On the measures of wave packet revivals and fractional revivals

by

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Dedicated to

My Loving Parents and my Siblings for their Love, Endless support & Encouragement.

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In the name of Allah (S.W.T), the most Merciful, the most Gracious. All praise is due to Allah (S.W.T), we praise him, seek his help and ask for His forgiveness. I am thankful to Allah (S.W.T), who supplied me with the courage, the guidance and the love to complete this research. Also, I cannot forget the ideal man of the world and the most respectable personality for whom Allah (S.W.T) created the whole universe, Prophet Hazrat Muhammad (P.B.U.H).
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Abstract

The wave packet dynamics in a bounded one-dimensional system exhibits fascinating phenomena of quantum revivals and fractional revivals which is a manifestation of self-interference of the wave packet. An initially well-localized wave packet in a bounded system follows classical evolution in its short time dynamics and displays reconstruction after a classical period. However, after many classical periods phase difference between constituent wavelets leads to destructive interference and results a collapse when the phase difference is maximum. Later, constructive interference dominates and supports quantum revivals and fractional revivals.

In this thesis, different measures on the phenomena of quantum recurrences, namely full revivals and fractional revivals, are investigated. In particular, we calculate the classical periodicity, quantum revival and super-revival times for general one-dimensional systems by means of autocorrelation function. The autocorrelation function is plotted, to display the structure of full revivals and fractional revivals, for some model systems such as infinite square well, harmonic oscillator, quantum bouncer and Rydberg atom. We extend our discussion to visualize these periodicities by means of quantum carpets which are self-interference patterns formed by temporal evolution of position-space and momentum-space probability densities. Finally we investigate the phenomena of quantum revivals and fractional revivals by means of information entropy and compare the results with the temporal evolution of variance-based uncertainty product. We found that information entropy is a better measure for the identification of fractional revivals.

Contents

Chapter 1

Introduction

The Quantum world shows very interesting phenomena which do not have corresponding analogue in classical world. The agreement between quantum physics and classical physics has been intensively studied in a variety of systems by the quantum-classical correspondence [1]. Therefore the precise distinction between these two theories is one of the philosophical question from the very binging of quantum theory. How we can visualize the computability between a microscopic probabilistic world and deterministic macroscopic world is the most debated philosophical question. Bohr correspondence principle provides a cornerstone for the philosophical interpretation between the classical theory and the quantum theory. According to Bohr correspondence principle the predictions of quantum theory must corresponds to the predictions of classical theory.

In classical mechanics, the position and momentum of a particle at particular instant of time t is marked by some point $x(t)$ and $p(t)$ respectively. The equation of motion of the particle are used to describe the position and momentum variable and these variables are used to determine the state of the system. In quantum mechanics, to study the dynamical behaviour of a wave packet $\psi(x,t)$ we need to construct its wave function, which describes the position and momentum of a particle. A localized wave packet $\psi(x,t)$ can be produced in many physical system. A localized wave packet is the result of the superposition of quantum mechanical eigenstates of the system. The time evolution and dynamics of wave packet are the subject of current research in molecular, atomic, chemical and condensed matter physics [2].

This wave packet $\psi(x,t)$ displays a variety of non-classical effects. These effects can be observed through several interesting phenomena like revivals, fractional revivals and super-revivals of the wave packet at some particular instant of time.

The wave packet dynamics manifests several interesting phenomena that do not have corresponding classical analogous, such as quantum revival and fractional revivals. In this thesis we study the dynamics of wave packet for different model systems. We investigate the revivals and fractional revivals of a wave packet by means of different theoretical tools. In our study, we use autocorrelation function, quantum carpets and information entropy to investigate the dynamical behaviour of quantum wave packet. In the following, we give a brief introduction of what we are going to present in thesis.

1.1 Quantum revivals

The revival phenomena [2] can be understand by the precise nature of the transition from quantum to classical dynamics and the deviation from classical predictions that are displayed during the long- term time evolution of wave packet [3]. According to Bohr correspondence principle, for high quantum numbers $n \to \infty$ the quantum theory reproduce classical mechanics. However, the harmonic oscillator violates the Bohr correspondence principle [7]. The quantum-to-classical transition can be also studied by decoherence in quantum wave packet. The time evolution of a wave packet and interactions with some hard boundaries destroy the coherence of wave packet and the quantum system collapse to a classical world. The Rydberg wave packets are also used to prob the correspondence between quantum and classical mechanics [12] because initially the motion of wave packet is periodic with the same classical period T_{cl} as the motion of the charged particle in coulomb field. This periodicity is lost after few cycle due to the quantum interference effect, the wave packet collapses and then undergoes a sequence of revivals and fractional revivals.

Schrodinger said that "*the wave groups can be constructed which move round highly quantized Kepler ellipses and are the representations by wave mechanics of the hydrogen electron*"[5]. From the early investigation it is found that such dispersion [15] was a natural features of wave packet for coulomb potential. To calculate the semiclassical solution of wave packet for coulomb potential under schrodinger suggestion leads to many theoretical results. The time development of such wave packet leads to the prediction of quantum wave packet revival. For the first time, Parker and Stroud [21] study the revival behaviour in numerical studies of Rydberg atoms, while Yeazell and Stroud [19, 38] experimentally confirmed their prediction. Averbukh and Perelman [37] discussed the dynamics of wave packets of highly excited states of atoms, and also discussed the revival and fractional revivals of these highly excited state atoms. While Kostelecký and Bluhm [2] have discussed the general revival structure of a localized wave packets.

The experimental realization of quantum wave packet revival is first found in Rydberg atom [36]. The study of the quantum revivals in atoms has focused on the excitation of Rydberg wave packets [44]. The coherent superposition of Rydberg atomic states is possible by the interaction of ultrashort laser pulses. Excitation of Rydberg atom from its ground state by the interaction of short laser pulse produces a superposition of the state which are centered on a mean value $\bar{n}$ and this mean value depends on the frequency of the laser pulse. The periodicity of the wave packet is directly determined by the energy of the Rydberg atoms.

Over the past few years the theoretical analysis and numerical predictions of quantum wave packet revival phenomenon has been developed in a rigorous way. We can study the characteristic of quantum revival of a wave packet by means of initially localized wave packet which have a short term time evolution and also have a significant spreads over a several orbits [4]. For shot term time evolution, the spread of wave packet reverse itself and the wave packet relocalize. The relocalization of the wave packet is called revival of the wave packet. There is also a possibility of smaller copies of initial wave packet or clones of initial wave packet, this give rise to the fractional revival of the initial wave packet.

The revival phenomenon arises because of self interference of a time evolved wave packet. The systems which are exhibiting revival behaviour are important for the fundamental realization of the time-independent interference for bound state system [4], with quantized energy. The revival phenomenon has a wide use in chemistry and physics to study the dynamical behaviour of a system.

1.2 Autocorrelation function

To study the revival behaviour we have different theoretical tools, one of them is autocorrelation function. The autocorrelation function is use to compare the wave packet with time delayed version of itself. In statistics the autocorrelation of a random process is described by the correlation between different values at different times. The statistical definition of autocorrelation function is given as,

$$R(T,t) = \frac{E[(X_t - \mu_t)(X_T - \mu_T)]}{\sigma_t \sigma_T},$$

where, E is the expected value operator. In quantum mechanics, Nauenberg introduce the autocorrelation function and is use to investigate the revival and fractional of a wave packet during its sort-term and long-term time evolution. The autocorrelation function is basically the overlap of a initial wave packet with time evolved wave packet and it is given as,

$$A(t) = \langle \psi(0) | \psi(t) \rangle. \tag{1.2.1}$$

or,

$$A(t) = \int_{-\infty}^{\infty} \psi^*(x,t)\psi(x,0)dx. \tag{1.2.2}$$

In momentum space,

$$A(t) = \int_{-\infty}^{\infty} \phi^*(p,t)\phi(p,0)dp. \tag{1.2.3}$$

At $t = T_{rev}$, the $\psi(t)$ completely relocalize itself and at that time we have a maximum value of autocorrelation function. T_{rev} is the revival time of the wave packet and it is defined in terms of the second derivative of the eigenenergy of the system.

If $t = \frac{pT_{rev}}{q}$, the wave packet $\psi(t)$ relocalize itself with some fraction of initial wave packet, is called the fractional revival. Where p and q are the correlated numbers with common deviser one. At fractional revival, we have clone of initial wave packet. At fractional revival time the autocorrelation function also does not have its maximum value.

1.2.1 Expectation value analysis

The analysis of quantum wave packet revival during short-term and long-term time evolution by using the autocorrelation function is directly comparable with the experimental observable. But the visualization of quantum wave packet evolution through their time dependent expectation values in position space as well as in momentum space is also a very valuable tool for the understanding of quantum mechanical effects [4].

For the infinitely large number of particles, the expectation value in position space is defined as,

$$\langle x(t)\rangle = \int_{-\infty}^{\infty} xP(x,t)dx = \int_{-\infty}^{\infty} x|\psi(x,t)|^2 dx, \tag{1.2.4}$$

and the uncertainty in the position of the particle is given as,

$$(\Delta x)^2 = \langle x^2\rangle_t - \langle x\rangle_t^2. \tag{1.2.5}$$

As we know the Heisenberg uncertainty principle,

$$\Delta x \Delta p \geq \frac{\hbar}{2},$$

is use to distinguish regimes of classical and non-classical behavior of wave packet [4]. The expectation values are use to determine the uncertainty. Smaller the uncertainty , larger the probability of wave packet relocalization.

1.3 Quantum carpets

The measurement and controlled preparation of wave packets has emerged as a extremely active field of research [13]. The quantum carpet can be constructed by means of space-time probability density. Highly regular spatiotemporal or multidimensional pattrens in the quantum mechanical probability can appear due to interference of a wave packet and the regular pattern drawn by the spatiotemporal probability density is called the quantum carpets. Quantum carpets is the result of self interference of wave packet. The quantum carpets are characterized by the regular net of ridges, maximum probability area and canals, minimum probability areas and it varies along a straight lines in space-time (x, t).

Quantum carpets appear in many fields of wave physics ranging from quantum mechanics, with application in Boss-Einstein condensation [22] and nuclear physics

[23] to electromagnetic waves and wave guide [25]. The quantum carpets is also use to express the quasi-probability distribution which is expressed by the Wigner function. Wigner function was introduced by Eugene Wigner in 1932 and it is use to study the quantum corrections to classical statistical mechanics.

1.4 Wigner distribution function

The Wigner distribution function is a quasi-probability distribution function which is introduced by Eugene Wigner in 1932. The Wigner finction is use to study the quantum corrections for classical statistical mechanics [6]. The Wigner function is a joint probability distribution function in phase space. In 1927, Hermann Weyl introduced the quantum classical correspondence by the average value of real phase-space and Hermitian operator. Latter on the Wigner-Weyl transformation provides a realization of operator in phase space.

In 1949, José Enrique Moyal use the demostration of Weyl to calculate the expectation value of observable in phase space. Their is also many other quasi-probability distribution in quantum mechanics, like P-representation which is introduced by Glauber and Sudarshan [9]. The phase space distribution function with application is also explained by Hai-woong Lee [11].

The Wigner function is use to study the classical limit of a system. As we know in classical mechanics the particle has a definite position and momentum and it can be represented by a point in phase space. But in quantum mechanics, the position and momentum of any particle can not be exactly known simultaneously because of the Hesienberg uncertainty principle, if the position of a particle is exactly known then its momentum is totaly uncertain. The Wigner function provides a comparison of classical and quantum dynamics in phase space.

1.5 Information entropy for quantum system

In early 1850s, Rudolf Clausius introduced the entropy which plays a very important role in thermodynamics and in classical statistical mechanics. Clausius introduced an extensive thermodynamics variable, which is used to measure the entropy of an isolated system in thermodynamics. Later on, in 1870s the Ludwig Boltzman introduced the entropy,

$$S_B = K_B ln\Omega, \tag{1.5.1}$$

and he linked entropy with the lack of information about the system.

In statistical mechanics, entropy is related with the measure of uncertainty of macroscopic properties of a system, such as, pressure, temperature, and volume. The statistical entropy measure the probability of the spread of the system over different possible microstate [29].

After Boltzman theorem, Shannon introduce the entropy [30] which is defined in terms of the discrete random variable X have possible values $\{x_1, x_2, ..., x_n\}$, and its is given as,

$$H(X) = E[\frac{1}{X}] = E[-lnP_X], \tag{1.5.2}$$

where, E is the expected value operator and $\frac{1}{X}$ is the information content of X [31]. The entropy for a finite system can be express as,

$$H(X) = -\sum_{i=1}^{n} P(x_i) lnP(x_i), \tag{1.5.3}$$

where, $P(x_i)$ is the probability of the system.

The quantum correspondence of classical definition of Shannon's entropy was provided by the Jhon Von Neumann. He described the collapse of wave packet are irreversible process. As in classical mechanics the state of a system is described by the microstate, similarly the quantum mechanical state of a system described by the wave function which contains all the information of the system. Von Neumann introduced the classical statistical entropy in quantum domain. Von Neumann defined the entropy [27],

$$S(\rho) = -Tr(\rho ln\rho), \tag{1.5.4}$$

where ρ is the density matrix and it is defined as,

$$\rho = \sum_j \eta_j |j\rangle\rangle j|,$$

by using the value of ρ we have,

$$S(\rho) = -\sum_j \eta_j ln\eta_j. \tag{1.5.5}$$

R*é*nyi introduced the generalization of Shannon's entropy in quantum information theory. R*é*nyi entropy is very important in quantum information, where it is used to measure the entanglement of the systems. The R*é*nyi entropy as a function of α is given as,

$$H_\alpha(X) = \frac{1}{1-\alpha} log\Big(\sum_{i=1}^{n} P_i^\alpha\Big), \tag{1.5.6}$$

where $\alpha \geq 0$ and $\alpha \neq 1$ and X is a discrete random variable.

In quantum information theory, Ronald Fisher introduced a quantity called Fisher information [28] which is used to measure the information of a random variable X carries about an unknown parameter θ, and it is defined as,

$$I_F(f_{x;\theta}) = \int f_{x;\theta}(X)\Big(\frac{\partial ln f_{x;\theta}(X)}{\partial\theta_k}\Big)^2 dk, \tag{1.5.7}$$

where $f_{x;\theta} = f_{x;\theta}(X)$ is the density function and θ is the parameter which use to defined the density function. From the defination, it is clear that $I_F(f_{x;\theta})_{\theta_k} \geq 0$ and if f_X does not depend on θ_k, then $I_F(f_{x;\theta})_{\theta_k} = 0$ [24].

The revival and fractional revival of a wave packet can be also observed by using the information entropy. Fractional revival usually observed by using the autocorrelation function, but at ceratin fractional revivals, the autocorrelation function may not be helpful because the wave packet relocalize in a location does not generally coincide with initial position [58]. The information entropy is use to fully detect the fractional revivals. As we know, the fractional revival is a relevant feature in long-time evolution of a time evolved wave packet. The entropy is defined in terms of probability as,

$$S_\gamma = -\int_{-\infty}^{\infty} P(x) ln P(x) dx, \tag{1.5.8}$$

and in momentum space,

$$S_\rho = -\int_{-\infty}^{\infty} P(p) ln P(p) dp. \tag{1.5.9}$$

The entropy is related with the uncertainty in the position or momentum of the particle. If the particle is more uncertain in position in some particular region, it will have a low probability and the particle will have the large value of entropy in that particular region. If the particle has the large value of entropy, it will have more uncertainty in the relocalization of wave packet. Thus, the information entropy gives an account of the spreading, high entropy values and the regenerating , low entropy values of initially well localized wave packets during the time evolution [58]. The sum of information entropies in conjugate space provides a relative minima at the fractional revival times. So we can easily detect the fractional revival by sum of entropies which are difficult to observe by means of autocorrelation function. The description of fractional revival in terms of information entropies complements the usual method in terms of autocorrelation function.

1.6 Thesis Outline

The work in this dissertation is organized as: In second chapter we discuss the general construction of a wave packet, wave packet is given by the superposition of all possible eigenstate of the system,

$$\psi(x) = \sum_n c_n u_n(x), \tag{1.6.1}$$

c_n is the probability amplitude and $u_n(x)$ is the energy eigenstate of the system. We observe the behaviour of that wave packet under time evolution operator and discuss the different periodicities of wave packet like classical , quantum revival and super-revival. We also discuss the different theoretical tools like autocorrelation function, space-time probability density and information entropy, which are use to observe the revival behaviour of the time evolved wave packet.

In third chapter, we will see the application of general formulism which is constructed in chapter two by applying on different model system like infinite potential well, harmonic oscillator, the quantum bouncer and on a one dimensional Rydberg atom. We will construct their wave packet and will observe its behaviour under time evolution operator. Then we discuss the revival behaviour of wave packet by means of autocorrelation function which is the overlap of initial wave packet and time evolved wave packet. We also discussed the quantum carpets of infinite square well, harmonic oscillator and the quantum bouncer in third chapter. The quantum carpets woven by the space-time representation of the probability density [18]. We have discuss the quantum carpets woven by the position space-time probability and momentum-space probability density. We can observe the revival behaviour by ridges and valleys in quantum carpets.

In fourth chapter we study the revival behaviour by means of entropy and time dependent expectation values in position space and momentum space. We have also discussed the relation between information entropy and uncertainty in the wave packet relocalization. We have shown, at certain fractional revival time the autocorrelation function may not be helpful because the wave packet might be relocalize itself at the scale where the location does not generally conside with its initial position [58], so at that position we can detect the fractional revival of the wave packet by means of information entropy. At maximum value of entropy we have low probability of wave packet relocalization and at smaller value of entropy we have maximum probability of wave packet relocalization. At the end we conclude the whole work.

Chapter 2

Wave packet evolution and quantum recurrences

The wave packet revival phenomenon can be observed in many experiments. This phenomenon arises when a localized wave packet is produced by superposition of different wavelength and initially this wave packet exhibits a short term time evolution with classical periodicity T_{cl}. The spread of wave packet is significant after a number of orbits. After an initial excitation the wave packet relocalizes with much longer time ($T_{rev} > T_{cl}$) called quantum revival time. However, in many experiments, the wave packet revival has been observed with smaller periodicities found at fractional revival time pT_{rev}/q [37]. These wave packet shows periodicities at T_{cl}/q with $p = 1$. Fractional revival has been observed in many atomic [38] and molecular system [39].

To understand the revival behaviour we have different theoretical tools like auto-correlation function, space-time probability density and information entropy of the system. These tools are required to under stand many aspect of revival behaviour during short and long-term time evolution.

In this chapter we are going to discuss the general methods which are required for understanding the revival behaviour. We will start our discussion with time evolution of a wave packet. In this section we will see how a wave packet evolves with time and see the action of a time evolution operator on a time independent wave packet. We have also talked about the different periodicities, like quantum revival, fractional revival and super-revival of wave packet. Then we discussed the different theoretical tools like, autocorrelation function which is the overlap of initial wave packet and time evolved wave packet, space time probability of time evolved wave packet and information entropy of the system.

2.1 Time evolution and revival structure of quantum wave packet

In quantum mechanics, the Hamiltonian of a particle of mass is defined as

$$\hat{H} = \frac{\hat{P}}{2m} + V,$$

and corresponding time dependent Schrodinger equation is given as,

$$\hat{H}|\psi(t)\rangle = i\hbar\frac{\partial|\psi(t)\rangle}{\partial t},$$

and the time dependent quantum state of particles in any system is described by wave function which is given as,

$$\psi(x,t) = \sum_n c_n u_n(x) e^{-iE_n t/\hbar}. \tag{2.1.1}$$

where, c_n is the probability amplitude and E_n is the energy of the system. The probability amplitude c_n is use to describe the behaviour of the system and it is also provide a relationship between a quantum state and the result of the observations of that system. If the initial wave packet and the energy eigenstate of the system is known then the probability amplitude is defined as,

$$c_n = \int_{-\infty}^{\infty} \psi(x,0) u_n^*(x) dx,$$

where, $\psi(x,0)$ is the initial wave packet. In general we consider the initial wave packet as a Gaussian,

$$\psi(x,0) = \frac{1}{(\sigma\pi^2)^{1/4}} e^{-(x-x_0)^2/2\sigma^2} e^{ip_0 x/\hbar},$$

where, p_0 is the initial momentum.

The time dependence of time dependent wave packet Eq.(2.1.1) can be complex. However, if a wave packet is excited with energy which has a tight spread around central value of quantum number n_0 and $n_0 \gg \Delta n \gg 1$. For an initial wave packet which is narrowly peaked around the central quantum number n_0 with spread Δn, we can expand the individual eigenvalues of energy by using Taylor expansion about this value, and it is given as,

$$E_n = E_{n_0} + \sum_{r=1}^{\infty} \frac{1}{r!} \frac{\partial^r E_n}{\partial n^r}\bigg|_{n=n_0} (n - n_0)^r \,. \tag{2.1.2}$$

Each term in the series defined a characteristic time scale of periodicity for the wave packet and it is defined as,

$$T_{(r)} = 2\pi \left(\frac{1}{r!} \frac{\partial^r E_n}{\partial n^r}\bigg|_{n=n_0} \right)^{-1} .$$

Here, in the above equation the first order is associated with the classical period, the second derivative is associated with the revival time of the wave packet and they are defined as,

$$T_{cl} = \frac{2\pi\hbar}{\left|\frac{dE_n}{dn}\right|}, \tag{2.1.3}$$

$$T_{rev} = \frac{2\pi\hbar}{\left|\frac{d^2E_n}{dn^2}/2!\right|}, \tag{2.1.4}$$

and

$$T_{sr} = \frac{2\pi\hbar}{\left|\frac{d^3E_n}{dn^3}/3!\right|}, \tag{2.1.5}$$

we have,

$$\psi(t) = \sum_{n=0}^{\infty} a_n u_n exp\Big(- 2\pi i(n - n_0)\frac{t}{T_{cl}} - 2\pi i(n - n_0)^2\frac{t}{T_{rev}} - 2\pi i(n - n_0)^3\frac{t}{T_{sr}} + ...\Big). \tag{2.1.6}$$

Here, T_{cl} is the classical period, T_{rev} revival period and T_{sr} super revival period. The term $-i\omega_0$ in above equation is not important because it is just like a time dependent phase in the solution of stationary state and this term does not induce any interference in the spectrum. The second term is associated with the classical period of the wave packet motion in bound state. From Eq.(2.1.6) we can see the time dependence of the wave packet require three time scales which are controlled by the dependence of the energy on the quantum number n [2]. For very small value of t, the first term in Eq.(2.1.6) which depends on classical period T_{cl} is dominant and during this interval the wave packet shows classical periodicity.

As time t increases the term which depends on T_{rev} becomes prominent and this term gives rise to spread and collapse of wave packet. However at $t = T_{rev}$ the phase of wave packet is approximately equal to $-2\pi i$, which is irrelevant to the motion of wave packet because of first term in Eq.(2.1.6) and the wave packet regains its initial shape and its motion is periodic with the classical period T_{cl}. This is called a full revival [2]. When the time is equal to some rational fraction of T_{rev}, the wave packet regains into a series of mini wave packet called fractional revival. The classical component of the wave packet is defined as,

$$\psi_{cl}(t) = \sum_{n=0}^{\infty} a_n u_n e^{-2\pi i(n-n_0)E'_n t/\hbar},$$

$$\psi_{cl}(t) = \sum_{n=0}^{\infty} a_n u_n e^{-2\pi ikt/T_{cl}}, \tag{2.1.7}$$

here $k = (n - n_0)$. This component is helpful in discussing fractional revivals. The time is associated with the quantum revival. This time scale is responsible for the spreading of wave packet in same way. The anharmonic oscillator gives revival time which is related with a system which can be physically realizable such as vibrational motion of molecules. The different periodicity can be observed during the evolution of the wave packet, i.e., quantum revival, fractional revival and super-revival. In quantum mechanics, the quantum revival phenomena is mostly observed in wave packets which are well localized at the beginning of the time evolution. During time evolution, the wave packet regain its initial form called revival of wave packet, and the time at which wave packet re-localize is called revival time, T_{rev}. The Eq.(2.1.7), at time $t = T_{rev}$,

$$\psi(t \approx T_{rev}) = \sum_n c_n u_n e^{-\frac{i2\pi kt}{T_{cl}}} e^{-2\pi ik^2}. \tag{2.1.8}$$

The additional phase in above equation is equal to unity and the wave packet is said to be relocalized [4].

$$\psi(t \approx T_{rev}) = \psi_{cl}(t), \tag{2.1.9}$$

so we can say at revival time the wave packet regain its initial position.

During the time evolution when the wave packet relocalize with some fraction of initial wave packet, is called fractional revival. The wave packet revival has been observed with smaller periodicity at fractional revival time $\frac{pT_{rev}}{q}$ [37]. Fractional revival has been observed in many atomic [38] and molecular system [39]. At half revival time $t = \frac{T_{rev}}{2}$, the Eq.(2.1.7) is,

$$\psi(t \approx T_{rev}/2) = \sum_n c_n u_n e^{-\frac{i2\pi kt}{T_{cl}}} e^{-4\pi i k^2}, \tag{2.1.10}$$

At time, $t = t + T_{cl}/2$,

$$\psi(t + T_{cl}/2) = \sum_n c_n u_n e^{-\frac{i2\pi k(t+T_{cl}/2)}{T_{cl}}},$$

$$\psi(t + T_{cl}/2) = \sum_n c_n u_n e^{-\frac{i2\pi kt}{T_{cl}}} e^{-i\pi k}, \tag{2.1.11}$$

So we can say,

$$\psi(t \approx T_{rev}/2) = \psi(t + T_{cl}/2), \tag{2.1.12}$$

the wave packet relocalize near half revival time. From the above equation we can see that at half revival time the wave packet is not completely relocalizing itself and we can observe the collapse of the classical periods during the time evolution of the wave packet.

The quantum revival and fractional revival can only be observed in a system which has quadratic dependence of energy on a single quantum number [4], such as infinite square well. But super-revival can be observed in a system which has higher order dependence of energy on a single quantum number.

Bluhm and Kostelecký [53] have observed long term time evolution and revival in a Rydberg atom. They found a new patterns of revival and observed the similar structure like auto-correlation function plots at $t > T_{rev}$ and $t < T_{rev}$.

At super-revival time T_{sr} the wave packet behaviour is similar to the fractional revival behaviour observed on T_{rev} scale.

2.2 Tools to probe the revival structure

In quantum mechanics, there are many theoretical tools which are use to investigate the revival structure of a time evolved quantum wave packet, i.e, autocorrelation function, expectation value of the wave packet, spatiotemporal probability density, information entropy and Wigner distribution function. In this section we have given a brief overview of few of them.

2.2.1 Autocorrelation function

The projection of time dependent state $|\psi(t)\rangle$ on the initial state $|\psi_0\rangle$ referred to as autocorrelation function and can be express as,

$$A(t) = \langle\psi(t)|\psi_0\rangle = \int_{-\infty}^{+\infty} \psi^*(x,t)\psi(x,0)dx, \tag{2.2.1}$$

The position space representation of wave packet $\psi(x,t)$ Eq.(2.1.1) is given as,

$$\psi(x,t) = \sum_{n=0}^{\infty} c_n u_n(x) e^{-iE_n t/\hbar}, \tag{2.2.2}$$

here, $u_n(x)$ is the energy eigenfunction and E_n is the quantized energy eigenvalue. The useful form of the autocorrelation function [40] is given as,

$$A(t) = \sum_{n=0}^{\infty} |c_n|^2 e^{iE_n t/\hbar}. \tag{2.2.3}$$

The autocorrelation function is important because it is directly related with the study of revival behaviour in the pump-probe type experiment [19].

2.2.2 Expectation value analysis

The visualization of quantum wave packet evolution through their time dependent expectation value is a very valuable tool for the understanding of quantum mechanical effects [4]. The solution of time evolution of wave packet for bound state systems are discussed with increasing frequency in the pedagogical literature [10]. by using different methods, one of them is expectation value analysis.

The expectation value of position variable is given as,

$$\langle x(t)\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\hat{x}\psi(x,t)dx,$$

and the expectation value for momentum variable is given as,

$$\langle p(t)\rangle = \int_{-\infty}^{\infty} \psi^*(x,t)\hat{p}\psi(x,t)dx.$$

Also we know that the expectation value is associated with the uncertainty as,

$$(\Delta x)_t = \sqrt{\langle x^2\rangle_t - \langle x\rangle_t^2}.$$

During long-term evolution of wave packet the uncertainties in position and momentum variable gives the information of collapsed phase and several revivals. At different fractional revival time, the uncertainty of wave packet relocalization is greater then the probability of wave packet relocalization, but at the complete revival time when the wave packet is regain its initial shape the probability of wave packet relocalization is maximum and the uncertainty has its minimum value.

2.2.3 Quantum carpets

The quantum carpets are a regular pattern drawn by the probability density which is represented in space-time. The space-time probability density is defined as,

$$P(x,t) = |\psi(x,t)|^2 = \sum_n \sum_m c_n^* c_m u_n^*(x) u_m(x) e^{\frac{i(E_n - E_m)t}{\hbar}}. \tag{2.2.4}$$

From quantum carpet, we can study the revival and fractional revival of the time evolved wave packet. As the quantum carpet is the result of self interference of wave packet during its interaction with some reflecting boundaries. When the wave packet evolves with time and interact with the boundaries of the system and it will bounce back. The decoherence appears because of self-interference of wave packet. By the ridges and valleys of quantum carpets we can identify the fractional revival. From the Eq.(2.2.4) we can observed the interference terms at $m \neq n$ and because of these interference terms we can observe fractional revival by means of space-time probability density.

The quantum carpet can be also constructed in momentum space by calculating momentum space probability density. The momentum space wave packet can be calculated by taking the Fourier transform of Eq.(2.2.2),

$$\phi_n(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} u_n(x) e^{-\frac{ipx}{\hbar}} dx, \tag{2.2.5}$$

and the probability density in momentum space is given as,

$$P(p,t) = |\psi(p,t)|^2 = \sum_n \sum_m c_n^* c_m \phi_n^*(p) \phi_m(p) e^{\frac{i(E_n - E_m)t}{\hbar}}. \tag{2.2.6}$$

The temporal evolution of these probabilities undergoes a series of constructive and destructive interference which leads to the formation of regular interference patterns and from these interference terms we can identify revival and fractional revival in the system.

2.2.4 Information entropy

Entropy is the central concept of classical statistical mechanics and thermodynamics which is introduced by the Clausius in mid century. This is a phenomenological variable which quantifies the intrinsic irreversibility of thermodynamics process. In quantum mechanics, the probability distribution is use to described the complete information of a wave function in position as well in momentum space. The information by the position space probability density $P(x)$ is defined as,

$$S_x = -\int P(x) \ln P(x) dx.$$

The information entropy measures the uncertainty in the localization of a evolved wave packet. When the uncertainty in the localization of wave pacekt os minimum, the entropy is also minimum and the we can say that the wave packet os well localized. When the wave packet spread in space, the uncertainty in the position of wave packet is increases which implies that the information entropy will also increase. Thus the information entropy gives the information of spreading and relocalization of an initial well localized wave packet during its time evolution. Similarly in momentum space, the entropy is defined as,

$$S_p = -\int P(p) \ln P(p) dp.$$

The entropic uncertainty relation by using the position space entropy and momentum space entropy is defined as,

$$S_x + S_p \geq 1 + \ln \pi, \tag{2.2.7}$$

this inequality is a generalization of the variance based Heisenberg uncertainty relation. As we know that, at fractional revival time the uncertainty has its maximum value, so we can say that entropy might be a better tool to identify fractional revival because entropy is related withe uncertainty of the wave packet relocalization.

Chapter 3

Quantum wave packet revivals in one-dimensional model systems

As in previous chapter, we have talked about the time evolution of wave packet and revival of a wave packet. We can observed a full revival of a wave packet by means of autocorrelation function and quantum carpets. When the inner product of time evolved state and initial state is equal to one, we have a maximum value of autocorrelation function and a full revival of wave packet can be observed at that time. The fractional revival can be observe when the time evolved state is not exactly equal to the initial state but some fraction of initial state and at fractional revival time the autocorrelation is also not at its maximum value.

Quantum carpet is a regular pattern drawn by the probability density which is represented in space-time. Quantum carpet is the result of self interference of wave packet during its interaction with some reflecting boundaries. For example, when an intially localized Gaussian wave packet inside an infinite potential well starts spreading in the center of the well, different pieces of wave packet starts to interfere with each other after reflecting from the boundaries of infinite potential well. Quantum carpets are used to study the dechorence of wave packet. In quantum mechanics, when probability of a particle is represented in space-time, it displays a characteristic ridges and valleys [56]. The valleys and ridges do not follow any classical trajectories. They follow the world line which is created by the interference and the design of the carpets are purely an effect of interference and sensitive to dechorence [57]. The quantum fractional revival can be determined by the structure of quantum carpets.

In this chapter, we will study the revival structure in different one dimensional model system i.e. infinite square well, harmonic oscillator, a quantum bouncer, and one dimensional Rydberg atom. We will discuss the quantum wave packet revival in all system by means of autocorrelation function and quantum carpets.

3.1 Harmonic oscillator

The harmonic oscillator is a most useful example which provides a straight forward solution of wave packet of a bound state system. The Hamiltonian of a particle with mass m, oscillating under the influence of one dimensional harmonic potential with angular frequency ω is,

$$\hat{H} = \frac{\hat{P}^2}{2m} + \frac{1}{2}m\omega\hat{X}^2. \tag{3.1.1}$$

The hamiltonian for this system can be solve by using two methods, one is analytic method and other is ladder or algebraic method. We have solved the time-independent schrodinger equation analytically to find the eigenstate and energy eigenvalues of hamiltonian eq.(3.1.1). In algebraic method we deal with the ladder operators.

The analytic method

Consider the one dimensional time-independent schrodinger equation,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} + \frac{1}{2}m\omega^2x^2\psi_n(x) = E_n\psi_n(x),$$

or,

$$\frac{d^2\psi_n(x)}{dx^2} + \Big(\frac{2mE_n}{\hbar^2} - \frac{x^2}{x_0^4}\Big)\psi_n(x) = 0, \tag{3.1.2}$$

where,

$$x_0 = \sqrt{\frac{\hbar}{m\omega}},$$

is a constant and has the dimension of length [8]. Now consider,

$$u = \frac{x}{x_0},$$

and,

$$\varepsilon = \frac{2E_n}{\hbar\omega},$$

after using the above values and rearranging we have,

$$\frac{d^2\psi_n}{du^2} = (u^2 - \varepsilon)\psi. \tag{3.1.3}$$

This differential equation is not easy to solve, we can solve this equation by considering asymptotic analysis. The asymptotic result of the above equation is,

$$\psi_n(x) = \frac{1}{\sqrt{\sqrt{\pi}2^n n! x_0}} e^{-x^2/2x_0^2} H_n\Big(\frac{x}{x_0}\Big), \tag{3.1.4}$$

where, $H_n\big(\frac{x}{x_0}\big)$ is the Hermite polynomial and its defined as,

$$H_n(y) = (-1)^n e^{y^2} \frac{d^n}{dy^n} e^{-y^2}.$$

The energy eigenvalues for the harmonic oscillator is given as,

$$E_n = \Big(n + \frac{1}{4}\Big)\hbar\omega.$$

For the gaussian type probability amplitude the time dependent wave packet Eq.(2.2.2) for harmonic oscillator is given as,

$$\Psi(x,t) = \sum_n c_n \frac{1}{\sqrt{\sqrt{\pi}2^n n! x_0}} e^{-x^2/2x_0^2} H_n\Big(\frac{x}{x_0}\Big) e^{-iE_n t/\hbar}.$$

Where, c_n is the Gaussian type probability amplitude is,

$$c_n = \frac{1}{\sigma\sqrt{\pi}} e^{-(n-n_0)^2/2\sigma^2},$$

centered at the n_0 central quantum number and σ is the width of the wave packet which is centered at n_0. The momentum space wave packet can be calculated by taking the Fourier transform of above equation or by solving the momentum space Schrodinger equation. The Schrodinger equation for harmonic oscillator in momentum space has the form,

$$\frac{p^2}{2m}\varphi_n(p) - \frac{\hbar^2 m\omega^2}{2}\frac{d^2\varphi_n(p)}{dp^2} = E_n\varphi_n(p), \tag{3.1.5}$$

and we know,

$$\hat{p} = -i\hbar\frac{\partial}{\partial x} \qquad \hat{x} = i\hbar\frac{\partial}{\partial p}.$$

The solution of above equation is,

$$\varphi_n(p) = \Big(\frac{1}{n!2^n\sqrt{m\omega\hbar\pi}}\Big)e^{-q^2/2}H_n(q), \tag{3.1.6}$$

where,

$$q = \frac{p}{\sqrt{m\omega\hbar}},$$

The momentum space wave packet of harmonic oscillator is given as,

$$\Psi(p,t) = \sum_n c_n\Big(\frac{1}{n!2^n\sqrt{m\omega\hbar\pi}}\Big)e^{-q^2/2}H_n(q)e^{-iE_nt/\hbar}. \tag{3.1.7}$$

The classical period of the harmonic oscillator is,

$$T_{cl} = \frac{2\pi}{\omega},$$

depends on the angular frequency of the oscillating particle. If we plot the au-

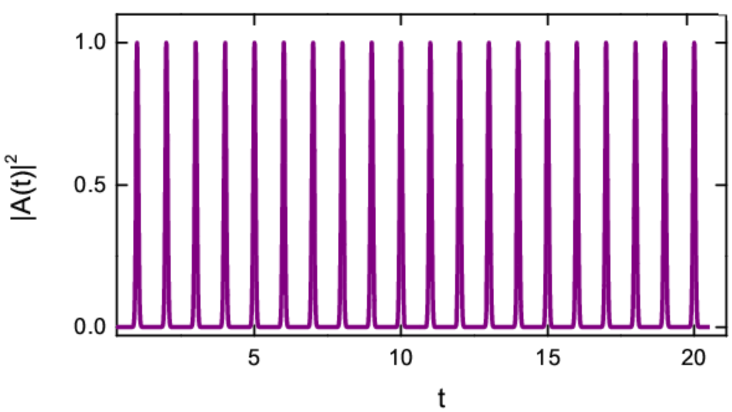

Figure 3.1: Plot of autocorrelation function $|A(t)|^2$ over the 20 classical period for particle oscillating in harmonic oscillator. The wave packet is centered at $n_0 = 5$, with $\sigma = 1$.

tocorrelation function Eq.(2.2.3) for the harmonic oscillator, we can see from the Fig.(3.1), the wave packet relocalize itself after every classical period and we have the maximum value of the autocorrelation function.

Here we can see from the figure the harmonic oscillator showing a classical like behaviour, after every classical period the system goes to its initial state and relocalize itself.

3.1.1 Spatiotemporal probability density

Temporal evolution of position space probability density

The probability density Eq.(2.2.4) for harmonic oscillator is,

$$P(x,t) = \sum_m \sum_n c_n c_m^* u_n(x) u_m^* e^{\frac{-i(E_n - E_m)t}{\hbar}}, \tag{3.1.8}$$

where,

$$u_n(x) = \frac{1}{\sqrt{\sqrt{\pi} 2^n n! x_0}} e^{-x^2/2x_0^2} H_n\Big(\frac{x}{x_0}\Big).$$

If we plot the probability density eq.(3.1.9) as a function of position and time, from Fig.(3.2) we can see the classical behaviour of harmonic oscillator because we know the energy of harmonic oscillator has linear dependence on principle quantum number, so over the classical period the wave packet has maximum probability. In figure we have ridges with maximum probability of wave packet and valleys with minimum probability. Similarly the momentum space probability density is given as,

$$P(p,t) = \sum_k \sum_n c_n c_m^* \varphi_n(p) \varphi_m^*(p) e^{\frac{-i(E_n - E_k)t}{\hbar}}, \tag{3.1.9}$$

where,

$$\varphi_n(p) = \Big(\frac{1}{n! 2^n \sqrt{m\omega\hbar\pi}}\Big) e^{-q^2/2} H_n(q).$$

3.2 The infinite square well

The infinite square well is frequently used as an example for bound state system. In this system many aspects of wave packet propagation and particularly quantum revival have been studied [2, 50].

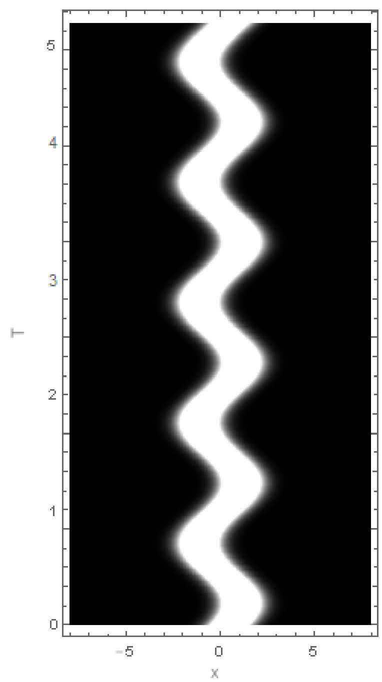

Figure 3.2: The quantum carpet for linear harmonic oscillator with central quantum number n_0.

Consider a particle of mass m moving in the potential,

$$V(x) = \begin{cases} 0, & 0 < x < L \\ \infty, & \text{otherwise} \end{cases}$$

The energy eigenvalue is given as,

$$E_n = \frac{p_n^2}{2m} = \frac{n^2\pi^2\hbar^2}{2mL^2}, \tag{3.2.1}$$

where, $p_n = \frac{n\hbar\pi}{L}$, and energy eigenfunction is given as,

$$u_n(x) = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi x}{L}\right). \tag{3.2.2}$$

The classical period is defined as,

$$T_{cl} = \frac{2mL^2}{n\pi\hbar},$$

$$T_{cl} = \frac{2L}{(n\pi\hbar/L)/m} = \frac{2L}{v_n},$$

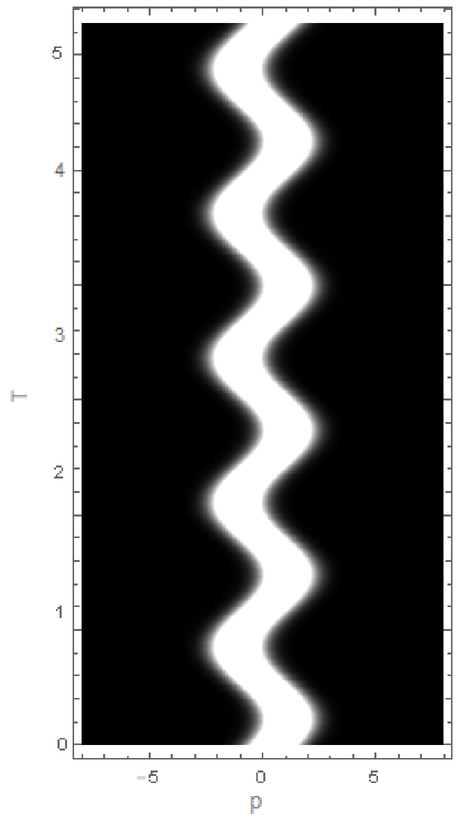

Figure 3.3: The quantum carpet for linear harmonic oscillator in momentum space which is centered at $n_0 = 5$.

here, $v_n = \frac{p_n}{m} = \frac{n\pi\hbar/L}{m}$, is the classical speed. The revival time is given as,

$$T_{rev} = \frac{4mL^2}{\pi\hbar} = 2nT_{cl},$$

for $n \gg 1$, $T_{rev} \gg T_{cl}$.

3.2.1 Gaussian wave packet

At time $t = 0$ the Gaussian wave packet is given in position or initial momentum-space as,

$$\psi_G(x,0) = \frac{1}{\sqrt{b\sqrt{\pi}}} e^{-\frac{(x-x_0)^2}{2b^2}} e^{ip_0(x-x_0)/\hbar}. \tag{3.2.3}$$

where, $b = \alpha\hbar$. Consider this wave packet is contained within the well and neglect the overlap with the region outside of the walls. The expansion coefficient [49] is given as,

$$c_n \approx \int_{-\infty}^{\infty} u_n(x)\psi_G(x,0)dx$$

$$c_n \approx \int_{-\infty}^{\infty} \sqrt{\frac{2}{L}}\sin(\frac{n\pi x}{L})\frac{1}{\sqrt{b\sqrt{\pi}}}e^{-\frac{(x-x_0)^2}{2b^2}}e^{ip_0(x-x_0)/\hbar},$$

by using the identity,

$$\int_{-\infty}^{\infty} e^{-ax^2-bx} = \sqrt{\frac{\pi}{a}}e^{b^2/4a},$$

we have,

$$c_n = \frac{1}{2i}\sqrt{\frac{4b\pi}{L\sqrt{\pi}}}\Big(e^{in\pi x_0/L}e^{-b^2(p_0+n\pi\hbar/L)^2/2\hbar^2} - e^{-in\pi x_0/L}e^{-b^2(p_0-n\pi\hbar/L)^2/2\hbar^2}\Big), \quad (3.2.4)$$

here,

$$\sin(\frac{n\pi x}{L}) = \frac{1}{2i}\Big(e^{in\pi x/L-e^{-in\pi x/L}}\Big).$$

This expression of c_n is very important for the evaluation of $A(t)$. For $(x_0, L-x_0 \gg b)$, the energy expectation value and normalization condition is given as,

$$\sum_{n=1}^{\infty}|c_n|^2 = 1,$$

and,

$$\sum_{n=1}^{\infty}c_n|^2E_n = \langle E_n\rangle = \frac{1}{2m}\Big(p_0^2 + \frac{\hbar^2}{2b^2}\Big).$$

For, $p_0 = 0$ the Eq.(3.2.4) becomes,

$$c_n = \frac{1}{2i}\sqrt{\frac{4b\pi}{L\sqrt{\pi}}}\Big(e^{in\pi x_0/L}e^{\frac{-b^2n^2\pi^2\hbar^2}{L^2}/2\hbar^2} - e^{-in\pi x_0/L}e^{\frac{-b^2n^2\pi^2\hbar^2}{L^2}/2\hbar^2}\Big),$$

or,

$$c_n = \sqrt{\frac{4b\pi}{L\sqrt{\pi}}}e^{\frac{-b^2n^2\pi^2\hbar^2}{L^2}/2\hbar^2}\sin\big(\frac{n\pi x_0}{L}\big). \quad (3.2.5)$$

The above equation shows that the expansion coefficient vanish for several values of x_0.

3.2.2 Short-term time evolution

As from the Fig.(3.1), the harmonic oscillator shows classical like behaviour over a few classical period. But for the infinite square well, if we plot the autocorrelation function over the few classical period, we observe the wave packet do not have classical like behaviour. As we can see from the Fig.(3.4) during the time evolution

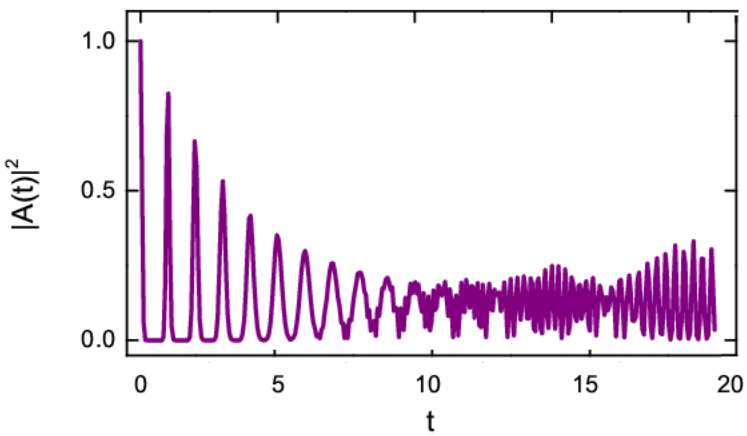

Figure 3.4: The modulus of autocorrelation function $|A(t)|^2$ for the particle inside the infinite potential well is plotted over few classical period.

of the wave packet, first the wave packet collapse and then undergoes to the revival of wave packet but after every classical period the wave packet never gain its initial position. It is because of the higher periodicity T_{rev} can be observe for a particle oscillating in infinite square well. This higher periodicity comes from the quadratic dependence of energy on quantum number. The classical behaviour of wave packet can be observe for a long-term time evolution.

3.2.3 Structure of quantum revival and fractional revival

As we know the probability amplitude c_n Eq.(3.2.4) is very important for the evolution of autocorrelation function $A(t)$ and depends on the initial momentum of the particle inside the well. The probability amplitude is used to describe the behaviour of system. The Fig.(3.5) is the plot for the modulus of the autocorrelation function $|A(t)|^2$ for the Gaussian wave packet with different values of initial momentum p_0

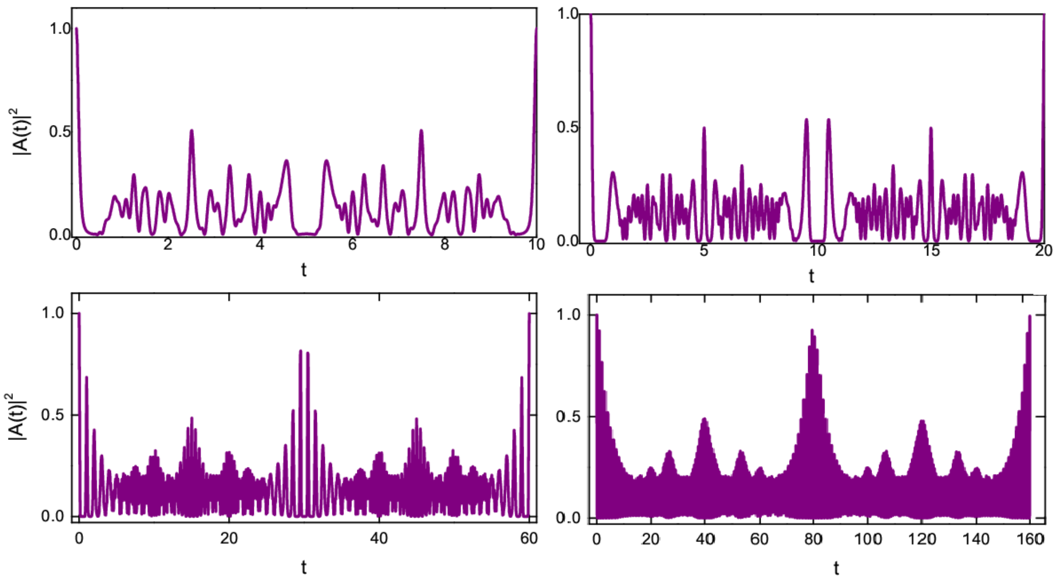

Figure 3.5: The $|A(t)|^2$ is plotted over revival time for the different values of initial momentum, for $p_0 = 5\pi, 10\pi, 30\pi, 60\pi$, where the horizontal axis is rescaled by classical period such that t/T_{cl}.

are plotted for one revival time, where the horizontal axis is rescaled by the classical period such that $t = t/T_{cl}$. It is obvious from the plots that the quantum wave packet occurs after larger number classical periods as the value of p_0 decreased or increased. This is due to the fact that $T_{rev}/T_{cl} = 2n_0$. The initial momentum of the particle,

$$p_0 = \frac{n\pi\hbar}{L},$$

depends on the quantum number n and L is the length of the well. If we increase the value of initial momentum the corresponding value of principle quantum number also increases. If we plot the autocorrelation function of the particle inside the potential well, we can see the behaviour of wave packet evolution for the different values of initial momentum.

As we can see from Fig.(3.6) with the increase in initial momentum the number of periodicities of wave packet is also increases. It's because the classical period,

$$T_{cl} = \frac{2mL^2}{n\pi\hbar},$$

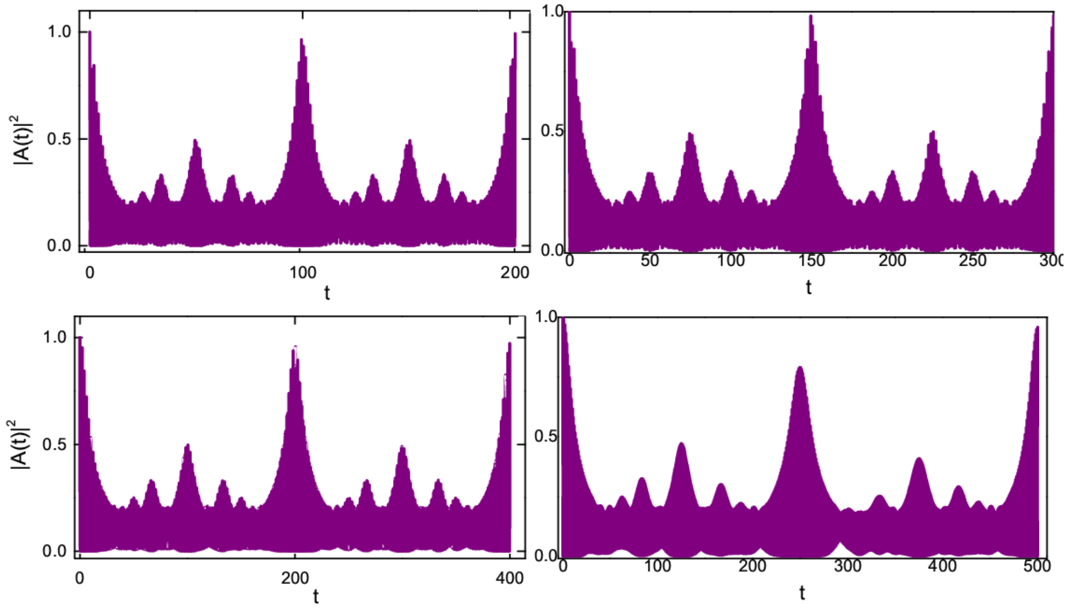

Figure 3.6: The $|A(t)|^2$ is plotted over revival time for the different values of initial momentum, for $p_0 = 100\pi, 150\pi, 200\pi, 250\pi$, where the horizontal axis is rescaled by classical period such that t/T_{cl}.

has the inverse relation with the quantum number, with the increase in n the duration of classical period T_c is decreases and we have more mini wave packets in long time evolution of wave packet.

3.2.4 Quantum carpets for Infinite potential well

Position space probability density

The probability of density of the particle is given as,

$$P(x,t) = \frac{2}{L} \sum_m \sum_n c_n c_m^* \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) e^{\frac{-i(E_n - E_m)t}{\hbar}}, \tag{3.2.6}$$

If we plot this probability for initial momentum $p_0 = 30\pi$ over different fractional revival and over complete half revival period, we can observe the decoherence in wave packet with time. During time evolution of wave packet the decoherence appear in wave packet after interacting with a reflecting boundaries. As we can see from Fig.(3.7), the wave packet starts its evolution and we can see the wave

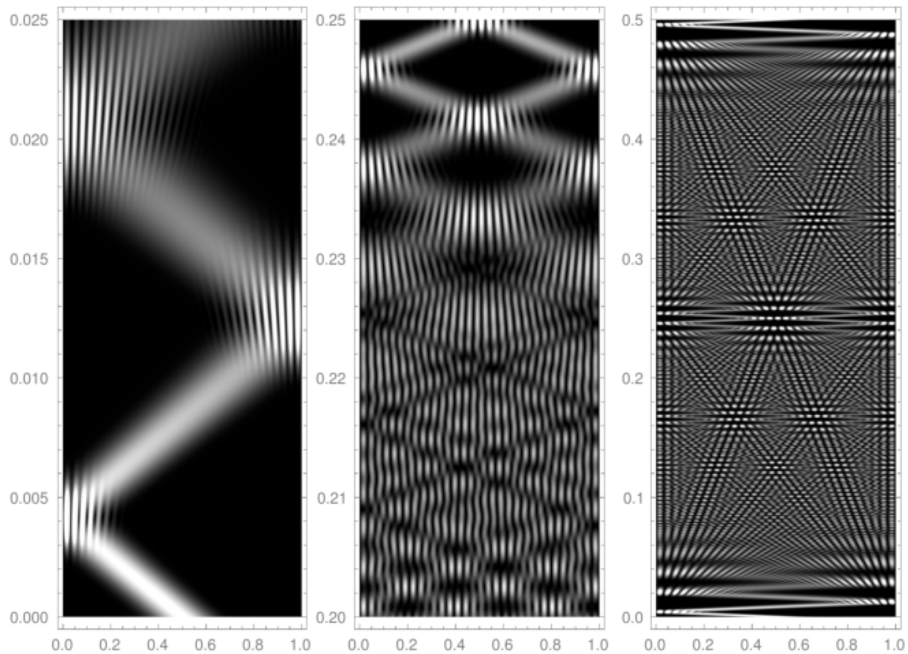

Figure 3.7: The quantum carpet woven for the particle inside infinite potential well has the value of initial momentum $p_0 = 30\pi$ indicating the classical-like short time evolution which experiences collapse during its evolution and relocalize itself with fractional copies od different order.

packet splits into many peaks when it interact with the reflecting boundaries. But in second graph, the decoherence appear in wave packet due to self interference of wave packet. And if we plot the probability density over a complete revival period, we can observe a pattern of self interference of mini wave packets. From Fig.(3.7), we can observe the evolution of wave packet at $2T_{cl}$ more clearly and with the increase in time the revival structure becomes more dense. It is obvious from the plot that a well localized single peaked probability density of the wave packet evolves quasi-classically during its early time evolution and splits in to multiple sub-peaks after successive bounces with the walls of the infinite potential well. These multiple peaks evolve with time with their own phases and undergoes a series of constructive and destructive interference.

During the time evolution the fractional copies of original wave packet also appear at time $t = pT_{rev}/q$, where p and q are mutually prime numbers. From Fig.(3.8),

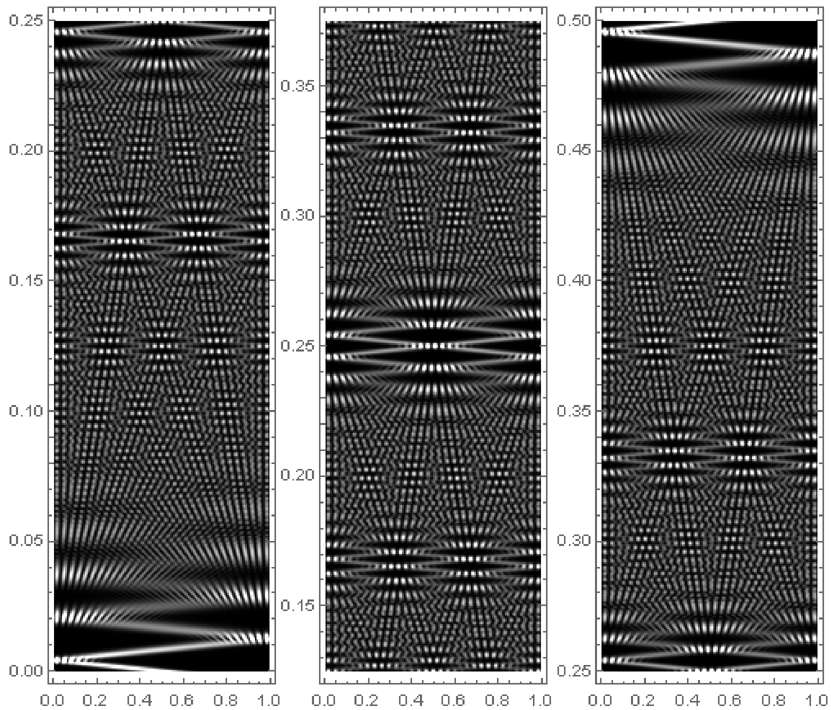

Figure 3.8: The probability density $P(x,t)$ is plotted over different fractional revival time $t = T_{rev}/4,\ 3T_{rev}/8,\ T_{rev}/2,\ 30\pi$.

we can observed different fractional revival for infinite potential well the first graph the plotted for the $t = 0\ to\ \frac{T_{rev}}{4}$, second is plotted for the $t = \frac{1}{8}\ to\ \frac{3T_{rev}}{8}$ and the last one is plotted for the $t = \frac{1}{4}\ to\ \frac{T_{rev}}{2}$.

If we plot the probability density as a function of x and t for the different values of momentum , the carpet woven by a particle inside a infinite potential well for the different values of initial momentum p_0 is shown. As we can see from the figure by increasing the value of initial momentum we have a more dense pattern of probability density. With the increase in the value of initial momentum the corresponding value of classical period decreases, we have more mini wave packets for one revival period. The dark areas in the carpet correspond to the low probability whereas the brighter areas have high probability. From Fig.(3.9) we can see, the fractional revivals are more prominent for very small value of initial momentum, i.e., $p_0 = 5\pi$ and 10π, but for the high value of initial momentum $p_0 = 20\pi$ and 30π, its hard to clearly identify the fractional revival.

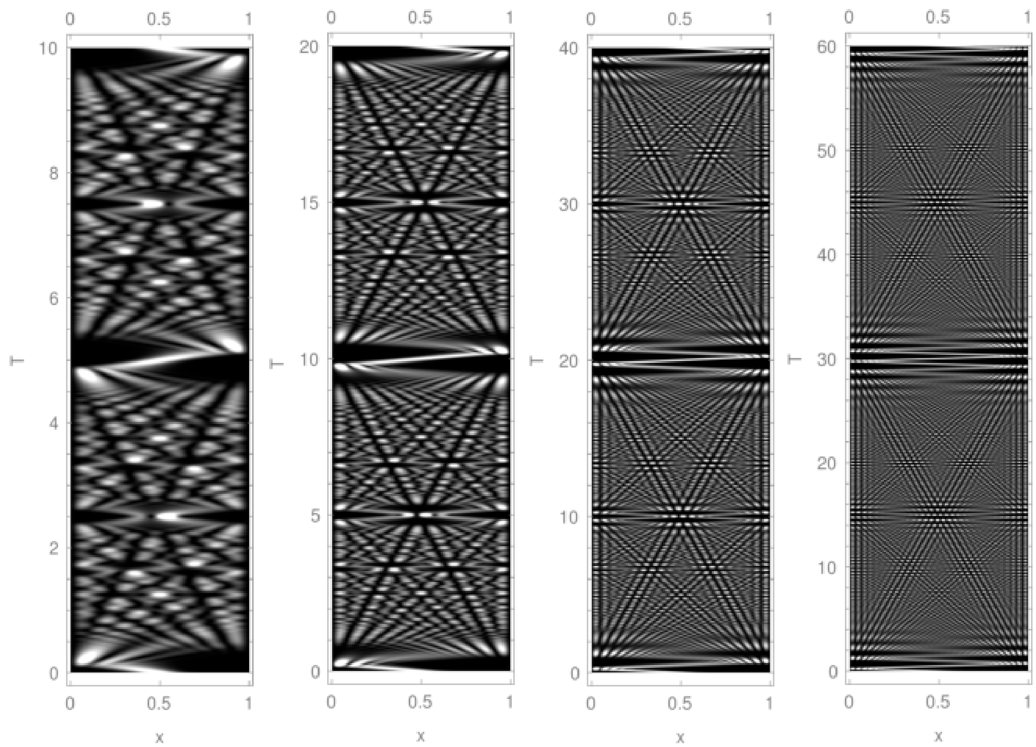

Figure 3.9: The quantum carpet for the particle inside infinite potential well for different values of initial momentum, $p_0 = 5\pi,\ 10\pi,\ 20\pi,\ 30\pi$.

Momentum space quantum carpets

The probability in momentum space can be calculated by taking the fourier transform of position space eigen function. Let's consider,

$$\varphi_n(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi_n(x) dx, \tag{3.2.7}$$

$$\varphi_n(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{0}^{L} e^{-ipx/\hbar} \sin\left(\frac{n\pi x}{L}\right)(x) dx,$$

$$\varphi_n(p) = \sqrt{\frac{\hbar}{\pi L}} \frac{p_n}{p^2 - p_n^2} [(-1)^n e^{ipL/\hbar} - 1]. \tag{3.2.8}$$

and the momentum space probability is given as,

$$P(p,t) = \frac{\hbar}{\pi L} \sum_m \sum_n c_n c_m^* \left(\frac{p_n}{p^2 - p_n^2} [(-1)^n e^{ipL/\hbar} - 1] \right) \left(\frac{p_m}{p^2 - p_m^2} [(-1)^n e^{-ipL/\hbar} - 1] \right) e^{\frac{-i(E_n - E_m)t}{\hbar}}. \tag{3.2.9}$$

If we plot momentum space-time probability over a different fractional revival time,

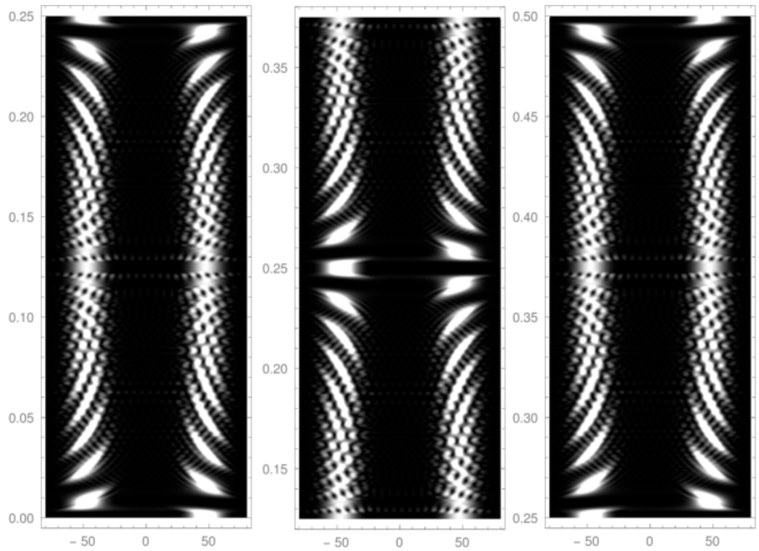

Figure 3.10: Time evolution of the momentum space probability density $P(p,t)$ for an initial gaussian wave packet with initial momentum $p_0 = 15\pi$.

The Fig.(3.10) is plotted for the different fractional revival in a time window equal to $t = T_{rev}/2$, for this Fig.(3.10) we can see that the fractional revival structure is not as prominent as for the position space quantum carpets, but somehow we can identify some fractional revival from the momentum space carpet as well. The quantum carpets drawn by the momentum space probability for the different value of initial momentum is shown in Fig.(3.11). As we can see for the very small value of initial momentum, $p_0 = 5\pi$, the particle is peaked at zero but with the increase in initial momentum, we have zero probability at zero.

As we can see from figure, the momentum space probability density of the particle shows two peaks in momentum centered on the classical values of probability density $P_{cl} = 1/L$. It is because the momentum of the particle,

$$p_n^2 = 2mE_n,$$

or

$$p_n = \pm\sqrt{2mE_n},$$

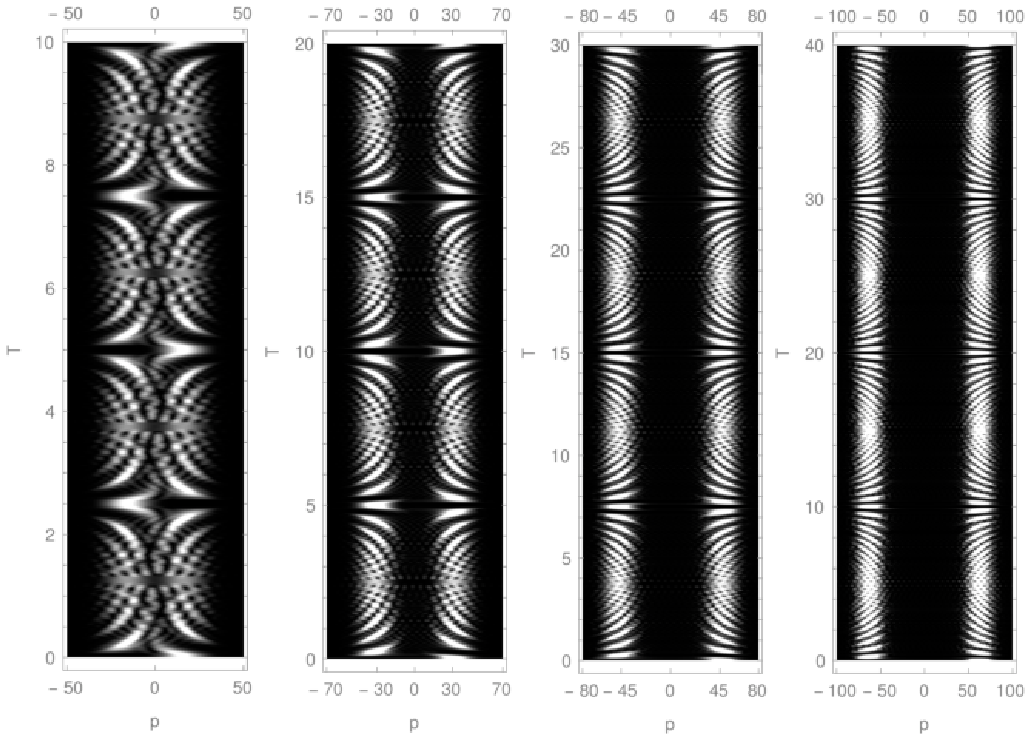

Figure 3.11: The quantum carpet for values of initial momentum is shown, $p_0 = 5\pi, 10\pi, 15\pi, 20\pi$

the momentum changes its sign when the particle reverse its direction. For high value of quantum number the quantum momentum space probability approaches to classical probability density.

3.3 The quantum bouncer

The quantum bouncer is considered as a point mass m which is fallen under the influence of gravity on a perfectly elastic surface and bounce back with no loss of kinetic energy, the potential energy of this bouncing mass is given as [41],

$$V(x) = \begin{cases} mgx, & x > 0 \\ \infty, & x < 0 \end{cases}$$

The development of different techniques to manipulate and cool the atoms with high precision has made the experimental realization of quantum bouncer possible

[41]. The schematic diagram is shown.Initially the atoms are trapped and cooled in

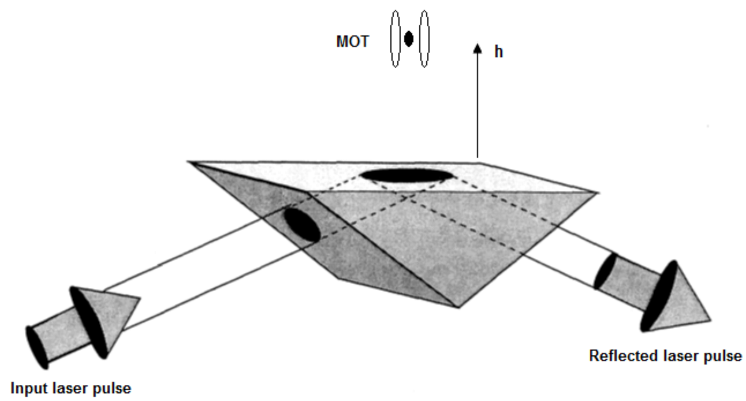

Figure 3.12: The Schematic diagram of the quantum bouncer

magneto-optical trap (MOT) and are dropped from a height of few millimeters on the atomic mirror. Atomic mirror is a device which can reflect atoms and it can be made of electric or magnetic field. The atomic mirror is formed by the interaction of laser pulse with the concave glass prism and after interaction the laser pulse reflect at some angle. The atoms moves under the influence of gravity and reflect back when they interact with the evanescent field [35].

The quantum bouncer gives a clear correspondence between classical and quantum limits because the classical motion of a bouncing ball is periodic and its quantum motion is aperiodic. The quantum motion of a bouncing ball exhibits the collapses and revival of the oscillations. To understand its classical motion consider a ball is dropped on a perfectly elastic and reflecting surface. The ball will bounce back to its initial height because there is no loss of energy, the classical motion of the ball is shown in Fig.(3.13), which is equally probable.

In quantum mechanics, due to the Heisenberg uncertainty principle the position and velocity of the particle is not well defined at any instant of time. To study the dynamics of a bouncing ball in quantum mechanics, we need to solve the schrodinger equation for the quantum bouncer and it is given as,

$$-\frac{\hbar^2}{2m}\frac{d^2\psi_n(x)}{dx^2} + mgx\psi_n(x) = E_n\psi_n(x), \tag{3.3.1}$$

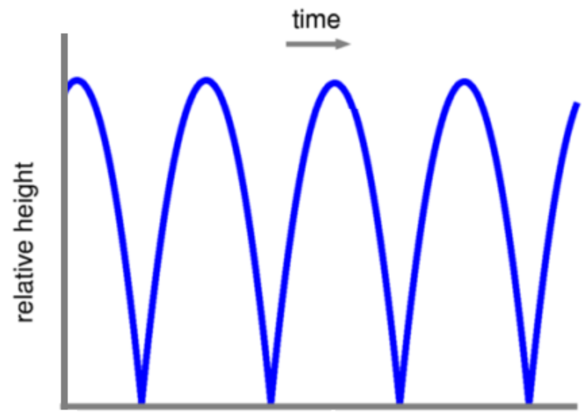

Figure 3.13: The classical motion of a bouncing ball

with the boundary condition,

$$\psi_n(0) = 0.$$

To solve the above equation we need to rescale the position and energy variable. Consider,

$$l_g = \Big(\frac{\hbar^2}{2gm^2}\Big)^{1/3}, \tag{3.3.2}$$

l_g is the gravitational length.

Let $x' = x/l_g$ and $E'_n = E_n/(mgl_g)$, or,

$$E'_n = E_n\Big(\frac{2}{\hbar^2 mg^2}\Big)^{1/3}. \tag{3.3.3}$$

By using Eq.(3.3.3) and Eq.(3.3.2) in Eq.(3.3.1) and rearranging the equation, we have,

$$\frac{d^2\psi_n}{dx'^2} - (x' - E'_n)\psi = 0. \tag{3.3.4}$$

The solution of this equation is given by Airy function, Ai or Bi, of the variable $(x' - E')$. The Airy function Bi is not acceptable here because it goes to infinity as the argument grows [41]. From the boundary condition $\psi(0) = 0$, we can say $Ai(-E') = 0$. If the zeros of airy is denoted by $-x_n$ with $n = 1, 2, 3.......$, the solution of abve equation is written as,

$$\psi_n(x') = N_n Ai(x' - x_n), \tag{3.3.5}$$

where, N_n is the normalization constant. The zeros of airy x_n is define as,

$$x_n = \left[\frac{3\pi}{2}\left(n - \frac{1}{4}\right)\right]^{2/3}. \tag{3.3.6}$$

For normalization constant, we have not found any approximate expression in literature because for normalization constant we need to solve,

$$N_n \simeq \left[\int_0^\infty Ai^2(x - x_n)dx\right]^{-1/2}, \tag{3.3.7}$$

for large value of n and by using the asymptotic expansion of Airy function we have,

$$N_n \simeq \left(\frac{\pi}{\sqrt{x_n}}\right)^{1/2} \simeq \left(\frac{2\pi^2}{3(n - 1/4)}\right)^{1/6}. \tag{3.3.8}$$

To calculate the probability amplitude c_n, consider an initial gaussian function with zero initial momentum given as,

$$\Psi(x, 0) = \left(\frac{2}{\pi\sigma^2}\right)^{1/4} e^{-(x-x_0)^2/\sigma^2}, \tag{3.3.9}$$

where, σ is the full width of the wave packet.

The probability amplitude c_n is given as,

$$c_n = N_n\left(\frac{2}{\pi\sigma^2}\right)^{1/4} \int_0^\infty Ai(x - x_n)e^{-(x-x_0)^2/\sigma^2}dx,$$

the approximate expression for c_n is given as,

$$c_n \simeq N_n\left(\frac{2}{\pi\sigma^2}\right)^{1/4}\left(1 - 4\frac{x_0 - x_n}{\sigma^4} + \frac{8}{3}\frac{(x_0 - x_n)^3}{\sigma^6}\right)e^{-(x_0-x_n)^2/\sigma^2}. \tag{3.3.10}$$

The time evolved wave packet is given as,

$$\Psi(x, t) = \sum_n c_n N_n Ai(x' - x_n)e^{-\frac{iE_n t}{\hbar}},$$

where, E_n is the energy of the bouncing ball and it is given as,

$$E_n = x_n\left(\frac{\hbar^2 mg^2}{2}\right)^{1/3}. \tag{3.3.11}$$

The energy of quantum bouncer can associate with the central value of n_0 in the eigenstate expansion with the initial value x_0 [4]. The autocorrelation function

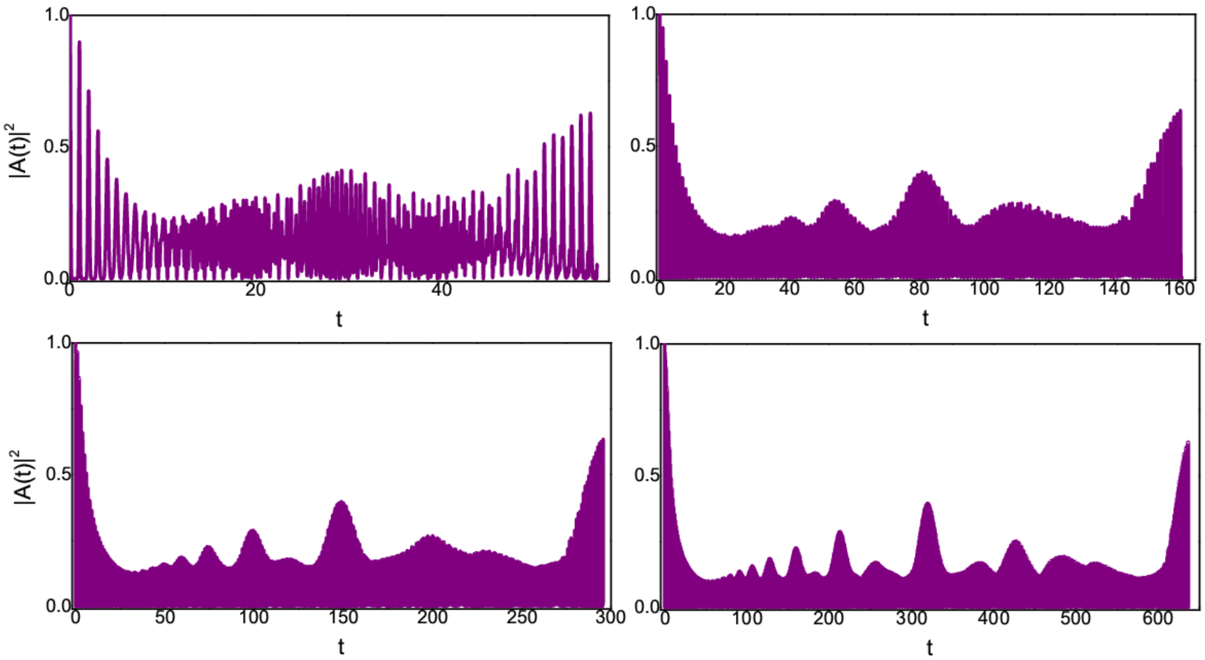

Figure 3.14: The mod square of autocorrelation function $|A(t)|^2$ is plotted for the different value of initial height $x_0 = 20,\ 40,\ 60\ \ and\ \ 100$.

Eq.(2.2.3) for quantum bouncer is, The classical period of the bouncing ball is given as,

$$T_{cl} = \frac{2\pi\hbar}{|E'(n)|} = \frac{3\pi\hbar(n+3/4)}{E(n)}, \tag{3.3.12}$$

and revival time is,

$$T_{rev} = \frac{16mx_0^2}{\pi\hbar}. \tag{3.3.13}$$

From the Fig.(3.14) we can observe collapse of revival with the increase in time. The above figure is plotted for the different value of initial height, from this figure we can see that the structure of fractional revival becomes prominent with the increase in initial height. This is because their is high periodicity "super revival" exists in quantum bouncer. As we can see from the figure even at $t = T_{rev}$ the time evolved wave packet is not exactly equal to its initial wave packet and with the increase in time the collapse of wave packet relocalization becomes more prominent.

3.3.1 Super-revival in quantum bouncer

As we know the energy eigenvalue of the quantum bouncer,

$$E_n = \Big(\frac{3\pi}{2}(n - \frac{1}{4})\Big)^{2/3} mgl_g,$$

is not have the linear dependence on quantum number. As in the previous section we observed revival and fractional revival for long time evolution of wave packet. Due to the fractional dependence of energy on quantum number we can even observe super-revival in quantum bouncer. As we know the revival time for quantum bouncer is given as,

$$T_{rev} = \frac{2\pi\hbar}{|E_n''/2!|},$$

for $m = g = \hbar = l_g = 1$

$$T_{rev} = \frac{4\hbar\Big(\frac{3\pi}{2}\Big(n - \frac{1}{4})\Big)^{4/3}}{\pi},$$

As we know the super-revival is defined as,

$$T_{sr} = \frac{2\pi\hbar}{|E_n'''/3!|},$$

$$\frac{d^3 E_n}{dn^3} = \pi^3\Big(\frac{3\pi}{2}(n - \frac{1}{4})\Big)^{-7/3}.$$

Using E_n''', we have,

$$T_{sr} = \frac{12\hbar\Big(\frac{3\pi}{2}(n - \frac{1}{4})\Big)^{7/3}}{\pi^2}, \tag{3.3.14}$$

$$T_{sr} = \frac{9}{2}\Big(n - \frac{1}{4}\Big)T_{rev}. \tag{3.3.15}$$

As we can see for a very long-term time evolution we can observe super-revival in quantum bouncer. The Fig.(3.16) is plotted for the different revival time, we can see from this figure the wave packet is not completely relocalize itself even after $5T_{rev}$, it is because of the higher periodicity exists in quantum bouncer, it refers that we may observe a full revival at super revival time and which is a very high time.

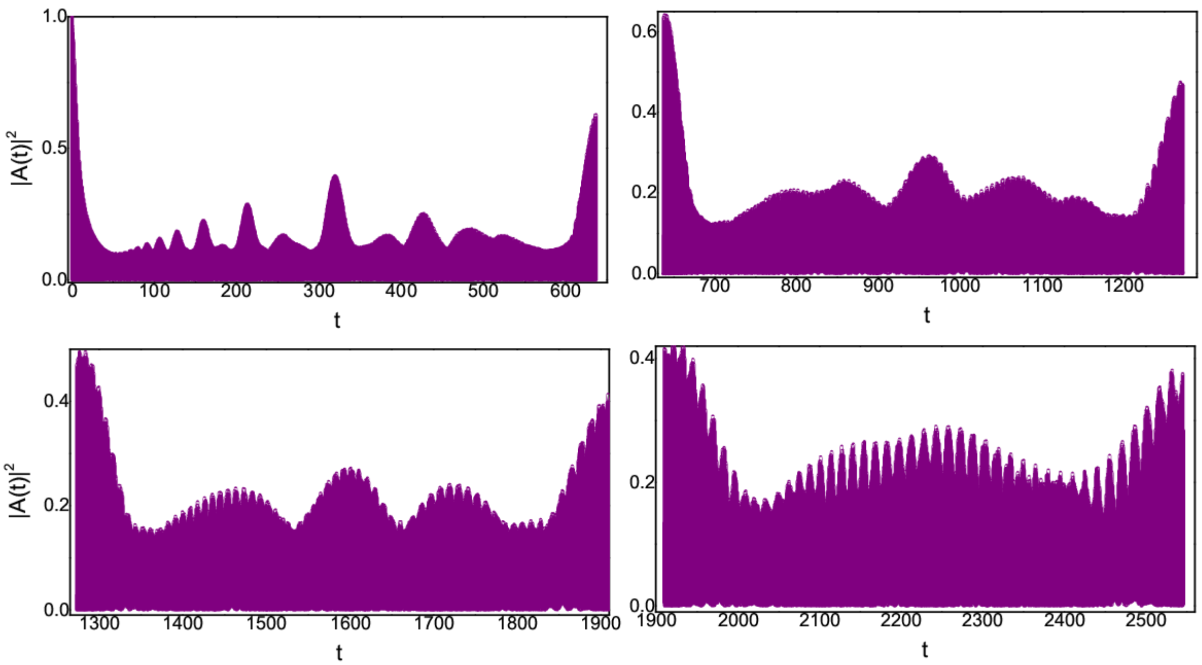

Figure 3.15: The mod square of autocorrelation function $|A(t)|^2$ is plotted over, $2T_{rev},\ 3T_{rev},\ 4T_{rev},\ and\ 5T_{rev}$.

3.3.2 Position space quantum carpets

The probability density Eq.(2.2.4) can be expressed for quantum bouncer as,

$$P(x,t) = |\psi(x,t)|^2 = \sum_n \sum_m c_n^* c_m N_n^* Ai^*(x' - x_n) N_m Ai(x' - x_m) e^{\frac{i(E_n - E_m)t}{\hbar}}.$$

If we plot this probability density as a function of position and time we have a structure of quantum carpet shown in Fig.(3.16), these plots are for the different fractional revival time. As we can see from the Fig.(3.16), initially when the wave packet interact with the hard boundary of the gravitational cavity the wave packet collapse and split into multiple peaks but regain its initial form when interact with the soft boundary of the cavity. But with the increase in time the wave packet continuously interact with the hard boundary of the triangular well and complectly collapse and we can observe a number of mini wave packets. Even at time $t = T_{rev}$, we do not have the maximum probability of the time evolved wave packet. It is because of the higher periodicity, super revival exists for quantum bouncer. We

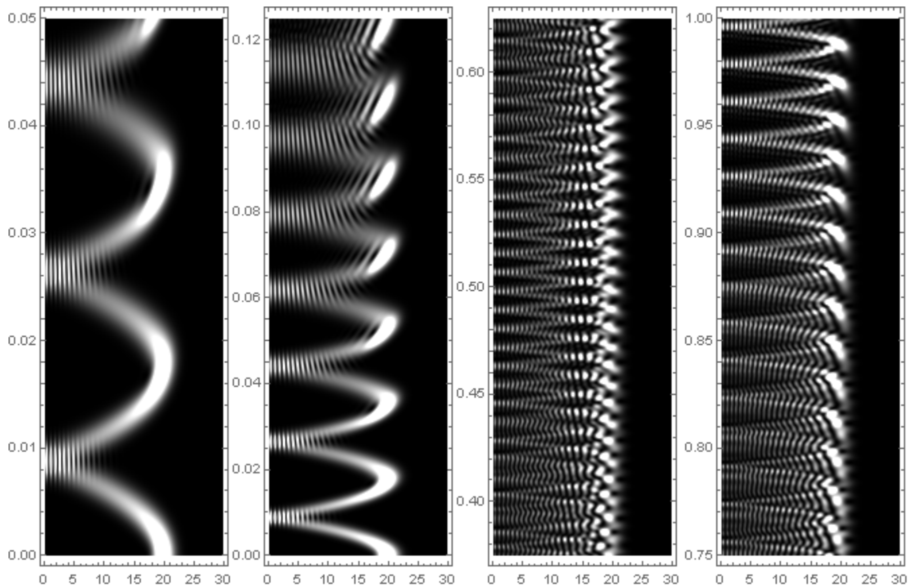

Figure 3.16: The quantum carpet of a bouncing ball for the initial height $x_0 = 20$.

may observe a collapse of revival in the wave packet of quantum bouncer, like in infinite potential well we have observed the collapse of classical period during wave packet evolution. The wave packet of quantum bouncer may be completely relocalize itself at time $t = T_{sr}$ and we can observed the maximum probability of wave packet relocalization at time $t = T_{sr}$.

Chapter 4

Wave packet revivals through variance-based uncertainty and information entropy

The information and entropy of the system can be considered as measure of uncertainty of probability distribution. There are many relationships established [60] between entropy and probability density on the basis of the properties of entropy. The information entropy is related with the evolution and relocalization of a initially well localized wave packet. As we know that the probability density in position and momentum space use to describe the time evolution of a quantum system. Position and momentum space probability density together define the phase space of the system. The phase space probability density is helped to develop different measuring tool, such as, quantum carpets, Wigner function and information entropy.

In this chapter we discuss the revival of wave packet by means of information entropy. If we measure the information entropy in position space, we can also measure the uncertainty in the localization of the particle in that space [58]. If the entropy is small then we have a higher accuracy of predicting the localization of the particle in particular region of the space. Similarly in momentum space, the momentum space entropy gives the uncertainty in the momentum of the particle. In this chapter we are going to discuss the revival behaviour by means of information entropy for the infinite square well.

4.1 The information entropy and uncertainty principle

Wehrl and Shannon were the first who describe the information entropy of a quantum mechanical state in terms of its probability distributions [64]. The information entropy is also compaitable with the Heisenberg uncertainty principle. Mathematically the uncertainty principle is express as,

$$\Delta A \Delta B \geq \frac{1}{2}|\langle [A, B] \rangle|, \tag{4.1.1}$$

where, A and B are the operators and,

$$[A, B] = AB - BA,$$

and ΔA is the standard deviation in operator A, given as,

$$(\Delta A)^2 = \langle A^2 \rangle - \langle A \rangle^2. \tag{4.1.2}$$

In 1956, Brillouin was shown that their is some relation exists between the uncertainty principle and information entropy. In quantum mechanics, Partovi [59] and Deutsch [61] was introduce the uncertainty relation which can be expressed in terms of information entropy [65]. The entropic uncertainty relation is given as,

$$H(X) + H(P) \geq log(\frac{e}{2}), \tag{4.1.3}$$

where,

$$H(X) = -\int |\psi(x)|^2 log|\psi(x)|^2 dx,$$

or,

$$H = -\sum_i P_i log P_i,$$

is the information entropy in position space and P_i is the probability of the system. The information entropy may use as precise measurements of uncertainty. The information entropy is a better measure of uncertainty than the standard deviation in many ways. The confirmation of entropy as a natural measure of uncertainty is found in experimental psychology.

In 1955, Hyman establish as experiment in which he purpose that the reaction time of a human as a function of stimulus uncertainty is proportional to entropy H. In this experiment a person were asked to react in a certain specified way to each pattern [65]. The result of this experiment that the reaction time is a linear function of entropy.

In physics, the entropy is important for the physical application to account the disorder in the system. Entropy is first introduce by Clausius in mid $19th$ century. It is an important concept in thermodynamics and in classical statistical mechanics. Later on the Boltzman defined entropy as,

$$S_B = K_b ln \Omega,$$

where K_B is the Boltzman constant and Ω is the number of microstate. Shanon's construct the entropy by using the idea of information obtaind from a system with a large vagueness has been highly profitable and Shanon's entropy is useful for the discrete probability theory. He introduce the entropy measuring the amount of information of the state of a system and mutual entropy [62] both represents the correct amount of information which transmitted from initial to final system through a channel.

Shannon's entropy is basically use for the measure of uncertainty in the system. Let's consider a random variable Y, which have the discrete finite number of possible values $y_1, y_2, ..., y_n$ with probabilities $p_1, p_2, ..., p_n$, and total probability,

$$\sum_i p_i = 1.$$

The entropy by the position space probability density is defined as,

$$S_\rho = -\int P(x) \ln P(x) dx, \tag{4.1.4}$$

where,

$$P(x) = |\psi(x)|^2,$$

The momentum space entropy is,

$$S_\gamma = -\int P(p) \ln P(p) dp, \tag{4.1.5}$$

where,

$$P(p) = |\varphi(p)|^2,$$

is the momentum space probability density. The uncertainty relation for the information entropy is,

$$S_\rho + S_\gamma \geq 1 + \ln \pi. \tag{4.1.6}$$

4.1.1 Uncertainty and expectation value analysis

As from the Eq.(4.1.2) the uncertainty in position space can be express as,

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2,$$

and in momentum space,

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2.$$

To calculate the uncertainties in position and momentum space, we need to calculate the expectation value in position space as well as in momentum space. The time dependent expectation value in position space is given as,

$$\langle x \rangle_t = \int_0^L \psi^*(x,t) x \psi(x,t) dx,$$

as we know $\psi(x,t)$ is,

$$\psi(x,t) = \sqrt{\frac{2}{L}} \sum_n c_n \sin\left(\frac{n\pi x}{L}\right) e^{-iE_n t/\hbar},$$

For $m = n$, the matrix element is

$$\langle x \rangle_t = \frac{2}{L} \sum_n |c_n|^2 \int_0^L x \sin^2\left(\frac{n\pi x}{L}\right) dx, \tag{4.1.7}$$

$$\langle x \rangle_t = \sum_n |c_n|^2 \frac{\pi}{2}. \tag{4.1.8}$$

The cross-matrix element of the expectation value can be express as,

$$\langle n|x|m\rangle_t = \frac{2}{L}\sum_n\sum_m |c_n||c_m|e^{-i(E_n-E_m)t/\hbar}\int_0^L x\sin\Big(\frac{n\pi x}{L}\Big)sin\Big(\frac{m\pi x}{L}\Big)dx, \quad (4.1.9)$$

by using the trigonometric identity,

$$\sin(u)\sin(v) = \frac{1}{2}(\cos(u-v) - \cos(u+v)), \quad (4.1.10)$$

$$\langle n|x|m\rangle_t = \frac{1}{L}\sum_n\sum_m |c_n||c_m|e^{-i(E_n-E_m)t/\hbar}\Big[\int_0^L x \times\Big(\cos\Big(\frac{(m-n)\pi x}{L}\Big) - \cos\Big(\frac{(m+n)\pi x}{L}\Big)\Big).$$

After computing this integral we have,

$$\langle n|x|m\rangle_t = \frac{1}{L}\sum_n\sum_m |c_n||c_m|e^{-i(E_n-E_m)t/\hbar} \times \frac{4mn[(-1)^{m+n}-1]}{\pi(m^2-n^2)^2} \quad (4.1.11)$$

$$\langle n|x|m\rangle = \begin{cases} \frac{\pi}{2}, & m = n \\ \frac{4mn[(-1)^{(m+n)}-1]}{\pi(m^2-n^2)^2}, & m \neq n \end{cases} \quad (4.1.12)$$

Similarly for $\langle n|x^2|m\rangle$, we have,

$$\langle n|x^2|m\rangle = \begin{cases} \frac{2n^2\pi^2-3}{6n^n}, & m = n \\ \frac{2\cos[(m-n)\pi]}{(m-n)^2} - \frac{2\cos[(m+n)\pi]}{(m+n)^2}, & m \neq n \end{cases} \quad (4.1.13)$$

The uncertainty in position space is given as,

$$(\Delta x)^2 = \langle x^2\rangle - \langle x\rangle^2, \quad (4.1.14)$$

As we know the momentum in terms of position is express as,

$$\hat{p} = -i\hbar\frac{\partial}{\partial x},$$

by using the position representation of momentum, we obtain the matrix of the momentum p as,

$$\langle n|p|m\rangle_t = -i\hbar\int_0^L \psi^*(x,t)\frac{\partial}{\partial x}\psi(x,t)dx,$$

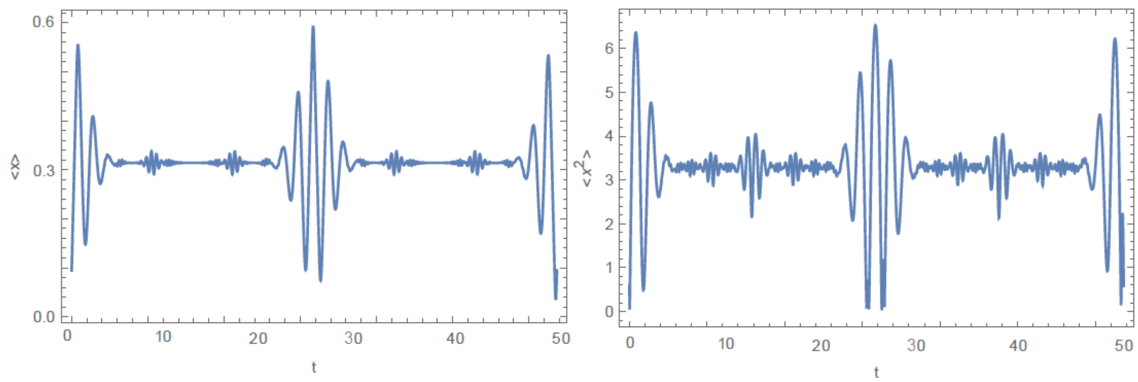

Figure 4.1: The plot for the $\langle x^2 \rangle$ and $\langle x \rangle$ over $t = T_{rev}$

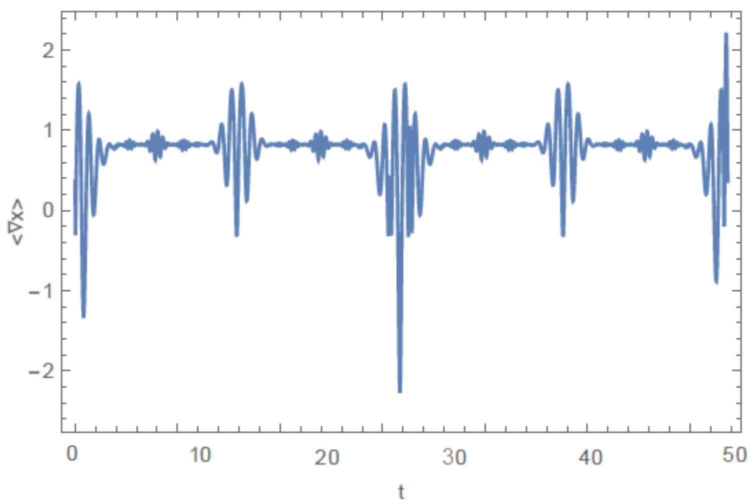

Figure 4.2: The uncertainty in position space is plotted as function of time.

$$\langle n|p|m\rangle = \begin{cases} 0, & m = n \\ \frac{i2mn[1-(-1)^{(m+n)}]}{\pi(m^2-n^2)^2}, & m \neq n \end{cases} \tag{4.1.15}$$

Similarly for p^2, we have,

$$\langle n|p^2|m\rangle = \begin{cases} n^2, & m = n \\ 0, & m \neq n \end{cases} \tag{4.1.16}$$

The uncertainty in momentum space is given as,

$$(\Delta p)^2 = \langle p^2 \rangle - \langle p \rangle^2, \tag{4.1.17}$$

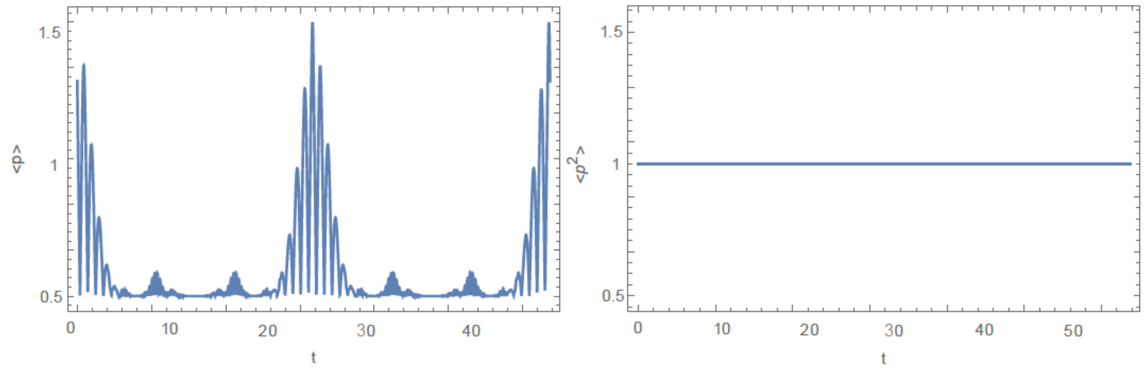

Figure 4.3: The $\langle p^2 \rangle$ and $\langle p \rangle$ is plotted over $t = T_{rev}$

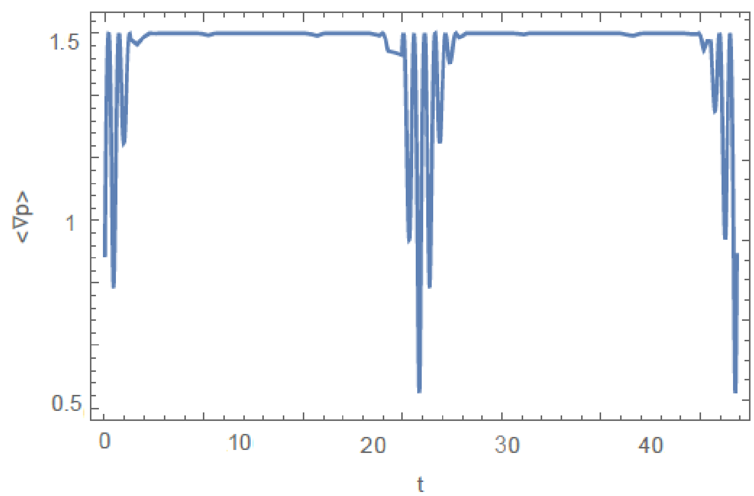

Figure 4.4: The uncertainty in momentum space is plotted as function of time.

The product of the position space uncertainty and momentum space uncertainty $\Delta x \Delta p$ is shown in figure as,

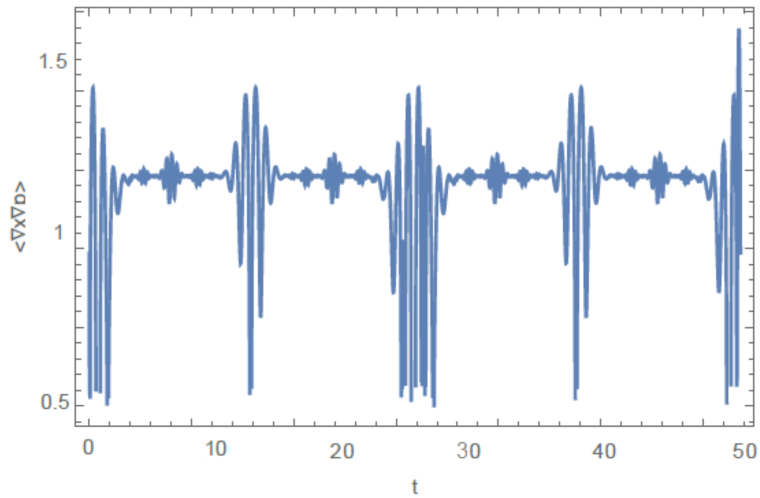

Figure 4.5: The uncertainty in momentum space and in position space is plotted as function of time.

4.1.2 Investigation of revivals and fractional revivals by information entropy

The information entropy is defined as,

$$S(t) = -\int P(x,t)\ln P(x,t)dx - \int P(p,t)\ln P(p,t)dp. \tag{4.1.18}$$

$$S(t) = S_\rho(t) + S_\gamma(t).$$

The Eq.(4.1.18) is the sum of the position space entropy $S_\rho(t)$ and momentum space entropy $S_\gamma(t)$. The information entropy is used to probe the fractional revival which are are hard to identify through autocorrelation function.

A localize wave packet with high probability density around its mean position in position space have smaller unceratnity and leads to smaller value of entropy. As the wave packet evolve with time in position space, it will interfere constructively/destructively with itself during short-term time evolution. The related uncertainty in position increases which implies an increase in information entropy [32]. Thus during time evolution, the information entropy gives order and disorder of an initially localized wave packet.

The entropic uncertainty [33] is defined as,

$$S_\rho + S_\gamma \geq 1 + \ln \pi. \tag{4.1.19}$$

This inequality is a generalization of variance based Heisenberg uncertainty relation.

We can understand the revivals and fractional revivals during long time evolution of an initial localize wave packet by means of information entropy. If the sum of the entropies in conjugate space has a minimum value at some particular time, we have a maximum information of wave packet relocalization at this particular time. For larger vale of entropic sum. we have smaller information about wave packet relocalization. The maxima and minima of sum of entropies in conjugate space is opposite to the autocorrelation function, because the autocorrelation function has its maximum value in case of revival and zero in case of collapse.

To identify the revival and fractional revival of a particle moving inside the infinite potential well [58], we will calculate the position space entropy and momentum space entropy. The time-dependent wave packet for the particle moving inside the infinite potential well is,

$$\psi(x,t) = \sum_n c_n \sqrt{\frac{2}{L}} \sin\left(\frac{n\pi x}{L}\right) e^{-iE_n t/\hbar},$$

is the superposition of all the possible eigenstate of the system.

The information entropy is,

$$S_\rho(t) = -\int P(x,t) lnP(x,t) dx, \tag{4.1.20}$$

where, $P(x,t)$ is the position space probability density Eq.(3.2.6).

The entropy for the momentum space probability is given as,

$$S_\gamma(t) = \int P(p,t) lnP(p,t) dp, \tag{4.1.21}$$

where, $P(p,t)$ is the momentum space probability density Eq.(3.2.9).

The sum of the position and momentum space entropy is given as,

$$S(t) = S_\rho(t) + S_\gamma(t). \tag{4.1.22}$$

For long time evolution, a large amplitude modulation is superimposed on the quasi-periodic oscillation [58]. During the time evolution, the initial wave packet evolve with time and delocalize and gives a sequence of fractional revivals. If we plot the Eq.(4.1.20), Eq.(4.1.21) and Eq.(4.1.22) and by comparing it with the autocorrelation function, we found that the sum of entropies indicate the fractional revival, but either of them separately not helpful. It is because the sum of the entropies in both configurational space provides a complete information of wave packet dynamics in position and momentum space. from above Fig.(4.6) we observe at the fractional revival time the sum of entropies reaches a relative minimum value and the modulus square of autocorrelation function is at a relation maximum value. Even for the higher value of quantum number, the fractional revivals are difficult to detect by autocorrelation function we can easily identify them by means of sum of entropies in conjugate space. The sum of entropies shows a relative minima at the fractional revival time and it provide a very useful tool to observe the fractional revival even for high value of initial momentum [58]. These minima in the plot of sum of entropies is because of the mini wave packet appear independently during the time evolution of the wave packet. From the Fig.(4.6) we can see, only the sum of entropies can indicate the fractional revivals, not by either of them separately because the sum of entropies have the information of momentum and position of the wave packet. So we can say only the sum of entropies in conjugate space have the importance over the autocorrelation function to study the behaviour of fractional revivals of quantum wave packet.

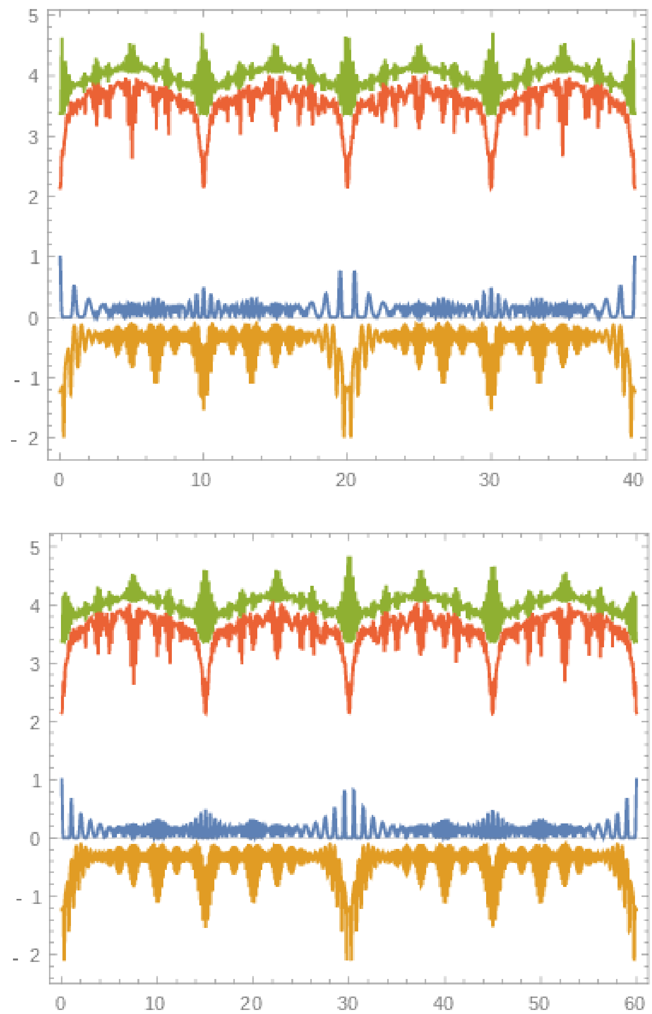

Figure 4.6: The plot for the initial momentum $p_0 = 20\pi, 30\pi$, the yellow line is the position space probability entropy, the green line is the momentum space probability entropy, red line is the sum of entropies in conjugate space and blue line is the modulus of autocorrelation function.

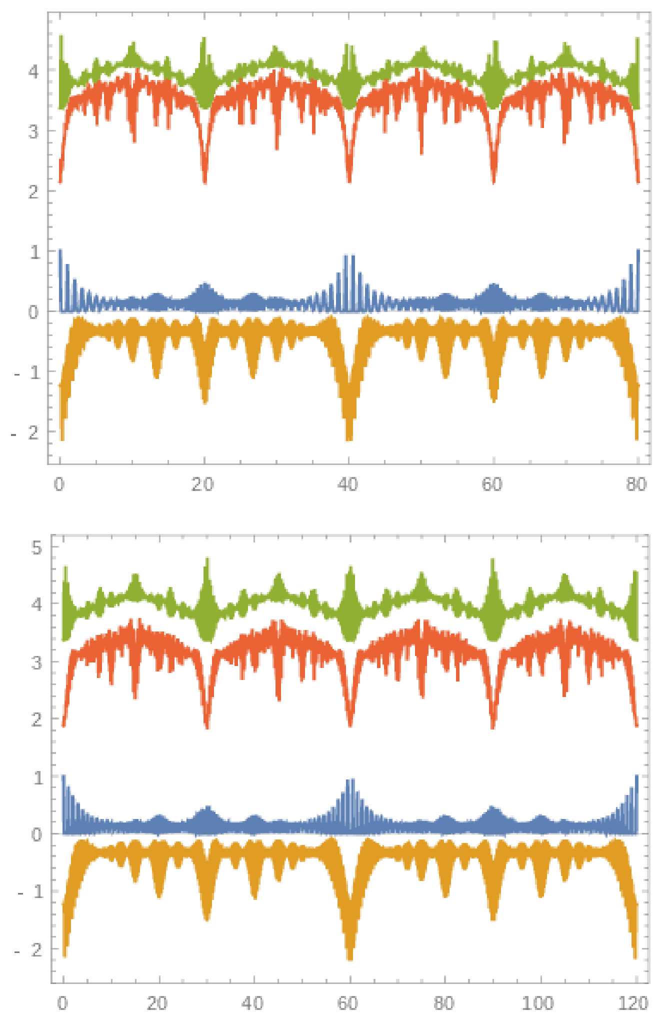

Figure 4.7: The plot for the initial momentum $p_0 = 40\pi, 60\pi$, the yellow line is the position space probability entropy, the green line is the momentum space probability entropy, red line is the sum of entropies in conjugate space and blue line is the modulus of autocorrelation function.

Chapter 5

Summary and conclusion

In this thesis we have studied the dynamics of wave packet in context of wave packet revival. In our work, we have studied the general construction of localized wave packet, a localized wave packet is the superposition of all possible eigenstate of the system. We study the time evolution of these states and analyze their temporal characteristics. The time evolved state is the result of the interaction of time evolution operator on the initial state of the system. The characteristic of quantum revival of a wave packet can be studied by means of initially localized wave packet which has a short term time evolution. During the short time evolution the spread of wave packet reverse itself and relocalize. The relocalization of wave packet is called the revival of wave packet. Full revival, fractional revival and super-revival can be identify by the autocorrelation function. We studied the autocorrelation function which is equal to the overlap of time evolved state with initial state. If the autocorrelation function has its maximum value, then we can say the wave packet complectly relocalize itself. The fractional revival appears if the wave packet relocalize into a number of smaller copies of initial wave packet.

Furthermore, we studied the model system including the infinite square well, harmonic oscillator and quantum bouncer by applying that general formalism. We study the revival and fractional revival behaviour by constructing the autocorrelation function. Then we constructed the probability density as a function of space and time for all model system and we extended our work to momentum space quantum carpet for the infinite potential well. The time evolution of these probability density results in the constructive and destructive interference. The time evolved probability densities lead to the formation of quantum carpets.

Furthermore, we constructed the time dependent expectation value for the infinite potential well in position space as well as in momentum space. We have discussed the relation between information entropy and uncertainty of the wave packet relocalization during time evolution. We have also calculated the information entropy by using the position space probability density and momentum space probability density. The information entropy is basically used for the measure of uncertainty in the system. We have shown, the uncertainty in wave packet relocalization by means of information entropy. By comparing the uncertainty in position and momentum space with the information entropy we have seen that, the entropy also has its maximum value when the uncertainty of wave packet relocalization is maximum. We studied that by calculating the sum of position space and momentum space information entropies, we can clearly identify the fractional revival which can not be easily distinguished by means of autocorrelation function.

Bibliography

[1] M. Kryvohuz, J. Cao, Quantum-classical correspondance in responce theory, *Phys. Rev. Lett.*, **95**, 180405 (2005)

[2] R. Bluhm, V. A. Kostelecky and James A. Portes, The evolution and revival structure of localized quantum wave packets, *American J. Phys.*, **64**, 944 (1996).

[3] C. Sudhees, S. Lakshmibala, and V. Balakrishnan, Signature of wave packet revival phenomena in the expectation values of observables, *arXiv:quant-ph/0401181v2*, (2004)

[4] R. W. Robinett, Quantum wave packet revivals, *Phys. Rep.* **392**, 1-119 (2004).

[5] E. Schrödinger, The continuous transition from micro to macro mechanics, collected papers on wave mechanics, *Chelsea publishing*, New York, (1982).

[6] E. P. Wigner, On the quantum correction for thermodynamics equilibrium, *Phys. Rev.*, **40**, 749 (1932).

[7] A. O. Bolivar, Quantum-classical correspondence dynamical quantization and the classical limit, *The Frontiers Collection: Springer*, (2013).

[8] N. Zettili, Quantum mechanics: Concepts and Applications, **2nd Edition**, *WILEY*, (2009).

[9] E. C. G. Sudarshan, Equivalence of semiclassical and quantum mechanical description of statistical light beams, *J. Math. Phys.*, **7**, 1863 (1976).

[10] M. A. Doncheski, R. W. Robinett, Expectation vaue analysis of wvae packet solutions for the quantum bouncer: short-term classical and long-term revival behaviour, *Am. J. Phys.*, **69**, 1084 (2001).

[11] Hai-Woong Lee, M. O. Scully, The wigner phase space description of collision process, *Foundation of Phys.*, **13**, 61 (1983).

[12] S. Chatterjee, A. Saha and B. Talukdar, Fractional revival of Rydberg wave packets in twice-kicked one-dimensional atoms, *Acta Phys. Polonica A*, **120**, 438 (2011).

[13] A. E. Kaplan, I. Marzoli, W. E. Lamb, Jr., and W. P. Scheleich, Multimode interference: Highly regular pattern formation in quantum wave packet evolution, *Phys. Rev. A,* **61**, 032101 (200).

[14] M. Belloni, M. A. Doncheski, R. W. Robinett, Wigner quasi-probability distribution for the infinite square well: Energy eigenstate and time dependent wave packet, *Am. J. Phys.* **72**, 1183 (2004)

[15] C. G. Darwin, Free motion in the wave mechanics, *Proc. R. Soc(London) A*, **117**, 258 (1928).

[16] J. A. Yeazell, M. Mallalieu, C. R. Stroud Jr., Observation of the collapse and revival of a Rydberg electronic wave packet, *Phys. Rev. Lett.* **64**, 2007 (1990).

[17] A. Pukhov, Strong field interaction of laser radiation, *Rep. Prog. Phys.* **65**, R1R55 (2002).

[18] I. Marzoli, F. Saif, I. Bialynicki-Biruta, O. M. Friesh, A. E. Kaplan, and W. P. Schleich, Quantum carpets made simple, *Acta Physics Slovaca*, **48**, 323, (1998).

[19] J. A. Yeazell ,M. Mallalieu and C. R. Stroud Jr. (1990), Observation of the Collapse and Revival of a Rydberg Electronic Wave Packet, *Phys. Rev. Lett.* **64**, 2007 (1990).

[20] J. A. Yeazell, C. R. Stroud Jr., Observation of fractional revivals in the evolution of a Rydberg atomic wave packet, *Phys. Rev. A*, **43**, 5153 (1991).

[21] J. Parker, C. R. Stroud Jr., Coherence and decay of Rydberg wave packets, *Phys. Rev. Lett.*, **56**, 716 (1986).

[22] E. M. Wright, T. Wong, M. J. Collett, S. M. Tang and D. F. Walls, Collapse and revivals in the interference between two Bose-Einstein condensate formed in small atomic samples, *Phys. Rev. A*, **56**, 591 (1997).

[23] P. Rozmej and R. Arvieu, Clones and other interference effects in the evolution of angular momentum coherent states, *Phys. Rev. A*, **58**, 4314 (1998).

[24] P. Zegers, Fisher information properties, *Entropy,* **17**, 4918 (2015).

[25] O. M. Friesch, I. Marzoli, and W. P. Schleich, Quantum carpets woven by wigner function, *New J. Phys.* **2**, 4.1 (2000).

[26] I. Sh. Averbukh, N. F. Perelman, The dynamics of wave packets of highely excited atoms and molecules, *Usp. Fiz. Nauk*, **161**, 41 (1991).

[27] I. Bengtsson, K. Zyczkowski, Geometry of quantum states: An intorduction to quantum entanglement, *Cambridge unversity press*, (2008).

[28] R. Fisher, Theory of statistical estimation, *Proc. Camb. Philos. Soc.* **22**, 700 (1925).

[29] J. Sethna, Statistical mechanics, *Oxford university press*, (2006).

[30] C. E. Shannon, A mathematical theory of communication, *Bell system technical Journal*, (1948).

[31] M. Borda, Fundamentals in information theory and coding, *Engineering: Signals and Communication, Springer*, (2011).

[32] T. Abbas, F. Saif, Information entropy to probe revivals in dynamical systems, *J. Theor. Phys.*, **53**, 196 (2014).

[33] I. Bialynicki-Birula, J. Mycielski, Uncertainty mechanics, *Commun. Math. Phys.*, **44**, 129 (1975).

[34] P. Rivier, S. Iqbal, and J. M. Rost, Electronic excitation by short X-ray pulses: from quantum beats to wave packet revival, *J. Phys. B: At. Mol. Opt. Phys.* **47**, 124039 (2014).

[35] F. Saif, Atomic evolution in a driven gravitational cavity, *Chinese J. Phys.*, **39**, 311 (2001).

[36] G. Alber, P. Zoller, Laser excitation of electronic wave packets in Rydberg atoms, *Phys. Rep.* **199**, 231 (1991).

[37] I. Sh. Averbukh, N. F. Perelman, Fractional revivals: universality in the long-term evolution of quantum wave packets beyond the correspondence principle dynamics, *Phys. Lett. A*, **139**, 449 (1989).

[38] J. A. Yeazell, C. R. Stroud Jr., Observation of fractional revivals in the evolution of a Rydberg atomic wave packet, *Phys. Rev. A*, **43**, 5153 (1991).

[39] M. J. J. Vrakking, D. M. Villeneuve, A. Stolow, Observation of fractional revivals of a molecular wave packet, *Phys. Rev. A*, **54**, 37 (1996).

[40] M. Nauenberg, Autocorrelation function and quantum recurrence of wave packets, *J. Phys. B. At. Mol. Opt. Phys.*, **23**, L385 (1990).

[41] J. Gea-Banacloche, A quantum bouncing ball, *American J. Phys.*, **67**, 776 (1999).

[42] G. Albert, P. Zoller, Laser excitation of electronic wave packets in rydberg atoms, *Phys. Rep.*, **199**, 231 (1991).

[43] C. Cohen-Tannoudji, D. Diu, F. LaloKe, Quantum Mechanics, *Wiley, New York,* **Vol. II**, (1977).

[44] P. Rivi*ére*, S. Iqbal and J. M. Rost, Electronic excitation by short x-ray pulses: from quantum beats to wave packet revivals, *J. Phys. B: At. Mol. Opt. Phys.*, **47**, 124039 (2014).

[45] S. Knoop, M. Keim, H. J. Ludde, T. Kirchner, R. Morgenstern and R. Hoekstra, Inner- and outer-shell electron dynamics in proton collisions with sodium atoms, *J. Phys. B: At. Mol. Opt. Phys.* **38**, 2353 (2005).

[46] P. M. Morse, Diatomic molecules according to the wave mechanics II. Vibrational levels. *Physical Review,* **34**, (1929).

[47] H. Lefebvre-Brion, R. W. Field, The spectra and dynamics of diatomic molecules. 2nd ed., *Academic press*, (2004)

[48] R. Veilande, I. Bersons, Wave packet fractional revivals in a one-dimensional Rydberg atom, *J. Phys. B: At. Mol. Opt. Phys.* **40**, 2111 (2007).

[49] M. A. Doncheski, S. Heppelmann, R. W. Robinett, D. C. Tussey, Wave packet construction in two-dimensional quantum billiards: blueprints for the square, equilateral triangle, and circular cases, *Am. J. Phys.* **71**, 541 (2003).

[50] C. U. Segre, J. D. Sullivan, Bound-state wave packets, *Am. J. Phys.* **44**, 729 (1976). bibitemPOUP. Riviére, O. Uhden, U. Saalmann and J. M. Rost, Strong field dynamics with ultra-short electron wavepacket replicas *New J. Phys.* **11**, 053011 (2009).

[51] I. S. Averbukh, N. F. Perelman, Fractional revivals: Universality in the long-term evolution of quantum wave packets beyond the correspondence principle dynamics *Phys. Lett. A*, **139**, 449 (1989).

[52] N. B. Delone, S. P. Goreslavsky and V. P. Krainov, Quasiclassical dipole matrix elements for atomic continuum states, *J. Phys. B: At. Mol. Opt. Phys.*, **27**, 2941 (1994).

[53] R. Bluhm, V.A. Kostelecký, Long-term evolution and revival structure of Rydberg wave packets for hydrogen and alkali-metal atoms, *Phys. Rev. A*, **51**, 4767 (1995).

[54] G. Alber, P. Zoller, Laser excitation of electronic wave packets in Rydberg atoms, *Phys. Rep.*, **199**, 231 (1991).

[55] L. D. Noordam, D. I. Duncan, T. F. Gallagher, Ramsey fringers in atomic Rydberg wave packets, *Phys. Rev. A,* **45**, 4734 (1992).

[56] M. V. Berry, I. Marzoli and W. P. Schleich, Wave packet dynamics and factorization of numbers, *Phys. world*, **14**, 39 (2001).

[57] P. Kazemi, S. Chaturvedi, I. Marzoli, R. F. O′Connell and W. P. Schelich, Quantum carpets: a tool to observe decoherence, *New J. Phys.*, **15**, 013052 (2013).

[58] E. Romera, F. de los santos, Identifying wave packet fractional revival by means of information entropy, *Phys. Rev. Lett.*, **99**, 263601 (2007).

[59] M. H. Partovi, Entropic formulation of uncertainty for quantum measurements, *Phys. Rev. Lett.* **50**, 24 (1983).

[60] Q. A. wang, probability distribution and entropy as a measure of uncertanity, *J. of Phys.*, **41**, 065004 (2008).

[61] D. Deutsch, Uncertainty in quantum measurements, *Phys. Rev. Lett.* **50**, 631 (1983).

[62] M. Ohya and N. Watanabe, quantum entropy and its applications to quantum communication and statistical physics, *Entropy,* **12**, 1194 (2010).

[63] G. Manfredi, M. R. Feix, Entropy and wigner functions, *Phys. Rev. E*, **62**, 4665 (2000).

[64] C. E. Shannon, W. Weaver, The methematical theory of communication, *Rev. Mod. Phys.* **50**, 221 (1978).

[65] I. Bialynicki-Birula, Entropic uncertainty relations, *Phys. Lett. A*, **103**, 253 (1984).