The Approximate Mei Symmetries and their Applications

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To my parents

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ABSTRACT

The Mei symmetries, a class of symmetries, correspond to conserved quantities just like Noether symmetries. However, the two sets of symmetries result in different conserved quantities.

The formulation of first-order approximate Mei symmetries of the associated perturbed Lagrangian is presented in this thesis. Theorems and determining equations are given to evaluate approximate Mei symmetries, as well as approximate Mei invariants relative to each symmetry of the associated Lagrangian. The stated approach is illustrated using the linear equation of motion of a damped harmonic oscillator (DHO).

Furthermore, a method for determining approximate Mei symmetries and invariants of the perturbed Hamiltonian is described, which can be employed in various fields of study where approximate Hamiltonian are considered. The Legendre transformation is used to convert Lagrangian into Hamiltonian. The results are provided as theorems with proof. To elaborate on the method of determining these symmetries and the related Mei invariants, a basic example of DHO is presented. Moreover, a comparison of approximate Mei symmetries with approximate Noether symmetries is provided. The comparison indicates that both sets of symmetries have only one common symmetry. Furthermore, the number of approximate Mei symmetries exceeds the number of approximate Noether symmetries. As a result, the remaining symmetries in the two sets correspond to two distinct sets of conserved quantities. The Mei symmetries associated with the Lagrangian and Hamiltonian of DHO are compared.

First-order approximate Mei symmetries of the geodesics Lagrangian are determined as an application of approximate Mei symmetries for particular classes of pp-wave spacetimes. These classes of pp-wave spacetimes include plane wave spacetimes in which (i).$A(u) = \alpha^2$(ii). $A(u) = \alpha u^{-2}$ (iii). $A(u) = \alpha^2 u^{-4}$ and for pp-wave spacetimes (iv). $h = \alpha x^n$ (where h is called scale factor and α is a constant). After that, approximate Mei invariants are calculated corresponding to each case.

Keywords: Noether symmetries, Conserved quantities, Hamiltonian, Lagrangian, Damped Harmonic Oscillator

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Chapter 1

Introduction

In mathematics, the theory of differential equations (DEs) holds a prominent place and gives rise to numerous directions of studies. A pure analytical direction includes series solution of DEs, theory of function spaces, and existence theorems regarding the solutions of DEs etc. The other direction has only geometric implications and is connected to the theory of surfaces and curves. The theory of continuous group of transformations falls in between these two directions.

DEs play a significant role in both pure and applied mathematics. Both of these fields are concerned with the characteristics of various types of DEs. Pure mathematics deals with the existence and uniqueness of solutions, whereas applied mathematics requires a strict justification of how to get solutions. They were first introduced in the 17^{th} century by Newton and Leibniz. Many real-world phenomena are modeled in the form of DEs (or system of DEs). These are usually difficult to solve. There are numerous established techniques for solving various types of DEs [1, 2]. The majority of the developed methods, however, are for particular classes of DEs.

Since nonlinear DEs are actually difficult to solve, so one may try symmetry methods for the solutions of DEs [3]. Sophus Lie, a Norwegian mathematician, developed symmetry methods for solving DEs in 1867. The beauty of this technique is that it works for any type of DEs, including homogeneous, non-homogeneous, linear, nonlinear, ordinary differential equation (ODEs), and partial differential equations (PDEs) of any order. Later, he used this technique for the linearization of nonlinear DEs, the group classification of DEs, and for finding invariant solutions to DEs.

Symmetries offer a geometry-based shortcut for accessing some of nature's deepest secrets. It is a helpful tool for solving DEs. A symmetry can be regarded as a transformation that, when applied to a particular structure, preserves the structure's properties. Scientists employ symmetry as a tool to comprehend real-world problems. The most well-known and well-established technique for determining the exact solutions of DEs is the classical symmetry approach, often known as group analysis.

Symmetry methods is applicable in many fields, including mathematics, social sciences, natural sciences, engineering, and so on. Now, here we try to understand what is a symmetry? Symmetry is defined as harmonious, perfect proportion and balance of an object. Beautiful balance and proportion are described mathematically as patterned self similarity. The simplest symmetry is the reflection symmetry, often known as line symmetry or mirror symmetry in mathematics. In reflection symmetry, one half of the object reflects the other half. Mathematically, symmetry is formally defined as a transformation that leaves the original object unaltered. Symmetries of functions, DEs, integral equations, etc., are transformations of the variables that do not alter the functions, DEs, integral equations, etc. Simple examples are provided for each case in the table below. All the transformations given in the above Table 1.1 depend upon a

Type	**Examples**	**Symmetry Transformations**
Algebraic Expressions	$x^2 + y^2$	$(x, y) \to (x \cos\delta - y \sin\delta, x \sin\delta + y \cos\delta)$
Differential Equations	$\frac{dy}{dx} = f(x, y)$	$(x, y) \to (x + \delta_1, y + \delta_2)$
Integral Equations	$I = \int_0^x (x - t) y(t) dt$	$(x, y) \to (x \cos\delta - y \sin\delta, x \sin\delta + y \cos\delta)$

Table 1.1: Symmetries of functions, DEs and integral equations

parameter $\delta \in \mathbb{R}$, form groups known as Lie groups of point transformations. The Lie group of point transformation of a DEs is a symmetry transformation that maps the set of solutions of a system on to the set of solutions of the same DEs. Examples of Lie point transformations include translations, rotations, and scaling [4]. The order of DEs is reduced by one, if DEs are unchanged under one parameter Lie group of point transformations.

The set S that contains all invertible transformations T that leaves an object O [5] invariant is referred to as the symmetry group of the object O that is

$$T : O \to O,$$

such that the set S contains identity I, the inverse transformations T^{-1}; for all $T \in S$ and the composition $T_1 T_2 \in S$ of the transformations $T_1, T_2 \in S$.

Lie developed his group theoretic method for solving DEs using Galois' idea of groups. Although Lie symmetry groups contain an infinite number of transformations that rely on continuous parameters. However, Galois groups are finite. A Lie group of transformations uses infinitesimal generators to form a vector space that is closed under a Lie algebra defined later in Chapter 2. Contrary to more general topological groups, Lie groups are smooth and twice differentiable manifolds that may be studied using differential calculus. One of the fundamental concepts in the Lie theory of groups is the replacement of the global object, the group, with its local or linearized version, which Lie originally referred to as its infinitesimal group and is today known as its Lie algebra.

The Lie symmetries are the invariance of DEs under infinitesimal transformations of a group. The Lie point symmetries that leave the action invariant for DEs arising from a variational principle, i.e., following from a Lagrangian (difference of kinetic energy and potential energy), are known as Noether symmetries. All Lie point symmetries including Noether symmetries span Lie algebras. According to Noether's theorem, each Noether symmetry corresponds to a conservation law [6, 7, 8]. Noether symmetries are a subclass of Lie symmetries that are used to investigate dynamical systems defined by a point Lagrangian while keeping the action integral invariant. Different fields of study e.g, classical mechanics, general relativity, field theory, and the study of dynamical systems, in general, all mainly depend on the conservation laws [9, 10].

The Lie symmetry method [12, 13, 14] and the Mei symmetry method [11] are two useful tools for investigating dynamical systems other than Noether symmetry method. The form invariance, often referred to as Mei symmetries, was first described by Feng-Xiang [11] in 2000. It is defined as an invariance of the equations of motion under infinitesimal transformation of a group. The transformed dynamical functions replace the Lagrangian, Hamiltonian, and other dynamical functions in Mei symmetries. Furthermore, after performing some infinitesimal transformation of a group, the equations of motion are satisfied. In particular, Mei symmetries preserve the form of *equations of motion*.

The form invariance of the Appell equations is calculated under infinitesimal transformation of a group in [15]. The Lagrangian which is obtained from the Appell equations is used to compute the Noether symmetries. After that, Noether symmetries are compared to form invariance, and different conserved quantities are found. Shu-Yong and Feng-Xiang [16] studied the form invariance and the Lie symmetries of the non-holonomic system. In [16], structure equations and form invariance are constructed which are analogous to Lie symmetries. The Mei symmetries of the rotational relativistic mass variable system are studied [18] with a focus on the connection between Mei and Lie symmetries. Jiang et al. [19] constructed the Mei symmetries for non-material volumes. In order to determine the conserved quantities, a non-material volume with a single degree of freedom is used as an example. In [20, 21], the Mei symmetries on time scale are calculated using Lagrangian and Birkhoffian system. Its connection to the Noether symmetries is explained in detail in these papers. The Hamiltonian canonical equations are taken into consideration as a specific case in the construction of Mei symmetries of Birkhoffian systems.

1.1 Plan of Thesis

This thesis is arranged as follows: The detail description of manifolds along with related concepts such as sub-manifolds, tangent bundles, tangent spaces, Lie groups, Lie algebra, Lie derivatives, Isometries, homotheties are discussed in Chapter 2. The Legendre transformation is then thoroughly explained.

In Chapter 3, the Lagrangian of damped harmonic oscillator (DHO) is transformed into Hamiltonian by using Legendre transformation. The method for finding approximate Mei invariants and Mei symmetries of the first order perturbed Hamiltonian is presented. At the end, the approximate Mei symmetries for the Lagrangian and Hamiltonian are compared.

In Chapter 4, approximate Mei symmetries and approximate Mei invariants corresponding to Lagrangian of linear equation of DHO are developed with the help of theorems. At the end, approximate Mei symmetries and approximate Noether symmetries are compared.

In Chapter 5, first order approximate Mei symmetries of the geodesic Lagrangian for some classes of the pp-wave spacetimes are obtained. These classes of pp-wave spacetimes include plane wave spacetimes for metric coefficient $A(u)$ (i). $A(u) = \alpha^2$(ii). $A(u) = \alpha u^{-2}$ (iii). $A(u) = \alpha^2 u^{-4}$ and for pp-waves spacetimes with (iv). $h(x) = \alpha x^n$ (where h is called scale factor and α is a constant).

Chapter 2

Preliminaries

2.1 Introduction

To study Lie groups and Lie algebras, it is required to understand the notions of manifold, tangent space, tangent bundle, and so on, which are all briefly introduced here. Manifolds, the fundamental concept in the study of differential geometry, generalize the familiar concepts of curves and surfaces in three-dimensional space. Manifolds are the spaces that locally look like some Euclidean space $\mathbb{R}^n$, and on which calculus can be performed. Apart from Euclidean spaces, smooth plane curves like circles and parabolas, as well as smooth surfaces like spheres, tori, paraboloids, ellipsoids, and hyperboloids are the most familiar examples of manifolds. Graphs of smooth maps between Euclidean spaces and the set of points in $\mathbb{R}^{n+1}$ at a uniform distance from the origin (an n-sphere) are two examples of higher-dimensional manifolds [23].

Lie groups are actually manifolds because they satisfy all criteria of the manifolds [24]. The idea of symmetry transformations, often known as Lie symmetry transformations, gives rise to Lie groups as an algebraic abstraction. An important example of a Lie group is the group of rotations in the plane or in the surface. Both the algebraic group theory methods and the multi-variable calculus used in analytic geometry are combined and substantially extended by the merging of these two seemingly different mathematical concepts. The resulting theory, particularly infinitesimal symmetry generator techniques, can then be applied to a variety of physical and mathematical problems.

There are several vector fields on Lie group G that are distinguished by their invariance under group multiplication. The Lie algebra of G, denoted by g, is made up of these invariant vector fields, which are precisely "infinitesimal generators" of G. The Lie algebra of Lie group G contains all of its generators, which is fundamental for the foundation of Lie group theory. It allows us to replace complicated nonlinear invariance conditions under a group action with relatively simple linear conditions. It is impossible to overestimate the efficiency of this technique; in fact, practically all Lie group applications to DEs are based on this single construction.

Since DEs are used to formulate the mathematical models of many real-world phenomena. It turns out that the general theory of DEs is one of the most significant applications of Lie group theory. It is important to note that Sophus Lie's first purpose was to provide a way of integrating ODEs that was comparable to the Abelian method for solving algebraic equations. In this context, he defined the group admitted by the given system of DEs. At the present, group analysis of DEs refers to mathematical trend whose purpose is a common treatment of the Lie group of transformations and DEs admitted by these groups [31].

In this Chapter, we focus on the manifolds, tangent bundles, tangent spaces and sub-manifolds, Lie brackets and its properties. Definition of Lie group and Lie algebra along with their examples are considered. After that, Lie point transformation and their particular cases are discussed. Lie point transformation, Lie symmetry generators for DEs and their examples are discussed with their Lie groups. At the end, Mei symmetries and their corresponding conserved quantities are presented.

2.1.1 Manifold

A fundamental concept in mathematical physics is the manifold [24]. It locally looks like a Euclidean space $\mathbb{R}^n$, and represents a space that may be curved and have a complex topology, but (it does not mean that both have the same metric). A manifold is divided into coordinate charts that make up an atlas in order to do calculus. Union of these coordinate charts is the manifold.

Definition 2.1.1. *An n-dimensional manifold is a non empty set U, with countable subsets $U_i \subset U$, called the coordinate chart, and one to one mapping $g_i : U_i \to V_i$ onto connected subsets $V_i \subset \mathbb{R}^n$, called the local coordinate mapping which satisfy the following conditions:*

- *The coordinate charts cover U; that is*

$$\bigcup_i U_i = U.$$

- *On the intersection of coordinate charts, $U_i \bigcap U_j$, the composite mapping*

$$g_i \circ g_j^{-1} : g_i(U_i \bigcap U_j) \to g_j(U_i \bigcap U_j)$$

  is smooth (infinitely differentiable).

- *For distinct points $p \in U_i$ and $q \in U_j$ in U, there exist open subsets $R \subset V_i$ and $S \subset V_j$, with the property that $g_i(p) \in R$, $g_j(q) \in S$, satisfying*

$$g_i^{-1}(R) \bigcap g_j^{-1}(S) = \phi.$$

Example 1

The simplest n-dimensional manifold is the Euclidean space $\mathbb{R}^n$. A single coordinate chart $U = \mathbb{R}^n$ covers it, with local coordinate identity map provided by

$$g = 1 : \mathbb{R}^n \to \mathbb{R}^n$$

More generally, any open subset U of $\mathbb{R}^n$ is an n-dimensional manifold with a coordinate chart given by U itself, and with local coordinate identity map

$$g : U \to V \subset \mathbb{R}^n.$$

Sub-manifold

A sub-manifold $N \subset U$ of a smooth manifold U is a subset of U if it satisfies all the conditions of a manifold [24]. The unit circle $S^1 = \{(x, y) : x^2 + y^2 = 1\}$ and unit sphere $S^2 = \{(x, y, z) : x^2 + y^2 + z^2 = 1\}$ are examples of one dimensional and two dimensional sub-manifolds of $\mathbb{R}^n$, $n \geq 2$, respectively. More specifically, we have the definition of sub-manifold as follow

Definition 2.1.2. *Let U represents the smooth manifold. A sub-manifold $N \subset U$ is a subset of U that includes a smooth one-to-one map*

$$\phi : \tilde{N} \to N \subset U,$$

satisfying the maximal rank condition every where, and $\tilde{N}$ is another manifold where $N = \phi(\tilde{N})$. The dimension of N, in particular, is the same as that of $\tilde{N}$ and does not exceed the dimension of U. The greatest rank of the map indicates that there are no singularities on the manifold N.

Tangent space to a manifold

Tangent space to manifold U at point p denoted by $TU|_p$ is the collection of all tangent vectors at point $p \in U$ to all possible curves passing through this point. If U is an n-dimensional manifold, then $TU|_p$ is a n-dimensional vector space with the basis vectors $\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}, ..., \frac{\partial}{\partial y^n}$ in the local coordinates [24].

Tangent bundle

The tangent bundle of a manifold U is the collection of all possible tangent spaces over the manifold [24] i.e.

$$TU := \bigcup_{p\epsilon U} TU|_p.$$

Vector field

A tangent vector $\mathbf{V}|_p$ is assigned to each point $p \in U$ by a vector field $\mathbf{V}$ on U, where $\mathbf{V}|_p$ varies smoothly from point to point on the manifold U. In local coordinates $\mathbf{y} = (y^1, y^2, ..., y^n)$, the vector field has the following form

$$\mathbf{V}|_p = \alpha^1(\mathbf{y})\frac{\partial}{\partial y^1} + \alpha^2(\mathbf{y})\frac{\partial}{\partial y^2} + ... + \alpha^n(\mathbf{y})\frac{\partial}{\partial y^n},$$

where all α^i are the function of $\mathbf{y}$ [24].

Lie bracket

For the vector fields $\mathbf{V_1}$ and $\mathbf{V_2}$ on the manifold U [32], the Lie bracket is defined as

$$[\mathbf{V_1}, \mathbf{V_2}].\phi = \mathbf{V_1}(\mathbf{V_2}.\phi) - \mathbf{V_2}(\mathbf{V_1}.\phi),$$

for all smooth function

$$\phi : U \to \mathbb{R}.$$

The Lie bracket satisfies the following properties

Bi-linearity:

For vector fields $\mathbf{V}_1$, $\mathbf{V}_2$, $\mathbf{V}_3$ and $\mathbf{V}_4$ on any manifold with the constants k_1, k_2, k_3, k_4, the bilinearity condition is

$$[k_1\mathbf{V}_1 + k_2\mathbf{V}_2, k_3\mathbf{V}_3 + k_4\mathbf{V}_4] = k_1k_3[\mathbf{V}_1, \mathbf{V}_3] + k_1k_4[\mathbf{V}_1, \mathbf{V}_4] + k_2k_3[\mathbf{V}_2, \mathbf{V}_3] + k_2k_4[\mathbf{V}_2, \mathbf{V}_4].$$

Skew symmetry

For vector fields $\mathbf{V}_1$ and $\mathbf{V}_2$ on a manifold U, then the following condition

$$[\mathbf{V}_1, \mathbf{V}_2] = -[\mathbf{V}_2, \mathbf{V}_1],$$

holds and is called skew symmetry.

Jacobi identity

If $\mathbf{V}_1$, $\mathbf{V}_2$ and $\mathbf{V}_3$ are vector fields on a manifold, and satisfy the condition

$$[\mathbf{V}_1, [\mathbf{V}_2, \mathbf{V}_3]] + [\mathbf{V}_2, [\mathbf{V}_3, \mathbf{V}_1]] + [\mathbf{V}_3, [\mathbf{V}_1, \mathbf{V}_2]] = 0,$$

which is called the Jacobi identity.

2.1.2 Lie Groups

Definition 2.1.3. *An r-dimensional Lie group is defined as a group G which carries the structure of an r-dimensional manifold in such a way that the following composition function I and inversion function k are smooth for all elements of G*

$$I : G \times G \to G,$$
$$I(g, h) = g.h, \quad g, h \in G.$$
$$k : G \to G,$$
$$k(g) = g^{-1}, \quad g \in G.$$

Lie groups most frequently arises as subgroups of larger groups, for example orthogonal group $SO(2; \mathbb{R})$ of 2×2 matrices with unit determinant, is the subgroup of the general linear group $GL(2; \mathbb{R})$ of all invertible 2×2 matrices. Similarly, the orthogonal group $SO(n; \mathbb{R})$ is a subgroup of general linear invertible matrices $GL(n; \mathbb{R})$. Lie subgroups are groups in their own right [24].

Examples of Lie groups

- (i). The simple example of a Lie group is $G = \mathbb{R}$ i.e., the set of all real numbers which satisfy all the conditions of Lie groups under addition.

- (ii). The group $G = GL(n; \mathbb{R})$ is the set of all $n \times n$ non-singular matrices with real entries form a Lie group under matrix multiplication. The product of two non-singular matrices is again a non-singular matrix, the inverse of each matrix exists as it is non-singular, the identity matrix is the identity of the group and matrix multiplication is always associative.

- (iii). The set $SO(2; \mathbb{R})$ is the set of 2×2 special orthogonal matrices of the form

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

  which is the rotation group in $\mathbb{R}^2$. These matrices form a Lie group.

2.1.3 Lie Algebra

Definition 2.1.4. *An r-dimensional Lie algebra denoted by "g", forms a vector space corresponds to r-parameter Lie group G. It contains all the generators of r-dimensional Lie group, satisfying the conditions of* ***bilinearity, skew symmetry and Jacobi identity*** *defined above. This algebra is said to be abelian if* $[\mathbf{V_i}; \mathbf{V_j}] = 0$ *for all* $\mathbf{V_i}, \mathbf{V_j} \in g$ *[24].*

2.1.4 The Lie Derivatives

Let $\mathbf{v}$ is a vector field on a manifold U [24]. We're often curious about how specific geometric objects on U, such as functions, tensors, differential forms, and other vector fields, change as the flow, $exp(\epsilon\mathbf{v})$, generated by $\mathbf{v}$ varies. So the Lie derivative of an object will clearly inform us about its infinitesimal change when subjected to the flow.

(Using our standard integration methods, we can reconstruct the variation under the flow from this infinitesimal version.) For example, the behavior of a function f under the flow induced by a vector field $\mathbf{v}$ is $\mathbf{v}(f)$, and called the "Lie derivative" of function f with respect to $\mathbf{v}$.

If $\mathbf{w}$ is another field or differential form and $\mathbf{V}$ is a vector field on the manifold U, then the Lie derivative of $\mathbf{w}$ with respect to $\mathbf{V}$ at point $p \in U$ must satisfy the following limit:

$$\mathcal{L}_{\mathbf{V}}(\mathbf{w}) = \mathbf{V}(\mathbf{w})|_p = \lim_{\epsilon \to 0} \frac{\Phi(\mathbf{w}|_{(exp\epsilon\mathbf{V})p}) - \mathbf{w}|_p}{\epsilon}.$$

For two vector fields $\mathbf{V_1}$ and $\mathbf{V_2}$, the Lie derivative of $\mathbf{V_2}$ with respect to $\mathbf{V_1}$ is in fact the Lie bracket

$$[\mathbf{V_1}, \mathbf{V_2}] = \mathbf{V_1}(\mathbf{V_2}) - \mathbf{V_2}(\mathbf{V_1}).$$

Killing Vectors

If the Lie derivative of the metric tensor $\mathbf{g} = g_{ab}\mathbf{e^a} \otimes \mathbf{e^b}$, (additional structure on the manifold which defines distances and angles) whose components g_{ab}, vanishes with respect to vector field $\mathbf{V}$, then $\mathbf{V}$ on the manifold U is Killing vectors, that is [25]

$$\mathcal{L}_{\mathbf{V}}(g_{ab}) := g_{ab,c}\mathbf{V}^c + g_{ac}\mathbf{V}^c_b + g_{cb}\mathbf{V}^c_a = 0, \qquad (a, b, c = 1, 2, ..., n).$$

Homothetic Vectors

If the Lie derivative of the metric tensor whose components g_{ab}, with respect to the vector field $\mathbf{V}$ is equal to a constant multiple of g_{ab}, then $\mathbf{V}$ on the manifold U is a homothety that is [25]

$$\mathcal{L}_{\mathbf{V}}(g_{ab}) := g_{ab,c}\mathbf{V}^c + g_{ac}\mathbf{V}^c_b + g_{cb}\mathbf{V}^c_a = cg_{ab}, \qquad (a, b, c =, 1, 2, ..., n).$$

where c is a constant.

Conformal Killing Vectors

If the Lie derivative of the metric tensor whose components g_{ab}, with respect to the vector field $\mathbf{V}$ is function time g_{ab}, then $\mathbf{V}$ on the manifold U is a conformal Killing vector that is [25]

$$\mathcal{L}_{\mathbf{V}}(g_{ab}) := g_{ab,c}\mathbf{V}^c + g_{ac}\mathbf{V}^c_b + g_{cb}\mathbf{V}^c_a = \phi(x^i)g_{ab}, \qquad (i, a, b, c =, 1, 2, ..., n).$$

2.2 Lie Point Transformation

For coordinates (x, y) where x is independent and y is dependent variable, then the transformation of the form [32]

$$\tilde{x} = \tilde{x}(x, y; \delta) = x + \delta\alpha(x, y) + ..., \tag{2.2.1}$$

$$\tilde{y} = \tilde{y}(x, y; \delta) = y + \delta\beta(x, y) + ..., \tag{2.2.2}$$

where the function α and β are defined as

$$\alpha(x, y) = \frac{\partial \tilde{x}}{\partial \delta}|_{\delta=0}, \qquad \beta(x, y) = \frac{\partial \tilde{y}}{\partial \delta}|_{\delta=0}, \tag{2.2.3}$$

is known as Lie point transformation, where δ is a parameter. These transformations can be extended to the order of DE. For example if we have a DE of the form

$$E(x, y, \dot{y}, \ddot{y}, ..., y^k) = 0,$$

where y^k denotes k^{th} derivative with respect to x, then the Lie point transformation takes the form

$$\tilde{x} = \tilde{x}(x, y; \delta) = x + \delta\alpha(x, y) + ...,$$

$$\tilde{y} = \tilde{y}(x, y; \delta) = y + \delta\beta(x, y) + ...,$$

$$\tilde{\dot{y}} = \tilde{\dot{y}}(x, y, \dot{y}; \delta) = \dot{y} + \delta\beta^1(x, y, \dot{y}) + ...,$$

$$\vdots$$

$$\tilde{y}^k = \tilde{y}^k(x, y, \dot{y}, ..., y^k; \delta) = y^k + \delta\beta^k(x, y, \dot{y}, ..., y^k) + ...$$

If we have n-independent variables $\mathbf{x} = (x^1, x^2, ..., x^n)$ and m-dependent variables $\mathbf{y} = (y^1, y^2, ..., y^m)$ then the above transformation takes the form

$$\begin{aligned}
&\tilde{x}^i = \tilde{x}^i(x, y; \delta) = x^i + \delta\alpha(x, y) + ..., \\
&\tilde{y}^j = \tilde{y}^j(x, y; \delta) = y^j + \delta\beta^j(x, y) + ..., \\
&\tilde{y}^j_{i_1} = \tilde{y}^j_{i_1}(x, y; \delta) = y^j_{i_1} + \delta\beta^j_{i_1}(x, y, y_{i_1}) + ..., \\
&\vdots \\
&\tilde{y}_{i_1,i_2,...,i_k} = \tilde{y}_{i_1,i_2,...,i_k}(x, y, y_{i_1}, ..., y_{i_1,i_2,...,i_k}; \delta) = \\
&y^j_{i_1,i_2,...,i_k} + \delta\beta^j_{i_1,i_2,...,i_k}(x, y, y_{i_1}, ..., y_{i_1,i_2,...,i_k}) + ...,
\end{aligned}$$

where the subscripts denote derivatives with respect to x and superscripts denote coordinates.

2.2.1 Some Cases of Lie Point Transformations

(i). Translation: The transformation

$$\tilde{x} = x + \delta_1 \qquad \tilde{y} = y + \delta_2,$$

is called translation in x and y axis, these transformations form groups called two parameter Lie groups of point transformation.

(ii): Rotation: The transformation of the form

$$\tilde{x} = x\cos\delta_1 - y\sin\delta_2, \qquad \tilde{y} = x\sin\delta_1 + y\cos\delta_2,$$

is called rotation. The area of geometric objects remain invariant under translation and rotation.

(iii): Scaling: The transformation

$$\tilde{x} = e^{\delta_1}x, \qquad \tilde{y} = e^{\delta_2}y,$$

is called scaling, similarity transformation or dilation. The geometric objects are expanded or contracted by this kind of transformation. The expansion or contraction is said to be uniform if $\delta_1 = \delta_2$ and non-uniform otherwise.

Definition 2.2.1. *Two geometrical objects are said to be similar if one is obtained from the other by translation, rotation or scaling transformation on the plane [25].*

Under specific transformation, rectangle is similar to unit square

Any rectangle $\{0 \leq \tilde{x} \leq a, 0 \leq \tilde{y} \leq b\}$ is similar to the unit square: $\{0 \leq \tilde{x} \leq 1, 0 \leq \tilde{y} \leq 1\}$. We can see that the stretching

$$\tilde{x} = \frac{x}{a}, \qquad\qquad \tilde{y} = \frac{y}{b}, \tag{2.2.4}$$

converts the rectangular region $\{0 \leq \tilde{x} \leq a, 0 \leq \tilde{y} \leq b\}$, whose area is ab into a unit square $\{0 \leq \tilde{x} \leq 1, 0 \leq \tilde{y} \leq 1\}$. Using the transformation given by Eq. (2.2.4), one can find the relation between the areas of the two figures as

$$Area = \tilde{x}\tilde{y} = \frac{xy}{ab} \Rightarrow 1 = \frac{xy}{ab} \Rightarrow ab = xy.$$

Hence, rectangle is similar to square by using Eq. (2.2.4). Similarly the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

can be transformed into the unit circle

$$\tilde{x}^2 + \tilde{y}^2 = 1,$$

by the similarity transformation given in Eq. (2.2.4) and their areas are related by

$$\tilde{A} = \frac{A}{ab} \Rightarrow ab\tilde{A} = A \Rightarrow ab\pi = A,$$

where $\tilde{A} = \pi$ is the area of the unit circle and A is the area of the ellipse.

2.3 Lie Symmetries of Ordinary Differential Equations (ODEs)

If x and y are independent and dependent variable respectively, and underlying space is $X \times Y \cong \mathbb{R}^2$ with coordinate (x, y), then corresponding jet space y^k of order k for k^{th}-order DE is given as [32]

$$E(x, y, \dot{y}, \ddot{y}, ..., y^k) = 0, \tag{2.3.1}$$

is $X \times Y^{n+1} \cong \mathbb{R}^{n+2}$ with coordinates $(x, y, \dot{y}, \ddot{y}, ..., y^k)$ where y^k is the k^{th}-derivative of y with respective x. The Lie symmetry generator for the space $X \times Y \cong \mathbb{R}^2$ is

$$\mathbf{V} = \alpha(x, y)\frac{\partial}{\partial x} + \beta(x, y)\frac{\partial}{\partial y}.$$

The Lie generator for the space $X \times Y^{n+1} \cong \mathbb{R}^{n+2}$, for any k^{th} order ODE is

$$\mathbf{V}^{[k]} = \alpha(x, y)\frac{\partial}{\partial x} + \beta(x, y)\frac{\partial}{\partial y} + \beta_x(x, y, \dot{y})\frac{\partial}{\partial \dot{y}} + \beta_{xx}(x, y, \dot{y}, \ddot{y})\frac{\partial}{\partial \ddot{y}} + ...+ \beta_{\underbrace{x, x, ..., x}_{n-times}}(x, y, \dot{y}, \ddot{y}, ..., y^k)\frac{\partial}{\partial y^k}, \tag{2.3.2}$$

above expression is called k^{th} order prolongation of the infinitesimal generator. Where

$$\beta_x(x, y, \dot{y}) = \frac{d}{dx}\beta(x, y) - \dot{y}\frac{d}{dx}\alpha(x, y),$$
$$\beta_{xx}(x, y, \dot{y}) = \frac{d}{dx}\beta_x(x, y, \dot{y}) - \ddot{y}\frac{d}{dx}\alpha(x, y),$$
$$\vdots$$
$$\beta_{\underbrace{x, x, ..., x}_{n-times}}(x, y, \dot{y}, ..., y^k) = \frac{d}{dx}\beta_{\underbrace{x, x, ..., x}_{n-times}}(x, y, \dot{y}, ..., y^{k-1}) - y^k\frac{d}{dx}\alpha(x, y),$$

and the total derivative operator is

$$D_x = \frac{d}{dx} = \frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial \dot{y}} + \ddot{y}\frac{\partial}{\partial \ddot{y}} + ... + y^k\frac{\partial}{\partial y^{k-1}}. \tag{2.3.3}$$

Exmple (Lie symmetries of second order ODE)

Now, we find the Lie symmetries of second order ODE i.e.,

$$\ddot{y} + y = 0. \tag{2.3.4}$$

The jet space for this differential equation is $X \times Y^3 = \mathbb{R}^4$, its jet coordinates are $(x, y, \dot{y}, \ddot{y})$. The infinitesimal generator takes the form

$$\mathbf{V} = \alpha(x,y)\frac{\partial}{\partial x} + \beta(x,y)\frac{\partial}{\partial y},$$

and the corresponding second order prolongation of symmetry generator is

$$\mathbf{V}^{[2]} = \alpha(x,y)\frac{\partial}{\partial x} + \beta(x,y)\frac{\partial}{\partial y} + \beta_x(x,y,\dot{y})\frac{\partial}{\partial \dot{y}} + \beta_{xx}(x,y,\dot{y},\ddot{y})\frac{\partial}{\partial \ddot{y}}. \tag{2.3.5}$$

Apply the generator given by Eq. (2.3.5) to the ODE given by Eq. (2.3.1) and using the values of β_x and β_{xx} in terms of β we have the following system of PDEs,

$$\begin{aligned}
&\alpha_{yy} = 0, && \alpha_{xxy} + \alpha_y = 0,\\
&\alpha_{xxx} - 3y\alpha_{xy} + 4\alpha_x = 0, && \beta_{yy} - 2\alpha_{xy} = 0,\\
&\beta_{yx} + 3y\alpha_y - \alpha_{xx} = 0, && \beta_{xx} - y\beta_y + 2y\alpha_x + \beta = 0.
\end{aligned}$$

Solution of above system is given by

$$\begin{aligned}
\alpha(x,y) &= Ay\sin x + By\cos x + C\sin 2x + D\cos 2x + E,\\
\beta(x,y) &= Ay^2\cos x - By^2\sin x + Cy\cos 2x - Dy\sin 2x + Fy + G\sin x + H\cos x.
\end{aligned}$$

This solution represents eight parameter Lie group of transformation. The corresponding symmetry generators are

$$\begin{aligned}
\mathbf{V}_1 &= y\sin x\frac{\partial}{\partial x} + y^2\cos x\frac{\partial}{\partial y}, & \mathbf{V}_2 &= y\cos x\frac{\partial}{\partial x} - y^2\sin x\frac{\partial}{\partial y},\\
\mathbf{V}_3 &= \sin 2x\frac{\partial}{\partial x} + y\cos 2x\frac{\partial}{\partial y}, & \mathbf{V}_4 &= \cos 2x\frac{\partial}{\partial x} - y\sin 2x\frac{\partial}{\partial y},\\
\mathbf{V}_5 &= \cos x\frac{\partial}{\partial y}, & \mathbf{V}_6 &= \sin x\frac{\partial}{\partial y},\\
\mathbf{V}_7 &= y\frac{\partial}{\partial y}, & \mathbf{V}_8 &= \frac{\partial}{\partial x}.
\end{aligned} \tag{2.3.6}$$

The Lie algebra is

$$[\mathbf{V}_1,\mathbf{V}_2]=-\mathbf{V}_1,\quad [\mathbf{V}_1,\mathbf{V}_3]=-\mathbf{V}_2,\quad [\mathbf{V}_1,\mathbf{V}_5]=-\frac{1}{2}\mathbf{V}_3-\frac{3}{2}\mathbf{V}_7,$$
$$[\mathbf{V}_1,\mathbf{V}_6]=-\frac{1}{2}\mathbf{V}_4-\frac{1}{2}\mathbf{V}_8,\quad [\mathbf{V}_1,\mathbf{V_7}]=-\mathbf{V_1},\quad [\mathbf{V_1},\mathbf{V}_8]=-\mathbf{V}_2,$$
$$[\mathbf{V}_2,\mathbf{V}_3]=\mathbf{V}_2,\quad [\mathbf{V}_2,\mathbf{V}_4]=-\mathbf{V}_1,\quad [\mathbf{V}_2,\mathbf{V}_5]=-\frac{1}{2}\mathbf{V}_4-\frac{1}{2}\mathbf{V}_8,$$
$$[\mathbf{V}_2,\mathbf{V}_6]=-\frac{3}{2}\mathbf{V}_7-\frac{1}{2}\mathbf{V}_3,\quad [\mathbf{V}_2,\mathbf{V}_7]=\mathbf{V}_2,\quad [\mathbf{V}_2,\mathbf{V}_8]=-\mathbf{V}_1,$$
$$[\mathbf{V}_3,\mathbf{V}_4]=-\mathbf{V}_8,\quad [\mathbf{V}_3,\mathbf{V}_5]=-\mathbf{V}_5,\quad [\mathbf{V}_3,\mathbf{V}_6]=\mathbf{V}_6,$$
$$[\mathbf{V}_3,\mathbf{V}_8]=-2\mathbf{V}_4,\quad [\mathbf{V}_4,\mathbf{V}_5]=\mathbf{V}_6,\quad [\mathbf{V}_4,\mathbf{V}_6]=\mathbf{V}_5,$$
$$[\mathbf{V}_4,\mathbf{V}_8]=2\mathbf{V}_3,\quad [\mathbf{V}_5,\mathbf{V}_7]=\mathbf{V}_5,\quad [\mathbf{V}_5,\mathbf{V}_8]=\mathbf{V}_6,$$
$$[\mathbf{V}_6,\mathbf{V}_7]=\mathbf{V}_6\quad [\mathbf{V}_6,\mathbf{V}_8]=\mathbf{V}_5\quad [\mathbf{V}_i,\mathbf{V}_j]=0,\quad otherwise.$$

The Lie group corresponding to $\mathbf{V}_8=\dfrac{\partial}{\partial x}$ is calculated as

$$\frac{\partial\tilde{x}}{\partial\delta}|_{\delta=0}=1,\qquad \frac{\partial\tilde{y}}{\partial\delta}|_{\delta=0}=0,$$
$$\tilde{x}(0)=x,\qquad \tilde{y}(0)=y.$$

The solution of above DEs gives the Lie group G_8. The other one parameter Lie groups are (δ is the parameter)

$$G_1:\quad \left[x+\delta y\sin x,\quad \frac{y}{1-\delta y\cos x}\right],$$
$$G_2:\quad \left[x+\delta y\cos x,\quad \frac{y}{1+\delta y\sin x}\right],$$
$$G_3:\quad [arc\tan(\tan x.\exp(2\delta)),\quad y\exp(\delta\cos 2x)],$$
$$G_4:\quad [arc\tan(\tan(x-\frac{\pi}{4}).\exp(2\delta))+\frac{\pi}{4},\quad y\exp(-\delta\sin 2x)],$$
$$G_5:\quad [x,\quad y+\delta\cos x],$$
$$G_6:\quad [x,\quad y+\delta\sin x],$$
$$G_7:\quad [x,\quad \exp(\delta y)],$$
$$G_8:\quad [x+\delta,\quad y].$$

2.3.1 The Lie Symmetries of Partial Differential Equations (PDEs)

Let $\mathbf{x} = (x^1, x^2, x^3, ..., x^n)$ be independent and $\mathbf{z} = (z^1, z^2, z^3, ..., z^m)$ be dependent variables. Then the Euclidean space is $\mathbf{x} \times \mathbf{z} = \mathbb{R}^{n+m}$, and in coordinate form (x, z), corresponding k^{th} order jet space is $\mathbf{x} \times \mathbf{z}_{i_1,i_2,...,i_k}$ having coordinate $(x, z, z_{i_1}, ..., z_{i_1,i_2,...,i_k})$, where subscript shows the derivatives. The PDE in k^{th} order is given below

$$g(\mathbf{x}, \mathbf{z}, \mathbf{z}_{i_1}, \mathbf{z}_{i_1,i_2}, ..., \mathbf{z}_{i_1,i_2,...,i_k}) = 0.$$

The Lie generator for this PDE is

$$\mathbf{V} = \alpha^i \frac{\partial}{\partial x^i} + \beta^j \frac{\partial}{\partial z^j},$$

and prolongation upto k^{th} order is

$$\mathbf{V}^{[k]} = \alpha^i \frac{\partial}{\partial x^i} + \beta^j \frac{\partial}{\partial z^j} + \beta^j_{i_1} \frac{\partial}{\partial z^j_{i_1}} + \beta^j_{i_1,i_2} \frac{\partial}{\partial z^j_{i_1,i_2}} + ... + \beta^j_{i_1,i_2,...,i_k} \frac{\partial}{\partial z^j_{i_1,i_2,...,i_k}}.$$

Example 5

To illustrate the Lie point symmetries, consider one-dimensional heat equation

$$\phi_t(t, x) - \phi_{xx}(t, x) = 0. \tag{2.3.7}$$

It is linear second order PDE, its solution space $\mathbf{X} \times \Phi$ with coordinate (t, x, ϕ) and corresponding jet space is $\mathbf{X} \times \Phi^2$ having jet coordinate $(t, x, \phi, \phi_t, \phi_x.\phi_{tt}, \phi_{tx}, \phi_{xx})$. The symmetry generator for the solution space will be

$$\mathbf{V} = \alpha^1(t, x, \phi) \frac{\partial}{\partial t} + \alpha^2(t, x, \phi) \frac{\partial}{\partial x} + \beta(t, x, \phi) \frac{\partial}{\partial \phi},$$

and the second order prolonged generator is

$$\mathbf{V}^{[2]} = \alpha^1 \frac{\partial}{\partial t} + \alpha^2 \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial \phi} + \beta_t \frac{\partial}{\partial \phi_t} + \beta_x \frac{\partial}{\partial \phi_x} + \beta_{tt} \frac{\partial}{\partial \phi_{tt}} + \beta_{tx} \frac{\partial}{\partial \phi_{tx}} + \beta_{xx} \frac{\partial}{\partial \phi_{xx}}. \tag{2.3.8}$$

Apply the generator given by Eq. (2.3.8) on the PDE given by Eq. (2.3.7) and using the values of β_t, β_{xx} in terms of β and α^i we have the following system of determining equations,

$$
\begin{aligned}
&\alpha^2_\phi = 0, && \alpha^2_x = 0, && \alpha^2_{\phi\phi} = 0,\\
&\alpha^2_{x\phi} + \alpha^1_\phi = 0, && 2\alpha^2_{x\phi} - \beta_{\phi\phi} = 0, && \alpha^1_{\phi\phi} = 0,\\
&\alpha^1_{xx} - \alpha^1_t - 2\beta_{x\phi} = 0, && \beta_t - \beta_{xx} = 0, &&\\
&2\alpha^1_x - \alpha^2_t + \alpha^2_{xx} - \beta_\phi = 0. && &&
\end{aligned} \tag{2.3.9}
$$

The solution of this system is

$$
\begin{aligned}
&\alpha^1 = 2d_1t + 4d_3t^2 + d_4,\\
&\alpha^2 = d_1x + 2d_2t + 4d_3xt + d_5,\\
&\beta = -d_2x\phi - 2d_3t\phi - d_3\phi x^2 + d_6\phi + f(t,x).
\end{aligned} \tag{2.3.10}
$$

Since, we have an arbitrary function $f(t,x)$ in the solution, therefore algebra is infinite dimensional here. The Lie symmetry generators are

$$
\begin{aligned}
&\mathbf{V}_1 = x\frac{\partial}{\partial x} + 2t\frac{\partial}{\partial t}, && \mathbf{V}_2 = 2t\frac{\partial}{\partial x} - x\phi\frac{\partial}{\partial \phi},\\
&\mathbf{V}_3 = 4tx\frac{\partial}{\partial x} + 4t^2\frac{\partial}{\partial t} - \phi(2t + x^2)\frac{\partial}{\partial \phi}, && \mathbf{V}_4 = \frac{\partial}{\partial t},\\
&\mathbf{V}_5 = \frac{\partial}{\partial x}, && \mathbf{V}_6 = \phi\frac{\partial}{\partial \phi},\\
&\mathbf{V}_\beta = f(t,x)\frac{\partial}{\partial \phi}, &&
\end{aligned} \tag{2.3.11}
$$

where β is an arbitrary constant. The Lie groups are

$$
\begin{aligned}
&G_1: && [\exp(2\delta)t, && \exp(\delta)x, && \phi],\\
&G_2: && [t, && x + 2\delta t, && \phi\exp(-\delta x - \delta^2 t)],\\
&G_3: && \left[\frac{t}{1-4\delta t},\right. && \frac{t}{1+\delta t}, && \left.\phi\sqrt{1-4\delta t}\exp\left(\frac{-\delta x^2}{1-\delta t}\right)\right],\\
&G_4: && [t+\delta, && x, && \phi],
\end{aligned}
$$

$$G_5: \quad [t, \quad x+\delta, \quad \phi],$$
$$G_6: \quad [t, \quad x, \quad \exp(\delta\phi)],$$
$$G_\beta: \quad [t, \quad x, \quad \phi+\delta f(t,x)].$$

2.3.2 Approximate Lie Groups and their Symmetry Generators

Sometimes physical problems admit approximation. For instance, in free fall, we disregard air friction. Similar to how air friction affects simple pendulums, which lose their motion. Friction on the surface that a body coupled to a spring is moving on, slows down simple harmonic motion. The exact Lie group theoretic approach to the solution of DEs is highly sensitive to very small perturbations in physical systems (in this case, the air resistance and friction between the spring and surface on which it moves) [30]. Thus, using the Lie group technique to solve DEs in these physical problems is not effective. The instability of the Lie group theoretic approach to the solution of DEs was fortunately reduced by the development and application of an approximate Lie group method [34, 35, 36].

Definition 2.3.1. *An approximate transformation of order p in $\mathbb{R}^n$ can be written as [5]*

$$x^i \to \tilde{x}^i \approx \tilde{x}^i_0(x^m,\delta) + \epsilon\tilde{x}^i_1(x^m,\delta) + ... + \epsilon^p\tilde{x}^i_p(x^m,\delta), \tag{2.3.12}$$

which obey the initial conditions

$$\tilde{x}^i_j|_{\delta=0} \approx x^i_j, \quad \forall i = 1,2,...,n.$$

Approximate symmetry group generators of order p

The generator of the approximate transformation given in equation Eq. (2.3.12) is of the form

$$\mathbf{V} = \alpha^i(\mathbf{x}, \delta)\frac{\partial}{\partial x^i}, \tag{2.3.13}$$

such that

$$\begin{aligned} &\alpha^i(\mathbf{x}, \delta) \approx \alpha_0^i(\mathbf{x}) + \delta\alpha_1^i(\mathbf{x}) + ... + \delta^p\alpha_p^i(\mathbf{x}), \\ &\alpha_j^i(x) = \frac{\partial}{\partial\delta}\tilde{x}_j^i|_{\delta=0}, \end{aligned} \tag{2.3.14}$$

then the generator given in equation Eq. (2.3.13) takes the form

$$\mathbf{V} = (\alpha_0^i(\mathbf{x}) + \delta\alpha_1^i(\mathbf{x}) + ... + \delta^p\alpha_p^i(\mathbf{x}))\frac{\partial}{\partial x^i}, \tag{2.3.15}$$

where δ is small parameter.

First order approximation

The symmetry generator of the form

$$\mathbf{V} = \alpha^i\frac{\partial}{\partial x^i}, \tag{2.3.16}$$

is said to be of the first order if $\alpha^i = \alpha_0^i + \delta\alpha_1^i$, where δ is a small arbitrary parameter. The generator in Eq. (2.3.15) splits into two parts as

$$\begin{aligned} \mathbf{V}^0 &= \alpha_0^i\frac{\partial}{\partial x^i}, \\ \mathbf{V}^1 &= \alpha_1^i\frac{\partial}{\partial x^i}. \end{aligned}$$

where $\mathbf{V}^0$ is the exact and $\mathbf{V}^1$ is the approximate part of the symmetry generator given in Eq. (2.3.16). The corresponding approximate transformation group of point x into $\tilde{x}$ is

$$x^i \to \tilde{x}^i \approx \tilde{x}_0^i(x^m, \delta) + \epsilon\tilde{x}_1^i(x^m, \delta). \tag{2.3.17}$$

Example 6

Now we find the approximate Lie group of first order approximate (one dimensional) symmetry generator

$$\mathbf{V} = (y^2 + \epsilon y)\frac{\partial}{\partial y}, \tag{2.3.18}$$

has two parts as

$$\mathbf{V}^0 = y^2\frac{\partial}{\partial y}, \qquad \mathbf{V}^1 = y\frac{\partial}{\partial y},$$

where $\mathbf{V}^0$ is the exact and $\mathbf{V}^1$ is the approximate symmetry. Here $\alpha_0 = y^2$ and $\alpha_1 = y$. The corresponding approximate Lie equations are

$$\begin{aligned} \frac{d\tilde{y}_0}{d\delta} &= y^2, \qquad \tilde{y}_0|_{\delta=0} = y, \\ \frac{d\tilde{y}_1}{d\delta} &= \tilde{y}_0, \qquad \tilde{y}_1|_{\delta=0} = 0, \end{aligned} \tag{2.3.19}$$

then the solution to this system is

$$\tilde{y}_0 = \delta y^2 + y, \qquad \tilde{y}_1 = \frac{\delta^2 y^2}{2} + \delta y,$$

and

$$\tilde{y} = \delta y^2 + y + \epsilon\left(\frac{\delta^2 y^2}{2} + \delta y\right).$$

is the approximate Lie group of transformation.

Example 7

Now consider a two dimensional first order approximate symmetry generator

$$\mathbf{V} = (1 + \epsilon x)\frac{\partial}{\partial x} + \epsilon y\frac{\partial}{\partial y},$$

which can be splits into two parts as

$$\mathbf{V}^0 = \frac{\partial}{\partial x}, \qquad \qquad \mathbf{V}^1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y},$$

where $\mathbf{V}^0$ is the exact and $\mathbf{V}^1$ is the approximate symmetry. The corresponding approximate Lie equations are

$$\begin{aligned} &\frac{d\tilde{x}_0}{d\delta} = 1, \quad \tilde{x}_0|_{\delta=0} = x, && \frac{d\tilde{y}_0}{d\delta} = 0, \quad \tilde{y}_0|_{\delta=0} = y, \\ &\frac{d\tilde{x}_1}{d\delta} = \tilde{x}_0, \quad \tilde{x}_1|_{\delta=0} = 0, && \frac{d\tilde{y}_1}{d\delta} = 0, \quad \tilde{y}_1|_{\delta=0} = 0. \end{aligned} \tag{2.3.20}$$

The solution of this system is

$$\begin{aligned} &\tilde{x}_0 = x + \delta, && \tilde{x}_1 = \delta x + \frac{\delta^2}{2}, \\ &\tilde{y}_0 = y, && \tilde{y}_1 = \delta y, \end{aligned}$$

and

$$\tilde{x} = x + \delta + \epsilon\left(\delta x + \frac{\delta^2}{2}\right), \qquad \qquad \tilde{y} = y + \delta y.$$

are the approximate Lie groups corresponding to above system.

2.4 The Euler-Lagrange Equation

A class of Lie symmetries that correspond to conserved quantities are the Noether symmetries. The Noether symmetry is an invariance of action integral under infinitesimal transformation of a group. Noether symmetries are the symmetries of a variational problem, which represents a physical system and can be written in an integral form, which is known as the action of problem [26]. For example, consider length of a curve of function $q(t)$; from a point $(a, q(a))$ to another point $(b, q(b))$ that is

$$S = \int_a^b \sqrt{1 + \dot{q}(t)^2} \; dt, \tag{2.4.1}$$

where "." denotes differentiation with respect to t. The extremal value for S, $q(t)$ must be a straight line. In order to show that $q(t)$ is a straight line, we shift $q(t)$ from its minimum value by $\epsilon u(t)$ that is

$$q(t) \to q(t) + \epsilon u(t), \tag{2.4.2}$$

ϵ is an arbitrary small parameter, and $u(t)$ is arbitrary function satisfying $u(a) = u(b) = 0$. The integral given in Eq. (2.4.1) takes the form

$$S = \int_a^b \sqrt{1 + (\dot{q}(t) + \epsilon u(t))^2}\ \ dt. \tag{2.4.3}$$

Taking derivative of Eq. (2.4.3) with respect to ϵ, we have

$$\begin{aligned} \frac{d}{d\epsilon} S &= \frac{d}{d\epsilon} \int_a^b \sqrt{1 + (\dot{q}(t) + \epsilon u(t))^2} \quad dt, \\ &= \int_a^b \frac{\dot{q} + \epsilon \dot{u}}{\sqrt{1 + (\dot{q}(t) + \epsilon u(t))^2}} \dot{u}|_{\epsilon=0} \quad dt, \\ &= \int_a^b \frac{\dot{q}(t)}{\sqrt{1 + \dot{q}(t)^2}} \dot{u} \quad dt. \end{aligned} \tag{2.4.4}$$

Integrating the right hand side with respect to t, we have

$$\begin{aligned} \frac{d}{d\epsilon} S &= \frac{\dot{q}(t)}{\sqrt{1 + \dot{q}(t)^2}} u|_a^b - \int_a^b \frac{d}{dt} \left[\frac{\dot{q}(t)}{\sqrt{1 + \dot{q}(t)^2}} \right] u dt, \\ &= - \int_a^b \frac{d}{dt} \left[\frac{\dot{q}(t)}{\sqrt{1 + \dot{q}(t)^2}} \right] u dt. \end{aligned} \tag{2.4.5}$$

In order to get extremal values of S, we impose conditions on the function $q(t)$ through variation.

$$\frac{d}{d\epsilon} S = -\frac{d}{dt} \left[\frac{\dot{q}(t)}{\sqrt{1 + \dot{q}(t)^2}} \right] = 0,$$

Since $u(t)$ is continuous, therefore, by the fundamental theorem of calculus of variations gives

$$-\frac{d}{dt} \left[\frac{\dot{q}(t)}{\sqrt{1 + \dot{q}(t)^2}} \right] = 0, \tag{2.4.6}$$

as a result, $q(t)$ must be a straight line. If we take

$$\mathcal{L}(t, q(t), \dot{q}(t)) = \sqrt{1 + \dot{q}(t)^2},$$

then Eq. (2.4.6) can be written as

$$\frac{\partial \mathcal{L}}{\partial q(t)} - \frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}(t)}\right) = 0, \tag{2.4.7}$$

where the function $\mathcal{L}$ is called Lagrangian density, Eq. (2.4.7) is the Euler-Lagrange equation. The calculation might be applied to any general Lagrangian in this specific variational problem given general coordinates and derivatives of any order. The Euler-Lagrange equation impose limitations on the variables that can be used in the action. This equation is solved to give an extremal of action, can be obtained from the variation in the action.

2.4.1 Geodesics

Assume that Σ is a surface and x_0 and x_1 are two points on it. Finding curves on Σ with endpoints x_0 and x_1 having minimum arc length is the focus of the geodesics problem. A curve with this characteristic is referred to as a geodesic. The theory of geodesics is one of the most advanced concepts in differential geometry. The general theory is complicated by the fact that common, simple surfaces such as the sphere require many vector functions to represent them analytically. The sphere is a manifold in geometry that requires at least two charts.

Geodesics on the plane

Let $(x_0, y_0) = (0, 0)$ and $(x_1, y_1) = (1, 1)$. The arc length of a curve described by $y(t)$, $t \in [0, 1]$ is given by

$$J(y) = \int_0^1 \sqrt{(1 + \dot{y}^2)}dx. \tag{2.4.8}$$

The geodesic problem on the plane consists of getting the function y that minimizes the arc length. Our investigation is limited to functions in $\mathbb{C}^2[0,1]$ such that

$$y(0) = 0, \qquad y(1) = 1. \tag{2.4.9}$$

The Euler-Lagrange equation must be satisfied if y is an extremal for J; hence,

$$\frac{d}{dt}\left(\frac{\partial J}{\partial \dot{y}}\right) - \frac{\partial J}{\partial y} = \frac{d}{dt}\left(\frac{\dot{y}}{\sqrt{1+\dot{y}^2}}\right) - 0 = 0, \tag{2.4.10}$$

$$\frac{\dot{y}}{\sqrt{1+\dot{y}^2}} = constant.$$

The last equation is equivalent to the condition that $\dot{y} = c_1$, where c_1 is a constant. As a result, an extremal for J must be of the form

$$y(t) = At + B \tag{2.4.11}$$

where B is another constant of integration. Since $y(0) = 0$, we see that $B = 0$, and since $y(1) = 1$, we see that $A = 1$. Consequently, $y(t) = t$ which describes the line segment from $(0,0)$ to $(1,1)$ in the plane, is the only extreme value of y [26].

2.5 Legendre Transformation

The contact transformations are defined by functions that depend on the derivatives of the dependent variable. These transformation are significant in DEs and geometry [26]. The Legendre transformation is one of the most simple and effective contact transformations. This transformation has a remarkable property that it links Euler-Lagrange equations and Hamilton's equations ($\dot{p}_a = -\frac{\partial H}{\partial \dot{q}_a}, \dot{q}_a = \frac{\partial H}{\partial \dot{p}_a}$), which also has several remarkable characteristics. We start by thinking about the most basic Legendre transformation with a single independent variable.

Let $y : [x_0, x_1] \to \mathbb{R}$ be a smooth function, and define the new variable by

$$q = \dot{y}(t). \tag{2.5.1}$$

If $\ddot{y}(t) \neq 0$, then Eq. (2.5.1) can be used to define the variable t in terms of q. For definiteness, let us suppose that

$$\ddot{y}(t) > 0, \tag{2.5.2}$$

for all $t \in [x_0, x_1]$. Inequality given in Eq. (2.5.2) implies that the curve described by $\mathbf{r}(t) = (t, y(t)), t \in [x_0, x_1]$ is strictly convex upwards in shape. The slope of the tangent line is represented by the new variable q. Geometrically, any point on a curve is under these conditions uniquely specified by the slope of its tangent line. Suppose that, we introduce the function

$$\mathcal{H}(q) = -y(t) + qt. \tag{2.5.3}$$

The transformation from the pair $(t, y(t))$ to $(q, \mathcal{H}(q))$ is given by Eqs. (2.5.1) and (2.5.3). This is an example of a Legendre transformation. This transformation has its own inverse i.e., it is an involution. For this, note that

$$\begin{aligned}
\frac{d\mathcal{H}}{dq} &= -\frac{d}{dq}y(t) + \frac{d}{dq}(qt), \\
&= \frac{dy}{dt}\frac{dt}{dq} + q\frac{dt}{dq} + t, \\
&= -(\dot{y}(t) + q)\frac{dt}{dq} + t = t,
\end{aligned}$$

where we have used Eq. (2.5.1). Also note that

$$-\mathcal{H}(q) + tq = -(-y(t) + qt) + qt = y(t). \tag{2.5.4}$$

From Eqs. (2.5.3) and (2.5.4), it is clear that one can obtain the original pair $(t, y(t))$ of transformation by applying the Legendre transformation to the pair $(q, \mathcal{H}(q))$.

Now, we apply Legendre transformation on the function $y(t)$. Suppose

$$y(t) = \frac{t^4}{4}. \tag{2.5.5}$$

Then using Eq. (2.5.1)

$$q = \frac{dy}{dt} = t^3. \tag{2.5.6}$$

So that, $t = q^{\frac{1}{3}}$. The function $\mathcal{H}$ is obtained by Eq. (2.5.3)

$$\mathcal{H}(q) = -\frac{t^4}{4} + qt = \frac{3}{4}q^{\frac{4}{3}}.$$

Taking derivative, we get

$$\dot{\mathcal{H}}(q) = \frac{4}{3}\left(\frac{3}{4}q^{\frac{4}{3}}\right) = q^{\frac{1}{3}} = t,$$

and that

$$\begin{aligned} -\mathcal{H}(q) + tq &= \frac{3}{4}q^{\frac{4}{3}} + tq, \\ &= -\frac{3}{4}t^4 + t^4 = y(t). \end{aligned}$$

Hence proved that $y(t)$ and $\mathcal{H}(q)$ are inverse of each other.

2.6 The Mei Symmetries

A technique to find the Mei symmetries of Euler Lagrange equation of the Lagrangian was described by Zhai and Zhang [20]. The summary of the method is outlined here. Consider the Euler Lagrange equation of the corresponding Lagrangian $\mathcal{L}(t, x^a, \dot{x}^a)$,

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right) - \frac{\partial \mathcal{L}}{\partial x^a} = 0, \qquad (a = 1, 2, ..., n). \tag{2.6.1}$$

Writing

$$E^a = \frac{d}{dt}\left(\frac{\partial}{\partial \dot{x}^a}\right) - \frac{\partial}{\partial x^a}, \qquad (a = 1, 2, ..., n). \tag{2.6.2}$$

Eq. (2.6.1) takes the form

$$E^a(\mathcal{L}) = 0, \qquad (a = 1, 2, ..., n). \tag{2.6.3}$$

Consider the infinitesimal group of transformations associated with a single parameter

$$t^* = T(t, x^i(t), \delta) = t + \delta\alpha(t, x^i(t)),$$
$$x^{a*} = X^a(t, x^i(t), \delta) = x^a(t) + \delta\beta^a(t, x^i(t)), \ (a, i = 1, 2, ..., n), \tag{2.6.4}$$

corresponding generator $\mathbf{V}$ is given as

$$\mathbf{V} = \alpha\frac{\partial}{\partial t} + \beta^a\frac{\partial}{\partial x^a}. \tag{2.6.5}$$

The Lagrangian $\mathcal{L}$ is replaced by a new Lagrangian $\mathcal{L}^*$ under the transformation

$$\mathcal{L}^* = \mathcal{L}^*(t^*, x^{a*}, \frac{dx^{a*}}{dt^*}) = \mathcal{L}\bigg(t + \delta\alpha, x^a + \delta\beta^a, \frac{\dot{x}^a + \delta\dot{\beta}^a}{1 + \delta\dot{\alpha}}\bigg), \tag{2.6.6}$$

where "." represents the derivative with respect to independent variable t. After that, we apply Taylor's series expansion at $\delta = 0$ to get

$$\mathcal{L}^* = \mathcal{L}(t, x^a, \dot{x}^a) + \delta\mathbf{V}^{[1]}\mathcal{L} + O(\delta^2), \tag{2.6.7}$$

where

$$\mathbf{V}^{[1]} = \alpha\frac{\partial}{\partial t} + \beta^a\frac{\partial}{\partial x^a} + (\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{\partial}{\partial \dot{x}^a}, \tag{2.6.8}$$

is the first extension of the infinitesimal generator [20].

Definition 2.6.1. *Mathematically, Mei symmetries of the Lagrangian are defined as*

$$E^a(\mathcal{L}^*) = 0, \qquad (a = 1, 2, ..., n). \tag{2.6.9}$$

When the Lagrangian $\mathcal{L}$ is replaced with transformed Lagrangian $\mathcal{L}^*$, and Eq. (2.6.3) remains the same, then it is referred to as the Mei symmetries of the relative Lagrangian [20].

Definition 2.6.2. *If α and β^a satisfy the given condition*

$$E^a[\mathbf{V}^{[1]}\mathcal{L}] = 0, \qquad (a = 1, 2, ..., n). \tag{2.6.10}$$

The given invariance is called the Mei symmetry. After solving Eq. (2.6.10), we get a system of PDEs. The solution of this system gives α and β^a which satisfy the given criterion of Mei symmetries [20].

2.7 Relation between Lie and Mei Symmetries

To develop the relation between Lei and Mei symmetries [16], consider $\mathbf{V}^{[1]}\mathcal{L}$

$$\mathbf{V}^{[1]}\mathcal{L} = \alpha\frac{\partial\mathcal{L}}{\partial t} + \beta^a\frac{\partial\mathcal{L}}{\partial x^a} + (\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{\partial\mathcal{L}}{\partial\dot{x}^a}. \tag{2.7.1}$$

Now, applying Euler operator on Eq. (2.7.1), we get

$$\frac{\partial}{\partial\dot{x}^k}(\mathbf{V}^{[1]}\mathcal{L}) = \frac{\partial\alpha}{\partial\dot{x}^k}\frac{\partial\mathcal{L}}{\partial t} + \alpha\frac{\partial^2\mathcal{L}}{\partial\dot{x}^k\partial t} + \beta^a\frac{\partial^2\mathcal{L}}{\partial\dot{x}^k\partial x^a} + \frac{\partial\beta^a}{\partial\dot{x}^k}\frac{\partial\mathcal{L}}{\partial x^a} + \frac{\partial}{\partial\dot{x}^k}(\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{\partial\mathcal{L}}{\partial\dot{x}^a}$$
$$+ (\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{\partial}{\partial\dot{x}^k}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right),$$
$$\frac{d}{dt}\frac{\partial}{\partial\dot{x}^k}(\mathbf{V}^{[1]}\mathcal{L}) = \frac{d}{dt}\left(\frac{\partial\alpha}{\partial\dot{x}^k}\right)\frac{\partial\mathcal{L}}{\partial t} + \left(\frac{\partial\alpha}{\partial\dot{x}^k}\right)\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial t}\right) + \dot{\alpha}\frac{\partial^2\mathcal{L}}{\partial\dot{x}^k\partial t} + \alpha\frac{d}{dt}\left(\frac{\partial^2\mathcal{L}}{\partial\dot{x}^k\partial t}\right)$$
$$+ \dot{\beta}^a\frac{\partial^2\mathcal{L}}{\partial\dot{x}^k\partial x^a} + \beta^a\frac{d}{dt}\left(\frac{\partial^2\mathcal{L}}{\partial\dot{x}^k\partial x^a}\right) + \frac{d}{dt}\left(\frac{\partial\beta^a}{\partial\dot{x}^k}\right)\frac{\partial\mathcal{L}}{\partial x^a} + \frac{\partial\beta^a}{\partial\dot{x}^k}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial x^a}\right) + \frac{\partial\mathcal{L}}{\partial\dot{x}^a}\left[\frac{d}{dt}\frac{\partial}{\partial\dot{x}^k}\right.$$
$$\left.(\dot{\beta}^a - \dot{x}^a\dot{\alpha})\right] + \frac{\partial}{\partial\dot{x}^k}(\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right) + \frac{d}{dt}(\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{\partial}{\partial\dot{x}^k}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right) + (\dot{\beta}^a - \dot{x}^a\dot{\alpha})$$
$$\frac{d}{dt}\left[\frac{\partial}{\partial\dot{x}^k}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right)\right],$$

and

$$\frac{\partial}{\partial x^k}(\mathbf{V}^{[1]}\mathcal{L}) = \frac{\partial\alpha}{\partial x^k}\frac{\partial\mathcal{L}}{\partial t} + \alpha\frac{\partial^2\mathcal{L}}{\partial x^k\partial t} + \frac{\partial\beta^a}{\partial x^k}\frac{\partial\mathcal{L}}{\partial x^a} + \beta^a\frac{\partial^2\mathcal{L}}{\partial x^k\partial x^a} + \frac{\partial}{\partial x^k}(\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{\partial\mathcal{L}}{\partial\dot{x}^a}$$
$$+ (\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{\partial}{\partial x^k}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right).$$
$$E^k(\mathbf{V}^{[1]}\mathcal{L}) = E^k(\alpha)\frac{\partial\mathcal{L}}{\partial t} + E^k(\beta^a)\frac{\partial\mathcal{L}}{\partial x^a} + E^k(\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{\partial\mathcal{L}}{\partial\dot{x}^a} + \alpha E^k\left(\frac{\partial\mathcal{L}}{\partial t}\right)$$
$$+ \frac{\partial\alpha}{\partial\dot{x}^k}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial t}\right) + \dot{\alpha}\frac{\partial^2\mathcal{L}}{\partial\dot{x}^k\partial t} + \dot{\beta}^a\frac{\partial^2\mathcal{L}}{\partial\dot{x}^k\partial x^a} + \frac{\partial\beta^a}{\partial\dot{x}^k}\left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial x^a}\right)\right] + \frac{\partial}{\partial\dot{x}^k}(\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right)$$
$$+ \frac{d}{dt}(\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{\partial}{\partial\dot{x}^k}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right) + (\dot{\beta}^a - \dot{x}^a\dot{\alpha})E^k\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right) + \beta^a E^k\left(\frac{\partial\mathcal{L}}{\partial x^a}\right), \quad (k = 1, 2, ..., n)$$
$$E^k(\mathbf{V}^{[1]}\mathcal{L}) = \mathbf{V}^{[2]}(E^k(\mathcal{L})) + \mathbf{Z}^{[1]}\mathcal{L} + \frac{\partial\alpha}{\partial\dot{x}^k}\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial t}\right) + \frac{\partial\beta^a}{\partial\dot{x}^k}\left[\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial x^a}\right)\right] + \frac{\partial}{\partial\dot{x}^k}(\dot{\beta}^a - \dot{x}^a\dot{\alpha})$$
$$\frac{d}{dt}\left(\frac{\partial\mathcal{L}}{\partial\dot{x}^a}\right),$$

where

$$\mathbf{Z}^{[1]}\mathcal{L} = E^k(\alpha)\frac{\partial \mathcal{L}}{\partial t} + E^k(\beta^a)\frac{\partial \mathcal{L}}{\partial x^a} + E^k(\dot{\beta}^a - \dot{x^a}\dot{\alpha})\frac{\partial \mathcal{L}}{\partial \dot{x^a}},$$

under infinitesimal transformation given in Eq. (2.6.4), if the equation of motions are form invariant, and the following relations hold

$$\mathbf{Z}^{[1]}\mathcal{L} + \frac{\partial \alpha}{\partial \dot{x}^k}\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial t}\right) + \frac{\partial \beta^a}{\partial \dot{x}^k}\left[\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial x^a}\right)\right] + \frac{\partial}{\partial \dot{x}^k}(\dot{\beta}^a - \dot{x}^a\dot{\alpha})\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^a}\right) = 0,$$
$$(k = 1, 2, ..., n),$$

which yields

$$E^k(\mathbf{V}^{[1]}\mathcal{L}) = \mathbf{V}^{[2]}(E^k(\mathcal{L})).$$

The above expression shows a relation between Lie and the Mei symmetries.

2.8 Relation between Mei Symmetries and Noether Symmetries

Jian-Hui *et al.* in [17] developed the criteria of finding symmetries named as Noether-Mei symmetries of mechanical system in phase space. If we have gauge function $B = B(t, \mathbf{q}, \mathbf{p})$, then the infinitesimal generators $\alpha_0, \beta_0^a, \beta_1^a$ satisfy the given condition

$$\left[H\dot{\alpha_0} + \frac{\partial H}{\partial t}\alpha_0 + \frac{\partial H}{\partial q_s}\beta_0^a - p_s\dot{\beta_0^a} - Q_s(\beta_0^a - \dot{q}_s\beta_1^a) - \dot{B}\right]^2 + \left[\frac{\partial}{\partial p_s}[\mathbf{V}H]\right]^2$$
$$+ \left[\frac{\partial}{\partial q_s}[\mathbf{V}H] - \mathbf{V}Q_s\right]^2 = 0, \tag{2.8.1}$$

then the Eq. (2.8.1) is called Noether-Mei symmetries of the mechanical system in phase space.

2.9 Mei Conserved Quantities corresponding to Lagrangian

The Mei conserved quantities induced by Mei symmetries can be obtained by **Theorem 2.9.1** and **Theorem 2.9.2**.

Theorem 2.9.1. *There exists Mei conserved quantity*

$$I = (\mathbf{V}^{[1]}\mathcal{L})\alpha + (\beta^a - \dot{x}^a\alpha)\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial\dot{x}^a} + B, \tag{2.9.1}$$

corresponding to the Mei symmetry generator $\mathbf{V}^{[1]}$*, that satisfies the condition*

$$(\mathbf{V}^{[1]}\mathcal{L})\dot{\alpha} + \mathbf{V}^{[1]}(\mathbf{V}^{[1]}\mathcal{L}) + \dot{B} = 0, \tag{2.9.2}$$

where $B = B(t, x^a, \dot{x}^a)$ *is a gauge function [22].*

Proof. Taking derivative of I with respect to $'t'$, we have

$$\begin{aligned}
\frac{dI}{dt} &= \left(\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial t} + \dot{x}^a\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial x^a} + \ddot{x}^a\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial\dot{x}^a}\right)\alpha + (\mathbf{V}^{[1]}\mathcal{L})\dot{\alpha} + \frac{d}{dt}\left(\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial\dot{x}^a}\right)(\beta^a - \dot{x}^a\alpha) \\
&+ \left(\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial\dot{x}^a}\right)(\dot{\beta}^a - \dot{x}^a\dot{\alpha} - \ddot{x}^a\alpha) + \dot{B}, \\
\frac{dI}{dt} &= \left(\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial t} + \dot{x}^a\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial x^a} + \ddot{x}^a\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial\dot{x}^a}\right)\alpha + (\mathbf{V}^{[1]}\mathcal{L})\dot{\alpha} + \frac{d}{dt}\left(\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial\dot{x}^a}\right)(\beta^a - \dot{x}^a\alpha) \\
&+ \left(\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial\dot{x}^a}\right)(\dot{\beta}^a - \dot{x}^a\dot{\alpha} - \ddot{x}^a\alpha) - (\mathbf{V}^{[1]}\mathcal{L})\dot{\alpha} - \alpha\left(\frac{\partial\mathbf{V}^{[1]}\mathcal{L}}{\partial t}\right) - \beta^a\left(\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial x^a}\right) \\
&- (\dot{\beta}^a - \dot{x}^a\dot{\alpha})\left(\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial\dot{x}^a}\right), \\
\frac{dI}{dt} &= \left[\frac{d}{dt}\left(\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial\dot{x}^a}\right) - \left(\frac{\partial(\mathbf{V}^{[1]}\mathcal{L})}{\partial x^a}\right)\right](\beta^a - \dot{x}^a\alpha),
\end{aligned}$$

using Eq. (2.6.10), we get

$$\begin{aligned}
\frac{dI}{dt} &= E^a(\mathbf{V}^{[1]}\mathcal{L})(\beta^a - \dot{x}^a\alpha), \\
\frac{dI}{dt} &= 0,
\end{aligned}$$

hence proved that I is a conserved quantity. □

Theorem 2.9.2. *There exists Mei conserved quantity*

$$I = \mathbf{V}^{[1]}\mathcal{L} - \dot{x}^a \frac{\partial}{\partial \dot{x}^a}\mathbf{V}^{[1]}\mathcal{L} + B, \tag{2.9.3}$$

corresponding to the Mei symmetry generator $\mathbf{V}^{[\mathbf{1}]}$*, that satisfies the condition*

$$\frac{\partial}{\partial t}\mathbf{V}^{[1]}\mathcal{L} + \dot{B} = 0, \tag{2.9.4}$$

where $B = B(t, x^a, \dot{x}^a)$ *is a gauge function [22].*

Proof. Taking derivative of I with respect to $'t'$, we get

$$\frac{dI}{dt} = \left(\frac{\partial \mathbf{V}^{[1]}\mathcal{L}}{\partial t} + \dot{x}^a\frac{\partial \mathbf{V}^{[1]}\mathcal{L}}{\partial x^a} + \ddot{x}^a\frac{\partial \mathbf{V}^{[1]}\mathcal{L}}{\partial \dot{x}^a}\right) - \frac{d}{dt}\left(\frac{\partial \mathbf{V}^{[1]}\mathcal{L}}{\partial \dot{x}^a}\right)\dot{x}^a - \ddot{x}^a\frac{\partial \mathbf{V}^{[1]}\mathcal{L}}{\partial \dot{x}^a} + \dot{B},$$

using Eq. (2.9.3), we get

$$\begin{aligned}
\frac{dI}{dt} &= \left[-\frac{d}{dt}\left(\frac{\partial \mathbf{V}^{[1]}\mathcal{L}}{\partial \dot{x}^a}\right) + \frac{\partial \mathbf{V}^{[1]}\mathcal{L}}{\partial x^a}\right]\dot{x}^a, \\
\frac{dI}{dt} &= E^a(\mathbf{V}^{[1]}\mathcal{L})\dot{x}^a, \\
\frac{dI}{dt} &= 0,
\end{aligned}$$

this completes the proof. □

Chapter 3

Approximate Mei Symmetries and Invariants of the Hamiltonian

3.1 Introduction

This chapter presents a procedure for determining approximate Mei symmetries and invariants of perturbed Hamiltonian that can be applied in a variety of disciplines of study where approximate Hamiltonian are of interest. This Hamiltonian is calculated from the Lagrangian of DHO which is given in [33]. The Legendre transformations are used to convert the Lagrangian to Hamiltonian. The results are provided as theorems, accompanied with proofs in Section 3.2 and Section 3.4. To elaborate the method of determining these symmetries and the associated Mei invariants, a basic example of mechanics is presented in Section 3.3. Finally, a comparison of approximate Mei and approximate Noether symmetries is provided. The comparison reveals that both sets of symmetries have only one common symmetry. As a result, the remaining symmetries in the two sets correspond to two distinct sets of conserved quantities.

The applications of symmetry methods and conserved quantities are significant in a wide range of academic fields, including mathematics, social sciences, natural sciences, engineering, etc. In her well-known theorem presented in 1918, Noether [8] connected symmetry with conservation laws. In addition, an action integral of a functional (Lagrangian) is invariant under a infinitesimal transformations of a group. This set of transformations is known as Noether symmetries. Feng-Xiang [11], introduced Mei symmetries in 2000 (also known as form invariance), which are equations of motion under infinitesimal transformation of a group. However, in approximate Mei symmetries dynamical functions such as perturbed Lagrangian, perturbed Hamiltonian etc., are replaced by transformed dynamical functions. Furthermore, after performing some infinitesimal transformation of a group, the equations of motion are satisfied. The approximate Mei symmetries, in particular, preserve the form of *equations of motion*.

It is well known that there is a conserved quantity associated with each Noether symmetry. Similarly, Mei symmetries are another class of symmetries that corresponds to conserved quantities. Therefore, the conserved quantities may differ between the two sets of symmetries.

DEs with small parameters, known as the *approximate/perturbed term*, emerge frequently as mathematical models of real-world problems. In general, the parameter refers to some kind of error or correction. To solve equations including perturbed/approximate terms, various approaches have been proposed, including the homotopy perturbation method, the A-domain decomposition method, the inverse scattering transformation method, and the approximate symmetry method. The approximate groups for perturbed DEs were initially studied by Baikov *et al.* [34], who also developed theory based on approximate groups. A method to determine the approximate symmetries of perturbed DEs is also provided by the approximate Lie theorem. Gazizov [37] developed an approach to determine the approximate invariants and defined some properties of approximate symmetries.

The approximate symmetry generators and invariants of the perturbed ODEs were used by Feroze and Kara [38] to construct the Lagrangians. The similar method was used by Johnpillai and Kara [39] to create approximate Lagrangians for perturbed

PDEs. The approximate symmetries and conserved quantities of a system of DEs are discussed in [40]. For the perturbed K-dV equation, approximate symmetries are determined, and a one-dimensional subalgebra of an optimal system is constructed in [41]. Third order approximate Noether symmetries for Bardeen spacetime are presented by Camci [42], who has also created some new approximate Noether gauge symmetry relations for perturbed Lagrangians.

3.2 Mei Symmetries of Approximate Hamiltonian

The unperturbed Hamiltonian system has been discussed in terms of the Mei symmetries and their related first integrals in [18]. Here, we present approximate Mei symmetries and invariants corresponding to perturbed Hamiltonian in the following theorems.

Theorem 3.2.1. *Let* $\mathbf{V} = \mathbf{V}^0 + \epsilon\mathbf{V}^1$ *be approximate symmetry generator and* $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1$ *be the first order perturbed Lagrangian, where* $\mathbf{V}^0 = \alpha_0\frac{\partial}{\partial t} + \beta_0^a\frac{\partial}{\partial x^a}$ *and* $\mathbf{V}^1 = \alpha_1\frac{\partial}{\partial t} + \beta_1^a\frac{\partial}{\partial x^a}$, *then*

$$E^a(\mathbf{V}^{0[1]}H_0) = 0, \qquad (a = 1, 2, ..., n). \tag{3.2.1}$$

$$E^a(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) = 0, \qquad (a = 1, 2, ..., n). \tag{3.2.2}$$

Here E^a *is called the Euler operator defined in Eq. (2.6.2),* $\mathbf{V}^{0[1]}$ *and* $\mathbf{V}^{0[1]}$ *are first-order prolongation of symmetry generator and approximate symmetry generator.*

Proof. The Legendre transformation which forms the bridge between Lagrangian and Hamiltonian as

$$H(t, x^a, \dot{x^a}) = p_a\dot{x^a} - \mathcal{L}(t, x^a, \dot{x^a}), \qquad p_a = \frac{\partial\mathcal{L}}{\partial\dot{x^a}}, \tag{3.2.3}$$

applying $\mathbf{V}^{[1]}$ on Eq. (3.2.3), we get

$$\mathbf{V}^{[1]}H = (\mathbf{V}^{0[1]} + \epsilon\mathbf{V}^{1[1]})(p_a\dot{x^a} - (\mathcal{L}_0 + \epsilon\mathcal{L}_1)), \tag{3.2.4}$$

where

$$p_a = \frac{\partial}{\partial \dot{x^a}}(\mathcal{L}_0 + \epsilon \mathcal{L}_1),$$

then Eq. (3.2.4) becomes

$$\mathbf{V}^{[\mathbf{1}]}H = (\mathbf{V}^{0[1]} + \epsilon \mathbf{V}^{1[1]})\left[p_a\left(\frac{\partial}{\partial \dot{x^a}}(\mathcal{L}_0 + \epsilon \mathcal{L}_1)\right) - (\mathcal{L}_0 + \epsilon \mathcal{L}_1)\right],$$
$$\mathbf{V}^{[\mathbf{1}]}H = (\mathbf{V}^{0[1]} + \epsilon \mathbf{V}^{1[1]})\left[\left(p_a\frac{\partial \mathcal{L}_0}{\partial \dot{x^a}} - \mathcal{L}_0\right) + \epsilon(p_a\frac{\partial \mathcal{L}_1}{\partial \dot{x^a}} - \mathcal{L}_1)\right],$$
$$\mathbf{V}^{[\mathbf{1}]}H = (\mathbf{V}^{0[1]} + \epsilon \mathbf{V}^{1[1]})(H_0 + \epsilon H_1),$$

To prove the above relations, Eqs. (3.2.1)-(3.2.2), apply first order prolongation of $\mathbf{V}$ i.e. $\mathbf{V}^{[1]} = \mathbf{V}^{0[1]} + \epsilon \mathbf{V}^{1[1]}$ on $H = H_0 + \epsilon H_1$ to have

$$\mathbf{V}^{[1]}H = (\mathbf{V}^{0[1]} + \epsilon \mathbf{V}^{1[1]})(H_0 + \epsilon H_1), \tag{3.2.5}$$

neglecting the higher order terms in ϵ yields

$$\mathbf{V}^{[1]}H = (\mathbf{V}^{0[1]}H_0) + \epsilon(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0). \tag{3.2.6}$$

Applying operator E^a, given in Eq. (2.6.2), on Eq. (4.1.3) and requiring the invariance $E^a(\mathbf{V}^{[1]}H) = 0$, we have

$$E^a(\mathbf{V}^{0[1]}H_0) + \epsilon E^a(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) = 0, \tag{3.2.7}$$

comparing the coefficients of different powers of ϵ leads to Eqs. (3.2.1) and (3.2.2). This completes the proof. □

3.3 Mei Symmetries of Approximate Hamiltonian of DHO

The linear equation of motion of DHO is taken as an example. The Hamiltonian of DHO is obtained by using Legendre transformations defined in Eq. (3.2.3) using

Lagrangian of DHO given in Eq. (4.2.2) in Chapter 4.

$$H_0 + \epsilon H_1 = \left[\left(\dot{y}\frac{\partial \mathcal{L}_0}{\partial \dot{y}} - \mathcal{L}_0\right) + \epsilon(\dot{y}\frac{\partial \mathcal{L}_1}{\partial \dot{y}} - \mathcal{L}_1)\right], \tag{3.3.1}$$

where

$$\mathcal{L}_0 = \frac{1}{2}(\dot{y}^2 - y^2),$$
$$\mathcal{L}_1 = t(\dot{y}^2 - y^2),$$

after solving Eq. (3.3.1), we get

$$H_0 + \epsilon H_1 = \frac{1}{2}(\dot{y}^2 + y^2) + \epsilon t(\dot{y}^2 + y^2). \tag{3.3.2}$$

Now, writing H by separating the powers of ϵ, we obtain

$$H_0 = \frac{1}{2}(\dot{y}^2 + y^2), \tag{3.3.3a}$$
$$H_1 = t(\dot{y}^2 + y^2). \tag{3.3.3b}$$

Applying $\mathbf{V}^{0[1]}$ on Eq. (3.3.3a), we get

$$\mathbf{V}^{0[1]}H_0 = \dot{y}\beta_{0,t} + \dot{y}^2\beta_{0,y} - \dot{y}^2\alpha_{0,t} - \dot{y}^3\alpha_{0,y} + y\beta_0. \tag{3.3.4}$$

Now Eq. (3.2.1) (for $a = 1$), for the above Eq. (3.3.4), gives

$$E^1(\mathbf{V}^{0[1]}H_0) = 0. \tag{3.3.5}$$

Alternatively,

$$\frac{d}{dt}\left(\frac{\partial \mathbf{V}^{0[1]}H_0}{\partial \dot{y}}\right) - \left(\frac{\partial \mathbf{V}^{0[1]}H_0}{\partial y}\right) = 0. \tag{3.3.6}$$

The above Eq. (3.3.6) gives the following expression

$$\beta_{0,tt} + 2\dot{y}^2\beta_{0,ty} + 2\ddot{y}\beta_{0,y} - 2\ddot{y}\alpha_{0,t} + \dot{y}^2\beta_{0,yy} - 2\dot{y}\alpha_{0,tt} - 4\dot{y}^2\alpha_{0,ty} - 6\ddot{y}\dot{y}\alpha_{0,y} - 2\dot{y}^3\alpha_{0,yy}$$
$$- \beta_0 - y\beta_{0,y} = 0, \tag{3.3.7}$$

since first-order prolongation of symmetry generator is used therefore, we used $\ddot{y}-y=0$ in Eq. (3.3.7), we get

$$\beta_{0,tt}+2\dot{y}\beta_{0,ty}+2y\beta_{0,y}-2y\alpha_{0,t}+\dot{y}^2\beta_{0,yy}-2\dot{y}\alpha_{0,tt}-4\dot{y}^2\alpha_{0,ty}-6y\dot{y}\alpha_{0,y}-2\dot{y}^3\alpha_{0,yy}$$
$$-\beta_0-y\beta_{0,y}=0. \tag{3.3.8}$$

The coefficients of different powers of $\dot{y}$ provide the following system of PDEs

$$\beta_{0,tt}-\beta_0+y\beta_{0,y}-2y\alpha_{0,t}=0, \tag{3.3.9}$$
$$\beta_{0,ty}-\alpha_{0,tt}-3y\alpha_{0,y}=0, \tag{3.3.10}$$
$$\beta_{0,yy}-4\alpha_{0,ty}=0, \tag{3.3.11}$$
$$\alpha_{0,yy}=0. \tag{3.3.12}$$

Eq. (3.3.11) and Eq. (3.3.12) respectively imply

$$\alpha_0(t,y)=yf(t)+g(t), \tag{3.3.13}$$
$$\beta_0(t,y)=\frac{y^2}{2}f_{,t}+y\delta(t)+\gamma(t). \tag{3.3.14}$$

Using Eqs. (3.3.13) and (3.3.14) in Eqs. (3.3.9) and (3.3.10), we obtain the following system

$$f_{,ttt}=0, \qquad \delta_{,tt}-2g_{,t}=0, \tag{3.3.15}$$
$$\gamma_{,tt}-\gamma(t)=0, \qquad \delta_{,t}-g_{,tt}=0.$$

Solving the above system given in Eq. (3.3.15) and substituting the solution in Eqs. (3.3.13) and (3.3.14) yields

$$\alpha_0=C_1+e^{\sqrt{2}t}C_2+e^{-\sqrt{2}t}C_3,$$
$$\beta_0=y\sqrt{2}e^{\sqrt{2}t}C_2-y\sqrt{2}e^{-\sqrt{2}t}C_3+e^{-t}C_4+e^{t}C_5+yC_6. \tag{3.3.16}$$

Eq. (3.3.16) provides the following list of Mei symmetries viz.,

$$\mathbf{V}_1^0 = \frac{\partial}{\partial t}, \tag{3.3.17}$$

$$\mathbf{V}_2^0 = e^{\sqrt{2}t}\frac{\partial}{\partial t} + y\sqrt{2}e^{\sqrt{2}t}\frac{\partial}{\partial y}, \tag{3.3.18}$$

$$\mathbf{V}_3^0 = e^{-\sqrt{2}t}\frac{\partial}{\partial t} - y\sqrt{2}e^{-\sqrt{2}t}\frac{\partial}{\partial y}, \tag{3.3.19}$$

$$\mathbf{V}_4^0 = e^{-t}\frac{\partial}{\partial y}, \tag{3.3.20}$$

$$\mathbf{V}_5^0 = e^{t}\frac{\partial}{\partial y}, \tag{3.3.21}$$

$$\mathbf{V}_6^0 = y\frac{\partial}{\partial y}. \tag{3.3.22}$$

Approximate Mei symmetries are calculated using the above exact symmetries given by Eqs. (3.3.17-3.3.22). For this, we consider $\mathbf{V}_3^0$, to illustrate the method, where $\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0$ is expressed as

$$\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0 = \dot{y}\beta_{1,t} + \dot{y}^2\beta_{1,y} - \dot{y}^2\alpha_{1,t} - \dot{y}^3\alpha_{1,y} + y\beta_1 + \dot{y}^2e^{-\sqrt{2}t} + y^2e^{-\sqrt{2}t} + 2\sqrt{2}ty^2$$
$$e^{-\sqrt{2}t} + 4ty\dot{y}e^{-\sqrt{2}t}. \tag{3.3.23}$$

Now using Eq. (3.3.23) in Eq. (3.2.2), we have

$$\frac{d}{dt}\left(\frac{\partial(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)}{\partial \dot{y}}\right) - \left(\frac{\partial(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)}{\partial y}\right) = 0, \tag{3.3.24}$$

or

$$\beta_{1,tt} + 2\dot{y}\beta_{1,ty} + 2\ddot{y}\beta_{1,y} - 2\ddot{y}\alpha_{1,t} + \dot{y}^2\beta_{1,yy} - 2\dot{y}\alpha_{1,tt} - 4\dot{y}^2\alpha_{1,ty} - 6\dot{y}\ddot{y}\alpha_{1,y} - 2\dot{y}^3\alpha_{1,yy}$$
$$- \beta_1 - y\beta_{1,y} - 2\sqrt{2}\dot{y}e^{-\sqrt{2}t} + 4ye^{-\sqrt{2}t} = 0. \tag{3.3.25}$$

Using standard procedure of comparing the coefficients of different powers of $\dot{y}$, the obtained system is

$$\beta_{1,tt} - \beta_1 + y\beta_{1,y} - 2y\alpha_{1,t} + 4ye^{-\sqrt{2}t} = 0, \tag{3.3.26}$$

$$\beta_{1,ty} - \alpha_{1,tt} - 3y\alpha_{1,y} - \sqrt{2}e^{-\sqrt{2}t} = 0, \tag{3.3.27}$$

$$\beta_{1,yy} - 4\alpha_{1,ty} = 0, \tag{3.3.28}$$

$$\alpha_{1,yy} = 0. \tag{3.3.29}$$

Eq. (3.3.28) and Eq. (3.3.29) respectively imply

$$\alpha_1(t,y) = yf(t) + g(t), \tag{3.3.30}$$

$$\beta_1(t,y) = \frac{y^2}{2} f_{,t} + y\delta(t) + \gamma(t). \tag{3.3.31}$$

Using Eqs. (3.3.30) and (3.3.31) in Eqs. (3.3.26) and (3.3.27), we obtain the following system

$$\begin{aligned} f_{,ttt} &= 0, & \delta_{,tt} - 2g_{,t} + 4e^{-\sqrt{2}t} &= 0, \\ \gamma_{,tt} - \gamma(t) &= 0, & \delta_{,t} - g_{,tt} - \sqrt{2}e^{-\sqrt{2}t} &= 0. \end{aligned} \tag{3.3.32}$$

Solving the above system given in Eq. (3.3.32) and substituting the solution in Eqs. (3.3.30) and (3.3.31) yields

$$\alpha_1 = C_1 + e^{\sqrt{2}t}C_2 + e^{-\sqrt{2}t}C_3 - \frac{1}{2}te^{-\sqrt{2}t}, \tag{3.3.33}$$

$$\beta_1 = y\sqrt{2}e^{\sqrt{2}t}C_2 - y\sqrt{2}e^{-\sqrt{2}t}C_3 + e^{-t}C_4 + e^{t}C_5 + yC_6 - \frac{3}{2}ye^{-\sqrt{2}t} + \frac{1}{\sqrt{2}}yte^{-\sqrt{2}t}.$$

Now assigning value of any constant equal to one, say $C_3 = 1$, and remaining constants equal to zero, we obtain the generator $\mathbf{V}_3^0$ given in Eq. (3.3.19). Then, $\mathbf{V}_3^1$ can be written as

$$\mathbf{V}_3^1 = -\frac{1}{2}te^{-\sqrt{2}t}\frac{\partial}{\partial t} + \left(-\frac{3}{2}ye^{-\sqrt{2}t} + \frac{1}{\sqrt{2}}yte^{-\sqrt{2}t} \right)\frac{\partial}{\partial y}. \tag{3.3.34}$$

The nontrivial approximate Mei symmetry of Eq. (3.2.2) has the form

$$\mathbf{V}_3 = \mathbf{V}_3^0 + \epsilon\mathbf{V}_3^1 = \left(e^{-\sqrt{2}t}\frac{\partial}{\partial t} - y\sqrt{2}e^{-\sqrt{2}t}\frac{\partial}{\partial y} \right) + \epsilon\Bigg(-\frac{1}{2}te^{-\sqrt{2}t}\frac{\partial}{\partial t} + \bigg(-\frac{3}{2}ye^{-\sqrt{2}t} + \frac{1}{\sqrt{2}}yte^{-\sqrt{2}t} \bigg)\frac{\partial}{\partial y} \Bigg). \tag{3.3.35}$$

In a similar way, the remaining approximate Mei symmetries are obtained as

$$\mathbf{V}_1 = \mathbf{V}_1^0 + \epsilon \mathbf{V}_1^1 = \frac{\partial}{\partial t} - \epsilon y \frac{\partial}{\partial t}, \tag{3.3.36}$$

$$\mathbf{V}_2 = \mathbf{V}_2^0 + \epsilon \mathbf{V}_2^1 = \left(e^{\sqrt{2}t} \frac{\partial}{\partial t} + y\sqrt{2} e^{\sqrt{2}t} \frac{\partial}{\partial y} \right) + \epsilon \Bigg(- \frac{1}{2} t e^{\sqrt{2}t} \frac{\partial}{\partial t} + \Bigg(- \frac{3}{2} y e^{\sqrt{2}t} - \frac{1}{\sqrt{2}} y t e^{\sqrt{2}t} \Bigg) \frac{\partial}{\partial y} \Bigg), \tag{3.3.37}$$

$$\mathbf{V}_4 = \mathbf{V}_4^0 + \epsilon \mathbf{V}_4^1 = e^{-t} \frac{\partial}{\partial y} - \epsilon t e^{-t} \frac{\partial}{\partial y}, \tag{3.3.38}$$

$$\mathbf{V}_5 = \mathbf{V}_5^0 + \epsilon \mathbf{V}_5^1 = e^{t} \frac{\partial}{\partial y} - \epsilon t e^{t} \frac{\partial}{\partial y}, \tag{3.3.39}$$

$$\mathbf{V}_6 = \mathbf{V}_6^0 + \epsilon \mathbf{V}_6^1 = y \frac{\partial}{\partial y} - \epsilon 4 t y \frac{\partial}{\partial y}. \tag{3.3.40}$$

3.4 Approximate Mei Invariants corresponding to Hamiltonian

The approximate Mei invariants are formulated in **Theorem** 3.4.1, **Theorem** 3.4.2 and **Theorem** 3.4.3.

Theorem 3.4.1. *There exist Mei conserved quantities*

$$I^0 = \alpha_0(\mathbf{V}^{0[1]} H_0) + (\beta_0^a - \dot{x}^a \alpha_0) \frac{\partial(\mathbf{V}^{0[1]} H_0)}{\partial \dot{x}^a} + B_0, \tag{3.4.1}$$

$$I^1 = \alpha_0(\mathbf{V}^{0[1]} H_1 + \mathbf{V}^{1[1]} H_0) + \alpha_1(\mathbf{V}^{0[1]} H_0) + (\beta_1^a - \dot{x}^a \alpha_1) \frac{\partial(\mathbf{V}^{0[1]} H_0)}{\partial \dot{x}^a} + (\beta_0^a - \dot{x}^a \alpha_0) \frac{\partial(\mathbf{V}^{0[1]} H_1 + \mathbf{V}^{1[1]} H_0)}{\partial \dot{x}^a} + B_1, \tag{3.4.2}$$

corresponding to the Mei symmetry generators $\mathbf{V}^0 = \alpha_0 \frac{\partial}{\partial t} + \beta_0^a \frac{\partial}{\partial x^a}$ *and* $\mathbf{V}^1 = \alpha_1 \frac{\partial}{\partial t} + \beta_1^a \frac{\partial}{\partial x^a}$*, that satisfy the conditions*

$$\dot{\alpha}_0(\mathbf{V}^{0[1]} H_0) + \mathbf{V}^{0[1]}(\mathbf{V}^{0[1]} H_0) + \dot{B}_0 = 0, \tag{3.4.3}$$

$$\dot{\alpha}_0(\mathbf{V}^{0[1]} H_1 + \mathbf{V}^{1[1]} H_0) + \dot{\alpha}_1(\mathbf{V}^{0[1]} H_0) + \mathbf{V}^{0[1]}(\mathbf{V}^{0[1]} H_1 + \mathbf{V}^{1[1]} H_0) + \mathbf{V}^{1[1]}(\mathbf{V}^{0[1]} H_0) + \dot{B}_1 = 0, \tag{3.4.4}$$

where $B_0 = B_0(t, x^a, \dot{x^a})$ *and* $B_1 = B_1(t, x^a, \dot{x^a})$ *are the gauge functions and* $H = H_0 + \epsilon H_1$ *is first-order perturbed Hamiltonian.*

Proof. First, we have to show that $\frac{dI^0}{dt} = 0$ to ensure that I_0 is a conserved quantity.

$$\frac{dI^0}{dt} = \bigg(\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial t} + \dot{x}^a \frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial x^a} + \ddot{x}_a \frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial \dot{x}_a}\bigg)\alpha_0 + (\mathbf{V}^{0[1]}H_0)\dot{\alpha_0}$$
$$+ \frac{d}{dt}\bigg(\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial \dot{x}^a}\bigg)(\beta_0^a - \dot{x}^a\alpha_0) + \bigg(\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial \dot{x}^a}\bigg)\frac{d}{dt}(\beta_0^a - \dot{x}^a\alpha_0) + \dot{B}_0,$$
$$\frac{dI^0}{dt} = \bigg(\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial t} + \dot{x}^a \frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial x^a} + \ddot{x}^a \frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial \dot{x}^a}\bigg)\alpha_0 + (\mathbf{V}^{0[1]}H_0)\dot{\alpha_0}$$

$$+ \frac{d}{dt}\bigg(\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial \dot{x}^a}\bigg)(\beta_0^a - \dot{x}^a\alpha_0) + \bigg(\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial \dot{x}^a}\bigg)(\dot{\beta_0^a} - \dot{x}^a\dot{\alpha_0} - \ddot{x}^a\alpha_0)$$
$$- (\mathbf{V}^{0[1]}H_0)\dot{\alpha_0} - \alpha_0\bigg(\frac{\partial\mathbf{V}^{0[1]}H_0}{\partial t}\bigg) - \beta_0^a\bigg(\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial x^a}\bigg) - (\dot{\beta_0^a} - \dot{x}^a\dot{\alpha_0})\bigg(\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial \dot{x}^a}\bigg),$$
$$\frac{dI^0}{dt} = \bigg[\frac{d}{dt}\bigg(\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial \dot{x}^a}\bigg) - \bigg(\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial x^a}\bigg)\bigg](\beta_0^a - \dot{x}^a\alpha_0),$$
$$\frac{dI^0}{dt} = 0.$$

To establish the aforementioned statement, we consider Eq. (3.4.1). Taking perturbed invariant up to the first order of ϵ i.e. $I^0 + \epsilon I^1$, $\mathbf{V} = \mathbf{V}^0 + \epsilon\mathbf{V}^1$, $B_0 + \epsilon B_1$ and $H = H_0 + \epsilon H_1$ we have

$$I^0 + \epsilon I^1 = (\alpha_0 + \epsilon\alpha_1)[(\mathbf{V}^{0[1]} + \epsilon\mathbf{V}^{1[1]})(H_0 + \epsilon H_1)] + [(\beta_0^a + \epsilon\beta_1^a) - \dot{x}^a(\alpha_0 + \epsilon\alpha_1)]$$
$$\frac{\partial(\mathbf{V}^{0[1]} + \epsilon\mathbf{V}^{1[1]})}{\partial \dot{x}^a}(H_0 + \epsilon H_1) + (B_0 + \epsilon B_1), \tag{3.4.5}$$

rearranging the above expression as

$$I^0 + \epsilon I^1 = \alpha_0(\mathbf{V}^{0[1]}H_0) + \epsilon[\alpha_0(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) + \alpha_1(\mathbf{V}^{0[1]}H_0)] + (\beta_0^a - \dot{x}^a\alpha_0)$$
$$\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial \dot{x}^a} + \epsilon\bigg[(\beta_1^a - \dot{x}^a\alpha_1)\frac{\partial(\mathbf{V}^{0[1]}H_0)}{\partial \dot{x}^a} + (\beta_0^a - \dot{x}^a\alpha_0)\frac{\partial(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)}{\partial \dot{x}^a}\bigg]$$
$$+ (B_0 + \epsilon B_1), \tag{3.4.6}$$

separating powers of ϵ up to first order, we obtain the expressions given in Eq. (3.4.2). Now, we need to prove the $\frac{dI^1}{dt}=0$. Taking the derivative of I^1, we get

$$\frac{dI^1}{dt} = \dot{\alpha_0}(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) + \dot{\alpha_1}(\mathbf{V}^{0[1]}H_0) + \alpha_0\left[\frac{\partial}{\partial t}(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) + \dot{x^a}\frac{\partial}{\partial x^a}\right.$$
$$(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) + \left.\ddot{x^a}\frac{\partial}{\partial \dot{x^a}}(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)\right] + \alpha_1\left[\frac{\partial}{\partial t}(\mathbf{V}^{0[1]}H_0) + \dot{x^a}\frac{\partial}{\partial x^a}(\mathbf{V}^{0[1]}H_0)\right.$$
$$\left. + \ddot{x^a}\frac{\partial}{\partial \dot{x^a}}(\mathbf{V}^{0[1]}H_0)\right] + (\beta_0^a - \dot{x^a}\alpha_0)\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x^a}}(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)\right) + (\dot{\beta_0^a} - \dot{x^a}\dot{\alpha_0} - \ddot{x^a}\alpha_0)$$
$$\frac{\partial}{\partial \dot{x^a}}(\mathbf{V}^{0[1]}H_1 + (\mathbf{V}^{1[1]}H_0) + (\beta_1^a - \dot{x^a}\alpha_1)\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x^a}}(\mathbf{V}^{1[1]}H_0)\right) + (\dot{\beta_1^a} - \dot{x^a}\dot{\alpha_1} - \ddot{x^a}\alpha_1)\frac{\partial}{\partial \dot{x^a}}$$
$$(\mathbf{V}^{0[1]}H_0) - \dot{\alpha_0}(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) - \dot{\alpha_1}(\mathbf{V}^{0[1]}H_0) - \alpha_0\frac{\partial}{\partial t}(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) - \beta_0^a\frac{\partial}{\partial x^a}$$
$$(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) - (\dot{\beta_0^a} - \dot{x^a}\dot{\alpha_0})\frac{\partial}{\partial \dot{x^a}}(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) - \alpha_1\frac{\partial}{\partial t}(\mathbf{V}^{0[1]}H_0) - \beta_1^a\frac{\partial}{\partial x^a}$$
$$(\mathbf{V}^{0[1]}H_0) - (\dot{\beta_1^a} - \dot{x^a}\dot{\alpha_1})\frac{\partial}{\partial \dot{x^a}}(\mathbf{V}^{0[1]}H_0),$$
$$\frac{dI^1}{dt} = (\beta_1^a - \dot{x^a}\alpha_1)\left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x^a}}(\mathbf{V}^{0[1]}H_0)\right) - \frac{\partial}{\partial x^a}(\mathbf{V}^{0[1]}H_0)\right)\right] + (\beta_0^a - \dot{x^a}\alpha_0)$$
$$\left[\frac{d}{dt}\left(\frac{\partial}{\partial \dot{x^a}}(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)\right) - \frac{\partial}{\partial x^a}(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)\right)\right],$$
$$\frac{dI^1}{dt} = (\beta_1^a - \dot{x^a}\alpha_1)E^a(\mathbf{V}^{0[1]}H_0) + (\beta_0^a - \dot{x^a}\alpha_0)E^a(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0),$$
$$\frac{dI^1}{dt} = 0,$$

this completes the proof. □

To elaborate the results found in **Theorem** 3.4.1 and **Theorem** 3.4.2, we find Mei invariants corresponding to symmetry generator $\mathbf{V}_3^0$.

$$\mathbf{V}_3^{0[1]} = e^{-\sqrt{2}t}\frac{\partial}{\partial t} - y\sqrt{2}e^{-\sqrt{2}t}\frac{\partial}{\partial y} + 2ye^{-\sqrt{2}t}\frac{\partial}{\partial \dot{y}},$$

and

$$\mathbf{V}_3^{0[1]}H_0 = -\sqrt{2}te^{-\sqrt{2}t} + 2y\dot{y}e^{-\sqrt{2}t}.$$

Using Eq. (3.4.3)

$$\dot{\alpha}_0(\mathbf{V}^{0[1]}H_0) + \mathbf{V}^{0[1]}(\mathbf{V}^{0[1]}H_0) + \dot{B}_0 = 0,$$

which gives $\dot{B}_0$

$$\dot{B}_0 = 6\sqrt{2}y\dot{y}e^{-2\sqrt{2}t} - 12y^2e^{-2\sqrt{2}t}, \tag{3.4.7}$$

Using Eq. (3.4.1)

$$I_3^0 = -3\sqrt{2}y^2e^{-2\sqrt{2}t} + B_0, \tag{3.4.8}$$

for conserved quantity, we need to show $\frac{dI_0}{dt} = 0$

$$\frac{dI_3^0}{dt} = 12y^2e^{-2\sqrt{2}t} - 6\sqrt{2}y\dot{y}e^{-2\sqrt{2}t} + \dot{B}_0,$$

putting $\dot{B}_0$, we get the require result i.e.,

$$\frac{dI_3^0}{dt} = 0.$$

Consider $\mathbf{V}_3^{1[1]}$

$$\mathbf{V}_3^{1[1]} = -\frac{1}{2}te^{-\sqrt{2}t}\frac{\partial}{\partial t} + \left(-\frac{3}{2}ye^{-\sqrt{2}t} + \frac{1}{\sqrt{2}}yte^{-\sqrt{2}t}\right)\frac{\partial}{\partial y} + \left(-\dot{y}e^{-\sqrt{2}t} - tye^{-\sqrt{2}t} + 2\sqrt{2}ye^{-\sqrt{2}t}\right)\frac{\partial}{\partial \dot{y}},$$

and

$$\mathbf{V}_3^{0[1]}H_1 + \mathbf{V}_3^{1[1]}H_0 = -\frac{y^2}{2}e^{-\sqrt{2}t} - \frac{3}{\sqrt{2}}ty^2e^{-\sqrt{2}t} + 3ty\dot{y}e^{-\sqrt{2}t} + 2\sqrt{2}y\dot{y}e^{-\sqrt{2}t},$$
$$\mathbf{V}_3^{0[1]}H_0 = -\sqrt{2}te^{-\sqrt{2}t} + 2y\dot{y}e^{-\sqrt{2}t}.$$

Also

$$\mathbf{V}_3^{0[1]}(\mathbf{V}_3^{0[1]}H_1 + \mathbf{V}_3^{1[1]}H_0) = 4\sqrt{2}y^2e^{-2\sqrt{2}t} + 15ty^2e^{-2\sqrt{2}t} - 6\sqrt{2}ty\dot{y}e^{-2\sqrt{2}t} - 5y\dot{y}e^{-2\sqrt{2}t},$$
$$\mathbf{V}_3^{1[1]}(\mathbf{V}_3^{0[1]}H_0) = 7\sqrt{2}y^2e^{-2\sqrt{2}t} - 5ty^2e^{-2\sqrt{2}t} + 2\sqrt{2}ty\dot{y}e^{-2\sqrt{2}t} - 5y\dot{y}e^{-2\sqrt{2}t},$$
$$\dot{\alpha}_0(\mathbf{V}_3^{0[1]}H_1 + \mathbf{V}_3^{1[1]}H_0) = \frac{y^2}{\sqrt{2}}e^{-2\sqrt{2}t} + 3ty^2e^{-2\sqrt{2}t} - 3\sqrt{2}ty\dot{y}e^{-2\sqrt{2}t} - 4y\dot{y}e^{-2\sqrt{2}t},$$
$$\dot{\alpha}_1(\mathbf{V}_3^{0[1]}H_0) = \frac{y^2}{\sqrt{2}}e^{-2\sqrt{2}t} - ty^2e^{-2\sqrt{2}t} - \sqrt{2}ty\dot{y}e^{-2\sqrt{2}t} - y\dot{y}e^{-2\sqrt{2}t}.$$

By Eq. (3.4.4), we get $\dot{B}_0$

$$\dot{B}_0 = -12\sqrt{2}y^2e^{-2\sqrt{2}t} - 12ty^2e^{-2\sqrt{2}t} + 6\sqrt{2}ty\dot{y}e^{-2\sqrt{2}t} + 15y\dot{y}e^{-2\sqrt{2}t}. \tag{3.4.9}$$

Then I_3^1 takes the form

$$I_3^1 = -\frac{15}{2}y^2e^{-2\sqrt{2}t} - 3\sqrt{2}ty^2e^{-2\sqrt{2}t} + B_1, \tag{3.4.10}$$

and

$$\frac{dI_3^1}{dt} = 12\sqrt{2}y^2e^{-2\sqrt{2}t} + 12ty^2e^{-2\sqrt{2}t} - 6\sqrt{2}ty\dot{y}e^{-2\sqrt{2}t} - 15y\dot{y}e^{-2\sqrt{2}t} + \dot{B}_1,$$

after plugging $\dot{B}_0$, we get

$$\frac{dI_3^1}{dt} = 0.$$

Theorem 3.4.2. *There exists Mei conserved quantity*

$$I^0 = \mathbf{V}^{0[1]}H_0 - \dot{x^a}\frac{\partial}{\partial\dot{x^a}}\mathbf{V}^{0[1]}H_0 + B_0, \tag{3.4.11}$$

corresponding to Mei symmetry generator $\mathbf{V}^0 = \alpha_0\frac{\partial}{\partial t} + \beta_0^a\frac{\partial}{\partial x^a}$, *that satisfies the condition*

$$\frac{\partial}{\partial t}\mathbf{V}^{0[1]}H_0 + \dot{B}_0 = 0, \tag{3.4.12}$$

where $B_0 = B_0(t, x^a, \dot{x}^a)$ *is gauge function.*

Proof. Taking derivative of I_0 with respect to $'t'$, we get

$$\frac{dI^0}{dt} = \left(\frac{\partial\mathbf{V}^{0[1]}H_0}{\partial t} + \dot{x^a}\frac{\partial\mathbf{V}^{0[1]}H_0}{\partial x^a} + \ddot{x^a}\frac{\partial\mathbf{V}^{0[1]}H_0}{\partial\dot{x^a}}\right) - \frac{d}{dt}\left(\frac{\partial\mathbf{V}^{0[1]}H_0}{\partial\dot{x^a}}\right)\dot{x^a} - \ddot{x^a}\frac{\partial\mathbf{V}^{0[1]}H_0}{\partial\dot{x^a}} + \dot{B}_0,$$

$$\frac{dI^0}{dt} = \dot{x^a}\left[-\frac{d}{dt}\left(\frac{\partial\mathbf{V}^{0[1]}H_0}{\partial\dot{x^a}}\right) + \frac{\partial\mathbf{V}^{0[1]}H_0}{\partial x^a}\right],$$

$$\frac{dI^0}{dt} = \dot{x^a}E^a(\mathbf{V}^{0[1]}H_0),$$

$$\frac{dI^0}{dt} = 0,$$

this completes the proof. □

Theorem 3.4.3. *There exists Mei conserved quantity*

$$I^1 = (\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) - \dot{x^a}\frac{\partial}{\partial \dot{x^a}}(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) + B_1, \tag{3.4.13}$$

corresponding to Mei symmetry generator $\mathbf{V}^1 = \alpha_1\frac{\partial}{\partial t} + \beta_1^a\frac{\partial}{\partial x^a}$*, that satisfies the condition*

$$\frac{\partial}{\partial t}(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0) + \dot{B}_1 = 0, \tag{3.4.14}$$

where $B_1 = B_1(t, x^a, \dot{x}^a)$ *is gauge function.*

Proof. Differentiating I^1 with respect to t, we get

$$\frac{dI^1}{dt} = \left(\frac{\partial(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)}{\partial t} + \dot{x^a}\frac{\partial(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)}{\partial x^a} + \ddot{x^a}\frac{\partial(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)}{\partial \dot{x^a}}\right)$$
$$- \frac{d}{dt}\left(\frac{\partial(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)}{\partial \dot{x^a}}\right)\dot{x^a} - \ddot{x^a}\frac{\partial(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)}{\partial \dot{x^a}} + \dot{B}_1,$$
$$\frac{dI_1}{dt} = \left[-\frac{d}{dt}\left(\frac{\partial(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)}{\partial \dot{x^a}}\right) + \frac{\partial(\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0)}{\partial x^a}\right]\dot{x^a},$$
$$\frac{dI^1}{dt} = E^a((\mathbf{V}^{0[1]}H_1 + \mathbf{V}^{1[1]}H_0))\dot{x^a},$$
$$\frac{dI^1}{dt} = 0,$$

this completes the proof. □

Concluding Remarks:

The approximate Mei symmetries of DHO corresponding to Hamiltonian are calculated. These approximate Mei symmetries are more than the approximate Noether symmetries. Their comparison showed that $\mathbf{V}^1$ is common in both sets, whereas the remaining Mei symmetries differ from Noether symmetries resulting in new conserved quantities.

Mei symmetries of DHO	Mei Invariants of DHO corresponding to Hamiltonian
$\mathbf{V}_1^0 = \frac{\partial}{\partial t}$	$I_1^0 = 0, \dot{B}_0 = 0.$
$\mathbf{V}_2^0 = \left(e^{\sqrt{2}t}\frac{\partial}{\partial t} + y\sqrt{2}e^{\sqrt{2}t}\frac{\partial}{\partial y}\right)$	$\dot{B}_0 = -6\sqrt{2}y\dot{y}e^{2\sqrt{2}t} - 12y^2e^{2\sqrt{2}t},$ $I_2^0 = 3\sqrt{2}y^2e^{2\sqrt{2}t} + B_0.$
$\mathbf{V}_3^0 = \left(e^{-\sqrt{2}t}\frac{\partial}{\partial t} + y\sqrt{2}e^{-\sqrt{2}t}\frac{\partial}{\partial y}\right)$	$\dot{B}_0 = 6\sqrt{2}y\dot{y}e^{-2\sqrt{2}t} - 12y^2e^{-2\sqrt{2}t},$ $I_3^0 = -3\sqrt{2}y^2e^{-2\sqrt{2}t} + B_0.$
$\mathbf{V}_4^0 = e^{-t}\frac{\partial}{\partial y}$	$\dot{B}_0 = -2e^{-2t},$ $I_4^0 = -e^{-2t} + B_0.$
$\mathbf{V}_5^0 = e^{t}\frac{\partial}{\partial y}$	$\dot{B}_0 = 2e^{2t},$ $I_5^0 = -e^{2t} + B_0.$
$\mathbf{V}_6^0 = y\frac{\partial}{\partial y}$	$\dot{B}_0 = -2y^2 - 2\dot{y}^2,$ $I_6^0 = 2y\dot{y} + B_0.$

Table 3.1: Mei Invariants of DHO corresponding to Hamiltonian using Eq. (3.4.1)

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Approximate Mei symmetries of DHO	**Approximate Mei Invariants of DHO**
$\mathbf{V}_1^1 = y\frac{\partial}{\partial t}$	$I_1^1 = 0, \dot{B}_1 = 0.$
$\mathbf{V}_2^1 = \left(-\frac{1}{2}te^{\sqrt{2}t}\frac{\partial}{\partial t}\right) + \left(-\frac{3}{2}ye^{\sqrt{2}t} - \frac{1}{\sqrt{2}}yte^{\sqrt{2}t}\right)\frac{\partial}{\partial y}$	$\dot{B}_1 = 12\sqrt{2}y^2e^{2\sqrt{2}t} - 12ty^2e^{2\sqrt{2}t} - 6\sqrt{2}ty\dot{y}e^{2\sqrt{2}t} + 15y\dot{y}e^{2\sqrt{2}t},$ $I_2^1 = -\frac{15}{2}y^2e^{2\sqrt{2}t} + 3\sqrt{2}ty^2e^{2\sqrt{2}t} + B_1.$
$\mathbf{V}_3^1 = \left(-\frac{1}{2}te^{-\sqrt{2}t}\frac{\partial}{\partial t}\right) + \left(-\frac{3}{2}ye^{\sqrt{2}t} + \frac{1}{\sqrt{2}}yte^{\sqrt{2}t}\right)\frac{\partial}{\partial y}$	$\dot{B}_1 = -12\sqrt{2}y^2e^{-2\sqrt{2}t} - 12ty^2e^{-2\sqrt{2}t} + 6\sqrt{2}ty\dot{y}e^{-2\sqrt{2}t} + 15y\dot{y}e^{-2\sqrt{2}t},$ $I_3^1 = -\frac{15}{2}y^2e^{-2\sqrt{2}t} - 3\sqrt{2}ty^2e^{-2\sqrt{2}t} + B_1.$
$\mathbf{V}_4^1 = -te^{-t}\frac{\partial}{\partial y}$	$\dot{B}_1 = -2e^{-2t},$ $I_4^1 = -e^{-2t} + B_1.$
$\mathbf{V}_5^1 = -te^{t}\frac{\partial}{\partial y}$	$\dot{B}_1 = 2e^{2t},$ $I_5^1 = -e^{2t} + B_1.$
$\mathbf{V}_6^1 = -4ty\frac{\partial}{\partial y}$	$\dot{B}_1 = 12ty^2 + 12t\dot{y}^2 + 12y\dot{y},$ $I_6^1 = -4y^2 - 12y\dot{y} + B_1.$

Table 3.2: Approximate Mei Invariants of DHO corresponding to Hamiltonian using Eq. (3.4.2)

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Mei symmetries of DHO	**Mei Invariants of DHO corresponding to Hamiltonian**
$\mathbf{V}_1^0 = \frac{\partial}{\partial t}$	$I_1^0 = 0, \dot{B}_0 = 0.$
$\mathbf{V}_2^0 = \left(e^{\sqrt{2}t}\frac{\partial}{\partial t} + y\sqrt{2}e^{\sqrt{2}t}\frac{\partial}{\partial y}\right)$	$\dot{B}_0 = -2\sqrt{2}y\dot{y}e^{\sqrt{2}t} - 2y^2e^{\sqrt{2}t},$ $I_2^0 = \sqrt{2}y^2e^{\sqrt{2}t} + B_0.$
$\mathbf{V}_3^0 = \left(e^{-\sqrt{2}t}\frac{\partial}{\partial t} + y\sqrt{2}e^{-\sqrt{2}t}\frac{\partial}{\partial y}\right)$	$\dot{B}_0 = 2\sqrt{2}y\dot{y}e^{-\sqrt{2}t} - 2y^2e^{-\sqrt{2}t},$ $I_3^0 = -\sqrt{2}y^2e^{-\sqrt{2}t} + B_0.$
$\mathbf{V}_4^0 = e^{-t}\frac{\partial}{\partial y}$	$\dot{B}_0 = ye^{-t} - \dot{y}e^{-t},$ $I_4^0 = ye^{-t} + B_0.$
$\mathbf{V}_5^0 = e^{t}\frac{\partial}{\partial y}$	$\dot{B}_0 = -ye^{t} - \dot{y}e^{t},$ $I_5^0 = ye^{t} + B_0.$
$\mathbf{V}_6^0 = y\frac{\partial}{\partial y}$	$\dot{B}_0 = 0,$ $I_6^0 = y^2 - \dot{y}^2 + B_0.$

Table 3.3: Mei Invariants of DHO corresponding to Hamiltonian using **Theorem 3.4.2**.

Approximate Mei symmetries of DHO	Approximate Mei Invariants of DHO
$\mathbf{V}_1^1 = y\frac{\partial}{\partial t}$	$I_1^1 = 0, \dot{B}_1 = 0.$
$\mathbf{V}_2^1 = \Big(-\frac{1}{2}te^{\sqrt{2}t}\frac{\partial}{\partial t}\Big) + \Big(-\frac{3}{2}ye^{\sqrt{2}t} - \frac{1}{\sqrt{2}}yte^{\sqrt{2}t}\Big)\frac{\partial}{\partial y}$	$\dot{B}_1 = -\sqrt{2}y^2e^{\sqrt{2}t} - 3ty^2e^{\sqrt{2}t} - 3\sqrt{2}ty\dot{y}e^{\sqrt{2}t} + y\dot{y}e^{\sqrt{2}t}$, $I_2^1 = -\frac{1}{2}y^2e^{\sqrt{2}t} + \frac{3}{\sqrt{2}}ty^2e^{\sqrt{2}t} + B_1.$
$\mathbf{V}_3^1 = \Big(-\frac{1}{2}te^{-\sqrt{2}t}\frac{\partial}{\partial t}\Big) + \Big(-\frac{3}{2}ye^{\sqrt{2}t} + \frac{1}{\sqrt{2}}yte^{\sqrt{2}t}\Big)\frac{\partial}{\partial y}$	$\dot{B}_1 = \sqrt{2}y^2e^{-\sqrt{2}t} - 3ty^2e^{-\sqrt{2}t} + 3\sqrt{2}ty\dot{y}e^{-\sqrt{2}t} + y\dot{y}e^{-\sqrt{2}t}$, $I_3^1 = -\frac{1}{2}y^2e^{-\sqrt{2}t} - \frac{3}{\sqrt{2}}ty^2e^{-\sqrt{2}t} + B_1.$
$\mathbf{V}_4^1 = -te^{-t}\frac{\partial}{\partial y}$	$\dot{B}_1 = -ye^{-t} + tye^{-t} - t\dot{y}e^{-t}$, $I_4^1 = tye^{-t} + B_1.$
$\mathbf{V}_5^1 = -te^{t}\frac{\partial}{\partial y}$	$\dot{B}_1 = -ye^{t} - tye^{t} - t\dot{y}e^{t}$, $I_5^1 = tye^{t} + B_1.$
$\mathbf{V}_6^1 = -4ty\frac{\partial}{\partial y}$	$\dot{B}_1 = 2y^2 + 2\dot{y}^2$, $I_6^1 = -2ty^2 - 2t\dot{y}^2 + B_1.$

Table 3.4: Approximate Mei Invariants of DHO corresponding to Hamiltonian using **Theorem 3.4.3**.

Approximate Noether symmetries	Approximate Mei symmetries
$\mathbf{Z}_1 = \frac{\partial}{\partial t} - \epsilon y \frac{\partial}{\partial t}$	$\mathbf{V}_1 = \frac{\partial}{\partial t} - \epsilon y \frac{\partial}{\partial t}$
$\mathbf{Z}_2 = \left(\cos 2t \frac{\partial}{\partial t} - y \sin 2t \frac{\partial}{\partial y}\right) + \epsilon\left(-y \cos 2t \frac{\partial}{\partial y}\right)$	$\mathbf{V}_2 = \left(e^{\sqrt{2}t} \frac{\partial}{\partial t} + y\sqrt{2} e^{\sqrt{2}t} \frac{\partial}{\partial y}\right) + \epsilon\left(-\frac{1}{2} t e^{\sqrt{2}t} \frac{\partial}{\partial t}\right) + \left(-\frac{3}{2} y e^{\sqrt{2}t} - \frac{1}{\sqrt{2}} y t e^{\sqrt{2}t}\right)\frac{\partial}{\partial y}$
$\mathbf{Z}_3 = \left(\sin 2t \frac{\partial}{\partial t} + y \cos 2t \frac{\partial}{\partial y}\right) + \epsilon\left(-y \sin 2t \frac{\partial}{\partial y}\right)$	$\mathbf{V}_3 = \left(e^{-\sqrt{2}t} \frac{\partial}{\partial t} + y\sqrt{2} e^{-\sqrt{2}t} \frac{\partial}{\partial y}\right) + \epsilon\left(-\frac{1}{2} t e^{-\sqrt{2}t} \frac{\partial}{\partial t}\right) + \left(-\frac{3}{2} y e^{\sqrt{2}t} + \frac{1}{\sqrt{2}} y t e^{\sqrt{2}t}\right)\frac{\partial}{\partial y}$
$\mathbf{Z}_4 = \sin t \frac{\partial}{\partial y} - \epsilon t \sin t \frac{\partial}{\partial y}$	$\mathbf{V}_4 = e^{-t} \frac{\partial}{\partial y} - \epsilon t e^{-t} \frac{\partial}{\partial y}$
$\mathbf{Z}_5 = \cos t \frac{\partial}{\partial y} - \epsilon t \cos t \frac{\partial}{\partial y}$	$\mathbf{V}_5 = e^{t} \frac{\partial}{\partial y} - \epsilon t e^{t} \frac{\partial}{\partial y}$
	$\mathbf{V}_6 = y \frac{\partial}{\partial y} - \epsilon 4 t y \frac{\partial}{\partial y}$

Table 3.5: Comparison between approximate Noether symmetries and approximate Mei symmetries for DHO.

Chapter 4

Approximate Mei Symmetries Corresponding to the Lagrangian

This chapter presents the formulation of first order approximate Mei symmetries and Mei invariants of the associated Lagrangian. Theorems and determining equations are provided for determining approximate Mei symmetries and approximate first integrals corresponding to each symmetry of the related Lagrangian, which are given in Section 4.1 and Section 4.3. The linear equation of motion of the DHO is used to explain the method in Section 4.2. The Mei symmetries that correspond to the Lagrangian and Hamiltonian of the DHO are compared.

4.1 Approximate Mei Symmetries

Here, approximate Mei symmetries of DHO corresponding to perturbed Lagrangian are formulated in **Theorem 4.1.1** respectively.

Theorem 4.1.1. *Let* $\mathbf{V} = \mathbf{V}^0 + \epsilon \mathbf{V}^1$ *is an approximate symmetry generator and* $\mathcal{L} = \mathcal{L}_0 + \epsilon \mathcal{L}_1$ *is the first order approximate Lagrangian, where* $\mathbf{V}^0 = \alpha_0 \frac{\partial}{\partial t} + \beta_0^a \frac{\partial}{\partial x^a}$ *and* $\mathbf{V}^1 = \alpha_1 \frac{\partial}{\partial t} + \beta_1^a \frac{\partial}{\partial x^a}$. *Then*

$$E^a(\mathbf{V}^{0[1]}\mathcal{L}_0) = 0, \qquad a = 1, 2, ..., n. \tag{4.1.1}$$

$$E^a(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) = 0, \qquad a = 1, 2, ..., n. \tag{4.1.2}$$

Proof. With $\mathbf{V} = \mathbf{V}^0 + \epsilon\mathbf{V}^1$ and $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1$, we have

$$\mathbf{V}^{[1]}\mathcal{L} = \mathbf{V}^{0[1]}\mathcal{L}_0 + \epsilon(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) + O(\epsilon^2), \tag{4.1.3}$$

neglecting higher powers of ϵ, we get

$$\mathbf{V}^{[1]}\mathcal{L} = \mathbf{V}^{0[1]}\mathcal{L}_0 + \epsilon(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0), \tag{4.1.4}$$

applying Euler operator E^a, we get

$$E^a(\mathbf{V}^{[1]}\mathcal{L}) = E^a(\mathbf{V}^{0[1]}\mathcal{L}_0) + \epsilon E^a(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0), \tag{4.1.5}$$

comparing powers of ϵ, we get Eqs. (4.1.1) and (4.1.2). This completes the proof. □

4.2 Example

Consider DHO equation, which is linear in this case

$$\ddot{y} + 2\epsilon\dot{y} + y = 0. \tag{4.2.1}$$

The Lagrangian of DHO is $\mathcal{L} = \frac{1}{2}e^{2\epsilon t}(\dot{y}^2 - y^2)$ is given in [33], with $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1$, then $\mathcal{L}$ takes the following form

$$\mathcal{L} = \frac{1}{2}(\dot{y}^2 - y^2) + \epsilon t(\dot{y}^2 - y^2). \tag{4.2.2}$$

Now, writing $\mathcal{L}$ by separating the powers of ϵ, we obtain

$$\mathcal{L}_0 = \frac{1}{2}(\dot{y}^2 - y^2), \tag{4.2.3a}$$

$$\mathcal{L}_1 = t(\dot{y}^2 - y^2). \tag{4.2.3b}$$

4.2.1 The Mei Symmetries of DHO

To find Mei symmetries, we take first order prolonged infinitesimal generator given in Eq. (2.6.8), is applied to first order perturbed Lagrangian given in Eq. (4.2.3a), yields

$$\mathbf{V}^{0[1]}\mathcal{L}_0 = -y\beta_0 + \dot{y}\beta_{0,t} + \dot{y}^2\beta_{0,y} - \dot{y}^2\alpha_{0,t} - \dot{y}^3\alpha_{0,y}. \tag{4.2.4}$$

Then, applying Euler operator E^a for $a = 1$ on Eq. (4.2.4), gives

$$E^1(\mathbf{V}^{0[1]}\mathcal{L}_0) = 0. \tag{4.2.5}$$

Or,

$$\frac{d}{dt}\left(\frac{\partial \mathbf{V}^{0[1]}\mathcal{L}_0}{\partial \dot{y}}\right) - \left(\frac{\partial \mathbf{V}^{0[1]}\mathcal{L}_0}{\partial y}\right) = 0. \tag{4.2.6}$$

Eq. (4.2.6) gives the following expression

$$\begin{aligned}&\beta_{0,tt} + 2\dot{y}\beta_{0,ty} + 2\ddot{y}\beta_{0,y} - 2\ddot{y}\alpha_{0,t} + \dot{y}^2\beta_{0,yy} - 2\dot{y}\alpha_{0,tt} - 4\dot{y}^2\alpha_{0,ty} - 6\dot{y}\ddot{y}\alpha_{0,y} - 2\dot{y}^3\alpha_{0,yy}\\&+ \beta_0 + y\beta_{0,y} = 0.\end{aligned} \tag{4.2.7}$$

Substituting $\ddot{y} + y = 0$ in above Eq. (4.2.7), we obtain

$$\begin{aligned}&\beta_{0,tt} + 2\dot{y}\beta_{0,ty} - y\beta_{0,y} + 2y\alpha_{0,t} + \dot{y}^2\beta_{0,yy} - 2\dot{y}\alpha_{0,tt} - 4\dot{y}^2\alpha_{0,ty} + 6y\dot{y}\alpha_{0,y} - 2\dot{y}^3\alpha_{0,yy}\\&+ \beta_0 = 0.\end{aligned} \tag{4.2.8}$$

Comparing the coefficients of different powers of $\dot{y}$, we obtain a system of PDEs

$$\beta_{0,tt} + \beta_0 - y\beta_{0,y} + 2y\alpha_{0,t} = 0, \tag{4.2.9}$$

$$\beta_{0,ty} - \alpha_{0,tt} + 3y\alpha_{0,y} = 0, \tag{4.2.10}$$

$$\beta_{0,yy} - 4\alpha_{0,ty} = 0, \tag{4.2.11}$$

$$\alpha_{0,yy} = 0. \tag{4.2.12}$$

Solving Eqs. (4.2.9-4.2.12), we get

$$\alpha_0 = C_1 + \sin\sqrt{2}tC_2 + \cos\sqrt{2}tC_3,$$
$$\beta_0 = -\sqrt{2}y\sin\sqrt{2}tC_3 + \sqrt{2}y\cos\sqrt{2}tC_2 + \cos tC_4 + \sin tC_5 + C_6 y, \tag{4.2.13}$$

corresponding Mei symmetries are listed below

$$\mathbf{V}_1^0 = \frac{\partial}{\partial t}, \tag{4.2.14}$$
$$\mathbf{V}_2^0 = \sin\sqrt{2}t\frac{\partial}{\partial t} + \sqrt{2}y\cos\sqrt{2}t\frac{\partial}{\partial y}, \tag{4.2.15}$$
$$\mathbf{V}_3^0 = \cos\sqrt{2}t\frac{\partial}{\partial t} - \sqrt{2}y\sin\sqrt{2}t\frac{\partial}{\partial y}, \tag{4.2.16}$$
$$\mathbf{V}_4^0 = \cos t\frac{\partial}{\partial y}, \tag{4.2.17}$$
$$\mathbf{V}_5^0 = \sin t\frac{\partial}{\partial y}, \tag{4.2.18}$$
$$\mathbf{V}_6^0 = y\frac{\partial}{\partial y}. \tag{4.2.19}$$

4.2.2 Approximate Mei Symmetries of DHO

Now, we calculate the approximate Mei symmetries up to first order of ϵ by using the exact symmetries given in Eqs. (4.2.14)-(4.2.19). First of all taking $\mathbf{V}_2^0 = \sin\sqrt{2}t\frac{\partial}{\partial t} + \sqrt{2}y\cos\sqrt{2}t\frac{\partial}{\partial y}$, where $\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0$ is expressed as

$$\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0 = -y\beta_1 + \dot{y}\beta_{1,t} + \dot{y}^2\beta_{1,y} - \dot{y}^2\alpha_{1,t} - \dot{y}^3\alpha_{1,y} + \dot{y}^2\sin\sqrt{2}t - y^2\sin\sqrt{2}t$$
$$- 2\sqrt{2}y^2t\cos\sqrt{2}t - 4ty\dot{y}\sin\sqrt{2}t. \tag{4.2.20}$$

Now using Eq. (4.2.20) in Eq. (4.1.2), we have

$$\frac{d}{dt}\left(\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial\dot{y}}\right) - \left(\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial y}\right) = 0. \tag{4.2.21}$$

Putting Eq. (4.2.3a) and (4.2.3b) into Eq. (4.1.2), we get

$$\beta_{1,tt} + 2\dot{y}\beta_{1,ty} + 2\ddot{y}\beta_{1,y} - 2\ddot{y}\alpha_{1,t} + \dot{y}^2\beta_{1,yy} - 2\dot{y}\alpha_{1,tt} - 4\dot{y}^2\alpha_{1,ty} - 6\dot{y}\ddot{y}\alpha_{1,y} - 2\dot{y}^3\alpha_{1,yy}$$
$$+ \beta_1 + y\beta_{1,y} + 2\ddot{y}\sin\sqrt{2}t + 2\sqrt{2}\cos\sqrt{2}t\dot{y} - 2y\sin\sqrt{2}t = 0. \tag{4.2.22}$$

After plugging $\ddot{y}+y=0$, we get

$$\beta_{1,tt}+2\dot{y}\beta_{1,ty}-y\beta_{1,y}+2y\alpha_{1,t}+\dot{y}^2\beta_{1,yy}-2\dot{y}\alpha_{1,tt}-4\dot{y}^2\alpha_{1,ty}+6y\dot{y}\alpha_{1,y}-2\dot{y}^3\alpha_{1,yy}$$
$$+\beta_1+2\sqrt{2}\dot{y}\cos\sqrt{2}t-2y\sin\sqrt{2}t=0. \tag{4.2.23}$$

Again using standard procedure of comparing coefficients of different powers of $\dot{y}$, the obtained system of PDEs is

$$\beta_{1,tt}+\beta_1-y\beta_{1,y}+2y\alpha_{1,t}-4y\sin\sqrt{2}t=0, \tag{4.2.24}$$
$$\beta_{1,ty}-\alpha_{1,tt}+3y\alpha_{0,y}+\sqrt{2}\cos\sqrt{2}t=0, \tag{4.2.25}$$
$$\beta_{1,yy}-4\alpha_{1,ty}=0, \tag{4.2.26}$$
$$\alpha_{1,yy}=0. \tag{4.2.27}$$

Solving above system yields

$$\alpha_1=C_1+\sin\sqrt{2}tC_2+\cos\sqrt{2}tC_3-\frac{1}{2\sqrt{2}}\cos\sqrt{2}t-\frac{1}{2}t\sin\sqrt{2}t,$$
$$\beta_1=-\sqrt{2}y\sin\sqrt{2}tC_3+\sqrt{2}y\cos\sqrt{2}tC_2+\cos tC_4+\sin tC_5+yC_6\frac{1}{\sqrt{2}}ty\cos\sqrt{2t}-y\sin\sqrt{2}t.$$

Substituting any constant equal to one say, $C_2=1$, and all the remaining constants equal to zero gives $\mathbf{V}_2^0$, and $\mathbf{V}_2^1$ is given below

$$\mathbf{V}_2^1=\left(-\frac{1}{2\sqrt{2}}\cos\sqrt{2}t-\frac{1}{2}t\sin\sqrt{2}t\right)\frac{\partial}{\partial t}+\epsilon\left(-\frac{1}{\sqrt{2}}ty\cos\sqrt{2t}-y\sin\sqrt{2}t\right)\frac{\partial}{\partial y}.$$

The nontrivial approximate Mei symmetry of Eq. (4.1.2) have the form

$$\mathbf{V}_2=\mathbf{V}_2^0+\epsilon\mathbf{V}_2^1=\left(\sin\sqrt{2}t\frac{\partial}{\partial t}+\sqrt{2}y\cos\sqrt{2}t\frac{\partial}{\partial y}\right)+\epsilon\left[\left(-\frac{1}{2\sqrt{2}}\cos\sqrt{2}t-\frac{1}{2}t\sin\sqrt{2}t\right)\frac{\partial}{\partial t}\right.$$
$$\left.+\left(-\frac{1}{\sqrt{2}}ty\cos\sqrt{2t}-y\sin\sqrt{2}t\right)\frac{\partial}{\partial y}\right]. \tag{4.2.28}$$

The remaining approximate Mei symmetries are obtained in a similar way as described above. The list of symmetries are given as

$$\mathbf{V}_1=\mathbf{V}_1^0+\epsilon\mathbf{V}_1^1=\frac{\partial}{\partial t}-\epsilon y\frac{\partial}{\partial t}, \tag{4.2.29}$$

$$\mathbf{V}_3=\mathbf{V}_3^0+\epsilon\mathbf{V}_3^1=\left(\cos\sqrt{2}t\frac{\partial}{\partial t}-\sqrt{2}y\sin\sqrt{2}t\frac{\partial}{\partial y}\right)+\epsilon\Bigg[\left(-\frac{1}{2}t\cos\sqrt{2}t+\frac{1}{\sqrt{2}}\sin\sqrt{2}t\right)\frac{\partial}{\partial t}$$
$$+\left(\frac{1}{\sqrt{2}}ty\sin\sqrt{2}t-\frac{1}{2}y\cos\sqrt{2}t\right)\frac{\partial}{\partial y}\Bigg], \tag{4.2.30}$$
$$\mathbf{V}_4=\mathbf{V}_4^0+\epsilon\mathbf{V}_4^1=\sin t\frac{\partial}{\partial y}-\epsilon t\sin t\frac{\partial}{\partial y}, \tag{4.2.31}$$
$$\mathbf{V}_5=\mathbf{V}_5^0+\epsilon\mathbf{V}_5^1=\cos t\frac{\partial}{\partial y}-\epsilon t\cos t\frac{\partial}{\partial y}, \tag{4.2.32}$$
$$\mathbf{V}_6=\mathbf{V}_6^0+\epsilon\mathbf{V}_6^1=y\frac{\partial}{\partial y}-\epsilon 2ty\frac{\partial}{\partial y}. \tag{4.2.33}$$

4.3 Mei Invariants of DHO

To calculate the approximate Mei invariants corresponding to the above symmetries, consider **Theorem 4.3.1** and **Theorem 4.3.2.**

Theorem 4.3.1. *There exist Mei conserved quantities*

$$I^0=\alpha_0(\mathbf{V}^{0[1]}\mathcal{L}_0)+(\beta_0^a-\dot{x}^a\alpha_0)\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial\dot{x}^a}+B_0, \tag{4.3.1}$$
$$I^1=\alpha_0(\mathbf{V}^{0[1]}\mathcal{L}_1+\mathbf{V}^{1[1]}\mathcal{L}_0)+\alpha_1(\mathbf{V}^{0[1]}\mathcal{L}_0)+(\beta_1^a-\dot{x}^a\alpha_1)\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial\dot{x}^a}+(\beta_0^a-\dot{x}^a\alpha_0)$$
$$\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1+\mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial\dot{x}^a}+B_1, \tag{4.3.2}$$

corresponding to Mei symmetry generator $\mathbf{V}^0=\alpha_0\frac{\partial}{\partial t}+\beta_0^a\frac{\partial}{\partial x^a}$ *and* $\mathbf{V}^1=\alpha_1\frac{\partial}{\partial t}+\beta_1^a\frac{\partial}{\partial x^a}$ *that satisfy the conditions*

$$\dot{\alpha_0}(\mathbf{V}^{0[1]}\mathcal{L}_0)+\mathbf{V}^{0[1]}(\mathbf{V}^{0[1]}\mathcal{L}_0)+\dot{B}_0=0, \tag{4.3.3}$$
$$\dot{\alpha_0}(\mathbf{V}^{0[1]}\mathcal{L}_1+\mathbf{V}^{1[1]}\mathcal{L}_0)+\dot{\alpha_1}(\mathbf{V}^{0[1]}\mathcal{L}_0)+\mathbf{V}^{0[1]}(\mathbf{V}^{0[1]}\mathcal{L}_1+\mathbf{V}^{1[1]}\mathcal{L}_0)+\mathbf{V}^{1[1]}(\mathbf{V}^{0[1]}\mathcal{L}_0)$$
$$+\dot{B}_1=0, \tag{4.3.4}$$

where $B_0=B_0(t,x^a,\dot{x^a})$ *and* $B_1=B_1(t,x^a,\dot{x^a})$ *are gauge functions.*

Proof. To prove the above expression, consider the Mei invariant given in Eq. (2.9.1). Introducing first-order perturbed invariant by taking $I=I^0+\epsilon I^1$, $\mathbf{V}^{[1]}=\mathbf{V}^{0[1]}+\epsilon\mathbf{V}^{1[1]}$,

$B = B_0 + \epsilon B_1$ and $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1$ in Eq. (2.9.1), we obtain

$$I^0 + \epsilon I^1 = (\alpha_0 + \epsilon\alpha_1)[(\mathbf{V}^{0[1]} + \epsilon\mathbf{V}^{1[1]})(\mathcal{L}_0 + \epsilon\mathcal{L}_1)] + [(\beta_0^a + \epsilon\beta_1^a) - \dot{x}^a(\alpha_0 + \epsilon\alpha_1)] \frac{\partial(\mathbf{V}^{0[1]} + \epsilon\mathbf{V}^{1[1]})}{\partial\dot{x}^a}(\mathcal{L}_0 + \epsilon\mathcal{L}_1) + (B_0 + \epsilon B_1), \tag{4.3.5}$$

after simplifying the above Eq. (4.3.5), we get

$$I^0 + \epsilon I^1 = \alpha_0(\mathbf{V}^{0[1]}\mathcal{L}_0) + \epsilon[\alpha_0(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) + \alpha_1(\mathbf{V}^{0[1]}\mathcal{L}_0)] + (\beta_0^a - \dot{x}^a\alpha_0)\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial\dot{x}^a} + \epsilon\left[(\beta_1^a - \dot{x}^a\alpha_1)\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial\dot{x}^a} + (\beta_0^a - \dot{x}^a\alpha_0)\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial\dot{x}^a}\right] + (B_0 + \epsilon B_1), \tag{4.3.6}$$

comparing powers of ϵ up-to first order and neglecting higher powers give Eq. (4.3.2)

$$I^1 = \alpha_0(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) + \alpha_1(\mathbf{V}^{0[1]}\mathcal{L}_0) + (\beta_1^a - \dot{x}^a\alpha_1)\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial\dot{x}^a} + (\beta_0^a - \dot{x}^a\alpha_0) \frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial\dot{x}^a} + B_1.$$

Since $\frac{dI^0}{dt} = 0$ is proved in **Theorem 2.9.1** of Chapter 1. Now, we need to prove that $\frac{dI^1}{dt} = 0$. Taking derivative of I^1 with respect to t, we get

$$\frac{dI^1}{dt} = \dot{\alpha_0}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) + \dot{\alpha_1}(\mathbf{V}^{0[1]}\mathcal{L}_0) + \alpha_0\left[\frac{\partial}{\partial t}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) + \dot{x^a}\frac{\partial}{\partial x^a} (\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) + \ddot{x^a}\frac{\partial}{\partial\dot{x^a}}(\mathbf{V}^{1[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)\right] + \alpha_1\left[\frac{\partial}{\partial t}(\mathbf{V}^{0[1]}\mathcal{L}_0) + \dot{x^a}\frac{\partial}{\partial x^a}(\mathbf{V}^{0[1]}\mathcal{L}_0) + \ddot{x^a}\frac{\partial}{\partial\dot{x^a}}(\mathbf{V}^{0[1]}\mathcal{L}_0)\right] + (\beta_0^a - \dot{x^a}\alpha_0)\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x^a}}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)\right) + (\dot{\beta_0^a} - \dot{x^a}\dot{\alpha_0} - \ddot{x^a}\alpha_0) \frac{\partial}{\partial\dot{x^a}}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) + (\beta_1^a - \dot{x^a}\alpha_1)\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x^a}}(\mathbf{V}^{0[1]}\mathcal{L}_0)\right) + (\dot{\beta_1^a} - \dot{x^a}\dot{\alpha_1} - \ddot{x^a}\alpha_1)\frac{\partial}{\partial\dot{x^a}} (\mathbf{V}^{0[1]}\mathcal{L}_0) - \dot{\alpha_0}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) - \dot{\alpha_1}(\mathbf{V}^{0[1]}\mathcal{L}_0) - \alpha_0\frac{\partial}{\partial t}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) - \beta_0^a \frac{\partial}{\partial x^a}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) - (\dot{\beta_0^a} - \dot{x^a}\dot{\alpha_0})\frac{\partial}{\partial\dot{x^a}}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) - \alpha_1\frac{\partial}{\partial t}(\mathbf{V}^{0[1]}\mathcal{L}_0) - \beta_1^a\frac{\partial}{\partial x^a} (\mathbf{V}^{0[1]}\mathcal{L}_0) - (\dot{\beta_1^a} - \dot{x^a}\dot{\alpha_1})\frac{\partial}{\partial\dot{x^a}}(\mathbf{V}^{0[1]}\mathcal{L}_0),$$

$$\frac{dI^1}{dt} = (\beta_1^a - \dot{x^a}\alpha_1)\left[\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x^a}}(\mathbf{V}^{0[1]}\mathcal{L}_0)\right) - \frac{\partial}{\partial x^a}(\mathbf{V}^{0[1]}\mathcal{L}_0)\right)\right] + (\beta_0^a - \dot{x^a}\alpha_0) \left[\frac{d}{dt}\left(\frac{\partial}{\partial\dot{x^a}}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)\right) - \frac{\partial}{\partial x^a}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)\right)\right],$$

$$\frac{dI^1}{dt} = (\beta_1^a - \dot{x^a}\alpha_1)E^a(\mathbf{V}^{0[1]}\mathcal{L}_0) + (\beta_0^a - \dot{x^a}\alpha_0)E^a(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0),$$
$$\frac{dI^1}{dt} = 0,$$

this completes the proof. □

First we calculate Mei invariant corresponding to $\mathbf{V}_2^0$ using Eqs. (4.3.1) and (4.3.3). Consider

$$(\mathbf{V}_2^0)^{[1]} = \left(\cos\sqrt{2}t\frac{\partial}{\partial t} - \sqrt{2}y\sin\sqrt{2}t\frac{\partial}{\partial y} - 2y\cos\sqrt{2}t\frac{\partial}{\partial \dot{y}}\right),$$

and $\mathcal{L}_0 = \frac{1}{2}(\dot{y}^2 - y^2)$, then

$$(\mathbf{V}_2^0)^{[1]}\mathcal{L}_0 = \sqrt{2}y^2\sin\sqrt{2}t - 2y\dot{y}\cos\sqrt{2}t,$$

using Eq. (4.3.3), we get $\dot{B}_0$

$$\dot{B}_0 = 6y^2\sin^2\sqrt{2}t - 6\sqrt{2}y\dot{y}\cos\sqrt{2}t\sin\sqrt{2}t - 6y^2\cos\sqrt{2}t\cos\sqrt{2}t, \quad (4.3.7)$$

we calculate the Mei invariant using Eq. (4.3.1), we obtain

$$I_2^0 = 3\sqrt{2}y^2\cos\sqrt{2}t\sin\sqrt{2}t + B_0. \quad (4.3.8)$$

To prove that Eq. (4.3.8) is a conserved quantity, we need to show that $\frac{dI_2^0}{dt} = 0$. Taking derivative of I_2^0 with respect to t, we get

$$\frac{dI_2^0}{dt} = -6y^2\sin^2\sqrt{2}t + 6\sqrt{2}y\dot{y}\cos\sqrt{2}t\sin\sqrt{2}t + 6y^2\cos\sqrt{2}t\cos\sqrt{2}t + \dot{B}_0,$$

substituting $\dot{B}_0$ from Eq. (4.3.7), we get

$$\frac{dI_2^0}{dt} = 0.$$

Again using **Theorem 4.3.1**, approximate Mei invariants are calculated with the help of Eqs. (4.3.2) and (4.3.4). Consider

$$\mathbf{V}_2^1 = \left[\left(-\frac{1}{2}t\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\right)\frac{\partial}{\partial t} + \left(\frac{1}{\sqrt{2}}ty\sin\sqrt{2t} - \frac{1}{2}y\cos\sqrt{2}t\right)\frac{\partial}{\partial y}\right],$$

and

$$(\mathbf{V}_2^{0[1]}\mathcal{L}_1+\mathbf{V}_2^{1[1]}\mathcal{L}_0)=-\frac{y^2}{2}\cos\sqrt{2}t+\frac{3}{\sqrt{2}}y^2t\sin\sqrt{2}t+\frac{2}{\sqrt{2}}y\dot{y}\sin\sqrt{2}t-3ty\dot{y}\cos\sqrt{2}t,$$
$$(\mathbf{V}_2^{0})^{[1]}\mathcal{L}_0=\sqrt{2}y^2\sin\sqrt{2}t-2y\dot{y}\cos\sqrt{2}t.$$

Now using Eq. (4.3.4) to calculate $\dot{B}_1$, where

$$\mathbf{V}_2^{0[1]}(\mathbf{V}_2^{0[1]}\mathcal{L}_1+\mathbf{V}_2^{1[1]}\mathcal{L}_0)=\frac{2}{\sqrt{2}}y^2\sin\sqrt{2}t\cos\sqrt{2}t-y\dot{y}\cos^2\sqrt{2}t+6\sqrt{2}ty\dot{y}\sin\sqrt{2}t\cos\sqrt{2}t$$
$$-2y\dot{y}\sin^2\sqrt{2}t-6ty^2\sin^2\sqrt{2}t+9ty^2\cos^2\sqrt{2}t,$$
$$\mathbf{V}_2^{1[1]}(\mathbf{V}_2^{0[1]}\mathcal{L}_0)=-2\sqrt{2}ty\dot{y}\sin\sqrt{2}t\cos\sqrt{2}t+2y\dot{y}\sin^2\sqrt{2}t+y\dot{y}\cos^2\sqrt{2}t-3ty^2\cos^2\sqrt{2}t$$
$$+2ty^2\sin^2\sqrt{2}t,$$
$$\dot{\alpha}_0(\mathbf{V}_2^{0[1]}\mathcal{L}_1+\mathbf{V}_2^{1[1]}\mathcal{L}_0)=\frac{1}{\sqrt{2}}y^2\sin\sqrt{2}t\cos\sqrt{2}t-3ty^2\sin^2\sqrt{2}t-2y\dot{y}\sin^2\sqrt{2}t+3\sqrt{2}ty\dot{y}$$
$$\sin\sqrt{2}t\cos\sqrt{2}t,$$
$$\dot{\alpha}_1(\mathbf{V}_0^{2[1]}\mathcal{L}_0)=\frac{1}{\sqrt{2}}y^2\sin\sqrt{2}t\cos\sqrt{2}t+ty^2\sin^2\sqrt{2}t-y\dot{y}\cos^2\sqrt{2}t-\sqrt{2}ty\dot{y}\sin\sqrt{2}t\cos\sqrt{2}t$$

substituting above expressions in Eq. (4.3.4), we get

$$\dot{B}_1=-2\sqrt{2}y^2\sin\sqrt{2}t\cos\sqrt{2}t+y\dot{y}\cos^2\sqrt{2}t-6\sqrt{2}ty\dot{y}\sin\sqrt{2}t\cos\sqrt{2}t+2y\dot{y}\sin^2\sqrt{2}t$$
$$+6ty^2\sin^2\sqrt{2}t-6ty^2\cos^2\sqrt{2}t. \tag{4.3.9}$$

Using Eq. (4.3.2), we get

$$I_2^1=-\frac{y^2}{2}\cos^2\sqrt{t}-y^2\sin^2\sqrt{t}+3\sqrt{2}ty^2\sin\sqrt{2}t\cos\sqrt{2}t+B_1, \tag{4.3.10}$$

to prove Eq. (4.3.10) is a conserved quantity, we need to show that $\frac{dI_2^1}{dt}=0$.

$$\frac{dI_2^1}{dt}=2\sqrt{2}y^2\sin\sqrt{2}t\cos\sqrt{2}t-y\dot{y}\cos^2\sqrt{2}t+6\sqrt{2}ty\dot{y}\sin\sqrt{2}t\cos\sqrt{2}t-2y\dot{y}\sin^2\sqrt{2}t$$
$$-6ty^2\sin^2\sqrt{2}t+6ty^2\cos^2\sqrt{2}t+\dot{B}_1,$$

plugging $\dot{B}_1$ into above expression, we get

$$\frac{dI_2^1}{dt}=0.$$

Mei symmetries of DHO corresponding to Lagrangian	Mei Invariants of DHO corresponding to Lagrangian
$\mathbf{V}_1^0 = \frac{\partial}{\partial t}.$	$I_1^0 = 0, \dot{B}_0 = 0$
$\mathbf{V}_2^0 = \left(\cos\sqrt{2}t\frac{\partial}{\partial t} - \sqrt{2}y\sin\sqrt{2}t\frac{\partial}{\partial y}\right).$	$\dot{B}_0 = 6y^2\sin^2\sqrt{2}t - 6\sqrt{2}y\dot{y}\cos\sqrt{2}t\sin\sqrt{2}t$ $- 6y^2\cos\sqrt{2}t\cos\sqrt{2}t,$ $I_2^0 = 3\sqrt{2}y^2\cos\sqrt{2}t\sin\sqrt{2}t + B_0.$
$\mathbf{V}_3^0 = \left(\sin\sqrt{2}t\frac{\partial}{\partial t} + \sqrt{2}y\cos\sqrt{2}t\frac{\partial}{\partial y}\right).$	$\dot{B}_0 = -6y^2\sin^2\sqrt{2}t + 6\sqrt{2}y\dot{y}\cos\sqrt{2}t\sin\sqrt{2}t$ $+ 6y^2\cos\sqrt{2}t\cos\sqrt{2}t,$ $I_3^0 = -3\sqrt{2}y^2\cos\sqrt{2}t\sin\sqrt{2}t + B_0.$
$\mathbf{V}_4^0 = \sin t\frac{\partial}{\partial y}.$	$\dot{B}_0 = \sin^2 t - \cos^2 t,$ $I_4^0 = \sin t\cos t + \dot{B}_0.$
$\mathbf{V}_5^0 = \cos t\frac{\partial}{\partial y}.$	$\dot{B}_0 = -\sin^2 t + \cos^2 t,$ $I_5^0 = -\sin t\cos t + \dot{B}_0.$
$\mathbf{V}_6^0 = y\frac{\partial}{\partial y}.$	$\dot{B}_0 = 2y^2 - 2\dot{y}^2,$ $I_6^0 = 2y\dot{y}.$

Table 4.1: Mei Invariants of DHO using Eq. (4.3.1)

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Approximate Mei symmetries of DHO	**Approximate Mei Invariants of DHO**
$\mathbf{V}_1^1 = y\frac{\partial}{\partial t}$	$I_1^1 = 0, \dot{B}_1 = 0$
$\mathbf{V}_2^1 = \Big[\Big(-\frac{1}{2}t\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\Big)\frac{\partial}{\partial t} + \Big(\frac{1}{\sqrt{2}}ty\sin\sqrt{2}t - \frac{1}{2}y\cos\sqrt{2}t\Big)\frac{\partial}{\partial y}\Big]$	$\dot{B}_1 = -2\sqrt{2}y^2\sin\sqrt{2}t\cos\sqrt{2}t + y\dot{y}\cos^2\sqrt{2}t - 6\sqrt{2}ty\dot{y}\sin\sqrt{2}t\cos\sqrt{2}t + 2y\dot{y}\sin^2\sqrt{2}t + 6ty^2\sin^2\sqrt{2}t - 6ty^2\cos^2\sqrt{2}t,$ $I_2^1 = -\frac{y^2}{2}\cos^2\sqrt{t} - y^2\sin^2\sqrt{t} + 3\sqrt{2}ty^2\sin\sqrt{2}t\cos\sqrt{2}t + B_1.$
$\mathbf{V}_3^1 = \Big[\Big(-\frac{1}{2\sqrt{2}}\cos\sqrt{2}t - \frac{1}{2}t\sin\sqrt{2}t\Big)\frac{\partial}{\partial t} + \Big(-\frac{1}{\sqrt{2}}ty\cos\sqrt{2}t - y\sin\sqrt{2}t\Big)\frac{\partial}{\partial y}\Big]$	$\dot{B}_1 = -6\sqrt{2}y^2\sin\sqrt{2}t\cos\sqrt{2}t + 5y\dot{y}\cos^2\sqrt{2}t + 6\sqrt{2}ty\dot{y}\sin\sqrt{2}t\cos\sqrt{2}t - 4y\dot{y}\sin^2\sqrt{2}t - 6ty^2\sin^2\sqrt{2}t + 6ty^2\cos^2\sqrt{2}t,$ $I_3^1 = -\frac{5}{2}y^2\cos^2\sqrt{t} + 2y^2\sin^2\sqrt{t} - 3\sqrt{2}ty^2\sin\sqrt{2}t\cos\sqrt{2}t + B_1.$
$\mathbf{V}_4^1 = -t\sin t\frac{\partial}{\partial y}$	$\dot{B}_1 = 2\cos t\sin t,$ $I_4^1 = -\sin^2 t + B_1.$
$\mathbf{V}_5^1 = -t\cos t\frac{\partial}{\partial y}$	$\dot{B}_1 = -2\cos t\sin t,$ $I_5^1 = -\cos^2 t + B_1.$
$\mathbf{V}_6^1 = -2ty\frac{\partial}{\partial y}$	$\dot{B}_1 = 8y\dot{y} + 4t\dot{y}^2 - 4ty^2,$ $I_6^1 = -4ty\dot{y} - 2y^2 + B_1.$

Table 4.2: Approximate Mei Invariants of DHO corresponding to Lagrangian using Eq. (4.3.2)
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Theorem 4.3.2. *There exist Mei conserved quantities*

$$I^0 = (\mathbf{V}^{0[1]}\mathcal{L}_0) - \dot{x^a}\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial \dot{x}^a} + B_0, \tag{4.3.11}$$

$$I^1 = (\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) - \dot{x^a}\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial \dot{x}^a} + B_1, \tag{4.3.12}$$

corresponding to Mei symmetry generators $\mathbf{V}^0 = \alpha_0\frac{\partial}{\partial t} + \beta_0^a\frac{\partial}{\partial x^a}$ *and* $\mathbf{V}^1 = \alpha_1\frac{\partial}{\partial t} + \beta_1^a\frac{\partial}{\partial x^a}$ *that satisfy the conditions*

$$\frac{\partial}{\partial t}(\mathbf{V}^{0[1]}\mathcal{L}_0) + \dot{B}_0 = 0, \tag{4.3.13}$$

$$\frac{\partial}{\partial t}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) + \dot{B}_1 = 0, \tag{4.3.14}$$

where $B_0 = B_0(t, x^a, \dot{x}^a)$ *and* $B_1 = B_1(t, x^a, \dot{x}^a)$ *are gauge functions.*

Proof. To prove the above expressions, consider the Mei invariant given in Eq. (2.9.3). Introducing first-order perturbed invariant by taking $I = I^0 + \epsilon I^1$, $\mathbf{V}^{[1]} = \mathbf{V}^{0[1]} + \epsilon\mathbf{V}^{1[1]}$, $B = B_0 + \epsilon B_1$ and $\mathcal{L} = \mathcal{L}_0 + \epsilon\mathcal{L}_1$ in Eq. (2.9.3), we obtain

$$I^0 + \epsilon I^1 = [(\mathbf{V}^{0[1]} + \epsilon\mathbf{V}^{1[1]})(\mathcal{L}_0 + \epsilon\mathcal{L}_1)] - \dot{x^a}\frac{\partial}{\partial\dot{x^a}}[(\mathbf{V}^{0[1]} + \epsilon\mathbf{V}^{1[1]})(\mathcal{L}_0 + \epsilon\mathcal{L}_1)]$$
$$+ (B_0 + \epsilon B_1), \tag{4.3.15}$$

after simplifying the above Eq. (4.3.15), we get

$$I^0 + \epsilon I^1 = [(\mathbf{V}^{0[1]}\mathcal{L}_0) + \epsilon[(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)] - \frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial \dot{x}^a} + (B_0 + \epsilon B_1), \tag{4.3.16}$$

comparing powers of ϵ up-to first order and neglecting higher powers, give Eqs. (4.3.11) and (4.3.12). This completes the proof. □

Since I^0 is a conserved quantity proved in **Theorem 2.9.2** of Chapter 1. Now we need to show that I^1 is a conserved quantity. For this, taking derivative of I^1 with respect to $'t'$, we get

$$\frac{dI^1}{dt} = \left(\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial t} + \dot{x^a}\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial x^a} + \ddot{x^a}\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial \dot{x^a}}\right)$$
$$- \frac{d}{dt}\left(\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial \dot{x^a}}\right)\dot{x^a} - \ddot{x^a}\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial \dot{x^a}} + \dot{B}_1,$$
$$\frac{dI_1}{dt} = \left[-\frac{d}{dt}\left(\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial \dot{x^a}}\right) + \frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0)}{\partial x^a}\right]\dot{x^a},$$
$$\frac{dI^1}{dt} = E^a((\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0))\dot{x^a},$$
$$\frac{dI^1}{dt} = 0,$$

hence I^1 is a conserved quantity.

Now, we calculate Mei invariant corresponding to $\mathbf{V}_2^0$ using Eqs. (4.3.11) and (4.3.13).

$$(\mathbf{V}_2^0)^{[1]} = \left(\cos\sqrt{2}t\frac{\partial}{\partial t} - \sqrt{2}y\sin\sqrt{2}t\frac{\partial}{\partial y} - 2y\cos\sqrt{2}t\frac{\partial}{\partial \dot{y}}\right),$$

and $\mathcal{L}_0 = \frac{1}{2}(\dot{y}^2 - y^2)$, then

$$(\mathbf{V}_2^0)^{[1]}\mathcal{L}_0 = \sqrt{2}y^2\sin\sqrt{2}t - 2y\dot{y}\cos\sqrt{2}t,$$

using Eq. (4.3.13), we get $\dot{B}_0$

$$\dot{B}_0 = -2\sqrt{2}y\dot{y}\sin\sqrt{2}t - 2y^2\cos\sqrt{2}t, \tag{4.3.17}$$

we calculate the Mei invariant using Eq. (4.3.11), we obtain

$$I_2^0 = \sqrt{2}y^2\sin\sqrt{2}t + B_0. \tag{4.3.18}$$

To prove that Eq. (4.3.18) is a conserved quantity, we need to show that $\frac{dI_0^2}{dt} = 0$.

$$\frac{dI_2^0}{dt} = 2y^2\cos^2\sqrt{2}t + 2\sqrt{2}y\dot{y}\sin\sqrt{2}t + \dot{B}_0,$$

substituting $\dot{B}_0$ from Eq. (4.3.17), we get

$$\frac{dI_2^0}{dt} = 0.$$

Now, we calculate approximate Mei invariants by using Eqs. (4.3.12) and (4.3.14). Consider

$$\mathbf{V}_2^1 = \Bigg[\bigg(-\frac{1}{2}t\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\bigg)\frac{\partial}{\partial t} + \bigg(\frac{1}{\sqrt{2}}ty\sin\sqrt{2t} - \frac{1}{2}y\cos\sqrt{2}t\bigg)\frac{\partial}{\partial y}\Bigg],$$

and

$$(\mathbf{V}_2^{0[1]}\mathcal{L}_1 + \mathbf{V}_2^{1[1]}\mathcal{L}_0) = -\frac{y^2}{2}\cos\sqrt{2}t + \frac{3}{\sqrt{2}}y^2 t\sin\sqrt{2}t + \frac{2}{\sqrt{2}}y\dot{y}\sin\sqrt{2}t - 3ty\dot{y}\cos\sqrt{2}t.$$

Now for $\dot{B}_1$, we consider Eq. (4.3.13)

$$\dot{B}_1 = \frac{3}{\sqrt{2}}y^2\cos\sqrt{2}t - 3ty^2\sin\sqrt{2}t + 3\sqrt{2}ty\dot{y}\cos\sqrt{2}t,$$

then I_2^1 takes the form

$$I_2^1 = -\frac{3}{\sqrt{2}}y^2\cos\sqrt{2}t + B_1.$$

Now we need to prove $\frac{dI_2^1}{dt} = 0$. Taking derivative of I_2^1 with respect to t, we obtain

$$\frac{dI_2^1}{dt} = -\frac{3}{\sqrt{2}}y^2\cos\sqrt{2}t + 3ty^2\sin\sqrt{2}t - 3\sqrt{2}ty\dot{y}\cos\sqrt{2}t + \dot{B}_1,$$

substituting $\dot{B}_1$, we get

$$\frac{dI_2^1}{dt} = 0.$$

Mei symmetries of DHO corresponding to Lagrangian	Mei Invariants of DHO corresponding to Lagrangian
$\mathbf{V}_1^0 = \frac{\partial}{\partial t}$	$I_1^0 = 0, \dot{B}_0 = 0.$
$\mathbf{V}_2^0 = \left(\cos\sqrt{2}t\frac{\partial}{\partial t} - \sqrt{2}y\sin\sqrt{2}t\frac{\partial}{\partial y} \right)$	$\dot{B}_0 = -2\sqrt{2}y\dot{y}\sin\sqrt{2}t - 2y^2\cos\sqrt{2}t,$ $I_2^0 = \sqrt{2}y^2\sin\sqrt{2}t + B_0.$
$\mathbf{V}_3^0 = \left(\sin\sqrt{2}t\frac{\partial}{\partial t} + \sqrt{2}y\cos\sqrt{2}t\frac{\partial}{\partial y} \right)$	$\dot{B}_0 = 2\sqrt{2}y\dot{y}\cos\sqrt{2}t - 2y^2\sin\sqrt{2}t,$ $I_3^0 = -\sqrt{2}y^2\cos\sqrt{2}t + B_0.$
$\mathbf{V}_4^0 = \sin t\frac{\partial}{\partial y}$	$\dot{B}_0 = y\cos t + \dot{y}\sin t,$ $I_4^0 = -y\sin t + B_0.$
$\mathbf{V}_5^0 = \cos t\frac{\partial}{\partial y}$	$\dot{B}_0 = -y\sin t + \dot{y}\cos t,$ $I_5^0 = -y\cos t + B_0.$
$\mathbf{V}_6^0 = y\frac{\partial}{\partial y}$	$\dot{B}_0 = 0,$ $I_6^0 = y^2 - \dot{y}^2 + B_0.$

Table 4.3: Mei Invariants of DHO using Eq. (4.3.11)

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Approximate Mei symmetries of DHO	Approximate Mei Invariants of DHO
$\mathbf{V}_1^1 = y\frac{\partial}{\partial t}$	$I_1^1 = 0, \dot{B}_1 = 0.$
$\mathbf{V}_2^1 = \Big[\Big(-\frac{1}{2}t\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\Big)\frac{\partial}{\partial t} + \Big(\frac{1}{\sqrt{2}}ty\sin\sqrt{2t} - \frac{1}{2}y\cos\sqrt{2}t\Big)\frac{\partial}{\partial y}\Big]$	$\dot{B}_1 = -2\sqrt{2}y^2\sin\sqrt{2}t - 3ty^2\cos\sqrt{2}t + y\dot{y}\cos\sqrt{2}t - 3\sqrt{2}ty\dot{y}\sin\sqrt{2}t,$ $I_1^2 = -\frac{y^2}{2}\cos\sqrt{t} + 3\sqrt{2}y^2\sin\sqrt{2}t + B_1.$
$\mathbf{V}_3^1 = \Big[\Big(-\frac{1}{2\sqrt{2}}\cos\sqrt{2}t - \frac{1}{2}t\sin\sqrt{2}t\Big)\frac{\partial}{\partial t} + \Big(-\frac{1}{\sqrt{2}}ty\cos\sqrt{2t} - y\sin\sqrt{2}t\Big)\frac{\partial}{\partial y}\Big]$	$\dot{B}_1 = -\frac{3}{\sqrt{2}}y^2\cos\sqrt{2}t - 3ty^2\sin\sqrt{2}t + 3\sqrt{2}ty\dot{y}\cos\sqrt{2}t,$ $I_3^1 = -\frac{3}{\sqrt{2}}ty^2\cos\sqrt{t} + B_1.$
$\mathbf{V}_4^1 = -t\sin t\frac{\partial}{\partial y}$	$\dot{B}_1 = y\sin t + ty\cos t + t\dot{y}\sin t,$ $I_4^1 = -ty\sin t + B_1.$
$\mathbf{V}_5^1 = -t\cos t\frac{\partial}{\partial y}$	$\dot{B}_1 = y\cos t - ty\sin t + t\dot{y}\cos t,$ $I_5^1 = -ty\cos t + B_1.$
$\mathbf{V}_6^1 = -2ty\frac{\partial}{\partial y}$	$\dot{B}_1 = 0,$ $I_6^1 = -2y^2 - 2\dot{y}^2 + B_1.$

Table 4.4: Approximate Mei Invariants of DHO corresponding to Lagrangian using Eq. 4.3.12

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4.3.1 Comparison between Mei Symmetries of the Hamiltonian and the Lagrangian

The Mei symmetries relating to the Hamiltonian and Lagrangian of DHO are compared in Table 4.5. The approximate Mei symmetries corresponding to Hamiltonian are discussed in detail in Section 3.3 of Chapter 3. Both sets of symmetries give rise to different conserved quantities. The number of Mei symmetries corresponding to both Hamiltonian and the Lagrangian is the same. In the two sets of symmetries, $\mathbf{V}_1$ is common, approximate part of $\mathbf{V}_2$ is slightly different, whereas other Mei symmetries i.e., $\mathbf{V}_3$, $\mathbf{V}_4$, $\mathbf{V}_5$ and $\mathbf{V}_6$ are completely different.

Concluding Remarks:

The approximate Mei symmetries and invariants corresponding to Lagrangian of DHO are constructed in this chapter. The Mei symmetries corresponding to Lagrangian and Hamiltonian are compared. In both sets, the number of Mei symmetries is equal.

Mei symmetries of the Hamiltonian	**Mei symmetries of the Lagrangian**
$\mathbf{V}_1 = \frac{\partial}{\partial t} - \epsilon y \frac{\partial}{\partial t}$	$\mathbf{V}_1 = \frac{\partial}{\partial t} - \epsilon y \frac{\partial}{\partial t}$
$\mathbf{V}_2 = \left(e^{\sqrt{2}t}\frac{\partial}{\partial t} + y\sqrt{2}e^{\sqrt{2}t}\frac{\partial}{\partial y}\right) + \epsilon\left(-\frac{1}{2}te^{\sqrt{2}t}\frac{\partial}{\partial t}\right) + \left(-\frac{3}{2}ye^{\sqrt{2}t} - \frac{1}{\sqrt{2}}yte^{\sqrt{2}t}\right)\frac{\partial}{\partial y}$	$\mathbf{V}_2 = \left(\cos\sqrt{2}t\frac{\partial}{\partial t} - \sqrt{2}y\sin\sqrt{2}t\frac{\partial}{\partial y}\right) + \epsilon\Big[\Big(-\frac{1}{2}t\cos\sqrt{2}t + \frac{1}{\sqrt{2}}\sin\sqrt{2}t\Big)\frac{\partial}{\partial t} + \left(\frac{1}{\sqrt{2}}ty\sin\sqrt{2t} - \frac{1}{2}y\cos\sqrt{2}t\right)\frac{\partial}{\partial y}\Big]$
$\mathbf{V}_3 = \left(e^{-\sqrt{2}t}\frac{\partial}{\partial t} + y\sqrt{2}e^{-\sqrt{2}t}\frac{\partial}{\partial y}\right) + \epsilon\left(-\frac{1}{2}te^{-\sqrt{2}t}\frac{\partial}{\partial t}\right) + \left(-\frac{3}{2}ye^{\sqrt{2}t} + \frac{1}{\sqrt{2}}yte^{\sqrt{2}t}\right)\frac{\partial}{\partial y}$	$\mathbf{V}_3 = \left(\sin\sqrt{2}t\frac{\partial}{\partial t} + \sqrt{2}y\cos\sqrt{2}t\frac{\partial}{\partial y}\right) + \epsilon\Big[\Big(-\frac{1}{2\sqrt{2}}\cos\sqrt{2}t - \frac{1}{2}t\sin\sqrt{2}t\Big)\frac{\partial}{\partial t} + \left(-\frac{1}{\sqrt{2}}ty\cos\sqrt{2t} - y\sin\sqrt{2}t\right)\frac{\partial}{\partial y}\Big]$
$\mathbf{V}_4 = e^{-t}\frac{\partial}{\partial y} - \epsilon te^{-t}\frac{\partial}{\partial y}$	$\mathbf{V}_4 = \sin t\frac{\partial}{\partial y} - \epsilon t\sin t\frac{\partial}{\partial y}$
$\mathbf{V}_5 = e^{t}\frac{\partial}{\partial y} - \epsilon te^{t}\frac{\partial}{\partial y}$	$\mathbf{V}_5 = \cos t\frac{\partial}{\partial y} - \epsilon t\cos t\frac{\partial}{\partial y}$
$\mathbf{V}_6 = y\frac{\partial}{\partial y} - \epsilon 4ty\frac{\partial}{\partial y}.$	$\mathbf{V}_6 = y\frac{\partial}{\partial y} - \epsilon 2ty\frac{\partial}{\partial y}.$

Table 4.5: Comparison between Mei Symmetries corresponding to the Hamiltonian and the Lagrangian

Chapter 5

Approximate Mei Symmetries of pp-Waves Spacetime

The approximate Mei symmetries are constructed for DHO in previous chapters. Here we discuss an application of approximate Mei symmetries of pp-waves spacetime from general theory of relativity in Section 5.1 and Section 5.2. The line element of the pp-waves spacetimes have the following form [43]

$$ds^2 = -2h(u,x,y)du^2 - 2dudv + dx^2 + dy^2, \tag{5.0.1}$$

where h is called the metric function. Eq. (5.0.1) is called vacuum pp-waves if $h_{xx} + h_{yy} = 0$ and it is called conformally flat if $h_{xx} = h_{yy}$ and $h_{xy} = 0$. These spacetimes are known as plane-fronted gravitational waves with parallel rays (pp-waves), and discovered by Brinkmann [44].

If h has the form

$$2h(u,x,y) = x^2 A(u) + 2xB(u)y + y^2 C(u), \tag{5.0.2}$$

then Eq. (5.0.2) is called the plane wave spacetimes. A conformal Killing vector (CKV) field $\mathbf{V}$ produces a group of conformal motions that is defined by [46]

$$\mathcal{L}_{\mathbf{V}} g_{ab} = 2\Psi g_{ab} \iff g_{ab,c}\mathbf{V}^c + g_{ac}\mathbf{V}^c_b + g_{cb}\mathbf{V}^c_a = 2\Psi g_{ab}, \tag{5.0.3}$$

where $\mathcal{L}_{\mathbf{V}}$ is called Lie derivative operator along the direction of vector field $\mathbf{V}$ and $\Psi = \Psi(x^a)$ is known as conformal factor. The CKV field is said to be proper if $\Psi_{;ab} \neq 0$. If $\Psi_{;ab} = 0$, then vector field $\mathbf{V}$ is called special conformal Killing vector (SCKV), $\mathbf{V}$ is called homothetic vector (HV) field if $\Psi_{,a} = 0$ and Killing vector (KV) field if $\Psi = 0$. Keane and Tupper [47], determined the conformal symmetry classes of the pp-waves spacetimes given in Eq. (5.0.3). According to their classification, the following plane waves spacetime

$$ds^2 = -2A(u)x^2du^2 - 2dudv + dx^2 + dy^2, \tag{5.0.4}$$

is a special case of a plane wave spacetime. This spacetime admits a 6-dimensional homothetic Lie algebra for arbitrary $A(u)$. The five KVs are

$$\mathbf{V}_1 = \frac{\partial}{\partial v}, \qquad \mathbf{V}_2 = I(u)\frac{\partial}{\partial x} + I'(u)x\frac{\partial}{\partial v},$$
$$\mathbf{V}_3 = K(u)\frac{\partial}{\partial x} + K'(u)x\frac{\partial}{\partial v}, \qquad \mathbf{V}_4 = y\frac{\partial}{\partial v} + u\frac{\partial}{\partial y},$$
$$\mathbf{V}_5 = \frac{\partial}{\partial y},$$

where $I(u)$ and $K(u)$ are independent solutions of the differential equation

$$H''(u) + A(u)k(u) = 0,$$

and HV is

$$\mathbf{V}_6 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + v\frac{\partial}{\partial v}. \tag{5.0.5}$$

For a (1+3)-dimensional pp-waves spacetime, Carot *et al* [48] find that the only choices for $A(u)$ leading to an additional symmetry are (i). $A(u) = \alpha^2$(ii). $A(u) = \alpha u^{-2}$ (iii). $A(u) = \alpha^2 u^{-4}$. The pp-waves spacetimes with those $A(u)$ are the plane wave spacetimes.

Furthermore, Carot *et al* [48] consider another pp-waves spacetime

$$ds^2 = -2h(x)du^2 - 2dudv + dx^2 + dy^2, \tag{5.0.6}$$

This spacetime admits four-dimensional Killing algebra with basis

$$\mathbf{V}_1 = \frac{\partial}{\partial v}, \qquad \mathbf{V}_2 = \frac{\partial}{\partial u},$$
$$\mathbf{V}_3 = \frac{\partial}{\partial y}, \qquad \mathbf{V}_4 = y\frac{\partial}{\partial v} + u\frac{\partial}{\partial y}.$$

An additional HV is obtained by considering (iv). $h(x) = x^n$ as a special case of pp-waves spacetimes, where n is any constant

$$\mathbf{V}_5 = \frac{1}{2}u(2-n)\frac{\partial}{\partial u} + \frac{1}{2}v(2-n)\frac{\partial}{\partial v} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}.$$

Therefore, this special pp-waves spacetime admits a 5-dimensional Homothetic algebra.

Ehlers and Kundt in [49], provided the symmetry classes of vacuum plane fronted gravitational waves with parallel-ray also known as pp-waves, that satisfy the vacuum field equations of general relativity. Podolsky and Vesely in [50], used rigorous analytical and numerical techniques to investigate the chaotic behavior of geodesics in non-homogeneous vacuum pp-waves spacetimes. Keane and Tupper in [47], examined the conformal symmetry classes of the pp-waves spacetimes and reviewed Sippel and Goenners' isometry classification. Camci found the Noether gauge symmetries of the geodetic Lagrangian in [51]. He discovered that, conformally flat pp-waves spacetimes admit ten Noether gauge symmetry, while type N pp-waves spacetimes admit three. Jamal and Shabbir in [45], used pp-waves spacetimes to calculate the Noether symmetry algebra, which is admitted by the wave equation. Moreover, they proposed another metric function to be the generator of the wave equation. In this chapter, Mei and approximate Mei symmetries of pp-waves spacetimes corresponding to Lagrangian are calculated [55].

5.1 Mei Symmetries of pp-Waves Spacetimes

The Lagrangian of the pp-waves spacetimes, expressed in Eq. (5.0.1), is given as

$$\mathcal{L}_0 = -h(u,x,y)\dot{u}^2 - \dot{u}\dot{v} + \frac{1}{2}(\dot{x}^2 + \dot{y}^2). \tag{5.1.1}$$

Consider the symmetry generator

$$\mathbf{V}^0 = \alpha_0 \frac{\partial}{\partial s} + \beta_0^1 \frac{\partial}{\partial u} + \beta_0^2 \frac{\partial}{\partial v} + \beta_0^3 \frac{\partial}{\partial x} + \beta_0^4 \frac{\partial}{\partial y}, \tag{5.1.2}$$

where α_0, β_0^1, β_0^2, β_0^3, β_0^4 depend upon s, u, v, x, y. The first prolongation of $\mathbf{V}^0$ is given as

$$\mathbf{V}^{0[1]} = \mathbf{V}^0 + \dot{\beta}_0^1 \frac{\partial}{\partial \dot{u}} + \dot{\beta}_0^2 \frac{\partial}{\partial \dot{v}} + \dot{\beta}_0^3 \frac{\partial}{\partial \dot{x}} + \dot{\beta}_0^4 \frac{\partial}{\partial \dot{y}}, \tag{5.1.3}$$

where

$$\dot{\beta}_0^i = \frac{d\beta_0^i}{ds} - \dot{x}^i \frac{d\alpha_0}{ds}, \qquad ((x^1, x^2, x^3, x^4) = (u, v, x, y)), \tag{5.1.4}$$

and total differential operator $\frac{d}{ds}$ is defined as

$$\frac{d}{ds} = \frac{\partial}{\partial s} + \dot{u}\frac{\partial}{\partial u} + \dot{v}\frac{\partial}{\partial v} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y}. \tag{5.1.5}$$

The symmetry generator given in Eq. (5.1.2) is called the Mei symmetry of the Lagrangian $\mathcal{L}(s, u, v, x, y, \dot{u}, \dot{v}, \dot{x}, \dot{y})$, defined in Eq. (2.6.2). Now, we calculate the Mei symmetries of the Lagrangian Eq. (5.1.1) by considering the cases [51, 52] (i). $A(u) = \alpha^2$ (ii). $A(u) = \alpha u^{-2}$ (iii). $A(u) = \alpha^2 u^{-4}$ and for pp-wave spacetimes (iv). $h(x) = \alpha x^n$.

5.1.1 Case (i): $A(u) = \alpha^2$

Consider the line element of plane pp-waves spacetimes

$$ds^2 = -2A(u)x^2du^2 - 2dudv + dx^2 + dy^2. \tag{5.1.6}$$

The Lagrangian for $A(u) = \alpha^2$, takes the form

$$\mathcal{L}_0 = -\alpha^2 x^2 \dot{u}^2 - \dot{u}\dot{v} + \frac{1}{2}(\dot{x}^2 + \dot{y}^2). \tag{5.1.7}$$

The corresponding geodesics equations are given by

$$\ddot{u} = 0, \qquad \ddot{v} + 4\alpha^2 x \dot{u}\dot{x} = 0,$$
$$\ddot{x} + 2\alpha^2 x \dot{u}^2 = 0, \qquad \ddot{y} = 0.$$

Now applying $\mathbf{V}^{0[1]}$ on Eq. (5.1.7), yields

$$\mathbf{V}^{0[1]}\mathcal{L}_0 = -2\alpha^2 x^2 \dot{u}\dot{\beta}_0^1 - 2\alpha^2 x \dot{u}^2 \beta_0^3 - \dot{\beta}_0^1 \dot{v} - \dot{\beta}_0^2 \dot{u} + \dot{x}\dot{\beta}_0^3 + \dot{y}\dot{\beta}_0^4. \tag{5.1.8}$$

Now substituting $\dot{\beta}_0^1$, $\dot{\beta}_0^2$, $\dot{\beta}_0^3$, $\dot{\beta}_0^4$ from Eq. (5.1.4) into Eq. (5.1.8), we get

$$\begin{aligned}
\mathbf{V}^{0[1]}\mathcal{L}_0 = {} & -2\alpha^2 x \dot{u}^2 \beta_0^3 - 2\alpha^2 x^2 \dot{u}\beta_{0,s}^1 - 2\alpha^2 x^2 \dot{u}^2 \beta_{0,u}^1 - 2\alpha^2 x^2 \dot{u}\dot{v}\beta_{0,v}^1 - 2\alpha^2 x^2 \dot{u}\dot{x}\beta_{0,x}^1 \\
& - 2\alpha^2 x^2 \dot{u}\dot{y}\beta_{0,y}^1 + 2\alpha^2 x^2 \dot{u}^2 \alpha_{0,s} + 2\alpha^2 x^2 \dot{u}^3 \alpha_{0,u} + 2\alpha^2 x^2 \dot{u}^2 \dot{v}\alpha_{0,v} + 2\alpha^2 x^2 \dot{u}^2 \dot{x}\alpha_{0,x} \\
& + 2\alpha^2 x^2 \dot{u}^2 \dot{y}\alpha_{0,y} - \dot{v}\beta_{0,s}^1 - \dot{u}\dot{v}\beta_{0,u}^1 - \dot{v}^2 \beta_{0,v}^1 - \dot{v}\dot{x}\beta_{0,x}^1 - \dot{v}\dot{y}\beta_{0,y}^1 + 2\dot{u}\dot{v}\alpha_{0,s} + 2\dot{u}^2 \dot{v}\alpha_{0,u} \\
& + 2\dot{u}\dot{v}^2 \alpha_{0,v} + 2\dot{u}\dot{v}\dot{x}\alpha_{0,x} + 2\dot{u}\dot{v}\dot{y}\alpha_{0,y} - \dot{u}\beta_{0,s}^2 - \dot{u}^2 \beta_{0,u}^2 - \dot{u}\dot{v}\beta_{0,v}^2 - \dot{u}\dot{x}\beta_{0,x}^2 - \dot{u}\dot{y}\beta_{0,y}^2 \\
& + \dot{x}\beta_{0,s}^3 + \dot{u}\dot{x}\beta_{0,u}^3 + \dot{v}\dot{x}\beta_{0,v}^3 + \dot{x}^2 \beta_{0,x}^3 + \dot{x}\dot{y}\beta_{0,y}^3 - \dot{x}^2 \alpha_{0,s} - \dot{u}\dot{x}^2 \alpha_{0,u} - \dot{v}\dot{x}^2 \alpha_{0,v} - \dot{x}^3 \alpha_{0,x} \\
& - \dot{x}^2 \dot{y}\alpha_{0,y} + \dot{y}\beta_{0,s}^4 + \dot{u}\dot{y}\beta_{0,u}^4 + \dot{v}\dot{y}\beta_{0,v}^4 + \dot{x}\dot{y}\beta_{0,x}^4 + \dot{y}^2 \beta_{0,y}^4 - \dot{y}^2 \alpha_{0,s} - \dot{u}\dot{y}^2 \alpha_{0,u} - \dot{v}\dot{y}^2 \alpha_{0,v} \\
& - \dot{x}\dot{y}^2 \alpha_{0,x} - \dot{y}^3 \alpha_{0,y}.
\end{aligned} \tag{5.1.9}$$

Now, applying Euler operator for $a = 1$ defined in Eq. (2.6.2) on Eq. (5.1.9), we get

$$E^1(\mathbf{V}^{0[1]}\mathcal{L}_0) = 0, \tag{5.1.10}$$

above expression can also be written as

$$\frac{d}{ds}\left(\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial \dot{v}}\right) - \frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial v} = 0. \tag{5.1.11}$$

After applying Eq. (5.1.11) on Eq. (5.1.9), we get the following expression

$$
\begin{aligned}
&-\beta^1_{0,ss} - 2\dot{u}\beta^1_{0,su} - 2\dot{x}\beta^1_{0,sx} - 2\dot{y}\beta^1_{0,sy} - \dot{u}^2\beta^1_{0,uu} - 2\dot{u}\dot{y}\beta^1_{0,uy} + 4\alpha^2 x\dot{u}\dot{x}\beta^1_{0,v}\\
&- 2\dot{v}\beta^1_{0,vs} - 2\dot{u}\dot{v}\beta^1_{0,uv} - \dot{v}^2\beta^1_{0,vv} - 2\dot{v}\dot{x}\beta^1_{0,vx} - 2\dot{v}\dot{y}\beta^1_{0,vy} + 2\alpha^2 x\dot{u}^2\beta^1_{0,x} - 2\dot{u}\dot{x}\beta^1_{0,ux}\\
&- \dot{x}^2\beta^1_{0,xx} - 2\dot{x}\dot{y}\beta^1_{0,xy} - \dot{y}^2\beta^1_{0,yy} + 2\dot{u}^2\alpha_{0,ss} + 4\dot{u}^2\alpha_{0,su} + 2\dot{u}\dot{v}\alpha_{0,sv} + 4\dot{u}\dot{x}\alpha_{0,sx}\\
&+ 4\dot{u}\dot{y}\alpha_{0,sy} + 2\dot{u}^2\alpha_{0,su} + 4\dot{u}\dot{y}\alpha_{0,sy} + 2\dot{u}^3\alpha_{0,uu} + 4\dot{u}^2\dot{v}\alpha_{0,uv} + 4\dot{u}^2\dot{x}\alpha_{0,ux}\\
&+ 4\dot{u}^2\dot{y}\alpha_{0,uy} + 2\dot{u}\dot{v}^2\alpha_{0,vv} + 4\dot{u}\dot{v}\dot{x}\alpha_{0,vx} + 4\dot{u}\dot{v}\dot{y}\alpha_{0,vy} + 2\dot{x}^2\dot{u}\alpha_{0,xx} + 4\dot{u}\dot{x}\dot{y}\alpha_{0,xy}\\
&+ 2\dot{u}\dot{y}^2\alpha_{0,yy} = 0. \qquad (5.1.12)
\end{aligned}
$$

Comparing coefficients of different powers of $\dot{u}, \dot{v}, \dot{x}, \dot{y}$, yield the system of PDEs

$$
\begin{aligned}
\text{constant}:&\quad \beta^1_{0,ss} = 0, & \dot{u}:&\quad \beta^1_{0,su} - \alpha_{0,ss} = 0,\\
\dot{x}:&\quad \beta^1_{0,sx} = 0, & \dot{y}:&\quad \beta^1_{0,sy} = 0,\\
\dot{u}^2:&\quad \beta^1_{0,uu} - 4\alpha_{0,su} - 2\alpha^2\beta^1_{0,x} = 0, & \dot{v}:&\quad \beta^1_{0,vs} = 0,\\
\dot{u}\dot{x}:&\quad \beta^1_{0,ux} - 2\alpha_{0,sx} - 2\alpha^2 x\beta^1_{0,v} = 0, & \dot{u}\dot{y}:&\quad \beta^1_{0,uy} - 2\alpha_{0,sy} = 0,\\
\dot{u}\dot{v}:&\quad \beta^1_{0,uv} - \alpha_{0,sv}, & \dot{v}^2:&\quad \beta^1_{0,vv} = 0,\\
\dot{v}\dot{x}:&\quad \beta^1_{0,vx} = 0, & \dot{v}\dot{y}:&\quad \beta^1_{0,vy} = 0,\\
\dot{x}^2:&\quad \beta^1_{0,xx} = 0, & \dot{x}\dot{y}:&\quad \beta^1_{0,xy} = 0,\\
\dot{y}^2:&\quad \beta^1_{0,yy} = 0, & \dot{u}^3:&\quad \alpha_{0,uu} = 0,\\
\dot{u}^2\dot{v}:&\quad \alpha_{0,uv} = 0, & \dot{u}^2\dot{x}:&\quad \alpha_{0,ux} = 0,\\
\dot{u}^2\dot{y}:&\quad \alpha_{0,uy} = 0, & \dot{u}\dot{v}\dot{x}:&\quad \alpha_{0,vx} = 0,\\
\dot{u}\dot{v}\dot{y}:&\quad \alpha_{0,vy} = 0, & \dot{x}^2\dot{u}:&\quad \alpha_{0,xx} = 0,\\
\dot{u}\dot{y}\dot{x}:&\quad \alpha_{0,yx} = 0, & \dot{u}\dot{y}^2:&\quad \alpha_{0,yy} = 0.
\end{aligned}
$$

Again applying Euler operator for $a = 2$ defined on Eq. (5.1.9), we get

$$E^2(\mathbf{V}^{0[1]}\mathcal{L}_0) = 0, \qquad (5.1.13)$$

above expression can also be written as

$$\frac{d}{ds}\bigg(\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial \dot{u}}\bigg) - \frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial u} = 0. \tag{5.1.14}$$

The above Eq. (5.1.14) gives the following expression

$$\begin{aligned}
&-4\alpha^2\dot{u}\dot{x}\beta^3_0 - 4\alpha^2x\dot{u}\beta^3_{0,s} - 4\alpha^2x\dot{u}^2\beta^3_{0,u} - 4\alpha^2x\dot{u}\dot{v}\beta^3_{0,v} - 4\alpha^2x\dot{u}\dot{x}\beta^3_{0,x} - 4\alpha^2x\dot{u}\dot{y}\beta^3_{0,y}\\
&-4\alpha^2x\dot{x}\beta^1_{0,s} - 2\alpha^2x^2\beta^1_{0,ss} - 2\alpha^2x^2\dot{v}\beta^1_{0,sv} - 2\alpha^2x^2\dot{x}\beta^1_{0,sx} - 2\alpha^2x^2\dot{y}\beta^1_{0,sy} - 4\alpha^2x\dot{u}\dot{x}\beta^1_{0,u}\\
&-4\alpha^2x^2\dot{u}\beta^1_{0,su} - 2\alpha^2x^2\dot{u}^2\beta^1_{0,uu} - 2\alpha^2x^2\dot{u}\dot{v}\beta^1_{0,uv} - 4\alpha^2x^2\dot{u}\dot{x}\beta^1_{0,ux} - 2\alpha^2x^2\dot{u}\dot{y}\beta^1_{0,uy}\\
&-2\alpha^2x\dot{v}\dot{x}\beta^1_{0,v} + 8\alpha^4x^3\dot{u}\dot{x}\beta^1_{0,v} - 2\alpha^2x^2\dot{v}\beta^1_{0,sv} - 2\alpha^2x^2\dot{v}\dot{u}\beta^1_{0,uv} - 2\alpha^2x^2\dot{v}^2\beta^1_{0,vv}\\
&-2\alpha^2x^2\dot{v}\dot{x}\beta^1_{0,xv} - 2\alpha^2x^2\dot{v}\dot{y}\beta^1_{0,yv} - 8\alpha^2x\dot{x}^2\beta^1_{0,x} + 4\alpha^4x^3\dot{u}^2\beta^1_{0,x} - 2\alpha^2x^2\dot{x}\beta^1_{0,sx}\\
&-2\alpha^2x^2\dot{v}\dot{x}\beta^1_{0,vx} - 2\alpha^2x^2\dot{x}^2\beta^1_{0,xx} - 2\dot{x}^2\beta^2_{0,xx} + 4\alpha^2\dot{u}\dot{x}\beta^2_{0,v} - 2\alpha^2x^2\dot{y}\dot{x}\beta^1_{0,xy}\\
&-4\alpha^2x\dot{x}\dot{y}\beta^1_{0,y} - 2\alpha^2x^2\dot{y}\beta^1_{0,sy} - 2\alpha^2x^2\dot{u}\dot{y}\beta^1_{0,uy} - 2\alpha^2x^2\dot{v}\dot{y}\beta^1_{0,vy} - 2\alpha^2x^2\dot{y}^2\beta^1_{0,yy}\\
&-\beta^2_{0,ss} - \dot{u}\beta^2_{0,su} - \dot{u}^2\beta^2_{0,uu} - 2\dot{u}\dot{v}\beta^2_{0,vu} - 2\dot{u}\dot{x}\beta^2_{0,ux} - \dot{u}\dot{y}\beta^2_{0,uy} + 4\alpha^2x\dot{x}\dot{u}\beta^2_{0,v}\\
&-\dot{v}\beta^2_{0,vs} - \dot{v}^2\beta^2_{0,vv} - 2\dot{x}\dot{v}\beta^2_{0,vx} - 2\dot{y}\dot{v}\beta^2_{0,vy} + 2\alpha^2x\dot{u}^2\beta^2_{0,x} - \dot{x}\beta^2_{0,sx} - 2\dot{y}\dot{x}\beta^2_{0,xy}\\
&-\dot{y}\beta^2_{0,sy} - \dot{y}^2\beta^2_{0,yy} = 0.
\end{aligned} \tag{5.1.15}$$

From the above expression, we obtain the system of PDEs as

$$\begin{aligned}
\dot{u}\dot{x}:&\quad 2\alpha^2\beta^3_0 + 2\alpha^2x\beta^3_{0,x} + 2\alpha^2x\beta^1_{0,u} + 2\alpha^2x^2\beta^1_{0,ux} + \beta^2_{0,ux} - 4\alpha^2x^3\beta^1_{0,v}\\
&\quad - 2\alpha^2x\beta^2_{0,v} = 0,\\
\dot{u}:&\quad 2\alpha^2x\beta^3_{0,s} + 2\alpha^2x^2\beta^1_{0,us} + \beta^2_{0,su} = 0,\\
\text{constant}:&\quad 2\alpha^2x^2\beta^1_{0,ss} + \beta^2_{0,ss} = 0,\\
\dot{u}^2:&\quad 4\alpha^2x\beta^3_{0,u} + 2\alpha^2x\beta^1_{0,uu} - 2\alpha^2x\beta^2_{0,u} - 4\alpha^3x^2\beta^1_{0,x} + \beta^2_{0,uu} = 0,\\
\dot{u}\dot{v}:&\quad 2\alpha^2x\beta^3_{0,v} + 2\alpha^2x^2\beta^1_{0,uv} + \beta^2_{0,uv} = 0,\\
\dot{u}\dot{y}:&\quad 2\alpha^2x\beta^3_{0,y} + 2\alpha^2x^2\beta^1_{0,uy} + \beta^2_{0,uy} = 0,\\
\dot{v}:&\quad 2\alpha^2x^2\beta^1_{0,sv} + \beta^2_{0,sv} = 0,
\end{aligned}$$

$$
\begin{aligned}
\dot{x}: &\qquad 2\alpha^2x^2\beta^1_{0,sx} + \beta^2_{0,sx} + 2\alpha^2x\beta^1_{0,s} = 0,\\
\dot{y}: &\qquad 2\alpha^2x^2\beta^1_{0,sy} + \beta^2_{0,sy} = 0,\\
\dot{v}\dot{x}: &\qquad 2\alpha^2x\beta^1_{0,v} + 2\alpha^2x^2\beta^1_{0,vx} - \beta^2_{0,vx} = 0,\\
\dot{v}^2: &\qquad 2\alpha^2x^2\beta^1_{0,vv} + \beta^2_{0,vv} = 0,\\
\dot{v}\dot{y}: &\qquad 2\alpha^2x^2\beta^1_{0,vy} + \beta^2_{0,vy} = 0,\\
\dot{x}^2: &\qquad 4\alpha^2x\beta^1_{0,x} + \alpha^2x^2\beta_{0,xx} + \beta^2_{0,xx} = 0,\\
\dot{x}\dot{y}: &\qquad 2\alpha^2x^2\beta^1_{0,xy} + 2\alpha^2x\beta^1_{0,y} + \beta^2_{0,xy} = 0,\\
\dot{y}^2: &\qquad 2\alpha^2x^2\beta^1_{0,yy} + \beta^2_{0,yy} = 0.
\end{aligned} \tag{5.1.16}
$$

Again, applying Euler operator for $a = 3$ on Eq. (5.1.9), we get

$$
E^3(\mathbf{V}^{0[1]}\mathcal{L}_0) = 0, \tag{5.1.17}
$$

Alternatively,

$$
\frac{d}{ds}\left(\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial\dot{x}}\right) - \frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial x} = 0. \tag{5.1.18}
$$

As a result, we obtain the following equation

$$
\begin{aligned}
&\beta^3_{0,ss} + 2\dot{u}\beta^3_{0,su} + 2\dot{v}\beta^3_{0,sv} + 2\dot{y}\beta^3_{0,sy} + \dot{u}^2\beta^3_{0,uu} + 2\dot{u}\dot{v}\beta^3_{0,uv} + 2\dot{u}\dot{y}\beta^3_{0,uy}\\
&- 4\alpha^2x\dot{u}\dot{x}\beta^3_{0,v} + \dot{v}^2\beta^3_{0,vv} + 2\dot{v}\dot{x}\beta^3_{0,vx} + 2\dot{v}\dot{y}\beta^3_{0,vy} - 2\alpha^2x\dot{u}^2\beta^3_{0,x} + 2\dot{x}\beta^3_{0,sx}\\
&+ 2\dot{u}\dot{x}\beta^3_{0,ux} + \dot{x}^2\beta^3_{0,xx} + 2\dot{x}\dot{y}\beta^3_{0,xy} + 2\alpha^2\dot{u}^2\beta^3_{0} + \dot{y}^2\beta^3_{0,yy} + 4\alpha^2x\dot{u}\dot{x}\beta^1_{0,y} = 0.
\end{aligned} \tag{5.1.19}
$$

The corresponding system of PDEs are

$$
\begin{aligned}
constant: &\quad \beta^3_{0,ss} = 0, &\quad \dot{u}: &\quad \beta^3_{0,su} = 0,\\
\dot{v}^2: &\quad \beta^3_{0,vv} = 0, &\quad \dot{u}^2: &\quad \beta^3_{0,uu} - 2\alpha^2x\beta^3_{0,x} + 2\alpha^2\beta^3_0 = 0,\\
\dot{u}\dot{y}: &\quad \beta^3_{0,uy} = 0, &\quad \dot{v}: &\quad \beta^3_{0,vs} = 0,\\
\dot{v}\dot{x}: &\quad \beta^3_{0,vx} = 0, &\quad \dot{v}\dot{y}: &\quad \beta^3_{0,vy} = 0,\\
\dot{x}: &\quad \beta^3_{0,sx} = 0, &\quad \dot{u}\dot{x}: &\quad \beta^3_{0,xu} - 2\alpha^2x\beta^3_{0,v} = 0,
\end{aligned}
$$

$$\dot{x}^2: \quad \beta^3_{0,xx} = 0, \qquad \dot{x}\dot{y}: \quad \beta^3_{0,xy} = 0,$$

$$\dot{y}^2: \quad \beta^3_{0,yy} = 0, \qquad \dot{u}\dot{v}: \quad \beta^3_{0,uv} = 0,$$

$$\dot{y}: \quad \beta^3_{0,sy} = 0.$$

Again, applying Euler operator for $a = 4$ on Eq. (5.1.9), we get

$$E^4(\mathbf{V}_0^{[1]}\mathcal{L}_0) = 0, \tag{5.1.20}$$

Alternatively,

$$\frac{d}{ds}\left(\frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial\dot{y}}\right) - \frac{\partial(\mathbf{V}^{0[1]}\mathcal{L}_0)}{\partial y} = 0. \tag{5.1.21}$$

As a result, we obtain the following equation

$$\begin{aligned}
&\beta^4_{0,ss} + 2\dot{u}\beta^4_{0,su} + 2\dot{v}\beta^4_{0,sv} + 2\dot{x}\beta^4_{0,sx} + 2\dot{y}\beta^4_{0,sy} + \dot{u}^2\beta^4_{0,uu} + 2\dot{u}\dot{v}\beta^4_{0,uv} + 2\dot{u}\dot{x}\beta^4_{0,ux} \\
&+ 2\dot{u}\dot{y}\beta^4_{0,uy} - 4\alpha^2 x\dot{u}\dot{x}\beta^4_{0,v} + \dot{v}^2\beta^4_{0,vv} + 2\dot{v}\dot{x}\beta^4_{0,vx} + 2\dot{v}\dot{y}\beta^4_{0,vy} - 2\alpha^2 x\dot{u}^2\beta^4_{0,x} \\
&+ \dot{x}^2\beta^4_{0,xx} + 2\dot{x}\dot{y}\beta^4_{0,xy} + \dot{y}^2\beta^4_{0,yy} = 0.
\end{aligned} \tag{5.1.22}$$

The system of PDEs is given by

$$\text{constant}: \quad \beta^4_{0,ss} = 0, \qquad \dot{u}: \quad \beta^4_{0,su} = 0,$$

$$\dot{v}: \quad \beta^4_{0,sv} = 0, \qquad \dot{x}: \quad \beta^4_{0,sx} = 0,$$

$$\dot{v}^2: \quad \beta^4_{0,vv} = 0, \qquad \dot{x}\dot{y}: \quad \beta^4_{0,xy} = 0,$$

$$\dot{y}^2: \quad \beta^4_{0,yy} = 0, \qquad \dot{y}: \quad \beta^4_{0,sy} = 0,$$

$$\dot{u}^2: \quad \beta^4_{0,uu} - 2\alpha^2 x^2\beta^4_{0,x} = 0, \qquad \dot{u}\dot{v}: \quad \beta^4_{0,uv} = 0,$$

$$\dot{u}\dot{x}: \quad \beta^4_{0,ux} - 2\alpha^2\beta^4_{0,v} + 2\alpha^2 x\beta^1_{0,y} = 0, \qquad \dot{u}\dot{y}: \quad \beta^4_{0,uy} = 0,$$

$$\dot{v}\dot{x}: \quad \beta^4_{0,vx} = 0, \qquad \dot{v}\dot{y}: \quad \beta^4_{0,vy} = 0,$$

$$\dot{x}^2: \quad \beta^4_{0,xx} = 0.$$

Solving system of PDEs simultaneously for $\beta_0^1, \beta_0^2, \beta_0^3, \beta_0^4$, and α_0 with the help of **MAPLE** package **PDE**, we get

$$\alpha_0 = A,$$
$$\beta_0^1 = B,$$
$$\beta_0^2 = Cy - D\sqrt{2}\alpha x \sin(\sqrt{2}\alpha u) + E\sqrt{2}\alpha x \cos(\sqrt{2}\alpha u) + Fs + Hu + G,$$
$$\beta_0^3 = E\sin(\sqrt{2}\alpha u) + D\cos(\sqrt{2}\alpha u),$$
$$\beta_0^4 = Iu + Rs + K.$$

The set of the Mei symmetries are given below

$$\mathbf{V_1^0} = \frac{\partial}{\partial s}, \qquad \mathbf{V_2^0} = \frac{\partial}{\partial u},$$
$$\mathbf{V_3^0} = y\frac{\partial}{\partial v}, \qquad \mathbf{V_4^0} = -\sqrt{2}\alpha x \sin(\sqrt{2}\alpha u)\frac{\partial}{\partial v} + \cos(\sqrt{2}\alpha u)\frac{\partial}{\partial x},$$
$$\mathbf{V_5^0} = \sqrt{2}\alpha x \cos(\sqrt{2}\alpha u)\frac{\partial}{\partial v} + \sin(\sqrt{2}\alpha u)\frac{\partial}{\partial x}, \qquad \mathbf{V_6^0} = s\frac{\partial}{\partial v},$$
$$\mathbf{V_7^0} = \frac{\partial}{\partial v}, \qquad \mathbf{V_8^0} = u\frac{\partial}{\partial y},$$
$$\mathbf{V_9^0} = s\frac{\partial}{\partial y}, \qquad \mathbf{V_{10}^0} = \frac{\partial}{\partial y},$$
$$\mathbf{V_{11}^0} = u\frac{\partial}{\partial v}.$$

5.2 Approximate Mei Symmetries of pp-Waves Spacetimes

The approximate Mei symmetries and related conserved quantities of Lagrangian up to first order ϵ are formulated in [54]. The approximate Mei symmetries determining equations are given as

$$E^a(\mathbf{V}^{1[1]}\mathcal{L}_0 + \mathbf{V}^{0[1]}\mathcal{L}_1) = 0, \qquad (a = 1, 2, 3, 4). \tag{5.2.1}$$

The first order perturbed part of the Lagrangian is given by

$$\mathcal{L}_1 = -2\alpha^2 x^2 \dot{u}^2 s - 2\dot{u}\dot{v}s + \dot{x}^2 s + \dot{y}^2 s. \tag{5.2.2}$$

The first prolongation of the perturbed symmetry generator is given below

$$\mathbf{V}^{1[1]} = \mathbf{V}^1 + \dot{\beta}_1^1 \frac{\partial}{\partial \dot{u}} + \dot{\beta}_1^2 \frac{\partial}{\partial \dot{v}} + \dot{\beta}_1^3 \frac{\partial}{\partial \dot{x}} + \dot{\beta}_1^4 \frac{\partial}{\partial \dot{y}}. \tag{5.2.3}$$

Now, we calculate the approximate Mei symmetries up to first order of ϵ by using exact symmetry $\mathbf{V}_0^8 = u\frac{\partial}{\partial y}$.

$$\begin{aligned}
\mathbf{V}^{1[1]}\mathcal{L}_0 + \mathbf{V}^{0[1]}\mathcal{L}_1 &= -2\alpha^2 x\dot{u}^2\beta_1^3 - 2\alpha^2 x^2\dot{u}\beta_{1,s}^1 - 2\alpha^2 x^2\dot{u}^2\beta_{1,u}^1 - 2\alpha^2 x^2\dot{u}\dot{v}\beta_{1,v}^1 \\
&- 2\alpha^2 x^2\dot{u}\dot{x}\beta_{1,x}^1 - 2\alpha^2 x^2\dot{u}\dot{y}\beta_{1,y}^1 + 2\alpha^2 x^2\dot{u}^2\alpha_{1,s} + 2\alpha^2 x^2\dot{u}^3\alpha_{1,u} + 2\alpha^2 x^2\dot{u}^2\dot{v}\alpha_{1,v} \\
&+ 2\alpha^2 x^2\dot{u}^2\dot{x}\alpha_{1,x} + 2\alpha^2 x^2\dot{u}^2\dot{y}\alpha_{1,y} - \dot{v}\beta_{1,s}^1 - \dot{u}\dot{v}\beta_{1,u}^1 - \dot{v}^2\beta_{1,v}^1 - \dot{v}\dot{x}\beta_{1,x}^1 - \dot{v}\dot{y}\beta_{1,y}^1 \\
&+ 2\dot{u}\dot{v}\alpha_{1,s} + 2\dot{u}^2\dot{v}\alpha_{1,u} + 2\dot{u}\dot{v}^2\alpha_{1,v} + 2\dot{u}\dot{v}\dot{x}\alpha_{1,x} + 2\dot{u}\dot{v}\dot{y}\alpha_{1,y} - \dot{u}\beta_{1,s}^2 - \dot{u}^2\beta_{1,u}^2 - \dot{u}\dot{v}\beta_{1,v}^2 \\
&- \dot{u}\dot{x}\beta_{1,x}^2 - \dot{u}\dot{y}\beta_{1,y}^2 + \dot{x}\beta_{1,s}^3 + \dot{u}\dot{x}\beta_{1,u}^3 + \dot{v}\dot{x}\beta_{1,v}^3 + \dot{x}^2\beta_{1,x}^3 + \dot{x}\dot{y}\beta_{1,y}^3 - x^2\alpha_{1,s} - \dot{u}\dot{x}^2\alpha_{1,u} \\
&- \dot{v}\dot{x}^2\alpha_{1,v} - \dot{x}^3\alpha_{1,x} - \dot{x}^2\dot{y}\alpha_{1,y} + \dot{y}\beta_{1,s}^4 + \dot{u}\dot{y}\beta_{1,u}^4 + \dot{v}\dot{y}\beta_{1,v}^4 + \dot{x}\dot{y}\beta_{1,x}^4 + \dot{y}^2\beta_{1,y}^4 \\
&- \dot{y}^2\alpha_{1,s} - \dot{u}\dot{y}^2\alpha_{1,u} - \dot{v}\dot{y}^2\alpha_{1,v} - \dot{x}\dot{y}^2\alpha_{1,x} - \dot{y}^3\alpha_{1,y} + 2s\dot{u}\dot{y}.
\end{aligned} \tag{5.2.4}$$

The Eq. (5.2.1) for $a = 1$, takes the form

$$\frac{d}{ds}\left(\frac{\partial(\mathbf{V}^{1[1]}\mathcal{L}_0 + \mathbf{V}^{0[1]}\mathcal{L}_1)}{\partial \dot{u}}\right) - \frac{\partial(\mathbf{V}^{1[1]}\mathcal{L}_0 + \mathbf{V}^{0[1]}\mathcal{L}_1)}{\partial u} = 0. \tag{5.2.5}$$

Applying Eq. (5.2.5) on Eq. (5.2.4), we obtain

$$\begin{aligned}
&- 4\alpha^2\dot{u}\dot{x}\beta_1^3 - 4\alpha^2 x\dot{u}\beta_{1,s}^3 - 4\alpha^2 x\dot{u}^2\beta_{1,u}^3 - 4\alpha^2 x\dot{u}\dot{v}\beta_{1,v}^3 - 4\alpha^2 x\dot{u}\dot{x}\beta_{1,x}^3 - 4\alpha^2 x\dot{u}\dot{y}\beta_{1,y}^3 \\
&- 4\alpha^2 x\dot{x}\beta_{1,s}^1 - 2\alpha^2 x^2\beta_{1,ss}^1 - 2\alpha^2 x^2\dot{v}\beta_{1,sv}^1 - 2\alpha^2 x^2\dot{x}\beta_{1,sx}^1 - 2\alpha^2 x^2\dot{y}\beta_{1,sy}^1 - 4\alpha^2 x\dot{u}\dot{x}\beta_{1,u}^1 \\
&- 4\alpha^2 x^2\dot{u}\beta_{1,su}^1 - 2\alpha^2 x^2\dot{u}^2\beta_{1,uu}^1 - 2\alpha^2 x^2\dot{u}\dot{v}\beta_{1,uv}^1 - 2\alpha^2 x^2\dot{u}\dot{x}\beta_{1,ux}^1 - 2\alpha^2 x^2\dot{u}\dot{y}\beta_{1,uy}^1 \\
&- 2\alpha^2 x\dot{v}\dot{x}\beta_{1,v}^1 + 8\alpha^4 x^3\dot{u}\dot{x}\beta_{1,v}^1 - 2\alpha^2 x^2\dot{v}\beta_{1,sv}^1 - 2\alpha^2 x^2\dot{v}\dot{u}\beta_{1,uv}^1 - 2\alpha^2 x^2\dot{v}^2\beta_{1,vv}^1 \\
&- 2\alpha^2 x^2\dot{v}\dot{x}\beta_{1,xv}^1 - 2\alpha^2 x^2\dot{v}\dot{y}\beta_{1,yv}^1 - 4\alpha^2 x\dot{x}^2\beta_{1,x}^1 + 4\alpha^4 x^3\dot{u}^2\beta_{1,x}^1 - 2\alpha^2 x^2\dot{x}\beta_{1,sx}^1
\end{aligned}$$

$$
\begin{aligned}
&- 2\alpha^2x^2\dot{v}\dot{x}\beta^1_{1,vx} - 2\alpha^2x^2\dot{x}^2\beta^1_{1,xx} - 2\dot{x}^2\beta^2_{1,xx} + 4\alpha^2x\dot{u}\dot{x}\beta^2_{1,v} - 2\alpha^2x^2\dot{y}\dot{x}\beta^1_{1,xy} \\
&- 4\alpha^2x\dot{x}\dot{y}\beta^1_{1,y} - 2\alpha^2x^2\dot{y}\beta^1_{1,sy} - 2\alpha^2x^2\dot{u}\dot{y}\beta^1_{1,uy} - 2\alpha^2x^2\dot{v}\dot{y}\beta^1_{1,vy} - 2\alpha^2x^2\dot{y}^2\beta^1_{1,yy} \\
&- \beta^2_{1,ss} - \dot{u}\beta^2_{1,su} - \dot{u}^2\beta^2_{1,uu} - 2\dot{u}\dot{v}\beta^2_{1,vu} - 2\dot{u}\dot{x}\beta^2_{1,ux} - \dot{u}\dot{y}\beta^2_{1,uy} + 4\alpha^2x\dot{x}\dot{u}\beta^2_{1,v} \\
&- \dot{v}\beta^2_{1,vs} - \dot{v}^2\beta^2_{1,vv} - 2\dot{x}\dot{v}\beta^2_{1,vx} - 2\dot{y}\dot{v}\beta^2_{1,vy} + 2\alpha^2x\dot{u}^2\beta^2_{1,x} - \dot{x}\beta^2_{1,sx} - 2\dot{y}\dot{x}\beta^2_{1,xy} \\
&- \dot{y}\beta^2_{1,sy} - \dot{y}^2\beta^2_{1,yy} + 2\dot{y} = 0. \qquad (5.2.6)
\end{aligned}
$$

The obtain system of PDEs is given by

$$
\begin{aligned}
\dot{u}\dot{x}: &\quad 2\alpha^2\beta^3_1 + 2\alpha^2x\beta^3_{1,x} + 2\alpha^2x\beta^1_{1,u} + 2\alpha^2x^2\beta^1_{1,ux} + \beta^2_{1,ux} \\
&\quad - 4\alpha^2x^3\beta^1_{1,v} - 2\alpha^2x\beta^2_{1,v} = 0, \\
\dot{u}: &\quad 2\alpha^2x\beta^3_{1,s} + 2\alpha^2x^2\beta^1_{1,us} + \beta^2_{1,su} = 0, \\
\text{constant}: &\quad 2\alpha^2x^2\beta^1_{1,ss} + \beta^2_{1,ss} = 0, \\
\dot{u}^2: &\quad 4\alpha^2x\beta^3_{1,u} + 2\alpha^2x\beta^1_{1,uu} - 2\alpha^2x\beta^2_{1,x} - 4\alpha^3x^2\beta^1_{1,x} + \beta^2_{1,uu} = 0, \\
\dot{u}\dot{v}: &\quad 2\alpha^2x\beta^3_{1,v} + 2\alpha^2x^2\beta^1_{1,uv} + \beta^2_{1,uv} = 0, \\
\dot{u}\dot{y}: &\quad 2\alpha^2x\beta^3_{1,y} + 2\alpha^2x^2\beta^1_{1,uy} + \beta^2_{1,uy} = 0, \\
\dot{u}\dot{y}: &\quad 2\alpha^2x\beta^3_{1,y} + 2\alpha^2x^2\beta^1_{1,uy} + \beta^2_{1,uy} = 0, \\
\dot{v}: &\quad 2\alpha^2x^2\beta^1_{1,sv} + \beta^2_{1,sv} = 0, \\
\dot{x}: &\quad 2\alpha^2x^2\beta^1_{1,sx} + \beta^2_{1,sx} + 2\alpha^2x\beta^1_{1,s} = 0, \\
\dot{y}: &\quad 2\alpha^2x^2\beta^1_{1,sy} + \beta^2_{1,sy} - 1 = 0, \\
\dot{v}\dot{x}: &\quad 2\alpha^2\beta^1_{1,v} + 2\alpha^2x^2\beta^1_{1,vx} - \beta^2_{1,vx} = 0, \\
\dot{v}^2: &\quad 2\alpha^2x^2\beta^1_{1,vv} + \beta^2_{1,vv} = 0, \\
\dot{v}\dot{y}: &\quad 2\alpha^2x^2\beta^1_{1,vy} + \beta^2_{1,vy} = 0, \\
\dot{x}^2: &\quad 4\alpha^2x\beta^1_{1,x} + \alpha^2x^2\beta_{1,xx} + \beta^2_{1,xx} = 0, \\
\dot{x}\dot{y}: &\quad 2\alpha^2x^2\beta^1_{1,xy} + 2\alpha^2x\beta^1_{1,y} + \beta^2_{1,xy} = 0, \\
\dot{y}^2: &\quad 2\alpha^2x^2\beta^1_{1,yy} + \beta^2_{1,yy} = 0. \qquad (5.2.7)
\end{aligned}
$$

Again, repeating the same process, by applying E^2 (for $a = 2$), yields

$$\frac{d}{ds}\left(\frac{\partial(\mathbf{V}^{1[1]}\mathcal{L}_0 + \mathbf{V}^{0[1]}\mathcal{L}_1)}{\partial \dot{v}}\right) - \frac{\partial(\mathbf{V}^{1[1]}\mathcal{L}_0 + \mathbf{V}^{0[1]}\mathcal{L}_1)}{\partial v} = 0. \tag{5.2.8}$$

The following expression is obtained

$$\begin{aligned}
&-\beta^1_{1,ss} - 2\dot{u}\beta^1_{1,su} - 2\dot{x}\beta^1_{1,sx} - 2\dot{y}\beta^1_{1,sy} - \dot{u}^2\beta^1_{1,uu} - 2\dot{u}\dot{y}\beta^1_{1,uy} + 4\alpha^2 x\dot{u}\dot{x}\beta^1_{1,v}\\
&-2\dot{v}\beta^1_{1,vs} - 2\dot{u}\dot{v}\beta^1_{1,uv} - \dot{v}^2\beta^1_{1,vv} - 2\dot{v}\dot{x}\beta^1_{1,vx} - 2\dot{v}\dot{y}\beta^1_{1,vy} + 2\alpha^2 x\dot{u}^2\beta^1_{1,x} - 2\dot{u}\dot{x}\beta^1_{1,ux}\\
&-\dot{x}^2\beta^1_{1,xx} - 2\dot{x}\dot{y}\beta^1_{1,xy} - \dot{y}^2\beta^1_{1,yy} + 2\dot{u}^2\alpha_{1,ss} + 4\dot{u}^2\alpha_{1,su} + 2\dot{u}\dot{v}\alpha_{1,sv} + 4\dot{u}\dot{x}\alpha_{1,sx}\\
&+4\dot{u}\dot{y}\alpha_{1,sy} + 2\dot{u}^2\alpha_{1,su} + 4\dot{u}\dot{y}\alpha_{1,sy} + 2\dot{u}^3\alpha_{1,uu} + 4\dot{u}^2\dot{v}\alpha_{1,uv} + 4\dot{u}^2\dot{x}\alpha_{1,ux} + 4\dot{u}^2\dot{y}\alpha_{1,uy}\\
&+2\dot{u}\dot{v}^2\alpha_{1,vv} + 4\dot{u}\dot{v}\dot{x}\alpha_{1,vx} + 4\dot{u}\dot{v}\dot{y}\alpha_{1,vy} + 2\dot{x}^2\dot{u}\alpha_{1,xx} + 4\dot{u}\dot{x}\dot{y}\alpha_{1,xy} + 2\dot{u}\dot{y}^2\alpha_{1,yy} = 0.
\end{aligned} \tag{5.2.9}$$

The obtained system of PDEs is given by

$$\begin{aligned}
\text{constant}: &\quad \beta^1_{1,ss} = 0, & \dot{u}: &\quad \beta^1_{0,su} - \alpha_{1,ss} = 0,\\
\dot{x}: &\quad \beta^1_{1,sx} = 0, & \dot{y}: &\quad \beta^1_{1,sy} = 0,\\
\dot{u}^2: &\quad \beta^1_{1,uu} - 4\alpha_{1,su} - 2\alpha^2\beta^1_{1,x} = 0, & \dot{v}: &\quad \beta^1_{1,vs} = 0,\\
\dot{u}\dot{x}: &\quad \beta^1_{1,ux} - 2\alpha_{1,sx} - 2\alpha^2 x\beta^1_{1,v} = 0, & \dot{u}\dot{y}: &\quad \beta^1_{1,uy} - 2\alpha_{1,sy} = 0,\\
\dot{u}\dot{v}: &\quad \beta^1_{1,uv} - \alpha_{1,sv} = 0, & \dot{v}^2: &\quad \beta^1_{1,vv} = 0,\\
\dot{v}\dot{x}: &\quad \beta^1_{1,vx} = 0, & \dot{v}\dot{y}: &\quad \beta^1_{1,vy} = 0,\\
\dot{x}^2: &\quad \beta^1_{1,xx} = 0, & \dot{x}\dot{y}: &\quad \beta^1_{1,xy} = 0,\\
\dot{y}^2: &\quad \beta^1_{1,yy} = 0, & \dot{u}^3: &\quad \alpha_{1,uu} = 0,\\
\dot{u}^2\dot{v}: &\quad \alpha_{1,uv} = 0, & \dot{u}^2\dot{x}: &\quad \alpha_{1,ux} = 0,\\
\dot{u}^2\dot{y}: &\quad \alpha_{1,uy} = 0, & \dot{u}\dot{v}\dot{x}: &\quad \alpha_{1,vx} = 0,\\
\dot{u}\dot{v}\dot{y}: &\quad \alpha_{1,vy} = 0, & \dot{x}^2\dot{u}: &\quad \alpha_{1,xx} = 0,\\
\dot{u}\dot{y}\dot{x}: &\quad \alpha_{1,yx} = 0, & \dot{u}\dot{y}^2: &\quad \alpha_{1,yy} = 0.
\end{aligned} \tag{5.2.10}$$

Now, applying E^3 (for $a=3$), on Eq. (5.2.4)

$$\frac{d}{ds}\bigg(\frac{\partial(\mathbf{V}^{1[1]}\mathcal{L}_0+\mathbf{V}^{0[1]}\mathcal{L}_1)}{\partial\dot{x}}\bigg)-\bigg(\frac{\partial(\mathbf{V}^{1[1]}\mathcal{L}_0+\mathbf{V}^{0[1]}\mathcal{L}_1)}{\partial x}\bigg)=0. \tag{5.2.11}$$

Then we get the following expression

$$\begin{aligned}&\beta^3_{1,ss}+2\dot{u}\beta^3_{1,su}+2\dot{v}\beta^3_{1,sv}+2\dot{y}\beta^3_{1,sy}+\dot{u}^2\beta^3_{1,uu}+2\dot{u}\dot{v}\beta^3_{1,uv}+2\dot{u}\dot{y}\beta^3_{1,uy}-4\alpha^2x\dot{u}\dot{x}\beta^3_{1,v}\\&+\dot{v}^2\beta^3_{1,vv}+2\dot{v}\dot{x}\beta^3_{1,vx}+2\dot{v}\dot{y}\beta^3_{1,vy}-2\alpha^2x\dot{u}^2\beta^3_{1,x}+2\dot{x}\beta^3_{1,sx}+2\dot{u}\dot{x}\beta^3_{1,ux}+\dot{x}^2\beta^3_{1,xx}\\&+2\dot{x}\dot{y}\beta^3_{1,xy}+2\alpha^2\dot{u}^2\beta^3_1+\dot{y}^2\beta^3_{1,yy}=0.\end{aligned} \tag{5.2.12}$$

The corresponding system is given below

$$\begin{aligned}
constant:&\quad \beta^3_{1,ss}=0, &\dot{u}:&\quad \beta^3_{1,su}=0,\\
\dot{v}^2:&\quad \beta^3_{1,vv}=0, &\dot{u}^2:&\quad \beta^3_{uu}-2\alpha^2x\beta^3_{1,x}+2\alpha^2\beta^3_1=0,\\
\dot{u}\dot{y}:&\quad \beta^3_{1,uy}=0, &\dot{v}:&\quad \beta^3_{1,vs}=0,\\
\dot{v}\dot{x}:&\quad \beta^3_{1,vx}=0, &\dot{v}\dot{y}:&\quad \beta^3_{1,vy}=0,\\
\dot{x}:&\quad \beta^3_{1,sx}=0, &\dot{u}\dot{x}:&\quad \beta^3_{1,xu}-2\alpha^2x\beta^3_{1,v}=0,\\
\dot{x}^2:&\quad \beta^3_{1,xx}=0, &\dot{x}\dot{y}:&\quad \beta^3_{1,xy}=0,\\
\dot{y}^2:&\quad \beta^3_{1,yy}=0, &\dot{u}\dot{v}:&\quad \beta^3_{1,uv}=0,\\
\dot{y}:&\quad \beta^3_{1,sy}=0.
\end{aligned} \tag{5.2.13}$$

After applying E^4 on Eq. (5.2.4), we get

$$\frac{d}{ds}\bigg(\frac{\partial(\mathbf{V}^{1[1]}\mathcal{L}_0+\mathbf{V}^{0[1]}\mathcal{L}_1)}{\partial\dot{y}}\bigg)-\frac{\partial(\mathbf{V}^{1[1]}\mathcal{L}_0+\mathbf{V}^{0[1]}\mathcal{L}_1)}{\partial y}=0. \tag{5.2.14}$$

Then the following expression is obtained

$$\begin{aligned}&\beta^4_{1,ss}+2\dot{u}\beta^4_{1,su}+2\dot{v}\beta^4_{1,sv}+2\dot{x}\beta^4_{1,sx}+2\dot{y}\beta^4_{1,sy}+\dot{u}^2\beta^4_{1,uu}+2\dot{u}\dot{v}\beta^4_{1,uv}+2\dot{u}\dot{x}\beta^4_{1,ux}\\&+2\dot{u}\dot{y}\beta^4_{1,uy}-4\alpha^2x\dot{u}\dot{x}\beta^4_{1,v}+\dot{v}^2\beta^4_{1,vv}+2\dot{v}\dot{x}\beta^4_{1,vx}+2\dot{v}\dot{y}\beta^4_{1,vy}-2\alpha^2x\dot{u}^2\beta^4_{1,x}+\dot{x}^2\beta^4_{1,xx}\\&+2\dot{x}\dot{y}\beta^4_{1,xy}+\dot{y}^2\beta^4_{1,yy}+4\alpha^2x\dot{u}\dot{x}\beta^1_{1,y}+2\dot{u}=0.\end{aligned} \tag{5.2.15}$$

After comparing the coefficients of $\dot{u}, \dot{v}, \dot{x}, \dot{y}$, the following system of PDEs is obtained

$$
\begin{aligned}
\text{constant}: &\quad \beta^4_{1,ss} = 0, & \dot{u}: &\quad \beta^4_{1,su} + 1 = 0,\\
\dot{v}: &\quad \beta^4_{1,sv} = 0, & \dot{x}: &\quad \beta^4_{1,sx} = 0,\\
\dot{v}^2: &\quad \beta^4_{1,vv} = 0, & \dot{x}\dot{y}: &\quad \beta^4_{1,xy} = 0,\\
\dot{y}^2: &\quad \beta^4_{1,yy} = 0, & \dot{y}: &\quad \beta^4_{1,sy} = 0,\\
\dot{u}^2: &\quad \beta^4_{1,uu} - 2\alpha^2 x^2 \beta^4_{1,x} = 0, & \dot{u}\dot{v}: &\quad \beta^4_{1,uv} = 0,\\
\dot{u}\dot{x}: &\quad \beta^4_{1,ux} - 2\alpha^2 \beta^4_{1,v} + 2\alpha^2 x \beta^1_{1,y} = 0, & \dot{u}\dot{y}: &\quad \beta^4_{1,uy} = 0,\\
\dot{v}\dot{x}: &\quad \beta^4_{1,vx} = 0, & \dot{v}\dot{y}: &\quad \beta^4_{1,vy} = 0,\\
\dot{x}^2: &\quad \beta^4_{1,xx} = 0.
\end{aligned}
\tag{5.2.16}
$$

Solving system of PDEs, (Eqs. ((5.2.7),(5.2.10),(5.2.13),(5.2.16))) with the help of **MAPLE**, we get

$$
\begin{aligned}
&\alpha_1 = A,\\
&\beta^1_1 = B,\\
&\beta^2_1 = sy + Cy - D\sqrt{2}\alpha x \sin(\sqrt{2}\alpha u) + E\sqrt{2}\alpha x \cos(\sqrt{2}\alpha u) + Fs + Hu + G,\\
&\beta^3_1 = E\sin(\sqrt{2}\alpha u) + D\cos(\sqrt{2}\alpha u),\\
&\beta^4_1 = -su + Iu + Rs + K.
\end{aligned}
$$

Now assigning value of any constant equal to one, say $I = 1$, and the remaining constants equal to zero, we obtain the generator $\mathbf{V}^0_8$. Then $\mathbf{V}^1_8$ can be written as

$$
\mathbf{V}^1_8 = sy\frac{\partial}{\partial v} - su\frac{\partial}{\partial y}. \tag{5.2.17}
$$

The nontrivial approximate Mei symmetry of Eq. (5.2.6) has the form

$$
\mathbf{V}^0_8 + \epsilon\mathbf{V}^1_8 = u\frac{\partial}{\partial y} + \epsilon\left(sy\frac{\partial}{\partial v} - su\frac{\partial}{\partial y}\right). \tag{5.2.18}
$$

[,]	$\mathbf{V}_3^1$	$\mathbf{V}_6^1$	$\mathbf{V}_8^1$	$\mathbf{V}_9^1$	$\mathbf{V}_{11}^1$
$\mathbf{V}_3^1$	0	0	0	0	0
$\mathbf{V}_6^1$	0	0	0	0	0
$\mathbf{V}_8^1$	0	0	0	0	0
$\mathbf{V}_9^1$	0	0	0	0	0
$\mathbf{V}_{11}^1$	0	0	0	0	0

Table 5.1: The commutator relations of approximate Mei symmetries for case (i). $A(u) = \alpha^2$

In a similar way, the remaining approximate Mei symmetries are obtain as

$$\begin{aligned}
&\mathbf{V}_1^0 + \epsilon \mathbf{V}_1^1 = \frac{\partial}{\partial s},\\
&\mathbf{V}_2^0 + \epsilon \mathbf{V}_2^1 = \frac{\partial}{\partial u},\\
&\mathbf{V}_3^0 + \epsilon \mathbf{V}_3^1 = y\frac{\partial}{\partial v} + \epsilon\left(-sy\frac{\partial}{\partial v} + su\frac{\partial}{\partial y}\right),\\
&\mathbf{V_4^0} + \epsilon \mathbf{V_4^1} = -\sqrt{2}\alpha x \sin(\sqrt{2}\alpha u)\frac{\partial}{\partial v} + \cos(\sqrt{2}\alpha u)\frac{\partial}{\partial x},\\
&\mathbf{V_5^0} + \epsilon \mathbf{V_5^1} = \sqrt{2}\alpha x \cos(\sqrt{2}\alpha u)\frac{\partial}{\partial v} + \sin(\sqrt{2}\alpha u)\frac{\partial}{\partial x},\\
&\mathbf{V_6^0} + \epsilon \mathbf{V_1^6} = s\frac{\partial}{\partial v} - \epsilon s^2\frac{\partial}{\partial v},\\
&\mathbf{V_7^0} + \epsilon \mathbf{V_1^7} = \frac{\partial}{\partial v},\\
&\mathbf{V_8^0} + \epsilon \mathbf{V_8^1} = u\frac{\partial}{\partial y} + \epsilon\left(sy\frac{\partial}{\partial v} - su\frac{\partial}{\partial y}\right),\\
&\mathbf{V_9^0} + \epsilon \mathbf{V_9^1} = s\frac{\partial}{\partial y} - \epsilon s^2\frac{\partial}{\partial y},\\
&\mathbf{V_{10}^0} + \epsilon \mathbf{V_{10}^1} = \frac{\partial}{\partial y},\\
&\mathbf{V_{11}^0} + \epsilon \mathbf{V_{11}^1} = u\frac{\partial}{\partial v} - \epsilon 2su\frac{\partial}{\partial v}.
\end{aligned}$$

The Lie algebra of Mei symmetry generators is given in **Table 5.2**, approximate Mei symmetries up to first order of precision is given in **Table 5.1**.

[,]	$\mathbf{V}_1^0$	$\mathbf{V}_2^0$	$\mathbf{V}_3^0$	$\mathbf{V}_4^0$	$\mathbf{V}_5^0$	$\mathbf{V}_6^0$	$\mathbf{V}_7^0$	$\mathbf{V}_8^0$	$\mathbf{V}_9^0$	$\mathbf{V}_{10}^0$	$\mathbf{V}_{11}^0$
$\mathbf{V}_1^0$	0	0	0	0	0	$\mathbf{V}_7^0$	0	0	$\mathbf{V}_{10}^0$	0	0
$\mathbf{V}_2^0$	0	0	0	$-\sqrt{2}\alpha\mathbf{V}_5^0$	$\sqrt{2}\alpha\mathbf{V}_4^0$	0	0	$\mathbf{V}_{10}^0$	0	0	$\mathbf{V}_7^0$
$\mathbf{V}_3^0$	0	0	0	0	0	0	0	$-\mathbf{V}_{11}^0$	$-\mathbf{V}_6^0$	$-\mathbf{V}_7^0$	0
$\mathbf{V}_4^0$	0	$\sqrt{2}\alpha\mathbf{V}_5^0$	0	0	$\sqrt{2}\alpha\mathbf{V}_7^0$	0	0	0	0	0	0
$\mathbf{V}_5^0$	0	$-\sqrt{2}\alpha\mathbf{V}_4^0$	0	$-\sqrt{2}\alpha\mathbf{V}_7^0$	0	0	0	0	0	0	0
$\mathbf{V}_6^0$	$-\mathbf{V}_7^0$	0	0	0	0	0	0	0	0	0	0
$\mathbf{V}_7^0$	0	0	0	0	0	0	0	0	0	0	0
$\mathbf{V}_8^0$	0	$-\mathbf{V}_{10}^0$	$\mathbf{V}_{11}^0$	0	0	0	0	0	0	0	0
$\mathbf{V}_9^0$	$-\mathbf{V}_{10}^0$	0	$\mathbf{V}_6^0$	0	0	0	0	0	0	0	0
$\mathbf{V}_{10}^0$	0	0	$\mathbf{V}_7^0$	0	0	0	0	0	0	0	0
$\mathbf{V}_{11}^0$	0	$-\mathbf{V}_7^0$	0	0	0	0	0	0	0	0	0

Table 5.2: The commutator relations of Mei symmetries for case (i). $A(u) = \alpha^2$

For $\mathbf{V}_8$, Mei and approximate Mei invariants are calculated.

$$\mathbf{V}_8^0 + \epsilon \mathbf{V}_8^1 = u\frac{\partial}{\partial y} + \epsilon\left(sy\frac{\partial}{\partial v} - su\frac{\partial}{\partial y}\right),$$

and

$$\mathbf{V}_8^{0[1]}\mathcal{L}_0 = \dot{u}\dot{y},$$
$$\mathbf{V}_8^{0[1]}\mathcal{L}_1 + \mathbf{V}_8^{1[1]}\mathcal{L}_0 = -\dot{u}y - u\dot{y}.$$

Then gauge functions $\dot{B}_0$ is given by

$$\dot{\alpha}_0(\mathbf{V}^{0[1]}\mathcal{L}_0) + \mathbf{V}^{0[1]}(\mathbf{V}^{0[1]}\mathcal{L}_0) + \dot{B}_0 = 0,$$
$$0 + \dot{u}^2 + \dot{B}_0 = 0,$$
$$\dot{B}_0 = -\dot{u}^2.$$

and $\dot{B}_1$ is

$$\dot{\alpha}_0(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) + \dot{\alpha}_1(\mathbf{V}^{0[1]}\mathcal{L}_0) + \mathbf{V}^{0[1]}(\mathbf{V}^{0[1]}\mathcal{L}_1 + \mathbf{V}^{1[1]}\mathcal{L}_0) + \mathbf{V}^{1[1]}(\mathbf{V}^{0[1]}\mathcal{L}_0) + \dot{B}_1 = 0,$$
$$0 + 0 - 3u\dot{u} - s\dot{u}^2 + \dot{B}_1 = 0,$$
$$\dot{B}_1 = 3u\dot{u} + s\dot{u}^2.$$

Then corresponding Mei invariant is given below

$$I_8^0 = u\dot{u} + B_0,$$
$$\frac{dI_8^0}{ds} = \dot{u}^2 + \dot{B}_0, \qquad \Longrightarrow \frac{dI_8^0}{ds} = 0.$$

and approximate invariant is

$$I_8^1 = -su\dot{u} - u^2 + B_1,$$
$$\frac{dI_8^1}{ds} = -3u\dot{u} - s\dot{u}^2 + \dot{B}_1,$$
$$\frac{dI_8^1}{ds} = 0.$$

The remaining approximate Mei invariants corresponding to approximate Mei symmetries are given in **Table 5.3.**

	Mei/Approximate Mei Invariants of pp-waves spacetimes
$I_1^0 + \epsilon I_1^1$	$I_1^0 = 0, \dot{B}_0 = 0.$
$I_2^0 + \epsilon I_2^1$	$I_2^0 = 0, \dot{B}_0 = 0.$
$I_0^3 + \epsilon I_1^3$	$I_3^0 = 0, \dot{B}_0 = 0,$ $I_3^1 = -su\dot{u} + B_1, \dot{B}_1 = u\dot{u} + s\dot{u}^2.$
$I_4^0 + \epsilon I_4^1$	$I_4^0 = s + B_0, \dot{B}_0 = -1,$ $I_4^1 = -s^2 + B_1, \dot{B}_1 = 2s.$
$I_5^0 + \epsilon I_5^1$	$I_5^0 = 0, \dot{B}_0 = 0.$
$I_6^0 + \epsilon I_6^1$	$I_6^0 = 0, \dot{B}_0 = 0,$ $I_6^1 = 0, \dot{B}_1 = 0.$
$I_7^0 + \epsilon I_7^1$	$I_7^0 =, \dot{B}_0 = 0.$
$I_8^0 + \epsilon I_8^1$	$I_8^0 = u\dot{u} + B_0, \dot{B}_0 = -\dot{u}^2,$ $I_8^1 = -su\dot{u} - u^2 + B_1, \dot{B}_1 = 3u\dot{u} + s\dot{u}^2.$
$I_9^0 + \epsilon I_9^1$	$I_9^0 = s + B_0, \dot{B}_0 = -1,$ $I_9^1 = -s^2 + B_1, \dot{B}_1 = 2s.$
$I_{10}^0 + \epsilon I_{10}^1$	$I_{10}^0 = 0, \dot{B}_0 = 0.$

Table 5.3: Mei Invariants of case (i): $A(u) = \alpha^2$

5.2.1 Case (ii). $A(u) = \alpha u^{-2}$

The Lagrangian of the pp-wave spacetimes for $A(u) = \alpha u^{-2}$ is given by

$$\mathcal{L}_0 = -\alpha u^{-2} x^2 \dot{u}^2 - \dot{u}\dot{v} + \frac{1}{2}(\dot{x}^2 + \dot{y}^2),$$

The first order perturbed part of the Lagrangian is given below

$$\mathcal{L}_1 = -2s\alpha u^{-2} x^2 \dot{u}^2 - 2s\dot{u}\dot{v} + (s\dot{x}^2 + s\dot{y}^2).$$

The geodesics equations are

$$\ddot{u} = 0, \qquad \ddot{v} + 2\alpha u^{-3} x^2 \dot{u}^2 - 4\alpha u^{-2} x\dot{u}\dot{x} = 0,$$

$$\ddot{x} + 4\alpha u^{-2} x \dot{u}^2 = 0, \qquad \ddot{y} = 0.$$

The approximate Mei symmetries for this case, viz.

$$\mathbf{V}_1^0 + \epsilon \mathbf{V}_1^1 = \frac{\partial}{\partial s}, \qquad \mathbf{V}_2^0 + \epsilon \mathbf{V}_2^1 = y\frac{\partial}{\partial v} + u\frac{\partial}{\partial y},$$
$$\mathbf{V}_3^0 + \epsilon \mathbf{V}_3^1 = s\frac{\partial}{\partial y} - \epsilon s^2\frac{\partial}{\partial y}, \qquad \mathbf{V}_4^0 + \epsilon \mathbf{V}_4^1 = \frac{\partial}{\partial y},$$
$$\mathbf{V}_5^0 + \epsilon \mathbf{V}_5^1 = s\frac{\partial}{\partial v} - \epsilon s^2\frac{\partial}{\partial v}, \qquad \mathbf{V}_6^0 + \epsilon \mathbf{V}_6^1 = \frac{\partial}{\partial v}.$$

The vanishing and non-vanishing Lie algebra of case (ii). $A(u) = \alpha u^{-2}$ is given in **Table** 5.4.

[,]	$\mathbf{V}_1^0$	$\mathbf{V}_2^0$	$\mathbf{V}_3^0$	$\mathbf{V}_4^0$	$\mathbf{V}_5^0$	$\mathbf{V}_6^0$
$\mathbf{V}_1^0$	0	0	$\mathbf{V}_4^0$	0	$\mathbf{V}_6^0$	0
$\mathbf{V}_2^0$	0	0	$-\mathbf{V}_5^0$	$-\mathbf{V}_6^0$	0	0
$\mathbf{V}_3^0$	$-\mathbf{V}_4^0$	$\mathbf{V}_5^0$	0	0	0	0
$\mathbf{V}_4^0$	0	$\mathbf{V}_6^0$	0	0	0	0
$\mathbf{V}_5^0$	$-\mathbf{V}_6^0$	0	0	0	0	0
$\mathbf{V}_6^0$	0	0	0	0	0	0

Table 5.4: The commutator relations of Mei symmetries for case (ii). $A(u) = \alpha u^{-2}$

	Mei/Approximate Mei Invariants of pp-waves spacetimes
$I_1^0 + \epsilon I_1^1$	$I_1^0 = 0, \dot{B}_0 = 0.$
$I_2^0 + \epsilon I_2^1$	$I_2^0 = 0, \dot{B}_0 = 0.$
$I_3^0 + \epsilon I_3^1$	$I_3^0 = 0, \dot{B}_0 = 0,$ $I_3^1 = 0, \dot{B}_1 = 0.$
$I_4^0 + \epsilon I_4^1$	$I_4^0 = 0, \dot{B}_0 = 0.$
$I_5^0 + \epsilon I_5^1$	$I_5^0 = 0, \dot{B}_0 = 0,$ $I_5^1 = 0, \dot{B}_1 = 0.$
$I_6^0 + \epsilon I_6^1$	$I_0^6 = 0, \dot{B}_0 = 0.$

Table 5.5: Mei Invariants of case (ii): $A(u) = \alpha u^{-2}$

5.2.2 Case (iii). $A(u) = \alpha^2 u^{-4}$

The Lagrangian of the pp-waves spacetimes for $A(u) = \alpha^2 u^{-4}$ is given by

$$\mathcal{L}_0 = -\alpha^2 u^{-4} x^2 \dot{u}^2 - \dot{u}\dot{v} + \frac{1}{2}(\dot{x}^2 + \dot{y}^2),$$

The first order perturbed part of the Lagrangian is given below

$$\mathcal{L}_1 = -2s\alpha^2 u^{-4} x^2 \dot{u}^2 - 2s\dot{u}\dot{v} + (s\dot{x}^2 + s\dot{y}^2).$$

The geodesics equations are

$$\ddot{u} = 0, \qquad \ddot{v} - 4\alpha^2 u^{-5} x^2 \dot{u}^2 + 4\alpha^2 u^{-4} x\dot{u}\dot{x} = 0,$$

$$\ddot{x} + 2\alpha^2 u^{-4} x\dot{u}^2 = 0, \qquad \ddot{y} = 0.$$

The following list consist of approximate Mei symmetries viz.

$$\mathbf{V}_1^0 + \epsilon \mathbf{V}_1^1 = s\frac{\partial}{\partial s}, \qquad \mathbf{V}_2^0 + \epsilon \mathbf{V}_2^1 = \frac{\partial}{\partial s},$$

$$\mathbf{V}_3^0 + \epsilon \mathbf{V}_3^1 = s\frac{\partial}{\partial v} - \epsilon s^2 \frac{\partial}{\partial v}, \qquad \mathbf{V}_4^0 + \epsilon \mathbf{V}_4^1 = y\frac{\partial}{\partial v} + \epsilon\left(-sy\frac{\partial}{\partial v} + su\frac{\partial}{\partial y}\right),$$

$$\mathbf{V}_5^0 + \epsilon \mathbf{V}_5^1 = \frac{\partial}{\partial v}, \qquad \mathbf{V}_6^0 + \epsilon \mathbf{V}_6^1 = u\frac{\partial}{\partial y} + \epsilon\left(sy\frac{\partial}{\partial v} - su\frac{\partial}{\partial y}\right),$$

$$\mathbf{V}_7^0 + \epsilon \mathbf{V}_7^1 = s\frac{\partial}{\partial y} - \epsilon s^2 \frac{\partial}{\partial y}, \qquad \mathbf{V}_8^0 + \epsilon \mathbf{V}_8^1 = y\frac{\partial}{\partial y} - \epsilon 2sy\frac{\partial}{\partial y},$$

$$\mathbf{V}_9^0 + \epsilon \mathbf{V}_9^1 = u\frac{\partial}{\partial v} - \epsilon 2su\frac{\partial}{\partial v}, \qquad \mathbf{V}_{10}^0 + \epsilon \mathbf{V}_{10}^1 = \frac{\partial}{\partial y}.$$

5.2.3 Case (iv) $h(x) = \alpha x^n$

The Lagrangian of the pp-waves spacetimes for $h(x) = \alpha x^n$ is given by

$$\mathcal{L}_0 = -\alpha x^n \dot{u}^2 - \dot{u}\dot{v} + \frac{1}{2}(\dot{x}^2 + \dot{y}^2),$$

[,]	$\mathbf{V}_3^1$	$\mathbf{V}_4^1$	$\mathbf{V}_6^1$	$\mathbf{V}_7^1$	$\mathbf{V}_8^1$	$\mathbf{V}_9^1$
$\mathbf{V}_3^1$	0	0	0	0	0	0
$\mathbf{V}_4^1$	0	0	0	0	0	0
$\mathbf{V}_6^1$	0	0	0	0	0	0
$\mathbf{V}_7^1$	0	0	0	0	0	0
$\mathbf{V}_8^1$	0	0	0	0	0	0
$\mathbf{V}_9^1$	0	0	0	0	0	0

Table 5.6: The commutator relations of approximate Mei symmetries for case (iii). $A(u) = \alpha^2 u^{-4}$

The first order perturbed part of the Lagrangian is given below

$$\mathcal{L}_1 = -2s\alpha x^n \dot{u}^2 - 2s\dot{u}\dot{v} + (s\dot{x}^2 + s\dot{y}^2).$$

The geodesics equations are

$$\ddot{u} = 0, \qquad \ddot{v} + 2\alpha n x^{n-1}\dot{u}\dot{x} = 0,$$
$$\ddot{x} + \alpha n x^{n-1}\dot{u}^2 = 0, \qquad \ddot{y} = 0.$$

The approximate Mei symmetries of Lagrangian for this case is given as follows

$$\mathbf{V}_1^0 + \epsilon\mathbf{V}_1^1 = \frac{\partial}{\partial s}, \qquad \mathbf{V}_2^0 + \epsilon\mathbf{V}_2^1 = \frac{\partial}{\partial u},$$
$$\mathbf{V}_3^0 + \epsilon\mathbf{V}_3^1 = u\frac{\partial}{\partial y} + \epsilon\left(sy\frac{\partial}{\partial v} - su\frac{\partial}{\partial y}\right), \qquad \mathbf{V}_4^0 + \epsilon\mathbf{V}_4^1 = s\frac{\partial}{\partial y} - \epsilon s^2\frac{\partial}{\partial y},$$
$$\mathbf{V}_5^0 + \epsilon\mathbf{V}_5^1 = \frac{\partial}{\partial y}, \qquad \mathbf{V}_6^0 + \epsilon\mathbf{V}_6^1 = \frac{\partial}{\partial v},$$
$$\mathbf{V}_7^0 + \epsilon\mathbf{V}_7^1 = y\frac{\partial}{\partial y} - \epsilon 2sy\frac{\partial}{\partial y}, \qquad \mathbf{V}_8^0 + \epsilon\mathbf{V}_8^1 = u\frac{\partial}{\partial v} - \epsilon 2su\frac{\partial}{\partial v},$$
$$\mathbf{V}_9^0 + \epsilon\mathbf{V}_9^1 = s\frac{\partial}{\partial v} - \epsilon s^2\frac{\partial}{\partial v}, \qquad \mathbf{V}_{10}^0 + \epsilon\mathbf{V}_{10}^1 = y\frac{\partial}{\partial v} + \epsilon\left(-sy\frac{\partial}{\partial v} + su\frac{\partial}{\partial y}\right).$$

The Lie algebra of Mei and approximate Mei symmetry generators are given in **Table 5.9** and **Table 5.10** respectively.

[,]	$\mathbf{V}_1^0$	$\mathbf{V}_2^0$	$\mathbf{V}_3^0$	$\mathbf{V}_4^0$	$\mathbf{V}_5^0$	$\mathbf{V}_6^0$	$\mathbf{V}_7^0$	$\mathbf{V}_8^0$	$\mathbf{V}_9^0$	$\mathbf{V}_{10}^0$
$\mathbf{V}_1^0$	0	$\mathbf{V}_2^0$	$\mathbf{V}_3^0$	0	0	0	$\mathbf{V}_7^0$	0	0	0
$\mathbf{V}_2^0$	$-\mathbf{V}_2^0$	0	$\mathbf{V}_5^0$	0	0	0	$\mathbf{V}_{10}^0$	0	0	0
$\mathbf{V}_3^0$	$-\mathbf{V}_3^0$	$-\mathbf{V}_5^0$	0	0	0	0	0	0	0	0
$\mathbf{V}_4^0$	0	0	0	0	0	$-\mathbf{V}_9^0$	$-\mathbf{V}_3^0$	$-\mathbf{V}_4^0$	0	$-\mathbf{V}_5^0$
$\mathbf{V}_5^0$	0	0	0	0	0	0	0	0	0	0
$\mathbf{V}_6^0$	0	0	0	$\mathbf{V}_9^0$	0	0	0	$\mathbf{V}_6^0$	0	0
$\mathbf{V}_7^0$	$-\mathbf{V}_7^0$	$-\mathbf{V}_{10}^0$	0	$\mathbf{V}_3^0$	0	0	0	$\mathbf{V}_7^0$	0	0
$\mathbf{V}_8^0$	0	0	0	$\mathbf{V}_4^0$	0	$-\mathbf{V}_6^0$	0	0	0	$-\mathbf{V}_{10}^0$
$\mathbf{V}_9^0$	0	0	0	0	0	0	0	0	0	0
$\mathbf{V}_{10}^0$	0	0	0	$\mathbf{V}_5^0$	0	0	0	$\mathbf{V}_{10}^0$	0	0

Table 5.7: The commutator relations of Mei symmetries for case (iii). $A(u) = \alpha^2 u^{-4}$

	Mei/Approximate Mei Invariants of pp-waves spacetimes
$I_1^0 + \epsilon I_1^1$	$I_1^0 = 0, \dot{B}_0 = 0.$
$I_2^0 + \epsilon I_2^1$	$I_2^0 = 0, \dot{B}_0 = 0.$
$I_3^0 + \epsilon I_3^1$	$I_3^0 = 0, \dot{B}_0 = 0,$ $I_3^1 = 0, \dot{B}_1 = 0.$
$I_4^0 + \epsilon I_4^1$	$I_4^0 = 0, \dot{B}_0 = 0,$ $I_4^1 = -su\dot{u} + B_1, \dot{B}_1 = u\dot{u} + s\dot{u}^2.$
$I_5^0 + \epsilon I_5^1$	$I_5^0 = 0, \dot{B}_0 = 0.$
$I_6^0 + \epsilon I_6^1$	$I_6^0 = u\dot{u} + B_0, \dot{B}_0 = -\dot{u}^2,$ $I_6^1 = -su\dot{u} - u^2 + B_1, \dot{B}_1 = 3u\dot{u} + s\dot{u}^2.$
$I_7^0 + \epsilon I_7^1$	$I_7^0 = 0, \dot{B}_0 = 0,$ $I_7^1 = 0, \dot{B}_1 = 0.$
$I_8^0 + \epsilon I_8^1$	$I_8^0 = 2y\dot{y} + B_0, \dot{B}_0 = -2\dot{y}^2,$ $I_8^1 = -4sy\dot{y} - 2y^2 + B_1, \dot{B}_1 = 8y\dot{y} + 4s\dot{y}^2.$
$I_9^0 + \epsilon I_9^1$	$I_9^0 = 0, \dot{B}_0 = 0,$ $I_9^1 = 0, \dot{B}_1 = 0.$
$I_{10}^0 + \epsilon I_{10}^1$	$I_{10}^0 = 0, \dot{B}_0 = 0,$ $I_{10}^1 = 0, \dot{B}_1 = 0.$

Table 5.8: Mei Invariants of case (iii): $A(u) = \alpha^2 u^{-4}$

[,]	$\mathbf{V}_1^0$	$\mathbf{V}_2^0$	$\mathbf{V}_3^0$	$\mathbf{V}_4^0$	$\mathbf{V}_5^0$	$\mathbf{V}_6^0$	$\mathbf{V}_7^0$	$\mathbf{V}_8^0$	$\mathbf{V}_9^0$	$\mathbf{V}_{10}^0$
$\mathbf{V}_1^0$	0	0	0	$\mathbf{V}_5^0$	0	0	0	0	$\mathbf{V}_6^0$	0
$\mathbf{V}_2^0$	0	0	$\mathbf{V}_5^0$	0	0	0	0	$\mathbf{V}_6^0$	0	0
$\mathbf{V}_3^0$	0	$-\mathbf{V}_5^0$	0	0	0	0	$\mathbf{V}_3^0$	0	0	$\mathbf{V}_8^0$
$\mathbf{V}_4^0$	$-\mathbf{V}_5^0$	0	0	0	0	0	$\mathbf{V}_4^0$	0	0	$\mathbf{V}_9^0$
$\mathbf{V}_5^0$	0	0	0	0	0	0	$\mathbf{V}_5^0$	0	0	$\mathbf{V}_6^0$
$\mathbf{V}_6^0$	0	0	0	0	0	0	0	0	0	0
$\mathbf{V}_7^0$	0	0	$-\mathbf{V}_3^0$	$-\mathbf{V}_4^0$	$-\mathbf{V}_5^0$	0	0	0	0	$\mathbf{V}_{10}^0$
$\mathbf{V}_8^0$	0	$-\mathbf{V}_6^0$	0	0	0	0	0	0	0	0
$\mathbf{V}_9^0$	$-\mathbf{V}_6^0$	0	0	0	0	0	0	0	0	0
$\mathbf{V}_{10}^0$	0	0	$-\mathbf{V}_8^0$	$-\mathbf{V}_9^0$	$-\mathbf{V}_6^0$	0	$-\mathbf{V}_{10}^0$	0	0	0

Table 5.9: The commutator relations of Mei symmetries for case (iv). $h(x) = \alpha x^n$

[,]	$\mathbf{V}_3^1$	$\mathbf{V}_4^1$	$\mathbf{V}_7^1$	$\mathbf{V}_8^1$	$\mathbf{V}_9^1$	$\mathbf{V}_{10}^1$
$\mathbf{V}_3^1$	0	0	0	0	0	0
$\mathbf{V}_4^1$	0	0	0	0	0	0
$\mathbf{V}_7^1$	0	0	0	0	0	0
$\mathbf{V}_8^1$	0	0	0	0	0	0
$\mathbf{V}_9^1$	0	0	0	0	0	0
$\mathbf{V}_{10}^1$	0	0	0	0	0	0

Table 5.10: The commutator relations of approximate Mei symmetries for case (iv). $h(x) = \alpha x^n$

Concluding Remarks:

Mei symmetries/approximate Mei symmetries of geodetic Lagrangian of pp-waves spacetimes are calculated for various classes in this chapter. The different cases include plane wave spacetimes in which (i). $A(u) = \alpha^2$(ii). $A(u) = \alpha u^{-2}$ (iii). $A(u) = \alpha^2 u^{-4}$ and for pp-wave spacetimes (iv). $h(x) = \alpha x^n$. We obtained eleven Mei symmetries for case (i), ten for case (iii) and case (iv) and five for (ii).

	Mei/Approximate Mei Invariants of pp-waves spacetimes
$I_1^0 + \epsilon I_1^1$	$I_1^0 = 0, \dot{B}_0 = 0.$
$I_2^0 + \epsilon I_2^1$	$I_2^0 = 0, \dot{B}_0 = 0.$
$I_3^0 + \epsilon I_3^1$	$I_3^0 = u\dot{u} + B_0, \dot{B}_0 = -\dot{u}^2,$ $I_3^1 = -su\dot{u} - u^2 + B_1, \dot{B}_1 = 3u\dot{u} + s\dot{u}^2.$
$I_4^0 + \epsilon I_4^1$	$I_4^0 = 0, \dot{B}_0 = 0,$ $I_4^1 = 0, \dot{B}_1 = 0$
$I_5^0 + \epsilon I_5^1$	$I_5^0 = 0, \dot{B}_0 = 0.$
$I_6^0 + \epsilon I_6^1$	$I_6^0 = 0, \dot{B}_0 = 0,$ $I_6^1 = 0, \dot{B}_1 = 0.$
$I_7^0 + \epsilon I_7^1$	$I_7^0 = 2y\dot{y} + B_0, \dot{B}_0 = -2\dot{y}^2,$ $I_7^1 = -4sy\dot{y} - 2y^2 + B_1, \dot{B}_1 = 8y\dot{y} + 4s\dot{y}^2.$
$I_8^0 + \epsilon I_8^1$	$I_8^0 = 0, \dot{B}_0 = 0,$ $I_8^1 = 0, \dot{B}_1 = 0.$
$I_9^0 + \epsilon I_9^1$	$I_9^0 = 0, \dot{B}_0 = 0,$ $I_9^1 = 0, \dot{B}_1 = 0.$
$I_{10}^0 + \epsilon I_{10}^1$	$I_{10}^0 = 0, \dot{B}_0 = 0,$ $I_{10}^1 = -su\dot{u} + B_1, \dot{B}_1 = u\dot{u} + s\dot{u}^2.$

Table 5.11: Mei Invariants of case (iv): $h(x) = \alpha x^n$

Chapter 6

Conclusions

In this thesis, approximate Mei symmetries and related Mei invariants corresponding to the approximate Hamiltonian are studied using [53]. Formulae of obtaining these symmetries of approximate Hamiltonian are given in **Theorems 3.2.1** and Mei invariants are given in **Theorem 3.4.1**, **Theorem 3.4.2** and **Theorem 3.4.3** respectively. The given example of DHO demonstrates the developed procedure in detail. A comparison of approximate Noether and approximate Mei symmetries for DHO is given in Table 3.5 that shows:

- the number of approximate Mei symmetries is more than the number of approximate Noether symmetries;
- the Mei symmetry $\mathbf{V}_1$ is also contained in the set of Noether symmetries;
- the other Mei symmetries, $\mathbf{V}_2$, $\mathbf{V}_3$, $\mathbf{V}_4$, $\mathbf{V}_5$, and $\mathbf{V}_6$ are different from the Noether symmetries, therefore, there are new corresponding conserved quantities.

Further, approximate Mei symmetries and invariants corresponding to the Lagrangian are formulated. First of all, definition and criterion to develop the Mei symmetries are explained [11, 20]. Then, these exact Mei symmetries are used to construct approximate Mei symmetries and invariants, which are discussed in **Theorem 4.1.1**, **Theorem**

4.3.1, **Theorem** 4.3.2. At the end, approximate Mei symmetries and invariants of Lagrangian of DHO are obtained as an example. A comparison of approximate Mei symmetries corresponding to the Lagrangian and Hamiltonian are given in Table 4.5. From this comparison, it is noticed that

- $\mathbf{V}_1$ is common in both i.e., related to the Hamiltonian and the Lagrangian
- a minor difference in approximate part of $\mathbf{V}_2$ is noted
- Mei symmetries $\mathbf{V}_3$, $\mathbf{V}_4$, $\mathbf{V}_5$, and $\mathbf{V}_6$ of both sets are completely different from each other. These new Mei symmetries related to the Lagrangian lead to new Mei invariants of DHO

After that, an application from general theory of relativity is taken. Mei symmetries/approximate Mei symmetries of geodetic Lagrangian of pp-waves spacetimes are obtained for several classes that are listed in [51, 52]. The different cases include plane wave spacetimes in which (i). $A(u) = \alpha^2$(ii). $A(u) = \alpha u^{-2}$ (iii). $A(u) = \alpha^2 u^{-4}$ and for pp-wave spacetimes (iv). $h(x) = \alpha x^n$. We obtained eleven Mei symmetries for case (i), ten for case (iii) and case (iv) and five for (ii). For the cases (i)-(iv), Noether gauge symmetries (NGS) are also known [52]. From the comparison of NGS and Mei symmetries of pp-waves, following result are drawn

- For case (i), $\mathbf{V}_3^0, \mathbf{V}_6^0, \mathbf{V}_8^0, \mathbf{V}_9^0$ are the proper Mei symmetries, i.e., other than NGS. According to [47], for arbitrary $A(u)$, $\mathbf{V}_4^0, \mathbf{V}_5^0, \mathbf{V}_7^0, \mathbf{V}_{10}^0$ are Killing vectors (KVs).
- For case (ii), $\mathbf{V}_2^0, \mathbf{V}_3^0, \mathbf{V}_5^0$ are proper Mei symmetries while the other develop a equivalence relation with NGS. In addition, $\mathbf{V}_4^0, \mathbf{V}_6^0$ are KVs.
- For case (iii), $\mathbf{V}_3^0, \mathbf{V}_6^0, \mathbf{V}_7^0, \mathbf{V}_8^0$ are the proper Mei symmetries and $\mathbf{V}_{\mathbf{2}}^{\mathbf{0}}, \mathbf{V}_{\mathbf{10}}^{\mathbf{0}}$ are Noether symmetries as well as KVs.

- For case (iv), symmetry generators $\mathbf{V}_3^0, \mathbf{V}_4^0, \mathbf{V}_7^0, \mathbf{V}_8^0, \mathbf{V}_9^0, \mathbf{V}_{10}^0$ are the proper Mei symmetries, while $\mathbf{V}_1^0, \mathbf{V}_2^0, \mathbf{V}_4^0, \mathbf{V}_5^0, \mathbf{V}_6^0$ satisfy the NGS conditions. Also, we have two Killing vectors i.e., $\mathbf{V}_5^0$ and $\mathbf{V}_6^0$.

- In case (iv), a non-Homothetic affine collination (AC) vector $\mathbf{V}_8^0 = u\frac{\partial}{\partial v}$ is obtained [56].

- The first order approximate Mei symmetries of perturbed Lagrangian of all the above mentioned cases are also calculated using the method developed in [54]. In case (i) we have obtained five dimensional approximate symmetries up to first order of ϵ, two for case (ii), five for (iii) and six for case (iv).

Future Work

The Mei symmetry of several dynamical systems has been discussed in the literature. The approximate Mei symmetries of the perturbed Lagrangian and perturbed Hamiltonian, that correspond to some spacetimes, such as the Friedman-Robertson-Walker (FRW) spacetime, the Bardeen spacetime, etc., have not yet been studied. Future studies will also take into account the maximum dimension of approximate Mei symmetries and the approximate Mei symmetries of PDEs.

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