Engineering non-classical continuous variable field states via superposition and excitation of coherent states

by

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Dedicated to

My Loving Parents

for their Love, endless support &

Encouragement.

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Abstract

The nonclassical states of radiation field have acquired a central position in quantum optics because of their potential applications in various areas of research such as quantum information, quantum computation and quantum metrology. For example, the entanglement generation between various modes of a subsystem do essentially require the nonclassical nature of at least either of the individual modes. On the other hand, a coherent state of radiation field is known as a quantum state that can be regarded in many ways as being classical with quantum noise added. Therefore, coherent states, despite of their much more other applications, cannot be used in phenomena which require nonclassicality as a prerequisite, such as entanglement generation.

In this thesis we review various techniques to construct nonclassical field states by using coherent states of light and analyze their nonclassical features. A convenient approach in this regard is to construct a linear superposition of two coherent states which are equal in amplitude but opposite in phase. Such superpositions of (single-degree-of-freedom) coherent states are often called Schrödinger cat states. Another, approach is to excite a coherent state by discrete number of photons. These states are know as excited coherent states or photon-added coherent states. These states are obtained, mathematically, by repeated action of creation operator on continously parameterized coherent states. Finally, we present excitations of Superposition of coherent states. In all these cases, we analyze the signatures of nonclassicality by means of sub-Poissonian photon counting statistics and negativity of the Wigner function.

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Chapter 1

Introduction and Outline

The quantum-classical correspondence of a physical system is one of the most interesting questions since the very binging of quantum mechanics. However, the quantum world manifests a lot of phenomena which do not have analogous description in classical world, commonly referred to as nonclassical phenomena. Such nonclassical phenomena are essentially required to build various quantum technologies such as quantum information, quantum computation, quantum metrology [1, 2]. Due to their potential applications in such advanced research areas, the nonclassical states attracted a lot of attention in the field of quantum optics. In this thesis we present various techniques to engineer nonlcassical field states by using classical-like states of light, namely coherent states. In the following, first we introduce briefly what we are going to present in thesis and then present the outline of the thesis.

1.1 Nonclassical continuous-variable field states

The very well known continuous-variable quantum optical field states are coherent states which have an important role in the description of coherent electromagnetic field. The coherent states are regarded in many ways as being classical with quantum noise added. Therefore, despite of their numerous applications in quantum optics and many other areas, these states cannot be used directly to partake the quantum information processing due to their classical-like behavior. However, we will use the coherent states as a key ingredient to construct the nonclassical field states.

Nevertheless, before introducing the techniques to construct nonclassical states, we first introduce the coherent states.

1.1.1 Coherent states: classical-like states

The history of the coherent states goes back to 1926, when Erwin Schrödinger tried to build quantum mechanical states which can exhibit dynamics analogous to classical one. He succeeded only to construct such states for harmonic oscillator [3] and failed to generalize the idea for other quantum systems. Using the concepts of his wave mechanics, it was shown that centers of the Gaussian wave packets follow the trajectories of classical harmonic oscillator, hence, providing the correspondence between classical and quantum mechanics. Moreover, these states minimize the uncertainty relation which remains minimum during time evolution [3].

After a dormant period of more than three decades, Roy Glauber in 1963 reformulated these states using the ladder operators of harmonic oscillator. He used these quantum mechanical states to express coherent electromagnetic field [4, 5, 6]. He defined coherent states in three but equivalent ways: i) as eigenstates of annihilation operator of harmonic oscillator; ii) as the displaced vacuum states; and iii) the states that minimize the uncertainty relation. Right after the seminal work of Glauber, these coherent states become indispensable in many areas such as, quantum mechanics [7] quantum optics [8], mathematical physics [9] many other areas [10]. The importance and usefulness of these states was acknowledged by awarding the Physics Nobel Prize 2005 to the inventor Roy Glauber.

Because of their a lot of applications, there have been continuous efforts to generalize the notion of coherent states for systems other than harmonic oscillator [11]. In fact most of the generalizations were made by making use of underlying algebras of the general systems [12, 13] (beyond the Heisenberg-Weyl in the case of harmonic oscillator). However, some algebraic independent approaches for the generalized coherent do also exist [14, 15]. Currently, the construction of coherent states have been done in large variety of physical systems [16, 17, 18, 19].

In the context of quantum optics, the coherent states provide an excellent description to coherent electromagnetic field such as ideal lasers. Therefore, in many ways these coherent states can be regarded as classical-like state with quantum noise added [8]. For example, the expectation value of electric field (as well as magnetic field) evolves in time just like classical field. Moreover, it has been observed that the coherent states are temporally stable, i.e., a coherent state remains coherent under the time evolution. The effect of time evolution is only to rotate a coherent state in complex plan. In addition to their classical-like behavior, these coherent states adhere a set of mathematical properties, such as, non-orthogonality, over-completeness, continuity of parameters and resolution of unity which make them important from the mathematical physics' point of view. Furthermore, the coherent states have played an important role in the understanding of many phenomena, such as, quantum interferometry, quantum entanglement and precision measurement [20].

1.1.2 Constructing the nonclassical states via coherent states

Despite of the fact that coherent states have numerous applications in different research areas, there are phenomena (having no classical analogue therefore known as nonclassical phenomena) of great importance which cannot be performed with classical-like coherent states. For example, it is not possible to generate entanglement, a nonclassical correlation, between the output modes of a beam splitter with coherent states at both input modes [20]. The entanglement generation does essentially require nonclassical nature, at least, of one of the subsystems [21]. There are a lot of methods available in literature [22] to generate nonclassical field states. However, in the following we discuss only three of them which make use of coherent states as their key ingredient.

Superpositions of coherent states

Quantum superposition is regarded as one of the biggest mysteries of the wold which lies in the heart of quantum mechanics [23]. For a set of quantum states represented by $\{|\phi_i\rangle\}$, the superposition state can be written, mathematically, as

$$|\psi\rangle = \sum_{i=0}^{\infty} c_i |\phi_i\rangle \tag{1.1.1}$$

where $|\phi_i\rangle$ represents the ith state of the system with c_i as the corresponding probability amplitude. The probability of finding the system in state $|\phi_i\rangle$ can then be calculated by $|c_i|^2$. However, such superposition states can never be observed on performing measurement. In fact, the measurement collapses the superposition and the outcome appears as one of the eigenstates of the system.

In our work, we consider the superpositions of two coherent states with equal amplitude but separated in phase by 180^o. Such superpositions are also known as Schrödinger cat states [24]. The name is attributed due to their analogy (of being a superposition of macroscopically distinguishable states) to the original paradoxical thought experiment proposed by Erwin Schrödinger in 1935 [25]. Despite of the fact that coherent states are classical-like states, their superpositions are highly nonclassical. The idea of such superpositions has been first introduced in [26]. Later on explicit constructions have been introduced theoretically [27, 28, 29] and realized experimentally [30].

Excitations of coherent states

It is very well known that the energy eigenstates are also the eigenstates of photon number operator. In that sense, the number states (the simultaneous eigenstates of the Hamiltonian and the number operator) can also be considered as field states in the context of quantum optics. However, these states cannot be treated as definite states for electric field operator because its expectation values with respect to these states do vanish. Nevertheless, in contrast to the coherent states, these states are highly nonclassical in nature. Agarwal and Tara proposed [31], mathematically, a new kind of field states by the repeated action of creation operator on the coherent states and named them as exited coherent states, also named as photon added coherent states (we will be using these two terminologies intermittently through out the thesis).

The photon added coherent states, $|\alpha, n\rangle$, (α being complex continuous parameter and n is an integer) represent an interesting combination of classical-like coherent states and highly nonclassical number states such that for $\alpha \rightarrow 0$, the state $|\alpha, n\rangle$ reduces the Fock state and when $n \rightarrow 0$, the state $|\alpha, n\rangle$ reduces coherent state.

Hence controlling the values of α and n one can control the extent of nonclassicality of the field states. Due to their interesting features and better quantum control, the photon added coherent states attracted considerable attention [32, 33, 34]. The idea of photon addition was extended to large variety of quantum states, such as, generalized photon added states [32, 33, 34] and photon-added squeezed states [35].

Excitations of coherent states superpositions

Inspired by the usefulness of Schrödinger-cat states and photon-added coherent states, it is reasonable to construct photon-added Schrödinger-cat states or excited superpositions of coherent states. These states can be obtained by repeated action of creation operator on superposition of coherent states. The idea of excited even and excited odd coherent states was proposed by Dodonov et al [36] and its nonclassical properties have been discussed. Photon-added superpositions of coherent states provide an extra control parameter m, namely, number of photon added. These states exhibit different behavior as far as nonclassical features are concerned. The idea of excitation can also be extended from the superpositions of coherent states to the superpositions of squeezed state [37].

1.2 Non-classicality criteria

As mentioned earlier, a phenomenon is known as nonclassical if its classical analogue does not exist. In the context of quantum optics, any field state is regards as nonclassical if its characteristics behavior does not resemble in anyway to classical one. However, it is very important question to ask how to distinguish nonclassical states from classical ones, both qualitatively as well quantitatively. In order to answer this question, various parameters have been proposed to analyze the nonclassicality of quantum optical states, such as, Glauber-Sudarshan P-function, amplitude and intensity squeezing, anti-bunching, sub-Poissonian photon statistics and negativity of the Wigner function [22]. The choice of a particular parameter depends on the nature of the field state under consideration. For our particular choice of field states, we will use Wigner function and sub-Poissonian photon statistics.

The Wigner function is defined as phase-space joint probability distribution function. According classical statistical mechanics, this function, being a probability function, should be positive definite. However, in quantum mechanics it came out that Wigner function may take negative values in some region of phase-phase for some states. Such states can be regarded as nonclassical states and negativity of Wigner function can be regarded as an indicator of nonclassicality. Secondly, the photon counting statistics of coherent states is Poissonian. Furthermore, the photon counting statistics of all classical light sources, such as, thermal light sources or chaotic sources is super-Poissonian (broader than Poissonian) [38]. Interestingly, any source that has no classical analogue exhibits sub-Poissonian (narrower than Poissonian) statistics[39]. In order to investigate the underlying photon counting statistics of a given field state, a parameter, originally defined by Mandel [40], which gives the ratio of mean photon number to its variance minus one. This actually distinguishes whether the photon counting probability distribution is broader or narrower than Poisson distribution.

1.3 Outline of Thesis

The thesis is organized as the following.

In the second chapter, we present a brief review regarding construction of coherent states and their various properties. The first part is devoted to the construction of coherent states followed by their basic properties and the second part of this chapter is focused on the classical-like properties of these states. At the end, the concept of phase-space probability distribution functions will be discussed, especially in the form of Wigner functions.

The third chapter is devoted to the construction of nonclassical field states via various operations on coherent states. First, the construction of superpositions of coherent states, including even coherent states, odd coherent states and Yorke-Stoler coherent states, will be presented. Secondly, the explicit construction of excited coherent states will be presented. Finally, we will present the excited superpositions of coherent states.

The forth chapter is dedicated to the analysis of nonclassical properties of the constructed field states. First we will present the criteria to probe the nonclassicality which include the indication of either negativity of Wigner function or sub-Poissonian photon counting statistics. Then we will discuss the computation of Wigner function and Mandel's parameter analytically which will be followed by their numerical computation. Finally we will discuss the results of this chapter.

Finally, the fifth chapter is reserved for the summary of the whole work and conclusions of the thesis.

Chapter 2

Single-mode radiation field states

Before we proceed to the construction of nonclassical field states, we first introduced the continuously parameterized field states, namely, coherent states. It is well known that coherent states provide the classical-quantum correspondence of electromagnetic field in many ways. In this chapter, we first review the construction and some basic properties of coherent states and then discuss various classical-like characteristics. In particular, we focus on the photon counting statistic and phase-space properties of these coherent states which will help us to analyze the nonclassical states that we are intending to discuss in the next chapters.
The chapter is organized as fallowing. In section 2.1 we discuss the quantization of electromagnetic radiation field and present various field operators. The construction of coherent states and analysis of their properties has been presented in section 2.2. Finally, we discuss classical-like characteristics in section 2.3.

2.1 Single-mode quantized field

We consider one dimensional cavity whose walls are perfectly conducting at $z = 0$ and $z = L$ and radiation are confined in it such that at boundaries it become zero. The electric field component is given as,

$$E_x\left(z,t\right) = \sqrt{\frac{2\omega^2}{\epsilon_o V}} q\left(t\right) \sin kz, \tag{2.1.1}$$

where V is volume of the cavity, ϵ_o is permittivity of free space, k is wave vector and ω is the frequency. Amplitude of the electric field depends on $q(t)$ which has dimension of length and changing with time. We can calculate the amplitude of magnetic field component by using Maxwell equation

$$\nabla \times B = \mu_o \epsilon_o \frac{\partial E}{\partial t}, \tag{2.1.2}$$

that is by putting Eq.(2.1.1) in Eq.(2.1.2) we get,

$$B_y(z,t) = \left(\frac{\mu_o \epsilon_o}{k}\right) \sqrt{\frac{2\omega^2}{V\epsilon_o}} p(t) \cos kz, \tag{2.1.3}$$

where $p(t)$ is momentum of particle with unit mass. Energy of field is given by the Hamiltonian as,

$$H = \frac{1}{2} \int \left(E_x^2(z,t)\, \epsilon_o + \frac{1}{\mu_o} B_y^2(z,t) \right) dV. \tag{2.1.4}$$

Using the value of $E_x(z,t)$ from Eq.(2.1.1) and $B_y(z,t)$ from Eq.(2.1.2) in Eq.(2.1.4) the Hamiltonian "H" takes the form,

$$H = \frac{1}{2} \left(p^2 + \omega^2 q^2\right). \tag{2.1.5}$$

It is interesting to note that Hamiltonian given in Eq.(2.1.5) of the single mode field is identical to that of harmonic oscillator with unit mass, where electric and magnetic fields play the role of canonical position and momentum.

For quantization we replace canonical variables, p and q, by their equivalent operators, $\hat{q}$ and $\hat{q}$. These operators satisfy commutation relation,

$$[\hat{q}, \hat{p}] = i\hbar. \tag{2.1.6}$$

In terms of these operators the electric field given in Eq.(2.1.1) becomes,

$$\hat{E}_x(z,t) = \sqrt{\frac{2\omega^2}{\epsilon_o V}} \hat{q}(t) \sin kz, \tag{2.1.7}$$

and magnetic field given in Eq.(2.1.2) becomes,

$$\hat{B}_y(z,t) = \left(\frac{\mu_o \epsilon_o}{k}\right) \sqrt{\frac{2\omega^2}{V\epsilon_o}} \hat{p}(t) \cos kz. \tag{2.1.8}$$

Similarly by using Eq.(2.1.7) and Eq.(2.1.8), quantum Hamiltonian becomes,

$$\hat{H} = \frac{1}{2}\left(\hat{p}^2 + \omega^2 \hat{q}^2\right). \tag{2.1.9}$$

Operators $\hat{q}$ and $\hat{p}$, which are used to characterize the electric and magnetic field, are Hermitian operators. We can define the non-Hermitian operators $\hat{a}$ and $\hat{a}^\dagger$ in terms of $\hat{q}$ and $\hat{p}$ as,

$$\begin{aligned} \hat{a} &= (2\hbar\omega)^{-\frac{1}{2}}(\omega\hat{q} + i\hat{p}) \\ \hat{a}^\dagger &= (2\hbar\omega)^{-\frac{1}{2}}(\omega\hat{q} - i\hat{p}). \end{aligned} \tag{2.1.10}$$

The operators $\hat{a}$ and $\hat{a}^\dagger$ are known as annihilation and creation operator respectively. Using Eq.(2.1.10) the electric and magnetic field can be written in terms of these operators given, respectively, as,

$$\hat{E}_x(z,t) = E_o(\hat{a} + \hat{a}^\dagger)\sin kz, \qquad E_o = \sqrt{\frac{\hbar\omega}{V\epsilon_o}}. \tag{2.1.11}$$

$$\hat{B}_y(z,t) = \frac{B_o}{i}(\hat{a} - \hat{a}^\dagger)\cos kz, \qquad B_o = \frac{\mu_o}{k}\sqrt{\frac{\epsilon_o\hbar\omega^3}{V}}. \tag{2.1.12}$$

Similarly the quantum Hamiltonian, in terms of these creation and annihilation operators, is given as,

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right). \tag{2.1.13}$$

Here $\hat{a}^\dagger\hat{a} = \hat{n}$ is known as number operator satisfying the eigenvalue equation,

$$\hat{n}|n\rangle = n|n\rangle. \tag{2.1.14}$$

where eigenvalue $n = 0, 1, 2, 3....$ and eigenvector $|n\rangle$ are known as number states. As $\hat{H}$ and $\hat{n}$ are Hermitian operators such that $|n\rangle$ are their simultaneous eigenstates.

2.2 Coherent states and their properties

We know that classical field oscillates sinusoidally in time at any fixed point in space. But when we calculate expectation value of field operator for a number state it gives $\langle n|\hat{E}_x|n\rangle = 0$. Therefore we introduce another set of states, namely, the coherent states.

2.2.1 Construction

Three equivalent definitions were proposed by Roy Glauber in his seminal papers [4, 5, 6].

Eigenstate of the annihilation operator

The eigenstate of annihilation operator $\hat{a}$ is known as coherent state and given as,

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle. \tag{2.2.1}$$

For explicit expression of coherent states we expand them in number states as,

$$|\alpha\rangle = \sum_{m=0}^{\infty} |m\rangle\langle m|\alpha\rangle, \tag{2.2.2}$$

to determine $\langle m|\alpha\rangle$, we project $\langle m|$ on Eq.(2.2.1) and get,

$$\langle m|\hat{a}|\alpha\rangle = \alpha\langle m|\alpha\rangle. \tag{2.2.3}$$

As we know that creation operator act on number state as, $\hat{a}^{\dagger}|m\rangle = \sqrt{m+1}|m+1\rangle$ and its complex conjugate is $\langle m|\hat{a} = \sqrt{m+1}\langle m+1|$. By using this result Eq.(2.2.3) become,

$$\sqrt{m+1}\langle m+1|\alpha\rangle = \alpha\langle m|\alpha\rangle, \tag{2.2.4}$$

by replacing $m+1$ with m we get,

$$\langle m|\alpha\rangle = \frac{\alpha}{\sqrt{m}}\langle m-1|\alpha\rangle,$$

m times repetition of this step give us,

$$\langle m|\alpha\rangle = \frac{\alpha^m}{\sqrt{m!}}\langle 0|\alpha\rangle, \tag{2.2.5}$$

by using this relation Eq.(2.2.1) become,

$$|\alpha\rangle = \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}}\langle 0|\alpha\rangle|m\rangle, \tag{2.2.6}$$

where $\langle 0|\alpha\rangle$ can be determined by normalization of above equation as,

$$\begin{aligned}
\langle\alpha|\alpha\rangle &= \sum_{m=0}^{\infty} \frac{\alpha^m}{\sqrt{m!}} \sum_{n=0}^{\infty} \frac{\alpha^{*n}}{\sqrt{n!}} |\langle 0|\alpha\rangle|^2 \langle n|m\rangle, \\
\langle 0|\alpha\rangle &= \exp(-\frac{1}{2}|\alpha|^2),
\end{aligned}$$

thus normalized coherent state is given as,

$$|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}|m\rangle. \tag{2.2.7}$$

Displaced vacuum states

Coherent states are also obtained by applying displacement operator on vacuum state. so,

$$|\alpha\rangle = D\left(\alpha\right)|0\rangle. \tag{2.2.8}$$

Where displacement operator $D\left(\alpha\right)$ is defined as,

$$D\left(\alpha\right) = \exp\left(\alpha a^{\dagger} - \alpha^{*}\hat{a}\right).$$

Applying this operator on vacuum state we get,

$$\begin{aligned} D\left(\alpha\right)|0\rangle &= \exp\left(\alpha a^{\dagger} - \alpha^{*}\hat{a}\right)|0\rangle, \\ |\alpha\rangle &= \exp\left(\alpha a^{\dagger}\right)\exp\left(-\alpha^{*}\hat{a}\right)\exp\left(\frac{-|\alpha|^2}{2}\right)|0\rangle, \end{aligned} \tag{2.2.9}$$

where,

$$\begin{aligned} \exp\left(-\alpha^{*}\hat{a}\right)|0\rangle &= \sum_{m=0}^{\infty}\frac{\left(-\alpha^{*}\hat{a}\right)^m}{m!}|0\rangle, \\ &= 0, \quad \text{except for} \quad m = 0, \end{aligned} \tag{2.2.10}$$

and,

$$\exp\left(\alpha a^{\dagger}\right)|0\rangle = \sum_{m=0}^{\infty}\frac{\left(\alpha a^{\dagger}\right)^m}{m!}|0\rangle,$$

when $a^{\dagger}$ acts m times on vacuum state it gives,

$$\begin{aligned} \exp\left(\alpha a^{\dagger}\right)|0\rangle &= \sum_{m=0}^{\infty}\frac{\left(\alpha\right)^m}{m!}\sqrt{m!}|0\rangle, \\ &= \sum_{n=0}^{\infty}\frac{\left(\alpha\right)^n}{\sqrt{n!}}|m\rangle, \end{aligned} \tag{2.2.11}$$

Using Eq.(2.2.10) and Eq.(2.2.11) in Eq.(2.2.9) we get,

$$|\alpha\rangle = \exp\left(\frac{-|\alpha|^2}{2}\right)\sum_{m=0}^{\infty}\frac{\left(\alpha\right)^m}{\sqrt{m!}}|m\rangle, \tag{2.2.12}$$

which is the same as obtained in Eq.(2.2.7).

Minimum uncertainty states

Coherent states minimize the uncertainty relation for any given canonical variables. We introduce dimensionless operators, also known as quadrature operators,

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \qquad \hat{X}_2 = \frac{1}{2i}(\hat{a} - \hat{a}^\dagger). \tag{2.2.13}$$

We can determine the uncertainty in both quadratures with respect to coherent states $|\alpha\rangle$ by using the following relation,

$$\langle(\triangle X_1)^2\rangle = \langle X_1^2\rangle - \langle X_1\rangle^2. \tag{2.2.14}$$

To calculate the uncertainty we first commute the $\langle X_1\rangle$ given as,

$$\langle X_1\rangle = \frac{1}{2}(\alpha + \alpha^*), \tag{2.2.15}$$

where we have used the expression $\hat{a}|\alpha\rangle = \alpha|\alpha\rangle$, $\langle\alpha|\hat{a}^\dagger = \alpha^*\langle\alpha|$ and $[\hat{a}, \hat{a}^\dagger] = 1$. Similarly we can calculate,

$$\begin{aligned}\langle X_1^2\rangle = \langle X_1 X_1\rangle &= \frac{1}{4}(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger + 1 + 2\hat{a}^\dagger\hat{a}) \\ &= \frac{1}{4}(\alpha^2 + (\alpha^*)^2 + 1 + 2|\alpha|^2).\end{aligned} \tag{2.2.16}$$

Using Eq.(2.2.15) and Eq.(2.2.16) in Eq.(2.2.14) we get,

$$\langle(\triangle X_1)^2\rangle = \frac{1}{4}(\alpha^2 + (\alpha^*)^2 + 1 + 2|\alpha|^2) - \frac{1}{4}(\alpha + \alpha^*)^2.$$

After some calculations the uncertainty in X_1 takes the form,

$$\langle\triangle X_1\rangle = \frac{1}{2}. \tag{2.2.17}$$

Similarly uncertainty in second quadrature can be calculated by using the relation,

$$\langle(\triangle X_2)^2\rangle = \langle X_2^2\rangle - \langle X_2\rangle^2. \tag{2.2.18}$$

We separately calculate $\langle X_2\rangle$ and $\langle X_2^2\rangle$,

$$\langle X_2\rangle = \frac{1}{2i}(\alpha - \alpha^*), \tag{2.2.19}$$

$$\begin{aligned}\langle X_2^2\rangle &= -\frac{1}{4}(\hat{a}\hat{a} + \hat{a}^\dagger\hat{a}^\dagger - 1 - 2\hat{a}^\dagger\hat{a}), \\ &= -\frac{1}{4}(\alpha^2 + (\alpha^*)^2 - 1 - 2|\alpha|^2),\end{aligned} \tag{2.2.20}$$

Putting values from Eq.(2.2.19) in Eq.(2.2.18) we get,

$$\begin{aligned}\langle(\triangle X_2)^2\rangle &= -\frac{1}{4}(\alpha^2+(\alpha^*)^2-1-2|\alpha|^2)+\frac{1}{4}(\alpha-\alpha^*)^2,\\ \langle(\triangle X_2)\rangle &= \frac{1}{2}.\end{aligned} \tag{2.2.21}$$

By multiplying Eq.(2.2.17) and Eq.(2.2.21) we get,

$$\triangle X_1 \triangle X_2 = \frac{1}{4}, \tag{2.2.22}$$

which is condition of minimum uncertainty. So coherent states are known as minimum uncertainty states.

2.2.2 Physical significance of coherent state

After discussing various equivalent mathematical definitions of coherent states we are interested to understand the physical meaning of coherent states. In that perspective, let us ask *what is the physical meaning of the complex coherent state parameter* α? It can be shown (by calculating the expectation value of field operator with respect to coherent states) that $|\alpha|$ is related to the field amplitude. Moreover, the expectation value of the photon number operator $\hat{n} = \hat{a}^\dagger \hat{a}$ is

$$\bar{n} = \langle \hat{n} \rangle = |\alpha|^2. \tag{2.2.23}$$

Thus $|\alpha|^2$ gives the average number of photon in electromagnetic field.

Photon counting statistics

In a given measurement experiment for the number of photons in the coherent electromagnetic field, represented by coherent state defined in Eq.(2.2.7), the probability of detecting n photons is given as,

$$P_n = |\langle n|\alpha\rangle|^2. \tag{2.2.24}$$

Using Eq.(2.2.7), we get,

$$\langle n|\alpha\rangle = \exp\left(-\frac{1}{2}|\alpha|^2\right)\sum_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}\langle n|m\rangle,$$

and

$$|\langle n|\alpha\rangle|^2 = \left|\exp\left(-\frac{1}{2}|\alpha|^2\right)\frac{\alpha^n}{\sqrt{n!}}\right|^2.$$

The probability of finding n photons takes the form,

$$P_n = \exp\left(-|\alpha|^2\right)\frac{|\alpha|^{2n}}{n!} = e^{-\bar{n}}\frac{\bar{n}^n}{n!}$$

which is a Poisson distribution with mean $\bar{n} = |\alpha|^2$.

2.2.3 Properties of coherent states

In this subsection we discuss various properties of the coherent states.

- **Orthogonality**

  Coherent states are non-orthogonal states and inner product of states can be used to prove this statement. Let $|\alpha\rangle$ and $|\beta\rangle$ are two coherent states then,

  $$\langle\beta|\alpha\rangle = \exp\left(-\frac{1}{2}(|\beta|^2 + |\alpha|^2)\right)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{(\beta^*)^n\,\alpha^m}{\sqrt{n!m!}}\langle n|m\rangle.$$

  This inner product is one only when $n = m$,

  $$\langle\beta|\alpha\rangle = \exp\left(-\frac{1}{2}(|\beta|^2 + |\alpha|^2)\right)\sum_{n=0}^{\infty}\frac{(\beta^*\alpha)^n}{n!}.$$

  By writing sum in term of exponential series,

  $$\langle\beta|\alpha\rangle = \exp\left(-\frac{1}{2}(|\beta|^2 + |\alpha|^2) + \beta^*\alpha\right),$$

  we can write it as,

  $$\langle\beta|\alpha\rangle = \exp\left(\frac{1}{2}\left(\beta^*\alpha - \beta\alpha^*\right)\right)\exp\left(-\frac{1}{2}|\beta - \alpha|^2\right),$$

  first term is just complex phase so that,

  $$|\langle\beta|\alpha\rangle|^2 = \exp\left(-\frac{1}{2}|\beta - \alpha|^2\right) \neq 0. \tag{2.2.25}$$

  Thus the coherent states are not orthogonal but if $|\beta - \alpha|^2$ is large, they are nearly orthogonal.

- **Completeness**

  Completeness is expressed by taking integral of projection operator over complex plane. Mathematical expression is given as,

$$\int |\beta\rangle\langle\beta| \frac{d^2\beta}{\pi} = 1, \tag{2.2.26}$$

where $d^2\beta = dRe(\beta)dIm(\beta)$. The proof of above relation is as follows,

$$\int |\beta\rangle\langle\beta| d^2\beta = \int \exp(-|\beta|^2) \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(\beta^*)^n}{\sqrt{n!}} \frac{\beta^m}{\sqrt{m!}} |n\rangle\langle m| d^2\beta, \tag{2.2.27}$$

in poler coordinate system $\beta = re^{i\phi}$ and $d^2\beta = rdrd\phi$,

$$\int |\beta\rangle\langle\beta| d^2\beta = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|n\rangle\langle m|}{\sqrt{n!m!}} \int_0^{\infty} r^{m+n+1} \exp(r^2) dr \int_0^{2\pi} e^{(n-m)i\phi} d\phi, \tag{2.2.28}$$

where $\int_0^{2\pi} e^{(n-m)i\phi} d\phi = 2\pi\delta_{nm}$. For simplicity substituting $r^2 = y$, $dy = 2rdr$,

$$\int |\beta\rangle\langle\beta| d^2\beta = \pi \sum_{m=0}^{\infty} \frac{|m\rangle\langle m|}{m!} \int_0^{\infty} e^{-y} y^m dy. \tag{2.2.29}$$

Where $\int_0^{\infty} e^{-y} y^m dy = m!$,

$$\int |\beta\rangle\langle\beta| d^2\beta = \pi \sum_{m=0}^{\infty} |m\rangle\langle m|,$$

by putting $\sum_{m=0}^{\infty} |m\rangle\langle m| = 1$ we get Eq.(2.2.26).

$$\int |\beta\rangle\langle\beta| d^2\beta = \pi,$$

which is completeness relation of coherent states.

2.3 Classical-like characteristics

A coherent state is known as a quantum state that can be regarded in many ways as being classical with quantum noise added. In this section we discuss some classical-like characteristics of continuous variable states. Following parameters show the classical-like nature of coherent states.

2.3.1 Expectation value of field operators

In section 2.1 we have introduced the operator concerning electric field $\hat{E}$, given by Eq.(2.1.12). We can calculate the expectation value of $\hat{E}$ as,

$$\langle\alpha|\hat{E}|\alpha\rangle = 2E_o|\alpha|\sin kz\cos(\theta-\omega t), \qquad \hat{E} = E_o(\hat{a}+\hat{a}^{\dagger})\sin kz,$$

where $E_o = (\hbar\omega/\epsilon_o V)^{\frac{1}{2}}$. This shows that expectation value of electric field is sinusoidally varying just analogous to its classical description.

2.3.2 Minimum uncertainty

While defining the coherent states in section 2.2, it has been pointed out that the coherent states are minimum uncertainty states, i.e,

$$\triangle X_1 \triangle X_2 = \frac{1}{4} \quad \text{since} \quad \hat{X}_1 = \frac{1}{2}(\hat{a}+\hat{a}^{\dagger}), \qquad \hat{X}_2 = \frac{1}{2i}(\hat{a}-\hat{a}^{\dagger}).$$

This implies that coherent states are quantum mechanical states which are closet to the classical description.

2.3.3 Temporal stability

These states are temporally stable i.e coherent states remain coherent under time evolution. Let us consider, at $t=0$ coherent state is $|\alpha\rangle$. The time evolved coherent state can be obtained

$$|\alpha, t\rangle = \hat{U}(t)|\alpha\rangle \tag{2.3.1}$$

where unitary time evolution operator $U(t)$ is defined as,

$$\hat{U}(t) = \exp\left(\frac{-it\hat{H}}{\hbar}\right).$$

where coherent state $|\alpha\rangle$ is given in Eq.(2.2.7) and $\hat{H}$ is given in Eq.(2.1.13), by using these two Eq.(2.3.1) becomes,

$$|\alpha, t\rangle = e^{(-\frac{1}{2}|\alpha|^2)} \sum_{m=0}^{\infty} \frac{(\alpha)^m}{\sqrt{m!}} e^{-i\omega t(a^{\dagger}a+1/2)}|m\rangle.$$

After some calculations, the time evolved state can be written as,

$$|\alpha,t\rangle = e^{-\frac{1}{2}i\omega t}e^{-\frac{1}{2}|\alpha|^2}\sum_{m=0}^{\infty}\frac{(\alpha)^m}{\sqrt{m!}}e^{-i\omega tm}|m\rangle.$$

Let us define $\alpha^{'}=\alpha e^{-i\omega t}$ and $|\alpha|=|\alpha^{'}|$,

$$|\alpha,t\rangle = e^{-\frac{1}{2}i\omega t}\left(e^{-\frac{1}{2}|\alpha^{'}|^2}\sum_{m=0}^{\infty}\frac{(\alpha^{'})^m}{\sqrt{m!}}|m\rangle\right), \tag{2.3.2}$$

which is another coherent state with some phase factor $e^{-\frac{1}{2}i\omega t}$. It mean coherent states are temporally stable.

2.3.4 Phase-space probability distribution: Wigner function

In quantum optics various probability distribution functions are being used to analyze the phase space characteristics such as Glauber Sudarshan P function, Q function and Wigner function. However keeping in view our latter discussion on nonclassical states here we discuss only Wigner function initially introduced by Wigner in 1932 [41]. The Wigner function, as a phase space probability distribution function, can be defined [41] as,

$$W(x,p) = \int_{-\infty}^{-\infty}\exp(-iyp)\langle x+\frac{1}{2}y|\hat{\rho}|x-\frac{1}{2}y\rangle dy, \tag{2.3.3}$$

where the vectors $|x\pm\frac{1}{2}y\rangle$ are the eigenvectors of the position operators and $\hat{\rho}$ is the density operator of the state in question. Various bounds on the Wigner function have been discussed in literature [42, 43, 44, 45].

For quantum optical field states the Wigner function given in Eq.(2.3.3) can be written [20] as,

$$W(\beta) = \frac{1}{\pi^2}\int\exp(\lambda^*\beta-\lambda\beta^*)\,C_w d^2\lambda, \tag{2.3.4}$$
$$C_w(\lambda) = Tr[\hat{\rho}D(\lambda)],$$

where $\hat{\rho}=|\psi\rangle\langle\psi|$ is the density operator and $D(\lambda)$ is displacement operator, given as,

$$D(\lambda) = \exp\left(\lambda\hat{a^{\dagger}}-\lambda^*\hat{a}\right). \tag{2.3.5}$$

The equivalence of Eq.(2.3.3) and Eq.(4.1.1) is given in Appendix A. Using the identity (so-called disentangling theorem),

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{\frac{-1}{2}[\hat{A},\hat{B}]}$$

in Eq.(2.3.5), the characteristic function $C_w(\lambda)$ becomes,

$$C_w(\lambda) = e^{-|\lambda|^2/2} Tr(\hat{\rho} e^{\lambda a^\dagger} e^{-\lambda^* \hat{a}}). \tag{2.3.6}$$

A comprehensive discussion on the phase space in the context of quantum optics has been given by Schleich [46].

In our later discussion we will use the characteristic function, given above in Eq.(2.3.7), to calculate Wigner function of various continuous variable field states. As an example, in the following we calculate Wigner function of coherent states.

Wigner function of coherent states

For pure coherent state $|\alpha\rangle$, the density operator $\hat{\rho} = |\alpha\rangle\langle\alpha|$ and corresponding characteristic function, $C_w(\lambda)$, takes the form,

$$\begin{aligned} C_w(\lambda) &= e^{-|\lambda|^2/2} \langle\alpha|(e^{\lambda a^\dagger} e^{-\lambda^* \hat{a}})|\alpha\rangle, \\ &= e^{-|\lambda|^2/2} \exp(\alpha^* \lambda) \exp(-\alpha \lambda^*). \end{aligned} \tag{2.3.7}$$

Substituting the value from Eq.(2.3.7) to Eq.(4.1.1), we get the Wigner function for coherent states,

$$W(\beta) = \frac{1}{\pi^2} \int \exp(\beta\lambda^* - \beta^*\lambda) \exp(\alpha^*\lambda) \exp(-\alpha\lambda^*) \exp\left(\frac{-|\lambda|^2}{2}\right) d^2\lambda,$$

which on further simplification takes the form,

$$W(\beta) = \frac{1}{\pi^2} \int \exp\left((\beta - \alpha)\lambda^* - (\beta^* - \alpha^*)\lambda - \frac{|\lambda|^2}{2}\right) d^2\lambda. \tag{2.3.8}$$

It is important to note that the Wigner function for coherent state is given in terms of an integral over complex plane. We can compute this complex integral using the identity [47],

$$\int \exp\left(x\lambda^* - x^*\lambda - z|\lambda|^2\right) d^2\lambda = \frac{\pi}{z} \exp\left(-|x|^2 z^{-1}\right), \tag{2.3.9}$$

such that,

$$x = \beta - \alpha, \quad x^* = \beta^* - \alpha^*, \quad z = 1/2.$$

As a result the Wigner function for coherent state, given in Eq.(2.3.8), comes out to be,

$$\begin{aligned} W(\beta) &= \frac{\pi}{2\pi^2} \exp\left(-2\left(\beta - \alpha\right)\left(\beta^* - \alpha^*\right)\right), \\ &= \frac{2}{\pi} \exp\left(-2|\beta - \alpha|^2\right). \end{aligned} \tag{2.3.10}$$

It is seen from Eq.(2.3.10) that Wigner function for the coherent states is a two-

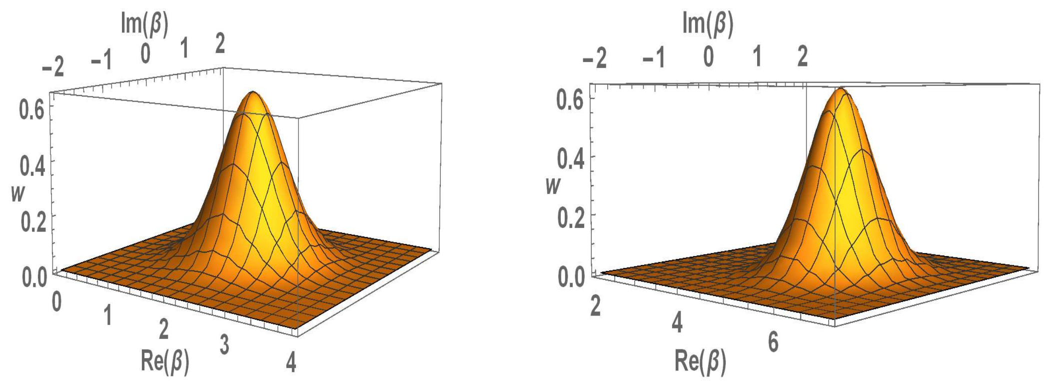

Figure 2.1: Wigner function for coherent states. left: $\alpha = 2$ and Right: $\alpha = 5$

dimensional Gaussian centered at coherent state parameter α. In Fig.(2.1), we plot $W(\beta)$ as a function of β for different values ofα. It is obvious from the Eq.(2.3.10) and Fig.(2.1) that Wigner function is the positive definite just like classical phase space distribution function. Therefore we conclude that coherent states are quantum mechanical states that have phase space description analogous to classical one.

Chapter 3

Construction of Nonclassical Field States via Coherent States

Nonclassical states have numerous applications in quantum information and quantum computation. In quantum optics non-classical states of different types were studies. For example, generalized coherent states [48], squeezed states[49], squeezed and displaced number states[50], Binomial states[51], odd and even coherent states[52] and some new families in [53]. In literature, the review of experimental approaches which may realize these states is given [54].
In previous chapter we discuss classical-like field states, the coherent states, and their properties in detail. Despite of their many uses in quantum optics we can not use them in the phenomena which require nonclassicality as a prerequisite. For example, generation of entanglement, a key resource for quantum information processing, does require nonclassicality as a pre-requisite. Here we discuss various approaches to construct nonclassical eld states using very well known coherent states. In section (3.1) Our first approach is to construct superpositions of coherent states and then construct excitations of coherent states in section (3.2). At the end in section (3.3) we discuss excitations of superpositions of coherent states.

3.1 Construction of superpositions of coherent states

Let us take the superposition of coherent state $|\alpha\rangle$ and $|-\alpha\rangle$. These states have same amplitude but separated in phase by 180^o.

$$|\psi\rangle = N(|\alpha\rangle + e^{i\phi}|-\alpha\rangle), \tag{3.1.1}$$

where N is normalization constant. To find this normalization constant, we have to find inner product $\langle\psi|\psi\rangle$. By expending coherent state $|\alpha\rangle$ in number state basis we get,

$$|\psi\rangle = N\exp\left(-\frac{1}{2}|\alpha|^2\right)\left(\sum_{m=0}^{\infty}\frac{(\alpha)^m}{\sqrt{m!}}|m\rangle + e^{i\phi}\sum_{m=0}^{\infty}\frac{(-\alpha)^m}{\sqrt{m!}}|m\rangle\right).$$

Now multiply $|\psi\rangle$ with its complex conjugate $\langle\psi|$ and get,

$$\langle\psi|\psi\rangle = |N|^2[\exp(-|\alpha|^2)\sum_{m=0}^{\infty}\frac{(\alpha\alpha^*)^m}{\sqrt{m!}} + e^{i\phi}\exp(-|\alpha|^2)\sum_{m=0}^{\infty}\frac{(-\alpha\alpha^*)^m}{\sqrt{m!}}$$
$$+\exp(i\phi)\exp(-|\alpha|^2)\sum_{m=0}^{\infty}\frac{(-\alpha\alpha^*)^m}{\sqrt{m!}} + \exp(-|\alpha|^2)\sum_{m=0}^{\infty}\frac{(\alpha\alpha^*)^m}{\sqrt{m!}}].$$

After simplification and using $\langle\psi|\psi\rangle = 1$ we get normalization constant as,

$$|N|^2 = \frac{1}{2 + 2\cos\phi\exp(-2|\alpha|^2)}. \tag{3.1.2}$$

By taking different values of ϕ we can construct different type of superposition but here we discuss only three cases.

3.1.1 Even coherent states

If we take $\phi = 0$ Eq.(3.1.1) becomes,

$$|\alpha_+\rangle = N_e(|\alpha\rangle + |-\alpha\rangle), \qquad \exp(i\phi) = 1, \tag{3.1.3}$$

where N_e is constant of normalization. These states, $|\alpha_+\rangle$, are named as even coherent state [27]. By putting $\phi = 0$ in Eq.(3.1.2) we get,

$$|N_e|^2 = \frac{1}{2 + 2\exp(-2|\alpha|^2)}. \tag{3.1.4}$$

By putting Eq.(3.1.3) in Eq.(3.1.4) we get,

$$|\alpha_+\rangle = \frac{1}{2 + 2\exp(-2|\alpha|^2)}\left(|\alpha\rangle + |-\alpha\rangle\right). \tag{3.1.5}$$

Photon counting statistic of even coherent states can be obtain as,

$$P_+(n) = |\langle n|\alpha_+\rangle|^2. \tag{3.1.6}$$

We start from projecting $\langle n|$ on it as,

$$\langle n|\alpha_+\rangle = N_e(\langle n|\alpha\rangle + \langle n| - \alpha\rangle).$$

By expanding $|\alpha\rangle$ in Fock basis we get,

$$\langle n|\alpha_+\rangle = N_e\left(\exp\left(-\frac{1}{2}|\alpha|^2\right)\frac{\alpha^n}{\sqrt{n!}} + \exp\left(-\frac{1}{2}|\alpha|^2\right)\frac{(-\alpha)^n}{\sqrt{n!}}\right).$$

Using value of normalization constant and taking modulus square on both sides we get,

$$P_+(n) = \frac{\exp\left(-|\alpha|^2\right)}{2(1+\exp\left(-2|\alpha|^2\right))}\left|\left(\frac{\alpha^n}{\sqrt{n!}} + \frac{(-\alpha)^n}{\sqrt{n!}}\right)\right|^2. \tag{3.1.7}$$

$P_+(n)$ have different value for different value of n.
For odd values of n,

$$P_+(n) = 0. \tag{3.1.8}$$

But for even value of n,

$$P_+(n) = \frac{\exp\left(-|\alpha|^2\right)}{2(1+\exp[-2|\alpha|^2])}\left|\left(\frac{\alpha^n}{\sqrt{n!}} + \frac{\alpha^n}{\sqrt{n!}}\right)\right|^2,$$

$$P_+(n) = \frac{2\exp\left(-|\alpha|^2\right)}{(1+\exp[-2|\alpha|^2])}\frac{|\alpha|^{2n}}{n!}, \tag{3.1.9}$$

where $|\alpha|^2 = \bar{n}$. Now by combining Eq.(3.1.8) and Eq.(3.1.9) we can write,

$$P_+(n) = \begin{cases} 0 & \text{for odd } n \\ \frac{2\exp\left(-|\alpha|^2\right)}{(1+\exp(-2\bar{n}))}\frac{\bar{n}^n}{n!} & \text{for even } n \end{cases}$$

As probability of finding odd number of photon vanishes therefore we name these states as "Even coherent states ".

3.1.2 Odd coherent states

If we take $\phi = \pi$ Eq.(3.1.1) become,

$$|\alpha_-\rangle = N_o(|\alpha\rangle - |-\alpha\rangle), \quad \exp(i\phi) = -1, \tag{3.1.10}$$

where N_o is normalization constant. These states are named as odd coherent state. By putting $\phi = \pi$ in Eq.(3.1.2) we get,

$$|N_o|^2 = \frac{1}{2 - 2\exp(-2|\alpha|^2)}. \tag{3.1.11}$$

Using Eq.(3.1.11) in Eq.(3.1.10) we get,

$$|\alpha_-\rangle = \frac{1}{2 - 2\exp(-2|\alpha|^2)} \left(|\alpha\rangle - |-\alpha\rangle\right). \tag{3.1.12}$$

To calculate photon statistic of odd coherent states, Eq.(3.1.6) can be written as,

$$P_-(n) = |\langle n|\alpha_-\rangle|^2. \tag{3.1.13}$$

Now project $\langle n|$ on odd coherent states given as in Eq.(3.1.10) as,

$$\langle n|\alpha_-\rangle = N_o \left(\langle n|\alpha\rangle - \langle n|-\alpha\rangle\right).$$

After some calculations we get,

$$P_-(n) = \frac{\exp\left(-|\alpha|^2\right)}{2(1 - \exp[-2|\alpha|^2])} \left| \left(\frac{\alpha^n}{\sqrt{n!}} - \frac{(-\alpha^n)}{\sqrt{n!}} \right) \right|^2. \tag{3.1.14}$$

Probability of finding n photon is different for different value of n.
For even n,

$$P_-(n) = 0. \tag{3.1.15}$$

But for odd value of n,

$$P_-(n) = \frac{\exp\left(-|\alpha|^2\right)}{2(1 - \exp[-2|\alpha|^2])} \left| \left(\frac{\alpha^n}{\sqrt{n!}} + \frac{\alpha^n}{\sqrt{n!}} \right) \right|^2,$$

$$P_-(n) = \frac{4\exp\left(-|\alpha|^2\right)}{2(1 - \exp[-2|\alpha|^2])} \frac{|\alpha|^{2n}}{n!}.$$

By putting $|\alpha|^2 = \bar{n}$ we get,

$$P_{-}(n) = \frac{2\exp\left(-|\alpha|^2\right)}{(1-\exp\left(-2\bar{n}\right))}\frac{\bar{n}^n}{n!}. \tag{3.1.16}$$

By combining Eq.(3.1.15) and Eq.(3.1.16) we can write,

$$P_{-}(n) = \begin{cases} 0 & \text{for even} \quad n \\ \frac{2\exp\left(-|\alpha|^2\right)}{(1+\exp(-2\bar{n}))}\frac{\bar{n}^n}{n!} & \text{for odd} \quad n \end{cases}$$

As probability of finding even number of photon vanishes therefore we name them as "Odd coherent states ".

3.1.3 Yurke-Stoler states

If we take $\phi = \frac{\pi}{2}$, we get special kind of states which are known as Yurke-Stoler states, on the name of scientist[55]. With this value of ϕ, and Eq.(3.1.1) become,

$$|\alpha_{ys}\rangle = N_{ys}(|\alpha\rangle + i|-\alpha\rangle), \quad \exp(i\phi) = i, \tag{3.1.17}$$

N_{ys} is normalization constant. By putting $\phi = \frac{\pi}{2}$ in Eq.(3.1.2) we get,

$$|N_{ys}|^2 = \frac{1}{\sqrt{2}}. \tag{3.1.18}$$

Using Eq.(3.1.17) in Eq.(3.1.18) we get,

$$|\alpha_{ys}\rangle = \frac{1}{\sqrt{2}}\left(|\alpha\rangle + i|\alpha\rangle\right). \tag{3.1.19}$$

For this state photon counting statistic $P(n)$ is same as for coherent states, i.e, Poisson distribution.

3.2 Construction of excitations of coherent states

In our second approach we discuss the states whose properties are in between the number state $|m\rangle$, pure quantum mechanical field states, and coherent states. Initially excited coherent states of light were discussed in [56, 57]. In this paper their non-classical properties were also probe.

3.2.1 Excited coherent states

By the n time application of the creation operator on continuous-variable field states, coherent states, we get excited coherent states. These states are also known as photon added coherent states.

$$|\alpha, n\rangle = N a^{\dagger^n}|\alpha\rangle. \tag{3.2.1}$$

We can calculate normalization constant N as,

$$\langle\alpha, n|\alpha, n\rangle = |N|^2\langle\alpha|a^n a^{\dagger^n}|\alpha\rangle,$$

$$|N|^2 = \frac{1}{\langle\alpha|a^n a^{\dagger^n}|\alpha\rangle}.$$

$$|N|^2 = \frac{1}{L_n(-|\alpha|^2)n!}, \qquad \text{Since} \quad \langle\alpha|a^n a^{\dagger^n}|\alpha\rangle = L_n(-|\alpha|^2)n!, \tag{3.2.2}$$

where $L_n(-|\alpha|^2)$ is Laguerre polynomial of order n,

$$|\alpha, n\rangle = \frac{a^{\dagger^n}|\alpha\rangle}{[L_n(-|\alpha|^2)n!]^{\frac{1}{2}}}. \tag{3.2.3}$$

It is interesting to note that when $n \longrightarrow 0$, we get coherent state and when $\alpha \longrightarrow 0$ we get Fock states. Thus, we name these state as "photon added coherent states ". By using Eq.(2.2.12), excited coherent states $|\alpha, n\rangle$ can be written in term of number states as,

$$|\alpha, n\rangle = \frac{a^{\dagger^n}\exp\left(-\frac{1}{2}|\alpha|^2\right)\sum\limits_{m=0}^{\infty}\frac{\alpha^m}{\sqrt{m!}}|m\rangle}{[L_n(-|\alpha|^2)n!]^{\frac{1}{2}}}, \tag{3.2.4}$$

and action of creation operator is,

$$a^{\dagger^n}|m\rangle = \frac{\sqrt{(m+n)!}}{\sqrt{m!}}|n+m\rangle. \tag{3.2.5}$$

Putting Eq.(3.2.5) in Eq.(3.2.4) and get,

$$|\alpha, n\rangle = \frac{\exp\left(-\frac{1}{2}|\alpha|^2\right)}{[L_n(-|\alpha|^2)n!]^{\frac{1}{2}}}\sum_{m=0}^{\infty}\alpha^n\frac{\sqrt{(n+m)!}}{m!}|n+m\rangle. \tag{3.2.6}$$

We can also calculate the scaler product $\langle\alpha, n|\alpha, m\rangle$ as,

$$\langle\alpha, n|\alpha, m\rangle = \frac{\exp(-|\alpha|^2)}{[L_n(-|\alpha|^2)n!]^{\frac{1}{2}}[L_m(-|\alpha|^2)m!]^{\frac{1}{2}}}\sum_{p=0}^{\infty}\sum_{r=0}^{\infty}\frac{(\alpha^*)^p\alpha^r\sqrt{(p+n)!(r+m)!}}{p!r!}$$
$$\langle r+m|p+n\rangle,$$

Let $r = p + n - m$,

$$\langle \alpha, n | \alpha, m \rangle = \frac{\exp(-|\alpha|^2)}{[L_n(-|\alpha|^2)n! L_m(-|\alpha|^2)m!]^{\frac{1}{2}}} \sum_{p=0}^{\infty} \frac{|\alpha|^{2p}(p+m)!\alpha^{n-m}}{p!(p+n-m)!}. \tag{3.2.7}$$

Probability distribution of the field

The probability distribution in the field can be determined as,

$$P(m) = |\langle m | \alpha, n \rangle|^2. \tag{3.2.8}$$

We start from taking inner product and by using Eq.(3.2.6) we get,

$$\langle m | \alpha, n \rangle = \frac{\exp\left(-\frac{1}{2}|\alpha|^2\right)}{[L_n(-|\alpha|^2)n!]^{\frac{1}{2}}} \sum_{m=0}^{\infty} \alpha^m \frac{\sqrt{(n+m)!}}{m!} \langle m | m+n \rangle.$$

By multiplying it with its complex conjugate we get,

$$P(m) = \left| \frac{\exp\left(-\frac{1}{2}|\alpha|^2\right)}{[L_n(-|\alpha|^2)n!]^{\frac{1}{2}}} \sum_{m=0}^{\infty} \alpha^m \frac{\sqrt{(m+n)!}}{m!} \langle m | m+n \rangle \right|^2,$$

Let $m = m - n$,

$$P(m) = \left| \frac{\exp\left(-\frac{1}{2}|\alpha|^2\right)}{[L_n(-|\alpha|^2)n!]^{\frac{1}{2}}} \alpha^{m-n} \frac{\sqrt{(m)!}}{(m-n)!} \right|^2, \tag{3.2.9}$$

after simplifying we get,

$$P(m) = \frac{\exp\left(-|\alpha|^2\right)}{[L_n(-|\alpha|^2)n!]} \frac{|\alpha|^{2(m-n)}(m)!}{[(m-n)!]^2}. \tag{3.2.10}$$

which is zero for $m < n$. Variance of this distribution is less than that for Poisson distribution which we calculate in next chapter.

3.3 Construction of Excitations of coherent states superpositions

In this section we discuss the excitations of coherent state superpositions. These states are obtain by repeated action of creation operator on coherent state superpositions[36]. when we apply creation operator on even coherent state we get excited even coherent state and when we apply creation operator on odd coherent state we get excited odd coherent state.

3.3.1 Excited even coherent states

Photon added even coherent state is defined as,

$$|\alpha_+, n\rangle = N a^{\dagger^n}|\alpha_+\rangle, \tag{3.3.1}$$

where even coherent state is $|\alpha_+\rangle$ and N is normalization constant and can be calculated as,

$$\langle\alpha_+, n|\alpha_+, n\rangle = |N|^2\langle\alpha_+|a^n a^{\dagger^n}|\alpha_+\rangle. \tag{3.3.2}$$

We first calculate $\langle\alpha_+|a^n a^{\dagger^n}|\alpha_+\rangle$ by putting value of even coherent states, given in Eq.(3.1.3), as,

$$\langle\alpha_+|a^n a^{\dagger^n}|\alpha_+\rangle = \frac{\exp\left(|\alpha|^2\right)}{4\cosh|\alpha|^2}\left((\langle\alpha| + \langle-\alpha|)a^n a^{\dagger^n}(|\alpha\rangle + |-\alpha\rangle)\right).$$

After simplification we get,

$$\langle\alpha_+|a^n a^{\dagger^n}|\alpha_+\rangle = \frac{n!}{2\cosh|\alpha|^2}L_n^+(|\alpha|^2). \tag{3.3.3}$$

Using Eq.(3.3.3) in Eq.(3.3.2) normalization constant becomes,

$$|N|^2 = \frac{2\cosh|\alpha|^2}{n!L_n^+(|\alpha|^2)}, \tag{3.3.4}$$

where $L_n^+(|\alpha|^2)$ is given as,

$$L_n^+(|\alpha|^2) = \exp(|\alpha|^2)L_n(-|\alpha|^2) + \exp(-|\alpha|^2)L_n(|\alpha|^2). \tag{3.3.5}$$

by putting value of normalization constant Eq. (3.3.1) becomes,

$$|\alpha_+, n\rangle = \sqrt{\frac{2\cosh|\alpha|^2}{L_n^+(|\alpha|^2)n!}}a^{\dagger^n}|\alpha_+\rangle. \tag{3.3.6}$$

After putting value of even coherent states, given in Eq.(3.1.3), our state becomes,

$$|\alpha_+, n\rangle = \frac{\exp\left(\frac{1}{2}|\alpha|^2\right)}{\sqrt{2L_n^+(|\alpha|^2)n!}}[a^{\dagger^n}|\alpha\rangle + a^{\dagger^n}|-\alpha\rangle],$$

when creation operator, $a^\dagger$, act n times on coherent state $|\alpha\rangle$ and $|-\alpha\rangle$, our state becomes,

$$|\alpha_+, n\rangle = \frac{\exp\left(\frac{1}{2}|\alpha|^2\right)}{\sqrt{2L_n^+(|\alpha|^2)n!}}\left(\sqrt{n!L_n(|-\alpha|^2)}|\alpha, n\rangle + \sqrt{n!L_n(|-\alpha|^2)}|-\alpha, n\rangle\right).$$

By taking like terms common,

$$|\alpha_+, n\rangle = \sqrt{\frac{\exp\left(|\alpha|^2\right) L_n(|-\alpha|^2)}{2L_n^+(|\alpha|^2)}} \left(|\alpha, n\rangle + |-\alpha, n\rangle\right), \tag{3.3.7}$$

where $|\alpha_+, n\rangle$ is photon added even coherent state.

3.3.2 Excited odd coherent states

When creation operator, $a^\dagger$, act n time on odd coherent state we get excited odd coherent states $|\alpha_-, n\rangle$, which is also known as photon added odd coherent state. By following same steps as we follow for excited even coherent state we can get $|\alpha_-, n\rangle$ as,

$$|\alpha_-, n\rangle = \sqrt{\frac{\exp\left(|\alpha|^2\right) L_n(|-\alpha|^2)}{2L_n^-(|\alpha|^2)}} \left(|\alpha, n\rangle - |-\alpha, n\rangle\right), \tag{3.3.8}$$

where $L_n^-(|\alpha|^2)$ is given as,

$$L_n^-(|\alpha|^2) = \exp(|\alpha|^2)L_n(-|\alpha|^2) - \exp(-|\alpha|^2)L_n(|\alpha|^2). \tag{3.3.9}$$

In Fock basis

The photon added even/odd coherent state can be written in term of number state as,

$$|\alpha_\pm, n\rangle = \frac{1}{\sqrt{2n!L_n^\pm(|\alpha|^2)}} \sum_{m=0}^{\infty} \frac{(1 \pm (-1)^m)}{m!} (\alpha^m \sqrt{(m+n)!}|m+n\rangle). \tag{3.3.10}$$

By introducing the notation $L_n^\pm$, which is combine form of Eq.(3.3.5) and Eq.(3.3.9), we get,

$$L_n^\pm(|\alpha|^2) = \exp(|\alpha|^2)L_n(-|\alpha|^2) \pm \exp(-|\alpha|^2)L_n(|\alpha|^2). \tag{3.3.11}$$

Photon statistic of the excitations of coherent state superpositions

Probability distribution of excitations of coherent state superpositions is given as,

$$P_\pm(m) = |\langle m|\alpha\pm, n\rangle|^2. \tag{3.3.12}$$

Using excitations of coherent state superpositions in number state basis, given in Eq.(3.3.10), we can write,

$$\langle m|\alpha\pm,n\rangle = \frac{\exp(-\frac{1}{2}|\alpha|^2)}{[2L_n^{\pm}(|\alpha|^2)n!]^{\frac{1}{2}}}\sum_{m=0}^{\infty}\alpha^m\frac{\sqrt{(m'+n)!}}{m'!}\langle m|m'+n\rangle.$$

Multiplying it with its complex conjugate we get,

$$P_{\pm}(m) = \left|\frac{(1\pm(-1)^{m'})}{[2L_n^{\pm}(|\alpha|^2)n!]^{\frac{1}{2}}}\sum_{m=0}^{\infty}\alpha^{m'}\frac{\sqrt{(m'+n)!}}{m'!}\langle m|m'+n\rangle\right|^2.$$

Let $m' = m-n$,

$$P_{\pm}(m) = \left|\frac{(1\pm(-1)^{m-n})}{[2L_n^{\pm}(|\alpha|^2)n!]^{\frac{1}{2}}}\alpha^{m-n}\frac{\sqrt{(m)!}}{(m-n)!}\right|^2. \tag{3.3.13}$$

After simplifying we get,

$$P_{\pm}(m) = \frac{(1\pm(-1)^{m-n})}{[2L_n^{\pm}(|\alpha|^2)n!]}\frac{|\alpha|^{2(m-n)}(m)!}{[(m-n)!]^2}. \tag{3.3.14}$$

where $m \geq n$ and $P_{\pm}(m) = 0$ for $m < n$. This distribution have variance less than mean. Mandel Q parameter is use to probe this.

Chapter 4

Analyzing Nonclassicality of the Continuous-Variable Field States

In previous chapter we discuss different approaches to construct the continuously parameterized field states, supposedly, having nonclassical features. However, for practical purposes, it is mandatory to analyze their noncassicality, quanlitatively and well as quantitatively. There are various parameters that can be used to analyze nonclassical properties of the field state in different situations, such as, Glauber-Sudarshan P-function, Wigner function, amplitude and intensity squeezing, Anti-bunching and sub-Poissonian photon statistics. However, in our discussion we will use Wigner function and characterization of photon statistics via Mandel Q-parameter.

The chapter is organized as the following. In section 4.1 we discuss the nonclassical criteria by introducing the negativity of Wigner function and Mandel Q-parameter. The section 4.2 is dedicated to the analysis of nonclassicality of our constructed field states. Finally, in 4.3, we present the results and discussions.

4.1 Criteria of Non-classicality

As pointed out earlier, for a given field state, the occurrence of any characteristic that has no classical analogue can used as an indicator of nonclassicality. Therefore, various parameters that can be used to analyze nonclassical properties of the field state in different situations [58, 59]. However, the choice of a suitable parameter as an indicator of nonclassicality varies depending on the physical situation of field states. For our constructed continuous-variable field states we use the following parameters as our criteria for probing the nonclassicality.

4.1.1 Negativity of the Winger function

As introduced in Chapter 2, the Wigner function is defined as phase-space joint probability distribution function. According classical statistical mechanics, this function, being a probability function, should be positive definite. However, in quantum mechanics it came out that Wigner function may take negative values in some region of phase-phase for some states. Such states can be regarded as nonclassical states and negativity of Wigner function can be regarded as an indicator of nonclassicality [60, 61].

In the context of quantum optics, we have defined the Wigner function in Eq.(4.1.1) of Chapter 2 which we rewrite here for our convenience as

$$W(\beta) = \frac{1}{\pi^2} \int \exp\left(\lambda^* \beta - \lambda \beta^*\right) C_w d^2\lambda, \tag{4.1.1}$$

where $C_w(\lambda)$ is given as,

$$C_w(\lambda) = e^{-|\lambda|^2/2} Tr(\hat{\rho} e^{\lambda a^\dagger} e^{-\lambda^* \hat{a}}).$$

4.1.2 Sub-Poissonian Photon Statistics

As we have seen earlier in Chapter 2, the photon counting statistics of coherent states, classical-like states, is Poissonian. Furthermore, the photon counting statistics of all classical light sources, such as, thermal light sources or chaotic sources is super-Poissonian (broader than Poissonian) [38, 62]. Interestingly, any source that has no classical analogue exhibits sub-Poissonian (narrower than Poissonian) statistics[39].

In order to investigate the underlying photon counting statistics of a given field state, a parameter, originally defined by Mandel, has been used, which is defined

[40] as,

$$Q = \frac{\sigma^2}{\langle n \rangle} - 1, \tag{4.1.2}$$

where $\langle n \rangle$ is the mean and σ^2 is the variance of photon counting probability distribution. According to this parameter, the distribution is:

- Super-Poissonian if $Q > 0$,
- Poisson if $Q = 0$,
- Sub-Poissonian if $Q < 0$.

For our later calculations, we rearrange the terms in Eq.(4.1.2), and write the Mandel's parameter as

$$Q = \frac{\langle n^2 \rangle - \langle n \rangle^2}{\langle n \rangle} - 1, \quad \text{where} \quad \sigma^2 = \langle n^2 \rangle - \langle n \rangle^2. \tag{4.1.3}$$

Furthermore, substituting the value of number operator $n = a^\dagger \hat{a}$,

$$Q = \frac{\langle a^\dagger \hat{a} \rangle + \langle a^{\dagger^2} \hat{a}^2 \rangle - \langle a^\dagger \hat{a} \rangle^2}{\langle a^\dagger \hat{a} \rangle} - 1, \quad \hat{a} a^\dagger = 1 + a^\dagger \hat{a}. \tag{4.1.4}$$

We will use this form of Mandel's parameter to investigate the photon counting statistics of our constructed field states.

4.2 Analysis of Nonclassicality

In this section we apply the nonclassicality criteria, defined above, to our constructed field states and analyze their nonclassical properties. In particular, first we calculate the Wigner function and then Mandel parameter as indicators of nonclassicality.

4.2.1 Wigner Function

Using the explicit form of Wigner function, given in Eq.(4.1.1), we proceed to calculate the complex integral for various field states.

Superpositions of coherent states

As discussed in previous chapter, we consider three kinds of coherent state superpositions, for which Wigner functions are being calculated below.

- **Even/odd coherent states:**

  As shown by Eq.(4.1.1), in order to compute the Wigner function for a given state, one needs to have corresponding density operator. For even/odd coherent states, given in Eqs.(3.1.3) and (3.1.3), the density operator $\hat{\rho}$ is given as,

$$\hat{\rho} = |\alpha_{\pm}\rangle\langle\alpha_{\pm}|.$$

  Using this value of $\hat{\rho}$, the characteristic function, given in Eq.(2.3.7), becomes,

$$\begin{aligned} C_w(\lambda) = & \frac{\exp\left(-\frac{1}{2}|\lambda|^2\right)\exp(-|\alpha|^2)}{2(1\pm\exp(-2|\alpha|^2))}[\exp(|\alpha|^2)(\exp(\alpha^*\lambda - \lambda^*\alpha + \exp(\alpha^*\lambda) + \lambda^*\alpha)) \\ & \pm \exp(-|\alpha|^2)(\exp(-\alpha^*\lambda - \lambda^*\alpha) + \exp(\alpha^*\lambda + \lambda^*\alpha)). \end{aligned}$$

  After putting value of $C_w(\lambda)$ in Eq.(4.1.1) and evaluating the integral by using identity given in Eq.(2.3.7) we get,

$$\begin{aligned} W(\beta) = & \frac{1}{\pi(1\pm\exp(-2|\alpha|^2)}[\exp(-2|\beta-\alpha|^2) + \exp(-2|\beta+\alpha|^2) \\ & \pm 2\exp(-2|\beta|^2)\cos(2i(\alpha^*\beta - \beta^*\alpha))]. \end{aligned} \tag{4.2.1}$$

  Detail calculations are given in Appendix B. As β is complex number so by putting $\beta = p - iq$ we get,

$$\begin{aligned} W(p,q) = & \frac{1}{\pi(1\pm\exp(-2|\alpha|^2)}[\exp(-2(p-\alpha)^2 - 2q^2) + \exp(-2(p+\alpha)^2 - 2q^2) \\ & \pm 2\exp(-2(p^2+q^2)\cos(4q\alpha)]. \end{aligned}$$

  When we plot this Wigner function $W(p,q)$, we observe two Gaussian peaks that are always positive and between them there is interference pattern. As a result of this interference term Wigner function become highly oscillatory, as shown by Fig (4.1,4.2)

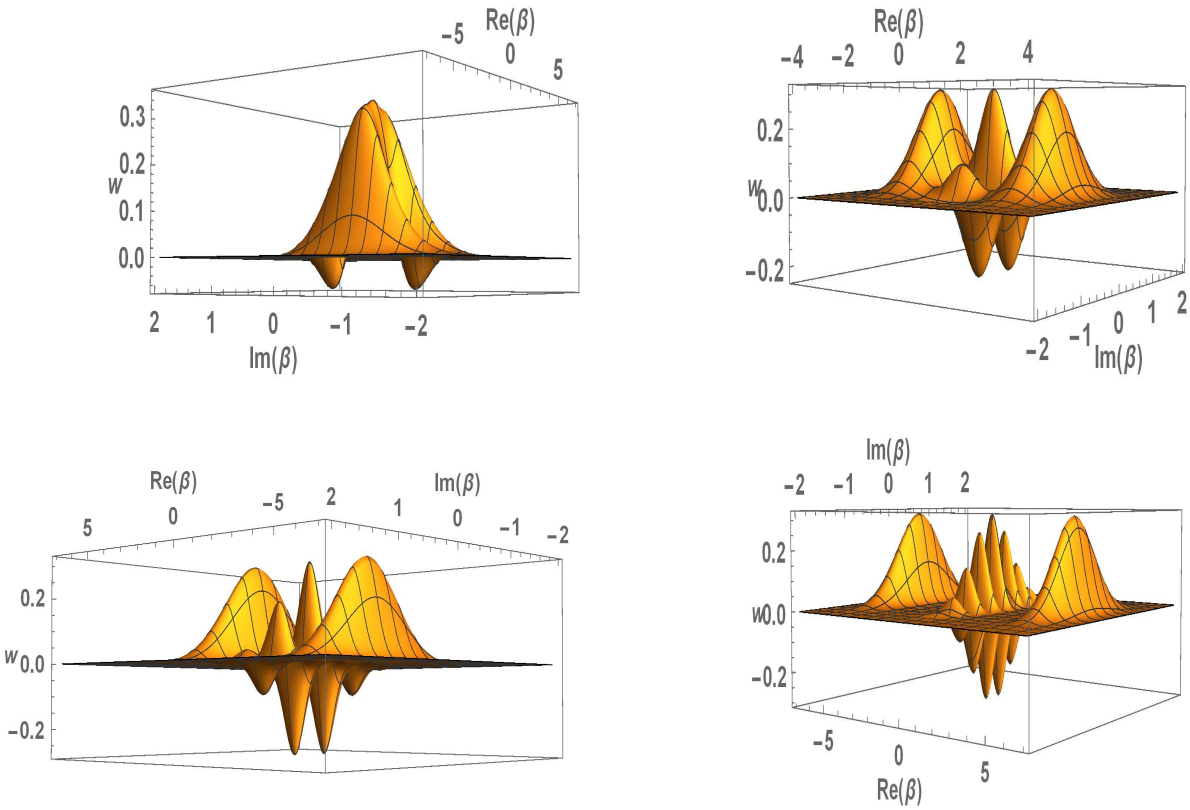

Figure 4.1: Wigner function of even coherent states for different values α i.e in $\alpha = 1, 2, 3$ and 5 respectively.

- **Yurke-Stoler states:**

  For the sake of completeness, we give the corresponding Wigner function of the Yurke Stoler states. Using the density operator $\hat{\rho}$ is given as,

$$\hat{\rho} = |\alpha_{ys}\rangle\langle\alpha_{ys}|.$$

and Same steps are followed to get the Wigner function of Yurke-Stoler state states.

$$\begin{aligned} W(\beta) = & \frac{1}{\pi(1+\exp(-2|\alpha|^2)}[\exp(-2|\beta-\alpha|^2) + \exp(-2|\beta+\alpha|^2) \\ & - 2\exp(-2|\beta|^2)\sin(2i(\alpha^*\beta - \beta^*\alpha))]. \end{aligned} \tag{4.2.2}$$

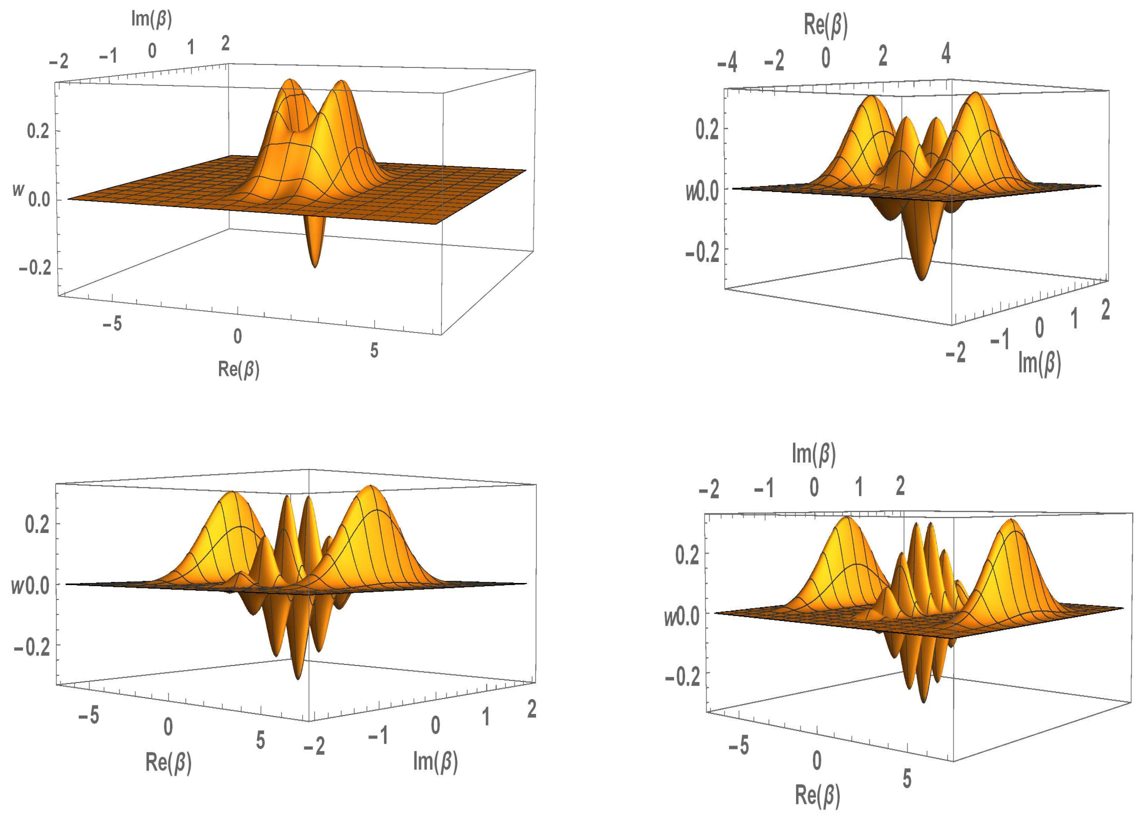

Figure 4.2: Wigner function of odd coherent states for $\alpha = 1, 2, 4$ and 5 respectively.

By putting $\beta = p + iq$ we get,

$$\begin{aligned} W(p,q) = \; & \tfrac{1}{\pi}[\exp(-2(p-\alpha)^2 - 2y^2) + \exp(-2(p+\alpha)^2 - 2q^2) \\ & -2\exp(-2(p^2+q^2)\sin(4q\alpha). \end{aligned} \tag{4.2.3}$$

Negativity of Wigner function of even/odd coherent states and Yurke-Stoler states shows the non-classicality of these states [63].

Wigner function of excited coherent states

The definition of Wigner function, given in Eq.(4.1.1), can be calculated by using density operator $\hat{\rho}$, given as,

$$\hat{\rho} = |\alpha, n\rangle\langle\alpha, n|. \tag{4.2.4}$$

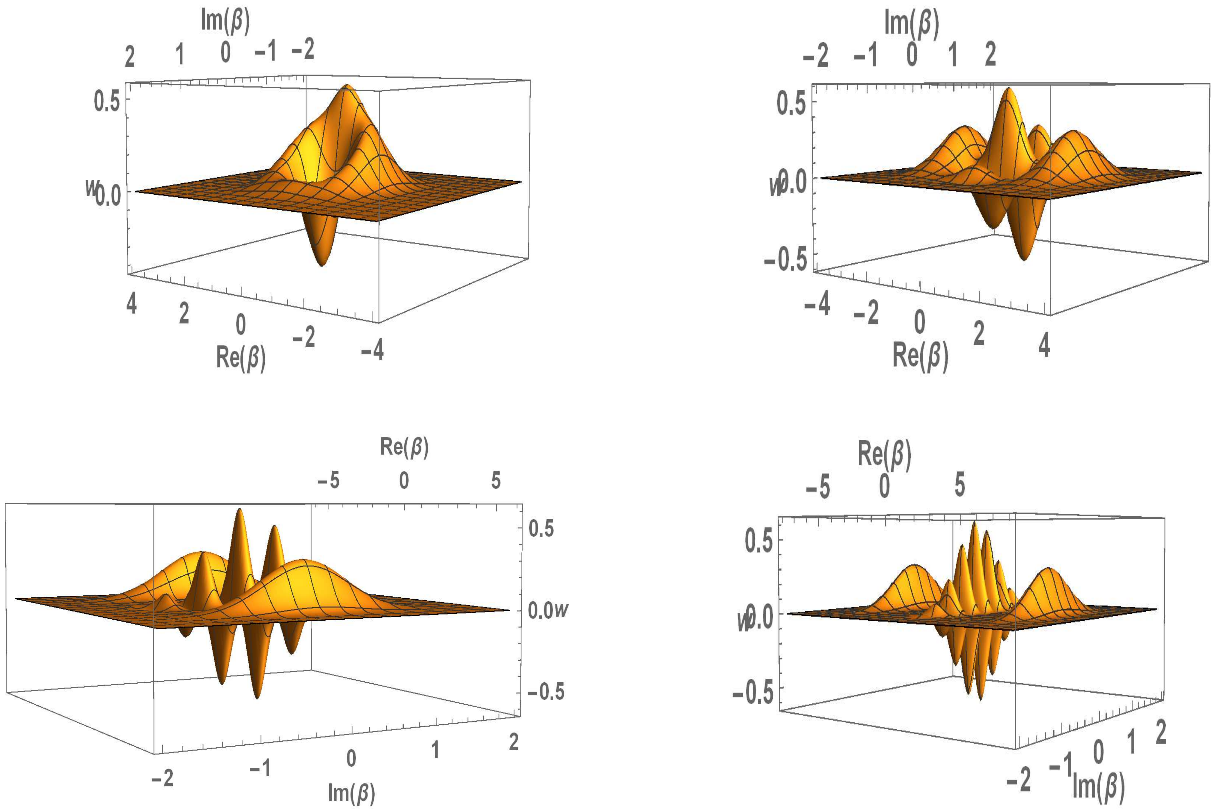

Figure 4.3: Wigner function of Yurke-Stoler states for $\alpha = 1, 2, 3.5$ and 5 respectively.

We calculate characteristic function $C_w(\lambda)$, given in Eq.(2.3.7) and after some calculations we get,

$$W(\beta) = \frac{2(-1)^n L_n(|2\beta - \alpha|^2)}{\pi[L_n(-|\alpha|^2)]} \exp(-2|\beta - \alpha|^2), \tag{4.2.5}$$

where β is complex number. Let $\beta = p + iq$ and $\alpha = a_i + ia_j$ and then plot Wigner function as shown in Fig.(4.4). Negativity of Wigner function indicate nonclassicality of these states.

Excitations of coherent state superpositions

The Wigner function of excitations of coherent state superpositions can be calculated

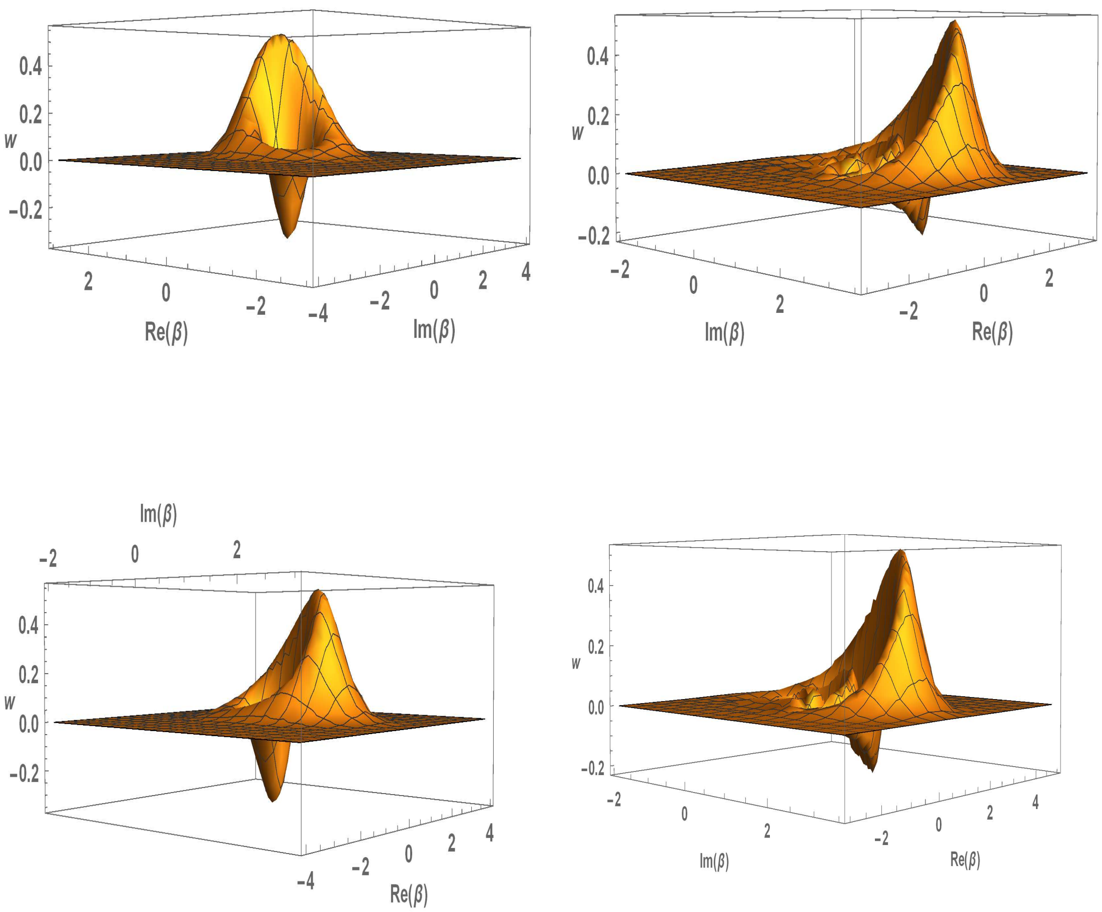

Figure 4.4: Wigner function of photon added coherent states. First two plots are for $\alpha = 1$ and other two plots are for $\alpha = 0.5 + i0.5$. In both cases we use $n = 1, 5$.

Excited even/odd coherent states

The general form of Wigner function is given in Eq.(4.1.1). But Wigner function of excited even/odd coherent state $|\alpha_\pm, n\rangle$ can be obtained by using density operator $\hat{\rho}$, given as,

$$\hat{\rho} = |\alpha_\pm, n\rangle\langle\alpha_\pm, n|. \tag{4.2.6}$$

So Eq.(4.1.1)can be written as,

$$W(\beta) = \frac{2}{\pi^2}\exp(2|\beta|^2)\int \exp\left(\lambda^*\beta - \lambda\beta^*\right)\langle-\lambda|\alpha_\pm, n\rangle\langle\alpha_\pm, n|\lambda\rangle d^2\lambda. \tag{4.2.7}$$

We can calculate characteristic function, given in Eq.(2.3.7), by using $\hat{\rho}$ as,

$$\begin{aligned} C_w(\lambda) &= Tr(\rho D(\lambda)), \\ &= \langle \alpha_\pm, n | D(\lambda) | \alpha_\pm, n \rangle, \\ &= N^2 \left(\langle \alpha, n| \pm \langle -\alpha, n| \right) D(\lambda) \left(|\alpha, n\rangle \pm |-\alpha, n\rangle \right), \end{aligned} \tag{4.2.8}$$

where N is normalization constant and given as ,

$$N = \sqrt{\frac{\exp\left(|\alpha|^2\right) L_n(|-\alpha|^2)}{2L_n^{\pm}(|\alpha|^2)}}.$$

After some calculations we get C_w and by putting its value in Eq.(4.1.1) and solving the integral we get,

$$\begin{aligned} W_\pm(\beta) = \frac{(-1)^n \exp(|\alpha|^2)}{[L_n^{\pm}(|\alpha|^2)]} & [\exp(-2|\beta-\alpha|^2) L_n(|2\beta-\alpha|^2) + \exp(-2|\beta-\alpha|^2) \\ & L_n(|2\beta-\alpha|^2) \pm 2\exp(-2|\alpha|^2) Re[\exp(-2(\beta-\alpha)(\beta^*+\alpha^*)) L_n((2\beta-\alpha)(2\beta^*+\alpha^*)]]. \end{aligned} \tag{4.2.9}$$

We plot $W_\pm(\beta)$ as function of β by fixing different value of α.

4.2.2 Mandel's Q-Parameter

As mentioned above in section (4.1), the underlying photon counting statistics of a given field state can be explored conveniently by means of Mandel's Q-parameter defined in Eq.(4.1.2). Here we use Eq. (4.1.4) for the calculation of Mandel's Q-parameter for our constructed field states.

Superpositions of coherent states

- **For even/odd coherent states:** To find Mandel Q parameter for even coherent states we use the definition which is given in Eq.(4.1.4). We start from expectation value of operator as,

$$\langle a^\dagger \hat{a} \rangle = \frac{|\alpha|^2(1-\exp(-2|\alpha|^2))}{1 \pm \exp(-2|\alpha|^2)}. \tag{4.2.10}$$

and

$$\langle a^{\dagger^2} \hat{a}^2 \rangle = |\alpha|^4. \tag{4.2.11}$$

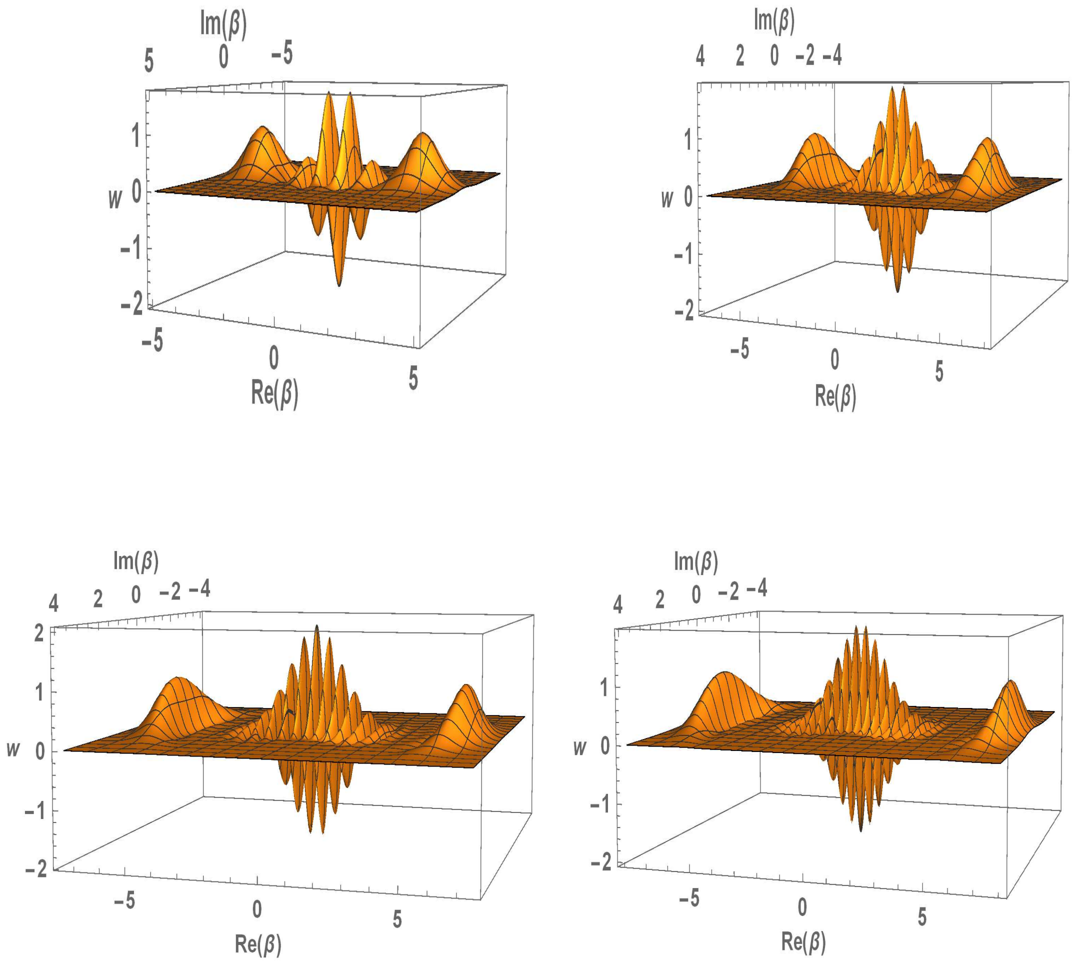

Figure 4.5: Wigner function of excited even coherent state for $\alpha = 2$ and $n = 1, 5, 10$ and 15 respectively.

By putting Eq.(C.0.4) and Eq.(4.2.11) in Eq.(4.1.4) we get,

$$Q = \frac{\pm 4|\alpha|^2 \exp(-2|\alpha|^2)}{(1 - \exp(-4|\alpha|^2))}. \tag{4.2.12}$$

$Q > 0$ indicate that Super-Poissonian statistic of even coherent states.. Detail solution is given in Appendix C $Q < 0$ indicate that Sub-Poissonian statistic of odd coherent states.

- **For Yurke-Stoler states:**

Now for Yurke-Stoler states, we follow same steps, starting from the calcula-

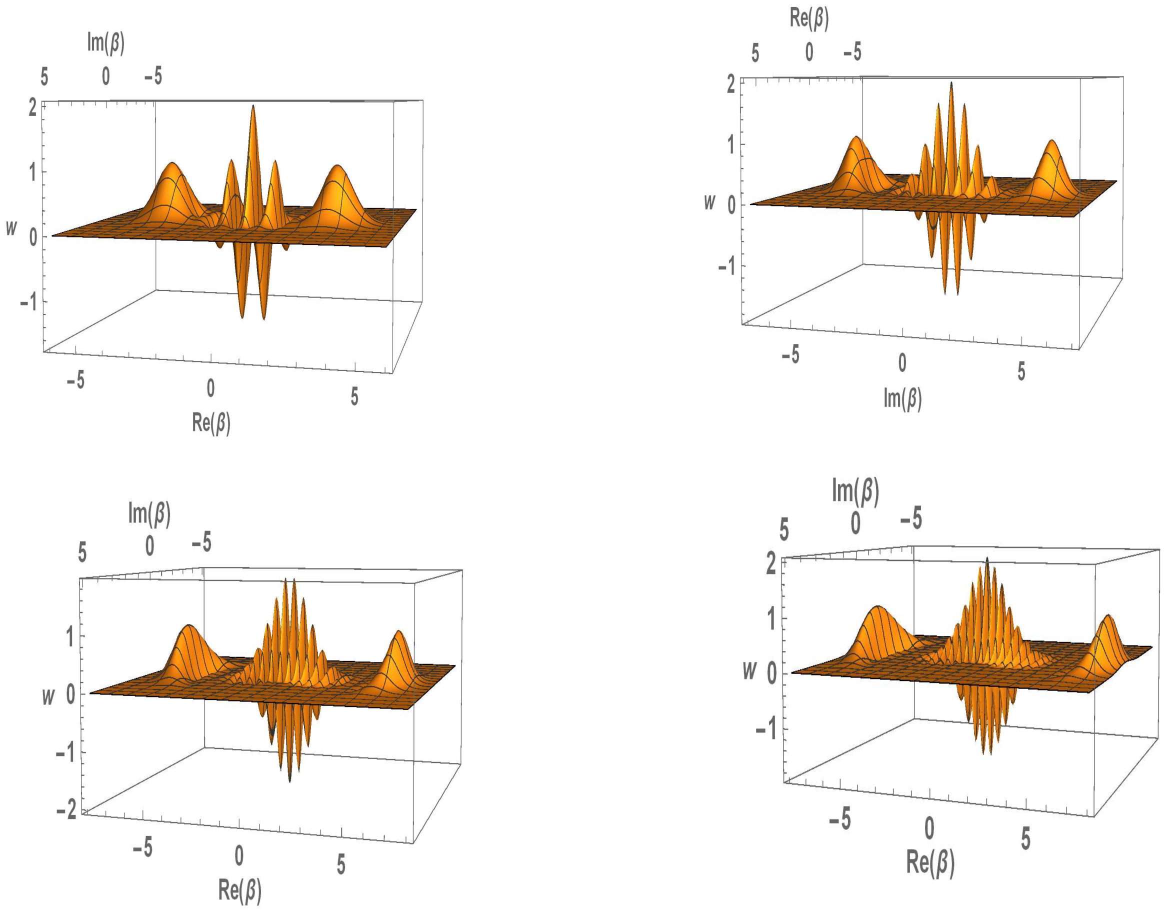

Figure 4.6: Wigner function of excited odd coherent state by fixing the value of $\alpha = 2$ and for different $n = 1, 5, 10$ and 15 respectively.

tions of expectation value $\langle a^{\dagger}\hat{a}\rangle = \langle\psi_{ys}|a^{\dagger}\hat{a}|\psi_{ys}\rangle$, as,

$$\langle a^{\dagger}\hat{a}\rangle = |\alpha|^{2}. \tag{4.2.13}$$

and

$$\langle(a^{\dagger^{2}}\hat{a}^{2}\rangle = |\alpha|^{4}, \tag{4.2.14}$$

by putting Eq.(4.2.13) and Eq.(4.2.14) in Eq.(4.1.4) we get,

$$Q = 0. \tag{4.2.15}$$

Which indicate the Poisson distribution of Yurke-Stoler states.

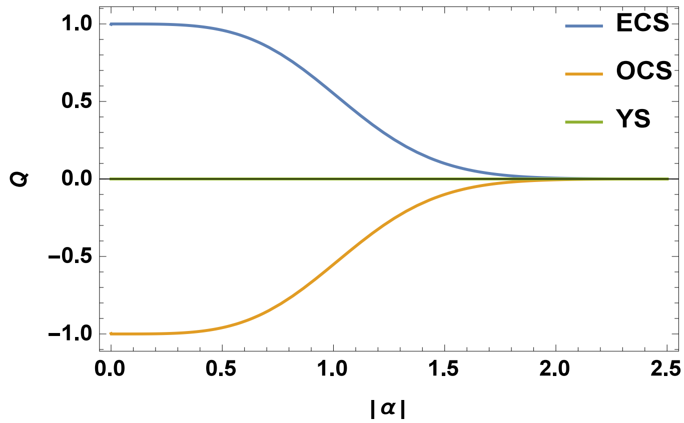

Figure 4.7: Mandel Q parameter as a function of $|\alpha|$ for superpositions of coherent states. ECS/OCS stand for even/odd coherent states respectively and ys is for Yurke-Stoler states.

Excitations of coherent states

We start from expectation value of number operator in the basis of excited coherent states as,

$$\begin{aligned}\langle a^\dagger a\rangle &= \frac{(n+1)!L_{n+1}(-|\alpha|^2)}{n!L_n(-|\alpha|^2)} - 1,\\ &= \frac{(n+1)L_{n+1}(-|\alpha|^2)}{L_n(-|\alpha|^2)} - 1.\end{aligned} \tag{4.2.16}$$

and

$$\langle\alpha, n|a^2 a^{\dagger^2}|\alpha, n\rangle = \frac{(n+2)!L_{n+2}(-|\alpha|^2)}{n!L_n(-|\alpha|^2)}, \tag{4.2.17}$$

By using Eq.(4.2.16) and Eq.(4.2.17) Eq.(4.1.4) becomes,

$$Q(\alpha, n) = \frac{[(n+2)(n+1)L_{n+2} - 4(n+1)L_{n+1} + 2L_n)][L_n] - [(n+1)L_{n+1} - L_n]^2}{[(n+1)!L_{n+1} - L_n]L_n}. \tag{4.2.18}$$

We note that for $n = 0$, $Q = 0$. In fig(4.8) we plot Q as a function of $|\alpha|$, but for different value of n. For $\alpha = 0$, $Q(0, n) < 0$ but for $\alpha \neq 0$ $Q(0, n) \neq 0$ field in the state $|\alpha, n\rangle$ shows sub-Poissonian statistics.

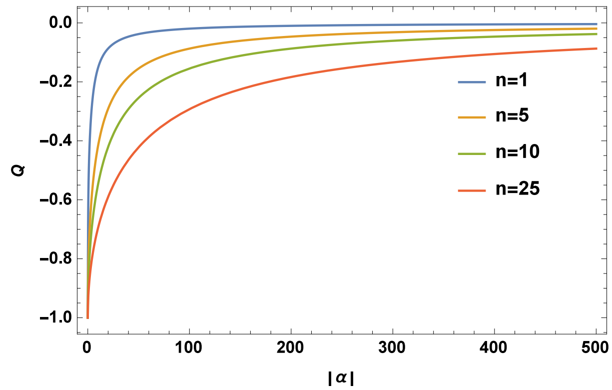

Figure 4.8: Mandel Q-parameter of photon added coherent state as a function of $|\alpha|$. we get different curves for different values of n.

Excitations of coherent state superpositions

Expectation value $\langle \alpha_{\pm}, n|aa^{\dagger}|\alpha_{\pm}, n\rangle$ is given as,

$$\langle \alpha_{\pm}, n|aa^{\dagger}|\alpha_{\pm}, n\rangle = \frac{(n+1)!L^{\pm}_{n+1}(|\alpha|^2)}{2\cosh|\alpha|^2}, \tag{4.2.19}$$

and

$$\langle \alpha_{\pm}, n|a^2a^{\dagger^2}|\alpha_{\pm}, n\rangle = \frac{(n+2)!L^{\pm}_{n+2}(|\alpha|^2)}{2\cosh|\alpha|^2}, \tag{4.2.20}$$

Putting Eq.(4.2.19) and Eq.(4.2.20) in Eq.(4.1.4) we get,

$$Q(\alpha_{\pm}, n) = \frac{[(n+2)(n+1)L^{\pm}_{n+2} - 4(n+1)L^{\pm}_{n+1} + 2L^{\pm}_{n})][L^{\pm}_{n}] - [(n+1)L^{\pm}_{n+1} - L^{\pm}_{n}]^2}{[(n+1)!L^{\pm}_{n+1} - L^{\pm}_{n}]L^{\pm}_{n}}. \tag{4.2.21}$$

Graph on the left side represent Mandel Q-parameter for excited even coherent state and graph on the right side is for excited odd coherent states. These graph shows that with increasing number of added photon, the negativity of states increases.

4.3 Results and Discussion

In previous section, we have calculated the Wigner function and Mandel's Q-parameter analytically and plotted graphically, for three kinds of field states, constructed in

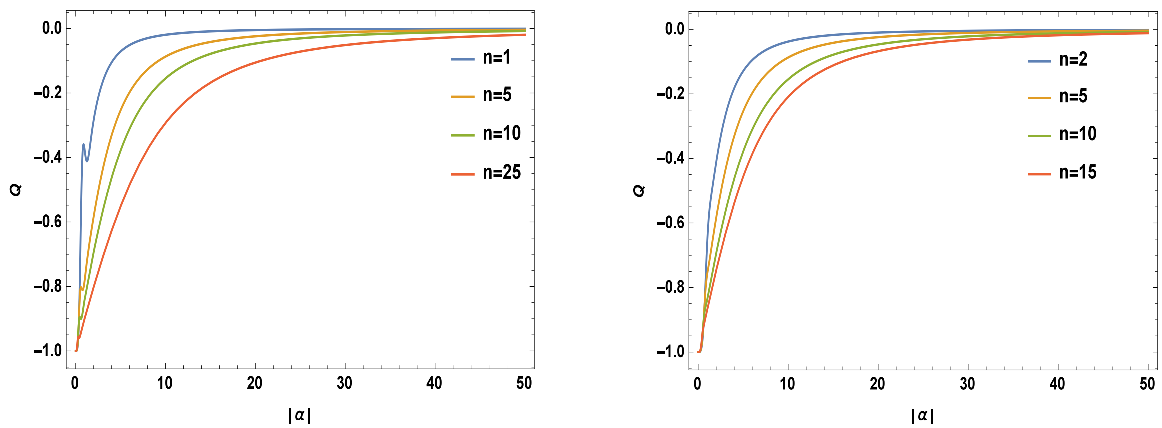

Figure 4.9: Mandel Q parameter as a function of $|\alpha|$ for excitations of coherent state superpositions. Different curves are obtain for different values of n.

the previous chapter. The analytical expressions for Wigner function are given by Eqs. and those for Mandel Q-parameter are given by Eqs. .

The Wigner for even coherent state, odd coherent states and Yorke-Stoler coherent states are given in Figs. (4.1,4.2,4.3), respectively. These plots plots indicate similar behavior such that, there is interference in phase-space in between the two Gaussian peaks. Due to this quantum interference, Wigner function shows negativity in phase-space which is a clear indication of nonclassicality. The Wigner function of excited coherent states has plotted in Fig. (4.4). It can be seen from this plot that the behavior of Wigner function of excited coherent states is different that of Wigner function of superposition of coherent states. However, clearly the negativity of the Wigner function can be seen in phase-space for this case as well. Finally, the Wigner function for excited superposition has been presented in Figs. (4.5,4.6) which again exhibit the interference and negativity in phase-space. These facts lead us to conclude that all of our constructed field states are nonclassical.

Moreover, the Mandel's Q-parameter for superposition coherent states has been plotted in Fig. (4.7). In this case Mandel Q-parameter does not portray the correct picture as far as nonclassicality is concerned because super-posionian distribution for even coherent states and Poisson distribution for Yurke-Stoler do not suggest nonclassicality which is in contradiction to our earlier results suggested by Wigner function. Futhermore, the Mandel Q-parameter for excitation of coherent states, is given in Fig(4.8), which shows that increasing the number of added photons, the nonclassicality of the states increases. Similar behavior can be seen in the case of excitations of coherent state superpositions, given in in Fig(4.9), for different value of n. Hence, we conclude that nonclassicality of the states increases by increasing the number of added photons.

Chapter 5

Summary and Conclusions

In this thesis we have studied, theoretically, various approaches to construct non-classical field states using very well known continuously parameterized field states, namely, coherent states. Such non-classical states have numerous applications in various quantum technologies such as quantum computing, quantum information processing and quantum metrology. For example, the implementation of these technologies does require nonclassical correlations, such as entanglement, as a key resource, while, the generation of entanglement does require nonclassicality of any subsystem as a pre-requisite. Moreover, continuous parameterization of field states carries several advantages, for instance, in the implementation of quantum logic gates. However, it is very well known that coherent states themselves are classical-like. Therefore, despite of their a lot of other usefulness in various areas, coherent states cannot be used for these practical applications.

Our first approach in this regard is to construct a linear superposition of two coherent states which are equal in amplitude but separated in phase by 180^o. Such superpositions of (single- degree-of-freedom) coherent states are often called Schrödinger-cat states. Choosing various values of the relative phase, we obtain various kinds of superposition states. For example, we get even coherent states, odd coherent states and Yurke-Stoler coherent states when we choose relative phase equal to 0, π and $\pi/2$, respectively. Another, approach is to excite a coherent state by discrete number of photons. These states are know as excited coherent states or photon-added coherent states. These states are obtained, mathematically, by repeated action of creation operator on continously parameterized coherent states. Finally, we present excited Schrödinger-cat states which are obtained by adding photons to the superpositions of coherent states.

For these constructed coherent states, we analyze the signatures of nonclassicality by means of various indicators such as, sub- Poissonian photon counting statistics and negativity of the Wigner function. We calculate the Wigner function and Mandel's Q-parameter for our constructed field states analytically and visualize them graphically. The plots of Wigner function show that all our constructed coherent states exhibit some negativity in phase space which is an indicator nonclassicality. Moreover, from Mandel Q parameter, we have shown that the non-classicality of the constructed states increases as the number of added photons increases.

Appendices

Appendix A

Equivalence of Wigner function

$$W(\beta) = \frac{2}{\pi^2}\exp(2|\beta|^2)\int \exp\left(\lambda^*\beta - \lambda\beta^*\right) C_w(\lambda) d^2\lambda, \tag{A.0.1}$$

where characteristic function $C_w(\lambda)$ is given as,

$$C_w(\lambda) = Tr(\hat{\rho}\exp\left(\lambda a^\dagger - \lambda^* a\right).$$

Let us consider that

$$\lambda = \frac{1}{\sqrt{2}}(x+iy), \quad \hat{a} = \frac{1}{\sqrt{2}}(\hat{p}+i\hat{q}), \quad d^2\lambda = dxdy. \tag{A.0.2}$$

By using these values our integral becomes,

$$W = \frac{1}{\pi^2}\int \exp(-i(yq - xp) - ixy/2) Tr(\hat{\rho}\exp(-ix\hat{p})\exp(-iy\hat{q}))\mathrm{dxdy} \tag{A.0.3}$$

In continuous basis, we take "Tr " as,

$$W = \frac{1}{\pi^2}\int \exp(-i(yq-xp)-ixy/2)\langle q^{'}|\exp(-ix\hat{p})\exp(-iy\hat{q})\hat{\rho})|q^{'}\rangle \mathrm{dxdy}dq^{'} \tag{A.0.4}$$

After rearranging we grt,

$$W = \frac{1}{\pi^2}\int \exp(-i(yq-xp)-ixy/2)\langle q^{'}|\exp(-ix\hat{p}/2)\exp(-iy\hat{q})\hat{\rho})\exp(-ix\hat{p}/2)|q^{'}\rangle \mathrm{dxdy}dq^{'}, \tag{A.0.5}$$

where $\exp(-ix\hat{p}/2)$ is translational operator. As a result we get,

$$W = \frac{1}{\pi^2}\int \exp(-i(yq - xp) - ixy/2)\langle q^{'} + \frac{x}{2}|\exp(-iy\hat{q})\hat{\rho})|q^{'} - \frac{x}{2}\rangle \mathrm{dxdy}dq^{'}.$$

Now by the action of position operator $\exp(-ix\hat{q})$ we get,

$$W = \frac{1}{\pi^2}\int \exp(-i(yq - xp) - ixy/2)\exp(-ix(q^{'} + \frac{x}{2})y)\langle q^{'} + \frac{x}{2}|\hat{\rho}|q^{'} - \frac{x}{2}\rangle \mathrm{dxdy}dq^{'}.$$

After simplification we get

$$W = \frac{1}{\pi^2}\int \exp(ixp)\exp(-iy(q - q^{'})\langle q^{'} + \frac{x}{2}|\hat{\rho}|q^{'} - \frac{x}{2}\rangle \mathrm{dxdy}dq^{'}.$$

As $\frac{1}{\pi^2}\int \exp(-iy(q - q^{'})dy = \delta(q - q^{'})$

$$W = \frac{1}{\pi}\int \exp(ixp)\delta(q - q^{'})\langle q^{'} + \frac{x}{2}|\hat{\rho}|q^{'} - \frac{x}{2}\rangle \mathrm{dx}dq^{'}.$$

$$W = \frac{1}{\pi}\int \exp(ixp)\langle q^{'} + \frac{x}{2}|\hat{\rho}|q^{'} - \frac{x}{2}\rangle \mathrm{dx}. \tag{A.0.6}$$

Appendix B

Derivation of Wigner function

Calculation for Wigner function of superpositions of coherent states

Wigner function for even coherent states can be obtain by using this $\hat{\rho}$ characteristic function, given above in Eq.(2.3.7), becomes,

$$\begin{aligned} C_w(\lambda) &= Tr(\hat{\rho}D(\lambda)), \\ &= \frac{(\langle\alpha| + \langle-\alpha|)D(\lambda)(|\alpha\rangle + |-\alpha\rangle)}{2(1+\exp(-2|\alpha|^2))}. \end{aligned} \tag{B.0.1}$$

When displacement operator act on coherent state,,

$$D(\lambda)|\alpha\rangle = D(\lambda)D(\alpha)|0\rangle.$$

Using the value of Displacement operator, given above in Eq.(2.3.5), we get,

$$D(\lambda)|\alpha\rangle = \exp(\lambda a^\dagger - \lambda^*\hat{a})\exp\left(\alpha a^\dagger - \alpha^*\hat{a}\right)|0\rangle.$$

For simplification let us consider,

$$A = \lambda a^\dagger - \lambda^*\hat{a} \quad \text{and} \quad B = \alpha a^\dagger - \alpha^*\hat{a}. \tag{B.0.2}$$

With these substitutions our displacement operator becomes,

$$D(\lambda)|\alpha\rangle = \exp\left(A + B + \frac{[A,B]}{2}\right)|0\rangle, \tag{B.0.3}$$

where commutation relation is given as,

$$\frac{1}{2}[A,B] = \frac{1}{2}(AB - BA). \tag{B.0.4}$$

Putting value of A and B, given in Eq.(B.0.2), Eq.(B.0.4) become,

$$\frac{1}{2}[A, B] = \frac{1}{2}((\lambda a^\dagger - \lambda^* \hat{a})(\alpha a^\dagger - \alpha^* \hat{a}) - (\alpha a^\dagger - \alpha^* \hat{a})(\lambda a^\dagger - \lambda^* \hat{a})). \tag{B.0.5}$$

After simplification we get,

$$\begin{aligned} \frac{[A, B]}{2} &= \frac{1}{2}(\alpha^* \lambda - \alpha \lambda^*), \\ &= Im(\lambda \alpha^*). \end{aligned} \tag{B.0.6}$$

By substituting Eq.(B.0.6) in Eq.(B.0.3), we get,

$$D(\lambda)|\alpha\rangle = \exp(a^\dagger \lambda - \lambda^* \hat{a} + a^\dagger \alpha - \alpha^* \hat{a}) \exp(Im(\lambda \alpha^*))|0\rangle.$$

Combining like terms of creation and annihilation operator,

$$D(\lambda)|\alpha\rangle = \exp((\lambda + \alpha)a^\dagger - (\lambda^* + \alpha^*)\hat{a}) \exp(Im(\lambda \alpha^*))|0\rangle.$$

When this displacement operator act on vacuum state then we get,

$$D(\lambda)|\alpha\rangle = \exp(Im(\lambda \alpha^*))|\lambda + \alpha\rangle. \tag{B.0.7}$$

Similarly for coherent state $|-\alpha\rangle$,

$$D(\lambda)|-\alpha\rangle = \exp(-Im(\lambda \alpha^*))|\lambda - \alpha\rangle. \tag{B.0.8}$$

Using Eq.(B.0.7) and Eq.(B.0.8) in Eq.(B.0.1) we get,

$$C_w(\lambda) = \frac{(\langle\alpha| + \langle-\alpha|)(\exp(Im(\lambda\alpha^*)|\lambda + \alpha\rangle + \exp(-Im(\lambda\alpha^*))|\lambda - \alpha\rangle))}{2(1 + \exp(-2|\alpha|^2))}.$$

After multiplication we get,

$$\begin{aligned} C_w(\lambda) = \frac{1}{2(1 + \exp(-2|\alpha|^2))}&[\exp(Im(\lambda\alpha^*))\langle\alpha|\lambda + \alpha\rangle + \exp(-Im\lambda\alpha^*) \\ \langle\alpha|\lambda - \alpha\rangle + \exp(Im\lambda\alpha^*)&\langle-\alpha|\lambda + \alpha\rangle + \exp(-Im\lambda\alpha^*)\langle-\alpha|\lambda - \alpha\rangle]. \end{aligned} \tag{B.0.9}$$

Now we calculate the inner product $\langle\alpha|\lambda+\alpha\rangle, \langle\alpha|\lambda-\alpha\rangle, \langle-\alpha|\lambda+\alpha\rangle$ and $\langle-\alpha|\lambda-\alpha\rangle$ separately. Firstly,

$$\begin{aligned} \exp(Im(\lambda\alpha^*))\langle\alpha|\lambda + \alpha\rangle = \exp\frac{1}{2}(\alpha^*\lambda - \alpha\lambda^*) \sum_{n=0}^{\infty} \frac{(\alpha^*)^n}{\sqrt{n!}} \exp\left(\frac{-|\alpha|^2}{2}\right) \\ \langle n|n\rangle \sum_{n=0}^{\infty} \frac{(\lambda + \alpha)^n}{\sqrt{n!}} \exp\frac{-|\lambda + \alpha|^2}{2}. \end{aligned}$$

After solving we get,

$$\exp(Im(\lambda\alpha^*))\langle\alpha|\lambda+\alpha\rangle = \exp(\alpha^*\lambda-\lambda^*\alpha+|\alpha|^2)\exp(-|\alpha|^2)\exp\left(\frac{-|\lambda|^2}{2}\right), \quad (B.0.10)$$

Second inner product is calculated as,

$$\begin{aligned}\exp(-Im(\lambda\alpha^*))\langle\alpha|\lambda-\alpha\rangle = \exp\frac{1}{2}(-\alpha^*\lambda+\alpha\lambda^*)\sum_{n=0}^{\infty}\frac{(\alpha^*)^n}{\sqrt{n!}}\exp\left(\frac{-|\alpha|^2}{2}\right)\\ \langle n|n\rangle\sum_{n=0}^{\infty}\frac{(\lambda-\alpha)^n}{\sqrt{n!}}\exp\left(\frac{-|\lambda-\alpha|^2}{2}\right).\end{aligned}$$

After simplification we get,

$$\exp(-Im(\lambda\alpha^*))\langle\alpha|\lambda-\alpha\rangle = \exp(\alpha^*\lambda+\lambda^*\alpha-|\alpha|^2)\exp(-|\alpha|^2)\exp\left(\frac{-|\lambda|^2}{2}\right). \quad (B.0.11)$$

Inner product of third term is obtain as,

$$\begin{aligned}\exp(Im(\lambda\alpha^*))\langle-\alpha|\lambda+\alpha\rangle = \exp\frac{1}{2}(\alpha^*\lambda-\alpha\lambda^*)\sum_{n=0}^{\infty}\frac{(-\alpha^*)^n}{\sqrt{n!}}\exp\left(\frac{-|\alpha|^2}{2}\right)\\ \langle n|n\rangle\sum_{n=0}^{\infty}\frac{(\lambda+\alpha)^n}{\sqrt{n!}}\exp\left(\frac{-|\lambda+\alpha|^2}{2}\right),\end{aligned}$$

After simplification we get,

$$\exp(-Im(\lambda\alpha^*))\langle-\alpha|\lambda+\alpha\rangle = \exp(-\alpha^*\lambda-\lambda^*\alpha-|\alpha|^2)\exp(-|\alpha|^2)\exp\left(\frac{-|\lambda|^2}{2}\right). \quad (B.0.12)$$

Fourth term is given as,

$$\begin{aligned}\exp(-Im(\lambda\alpha^*))\langle-\alpha|\lambda-\alpha\rangle &= \exp\tfrac{1}{2}(\alpha^*\lambda-\alpha\lambda^*)\sum_{n=0}^{\infty}\tfrac{(-\alpha^*)^n}{\sqrt{n!}}\exp\left(\tfrac{-|\alpha|^2}{2}\right)\\ &\quad\langle n|n\rangle\sum_{n=0}^{\infty}\tfrac{(\lambda-\alpha)^n}{\sqrt{n!}}\exp\left(\tfrac{-|\lambda-\alpha|^2}{2}\right),\end{aligned}$$

after simplification we get,

$$\exp(-Im(\lambda\alpha^*))\langle-\alpha|\lambda-\alpha\rangle = \exp(-\alpha^*\lambda+\lambda^*\alpha+|\alpha|^2)\exp(-|\alpha|^2)\exp\left(\frac{-|\lambda|^2}{2}\right). \quad (B.0.13)$$

By using Eq.(B.0.10), (B.0.11), (B.0.12) and (B.0.13) in Eq.(B.0.9) we get,

$$C_w(\lambda) = \frac{\exp(-|\alpha|^2)\exp\left(-\frac{1}{2}|\lambda|^2\right)}{2(1+\exp(-2|\alpha|^2)})[\exp(\alpha^*\lambda - \lambda^*\alpha + |\alpha|^2) + \exp(\alpha^*\lambda + \lambda^*\alpha - |\alpha|^2)$$
$$+ \exp(-\alpha^*\lambda - \lambda^*\alpha - |\alpha|^2) + \exp(-\alpha^*\lambda + \lambda^*\alpha + |\alpha|^2)],$$

Combining the exponential terms as,

$$C_w(\lambda) = \frac{\exp\left(-\frac{1}{2}|\lambda|^2\right)\exp(-|\alpha|^2)}{2(1+\exp(-2|\alpha|^2))}[\exp(|\alpha|^2)(\exp(\alpha^*\lambda - \lambda^*\alpha + \exp(\alpha^*\lambda) + \lambda^*\alpha))$$
$$+ \exp(-|\alpha|^2)(\exp(-\alpha^*\lambda - \lambda^*\alpha) + \exp(\alpha^*\lambda + \lambda^*\alpha)).$$

By putting value of $C_w(\lambda)$ Wigner function becomes, after simplification we get,

$$W(\beta) = \frac{1}{2\pi^2(1+\exp(-2|\alpha|^2)}[\int \exp(\lambda^*(\beta-\alpha) - \lambda(\beta^*-\alpha^*) - \frac{|\lambda|^2}{2})d^2\lambda$$
$$+ \int \exp(\lambda^*(\beta+\alpha) - \lambda(\beta^*+\alpha^*) - \frac{|\lambda|^2}{2})d^2\lambda$$
$$+ \exp(-2|\alpha|^2)[\int (\exp(\lambda^*(\beta+\alpha) - \lambda(\beta^*-\alpha^*) - \frac{|\lambda|^2}{2})d^2\lambda$$
$$+ \int \exp(\lambda^*(\beta-\alpha) - \lambda(\beta^*+\alpha^*) - \frac{|\lambda|^2}{2})d^2\lambda]].$$

We evaluate these integral by using identity given in Eq.(2.3.9) and after simplifying we get,

$$\begin{aligned} W(\beta) = \frac{1}{\pi(1+\exp(-2|\alpha|^2)}&[\exp(-2|\beta-\alpha|^2) + \exp(-2|\beta+\alpha|^2) \\ &+ 2\exp(-2|\beta|^2)\cos(2i(\alpha^*\beta - \beta^*\alpha))]. \end{aligned} \tag{B.0.14}$$

Which is Wigner function of even coherent states. The first two terms are Gaussian centered at α and the last term is due to interference between $|\alpha\rangle$ and $|-\alpha\rangle$.
As β is complex number so we put $\beta = p - iq$ in each term separately,

$$\begin{aligned} \exp(-2|\beta-\alpha|^2) &= \exp(-2(|\alpha|^2 + p^2 + q^2 - \alpha(p-iq) - \alpha^*(p-iq)) \\ &= \exp(-2(p-\alpha)^2)\exp(-2q^2). \end{aligned} \tag{B.0.15}$$

Second term become,

$$\begin{aligned} \exp(-2|\beta+\alpha|^2) &= \exp(-2(|\alpha|^2 + p^2 + q^2 + \alpha(p-iq) + \alpha^*(p-iq)) \\ &= \exp(-2(p+\alpha)^2)\exp(-2q^2). \end{aligned} \tag{B.0.16}$$

Our third term is,

$$\begin{aligned}\cos(2i(\alpha^*\beta-\beta^*\alpha)) &= \cos(2i(\alpha^*(p+iq)-(p-iq)\alpha)\\ &= \cos(-4q\alpha)).\end{aligned} \tag{B.0.17}$$

Putting Eq.(B.0.15), (B.0.16) and (B.0.17) in Eq.(B.0.14) we get,

$$\begin{aligned}W(p,q) =& \frac{1}{\pi(1+\exp(-2|\alpha|^2)}[\exp(-2(p-\alpha)^2-2q^2)+\exp(-2(p+\alpha)^2-2q^2)\\ &+2\exp(-2(p^2+q^2)\cos(4q\alpha)].\end{aligned}$$

For excitation of coherent states

$$W(\beta)=\frac{2}{\pi^2}\exp(2|\beta|^2)\int\exp\left(\lambda^*\beta-\lambda\beta^*\right)\langle-\lambda|\alpha,n\rangle\langle\alpha,n|\lambda\rangle d^2\lambda. \tag{B.0.18}$$

We first calculate $\langle\alpha,n|\lambda\rangle$ and $\langle-\lambda|\alpha,n\rangle$ as,

$$\begin{aligned}\langle\alpha,n|\lambda\rangle &= \frac{\langle\alpha|a^n|\lambda\rangle}{[n!L_n(-|\alpha|^2)]^{\frac{1}{2}}}\\ &= \frac{\lambda^n}{[n!L_n(-|\alpha|^2)]^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(|\alpha|^2+|\lambda|^2)+\alpha^*\lambda\right),\end{aligned} \tag{B.0.19}$$

and

$$\begin{aligned}\langle-\lambda|\alpha,n\rangle &= \frac{\langle-\lambda|a^{\dagger^n}|\alpha\rangle}{[n!L_n(-|\alpha|^2)]^{\frac{1}{2}}}\\ &= \frac{(-\lambda^*)^n}{[n!L_n(-|\alpha|^2)]^{\frac{1}{2}}}\exp\left(-\frac{1}{2}(|\alpha|^2+|\lambda|^2)-\lambda^*\alpha\right).\end{aligned} \tag{B.0.20}$$

By substituting Eq.(B.0.19) and Eq.(B.0.20) in Eq.(B.0.18) we get,

$$W(\beta)=\frac{2\exp(2|\beta|^2)}{\pi^2[n!L_n(-|\alpha|^2)]}\int(-\lambda\lambda^*)^n\exp\left(-|\alpha|^2-|\lambda|^2-\lambda^*\alpha+\alpha^*\lambda\right)\exp\left(2(\lambda^*\beta-\lambda\beta^*)\right)d^2\lambda.$$

$$W(z)=\frac{2\exp(2|z|^2-|\alpha|^2)}{\pi^2[m!L_m(-|\alpha|^2)]}\int(-\beta\beta^*)^m\exp[-|\beta|^2+\beta^*(2z-\alpha)-\beta(2z^*-\alpha^*)]d^2\beta \tag{B.0.21}$$

Now rearrange the terms and let $2\beta-\alpha=\xi$,

$$W(\beta)=\frac{2\exp(2|\beta|^2-|\alpha|^2)}{\pi^2[n!L_n(-|\alpha|^2)]}\int(-\lambda\lambda^*)^n\exp\left(-|\lambda|^2+\lambda^*\xi-\lambda\xi^*\right)d^2\lambda.$$

After simplification

$$W(z) = \frac{2\exp(2|z|^2 - |\alpha|^2)}{\pi^2[m!L_m(-|\alpha|^2)]}\frac{\partial^{2m}}{\partial(\xi^*)^m\partial\xi^m}\int \exp[-|\beta|^2 + \beta^*\xi - \beta\xi^*]d^2\beta \quad \text{(B.0.22)}$$

we will also make use of the identity

$$\exp[-|2z|^2] = \frac{1}{2\pi}\int \exp[-\frac{1}{2}|\alpha|^2]\exp[z\alpha^* - z^*\alpha]d^2\alpha \quad \text{(B.0.23)}$$

$$W(z) = \frac{2\exp(2|z|^2 - |\alpha|^2)}{\pi[m!L_m(-|\alpha|^2)]}\frac{\partial^{2m}}{\partial(\xi^*)^m\partial\xi^m}\exp[-|\xi|^2] \quad \text{(B.0.24)}$$

Now we use following identity

$$\frac{\partial^{2m}}{\partial(z^*)^m\partial z^m}\exp(-\beta z^* z) = (-\beta)^m\exp[-\beta|z|^2]L_m(\beta|z|^2) \quad \text{(B.0.25)}$$

In our case $\beta = 1$ and $z = \xi$

$$\frac{\partial^{2m}}{\partial(\xi^*)^m\partial\xi^m}\exp(-\xi^*\xi) = (-1)^m\exp[-|\xi|^2]L_m(|\xi|^2) \quad \text{(B.0.26)}$$

$W(\beta)$ becomes,

$$= \frac{2\exp(2|\beta|^2 - |\alpha|^2)}{\pi[n!L_n(-|\alpha|^2)]}(-1)^n n!\exp\left(-|\xi|^2\right)L_n(|\xi|^2).$$

Putting the value of ξ we get,

$$= \frac{2(-1)^n\exp\left(2|\beta|^2 - |\alpha|^2\right)}{\pi[n!L_n(-|\alpha|^2)]}m!\exp\left(-|2\beta - \alpha|^2\right)L_m(|2\beta - \alpha|^2).$$

After some calculations we get, $|\alpha, n\rangle$ is,

$$W(z) = \frac{2(-1)^m L_m(|2z - \alpha|^2)}{\pi[L_m(-|\alpha|^2)]}\exp[-(2z - \alpha)(2z^* - \alpha^*)]\exp(2|z|^2 - |\alpha|^2) \quad \text{(B.0.27)}$$

$$W(z) = \frac{2(-1)^m L_m(|2z - \alpha|^2)}{\pi[L_m(-|\alpha|^2)]}\exp(-(4|z|^2 - 2\alpha^* z - 2z^*\alpha + |\alpha|^2) + 2|z|^2 - |\alpha|^2) \quad \text{(B.0.28)}$$

$$W(z) = \frac{2(-1)^m L_m(|2z - \alpha|^2)}{\pi[L_m(-|\alpha|^2)]}\exp(-2|z|^2 + 2\alpha^* z + 2z^*\alpha - 2|\alpha|^2) \quad \text{(B.0.29)}$$

the Wigner function for the state,

$$W(\beta) = \frac{2(-1)^n L_n(|2\beta - \alpha|^2)}{\pi[L_n(-|\alpha|^2)]}\exp(-2|\beta - \alpha|^2), \quad \text{(B.0.30)}$$

Appendix C

Derivation of Mandel Q-Parameter

Solution of Mandel Q-parameter for even/odd coherent states

To find Mandel Q-parameter for even/odd coherent states we first calculate,

$$\langle a^{\dagger}\hat{a}\rangle = \langle \alpha_{\pm}|a^{\dagger}\hat{a}|\alpha_{\pm}\rangle, \tag{C.0.1}$$

put value of $|\alpha_{\pm}\rangle$ as given in Eq.(3.1.3) and (3.1.10) ,

$$\langle a^{\dagger}\hat{a}\rangle = N^2 \left[(\langle\alpha| \pm \langle-\alpha|)a^{\dagger}\hat{a}(|\alpha\rangle \pm |-\alpha\rangle)\right],$$

by simplifying,

$$\begin{aligned}
\langle a^{\dagger}\hat{a}\rangle &= |N|^2[\langle\alpha|a^{\dagger}\hat{a}|\alpha\rangle \pm \langle\alpha|a^{\dagger}\hat{a}|-\alpha\rangle \pm \langle-\alpha|a^{\dagger}\hat{a}|\alpha\rangle \pm \langle-\alpha|a^{-\dagger}\hat{a}|-\alpha\rangle], \\
&= |N_e|^2[\alpha^*\alpha\langle\alpha|\alpha\rangle - \alpha^*\alpha\langle\alpha|-\alpha\rangle - \alpha^*\alpha\langle-\alpha|\alpha\rangle + \alpha^*\alpha\langle-\alpha|-\alpha\rangle],
\end{aligned}$$

Where,

$$\langle\alpha|\alpha\rangle = \langle-\alpha|-\alpha\rangle = 1, \tag{C.0.2}$$

$$\langle-\alpha|\alpha\rangle = \langle\alpha|-\alpha\rangle = \exp(-2|\alpha|^2). \tag{C.0.3}$$

Using these relations we get,

$$\begin{aligned}
\langle a^{\dagger}\hat{a}\rangle &= |N_e|^2[\alpha^*\alpha - \alpha^*\alpha\exp[-2|\alpha|^2] - \alpha^*\alpha\exp[-2|\alpha|^2] + \alpha^*\alpha], \\
&= |N_e|^2[2\alpha^*\alpha - 2\alpha^*\alpha\exp(-2|\alpha|^2)],
\end{aligned}$$

Putting value of normalization constant we get,

$$\langle a^{\dagger}\hat{a}\rangle = \frac{2\alpha^{*}\alpha - 2\alpha^{*}\alpha\exp(-2|\alpha|^{2})}{2 \pm 2\exp(-2|\alpha|^{2})},$$

Taking like term common we get,

$$\langle a^{\dagger}\hat{a}\rangle = \frac{|\alpha|^{2}(1 \pm \exp(-2|\alpha|^{2}))}{1 \pm \exp(-2|\alpha|^{2})}. \tag{C.0.4}$$

Now,

$$\langle a^{\dagger^{2}}\hat{a}^{2}\rangle = \langle\alpha_{\pm}|a^{\dagger^{2}}\hat{a}^{2}|\alpha_{\pm}\rangle, \tag{C.0.5}$$

Putting value of even coherent state $|\alpha_{\pm}\rangle$ as,

$$\begin{aligned}\langle a^{\dagger^{2}}\hat{a}^{2}\rangle &= |N|^{2}[(\langle\alpha| \pm \langle-\alpha|)a^{\dagger^{2}}\hat{a}^{2}(|\alpha\rangle \pm |-\alpha\rangle)] \\ &= |N|^{2}[(\alpha^{*})^{2}\alpha^{2} \pm (\alpha^{*})^{2}\alpha^{2}\langle\alpha|-\alpha\rangle \pm (\alpha^{*})^{2}\alpha^{2}\langle-\alpha|\alpha\rangle \pm (\alpha^{*})^{2}\alpha^{2}]\end{aligned} \tag{C.0.6}$$

By using Eq.(C.0.2) and Eq.(C.0.3) we get,

$$\begin{aligned}\langle a^{\dagger^{2}}\hat{a}^{2}\rangle &= |N|^{2}[(\alpha^{*})^{2}\alpha^{2} + (\alpha^{*})^{2}\alpha^{2}\exp(-2|\alpha|^{2}) + (\alpha^{*})^{2}\alpha^{2}\exp(-2|\alpha|^{2}) + (\alpha^{*})^{2}\alpha^{2}], \\ &= |N|^{2}[2(\alpha^{*})^{2}\alpha^{2} + 2(\alpha^{*})^{2}\alpha^{2}\exp(-2|\alpha|^{2})],\end{aligned}$$

Using value of normalization constant,

$$\begin{aligned}\langle a^{\dagger^{2}}\hat{a}^{2}\rangle &= \frac{2|\alpha|^{4} \pm 2|\alpha|^{4}\exp(-2|\alpha|^{2})}{2 \pm 2\exp(-2|\alpha|^{2})}, \\ \langle a^{\dagger^{2}}\hat{a}^{2}\rangle &= |\alpha|^{4}.\end{aligned} \tag{C.0.7}$$

Now by putting equation (C.0.4) and (C.0.7) in (4.1.4) we get,

$$Q = [|\alpha|^{4} + \frac{(|\alpha|^{2})(1 \pm \exp(-2|\alpha|^{2}))}{1 \pm \exp(-2|\alpha|^{2})} - \frac{(|\alpha|^{4})(1 \pm \exp(-2|\alpha|^{2}))^{2}}{(1 \pm \exp(-2|\alpha|^{2}))^{2}}][\frac{(1 \pm \exp(-2|\alpha|^{2}))}{(1 \pm \exp(-2|\alpha|^{2}))|\alpha|^{2}}] - 1,$$

by solving we get,

$$Q = \frac{\pm 4|\alpha|^{2}\exp(-2|\alpha|^{2})}{(1 - \exp(-4|\alpha|^{2}))}. \tag{C.0.8}$$

For Yurke-Stoler states

$$\langle a^{\dagger}\hat{a}\rangle = \langle\psi_{ys}|a^{\dagger}\hat{a}|\psi_{ys}\rangle, \tag{C.0.9}$$

by putting value of $|\psi_{ys}\rangle$,

$$\begin{aligned}\langle a^\dagger \hat{a}\rangle &= |N|^2[(\langle\alpha| - i\langle -\alpha|)a^\dagger \hat{a}(|\alpha\rangle + i|-\alpha\rangle)],\\ \hat{a}|-\alpha\rangle], & \\ &= |N|^2[\alpha^*\alpha\langle\alpha|\alpha\rangle - i\alpha^*\alpha\langle\alpha|-\alpha\rangle + i\alpha^*\alpha\langle -\alpha|\alpha\rangle + \alpha^*\alpha\langle -\alpha|-\alpha\rangle],\end{aligned}$$

by using eq.(C.0.2) and eq.(C.0.3) we get,

$$= |N|^2[\alpha^*\alpha - i\alpha^*\alpha\exp(-2|\alpha|^2) + i\alpha^*\alpha\exp(-2|\alpha|^2) + \alpha^*\alpha],$$

by using value of normalization constant and after solving we get,

$$\langle a^\dagger \hat{a}\rangle = N^2[2|\alpha|^2]$$

$$\langle a^\dagger \hat{a}\rangle = |\alpha|^2. \tag{C.0.10}$$

Now,

$$\begin{aligned}\langle a^{\dagger^2}\hat{a}^2\rangle &= \langle\psi_{ys}|a^{\dagger^2}\hat{a}^2|\psi_{ys}\rangle,\\ &= |N|^2[a^{\dagger^2}\alpha^2\langle\alpha|\alpha\rangle + ia^{\dagger^2}\alpha^2\langle\alpha|-\alpha\rangle - ia^{\dagger^2}\alpha^2\langle -\alpha|\alpha\rangle + a^{\dagger^2}\alpha^2\langle -\alpha|-\alpha\rangle],\end{aligned}$$

By using Eq.(C.0.2) and Eq.(C.0.3) we get,

$$\langle a^{\dagger^2}\hat{a}^2\rangle = |N|^2[(\alpha^*)^2\alpha^2 + i(\alpha^*)^2\alpha^2\exp[-2|\alpha|^2] - i(\alpha^*)^2\alpha^2\exp(-2|\alpha|^2) + (\alpha^*)^2\alpha^2],$$

after simplifications we get,

$$\langle(a^{\dagger^2}\hat{a}^2\rangle = |\alpha|^4, \tag{C.0.11}$$

by putting Eq.(C.0.10) and Eq.(C.0.11) in Eq.(4.1.4) we get,

$$Q = \frac{|\alpha|^4 + |\alpha|^2 - |\alpha|^4}{|\alpha|^2} - 1.$$

After simplifying we get,

$$Q = 0. \tag{C.0.12}$$

For excitations of coherent states

Mandel Q parameter is define as in equation (4.1.3) where,

$$Q(\alpha, m) = \frac{\langle(a\dagger a)^2\rangle - \langle a^\dagger a\rangle^2}{\langle a^\dagger a\rangle} - 1 \tag{C.0.13}$$

$$\langle a^\dagger a\rangle = \langle aa^\dagger\rangle - 1.$$

By using photon added coherent state as given in equation (3.2.3) we get,

$$\langle \alpha, n|aa^\dagger|\alpha, n\rangle = \frac{\langle\alpha|a^n(aa^\dagger)a^{\dagger^n}|\alpha\rangle}{n!L_n(-|\alpha|^2)} - 1. \tag{C.0.14}$$

As we know that,

$$\begin{aligned}\langle \alpha, n|a^m a^{\dagger^m}|\alpha, n\rangle &= \frac{\langle\alpha|a^{m+n}a^{\dagger^{(m+n)}}|\alpha\rangle}{n!L_n(-|\alpha|^2)}, \\ &= \frac{(n+m)!L_{m+n}(-|\alpha|^2)}{n!L_n(-|\alpha|^2)},\end{aligned} \tag{C.0.15}$$

with $m = 1$ we get,

$$\langle \alpha, m|aa^\dagger|\alpha, m\rangle = \langle\alpha|a^{m+1}a\ {\dagger}^{m+1}\ |\alpha\rangle \tag{C.0.16}$$

$$\langle \alpha, n|aa^\dagger|\alpha, n\rangle = \frac{(n+1)!L_{n+1}(-|\alpha|^2)}{n!L_n(-|\alpha|^2)}, \tag{C.0.17}$$

by putting this value in Eq.(C.0.14) we get,

$$\begin{aligned}\langle a^\dagger a\rangle &= \frac{(n+1)!L_{n+1}(-|\alpha|^2)}{n!L_n(-|\alpha|^2)} - 1, \\ &= \frac{(n+1)L_{n+1}(-|\alpha|^2)}{L_n(-|\alpha|^2)} - 1.\end{aligned} \tag{C.0.18}$$

Now expectation value of square of number operator,

$$\begin{aligned}\langle(a^\dagger a)^2\rangle &= \langle(aa^\dagger - 1)^2\rangle, \\ &= \langle(aa^\dagger)^2 + 1 - 2\langle aa^\dagger\rangle, \\ &= \langle a(aa^\dagger - 1)a^\dagger\rangle + 1 - 2\langle aa^\dagger\rangle, \\ &= \langle a^2 a^{\dagger^2}\rangle - 3\langle aa^\dagger\rangle + 1.\end{aligned} \tag{C.0.19}$$

To calculate this put $m = 2$ in Eq.(C.0.15),

$$\langle \alpha, n|a^2 a^{\dagger^2}|\alpha, n\rangle = \frac{(n+2)!L_{n+2}(-|\alpha|^2)}{n!L_n(-|\alpha|^2)}, \tag{C.0.20}$$

$$\langle \alpha, m|aa\ {\dagger}\ |\alpha, m\rangle = \frac{(m+1)!L_{m+1}(-|\alpha|^2)}{m!L_m(-|\alpha|^2)} \tag{C.0.21}$$

By using Eq.(C.0.18) and Eq.(C.0.20) in Eq.(C.0.13), we get, after simplification we get,

$$Q(\alpha,n)=\frac{[\frac{(n+2)!L_{n+2}}{n!L_n}-3\frac{(n+1)!L_{n+1}}{n!L_n}+1]-[\frac{(n+1)!L_{n+1}}{n!L_n}-1]^2}{[\frac{(n+1)!L_{n+1}}{n!L_n}-1]}-1.$$

Let's suppose $L_{n+m}(-|\alpha|^2)=L_{n+m}$ and after simplification we get,

$$Q(\alpha,m)=\frac{[(m+2)L_{m+2}-L_{m+1}](m+1)[L_m]-[(m+1)L_{m+1}]^2}{[(m+1)!L_{m+1}-L_m]L_m}-1. \quad \text{(C.0.22)}$$

For excitations of coherent state superpositions

For excited even/odd coherent states, expectation value of number operator is calculated as,

$$\langle\alpha_\pm,n|aa^\dagger|\alpha_\pm,n\rangle=\frac{2\cosh|\alpha|^2\langle\alpha_\pm|a^n(aa^\dagger){a^\dagger}^n|\alpha_\pm\rangle}{n!L_n^+(|\alpha|^2)}-1 \quad \text{(C.0.23)}$$

As we know that,

$$\begin{aligned}\langle\alpha_\pm,m|a^n a\,{\dagger}^n\,|\alpha_\pm,m\rangle &= \frac{\langle\alpha_\pm|a^{n+m}(a^\dagger)^{n+m}|\alpha_\pm\rangle}{m!L_m^\pm(|\alpha|^2)}\\ &= \frac{(m+n)!L_{n+m}^\pm(|\alpha|^2)}{n!L_n^+(|\alpha|^2)}.\end{aligned} \quad \text{(C.0.24)}$$

By putting $m=1$ we get,

$$\langle\alpha_\pm,n|aa^\dagger|\alpha_\pm,n\rangle=\frac{(n+1)!L_{n+1}^\pm(|\alpha|^2)}{2\cosh|\alpha|^2}.$$

By putting this value in equation (C.0.23) we get,

$$\begin{aligned}\langle a^\dagger a\rangle &= \frac{(n+1)!L_{n+1}^\pm(|\alpha|^2)2\cosh|\alpha|^2}{n!L_n^\pm(|\alpha|^2)2\cosh|\alpha|^2}-1,\\ &= \frac{(n+1)L_{n+1}^\pm(|\alpha|^2)-L_n^\pm(|\alpha|^2)}{L_n^\pm(|\alpha|^2)}.\end{aligned}$$

As we know that,

$$\langle(a^\dagger a)^2\rangle=\langle a^2{a^\dagger}^2\rangle-3\langle aa^\dagger\rangle+1. \quad \text{(C.0.25)}$$

To solve this Eq.(C.0.25), put $m=2$ in Eq.(C.0.24) and get,

$$\langle\alpha_\pm,n|a^2{a^\dagger}^2|\alpha_+,n\rangle=\frac{(n+2)!L_{n+2}^\pm(|\alpha|^2)}{2\cosh|\alpha|^2}, \quad \text{(C.0.26)}$$

$$\langle\alpha_\pm, n|aa^\dagger|\alpha_\pm, n\rangle = \frac{(n+1)!L^\pm_{n+1}(|\alpha|^2)}{2\cosh|\alpha|^2}. \qquad \text{(C.0.27)}$$

By using Eq.(C.0.25), (C.0.25), (C.0.26) and Eq.(4.1.3) we get,

$$Q(\alpha_\pm, m) = \frac{[\frac{(m+2)!L^\pm_{m+2}(|\alpha|^2)}{2\cosh|\alpha|^2} - 3\frac{(m+1)!L^\pm_{m+1}(|\alpha|^2)}{2\cosh|\alpha|^2} + 1] - [\frac{(m+1)!L^\pm_{m+1}(|\alpha|^2)}{m!L^\pm_m(|\alpha|^2)} - 1]^2}{[\frac{(m+1)!L^\pm_{m+1}(|\alpha|^2)}{m!L^\pm_m(|\alpha|^2)} - 1]} - 1.$$

Let $L^\pm_m(|\alpha|^2) = L^\pm_m$ only to simplify our calculations,

$$Q(\alpha_\pm, m) = \frac{[(m+2)(m+1)L^\pm_{m+2} - 3(m+1)L^+_{m+1}][L^\pm_m] + [L^\pm_m]^2 - [(m+1)L^\pm_{m+1} - L^\pm_m]^2}{[(m+1)!L^\pm_{m+1} - L^\pm_m]L^\pm_m} - 1,$$

$$Q(\alpha_\pm, m) = \frac{[(m+2)L^\pm_{m+2} - L^\pm_{m+1}](m+1)[L^\pm_m] - [(m+1)L^\pm_{m+1}]^2}{[(m+1)!L^\pm_{m+1} - L^\pm_m]L^\pm_m} - 1. \qquad \text{(C.0.28)}$$

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