Generalized coherent state wavepacket dynamics by means of entropy

by

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Dedicated to

My Loving Parents

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In the name of Almighty Allah, Who bestowed on me His blessings and gave me courage and vision to accomplish this work successfully. I invoke peace for Holy Prophet Hazrat Muhammad (PBUH) Who is forever a symbol of guidance for humanity.
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Abstract

Wavepacket dynamics manifests several interesting features which do not have corresponding classical analogue, such as, phenomena of quantum wavepacket revivals and fractional revivals. In this thesis, we study the quantum dynamics specially engineered wavepackets, namely generalized coherent state wavepackets, by means of quantum information entropy. For our general study, we consider a general class of one-dimensional quantum systems, with discrete and bounded-below energy spectrum. As particular examples, the harmonic oscillator and the infinite square well have been explicitly discussed through out our analysis.

The generalized coherent state wavepackets are constructed in position space as well as in momentum space which are then used to calculate the corresponding probability densities. These probability densities lead us to calculate the expectation values and variances for position-momentum operators needed to express Heisenberg uncertainty relation. Moreover, these probability densities are used to calculate the information entropy for our constructed wavepackets. The information entropy is then used to expressed the Heisenberg uncertainty relation which is an alternative formalism to the conventional variance-based measurements in quantum mechanics.

Moreover, temporal evolution of these wavepackets is analyzed by means autocorrelation function, position space probability density and momentum space probability density as a function of time. It is shown that generalized coherent state wavepackets exhibit the phenomena of quantum revivals and fractional revivals. The time-evolution of position space as well as momentum space probability densities manifest interesting spatio-temporal and momento-temporal patterns known as quantum carpets. Finally, we study the phenomena of quantum revivals and fractional revivals by means of information entropy which provides an elegant method to identify the fractional revivals with greater resolution.

Contents

Chapter 1

Introduction and outline

1.1 Introduction

The quantum theory emerged as a consequence of the inability of the classical theory to explain several microscopic phenomena, such as, black body radiation spectrum, photo electric effect and atomic structure. The quantum world manifests interesting phenomena that do not have corresponding analogue in classical world. Therefore, quantum-classical correspondence has been an area of great interest for researchers since the very beginning of quantum theory. The precise demarcation between these two theories has been one of the most debated philosophical issues from the very inception of quantum theory. The core problem that surfaced from the very beginning: How to envision compatibility between thoroughly deterministic macroscopic world and completely probabilistic microscopic world. Bohr's famous correspondence principle was the sole solution but even that lacked the assignment of demarcation line.

Classically, the state of a dynamical system is uniquely determined by the position $x(t)$ and momentum $p(t)$ as a function of time t. Thus in classical physics, the equation of motion of a particle unify position and momentum variable and deterministically specify these state variables over all envisionable span of space-time. However, it is well-known that initial formulation of quantum theory in 1920s is connected with its classical predecessor through ad-hoc quantization rules with classical variables duly replaced with algebraic operators. These operators representing dynamical variables act upon a complex function $\psi(x, t)$, known as quantum mechanical wave function. Such wave function therefore stands for the new notion of the state of the physical system. Moreover, the wave function $\psi(x,t)$ has to obey the strict imposition of uncertainty principle. Therefore, the spatial localization of the wave packet during time evolution and its correspondence with the particle trajectory has been the key issue. Thus under the urge of classical romanticism, forefathers of the quantum theory tried to identify narrow wavepackets in position space with particles and their corresponding trajectories.

The wavepacket dynamics manifests several interesting phenomena that do not have corresponding classical analogues, such as the phenomena of quantum revivals and fractional revivals. In this thesis we study the dynamics of specially engineered wave packets, known as generalized coherent state wave packets, and investigate the phenomena and quantum revivals and fractional revivals. In our study, we use information entropy as a probe to investigate the dynamical characteristics of generalized coherent state wave packets. In the following, we give a brief introduction of what is being presented in the thesis.

1.1.1 The coherent states: a historical review

The history of coherent states goes back to the early days of quantum mechanics while Schrödinger was developing the theory of wave mechanics. He faced the heuristic problem of representing a well-localized particle through spatially extended waves. The theory clearly exhibits that unitary time evolution of a free particle (i.e., wave packet) is deemed to spread it coherently over extended regions of space. This coherent spreading posed a serious challenge for various approaches that aim to associate narrowly peaked wavepackets with particles. Thus the fundamental question at hand was: How to represent classically well-localized particles through quantum mechanical wave packets that tend to disperse inherently over macroscopic distances? In an attempt to build a one-to-one correspondence between wavepackets and particles Schrödinger, in a famous 1926 paper entitled "*The continuous transition from micromechanics to macromechanics*", considered the example of simple harmonic oscillator. He constructed a wave packet, through a particular linear superposition of spatial solutions of the Schrödinger equation, that is not only well-localized at time $t = 0$ but also remain well-localized over subsequent evolution for time $t > 0$.

Its peak oscillates back and forth bearing an evident similarity with a classical point particle [1], hence, provide the classical-quantum correspondence.

The minimum uncertainty quantum states (non-spreading minimum uncertainty wavepackets) remained dormant more than three decades since their early introduction. Roy Glauber in 1963 re-defined these quantum states of harmonic oscillator in terms of the ladder operator of harmonic oscillator [2, 3, 4]. He expressed the coherent electromagnetic field by means of these states and hence named as *coherent states*. He defined the coherent states in three different ways as: i) an eigen states of annihilation operator; ii) the displaced ground states of harmonic oscillator and; iii) minimum uncertainty states. These three definitions were shown to be mutually equivalent. The coherent states have been used extensively in many areas of quantum physics, especially in quantum optics and quantum information. The importance of these states were acknowledged by rewarding Roy Glauber with 2005 Nobel Prize in Physics.

The coherent states exhibit a set special properties. One of the most striking features of the coherent states is their temporal stability, which means that a coherent state remains coherent under time evolution. Moreover, these states are nonorthogonal but yet hold the completeness relation which results in an other important feature known as over-completeness, which means that any coherent state can be represented in terms of other coherent states. Hence there are more than enough states available to represent one coherent state in terms of others [5].

1.1.2 Generalized coherent states

The overwhelming success of the coherent states of harmonic oscillator (Glauber coherent states) in different areas of physics and mathematics [5], has motivated researchers to generalized the concept of coherent states for general systems beyond harmonic oscillator. The most commonly used procedure in this regard was to generalize, keeping in view a set of requirements, any one of the definitions of the Glauber's coherent states, i.e., the generalized coherent states should preserve some of the properties of coherent states of harmonic oscillator. The generalization techniques of this kind make use of the ladder operators and associated algebra of the pertaining system to construct corresponding coherent states.

The first step in this regard was presented by Klauder in 1963 when he developed a generalized formalism in which he described the relation between quantum dynamics and the classical dynamics [6, 7]. Afterwards, Klauder and Sudarshan presented the generalized coherent states based on Lie group algebra and Barut and Girardello developed the coherent states for non compact groups [8] which are known as Barut-Girardello coherent states. The concept was further generalized for all kind of Lie groups by Perelomov [9] and these states are known as Perelomov coherent states. The work on generalized coherent states was beautifully collected and arranger by Klauder and Skagerstam in the form of a book [10]. In this work the literature was classified on the basis of the applications of the coherent states in different fields of physics and mathematics.

In 1996, Klauder proposed a direct method to construct the generalized coherent states for a quantum mechanical system with degenerate spectrum [11], such as, hydrogen atom. This approach has no explicit dependence on underlying algebra of the system. Later on, this formalism was extended by Gazeau and Klauder for the systems with continuous and discrete, non-degenerate spectrum which is bounded below [12]. These states, known as Gazeau-Klauder coherent states (referred to as GK coherent states), hold a set of special properties. This formalism received a lot attention due to their algebraic independence. The GK coherent states were constructed for a vast range of Hamiltonian systems, such as, the infinite square well and Pöschl-Teller potential [13], the pseudoharmonic oscillator [14], the power-law potentials [15, 16], the triangular well potential [17], the Morse potential[18, 19], and single mode periodic potential systems [20]. Furthermore, another generalization technique for constructing coherent states was introduced by R. F. Fox [21], in which he used a Gaussian function to approximate the behavior of the coherent states. This formalism was named as Gaussian Klauder coherent states.

In contrast to the classical-like behavior of the coherent states of harmonic oscillator, the generalized coherent states exhibit several characteristics which do not have classical analogue. These properties are known as non-classical properties. The non-classicality of coherent states plays an important role in quantum physics and have many applications in quantum information and quantum communication such as quantum teleportation [22], quantum computation, quantum cryptography and interferometric measurements [23]. In this thesis we study the construction of generalized coherent states, following Gazeau-Klauder formalism, and then construct corresponding wave packets in position space and momentum space and analyze their dynamical properties.

1.1.3 Quantum revivals

As pointed out by Schrödinger in his seminal paper that the wave packets of harmonic oscillator, namely coherent state wavepackets, follow classical trajectories during their time evolution. However, this classical-like dynamics is, generally, not exhibited by wavepackets of general quantum systems. In general, wavepacket dynamics exhibits non-classical features, such as, quantum revivals and fractional revivals. A quantum wave packet in its early evolution follows classical mechanics and exhibits classical periodicity only for a short time following classical trajectories. Later on, it spreads following wave mechanics and observes a collapse during long time evolution. However, discreteness of the quantum system leads to the reconstruction of the initial wavepacket, i.e., quantum revivals. This phenomenon is purely quantum mechanical in nature and does not have its classical analogue. The revival dynamics of wave packets of highly-excited states of atoms and molecules, were discussed by Averbukh and Perelman [24] and then a general discussion by Bluhm and Kostelecky was appeared. There have been developments in the field since then, and many of the basic quantum mechanical concepts behind revival behavior have also appeared in literature pedagogically [25]. Therefore it seems appropriate to provide a brief review of some of the fundamental concepts behind the phenomena.

Over the last one and a half decades the theoretical analysis, numerical prediction and experimental verification of the occurrence of wavepacket revivals in quantum systems has flourished a lot. An important tool used to measure the phenomena of wave packet revivals [25] in coherent states is the autocorrelation function $A(t)$ which measure the overlap of time dependent coherent states with its initial one. The maximum value of autocorrelation function is unity which occurs when time evolved states resembles the initial state completely. However, in general it has the value less than unity when time-evolved states are significantly out of phase from initial ones. We can then describe quantum revivals as a periodic recurrence of the wave packets from its initially localized state. Apart from the complete revivals, fractions of the initial wavepackets appear at fractional multiples of the revival time which are known as fractional revivals.

1.1.4 Entropy of a quantum system

Entropy, introduced by Clausius in the mid 19th century, is an important concept in thermodynamics as well as in classical statistical mechanics. This phenomenological variable quantities the intrinsic irreversibility of a thermodynamical processes. Later on, Boltzmann linked it with the lack of information about a system. He defined entropy as, $S_B = K_B \ln \Gamma$, where K_B is the Boltzmann constant and Γ is the number of microstates which have the same macroscopic properties in a system. In classical representation of a statistical system, the microstates are defined as points in a continuous 2-dimensional phase space, where D represents the degrees of freedom of the system. Since these microstates cannot be counted in any meaningful sense therefore, he took the number as the ratio between the available phase space volume divided by the volume of a unit cell. Further, this volume of a unit cell is recognized as $\hbar^D$ where $\hbar$ is the Planck constant. Now consider the volume of phase space as Ω, the number of microstates come to be $\Gamma = \Omega/\hbar^D$. Later on, the concept of entropy has extensively been used in large variety of contexts. In the following we review some important types of entropy.

1.1.4.1 Shannon entropy

Shannon introduced another form of entropy [26], in the context of classical information theory of a random process. He linked the information gain from a system with the concept of entropy. For a discrete random variable X with possible outcomes $X_1, ...X_n$ having probability mass function $P(X)$, the Shannon entropy is defined as

$$H(X) = E[I(X)] = E[-\ln P(X)]. \tag{1.1.1}$$

Here $E[..]$ represents the expected value random variable, and $I(X)$ is the information content of X [27]. It is important to note that $I(X)$ itself is a random variable.

For a finite sample, the entropy can be written as:

$$H(X) = \sum_i P(x_i)I(x_i) = -\sum_i P(x_i)\ln(x_i). \tag{1.1.2}$$

1.1.4.2 von Neumann entropy

The classical definition of Shannon entropy was extended to quantum mechanical scenario by von Neumann with the help of density matrix approach. Quantum mechanically, the concept of microstates is described by the wave function of a system.The wave function contains all the information of the quantum system. There are a large number of microstates in a single macrostate of a system. The most general approach to describe a quantum mechanical system is density matrix approach. In this approach, a quantum mechanical system is described by a density matrix ρ. The standard definition of entropy in quantum mechanics, a generalized form of Boltzmanns expression, is termed as von Neumann entropy and expressed [28, 29] as

$$S(\rho) = -Tr(\rho \ln \rho), \tag{1.1.3}$$

where ρ must be diagonalizable in order to compute the von Neumann entropy. If we consider λ_i as the eigenvalues of the density matrix ρ, such that, $\lambda_i \geq 0$ and $\sum \lambda_i = 1$, then von neumann entropy is then given

$$S(\lambda) = -\sum_j \lambda \ln \lambda. \tag{1.1.4}$$

The von Neumann entropy is being extensively used in different forms (conditional entropies, relative entropies, etc.) in the framework of quantum information theory [30]. Entanglement measures are based on some quantity directly related to the von Neumann entropy. However, there have appeared in the literature several papers dealing with the possible inadequacy of the Shannon information measure, and consequently of the von Neumann entropy as an appropriate quantum generalization of Shannon entropy. The main argument is that in classical measurement the Shannon information measure is a natural measure of our ignorance about the properties of a system, whose existence is independent of measurement.

Conversely, quantum measurement cannot be claimed to reveal the properties of a system that existed before the measurement was made [31]. This controversy has encouraged some authors to introduce the non-additivity property of Tsallis entropy (a generalization of the standard Boltzmann Gibbs entropy) as the main reason for recovering a true quantal information measure in the quantum context, claiming that non-local correlations ought to be described because of the particularity of Tsallis entropy.

1.1.5 Differential entropy

As discussed above, the Shannon entropy is restricted to random variables with discrete values. An analogous formula for a continuous random variable with probability density function $f(x)$ can also be obtained as

$$h[f] = \mathrm{E}[-\ln(f(x))] = -\int_{\mathbb{X}} f(x)\ln(f(x))\,dx. \tag{1.1.5}$$

This formula is usually referred to as the continuous entropy, or differential entropy. We consider a finite measure as the bin size goes to zero. In the discrete case, the bin size is the (implicit) width of each of the n (finite or infinite) bins whose probabilities are denoted by p_n. As we generalize to the continuous domain, we must make this width explicit. To do this, start with a continuous function f discretized into bins of size Δ. By the mean-value theorem there exists a value x_i in each bin such that:

$$f(x_i)\Delta = \int_{i\Delta}^{(i+1)\Delta} f(x)\,dx, \tag{1.1.6}$$

and thus the integral of the function f can be approximated by

$$\int_{-\infty}^{\infty} f(x)\,dx = \lim_{\Delta\to 0}\sum_{i=-\infty}^{\infty} f(x_i)\Delta, \tag{1.1.7}$$

where this limit and “bin size goes to zero” are equivalent. We will denote

$$H^{\Delta} := -\sum_{i=-\infty}^{\infty} f(x_i)\Delta\log\left(f(x_i)\Delta\right), \tag{1.1.8}$$

and expanding the logarithm, we have

$$H^{\Delta} = -\sum_{i=-\infty}^{\infty} f(x_i)\Delta\log(f(x_i)) - \sum_{i=-\infty}^{\infty} f(x_i)\Delta\log(\Delta). \tag{1.1.9}$$

As $\triangle \to 0$, we have

$$\sum_{i=-\infty}^{\infty} f(x_i)\Delta \to \int_{-\infty}^{\infty} f(x)\,dx = 1, \tag{1.1.10}$$

$$\sum_{i=-\infty}^{\infty} f(x_i)\Delta \log(f(x_i)) \to \int_{-\infty}^{\infty} f(x)\log f(x)\,dx. \tag{1.1.11}$$

But note that $\log(\triangle) \to -\infty$ as $\triangle \to 0$, therefore we need a special definition of the differential or continuous entropy:

$$h[f] = \lim_{\Delta\to 0}\left(H^{\Delta} + \log\Delta\right) = -\int_{-\infty}^{\infty} f(x)\log f(x)\,dx, \tag{1.1.12}$$

which is referred to as the differential entropy.

1.2 Thesis outline

Our thesis work is organised as follow: In chapter (2) we discuss a general class of one-dimensional potentials with discrete energy spectrum, and as particular example the harmonic oscillator and the infinite square well have been discussed. First we get coherent states for harmonic oscillator by using the definition of Glauber and generalization of coherent states. The Gazeau-klauder a special type of coherent states is also discuss in this chapter, and we calculate GK Cs wavepackets for the harmonic oscillator and infinite square well. At the end of the chapter generalized coherent state wavepackets in position and momentum space calculate.

In chapter (3), we tell about the general formalism of probability densities and expectation values in quantum mechanics. After that we discuss heisenberg's uncertainty relation in general. As an example, we discuss the expectation values for infinite square well in position and momentum space, and also heisenberg uncertainty relations calculated for this system. In next section, the introduction of entropy in quantum mechanics. Furthermore, we discuss how to measure entropies for generalized coherent states wavepackets and heisenberg uncertainty relation in terms of entropies.

Our chapter (4), is dedicated to time evolution of coherent states and after that we calculate auto correlation function for quantum systems, as an example it is calculated for harmonic oscillator and infinite square well. And we gives there autocorrelation function plots for both harmonic oscillator and infinite square well. By understanding quantum mechanics in the phase space has led to the development of tools such as quantum carpets, the autocorrelation function. The position space and momentum space quantum carpets for harmonic oscillator and infinite square well as an example discuss in this chapter. And we discuss the quantum revivals by means of entropy and our main task is the comparison of auto correlation function with entropies. At the end of thesis, in chapter (5) we give summary and conclusions of our thesis.

Chapter 2

Generalized coherent state wavepackets

2.1 Introduction

The quantum wavepackets of harmonic oscillator, constructed by *Schrödinger* in 1926 [1], exhibit a dynamics which is closely related to the dynamical behaviour of classical harmonic oscillator and minimize the uncertainty relation. These specially constructed wavepackets were reformulated by Roy Glauber, three decades later, in terms of the ladder operators the harmonic oscillator and used them to express quantum state of coherent electromagnetic field, hence, named as coherent states.

The coherent states of the harmonic oscillator are very useful and have played an important role in many areas of physics. Because of their lot of applications, it important to generalize the idea of coherent states for other dynamical systems. Therefore, it is quite natural to inquire whether there are states that can preserve most of the useful properties of the coherent states and be utilizable to describe and simplify other quantum systems? In other words, how does one generalize the concept of coherent states in order to describe other dynamical systems? In this chapter, we discuss a formalism, proposed by Gazeau and Klauder, to generalized the concept of coherent states for a class of one-dimensional potentials. For the constructed generalized coherent states, quantum wavepackets are constructed in position space as well as in momentum space.

The Chapter is organized as follow: In section (2.2), the model of our study, generally one-dimensional potentials is discussed. The formalism of coherent states and their generalization is presented in section (2.3). Finally, we discuss the construction of generalized wavepackets for the coherent states of one-dimensional potentials in section (2.4).

2.2 One-dimensional potentials

The dynamics of a particle of mass m is governed by the Hamiltonian given as

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(x), \tag{2.2.1}$$

where $\hat{V}(x)$ represents a large class of potentials, The eigenvalue equation corresponding to above Hamiltonian is written as

$$\hat{H}|n\rangle = E_n|n\rangle; \quad n \geq 0, \tag{2.2.2}$$

where $|n\rangle$ and E_n are the eigenstates and eigenenergies of the pertaining system, respectively. The eigenenergies for the harmonic oscillator E_n can be obtained [32] as

$$E_n = \hbar\omega(n + \frac{1}{2}).$$

In the following, we discuss harmonic oscillator and infinite square well as particular examples of the one-dimensional potentials.

2.2.1 Harmonic Oscillator

The harmonic oscillator plays a very important role in classical and quantum mechanics. In this case, the Hamiltonian, given in Eq. (2.2.1) takes the form

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \tag{2.2.3}$$

where the constants are adjusted such that $V(x) = \frac{1}{2}m\omega^2\hat{x}^2$, with ω being the frequency of oscillator. The Hamiltonian (2.2.3) satisfies the Schrödinger equation

$$\hat{H}|n\rangle = E_n|n\rangle, \tag{2.2.4}$$

whose solution gives the eigenenergies as

$$E_n = \hbar\omega\left(n + 1/2\right), \quad n = 0, 1, 2, ... \tag{2.2.5}$$

and the position space eigenfunctions are expressed as

$$\psi_n(x) \equiv \langle x|n\rangle = \sqrt{\frac{\beta}{\sqrt{\pi}2^n n!}} H_n(\beta x)\exp\left(-\beta^2 x^2/2\right), \tag{2.2.6}$$

where, $H_n(\beta x)$ are the Hermite polynomials and $\beta = \sqrt{m\omega/\hbar}$.

2.2.2 Infinite Square Well

The dynamics of the particle of mass m confined in the one-dimensional infinite square well of length a is governed by the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}(x), \tag{2.2.7}$$

where, $\hat{V}(x)$, is an infinite square well potential such that $\hat{V}(x) = 0$ for $|x| < a$ and $\hat{V}(x) = \infty$ otherwise. By solving the corresponding Schrödinger equation, $\hat{H}|n\rangle = E_n|n\rangle$, the eigenenergies are obtained as

$$E_n = \frac{n^2\pi^2\hbar^2}{8ma^2}, \tag{2.2.8}$$

and the position space wavefunction as

$$\psi_n(x) \equiv \langle x|n\rangle = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right). \tag{2.2.9}$$

2.3 Coherent states and their generalizations

Before proceeding to the discussion of generalized coherent states and corresponding wavepackets, we first review the poineering work on the coherent states of harmonic oscillator. In the following, we first present the basic concepts and necessary mathematical details concerning these states and then present a technique, namely Gazeau-Klauder method, to generalize the notion of coherent states for general one-dimensional potentials.

2.3.1 Coherent states of harmonic oscillator

Initially, the coherent states were introduced for harmonic oscillator which have played a very important role many areas of physics. Glauber used these states in the context of quantum optics to describe the coherent electromagnetic field quantum mechanically. He defined these states by three different but equivalent ways. Before we go to the formal definitions of the coherent states, we first develop the necessary mathematical framework for the harmonic oscillator.

2.3.1.1 Algebraic structure of harmonic oscillator

The Hamiltonian of the harmonic oscillator, given in Eq. (2.2.3), can be expressed in terms of a pair of new operators $\hat{a}^\dagger\hat{a}$ as

$$\hat{H} = \left(\hat{a}^\dagger\hat{a} + 1/2\right)\omega, \tag{2.3.1}$$

where these new operators $(\hat{a}, \hat{a}^\dagger)$ are connected with the coordinate and momentum operators $(\hat{x}, \hat{p})$ as

$$\hat{a} = \frac{1}{\sqrt{2\omega}}\left(\omega\hat{x} + i\hat{p}\right) \text{ and } \hat{a}^\dagger = \frac{1}{\sqrt{2\omega}}\left(\omega\hat{x} + i\hat{p}\right). \tag{2.3.2}$$

The operators $\hat{a}^\dagger$, $\hat{a}$ and $\hat{a}^\dagger\hat{a} = \hat{N}$ are known as creation, annihilation and number operators [33], respectively.

The set of operators $\{\hat{a}^\dagger, \hat{a}, \hat{N}\}$ together with the unit operator $\hat{I}$ obey the following well known commutation relations

$$[\hat{a}, \hat{a}^\dagger] = \hat{I}, \ \ [\hat{a}, \hat{I}] = [\hat{a}^\dagger, \hat{I}] = 0. \tag{2.3.3}$$

The commutation relations of $\hat{N}$ with $\hat{a}$ and $\hat{a}^\dagger$ are given as

$$[\hat{a}, \hat{N}] = \hat{a}, \quad [\hat{a}^\dagger, \hat{N}] = -\hat{a}^\dagger. \tag{2.3.4}$$

The operators $\{\hat{a}^\dagger, \hat{a}, \hat{N}, \hat{I}\}$ span a Lie algebra, denoted as h_4. The corresponding Lie group is the Weyl-Heisenberg group H_4 [34].

2.3.1.2 Fock space

The Hilbert space for H_4 is known as Fock space which is spanned by the number eigenstates $\{|0\rangle, |1\rangle, |2\rangle, ...|n\rangle\}$ satisfying orthonormality $\langle n|\acute{n}\rangle = \delta_{n\acute{n}}$. The number operator obeys eigenvalue equation

$$\hat{N}\left|n\right\rangle = n\left|n\right\rangle. \tag{2.3.5}$$

Whereas the operators $\hat{a}^{\dagger}$, $\hat{a}$ act upon the number eigenstates as

$$\hat{a}^{\dagger}\left|n\right\rangle = \sqrt{n+1}\left|n+1\right\rangle \tag{2.3.6}$$

and

$$\hat{a}\left|n\right\rangle = \sqrt{n}\left|n-1\right\rangle, \tag{2.3.7}$$

where the condition

$$\hat{a}|0\rangle = 0 \tag{2.3.8}$$

defines the ground state $|0\rangle$ (also known as external state) of the oscillator. The Fock space $\{|n\rangle\}$ may be obtained by repeated application of the creation operator $\hat{a}^{\dagger}$ on the vacuum state $|0\rangle$. The Fock states thus obey completeness,

$$\sum_{n}\left|n\right\rangle\left\langle n\right| = \hat{I}, \tag{2.3.9}$$

with $\hat{I}$ being n-dimensional identity operator.

2.3.1.3 Construction of coherent states

Based on this algebra, the Glauber's coherent states (also known as field coherent states) can be constructed starting from any one of three but equivalent mathematical definitions [4]. In the following we briefly summarize all of the three definitions.

Definition 1: The coherent states $|z\rangle$ are the eigenstates of the harmonic oscillator annihilation operator $\hat{a}$, i.e.,

$$\hat{a}\left|z\right\rangle = z\left|z\right\rangle, \tag{2.3.10}$$

where z is a complex number.

Definition 2: They are generated by applying a displacement operator $\hat{D}\left(z\right)$ on the vacuum state $|0\rangle$ of the harmonic oscillator,

$$\left|z\right\rangle = \hat{D}\left(z\right)\left|0\right\rangle, \tag{2.3.11}$$

where the displacement operator $\hat{D}(z) = e^{z\hat{a}^\dagger - z^*\hat{a}}$, with $\hat{a}^\dagger$ being the harmonic oscillator creation operator.

Definition 3: They are the quantum states minimizing uncertainties relationship, i.e.,

$$\Delta x \Delta p = \frac{1}{2}. \tag{2.3.12}$$

This is easy to show by calculating the dispersions of position and momentum operators with respect to the coherent coherent states as

$$\begin{aligned} (\Delta x)^2 &= \langle z|\hat{x}^2|z\rangle - \langle z|\hat{x}|z\rangle^2, \\ (\Delta p)^2 &= \langle z|\hat{p}^2|z\rangle - \langle z|\hat{p}|z\rangle^2, \end{aligned}$$

where the position and momentum operators $\hat{x}$, $\hat{p}$ are expressed in terms of $\hat{a}, \hat{a}^\dagger$ as

$$\begin{aligned} \hat{x} &= \frac{1}{\sqrt{2}}(\hat{a} + \hat{a}^\dagger), \\ \hat{p} &= \frac{-i}{\sqrt{2}}(\hat{a} - \hat{a^\dagger}), \end{aligned}$$

respectively.

2.3.1.4 Fock space representation of coherent states

To find the states $|z\rangle$ we take the scalar product of both sides of the equation (2.3.10) with number state, $\langle n|$, i.e.,

$$\langle n|\hat{a}|z\rangle = z\langle n|z\rangle. \tag{2.3.13}$$

The Hermitian adjoint of equation (2.3.6) is given as,

$$\langle n|\hat{a} = \sqrt{n+1}\langle n+1|, \tag{2.3.14}$$

which leads to the recursion relation

$$\sqrt{n+1}\langle n+1|z\rangle = z\langle n|z\rangle \tag{2.3.15}$$

for the scalar products $\langle n|z\rangle$. We immediately note from equation (2.3.15) that

$$\langle n|z\rangle = \frac{z^n}{\sqrt{n!}}\langle 0|z\rangle. \tag{2.3.16}$$

These scalar products appear as the expansion coefficients of the state $|z\rangle$ in terms of Fock states, i.e.,

$$|z\rangle = \sum_n |n\rangle\langle n|z\rangle = \langle 0|z\rangle \sum_n \frac{z^n}{\sqrt{n!}}|n\rangle. \tag{2.3.17}$$

The arbitrary phase factor $\langle 0|z\rangle$ can be fixed by normalization condition, $\langle z|z\rangle = 1$, so that

$$\langle 0|z\rangle = \exp\left[-\frac{1}{2}|z|^2\right]. \tag{2.3.18}$$

The states $|z\rangle$ therefore take the form

$$|z\rangle = \exp\left[-\frac{1}{2}|z|^2\right] \sum_n \frac{z^n}{\sqrt{n!}}|n\rangle, \tag{2.3.19}$$

which is well-known representation of coherent states [5].

2.3.1.5 Properties of Coherent States

There are two basic properties which are termed as the minimum set of requirements for a set of states to be termed as coherent states, and which are in fact just the two properties shared by all kind of coherent states. The first property is the continuity in parameter space, and the second one is the completeness. In the following we comment on these two basic properties and other properties based on the definitions introduced in the previous section.

Continuity: The state vector $|z\rangle$ is a continuous function of the continuous complex parameter z, that is,

$$z \to \acute{z} \Rightarrow |z\rangle \to |\acute{z}\rangle\,. \tag{2.3.20}$$

Resolution of Unity: It can be shown that the unit operator may be expressed as an integral of the projection operators $|z\rangle\langle z|$ over the complex plane which means that these states provide a resolution of the identity with respect to a positive measure d^2z/π defined over the complex plane, that is,

$$\frac{1}{\pi}\int |z\rangle\langle z|\, d^2z = \sum_{n=0}^{\infty} |n\rangle\langle n| = \hat{I}. \tag{2.3.21}$$

This equation can be proved by expanding $|z\rangle$ in harmonic oscillator eigenstates, that is, using *definition 3*, and using the identity

$$\frac{1}{\pi}\int e^{-|z|^2}(z^*)^n(z)^m d^2z = n!\delta_{nm}. \qquad (2.3.22)$$

It appears that Eq. (2.3.21) is exactly like a resolution of unity, but we will see in the following that it differs from the conventional one, which involves mutually orthogonal states, in that the one dimensional projection operators $|z\rangle\langle z|$ are not in general mutually orthogonal.

Non-orthogonality: One property which is made clear by the *definition 3* is that two such states are not in general orthogonal to one another. The scalar product $\langle z|\,\acute{z}\rangle$ can be calculated more simply as

$$\begin{aligned}\langle z|\,\acute{z}\rangle &= e^{-\frac{1}{2}|z|^2-\frac{1}{2}|\acute{z}|^2}\sum_{n,m}\frac{(z^*)^n(\acute{z})^m}{\sqrt{n!m!}}\langle n|\,m\rangle \\ &= e^{-\frac{1}{2}|z|^2-\frac{1}{2}|\acute{z}|^2+z^*\acute{z}} \neq 0. \qquad (2.3.23)\end{aligned}$$

Note that

$$|\langle z|\,\acute{z}\rangle|^2 = \exp\left(-|z-\acute{z}|^2\right), \qquad ()$$

which shows that the coherent states tend to become approximately orthogonal in case z and $\acute{z}$ recede much from one another in the complex plane. This non-orthogonality shows that coherent states in fact provide an over complete set.

Stability Under Time Evolution: Under the dynamical evolution, a coherent state remains in the family of coherent states but under a different label. This can be seen as

$$\begin{aligned}e^{-i\hat{H}t}|z\rangle &= e^{-\frac{1}{2}|z|^2}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!}}e^{-in\omega t}|n\rangle \\ &= |\,ze^{-i\omega t}\rangle. \qquad (2.3.24)\end{aligned}$$

The term coherent reflects in fact this property that such states evolve coherently in time. They remain non-spreading wave packets remaining localized around the corresponding classical trajectory, thereby, minimizing the uncertainty product [5].

2.3.2 Generalized coherent states: Gazeau-Klauder type

Gazeau and Klauder proposed the construction of coherent states for Hamiltonian H with discrete spectrum which is bounded below and can be adjusted so that $H \geq 0$, with out any explicit dependence on group properties [12]. The eigenstates $|n\rangle$ of H are assumed to be non-degenerate and orthonormal that satisfy

$$H|n\rangle = E_n|n\rangle, \quad n \geq 0. \tag{2.3.25}$$

The energy spectrum is arranged in the increasing order such that

$$0 = E_0 < E_1 < E_2 < ... \tag{2.3.26}$$

Moreover, $E_n = \omega e_n$, where, e_n is the dimensionless real parameter for some fixed energy scale $\omega > 0$ and follows the sequence

$$0 = e_0 < e_1 < e_2 ... \tag{2.3.27}$$

for definiteness. We will use $m = \hbar = 1$ through out our discussion unless otherwise stated. The Gazeau-Klauder generalized coherent state are labelled by two real parameters J $(0 \leq J)$ and θ $(-\infty < \theta < \infty)$ and are described by the state

$$|J, \theta\rangle = N(J)^{-\frac{1}{2}} \sum_{n=0}^{\infty} \frac{J^{\frac{n}{2}}}{\sqrt{\rho_n}} e^{-ie_n\theta} |n\rangle, \tag{2.3.28}$$

where the positive constants ρ_n are defined as

$$\rho_n = e_1 e_2 ... e_n, \quad \rho_0 = 1. \tag{2.3.29}$$

The normalization factor $N(J)$ is defined as

$$N(J) = \sum_{n=0}^{\infty} \frac{J^n}{\rho_n}, \tag{2.3.30}$$

which guarantees that

$$\langle J, \theta | J, \theta\rangle = 1. \tag{2.3.31}$$

The domain of allowed values of J, is determined by the radius of convergence R in the series (2.3.30) which is defined as

$$R = \lim_{n\to\infty} \sqrt[n]{\rho_n}, \tag{2.3.32}$$

such that $0 \leq J < R$.

The coherent states (2.3.28) are assumed to satisfy the set of conditions given as
(i) continuity: $(J,\theta) \to (J',\theta') \quad \Rightarrow \quad |J,\theta\rangle \to |\,J',\theta'\rangle$.
(ii) Resolution of unity: $1 = \int d\mu\,(J,\theta)\,|J,\theta\rangle\,\langle J,\theta|$.
(iii)Temporal stability: $e^{-i\hat{H}t}\,|J,\theta\rangle = |\,J,\,\theta+\omega t\rangle$.
(iv)Action identity: $\langle J,\theta|\,\hat{H}\,|J,\theta\rangle = \omega J$.
However, we are not going to discuss the details of these properties in this thesis. In the following, we discuss the Gazeau-Klauder coherent states for power-law potentials [15].

2.3.2.1 Gazeau-Klauder coherent states for one-dimensional potentials

Following the Gazeau-Klauder formalism, given above, we present generalized coherent states for one-dimensional potentials. The Hamiltonian is $\hat{H}^{(k)}$, given in Eq. (2.2.1), can be factorized as

$$\hat{H} = \omega\hat{\chi}_N, \tag{2.3.33}$$

where $\hat{\chi}_N$ is a dimensionless Hamiltonian and ω is a constant with dimensions of energy. The Hamiltonian $\hat{\chi}_N$ obeys the eigenvalue equation

$$\hat{\chi}_N|n\rangle = e_n|n\rangle, \tag{2.3.34}$$

where e_n are dimensionless eigenvalues given by

$$e_n = \frac{E_n - E_0}{\omega}. \tag{2.3.35}$$

The energy spectrum e_n is an increasing function of n, $e_{n+1} > e_n$, with $e_0 = 0$. Hence, the generalized coherent states for one-dimensional potentials take the form as

$$|(J,\theta)\rangle = \frac{1}{\sqrt{N(J)}}\sum_{n=0}^{\infty}\frac{J^{\frac{n}{2}}e^{-ie_n\theta}}{\sqrt{\rho_n}}|n\rangle. \tag{2.3.36}$$

The quantities ρ_n are defined in terms of e_n, given by Eq. (2.3.35) , as

$$\rho_n \equiv \prod_{j=1}^{n} e_j. \tag{2.3.37}$$

It is important to note that the generalized coherent states for one-dimensional potentials, given in Eq. (2.3.36), are based on eigenenergies obtained by solving the schrodinger equations of concerning physical systems. However, in the following we discuss the particular cases for the harmonic oscillator and the infinite square well, respectively.

2.3.2.2 Harmonic oscillator

For harmonic oscillator, we find $e_n = n$ and $\rho_n = n!$, from Eqs. (2.3.35) and (2.3.37), respectively. Thus GK CS for the present system is obtained as,

$$|(J,\theta)\rangle = \frac{1}{\sqrt{N(J)}} \sum_{n=0}^{\infty} \frac{J^{\frac{n}{2}} e^{-in\theta}}{\sqrt{n!}} |n\rangle \,, \tag{2.3.38}$$

where the normalization constant $N(J)$ is,

$$N(J) = \sum_{n=0}^{\infty} \frac{J^n}{n!}, \tag{2.3.39}$$

and the weighting distribution,

$$|c_n|^2 = \frac{1}{N(J)} \frac{J^n}{n!}. \tag{2.3.40}$$

A transformation from (J,θ) to z, such that, $z = \sqrt{J} e^{-i\theta}$, introduces a complete equivalence between Gazeau-Klauder coherent states of the harmonic oscillator, given by Eq. (2.3.38), and Glauber's coherent states [1, 2, 4]. In this case, the normalization factor in Eq. (2.3.39) is $N(J) = e^{|z|^2}$, and weighting distribution in Eq. (2.3.40) is $|c_n|^2 = |z|^{2n} e^{-|z|^2}/n!$ which is Poisson distribution as for Glauber's coherent states.

2.3.2.3 Infinite Square Well

The infinite square well as seen from Eq. (2.2.2), as discussed above. Thus, eigenvalues $e_n = n(n+2)$, and the parameter $\rho_n = n!(n+2)!/2$ are obtained, respectively, from Eq. (2.3.35) and Eq. (2.3.37). The generalized coherent states for infinite square well are therefore, written as

$$|(J,\theta)\rangle = \frac{1}{\sqrt{N(J)}}\sum_{n=0}^{\infty}\frac{J^{\frac{n}{2}}e^{-in(n+2)\theta}}{\sqrt{n!(n+2)!/2}}|n\rangle\,, \tag{2.3.41}$$

with the normalization factor

$$N(J) = 2\sum_{n=0}^{\infty}\frac{J^n}{n!(n+2)!},$$

and weighting distribution

$$|c_n|^2 = \frac{2}{N(J)}\frac{J^n}{n!(n+2)!}.$$

It is important to note here that the weighting distribution of the coherent states is not Poisson in the case of infinite square well.

2.4 Generalized coherent state wave packets

In the above sections, so far we have introduced the basic theory of coherent states and their generalization using Gazeau-Klauder formalism. In general, the coherent states have been considered as a special kind of linear superposition of the energy eigenstates of corresponding systems, such that, the weighting distribution of this superposition is continuously parameterized. However, in our later discussion we are interested in finding the wavepackets corresponding these generalized coherent states in some Hilbert space of continous variable, such as, position and momentum.

The quantum wavepackets corresponding to the generalized coherent states can be obtained as

$$\langle\zeta|(J,\theta)\rangle = \frac{1}{\sqrt{N(J)}}\sum_{n=0}^{\infty}\frac{J^{\frac{n}{2}}e^{-ie_n\theta}}{\sqrt{\rho_n}}\phi_n(\zeta), \tag{2.4.1}$$

where $\phi_n(\zeta) \equiv \langle\zeta|n\rangle$. Here, $|\zeta\rangle$ are considered as eigenkets corresponding an observable $\hat{\zeta}$ with continous eigenspectrum, that is, $\hat{\zeta}|\zeta\rangle = \zeta|\zeta\rangle$, where ζ is continous variable. In our ongoing discussion, we are interested in position space and momentum space wavepackets of generalized Gezeau-Klauder coherent states.

2.4.1 Position space wavepackets

Following Eq. (2.4.1), the position space wavepackets for generalized coherent states can be given as

$$\langle x|(J,\theta)\rangle = \frac{1}{\sqrt{N(J)}}\sum_{n=0}^{\infty}\frac{J^{\frac{n}{2}}e^{-ie_n\theta}}{\sqrt{\rho_n}}u_n(x), \tag{2.4.2}$$

where $u_n(x)$ is the position space eigenfunction of the pertaining system and $|x\rangle$ are the eigenkets of position operator $\hat{X}$ satisfying $\hat{X}|x\rangle = x|x\rangle$, where x is a position variable.

2.4.2 Momentum space wavepackets

Following the same procedure mentioned above, the momentum space wavepackets for generalized coherent states take the form

$$\langle p|(J,\theta)\rangle = \frac{1}{\sqrt{N(J)}}\sum_{n=0}^{\infty}\frac{J^{\frac{n}{2}}e^{-ie_n\theta}}{\sqrt{\rho_n}}\varphi_n(p), \tag{2.4.3}$$

where $|p\rangle$ are the eigenkets of momentum operator $\hat{P}$ satisfying $\hat{P}|p\rangle = p|p\rangle$. The momentum space eigenfunction $\varphi_n(p)$ are obtained by Fourier transform of the corresponding position-space wave function $u_n(x)$.

Chapter 3

Measurement uncertainty and entropy in quantum mechanics

3.1 Introduction

In contrast to purely deterministic classical theory, the quantum theory is probabilistic in nature: a physically observable quantity (referred as *observable*) can only be measured probabilistically. Therefore, in order to get information about an observable, ensemble measurements are performed and then average of the measured values, known as expectation value, is calculated. By means of these expectation values and corresponding variances, the uncertainty associated with the measurement process can be calculated. Generally, for quantum mechanical system, the measurements concerning two different observables do not commute (The observables that satisfy this condition are known as incompatible observables). As a result, we cannot simultaneously measure two incompatible observables precisely. The uncertainties associated with the simultaneous measurements of these observables are expressed by means of a principle, known as Heisenberg uncertainty principle. In standard quantum mechanics the Heisenberg uncertainty principle is expressed in terms of the variances based on the expectation values of observables. However, there is an other very elegant method to express the Heisenberg uncertainty principle, that is, in terms of entropy. In this chapter, after discussing variance-based uncertainty principle, we will present the equivalent expressions in terms of entropy.

In section (3.2) we describe the general formalism for the expectation values, dispersions and Heisenberg uncertainty relation and a particular case of generalized coherent state wavepackets is discussed in section (3.3). In this section, we first the calculate probability densities in position and momentum space in order to calculate the corresponding expectation values and then calculate the uncertainties to express the Heisenberg uncertainty relation. In section (3.4) we present the entropic measurements for generalized coherent states wavepackets and obtain the equivalent form of Heisenberg uncertainty relation in terms of entropy.

3.2 Quantum measurements and Heisenberg uncertainty relation

It is a very basic postulate of standard quantum mechanics, that a general state of a physical system is represented by a vector, $|\psi\rangle$, in a Hilbert space which is known as states vector. All the retrievable information concerning the physical system is contained in the state vector. Moreover, an observable is represented by a self-adjoint operator, A, on the Hilbert space. The expectation value of the observable A is then calculated as

$$\langle A\rangle = \langle\psi|A|\psi\rangle, \tag{3.2.1}$$

where the state vector $|\psi\rangle$ is considered normalized, i.e., $\langle\psi|\psi\rangle = 1$. For a space of continuous variable ζ the expectation value can be written as

$$\langle A\rangle = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\psi^*(\zeta')\langle\zeta'|A|\zeta\rangle\psi(\zeta)d\zeta' d\zeta, \tag{3.2.2}$$

where the set of vectors $\{|\zeta\rangle\}$ spans the complete orthonormal basis for the Hilbert space, i.e., $\int_{-\infty}^{\infty}|\zeta\rangle\langle\zeta|d\zeta = 1$, and $\langle\zeta'|\zeta\rangle = \delta(\zeta-\zeta')$. If $|\zeta\rangle$ is an eigenket of operator A, then Eq. (3.2.2) can be written as

$$\begin{aligned}\langle A\rangle &= \int_{-\infty}^{\infty}a_\zeta\psi^*(\zeta)\psi(\zeta)d\zeta,\\ &= \int_{-\infty}^{\infty}a_\zeta|\psi(\zeta)|^2 d\zeta,\end{aligned} \tag{3.2.3}$$

where a_ζ is an eigenvalue of operator A, that is, $A|\zeta\rangle = a_\zeta|\zeta\rangle$. From Eq. (3.2.3) it is important to note that

$$f(\zeta) = |\psi(\zeta)|^2, \tag{3.2.4}$$

is a probability density function, such that, $f(\zeta)d\zeta$ represents the probability of measuring ζ in interval $\zeta + d\zeta$. For this probability density function, the dispersion from the measured average value (expectation value) is given as

$$(\Delta A)^2 = \langle A^2\rangle - \langle A\rangle^2. \tag{3.2.5}$$

The quantity ΔA measures the uncertainty associated with the measurement of observable A. In general, for two observables A and B, their measurement uncertainties are related as

$$\Delta A \Delta B \geq \frac{1}{2}|\langle[A, B]\rangle|, \tag{3.2.6}$$

which is the generalized form of Heisenberg uncertainty principle. It is important to note that the commutator $[A, B]$ vanishes if observables A and B are compatible.

For time-independent solutions of *Schrödinger* equation, the expectation values of position and momentum operators can be expressed, using Eq. (3.2.3), as

$$\langle X\rangle = \int_{-\infty}^{\infty} x\psi^*(x)\psi(x)dx = \int_{-\infty}^{\infty} x|\psi(x)|^2 dx, \tag{3.2.7}$$

$$\langle P\rangle = \int_{-\infty}^{\infty} p\phi^*(p)\phi(p)dp = \int_{-\infty}^{\infty} p|\phi(p)|^2 dp, \tag{3.2.8}$$

respectively, where the eigenvalue equations $X|x\rangle = x|x\rangle$ and $P|p\rangle = p|p\rangle$ have been used to obtain the above results. In this case the Heisenberg uncertainty relation takes the form

$$\Delta X \Delta P \geq \frac{\hbar}{2}, \tag{3.2.9}$$

where we have used the fact that $[X, P] = i\hbar$.

3.3 Uncertainty relation for generalized coherent state wavepackets

The expectation value is a probabilistic measurement, it may have zero probability to occur. We calculate the expectation values and uncertainties for position and momentum operators measured with respect to generalized coherent states. By means of these uncertainties we calculate the Heisenberg uncertainty relation. As an example we consider the case of infinite square well and calculate the analytic expressions for the expectation values and uncertainties with respect to corresponding generalized coherent states.

3.3.1 Probability density and expectation values in position space

As shown by Eq. (3.2.7), the expectation values can be calculated by means of probability density function. In order to calculate the expectation value of position operator, we need to calculate the probability densities in position space. From Eq. (2.4.2), the probability density in position space can be calculated as

$$P(x,J,\theta) = |\langle x|(J,\theta)\rangle|^2 . \tag{3.3.1}$$

The time independent probability density for the infinite square well in position space from Eq. (2.4.2), is written as

$$P(x,J,\theta) = \sqrt{\frac{1}{\mathcal{N}(J)}} \sum_{n=0}^{\infty} \frac{J^{n/2} e^{-in\theta}}{\sqrt{n!(n+2)!}} \psi_n(x). \tag{3.3.2}$$

Now the expectation value of in position space for generalized coherent states is,

$$\langle x\rangle = \int xP(x,J,\theta)dx. \tag{3.3.3}$$

For the infinite square well, by putting the value of probability density in the above equation, we get

$$\langle x\rangle = \frac{1}{N^{\infty}(J)} \sum_{n,m} \frac{J^{\frac{m+n}{2}} \exp^{i(m(m+2)-n(n+2))\Theta}}{\sqrt{m!\frac{(m+2)!}{2} n!\frac{(n+2)!}{2}}} \left[\frac{2}{L}\int_0^L x\sin(\frac{n\pi x}{L})\sin(\frac{m\pi x}{L})dx\right], \tag{3.3.4}$$

for m = n

$$\langle x\rangle = \frac{1}{N^{\infty}(J)} \sum_{n} \frac{J^{n}}{\sqrt{n!\frac{(n+2)!}{2}}} \left(\frac{\pi}{2}\right). \tag{3.3.5}$$

Now from Eq. (2.3.2.3)

$$N(J) = 2\sum_{n=0}^{\infty} \frac{J^n}{n!(n+2)!}, \tag{3.3.6}$$

putting Eq. (3.3.6) in Eq. (3.3.5), we have

$$\langle x \rangle = \frac{\pi}{2}. \tag{3.3.7}$$

For $n \neq m$, Eq. (3.3.5) becomes

$$\langle x \rangle = \frac{1}{N(J)} \sum_{n,m} \frac{J^{\frac{m+n}{2}} \exp^{i(m(m+2)-n(n+2))\Theta}}{\sqrt{m!\frac{(m+2)!}{2}n!\frac{(n+2)!}{2}}} \left[\frac{4mn[(-1)^{m+n}-1]}{\pi(m^2-n^2)^2}\right]. \tag{3.3.8}$$

To calculate the variance in position space we need to calculate the expectation value for x^2, and it is written as

$$\langle x^2 \rangle = \int x^2 P(x, J, \theta) dx, \tag{3.3.9}$$

$$\langle x^2 \rangle = \frac{1}{N(J)} \sum_{n,m} \frac{J^{\frac{m+n}{2}} \exp^{i(m(m+2)-n(n+2))\Theta}}{\sqrt{m!\frac{(m+2)!}{2}n!\frac{(n+2)!}{2}}} \left[\frac{2}{L}\int_0^L x^2 \sin(\frac{n\pi x}{L}) \sin(\frac{m\pi x}{L}) dx\right]. \tag{3.3.10}$$

For $m = n$

$$\langle x^2 \rangle = \frac{2n^2\pi^2 - 3}{6n^2}. \tag{3.3.11}$$

The Eq. (3.3.7) and Eq. (3.3.11) are use to calculate the dispersion relation in position space.

3.3.2 Probability density and expectation values in momentum space

The probability density in momentum space, for time independent wave function is written in general as:

$$P(p) = \phi^*(p)\phi(p). \tag{3.3.12}$$

The time independent probability density in momentum space for generalized coherent states is from Eq. (2.4.3)

$$P(p, J, \theta) = |\langle p|(J, \theta)\rangle|^2, \tag{3.3.13}$$

and also written as

$$P(p,J,\theta)=\sqrt{\frac{1}{\mathcal{N}(J)}}\sum_{n=0}^{\infty}\frac{J^{n/2}e^{-in\theta}}{\sqrt{n!(n+2)!}}\phi_n(p). \tag{3.3.14}$$

The expectation value in momentum space for the infinite square well is,

$$\langle p\rangle=\int pP(p,J,\theta)dp. \tag{3.3.15}$$

By applying the Fourier transform on position dependent wave function,the expectation value of p is

$$\langle p\rangle=\frac{1}{N(J)}\sum_{n,m}\frac{J^{\frac{m+n}{2}}\exp^{i(m(m+2)-n(n+2))\Theta}}{\sqrt{m!\frac{(m+2)!}{2}n!\frac{(n+2)!}{2}}}\Big[-i\hbar\int_0^L\sin(\frac{m\pi x}{L})\frac{d}{dx}\sin(\frac{n\pi x}{L})dx\Big]. \tag{3.3.16}$$

Now taking, m=n the expectation value is

$$\langle p\rangle=0, \tag{3.3.17}$$

and for $m\neq n$ is

$$\langle p\rangle=\frac{1}{N(J)}\sum_{n,m}\frac{J^{\frac{m+n}{2}}\exp^{i(m(m+2)-n(n+2))\Theta}}{\sqrt{m!\frac{(m+2)!}{2}n!\frac{(n+2)!}{2}}}\Big[\frac{2\cos[(m-n)\csc\pi]}{(m-n)^2}-\frac{2\cos[(m+n)\csc\pi]}{(m+n)^2}\Big]. \tag{3.3.18}$$

To calculate the dispersion in momentum space we need to calculate the expectation values for p and p^2. Now the expectation value of p^2 is

$$\langle p^2\rangle=\int p^2P(p,J,\theta)dp, \tag{3.3.19}$$

$$\langle p^2\rangle=\frac{1}{N(J)}\sum_{n,m}\frac{J^{\frac{m+n}{2}}\exp^{i(m(m+2)-n(n+2))\Theta}}{\sqrt{m!\frac{(m+2)!}{2}n!\frac{(n+2)!}{2}}}\Big[-\hbar^2\int_0^L\sin(\frac{m\pi x}{L})\frac{d^2}{(dx)^2}\sin(\frac{n\pi x}{L})dx\Big], \tag{3.3.20}$$

Now the expectation value of p^2 for m=n and $m \neq n$ is

$$\langle p^2 \rangle = \begin{cases} n^2 & \text{for} \quad m = n \\ 0 & \quad m \neq n. \end{cases} \tag{3.3.21}$$

By using the Eq. (3.3.17) and Eq. (3.3.21), dispersion relation in momentum space is calculated.

3.3.3 Variance-based uncertainty relation

The uncertainty principle, which is also known as Heisenberg's uncertainty principle. It is mathematical inequalities which exerts a fundamental limit to the precision on physical properties of a known particle. As we calculate dispersions for uncertainty relation in position and momentum space for infinite square well.

For calculating dispersion in position space we use expectation values of x and x^2

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}, \tag{3.3.22}$$

now, for $m = n$ putting values of $\langle x \rangle$ and $\langle x^2 \rangle$,

$$\Delta x = \sqrt{\frac{2n^2\pi^2 - 3}{6n^2} - \frac{\pi^2}{4}}, \tag{3.3.23}$$

$$\Delta x = \frac{1}{2\sqrt{3}}\sqrt{\pi^2 - \frac{6}{n^2}}. \tag{3.3.24}$$

Now here we use momentum expectation values of p and p^2 to calculate dispersion in momentum space. For momentum space,

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}, \tag{3.3.25}$$

For $m = n$, putting values of $\langle p \rangle$ and $\langle p^2 \rangle$ in above equation

$$\Delta p = \sqrt{n^2 - 0} = n, \tag{3.3.26}$$

so, now

$$\Delta p = n. \tag{3.3.27}$$

The heisenberg uncertainty relation is now written as: for m=n,

$$\Delta x \Delta p = \frac{1}{2\sqrt{3}} \sqrt{\pi^2 - \frac{6}{n^2}} \cdot n,$$

$$= \frac{1}{2\sqrt{3}} \sqrt{n^2\pi^2 - 6}. \tag{3.3.28}$$

The uncertainty relation for the infinite square well is described in above equation. This uncertainty relation depends only on n.

3.4 Entropic measurements for generalized coherent states

From the wave packet tunneling through a barrier are investigated by means of information entropy $S_x = -\int P(x) \ln P(x) dx$, [40, 41]. We investigate entropies for generalized coherent states in position and momentum space from the given position space and momentum space wave packets. Now as an example we consider the system infinite square well for to calculate the entropy in position and momentum space.

3.4.1 Entropy in position space

To calculate the entropy in position space from the definition of information entropy we use probability density from Eq. (3.3.1)

$$S(x, J, \theta) = -\int P(x, J, \theta) \ln P(x, J, \theta) dx. \tag{3.4.1}$$

The probability density for the infinite square well is,

$$P(x, J, \theta) = \sqrt{\frac{1}{\mathcal{N}(J)}} \sum_{n=0}^{\infty} \frac{J^{n/2}}{\sqrt{n!(n+2)!}} \psi_n(p). \tag{3.4.2}$$

By putting the value of momentum probability density from Eq. (3.4.2) to Eq. (3.4.1) and value of $J = 38.05$, After integrating value of position entropy is,

$$S(x, 38.05, \theta) = 5.71. \tag{3.4.3}$$

3.4.2 Entropy in momentum space

From the Eq. (3.3.13) we use now probability density in momentum space. Now entropy is written as:

$$S(p, J, \theta) = -\int P(p, J, \theta) \ln P(p, J, \theta) dp. \tag{3.4.4}$$

The probability density for infinite square well is,

$$P(p, J, \theta) = \sqrt{\frac{1}{\mathcal{N}(J)}} \sum_{n=0}^{\infty} \frac{J^{n/2}}{\sqrt{n!(n+2)!}} \phi_n(p). \tag{3.4.5}$$

By putting the value of momentum probability density from Eq. (3.4.5) to Eq. (3.4.4) and value of, momentum entropy is,

$$S(p, 38.05, \theta) = 0.15. \tag{3.4.6}$$

3.4.3 Uncertainty relation in terms of entropy

The entropic uncertainty is defined as sum of the temporal and spectral Shannon entropies. It turns out that Heisenberg's uncertainty principle can be expressed as a lower bound on the sum of these entropies. This is stronger than the usual statement of the uncertainty principle in terms of the product of standard deviations. The entropic formulation of uncertainty principle provides a lower bound on the sum of the information entropy of two distributions. For example, for position and momentum, there is the bound.

For infinite square well the heisenberg's uncertainty relation is written as from Eq. (3.4.3) and Eq. (3.4.6)

$$S(x) + S(p) > 1 + \ln \pi, \tag{3.4.7}$$

where,

$$S(x) = -\int P(x, J, \theta) \ln P(x, J, \theta) dx, \tag{3.4.8}$$

and

$$S(p) = -\int P(p, J, \theta) \ln P(p, J, \theta) dp. \tag{3.4.9}$$

In Eq. (3.4.8) and Eq. (3.4.9) the continuous information entropy of the distribution of x, as given by it's wave function and $S(p)$ is defined as information entropy of the distribution of p.

Chapter 4

Coherent state wavepacket revivals by means of entropy

4.1 Introduction

The quantum wavepacket revivals phenomenon has received much attention over last years. It has been investigated theoretically in atomic and molecular quantum systems , and observed experimentally in Rydberg wave packets in atoms and molecules, molecular vibrational states, and Bose-Einstein condensed sates. Revivals occur when a wavepacket solution of the Schrödinger equation evolves in time and it closely reproduces its initial state [44]. Now we are focusing on the temporal characteristics of time evolved coherent states. We explain it in next section in form of the autocorrelation function.

In this chapter our work is organized as, in section (4.2) we explain the time evolution of generalized coherent states. Furthermore, in section (4.2.1) autocorrelation function and revivals of quantum also discuss. The spatio-temporal and momento-temporal quantum carpets for harmonic oscillator and infinite square well given in sections (4.2.2) and (4.2.3,) respectively . Finally, in section (4.3)we discuss quantum revivals by using the entropy in both position and momentum space. We compare the sum of both entropies in position and momentum space with autocorrelation function.

4.2 Time evolution of generalized coherent states

In order to investigate the temporal behavior of the generalized coherent state wavepackets, we first study their time evolution. The time-evolved coherent states are obtained by applying the time evolution operator on initial coherent states. The time evolution operator for one-dimensional potentials is expressed as

$$\hat{U}(t) = \exp\left[-i\hat{H}t\right], \tag{4.2.1}$$

where $\hat{H}$ is the Hamiltonian for one-dimensional potentials. The operator $\hat{U}(t)$ transforms the coherent states from the initial time $t_0 = 0$ to any latter time, t, such that

$$\hat{U}(t)\,|(J,\theta)\rangle = |\,(J,\theta),t\rangle.$$

The time evolved coherent state wave packets can be expressed as

$$|\,(J,\theta),t\rangle \;=\; \frac{1}{\sqrt{N(J)}}\sum_{n=0}^{\infty}\frac{J^{\frac{n}{2}}e^{-ie_n(\theta+\omega t)}}{\sqrt{\rho_n}}\,|n\rangle\,. \tag{4.2.2}$$

It is important to note that for the special case when $e_n \propto n$, the time evolution of the coherent states $|\,(J,\theta)\rangle$ reduces to a rotation in complex plane, that is,

$$\hat{U}(t)\;|\,(J,\theta)\rangle = |\,(J,\theta)e^{-i\omega t}\rangle.$$

In Eq. (4.2.3), for the harmonic oscillator we obtained as

$$\begin{aligned} |(J,\theta),t\rangle \;&=\; \frac{1}{\sqrt{N(J)}}\sum_{n=0}^{\infty}\frac{J^{\frac{n}{2}}e^{-in(\theta+\omega t)}}{\sqrt{n!}}\,|n\rangle \\ &=\; |\,(J,\theta)e^{-i\omega t}\rangle. \end{aligned} \tag{4.2.3}$$

Thus the time evolved coherent state, $|(J,\theta)^{(k)},t\rangle$, is the same coherent state with a difference of constant phase [1, 2, 4] which a special characteristic of the coherent states of harmonic oscillator. On the other hand, the Eq. (4.2.3) gives the time evolution for the coherent states of infinite square well, expressed as

$$|(J,\theta),t\rangle = \frac{1}{\sqrt{N(J)}}\sum_{n=0}^{\infty}\frac{J^{\frac{n}{2}}e^{-in(n+2)(\theta+\omega t)}}{\sqrt{n!(n+2)!}}|n\rangle. \tag{4.2.4}$$

Eq. (4.2.4) shows that the time evolution of the coherent states of the infinite square well involves phases having nonlinear dependence on quantum number n. Therefore, these coherent states may exhibit the dynamical characteristics beyond the classical-like dynamics of the coherent states of harmonic oscillator. These characteristics can be investigated by means of auto-correlation function and probability density function in position and momentum space.

4.2.1 Autocorrelation and quantum revivals

Autocorrelation function is an important parameter that measures the behavior of quantum states during their time evolution.

$$A(t) = \langle (J,\theta)| \, e^{-i\hat{H}t} \, |(J,\theta)\rangle \, . \tag{4.2.5}$$

Using time evolution operator, given in Eq. (4.2.1), the autocorrelation function for the coherent states of one-dimensional potentials can be calculated as

$$A(t) = \sum_{n=0}^{\infty} |c_n|^2 \, e^{-iE_n t},$$

where, $|c_n|^2$ is weighting distribution which depends on the coherent state parameter J.

For generalized coherent states, initially narrowly peaked around the mean value $\langle n\rangle$ such that their spread $\Delta n << \langle n\rangle$, we can expand E_n by Taylor expansion around $\langle n\rangle$, expressed as

$$E_n - E_{\langle n\rangle} = \sum_{r=1}^{\infty} \frac{1}{r!} \frac{\partial^r E_n}{\partial n^r}\bigg|_{n=\langle n\rangle} (n - \langle n\rangle)^r \, . \tag{4.2.6}$$

Each derivative in the Eq. (4.2.6) defines a characteristic time scale [43],

$$T_{(r)} = 2\pi \left(\frac{1}{r!} \left| \frac{\partial^r E_n}{\partial n^r} \right|_{n=\langle n\rangle} \right)^{-1} . \tag{4.2.7}$$

The first derivative defines classical period, $T_{(1)} = T_{cl}$, and second derivative defines quantum revival time, $T_{(2)} = T_{rev}$, such that $T_{cl} < T_{rev}$ [43].

It is important to note that for harmonic oscillator, the quantum revival time T_{rev} approaches to infinity. Therefore, the coherent state wave packets for harmonic oscillator follow classical trajectories with a periodicity defined by T_{cl} with out dispersion. This non dispersive evolution is depicted in Fig. (4.2.1). In contrast, for the infinite square well, Eq. (4.2.7) gives the finite value of quantum revival time $T_{rev} = \frac{2}{\pi}$. Hence, the coherent states of infinite square exhibit the phenomena of quantum revivals which is shown in Fig. (4.2).

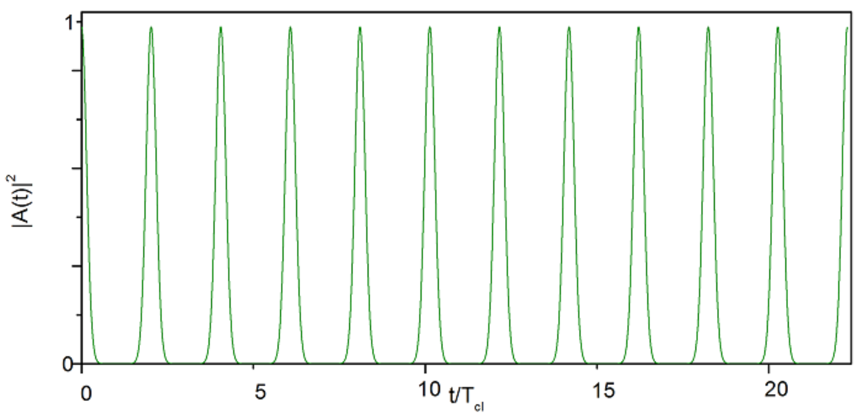

Figure 4.1: Modulus square of autocorrelation function $|A(t)|^2$ for J=5.

4.2.2 Spatio-temporal evolution: position space quantum carpets

The time evolution of coherent state wavepackets in position space can be characterized by means of probability density as function of time, namely, spatio-temporal evolution. In order to analyze spatio-temporal evolution of coherent states, we calculate the probability density, which is defined as,

$$P(x,t) = \left| \langle x| \left(J, \theta\right), t \rangle \right|^2 .$$

Using Eq. (4.2.3), we get

$$P(x,t) = \frac{1}{N(J)} \sum_{n=m}^{\infty} \frac{J^n}{\rho_n} \left|\psi_n(x)\right|^2 + 2Re\left[\frac{1}{\sqrt{N(J)}} \sum_{m \neq n}^{\infty} \frac{J^{\frac{(n+m)}{2}}}{\sqrt{\rho_n \rho_m}} \psi_n(x) \psi_m^*(x) e^{-i(e_n - e_m)(\theta + \omega t)}\right]. \tag{4.2.8}$$

Here, $\psi_n(x)$ are the eigenstates of the system for some particular member of one-dimensional potentials. In our analysis, we will discuss here the harmonic oscillator and the infinite square well.

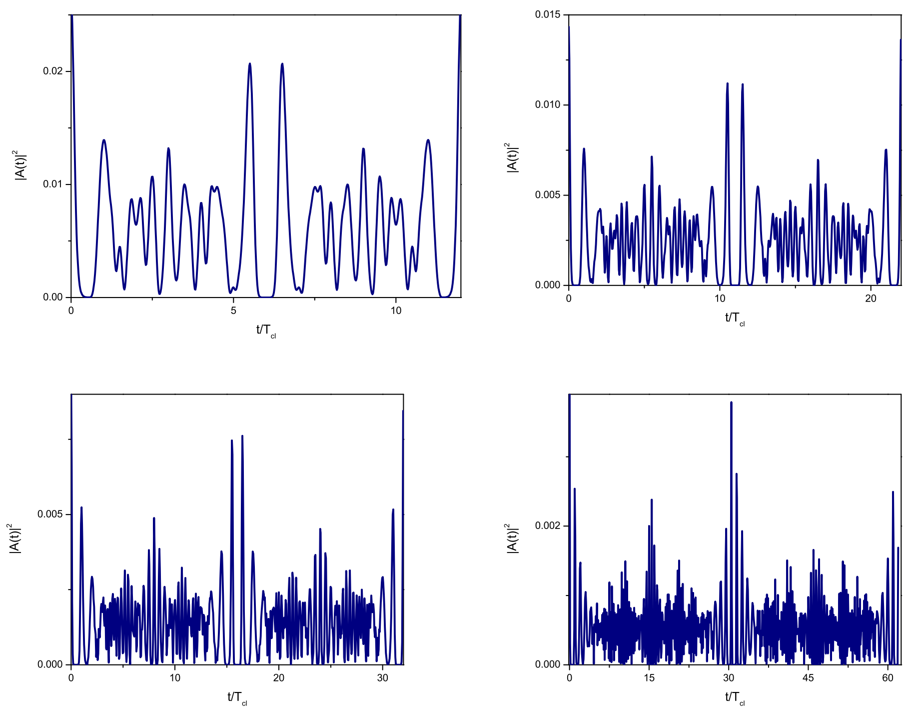

Figure 4.2: Autocorrelation function for different values of J, (1) $J = 38.05$, (2) $J = 125.6$ (3) $J = 263.1$ (4) $J = 975.61$.

Hence, the space-time evolution of the probability density has vital dependence on the nature of the eigenstates $\psi_n(x)$ and the structure of the energy spectrum e_n of the physical system and defines the interference behavior. A constant background independent of time is obtained from the second sum of probability density in Eq. (4.2.8).

The coherent states for harmonic oscillator are explained in chapter 2. The probability density in position space for harmonic oscillator is, from Eq. (4.2.8) In Fig. (4.2.2) plot shows that dispersion is not present and it gives a reconstruction at the classical time $T_{cl} = 2\pi$. This time is independent on J. In Fig. (4.2.2), we take snap shot at time $t = 0$, $t = T_{cl}$, $t = 3T_{cl}$, $t = T_{rev}$. At revival time($t = T_{rev}$) we again get the localized peak same as for the time $t = 0$.

Interference pattern and multiple splitting of time evolved probability density from Eq. (4.2.8) produce quantum carpets. As shown in Fig. (4.2.2). The infinite

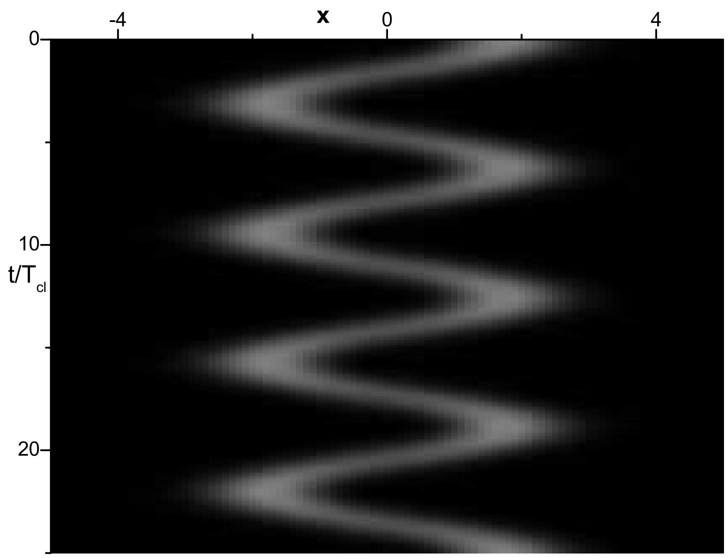

Figure 4.3: The contour plot of harmonic oscillator in position space for value of $J = 5$.

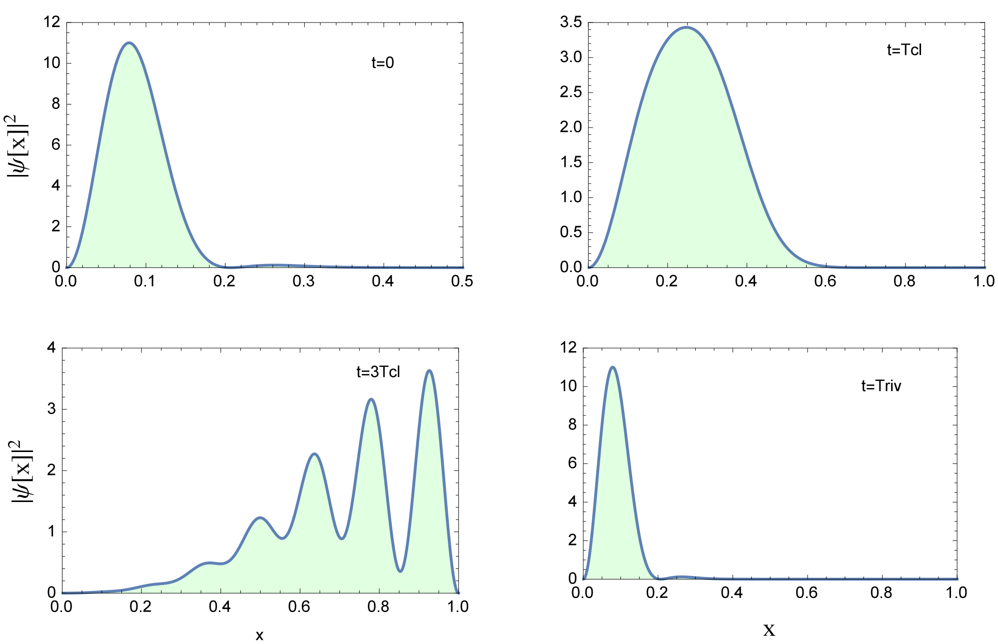

Figure 4.4: The position probability density plots at particular times: (a) $t = 0$; (b) $t = T_{cl}$; (c) $t = 3T_{cl}$; (d) $t = T_{rev}$.

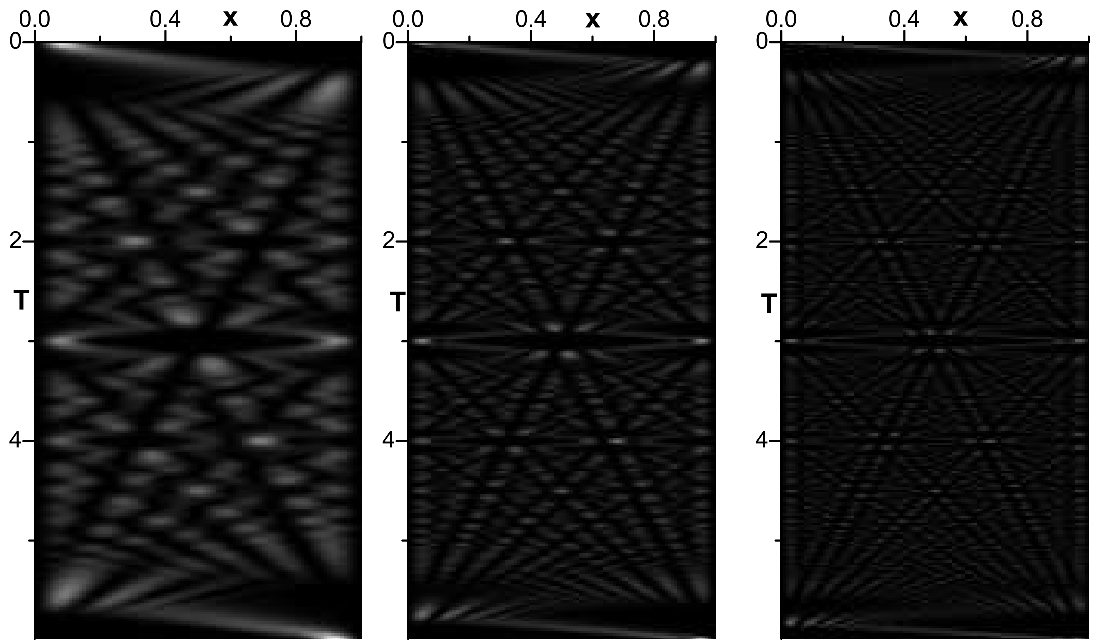

Figure 4.5: The contour plots of position probability density for different values of J, verses time $T = t/T_{cl}$.

square well show constructive and destructive pattern of interference the reason is time dependent term due to which interference occurs. From Fig. (4.2.2) we see that for infinite square well the Gazeau-Klauder coherent states exhibit collapse in later time appear as fractional revivals.

4.2.3 Momento-temporal evolution: momentum space quantum carpets

In order to analyze spatio-temporal evolution of GK CSs we study as well the probability density in momentum space and time, which is defined as,

$$P(p,t) = |\langle p| (J,\theta), t\rangle|^2 ,$$

and is obtained by Eq. (2.3.36) as

$$P(p,t)=\frac{1}{N(J)}\sum_{n=m}^{\infty}\frac{J^n}{\rho_n}\left|\phi_n(p)\right|^2+2Re\left[\frac{1}{\sqrt{N(J)}}\sum_{m\neq n}^{\infty}\frac{J^{\frac{(n+m)}{2}}}{\sqrt{\rho_n\rho_m}}\phi_n(p)\phi_m^*(p)e^{-i(e_n-e_m)(\theta+\omega t)}\right]. \tag{4.2.9}$$

Here, ϕ_n are the eigenstates for one-dimensional potentials. The dynamics of GK CS displays constructive and destructive interference which is governed by the second sum of probability density given in Eq. (4.2.9).

The momentum space probability density is calculated by taking the fourier of position wave function in Eq. (4.2.9),. The fourier of position wavepackets is written as: From Eq. (4.2.9), $\phi_n(p)$ is

$$\phi_n(p)=\frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\exp^{\frac{-ip.x}{\hbar}}\Psi_n(x)dx. \tag{4.2.10}$$

Now the momentum probability density plot for the harmonic oscillator is in Fig. (4.2.3) in Fig. (4.2.3) we see quantum carpet of the probability density $\phi(p,J,\theta,t)$

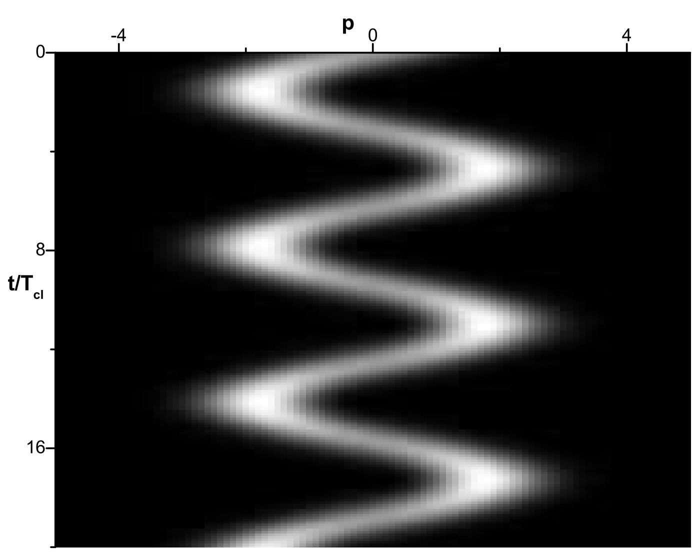

Figure 4.6: The contour plot of harmonic oscillator in momentum space for value of $J=5$.

for the harmonic oscillator in momentum space.

For the Gazeau Klauder coherent states probability density in momentum space for infinite square well is written as

$$P(p, J, \theta, t) = \sqrt{\frac{1}{\mathcal{N}(J)}} \sum_{n=0}^{\infty} \frac{J^{n/2} e^{-in(n+2)(\theta+\omega t)}}{\sqrt{n!(n+2)!}} \phi_n(p). \tag{4.2.11}$$

The snap shot of momentum probability for different times is given in Fig. (4.7). The complete representation for the time evolution of the probability density is

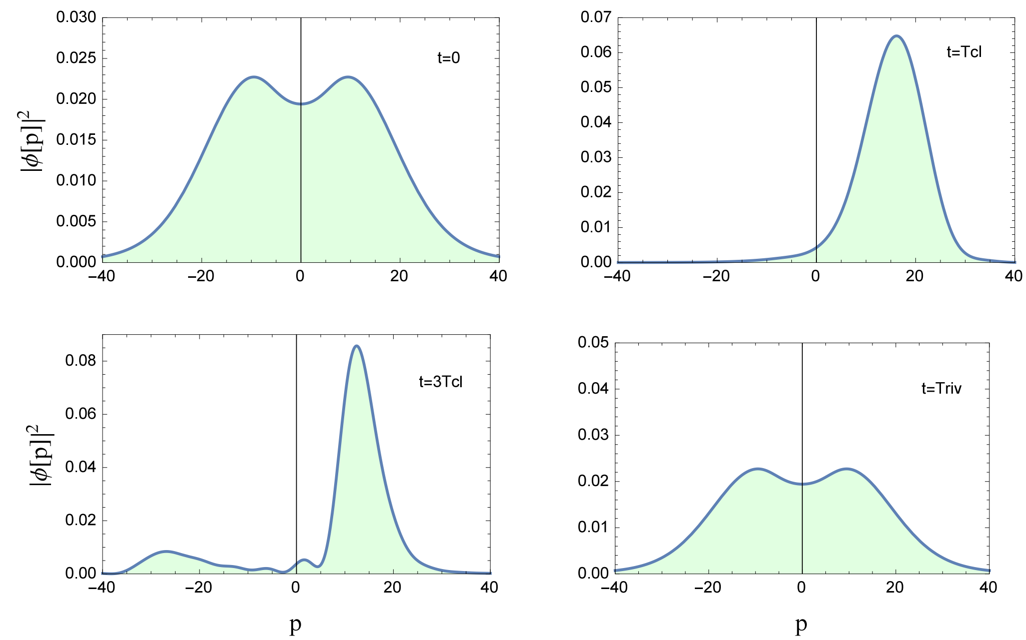

Figure 4.7: The momentum probability density at particular times: (a)$t = 0$; (b)$t = T_{cl}$; (c)$t = 3T_{cl}$; (d)$t = T_{rev}$.

shown in Fig. (4.8), by using the quantum carpet from the interference term in Eq. (4.2.9).

4.3 Quantum revivals by means of entropy

Dynamics of the wavepacket in a nonlinear media exhibits revivals and fractional revivals at specific instants of time, arising from the interference between the stationary states comprising the wavepacket. The revival phenomena has been investigated

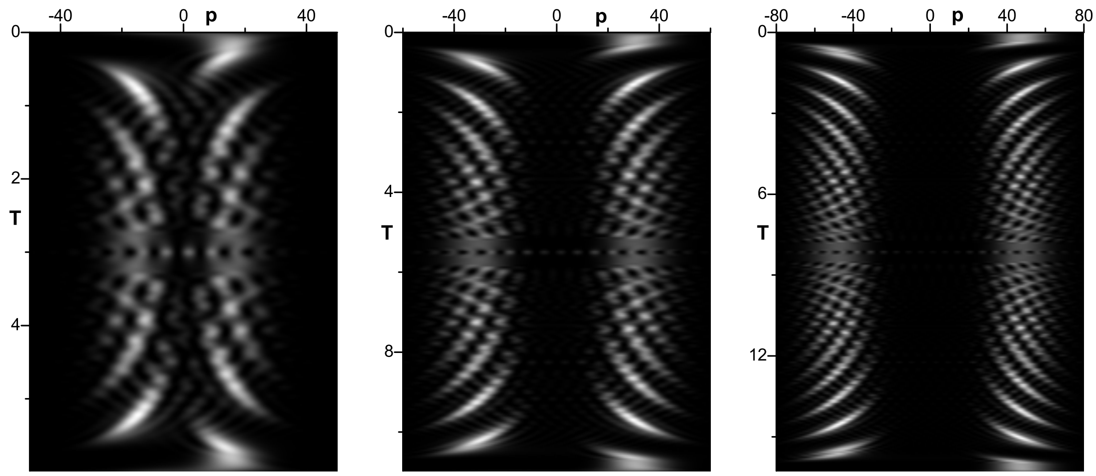

Figure 4.8: The contour plot of momentum probability density for different values of J, verses time $T = t/T_{cl}$.

both theoretically and experimentally in a wide class of systems. An initial well localized quantum state spreads during the propagation and after certain time T_{rev}, the revival time, the wavepacket localizes again giving rise to quantum wavepacket revival. Fractional revival occurs when the initial wavepacket evolves into a state that can be described as a collection of mini packets, each of which closely resembles the initial wave packet. The fractional revival phenomena has been observed experimentally in a variety of quantum systems such as Rydberg atomic wavepackets. The temporal evolution of wavepacket coherent states are performed by means of an autocorrelation function and the full revival properties are investigated in the usual time-domain analysis. This latter seems to be less useful for describing the fractional revivals due to the complicated nature of coherent wavepacket. Fortunately the autocorrelation function revels a little signature of fractional revivals at the vicinity of quarters of the revival time T_{rev} due to the quadratic energy spectrum and the use of the wavelet-based time-frequency analysis of the autocorrelation function provides an analytical and numerical observation of the fractional revivals at different orders of the system.

Fractional revivals appear as the temporal formation of structures that are given by a superposition of shifted and reproduced initial wavepackets[44]. It has been shown that the relevant time scales of wave function evolution are contained in the coefficients of the Taylor series of the energy spectrum, E_n, around the energy $E_{n_\circ}$ corresponding to the peak of the initial wavepacket. More precisely, the second-order, third, and fourth terms in this expansion are associated with, the classical period of motion T_{cl}, and the quantum revival T_{rev}, and the super-revival time. Fractional revival times can be given in terms of the quantum revival by $t = p\frac{Trev}{q}$, and are usually analyzed using the autocorrelation function $A(t)$, which is the overlap between the initial and the time evolved wave packet. At certain fractional revivals, autocorrelation function may be limited for help. The wavepacket reforms itself into a scaled copy of its original shape, that does not coincide with its initial position. An expectation value analysis of wavepacket evolution has been recently proposed, but it does not tell fully about the fractional revivals. We study the wavepacket dynamics by means of sum of information entropies which are of probability density of the wavepacket, in both position and momentum spaces. We shall show that it provides a framework for fractional revival phenomena. The position space information entropy measures the particle localization uncertainty in position space, so the lower is this entropy then the more concentrated is the wave function, the smaller is the uncertainty, and the higher is the accuracy in predicting the localization of a particle. Momentum space entropy measures the uncertainty in predicting momentum of the particle. So, information entropy gives an account of the spreading(high entropy values) and the regenerating (low entropy values) of initially well localized wavepackets during time evolution

$$\rho(x,t) = |\psi(x,t)|^2 \quad and \quad \gamma(p,t) = |\phi(p,t)|^2, \tag{4.3.1}$$

if equations are respectively the probability densities in position and momentum spaces (whereψ and ϕ are the position and momentum wavepackets), the information entropy implies:

$$S_x(t) + S_p(t) = -\int \rho(x,t)\ln\rho(x,t)dx - \int \gamma(p,t)\ln\gamma(p,t)dp. \tag{4.3.2}$$

This inequality is a generalization of the standard variance based Heisenberg uncertainty relation. It is satisfied as a strict equality only for Gaussian wavepackets and bounds from below the sum of the entropies to $1 + ln\pi$. During the evolution of a Gaussian wavepacket, the entropy sum decreases at the revival times to reach the above value, which is equal to unity in the autocorrelation function.

The formation of fractional revivals of the wave function, will correspond to the relative minima of the total entropy. And the sum of entropies that is as an indicator of the fractional revivals, not either of them separately, because only the sum describe both the configurational and the motion aspects of wavepacket dynamics. In this we tell that the sum of entropies for the phase and photon number has been used to study formation of macroscopic quantum superposition states from initially coherent state interaction with a Kerr-medium[45].

For infinite square well, the sum of position space and momentum space entropies and comparison with autocorrelation function for value of $J = 38.05$ is in Fig. (4.9) and for value of $J = 125.6$ and $= 263.1$ is in Fig. (4.10), and value of J is fixed for the graph. The autocorrelation function, as plotted in the top panel of Fig. (4.9), fails to show the fractional revivals occurring at, for example, $\frac{t}{T_{rev}} = \frac{1}{5}, \frac{2}{9}, \frac{2}{7}$ and $\frac{3}{10}$.

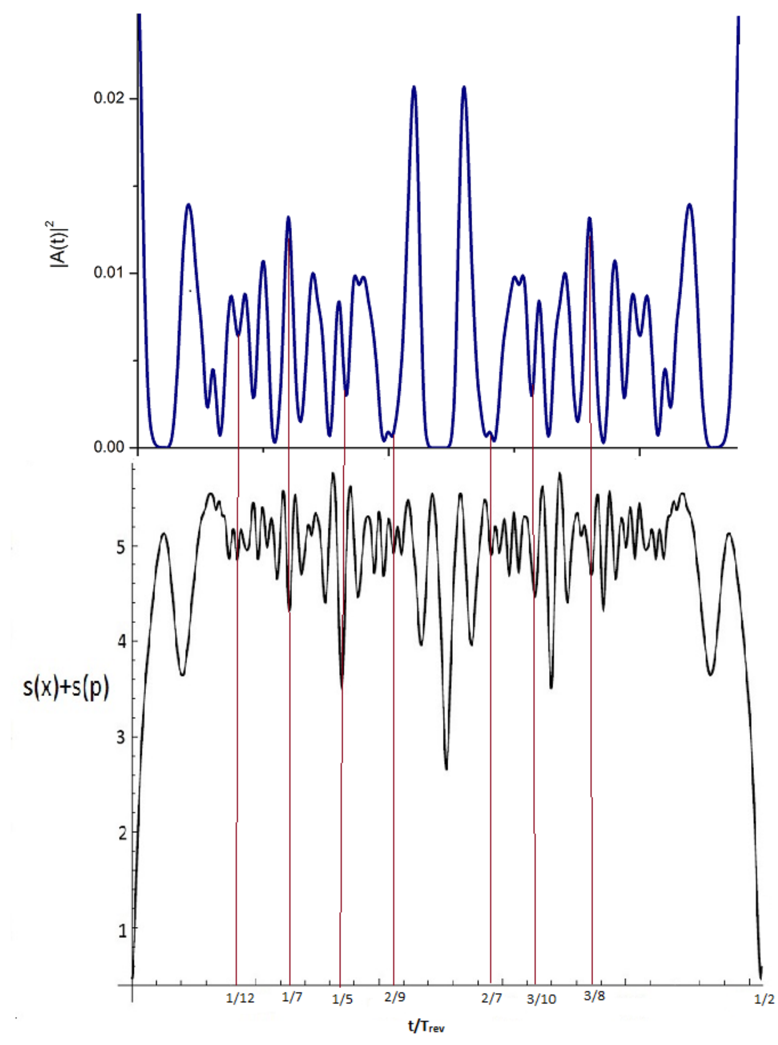

Figure 4.9: Comparison of autocorrelation function and entropy revivals for value of $J = 38.05$.

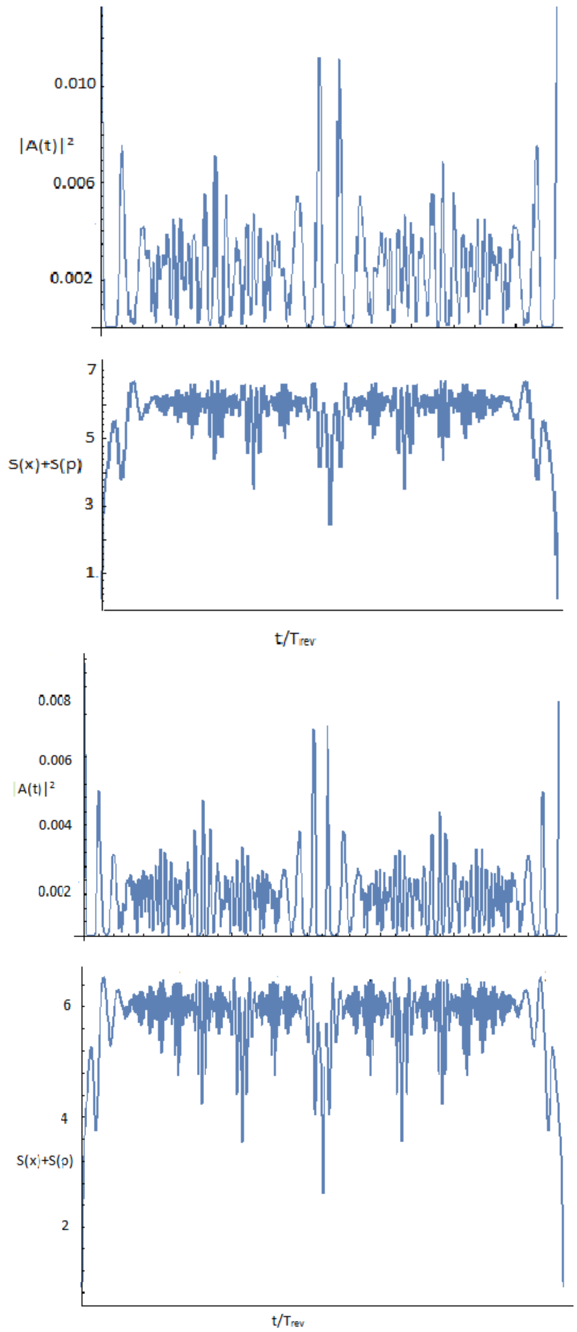

Figure 4.10: Comparison of autocorrelation function and entropy revivals for value of $J = 125.6$ and $J = 263.1$.

Chapter 5

Summary and Conclusion

In this dissertation we focus on measurement of entropy for generalized coherent states wavepackets. In this work, for the general study of quantum systems we introduce one-dimensional potentials. We study quantum systems, the harmonic oscillator and the infinite square well respectively. The coherent states of the harmonic oscillator studied and we construct generalized coherent states known as Gazeau-Klauder coherent states by using the eigen energies and eigen states. Further, we study how to construct generalized coherent state wavepackets, in both position and momentum space.

For the measurement of uncertainty relation we calculated the expectation values from the probability densities for time independent position wavepacket and for the momentum wavepacket. Firstly general formalism is described and then as an example expectation values for the infinite square well calculated. After that, uncertainty relation is defined as product of position dispersion and momentum dispersion for the infinite square well. Introduction of entropies in quantum mechanics is studied. The entropies for time independent generalized coherent states calculated in position and momentum space, and we write entropic uncertainty relation. This entropic uncertainty relation is defined as the sum of entropies of position entropy and momentum entropy.

Our main task in this thesis is to compare entropies of a dynamical system with autocorrelation function. Where, We have calculated the autocorrelation function for the comparison of the time evolved coherent state wavepackets with the initial states. The time evolved generalized coherent state wavepackets calculated for the harmonic oscillator and the infinite square well. The autocorrelation function for harmonic oscillator is shown, which shows that it follows the classical trajectory. For the infinite square well potential, autocorrelation function is also shown for different values of J in coherent state wavepackets. The contour plots of probability densities also shown for the harmonic oscillator and the infinite square well. The contour plots for the system the infinite square well shows quantum revivals. It is shown that the coherent states of infinite square well follows the classical evolution for their short time evolution. After that, quantum constructive and destructive interference occurs due to which the states collapsed and phenomenon of quantum revivals arises.

We discuss entropy and different types of entropy in introduction. At the end we compare autocorrelation function with the sum of entropies in both position and momentum space. It is shown that in the infinite square well information entropy structure provides us a better perception of fractional revivals as compared to the autocorrelation function. The hidden fractional revivals in autocorrelation functions are dominant by using entropy.

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