Ladder Operators, Inherent Algebra and Associated Coherent States of Entangled Bi-fermions

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Abstract

Ladder operators are the mathematical operators that have numerous applications in the field of quantum mechanics, considerably in the study of many-particle systems. The ladder operator approach is a very elegant way to deal with composite systems. The main focus in this thesis is on the quantum mechanical description of entangled bi-fermion forming a composite boson. Its composite behavior is studied using its ladder operators and comparing them to elementary bosonic operators. It is shown that entanglement between the constituents of coboson depends on the properties of cobosonic annihilation and creation operators. We focused on the entanglement in fermions and extracted their composite behavior. Moreover, a method to measure the level of entanglement is presented in this thesis. States of elementary particles i.e. pure, mixed states have been given a touch to develop a basic understanding. Separability criteria and Schmidt decomposition are also discussed. Whether the entangled bi-fermion will resemble a pure boson, depends on the correlation between the constituent particles. The inter-component entanglement determines the behavior of the composite boson. The effect of the Pauli exclusion principle, as the constituents particles in coboson are made of fermions, is also discussed. Having all the basic knowledge of composite bosons and behavior of their ladder operators the coherent states of these entangled bi-fermions are constructed. We derived coherent states as eigenstates of the annihilation operator of composite bosons. Moreover, the properties of composite bosonic operators like quadrature variance and Mendels Q-parameters are also discussed. In the end, the research work has been concluded.

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Chapter 1

Introduction

1.1 Historical Background

What exactly is light? In classical mechanics, we say that light is a wave or at least it behaves like a wave when we use it in certain experiments. For most of the 19th century, it seemed like the question had been settled. Physicists agreed: LIGHT IS A WAVE. But the new discoveries then made, started to question that. This started getting more and more clues that light could also behave like a particle which leads to the strange concept that light was both a particle and wave. This kick-started the development of a new theory called "Quantum Mechanics".

Quantum mechanics attempts to explain the behavior of subatomic particles at the nanoscopic level. It is one of the most successful branches of Physics. There are countless examples of scientific experiments confirming predictions made by the laws of quantum mechanics. In the early 19th century it was theorized that at the subatomic level energy can only be released and absorbed in discrete indivisible units called ”Quanta”. There is a famous experiment in quantum physics called ”double-slit experiment” which exposed something about particles that still surprises us today. Particles display both particle-like and wave-like behavior.

One of the basic disparity between Quantum and Classical Mechanics is that any physical variables which possess continuous values in classical mechanics for example angular momentum, energy, etc, can only have discrete or “quantized” values in quantum mechanics for example energy levels of electrons in atoms and the spin of elementary particles are all quantized, etc.

In the quantum world, we can not tell the characteristics of quantum particles unless we measure or observe them. Because before measurement or observation particles will be in all possible available states known as superposition state unless we observe or measure them. After measurement particle wave function collapses, as a result, we obtain only one state from all possible states and we perceive that the particle was present in this state already but this is not true. According to Bohr, the particle was in a superposition state before measurement and we do not know which state will come after measurement from all possible states. However, countless experiments have corroborated the results predicted by Quantum Mechanics.

1.2 Types of Particles

1.2.1 Distinguishable Particles

As the name suggests, distinguishable particles can be distinguished from one another. We can tell which is which particle in this case. They have an individualistic nature. Two particles may be identical yet treated as distinguishable as is done in classical statistical mechanics. Technically, such particles have thermal de Broglie wavelengths much smaller than average interparticle separation. This is what happens in the classical world. Even if we have two identical tennis balls we can physically distinguish. The Maxwell-Boltzmann Statistics works here. This doesn't hold in quantum mechanics. Two identical particles in quantum mechanics are indistinguishable in the sense that you can't tell which is which. Consider, for example, two electrons in the He atom. We can't talk about this electron or that electron. We only talk about an electron. There is no way to distinguish between the two electrons in the 1s orbital. Technically, the de Broglie wavelengths are of the order or larger than interparticle separations. Quantum statistical mechanics works here for counting distributions of systems with indistinguishable particles.

1.2.2 Indistinguishable (Identicle) particles

In quantum mechanics indistinguishable particles cannot be distinguished because we cannot label the particles as we do in classical mechanics The other reason is that we are uncertain about the particle trajectory due to Heisenberg uncertainty principle. When N-indistinguishable particles are mixed together, then we can not tell which particle have which coordinate. We can only specify probability of particles and that probability remain unchanged by interchanging coordinates, which means that particles may have symmetric or anti-symmetric wave-function under the interchange of particles. The particles will have symmetric wavefunction if they have integral spin, on the contrary, the wavefunction will be anti-symmetric if they have half integral spin. On the basis of symmetric and anti-symmetric wave-function we can divide the particles into two main classes, i.e. Bosons (symmetric wavefunction) and fermions (anti-symmetric wavefunction).

1.3 Statistics of the Elementary Particles and their Operators

1.3.1 Fermi-Dirac Statistics

Fermi-Dirac statistic is the statistics of indistinguishable (identical) particles. It applies to fermions; particles with half-integer spin, result of the half-integral spin of fermions is that this causes a restrain on the conduct of a system containing more then one fermion so it must obey the Pauli exclusion principle. i.e, no two fermions cannot exist in a single quantum state but only a single particle can occupy a single energy level. Let we have n_i number of particles and they are to be filled in E energy level then the appropriate probability distribution for the occupation of energy state E will be:

$$F_{FD}(E) = \frac{1}{\exp\frac{(E-E_F)}{KT}+1}, \tag{1.1}$$

The particles which follow fermi dirac statistics includes Protons, electrons, He^{3+} etc, they are collectively called fermions.

1.3.2 Bose-Einstein Statistics

Bose-Einstein statistics also belongs to indistinguishable (identical) particles, it applies to bosons; particles with integer spin. These are the particles which do not obey Pauli exclusion principle, so it is possible to have indefinate bosonic particles in same state. It means that unlimited number of particles can condense into a single level to form bose-einstein condensates. This behaviour gives rise to the remarkable properties of helium-4 when it is cooled to become a superfluid. The probability that the particle will have energy E is given by:

$$F_{BE}(E) = \frac{1}{Ae^{E/kT} - 1} \tag{1.2}$$

For Photons $A = 1$ so the occupation of very low energy states can increase without limit. The difference shown in the above equation arises due to the fact that particles are indistinguishable. Bosons includes mesons (e.g. pions), nuclei of even mass number (e.g, Helium-4) and the particles required to embody the field of Quantum optics (e.g. Photons and gluons).

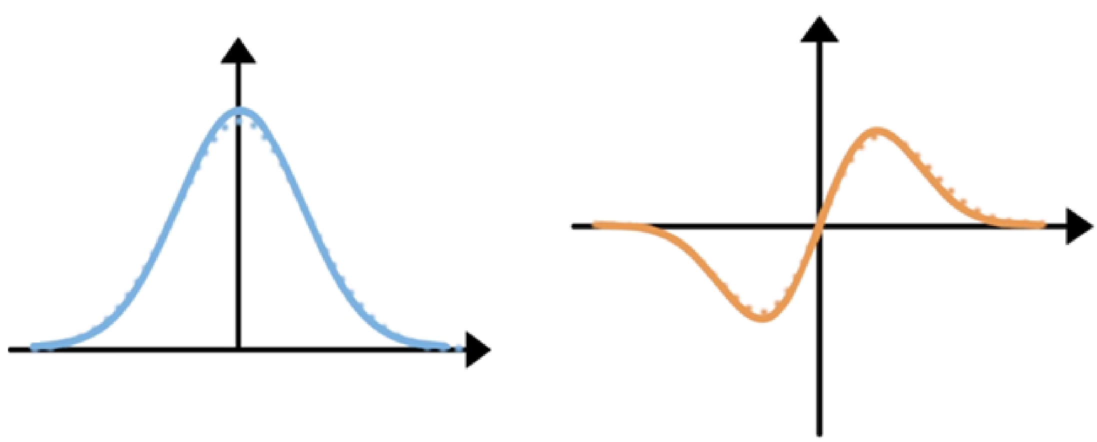

Figure 1.1: Symmetric and Anti-symmetric Wavefunction

1.3.3 Ladder operators for Bosons and Fermions

In quantum physics annihilation and creation operators, collectively called ladder operators, have numerous applications, considerably in the field of many-body physics. As the name suggests creation operator is used to add a particle in a system and annihilation operator, also called destruction operator, annihilates the particle from a field both operators are adjoint of each other.

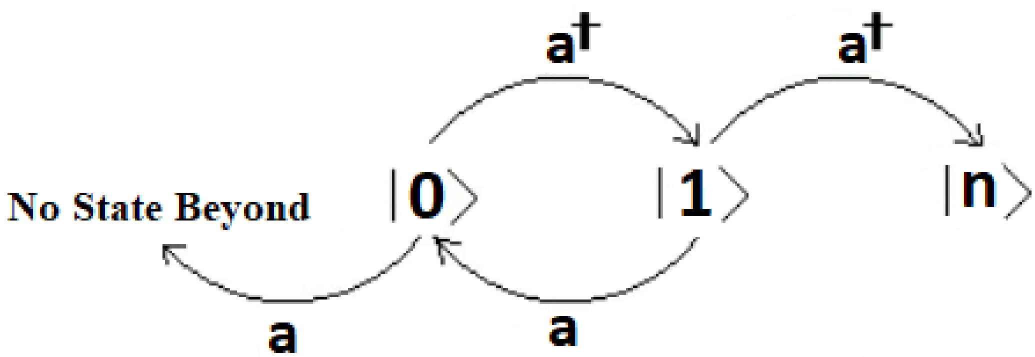

Figure 1.2: Illustration of creation and annihilation operator.

These operators follow different set of rules for different types of particles, as we shall see in the later section. The algebra of these operators for bosonic case is equal to the algebra of simple harmonic oscillator. The commutator of these operators of the same Boson state, equals one, all other commutators vanish. But, for the case of fermions the mathematics is different, involving anti-commutators instead of commutators. Electron states are acted upon by ladder operators in order to raise and annihilate electrons from the field.

Here we develop the concepts more generally, for both fermions and bosons. The importance of this new formalism is that it provides us a powerful way to deal with the symmetries of the states and also with operators for systems of many identical particles.

1.4 Entanglement

Entanglement is the purely quantum mechanical phenomenon and appears when we talk about the composite systems e.g, the system which is comprised by two or more then two subsystems. It is one of the unique aspect of quantum mechanics. The advantage of entanglement is taken in account for the purpose of quantum processing of information, its evaluation in quantum states is vital. In quantum mechanically entangled composite systems, the constituent subsystems have a strong correlation to each other even when they are spatially isolated such that they do not interact. The composite system can be regarded as a definite pure state, however this definition is not valid for the constituent subsystems states. Making a measurement on any of the subsystems will influence measurement on other sub-systems and this is the contradiction of local-realism, i.e. the quantum states of spatially isolated non-interacting particles are independent.

For the first time, this phenomenon was discussed by Einstein, Podolsky and Rosen in their seminal paper in 1935. They analyzed the incompatible measurements made about one subsystem of the two-particle composite system, which interacted previously but during measurement they are spatially separated. The contradiction pointed out by Einstein, made a question on the completeness of quantum mechanics and this problem was fixed in their need for local-realism. In 1960s, John S. Bell worked on the EPR argument and showed the correlation between measurements of entangled state predicted by quantum mechanics are out of scope for what local-realism based theories explain. The inequalities derived by Bell and others were experimentally tested for entangled photons and put a confirmation stamp on the predictions of quantum mechanics.

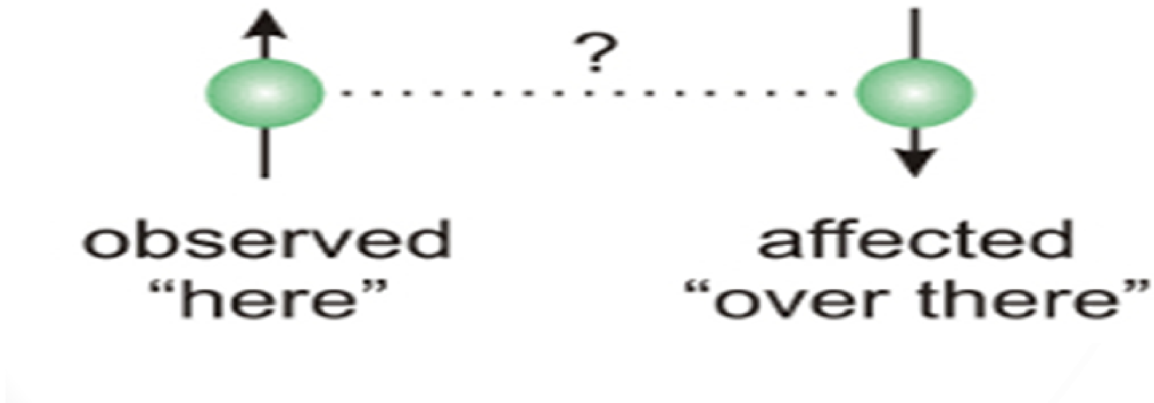

Figure 1.3: Illustration of Entanglement phenomena

Entanglement has been perceived as a phenomenon of no viable significance since its first appearance in 1935 until the mid of the 90s. With advances in quantum information science, entanglement has been seen as a resource for quantum information processing and communications. The applications of this vital resource include quantum cryptography, dense coding, teleportation of a quantum state and quantum algorithms that are faster than their classical counterparts.

1.5 Coherent/Glauber States

Coherent states are special kind of states of light. Back in 1926, Erwin Schrodinger successfully built such quantum states that were demonstrating physical behaviour very close to the classical one. He built such quantum mechanical states for the harmonic oscillator in which the uncertainty relation is minimized. A direct method of constructing generalized Glauber states for the degenerate spectrum of quantum mechanical systems, such as, hydrogen atom was given by Klauder in 1996 using ladder operator algebra.

One of the procedures concerning this was to generalize, fulfilling a set of requirements, any one of the coherent states definition given by Glauber, i.e., the generalization should preserve some properties of harmonic oscillator's coherent states. Ladder operators algebra is used for constructing coherent states of the system in this kind of generalization techniques. The first breakthrough in this regard was the development of a formalism relating quantum and classical dynamics, by Klauder in 1963. Klauder and Skagerstam organized the study on generalized coherent states in the form of a book.

1.6 Outline of the Thesis

The following chapter deals with basic concepts which provide us the necessary background for our work, this includes the discussion of Hilbert space, which is crucial in representing quantum mechanical state of a system. Having the concept of Hilbert space, we discussed the quantum mechanical states and its types. In the second part of this chapter a brief description regarding composite systems and entanglement in such systems is discussed in terms of Schmidt decomposition.

Chapter 3 deals with how fundamental particles can be added or subtracted in a Quantum state in terms of bosonic and fermionic creation and annihilation operator it also explains how are bosons and fermions, two fundamental type of particles, differ from each other. In the next part, the concept of entanglement between the constituent particles of a entangled bi-fermionic composite system is discussed. We found that the overall composite behavior is closely related to the quantum entanglement. With the help of creation and annihilation operators, the bosonic attribute are presented which will appear if the constituent particles become entangled. Basically, we showed that the entanglement between two fermions largely ascertain the extent to which the pair behaves like an elementary boson.

In Chapter 4, we discussed the Photonic Coherent states and its properties. In the next part we derived the coherent states of cobosons as an eigen states of cobosonic annihilation operator, then we estimated the resemblance between photonic coherent states and the coherent states of entangled bi-fermionic composite bosons using the measure of non-classicality, i.e, Quadrature Variance and Mandels Q- parameter. At the end our work have been concluded.

Chapter 2

Preliminaries

2.1 Hilbert Space of Composite System

Composite system is the one which is made up of two or more subsystems. While dealing with just one particle its state can be written by using a single particle Hilbert space but when dealing with the composite system it involves tensor product of all the subsystems of which the whole system is comprised. Consider a bipartite composite system in which state of one subsystem $|\alpha\rangle$ belongs to the hilbert space H_1, similarly state of the second subsystem $|\beta\rangle$ belongs to the Hilbert space H_2 then the overall state of a bipartite system is written as:

$$|\gamma\rangle = |\alpha\rangle \otimes |\beta\rangle , \tag{2.1}$$

here $|\gamma\rangle$ is the state of composite system belonging to Hilbert space H such that,

$$H = H_1 \otimes H_2. \tag{2.2}$$

Now, let's construct the basis of the Hilbert space of the composite system described above. Let the basis belonging to Hilbert space H_1 is $\{|i_n\rangle\}$, similerly the basis basis which belongs to Hilbert space H_2 is $\{|j_n\rangle\}$ then the basis belonging to the composite system Hilbert space will be given by:

$$\{|k_n\rangle\} = \{|i_n\rangle\} \otimes \{|j_n\rangle\}, \tag{2.3}$$

such that,

$$|\alpha\rangle \otimes |\beta\rangle = |\beta\rangle \otimes |\alpha\rangle \,. \tag{2.4}$$

The inner product of the states of two composite systems is given by:

$$\langle\gamma_1|\gamma_2\rangle = (|\alpha_1\rangle \otimes |\beta_1\rangle) \,.\, (|\alpha_2\rangle \otimes |\beta_2\rangle), \tag{2.5}$$

$$\langle\gamma_1|\gamma_2\rangle = \langle\alpha_1|\alpha_2\rangle \langle\beta_1|\beta_2\rangle \,, \tag{2.6}$$

such that,

$$|\gamma_1\rangle = |\alpha_1\rangle \otimes |\beta_1\rangle \,, \tag{2.7}$$

and

$$|\gamma_2\rangle = |\alpha_2\rangle \otimes |\beta_2\rangle \,. \tag{2.8}$$

2.2 Pure and Mixed States

Let's consider the state of polarization of a photon. We can write a general state of polarization as following type of expression;

$$|\psi\rangle = a\,|H\rangle + b\,|V\rangle \tag{2.9}$$

Here, $|H\rangle$ represents a horizontally polarized photon state and $|V\rangle$ represents a vertically polarized photon state. a and b are complex numbers.

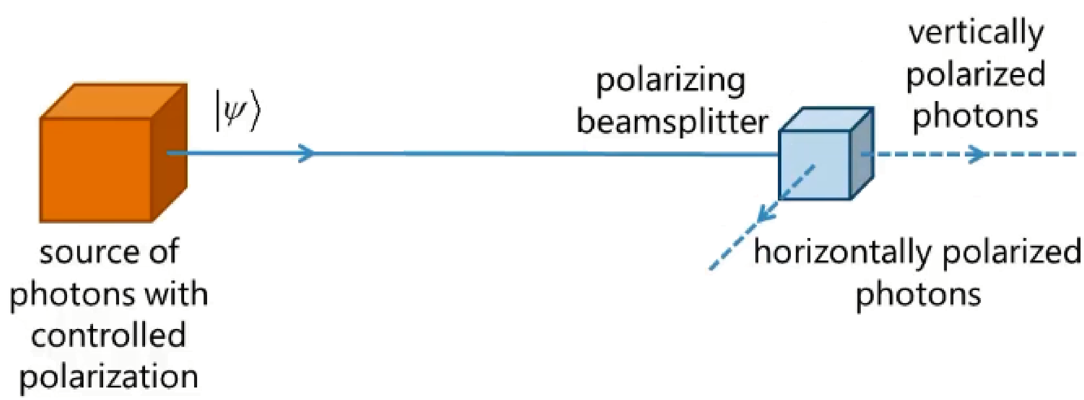

Figure 2.1: Polarization of Photons

Suppose we are going to measure the above state such that we have a source of photons with controlled polarization as shown in the figure 2.2 Specifically it's going to produce the above polarization state and we can then use a polarizing beam splitter that separate horizontal and vertical polarization Two different outputs with different detectors with apparatus like this we can perform a quantum mechanical measurement and put this state that we have we expect the following probabilities,
$|a|^2$ of measuring horizontal polarization.
$|b|^2$ of measuring vertical polarization.

Since we must have, by normalization condition;

$$|a|^2 + |b|^2 = 1 \tag{2.10}$$

We can also write,

$$a = cos\ \theta \tag{2.11}$$

$$b = exp\ (\iota\delta)\ \ sin\ \theta \tag{2.12}$$

$\theta = 0$ corresponds to linear polarization.
When $\theta \neq 0$, the field is in general elliptically polarized, which is the most general possible state of polarization. Also, $\delta = \pm\frac{\pi}{2}$ with $\theta = 45°$ gives right and left circular polarization.

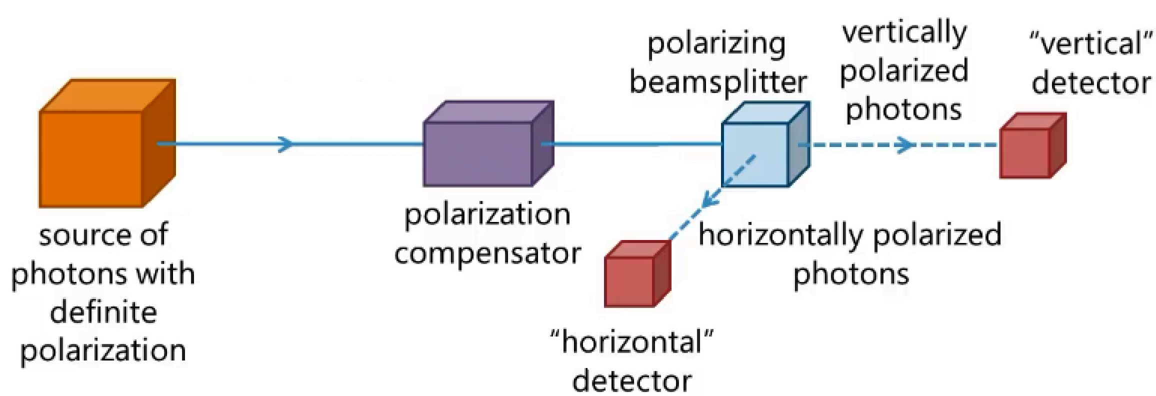

Figure 2.2: Polarization of Photons with a compensator

Now to allow the passage of a photon of any specific polarization we will build a polarizing filter which is also called a compensator the filter is built in such a way that it can polarized in a specific direction 100 percent of the time. We could arrange to delay only the horizontal polarization by a compensating amount $-\delta$ to make the photons linearly polarized. After that the polarization apparatus will be rotated by the angle θ so that the photons will always pass through the vertical detector. When we make a polarization filter or compensator so we get hundred percent of the photons to one detector we say that photons are in pure state as shown in figure 2.3.

All states considered so far has been pure States. A compensator could be made to pass any particles in any one specific quantum mechanical state with 100 percent efficiency to one detector. Suppose we have a beam that is a mixture of two different independent lasers 1 and 2 as shown in figure 2.4 we also have a non-polarizing beam splitter, it take some portion of photon from laser 1 and some from laser 2 passes through without changing the polarization states of either of these beams. We assume that laser 1 contributes a fraction P1 of the photons and Laser 2 contributes the fraction P2 the probability that a given Photon is from laser one is P1 and similarly there is probability P2 that the photon is from laser 2. We also assume that these two lasers give uncorrelated photons of two possible different polarization States $\psi 1$ and $\psi 2$ respectively.

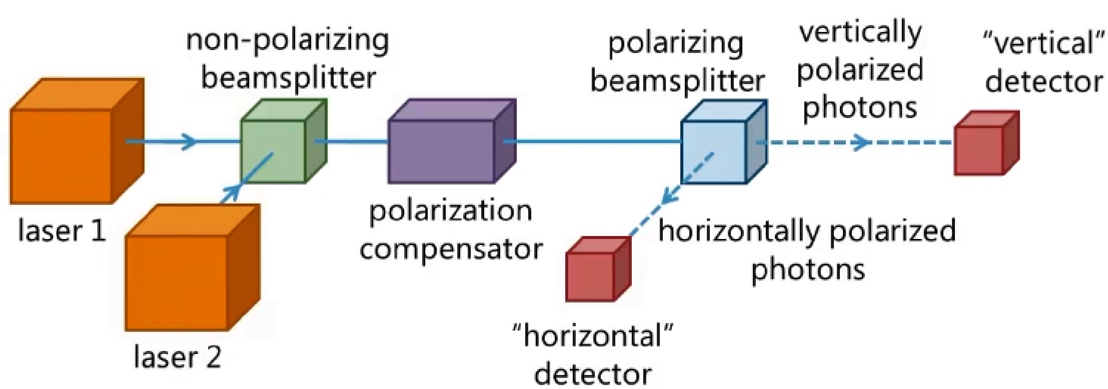

Figure 2.3: Polarization of Photons with two laser beams

There is now no single setting of the compensator that in general will pass all the photons from both lasers to the vertical detector hence we cannot simply write this state as some linear combination as in equation 2.9 of the two different polarization States. If you are able to do that we could construct a polarizing filter to pass hundred percent of the photons so the state of these photons is described differently as a mixed state.

2.3 Density Operator and Density Matrix

Some systems in quantum mechanics are completely described by state vector in such representation state vector contain all the information about the system. There is also an alternative and more general approach analogous to the state vector approach to describe a system, called density operator or density matrix. This is more easy way to thinking of some commonly encountered scenarios in quantum mechanics. In this section we will explain briefly about general properties and applications of density operator.

Pure States' Density Matrix

If a state of system is known exactly then it is called pure state of quantum system and the density operator for the pure state is given by

$$\rho = |\psi\rangle \langle\psi| \tag{2.13}$$

where $|\psi\rangle$ is a state of quantum system which can be represented as linear superposition of the basis vector $|n\rangle$ as $|\Psi\rangle=\sum_n c_n |n\rangle$. The density operator for this state is,

$$\rho = \sum_n \sum_m c_n c_m^* \langle n|m\rangle = \sum_{n,m} \rho_{nm} \langle n|m\rangle \tag{2.14}$$

where $\rho_{nm} = \langle n|\Psi\rangle \langle\psi|m\rangle = \langle n|m\rangle$ are the matrix element of density operator for the pure state.When we perform the measurement on state $|\psi\rangle$ then the probability

of getting the state $|n\rangle$ is $|c_n|^2$ .This provides the physical meaning to the diagonal elements of the density operator that is diagonal elements are necessarily non negative and hence it is positive operator.
The density matrix for the pure states have the following properties:

$$Tr(\rho) = \sum_n \rho_{nn} = \sum_n |c_n^2| = 1 \tag{2.15}$$

Since $\rho^2 = |\Psi\rangle \langle\Psi|\Psi\rangle \langle\Psi| = |\Psi\rangle \langle\Psi| = \rho$, therefore above equation becomes[3]

$$Tr(\rho^2 = 1). \tag{2.16}$$

General properties of the density operator

If the given operator satisfying the following given properties then it is said to be the valid density operator.

- The trace of the given density operator must be one. i.e Tr(ρ) = 1.
- It is always hermitian i.e $\rho^\dagger = \rho$.
- For all the given state $|\Psi\rangle$, the density matrix is positive i.e $|\Psi\rangle \rho \langle\Psi| \geq 0$.This follows from

$$\langle\psi| \rho |\psi\rangle = \sum_j P_j| \langle\psi|\psi_j\rangle |^2 \geq 0 \tag{2.17}$$

  We earlier said that the density operator is hermitian, it means that the eigenvalues of the given operators will be greater then or equal to zero, that's why the given density operator is positive.

2.4 Entanglement Detection

Initially entanglement was considered as qualitative feature of quantum theory but the development of Bell's inequalities in 1964 , made this distinction quantitative. In early years of the development, entanglement was considered as strange phenomenon but nowadays it is the resource of quantum information processing, enabling tasks like quantum cryptography, quantum computation, quantum teleportation, dense coding etc.

2.4.1 Schmidt Decomposition

Quantum systems comprised of interacting subsystems become highly correlated and their individual identities become entangled. This entanglement can be described using the Schmidt decomposition, in which a pair of preferred orthonormal bases can be constructed to emphasize the tight correlations between two quantum subsystems. It is one of the most important tools for analyzing bipartite pure states in quantum information theory. The Schmidt decomposition shows that it is possible to decompose any pure bipartite state as a superposition of corresponding states.
For any pure state $|\psi\rangle$ of a bipartite system $A \otimes B$ there exist orthonormal sets of states $|\phi\rangle_A^i$ and $|\phi\rangle_B^i$ for subsystems A and B respectively, such that:

$$|\psi\rangle = \sum_i \sqrt{\lambda_i}\, |\phi\rangle_A^i \otimes |\phi\rangle_B^i \tag{2.18}$$

where, $|\phi\rangle_A^i$ and $|\phi\rangle_B^i$ are known as Schmidt basis belonging to system A and B respectively. (for details see appendix 5.1)
The expansion coefficients λ_i are non-negative real numbers known as Schmidt Coefficient and for normalised state $|\psi\rangle$ we must have:

$$\sum_i \lambda_i^2 = 1$$

The expansion (1.1) is known as Schmidt decomposition.
Schmidt Decomposition is very important to detect the entanglement between composite systems.The Schmidt coefficients in eq(1.1) plays vital role to detect the entanglement.In order to calculate the Schmidt coefficients first we will find density matrix and then perform the partial trace on the one of sub system by fixing the other.

$$\mathrm{tr}(B) = |\psi\rangle\langle\psi|$$

The above matrix have eigenvalues λ_i^2.

The Schmidt number has nonzero eigenvalues λ_i and used to show entanglement. i.e, If a state is separable, then the Schmidt number is 1 and if a state is entangled, then the Schmidt number is > 1. Schmidt number only tell us that weather the state is entangled or not and unable to answer the strength of the entanglement i.e weather it is maximally entangled or minimum.That is why Schmidt measure is crude measure of entanglement. Schmidt decomposition is applicable only for distinguishable particles and provide wrong results for identical particles i.e nonzero Von Neumann entropy and the existence of entanglement for uncorrelated fermions.

Chapter 3

Ladder Operators and Inherent Algebra

3.1 Introduction

In this chapter, we are primarily interested in analyzing the algebra of ladder operators for fermions, bosons and composite bosons composed of fermionic constituents. In quantum physics annihilation and creation operators, collectively called ladder operators, have numerous applications, considerably in the field of many-body physics. As the name suggests creation operator is used to add a particle in a system and annihilation operator, also called destruction operator, annihilates the particle from a field both operators are adjoint of each other.

These operators follow different set of rules for different types of particles, as we shall see in the later section. Electron states of an acted upon by ladder operators in order to raise and annihilate electrons from the field the algebra of these operators for bosonic case is equal and to the algebra of simple harmonic oscillator. The commutator of these operators of the same Boson state, equals one, all other commutators vanish. But, for the case of fermions the mathematics is different, involving anti-commutators instead of commutators.

Here we develop the concepts more generally, for both fermions and bosons. The importance of this new formalism is that it provides us a powerful way to deal with the symmetries of the states and also with operators for systems of many identical particles.

We will also see that how much a composite particle behaves as a pure boson depends on the degree of entanglement between constituent particles using operator algebra. It suggests that quantum entanglement is some how the reason for the bonding between constituent particles. It is not necessary to have mechanical binding forces between particles they just provides physical means to execute quantum correlations.

3.2 Elementry Bosonic Algebra

To grab the idea of creation and annihilation operators of elementry bosons let's take an explicit example, consider we have a potential well, V(x), and single particle eigenstates $\psi_0, \psi_1, ...$ Now suppose we have a system of m (identical) bosons and all m bosons are in the lowest level, $|m\rangle$ let this state be $|m\rangle$ and we assume that $|m\rangle$ is normalized i.e,

$$\langle m|m\rangle = 1,$$

where, $m = 0, 1, 2, 3, ...$ and $|0\rangle$ is a state with no particle.

We define annihilation and creation operator for bosons as,

$$\hat{a}_b\,|m\rangle = \sqrt{m}\,|m-1\rangle \tag{3.1}$$

$$\hat{a}_b^\dagger\,|m\rangle = \sqrt{m+1}\,|m+1\rangle \tag{3.2}$$

These operators are basically used to destroy or create a particle to the system, in the state ψ_0. Now consider the commutator,

$$[\hat{a}_b, \hat{a}_b^\dagger]\,|m\rangle = (\hat{a}_b\hat{a}_b^\dagger - \hat{a}_b^\dagger\hat{a}_b)\,|m\rangle \tag{3.3}$$

$$= ((m+1) - m)\,|m\rangle = |m\rangle \tag{3.4}$$

Therefore, we can say that $[\hat{a}_b, \hat{a}_b^\dagger] = 1$. Now using these operators, we can also write m-particle state in term of the vacuum state as,

$$|m\rangle = \frac{(\hat{a}_b^\dagger)^m}{\sqrt{m!}}\,|0\rangle\,. \tag{3.5}$$

Here, $\hat{a}_b$ is a hermitian conjugate of $\hat{a}_b^\dagger$. It can be proved as following,

$$\langle m+1|\,\hat{a}_b^\dagger\,|m\rangle = \sqrt{m+1}, \tag{3.6}$$

therefore,

$$\langle m+1| \hat{a}_b^\dagger = \sqrt{m+1} \ \langle m| , \tag{3.7}$$

or

$$\langle m| \hat{a}_b^\dagger = \sqrt{m} \ \langle m-1| , \tag{3.8}$$

similarly, $\hat{a}_b$ also acts as a creation operator if it acts to the left side,

$$\langle m| \hat{a}_b = \sqrt{m+1} \ \langle m+1| , \tag{3.9}$$

Now consider two level system for bosons. Let two levels be ψ_0 and ψ_1 and the state $|m_0, m_1\rangle$ have m_0 bosons in ψ_0 and m_1 boson in ψ_1 According to the above definition we can write,

$$\hat{a}_b |m_0, m_1\rangle = \sqrt{m_0} |m_0 - 1, m_1\rangle , \tag{3.10}$$

$$\hat{a}_b^\dagger |m_0, m_1\rangle = \sqrt{m_0 + 1} |m_0 + 1, m_1\rangle , \tag{3.11}$$

and also,

$$\hat{a}_b |m_0, m_1\rangle = \sqrt{m_1} |m_0, m_1 - 1\rangle , \tag{3.12}$$

$$\hat{a}_b^\dagger |m_0, m_1\rangle = \sqrt{m_1 + 1} |m_0, m_1 + 1\rangle , \tag{3.13}$$

In addition to the commutation relations explained before, we can say that for different levels creation and annihilation operators commutes with each other:

$$[\hat{a}_{b0}, \hat{a}_{b1}] = 0, \tag{3.14}$$

$$\tag{3.15}$$

same is the case with creation operator.
We can generalize these results to spaces with an arbitrary number of single particle states. So, let us have vector in that space. Commutation relations are given by,

$$[\hat{a}_{bi}, \hat{a}_{bj}^\dagger] = \delta_{ij}, \tag{3.16}$$

$$\tag{3.17}$$

$$[\hat{a}_{bi}, \hat{a}_{bj}] = [\hat{a}_{bi}^\dagger, \hat{a}_{bj}^\dagger] = 0, \tag{3.18}$$

$$|m_0, m_1, ...\rangle = ... \frac{(a_0^{\dagger})^{m_0}}{\sqrt{m_0!}} \frac{(a_1^{\dagger})^{m_1}}{\sqrt{m_1!}} |0\rangle . \tag{3.19}$$

Above result show the Fock space representation of single particle state for bosons. Now we explicitly explain the formalism to add or subtract a particle in fermionic state.

3.3 Elementry Fermionic Algebra

For fermionic particles the creation and annihilation operators are given by,

$$\hat{a_f} |1\rangle = |0\rangle , \tag{3.20}$$

$$\hat{a_f} |0\rangle = 0, \tag{3.21}$$

$$\hat{a_f}^{\dagger} |1\rangle = 0, \tag{3.22}$$

$$\hat{a_f}^{\dagger} |1\rangle = |1\rangle . \tag{3.23}$$

If we consider the anticommutators, they will be,

$$\{\hat{a_f}, \hat{a_f}^{\dagger}\} |1\rangle = \{\hat{a_f}\hat{a_f}^{\dagger} + \hat{a_f}^{\dagger}\hat{a_f}\} |1\rangle = |1\rangle , \tag{3.24}$$

$$\{\hat{a_f}, \hat{a_f}^{\dagger}\} |0\rangle = |0\rangle . \tag{3.25}$$

Therefore,

$$\{\hat{a_f}, \hat{a_f}^{\dagger}\} = 1, \tag{3.26}$$

where,

$$\{\hat{a_f}, \hat{a_f}\} = 0, \tag{3.27}$$

and

$$\{\hat{a_f}^{\dagger}, \hat{a_f}^{\dagger}\} = 0. \tag{3.28}$$

For the case of fermionic two level system, we can have four possible states, which are: $|0,0\rangle , |0,1\rangle , |1,0\rangle$ and $|1,1\rangle$. The ladder operators for such systems are given by,

$$\hat{a_{f_0}}^{\dagger} |0,0\rangle = |0,0\rangle , \tag{3.29}$$

$$\hat{a_{f0}}^{\dagger} |0, 1\rangle = |1, 1\rangle , \tag{3.30}$$

$$\hat{a_{f0}}^{\dagger} |1, 0\rangle = \hat{a_{f0}}^{\dagger} |1, 1\rangle = 0, \tag{3.31}$$

$$\hat{a_{f0}} |0, 0\rangle = \hat{a_{f0}} |0, 1\rangle = 0, \tag{3.32}$$

$$\hat{a_{f0}} |1, 0\rangle = |0, 0\rangle , \tag{3.33}$$

$$\hat{a_{f0}} |1, 1\rangle = |0, 1\rangle , \tag{3.34}$$

also,

$$\hat{a_{f_1}}^{\dagger} |0, 0\rangle = |0, 1\rangle , \tag{3.35}$$

$$\hat{a_{f_1}}^{\dagger} |1, 0\rangle = |1, 1\rangle , \tag{3.36}$$

$$\hat{a_{f_1}}^{\dagger} |0, 1\rangle = \hat{a_{f0}}^{\dagger} |1, 1\rangle = 0, \tag{3.37}$$

$$\hat{a_{f_1}} |1, 1\rangle = |0, 1\rangle , \tag{3.38}$$

$$\hat{a_{f_1}} |0, 1\rangle = |0, 1\rangle , \tag{3.39}$$

$$\hat{a_{f_1}} |1, 0\rangle = \hat{a_{f_1}} |0, 1\rangle = 0. \tag{3.40}$$

The anticommutation relations for such two level systems is given by,

$$\{\hat{a_{f_0}}, \hat{a_{f_0}}^{\dagger}\} = \{\hat{a_{f_1}}, \hat{a_{f_1}}^{\dagger}\} = 1, \tag{3.41}$$

and

$$\{\hat{a_{f_0}}, \hat{a_{f_1}}^{\dagger}\} = \{\hat{a_{f_1}}, \hat{a_{f_0}}^{\dagger}\} = 0. \tag{3.42}$$

The above results can be generalized as:

$$\{\hat{a_{f_i}}, \hat{a_{f_j}}^{\dagger}\} = \delta_{ij} = 0, \tag{3.43}$$

$$\{\hat{a_{f_i}}, \hat{a_{f_j}}\} = \{\hat{a_{f_i}}^{\dagger}, \hat{a_{f_j}}^{\dagger}\} = 0, \tag{3.44}$$

$$|m_0, m_1, ...\rangle = ...(\hat{a_{f_1}}^{\dagger})^{m_1} \ (\hat{a_{f_0}}^{\dagger})^{m_0} |0\rangle . \tag{3.45}$$

After complete understanding of how particles behave as bosons or fermions, now we are in a position to understand the concept of composite particle.

3.4 Bi-partite Composite System

As discussed earlier, there are two types of elementary particles, fermions and bosons. Matter is generally composed of fermions and composite bosons, however, elementary bosons are mostly the exchange particles of the fields. To form a composite bi-particle there can have three possibilities. Either its composed of two bosons, or one boson and one fermion forming a composite fermion or it can be composed of two fermions forming a composite boson. The composite bosons made up of two fermions exists widely in nature like Hydrogen atom, Cooper pair etc, so we discussed bi-fermionic composite bosons. Composite system is the one that naturally decomposes into two or more subsystems. A composite system can be represented by:

$$H = H_1 \otimes H_2 \otimes H_3 \otimes ... \otimes H_n \tag{3.46}$$

We consider a bi-partite system (system having two subsystems)

$$H = H_1 \otimes H_2 \tag{3.47}$$

The composite system of both subsystems can be written as a direct product of two spaces $|\phi_i\rangle \otimes |\phi_j\rangle$ as:

$$|\psi\rangle = \sum_{ij} d_{ij} |\phi_i\rangle \otimes |\phi_j\rangle \tag{3.48}$$

where d_{ij} is the expansion coefficient.

$$d_{ij} = \sum_{ij} \langle\phi_i| \otimes \langle\phi_i|\psi\rangle \tag{3.49}$$

(For details see appendix 5.1).

Suppose a composite particle C which is made up of two elementary fermions A and B. Let the wavefunction of two particles is $\psi(m_A, n_B)$. Expressing this wave function in terms of Schmidt decomposition:

$$\psi(m_A, n_B) = \sum_{N=o}^{\infty} \sqrt{\lambda_N} \phi_N(m_A).\phi_N(m_B) \tag{3.50}$$

where $\phi_N(m_A)$ and $\phi_N(n_B)$ are schmidt modes. $\phi_N(m_A)$ make an orthonormal complete set for particle A. $\phi_N(m_A)$ and λ_N are defined by eigen vectors and reduced density matrix eigen values of particle A. 3.50 shows the pairing structure using quantum correlation. If A particle shows up in mode $\phi_N(m_A)$ then B will be in mode $\phi_M(m_B)$ with certainty.

3.5 Measure of Entanglement

The measure of entanglement is provided by the distribution of λ_M. Which is generally discussed using entanglement entropy, given by:

$$E = -\sum_M \lambda_M log_2 \lambda_M \tag{3.51}$$

Using Schmidt number κ we can have a more explicit way of measuring the entanglement i.e, by counting the mean number of schmidt modes that are involved actively. It is given by;

$$\kappa = \frac{1}{\sum_{M=0}^{\infty} \lambda_M^2} \tag{3.52}$$

Greater the value of κ higher will be the entanglement. If there is only one term in the Schimdt decomposition the state would be disentangled (product), this corresponds to $\kappa = 1$ If there are r terms (of equal weight) present in schmidt decomposition i.e

$$\lambda_M = \frac{1}{r}$$

this corresponds to

$$\kappa = r$$

which shows the exact number of mode pairs present.

3.6 Operator Algebra for Entangled Bi-fermionic Composite Boson

3.6.1 Creation/Annihilation operators

The state $\psi(m_A, n_B)$ in equation 3.9 is generated when $c^\dagger$ acts on vacuum.

$$c^\dagger = \sum_{M=0}^{\infty} \sqrt{\lambda_M} a_M^\dagger b_M^\dagger \tag{3.53}$$

where $a_M^\dagger$, $b_M^\dagger$ and $c_M^\dagger$ are the creation operators for particle A, B and C respectively. similarly,

$$c = \sum_{M=0}^{\infty} \sqrt{\lambda_M} b_M a_M \tag{3.54}$$

where a_M, b_M and c_M are the annihilation operators for particle$A$, B and C respectively.

3.6.2 Commutation relation

Now we will find commutation relation between c and $c^\dagger$

Derivation (For details see appendix (5.2))

$$[c, c^\dagger] = cc^\dagger - c^\dagger c \tag{3.55}$$

$$[c, c^\dagger] = \sum_{M=0}^{\infty} (\sqrt{\lambda_M} b_M a_M)(\sqrt{\lambda_M} a_M^\dagger b_M^\dagger) - \sum_{M=0}^{\infty} (\sqrt{\lambda_M} a_M^\dagger b_M^\dagger)(\sqrt{\lambda_M} b_M a_M)$$

$$[c, c^\dagger] = 1 - \triangle \tag{3.56}$$

where,

$$\triangle = \sum_{M=0}^{\infty} \lambda_M (a_M^\dagger a_M + b_M^\dagger b_M) \tag{3.57}$$

3.7 Properties of creation and annihilation operator for Entangled Bi-fermionic Coboson

Now we will investigate the properties of annihilation and creation operators. N-particle state for a system having two or more N-Particles is:

$$|M\rangle = \chi_M^{1/2} \frac{c^{\dagger M}}{\sqrt{M!}} |0\rangle \tag{3.58}$$

[for detailed calculations see Appendix 5.3]

$$\langle M|M\rangle = 1 \tag{3.59}$$

$$c^\dagger |0\rangle = |1\rangle$$

The action of c on $|M\rangle$ would be:

$$|M\rangle = \alpha_M \sqrt{M}\,|M-1\rangle + |\epsilon_M\rangle \tag{3.60}$$

where $|\epsilon_M\rangle$ is the correction term. c would be bosonic if it follows the following properties:

- $$\alpha_M \longrightarrow 1 \tag{3.61}$$

- $$\langle\epsilon_M|\epsilon_M\rangle \longrightarrow 0 \tag{3.62}$$

Therefore, the condition mentioned in equations 3.61 and 3.62 can be controlled by ratio of the normalisation constant which we will discuss in the next section. [Detailed calculations for α_M and $\langle\epsilon_M|\epsilon_M\rangle$ see appendix 5.4]

3.8 Normalization ratios for composite systems

For fermionic constituents normalization constant comes out to be: (For detailed calculations see Appendix 5.3)

$$\chi_M^F \;=\; M! \sum_{P_M > P_{M-1} > \ldots P_2 > P_1} \lambda_{P_1},\ \lambda_{P_2},\ \lambda_{P_3}, \ldots, \lambda_{P_M}. \tag{3.63}$$

Now we have a case of bi-particle wave function which permits exact close-form expressions of χ_M. This wave function is specified by the Schmidt eigenvalues:

$$\lambda_M \;=\; (1-x^2)\, x^{2M}, \qquad\qquad M = 0, 1, 2, \ldots \tag{3.64}$$

where x is defined in the range $0 < x < 1$.
(For detailed calculations see Appendix 5.5)
The normalization constants for fermions is,

$$\chi_M^F \;=\; \frac{M!\,(x)^{M(M-1)/2}(1-x)^M}{(1-x)(1-x^2)\ldots(1-x)^M} \qquad (For\ Fermion) \tag{3.65}$$

and hence the normalization ratios will be,

$$\frac{\chi^F_{M+1}}{\chi^F_M} = z^M \frac{(M+1)(1-x)}{(1-x)^{M+1}} \tag{3.66}$$

The above results show that;

$$\frac{\chi^F_{M+1}}{\chi^F_M} < 1 \tag{3.67}$$

The difference between fermionic and bosonic constituents is that bosons can be together within a same state, but fermions cannot do that due to Pauli exclusion principle. A larger number of particle would will require $x \simeq 1$ so that normailization ratio is maintained.

3.9 Calculating Schmidt Number

Now we are in a position for making an explicit link between schmidt number and quantum entanglement. The Schmidt eigen values are given as,

$$\lambda_M = (1-x)\, x^M \tag{3.68}$$

For these 3.68 eighen values the schmidt number given in 3.52 will become,

$$\kappa = \frac{1}{\sum_{M=0}^{\infty} \lambda_M^2} \tag{3.69}$$

$$= \frac{1}{(1-x)^2 \sum_{M=0}^{\infty} x^{2M}} \tag{3.70}$$

Applying the Power series,

$$\sum_{M=0}^{\infty} x^{2M} = \frac{1}{(1-x)^2}$$

So, Schmidt number will take the form,

$$\kappa = \frac{1-x^2}{(1-x)^2} \tag{3.71}$$

It is a monotonic function which increases in the range $0 < x < 1$. By writing x in terms of κ the normalization ratios given in 3.67 is explicitly connected to the degree of entanglement. As the schmidt number increases the normalization ratios approaches 1 for the whole range of κ. As we have discussed above, the effective number of schmidt modes is given by the value of schmidt number. The particle will be considered bosonic when the number of Schmidt modes involved is a lot higher than the total number of over all composite particles.

3.9.1 Effect of Pauli-Exclusion Principle on Coboson composed of Fermionic Constituents

As discussed in the previous section the normalization ratio is very important in determining the entanglement, the explicit relation for χ_M ratio for fermionic constituents is,

$$\frac{\chi_{M+1}}{\chi_M} = 1 - \frac{M}{\kappa}. \tag{3.72}$$

We can see from equation (3.72) that the χ_M ratio depends on the number of cobosons and schmidt number which is also the effective number of schmidt modes. The χ_M ratio approaches to 1 if $\frac{M}{\kappa}$ approaches to zero and it will only happen if the number of cobosons are much smaller than the active number of schmidt modes. It means that when we include more and more particles in the system, its deviation from ideal bosonic behaviour will increase. It is understandable as the concequence of Pauli Exclusion principle between fermionic constituents.

The quantum nature of composite bosons is way more subtle than the elementary composite bosons. This intricacy lies in the fact that we cannot associate a specific pair of fermions to a composite boson. That indistinguishability of fermions leads to the exchanges between composite bosons, those exchanges produce the dimensionless "Pauli scatterings" of the composite boson many-body formalism. The Pauli exclusion principle invokes the "moth-eaten effect", if composite boson is made up of fermionic constituents, that specifically inhibit stacking up greater number of composite bosons than the number of fermion-pair states with which the composite bosons are made of [34]. The Pauli exclusion principle is basically quenchless due to which the moth-eaten effect produced is incredibly strong and appears in all problems which includes composite bosons.

Chapter 4

Coherent States

4.1 Introduction to Coherent States

The coherent state was discovered by Schrödinger in the year 1926 and then were restudied by Glauber, Sudarshan and Klauder at the beginning of the 1960s. When passed from a beam splitter these states are separable and alse are pure states. The fact that coherent states are significant criterion for the study of non-classicality in the quantum optics serves as important motivation for doing research on it. Preparing coherent states of entangled bi-fermionic coboson will give new ideas for analyzing composite boson statistics.

In this chapter, we will estimate the resemblance between the eigen state that we will derive for elementary bosonic annihilation operator and the usual coherent state of entangled bi-fermionic coboson using the measure of non-classicality i.e Quadrature Variance and Mandels Q- parameter. Then we will show that eigen state of annihilation operator of coboson is useful in estimating the eigen value of number operator of composite boson.

4.2 Photonic Coherent States and its Properties

The coherent states consists of indefinite photon number that permits their phase to be more precise than the number state (in which case we have a completely random phase). The uncertainty product in phase and amplitude is minimum for these states. So, we can say that coherent states are quantum mechanical states closest to the classical description of the field. The basic properties of these states are outlined below.

Glauber states can be conveniently generated using (unitary)displacement operator,

$$\hat{d}(z) \;=\; e^{(za^\dagger \;-\; z^* a)}$$

where, z is any arbitrary complex number.
Using Baker-Housdarf Formula,

$$e^{X+Y} \;=\; e^X \, e^Y \, e^{-[X,Y]/2} \tag{4.1}$$

we can write $d(\hat{z})$ as,

$$d(z) \;=\; e^{za^\dagger} \, e^{-z^* a} \, e^{-|z|^2/2} \tag{4.2}$$

The displacement operator $d(z)$ has following properties;

$$(i) \;\; d^\dagger(z) \;=\; d^{-1}(z) \;=\; d(-z) \tag{4.3}$$

$$(ii) \;\; d^\dagger(z) \; ad(z) \;=\; a + z \tag{4.4}$$

$$(iii) \;\; d^\dagger(z) \; a^\dagger d(z) \;=\; a^\dagger + z^* \tag{4.5}$$

We can generate coherent state $|z\rangle$ by the operation of $d(z)$ on vacuum state. i.e,

$$|z\rangle \;=\; d(z)\,|0\rangle \tag{4.6}$$

The Glauber/coherent state is regarded as the eigen state of annihilation operator a. It can be proved in the following way,

$$d^\dagger(z) \; a\,|0\rangle \;=\; d^\dagger(z) \; ad(z)\,|0\rangle \tag{4.7}$$

$$d^\dagger(z) \; a\,|0\rangle \;=\; (a + z)\,|0\rangle \tag{4.8}$$

$$d^\dagger(z) \; a\,|0\rangle \;=\; z\,|0\rangle \tag{4.9}$$

If we multiply both sides by $\hat{d}(z)$ we will get the following eigen value equation,

$$a\,|z\rangle \;=\; z\,|z\rangle \tag{4.10}$$

Since a is non Hermitian operator its eigenvalue z wii be complex.
One more important property which follows using 4.1 is,

$$\hat{d}(z+z') = \hat{d}(z)\,\hat{d}(z')\,e^{(-\iota\ Im[zz'^*])} \tag{4.11}$$

Since they have indefinite photon number, we can make this fact apparent by expanding coherent states in fock space basis.

By taking scalar product of equation 4.10 with $\langle m|$, we get;

$$\sqrt{(m+1)}\ \langle m+1|z\rangle = z\,\langle m|z\rangle \tag{4.12}$$

Then,

$$\langle m|z\rangle = \frac{z^m}{(m!)^{1/2}}\ \langle 0|z\rangle \tag{4.13}$$

we can expand $|z\rangle$ in terms of fock state $|m\rangle$ as follows,

$$|z\rangle = \sum_m |m\rangle\,\langle m|z\rangle = \langle 0|z\rangle \sum_m \frac{z^m}{(m!)^{1/2}}\,|m\rangle \tag{4.14}$$

Thus, the length squared of the vector $|z\rangle$ is,

$$|\,\langle z|z\rangle\,|^2 = |\,\langle 0|z\rangle\,|^2 \sum_m \frac{z^{2m}}{(m!)} = |\,\langle 0|z\rangle\,|^2\ exp(|z|^2) \tag{4.15}$$

It can be seen that,

$$\langle 0|z\rangle = \langle 0|\,d\hat{(z)}\,|0\rangle = exp(\frac{-|z|^2}{2}) \tag{4.16}$$

Hence,

$$|\,\langle z|z\rangle\,|^2 = 1$$

and we normalized the coherent states.
We can then expand the coherent state in terms of number state as,

$$|z\rangle = exp(-|z|^2/2) \sum \frac{z^m}{(m!)^{1/2}}\,|m\rangle \tag{4.17}$$

In coherent state, the probability distribution of photons is a Poisson distribution, given by,

$$P(n) = |\,\langle m|z\rangle\,|^2 = \frac{|z|^{2m}\ \exp\{-|z|^2\}}{m!} \tag{4.18}$$

such that $|z|^2$ represents the average photon number, i.e, ($\tilde{m} = \langle z| a^\dagger \; a |z\rangle = |z|^2$)

Lets take the scalar product of two coherent states,

$$\langle z'|z\rangle = \langle 0| \hat{d}^\dagger(z') \; \hat{d}(z) |0\rangle \tag{4.19}$$

By using equation 4.2 it becomes,

$$\langle z'|z\rangle = e^{(-\frac{1}{2}(|z|^2+|z'|^2) \; + \; zz'^*)} \tag{4.20}$$

The magnitude of this scalar product would be,

$$|\langle z'|z\rangle|^2 = e^{(-|z-z'|^2)}. \tag{4.21}$$

From the above equation we see that coherent states are non-othogonal but in the limit $|z - z'| \gg 1$ they become orthogonal.

These states are over-complete and they make a 2- dimensional continuum of states. The completeness relation is given by,

$$\frac{1}{\pi} \int |z\rangle \langle z| \, d^2z = 1 \tag{4.22}$$

The physical significance of coherent states is that they form a field produced by extremely stabilized LASER operating over the threshold limit. These states form a useful basis for expanding the optical field in laser physics problems and non-linear optics.

4.2.1 Coherent states' correlation function

Consider a quantum field ψ for elementary boson (i.e, a photon), which has creation operator $\hat{a_b}^\dagger$ and the commutation relation of creation and annihilation operator is, $[\hat{a_b}, \hat{a_b}^\dagger] = 1$. For this field, the correlation function of 2nd order is given by,

$$g_2 = \frac{\langle\psi| \hat{a_b}^{\dagger 2} \hat{a_b}^2 |\psi\rangle}{\langle\psi|\psi\rangle} \left(\frac{\langle\psi|\psi\rangle}{\langle\psi| \hat{a_b}^\dagger \hat{a_b} |\psi\rangle} \right)^2, \tag{4.23}$$

Its value can be less than 1. Lets find out its lowest possible value for number state $\hat{a_b}^\dagger M |0\rangle$ where $|0\rangle$ is the vacuum state.

As discussed earlier, the ladder operators follow the following commutation relations,

$$[\hat{a}_b{}^{\dagger}, \hat{a}_b{}^{\dagger}] = 0, \qquad [\hat{a}_b, \hat{a}_b{}^{\dagger}] = 1 \tag{4.24}$$

also,

$$[\hat{a}_b, \hat{a}_b{}^{\dagger M}] = [\hat{a}_b, \hat{a}_b{}^{\dagger}]\ \hat{a}_b{}^{\dagger M-1} + \hat{a}_b{}^{\dagger}[\hat{a}_b, \hat{a}_b{}^{\dagger M-1}] \tag{4.25}$$

$$= M\hat{a}_b{}^{\dagger M-1} \tag{4.26}$$

the above equation gives,

$$\hat{a}_b\hat{a}_b{}^{\dagger M}|0\rangle = M\hat{a}_b{}^{\dagger N-1}|0\rangle \tag{4.27}$$

which leads to,

$$\langle 0|\,\hat{a}_b{}^{M}\hat{a}_b{}^{\dagger M}|0\rangle = M! \tag{4.28}$$

also,

$$\hat{a}_b|M\rangle = \sqrt{M}\,|M-1\rangle \tag{4.29}$$

from this equation, we get,

$$\hat{a}_b{}^{\dagger}\hat{a}_b|M\rangle = M|M\rangle \tag{4.30}$$

So, the eigenstate of the number operator $\hat{M} = \hat{a}_b{}^{\dagger}\hat{a}_b$ is $|M\rangle$ with eigen value M.

$$\hat{a}_b{}^{\dagger 2}\hat{a}_b{}^{2}|M\rangle = M(M-1)\,|M\rangle \tag{4.31}$$

using 4.31 on 4.23, we get,

$$g_2 = 1\ -\ \frac{1}{M}, \tag{4.32}$$

if $|\psi\rangle = |M\rangle$. Equation 4.32 gives the lowest possible g_2 value for n-elementary bosons.

The n-order correlation function equal to 1 can be achieved using a linear combination of number states, called coherent state, defined in terms of fock state as in equation 4.17.

$$|z\rangle = exp(-|z|^2/2)\ \sum\ \frac{z^M}{\sqrt{M!}}|M\rangle \tag{4.33}$$

As discussed earlier, for elementary bosons Coherent states have poissonian distribution over $|M\rangle$, their distribution is peaked at $|z|^2$, as a function of M where M is the average no. of composite boson i.e,

$$|z|^2 = \frac{\langle z|\, \hat{a_b}^{\dagger} \hat{a_b}\, |z\rangle}{\langle z|z\rangle}, \tag{4.34}$$

also,

$$|z|^4 = \frac{\langle z|\, \hat{a_b}^{\dagger 2} \hat{a_b}^{2}\, |z\rangle}{\langle z|z\rangle}, \tag{4.35}$$

From equation 4.23, 4.34 and 4.35 we find that the correlation function of second order for the state $|z\rangle$ is exactly equal to 1 whatever is the z, and in the same way for all the functions of higher order.

4.3 Coherent States of Entangled Bi-fermionic Co-bosons

Coherent state can be defined as the eigen state of annihilation operator, hence we can write:

$$c\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{4.36}$$

where α is a complex number.
let,

$$|\alpha\rangle = \sum_{m=0}^{\infty} C_m\,|m\rangle \tag{4.37}$$

from and we get:

$$\begin{aligned} \alpha\,|\alpha\rangle &= c\,|\alpha\rangle\,, \\ \alpha \sum C_m\,|m\rangle &= \sum_{n=o}^{\infty} C_m\; c\,|m\rangle\,, \\ \alpha \sum_{m=0}^{\infty} C_m\,|m\rangle &= \sum_{m=0}^{\infty} C_m\; (f_m\,|m-1\rangle + \epsilon_m\,|m\rangle). \end{aligned} \tag{4.38}$$

replacing m by $(m-1)$ on $L.H.S$ of equation,

$$f_m = \sqrt{\frac{\chi_m}{\chi_{m-1}}}\sqrt{m}, \tag{4.39}$$

$$\alpha\sum_{m=1}^{\infty} C_{m-1}\left|m-1\right\rangle = \sum_{m=1}^{\infty} C_m f_m \left|m-1\right\rangle + \sum_{m=0}^{\infty} C_m \left|\epsilon_m\right\rangle \tag{4.40}$$

applying $\langle m-1|$ on both sides, also $\langle m-1|\epsilon_m\rangle = 0$ and $\sum_{m=0}^{\infty}\langle m-1|m-1\rangle = I$

$$C_m = \frac{z}{f_m}C_{m-1} \tag{4.41}$$

$$C_1 = \frac{z}{f_1}\ C_0$$

$$C_2 = \frac{z}{f_2}C_1 = \frac{\alpha^2}{f_2\ f_1}\ C_0$$

$$\cdot$$

$$\cdot$$

$$\cdot$$

$$C_m = \frac{\alpha^m}{f_m f_{m-1}\cdots f_1}C_0$$

$$C_m = \frac{\alpha^m}{\Pi_{i=1}^m f_i}\ C_0, \tag{4.42}$$

so equation can be written as:

$$\left|\alpha\right\rangle = \sum_{m=0}^{\infty}\frac{\alpha^m}{\Pi_{i=1}^n f_i}\ C_0\left|m\right\rangle, \tag{4.43}$$

alternatively, the above equation can be written as:

$$\left|\alpha\right\rangle = \sum_{m=0}^{\infty}\frac{\alpha^m\sqrt{\chi_m}}{\sqrt{\chi_m m!}}\ C_0\left|m\right\rangle, \tag{4.44}$$

C_0 is the normalization constant.

Finding the normalization constant(C_0)

In order to find C_0 we make use of the fact that $\langle\alpha|\alpha\rangle = I$

$$\begin{aligned}
\langle\alpha|\alpha\rangle = I &= |C_0|^2 \sum_m \sum_{m'} \frac{\alpha^{*m}\alpha^{m'}}{\sqrt{m!\, m'!}}, \\
|C_0|^2 &= \left[\sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{m!} \frac{\chi_0}{\chi_m}\right]^{-1}, \\
|C_0| &= \left[\sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{m!} \frac{\chi_0}{\chi_m}\right]^{\frac{-1}{2}}.
\end{aligned}$$

now let,

$$\begin{aligned}
M(|\alpha|^2) &= \sum_{m=0}^{\infty} \frac{|\alpha|^{2m}}{m!} \frac{\chi_0}{\chi_m} \\
|\alpha\rangle &= \frac{1}{\sqrt{M(|\alpha|^2)}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \sqrt{\frac{\chi_0}{\chi_m}} \alpha_0 \, |m\rangle \\
|\alpha\rangle &= \frac{1}{\sqrt{M}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\Pi_{i=0}^{\infty} f_i} \, |m\rangle \qquad (4.45)
\end{aligned}$$

we know that,

$$\begin{aligned}
c^\dagger |m\rangle &= \alpha_{m+1}\sqrt{m+1}\, |m+1\rangle \qquad (4.46) \\
c^\dagger |m\rangle &= \sqrt{\frac{\chi_{m+1}}{\chi_m}} \sqrt{m+1}\, |m+1\rangle \qquad (4.47) \\
c^\dagger |m\rangle &= f_{m+1}\, |m+1\rangle \qquad (4.48)
\end{aligned}$$

4.3.1 Action of Annihilation operator on Coherent States of Entangled Bi-fermionic Coboson

Lets apply annihilation operator $c^\dagger$ on Coherent State $|\alpha\rangle$ of Composite Boson,

$$\begin{aligned}
c^\dagger |\alpha\rangle &= \frac{1}{\sqrt{M}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\Pi_{i=0}^{m} \; f_i} c^\dagger \, |m\rangle, \\
c^\dagger |\alpha\rangle &= \frac{1}{\sqrt{M}} \sum_{m=0}^{\infty} \frac{\alpha^m}{\Pi_{i=0}^{m} \; f_i} f_{m+1}\, |m+1\rangle, \qquad (4.49)
\end{aligned}$$

Alternatively

$$
\begin{aligned}
c^{\dagger}|\alpha\rangle &= \frac{1}{\sqrt{M(|\alpha|^2)}}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\sqrt{\frac{\chi_0}{\chi_m}}c^{\dagger}|m\rangle, \\
c^{\dagger}|\alpha\rangle &= \frac{1}{\sqrt{M}}\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{(n+1)!}}\frac{m+1}{\chi_m}\sqrt{\chi/\chi_{m+1}}\,|m+1\rangle.
\end{aligned} \tag{4.50}
$$

4.3.2 Deriving the Commutator $[c, c^{\dagger}]$

Depending upon effective composite Boson operators property we can derive the operator $[c, c^{\dagger}]$, The creation and annihilation operators of composite bosons are given by,

$$
c = \sum_{m=0}^{\infty}|m\rangle\langle m+1|\, f_{m+1}, \tag{4.51}
$$

and,

$$
c^{\dagger} = \sum_{m=0}^{\infty}|m+1\rangle\langle m|\, f_{m+1}, \tag{4.52}
$$

lets find the commutator,

$$
\begin{aligned}
[c, c^{\dagger}] &= cc^{\dagger} - c^{\dagger}c, \\
[c, c^{\dagger}] &= \left[\sum_{m=0}^{\infty}|m\rangle\langle m+1|\, f_{m+1}\right]\left[\sum_{m=0}^{\infty}|m+1\rangle\langle m|_{m+1}\right] - \left[\sum_{m=0}^{\infty}|m+1\rangle\langle m|\, f_{m+1}\right]\left[\sum_{m=0}^{\infty}|m\rangle\langle m
\end{aligned}
$$

putting $m-1=m$,

$$
\begin{aligned}
[c, c^{\dagger}] &= \sum_{m=0}^{\infty}f_{m+1}^2|m\rangle\langle m| - \sum_{m=0}^{\infty}f_{m+1}^2|m+1\rangle\langle m+1| \\
[c, c^{\dagger}] &= |f_1|^2|0\rangle\langle 0| - \sum_{m=1}^{\infty}(|f_m|^2 - |f_{m+1}|^2)|m+1\rangle\langle m+1|,
\end{aligned} \tag{4.55}
$$

This commutator is reduced to Identity when $f_m = \surd$ For the eigen state $|\alpha\rangle$ which is also the coherent state, the commutator comes out to be,

$$
\langle[c, c^{\dagger}]\rangle = \frac{1}{M}[|f_1|^2] - \sum_{m=1}^{\infty}\frac{|\alpha|^{2m}}{\Pi_{i+1}^{\infty}|f_i|^2}(|f_m|^2 - |f_{m+1}|^2). \tag{4.56}
$$

Let's consider the classical annihilation operator $\hat{a}$ as an example, in which the composite particles are distinguishable, that can be defined by placing $f_M = 1$ for all M. The commutator's expectation value is equal to $\langle [c, c^{\dagger}] \rangle \ = \ 1 - |\alpha|^2 \ \leq \ 1$ because of the relation $|\alpha| \leq \lim_{m \to \infty} |f_m| = 1$.

4.4 Non-classicality Measure of Entangled Bi-fermionic Coherent States

In this section, (i) Quadrature Variance and (ii) Mandel's Q-parmeter will be applied to the annihilation Operator of Composite Boson and corresponding Coherent State. These are the prominent measure of non-classicality of the Coherent States. We will derive these properties using expectation value of the commutator $\langle [c, c^{\dagger}] \rangle$.

4.4.1 Quadrature Variance

We use Quadrature variance to measure the non-classicality of coherent state. In position-momentum phase space quadrature variance is defined as:

$$(\triangle X)^2 = \left\langle \hat{X}^2 \right\rangle - \left\langle \hat{X} \right\rangle^2, \tag{4.57}$$

$$(\triangle P)^2 = \left\langle \hat{P}^2 \right\rangle - \left\langle \hat{P} \right\rangle^2, \tag{4.58}$$

where,

$$\hat{X} = \frac{\hat{a} + \hat{a^{\dagger}}}{2} \quad , \quad \hat{P} = \frac{\hat{a} - \hat{a^{\dagger}}}{2\iota},$$

using the above equations we get the quadrature variance of our coherent state as:

$$(\triangle X)^2 = (\triangle P)^2 = \frac{1}{4}, \tag{4.59}$$

using these quadrature variances the uncertainty is calculated as:

$$(\triangle X)^2 \ (\triangle P)^2 = \frac{1}{16}, \tag{4.60}$$

which is the minimum uncertainty as given by Heisenberg Uncertainty principle.

Squeezed State: A state would be called as a squeezed state if it has one of the quadrature variance lower than 1/4. i.e,

$$(\triangle X)^2 \quad < \quad \frac{1}{4}, \tag{4.61}$$

or,

$$(\triangle P)^2 \quad < \quad \frac{1}{4}. \tag{4.62}$$

so any state which has quadrature variance less than $\frac{1}{4}$ would be squeezed state. Vacuum state is not considered to be squeezed.

Quadrature Variance for Composite Boson operator

In this section, we calculated the variances for the effective co-boson annihilation operator $\hat{c}$,

$$(\triangle P_{eff})^2 \quad = \quad (\triangle X_{eff})^2, \tag{4.63}$$

also,

$$\hat{X}_{eff} = \frac{\hat{a} + \hat{a^{\dagger}}}{2} \tag{4.64}$$

Putting these values in quadrature variance equation, we get:

$$(\triangle X_{eff})^2 = \frac{1}{4}\langle \hat{c}^2 + \hat{c^{\dagger}}^2 + \hat{c}^2\ \hat{c^{\dagger}}^2 + \hat{c^{\dagger}}^2\ \hat{c}^2 \rangle \ - \ \frac{1}{4}\langle \hat{c} + \hat{c^{\dagger}} \rangle \tag{4.65}$$

using the identity,

$$\hat{c}\hat{c}^{\dagger} = \hat{c}^{\dagger}\hat{c} \ + \ [\hat{c}, \hat{c}^{\dagger}] \tag{4.66}$$

and

$$\hat{c}\,|\alpha\rangle = \alpha\,|\alpha\rangle\,, \tag{4.67}$$

we have

$$(\triangle X_{eff})^2 = \frac{1}{4}(\alpha^2 + \alpha^{\star 2} + 2|\alpha|^2 + \langle [c, c^{\dagger}] \rangle) - \frac{1}{4}(\alpha + \alpha^{\star})^2 \tag{4.68}$$

which comes out to be:

$$(\triangle X_{eff})^2 \quad = \quad \frac{1}{4}\,\langle [c, c^\dagger]\rangle \tag{4.69}$$

from this equation we can deduce that quadrature variance of effective co-boson can have the value of $\frac{1}{4}$ at $\langle [c, c^\dagger]\rangle \;=\; 1$.

We can see from equation (3.56) that for fermionic constituents the commutation relation $[c, c^\dagger] < 1$ so the coherent states of composite bosons will have have quadrature variance lower than $\frac{1}{4}$.

4.4.2 Mandel's Q-parameter

The Mandel's Q-parameter is also called "Photon Counting Statistics". It is an effective and simple measure of distinguishing classical states from non-classical states.
Considering number operator $\hat{m}$ such that: $\hat{m} \;=\; \hat{a}^\dagger\hat{a}$, the Mandel's Q-parameter is given by:

$$Q \quad = \quad \frac{\triangle m^2 \;-\; \langle m\rangle}{\langle m\rangle}, \tag{4.70}$$

$$\tag{4.71}$$

where,

$$(\triangle m)^2 \;=\; \langle \hat{m}^2\rangle \;-\; \langle \hat{m}\rangle^2.$$

If quantum states lie in the range $-1 \,\leq\, Q \,\leq\, 0$ then they have sub-poissonian distribution, on the other hand, states lying in the range $Q \,\geq\, 0$ have poissionian or super-poissionian distribution. Coherent States have Mandel's Q-parameter $Q \,=\, 0$ and they exhibit poissionian distribution. Coherent States form the standard for non-classicality.

Distribution Statistics for the Coherent States of Composite Boson

Effective Mandel's Q-parameter of the Glauber states of composite boson can be derived as follows:

$$Q \quad = \quad \frac{\langle \hat{m}^2 \rangle \ - \ \langle \hat{m} \rangle^2 \ - \ \langle \hat{m} \rangle}{\langle \hat{m} \rangle},$$

where,

$$\langle \hat{m}^2 \rangle = \langle \alpha | \, \hat{m}^2 \, | \alpha \rangle = \langle \alpha | \, \hat{c}^\dagger \ \hat{c} \ \hat{c}^\dagger \ \hat{c} \, | \alpha \rangle \,, \tag{4.72}$$

using the identity,

$$\hat{c}\hat{c}^\dagger = \hat{c}^\dagger \hat{c} \ + \ [\hat{c}, \hat{c}^\dagger] \tag{4.73}$$

and

$$\hat{c} \, | \alpha \rangle = \alpha \, | \alpha \rangle \,, \tag{4.74}$$

we have

$$\langle \hat{m}^2 \rangle \quad = \quad \langle \alpha | \, \hat{c}^\dagger \ (\hat{c}^\dagger \hat{c} + [\hat{c}, \hat{c}^\dagger]) \ \hat{c} \, | \alpha \rangle \,, \tag{4.75}$$

$$\quad = \quad \langle \alpha | \, \hat{c}^\dagger \hat{c}^\dagger \hat{c} \hat{c} \, | \alpha \rangle + \langle \alpha | \, \hat{c}^\dagger [\hat{c}, \hat{c}^\dagger] \hat{c} \, | \alpha \rangle \,, \tag{4.76}$$

$$\langle \hat{m}^2 \rangle \quad = \quad |\alpha|^4 + |\alpha|^2 \langle [c, c^\dagger] \rangle. \tag{4.77}$$

Also,

$$\langle m \rangle^2 = (\langle \alpha | \, m \, | \alpha \rangle)^2 = (\langle \alpha | \, c^\dagger \ c \, | \alpha \rangle)^2 = (\alpha^\star \alpha)^2 = |\alpha|^4. \tag{4.78}$$

using equation (4.77) and (4.78) in equation (4.72),

$$Q \quad = \quad \frac{|\alpha|^4 + |\alpha|^2 \langle [c, c^\dagger] \rangle - |\alpha|^4 + |\alpha|^2}{|\alpha|^2}, \tag{4.79}$$

$$Q \quad = \quad |\alpha|^2 \left[\frac{\langle [c, c^\dagger] \rangle \ - \ 1}{|\alpha|^2} \right], \tag{4.80}$$

$$Q \quad = \quad \langle [c, c^\dagger] \rangle \ - \ 1. \tag{4.81}$$

The value of Q will be zero if $\langle [c, c^\dagger] \rangle = 1$

We can see from equation (3.56) that for fermionic constituents the commutation relation $[c, c^\dagger] < 1$ so the coherent states of composite bosons will have negative values of Q which corresponds to sub-poissionion distribution.

Chapter 5

Discussion and Conclusion

We started our work with the general introduction of Ladder operators of elementary particles. There are two types of elementary particles, fermions and bosons. Matter is generally composed of fermions and composite bosons, however, elementary bosons are mostly the exchange particles of the fields. To form a composite bi-particle there can have three possibilities. Either its composed of two bosons, or one boson and one fermion forming a composite fermion or it can be composed of two fermions forming a composite boson. The composite bosons made up of two fermions exists widely in nature like Hydrogen atom, Cooper pair etc, so we discussed bi-fermionic composite bosons. We did a literature survey of the entanglement and the composite particles. We laid a quantum mechanical foundation of the elementary particles and discussed their operators and then compared them to the operators of composite particles. A composite boson composed of two entangled fermions can behave as ideal boson. There bosonic behavior depends on the degree of entanglement between the two entangled fermions. We discussed the separability criterion for quantum states and found that if $|\psi_i\rangle$ has only one Schmidt number, then the state will be separable. If it is greater than 1 than the state will be entangled. Hence, it is concluded that any pure state $|\psi_i\rangle$ is separable iff it has only one non-zero Schmidt coefficient.

We proceed by presenting the operators that depict bosons and fermions, before demonstrating some known results with respect to composite particles that are frameworks of correlated however distinguishable fermions. We likewise quickly talk about the evaluation of entanglement, where it is to be seen as a resource quantity, and give a proof that entanglement permits fermion pair to have the properties of a perfect boson operators. We have also discussed briefly the effect of Pauli exclusion principle, as our composite boson is made up of fermionic constituents, that specifically inhibit stacking up greater number of composite bosons than the number of fermion-pair states with which the composite bosons are made of.

Finally, we have analyzed systematically, the coherent states of entangled bi-fermionic composite boson. We defined an effective composite boson annihilation operator and then derived its commutator and eigenstate. We discussed the non-classical properties, like Quadrature Variance and Mandels Q parameter, of these Coherent states.

The quadrature variances of coherent States of bi-fermionic composite bosons attain lower value than the values for the coherent state of elementary bosons. Also the Mandel's Q parameter in this case is sub-poissonian where as elementary bosons' Glauber states have poissonian statistics.

Bibliography

[1] Dirac, Paul A.M. The principles of quantum mechanics. No. 27. *Oxford university press*, 1981.

[2] Buchleitner, Andreas, C.V., and Markus Tiersch, eds. Entanglement and decoherence: foundations and modern trends. Vol. 768. *Springer Science and Business Media*, 2008.

[3] Sakurai, Jun J., and Jim N. Modern quantum mechanics. *Vol. 185. Harlow: Pearson*, 2014.

[4] Morse, Philip M. "Thermal physics." *Physics Today 17* (1964).

[5] Gerry C., Peter K. Introductory quantum optics. *Cambridge university press*, 2005.

[6] Schleich, Wolfgang P. Quantum optics in phase space. *John Wiley and Sons*, 2011.

[7] Wiechers C, Lydersen L, Wittmann C, Elser D, Skaar J, Marquardt C, Makarov V, and Leuch G. Quantum optics in phase space. *Quantiki*, 2013.

[8] Law, C. K., Quantum entanglement as an interpretation of bosonic character in composite two-particle systems. *Physical Review A* **71** (2005) 034306.

[9] Nielsen, Michael A., and Guifré Vidal. Majorization and the interconversion of bipartite states. *Quantum Information and Computation 1, no. 1* (2001): 76-93.

[10] Mintert, F., C. Viviescas, and A. Buchleitner. "Basic concepts of entangled states." In Entanglement and Decoherence, pp. 61-86. Springer, Berlin, Heidelberg, 2009.

[11] Nielsen, Michael A., and Isaac L. Chuang. *Quantum Computation and Quantum Information* (10th Anniv. Version). (2010).

[12] Scully, Marlan O., and Zubairy M. *Quantum optics*. (1999).

[13] Tichy, M. C., Bouvrie, P. A., and Mølmer, K. Bosonic behavior of entangled fermions. *Physical Review A 86* (2012): 042317.

[14] Combescot, M. "Commutator formalism" for pairs correlated through Schmidt decomposition as used in Quantum Information. *EPL (Europhysics Letters)* (2011): 60002.

[15] Avancini, S.S., Marinelli, J.R. and Krein, G., Compositeness effects in the Bose–Einstein condensation. *Journal of Physics A: Mathematical and General* 36, no. 34 (2003): 9045.

[16] Ekert, A. K., and Knight, P. L. Correlations and squeezing of two-mode oscillations. *American Journal of Physics* 57, no. 8 (1989): 692-697.

[17] Combescot, M., Leyronas, X., and Tanguy, C. On the N-exciton normalization factor. *The European Physical Journal B-Condensed Matter and Complex Systems 31*, no. 1 (2003): 17-24.

[18] Johansen, Lars M. Nonclassical properties of coherent states. *Physics Letters A* 329, no. 3 (2004): 184-187.

[19] Eberly, J. H. Schmidt analysis of pure-state entanglement. *Laser physics* 16, no. 6 (2006): 921-926.

[20] Shor, Peter W. Polynomial-time algorithms for prime factorization and discrete logarithms on a quantum computer. *SIAM review* 41, no. 2 (1999): 303-332.

[21] Ekert, Artur K. Quantum cryptography based on Bell's theorem. *Physical review letters* 67, no. 6 (1991): 661.

[22] Aspect, A., Dalibard, J. and Roger, G., Experimental test of Bell's inequalities using time-varying analyzers. *Physical review letters* 49, no. 25 (1982): 1804. (1982)

[23] Bennett, C.H., Brassard, G., Popescu, S., Schumacher, B., Smolin, J.A. and Wootters, W.K., Purification of noisy entanglement and faithful teleportation via noisy channels. *Physical review letters* 76, no. 5 (1996): 722.

[24] Thaller, B. (2005). Advanced visual quantum mechanics. Springer Science and Business Media, 2005.

[25] Pathak, A. (2013). Elements of quantum computation and quantum communication. CRC Press, 2013.

[26] Bouwmeester, D., and Zeilinger, A. The physics of quantum information: basic concepts. In The physics of quantum information (pp. 1-14). In The physics of quantum information, pp. 1-14. Springer, Berlin, Heidelberg, 2000.

[27] Tittel, W., Brendel, J., Zbinden, H., and Gisin, N. Violation of Bell inequalities by photons more than 10 km apart. Physical Review Letters 81, no. 17 (1998): 3563.

[28] Klein, A., and Marshalek, E. R. Boson realizations of Lie algebras with applications to nuclear physics. Reviews of modern physics 63, no. 2 : 375.(1991).

[29] Law, C. K. "Quantum entanglement as an interpretation of bosonic character in composite two-particle systems." Physical Review A 71, no. 3 (2005): 034306.

[30] Einstein, A., B. Podolsky, and N. Rosen, 1935, "Can quantum-mechanical description of physical reality be considered complete?", Phys. Rev. 47, 777-780.

[31] Von Delft, J., and Schoeller, H. (1998). Bosonization for beginners—refermionization for experts. Annalen der Physik 7, no. 4 : 225-305.

[32] Werner, R. F. (1989). Quantum states with Einstein-Podolsky-Rosen correlations admitting a hidden-variable model. Physical Review A 40, no. 8 : 4277.

[33] Combescot, M., and Tanguy, C. (2001). New criteria for bosonic behavior of excitons. EPL (Europhysics Letters) 55, no. 3 : 390.

[34] Shiau, S. Y., and Combescot, M. (2017). Effect of Pauli blocking on coherent states of composite bosons. Physical Review A, 95(1), 013838.

Appendix

5.1 Schmidt Decomposition

A pure state $|\psi\rangle$ in Hilbert space H can be written in terms of corresponding product basis as:

$$|\psi\rangle = \sum_{ij} d_{ij} \left|\phi_i^A\right\rangle \otimes \left|\phi_j^B\right\rangle,$$

where d_{ij} are the expansion coefficients.

$$d_{ij} = \sum_{ij} \left\langle\phi_i^A\right| \otimes \left\langle\phi_j^B\middle|\psi\right\rangle,$$

Lets now rotate the basis:

$$\tilde{\phi}_i = \hat{U} \left|\phi_i^A\right\rangle, \tag{5.1}$$

$$\tilde{\phi}_j = \hat{V} \left|\phi_j^B\right\rangle, \tag{5.2}$$

where $\hat{U}$ and $\hat{V}$ are two unitary operators.

$$\tilde{d_{ij}} = \left\langle\tilde{\phi_i^A}\right| \otimes \left\langle\tilde{\phi_j^B}\middle|\psi\right\rangle,$$

$$\tilde{d_{ij}} = \left\langle\tilde{\phi_i^A}\right| \hat{U}^\dagger \otimes \left\langle\tilde{\phi_j^B}\right| \hat{V}^\dagger \left|\psi\right\rangle,$$

using resolution of identity, i.e, $\sum_i \left|\phi_i^A\right\rangle \left\langle\phi_i^A\right| = I$ and $\sum_j \left|\phi_j^B\right\rangle \left\langle\phi_j\right| = I$;

$$\tilde{d_{ij}} = \left\langle\phi_i^A\right| \hat{U}^\dagger \left|\phi_p\right\rangle \left\langle\phi_p\right| \otimes \left\langle\phi_j^B\right| \hat{V}^\dagger \left|\phi_q\right\rangle \left\langle\phi_q|\psi\right\rangle,$$

$$\tilde{d_{ij}} = [\hat{U}d\hat{V}]_{ij}.$$

So the state given in rotated basis will become:

$$|\psi\rangle = \sum_{ij} [\hat{U}d\hat{V}]_{ij} \left|\tilde{\phi_i^A}\right\rangle \otimes \left|\tilde{\phi_j^B}\right\rangle,$$

For every complex matrix $\hat{d}$, there always exists a unitary transformation $\hat{U}$ and $\hat{V}$ such that $[\hat{U}d\hat{V}]$ is diagonal [10]. This provides singular value decomposition of d with real, non-negative diagonal values, S_i , called singular values.

$$|\psi\rangle = \sum_i \sqrt{\lambda_i} \left|\phi_i^A\right\rangle \otimes \left|\phi_i^B\right\rangle.$$

This is called Schmidt Decomposition where, $\lambda_i = S_i^2$ which are called Schmidt Coefficients. Schmidt Decomposition is unique as $[\hat{U}d\hat{V}]$ is unique. If two schmidt coefficients are non-zero it would not be possible to write $|\psi\rangle$ in the form of separable state.

5.2 Commutation Relation for Ladder Operators of Cobosons

The state given in equation 3.50 can be generated when the creation operator $c^\dagger$ acts on vacuum, as;

$$\hat{c}^\dagger |0\rangle = \sum_{M=0}^{\infty} \sqrt{\lambda_M} \;\; \hat{a}_M^\dagger |0\rangle \otimes \hat{b}_M^\dagger |0\rangle \,,$$

where $\hat{a}_M^\dagger$ and $\hat{b}_M^\dagger$ creats the constituent particles A and B respectively and $\hat{c}^\dagger$ is the creation operator for the composite particle.
If we compare the coefficients

$$\hat{c}^\dagger = \sum_{M=0}^{\infty} \sqrt{\lambda_M} \hat{a}_M^\dagger \hat{b}_M^\dagger.$$

The state in equation (5.2) is the second quantization representation of the state given in equation (3.50).
The hermitian conjugate of $\hat{c}^\dagger$ is;

$$\hat{c} = \sum_{M=0}^{\infty} \sqrt{\lambda_M} \; \hat{b}_M \hat{a}_M.$$

Finding the commutation relation between c and $c^\dagger$:

$$[c, c^\dagger] = cc^\dagger - c^\dagger c,$$

$$[c, c^\dagger] = \sum_{M=0}^{\infty} (\sqrt{\lambda_M} b_M a_M)(\sqrt{\lambda_M} a_M^\dagger b_M^\dagger) - \sum_{M=0}^{\infty} (\sqrt{\lambda_M} a_M^\dagger b_M^\dagger)(\sqrt{\lambda_M} b_M a_M),$$

$$[c, c^\dagger] = \sum_{M=0}^{\infty} (\sqrt{\lambda_M} b_M a_M a_M^\dagger b_M^\dagger) - \sum_{M=0}^{\infty} (\sqrt{\lambda_M} a_M^\dagger b_M^\dagger b_M a_M),$$

using the commutation relation for elementry bosons;

$$
\begin{aligned}
[a, a^\dagger] &= 1, \\
aa^\dagger - a^\dagger a &= 1, \qquad (5.3) \\
aa^\dagger &= 1 + a^\dagger a. \qquad (5.4)
\end{aligned}
$$

we get,

$$[c, c^\dagger] = \sum_{M=0}^{\infty} \sqrt{\lambda_M}(1 + b_M^\dagger b_M)(1 + a_M^\dagger a_M) - \sum_{M=0}^{\infty}(\sqrt{\lambda_M} a_M^\dagger b_M^\dagger b_M a_M),$$

$$[c, c^\dagger] = \sum_{M=0}^{\infty} \sqrt{\lambda_M}(1 + a_M^\dagger a_M + b_M^\dagger b_M + b_M^\dagger b_M a_M^\dagger a_M - a_M^\dagger b_M^\dagger b_M a_M),$$

$$[c, c^\dagger] = \sum_{M=0}^{\infty} \sqrt{\lambda_M}(1 + a_M^\dagger a_M + b_M^\dagger b_M),$$

$$[c, c^\dagger] = 1 + \sum_{M=0}^{\infty} \sqrt{\lambda_M}(a_M^\dagger a_M + b_M^\dagger b_M),$$

$$[c, c^\dagger] = 1 + s \ \triangle .$$

where,

$$\triangle = \sum_{M=0}^{\infty} \lambda_M (a_M^\dagger a_M + b_M^\dagger b_M).$$

- $s = 1$ for bosons.
- $s = -1$ for fermions.

5.3 Fock State ladder from Ground State for Composite Boson

$$c^{\dagger}|0\rangle = |1\rangle .$$

$$c^{\dagger}|1\rangle = \sqrt{2}\,|2\rangle .$$

$$|2\rangle = \frac{c^{\dagger}}{\sqrt{2}}(c^{\dagger}|0\rangle),$$

$$|2\rangle = \frac{(c^{\dagger})^2}{\sqrt{2}}|0\rangle ,$$

when we keep applying $c^{\dagger}, M$ times we get;

$$|M\rangle = \frac{c^{\dagger M}}{\sqrt{M!\chi_M}}|0\rangle , \tag{5.5}$$

Re-normalizing the state in equation (5.5);

$$\langle M|M\rangle = 1,$$

$$\langle 0|\, c^{M} c^{\dagger M}\,|0\rangle = M!\chi_M,$$

Derivation of χ_M:

When the creation operator of composite bosons acts on the ground state it gives,

$$c^{\dagger M}|0\rangle \;=\; \sum_{P_M > P_{M-1} > \ldots} \sqrt{\lambda_{P_1}\,\lambda_{P_2}\,\lambda_{P_3}\ldots\lambda_{P_M}}\; F\,(P_1, P_2, \ldots, P_M)\,|P_1, P_2, \ldots, P_M\rangle ,$$

where $|P\rangle$ denotes that the state is filled with an A particle and a B particle in the Schmidt mode P and $F\,(P_1, P_2, \ldots, P_M)$ is the weight factor for state $|P_1, P_2, \ldots, P_M\rangle$. If all the $P's$ have d same terms and all the remaining terms are different then,

$$P_1 \;=\; P_2 \;=\; P_3 \;=\; \ldots \;=\; P_d \;=\; P,$$

so,

$$c^{\dagger M}|0\rangle \;=\; \sum_{P_M > P_{M-1} > \ldots} \sqrt{\lambda_{P_1}\,\lambda_{P_2}\,\lambda_{P_3}\ldots\lambda_{P_M}}\; F\,(P)\,|P\rangle ,$$

where,

$$|P\rangle \;=\; \frac{\chi_M^{-1/2}}{\sqrt{M!}}\; c^{\dagger M}\,|0\rangle\,,$$

$$c^{\dagger M}\,|0\rangle \;=\; \sum_{P_M>P_{M-1}>...} \sqrt{\lambda_{P_1}\;\lambda_{P_2}\;\lambda_{P_3}...\lambda_{P_M}}\; F(P)\; \frac{\chi_M^{-1/2}}{\sqrt{M!}}\; c^{\dagger M}\,|0\rangle\,,$$

$$\chi_M^{1/2} \;=\; \sum_{P_M>P_{M-1}>...} \sqrt{\lambda_{P_1}\;\lambda_{P_2}\;\lambda_{P_3}...\lambda_{P_M}}\;\; \frac{F(P)}{\sqrt{M!}},$$

Squaring on both sides,

$$\chi_M \;=\; \sum_{P_M>P_{M-1}>...} \lambda_{P_1}\;\lambda_{P_2}\;\lambda_{P_3}...\lambda_{P_M}\;\; \frac{F^2(P)}{M!},$$

$$F(P) \;=\; \frac{M!}{d!}\;\times\; d!,$$

and,

$$F(P) \;=\; M!,$$

so we can write,

$$\chi_M^B \;=\; M! \sum_{P_M>P_{M-1}>...} \lambda_{P_1},\;\lambda_{P_2},\;\lambda_{P_3},...,\lambda_{P_M}, \qquad (For\;\;Bosons) \quad (5.6)$$

similarly for fermions,

$$\chi_M^F \;=\; M! \sum_{P_M>P_{M-1}>...P_2>P_1} \lambda_{P_1},\;\lambda_{P_2},\;\lambda_{P_3},...,\lambda_{P_M}. \qquad (For\;\;Fermions) \; (5.7)$$

5.4 Derivation for α_M and $\langle \epsilon_M | \epsilon_M \rangle$

When we apply the creation operator on number state, we have;

$$c_M^\dagger \left| M \right\rangle \;=\; \alpha_{M+1}(M+1)^{1/2} \left| M+1 \right\rangle ,$$

the conjugate of equation 5.4 would be,

$$\left\langle M \right| \; c \;=\; \alpha_{M+1}(M+1)^{1/2} \left\langle M+1 \right| , \tag{5.8}$$

taking inner product of equation 5.4 and 5.8, we get;

$$\left\langle M \right| c \; c^\dagger \left| M \right\rangle \;=\; (\alpha_{M+1})^2 (M+1) \left\langle M+1 | M+1 \right\rangle ,$$

also, the number state $|N\rangle$ is given by,

$$\left| M \right\rangle \quad = \quad \chi_M^{-1/2} \; \frac{c^{\dagger M}}{\sqrt{M!}} \; \left| 0 \right\rangle , \tag{5.9}$$

$$\left\langle M \right| \quad = \quad \chi_M^{-1/2} \; \frac{c^{M}}{\sqrt{M!}} \; \left\langle 0 \right| , \tag{5.10}$$

$$\tag{5.11}$$

also, we have;

$$\left| M+1 \right\rangle \quad = \quad \chi_{M+1}^{-1/2} \; \frac{c^{\dagger M+1}}{\sqrt{(M+1)!}} \; \left| 0 \right\rangle ,$$

$$\left\langle M+1 \right| \quad = \quad \chi_{M+1}^{-1/2} \; \frac{c^{M+1}}{\sqrt{(M+1)!}} \; \left\langle 0 \right| , \tag{5.12}$$

taking inner product, we get;

$$
\begin{aligned}
\langle M+1|M+1\rangle &= \chi_{M+1}^{-1}\,\frac{\langle 0|\,c^{M+1}\;\, c^{\dagger(M+1)}\,|0\rangle}{M!},\\
\langle M|M\rangle &= \chi_{M}^{-1}\,\frac{\langle 0|\,c^{M}\;\, c\;c^{\dagger M}\;\, c^{\dagger}\,|0\rangle}{M!},\\
\langle M|\,c^{\dagger}c\,|M\rangle &= \chi_{M}^{-1}\,\frac{\langle 0|\,c^{M+1}\;\, c^{\dagger(M+1)}\,|0\rangle}{M!},\\
\langle M|\,c^{\dagger}c\,|M\rangle &= (\alpha_{M+1})^2(M+1)\,\langle M+1|M+1\rangle\,,
\end{aligned}
\tag{5.13}
$$

we can further write,

$$
\frac{\chi_{M+1}^{-1}}{M!}\,\langle 0|\,c^{\dagger(M+1)}c^{M+1}\,|0\rangle \quad = \quad (\alpha_{M+1})^2(M+1)\frac{\chi_{M+1}^{-1}}{(M+1)!}\,\langle 0|\,c^{\dagger(M+1)}c^{M+1}\,|0\rangle\,,
$$

$$
\frac{\chi_{M}^{-1}}{\chi_{M+1}^{-1}}\,\frac{(M+1)!}{M!(M+1)!} \;=\; (\alpha_{M+1})^2,
$$

now we will put $M = M-1$,

$$
\begin{aligned}
\frac{\chi_{M-1}^{-1}}{\chi_{M}^{-1}}\,\frac{(M)!}{M(M-1)!} &= {\alpha_M}^2,\\
{\alpha_M}^2 &= \frac{\chi_M}{\chi_M - 1},
\end{aligned}
\tag{5.14}
$$

so, α_M comes out to be,

$$
\alpha_M \;=\; \sqrt{\frac{\chi_M}{\chi_{M-1}}}.
$$

Now, lets find the inner product $\langle \epsilon_M | \epsilon_M \rangle$,

$$\begin{aligned}
\langle M| \, [c, c^\dagger] \, |M\rangle &= \langle M| \, [cc^\dagger - c^\dagger c] \, |M\rangle \, , \\
c^\dagger \, |M\rangle &= \alpha_{M+1} \sqrt{M+1} \, |M+1\rangle \, , \\
c \, \langle M| &= \alpha_{M+1} \sqrt{M+1} \, \langle M+1| \, , \\
\langle M| \, cc^\dagger \, |M\rangle &= {\alpha_{M+1}}^2 \; (M+1), \\
c \, |M\rangle &= \alpha_M . \sqrt{M} \, |M-1\rangle \; + \; |\epsilon_M\rangle \, , \\
\langle M| \, c^\dagger c \, |M\rangle &= {\alpha_M}^2 \; (M) \; + \; \langle \epsilon_M | \epsilon_M \rangle \, .
\end{aligned}$$

combining above set of equations, we get;

$$\langle M| \, [c, c^\dagger] \, |M\rangle \quad = \quad {\alpha_{M+1}}^2 \; (M+1) \; - \; {\alpha_M}^2 \; (M) \; + \; \langle \epsilon_M | \epsilon_M \rangle \, .$$

solving for $\langle \epsilon_M | \epsilon_M \rangle$, we get;

$$\langle \epsilon_M | \epsilon_M \rangle \quad = \quad {\alpha_{M+1}}^2 \; (M+1) \; - \; {\alpha_M}^2 \; (M) \; - \; \langle M| \, [c, c^\dagger] \, |M\rangle \, , \tag{5.15}$$

$$\langle \epsilon_M | \epsilon_M \rangle \quad = \quad 1 \; - \; M \frac{\chi_M}{\chi_{M-1}} \; + \; (M-1) \frac{\chi_{M+1}}{\chi_M} . \tag{5.16}$$

For Perfect Boson:

- $\alpha_M \longrightarrow 1$
- $\langle \epsilon_M | \epsilon_M \rangle \longrightarrow 0$

5.5 Gaussian Wavefunction

Consider an example of double Gaussian wavefunction, let x_A and x_B be some continuous variable of particle A and B and $\psi_{(x_A,x_B)}$ be a bipartite wavefunction.

The most general form of double gaussian wavefunction is,

$$\psi_{(x_A,x_B)} \;=\; N \exp\left[\frac{-(x_A+x_B)^2}{\sigma_c^2}\right] \; \exp\left[\frac{(x_A-x_B)^2}{\sigma_r^2}\right],$$

where, σ_c^2 is width along x_A+x_B, σ_c^2 is width along x_A+x_B and N is normalization constant. Now,

$$\psi_{(x_A,x_B)} \;=\; N \exp\left[\frac{\sigma_c^2 x_A^2 - \sigma_r^2 x_A^2 + \sigma_c^2 x_A^2 - \sigma_r^2 x_B^2 - 2x_A^2 x_B^2(\sigma_c^2+\sigma_r^2)}{\sigma_c^2\sigma_r^2}\right],$$

$$= \; N \exp\left[\frac{(\sigma_c^2-\sigma_r^2)(x_A^2+x_B^2) - 2x_A x_B(\sigma_c^2+\sigma_r^2)}{\sigma_c^2\sigma_r^2}\right],$$

$$= \; N \exp\left[\frac{-4}{\sigma_c\sigma_r}\frac{(\sigma_c+\sigma_r)^2+(\sigma_c-\sigma_r)^2}{(\sigma_c+\sigma_r)^2-(\sigma_c-\sigma_r)^2}x_A x_B\right] + [\frac{4}{\sigma_c\sigma_r}\frac{(\sigma_c+\sigma_r)+(\sigma_c-\sigma_r)}{\sigma_c^2+\sigma_r^2+2\sigma_c\sigma_r-\sigma_c^2 2\sigma_c\sigma_r-\sigma_r^2}(x_A^2+x_B^2)]$$

let, $(\frac{\sigma_c-\sigma_r}{\sigma_c+\sigma_r})^2 = z$ Then;

$$\psi_{(x_A,x_B)} \;=\; N \exp\frac{4}{\sigma_c\sigma_r}[\frac{-1+x^2}{1-x^2}x_A x_B + \frac{x}{1-x^2}(x_A^2+x_B^2)],$$

finally, we can write;

$$\psi_{(x_A,x_B)} \;=\; \sqrt{1-x^2}\sum_{n=0}^{\infty} x^n \phi_n^A(x_A)\;\phi_n^B(x_B),$$

Compairing equation 3.50 and 5.5, we get;

$$\sqrt{\lambda_M} \;=\; \sqrt{1-x^2}\;x^{2M},$$

$$\lambda_M \;=\; (1-x^2)\;x^{2M}.$$

where, $M = 0, 1, 2, 3, ...$

Which is the Schmidt eigen value as given in equation 3.68

5.6 Evaluation of Normalization ratios for fermions and bosons

In this section we will evaluate, χ_M^B and χ_M^F;
let,

$$\begin{aligned}
P_1 &= q_M, \\
P_2 &= q_M + q_{N-1}, \\
P_3 &= q_M + q_{M-1} + q_{M-2}, \\
&. \\
&. \\
&. \\
P_M &= q_M + q_{M-1} + ... + q_1.
\end{aligned}$$

For $M = 4$,

$$\begin{aligned}
P_1 &= q_4, \\
P_2 &= q_3 + q_4, \\
P_3 &= q_2 + q_3 + q_4, \\
P_4 &= q_1 + q_2 + q_3 + q_4.
\end{aligned}$$

putting $N = P_1$ in equation 3.68,

$$\begin{aligned}
\lambda_{P_1} &= (1-x)^{q_4}, \\
\lambda_{P_2} &= (1-x)^{q_3+q_4}, \\
\lambda_{P_3} &= (1-x)^{q_2+q_3+q_4}, \\
\lambda_{P_4} &= (1-x)^{q_1+q_2+q_3+q_4}.
\end{aligned}$$

so for Bosons we have,

$$\chi_M^B = M! \sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\sum_{q_3=0}^{\infty}\sum_{q_4=0}^{\infty} (1-x)^4 \; x^{q_1+2q_2+3q_3+4q_4},$$

$$\chi_M^B = M!(1-x)^M \sum_{q_1=0}^{\infty}\sum_{q_2=0}^{\infty}\sum_{q_3=0}^{\infty}\sum_{q_4=0}^{\infty} \dots \sum_{q_M=0}^{\infty} x^{q_1+2q_2+3q_3+\dots+Mq_M}, \tag{5.17}$$

Similarly, for fermions,

$$\chi_M^F = M! \; (1-x)^M \sum_{q_1=1}^{\infty}\sum_{q_2=1}^{\infty}\sum_{q_3=1}^{\infty} \dots \sum_{q_M=0}^{\infty} x^{q_1+2q_2+3q_3+\dots+Mq_M}. \tag{5.18}$$

Expanding the power series,

$$\sum_{q_1=0}^{\infty} z^{q_1} = 1 + x + x^2 + \dots = \frac{1}{1-x}$$

$$\sum_{q_2=0}^{\infty} x^{2q_2} = 1 + x^2 + x^4 \dots = \frac{1}{1-x^2}$$

$$\vdots$$

$$\sum_{q_M=0}^{\infty} x^{Mq_M} = 1 + x^M + x^{2M} \dots = \frac{1}{1-x^M}.$$

So we have,

$$\sum_{q_1}\sum_{q_2}\dots\sum_{q_{M-1}}\sum_{q_M} x^{q_1}x^{2q_2}\dots x^{Mq_M} = \frac{1}{(1-x)(1-x^2)\dots(1-x^M)},$$

which implies that,

$$\chi_M^B = \frac{M! \; (1-x)^M}{(1-x)(1-x^2)\dots(1-x^M)}, \tag{5.19}$$

similarly for fermions,

$$\chi_M^F = \frac{M! \; z^{M(M-1)/2} \; (1-x)^M}{(1-x)(1-x^2)\dots(1-x^M)}. \tag{5.20}$$

Now we can easily find out the normalization ratios for fermions and bosons using 5.19 and 5.20 respectively. For Bosons,

$$\chi^B_{M+1} = \frac{(M+1)(1-x)}{(1-x^{M+1})}, \tag{5.21}$$

Similarly for fermions,

$$\chi^F_{M+1} = \frac{x^M(M+1)(1-x)}{(1-x^{M+1})}. \tag{5.22}$$