Study of Ordinary Waves in Different Plasma Environments

by

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Abstract

The propagation of ordinary waves in different plasma environments is studied by using the plasma kinetic theory. The non-degenerate and the degenerate plasmas are focussed. The non-degenerate plasma environment is studied by employing the ultra-relativistic Maxwellian distribution function in different magnetic field limits i.e., the strong and weak field. However, the degenerate plasma is studied by employing the Fermi-Dirac distribution function. The non-relativistic degenerate and the ultra-relativistic degenerate cases are also presented. For this purpose a generalized expression for the conductivity tensor in spherical polar coordinates is derived by employing the relativistic Vlasov-Maxwell equations. In the strong field limit it is observed that with increase in the strength of the ambient magnetic field the damping is reduced whereas in the weak field limit, due to negligible effects of the magnetic field, the dispersion diagram is same like the ordinary waves propagating in the non-relativistic plasma. In the degenerate case the cut-off points are shifted to the lower values of the frequency due to the relativistic effects. The magnetic field effects that are not observed in the fluid approximation becomes significant when studied by the plasma kinetic theory.

Chapter 1

Introduction

1.1 What is a plasma?

A plasma is an ionized gas consisting of large number of positive and negative charged particles but as a whole it is electrically neutral and exhibits "collective behavior". However, a plasma is not so neutral that all the interesting electromagnetic phenomena vanish because the local concentration of positive and negative charged particles can give rise to the electric field. As long range coulombic interaction exist between the charged particles so the motions are influenced not only by the local concentration of the particles but also due to the particles at large distances, this is called the *collective behavior*.

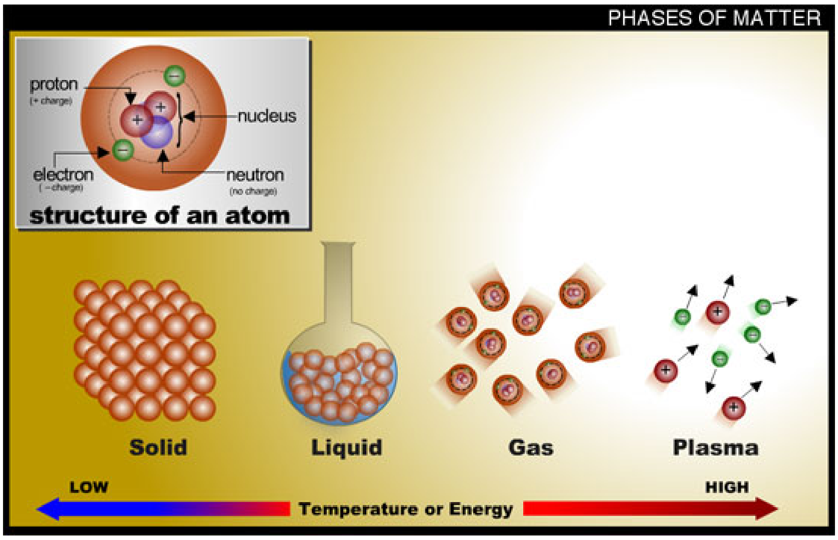

Figure 1.1: States of matter.

In the laboratory different methods are used to create the plasma. The most common methods for the plasma production are the gas discharge and photoionization. In the first (gas discharge) method, the free electrons of an ionized gas are accelerated by applying an electric field. These free electrons collide with the other atoms and ionize them. The electric field again accelerates these charged particles (ions and electrons) which produce further ionization. An example of the plasma produced by the gas discharge is the neon signs.

In the second (photoionization) method, high energy photons are incident on the gas atoms and ionize them. The photon's excess energy is converted to the kinetic energy of electron ion pairs formed [3]. A natural example of a photoionized plasma is the aurora borealis.

1.1.1 Various plasma environments

In the universe most of the matter visible to us is in the plasma state. A plasma can be described by the temperature T (in eV) and density n_0 (number of particles per unit volume) that have wide range of variation. A plasma can be found in various environments due to this variation.

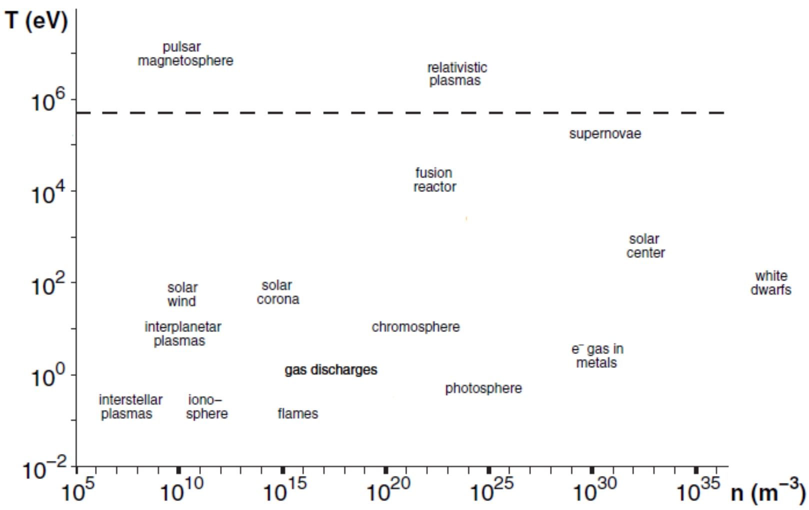

Figure 1.2: Various plasma environments.

1.2 Plasma characteristics

In order to understand the plasma behavior completely, we define some terms that are frequently used in describing plasma dynamics.

1.2.1 Plasma frequency (ω_p)

An important property of a plasma is its tendency to restore charge neutrality. Consider that the electrons and ions are uniformly distributed in a plasma. If the electrons are displaced from their equilibrium positions due to any external effect, extra ions will be left behind. As a result there will be a deficiency of the negative charge in that region. The ions will pull the electrons back to their respective equilibrium positions but due to inertia, the electrons will not stop exactly at their equilibrium positions and will overshoot. Again there will be a charge separation that will give rise to an electric field that pulls the electrons back to their initial positions. So the electrons will oscillate about their equilibrium positions. The frequency of these oscillations is called the *electron plasma frequency* [2].

$$\omega_p = \sqrt{\frac{4\pi n_0 e^2}{m}}, \tag{1.2.1}$$

where e is the charge and m is the mass of the electron. As the ions are more massive (at least 1836 times) than the electrons, so they respond very slowly. The same force acts on the electrons and ions but the acceleration of the ions is at least 1836 times less than that of the electron. As such they move slowly compared to time scale of our interest. That's why we take the ions to be stationary.

1.2.2 Debye shielding

The fundamental property of a plasma is its ability to minimize the effect of the applied electric fields. Consider a homogeneous (no density gradient ($\nabla n_0 = 0$)) plasma consisting of the positive ions and electrons. If we introduce a positive test charge Q_T it will attract the electrons and repel the ions. We assume that the density of the ions will remain unchanged i.e., $n_i = n_0$ (as before the introduction of Q_T). The electron density will increase near Q_T and a charge cloud will be formed. The electric field of these electrons will shield out the electric field of Q_T. Outside this region (cloud) there will be no electric field. If we assume that there is no thermal motion of the electrons then the electric field of Q_T will be completely screened. If the charged particles that are at the edge of the cloud (region of weak electric field), have thermal energy larger than the potential energy (that binds oppositely charged particle), then the shielding will not be perfect. The radius of this charge cloud is called the Debye length λ_{de} and this process is called Debye shielding. Consider that the electrons and ions in a plasma are in thermal equilibrium then according to the Maxwell Boltzmann law the electron number density is given by

$$n_e = n_0 \exp(e\phi/k_B T),$$

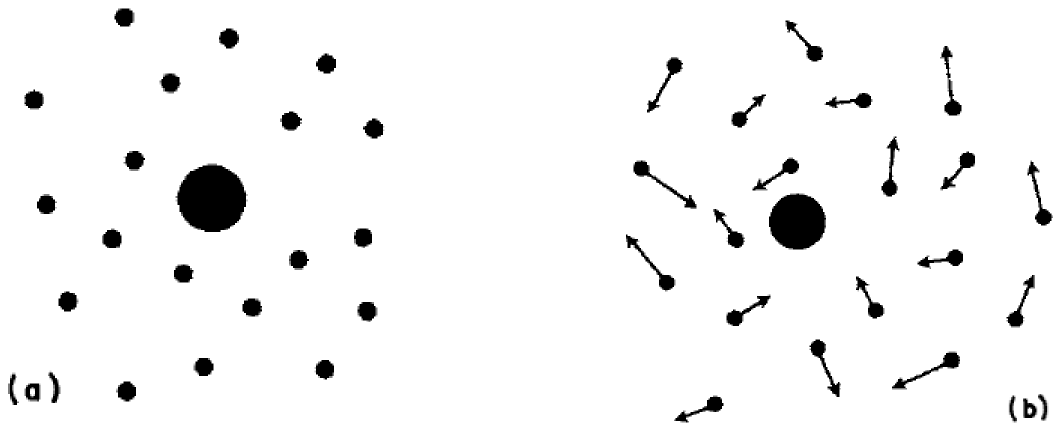

Figure 1.3: Comparison of Debye shielding at (a) temperature $T = 0$ and (b) temperature T.

where ϕ is the electrostatic potential. In order to find the expression for the electrostatic potential (ϕ) we will use the Poisson equation, given by

$$\nabla \cdot \mathbf{E} = 4\pi\rho, \tag{1.2.2}$$

where the charge density ρ is given by

$$\rho = e(n_i - n_e).$$

Now substituting $\mathbf{E} = -\nabla\phi$ in Eq. (1.2.2), we get

$$\nabla^2\phi = 4\pi e(n_e - n_0). \tag{1.2.3}$$

We assume that the electrostatic potential is weak i.e., $e\phi \ll k_B T$. So the Eq. (1.2.3) for the one dimensional case becomes

$$\frac{d^2\phi}{dx^2} = \frac{4\pi n_0 e^2}{k_B T}\phi. \tag{1.2.4}$$

We now take

$$\lambda_{de} = \left(\frac{k_B T}{4\pi n_0 e^2}\right)^{\frac{1}{2}}, \tag{1.2.5}$$

where k_B is the Boltzmann constant and T is the electron temperature [1]. The solution of the Eq. (1.2.4) (where $\phi \to 0$ for $x \to \infty$) is given by

$$\phi = \phi_0 \exp\left(\frac{-x}{\lambda_{de}}\right). \tag{1.2.6}$$

So the potential of Q_T will fall off exponentially outside the Debye length ($x > \lambda_{de}$) and the charges will feel the potential of Q_T inside λ_{de} (i.e., $x < \lambda_{de}$). From the Eq. (1.2.5), it is clear that if the density is small or temperature is high then λ_{de} will be large. This means that the electric field of the test charge will be screened at a large distance. The Debye length is different in different plasma environments e.g., the Debye length for the interstellar medium is $10^7 m$, whereas for the solar corona it is $0.07m$. The Debye length is related to the plasma frequency by the following expression (using Eq. 1.2.5)

$$\lambda_{de} = \frac{v_{th}}{\sqrt{2}\omega_p}, \tag{1.2.7}$$

where $v_{th} = \sqrt{2k_B T/m}$ is the thermal velocity of the electrons. If Debye length λ_{de} is smaller than the scale length L of our system i.e., $\lambda_{de} < L$ then upon introduction of any test charge (or potential) in a plasma, it will be screened within the dimension of the system. So bulk of the plasma will be free from electric potentials. The number of particles inside a Debye sphere is given by

$$N_{de} = \frac{4}{3} n_0 \pi \lambda_{de}^3,$$

this concept will be valid only if there is a large number of charged particles in the Debye sphere [2].

1.2.3 Collisionless plasma

In a plasma the charged particles interact with each other through the long range Coulomb force whereas gas atoms interact when they collide. So the collisions in a plasma are different from those in an ordinary gas. The following conditions must be satisfied for a collisionless plasma:
1. $\lambda_m \gg L$, where λ_m is the mean free path;
2. The collision frequency must be smaller than the plasma frequency, ω_p, i.e.

$$\omega_p \tau > 1,$$

where τ is the time between successive collisions. It means that in a collisionless plasma we are interested in those processes that are fast compared to the collision time.

1.2.4 Plasma criteria and parameters

An ionized gas can be called a plasma, if it satisfies the following conditions [2]:
(i) $\lambda_{de} \ll L$;

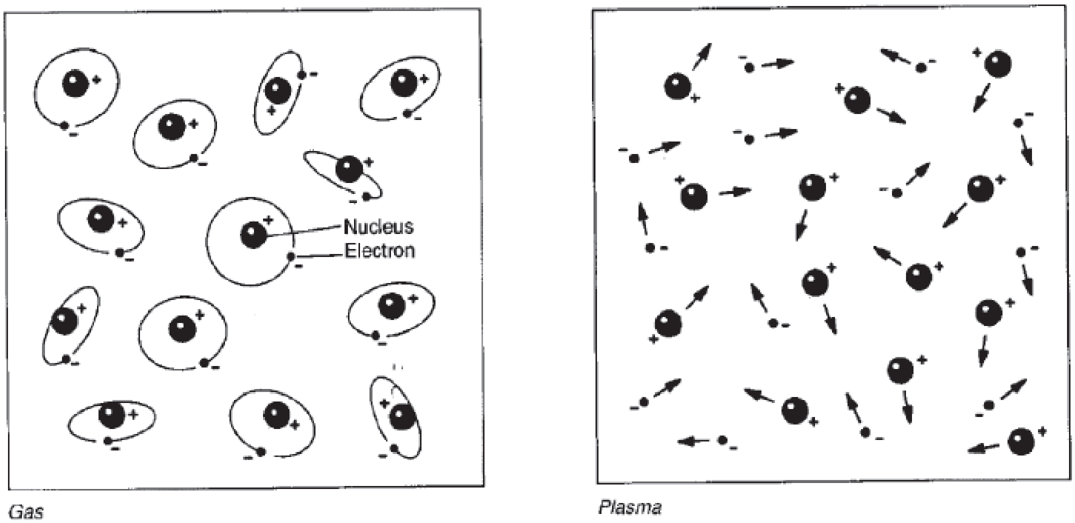

Figure 1.4: Comparison of an ordinary gas and a plasma.

(ii) $N_{de} \ggg 1$;
(iii) $\omega_p \tau > 1$;
and it is characterized by the following parameters [4]:
1. Density ($n_0 = N/V$);
2. Temperature T.
Consider that the particles in a gas are constrained to move in one direction. The Maxwellian distribution for such a one dimensional case is given by

$$f(v) = A \exp\Big(\frac{-mv^2}{2k_{\rm B}T}\Big), \tag{1.2.8}$$

where A is the normalization constant and v is the velocity of particles.
Now we can write

$$\int_{-\infty}^{\infty} f(v)dv = 1, \tag{1.2.9}$$

where $f(v)dv$ is the number of gas particles having velocities in the range between v and $v + dv$. For the Maxwellian distribution function, we get

$$A \int_{-\infty}^{\infty} \exp\Big(\frac{-mv^2}{2k_{\rm B}T}\Big) dv = 1.$$

In order to solve the above integral, we will use

$$\int_{-\infty}^{\infty} \exp(-a^2x^2)dx = \frac{\sqrt{\pi}}{a}.$$

Hence

$$A = \left(\frac{m}{2\pi k_{\mathrm{B}} T}\right)^{\frac{1}{2}}.$$

Now we calculate the average kinetic energy of all the particles present in this distribution.

$$E_{average} = \frac{\int_{-\infty}^{\infty} \frac{1}{2} m v^2 f(v) dv}{\int_{-\infty}^{\infty} f(v) dv}, \tag{1.2.10}$$

using the Maxwellian distribution function, we obtain

$$E_{average} = \frac{1}{2} k_{\mathrm{B}} T.$$

Thus the average kinetic energy is $k_B T/2$.

1.3 Different theoretical approaches

In order to understand various plasma processes different theoretical models are used. Depending on the process of interest, one can choose any model. Here we will discuss the fluid approximation and the kinetic theory of a plasma.

1.3.1 The fluid approximation and the kinetic theory of a plasma

In a plasma, there is a collection of charged particles and it is difficult to track the motion of each particle. The most common way to describe such a bulk behavior is the fluid approach that solves these problems by taking the average over collection of particles. In this approximation the individual particle identity is neglected and the collective properties (i.e., average velocity, density, temperature) are taken into account. This approach considers the fluid as a continuous medium, so all the quantities are functions of time, t, and position, $\mathbf{r}$. The fluid approximation gives the simplest plasma description and can explain most of the plasma phenomena but it has some limitations e.g., the information related to the distribution of particle velocities is lost in a fluid description because the fluid variables are not functions of the velocity [2]. So we define the velocity and position of the charged particles by using a distribution function. The plasma theory that is based on the description of the distribution function is known as the plasma kinetic theory. The kinetic theory of the plasma describes the variation of the distribution function with the time and it provides more information as compared to the fluid approximation. The probability of finding the particles at a time, t, in a volume element, d^3r, having the velocity in the range between $\mathbf{v}$ and $\mathbf{v}+d\mathbf{v}$ is given by $f(\mathbf{r},\mathbf{v},t)d^3rd^3v$. If we integrate it over all the possible velocities, we will get the density.

$$n(\mathbf{r},t) = \int_{-\infty}^{\infty} f(\mathbf{r},\mathbf{v},t)d^3v. \tag{1.3.1}$$

The distribution function do not give the information of the exact location and the velocity of any single charged particle but (it gives information) in a velocity range [2].

Now we will define the phase space which is described by the position and the velocity coordinates i.e., x, y, z, v_x, v_y, v_z (six coordinates). We take a small volume element $d^3r = dxdydz$ in the configuration space. When we say that a particle is inside the volume element d^3r, it means that the position coordinate is between x and $x+dx$, y and $y+dy$, z and $z+dz$. Similarly we can take a small volume element $d^3v = dv_xdv_ydv_z$ in the velocity space. So the volume element in the phase space is represented by $d^3rd^3v = dxdydzdv_xdv_ydv_z$. The distribution function gives

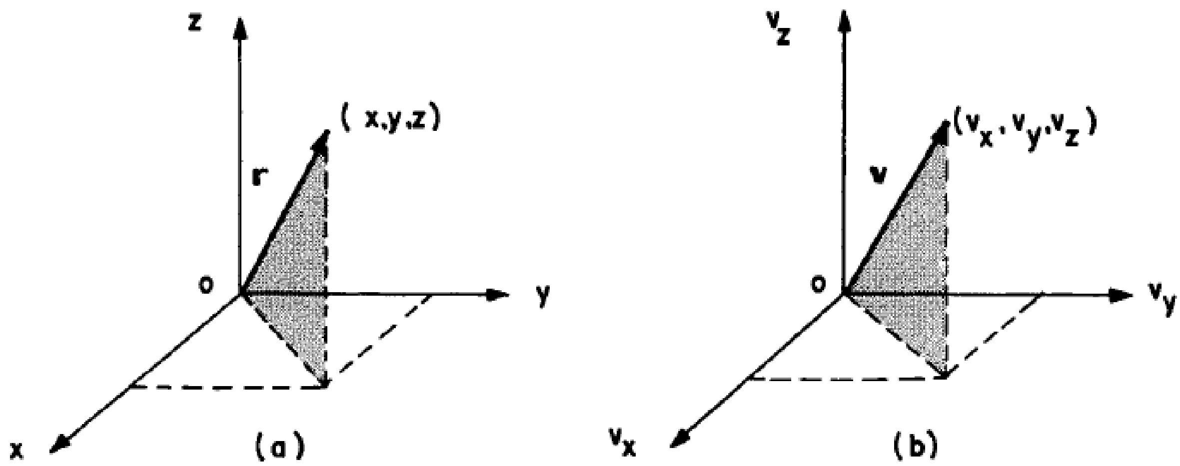

Figure 1.5: Volume elements in (a) Configuration space (b) Velocity space.

the density of the points (charged particles) in the phase space. Homogeneous plasmas are characterized by a position independent distribution function whereas in the inhomogeneous plasmas, the distribution function depends on the position of the particle. The distribution function in an isotropic plasma is independent of the direction of the particle velocity (v) but an anisotropic distribution function depends on the orientation of the velocity.

1.3.2 Examples of different distribution functions

There can be different distribution functions (isotropic, anisotropic, homogeneous, inhomogeneous) corresponding to various plasma environments. Here we will study the Maxwellian, the Fermi-Dirac and the Bose-Einstein distribution functions.

In a classical plasma, the inter-particle distance becomes greater than the de-Broglie's wavelength of the charged particles, so the energy distribution of the charged particles (in thermal equilibrium) is described by the Maxwell Boltzmann energy distribution function.

$$f_0(E) = n_0\Big(\frac{m}{2\pi k_B T}\Big)^{\frac{3}{2}} \exp\Big(\frac{-mv^2}{2k_B T}\Big).$$

The Maxwellian distribution function is a uniform, isotropic and it is independent of the time. This classical distribution is applicable when the temperature is high or the particle density is low. There is no limit on the number of particles in any energy state. Any number of particles can be found in any state [5]. An example of the Maxwellian distribution is the distribution of the particles velocities at the room temperature. For the relativistic case $m = \gamma m_0$, where $\gamma = 1/\sqrt{1 - \frac{v^2}{c^2}}$, where c is the speed of light. For the ultra-relativistic case, when the thermal energy dominates the rest mass energy we get the following distribution.

$$f_0(E) = \frac{n_0 c^3}{8\pi(k_B T)^3} \exp\Big(-\frac{cp}{k_B T}\Big). \tag{1.3.2}$$

In an environment with a high density and a low temperature the de-Broglie wavelength of the charged particles becomes greater than or comparable to the inter-particle distance. So the quantum effects become important and we cannot use the Maxwell Boltzmann distribution. According to the Pauli Exclusion principle, no two identical Fermions can occupy the same quantum state. The energy of the highest occupied level is called the *Fermi energy.* Such a system in which all the energy states below the Fermi energy are filled is called *degenerate.* So we use the Fermi-Dirac distribution which is applicable to the Fermions, the spin 1/2 particles, that obey the Pauli Exclusion principle [6]. In the thermal equilibrium, the energy distribution of the Fermions at temperature (T) is defined by the Fermi-Dirac distribution function.

$$f_0(E) = \frac{1}{e^{\frac{E-\mu}{k_B T}} + 1},$$

where μ is the chemical potential i.e., energy required to add a particle in the system. At absolute zero temperature ($T = 0$), the chemical potential is known as Fermi energy.

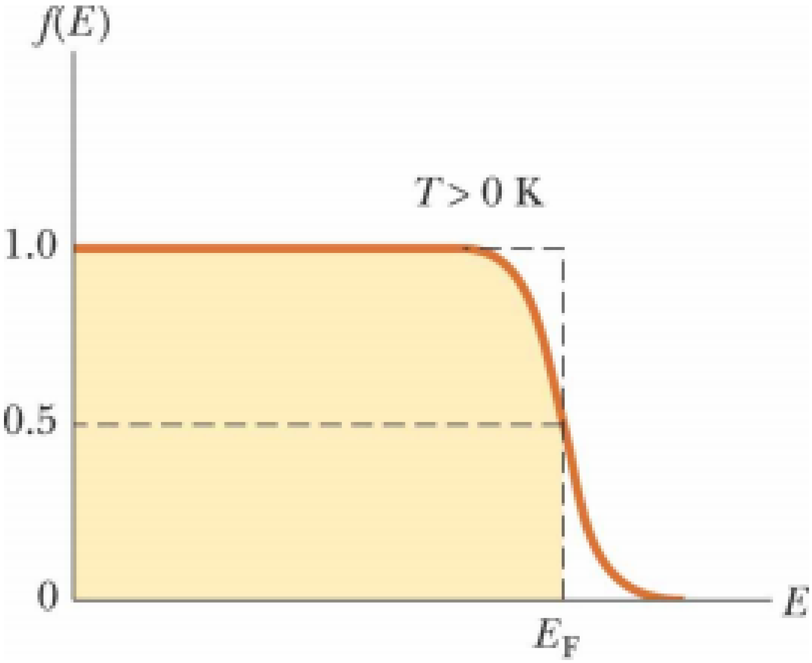

Figure 1.6: Fermi-Dirac distribution function at different temperatures.

$$\text{when} \quad T \to 0 \quad f_0(E) = \begin{cases} 1 & E < E_F \\ 0 & E > E_F. \end{cases} \tag{1.3.3}$$

This means that all the energy levels with $E < E_F$ (energy less than the Fermi energy) are filled and all above the Fermi energy $E > E_F$ are empty. At a high temperature, the Fermions lose quantum mechanical character and the Fermi distribution is reduced to the Maxwellian distribution [5].

The particles that have integer spin obey the Bose-Einstein statistics and are known as Bosons. Unlike the Fermions, the Bosons can accumulate in the same energy state as they do not obey the Pauli Exclusion principle. The Bose-Einstein distribution function is given by

$$f_0(E) = \frac{1}{e^{\frac{E-\mu}{k_B T}} - 1},$$

When $T \to 0$, the bosons move to the lowest energy state called the Bose-Einstein Condensate. With the increase in temperature the bosons will start to leave the lowest energy state and at a temperature T_c, there will be no particle in the lowest energy state. At a higher temperature the Bose-Einstein distribution function is reduced to the Maxwellian distribution function.

1.4 Examples of the relativistic plasma environments

When the thermal energy of the plasma particles becomes equivalent to their rest mass energy ($mc^2 \sim k_B T$) then the plasma is said to be relativistic. Here we will discuss some environments where the relativistic plasma can be found.

1.4.1 Jupiter

The magnetic field of the Jupiter is $0.42 Gauss$ which is 14 times stronger than the magnetic field of the Earth. So the magnetosphere of the Jupiter covers a large space as compared to the magnetosphere of the Earth. In Jupiter's magnetosphere, sulfur dioxide ejected by the moon of the Jupiter *"Io"* becomes ionized and forms a gaseous torous along the orbit of the *Io*. This ionized material (plasma) co-rotates with the Jupiter. It has been confirmed by measurements that the electrons having energies up to $20MeV$ exist in the Jupiter (high energy) radiation belts. Due to interaction with the plasma waves these electrons are accelerated and get energy from the waves [19].

1.4.2 Pulsar

A pulsar is a strongly magnetized neutron star. The magnetosphere of a pulsar provides a natural laboratory for examining the properties of a plasma immersed in a very strong magnetic field. So rotation of the pulsar along with the magnetospheric plasma takes place that results in the generation of an electric field, having a component in the direction of the magnetic field. This electric field ejects the charged particles from the surface of the pulsar. We have assumed here that these charged particles are the electrons. The electric field accelerates these electrons to the relativistic velocities. As the magnetic field lines of the pulsar are curved, so the electrons that are moving along the curved field lines radiate γ rays that will further convert into an electron positron pair (if its energy is two times greater than the rest mass energy of the electron), which will be accelerated by the electric field. This process continues and the pulsar's magnetosphere is filled with the relativistic electron positron plasma. It has two regions, a region with the open field lines and a region with the close field lines. In the first region, the plasma particles escape from the pulsar magnetosphere by following the open field lines. In the second region, the plasma cannot escape from the magnetosphere, the magnetic field confines it [14].

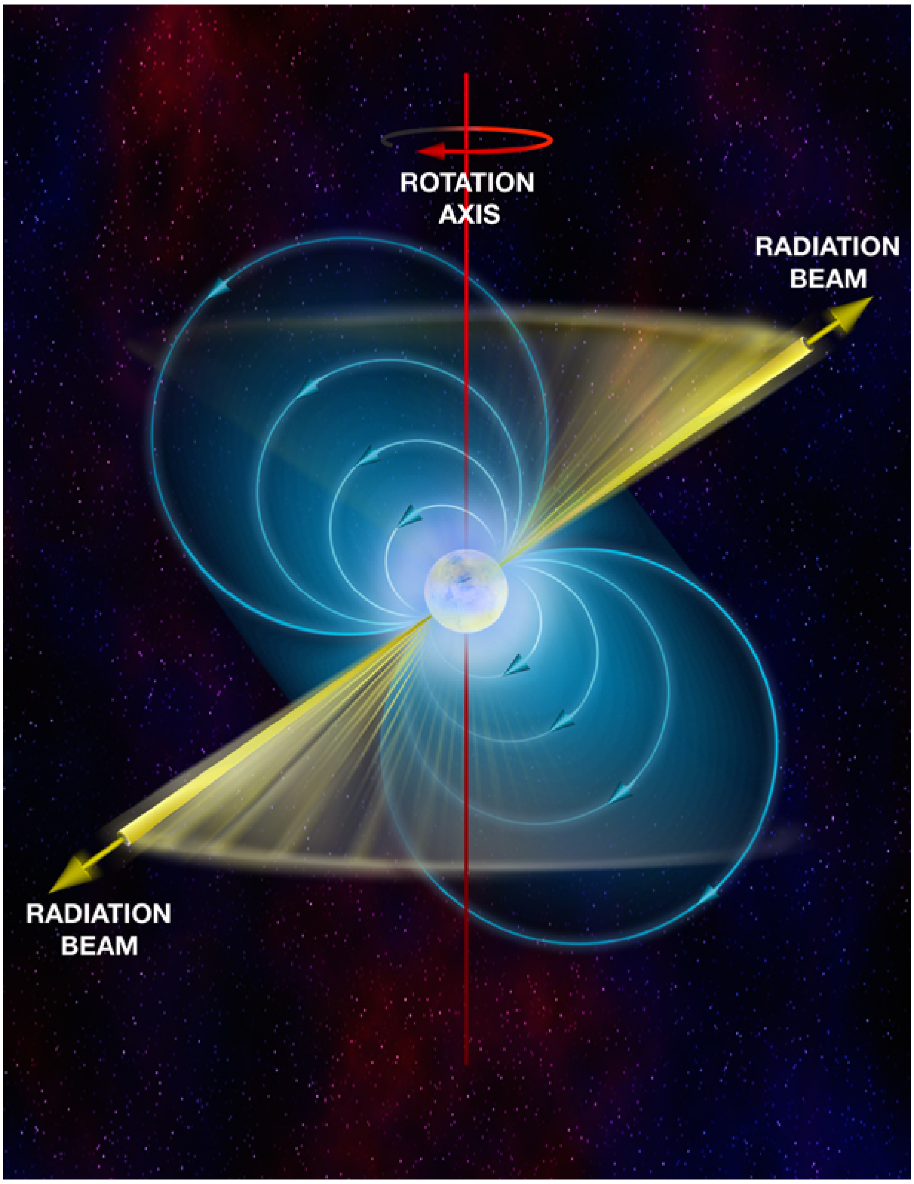

Figure 1.7: Pulsar.

1.4.3 White dwarf

A white dwarf is the best example of a relativistic partially degenerate plasma environment. Stars (like our Sun) spend maximum part of their lives by fusing

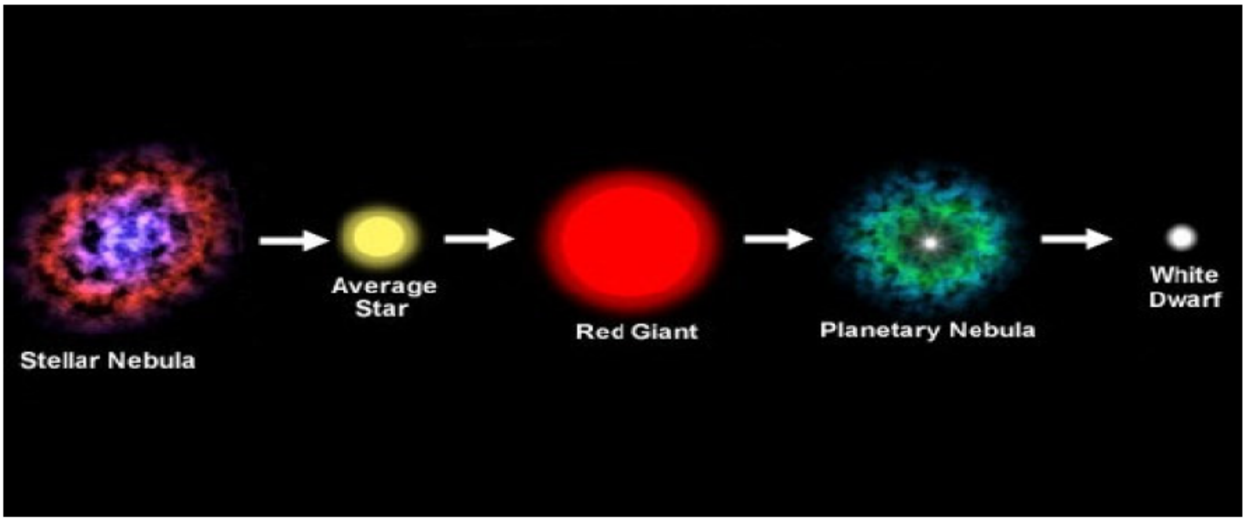

Figure 1.8: Stellar evolution.

hydrogen into helium in their core. Due to this fusion process heat and the thermal pressure is produced. This pressure is balanced by the gravitational attraction generated by the stellar mass. When nearly all the fuel i.e., hydrogen is burnt then the fusion reaction slows down and the thermal pressure will decrease. As a result, the gravitational attraction will dominate the thermal pressure and the gravitational collapse takes place. Due to the gravitational collapse the density becomes extremely high and the degeneracy effect becomes dominant. So collapse could not continue due to the introduction of the degeneracy pressure, which is a consequence of Pauli's exclusion principle. During this condensation process the star will heat up and its outer layers start to expand outward forming a star named as "red giant". If the mass of the star is greater than 1.44 times the solar mass then it can further convert helium into heavy elements (e.g., carbon). As a result the neutron star is formed. On the other hand, if the mass of the star is less than 1.44 times the solar mass then it will eject the outer layers, only a core will be left behind. It is the same core that forms the remanent white dwarf. Mass of a white dwarf is comparable to the mass of the Sun but its size is comparable to the size of the Earth. As it is a highly dense environment so the plasma found there will be partially degenerate [13, 15, 16, 20].

1.5 Waves in a plasma

In the presence of an ambient magnetic field $\mathbf{B_0}$ both the electromagnetic and the electrostatic waves can propagate in a plasma. For the high frequency electromagnetic waves (radio waves, light waves) it is assumed that the ions (being massive) remain in the background, so they do not participate in the dynamics and only the electrons will contribute. In an electromagnetic wave both the oscillating electric and magnetic fields ($\mathbf{E_1} \neq 0$,$\mathbf{B_1} \neq 0$) are present, so $\mathbf{k}$ must be perpendicular to $\mathbf{E_1}$ i.e., $\mathbf{k} \perp \mathbf{E_1}$ [2]. For the electromagnetic waves we use the following Maxwell equations.

$$\nabla \times \mathbf{E_1} = -\frac{1}{c}\frac{\partial \mathbf{B_1}}{\partial t}, \tag{1.5.1}$$

$$\nabla \times \mathbf{B_1} = \frac{4\pi \mathbf{J_1}}{c} + \frac{1}{c}\frac{\partial \mathbf{E_1}}{\partial t}, \tag{1.5.2}$$

where $\mathbf{J_1}$ is the current density. In an electrostatic wave the oscillating electric field ($\mathbf{E_1} \neq 0$) is present but the oscillating magnetic field ($\mathbf{B_1} = 0$) is zero, so $\mathbf{k} \parallel \mathbf{E_1}$. The relevant Maxwell equation for an electrostatic wave is

$$\nabla \cdot \mathbf{E_1} = 4\pi\rho,$$

where ρ is the charge density.

1.5.1 Ordinary waves

Ordinary waves are the high frequency electromagnetic waves that propagate perpendicular to the ambient magnetic field ($\mathbf{B_0}$).

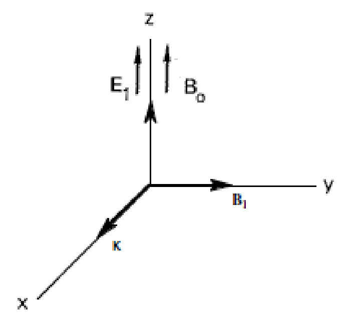

Figure 1.9: Geometry of the ordinary wave.

When frequency of the ordinary wave matches with the plasma frequency, the wave may encounter a cut-off. At the cut-off point, the wave propagation ceases.

These cut-offs are important for space and radio communication because they are directly related with the density of the plasma, so for the radio-communication (around earth), we send a wave with the frequency less than the plasma frequency. Such a wave will not penetrate the ionosphere and will be reflected. For the space-communication we send a wave having frequency greater than the plasma frequency, so that it can penetrate through the ionosphere [2].

1.6 Theoretical Background

Now we will derive a generalized expression for the conductivity tensor by using the (relativistic) Vlasov-Maxwell equations. We derive the dispersion relation from the expression of the conductivity tensor, which is important in studying the properties of different waves.

1.6.1 The Boltzmann and Vlasov equations

The Boltzmann equation is used when we are interested in the motion of the distribution of particles instead of the individual particles [31]. The distribution function, $f(\mathbf{r}, \mathbf{v}, t)$, depends on the position, $\mathbf{r}$, velocity, $\mathbf{v}$, and time, t, where,

$$\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}, \tag{1.6.1}$$

and

$$\mathbf{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}. \tag{1.6.2}$$

By taking the derivative of the distribution function, $f(x, y, z, v_x, v_y, v_z, t)$, with respect to time, we get

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial v_x}\frac{dv_x}{dt} + \frac{\partial f}{\partial v_y}\frac{dv_y}{dt} + \frac{\partial f}{\partial v_z}\frac{dv_z}{dt}. \tag{1.6.3}$$

If there are collisions then the variation of the distribution function, $f(\mathbf{r}, \mathbf{v}, t)$, with the time (t) is given by the Boltzmann equation

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_{coll}, \tag{1.6.4}$$

where,

$$\nabla = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}, \tag{1.6.5}$$

$$\frac{\partial}{\partial \mathbf{v}} = \hat{i}\frac{\partial}{\partial v_x} + \hat{j}\frac{\partial}{\partial v_y} + \hat{k}\frac{\partial}{\partial v_z}, \qquad (1.6.6)$$

and $(\partial f/\partial t)_{coll}$ in Eq. (1.6.4) represents the collision term. The acceleration, **a**, is given by

$$\mathbf{a} = \frac{\mathbf{F}}{m},$$

where **F** can be any force. As in the present case we are dealing with the charged particles, so **F** will be the Lorentz force, given by

$$\mathbf{F} = e\Big(\mathbf{E} + \frac{\mathbf{v}\times\mathbf{B}}{c}\Big), \qquad (1.6.7)$$

where **v** is the velocity, bold face symbols represent vector quantities.

As the charged particles are moving with different velocities and they interact through the long range Coulomb forces, their velocities get changed. For example in a short time interval, dt, due to the interaction (with the other charged particles), a charged particle that was initially inside a volume element within a particular velocity range (v_x, v_y etc) can come to another velocity range as it leaves that volume element. The collision term represents the gain or loss of the charged particles due to the interaction [2].

If we ignore the collision term in the Boltzmann equation, we get the Vlasov equation (collisionless Boltzmann equation)

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + \mathbf{a}\cdot\frac{\partial f}{\partial \mathbf{v}} = 0. \qquad (1.6.8)$$

This equation is used when we are interested in those processes that are rapid as compared to the collision time, $\tau > \frac{1}{\nu}$, or the mean free path is large as compared to the dimension of our system. Consider a small element, $d\mathbf{x}'d\mathbf{v}'$, described by a distribution function, $f(\mathbf{x}', \mathbf{v}', t')$, in the phase space. At a time, t', all the particles (with in this element) are located between $\mathbf{x}'$ and $\mathbf{x}' + d\mathbf{x}'$ and have the velocity between $\mathbf{v}'$ and $\mathbf{v}' + d\mathbf{v}'$. After some time, t, the particles will go to different positions and their velocities will also be slightly different (from the initial velocities) due to the Lorentz forces. The total number of particles will not change, that's why we say that if there are no collisions then the density in the phase space will remain constant [2], which is the Liouville's theorem i.e.,

$$f(\mathbf{x}', \mathbf{v}', t') = f(\mathbf{x}, \mathbf{v}, t).$$

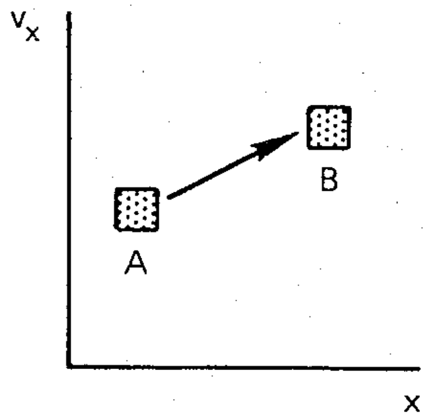

Figure 1.10: Movement of a group of particles.

1.6.2 The linearized Vlasov equation

If we use the electromagnetic force in the Eq. (1.6.8), we will get

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f + \frac{e}{m}\Big(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}\Big) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \quad (1.6.9)$$

Maxwell equations are given by

$$\nabla \cdot \mathbf{E} = 4\pi \sum e \int f d^3 v,$$

$$\nabla \times \mathbf{B} = \frac{4\pi}{c} \sum e \int \mathbf{v} f d^3 v + \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t},$$

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}.$$

Since the electric field, $\mathbf{E}$, and the magnetic field, $\mathbf{B}$, depend on the distribution function (f), so the Vlasov equation is a non-linear (partial differential) equation in f [32,33,35]. The last term in the Vlasov equation is a non-linear term. To linearize the Vlasov equation we assume that the amplitude of the perturbed quantities is small so we neglect the higher order perturbations. We consider a uniform plasma with an equilibrium distribution function $f_0(\mathbf{v})$ and a small perturbation in it [23]. To linearize the Vlasov equation, we consider

$$f = f_0 + f_1,$$

$$\mathbf{B} = \mathbf{B_0} + \mathbf{B_1},$$

$$\mathbf{E} = \mathbf{E_0} + \mathbf{E_1},$$

where the electric field, $\mathbf{E}$, and the magnetic field, $\mathbf{B}$, are functions of f. The zeroth order term represents the unperturbed part of the variable and the first order term represents the perturbed part of the variable. The linearized Vlasov equation will be

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \nabla f_1 + \frac{e}{m}\Big[\mathbf{E_0} + \frac{\mathbf{v} \times \mathbf{B_0}}{c}\Big] \cdot \frac{\partial f_1}{\partial \mathbf{v}} + \frac{e}{m}\Big[\mathbf{E_1} + \frac{\mathbf{v} \times \mathbf{B_1}}{c}\Big] \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0. \quad (1.6.10)$$

In our case, the equilibrium electric field is zero and the equilibrium magnetic field is along the z-direction i.e.,

$$\mathbf{E_0} = 0, \quad \mathbf{B_0} = B_0 \hat{z}.$$

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \frac{e}{m}\Big[\frac{\mathbf{v} \times \mathbf{B_0}}{c}\Big] \cdot \frac{\partial f_1}{\partial \mathbf{v}} + \frac{e}{m}\Big[\mathbf{E_1} + \frac{\mathbf{v} \times \mathbf{B_1}}{c}\Big] \cdot \frac{\partial f_0}{\partial \mathbf{v}} = 0. \quad (1.6.11)$$

In order to study the plasma dynamics, we have to solve the Maxwell equations along with the linearized Vlasov equation for the perturbed quantities $(\mathbf{E_1}, \mathbf{B_1})$. The charge and current density in terms of the perturbation is given as

$$\rho = e \int f_1 d^3p, \quad (1.6.12)$$

$$\mathbf{J} = e \int \mathbf{v} f_1 d^3p, \quad (1.6.13)$$

$$\frac{\partial f_1}{\partial t} + \mathbf{v} \cdot \frac{\partial f_1}{\partial \mathbf{x}} + \Omega\Big(\mathbf{p} \times \hat{z}\Big) \cdot \frac{\partial f_1}{\partial \mathbf{p}} + e\Big[\mathbf{E_1} + \frac{\mathbf{v} \times \mathbf{B_1}}{c}\Big] \cdot \frac{\partial f_0}{\partial \mathbf{p}} = 0, \quad (1.6.14)$$

where $\mathbf{v} = \mathbf{p}/\gamma m$ and $\Omega = eB_0/\gamma mc$ is called the relativistic cyclotron frequency and the relativistic factor is given by $\gamma = 1/\sqrt{1 - v^2/c^2}$.
In order to solve the above (differential) equation we introduce the cylindrical coordinate system, the parallel and perpendicular components of the momentum, $\mathbf{p}$, are given by

$$\begin{aligned} p_x &= p_\perp \cos\phi, \\ p_y &= p_\perp \sin\phi, \\ p_z &= p_\parallel. \end{aligned}$$

Using the values of p_x, p_y and p_z in the scalar triple product given below, we get

$$(\mathbf{p} \times \hat{z}) \cdot \frac{\partial f_1}{\partial \mathbf{p}} = \begin{vmatrix} p_x & p_y & p_z \\ 0 & 0 & 1 \\ \frac{\partial f_1}{\partial p_x} & \frac{\partial f_1}{\partial p_y} & \frac{\partial f_1}{\partial p_z} \end{vmatrix},$$

$$(\mathbf{p}\times\hat{z})\cdot\frac{\partial f_1}{\partial \mathbf{p}} = p_y\frac{\partial f_1}{\partial p_x} - p_x\frac{\partial f_1}{\partial p_y}. \tag{1.6.15}$$

In order to solve the above equation we use the chain rule

$$\frac{\partial f_1}{\partial p_x} = \frac{\partial f_1}{\partial p_\perp}\frac{\partial p_\perp}{\partial p_x} + \frac{\partial f_1}{\partial \phi}\frac{\partial \phi}{\partial p_x} + \frac{\partial f_1}{\partial p_\parallel}\frac{\partial p_\parallel}{\partial p_x},$$

substituting the values of $\partial p_\perp/\partial p_x$, $\partial\phi/\partial p_x$ and $\partial p_\parallel/\partial p_x$, we get

$$\frac{\partial f_1}{\partial p_x} = \frac{\partial f_1}{\partial p_\perp}(\cos\phi) + \frac{\partial f_1}{\partial \phi}\Big(\frac{-\sin\phi}{p_\perp}\Big).$$

Similarly,

$$\frac{\partial f_1}{\partial p_y} = \frac{\partial f_1}{\partial p_\perp}\frac{\partial p_\perp}{\partial p_y} + \frac{\partial f_1}{\partial \phi}\frac{\partial \phi}{\partial p_y} + \frac{\partial f_1}{\partial p_\parallel}\frac{\partial p_\parallel}{\partial p_y},$$

$$\frac{\partial f_1}{\partial p_y} = \frac{\partial f_1}{\partial p_\perp}(\sin\phi) + \frac{\partial f_1}{\partial \phi}\Big(\frac{\cos\phi}{p_\perp}\Big).$$

Using the values of p_x, p_y, $\partial f_1/\partial p_x$ and $\partial f_1/\partial p_y$ in Eq. (1.6.15), we get

$$(\mathbf{p}\times\hat{z})\cdot\frac{\partial f_1}{\partial \mathbf{p}} = -\frac{\partial f_1}{\partial \phi}.$$

Substituting values in Eq. (1.6.14), we get

$$\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{x}} - \Omega\frac{\partial f_1}{\partial \phi} + e\Big[\mathbf{E_1} + \frac{\mathbf{v}\times\mathbf{B_1}}{c}\Big]\cdot\frac{\partial f_0}{\partial \mathbf{p}} = 0, \tag{1.6.16}$$

$$\frac{\partial f_1}{\partial \phi} - \frac{1}{\Omega}\Big(\frac{\partial f_1}{\partial t} + \mathbf{v}\cdot\frac{\partial f_1}{\partial \mathbf{x}}\Big) = \frac{e}{\Omega}\Big[\mathbf{E_1} + \frac{\mathbf{v}\times\mathbf{B_1}}{c}\Big]\cdot\frac{\partial f_0}{\partial \mathbf{p}}. \tag{1.6.17}$$

Applying the Fourier transform in space and the Laplace transform in time, we get

$$\frac{\partial f_1}{\partial \phi} - \frac{1}{\Omega}\Big(-i\omega + i\mathbf{k}\cdot\mathbf{v}\Big)f_1 = \frac{e}{\Omega}\Big(\mathbf{E_1} + \frac{\mathbf{v}\times\mathbf{B_1}}{c}\Big)\cdot\frac{\partial f_0}{\partial \mathbf{p}}, \tag{1.6.18}$$

where we have used

$$(\mathbf{E},\mathbf{B},f) = \int_0^\infty dt\exp(-st)\int_{-\infty}^\infty \frac{d^3x}{(2\pi)^{3/2}}\exp(-i\mathbf{k}\cdot\mathbf{x})(\mathbf{E_1},\mathbf{B_1},f_1),$$

where $s=-i\omega$ and $Re(s)>0$. Eq. (1.6.18) is a 1st order inhomogeneous partial differential equation. Now we introduce a primed notation i.e.,

$$\mathbf{v}' = v_\perp\cos\phi'\hat{i} + v_\perp\sin\phi'\hat{j} + v_\parallel\hat{k}.$$

We first find the solution of the homogeneous differential equation

$$\frac{\partial G}{\partial \phi} - \frac{1}{\Omega}\Big(- i\omega + i\mathbf{k}\cdot\mathbf{v}\Big)G = 0, \tag{1.6.19}$$

the solution of the above equation is

$$G = \exp\Big[\frac{1}{\Omega}\int_{\phi'}^{\phi}(-i\omega + i\mathbf{k}\cdot\mathbf{v}'')d\phi''\Big]. \tag{1.6.20}$$

Now we can write the solution of the inhomogeneous equation as (the lower integration limit in the equation given below shows that there is no perturbation in the very beginning)

$$f_1 = \frac{1}{\Omega}\int_{-\infty}^{\phi}\Phi(\phi')Gd\phi', \tag{1.6.21}$$

where

$$\Phi(\phi') = e\Big(\mathbf{E_1} + \frac{\mathbf{v}'\times\mathbf{B_1}}{c}\Big). \tag{1.6.22}$$

As the particles are rotating about the z-axis, so ϕ' will be evolved with the time. The variable ϕ' is related to the time, t', by $\phi' = \Omega t'/\gamma$. As $t' \to -\infty$, this means we go back in infinite past where there is no perturbation, so lower integration limit must be $\phi' = \pm\infty$ [28]. If $e < 0$ then f_1 will converge at $\phi' \to -\infty$ and vice versa. As f_1 is periodic in ϕ so its limits of integration should be independent of ϕ, which can be easily seen by replacing $(\phi - \phi')$ by a new variable in the integrand of equation (1.6.21). Now by applying the Fourier Laplace transform on Eq. (1.6.10), we get

$$\mathbf{B_1} = \frac{c(\mathbf{k}\times\mathbf{E_1})}{\omega}. \tag{1.6.23}$$

Solving Eqs (1.6.10) and (1.6.10), we get

$$\omega^2\mathbf{E_1} - c^2k^2\mathbf{E_1} + c^2\mathbf{k}(\mathbf{k}\cdot\mathbf{E_1}) = -4\pi i\omega\mathbf{J_1}, \tag{1.6.24}$$

$$\omega^2 E_\alpha - c^2k^2E_\alpha + c^2k_\alpha(k_\beta E_\beta) = -4\pi i\omega J_\alpha, \tag{1.6.25}$$

by Ohm's law

$$\mathbf{J} = \underline{\sigma}\cdot\mathbf{E},$$

where $\underline{\sigma}$ is the conductivity. The Greek index notation represents the vector quantities. We can write in components form

$$J_\alpha = \sigma_{\alpha\beta}E_\beta, \tag{1.6.26}$$

where $\alpha, \beta = x, y, z$. We have used the Einstein summation convention that repeated indices are summed over and are using the Cartesian tensor, so that we do not need to distinguish between covariant and contravariant indices. Using Eq. (1.6.25)

$$[(\omega^2 - c^2k^2)\delta_{\alpha\beta} + c^2 k_\alpha k_\beta + 4\pi i\omega\sigma_{\alpha\beta}]E_\beta := R_{\alpha\beta}E_\beta = 0. \tag{1.6.27}$$

The current density and number density is given by

$$\begin{aligned} \mathbf{J} &= ne\mathbf{v}, \\ n &= \int f_1 d^3p, \\ J_\alpha &= e\int v_\alpha f_1 d^3p. \end{aligned} \tag{1.6.28}$$

Using the value of f_1, we get

$$\begin{aligned} J_\alpha &= e\int \frac{v_\alpha}{\Omega}\int_{-\infty}^{\phi} \Phi(\phi')G d\phi' d^3p, \\ &= e\int \frac{v_\alpha}{\Omega}\int_{-\infty}^{\phi} d\phi' \exp\Big[\frac{1}{\Omega}\int_{\phi'}^{\phi}(-i\omega + i\mathbf{k}\cdot\mathbf{v}'')d\phi''\Big]\Big[e(\mathbf{E_1} + \frac{\mathbf{v}'\times\mathbf{B_1}}{c})\cdot\frac{\partial f_0}{\partial\mathbf{p}'}\Big]d^3p. \end{aligned}$$

Substituting value of $\mathbf{B_1}$ from Eq. (1.6.23), we get

$$\begin{aligned} \Big[(\mathbf{E_1} + \frac{\mathbf{v}'\times\mathbf{B_1}}{c})\cdot\frac{\partial f_0}{\partial\mathbf{p}'}\Big] &= \mathbf{E_1}\cdot\frac{\partial f_0}{\partial\mathbf{p}'} + \frac{1}{\omega}\Big[\mathbf{k}(\mathbf{v}'\cdot\mathbf{E_1}) - \mathbf{E_1}(\mathbf{k}\cdot\mathbf{v}')\Big]\cdot\frac{\partial f_0}{\partial\mathbf{p}'}, \\ &= \mathbf{E_1}\cdot\frac{\partial f_0}{\partial\mathbf{p}'} + \frac{1}{\omega}(\mathbf{v}'\cdot\mathbf{E_1})\Big(\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{p}'}\Big) - \frac{1}{\omega}\Big(\mathbf{E_1}\cdot\frac{\partial f_0}{\partial\mathbf{p}'}\Big)(\mathbf{k}\cdot\mathbf{v}'), \end{aligned}$$

$$\begin{aligned} \Big[(\mathbf{E_1} + \frac{\mathbf{v}'\times\mathbf{B_1}}{c})\cdot\frac{\partial f_0}{\partial\mathbf{p}'}\Big] &= E_\beta\frac{\partial f_0}{\partial p'_\beta} + \frac{1}{\omega}(v'_\beta E_\beta)\Big(\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{p}'}\Big) - \frac{1}{\omega}\Big(E_\beta\frac{\partial f_0}{\partial p'_\beta}\Big)(\mathbf{k}\cdot\mathbf{v}'), \\ &= \Big[\frac{\partial f_0}{\partial p'_\beta} + \frac{v'_\beta}{\omega}\Big(\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{p}'}\Big) - \frac{1}{\omega}\frac{\partial f_0}{\partial p'_\beta}(\mathbf{k}\cdot\mathbf{v}')\Big]E_\beta. \end{aligned}$$

Using this value in expression of J_α, we get

$$J_\alpha = e^2\int \frac{v_\alpha}{\Omega}\int_{-\infty}^{\phi} d\phi' \exp\Big[\frac{1}{\Omega}\int_{\phi'}^{\phi}(-i\omega+i\mathbf{k}\cdot\mathbf{v}'')d\phi''\Big]\Big[\frac{\partial f_0}{\partial p'_\beta}+\frac{v'_\beta}{\omega}\Big(\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{p}'}\Big)-\frac{1}{\omega}\frac{\partial f_0}{\partial p'_\beta}(\mathbf{k}\cdot\mathbf{v}')\Big]E_\beta d^3p.$$

So we can write

$$\sigma_{\alpha\beta} = \frac{e^2}{\omega}\int \frac{v_\alpha}{\Omega}\int_{-\infty}^{\phi} d\phi' \exp\Big[\frac{1}{\Omega}\int_{\phi'}^{\phi}(-i\omega + i\mathbf{k}\cdot\mathbf{v}'')d\phi''\Big]\Big[(\omega - \mathbf{k}\cdot\mathbf{v}')\delta_{\beta l} + v'_\beta k_l\Big]\frac{\partial f_0}{\partial p'_l}d^3p.$$

This is a general expression for the conductivity tensor for any kind of distribution and for any coordinate system. Now we are going to use the spherical polar coordinates, so the volume element is given by

$$d^3p = p^2 dp \sin\theta d\theta d\phi. \tag{1.6.29}$$

So,

$$\begin{aligned}\sigma_{\alpha\beta} &= \frac{e^2}{\omega}\int_0^\infty\int_0^\pi\int_0^{2\pi} p^2\sin\theta dp d\theta d\phi \frac{v_\alpha}{\Omega}\int_{-\infty}^{\phi} d\phi' \exp\left[\frac{1}{\Omega}\int_{\phi'}^{\phi}(-i\omega + i\mathbf{k}\cdot\mathbf{v}'')d\phi''\right] \\ &\quad \left[(\omega - \mathbf{k}\cdot\mathbf{v}')\delta_{\beta l} + v'_\beta k_l\right]\frac{\partial f_0}{\partial p'_l}. \end{aligned} \tag{1.6.30}$$

In our case we are focussing on the ordinary waves for which $\mathbf{E_1} = E_1\hat{z}$ and $\mathbf{k} = k_x\hat{x}$. As we are interested in perpendicular propagation so we are left with σ_{zz} component of the conductivity tensor which specifies the dynamics of the ordinary wave [21].

The zz component of the conductivity tensor (σ_{zz}) is given by

$$\begin{aligned}\sigma_{zz} &= \frac{e^2}{\omega}\int_0^\infty\int_0^\pi\int_0^{2\pi} p^2\sin\theta dp d\theta d\phi \frac{v_z}{\Omega}\int_{-\infty}^{\phi} d\phi' \exp\left[\frac{1}{\Omega}\int_{\phi'}^{\phi}(-i\omega + i\mathbf{k}\cdot\mathbf{v}'')d\phi''\right] \\ &\quad \left[(\omega - \mathbf{k}\cdot\mathbf{v}')\delta_{zl} + v'_z k_l\right]\frac{\partial f_0}{\partial p'_l}. \end{aligned} \tag{1.6.31}$$

We can take the wave vector, $\mathbf{k}$, as

$$\mathbf{k} = (k_x, 0, k_z),$$

and the velocity $\mathbf{v}$ in the spherical coordinate system can be written as

$$\mathbf{v} = (v\sin\theta\cos\phi, v\sin\theta\sin\phi, v\cos\theta). \tag{1.6.32}$$

Now by substituting the value of v_z and simplifying Eq. (1.6.31), we get

$$\begin{aligned}\sigma_{zz} &= \frac{e^2}{\omega}\int_0^\infty\int_0^\pi\int_0^{2\pi} p^2\sin\theta dp d\theta d\phi \frac{v\cos\theta}{\Omega} \\ &\quad \int_{-\infty}^{\phi} d\phi' \exp\left[\frac{-i}{\Omega}(\omega - k_x v\sin\theta(\sin\phi - \sin\phi') - k_z v\cos\theta(\phi - \phi')\right] \\ &\quad \left[(\omega - k_x v\sin\theta\cos\phi' - k_z v\cos\theta)\cos\theta\frac{\partial f_0}{\partial p} + v\cos\theta(k_x\sin\theta\cos\phi' + k_z\cos\theta)\frac{\partial f_0}{\partial p}\right].\end{aligned}$$

As f_1 is periodic in ϕ, so we substitute $\phi - \phi' = \alpha$, such that the integration limits become independent of ϕ.

$$\text{Put} \quad \phi - \phi' = \alpha$$
$$d\phi' = -d\alpha$$

$$\sigma_{zz} = e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi \frac{v \sin\theta \cos^2\theta d\theta}{\Omega} \int_0^{2\pi} \exp\Big[\frac{ik_x v \sin\theta}{\Omega}[\sin\phi - \sin(\phi - \alpha)]\Big] d\phi$$
$$\int_0^\infty \exp\Big[\frac{-i}{\Omega}(\omega - k_z v \cos\theta)\alpha\Big] d\alpha.$$

In order to solve the above ϕ integration we will use the following identity.

$$\int_0^{2\pi} \exp\Big[i\xi(\sin\phi - \sin(\phi - \alpha))\Big] d\phi = 2\pi \sum_n \exp[in\alpha] J_n^2(\xi), \quad \text{where} \quad \xi = \Big(k_x v \sin\theta/\Omega\Big).$$

After performing ϕ integration, we get

$$\sigma_{zz} = 2\pi e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi \frac{v \sin\theta \cos^2\theta d\theta}{\Omega} \sum_n J_n^2\Big(\frac{k_x v \sin\theta}{\Omega}\Big)$$
$$\int_0^\infty \exp\Big[\frac{-i}{\Omega}(\omega - n\Omega - k_z v \cos\theta)\alpha\Big] d\alpha.$$

Now integrating over α, we get

$$\sigma_{zz} = -2\pi i e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi v \sin\theta \cos^2\theta \sum_n J_n^2\Big(\frac{k_x v \sin\theta}{\Omega}\Big) \frac{d\theta}{(\omega - k_z v \cos\theta - n\Omega)}. \tag{1.6.33}$$

We will use the above equation in the next chapter and will take $k_z = 0$ to find the dispersion relation of the ordinary waves.

Chapter 2

Propagation of ordinary waves in the non-degenerate and degenerate plasmas

In this chapter the propagation of the ordinary waves in the non-degenerate and degenerate plasmas has been studied. In section 2, the dispersion relation of the ordinary waves is derived by using the ultra-relativistic Maxwellian distribution function considering the strong and the weak magnetic field limits. In section 3, the dispersion relation of the ordinary waves in the relativistic degenerate electron plasma is derived by using the Fermi-Dirac distribution function.

2.1 Introduction

Consider the propagation of transverse ($\mathbf{k} \perp \mathbf{E_1}$) electromagnetic waves passing through a plasma. As we are interested in the perpendicular propagating waves so we have two options i.e., $\mathbf{E_1} \perp \mathbf{B_0}$ or $\mathbf{E_1} \parallel \mathbf{B_0}$. For the ordinary waves our coordinate system will be $\mathbf{E_1} = E_1\hat{z}$, $\mathbf{k} = k_x\hat{x}$ and $\mathbf{B_0} = B_0\hat{z}$. We will take $k_z = 0$ in Eq. (1.6.24) for the ordinary (perpendicular propagating) waves. So

$$\omega^2 - c^2k_x^2 = -4\pi i\omega \mathbf{J_1}. \tag{2.1.1}$$

The ion's dynamics can be ignored for the high frequency waves, so we get the current due to the motion of electrons.

$$\mathbf{J_1} = -n_0 e\mathbf{v_1}. \tag{2.1.2}$$

As we know that the linearized equation of motion for the electrons is

$$m\frac{\partial \mathbf{v_1}}{\partial t} = -e\mathbf{E_1}. \tag{2.1.3}$$

The above equation can be written as

$$\mathbf{v_1} = \frac{e\mathbf{E_1}}{im\omega}. \tag{2.1.4}$$

Using Eqs. (2.1.2, 2.1.4) in Eq. (2.1.1), we get

$$\omega^2 = \omega_p^2 + c^2 k_x^2, \tag{2.1.5}$$

where $\omega_p^2 = 4\pi n_0 e^2/m$ is the plasma frequency and k is the magnitude of the wave vector. The phase velocity ($v_\phi = \omega/k$) of the ordinary waves is given by

$$v_\phi = \frac{c}{\sqrt{1 - \frac{\omega_p^2}{\omega^2}}}. \tag{2.1.6}$$

In the fluid approximation, these waves remain unaffected by the ambient magnetic field, that's why these are named as the ordinary waves [2].

2.2 The non-degenerate ultra-relativistic Maxwellian electron plasma

When the thermal energy of the plasma particles becomes much greater than their rest mass energy ($m_0 c^2 \ll k_B T$) then the plasma is said to be ultra-relativistic. At a low density and a high temperature, the thermal effects dominate in a plasma and the plasma is said to be non-degenerate when the inter-particle distance becomes greater than the de-Broglie's wavelength of the charged particles [5].

Now we will focus on the propagation of the ordinary waves in the (non-degenerate) ultra-relativistic Maxwellian electron plasma by taking the strong and the weak magnetic field limits in a non-degenerate plasma.

2.2.1 The ultra-relativistic Maxwellian electron plasma in a strong magnetic field

A plasma is said to be strongly magnetized when the cyclotron frequency dominates on the plasma frequency. There are some plasma environments in nature with the low density and a strong magnetic field. In such environments the ratio of the cyclotron frequency to the plasma frequency is given by

$$\omega_c/\omega_p > 1, \tag{2.2.1}$$

the density (ω_p) is high but when we compare it with the magnetic field ($\omega_c = eB_0/mc$) its value is small. That's why we call it the strong field limit.

Consider the propagation of the ordinary waves through a plasma immersed in a strong magnetic field. Such plasma environment can be found in the magnetosphere of the pulsar, which is a natural laboratory (for studying the plasma) having a strong magnetic field of the order of 10^{12} Gauss [14]. The dispersion relation of the ordinary waves can be obtained by using Eq. (1.6.33) [21–23].

$$\sigma_{zz} = -2\pi i e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi v \sin\theta \cos^2\theta \sum_n J_n^2(\zeta) \frac{d\theta}{(\omega - n\Omega)}. \tag{2.2.2}$$

As we are considering that the magnetic field is very strong so for the relativistic case $\zeta = ck_x \sin\theta/\Omega \ll 1$. Under this assumption we can use the asymptotic value of the Bessel function i.e.,

$$J_n(\zeta) \approx \frac{1}{\Gamma(n+1)} \left(\frac{\zeta}{2}\right)^n, \tag{2.2.3}$$

where Γ is the Gamma function given by

$$\Gamma(n) = (n-1)!.$$

In order to perform θ integration, we will use the following result.

$$\int_0^\pi \sin^{2n+1}\theta \cos^2\theta d\theta = \Gamma(n+1)\Big[\frac{2^{2n+3}\Gamma(n+3)}{\Gamma(2n+4)} - \frac{\sqrt{\pi}}{\Gamma(n+\frac{3}{2})}\Big]. \tag{2.2.4}$$

So, we can write the Eq. (2.2.2) as

$$\sigma_{zz} = -2\pi i e^2 \int_0^\infty p^2 \frac{\partial f_0}{\partial p} dp \Big[\frac{2v}{3\omega} + \sum_{n=1}^{\infty} \left(\frac{k_x}{\Omega}\right)^{2n} v^{2n+1} \frac{4(n+1)}{\Gamma(2n+4)} \Big\{\frac{1}{\omega - n\Omega} + \frac{1}{\omega + n\Omega}\Big\}\Big]. \tag{2.2.5}$$

In the relativistic case, we can write

$$E^2 = p^2c^2 + {m_0}^2c^4, \tag{2.2.6}$$

and

$$\Omega = \frac{eB_0}{\sqrt{p^2 + m_0^2c^2}}. \tag{2.2.7}$$

When we consider the ultra-relativistic plasma then the rest mass energy of the plasma particles becomes so small as compared to the thermal energy that it can be neglected and we get $E = pc$. The velocity and the cyclotron frequency for the ultra-relativistic plasma is

$$v = c \quad \text{and} \quad \Omega = eB_0/p. \tag{2.2.8}$$

The ultra-relativistic Maxwellian distribution function for the electron plasma is [30]

$$f_0 = \frac{n_0 c^3 \exp\left(-cp/T\right)}{8\pi T^3}, \tag{2.2.9}$$

$$\frac{\partial f_0}{\partial p} = -\frac{n_0 c^4 \exp\left(-cp/T\right)}{8\pi T^4}, \tag{2.2.10}$$

using Eq. (2.2.8) and Eq. (2.2.10) in Eq. (2.2.5), we get

$$\sigma_{zz} = \frac{in_0e^2c^4}{4T^4}\int_0^\infty p^2\exp\Big(\frac{-cp}{T}\Big)\Big[\frac{2c}{3\omega}+\sum_{n=1}^{\infty}\Big(\frac{ck_xp}{eB_0}\Big)^{2n}\frac{4(n+1)c}{\Gamma(2n+4)}\Big\{\frac{1}{\omega-\frac{neB_0}{p}}+\frac{1}{\omega+\frac{neB_0}{p}}\Big\}\Big]dp,$$

$$\begin{aligned}\sigma_{zz} \quad = \quad & \frac{in_0e^2c^5}{4\omega T^4}\Big[\frac{2T^2}{3c^2}\int_0^\infty \frac{c^2p^2}{T^2}\exp\Big(\frac{-cp}{T}\Big)dp+\sum_{n=1}^{\infty}\frac{4(n+1)T^{2n+2}}{\Gamma(2n+4)c^{2n+4}}\Big(\frac{ck_x}{eB_0}\Big)^{2n}\\ & \Big\{\int_0^\infty\frac{(\frac{cp}{T})^{2n+3}\exp(\frac{-cp}{T})dp}{(\frac{cp}{T})+\frac{nB_0c}{\omega T}}+\int_0^\infty\frac{(\frac{cp}{T})^{2n+3}\exp(\frac{-cp}{T})dp}{(\frac{cp}{T})-\frac{nB_0c}{\omega T}}\Big\}\Big],\end{aligned}$$

on performing integration, we get

$$\begin{aligned}\sigma_{zz} \quad = \quad & \frac{in_0e^2c^2}{4\omega T}\Big[\frac{4}{3}+\sum_{n=1}^{\infty}\frac{4(n+1)}{\Gamma(2n+4)}\Big(\frac{ck_xT}{ceB_0}\Big)^{2n}\Big\{\exp\Big(\frac{nceB_0}{\omega T}\Big)E_{2n+4}\\ & \Big(\frac{nceB_0}{\omega T}\Big)\Gamma(2n+4)+\exp\Big(\frac{-nceB_0}{\omega T}\Big)E_{2n+4}\Big(\frac{-nceB_0}{\omega T}\Big)\Gamma(2n+4)\\ & +i\pi\exp\Big(\frac{-nceB_0}{\omega T}\Big)\Big(\frac{nceB_0}{\omega T}\Big)^{2n+3}\Big\}\Big].\end{aligned} \tag{2.2.11}$$

We have used these two integrals in order to solve the above calculation.

$$\int_0^\infty\frac{y^{2n+3}\exp(-y)}{y-nb}dy=\exp(-nb)\{E_{2n+4}(-nb)\Gamma(2n+4)+i\pi(nb)^{2n+3}\},$$

and

$$\int_0^\infty\frac{y^{2n+3}\exp(-y)}{y+nb}dy=\exp(nb)\{E_{2n+4}(nb)\Gamma(2n+4)+i\pi(nb)^{2n+3}\},$$

where E_{2n+4} is the Exponential integral defined as

$$E_n(x) = \int_1^\infty \frac{e^{-tx}dt}{t^n}.$$

In order to solve the momentum integration we have not used any approximation [24]. Substituting $\alpha = \beta = z$ in Eq. (1.6.27), we get

$$R_{zz} = \omega^2 - c^2k_x^2 + 4\pi i\omega\sigma_{zz}, \tag{2.2.12}$$

by substituting $R_{zz} = 0$ in the Eq. (2.2.12), we get

$$\omega^2 = c^2k_x^2 + 4\pi i\omega\sigma_{zz}, \tag{2.2.13}$$

substituting Eq. (2.2.11) in the Eq. (2.2.12), we get

$$\begin{aligned} R_{zz} &= \omega^2 - c^2k_x^2 - {\omega_p}^2 - 3{\omega_p}^2\sum_{n=1}^{\infty}(n+1)\Big(\frac{ck_x}{\omega_c}\Big)^{2n}\Big\{\exp\Big(\frac{n\omega_c}{\omega}\Big)E_{2n+4}\Big(\frac{n\omega_c}{\omega}\Big) \\ &+ \exp\Big(-\frac{n\omega_c}{\omega}\Big)E_{2n+4}\Big(-\frac{n\omega_c}{\omega}\Big) - \frac{i\pi}{\Gamma(2n+4)}\exp\Big(-\frac{n\omega_c}{\omega}\Big)\Big(\frac{n\omega_c}{\omega}\Big)^{2n+3}\Big\}, \end{aligned} \tag{2.2.14}$$

where the cyclotron frequency and the plasma frequency for the ultra-relativistic plasma are

$$\omega_c = \frac{eB_0c}{T}, \tag{2.2.15}$$

and

$$\omega_p^2 = \frac{4\pi n_0e^2c^2}{3T}, \tag{2.2.16}$$

by substituting $R_{zz} = 0$ in the Eq. (2.2.14), we get the real and imaginary parts of the dispersion relation, given by

$$\begin{aligned} {\omega_r}^2 &= c^2k_x^2 + {\omega_p}^2 + 3{\omega_p}^2\sum_{n=1}^{\infty}(n+1)\Big(\frac{ck_x}{\omega_c}\Big)^{2n}\Big\{\exp\Big(\frac{n\omega_c}{\omega_r}\Big)E_{2n+4}\Big(\frac{n\omega_c}{\omega_r}\Big) \\ &+ \exp\Big(-\frac{n\omega_c}{\omega_r}\Big)E_{2n+4}\Big(-\frac{n\omega_c}{\omega_r}\Big)\Big\}. \end{aligned} \tag{2.2.17}$$

$$\gamma_0 = -\frac{3\omega_p^2}{2\omega_r}\sum_{n=1}^{\infty}(n+1)\Big(\frac{ck_x}{\omega_c}\Big)^{2n}\frac{\pi}{\Gamma(2n+4)}\exp\Big(-\frac{n\omega_c}{\omega_r}\Big)\Big(\frac{n\omega_c}{\omega_r}\Big)^{2n+3}. \tag{2.2.18}$$

The complex part is not observed in the dispersion relation for the non-relativistic plasmas, but here due to the relativistic variation of the mass, the cyclotron frequency becomes momentum dependent, this gives an imaginary term in the dispersion relation.

2.2.2 The ultra-relativistic Maxwellian electron plasma in a weak magnetic field

A plasma is said to be weakly magnetized when the plasma frequency (ω_p) dominates on the cyclotron frequency (ω_c). There are some plasma environments in nature with the high density and a strong magnetic field. The strength of the magnetic field is although very high but when we take the ratio of the cyclotron frequency to the plasma frequency, we get

$$\omega_c/\omega_p < 1, \tag{2.2.19}$$

so the effects of magnetic field will become negligible and we can treat such a magnetic field as the *weak field* [26, 27].

In order to study the wave propagation through such an environment, we develop the following mathematical formalism. Using Eq. (1.6.20)

$$G(\phi') = \exp\left[\frac{1}{\Omega}(-i\omega + ivk_z\cos\theta)(\phi - \phi') + ivk_x\sin\theta(\sin\phi - \sin\phi')\right],$$

from the above equation, we can write

$$G(\phi') = \frac{-i\Omega}{\omega - \mathbf{k}\cdot\mathbf{v}'}\frac{\partial G(\phi')}{\partial\phi'},$$

using the above equation in Eq. (1.6.21) and integrating we get

$$f_1 = -i\frac{\Phi(\phi)}{\omega - \mathbf{k}\cdot\mathbf{v}} + i\int_{-\infty}^{\phi} G(\phi')\frac{\partial}{\partial\phi'}\frac{\Phi(\phi')}{(\omega - \mathbf{k}\cdot\mathbf{v}')}d\phi', \tag{2.2.20}$$

where

$$\Phi(\phi) = e\Big[\mathbf{E_1}\cdot\frac{\partial f_0}{\partial\mathbf{p}} + \frac{(\mathbf{v}\cdot\mathbf{E_1})}{(\omega - \mathbf{k}\cdot\mathbf{v})}\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{p}}\Big]. \tag{2.2.21}$$

We have used here

$$G(\phi') = 1 \quad \text{for} \quad \phi' = \phi,$$
$$G(\phi') = 0 \quad \text{for} \quad \phi' \to -\infty,$$

after performing integration by parts of Eq. (2.2.20), we get

$$f_1 = -i\frac{\Phi(\phi)}{(\omega - \mathbf{k}\cdot\mathbf{v})} - \frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\frac{\Phi(\phi)}{(\omega - \mathbf{k}\cdot\mathbf{v})} - \frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\frac{\Phi(\phi)}{(\omega - \mathbf{k}\cdot\mathbf{v})} + O(\Omega^3).$$

As the ambient magnetic field $\mathbf{B_0}$ is weak, so we will retain Ω up to the second order ($\omega > \Omega$) [23]. Now by substituting the value of $\Phi(\phi)$ from Eq. (2.2.21), we get

$$f_1 = -\frac{ie}{\omega}\Big[1+\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}+\frac{i\Omega}{(\omega - \mathbf{k}.\mathbf{v})}\frac{\partial}{\partial\phi}\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\Big]\times\Big[\mathbf{E_1}\cdot\frac{\partial f_0}{\partial\mathbf{p}}+\frac{(\mathbf{v}\cdot\mathbf{E_1})}{(\omega - \mathbf{k}\cdot\mathbf{v})}\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{p}}\Big]. \tag{2.2.22}$$

Substituting the above Eq. in Eq. (1.6.28), we get the expression for the current density

$$J_\alpha = -\frac{ie^2}{\omega}\int d^3p v_\alpha\Big[\Big(1+\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}+\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\Big)\Big] \times\Big[\mathbf{E_1}\cdot\frac{\partial f_0}{\partial\mathbf{p}}+\frac{(\mathbf{v}\cdot\mathbf{E_1})}{(\omega - \mathbf{k}\cdot\mathbf{v})}\mathbf{k}\cdot\frac{\partial f_0}{\partial\mathbf{p}}\Big],$$

on simplification, we get

$$J_\alpha = -\frac{ie^2}{\omega}\int d^3p v_\alpha\Big[\Big(1+\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}+\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\Big)\Big] \times\Big[\frac{k_l v_\beta}{(\omega - \mathbf{k}\cdot\mathbf{v})}+\delta_{l\beta}\Big]\frac{\partial f_0}{\partial p_l}E_\beta = \sigma_{\alpha\beta}E_\beta,$$

from the above expression we can write

$$4\pi i\omega\sigma_{\alpha\beta} = 4\pi e^2\int d^3p v_\alpha\Big[\Big(1+\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}+\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\Big)\Big] \times\Big[\frac{k_l v_\beta}{(\omega - \mathbf{k}\cdot\mathbf{v})}+\delta_{l\beta}\Big]\frac{\partial f_0}{\partial p_l}. \tag{2.2.23}$$

If we consider an isotropic distribution function [23] then

$$\frac{\partial f_0}{\partial p_\beta} = \frac{\partial f_0}{\partial p}\frac{v_\beta}{v}, \quad \text{where } \beta = x, y, z. \tag{2.2.24}$$

We can write

$$\Big[\frac{k_l v_\beta}{(\omega - \mathbf{k}\cdot\mathbf{v})} + \delta_{l\beta}\Big]\frac{\partial f_0}{\partial p_l} = \frac{\partial f_0}{\partial p}\frac{v_\beta}{v}\Big[\frac{\omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\Big]. \qquad (2.2.25)$$

Using the above value in Eq. (2.2.23), we get

$$\begin{aligned} 4\pi i\omega\sigma_{\alpha\beta} &= 4\pi e^2\omega\int\frac{1}{v}\frac{\partial f_0}{\partial p}d^3p v_\alpha\Big[\Big(1 + \frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi} + \frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\Big)\Big] \\ &\quad\times\Big[\frac{v_\beta}{(\omega - \mathbf{k}\cdot\mathbf{v})}\Big], \end{aligned}$$

which is a general expression for the conductivity tensor. Now by using spherical coordinates, we have $\mathbf{p} = (p\sin\theta\cos\phi, p\sin\theta\sin\phi, p\cos\theta)$.

$$\begin{aligned} 4\pi i\omega\sigma_{\alpha\beta} &= 4\pi e^2\omega\int_0^\infty\frac{p^2}{v}\frac{\partial f_0}{\partial p}dp\int_0^\pi\int_0^{2\pi}\sin\theta d\theta d\phi v_\alpha\times\Big[\Big(1 + \frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi} + \\ &\quad\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\Big)\Big]\times\Big(\frac{v_\beta}{(\omega - \mathbf{k}\cdot\mathbf{v})}\Big). \end{aligned}$$

Now we can write

$$4\pi i\omega\sigma_{\alpha\beta} = 4\pi i\omega(\sigma_{\alpha\beta})^{NM} + 4\pi i\omega(\sigma_{\alpha\beta})^{M} + 4\pi i\omega(\sigma_{\alpha\beta})^{2M},$$

(where NM stands for non-magnetized, M for first order magnetized and $2M$ for second order magnetized plasma), where

$$4\pi i\omega(\sigma_{\alpha\beta})^{NM} = 4\pi e^2\omega\int_0^\infty\frac{p^2}{v}\frac{\partial f_0}{\partial p}dp\int_0^\pi\int_0^{2\pi}\sin\theta d\theta d\phi v_\alpha\times\Big(\frac{v_\beta}{(\omega - \mathbf{k}\cdot\mathbf{v})}\Big), \qquad (2.2.26)$$

$$4\pi i\omega(\sigma_{\alpha\beta})^{M} = 4\pi e^2\omega\int_0^\infty\frac{p^2}{v}\frac{\partial f_0}{\partial p}dp\int_0^\pi\int_0^{2\pi}\sin\theta d\theta d\phi v_\alpha\times\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\times\frac{v_\beta}{(\omega - \mathbf{k}\cdot\mathbf{v})}, \qquad (2.2.27)$$

$$\begin{aligned} 4\pi i\omega(\sigma_{\alpha\beta})^{2M} &= 4\pi e^2\omega\int_0^\infty\frac{p^2}{v}\frac{\partial f_0}{\partial p}dp\int_0^\pi\int_0^{2\pi}\sin\theta d\theta d\phi v_\alpha\times \\ &\quad\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\frac{i\Omega}{(\omega - \mathbf{k}\cdot\mathbf{v})}\frac{\partial}{\partial\phi}\Big]\times\Big(\frac{v_\beta}{(\omega - \mathbf{k}\cdot\mathbf{v})}\Big). \end{aligned} \qquad (2.2.28)$$

To derive the dispersion relation of the ordinary wave we substitute $\alpha = \beta = z$, $\mathbf{k} = (k_x, 0, 0)$ and $v_z = v\cos\theta$ and after performing ϕ integration of Eq. (2.2.26), we get

$$4\pi i\omega(\sigma_{zz})^{NM} = 4\pi e^2\omega \int_0^\infty vp^2\frac{\partial f_0}{\partial p}dp\int_0^\pi \frac{2\pi\sin\theta\cos^2\theta d\theta}{\sqrt{\omega^2-k_x^2v^2\sin^2\theta}},$$

in order to solve the above integral, we have used

$$\int_0^{2\pi}\frac{dx}{(a+(-b)\cos x)^{n+1}} = \frac{2\pi}{\sqrt{a^2-b^2}}.$$

After performing θ integration

$$4\pi i\omega(\sigma_{zz})^{NM} = \frac{16\pi^2e^2\omega^2}{k_x^2}\int_0^\infty \frac{p^2}{v}\frac{\partial f_0}{\partial p}dp\Big[1-\frac{(\omega^2-k_x^2v^2)}{2k_xv\omega}\log\frac{(\omega+k_xv)}{(\omega-k_xv)}\Big], \quad (2.2.29)$$

on performing ϕ integration of Eq. (2.2.27), we get

$$4\pi i\omega(\sigma_{zz})^{M} = 0,$$

after ϕ integration of Eq. (2.2.28), we get

$$4\pi i\omega(\sigma_{zz})^{2M} = 4\pi^2 ie^2k_x^2\omega\int_0^\infty \Omega^2v^3p^2\frac{\partial f_0}{\partial p}dp\int_0^\pi \sin^3\theta\cos^2\theta d\theta\times\frac{(4\omega^2+k_x^2v^2\sin^2\theta)}{4(-\omega^2+{k_x}^2v^2\sin^2\theta)^{\frac{7}{2}}},$$

after performing θ integration,

$$4\pi i\omega(\sigma_{zz})^{2M} = -\frac{4\pi^2e^2\omega^2}{6{k_x}^2}\int_0^\infty \Omega^2\frac{p^2}{v}\frac{\partial f_0}{\partial p}dp\times\Big[\frac{(3\omega^2-5{k_x}^2v^2)}{(\omega^2-k_x^2v^2)^2}-\frac{3}{2\omega k_xv}\log\Big(\frac{\omega+k_xv}{\omega-k_xv}\Big)\Big]. \quad (2.2.30)$$

In order to solve the Eqs. (2.2.29, 2.2.30), we first perform the momentum integration. Using Eq. (2.2.10) and taking $v=c$, we get

$$16\pi^2e^2\int_0^\infty p^2\frac{\partial f_0}{\partial p}dp = -\frac{3}{c}\omega_p^2, \quad (2.2.31)$$

$$16\pi^2e^2\int_0^\infty p^2\Omega\frac{\partial f_0}{\partial p}dp = -\frac{3}{2c}\omega_c\omega_p^2, \quad (2.2.32)$$

$$16\pi^2e^2\int_0^\infty p^2\Omega^2\frac{\partial f_0}{\partial p}dp = -\frac{3}{2c}{\omega_c}^2\omega_p^2, \quad (2.2.33)$$

where $\omega_p^2 = \frac{4\pi n_0e^2c^2}{3T}$ and $\omega_c = \frac{eB_0c}{T}$

is the plasma and the cyclotron frequency in the ultra-relativistic regime. In order to solve Eqs. (2.2.31, 2.2.32, 2.2.33), we have used

$$\int_0^\infty x^n e^{-\mu x} = \frac{n!}{(\mu)^{n+1}}.$$

Now using value of Eq. (2.2.31, 2.2.33), in Eq. (2.2.29, 2.2.30), we get

$$4\pi i\omega(\sigma_{zz})^{NM} = \frac{-3{\omega_p}^2\omega^2}{c^2k_x^2}\Big[1 - \frac{(\omega^2 - c^2{k_x}^2)}{2ck_x\omega}\log\frac{(\omega + ck_x)}{(\omega - ck_x)}\Big], \tag{2.2.34}$$

$$4\pi i\omega(\sigma_{zz})^{2M} = \frac{{\omega_c}^2{\omega_p}^2\omega^2}{16c^2k_x^2}\Big[\frac{3\omega^2 - 5c^2k_x^2}{(\omega^2 - c^2k_x^2)^2} - \frac{3}{2\omega ck_x}\log\frac{(\omega + ck_x)}{(\omega - ck_x)}\Big], \tag{2.2.35}$$

using value of Eq. (2.2.34) and Eq. (2.2.35) in Eq. (2.2.13), we get the dispersion relation for the ordinary wave

$$\begin{aligned}\omega^2 &= c^2k_x^2 + \frac{3{\omega_p}^2\omega^2}{2c^2k_x^2}\Big[1 - \frac{(\omega^2 - c^2{k_x}^2)}{2ck_x\omega}\log\frac{(\omega + ck_x)}{(\omega - ck_x)}\Big] \\ &\quad - \frac{{\omega_c}^2{\omega_p}^2\omega^2}{16c^2k_x^2}\Big[\frac{3\omega^2 - 5c^2k_x^2}{(\omega^2 - c^2k_x^2)^2} - \frac{3}{2\omega ck_x}\log\frac{(\omega + ck_x)}{(\omega - ck_x)}\Big],\end{aligned} \tag{2.2.36}$$

so when $\omega \gg ck_x$ this implies $\omega \to \omega_p$, so we will expand logarithmic expression and retain terms up to order $c^2k_x^2/\omega^2$.

The dispersion relation of the ordinary wave in a weakly magnetized ultra-relativistic Maxwellian electron plasma is given by

$$\omega^2 = \omega_p^2 + \frac{6}{5}c^2k_x^2 + \frac{\omega_c^2}{10{\omega_p}^2}c^2k_x^2. \tag{2.2.37}$$

In order to solve the above logarithmic expression we have used

$$\log\Big(\frac{1+x}{1-x}\Big) = 2x + \frac{2}{3}x^3 + \frac{2}{7}x^7 + ...$$

No contribution from ω_c (1^{st} order magnetic field) is observed, only ω_c^2 (2^{nd} order magnetic field) contributes in the dispersion relation [25].

2.3 The relativistic degenerate electron plasma

The degeneracy effects must be taken into account when the de Broglie's wavelength (λ) of the plasma constituents become greater than or equal to the inter-particle distance. In a degenerate plasma the Fermi energy of the plasma particles becomes much greater than their thermal energy ($E_F \gg k_B T$) where the Fermi energy is given by $E_F = (3\pi^2 n_0)^{\frac{2}{3}}\hbar^2/2m$, it depends only on the density and is not effected by the temperature variation [7–12]. A degenerate plasma may lie in the non-relativistic ($p_F/m_0c \ll 1$), relativistic ($p_F/m_0c > 1$) and ultra-relativistic ($p_F/m_0c \gg 1$) regime, where p_F is the Fermi momentum.

By using the Fermi Dirac distribution function for the electrons, Chandrasekhar gave a mathematical criteria [17, 18]. According to this criteria one can consider a white dwarf as completely degenerate, but we know that it is not true because a white dwarf has partially degenerate interior. Still we can assume that the temperature of the plasma particles is finite but it is very less than the Fermi temperature. This assumption simplifies our problem and holds for many astrophysical environments. Now for a degenerate plasma, we use the Fermi-Dirac distribution function given by

$$f_0(E) = \left(1 + \exp\left[\frac{E - E_F}{k_B T}\right]\right)^{-1}, \tag{2.3.1}$$

where E is the relativistic energy, T is the thermal temperature, k_B is the Boltzmann constant and E_F is the Fermi energy. When $T \to 0$, the derivative of the above distribution function becomes the *step function*, it means that all energy states above the Fermi energy are vacant and all those below are filled [28].

$$\frac{\partial f_0}{\partial E} = \frac{-2}{(2\pi\hbar)^3}\delta(E - E_F), \tag{2.3.2}$$

where $\hbar$ is the *Planck's constant* and δ is the Dirac delta function [7]. In order to get the dispersion relation of the ordinary waves we use Eq. (1.6.33), we get

$$\begin{aligned} \sigma_{zz} &= -2\pi i e^2 \int_0^\infty v p^2 \frac{\partial f_0}{\partial p} dp \int_0^\pi \sin\theta \cos^2\theta d\theta \times \\ &\left[\frac{J_0^2(\zeta\sin\theta)}{\omega} + \sum_{n=1}^{\infty}\left\{\frac{J_n^2(\zeta\sin\theta)}{\omega - n\Omega} + \frac{J_n^2(\zeta\sin\theta)}{\omega + n\Omega}\right\}\right], \end{aligned} \tag{2.3.3}$$

where $\zeta = k_x v/\Omega$. In order to solve the above calculation, we have used the following integrals.

$$\begin{aligned} \int_0^\pi \sin\theta\cos^2\theta J_n^2(\zeta\sin\theta)d\theta &= \frac{2\zeta^{2n}}{(2n+3)\Gamma(2n+2)} \times \\ &\quad {}_1F_2\left[\left\{\frac{1}{2}+n\right\};\left\{\frac{5}{2}+n, 1+2n\right\};-\zeta^2\right], \\ \int_0^\pi \sin\theta\cos^2\theta J_0^2(\zeta\sin\theta)d\theta &= \frac{2}{3} \times {}_1F_2\left[\left\{\frac{1}{2}\right\};\left\{\frac{5}{2}, 1\right\};-\zeta^2\right], \end{aligned}$$

where ${}_pF_q\left[\{a_1,....a_p\};\{b_1,....b_q\};x\right]$ is a generalized Hypergeometric function [34].

Using Eq. (2.3.2) in Eq. (2.3.3) and simplifying, we get

$$\begin{aligned}\sigma_{zz} &= -\frac{4\pi ie^2}{3\omega}\int_{m_0c^2}^{\infty}\frac{(E^2-m_0^2c^4)^{\frac{3}{2}}}{cE}\frac{\partial f_0}{\partial E}dE \times \; {}_1F_2\left[\left\{\frac{1}{2}\right\};\left\{\frac{5}{2},1\right\};-\frac{k_x^2\,(E^2-m_0^2c^4)}{e^2B_0^2}\right]\\ &\quad -4\pi ie^2\int_{m_0c^2}^{\infty}\left(E^2-m_0^2c^4\right)^{\frac{3}{2}}\frac{\partial f_0}{\partial E}dE\sum_{n=1}^{\infty}\frac{(E^2-m_0^2c^4)^n}{cE(2n+3)\Gamma(2n+2)}\left(\frac{k_x}{eB_0}\right)^{2n}\\ &\quad \times\; {}_1F_2\left[\left\{\frac{1}{2}+n\right\};\left\{\frac{5}{2}+n,1+2n\right\};-\frac{k_x^2\,(E^2-m_0^2c^4)}{e^2B_0^2}\right]\\ &\quad \times\left\{\frac{1}{\omega-\frac{neB_0c}{E}}+\frac{1}{\omega+\frac{neB_0c}{E}}\right\}. \qquad (2.3.4)\end{aligned}$$

$$\begin{aligned}\sigma_{zz} &= \frac{8\pi ie^2}{3\omega\,(2\pi\hbar)^3}\frac{p_F^3}{\gamma m_0}\;\times\; {}_1F_2\left[\left\{\frac{1}{2}\right\};\left\{\frac{5}{2},1\right\};-\frac{c^2k_x^2}{\omega_c^2}\frac{p_F^2}{m_0^2c^2}\right]\\ &\quad +\frac{8\pi ie^2}{(2\pi\hbar)^3}\sum_{n=1}^{\infty}\frac{p_F^3}{(2n+3)\Gamma(2n+2)}\left(\frac{ck_x}{\omega_c}\right)^{2n}\left(\frac{p_F}{m_0c}\right)^{2n}\\ &\quad \times\; {}_1F_2\left[\left\{\frac{1}{2}+n\right\};\left\{\frac{5}{2}+n,1+2n\right\};-\frac{c^2k_x^2}{\omega_c^2}\frac{p_F^2}{m_0^2c^2}\right]\\ &\quad \times\;\frac{1}{\omega m_0}\left\{\frac{1}{\left(\sqrt{1+\frac{p_F^2}{m_0^2c^2}}-\frac{n\omega_c}{\omega}\right)}+\frac{1}{\left(\sqrt{1+\frac{p_F^2}{m_0^2c^2}}+\frac{n\omega_c}{\omega}\right)}\right\},\end{aligned}$$

where ω_c is the non-relativistic cyclotron frequency and the (equilibrium) number density in given by

$$n_0=\frac{p_F^3}{3\pi^2\hbar^3}.$$

Using the above value, we get

$$\begin{aligned}
\sigma_{zz} &= -\frac{n_0 i e^2}{\omega m_0\sqrt{1+\frac{p_F^2}{m_0 c^2}}} \times {}_1F_2\left[\left\{\frac{1}{2}\right\};\left\{\frac{5}{2},1\right\};-\frac{c^2k_x^2}{\omega_c^2}\frac{p_F^2}{m_0^2c^2}\right] \\
&+\frac{3n_0 i e^2}{\omega m_0}\sum_{n=1}^{\infty}\frac{1}{(2n+3)\Gamma(2n+2)}\left(\frac{ck_x}{\omega_c}\right)^{2n}\left(\frac{p_F}{m_0 c}\right)^{2n} \\
&\times {}_1F_2\left[\left\{\frac{1}{2}+n\right\};\left\{\frac{5}{2}+n,1+2n\right\};-\frac{c^2k_x^2}{\omega_c^2}\frac{p_F^2}{m_0^2c^2}\right] \\
&\times \left\{\frac{1}{\left(\sqrt{1+\frac{p_F^2}{m_0^2c^2}}-\frac{n\omega_c}{\omega}\right)}+\frac{1}{\left(\sqrt{1+\frac{p_F^2}{m_0^2c^2}}+\frac{n\omega_c}{\omega}\right)}\right\}. \qquad (2.3.5)
\end{aligned}$$

Substituting Eq. (2.3.5) in the Eq. (2.2.12), we get

$$\begin{aligned}
R_{zz} &= \omega^2 - c^2k_x^2 - \frac{\omega_p^2}{\sqrt{1+\frac{p_F^2}{m_0^2c^2}}} \times {}_1F_2\left[\left\{\frac{1}{2}\right\};\left\{\frac{5}{2},1\right\};-\frac{c^2k_x^2}{\omega_c^2}\frac{p_F^2}{m_0^2c^2}\right] \\
&-3\omega_p^2\sum_{n=1}^{\infty}\left(\frac{ck_x}{\omega_c}\right)^{2n}\left(\frac{p_F}{m_0 c}\right)^{2n}\frac{1}{(2n+3)\Gamma(2n+2)} \\
&\times {}_1F_2\left[\left\{\frac{1}{2}+n\right\};\left\{\frac{5}{2}+n,1+2n\right\};-\frac{c^2k_x^2}{\omega_c^2}\frac{p_F^2}{m_0^2c^2}\right] \\
&\times \left\{\frac{1}{\left(\sqrt{1+\frac{p_F^2}{m_0^2c^2}}-\frac{n\omega_c}{\omega}\right)}+\frac{1}{\left(\sqrt{1+\frac{p_F^2}{m_0^2c^2}}+\frac{n\omega_c}{\omega}\right)}\right\}. \qquad (2.3.6)
\end{aligned}$$

The dispersion relation of the ordinary waves in the relativistic degenerate electron plasma can be obtained by using Eq. (2.2.13).

$$\begin{aligned}\omega^2 &= \frac{\omega_p^2}{\sqrt{1+\frac{p_F^2}{m_0^2c^2}}} \times {}_1F_2\left[\left\{\frac{1}{2}\right\};\left\{\frac{5}{2},1\right\};-\frac{c^2k_x^2}{\omega_c^2}\frac{p_F^2}{m_0^2c^2}\right] \\ &+c^2k_x^2+3\omega_p^2\sum_{n=1}^{\infty}\left(\frac{p_F}{m_0c}\right)^{2n}\left(\frac{ck_x}{\omega_c}\right)^{2n}\frac{1}{(2n+3)\Gamma(2n+2)} \\ &\times {}_1F_2\left[\left\{\frac{1}{2}+n\right\};\left\{\frac{5}{2}+n,1+2n\right\};-\frac{c^2k_x^2}{\omega_c^2}\frac{p_F^2}{m_0^2c^2}\right] \\ &\times\left\{\frac{1}{\left(\sqrt{1+\frac{p_F^2}{m_0^2c^2}}-\frac{n\omega_c}{\omega}\right)}+\frac{1}{\left(\sqrt{1+\frac{p_F^2}{m_0^2c^2}}+\frac{n\omega_c}{\omega}\right)}\right\}. \end{aligned} \tag{2.3.7}$$

From the above equation, two special cases i.e., the non-relativistic and the ultra-relativistic degenerate case can be discussed.

2.3.1 The non-relativistic degenerate elecron plasma

As in the non-relativistic case ($p_\mathrm{F}/m_0c \ll 1$), so can write the dispersion relation of the ordinary waves for the non-relativistic degenerate electron plasma as

$$\begin{aligned}\omega^2 &= \omega_p^2 \times {}_1F_2\left[\left\{\frac{1}{2}\right\};\left\{\frac{5}{2},1\right\};-\frac{v_F^2k_x^2}{\omega_c^2}\right]+c^2k_x^2 \\ &+3\omega_p^2\sum_{n=1}^{\infty}\left(\frac{v_Fk_x}{\omega_c}\right)^{2n}\frac{1}{(2n+3)\Gamma(2n+2)} \\ &\times {}_1F_2\left[\left\{\frac{1}{2}+n\right\};\left\{\frac{5}{2}+n,1+2n\right\};-\frac{v_F^2k_x^2}{\omega_c^2}\right] \\ &\times\left\{\frac{1}{\left(1-\frac{n\omega_c}{\omega}\right)}+\frac{1}{\left(1+\frac{n\omega_c}{\omega}\right)}\right\}, \end{aligned} \tag{2.3.8}$$

where v_F is the Fermi velocity.

2.3.2 The ultra-relativistic degenerate electron plasma

As in the ultra-relativistic case $p_{\rm F}/m_0 c \gg 1$, so we can write the dispersion relation of the ordinary waves for the ultra-relativistic degenerate electron plasma as

$$\begin{aligned}
\omega^2 &= \omega_{pF}^2 \times \ {}_1F_2\left[\left\{\frac{1}{2}\right\};\left\{\frac{5}{2},1\right\};-\frac{c^2k_x^2}{\omega_{cF}^2}\right]+c^2k_x^2 \\
&\quad +3\omega_{pF}^2\sum_{n=1}^{\infty}\left(\frac{ck_x}{\omega_{cF}}\right)^{2n}\frac{1}{(2n+3)\Gamma(2n+2)} \\
&\quad \times \ {}_1F_2\left[\left\{\frac{1}{2}+n\right\};\left\{\frac{5}{2}+n,1+2n\right\};-\frac{c^2k_x^2}{\omega_{cF}^2}\right] \\
&\quad \times\left\{\frac{1}{(1-\frac{n\omega_{cF}}{\omega})}+\frac{1}{(1+\frac{n\omega_{cF}}{\omega})}\right\}, \qquad (2.3.9)
\end{aligned}$$

where $\omega_{\rm pF} = \sqrt{4\pi n_0 e^2 c/p_{\rm F}}$ and $\omega_{cF} = eB_0/p_{\rm F}$ are the plasma frequency and the cyclotron frequency for the ultra-relativistic degenerate electron plasma.

Chapter 3

Results and discussion

In this dissertation the propagation of ordinary waves in different plasma environments have been studied by using the plasma kinetic theory. We have derived the generalized expression for the conductivity tensor ($\sigma_{\alpha\beta}$) in spherical coordinates by using the linearized relativistic Vlasov-Maxwell equations. Ignoring dynamics of the ion, we have particulary focussed on the electron plasma.

So from the generalized expression we require only σ_{zz} component of the conductivity tensor which specifies the dynamics of the ordinary waves. Using different distribution functions (ultra-relativistic Maxwellian and relativistic Fermi-Dirac) we have studied the propagation characteristics of the ordinary waves in different (degenerate and non-degenerate) plasma environments. The non-degenerate plasma environment is further studied in a strong and a weak magnetic field limit. Now we will present the graphs of the ordinary waves in different plasma environments and discuss the results.

In Sec. 1 and 2, the propagation of the ordinary waves in the ultra-relativistic Maxwellian electron plasma is studied in a strong and a weak magnetic field limits. In Sec. 3, 4 and 5, the propagation of the ordinary waves in the non-relativistic, relativistic and ultra-relativistic degenerate cases has been studied by presenting graphs.

3.1 Non-degenerate ultra-relativistic Maxwellian electron plasma in the strong field limit

The dispersion characteristics of the ordinary waves in the non-degenerate ultra-relativistic electron plasma in a strong field limit can be studied by using the Eq. (2.2.17), given by

$$\omega_r^2 = c^2k_x^2 + {\omega_p}^2 + 3{\omega_p}^2\sum_{n=1}^{\infty}(n+1)\Big(\frac{ck_x}{\omega_c}\Big)^{2n}\Big\{\exp\Big(\frac{n\omega_c}{\omega_r}\Big)E_{2n+4}\Big(\frac{n\omega_c}{\omega_r}\Big)$$
$$+\exp\Big(-\frac{n\omega_c}{\omega_r}\Big)E_{2n+4}\Big(-\frac{n\omega_c}{\omega_r}\Big)\Big\}.$$

One can understand the analytical results by choosing the values of ck_x/ω_p and ω_c/ω_p. The summation is kept over all values of n.

In order to get a graph for the real part we plot $\omega_r^2/c^2{k_x}^2$ versus ω/ω_p. The graph in the Fig. 3.1 is the same as we get in the non-relativistic case [2]. In Fig. 3.1 it can be seen that unlike the non-relativistic case, the cut-off point is shifted to higher values of frequency due to the magnetic field effects. At this cut-off point the square of refractive index becomes zero. Beyond the cut-off point the wave cannot propagate further, so the wave will start propagating for $\omega_r > 44\omega_p$.

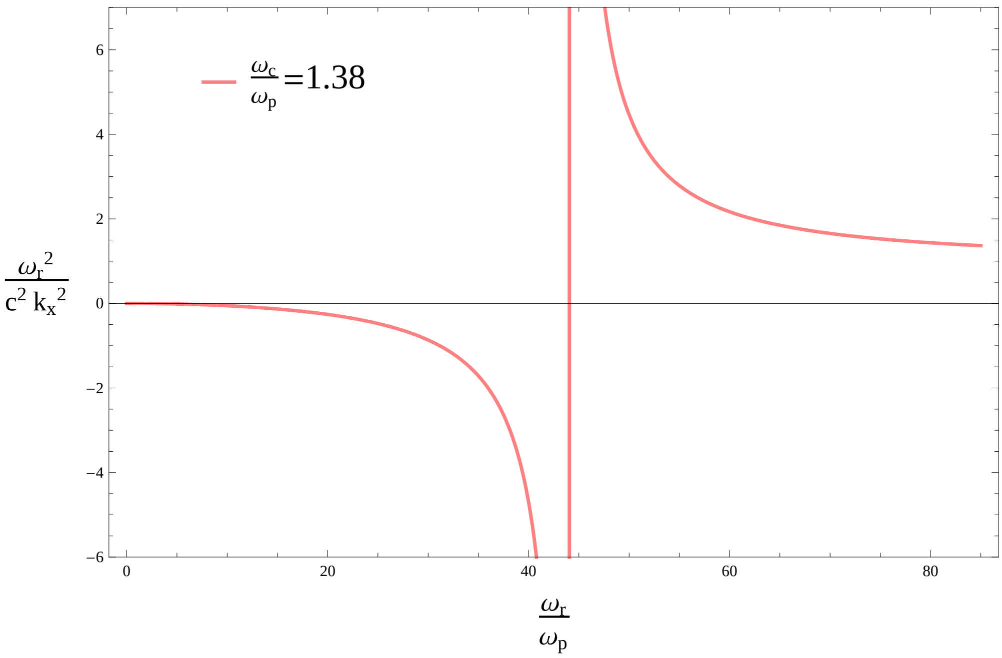

Figure 3.1: Graph for the ordinary wave in the strong field limit (${\omega_r}^2/c^2{k_x}^2$ versus ω_r/ω_p).

In order to a get graph for the imaginary part, we use Eq. (2.2.18), given by

$$\gamma_o = -\frac{3\omega_p^2}{\omega_r}\sum_{n=1}^{\infty}(n+1)\Big(\frac{ck_x}{\omega_c}\Big)^{2n}\frac{\pi}{\Gamma(2n+4)}\exp\Big(-\frac{n\omega_c}{\omega}\Big)\Big(\frac{n\omega_c}{\omega}\Big)^{2n+3}.$$

We plot γ_o/ω_p versus ck_x/ω_p, for this purpose we have used the first term i.e., $\omega = ck_x$ from the real part of the dispersion relation. It was noticed that with the increase in strength of the ambient magnetic field $\mathbf{B_0}$, the damping is reduced. Further for the limit $c^2{k_x}^2 \gg {\omega_p}^2$, the damping is significantly reduced and with further increase it vanishes.

The damping results from the wave-particle interaction. The perturbed magnetic field component $\mathbf{B_1}$ and parallel component of the electron velocity generate an electric field component $\mathbf{E_1}$ that is perpendicular to the ambient magnetic field $\mathbf{B_0}$ [29]. This perpendicular electric field will try to accelerate the gyrating particles in its own direction. The competition between the magnetic field (which gyrates the particles) and the electric field (which accelerate the particles) will give rise to the damping of wave. With the increase in strength of the magnetic field the damping reduces because the particles get more tightly bound with the magnetic field and the electric field cannot accelerate them in its own direction.

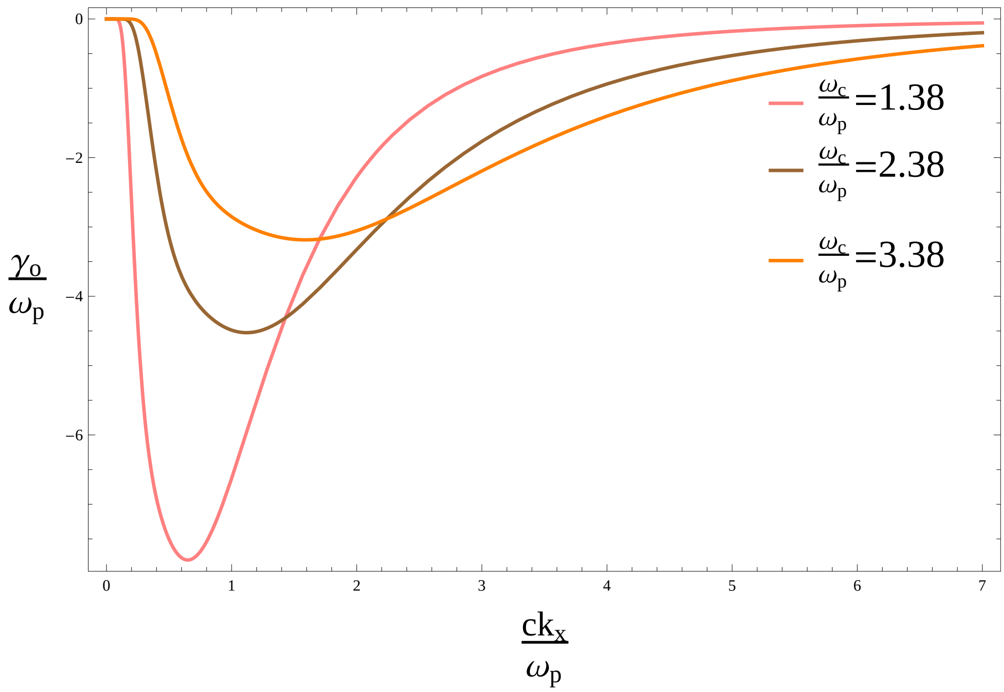

Figure 3.2: Graph for the ordinary wave in the strong field limit (γ_o/ω_p versus $c^2{k_x}^2/\omega_p^2$).

3.2 Non-degenerate ultra-relativistic Maxwellian electron plasma in the weak field limit

In order to study the propagation of the ordinary waves in the non-degenerate electron plasma in a weak magnetic field, we use the Eq. (2.2.37), given by

$$\omega^2 = \omega_p^2 + \frac{6}{5}c^2k_x^2 + \frac{\omega_c^2}{10{\omega_p}^2}c^2k_x^2.$$

In the present case we are interested in the weak magnetic field environment, so we have taken $\omega_c < \omega_p$ i.e., the $\omega_c = 0.1\omega_p$.

In Fig. 3.3, it can be seen that at the higher values of ω and k, the dispersion curve approaches the wave propagating through the free space (asymptote at $\omega = ck$). Physically, at very high frequencies, it become difficult for the electrons to respond at such a short time due to their inertia. Therefore, the electrons cannot affect the wave propagation. So we can say that the plasma influence goes on decreasing, as the frequency of the wave (ω) increases.

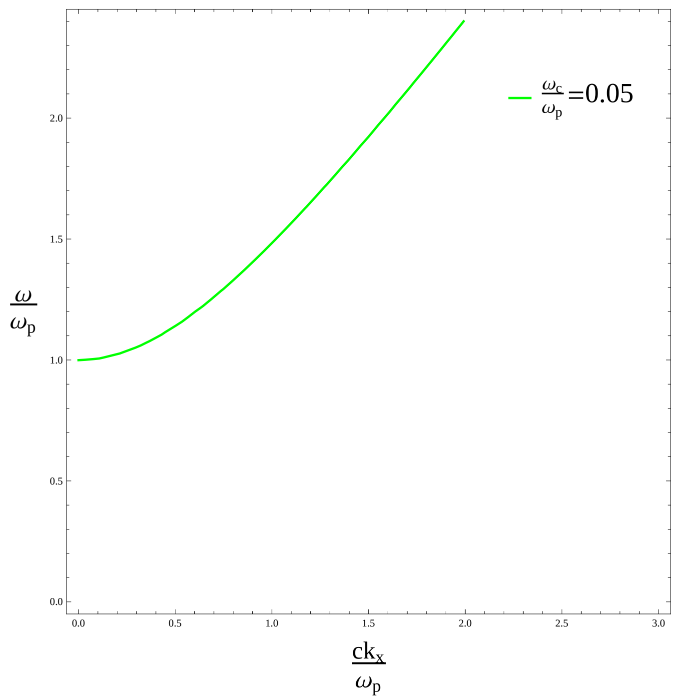

Figure 3.3: Graph for the ordinary wave in the weak field limit (ω/ω_p versus ck_x/ω_p).

In Fig. 3.4, we represent a plot between $\omega^2/c^2{k_x}^2$ versus ω/ω_p. As the strength of the magnetic field is very weak so we get the result like the ordinary wave propagating in the non-relativistic plasma. In Fig. 3.4, it can be seen that if frequency of the wave is less than plasma frequency (i.e., $\omega < \omega_p$), the wave will not propagate. For $\omega < \omega_p$, the electrons respond quickly and neutralize the effect of applied field. So the wave amplitude starts to decrease, that results in the wave damping. When frequency of the wave matches the plasma frequency i.e., $\omega = \omega_p$, a cut-off point will occur. At this point ($\omega = \omega_p$), $k \to 0$ and the refractive index becomes zero. When frequency of the wave exceeds the plasma frequency the wave will start propagating.

No contribution from ω_c (1^{st} order magnetic field) was observed, it was noticed that only ω_c^2 (2^{nd} order magnetic field) contributes in the dispersion relation.

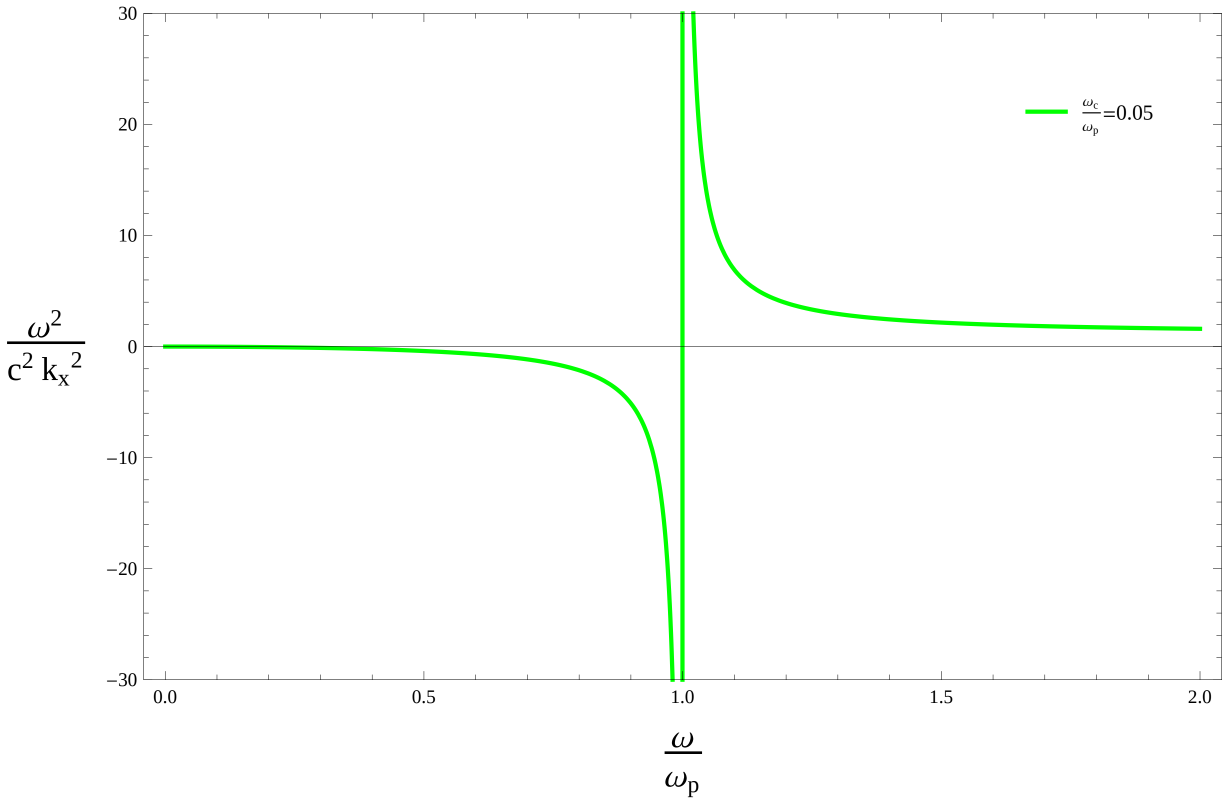

Figure 3.4: Graph for the ordinary wave in the weak field limit ($\omega^2/c^2{k_x}^2$ versus ω/ω_p).

3.3 Non-relativistic degenerate electron plasma

In order to study the propagation of the ordinary waves in the non-relativistic degenerate electron plasma, we use Eq. (2.3.8), given by

$$\begin{aligned}
\omega^2 &= \omega_p^2 \times {}_1F_2\left[\left\{\frac{1}{2}\right\};\left\{\frac{5}{2},1\right\};-\frac{v_F^2k_x^2}{\omega_c^2}\right]+c^2k_x^2 \\
&+3\omega_p^2\sum_{n=1}^{\infty}\left(\frac{v_Fk_x}{\omega_c}\right)^{2n}\frac{1}{(2n+3)\Gamma(2n+2)} \\
&\times\ {}_1F_2\left[\left\{\frac{1}{2}+n\right\};\left\{\frac{5}{2}+n,1+2n\right\};-\frac{v_F^2k_x^2}{\omega_c^2}\right] \\
&\times\left\{\frac{1}{\left(1-\frac{n\omega_c}{\omega}\right)}+\frac{1}{\left(1+\frac{n\omega_c}{\omega}\right)}\right\}.
\end{aligned}$$

The ordinary waves remain unaffected by the magnetic field in the fluid ap-

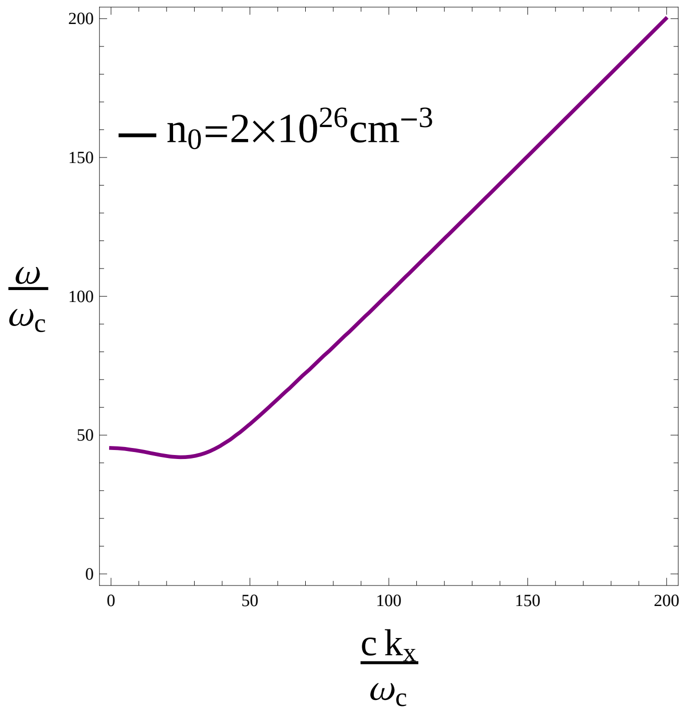

Figure 3.5: The ordinary wave in the non-relativistic degenerate plasma.

proximation. So to observe the fluid theory effects we neglect the contribution of the higher order harmonics and take $n = 0$ in Eq. (2.3.8). Taking $B_0 = 10^9$Gauss and $n_0 = 2 \times 10^{26} cm^{-3}$ we got the dispersion diagram (Fig. 3.5). In the Fig. 3.5, it can be seen that the ordinary wave in a non-relativistic degenerate plasma is propagating at a higher frequency.

In order to observe the kinetic theory effects we take the contribution of the higher order harmonics. The resulting cyclotron harmonic structure (for $n_0 = 2 \times 10^{27} cm^{-3} - 2 \times 10^{28} cm^{-3}$ and $B_0 = 10^9$Gauss) is shown in the Fig. 3.6. In the Fig. 3.6, it can be seen that the ordinary wave is propagating at the exact harmonics of the electron cyclotron frequency.

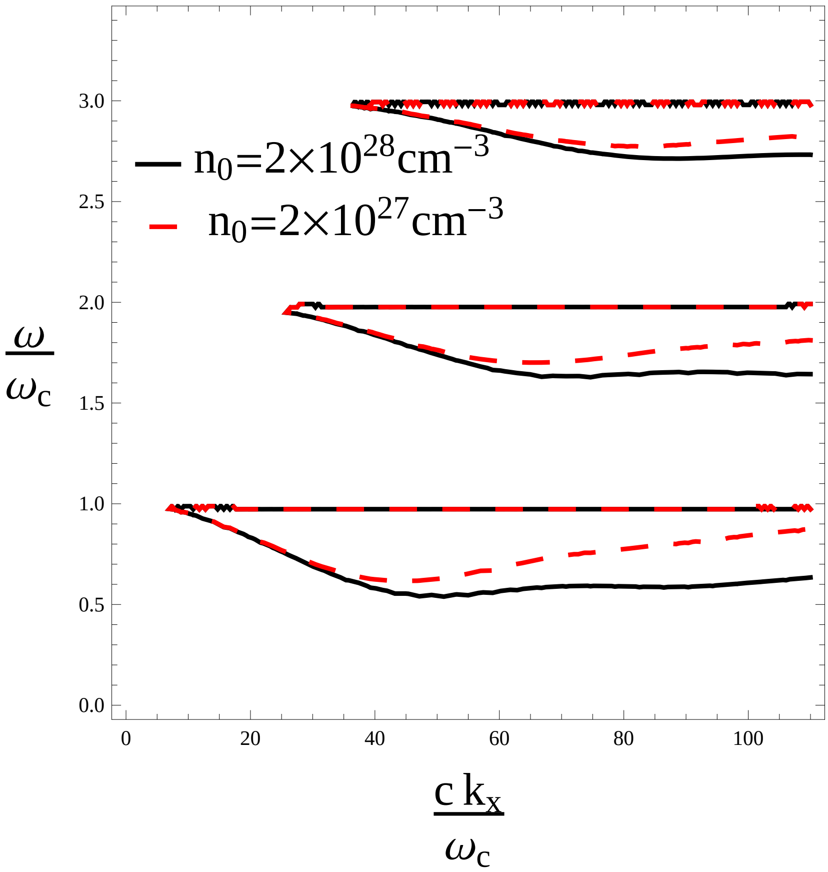

Figure 3.6: Harmonic structure of the ordinary wave in the non-relativistic degenerate plasma.

The dispersion diagram (Fig. 3.7) is same as for the non-degenerate plasma [2]. But when we reduce the range of the normalized frequency we get the resonances as shown in the Fig. 3.8. The points where sharp peaks are occurring cannot be treated as cut-offs i.e., $k = 0$ (because $\omega^2/c^2k_x^2$ is not approaching infinity).

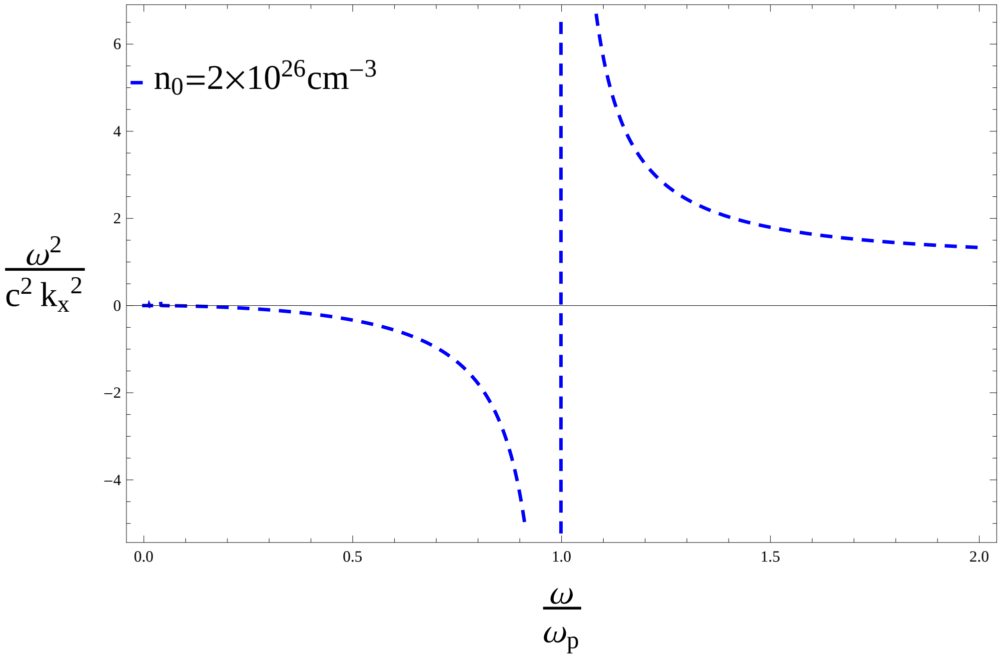

Figure 3.7: Plot between $\omega^2/c^2k_x^2$ and ω/ω_p (the ordinary wave in the non-relativistic degenerate plasma).

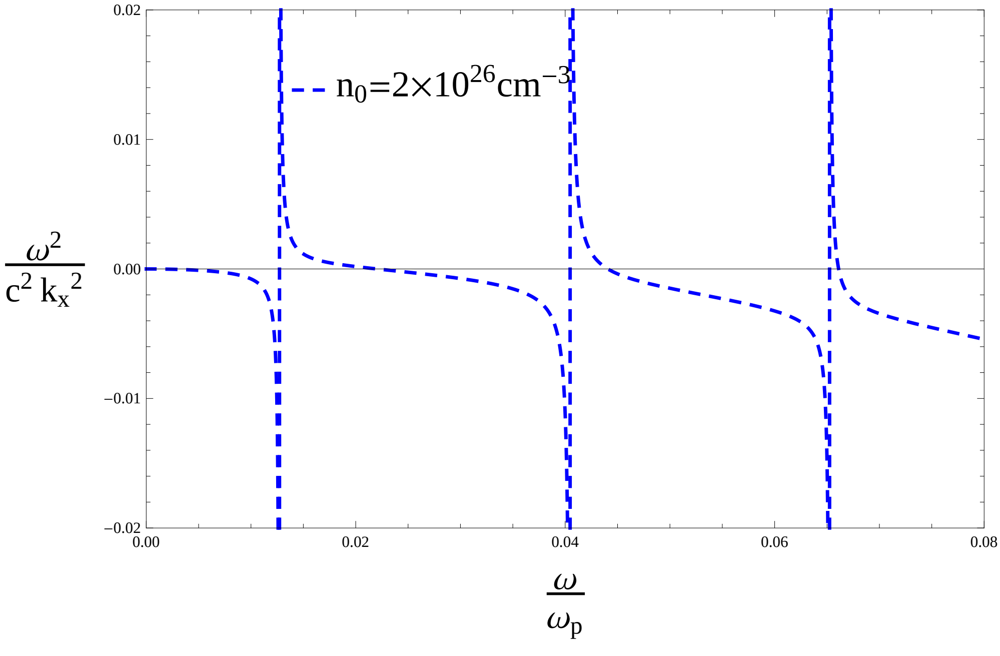

Figure 3.8: Plot between $\omega^2/c^2k_x^2$ and ω/ω_p (the ordinary wave in the non-relativistic degenerate plasma).

3.4 Relativistic degenerate electron plasma

In order to study the propagation of the ordinary waves in the relativistic degenerate electron plasma, we use Eq. (2.3.7), given by

$$\begin{aligned}
\omega^2 &= \frac{\omega_p^2}{\sqrt{1+\frac{p_F^2}{m_0^2c^2}}} \times {}_1F_2\left[\left\{\frac{1}{2}\right\};\left\{\frac{5}{2},1\right\};-\frac{c^2k_x^2}{\omega_c^2}\frac{p_F^2}{m_0^2c^2}\right] \\
&+c^2k_x^2+3\omega_p^2\sum_{n=1}^{\infty}\left(\frac{p_F}{m_0c}\right)^{2n}\left(\frac{ck_x}{\omega_c}\right)^{2n}\frac{1}{(2n+3)\Gamma(2n+2)} \\
&\times {}_1F_2\left[\left\{\frac{1}{2}+n\right\};\left\{\frac{5}{2}+n,1+2n\right\};-\frac{c^2k_x^2}{\omega_c^2}\frac{p_F^2}{m_0^2c^2}\right] \\
&\times\left\{\frac{1}{\left(\sqrt{1+\frac{p_F^2}{m_0^2c^2}}-\frac{n\omega_c}{\omega}\right)}+\frac{1}{\left(\sqrt{1+\frac{p_F^2}{m_0^2c^2}}+\frac{n\omega_c}{\omega}\right)}\right\}.
\end{aligned}$$

For the weakly-relativistic, relativistic and the strongly relativistic cases we have chosen $n_0 = 2\times10^{29} - 2\times10^{31}cm^{-3}$ and $B_0 = 10^{10}Gauss$.

Unlike non-relativistic degenerate case, in the relativistic case there is a shift in the harmonic structure. In moving from the weakly relativistic to the strongly relativistic regime the harmonic structure is shifted downward due to the increase in the gamma factor $\left(\gamma = \sqrt{1+\frac{p_F^2}{m_0^2c^2}}\right)$. Further, with the increase in ω_p/ω_c the propagation regime is reduced as shown in Figs. 3.9, 3.10.

In Figs. 3.11, 3.13, 3.15, we have used $ck_x/\omega_c = 4.5, 2.2, 0.9$. The increase in the plasma density results in considerable shift of the cut-off points to the lower frequency values because the refractive index depends on ω_p as shown in Figs. 3.11, 3.13, 3.15.

When we reduce the range of normalized frequency in Figs. 3.11, 3.13, 3.15, we get the resonances as shown in Figs. 3.12, 3.14, 3.16. Unlike non-relativistic degenerate case, the relativistic variation of the mass results in spreading and shifting the resonance points to the lower frequency values as shown in Figs. 3.12, 3.14, 3.16.

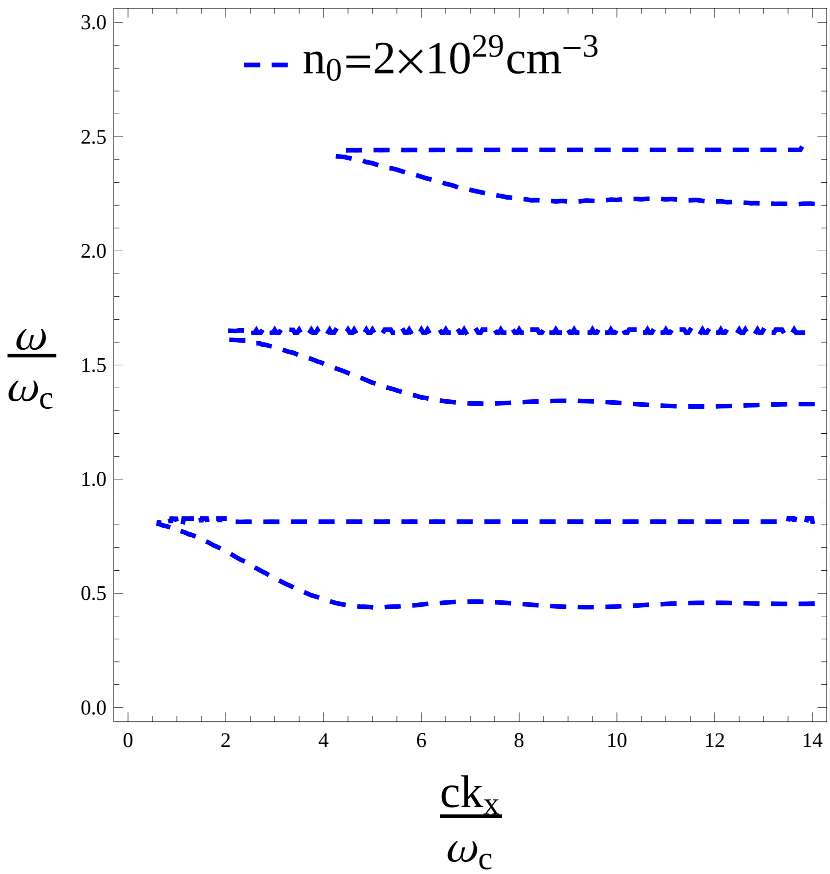

Figure 3.9: Harmonic structure of the ordinary wave in the weakly relativistic degenerate plasma.

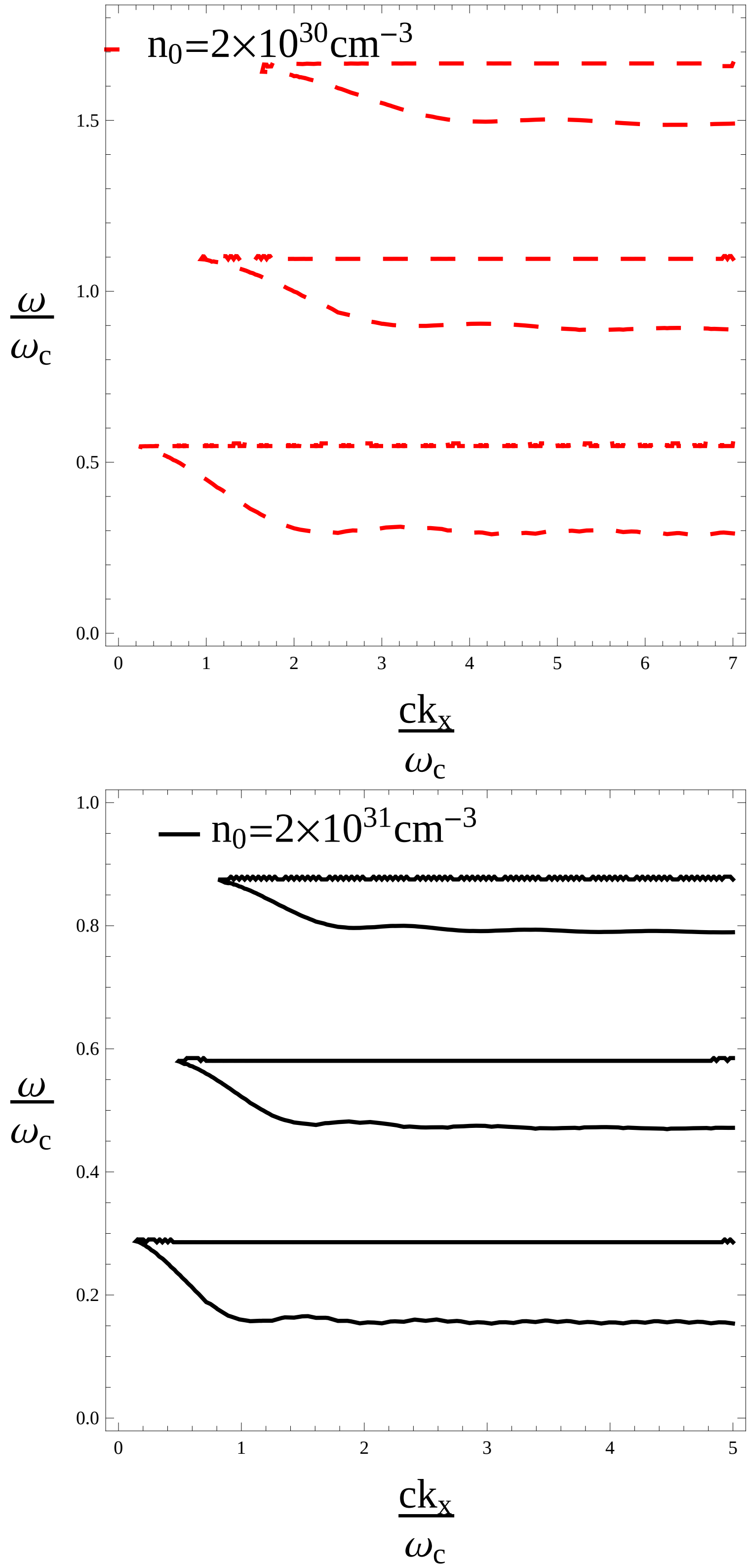

Figure 3.10: Harmonic structure of the ordinary wave in the relativistic and strongly relativistic degenerate plasma.

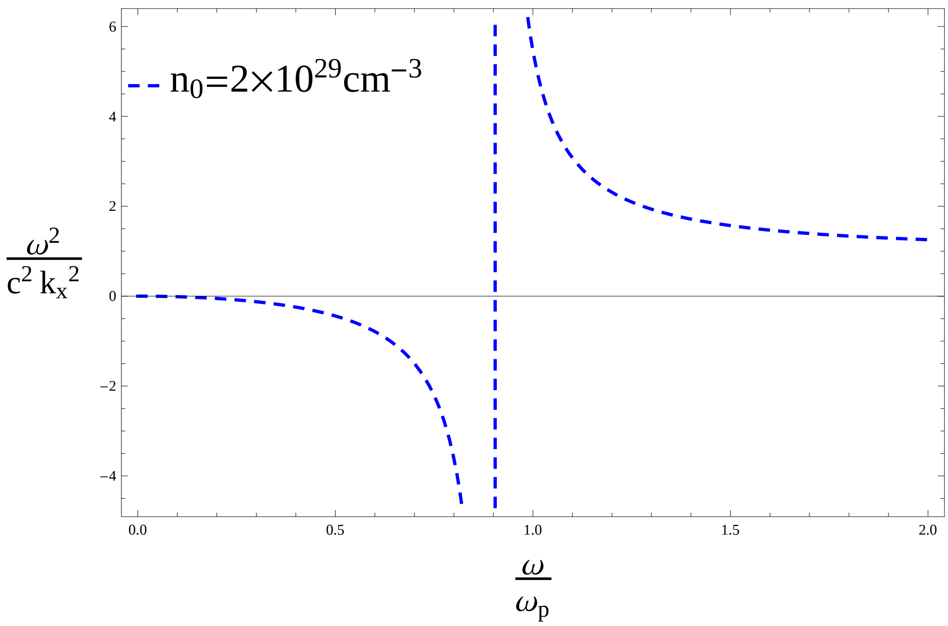

Figure 3.11: Plot between $\omega^2/c^2k_x^2$ and ω/ω_p (the ordinary wave in the weakly relativistic degenerate plasma).

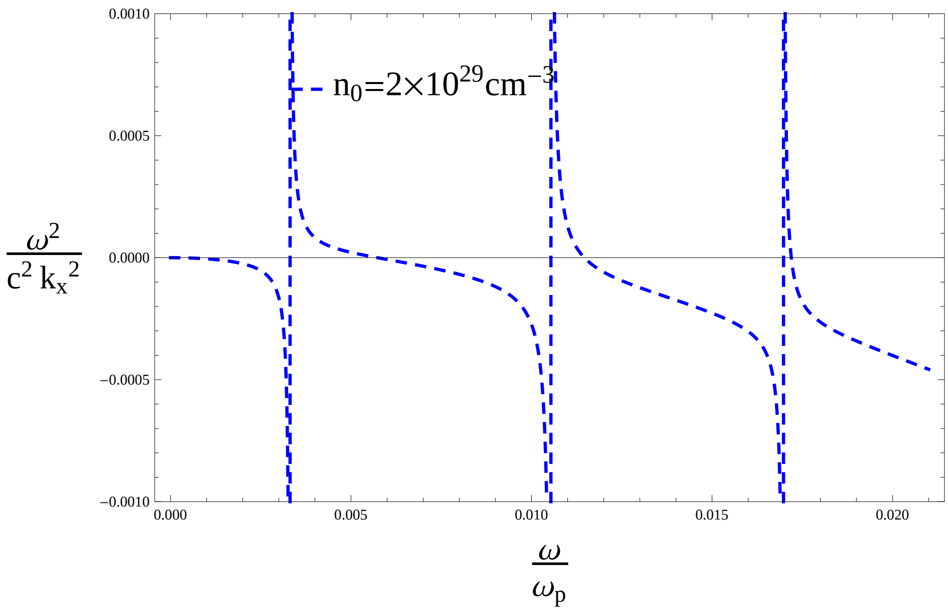

Figure 3.12: Plot between $\omega^2/c^2k_x^2$ and ω/ω_p (the ordinary wave in the weakly relativistic degenerate plasma).

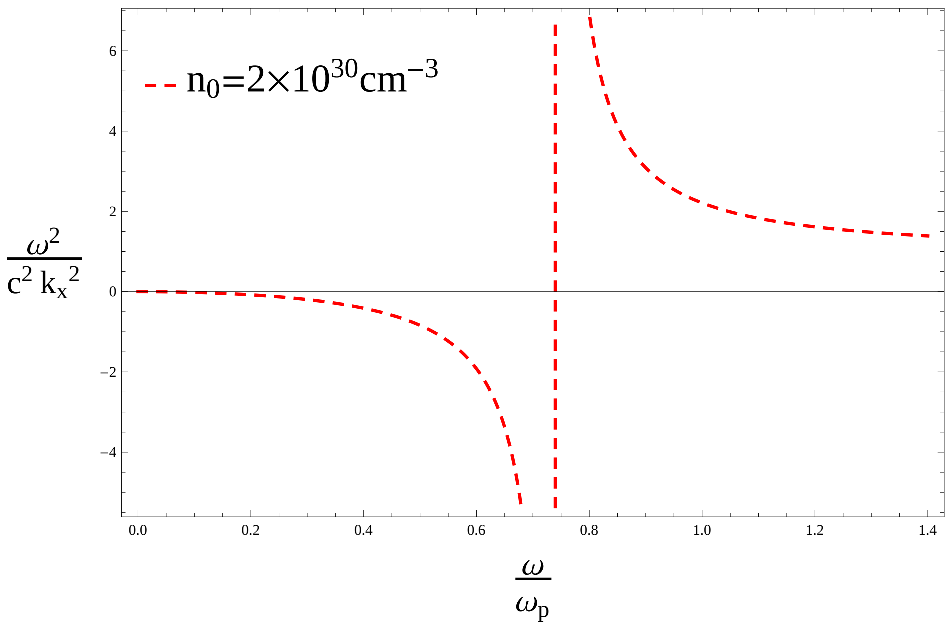

Figure 3.13: Plot between $\omega^2/c^2k_x^2$ and ω/ω_p (the ordinary wave in the relativistic degenerate plasma).

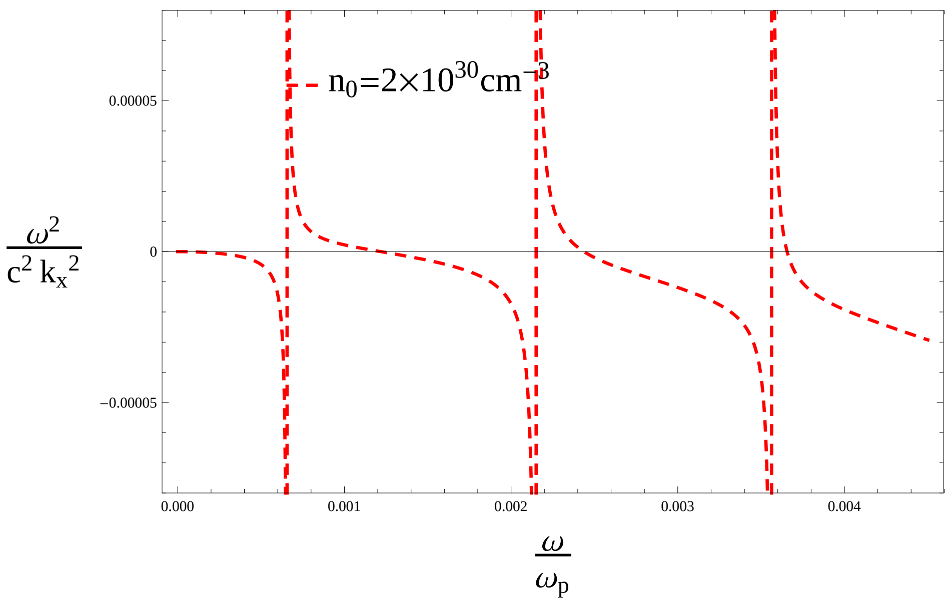

Figure 3.14: Plot between $\omega^2/c^2k_x^2$ and ω/ω_p (the ordinary wave in the relativistic degenerate plasma).

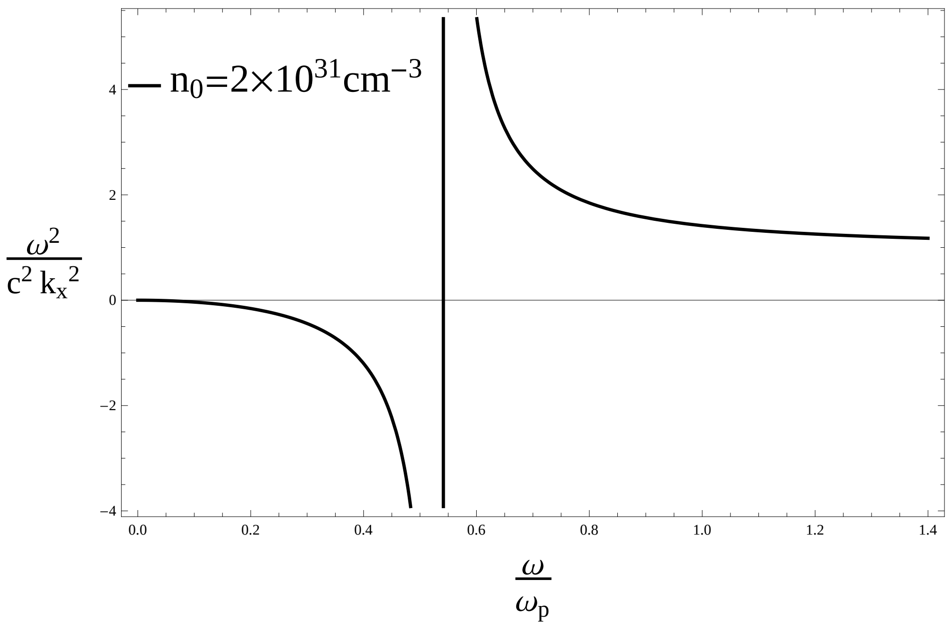

Figure 3.15: Plot between $\omega^2/c^2k_x^2$ and ω/ω_p (the ordinary wave in the strongly relativistic degenerate plasma).

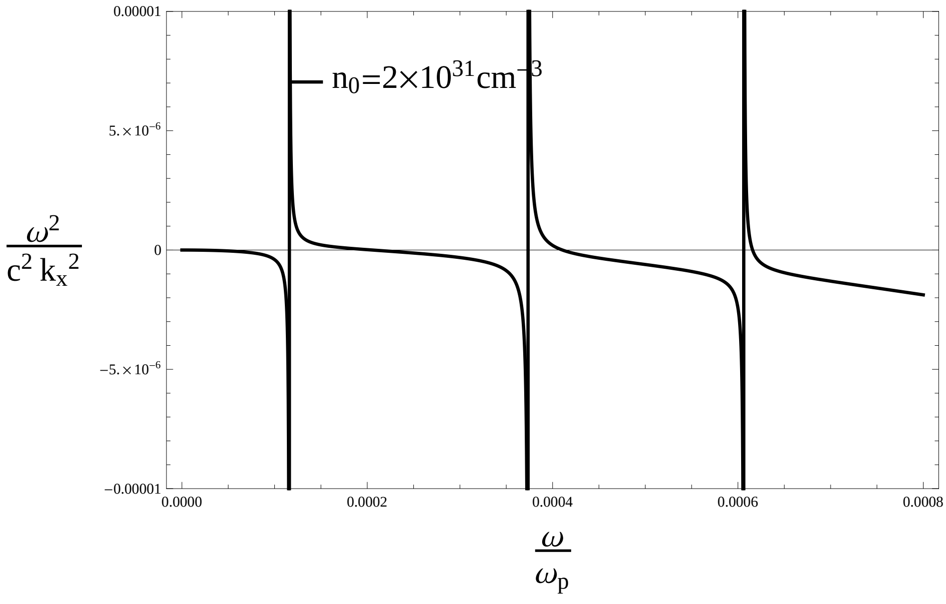

Figure 3.16: Plot between $\omega^2/c^2k_x^2$ and ω/ω_p (the ordinary wave in the strongly relativistic degenerate plasma).

3.5 Ultra-relativistic degenerate electron plasma

The dispersion relation of the ordinary waves in the ultra-relativistic degenerate electron plasma can be studied by using the Eq. (2.3.9), given by

$$\begin{aligned}\omega^2 \;=\; & \omega_{pF}^2 \times \, {}_1F_2\left[\left\{\frac{1}{2}\right\};\left\{\frac{5}{2},1\right\};-\frac{c^2k_x^2}{\omega_{cF}^2}\right] + c^2k_x^2 \\ & +3\omega_{pF}^2 \sum_{n=1}^{\infty}\left(\frac{ck_x}{\omega_{cF}}\right)^{2n} \frac{1}{(2n+3)\Gamma(2n+2)} \\ & \times \, {}_1F_2\left[\left\{\frac{1}{2}+n\right\};\left\{\frac{5}{2}+n,1+2n\right\};-\frac{c^2k_x^2}{\omega_{cF}^2}\right] \\ & \times \left\{\frac{1}{(1-\frac{n\omega_{cF}}{\omega})}+\frac{1}{(1+\frac{n\omega_{cF}}{\omega})}\right\}.\end{aligned}$$

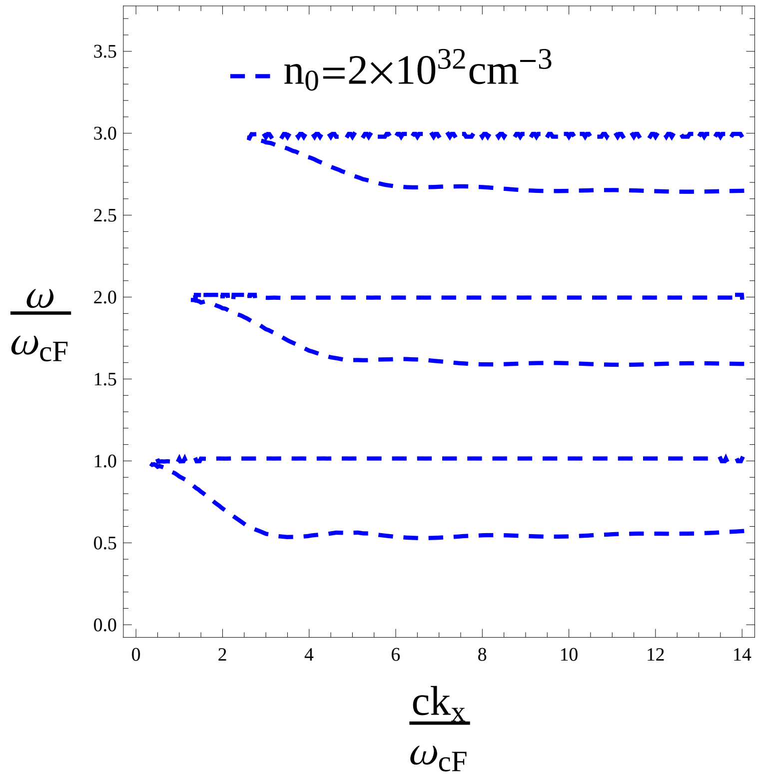

Figure 3.17: Harmonic structure of the ordinary wave in the ultra-relativistic degenerate plasma.

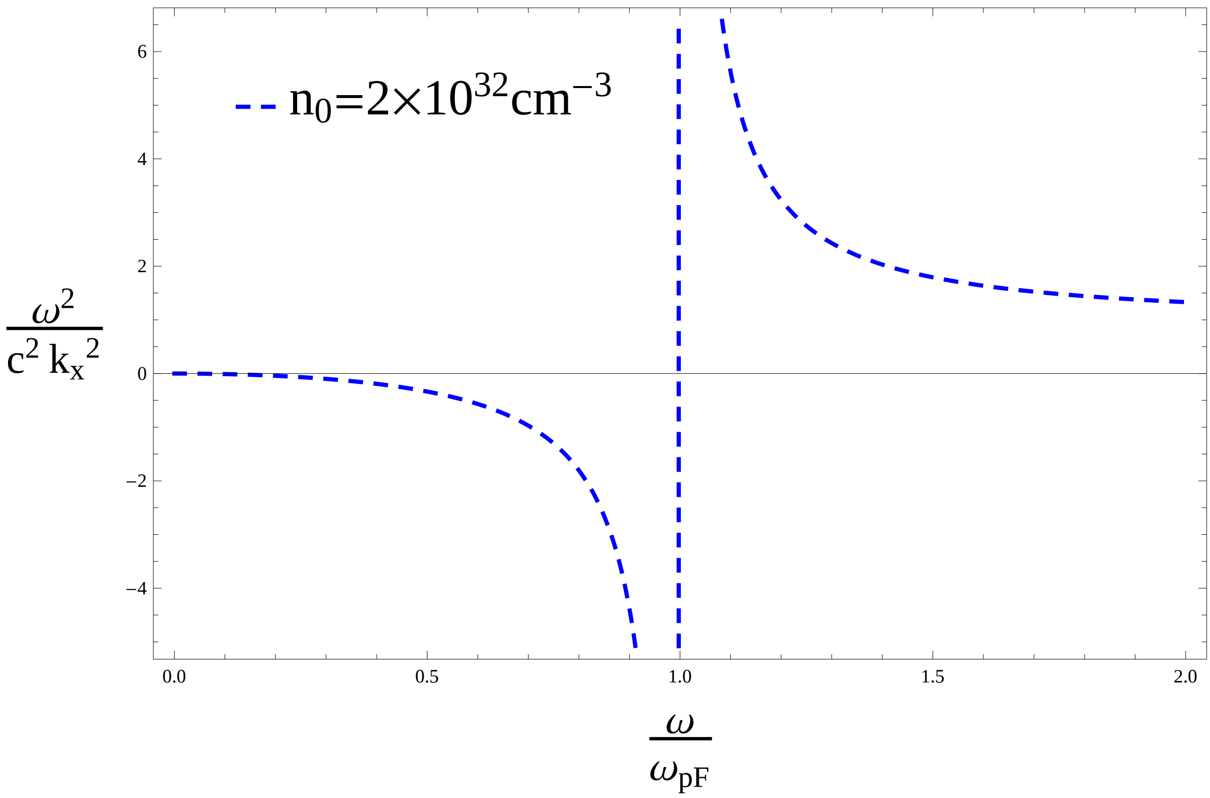

Figure 3.18: Plot between $\omega^2/c^2k_x^2$ and ω/ω_{pF} (ordinary wave in the ultra-relativistic degenerate plasma).

For the ultra-relativistic case we have chosen $n_0 = 2 \times 10^{32} cm^{-3} - 2 \times 10^{34} cm^{-3}$ and $B_0 = 10^{12}$ Gauss. It can be seen from the Fig. 3.17 that the dispersion curve for the ultra-relativistic degenerate electron plasma is exactly the same as for non-relativistic degenerate electron plasma (the dispersion curves occur at exact harmonics of cyclotron frequency) but in the present case we have normalized it with a different parameter.

In Fig. 3.18, it can be observed that when ω exceeds ω_{pF}, the wave will propagate. There is a cut point at $\omega = \omega_{pF}$, beyond the cut off there is no propagation of the wave. The dispersion diagram for the ultra-relativistic degenerate case (Fig. 3.18) is also the same as presented for the non-relativistic degenerate case.

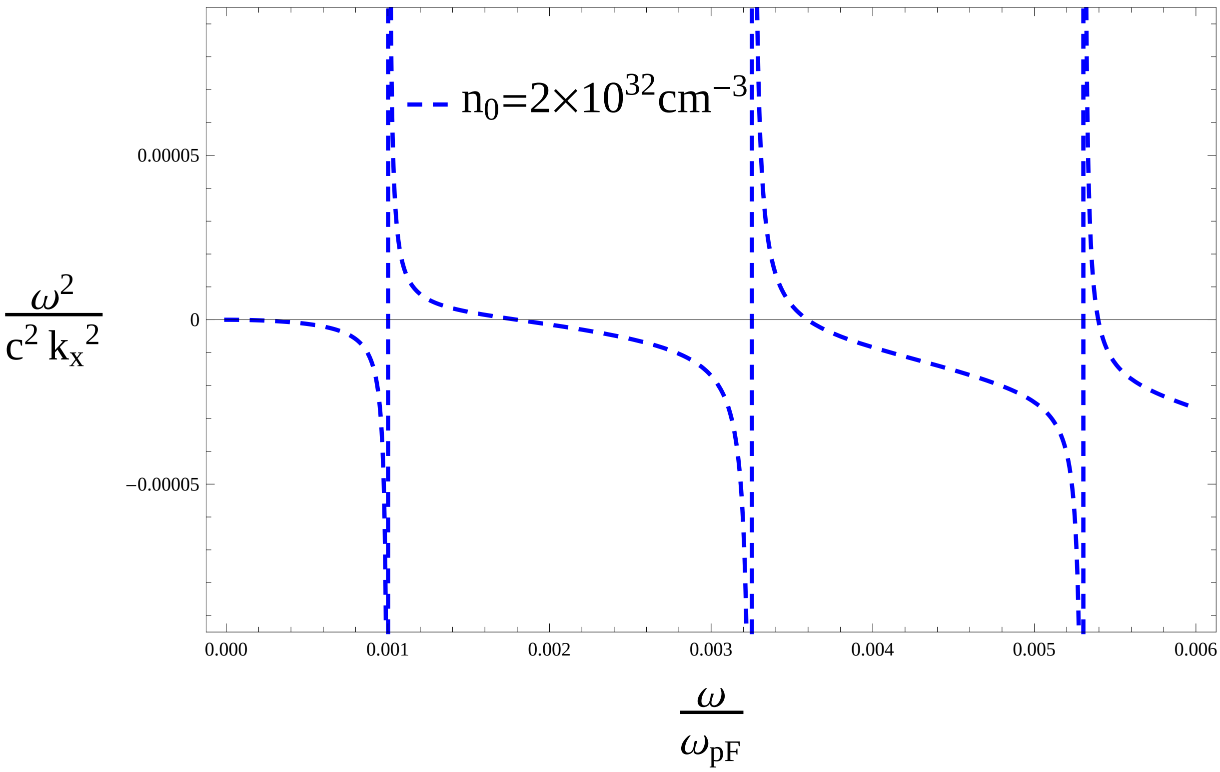

Figure 3.19: Plot between $\omega^2/c^2k_x^2$ and ω/ω_{pF} (ordinary wave in the ultra-relativistic degenerate plasma).

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