{1,ss}, \\ &\mu_3 = 0, \quad \mu_4 = d\eta_{2,v} - cv_s^2\eta_{2,vv} - \eta_{2,m} - 2cv_s\eta_{2,sv} - c\eta_{2,ss}. \end{aligned} \tag{4.32}$$

So the corresponding generator for these infinitesimal transformations becomes

$$\begin{aligned} \mathbf{X} = {} & \eta_1\frac{\partial}{\partial w} + \eta_2\frac{\partial}{\partial v} + (b\eta_{1,w} - aw_s^2\eta_{1,ww} - \eta_{1,m} - 2aw_s\eta_{1,sw} - a\eta_{1,ss})\frac{\partial}{\partial b} \\ & +(d\eta_{2,v} - cv_s^2\eta_{2,vv} - \eta_{2,m} - 2cv_s\eta_{2,sv} - c\eta_{2,ss})\frac{\partial}{\partial d}. \end{aligned} \tag{4.33}$$

The infinitesimal test for the zeroth order invariants

$$\mathbf{X}K(w,v,a,b,c,d) = 0, \tag{4.34}$$

leads to

$$\begin{aligned} &\eta_1\frac{\partial K}{\partial w} + \eta_2\frac{\partial K}{\partial v} + (b\eta_{1,w} - aw_s^2\eta_{1,ww} - \eta_{1,m} - 2aw_s\eta_{1,sw} - a\eta_{1,ss})\frac{\partial K}{\partial b} \\ &+(d\eta_{2,v} - cv_s^2\eta_{2,vv} - \eta_{2,m} - 2cv_s\eta_{2,sv} - c\eta_{2,ss})\frac{\partial K}{\partial d} = 0. \end{aligned} \tag{4.35}$$

Coefficients of η_1, η_2, $\eta_{1,m}$, $\eta_{2,m}$ in (4.35) provides

$$K_w = 0, \qquad K_v = 0, \qquad -K_b = 0, \qquad -K_d = 0. \tag{4.36}$$

Solution of the above system provide

$$K_1 = a \qquad K_2 = c. \tag{4.37}$$

In order to get invariants of first order i.e., $K(w,v,a,b,c,d,a_i,b_j,c_i,d_j)$ we extend the generator (4.33) up to order one, which is

$$\mathbf{X}^{[1]} = \eta_1\partial_w + \eta_2\partial_v + (b\eta_{1,w} - aw_s^2\eta_{1,ww} - \eta_{1,m} - 2aw_s\eta_{1,sw} - a\eta_{1,ss})\partial_b + (d\eta_{2,v}$$

$$-cv_s^2\eta_{2,vv} - \eta_{2,m} - 2cv_s\eta_{2,sv} - c\eta_{2,ss})\partial_d + (-a_w\eta_{1,m} - a_v\eta_{2,m})\partial_{a_m} + (-a_w\eta_{1,s}$$
$$-a_v\eta_{2,s})\partial_{a_s} - a_w\eta_{1,w}\partial_{a_w} - a_v\eta_{2,v}\partial_{a_v} + (b\eta_{1,mw} - aw_s^2\eta_{1,mww} - 2aw_s\eta_{1,msw}$$
$$-a\eta_{1,mss} - \eta_{1,mm} - a_mw_s^2\eta_{1,ww} - 2a_mw_s\eta_{1,sw} - a_m\eta_{1,ss} + b_m\eta_{1,w} - b_w\eta_{1,m}$$
$$-b_v\eta_{2,m} - b_{w_s}\eta_{1,ms} - b_{w_s}w_s\eta_{1,mw} - b_{v_s}\eta_{2,ms} - b_{v_s}v_s\eta_{2,mv})\partial_{b_m} + (b\eta_{1,sw}$$
$$-aw_s^2\eta_{1,sww} - 2aw_s\eta_{1,ssw} - a\eta_{1,sss} - \eta_{1,ms} - a_sw_s^2\eta_{1,ww} - 2a_sw_s\eta_{1,sw} - a_s\eta_{1,ss}$$
$$+b_s\eta_{1,w} - b_w\eta_{1,s} - b_v\eta_{2,s} - b_{w_s}\eta_{1,ss} - b_{w_s}w_s\eta_{1,sw} - b_{v_s}\eta_{2,ss} - b_{v_s}v_s\eta_{2,sv})\partial_{b_s}$$
$$+(b\eta_{1,ww} - aw_s^2\eta_{1,www} - 2aw_s\eta_{1,sww} - a\eta_{1,ssw} - \eta_{1,mw} - a_ww_s^2\eta_{1,ww} - 2a_ww_s\eta_{1,sw}$$
$$-a_w\eta_{1,ss} - b_{w_s}\eta_{1,sw} - b_{w_s}w_s\eta_{1,ww})\partial_{b_w} + (-a_vw_s^2\eta_{1,ww} - 2a_vw_s\eta_{1,sw} - a_v\eta_{1,ss}$$
$$+b_v\eta_{1,w} - b_v\eta_{2,v} - b_{v_s}\eta_{2,sv} - b_{v_s}v_s\eta_{2,vv})\partial_{b_v} + (-2aw_s\eta_{1,ww} - 2a\eta_{1,sw})\partial_{b_{w_s}}$$
$$+(b_{v_s}\eta_{1,w} - b_{v_s}\eta_{2,v})\partial_{b_{v_s}} + (-c_w\eta_{1,m} - c_v\eta_{2,m})\partial_{c_m} + (-c_w\eta_{1,s} - c_v\eta_{2,s})\partial_{c_s}$$
$$-c_w\eta_{1,w}\partial_{c_w} - c_v\eta_{2,v}\partial_{c_v} + (d\eta_{2,mv} - cv_s^2\eta_{2,mvv} - 2cv_s\eta_{2,msv} - c\eta_{2,mss} - \eta_{2,mm}$$
$$-c_mv_s^2\eta_{2,vv} - 2c_mv_s\eta_{2,sv} - c_m\eta_{2,ss} + d_m\eta_{2,v} - d_w\eta_{1,m} - d_v\eta_{2,m} - d_{w_s}\eta_{1,ms}$$
$$-d_{w_s}w_s\eta_{1,mw} - d_{v_s}\eta_{2,ms} - d_{v_s}v_s\eta_{2,mv})\partial_{d_m} + (d\eta_{2,sv} - cv_s^2\eta_{2,svv} - 2cv_s\eta_{2,ssv}$$
$$-c\eta_{2,sss} - \eta_{2,ms} - c_sv_s^2\eta_{2,vv} - 2c_sv_s\eta_{2,sv} - c_s\eta_{2,ss} + d_s\eta_{2,v} - d_w\eta_{1,s} - d_v\eta_{2,s}$$
$$-d_{w_s}\eta_{1,ss} - d_{w_s}w_s\eta_{1,sw} - d_{v_s}\eta_{2,ss} - d_{v_s}v_s\eta_{2,sv})\partial_{d_s} + (-c_wv_s^2\eta_{2,vv} - 2c_wv_s\eta_{2,sv}$$
$$-c_w\eta_{2,ss} + d_w\eta_{2,v} - d_w\eta_{1,w} - d_{w_s}\eta_{1,sw} - d_{w_s}w_s\eta_{1,ww})\partial_{d_w} + (d\eta_{2,vv} - cv_s^2\eta_{2,vvv}$$
$$-2cv_s\eta_{2,svv} - c\eta_{2,ssv} - \eta_{2,mv} - c_vv_s^2\eta_{2,vv} - 2c_vv_s\eta_{2,sv} - c_v\eta_{2,ss} - d_{v_s}\eta_{2,sv}$$
$$-d_{v_s}v_s\eta_{2,vv})\partial_{d_v} + (d_{w_s}\eta_{2,v} - d_{w_s}\eta_{1,w})\partial_{d_{w_s}} + (-2cv_s\eta_{2,vv} - 2c\eta_{2,sv})\partial_{d_{v_s}}.$$

The invariance criterion

$$\mathbf{X}^{[1]}K(w, v, a, b, c, d, a_i, b_j, c_i, d_j) = 0, \tag{4.38}$$

after equating the terms η_1, η_2, $\eta_{1,mm}$, $\eta_{1,sss}$, $\eta_{1,www}$, $\eta_{2,mm}$, $\eta_{2,sss}$, $\eta_{2,vvv}$ to zero, provides

$$\begin{aligned} &K_w = 0, \qquad K_v = 0, \qquad K_{b_m} = 0, \qquad K_{b_s} = 0, \\ &K_{b_w} = 0, \qquad K_{d_m} = 0, \qquad K_{d_s} = 0, \qquad K_{d_v} = 0. \end{aligned} \tag{4.39}$$

So, $K = K(a, b, c, d, a_m, a_s, a_w, a_v, b_v, b_{w_s}, b_{v_s}, c_m, c_s, c_w, c_v, d_w, d_{w_s}, d_{v_s})$. Further, the terms with $\eta_{1,m}$, $\eta_{1,s}$, $\eta_{2,m}$, $\eta_{2,s}$, $\eta_{1,ss}$, $\eta_{2,ss}$, $\eta_{2,sv}$, $\eta_{2,vv}$, $\eta_{1,sw}$, $\eta_{1,ww}$, $\eta_{1,w}$, $\eta_{2,v}$ provide following system of equations

$$
\begin{aligned}
&-a_w K_{a_m} - c_w K_{c_m} - K_b = 0,\\
&-a_w K_{a_s} - c_w K_{c_s} = 0,\\
&-a_v K_{a_m} - c_v K_{c_m} - J_d = 0,\\
&-a_v K_{a_s} - c_v K_{c_s} = 0,\\
&-a K_b - a_v K_{b_v} = 0,\\
&-c K_d - c_w K_{d_w} = 0,\\
&-2c v_s K_d - 2c_w v_s K_{d_w} - b_{v_s} K_{b_v} - 2c K_{d_{v_s}} = 0,\\
&-c v_s^2 K_d - c_w v_s^2 K_{d_w} - b_{v_s} v_s K_{b_v} - 2c v_s K_{d_{v_s}} = 0,\\
&-2a w_s K_b - d_{w_s} K_{d_w} - 2a_v w_s K_{b_v} - 2a K_{b_{w_s}} = 0,\\
&-a w_s^2 K_b - d_{w_s} w_s K_{d_w} - a_v w_s^2 K_{b_v} - 2a w_s K_{b_{w_s}} = 0,\\
&-c_w K_{c_w} + b K_b - d_w K_{d_w} - b_v K_{b_v} - d_{w_s} K_{d_{w_s}} + b_{v_s} K_{b_{v_s}} - a_w K_{a_w} = 0,\\
&-c_v K_{c_v} - a_v K_{a_v} + d K_d + d_w K_{d_w} - b_v K_{b_v} + d_{w_s} K_{d_{w_s}} - b_{v_s} K_{b_{v_s}} = 0. \qquad (4.40)
\end{aligned}
$$

Solving (4.40) simultaneously in MAPLE, we obtain following first order semi differential invariants along with K_1 and K_2,

$$
K_3 = \frac{c_w}{a_w}, \qquad K_4 = \frac{c_v}{a_v}, \qquad K_5 = \frac{a_v d_{w_s}}{a_w}, \qquad K_6 = \frac{a_w b_{v_s}}{a_v},
$$

$$
\begin{aligned}
K_7 &= \frac{1}{c d_{w_s}(a_w c_v - a_v c_w)}[-2a c_w^2 (a_v d - a_m) + [\{(2d c_v - 2c_m) a_w + 2a_v c d_w\}\, a\\
&\quad - a_v c b_{w_s} d_{w_s}] c_w + 2c a_w c_v (\frac{1}{2} b_{w_s} d_{w_s} - a d_w)],
\end{aligned}
$$

$$
\begin{aligned}
K_8 &= \frac{1}{a b_{v_s}(a_v c_w - a_w c_v)}[2c a_v^2 (c_w b - c_m) + [\{(-2b a_w + 2a_m) c_v - 2a c_w b_v\}\, c\\
&\quad + a c_w d_{v_s} b_{v_s}] a_v - 2a a_w c_v (\frac{1}{2} b_{v_s} d_{v_s} - c b_v)], \qquad (4.41)
\end{aligned}
$$

provided that a_w, a_v, b_{v_s}, c_w, c_v, d_{w_s} is not equal to zero.

4.2.1 Applications

To illustrate applications of the derived semi differential invariants for system of two second order nonlinear parabolic type PDEs, we present a few examples in this subsection.

Example 1

A second order coupled system of nonlinear parabolic type PDEs

$$\begin{aligned} &w_m + (\frac{w^2 s}{v})w_{ss} + (\frac{ws}{v})w_s^2 - (\frac{s}{2wv^2})v_s + \frac{1}{2wv} = 0, \\ &v_m + (\frac{w^2 v + s}{v})v_{ss} - (\frac{2w^2 v + 2s}{v^2})v_s^2 + (\frac{2w^2 v + 2s}{sv})v_s - (\frac{2wv^2}{s})w_s = 0, \end{aligned} \tag{4.42}$$

has the following semi differential invariants

$$\begin{aligned} &K_1 = \frac{w^2 s}{v}, \quad K_2 = (w^2 + \frac{s}{v}), \quad K_3 = \frac{v}{s}, \quad K_4 = \frac{1}{w^2}, \quad K_5 = \frac{w^2 v}{s}, \\ &K_6 = \frac{s}{w^2 v}, \quad K_7 = \frac{2w^2 s w_s}{(w^2 v + s)}, \quad K_8 = \frac{2(w^2 v + s)v_s}{s}. \end{aligned} \tag{4.43}$$

System (4.42) can be mapped into

$$\begin{aligned} &u_t + ucu_{xx} + c_x = 0, \\ &c_t + (u + c)v_{xx} + u_x = 0, \end{aligned} \tag{4.44}$$

using invertible transformations

$$m = t, \qquad s = x, \qquad w = \sqrt{u}, \qquad v = \frac{x}{c}. \tag{4.45}$$

Semi differential invariants for (4.44) are

$$\begin{aligned} &K_1 = uc, \quad K_2 = u + c, \quad K_3 = \frac{1}{c}, \quad K_4 = \frac{1}{u}, \quad K_5 = \frac{u}{c}, \\ &K_6 = \frac{c}{u}, \quad K_7 = \frac{2ucu_x}{(u + c)}, \quad K_8 = \frac{2(u + c)c_x}{c}, \end{aligned} \tag{4.46}$$

which are similar to (4.42), under (4.45).

Example 2

A coupled system of parabolic type PDEs

$$w_m + (\frac{1+wv^3}{w})w_{ss} - (\frac{2+2wv^3}{w^2})w_s^2 - (3w^2v^2)v_s = 0,$$
$$v_m + (\frac{v^3}{w})v_{ss} + (\frac{2v^2}{w})v_s^2 - (\frac{1}{3w^2v^2})w_s = 0, \tag{4.47}$$

having

$$a = (\frac{1+wv^3}{w}), \qquad b = -(\frac{2+2wv^3}{w^2})w_s^2 - (3w^2v^2)v_s,$$
$$c = (\frac{v^3}{w}), \qquad d = (\frac{2v^2}{w})v_s^2 - (\frac{1}{3w^2v^2})w_s,$$

can be transformed into

$$u_t + (u+c)u_{xx} + c_x = 0,$$
$$c_t + ucc_{xx} + u_x = 0, \tag{4.48}$$

by means of transformations

$$m = t, \qquad s = x, \qquad w = \frac{1}{u}, \qquad v = c^{1/3}. \tag{4.49}$$

The semi differential invariants for (4.47)

$$K_1 = \frac{1}{w} + v^3, \qquad K_2 = \frac{v^3}{w}, \qquad K_3 = v^3, \qquad K_4 = \frac{1}{w},$$
$$K_5 = K_6 = 1, \qquad K_7 = (2+2wv^3)w_s, \qquad K_8 = (\frac{2v^3}{1+wv^3})v_s, \tag{4.50}$$

that are same as (4.48) under transformations of dependent variables only.

Example 3

Consider a coupled system of PDEs

$$w_m + (\frac{mv\sqrt{w}+s}{v})w_{ss} - (\frac{mv\sqrt{w}+s}{2wv})w_s^2 + w_s - (\frac{2s\sqrt{w}}{mv^2})v_s + \frac{2wv+2\sqrt{w}}{mv} = 0,$$
$$v_m + (\frac{ms\sqrt{w}}{v})v_{ss} - (\frac{2ms\sqrt{w}}{v^2})v_s^2 + (\frac{2m\sqrt{w}}{v})v_s - \frac{mv^2}{2s\sqrt{w}}w_s = 0, \tag{4.51}$$

with semi differential invariants

$$K_1 = m\sqrt{w} + \frac{s}{v}, \qquad K_2 = \frac{ms\sqrt{w}}{v}, \qquad K_3 = \frac{s}{v}, \qquad K_4 = m\sqrt{w}, \qquad K_5 = 1,$$
$$K_6 = 1, \qquad K_7 = 2(\frac{s}{mv\sqrt{w}} + 1)w_s + 1, \qquad K_8 = \frac{2ms\sqrt{w}}{mv\sqrt{w} + s}(w_s + v_s). \quad (4.52)$$

It is transformable to

$$u_t + (u + c)u_{xx} + u_x + c_x = 0,$$
$$c_t + ucc_{xx} + u_x = 0, \quad (4.53)$$

by using invertible transformations

$$m = t, \qquad s = x, \qquad w = \frac{u^2}{t^2}, \qquad v = \frac{x}{c}. \quad (4.54)$$

Semi differential invariants of (4.53) is

$$K_1 = u + c, \qquad K_2 = uc, \qquad K_3 = c, \qquad K_4 = u, \qquad K_5 = 1,$$
$$K_6 = 1, \qquad K_7 = 2(\frac{c}{u} + 1)u_x + 1, \qquad K_8 = \frac{2uc}{u + c}(u_x + c_x). \quad (4.55)$$

Notice that the semi differential invariants of (4.51) and (4.53) are same by means of the transformations (4.54).

4.3 Characterization of a Few Classes of Systems of Two Parabolic Type PDEs

In this section we classify different subclasses of systems of two second order nonlinear parabolic type PDEs (3.1) by using Lie Symmetry Method. For this sake, we find both joint and semi differential invariants corresponding to each subclass with the help of their equivalence transformations. Furthermore, using deduced differential invariants we get canonical forms for our considered systems. Total number of symmetries of each simpler form of systems of nonlinear parabolic type PDEs are also given.

Case-I

A second order system of nonlinear parabolic type PDEs with a, b, c, d as its arbitrary coefficients

$$\begin{aligned} w_m + a(m,s,w)w_{ss} + b(m,s,w,v,w_s,v_s) = 0, \\ v_m + c(m,s,w)v_{ss} + d(m,s,w,v,w_s,v_s) = 0, \end{aligned} \tag{4.56}$$

have 1st oder joint differential invariants

$$J_1 = \frac{c}{a}, \qquad J_2 = \frac{c_w}{a_w}, \tag{4.57}$$

with the help of joint differential invariants (4.57) we get following canonical form of (4.56)

$$\begin{aligned} &w_m + ww_{ss} = 0, \\ &v_m = 0. \end{aligned} \tag{4.58}$$

Another simpler form of (4.56) can be obtained using 1st order joint differential invariants

$$\begin{aligned} &w_m + ww_{ss} = 0, \\ &v_m + wv_{ss} = 0. \end{aligned} \tag{4.59}$$

Semi differential invariants corresponding to the dependent variable for (4.56) are

$$\begin{aligned} &K_1 = a, \qquad K_2 = c, \qquad K_3 = b_{v_s}d_{w_s}, \qquad K_4 = \frac{a_w c_s - a_s c_w}{a_w}, \\ &K_5 = \frac{c_w}{a_w}, \qquad K_6 = \frac{a_w c_m - a_m c_w}{a_w}, \qquad K_7 = \frac{b_{v_s}d_{v_s} - 2cb_v}{b_{v_s}}. \end{aligned} \tag{4.60}$$

Canonical form of (4.56) under these semi differential invariants (4.60) is

$$\begin{aligned} &w_m + ww_{ss} + v_s = 0, \\ &v_m = 0. \end{aligned} \tag{4.61}$$

Equation (4.58) have four, (4.59) have nine and (4.61) have five symmetries.

Case-II

For a system of nonlinear parabolic type PDEs

$$\begin{aligned} w_m + a(m,s,w)w_{ss} + b(m,s,w,v,w_s,v_s) &= 0, \\ v_m + c(m,s,v)v_{ss} + d(m,s,w,v,w_s,v_s) &= 0, \end{aligned} \tag{4.62}$$

joint differential invariants of 1st kind are

$$J_1 = \frac{c}{a}, \qquad J_2 = \frac{ac_v}{a_w}, \tag{4.63}$$

which give the same canonical form (4.58). Simpler form of (4.62) under 1st order joint differential invariants is

$$\begin{aligned} &w_m + ww_{ss} = 0, \\ &v_m + vv_{ss} = 0. \end{aligned} \tag{4.64}$$

For (4.62) semi differential invariants derived are

$$\begin{aligned} &K_1 = a, \quad K_2 = c, \quad K_3 = \frac{c_v}{a_w}, \quad K_4 = \frac{b_{v_s} d_{w_s}}{2a}, \quad K_5 = \frac{a_w c_s - a_s c_v}{a_w}, \\ &K_6 = \frac{a_w c_m - a_m c_v}{a_w}, \quad K_7 = \frac{b_{v_s} d_{v_s} - 2cb_v}{b_{v_s}}, \quad K_8 = \frac{2ad_w - b_{w_s} d_{w_s}}{2ad_{w_s}}. \end{aligned} \tag{4.65}$$

By means of (4.65) least form for (4.62) is

$$\begin{aligned} &w_m + ww_{ss} + v_s = 0, \\ &v_m + w_s = 0. \end{aligned} \tag{4.66}$$

Both (4.64) and (4.66) have same number of Lie symmetries, i.e., four.

Case-III

Consider a system of nonlinear parabolic type PDEs

$$\begin{aligned} w_m + a(m,s,v)w_{ss} + b(m,s,w,v,w_s,v_s) &= 0, \\ v_m + c(m,s,v)v_{ss} + d(m,s,w,v,w_s,v_s) &= 0, \end{aligned} \tag{4.67}$$

along with

$$\begin{aligned} w_m + a(m,s,v)w_{ss} + b(m,s,w,v,w_s,v_s) &= 0, \\ v_m + c(m,s,w)v_{ss} + d(m,s,w,v,w_s,v_s) &= 0. \end{aligned} \tag{4.68}$$

Joint differential invariants of 1st order for (4.67) and (4.68) have the following forms

$$J_1 = \frac{c}{a}, \qquad J_2 = \frac{c_v}{a_v}, \tag{4.69}$$

$$J_1 = \frac{c}{a}, \qquad J_2 = \frac{c_w}{aa_v}, \tag{4.70}$$

respectively. Under these joint differential invariants we get same canonical form for both (4.67) and (4.68) i.e.,

$$\begin{aligned} &w_m + vw_{ss} = 0, \\ &v_m = 0. \end{aligned} \tag{4.71}$$

Semi differential invariants for (4.67) are

$$\begin{aligned} &K_1 = a, \quad K_2 = c, \quad K_3 = \frac{b_{v_s} d_{w_s}}{2a}, \quad K_4 = \frac{a_v c_s - a_s c_v}{a_v}, \\ &K_5 = \frac{c_v}{a_v}, \quad K_6 = \frac{a_v c_m - a_m c_v}{a_v}, \quad K_7 = \frac{2ad_w - b_{w_s} d_{w_s}}{2ad_{w_s}}, \end{aligned} \tag{4.72}$$

which generate a canonical form

$$\begin{aligned} &w_m + vw_{ss} = 0, \\ &v_m + w_s = 0. \end{aligned} \tag{4.73}$$

while semi differential invariants of (4.68) are

$$\begin{aligned} &K_1 = a, \quad K_2 = c, \quad K_3 = b_{v_s} d_{w_s}, \quad K_4 = \frac{c_w}{a_v}, \\ &K_5 = \frac{a_v c_s - a_s c_w}{a_v}, \quad K_6 = \frac{a_v c_m - a_m c_w}{a_v}. \end{aligned} \tag{4.74}$$

that also gives the same canonical form (4.71). Number of symmetries for (4.71) are eight while (4.73) have five symmetries.

Case-IV

A second order system of nonlinear parabolic type PDEs

$$\begin{aligned} w_m + a(w,v)w_{ss} + b(m,s,w,v,w_s,v_s) = 0, \\ v_m + c(w,v)v_{ss} + d(m,s,w,v,w_s,v_s) = 0, \end{aligned} \tag{4.75}$$

has canonical form

$$w_m + v w_{ss} = 0,$$
$$v_m + v v_{ss} = 0, \tag{4.76}$$

under 1st order joint differential invariants

$$J_1 = \frac{c}{a}, \qquad J_2 = \frac{b_{v_s} d_{w_s}}{(ac_v - ca_v)^2}. \tag{4.77}$$

Semi differential invariants of (4.75) are

$$K_1 = a, \qquad K_2 = c, \qquad K_3 = a_w, \qquad K_4 = c_w,$$
$$K_5 = a_v, \qquad K_6 = c_v, \qquad K_7 = b_{v_s} d_{w_s}, \tag{4.78}$$

which give the following simpler form for (4.75)

$$w_m + wv w_{ss} + v_s = 0,$$
$$v_m + wv v_{ss} + w_s = 0. \tag{4.79}$$

For (4.76) and (4.79) we have nine and three Lie symmetries.

Case-V

Consider the following systems of nonlinear parabolic type PDEs

$$\begin{aligned} w_m + a(m, w, v) w_{ss} + b(m, s, w, v, w_s, v_s) = 0, \\ v_m + c(m, w, v) v_{ss} + d(m, s, w, v, w_s, v_s) = 0, \end{aligned} \tag{4.80}$$

$$\begin{aligned} w_m + a(m, w, v) w_{ss} + b(m, s, w, v, w_s, v_s) = 0, \\ v_m + c(s, w, v) v_{ss} + d(m, s, w, v, w_s, v_s) = 0, \end{aligned} \tag{4.81}$$

$$w_m + a(s,w,v)w_{ss} + b(m,s,w,v,w_s,v_s) = 0,$$
$$v_m + c(s,w,v)v_{ss} + d(m,s,w,v,w_s,v_s) = 0, \tag{4.82}$$

$$w_m + a(s,w,v)w_{ss} + b(m,s,w,v,w_s,v_s) = 0,$$
$$v_m + c(m,w,v)v_{ss} + d(m,s,w,v,w_s,v_s) = 0, \tag{4.83}$$

$$w_m + a(m,s)w_{ss} + b(m,s,w,v,w_s,v_s) = 0,$$
$$v_m + c(w,v)v_{ss} + d(m,s,w,v,w_s,v_s) = 0, \tag{4.84}$$

$$w_m + a(s,w)w_{ss} + b(m,s,w,v,w_s,v_s) = 0,$$
$$v_m + c(s,w)v_{ss} + d(m,s,w,v,w_s,v_s) = 0, \tag{4.85}$$

$$w_m + a(s,v)w_{ss} + b(m,s,w,v,w_s,v_s) = 0,$$
$$v_m + c(s,v)v_{ss} + d(m,s,w,v,w_s,v_s) = 0. \tag{4.86}$$

For all the above system we get same 1st order joint differential invariants, i.e., $J = \frac{c}{a}$ which generates the following canonical form

$$w_m + w_{ss} = 0,$$
$$v_m = 0. \tag{4.87}$$

Semi differential invariants of 1st kind found for equation (4.80) to (4.86), yields

$$K = K(K_1, K_2, K_3, K_4, K_5, K_6),$$
$$K = K(K_1, K_2, K_3, K_4, K_5, K_7),$$
$$K = K(K_1, K_2, K_3, K_4, K_5, K_8),$$
$$K = K(K_1, K_2, K_3, K_4, K_5, K_9),$$
$$K = K(K_1, K_2, K_3, K_{10}, K_{11}, K_{12}, K_{13}, K_{14}),$$

$$K = K(K_1, K_2, K_3, K_{11}, K_{12}, K_{14}, K_{15}, K_{16}),$$
$$K = K(K_1, K_2, K_{12}, K_{13}, K_{15}, K_{17}, K_{18}, K_{19}), \tag{4.88}$$

where

$$\begin{aligned}
&K_1 := a, \quad K_2 := c, \quad K_3 := b_{v_s} d_{w_s}, \quad K_4 := \frac{c_v}{a_v},\\
&K_5 := \frac{a_v c_w - a_w c_v}{a_v}, \quad K_6 := \frac{a_v c_m - a_m c_v}{a_v},\\
&K_7 := \frac{a_v c_s - a_m c_v}{a_v}, \quad K_8 := \frac{a_v c_s - a_s c_v}{a_v},\\
&K_9 := \frac{a_v c_m - a_s c_v}{a_v}, \quad K_{10} := a_m, \quad K_{11} := c_w,\\
&K_{12} := a_s, \quad K_{13} := c_v, \quad K_{14} := \frac{b_{v_s} d_{v_s} - 2c b_v}{b_{v_s}},\\
&K_{15} := c_s, \quad K_{16} := a_w, \quad K_{17} := a_v,\\
&K_{18} := \frac{2a d_w - b_{w_s} d_{w_s}}{2a d_{w_s}}, \quad K_{19} := \frac{b_{v_s} d_{w_s}}{2a}.
\end{aligned} \tag{4.89}$$

Under these semi differential invariants equation (4.80) to (4.83) have (4.71) as a canonical form while least forms for (4.84), (4.85) and (4.86) are

$$\begin{aligned}
&w_m + v_s = 0,\\
&v_m = 0,
\end{aligned} \tag{4.90}$$

and

$$\begin{aligned}
&w_m + w_{ss} = 0,\\
&v_m + w_s = 0,
\end{aligned} \tag{4.91}$$

respectively. Here, (4.90) has one and (4.91) have six Lie symmetries.

Chapter 5

Conclusion

Lie infinitesimal method has been employed to find the set of equivalence transformations associated with the considered systems of nonlinear parabolic type PDEs. These equivalence transformations are used to derive differential invariants for systems of nonlinear parabolic type PDEs under transformations of both the dependent, independent and only dependent variables, that are called joint and semi differential invariants respectively.

By using derived joint and semi differential invariants, we reduce the nonlinearities of the highly nonlinear parabolic type systems of PDEs under point transformations. The forms of systems of PDEs attained here have some nonlinearities but they are solvable. For characterization different subclasses of systems of parabolic type PDEs are considered in which differential invariants are derived under transformations of both the dependent, independent and only dependent variables. Utilizing these differential invariants, canonical forms are obtained which provide a classification of these systems by putting them into the reducible and non-reducible classes under differential invariants.

The reductions achieved here through differential invariants are shown to solve many systems of PDEs from the considered classes. This idea can further be implemented to attempt double reductions of these systems. For double reduction one needs to investigate joint and semi differential invariants of canonical forms obtained here. Higher order differential invariants of systems of parabolic type PDEs can also be investigated in order to provide complete set of basis. Investigation of semi differential invariants under transformation of only the independent variables would also lead to reductions of system of parabolic type PDEs. Both the cases mentioned may require much more efficient algebraic computing tools and machines but can generate interesting results for systems of PDEs.

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