Modeling the Viscoelastic Behavior of Engine Oil in the Crankshaft-Journal-Bearing of High Torque Low-Speed Diesel Engine

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A thesis submitted in partial fulfillment of the requirements for the degree of

MS Mechanical Engineering

Thesis Supervisor:

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DECLARATION

I certify that this research work titled *"Modeling the Viscoelastic Behavior of Engine Oil in the Crankshaft-Journal-Bearing of High Torque Low-Speed Diesel Engine"* is my own work. The work has not been presented elsewhere for assessment. The material that has been used from other sources it has been properly acknowledged / referred.

Signature of

Student Talha Zia

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LANGUAGE CORRECTNESS CERTIFICATE

This thesis has been read by an English expert and is free of typing, syntax, semantic, grammatical and spelling mistakes. Thesis is also according to the format given by the university.

Signature of Student

Talha Zia

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Signature of Supervisor

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"All praise belongs to Allah, Lord of all the worlds [1:2]"

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To my beloved Parents, Teachers and Siblings…

ABSTRACT

Journal bearing is used to carry the cyclic and impact loads in many applications such as internal-combustion engine's crankshaft. During engine operation at lower speeds, the main end of the crankshaft is more vulnerable to inadequate film thickness generation resulting in adhesive wear. For high torque low speed engines, use of non-Newtonian lubricant gives more robust bearing design as compared to Newtonian lubricant. Polymer addition to mineral oils result in non-Newtonian viscoelastic behavior of the lubricant. There are three types of non-Newtonian lubricants out of which viscoelastic lubricant is the most prominent cause of improvement. In this research continuity and Navier-Stokes equations are solved to ensure conservation of mass and momentum of the lubricant flow. After determining the film thickness between the bearing and the crankshaft 2-D Reynolds equation is used to model the behavior of the lubricant. The constitutive equations exhibiting viscoelastic characteristics are coupled with the momentum conservation equations to generate simulation results for non-Newtonian lubricant response. The steady state wedging and transient squeeze effects are studied for a viscoelastic engine lubricant at low speed. Parametric studies at various engine speeds and radial clearances are studied for useful engine design. The simulation results for Newtonian and viscoelastic engine lubricants are compared. The results show that at a low initial speed the Newtonian lubricant does not produces sufficient hydrodynamic pressures due to thin hydrodynamic films. For the same configuration viscoelastic characteristics of the lubricant contribute towards improving the pressure and film thickness profiles of a high-torque low-speed engine. It reduces the chances of breakdown of lubricant film and enhances the life of crankshaft by preventing adhesive wear.

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CHAPTER 1

INTRODUCTION

IC Engines are used on daily basis for transporting people as well as goods which result in an increase in the air pollution and consequently lead to global warming. They also consume a large amount of fuels. Due to an increasing demand in the consumption of IC engines there is an increase in the usage of fossil fuels. It is feared that fossil fuels will become extinct in the next 50 years so it is the necessary to reduce the fuel consumption by optimizing the engine performance and make it economical for every person because economy is important for a nation to survive.

Usage of fuel in the engine results in low carbon emissions which has forced the automotive industry to shift its focus from the fossil fuels to other alternate fuel sources and also controlling the engine emissions. Oil and gas companies pay special attention and dedicate a huge percentage of their resources to the reduction in emission. Largest contribution towards greenhouse gases is from the emission of transport vehicles. Minimization of greenhouse gases can only be achieved through improving engine design which includes optimal lubricant design. There are many challenges in optimizing the lubricant in order to develop the lubricant of the future. The lubrication of journal bearing has gathered much interest from the major oil industries and engine industries. It is necessary to understand the performance of lubricants in detail. In classical hydrodynamic lubrication it is assumed that lubricant is Newtonian. Newtonian lubricant approximation is not always a satisfactory solution to the engineering problem as it is seen in many lubrication applications. Since 1950 addition of polymer additives in small amounts have resulted in producing desirable lubricants. When additives are added to the oil it acts as viscoelastic. Recently significant interest has been developed in studying the viscoelastic effects of the polymer added lubricant. Any factor which affects the load carrying capacity of the lubricant and results in reduction of wear is clearly of significant importance. Therefore it is logical to study the effect of viscoelasticity in lubrication.

Although mineral based lubricants are widely used in industries and they can satisfy many performance requirements related to lubrication but still there are many problems related to them in the future. The worlds now consumes almost 8 times more than what it consumed of mineral based oil in 1950. Mineral based lubricants are non-renewable and they will be consumed in 50 years. The challenge that we face today is to improve the efficiency of lubricants by controlling emission of gases or by designing new lubricants which are eco-friendly. Selection criteria of lubricant involves its testing over a wide range of operating conditions such as different engine speeds. This process of testing is expensive and requires a lot of time. Modeling and simulation of oil flow saves the cost as well as time. This research leads to modeling and simulation of Newtonian and non-Newtonian lubricants in the journal bearing which can result in improved engine design and better selection of lubricants based on their performance.

1.1 Lubrication

Lubrication of machine elements is done by introducing and maintaining fluids between the components moving with relative velocity. This reduces friction and also minimizes the chance of contact between the components. When the lubricant is subjected to shear, it generates very little resistance and deforms easily. Thus lubricant can separate the two surfaces in relative motion and can reduce friction and wear between these surfaces. In the engine operation as the operating condition varies, the lubricant film thickness between the surfaces varies giving rise to several lubrication regimes as shown in Fig 1.

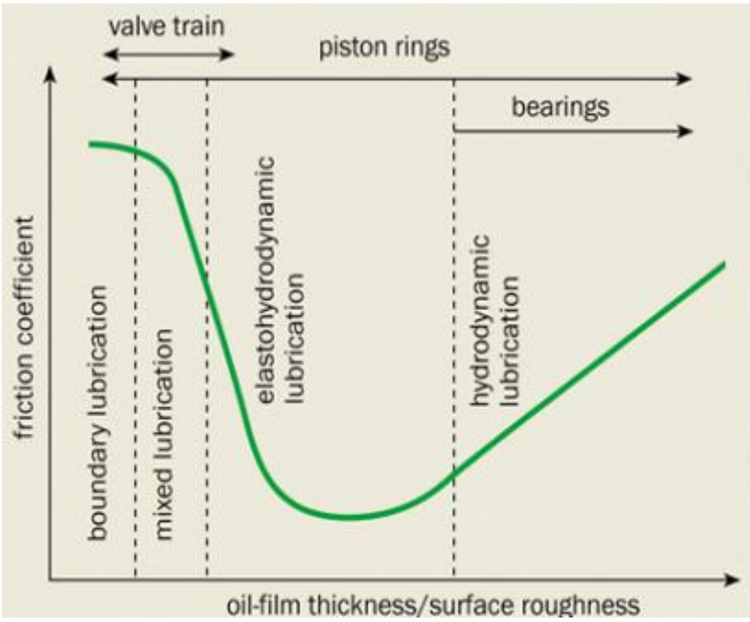


Fig 1. Lubrication Regimes for Journal Bearing

1.1.1 Hydrodynamic Lubrication

It is also known as thick film lubrication. This occurs when the two surfaces are completely separated by lubrication and the load is applied completely of the lubricant film. This can be understood by the example of a box moving on a table. The surface of box and table are separated by the lubrication present between them. If we move the box with a high velocity on the table a wedge shaped oil film will be formed between the box and the lubricant such that the area of the wedge shaped oil film is less at the tailing end than at the leading end. This will increase the pressure at the tailing end which enables it to carry the load of the box. Wedge shaped oil film is always formed in the hydrodynamic lubrication.

1.1.2 Boundary Lubrication

It is also known as thin film lubrication. This occurs when the pressure generated between the lubricant film and the moving surface is not sufficient enough to prevent the contact between the two surfaces. This occurs when the surface is moving at very low speed or the lubricant film is very thin. If the viscosity of lubricant is not high enough it can also cause contact. In this case the load bearing capacity of the lubricant is very low.

1.1.3 Elastohydrodynamic Lubrication

When the moving surface is under very high pressure the lubricant is unable to stick to the moving surface and it can decompose under such high pressure. Consider the example given in the hydrodynamic lubrication. The only difference is that now the block is placed under high pressure which increases the pressure generated in the lubricant film and this will result in deformation of the box. Such lubrication is possible only when the pressure generated in the oil film is high enough to elastically deform the moving surface.

1.1.4 Hydrostatic Lubrication

It is sort of hydrodynamic lubrication in which the two surfaces are separated by the lubricant film. The difference is that in hydrodynamic lubrication the pressure is produced within the lubricant film but in the hydrostatic lubrication it is provided externally with the help of some external source like a pump. Hydrostatic lubrication is dependent on the pressure of the pump and the clearance between the two surfaces. Hydrodynamic lubrication depends upon relative velocity of the surfaces.

1.2 Components of Internal Combustion Engine

The cylinder system consists of the following component in which each component play a significant role.

1. Ring Pack
2. Piston
3. Wrist-Pin
4. Connecting-Rod

5. Crankshaft
6. Journal Bearing

1.2.1 Crankshaft

The purpose of crankshaft is to convert lateral motion of piston into the rotary motion which smoothens the impulses produced in the combustion pressure. Crankshaft is a very important area in the lubrication of the engine due to its high speed. It has two ends namely big end and main end. At the main end journal bearing is attached with the crankshaft and at big end crankshaft is attached with the connecting rod. Loads during the expansion stroke are transmitted to the crankshaft by means of the connecting rod.

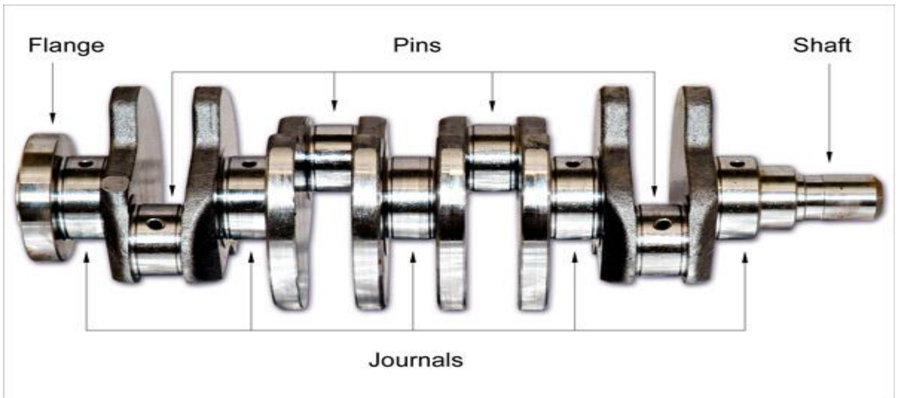


Fig 2(a): Crankshaft

1.2.2 Journal Bearing

A very common component that is being used in almost every machinery is journal bearing. They are used for carrying impact and cyclic loads and engines depend on them for high efficiency. It consist of a crankshaft which is rotating inside it and a lubricant film is present in between the two components. When the engine has not started the shaft lies on the bottom of the journal bearing. Before starting the engine lubricant is supplied through a pump and a film is produced between the crankshaft and journal bearing. This lubricant film lifts the shaft upwards. When the engine starts, shaft starts rotating and a hydrodynamic pressure is produced between the two surfaces which supports the load that crankshaft experiences. Load, speed, cavitation, vibration and frictional heating are some of the aspects for the analysis of journal bearing

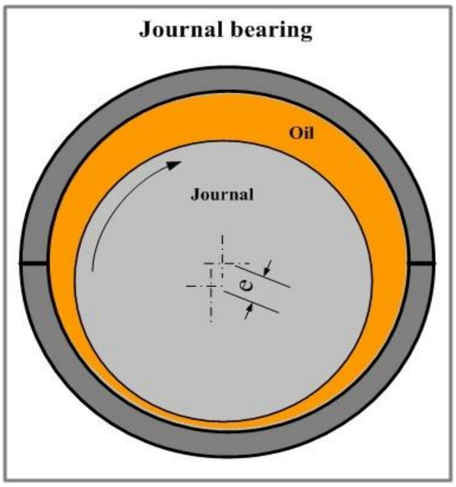


Fig 2(b): Journal Bearing

1.3 Types of Lubricants

Generally two types of fluids are used for the purpose of lubrication in journal bearing known as Newtonian and non-Newtonian Fluid

1.3.1 Newtonian Lubricant

Consider a thin layer of fluid between two parallel plates separated by a distance dy. If a force is applied to the fluid it will be subjected to shear. This force will be balanced by a reaction force in the fluid. For a Newtonian fluid the resulting shear stress is directly proportional to the rate of shear and the constant of proportionality between them is viscosity.

$$\tau_{yx} = \eta \left(\frac{-\partial V_x}{\partial y} \right)$$

Negative sign indicates that τ_{yx} is a measure of resistance to motion. η is the constant of proportionality which is also known as Newtonian viscosity. It is independent of shear rate and depends upon viscosity. Plot of shear rate against shear stress is a straight line.

1.3.2 Non-Newtonian Lubricant

For a non-Newtonian fluid the viscosity is not constant at a given temperature and pressure but it depends upon condition of flow for example shear rate. There are three types of non-Newtonian fluids

1.3.2.1 Time independent fluids

Fluids for which the rate of shear at any point is determined by the shear stress value at that point and instant of time are known as time independent fluids. These are also of three types.

(a) Shear thinning

(b) Shear thickening

(c) Viscoplastic

1.3.2.2 Time dependent fluids

Fluids for which viscosity not only depends upon rate of shear but also the time for which the fluid has been subjected to shear stress. For example cement paste when subjected to a long time shearing their viscosities change gradually. They are of two types

(a) Thixotropy

(b) Rheopexy or negative thixotropy

1.3.3 Viscoelastic Fluids

According to Hooke's law when a body is sheared, the stress is proportional to the strain.

$$\tau_{yx} = -G\left(\frac{dx}{dy}\right)$$

If a perfect solid is elastically deformed it regains its original position on the removal of the stress. For fluids Newtonian law states that shear stress is proportional to the rate of shear strain. Substances exhibiting characteristics of both elastic solids and viscous liquids are known as viscoelastic. Perfect elastic deformation and perfect viscous flow are limiting cases of viscoelastic behavior. For some materials only these limiting cases are observed for example elasticity of water and viscosity of ice may pass unnoticed.

1.4 Maxwell Model

The most striking feature connected with the deformation of viscoelastic substance is its simultaneous display of fluid like and solid like characteristics. Early attempts at modeling the viscoelastic behavior hinged at the linear combination of elastic and viscous properties by using mechanical analogues involving springs and dashpots. Maxwell model represents the cornerstone of the linear viscoelastic models. A mechanical analogue of this model is obtained by series combination of a spring and dashpot. If the individual strain rates of the spring and dashpot are $\dot{\gamma}_1$ and $\dot{\gamma}_2$ respectively then the total strain is given by the sum of these two components

$$\dot{\gamma} = \dot{\gamma}_1 + \dot{\gamma}_2 = \frac{d\gamma_1}{dt} + \frac{d\gamma_2}{dt}$$

For viscous fluid

$$\tau = \mu \frac{d\gamma_1}{dt} \longrightarrow \frac{d\gamma_1}{dt} = \frac{\tau}{\mu}$$

For elastic solid

$$\dot{\tau} = G \frac{d\gamma_2}{dt} \longrightarrow \frac{d\gamma_2}{dt} = \frac{\dot{\tau}}{G}$$

$$\dot{\gamma} = \frac{\tau}{\mu} + \frac{\dot{\tau}}{G}$$

$$\tau + \lambda \dot{\tau} = \mu \dot{\gamma}$$

Where $\lambda = \mu / G$

Consider a Maxwell model where spring is attached with dashpot in series. Apply instantaneously small strain which is maintained at constant level. For an ideal spring, the stress will follow the strain and reach a constant value. In the Maxwell model the spring will respond instantaneously in this experiment. On the other hand piston will move slowly through the dashpot and therefore stress in the spring will be gradually reduced with time. For constant value of strain

$\tau + \lambda \dot{\tau} = 0$ i.e. $\dot{\gamma} = 0$

$$\frac{d\tau}{dt} = -\frac{\tau}{\lambda}$$

Integrating w.r.t to time at $t = 0$ and $\tau = \tau_{\max}$

$$\tau = \tau_{\max} \exp\left(\frac{-t}{\lambda}\right)$$

It describes the decay of stress with time when rapid strain is applied to a Maxwell fluid. This is similar to a first order chemical reaction in which concentration of the reactant is depleted exponentially. When $t \to 0$, the term $\exp(-t/\lambda)$ tends to one and response is simply that of spring. When $t = \lambda$ the stress has dropped from $\tau_{\max}$ to $\tau_{\max}/e$.

1.5 Voigt Model

An important feature of Maxwell model is its fluid like behavior. A more solid like response is obtained by Voigt model. It is represented by parallel arrangement of spring and dashpot. In this case strain in the two components is equal and stress strain behavior of the system is obtained by adding individual stresses in the two elements.

$$\tau = G\gamma + \mu\dot{\gamma}$$

If the stress is constant at τ_0 and initial strain is zero, upon the removal of stress, the strain decays exponentially with a time constant $\lambda = \mu / G$. The more solid like response is clear from the fact that this model does not exhibit unlimited non-recoverable viscous flow and it will come to rest when spring has taken up the load.

1.6 Maxwell Voigt Model

This model consists of a Maxwell model in series with Voigt model. Let γ_1 and γ_2 represent strains in the Maxwell and Voigt model respectively. On application of force

$$\dot{\gamma}_1 = \tau + \frac{\mu}{G}\dot{\tau}$$

For Voigt Model

$$\tau = G\gamma + \mu\dot{\gamma}$$

1.7 Project Motivation

Diesel engine has a vast application in the industries. In the industry the horse power of diesel engine was increased from 580 to 730 without changing the design of the engine. The life of the crankshaft reduced to almost half of what it was earlier. Upon inspection it was revealed that there was a contact between crankshaft and journal bearing due to which wear occurs and crankshaft was damaged. The reason of this was the failure of lubricant which was not able to lift the crankshaft during the engine operation and hence the contact between crankshaft and journal bearing occurred.

In the industry Newtonian lubricant was used. When going through literature survey it was revealed that Newtonian lubricant produces low hydrodynamic pressure at low engine speeds and hence were unable to prevent the metal to metal contact due to which failure of crankshaft occurred. In the 1950s polymer added lubricants were introduced also known as non-Newtonian lubricant. These non-Newtonian lubricant produce more hydrodynamic pressure as compared to the Newtonian lubricant and can prevent the contact between journal bearing and crankshaft. That is why I chose my research on non-Newtonian lubricants to study their behavior and compare them with the Newtonian lubricants.

CHAPTER 2

LITERATURE REVIEW

At the low engine speed the hydrodynamic film generated between the crankshaft and journal bearing is not thick enough to carry out the hydrodynamic loads. Due to which low hydrodynamic pressure is produced and there can be instantaneous contact between the crankshaft and journal bearing. During the low speed the engine oil present between the components should be viscous enough to carry out the loads and at the same time it should be mobile enough so that the process of lubrication will not stop. G.I. Taylor investigated stability of the viscous fluids by doing experiments [1]. By using Newtonian fluid he also calculated temperature, velocity and fluid friction [2, 3]. B. S. Unlu and E. Atik measured friction coefficient in a journal bearing under dry and lubricated conditions using a journal bearing test rig [4]. High impact and cyclic loads cause excessive shear and heating due to which viscosity of oil reduces. This low viscosity generates thin film and low hydrodynamic pressures [5]. At the low speeds Newtonian lubricant is not able to produce enough hydrodynamic pressures to prevent the instantaneous contact between the crankshaft and the journal bearing due to which wear of the components occur. This wear leads to the failure of crankshaft. Upon investigation of different crankshafts it was observed that failure of crankshaft can be due to different reasons which include mechanical loading, failure of lubrication or thermal fatigue [6]. Changli Wang et al. analyzed the failure of crankshaft after it failed during testing for only 20 minutes in a strange manner. It was found that cause of failure was friction [7]. F. Jiménez Espadafor et al, Gul Cevik & Rıza Gurbuz, J.A. Becerra et al, M. Fonte et al and S.K. Bhaumik et al [8-12] studied failure of crankshaft and concluded that fatigue and loading were the main cause of failure for the crankshaft After the 1950 polymer added mineral oils are commonly used. This addition causes the lubricant to exhibit non-Newtonian characteristics [13]. A. Berker et al. experimentally studied the effect of polymer added lubricants in a journal bearing by selecting different multi-grade oils. It was seen that if polymer additive lubricant was used the size of eddy adjacent to the stationary wall i.e. the wall of the journal bearing can be reduced significantly [14]. These non-Newtonian characteristics exhibit different types of behavior one of which is viscoelastic behavior. Use of these oils result in high hydrodynamic pressures which leads to smooth running and enhanced engine life [15]. In some researches viscoelastic characteristics were neglected and thermal non-Newtonian Ree-Eyring lubrication model was used [16, 17]. Velocity of elastic shear thinning fluid was measured experimentally J. M. Nouri and J. H. Whitelaw. It was revealed that turbulence increased in the xi region of minimum gap between the two cylinders for Newtonian lubricant and decreased for the non-Newtonian lubricant [18]. Some researchers studied the performance of journal bearing using Power-law for non-Newtonian lubricants. They concluded that when dilatant fluid was used the load carrying ability was increased but when pseudo plastics were used the load carrying ability decreased [19]. M. P. Escudier compared experimental results with numerical calculations for fully developed laminar flow of Power-law and cross fluids through eccentric circles [20, 21]. It was found by B. P. Williamson et al. that viscoelasticity of the non-Newtonian oil does induce prominent and beneficial effects on the characteristics of lubrication one of them involves increased load carrying capacity [22, 23]. There have been attempts to model the non-Newtonian lubricants by different characteristics and the results have proved that viscoelasticity is the most likely cause for improvements in the performance [24, 25]. Experimental results show that viscoelastic lubricants result in an increase in the load carrying ability of the journal bearing [26]. Mechanical behavior of these lubricants is more complicated than those of the Newtonian lubricants. Maxwell model is used to characterize the viscoelastic response of the non-Newtonian lubricant. It exhibits two types of behaviors which are characterized by dissipation of energy and energy storage [27]. When Maxwell model was used it was realized that in order to increase the load carrying capacity of the journal bearing it was necessary to use the relaxation time [28]. In the case of lubricants this relaxation time ranges from 10-3 s to 10-6 s [29]. A. N. Beris et al. and X. Huang et al. studied upper convected Maxwell model in eccentrically rotating cylinders by neglecting inertia and considering low eccentricities. [30-32]. A special rheometer was designed for the purpose of studying at high shear rates i.e. up to 10_9 s-$_1$ [33]. S. Bair et al. measured characteristics of non-Newtonian lubricant at high shear rates [34-36]. W.B.Wan Nik et al presented rheological viscosity-shear results for edible oil [37]. This study models hydrodynamic lubrication of journal bearing at different speeds. Viscoelastic characteristics of a non-Newtonian lubricant are incorporated in the lubrication model. Multigrade oil, 80 crankshaft to journal bearing radial clearance and 0.4 eccentricity ratio are some of the inputs used in the model. Three numerical models are developed and results are obtained by simulation.

CHAPTER 3

MATHEMATICAL MODELING

3.1 Newtonian Lubricant

3.1.1 Navier Stokes Equation

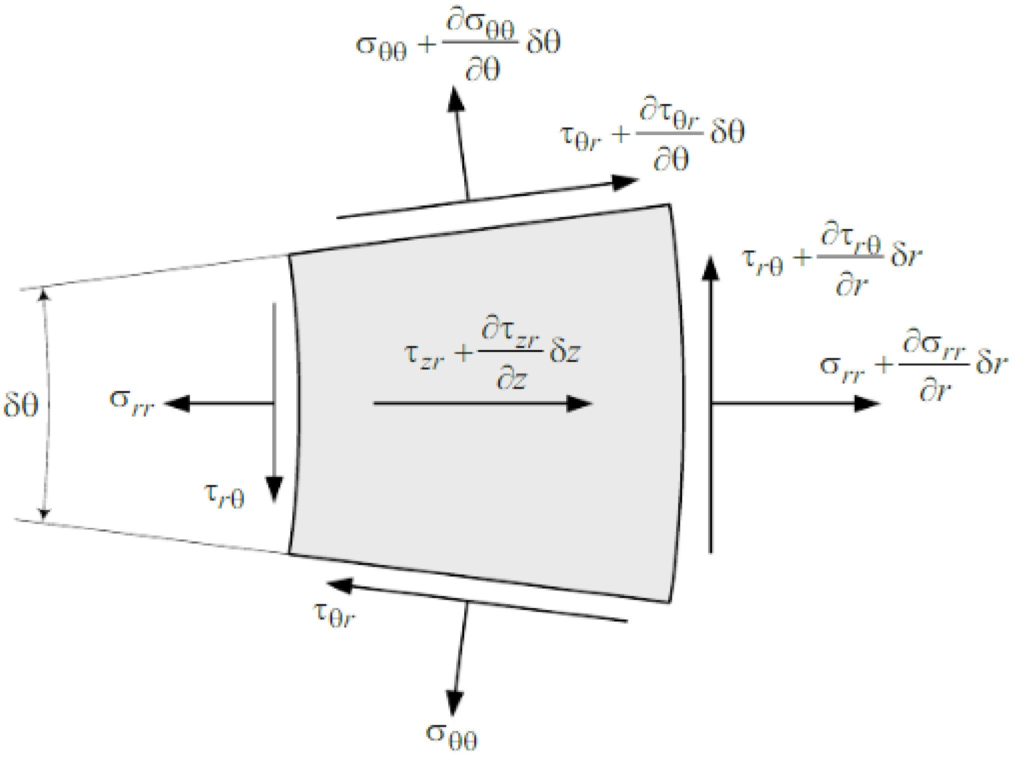


Fig 4: Stress distribution over fluid particle

$$\rho\frac{Du_r}{Dt} = F_r\rho - \frac{\partial p}{\partial r} + \mu\frac{\partial^2 u_r}{\partial r^2} + \frac{\mu}{r}\frac{\partial^2 u_r}{\partial \theta^2} + \mu\frac{\partial^2 u_r}{\partial z^2}$$

Similarly for θ and z direction

$$\rho\frac{Du_\theta}{Dt} = F_\theta\rho - \frac{\partial p}{\partial \theta} + \mu\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\mu}{r}\frac{\partial^2 u_\theta}{\partial \theta^2} + \mu\frac{\partial^2 u_\theta}{\partial z^2}$$

$$\rho\frac{Du_z}{Dt} = F_z\rho - \frac{\partial p}{\partial z} + \mu\frac{\partial^2 u_z}{\partial r^2} + \frac{\mu}{r}\frac{\partial^2 u_z}{\partial \theta^2} + \mu\frac{\partial^2 u_z}{\partial z^2}$$

3.1.2 Reynolds Equation

Reynolds derived the mathematical equation for hydrodynamic lubrication and it is known as Reynolds equation. It can be derived from the basis of continuity equation and Navier-Stokes momentum equation. It is also derived by taking equilibrium of an element when it is subjected to shear and applying the continuity equation. There are two conditions for hydrodynamic lubrication to occur.

1. Two surfaces must move with sufficient relative velocity for the load carrying film to be generated.
2. Surfaces must be at some angle to each other i.e. if the relative moving surfaces are parallel a pressure field will not be formed.

Consider two inclined surfaces as shown in Fig 3. It is assumed that the bottom surface is covered with lubricant and is moving with some velocity say U. The top surface is at an angle to the bottom surface. Lubricant will be dragged towards the converging wedge as the bottom surface moves. A pressure field is generated otherwise more lubricant will be entering the wedge than leaving it. Thus at the entry there will be an increase in pressure which will restrict the flow to enter and at the exit there will be a decrease in the pressure allowing more lubricant to leave the wedge. Due to this pressure gradient the velocity profile is bent inwards at the entrance of the wedge and bent outwards at the exit of the wedge. This generated pressure is known as hydrodynamic pressure and it separates the two surfaces.

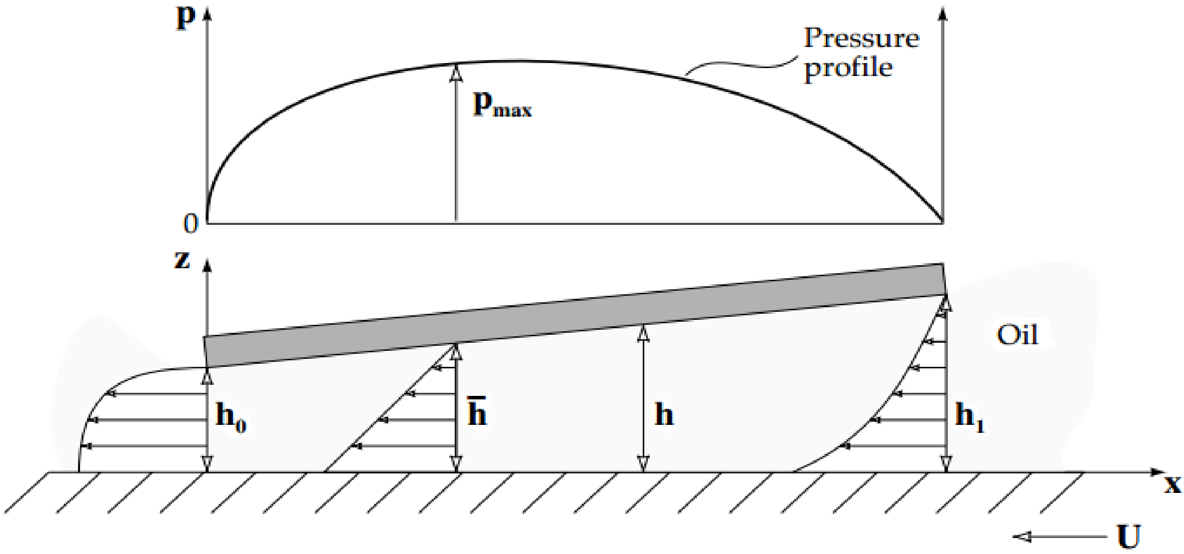


Fig 3. Principle of hydrodynamic pressure generation between non-parallel surfaces

Now apply the following assumptions on above equations

1. The fluid is inertia less.
2. Body forces are ignored.
3. Velocity gradient along the film thickness direction dominate the other two directions.
4. Lubricant is Newtonian.
5. No slip condition at the boundaries.
6. Pressure is not changing along the film thickness direction.
7. Adiabatic heating.

Final form of Reynolds Equation is

$$\frac{\rho}{r^2}\frac{\partial}{\partial\theta}\left[\frac{\rho h^3}{12\mu}\left(\frac{\partial p}{\partial\theta}\right)\right]+\frac{\partial}{\partial z}\left[\frac{\rho h^3}{12\mu}\left(\frac{\partial p}{\partial z}\right)\right]=\frac{U}{r}\frac{\partial}{\partial\theta}\left[\frac{\rho h}{2}\right]+V\frac{\partial}{\partial z}\left[\frac{\rho h}{2}\right]+\frac{\partial}{\partial t}\left[(\rho h)\right]$$

3.1.3 Non-Dimensionalization of Reynolds Equation

$$\frac{\partial}{\partial\hat{\theta}}\left[\frac{\hat{h}^3}{\hat{\mu}}\left(\frac{\partial\hat{p}}{\partial\hat{\theta}}\right)\right]+K^2\frac{\partial}{\partial\hat{z}}\left[\frac{\hat{h}^3}{\hat{\mu}}\left(\frac{\partial\hat{p}}{\partial\hat{z}}\right)\right]=\gamma_1\left[\frac{\partial}{\partial\hat{\theta}}\left(\hat{h}\right)\right]+\gamma_2\left[\frac{\partial}{\partial\hat{z}}\left(\hat{h}\right)\right]+\gamma\left[\frac{\partial}{\partial\hat{t}}\left(\hat{h}\right)\right]$$

3.1.4 Reynolds Equation with Vogelpohl parameter

$$\boxed{\begin{aligned}&\left(\frac{\partial^2 M_v}{\partial\hat{\theta}^2}+K^2\frac{\partial^2 M_v}{\partial\hat{z}^2}\right)+\frac{1}{\hat{\rho}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial M_v}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial M_v}{\partial\hat{z}}\right)-\frac{1}{\hat{\mu}}\left(\frac{\partial\hat{\mu}}{\partial\hat{\theta}}\frac{\partial M_v}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\mu}}{\partial\hat{z}}\frac{\partial M_v}{\partial\hat{z}}\right)\\&=M_v\left(F\right)+G\end{aligned}}$$

$$F=\frac{1.5}{\hat{\rho}\hat{h}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right)-\frac{1.5}{\hat{\mu}\hat{h}}\left(\frac{\partial\hat{\mu}}{\partial\hat{\theta}}\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\mu}}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right)+\frac{0.75}{\hat{h}^2}\left(\left(\frac{\partial\hat{h}}{\partial\hat{\theta}}\right)^2+K^2\left(\frac{\partial\hat{h}}{\partial\hat{z}}\right)^2\right)$$
$$+\frac{1.5}{\hat{h}}\left(\frac{\partial^2\hat{h}}{\partial\hat{\theta}^2}+K^2\frac{\partial^2\hat{h}}{\partial\hat{z}^2}\right)$$

$$G=\gamma_1\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial v}\right]+\gamma_2\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{z}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{z}}\right]+\gamma\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{t}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{t}}\right]$$

3.1.5 Discretization of Vogelpohl Equation

$$M_{v,i,j}=\frac{\left[\begin{aligned}&\left\{C_1\left(M_{v,i+1,j}+M_{v,i-1,j}\right)+C_2K^2\left(M_{v,i,j+1}+M_{v,i,j-1}\right)\right\}\\&+\frac{1}{\hat{\rho}}\left\{\begin{aligned}&\frac{C_1}{4}\left(\left(\hat{\rho}_{i+1,j}-\hat{\rho}_{i-1,j}\right)\left(M_{v,i+1,j}-M_{v,i-1,j}\right)\right)\\&+K^2\frac{C_2}{4}\left(\left(\hat{\rho}_{i,j+1}-\hat{\rho}_{i,j-1}\right)\left(M_{v,i,j+1}-M_{v,i,j-1}\right)\right)\end{aligned}\right\}\\&-\frac{1}{\hat{\mu}}\left\{\begin{aligned}&\frac{C_1}{4}\left(\left(\hat{\mu}_{i+1,j}-\hat{\mu}_{i-1,j}\right)\left(M_{v,i+1,j}-M_{v,i-1,j}\right)\right)\\&+K^2\frac{C_2}{4}\left(\left(\hat{\mu}_{i,j+1}-\hat{\mu}_{i,j-1}\right)\left(M_{v,i,j+1}-M_{v,i,j-1}\right)\right)\end{aligned}\right\}\end{aligned}\right]-G_{i,j}}{\left[2C_1+2K^2C_2+F_{i,j}\right]}$$

3.2 Non-Newtonian Lubricant

3.2.1 Reynolds Equation

$$\frac{\rho}{r^2}\frac{\partial}{\partial\theta}\left[\frac{\rho h^3}{12a_2\mu}\left(\frac{\partial p}{\partial\theta}\right)\right]+\frac{\partial}{\partial z}\left[\frac{\rho h^3}{12a_2\mu}\left(\frac{\partial p}{\partial z}\right)\right]=\frac{U}{r}\frac{\partial}{\partial\theta}\left[\frac{\rho h}{2}\right]+V\frac{\partial}{\partial z}\left[\frac{\rho h}{2}\right]+\frac{\partial}{\partial t}\left[\left(\rho h\right)\right]$$

3.2.2 Non-Dimensionalization of Reynolds Equation

$$\frac{\partial}{\partial\hat{\theta}}\left[\frac{\rho\hat{h}^3}{a_2\hat{\mu}}\left(\frac{\partial\hat{p}}{\partial\hat{\theta}}\right)\right]+K^2\frac{\partial}{\partial\hat{z}}\left[\frac{\rho\hat{h}^3}{a_2\hat{\mu}}\left(\frac{\partial\hat{p}}{\partial\hat{z}}\right)\right]=\gamma_1\left[\frac{\partial}{\partial\hat{\theta}}\left(\hat{h}\right)\right]+\gamma_2\left[\frac{\partial}{\partial\hat{z}}\left(\hat{h}\right)\right]+\gamma\left[\frac{\partial}{\partial\hat{t}}\left(\hat{h}\right)\right]$$

3.2.3 Reynolds Equation with Vogelpohl parameter

$$\boxed{\begin{aligned}&\left(\frac{\partial^2 M_v}{\partial\hat{\theta}^2}+K^2\frac{\partial^2 M_v}{\partial\hat{z}^2}\right)+\frac{1}{\hat{\rho}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial M_v}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial M_v}{\partial\hat{z}}\right)-\frac{\hat{h}^{-0.5}}{a_2}\left(a_3\frac{\partial M_v}{\partial\hat{\theta}}+K^2a_4\frac{\partial M_v}{\partial\hat{z}}\right)\\&=M_v\left(F\right)+G\end{aligned}}$$

$$F=\frac{1.5}{\hat{\rho}\hat{h}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right)-\frac{1.5}{a_2\hat{h}}\left(a_3\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2a_4\frac{\partial\hat{h}}{\partial\hat{z}}\right)+\frac{0.75}{\hat{h}^2}\left(\left(\frac{\partial\hat{h}}{\partial\hat{\theta}}\right)^2+K^2\left(\frac{\partial\hat{h}}{\partial\hat{z}}\right)^2\right)$$

$$+\frac{1.5}{\hat{h}}\left(\frac{\partial^2\hat{h}}{\partial\hat{\theta}^2}+K^2\frac{\partial^2\hat{h}}{\partial\hat{z}^2}\right)$$

$$G=\gamma_1\left[\frac{a_2}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+\frac{a_2}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{\theta}}\right]+\gamma_2\left[\frac{a_2}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{z}}+\frac{a_2}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{z}}\right]+\gamma\left[\frac{a_2}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{t}}+\frac{a_2}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{t}}\right]$$

Where

$$a_3=\lambda\tau_{rref}\frac{\partial\hat{\tau}_{rr}}{\partial\hat{\theta}}-\frac{\partial\hat{\mu}}{\partial\hat{\theta}}$$

$$a_4=\lambda\tau_{rref}\frac{\partial\hat{\tau}_{rr}}{\partial\hat{z}}-\frac{\partial\hat{\mu}}{\partial\hat{z}}$$

Lateral velocity of the crankshaft is calculated by [41]

$$V=U\sin\gamma_m$$

$$\gamma_m=\frac{PL^2}{16EJ}$$

The non-dimensional form of film thickness equation is given as [42]

$$\hat{h}=1+\varepsilon\cos(\hat{\theta})$$

Nomenclature			
U	Circumferential velocity of shaft	P_{ref}	Reference hydrodynamic pressure
V	Lateral velocity of shaft	$\hat{P}$	Hydrodynamic pressure
u_θ	Lubricant velocity in circumferential direction	h	Film thickness
u_z	Lubricant velocity in bearing width direction	r	Radius of journal bearing
u_r	Lubricant velocity in radial direction	η_{ref}	Dynamic viscosity of lubricant at ambient conditions
r,θ,z	Coordinates	t_{ref}	Reference starting time
ρ	Density of Lubricant at ambient conditions	M_v	Vogelpohl parameter
μ_{ref}	Initial viscosity of lubricant at ambient conditions	$\tau_{\theta\theta}$	Normal stress in circumferential direction
t_{ref}	Reference starting time	$\tau_{\theta z}$	Shear stress in circumferential and bearing width direction
γ_m	Angle of journal misalignment	$\tau_{\theta r}$	Shear stress in circumferential and radial direction
E	Modulus of elasticity of Shaft	τ_{zz}	Normal stress in bearing width direction
J	Moment of inertia of crank pin	τ_{zr}	Shear stress in bearing width and radial direction
P	Pressure produced inside the combustion chamber	τ_{rr}	Normal stress in radial direction
λ	Characteristic relaxation time		

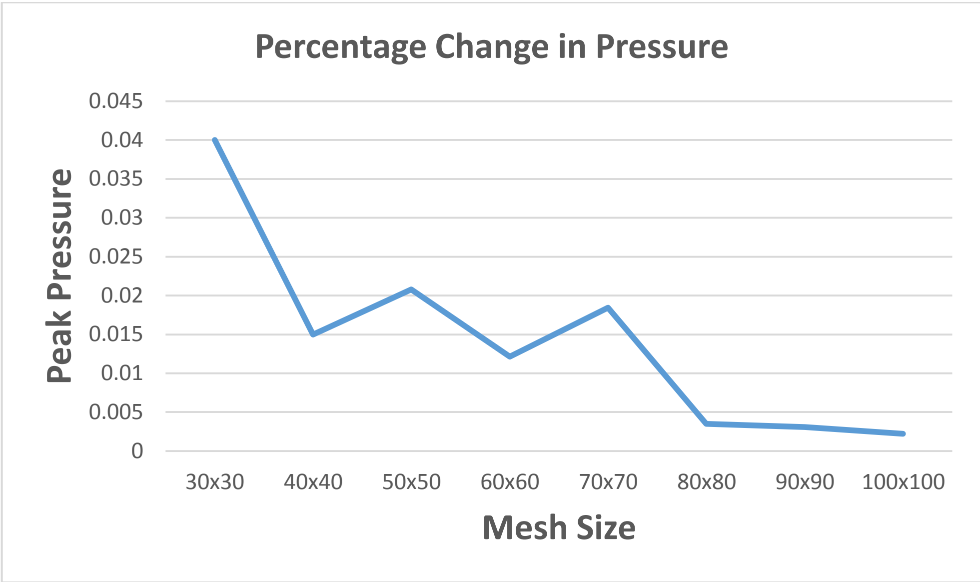
Table 1: Nomenclature

CHAPTER 4

RESULTS AND DISCUSSION

NUMERICAL SCHEME

On the commencement of numerical solution an initial guess of eccentricity ratio is taken to calculate the film thickness for the Newtonian lubricant. Reynolds equation is solved by finite difference method. For the initial step Vogelpohl parameter is calculated by using Gauss Seidel iterative solver and then FTCS is used to calculate the transient solution and to ensure the numerical stability of the solution. Error in the hydrodynamic pressure is assessed by setting the convergence criteria of 1e-6. When the error between two consecutive iteration is smaller than the stopping criteria the solution converges. The hydrodynamic pressure is coupled with the Navier-Stokes equation to calculate the flow rate in both the circumferential and the radial directions. New film thickness is calculated for the subsequent interval of time at which pressure is again calculated from the Reynolds equation. This process is repeated for the whole 720 degrees crank rotation cycle. For the viscoelastic behavior of the lubricant, film thickness is calculated by taking an initial guess of eccentricity ratio. From this film thickness flow rates in the circumferential and radial direction are calculated by coupling upper convected Maxwell model with Navier-Stokes equation. A stopping criteria is set so that error does not grow. From these flow rates and stresses the hydrodynamic pressures are calculated form the modified Reynolds equation given by equation for the non-Newtonian lubricant. At this pressure the viscosity of the lubricant is updated. New film thickness is calculated for the subsequent interval of time and at this time step flow rates, stresses and hydrodynamic pressures are updated. This process continues for the 720 degrees rotation of crankshaft. After completing the 720 degree rotation of crankshaft velocity of the crankshaft is updated and again the whole process is repeated for both Newtonian and non-Newtonian lubricant. Mesh independence is studied by setting the convergence for residual error to 1e-6. First mesh values are taken at initial grid size and then mesh is refined. Simulation results are analyzed for mesh size of up to 100x100. It is observed that there is not much change in the peak values by changing the mesh size as can be observed in the figure below.


Percentage Change in Pressure
Peak Pressure
0.045
0.04
0.035
0.03
0.025
0.02
0.015
0.01
0.005
0
30x30
40x40
50x50
60x60
70x70
80x80
90x90
100x100
Mesh Size

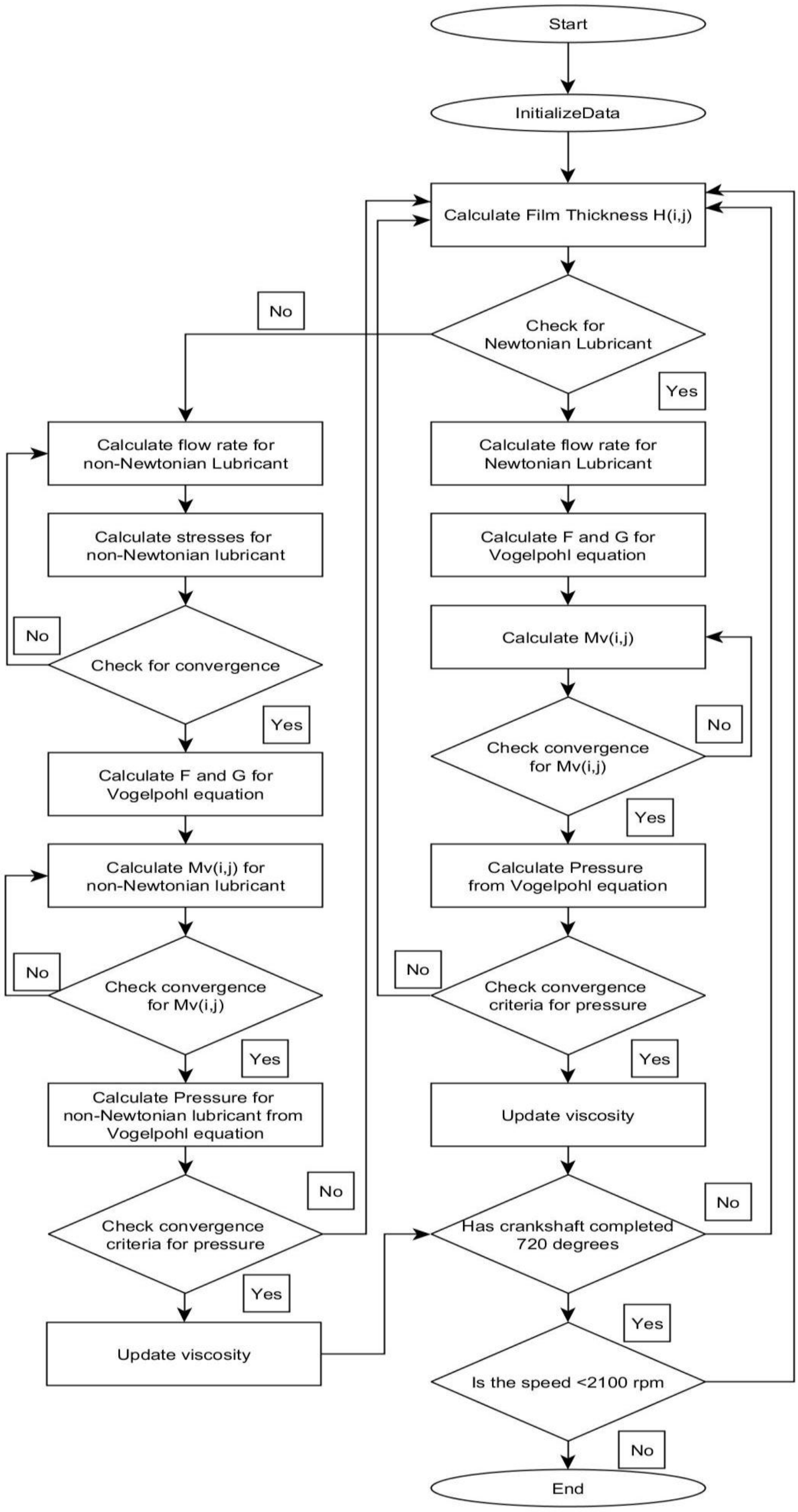
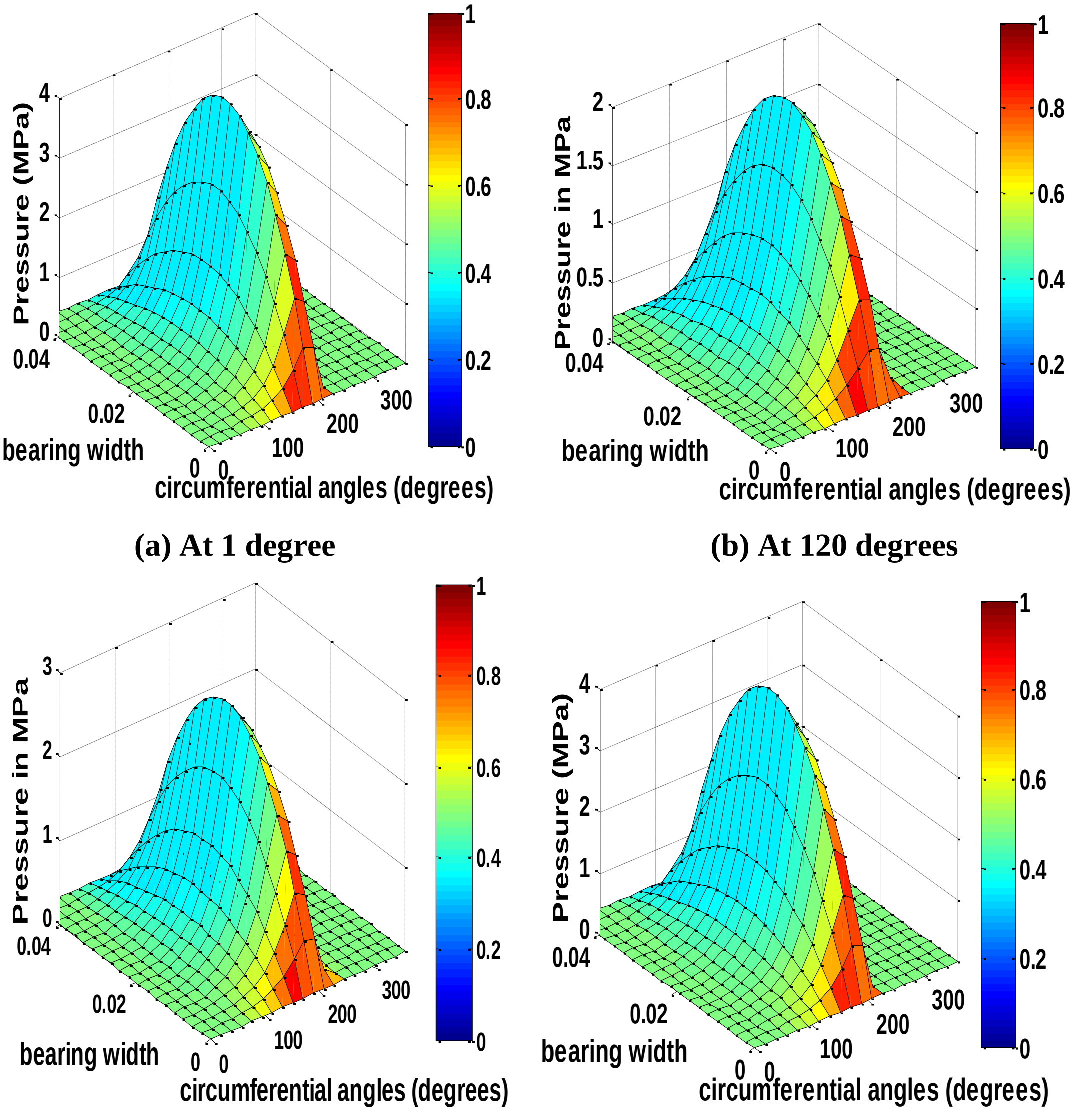


Fig 5: Flow Chart

4.1 Newtonian Lubricant

4.1.1 Hydrodynamic Pressure

Hydrodynamic pressures are generated due to the wedging action. In the Newtonian hydrodynamic model, the hydrodynamic pressures are plotted for the 720 degrees crank rotation angle. The 3-D pressure profiles are generated at the selected crank angles which are discussed. At 1 degree crank angle rotation during the intake stroke when piston starts moving from top dead center to bottom dead center, the hydrodynamic pressure for Newtonian lubricant is shown in Fig 6(a). At this point the crankshaft starts its rotation and the profile shows that minimum hydrodynamic pressure is produced. Along the circumferential direction from inlet to about 190 degrees the pressure reaches to a maximum point as shown in Fig 6(a). After 190 degrees the pressure decreases and drops to zero. Hydrodynamic pressure depends upon minimum film thickness as the pressure increases with the decrease in the minimum film thickness value. At this stage hydrodynamic film thickness is not sufficiently thick as the lubricant flows across the journal bearing under oil flooding conditions. A pressure field of low intensity is generated. This hydrodynamic pressure continues to increase up to 120 degree crank angle rotation when piston has moved past the mid of the intake stroke as shown in Fig 6(b). At this point the minimum film thickness reduces as compared to the previous instant. Fig 6(c) shows the pressure field when crankshaft has moved 240 degrees when piston is before the mid of the compression stroke. Minimum Film thickness reduces at this point indicating that intense hydrodynamic pressures are produced as compared to the previous instant. Fig 6(d) represents hydrodynamic pressure produced at the power stroke. The combustion loads are applied on the lubricant through the crankshaft. Due to this action the crankshaft moves downward and there is a rapid increase in the pressure field. This trend is also clearly visible in the profile of flow rate in the radial direction. Fig 8 shows a substantial increase in the flow rate during the power stroke. The pressure field shows intense pressures which implies enhanced load carrying capability of the lubricant.



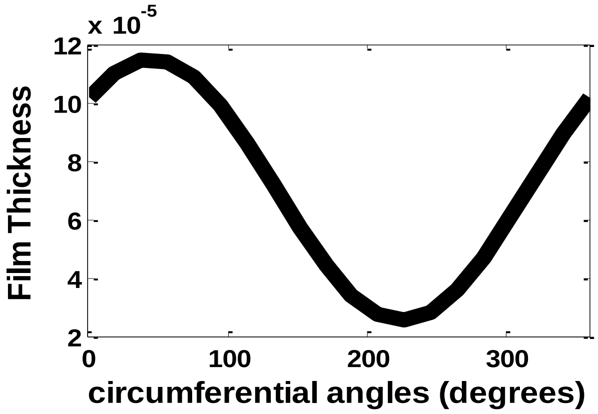
(a) At 1 degree **(b) At 120 degrees**

(c) At 240 degrees **(d) At 420 degrees**

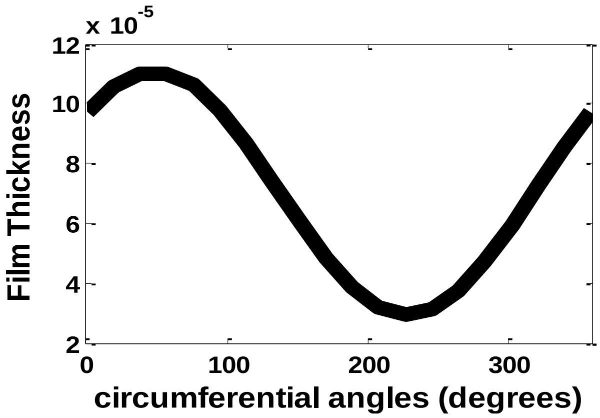
Fig 6: Hydrodynamic pressure for Newtonian lubricant at 600 rpm

4.1.2 Film Thickness

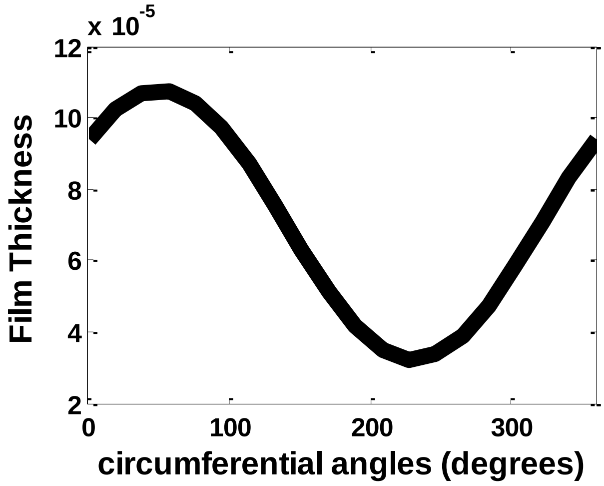
Film thickness for the different crank rotation angles are calculated at an initial guess of eccentricity ration 0.5 as shown in Fig 7. The profile shows that the minimum film thickness value reduces. At the beginning of crank rotation when the piston starts moving from top to bottom the value of film thickness is low. At this film thickness the hydrodynamic pressure profile produced is shown in Fig 7(a). These pressures try to lift the shaft upwards and make it concentric which increases the film thickness value at the start of curve but reduces to a point of minimum film thickness at about 210 degrees along the circumferential direction. At 240 degree of crank rotation angle when piston is just behind the mid of compression stroke the film thickness value increases from the previous instant at start but decreases at minimum film thickness point from the previous instant, due to which the hydrodynamic pressure increases with the film thickness as shown in Fig 7(c). With the further rotation of crankshaft the film thickness increases further. At the 420 degree crank angle when piston position is just before the mid of power stroke the hydrodynamic pressures at this instant of time are shown in Fig 6(d) which have increased from the previous instant of time due to decrease in minimum film thickness value. With this increased hydrodynamic pressure the film thickness is shown at 420 degree crank rotation angle. The analysis of film thickness curves at different crank degree rotation angles shows that film thickness at a minimum film thickness point reduces at each degree rotation of crankshaft due to which hydrodynamic pressures increase continuously.



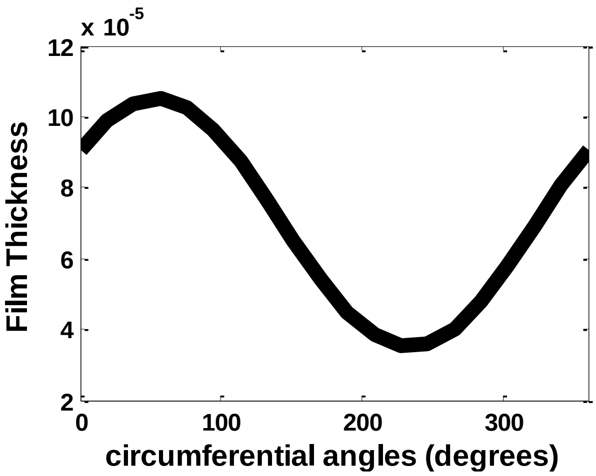
(a) At 1 degree



(b) At 120 degrees



(c) At 240 degrees

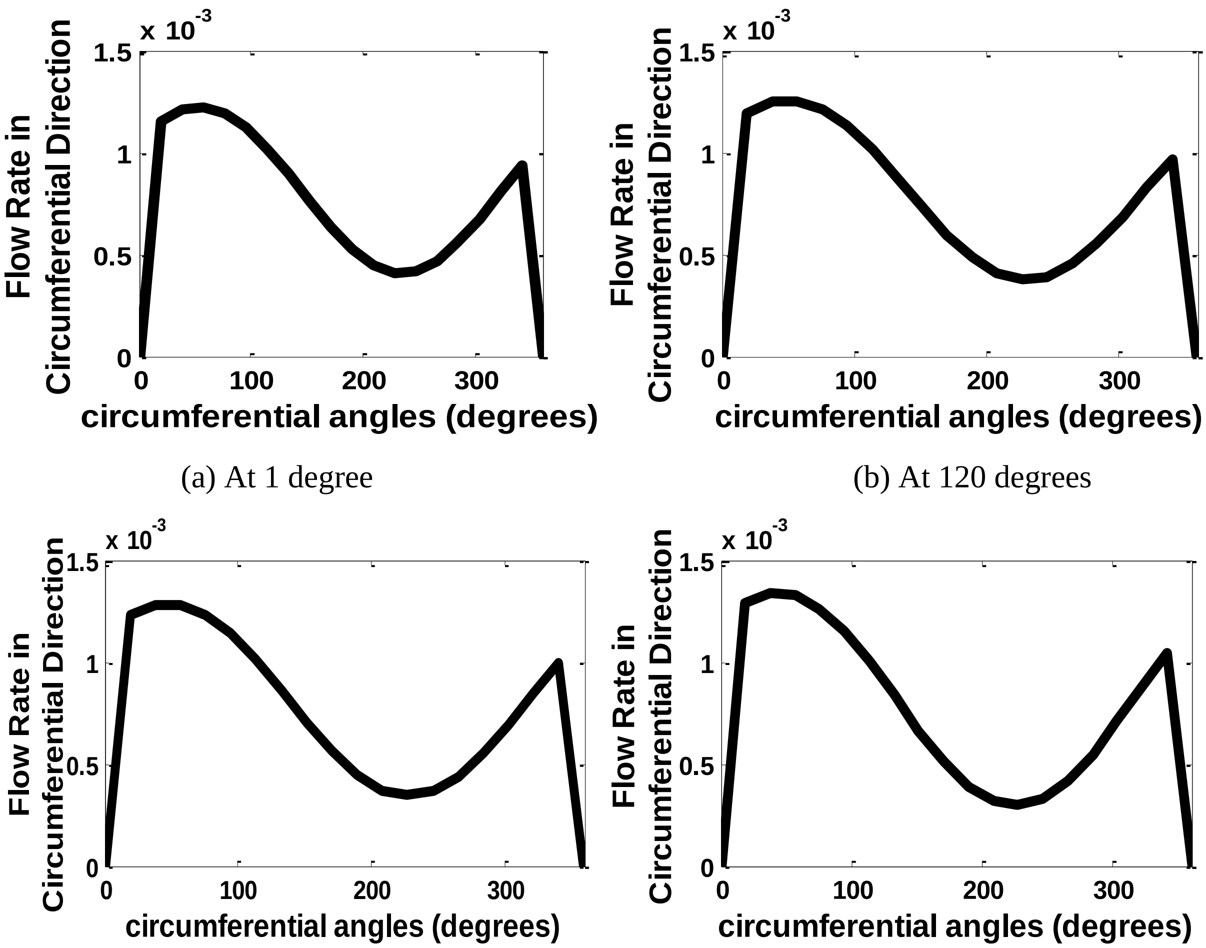


(d) At 420 degrees

Fig 7: Film thickness for Newtonian lubricant

4.1.3 Flow Rate

Flow rate is shown in Fig 8 for different crank rotations. When the crankshaft starts rotating it also plays its role in pumping the lubricant across the journal bearing. This lubricant is then pumped out of the bearing through the crankshaft and towards the piston after passing through the connecting rod. This flow rate also increases with an increase in the film thickness as shown in Fig 8. As the crankshaft starts rotating flow rate is minimum as shown in Fig 8(a). When crankshaft rotates further this flow rate increases with the increasing film thickness which leads to an increase in the hydrodynamic pressure. This flow rate increases when the crankshaft rotates further at 240 degrees with the increasing film thickness as shown in Fig 8(b). It increases further during the power stroke when the crankshaft is experiencing maximum load from the piston as shown in Fig 8(c).



(a) At 1 degree

(b) At 120 degrees

(c) At 240 degrees

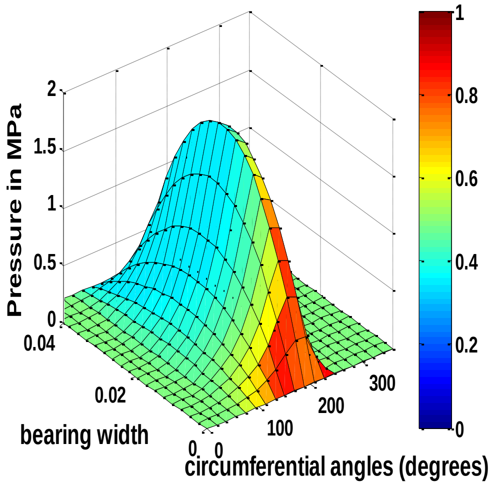
(d) At 420 degrees

Fig 8: Flow Rate for Newtonian lubricant

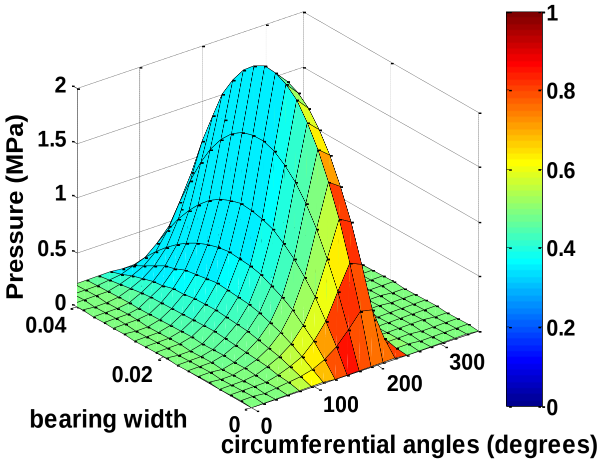
4.2 Non-Newtonian Lubricant

4.2.1 Hydrodynamic Pressure

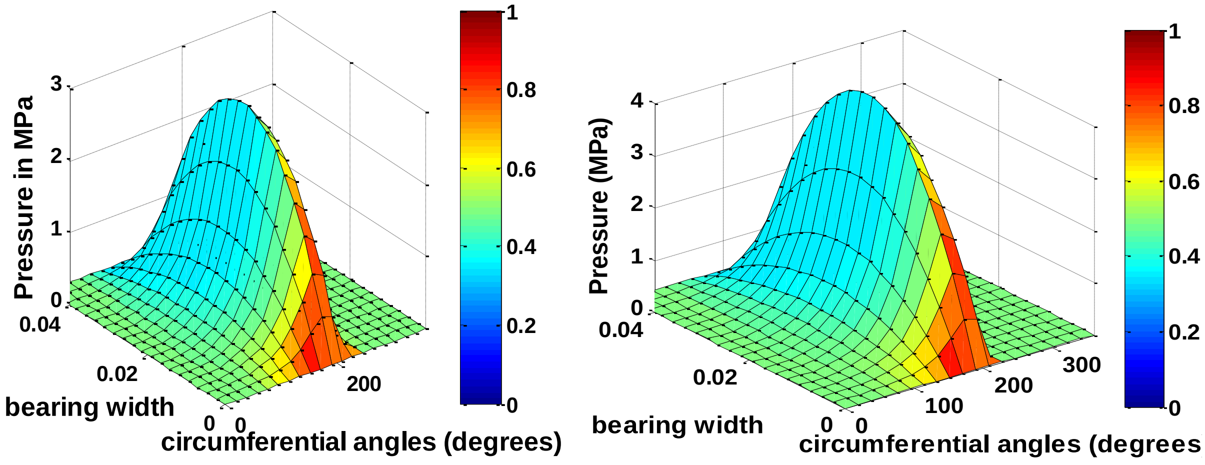
When the load is applied on the Newtonian lubricant it transfers the load as soon as it experiences it but that is not the case for the viscoelastic lubricant. As the load is applied on the lubricant it first stretches and the transfers a load causing a time delay in the transfer of the load. By using the non-Newtonian viscoelastic lubricant hydrodynamic pressures are generated over the complete 720 degrees rotation of crankshaft. For the sake of comparison the pressures are shown at the same angles as that in the case of the Newtonian lubricant. At 1 degree the crankshaft has just started its rotation as the piston starts its intake stroke, the hydrodynamic pressures developed for the non-Newtonian lubricant are low as shown in Fig 9(a). A Newtonian lubricant is unable to carry the hydrodynamic load at the low speed of engine. The pressures produced in the viscoelastic lubricant are greater than those in the Newtonian lubricant. As the crankshaft moves downward it compresses the lubricant but the viscosity of the lubricant does not increase significantly and more hydrodynamic pressure is developed in it. At 240 degrees rotation of crankshaft when piston is just before the mid of the compression stroke, hydrodynamic pressure increases with the decrease in the film thickness as shown in Fig 9(c). Fig 9(d) shows hydrodynamic pressure produced when the crankshaft experiences combustion load from the piston during the power stroke. Pressure rises subsequently in this cycle. The pressures developed for the non-Newtonian case are somewhat greater than those for the Newtonian lubricant which implies greater load carrying ability of the non-Newtonian lubricant.



(a) At 1 degree



(b) At 120 degrees

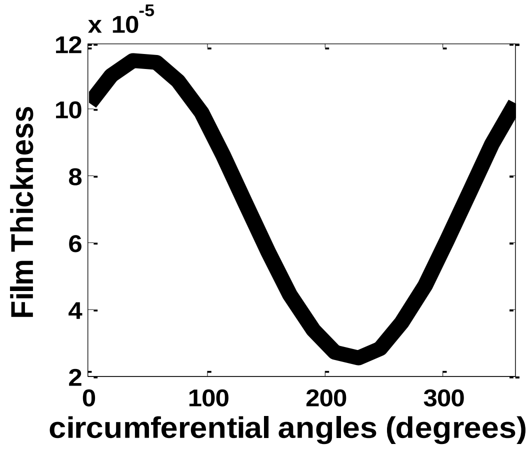


(c) At 240 Degrees (d) At 420 Degrees

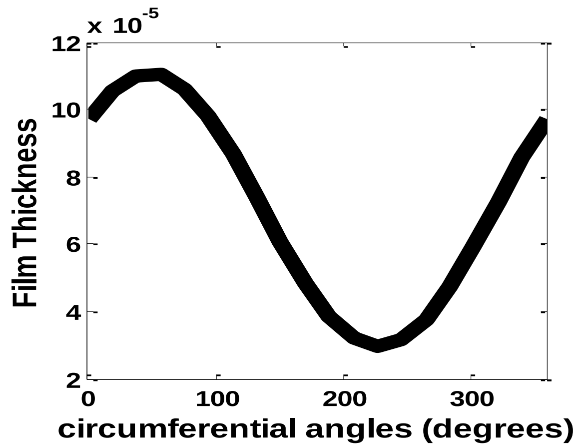
Fig 9: Hydrodynamic pressure for non-Newtonian lubricant at 600 rpm

4.2.2 Film Thickness

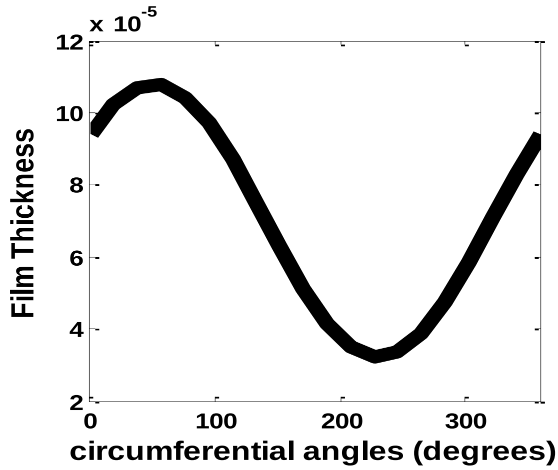
The crankshaft commences rotation the film thickness between the shaft and the bearing is thin and it is unable to carry the load for the Newtonian lubricant. For the non-Newtonian lubricant more hydrodynamic pressures are produced at the low values of film thickness. As the crankshaft rotates minimum film thickness decreases resulting in an increase in the hydrodynamic pressure across the journal bearing. Hydrodynamic pressure produced in the lubricant lifts the crankshaft upwards. Film thickness for the different crank rotation angles is shown in Fig 8. The profile shows that the minimum film thickness reduces along the circumferential direction of journal bearing. Film thickness is minimum at 1 degree crank rotation angle as shown in Fig 10. At this film thickness the hydrodynamic pressure produced is also minimum as shown in Fig 9(a). As the crankshaft rotates up to 120 degrees minimum film thickness decreases as a result more hydrodynamic pressure is produced inside the lubricant which lifts the crankshaft. The analysis of minimum film thickness shows that it is decreasing with the rotation of crankshaft due to which the hydrodynamic pressures increase continuously. Film thickness profile for the different crank degree rotation is shown in Fig 10.



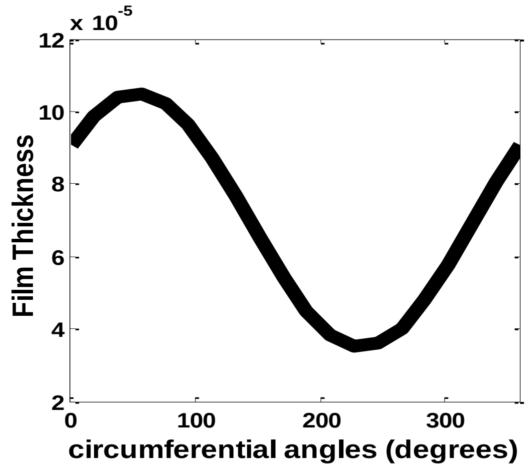
(a) At 1 degree



(b) At 120 degrees



(c) At 240 degrees

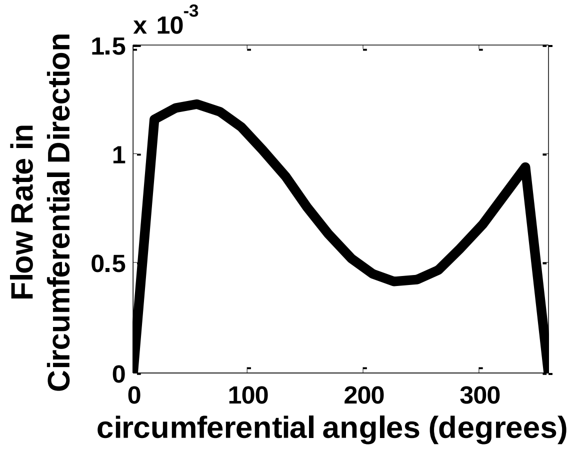


(d) At 420 degrees

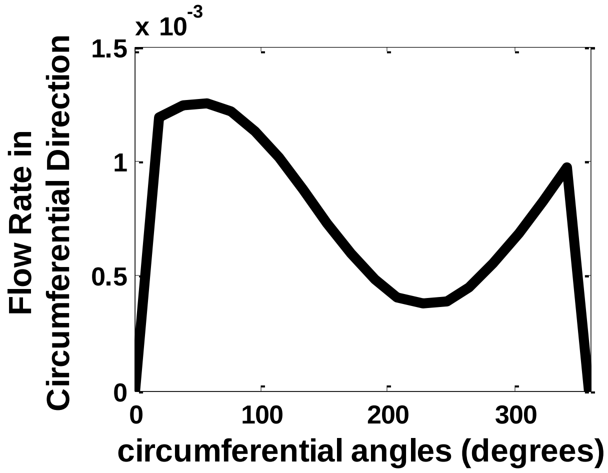
Fig 10: Film thickness for non-Newtonian lubricant

4.2.3 Flow Rate

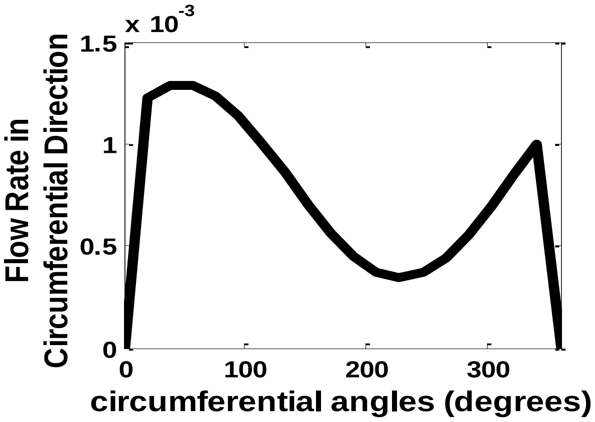
Viscosity of the lubricant increases with an increase in the pressure. This increase in viscosity reduces the flow rate. For the non-Newtonian lubricants flow rate needs to be maintained because as viscosity increases the flow rate is anticipated to decrease. If the flow rate falls significantly, lubricant will not be able to reach the piston resulting in the physical contact between the engine components that is where the pump plays its role and maintains the flow rate. Flow rate in the circumferential direction for selected crank angles is shown in Fig 10. When crank shaft starts rotating flow rate is minimum at that point as shown in Fig 11(a). This flow rate increases as the crankshaft rotates through 120 degrees when piston is past the mid of intake stroke as shown in Fig 11(b). The analysis at the different crank degree rotation angles shows that the flow rate increases as film get thicker continuously at each degree rotation of the crankshaft.



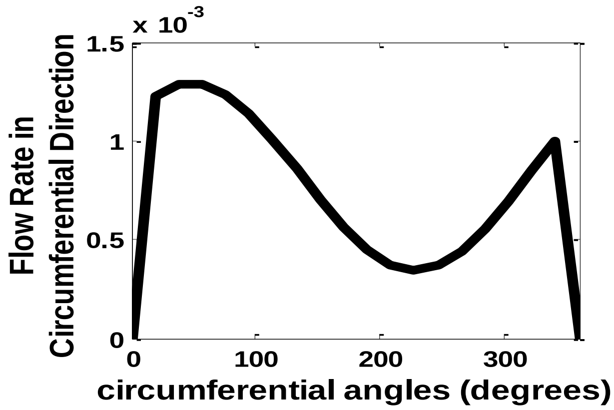
(a) At 1 degree



(b) At 120 degree



(c) At 240 degree



(d) At 420 degree

Fig 11: Flow rate for non-Newtonian lubricant

4.3 Experimental Validation

Results obtained for the viscoelastic lubricant and Newtonian lubricant are compared with those of obtained by Gertzos et. al [43]. Gertzos et. al. measured pressure distribution of viscoelastic liquid and Newtonian Liquid by using Fluent software. His results show that hydrodynamic pressures rise from inlet to 130 degrees and then pressures drop from 130 to 180 degrees. After 180 degrees hydrodynamic pressures become zero. We normalized the axis for the sake of comparison with Fluent software results. Our results are in good accordance with those of obtained by Fluent. The difference in the results obtained numerically and those calculated experimentally is because of the different eccentricity. Authors in [19] used 0.5 eccentricity ratio for both the Newtonian and non-Newtonian lubricant. In our case the eccentricity for Newtonian lubricant was 0.5853 and for non-Newtonian lubricant it was 0.5827.

CHAPTER 5

PARAMETRIC STUDIES

VARIABLE SPEED

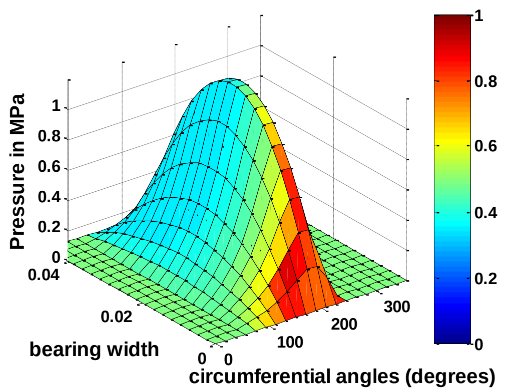
This study models hydrodynamic lubrication of journal bearing at different speeds. Viscoelastic characteristics of a non-Newtonian lubricant are incorporated in the lubrication model. Multigrade oil, 80 μm crankshaft to journal bearing radial clearance and 0.4 eccentricity ratio are some of the inputs used in the model. Various numerical models are developed and results are obtained by simulation. In the first case circumferential speed of 1000 rpm is considered whereas results at 600-2000 rpm are analyzed. The results obtained by simulation of these models are compared and analyzed to study the effect of viscoelastic oil at different speeds. Results for Newtonian model are also obtained at those speeds and compared with viscoelastic model to ensure the benefits of viscoelasticity and optimize the engine speed. Following assumptions are made in order to develop the mathematical model.

1. Thermal effects are ignored.
2. Pressure is not changing along film thickness direction.
3. Flow is laminar.
4. Fluid is inertia less.

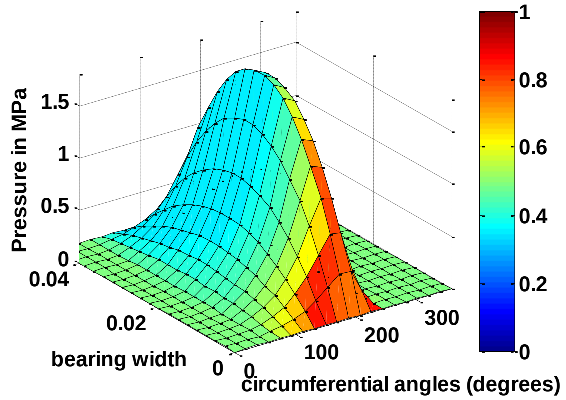
5.1 Newtonian Lubricant

5.1.1 Hydrodynamic Pressure

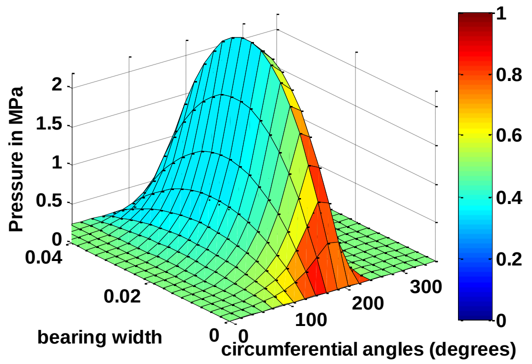
At low engine speed Newtonian lubricant is not able to produce enough hydrodynamic pressure and contact between crankshaft and journal bearing can occur which leads to the wear of crankshaft. Viscoelastic lubricant at low engine speeds produces more hydrodynamic pressure as compared to that of Newtonian lubricant thus preventing the contact between crankshaft and journal bearing. Hydrodynamic pressures develop over the bearing surface at each crank degree of 720^0 cycle. Few of the developed pressure profiles are plotted at the critical positions to analyze the intensities as shown in Fig 12 (d). At 1 degree crank angle rotation when piston has just started the induction stroke, low intensity pressures build up in the bearing. Very small film thickness and low effect of gas force are responsible for the buildup of such low pressures as shown in Fig 12(a). As the crankshaft rotates piston gets displaced in the induction stroke due to which film thickness increases and hydrodynamic pressure rises. At 240 crank degree rotation during the compression stroke the cyclic speed of piston is increasing before reaching the mid of compression stroke. This displacement of piston along with viscoelastic characteristics and decreased minimum film thickness give rise to increased pressure profiles as shown in Fig 12(b). At 420 crank angle rotation i.e. during the power stroke thrust is produced by combustion of gases which significantly increases the magnitude of gas force diagram. Due to this combustion extreme load is transferred to the crankshaft by piston due to which crankshaft moves downward and minimum film thickness reduces. Due to this increase in the magnitude of gas force diagram and decreased minimum film thickness hydrodynamic pressure profile increases significantly as shown in Fig 12(c). It can be seen from the pressure profiles that more hydrodynamic pressures are produced as compared to the Newtonian lubricant which indicates increased load carrying capacity of the viscoelastic lubricant thus minimizing the chance of contact between the two surfaces.



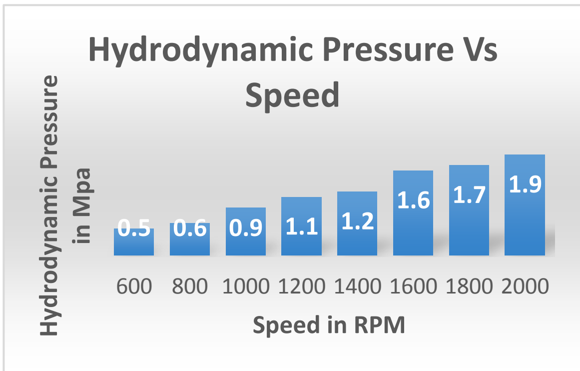
(a) At 1 degree



(b) At 240 degree



(c) At 420 degree



(d) Effect of speed on Hydrodynamic pressure

Fig 12: Hydrodynamic Pressure for Newtonian lubricant

5.1.2 Film Thickness

Hydrodynamic film thickness profile over the 720 degree crank angle rotation is shown in Fig 13 (a). Minimum film thickness is a critical parameter as it is responsible to carry out the hydrodynamic load. Viscoelastic characteristics of a lubricant prevent a substantial decrease in the film thickness as compared to the Newtonian lubricant. When the crankshaft starts rotating during the intake stroke the minimum film thickness has high value because of the almost negligible effect of gas force and low hydrodynamic pressures are produced as shown in Fig 12(a). As the crankshaft rotates through 240 degrees more hydrodynamic pressure is produced as shown in Fig 12(b) in the lubricant which forces the shaft to become eccentric with the journal bearing so maximum film thickness rises and minimum film thickness reduces. At 420 degree crank angle rotation when the piston is in the expansion stroke combustion of gases cause the piston to exert tremendous loads on the crankshaft due to which crankshaft is displaced and minimum film thickness further reduces but viscoelastic characteristics of the lubricant prevent the film thickness to fall to the same value as that of the Newtonian lubricant hence resulting in an improved film thickness and improved pressure fields. A comparison of hydrodynamic film thickness at 1 degree crank rotation for the various speeds is shown in Fig 13.

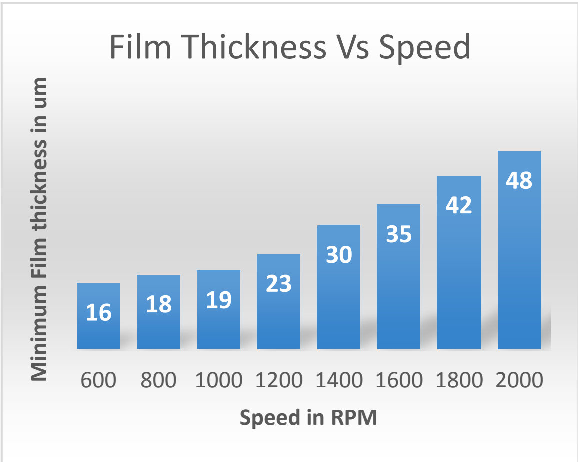
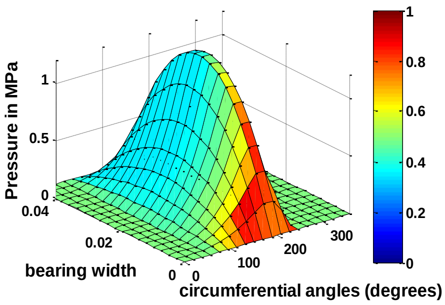


Fig 13: Effect of speed on Film thickness

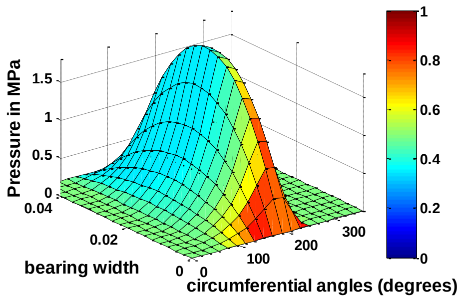
5.2 Non-Newtonian Lubricant

5.2.1 Hydrodynamic Pressure

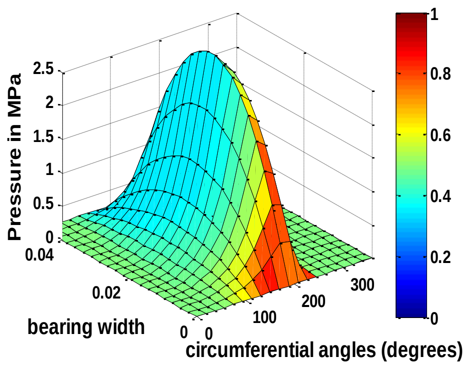
At low engine speed Newtonian lubricant is not able to produce enough hydrodynamic pressure and contact between crankshaft and journal bearing can occur which leads to the wear of crankshaft. Viscoelastic lubricant at low engine speeds produces more hydrodynamic pressure as compared to that of Newtonian lubricant thus preventing the contact between crankshaft and journal bearing. Hydrodynamic pressures develop over the bearing surface at each crank degree of 720^0 cycle. Few of the developed pressure profiles are plotted at the critical positions to analyze the intensities as shown in Fig 14. At 1 degree crank angle rotation when piston has just started the induction stroke, low intensity pressures build up in the bearing. Very small film thickness and low effect of gas force are responsible for the buildup of such low pressures as shown in Fig 14(a). As the crankshaft rotates piston gets displaced in the induction stroke due to which film thickness increases and hydrodynamic pressure rises. At 240 crank degree rotation during the compression stroke the cyclic speed of piston is increasing before reaching the mid of compression stroke. This displacement of piston along with viscoelastic characteristics and decreased minimum film thickness give rise to increased pressure profiles as shown in Fig 14(b). At 420 crank angle rotation i.e. during the power stroke thrust is produced by combustion of gases which significantly increases the magnitude of gas force diagram. Due to this combustion extreme load is transferred to the crankshaft by piston due to which crankshaft moves downward and minimum film thickness reduces. Due to this increase in the magnitude of gas force diagram and decreased minimum film thickness hydrodynamic pressure profile increases significantly as shown in Fig 14(c). It can be seen from the pressure profiles that more hydrodynamic pressures are produced as compared to the Newtonian lubricant which indicates increased load carrying capacity of the viscoelastic lubricant thus minimizing the chance of contact between the two surfaces.



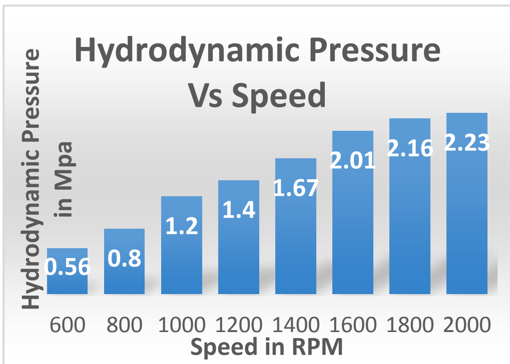
(a) At 1 degree



(b) At 240 Degrees



(c) At 420 degrees



(d) Effect of speed on Hydrodynamic Pressure

Fig 14: Hydrodynamic Pressure for non-Newtonian lubricant

5.2.2 Film Thickness

Hydrodynamic film thickness profile over the 720 degree crank angle rotation is shown in Fig

15 (a). Minimum film thickness is a critical parameter as it is responsible to carry out the hydrodynamic load. Viscoelastic characteristics of a lubricant prevent a substantial decrease in the film thickness as compared to the Newtonian lubricant. When the crankshaft starts rotating during the intake stroke the minimum film thickness has high value because of the almost negligible effect of gas force and low hydrodynamic pressures are produced as shown in Fig 14(a). As the crankshaft rotates through 240 degrees more hydrodynamic pressure is produced as shown in Fig 14(b) in the lubricant which forces the shaft to become eccentric with the journal bearing so maximum film thickness rises and minimum film thickness reduces. At 420 degree crank angle rotation when the piston is in the expansion stroke combustion of gases cause the piston to exert tremendous loads on the crankshaft due to which crankshaft is displaced and minimum film thickness further reduces but viscoelastic characteristics of the lubricant prevent the film thickness to fall to the same value as that of the Newtonian lubricant hence resulting in an improved film thickness and improved pressure fields. A comparison of hydrodynamic film thickness at 1 degree crank rotation for the various speeds is shown in Fig 15.

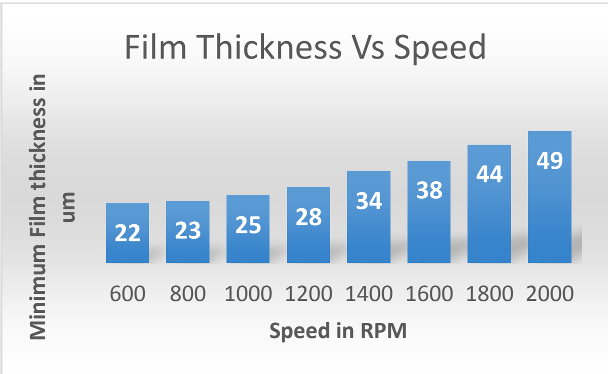


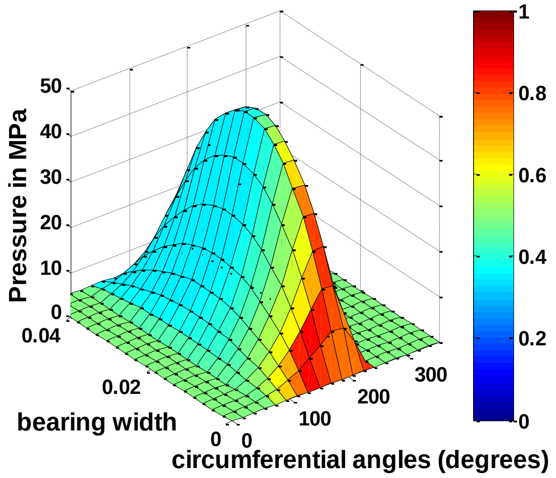
Fig 15: Effect of speed on Film thickness

Variable Clearance

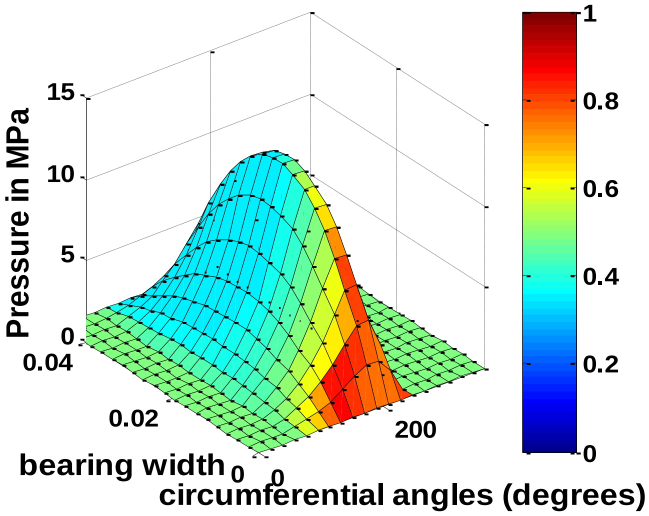
This study models the Newtonian and non-Newtonian behavior of lubricant in the crankshaft journal bearing at different radial clearance. The results obtained by simulation of these models are compared and analyzed to study the effect of viscoelastic oil at different clearances. Results for Newtonian model are also obtained at those clearances and compared with viscoelastic model to ensure the benefits of viscoelasticity and optimize the radial clearance. 600 rpm, 0.4 eccentricity ratio and 1 degree crank rotation angle are some of the parameters for this study.

5.3 Newtonian Lubricant

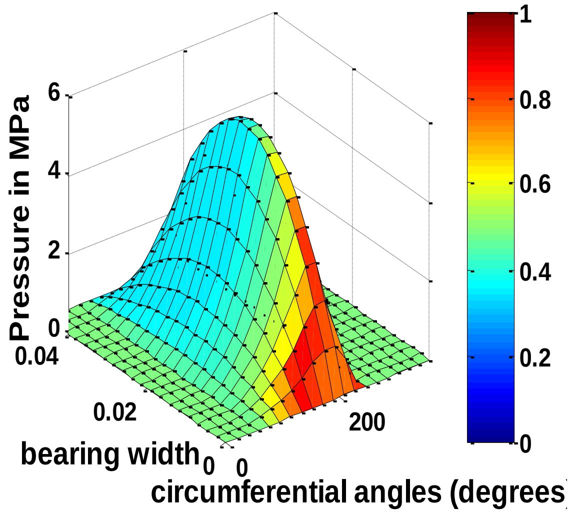
A Newtonian lubricant is used to model the hydrodynamic lubrication at different radial clearances. It is seen that magnitude of minimum film thickness increases with increasing the radial clearance which means that it is beneficial to have a greater radial clearance. But on the other hand the hydrodynamic pressure reduces which means that such a radial clearance should be set on which minimum film thickness is thick enough to carry out the hydrodynamic loads and also enough hydrodynamic pressures are produced so that metal to metal contact can be avoided to prevent failure of crankshaft. At 10 um a very thin film is observed which means that chances of physical contact are increased at very low radial clearance. As radial clearance increases minimum film thickness increases but hydrodynamic load reduces immediately which means that lubricant may not be able to lift up the crankshaft and during the expansion stroke when piston transfers load on to the crankshaft due to burning of gases, the lubrication might fail and adhesive wear can occur. Hydrodynamic pressures and minimum film thickness are given in the table below.



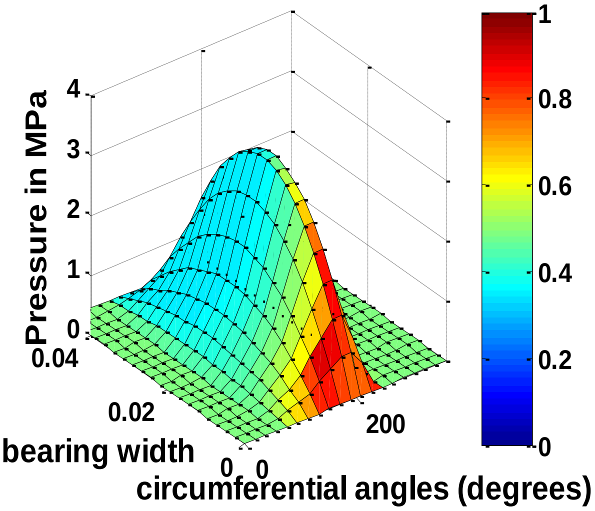
(a) At 10 um



(b) At 20 um



(c) At 30 um



(d) At 40 um

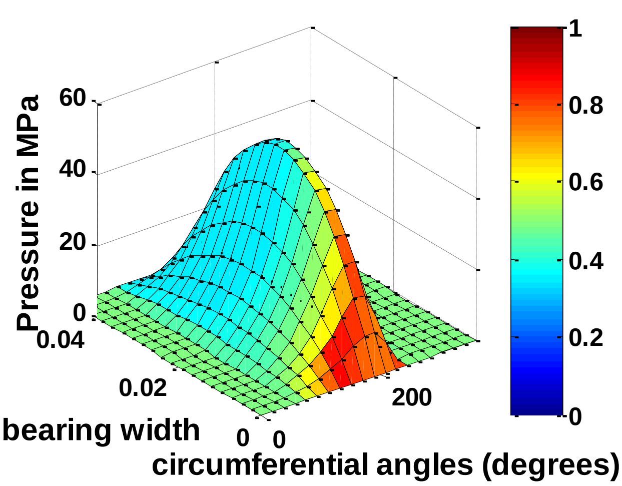
Fig 16: Hydrodynamic Pressure for Newtonian Lubricant

Clearance (um)	Hydrodynamic Pressure (Mpa)	Minimum Film Thickness (um)
10	45.35	6
20	11.4	13
30	5.46	16
40	3.017	22
50	1.8	30
60	1.26	35
70	0.9255	40
80	0.7086	44
90	0.5599	52
100	0.4535	59
110	0.3748	68
120	0.3149	73

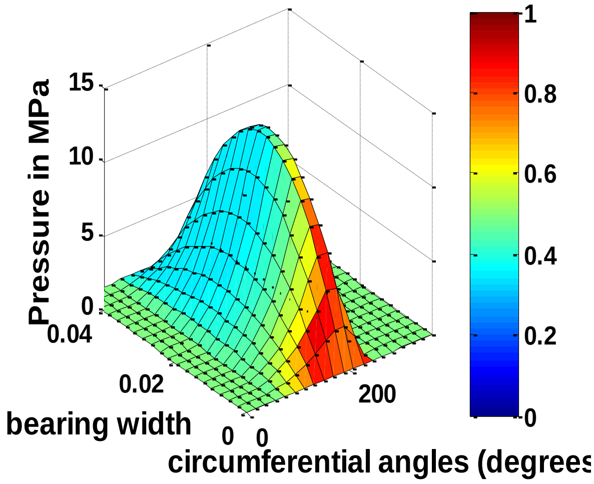
Table 2: Effect of Clearance on hydrodynamic pressure and film thickness for Newtonian lubricant

5.4 Non-Newtonian Lubricant

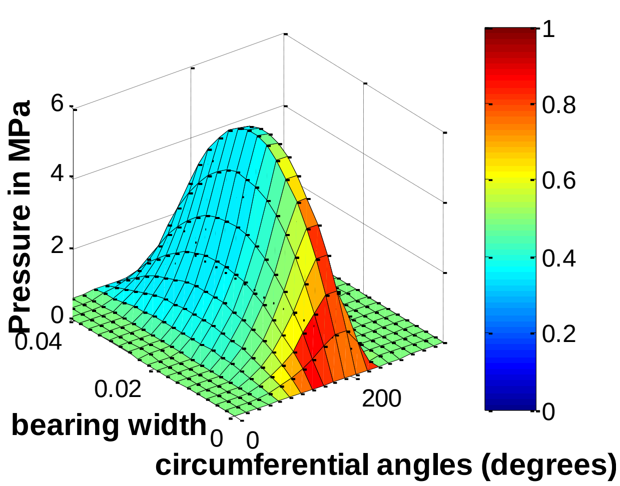
Non-Newtonian lubricant is used to model the hydrodynamic lubrication at the same radial clearances as those of the Newtonian lubricant. It is seen clearly that non-Newtonian lubricant produces more hydrodynamic pressure as compared to the Newtonian lubricant. Viscoelastic lubricant also does not allow the minimum film thickness to fall below a certain value resulting in thicker films and increased load carrying capacity. With the increase in radial clearance magnitude of minimum film thickness increases which means that increasing clearance is beneficial for the load carrying capacity but on the other hand hydrodynamic pressure reduces which means that when crankshaft is pushed downwards due to combustion of gases during the power stroke the lubricant might not be able to prevent the contact between crankshaft and journal bearing and as a result crankshaft might get damaged so an optimum clearance should be chosen at which hydrodynamic film thickness is also thick enough and sufficient hydrodynamic pressures are produced to prevent the adhesive wear. It is seen that more hydrodynamic pressures are produced for the non-Newtonian lubricant which implies beneficial effects of viscoelasticity. Hydrodynamic pressures and minimum film thickness for various clearances are shown in the table below.



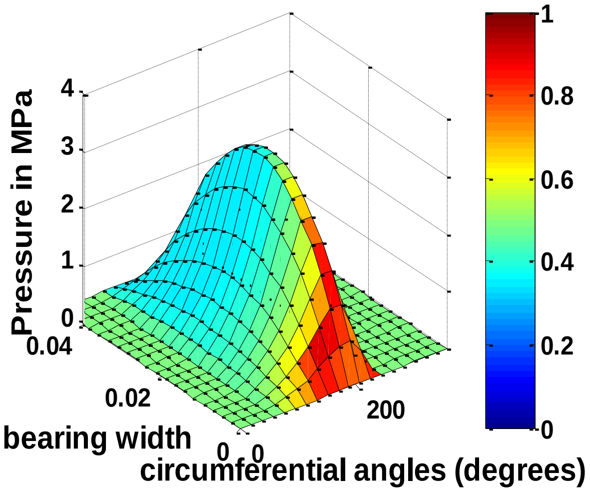
(a) At 10 um



(b) At 20 um



(c) At 30 um



(d) At 40 um

Fig 17: Hydrodynamic Pressure for Newtonian Lubricant

Clearance (um)	Hydrodynamic Pressure (Mpa)	Minimum Film Thickness (um)
10	48.27	8
20	12.07	15
30	5.68	21
40	3.47	27
50	1.931	33
60	1.341	40
70	0.9852	42
80	0.7543	50
90	0.596	57

100	0.4827	62
110	0.4	71
120	0.3352	75

Table 3: Effect of Clearance on hydrodynamic pressure and film thickness for non-Newtonian lubricant

CHAPTER 6

CONCLUSION

This study models the behavior of Newtonian and non-Newtonian lubricant at various speeds. Some important results are drawn by comparing the two models. While using the Newtonian lubricant there is an increased chance of contact between the bearing and crankshaft as not enough hydrodynamic pressure is produced which leads to wear of the crankshaft ultimately leading to failure of the crankshaft. When using Newtonian lubricant there is a significant reduction in the film thickness as the crankshaft rotates through 720 degrees. Engine operation at low speed while using Newtonian lubricant is not suitable as the contact reduces life of the crankshaft. Using a viscoelastic lubricant reduces the chance of journal bearing to crankshaft contact at low rotation speed of crankshaft since more pressures are produced as compared to the Newtonian lubricant indicating enhanced load carrying ability of the viscoelastic lubricant.

In the viscoelastic lubricant reduction in film thickness is less than that of the Newtonian lubricant. At 600 rpm low hydrodynamic pressures are produced for the Newtonian lubricant which is not suitable to carry out the hydrodynamic load. As the radial clearance increases hydrodynamic film thickness improves which means that a greater clearance is necessary for the improved engine design. An engine speed of 1600 rpm and 50 um radial clearance is proposed for the Newtonian lubricant since after which hydrodynamic pressures do not increase that much. A speed of 1400 rpm and 50 um radial clearance is proposed for the viscoelastic lubricant since they produce same amount of hydrodynamic pressure at 1400 rpm as that of Newtonian lubricant at 1600 rpm resulting in reduced fuel consumption.

FUTURE RECOMMENDATIONS

Viscoelastic lubricants produce higher hydrodynamic pressures but they also have shear thinning effects which may affect the beneficial effects of viscoelastic lubricants. This study needs to be extended to studying of shear thinning effects which will optimize the performance of viscoelastic lubricants in journal bearing of a high torque low speed diesel engine. The effects of viscous shear heating are ignored in this study which need to be considered for future studies.

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APPENDIX 1

Navier Stokes Equation

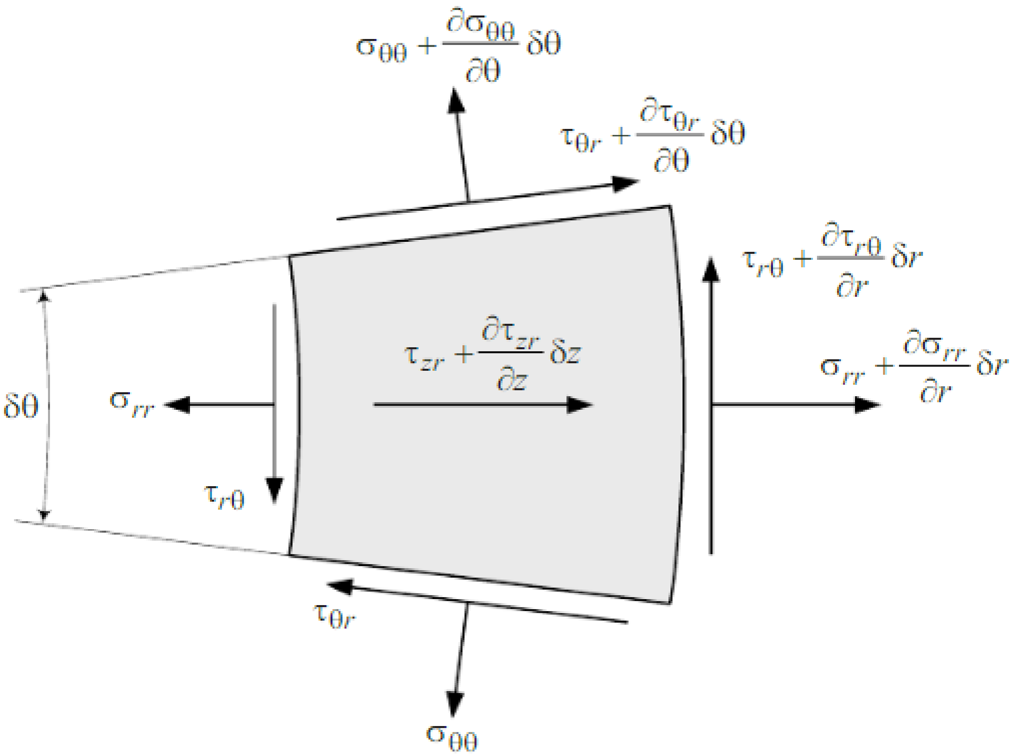

Fig 4: Stress distribution over fluid particle

Using Newton's second law, the equilibrium equations for an infinitesimal element in a cylindrical coordinates will be developed

$$m.\vec{a} = \sum \vec{F}$$

$$m.\frac{D\vec{V}}{Dt} = \sum \vec{F} \tag{1}$$

Total force is the sum of body force and surface force

$$\sum \vec{F} = \sum \vec{F}_{body} + \sum \vec{F}_{surface}$$

Surface Force

Surface forces are given as

$$\frac{\partial \sigma_i}{\partial x_i} dr d\theta dz$$

$$\frac{\partial \tau_{ij}}{\partial x_j} dr d\theta dz$$

Body Force

Body forces are given as

$$F_r drd\theta dz$$

$$F_\theta drd\theta dz$$

$$F_z drd\theta dz$$

Inertial Forces

$$V(r,\theta,z,t)=u_r(r,\theta,z,t)e_r+u_\theta(r,\theta,z,t)e_\theta+u_z(r,\theta,z,t)e_z$$

Time derivative of velocity is given as

$$\frac{DV}{Dt}=\frac{\partial V}{\partial t}+\frac{Dr}{Dt}\frac{\partial V}{\partial r}+\frac{D\theta}{Dt}\frac{\partial V}{\partial r}+\frac{Dz}{Dt}\frac{\partial V}{\partial r} \tag{2}$$

Where

$$\frac{\partial V}{\partial t}=\frac{\partial u_r}{\partial t}e_r+\frac{\partial u_\theta}{\partial t}e_\theta+\frac{\partial u_z}{\partial t}e_z$$

Rest of the terms is above equation are given as

$$\frac{Dr}{Dt}\frac{\partial V}{\partial r}=u_r\frac{\partial V}{\partial r}=u_r\left[\frac{\partial u_r}{\partial r}e_r+\frac{\partial u_\theta}{\partial r}e_\theta+\frac{\partial u_z}{\partial z}e_z\right]$$

$$\frac{D\theta}{Dt}\frac{\partial V}{\partial \theta}=\dot{\theta}\frac{\partial V}{\partial \theta}=\frac{u_\theta}{r}\left[\frac{\partial u_r}{\partial \theta}e_r+\frac{\partial u_\theta}{\partial \theta}e_\theta+\frac{\partial u_z}{\partial \theta}e_z\right]$$

$$\frac{Dz}{Dt}\frac{\partial V}{\partial z}=u_z\frac{\partial V}{\partial z}=u_z\left[\frac{\partial u_r}{\partial z}e_r+\frac{\partial u_\theta}{\partial z}e_\theta+\frac{\partial u_z}{\partial z}e_z\right]$$

Material derivative of the velocity is given as

$$\begin{aligned}\frac{DV}{Dt}&=u_r\left[\frac{\partial u_r}{\partial r}e_r+\frac{\partial u_\theta}{\partial r}e_\theta+\frac{\partial u_z}{\partial z}e_z\right]+\frac{u_\theta}{r}\left[\frac{\partial u_r}{\partial \theta}e_r+\frac{\partial u_\theta}{\partial \theta}e_\theta+\frac{\partial u_z}{\partial \theta}e_z\right]\\&+u_z\left[\frac{\partial u_r}{\partial z}e_r+\frac{\partial u_\theta}{\partial z}e_\theta+\frac{\partial u_z}{\partial z}e_z\right]\end{aligned} \tag{3}$$

Inertial Forces in r, θ and z direction are given as

$$\rho\frac{Du_r}{Dt}drd\theta dz,\rho\frac{Du_\theta}{Dt}drd\theta dz,\rho\frac{Du_z}{Dt}drd\theta dz$$

Equilibrium

Sum of boy forces and surface forces is equal to inertial forces

In the r direction

$$\rho\frac{Du_r}{Dt}drd\theta dz = F_r drd\theta dz + \frac{\partial\sigma_r}{\partial r}drd\theta dz + \frac{1}{r}\frac{\partial t_{r\theta}}{\partial\theta}drd\theta dz + \frac{\partial t_{rz}}{\partial z}drd\theta dz$$

$$\rho\frac{Du_r}{Dt} = F_r + \frac{\partial\sigma_r}{\partial r} + \frac{1}{r}\frac{\partial t_{r\theta}}{\partial\theta} + \frac{\partial t_{rz}}{\partial z} \quad (4)$$

According to Newton's Law

$$\sigma_r = -p + \mu\frac{\partial u_r}{\partial r}$$

$$\sigma_\theta = -p + \frac{\mu}{r}\frac{\partial u_\theta}{\partial\theta}$$

$$\sigma_z = -p + \mu\frac{\partial u_z}{\partial z}$$

$$\tau_{r\theta} = \mu\left(\frac{\partial u_\theta}{\partial r} + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)$$

$$\tau_{rz} = \mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)$$

$$\tau_{\theta z} = \mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial\theta}\right)$$

Putting values of stresses in equation (4)

$$\rho\frac{Du_r}{Dt} = F_r\rho - \frac{\partial p}{\partial r} + \mu\frac{\partial^2 u_r}{\partial r^2} + \left(\frac{1}{r}\frac{\partial}{\partial\theta}\left(\mu\left(\frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{\partial\theta}\right)\right) + \frac{\partial}{\partial z}\left(\mu\left(\frac{\partial u_r}{\partial z} + \frac{\partial u_z}{\partial r}\right)\right)\right)$$

$$\rho\frac{Du_r}{Dt} = F_r\rho - \frac{\partial p}{\partial r} + \mu\frac{\partial^2 u_r}{\partial r^2} + \frac{\mu}{r}\frac{\partial^2 u_r}{\partial\theta^2} + \frac{\mu}{r}\frac{\partial^2 u_\theta}{\partial r\partial\theta} + \mu\frac{\partial^2 u_r}{\partial z^2} + \mu\frac{\partial^2 u_z}{\partial r\partial z} \quad (5)$$

$$\rho\frac{Du_r}{Dt} = F_r\rho - \frac{\partial p}{\partial r} + \mu\frac{\partial^2 u_r}{\partial r^2} + \frac{\mu}{r}\frac{\partial^2 u_r}{\partial\theta^2} + \mu\frac{\partial^2 u_r}{\partial z^2} \quad (6)$$

Similarly for θ and z direction

$$\rho\frac{Du_\theta}{Dt} = F_\theta\rho - \frac{\partial p}{\partial\theta} + \mu\frac{\partial^2 u_\theta}{\partial r^2} + \frac{\mu}{r}\frac{\partial^2 u_\theta}{\partial\theta^2} + \mu\frac{\partial^2 u_\theta}{\partial z^2} \quad (7)$$

$$\rho\frac{Du_z}{Dt} = F_z\rho - \frac{\partial p}{\partial z} + \mu\frac{\partial^2 u_z}{\partial r^2} + \frac{\mu}{r}\frac{\partial^2 u_z}{\partial\theta^2} + \mu\frac{\partial^2 u_z}{\partial z^2} \quad (8)$$

Reynolds Equation

Now apply the following assumptions on above equations

1. The fluid is inertia less.
2. Body forces are ignored.
3. Velocity gradient along the film thickness direction dominate the other two directions.

4. Lubricant is Newtonian.
5. No slip condition at the boundaries.
6. Pressure is not changing along the film thickness direction.
7. Adiabatic heating.

$$\rho \frac{Du_z}{Dt} = F_z\rho - \frac{\partial p}{\partial r} + \mu \frac{\partial^2 u_z}{\partial r^2} + \frac{\mu}{r}\frac{\partial^2 u_z}{\partial \theta^2} + \mu \frac{\partial^2 u_z}{\partial z^2}$$

$$\frac{\partial p}{\partial z} = \frac{\partial}{\partial r}\left(\mu \frac{\partial u_z}{\partial r}\right) \tag{9}$$

$$\frac{\partial p}{\partial \theta} = \frac{\partial}{\partial r}\left(\mu \frac{\partial u_\theta}{\partial r}\right) \tag{10}$$

Integrating equation (9) and (10)

$$\frac{c}{\mu}\frac{\partial p}{\partial z} + \frac{A}{\mu} = \frac{\partial u_z}{\partial r} \tag{11}$$

$$\frac{c}{\mu}\frac{\partial p}{\partial \theta} + \frac{C}{\mu} = \frac{\partial u_\theta}{\partial r} \tag{12}$$

Again integrate equation (11) and (12)

$$u_z = \frac{\partial p}{\partial z}\frac{c^2}{2\mu} + \frac{A}{\mu}c + B \tag{13}$$

$$u_\theta = \frac{1}{r}\frac{\partial p}{\partial \theta}\frac{c^2}{\mu} + \frac{C}{\mu}c + D \tag{14}$$

Boundary conditions for velocity in circumferential direction are [39]

$u_\theta = U_b$ at $c = 0$ and $u_\theta = U_a$ at $c = h$

$u_z = V_a$ at $z = 0$ and $u_z = V_b$ at $c = h$

Applying these boundary conditions

at $c = 0$

$U_b = D$

At $c = h$

$$U_a = \frac{1}{r}\frac{\partial p}{\partial \theta}\frac{h^2}{\mu} + \frac{C}{\mu}h + U_b$$

$$C = -\frac{\mu}{rh}\left(\frac{h^2}{2\mu}\frac{\partial p}{\partial \theta}\right) + U_a\frac{\mu}{h} - U_b\frac{\mu}{h} \tag{15}$$

Putting value of equation (15) into (14)

$$u_\theta = \frac{z^2}{2\mu}\frac{\partial p}{\partial \theta} - \frac{ch}{2\mu}\frac{\partial p}{\partial \theta} - \frac{c}{h}\left(U_b - U_a\right) + U_b \tag{16}$$

Similarly in z-direction

$$u_z = \frac{c^2}{2\mu}\frac{\partial p}{\partial r} - \frac{ch}{2\mu}\frac{\partial p}{\partial r} - \frac{c}{h}\left(V_b - V_a\right) + V_b \tag{17}$$

To calculate changes in the velocity in film thickness direction we take derivative of the equation (16) and (17) w.r.t z

$$\frac{\partial u_\theta}{\partial r} = \frac{c}{\mu r}\frac{\partial p}{\partial \theta} - \frac{h}{2\mu}\frac{\partial p}{\partial \theta} - \left(\frac{U_b - U_a}{h}\right)$$

$$\frac{\partial u_\theta}{\partial r} = \left(\frac{2c-h}{2\mu r}\right)\frac{\partial p}{\partial \theta} - \left(\frac{U_b - U_a}{h}\right) \tag{18}$$

$$\frac{\partial u_z}{\partial r} = \left(\frac{2c-h}{2\mu}\right)\frac{\partial p}{\partial z} - \left(\frac{V_b - V_a}{h}\right) \tag{19}$$

Volumetric flow rate per unit length in the r and θ direction are

$$q_\theta = AV \qquad q_\theta = u_\theta dr dz$$

$$q_\theta' = u_\theta dz \qquad (20) \qquad q_\theta' = \frac{q_\theta}{dr}$$

Integrating equation 20 through whole film thickness

$$q_\theta' = \int_0^h u_\theta dz$$

$$q_z' = \int_0^h u_z dz$$

$$q_\theta' = \left(-\frac{h^3}{12\mu}\right)\frac{1}{r}\frac{\partial p}{\partial \theta} + \left(\frac{U_a + U_b}{2}\right)h \tag{21}$$

$$q_z' = \left(-\frac{h^3}{12\mu}\right)\frac{\partial p}{\partial z} + \left(\frac{V_a + V_b}{2}\right)h \tag{22}$$

Now consider continuity equation

$$\int_0^h \left(\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) + \frac{\partial}{\partial r}\left(\rho u_r\right) + \frac{\partial}{\partial z}\left(\rho u_z\right)\right) = 0 \tag{23}$$

$$\int_0^h \frac{\partial \rho}{\partial t} = \frac{\partial \rho}{\partial t} h$$

$$\int_0^h \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho u_\theta\right) dz = -\frac{1}{r}\left(\rho U_a\right)\frac{\partial h}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho \int_0^h u_\theta dz\right)$$

$$\int_0^h \frac{\partial}{\partial z}(\rho u_z)dz = -(\rho V_a)\frac{\partial h}{\partial z} + \frac{\partial}{\partial z}\left(\rho\int_0^h u_z dr\right)$$

$$\int_0^h \frac{\partial}{\partial r}(\rho u_r)dr = \rho(w_a - w_b)$$

Putting these values in equation (23)

$$\frac{\partial \rho}{\partial t}h - \frac{1}{r}(\rho U_a)\frac{\partial h}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho\int_0^h u_\theta dz\right) - (\rho V_a)\frac{\partial h}{\partial z} + \frac{\partial}{\partial z}\left(\rho\int_0^h u_z dr\right) + \rho(w_a - w_b) = 0$$

Putting flow rates in above equation

$$\frac{\partial \rho}{\partial t}h - \frac{1}{r}(\rho U_a)\frac{\partial h}{\partial \theta} + \frac{1}{r}\frac{\partial}{\partial \theta}\left(\rho\int_0^h q_\theta{}' dz\right) - (\rho V_a)\frac{\partial h}{\partial z} + \frac{\partial}{\partial z}\left(\rho\int_0^h q_r{}' dz\right) + \rho(w_a - w_b) = 0$$

After putting equation (21) and (22) final form of the Reynolds equation becomes

$$\frac{\rho}{r^2}\frac{\partial}{\partial \theta}\left[\frac{\rho h^3}{12\mu}\left(\frac{\partial p}{\partial \theta}\right)\right] + \frac{\partial}{\partial z}\left[\frac{\rho h^3}{12\mu}\left(\frac{\partial p}{\partial z}\right)\right] = \frac{U}{r}\frac{\partial}{\partial \theta}\left[\frac{\rho h}{2}\right] + V\frac{\partial}{\partial z}\left[\frac{\rho h}{2}\right] + \frac{\partial}{\partial t}\left[(\rho h)\right] \qquad (24)$$

Non-Dimensionalization of Reynolds Equation

Following are the parameters used to non-dimensionalize Reynolds equation

$$\hat{\theta} = \frac{\theta}{\theta_{ref}}, \hat{z} = \frac{z}{L}, \hat{h} = \frac{h}{c}, \hat{\mu} = \frac{\mu}{\mu_{ref}}, \hat{t} = \frac{t}{t_{ref}}, \hat{p} = \frac{p}{p_{ref}}, K = \frac{r\theta_{ref}}{L}, \mathrm{P}_{ref} = \frac{6\mu_{ref}UD}{c^2}$$

$$\gamma_1 = \frac{6\mu_{ref}\theta_{ref}V_\theta}{p_{ref}c^2}, \gamma_2 = \frac{6\mu_{ref}K\theta_{ref}RV_z}{p_{ref}c^2}, \gamma = \frac{12\mu_{ref}\theta_{ref}{}^2R^2}{p_{ref}c^2t_{ref}}$$

Putting these parameters into left hand side of equation (24)

$$\frac{\partial}{r^2\partial(\hat{\theta}\theta_{ref})}\left(\frac{(\rho\hat{h}c)^3}{12(\hat{\mu}\mu_{ref})}\frac{\partial p}{\partial \theta}\right) + \frac{\partial}{\partial(\hat{z}L)}\left(\frac{(\rho\hat{h}c)^3}{12(\hat{\mu}\mu_{ref})}\frac{\partial p}{\partial z}\right) \qquad (25)$$

We have a relation for pressure

$$p = (p_{ref} - p_c)\hat{p} + p_c$$

$$\frac{\partial p}{\partial \theta} = \frac{\partial}{\partial \theta}(p_{ref} - p_c)\hat{p} \qquad (26)$$

$$\frac{\partial p}{\partial z} = \frac{\partial}{\partial z}(p_{ref} - p_c)\hat{p} \qquad (27)$$

Putting equation (26) and (27) in (25)

$$\frac{\rho_{ref}\partial}{r^2\partial(\hat{\theta}\theta_{ref})}\left(\frac{\left(\hat{\rho}\hat{h}c\right)^3}{12\left(\hat{\mu}\mu_{ref}\right)}\frac{\partial\hat{p}}{\partial(\hat{\theta}\theta_{ref})}\left(p_{ref}-p_c\right)\right)+\frac{\rho_{ref}\partial}{\partial(\hat{z}L)}\left(\frac{\left(\hat{\rho}\hat{h}c\right)^3}{12\left(\hat{\mu}\mu_{ref}\right)}\frac{\partial\hat{p}}{\partial\hat{z}L}\left(p_{ref}-p_c\right)\right)$$

Simplifying above equation

$$\frac{\rho_{ref}c^3\left(p_{ref}-p_c\right)}{12r^2\mu_{ref}\theta_{ref}{}^2}\frac{\partial}{\partial\hat{\theta}}\left(\frac{\hat{\rho}\hat{h}^3}{\hat{\mu}}\frac{\partial\hat{p}}{\partial\hat{\theta}}\right)+\frac{\rho_{ref}c^3\left(p_{ref}-p_c\right)}{12\mu_{ref}L^2}\frac{\partial}{\partial(\hat{z}L)}\left(\frac{\hat{\rho}\hat{h}^3}{\hat{\mu}}\frac{\partial\hat{p}}{\partial\hat{z}}\right) \tag{28}$$

Now consider RHS of Reynolds equation

$$\frac{\rho_{ref}U}{r}\frac{\partial}{\partial\hat{\theta}\theta_{ref}}\left[\frac{\hat{\rho}\hat{h}c}{2}\right]+\rho_{ref}V\frac{\partial}{\partial\hat{z}L}\left[\frac{\hat{\rho}\hat{h}c}{2}\right]+\rho_{ref}\frac{\partial}{\partial\hat{t}t_{ref}}\left[\hat{\rho}\hat{h}c\right] \tag{29}$$

$$\frac{\rho cU}{2\theta_{ref}r}\frac{\partial}{\partial\hat{\theta}}\left[\hat{h}\right]+\frac{\rho cV}{2L}\frac{\partial}{\partial\hat{z}}\left[\hat{h}\right]+\frac{\rho c}{t_{ref}}\frac{\partial}{\partial\hat{t}}\left[\hat{h}\right] \tag{30}$$

Equating equations (28) and (30)

$$\begin{aligned}&\frac{\rho_{ref}c^3\left(p_{ref}-p_c\right)}{12r^2\mu_{ref}\theta_{ref}{}^2}\frac{\partial}{\partial\hat{\theta}}\left(\frac{\hat{\rho}\hat{h}^3}{\hat{\mu}}\frac{\partial\hat{p}}{\partial\hat{\theta}}\right)+\frac{\rho_{ref}c^3\left(p_{ref}-p_c\right)}{12\mu_{ref}L^2}\frac{\partial}{\partial(\hat{z}L)}\left(\frac{\hat{\rho}\hat{h}^3}{\hat{\mu}}\frac{\partial\hat{p}}{\partial\hat{z}}\right)\\&=\frac{\rho_{ref}cU}{2\theta_{ref}r}\frac{\partial}{\partial\hat{\theta}}\left[\hat{\rho}\hat{h}\right]+\frac{\rho cV}{2L}\frac{\partial}{\partial\hat{z}}\left[\hat{\rho}\hat{h}\right]+\frac{\rho_{ref}c}{t_{ref}}\frac{\partial}{\partial\hat{t}}\left[\hat{\rho}\hat{h}\right]\end{aligned} \tag{31}$$

Multiply above equation with $\dfrac{12\mu_{ref}D^2}{\rho_{ref}c^3\left(p_{ref}-p_c\right)}$ and solve [38]

$$\frac{\partial}{\partial\hat{\theta}}\left[\frac{\hat{\rho}\hat{h}^3}{\hat{\mu}}\left(\frac{\partial\hat{p}}{\partial\hat{\theta}}\right)\right]+\left(\frac{\theta}{L}\right)^2\frac{\partial}{\partial\hat{z}}\left[\frac{\hat{\rho}\hat{h}^3}{\hat{\mu}}\left(\frac{\partial\hat{p}}{\partial\hat{z}}\right)\right]=$$

$$\frac{6\mu_{ref}\theta_{ref}U}{\left(p_{ref}-p_c\right)c^2}\left[\frac{\partial}{\partial\hat{\theta}}\left(\hat{\rho}\hat{h}\right)\right]+\frac{6\mu_{ref}\theta^2{}_{ref}rV}{\left(p_{ref}-p_c\right)Lc^2}\left[\frac{\partial}{\partial\hat{z}}\left(\hat{\rho}\hat{h}\right)\right]+\frac{12\mu_{ref}\theta_{ref}{}^2r^2}{\left(p_{ref}-p_c\right)c^2t_{ref}}\left[\frac{\partial}{\partial\hat{t}}\left(\hat{\rho}\hat{h}\right)\right]$$

$$\frac{\partial}{\partial\hat{\theta}}\left[\frac{\hat{h}^3}{\hat{\mu}}\left(\frac{\partial\hat{p}}{\partial\hat{\theta}}\right)\right]+K^2\frac{\partial}{\partial\hat{z}}\left[\frac{\hat{h}^3}{\hat{\mu}}\left(\frac{\partial\hat{p}}{\partial\hat{z}}\right)\right]=\gamma_1\left[\frac{\partial}{\partial\hat{\theta}}\left(\hat{h}\right)\right]+\gamma_2\left[\frac{\partial}{\partial\hat{z}}\left(\hat{h}\right)\right]+\gamma\left[\frac{\partial}{\partial\hat{t}}\left(\hat{h}\right)\right] \tag{32}$$

Reynolds Equation with Vogelpohl parameter

The Vogelpohl parameter is defined as follows [40]

$$M_v=\hat{p}\hat{h}^{1.5}$$

$$\hat{p}=M_v\hat{h}^{-1.5}$$

$$\frac{\partial\hat{p}}{\partial\hat{\theta}}=\frac{\partial}{\partial\hat{\theta}}\left(M_v\hat{h}^{-1.5}\right)$$

$$\frac{\partial \hat{p}}{\partial \hat{\theta}} = \hat{h}^{-1.5} \frac{\partial M_v}{\partial \hat{\theta}} + M_v \frac{\partial \hat{h}^{-1.5}}{\partial \hat{\theta}}$$

$$\frac{\partial \hat{p}}{\partial \hat{\theta}} = \hat{h}^{-1.5} \frac{\partial M_v}{\partial \hat{\theta}} + M_v \left(-1.5 \hat{h}^{-2.5} \frac{\partial \hat{h}}{\partial \hat{\theta}} \right)$$

$$\frac{\partial \hat{p}}{\partial \hat{\theta}} = \hat{h}^{-1.5} \frac{\partial M_v}{\partial \hat{\theta}} - 1.5 M_v \hat{h}^{-2.5} \frac{\partial \hat{h}}{\partial \hat{\theta}}$$

Multiply above equation with $\hat{h}^3$

$$\hat{h}^3 \frac{\partial \hat{p}}{\partial \hat{\theta}} = \hat{h}^{1.5} \frac{\partial M_v}{\partial \hat{\theta}} - 1.5 M_v \hat{h}^{0.5} \frac{\partial \hat{h}}{\partial \hat{\theta}}$$

Multiply above equation with $\hat{\rho}$ and divide with $\hat{\mu}$

$$\frac{\hat{\rho}\hat{h}^3}{\hat{\mu}} \frac{\partial \hat{p}}{\partial \hat{\theta}} = \frac{\hat{\rho}\hat{h}^{1.5}}{\hat{\mu}} \frac{\partial M_v}{\partial \hat{\theta}} - \frac{1.5 \hat{\rho} M_v \hat{h}^{0.5}}{\hat{\mu}} \frac{\partial \hat{h}}{\partial \hat{\theta}} \tag{33}$$

Taking derivative of equation (33) w.r.t $\hat{\theta}$

$$\frac{\partial}{\partial \hat{\theta}} \left(\frac{\hat{\rho}\hat{h}^3}{\hat{\mu}} \frac{\partial \hat{p}}{\partial \hat{\theta}} \right) = \frac{\partial}{\partial \hat{\theta}} \left(\frac{\hat{\rho}\hat{h}^{1.5}}{\hat{\mu}} \frac{\partial M_v}{\partial \hat{\theta}} \right) - 1.5 \frac{\partial}{\partial \hat{\theta}} \left(\frac{\hat{\rho} M_v \hat{h}^{0.5}}{\hat{\mu}} \frac{\partial \hat{h}}{\partial \hat{\theta}} \right)$$

$$= \left[\left(\hat{h}^{1.5} \hat{\mu}^{-1} \frac{\partial \hat{\rho}}{\partial \hat{\theta}} + \rho \hat{h}^{1.5} \frac{\partial \hat{\mu}^{-1}}{\partial \hat{\theta}} + \hat{\mu}^{-1} \hat{\rho} \frac{\partial \hat{h}^{1.5}}{\partial \hat{\theta}} \right) \frac{\partial M_v}{\partial \hat{\theta}} + \frac{\hat{\rho}\hat{h}^{1.5}}{\hat{\mu}} \left(\frac{\partial^2 M_v}{\partial \hat{\theta}^2} \right) \right]$$

$$-1.5 \left[\left(M_v \hat{h}^{0.5} \hat{\mu}^{-1} \frac{\partial \hat{\rho}}{\partial \hat{\theta}} \hat{\rho} M_v \hat{h}^{0.5} \frac{\partial \hat{\mu}^{-1}}{\partial \hat{\theta}} + \hat{\rho} \hat{h}^{0.5} \hat{\mu}^{-1} \frac{\partial M_v}{\partial \hat{\theta}} + \hat{\rho} M_v \hat{\mu}^{-1} \frac{\partial \hat{h}^{0.5}}{\partial \hat{\theta}} \right) \frac{\partial \hat{h}}{\partial \hat{\theta}} + \frac{\hat{\rho} M_v \hat{h}^{0.5}}{\hat{\mu}} \left(\frac{\partial^2 \hat{h}}{\partial \hat{\theta}^2} \right) \right]$$

$$= \left[\left(\hat{h}^{1.5} \hat{\mu}^{-1} \frac{\partial \hat{\rho}}{\partial \hat{\theta}} \frac{\partial M_v}{\partial \hat{\theta}} - \rho \hat{h}^{1.5} \hat{\mu}^{-2} \frac{\partial \hat{\mu}}{\partial \hat{\theta}} \frac{\partial M_v}{\partial \hat{\theta}} + 1.5 \hat{\mu}^{-1} \rho \hat{h}^{0.5} \frac{\partial \hat{h}}{\partial \hat{\theta}} \frac{\partial M_v}{\partial \hat{\theta}} \right) + \frac{\rho \hat{h}^{1.5}}{\hat{\mu}} \left(\frac{\partial^2 M_v}{\partial \hat{\theta}^2} \right) \right]$$

$$-1.5 \left[\begin{pmatrix} M_v \hat{h}^{0.5} \hat{\mu}^{-1} \frac{\partial \hat{\rho}}{\partial \hat{\theta}} \frac{\partial \hat{h}}{\partial \hat{\theta}} - \rho M_v \hat{h}^{0.5} \hat{\mu}^{-2} \frac{\partial \hat{\mu}}{\partial \hat{\theta}} \frac{\partial \hat{h}}{\partial \hat{\theta}} + \rho \hat{h}^{0.5} \hat{\mu}^{-1} \frac{\partial M_v}{\partial \hat{\theta}} \frac{\partial \hat{h}}{\partial \hat{\theta}} \\ +0.5 \hat{h}^{-0.5} \rho M_v \hat{\mu}^{-1} \left(\frac{\partial \hat{h}}{\partial \hat{\theta}} \right)^2 \end{pmatrix} + \frac{\rho M_v \hat{h}^{0.5}}{\hat{\mu}} \left(\frac{\partial^2 \hat{h}}{\partial \hat{\theta}^2} \right) \right]$$

$$\frac{\partial}{\partial \hat{\theta}} \left(\frac{\hat{\rho}\hat{h}^3}{\hat{\mu}} \frac{\partial \hat{p}}{\partial \hat{\theta}} \right) = \begin{bmatrix} \hat{h}^{1.5} \hat{\mu}^{-1} \frac{\partial \hat{\rho}}{\partial \hat{\theta}} \frac{\partial M_v}{\partial \hat{\theta}} - \hat{\rho} \hat{h}^{1.5} \hat{\mu}^{-2} \frac{\partial \hat{\mu}}{\partial \hat{\theta}} \frac{\partial M_v}{\partial \hat{\theta}} + \frac{\hat{\rho}\hat{h}^{1.5}}{\hat{\mu}} \left(\frac{\partial^2 M_v}{\partial \hat{\theta}^2} \right) \\ -1.5 M_v \hat{h}^{0.5} \hat{\mu}^{-1} \frac{\partial \hat{\rho}}{\partial \hat{\theta}} \frac{\partial \hat{h}}{\partial \hat{\theta}} + 1.5 \hat{\rho} M_v \hat{h}^{0.5} \hat{\mu}^{-2} \frac{\partial \hat{\mu}}{\partial \hat{\theta}} \frac{\partial \hat{h}}{\partial \hat{\theta}} \\ -0.75 \hat{h}^{-0.5} \hat{\rho} M_v \hat{\mu}^{-1} \left(\frac{\partial \hat{h}}{\partial \hat{\theta}} \right)^2 - 1.5 \frac{\hat{\rho} M_v \hat{h}^{0.5}}{\hat{\mu}} \left(\frac{\partial^2 \hat{h}}{\partial \hat{\theta}^2} \right) \end{bmatrix} \tag{34}$$

Similarly in z direction

$$\frac{\partial}{\partial \hat{z}}\left(\frac{\hat{\rho}\hat{h}^{3}}{\hat{\mu}}\frac{\partial \hat{p}}{\partial \hat{z}}\right)=\begin{bmatrix} \hat{h}^{1.5}\hat{\mu}^{-1}\frac{\partial \hat{\rho}}{\partial \hat{z}}\frac{\partial M_{v}}{\partial \hat{z}}-\hat{\rho}\hat{h}^{1.5}\hat{\mu}^{-2}\frac{\partial \hat{\mu}}{\partial \hat{z}}\frac{\partial M_{v}}{\partial \hat{z}}+\frac{\hat{\rho}\hat{h}^{1.5}}{\hat{\mu}}\left(\frac{\partial^{2}M_{v}}{\partial \hat{z}^{2}}\right) \\ -1.5M_{v}\hat{h}^{0.5}\hat{\mu}^{-1}\frac{\partial \hat{\rho}}{\partial \hat{z}}\frac{\partial \hat{h}}{\partial \hat{z}}+1.5\hat{\rho}M_{v}\hat{h}^{0.5}\hat{\mu}^{-2}\frac{\partial \hat{\mu}}{\partial \hat{z}}\frac{\partial \hat{h}}{\partial \hat{z}} \\ -0.75\hat{h}^{-0.5}\hat{\rho}M_{v}\hat{\mu}^{-1}\left(\frac{\partial \hat{h}}{\partial \hat{z}}\right)^{2}-1.5\frac{\hat{\rho}M_{v}\hat{h}^{0.5}}{\hat{\mu}}\left(\frac{\partial^{2}\hat{h}}{\partial \hat{z}^{2}}\right) \end{bmatrix} \quad (35)$$

Now consider RHS of equation (32)

$$\gamma_{1}\frac{\partial\left(\hat{\rho}\hat{h}\right)}{\partial \hat{\theta}}=\gamma_{1}\left[\hat{h}\frac{\partial \hat{\rho}}{\partial \hat{\theta}}+\hat{\rho}\frac{\partial \hat{h}}{\partial \hat{\theta}}\right] \quad (36)$$

$$\gamma_{2}\frac{\partial\left(\hat{\rho}\hat{h}\right)}{\partial \hat{z}}=\gamma_{2}\left[\hat{h}\frac{\partial \hat{\rho}}{\partial \hat{z}}+\hat{\rho}\frac{\partial \hat{h}}{\partial \hat{z}}\right] \quad (37)$$

$$\gamma\frac{\partial\left(\hat{\rho}\hat{h}\right)}{\partial \hat{t}}=\gamma\left[\hat{h}\frac{\partial \hat{\rho}}{\partial \hat{t}}+\hat{\rho}\frac{\partial \hat{h}}{\partial \hat{t}}\right] \quad (38)$$

Putting equation (34) (35) (36) (37) and (38) into equation (32)

$$\begin{bmatrix} \hat{h}^{1.5}\hat{\mu}^{-1}\frac{\partial \hat{\rho}}{\partial \hat{\theta}}\frac{\partial M_{v}}{\partial \hat{\theta}}-\hat{\rho}\hat{h}^{1.5}\hat{\mu}^{-2}\frac{\partial \hat{\mu}}{\partial \hat{\theta}}\frac{\partial M_{v}}{\partial \hat{\theta}}+\frac{\hat{\rho}\hat{h}^{1.5}}{\hat{\mu}}\left(\frac{\partial^{2}M_{v}}{\partial \hat{\theta}^{2}}\right) \\ -1.5M_{v}\hat{h}^{0.5}\hat{\mu}^{-1}\frac{\partial \hat{\rho}}{\partial \hat{\theta}}\frac{\partial \hat{h}}{\partial \hat{\theta}}+1.5\hat{\rho}M_{v}\hat{h}^{0.5}\hat{\mu}^{-2}\frac{\partial \hat{\mu}}{\partial \hat{\theta}}\frac{\partial \hat{h}}{\partial \hat{\theta}} \\ -0.75\hat{h}^{-0.5}\hat{\rho}M_{v}\hat{\mu}^{-1}\left(\frac{\partial \hat{h}}{\partial \hat{\theta}}\right)^{2}-1.5\frac{\hat{\rho}M_{v}\hat{h}^{0.5}}{\hat{\mu}}\left(\frac{\partial^{2}\hat{h}}{\partial \hat{\theta}^{2}}\right) \end{bmatrix}+K^{2}\begin{bmatrix} \hat{h}^{1.5}\hat{\mu}^{-1}\frac{\partial \hat{\rho}}{\partial \hat{z}}\frac{\partial M_{v}}{\partial \hat{z}}-\hat{\rho}\hat{h}^{1.5}\hat{\mu}^{-2}\frac{\partial \hat{\mu}}{\partial \hat{z}}\frac{\partial M_{v}}{\partial \hat{z}}+\frac{\hat{\rho}\hat{h}^{1.5}}{\hat{\mu}}\left(\frac{\partial^{2}M_{v}}{\partial \hat{z}^{2}}\right) \\ -1.5M_{v}\hat{h}^{0.5}\hat{\mu}^{-1}\frac{\partial \hat{\rho}}{\partial \hat{z}}\frac{\partial \hat{h}}{\partial \hat{z}}+1.5\hat{\rho}M_{v}\hat{h}^{0.5}\hat{\mu}^{-2}\frac{\partial \hat{\mu}}{\partial \hat{z}}\frac{\partial \hat{h}}{\partial \hat{z}} \\ -0.75\hat{h}^{-0.5}\hat{\rho}M_{v}\hat{\mu}^{-1}\left(\frac{\partial \hat{h}}{\partial \hat{z}}\right)^{2}-1.5\frac{\hat{\rho}M_{v}\hat{h}^{0.5}}{\hat{\mu}}\left(\frac{\partial^{2}\hat{h}}{\partial \hat{z}^{2}}\right) \end{bmatrix}$$

$$=\gamma_{1}\left[\hat{h}\frac{\partial \hat{\rho}}{\partial \hat{\theta}}+\hat{\rho}\frac{\partial \hat{h}}{\partial \hat{\theta}}\right]+\gamma_{2}\left[\hat{h}\frac{\partial \hat{\rho}}{\partial \hat{z}}+\hat{\rho}\frac{\partial \hat{h}}{\partial \hat{z}}\right]+\gamma\left[\hat{h}\frac{\partial \hat{\rho}}{\partial \hat{t}}+\hat{\rho}\frac{\partial \hat{h}}{\partial \hat{t}}\right]$$

Divide this equation by $\frac{\hat{\rho}\hat{h}^{1.5}}{\hat{\mu}}$

$$\left(\frac{\partial^2 M_v}{\partial\hat{\theta}^2}+K^2\frac{\partial^2 M_v}{\partial\hat{z}^2}\right)+\frac{1}{\hat{\rho}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial M_v}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial M_v}{\partial\hat{z}}\right)-\frac{1}{\hat{\mu}}\left(\frac{\partial\hat{\mu}}{\partial\hat{\theta}}\frac{\partial M_v}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\mu}}{\partial\hat{z}}\frac{\partial M_v}{\partial\hat{z}}\right)$$

$$-\frac{1.5M_v}{\hat{\rho}\hat{h}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right)+\frac{1.5M_v}{\hat{\mu}\hat{h}}\left(\frac{\partial\hat{\mu}}{\partial\hat{\theta}}\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\mu}}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right)-\frac{0.75M_v}{\hat{h}^2}\left(\left(\frac{\partial\hat{h}}{\partial\hat{\theta}}\right)^2+K^2\left(\frac{\partial\hat{h}}{\partial\hat{z}}\right)^2\right)$$

$$-\frac{1.5M_v}{\hat{h}}\left(\frac{\partial^2\hat{h}}{\partial\hat{\theta}^2}+K^2\frac{\partial^2\hat{h}}{\partial\hat{z}^2}\right)=\gamma_1\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{\theta}}\right]+\gamma_2\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{z}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{z}}\right]$$

$$+\gamma\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{t}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{t}}\right]$$

$$\left(\frac{\partial^2 M_v}{\partial\hat{\theta}^2}+K^2\frac{\partial^2 M_v}{\partial\hat{z}^2}\right)+\frac{1}{\hat{\rho}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial M_v}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial M_v}{\partial\hat{z}}\right)-\frac{1}{\hat{\mu}}\left(\frac{\partial\hat{\mu}}{\partial\hat{\theta}}\frac{\partial M_v}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\mu}}{\partial\hat{z}}\frac{\partial M_v}{\partial\hat{z}}\right)$$

$$=M_v\begin{bmatrix}\frac{1.5}{\hat{\rho}\hat{h}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right)-\frac{1.5}{\hat{\mu}\hat{h}}\left(\frac{\partial\hat{\mu}}{\partial\hat{\theta}}\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\mu}}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right)\\+\frac{0.75}{\hat{h}^2}\left(\left(\frac{\partial\hat{h}}{\partial\hat{\theta}}\right)^2+K^2\left(\frac{\partial\hat{h}}{\partial\hat{z}}\right)^2\right)+\frac{1.5}{\hat{h}}\left(\frac{\partial^2\hat{h}}{\partial\hat{\theta}^2}+K^2\frac{\partial^2\hat{h}}{\partial\hat{z}^2}\right)\end{bmatrix}$$

$$+\gamma_1\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{\theta}}\right]+\gamma_2\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{z}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{z}}\right]+\gamma\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{t}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{t}}\right]$$

$$\boxed{\begin{aligned}&\left(\frac{\partial^2 M_v}{\partial\hat{\theta}^2}+K^2\frac{\partial^2 M_v}{\partial\hat{z}^2}\right)+\frac{1}{\hat{\rho}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial M_v}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial M_v}{\partial\hat{z}}\right)-\frac{1}{\hat{\mu}}\left(\frac{\partial\hat{\mu}}{\partial\hat{\theta}}\frac{\partial M_v}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\mu}}{\partial\hat{z}}\frac{\partial M_v}{\partial\hat{z}}\right)\\&=M_v\left(F\right)+G\end{aligned}} \qquad (39)$$

$$F=\frac{1.5}{\hat{\rho}\hat{h}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right)-\frac{1.5}{\hat{\mu}\hat{h}}\left(\frac{\partial\hat{\mu}}{\partial\hat{\theta}}\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\mu}}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right)+\frac{0.75}{\hat{h}^2}\left(\left(\frac{\partial\hat{h}}{\partial\hat{\theta}}\right)^2+K^2\left(\frac{\partial\hat{h}}{\partial\hat{z}}\right)^2\right)$$

$$+\frac{1.5}{\hat{h}}\left(\frac{\partial^2\hat{h}}{\partial\hat{\theta}^2}+K^2\frac{\partial^2\hat{h}}{\partial\hat{z}^2}\right)$$

$$G=\gamma_1\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial v}\right]+\gamma_2\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{z}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{z}}\right]+\gamma\left[\frac{\hat{\mu}}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{t}}+\frac{\hat{\mu}}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{t}}\right]$$

Discretization of Vogelpohl Equation

Equation number (39) is discretized by using central divided difference scheme

$$
\left(\frac{M_{v,i+1,j}-2M_{v,i,j}+M_{v,i-1,j}}{\left(\Delta\theta\right)^2}+K^2\frac{M_{v,i,j+1}-2M_{v,i,j}+M_{v,i,j-1}}{\left(\Delta z\right)^2}\right)
$$

$$
+\frac{1}{\hat{\rho}}\left(\frac{\hat{\rho}_{i+1,j}-\hat{\rho}_{i-1,j}}{2\Delta\theta}\frac{M_{v,i+1,j}-M_{v,i-1,j}}{2\Delta\theta}+K^2\frac{\hat{\rho}_{i,j+1}-\hat{\rho}_{i,j-1}}{2\Delta z}\frac{M_{v,i,j+1}-M_{v,i,j-1}}{2\Delta z}\right)
$$

$$
-\frac{1}{\hat{\mu}}\left(\frac{\hat{\mu}_{i+1,j}-\hat{\mu}_{i-1,j}}{2\Delta\theta}\frac{M_{v,i+1,j}-M_{v,i-1,j}}{2\Delta\theta}+K^2\frac{\hat{\mu}_{i,j+1}-\hat{\mu}_{i,j-1}}{2\Delta z}\frac{M_{v,i,j+1}-M_{v,i,j-1}}{2\Delta z}\right)
$$

$$
=M_{v,i,j}\left(F_{i,j}\right)+G_{i,j}
$$

$$
\left[C_1\left(M_{v,i+1,j}-2M_{v,i,j}+M_{v,i-1,j}\right)+C_2K^2\left(M_{v,i,j+1}-2M_{v,i,j}+M_{v,i,j-1}\right)\right]
$$

$$
+\frac{1}{\hat{\rho}}\left[\frac{C_1}{4}\left(\left(\hat{\rho}_{i+1,j}-\hat{\rho}_{i-1,j}\right)\left(M_{v,i+1,j}-M_{v,i-1,j}\right)\right)+K^2\frac{C_2}{4}\left(\left(\hat{\rho}_{i,j+1}-\hat{\rho}_{i,j-1}\right)\left(M_{v,i,j+1}-M_{v,i,j-1}\right)\right)\right]
$$

$$
-\frac{1}{\hat{\mu}}\left[\frac{C_1}{4}\left(\left(\hat{\mu}_{i+1,j}-\hat{\mu}_{i-1,j}\right)\left(M_{v,i+1,j}-M_{v,i-1,j}\right)\right)+K^2\frac{C_2}{4}\left(\left(\hat{\mu}_{i,j+1}-\hat{\mu}_{i,j-1}\right)\left(M_{v,i,j+1}-M_{v,i,j-1}\right)\right)\right]
$$

$$
=M_{v,i,j}\left(F_{i,j}\right)+G_{i,j}
$$

$$
\left[C_1\left(M_{v,i+1,j}+M_{v,i-1,j}\right)+C_2K^2\left(M_{v,i,j+1}+M_{v,i,j-1}\right)\right]
$$

$$
+\frac{1}{\hat{\rho}}\left[\frac{C_1}{4}\left(\left(\hat{\rho}_{i+1,j}-\hat{\rho}_{i-1,j}\right)\left(M_{v,i+1,j}-M_{v,i-1,j}\right)\right)+K^2\frac{C_2}{4}\left(\left(\hat{\rho}_{i,j+1}-\hat{\rho}_{i,j-1}\right)\left(M_{v,i,j+1}-M_{v,i,j-1}\right)\right)\right]
$$

$$
-\frac{1}{\hat{\mu}}\left[\frac{C_1}{4}\left(\left(\hat{\mu}_{i+1,j}-\hat{\mu}_{i-1,j}\right)\left(M_{v,i+1,j}-M_{v,i-1,j}\right)\right)+K^2\frac{C_2}{4}\left(\left(\hat{\mu}_{i,j+1}-\hat{\mu}_{i,j-1}\right)\left(M_{v,i,j+1}-M_{v,i,j-1}\right)\right)\right]
$$

$$
=M_{v,i,j}\left(F_{i,j}\right)+2C_1M_{v,i,j}+2K^2C_2M_{v,i,j}+G_{i,j}
$$

$$
\left[C_1\left(M_{v,i+1,j}+M_{v,i-1,j}\right)+C_2K^2\left(M_{v,i,j+1}+M_{v,i,j-1}\right)\right]
$$

$$
+\frac{1}{\hat{\rho}}\left[\frac{C_1}{4}\left(\left(\hat{\rho}_{i+1,j}-\hat{\rho}_{i-1,j}\right)\left(M_{v,i+1,j}-M_{v,i-1,j}\right)\right)+K^2\frac{C_2}{4}\left(\left(\hat{\rho}_{i,j+1}-\hat{\rho}_{i,j-1}\right)\left(M_{v,i,j+1}-M_{v,i,j-1}\right)\right)\right]
$$

$$
-\frac{1}{\hat{\mu}}\left[\frac{C_1}{4}\left(\left(\hat{\mu}_{i+1,j}-\hat{\mu}_{i-1,j}\right)\left(M_{v,i+1,j}-M_{v,i-1,j}\right)\right)+K^2\frac{C_2}{4}\left(\left(\hat{\mu}_{i,j+1}-\hat{\mu}_{i,j-1}\right)\left(M_{v,i,j+1}-M_{v,i,j-1}\right)\right)\right]
$$

$$
=M_{v,i,j}\left[2C_1+2K^2C_2+F_{i,j}\right]+G_{i,j}
$$

$$M_{v,i,j} = \frac{\left[\begin{array}{l} \left\{C_1\left(M_{v,i+1,j}+M_{v,i-1,j}\right)+C_2K^2\left(M_{v,i,j+1}+M_{v,i,j-1}\right)\right\} \\ +\dfrac{1}{\hat{\rho}}\left\{\begin{array}{l} \dfrac{C_1}{4}\left(\left(\hat{\rho}_{i+1,j}-\hat{\rho}_{i-1,j}\right)\left(M_{v,i+1,j}-M_{v,i-1,j}\right)\right) \\ +K^2\,\dfrac{C_2}{4}\left(\left(\hat{\rho}_{i,j+1}-\hat{\rho}_{i,j-1}\right)\left(M_{v,i,j+1}-M_{v,i,j-1}\right)\right) \end{array}\right\} \\ -\dfrac{1}{\hat{\mu}}\left\{\begin{array}{l} \dfrac{C_1}{4}\left(\left(\hat{\mu}_{i+1,j}-\hat{\mu}_{i-1,j}\right)\left(M_{v,i+1,j}-M_{v,i-1,j}\right)\right) \\ +K^2\,\dfrac{C_2}{4}\left(\left(\hat{\mu}_{i,j+1}-\hat{\mu}_{i,j-1}\right)\left(M_{v,i,j+1}-M_{v,i,j-1}\right)\right) \end{array}\right\} \end{array}\right] - G_{i,j}}{\left[2C_1+2K^2C_2+F_{i,j}\right]} \tag{40}$$

APENDIX 2

Non-Newtonian Lubricant

Upper convected Maxwell model is the simple viscoelastic model having small relaxation time. Continuity, momentum and constitutive equations are [42]

$$\nabla . V = 0$$

$$\rho(\frac{\partial V}{\partial t} + V.\nabla V) = -\nabla p + \nabla . \tau$$

$$\lambda \overset{\nabla}{\tau} + \tau = -2\mu d$$

(41)

Where

$$d = \frac{1}{2}(\nabla V + (\nabla V)^T)$$

$$\overset{\nabla}{\tau} = \frac{\partial \tau}{\partial t} + V.\nabla \tau - \nabla V . \tau - \tau . (\nabla V)^T$$

d is the rate of deformation tensor, $\overset{\nabla}{\tau}$ is the extra shear stress tensor and is the characteristic relaxation time.

Constitutive equation can be written in the polar coordinates as

$$\tau_{rr} + \lambda\left(V.\nabla \tau_{rr} - \frac{2\tau_{r\theta}}{r}\frac{\partial u_r}{\partial \theta} - 2\tau_{rz}\frac{\partial u_r}{\partial z} - 2\tau_{rr}\frac{\partial u_r}{\partial r}\right) = -2\mu\frac{\partial u_r}{\partial r} \tag{42}$$

$$\tau_{\theta\theta} + \lambda\left(V.\nabla \tau_{\theta\theta} - \frac{2\tau_{\theta\theta}}{r}\frac{\partial u_\theta}{\partial \theta} - 2\tau_{r\theta}\frac{\partial u_\theta}{\partial r} - 2\tau_{\theta z}\frac{\partial u_\theta}{\partial z}\right) = -\frac{2\mu}{r}\frac{\partial u_\theta}{\partial \theta} \tag{43}$$

$$\tau_{zz} + \lambda\left(V.\nabla \tau_{zz} - \frac{2\tau_{\theta z}}{r}\frac{\partial u_z}{\partial \theta} - 2\tau_{rz}\frac{\partial u_z}{\partial r} - 2\tau_{zz}\frac{\partial u_z}{\partial z}\right) = -2\mu\frac{\partial u_z}{\partial z} \tag{44}$$

$$\tau_{r\theta} + \lambda\left[V.\nabla \tau_{r\theta} - \frac{\tau_{\theta\theta}}{r}\frac{\partial u_r}{\partial \theta} - \tau_{rr}\frac{\partial u_\theta}{\partial r}\right] = -\mu\left(\frac{\partial u_\theta}{\partial r} + \frac{\partial u_r}{\partial \theta}\right) \tag{45}$$

$$\tau_{rz} + \lambda\left(V.\nabla \tau_{rz} - \tau_{rr}\frac{\partial u_z}{\partial r} - \tau_{zz}\frac{\partial u_r}{\partial z}\right) = -\mu\left(\frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z}\right) \tag{46}$$

$$\tau_{\theta z} + \lambda\left(V.\nabla \tau_{\theta z} - \frac{\tau_{\theta\theta}}{r}\frac{\partial u_z}{\partial \theta} - \tau_{zz}\frac{\partial u_\theta}{\partial z}\right) = -\mu\left(\frac{\partial u_\theta}{\partial z} + \frac{1}{r}\frac{\partial u_z}{\partial \theta}\right) \tag{47}$$

Consider Navier stokes equation for inertial less fluid

$$-\frac{1}{R}\frac{\partial p}{\partial \theta} + \frac{1}{R}\frac{\partial \tau_{\theta\theta}}{\partial \theta} + \frac{\partial \tau_{\theta z}}{\partial z} + \frac{\partial \tau_{\theta r}}{\partial r} = 0$$

Putting values of stresses in above equation

$$=-\frac{1}{R}\frac{\partial p}{\partial \theta}+\frac{1}{R}\frac{\partial}{\partial \theta}\left(-\lambda\left(V.\nabla\tau_{\theta\theta}-\frac{2\tau_{\theta\theta}}{r}\frac{\partial u_\theta}{\partial \theta}-2\tau_{r\theta}\frac{\partial u_\theta}{\partial r}-2\tau_{\theta z}\frac{\partial u_\theta}{\partial z}\right)-\frac{2\mu}{r}\frac{\partial u_\theta}{\partial \theta}\right)$$
$$+\frac{\partial}{\partial z}\left(-\lambda\left(V.\nabla\tau_{\theta z}-\frac{\tau_{\theta\theta}}{r}\frac{\partial u_z}{\partial \theta}-\tau_{zz}\frac{\partial u_\theta}{\partial z}\right)-\mu\left(\frac{\partial u_\theta}{\partial z}+\frac{1}{r}\frac{\partial u_z}{\partial \theta}\right)\right)$$
$$+\frac{\partial}{\partial r}\left(-\lambda\left[V.\nabla\tau_{r\theta}-\frac{\tau_{\theta\theta}}{r}\frac{\partial u_r}{\partial \theta}-\tau_{rr}\frac{\partial u_\theta}{\partial r}\right]-\mu\left(\frac{\partial u_\theta}{\partial r}+\frac{\partial u_r}{\partial \theta}\right)\right)=0$$

Consider velocity changes in R direction only

$$\frac{1}{r}\frac{\partial p}{\partial \theta}=\frac{\partial}{\partial r}\left(-\lambda\left[\frac{u_\theta}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+w\frac{\partial \tau_{r\theta}}{\partial r}-\tau_{rr}\frac{\partial u_\theta}{\partial r}\right]-\mu\left(\frac{\partial u_\theta}{\partial r}+\frac{\partial u_r}{\partial \theta}\right)\right)-\rho w\frac{\partial u_\theta}{\partial r}$$

Integrating from 0 to h

$$C_1+\frac{y}{r}\frac{\partial p}{\partial \theta}=-\lambda\left[\frac{u_\theta}{r}\frac{\partial \tau_{r\theta}}{\partial \theta}+w\frac{\partial \tau_{r\theta}}{\partial r}-\tau_{rr}\frac{\partial u_\theta}{\partial r}\right]-\mu\frac{\partial u_\theta}{\partial r}$$

$$\frac{\partial \tau_{r\theta}}{\partial \theta}=0$$

Integrating again

$$C_1 y+\frac{y^2}{2r}\frac{\partial p}{\partial \theta}+C_2=u_\theta(-\eta+\lambda\tau_{rr}) \tag{48}$$

At y = 0 $u_\theta=U_b$

$$C_2=U_b(-\eta+\lambda\tau_{rr})$$

At y = h $u_\theta=U_a$

$$C_1 h+\frac{h^2}{2r}\frac{\partial p}{\partial \theta}+C_2=u_\theta(-\eta+\lambda\tau_{rr})$$

Let's say that $a_2=-\eta+\lambda\tau_{rr}$

$$C_1=\frac{(U_a-U_b)a_2}{h}-\frac{h^2}{2r}\frac{\partial p}{\partial \theta}$$

Putting values in equation (48)

$$u_\theta=\frac{\frac{z^2}{2\mu}\frac{\partial p}{\partial \theta}-\frac{ch}{2\mu}\frac{\partial p}{\partial \theta}-\frac{c}{h}\left(U_b-U_a\right)+U_b}{a_2} \tag{49}$$

Similarly for u_z

$$u_z=\frac{\frac{c^2}{2\mu}\frac{\partial p}{\partial r}-\frac{ch}{2\mu}\frac{\partial p}{\partial r}-\frac{c}{h}\left(V_b-V_a\right)+V_b}{a_2} \tag{50}$$

Integrating equation 49 through whole film thickness

$$q_\theta{}' = \frac{\left(-\frac{h^3}{12\mu}\right)\frac{1}{r}\frac{\partial p}{\partial\theta} + \left(\frac{U_a + U_b}{2}\right)h}{a_2} \tag{51}$$

Similarly $q_z{}' = \dfrac{\left(-\frac{h^3}{12\mu}\right)\frac{\partial p}{\partial z} + \left(\frac{V_a + V_b}{2}\right)h}{a_2}$ (52)

Reynolds Equation

Now consider continuity equation

$$\int_0^h \left(\frac{\partial\rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho u_\theta) + \frac{\partial}{\partial r}(\rho u_r) + \frac{\partial}{\partial z}(\rho u_z)\right) = 0 \tag{23}$$

$$\int_0^h \frac{\partial\rho}{\partial t} = \frac{\partial\rho}{\partial t}h$$

$$\int_0^h \frac{1}{r}\frac{\partial}{\partial\theta}(\rho u_\theta)dz = -\frac{1}{r}(\rho U_a)\frac{\partial h}{\partial x} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho\int_0^h u_\theta dz\right)$$

$$\int_0^h \frac{\partial}{\partial z}(\rho u_z)dz = -(\rho V_a)\frac{\partial h}{\partial z} + \frac{\partial}{\partial z}\left(\rho\int_0^h u_z dr\right)$$

$$\int_0^h \frac{\partial}{\partial r}(\rho u_r)dr = \rho(w_a - w_b)$$

Putting these values in equation (23)

$$\frac{\partial\rho}{\partial t}h - \frac{1}{r}(\rho U_a)\frac{\partial h}{\partial\theta} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\rho\int_0^h u_\theta dz\right) - (\rho V_a)\frac{\partial h}{\partial z} + \frac{\partial}{\partial z}\left(\rho\int_0^h u_z dr\right) + \rho(w_a - w_b) = 0$$

$$\frac{\partial\rho}{\partial t}h - \frac{1}{r}(\rho U_a)\frac{\partial h}{\partial\theta} + \frac{1}{r}\frac{\partial}{\partial\theta}\left(\left(\left(-\frac{h^3}{12\mu}\right)\frac{1}{r}\frac{\partial p}{\partial\theta} + \left(\frac{U_a + U_b}{2}\right)h\frac{\rho}{a_2}\right) - (\rho V_a)\frac{\partial h}{\partial z}\right.$$

$$\left. + \frac{\partial}{\partial z}\left(\left(-\frac{h^3}{12\mu}\right)\frac{\partial p}{\partial z} + \left(\frac{V_a + V_b}{2}\right)h\frac{\rho}{a_2}\right) + \rho(w_a - w_b) = 0$$

$$\frac{1}{12r^2}\frac{\partial}{\partial\theta}\left(\frac{\rho h^3}{a_2}\frac{\partial p}{\partial\theta}\right) + \frac{1}{12}\frac{\partial}{\partial z}\left(\frac{\rho h^3}{a_2}\frac{\partial p}{\partial z}\right) = \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{\rho h(u_{\theta a} + u_{\theta b})}{2}\right) + \frac{\partial}{\partial z}\left(\frac{\rho h(u_{za} + u_{zb})}{2}\right)$$

$$+h\frac{\partial\rho}{\partial t} + \rho\left(w_1 - w_2 - \frac{u_{\theta a}}{r}\frac{\partial h}{\partial\theta} - u_{zb}\frac{\partial h}{\partial z}\right)$$

Final form of Reynolds equation is

$$\frac{\rho}{r^2}\frac{\partial}{\partial\theta}\left[\frac{\rho h^3}{12a_2\mu}\left(\frac{\partial p}{\partial\theta}\right)\right] + \frac{\partial}{\partial z}\left[\frac{\rho h^3}{12a_2\mu}\left(\frac{\partial p}{\partial z}\right)\right] = \frac{U}{r}\frac{\partial}{\partial\theta}\left[\frac{\rho h}{2}\right] + V\frac{\partial}{\partial z}\left[\frac{\rho h}{2}\right] + \frac{\partial}{\partial t}\left[(\rho h)\right] \tag{53}$$

Non-Dimensionalization of Reynolds Equation

Following are the parameters used to non-dimensionalize Reynolds equation

$$\hat{\theta}=\frac{\theta}{\theta_{ref}},\hat{z}=\frac{z}{L},\hat{h}=\frac{h}{c},\hat{\mu}=\frac{\mu}{\mu_{ref}},\hat{t}=\frac{t}{t_{ref}},\hat{p}=\frac{p}{p_{ref}},K=\frac{r\theta_{ref}}{L},\mathrm{P}_{ref}=\frac{6\mu_{ref}UD}{c^2}$$

$$\gamma_1=\frac{6a_2\mu_{ref}\theta_{ref}V_\theta}{p_{ref}c^2},\gamma_2=\frac{6\mu_{ref}K\theta_{ref}RV_z}{p_{ref}c^2},\gamma=\frac{12\mu_{ref}\theta_{ref}{}^2r^2}{p_{ref}c^2t_{ref}}$$

$$\frac{\partial}{\partial\hat{\theta}}\left[\frac{\rho\hat{h}^3}{a_2\hat{\mu}}\left(\frac{\partial\hat{p}}{\partial\hat{\theta}}\right)\right]+K^2\frac{\partial}{\partial\hat{z}}\left[\frac{\rho\hat{h}^3}{a_2\hat{\mu}}\left(\frac{\partial\hat{p}}{\partial\hat{z}}\right)\right]=\gamma_1\left[\frac{\partial}{\partial\hat{\theta}}\left(\hat{h}\right)\right]+\gamma_2\left[\frac{\partial}{\partial\hat{z}}\left(\hat{h}\right)\right]+\gamma\left[\frac{\partial}{\partial\hat{t}}\left(\hat{h}\right)\right] \quad (54)$$

Reynolds Equation with Vogelpohl parameter

The Vogelpohl parameter is defined as follows [40]

$$M_v=\hat{p}\hat{h}^{1.5}$$

$$\hat{p}=M_v\hat{h}^{-1.5}$$

$$\frac{\partial\hat{p}}{\partial\hat{\theta}}=\frac{\partial}{\partial\hat{\theta}}\left(M_v\hat{h}^{-1.5}\right)$$

$$\frac{\partial\hat{p}}{\partial\hat{\theta}}=\hat{h}^{-1.5}\frac{\partial M_v}{\partial\hat{\theta}}+M_v\frac{\partial\hat{h}^{-1.5}}{\partial\hat{\theta}}$$

$$\frac{\partial\hat{p}}{\partial\hat{\theta}}=\hat{h}^{-1.5}\frac{\partial M_v}{\partial\hat{\theta}}+M_v\left(-1.5\hat{h}^{-2.5}\frac{\partial\hat{h}}{\partial\hat{\theta}}\right)$$

$$\frac{\partial\hat{p}}{\partial\hat{\theta}}=\hat{h}^{-1.5}\frac{\partial M_v}{\partial\hat{\theta}}-1.5M_v\hat{h}^{-2.5}\frac{\partial\hat{h}}{\partial\hat{\theta}}$$

Multiply above equation with $\hat{h}^3\hat{\rho}$ and divide with a_2

$$\frac{\hat{\rho}\hat{h}^3}{a_2}\frac{\partial\hat{p}}{\partial\hat{\theta}}=\frac{\hat{\rho}\hat{h}^{1.5}}{a_2}\frac{\partial M_v}{\partial\hat{\theta}}-\frac{1.5\hat{\rho}M_v\hat{h}^{0.5}}{a_2}\frac{\partial\hat{h}}{\partial\hat{\theta}}$$

Taking derivative of above equation w.r.t $\hat{\theta}$

$$\frac{\partial}{\partial\hat{\theta}}\left(\frac{\hat{\rho}\hat{h}^3}{a_2}\frac{\partial\hat{p}}{\partial\hat{\theta}}\right)=\frac{\partial}{\partial\hat{\theta}}\left(\frac{\hat{\rho}\hat{h}^{1.5}}{a_2}\frac{\partial M_v}{\partial\hat{\theta}}\right)-1.5\frac{\partial}{\partial\hat{\theta}}\left(\frac{\hat{\rho}M_v\hat{h}^{0.5}}{a_2}\frac{\partial\hat{h}}{\partial\hat{\theta}}\right)$$

$$=\left[\left(\hat{h}^{1.5}a_2{}^{-1}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+a_2{}^{-1}\hat{\rho}\frac{\partial\hat{h}^{1.5}}{\partial\hat{\theta}}\right)\frac{\partial M_v}{\partial\hat{\theta}}+\frac{\hat{\rho}\hat{h}^{1.5}}{\hat{\mu}}\left(\frac{\partial^2 M_v}{\partial\hat{\theta}^2}\right)\right]$$

$$-1.5\left[\left(M_v\hat{h}^{0.5}a_2{}^{-1}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+\hat{\rho}\hat{h}^{0.5}a_2{}^{-1}\frac{\partial M_v}{\partial\hat{\theta}}+\hat{\rho}M_va_2{}^{-1}\frac{\partial\hat{h}^{0.5}}{\partial\hat{\theta}}\right)\frac{\partial\hat{h}}{\partial\hat{\theta}}+\frac{\hat{\rho}M_v\hat{h}^{0.5}}{a_2}\left(\frac{\partial^2\hat{h}}{\partial\hat{\theta}^2}\right)\right]$$

$$\frac{\partial}{\partial\hat{\theta}}\left(\frac{\hat{\rho}\hat{h}^3}{a_2}\frac{\partial\hat{p}}{\partial\hat{\theta}}\right)=-\frac{1.5}{a_2}\left[\hat{h}^{0.5}M_v\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+0.5\hat{h}^{-0.5}\hat{\rho}M_v\frac{\partial\hat{h}}{\partial\hat{\theta}}+\hat{\rho}\hat{h}^{0.5}\frac{\partial M_v}{\partial\hat{\theta}}-\hat{h}^{0.5}M_v\hat{\rho}\right]\frac{\partial\hat{h}}{\partial\hat{\theta}}$$
$$-\frac{1.5\hat{h}^{0.5}M_v\hat{\rho}}{a_2}\frac{\partial^2\hat{h}}{\partial\hat{\theta}^2}+\frac{1}{a_2}\left[1.5\hat{h}^{0.5}\hat{\rho}\frac{\partial\hat{h}}{\partial\hat{\theta}}+\hat{h}^{1.5}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}-\hat{h}\hat{\rho}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\right]\frac{\partial M_v}{\partial\hat{\theta}}+\frac{\hat{h}^{1.5}\hat{\rho}}{a_2}\frac{\partial^2 M_v}{\partial\hat{\theta}^2} \qquad (55)$$

Similarly in z direction

$$\frac{\partial}{\partial\hat{z}}\left(\frac{\hat{\rho}\hat{h}^3}{a_2}\frac{\partial\hat{p}}{\partial\hat{z}}\right)=-\frac{1.5}{a_2}\left[\hat{h}^{0.5}M_v\frac{\partial\hat{\rho}}{\partial\hat{z}}+0.5\hat{h}^{-0.5}\hat{\rho}M_v\frac{\partial\hat{h}}{\partial\hat{z}}+\hat{\rho}\hat{h}^{0.5}\frac{\partial M_v}{\partial\hat{z}}-\hat{h}^{0.5}M_v\hat{\rho}\right]\frac{\partial\hat{h}}{\partial\hat{z}}$$
$$-\frac{1.5\hat{h}^{0.5}M_v\hat{\rho}}{a_2}\frac{\partial^2\hat{h}}{\partial\hat{z}^2}+\frac{1}{a_2}\left[1.5\hat{h}^{0.5}\hat{\rho}\frac{\partial\hat{h}}{\partial\hat{z}}+\hat{h}^{1.5}\frac{\partial\hat{\rho}}{\partial\hat{z}}-\hat{h}\hat{\rho}\frac{\partial\hat{\rho}}{\partial\hat{z}}\right]\frac{\partial M_v}{\partial\hat{z}}+\frac{\hat{h}^{1.5}\hat{\rho}}{a_2}\frac{\partial^2 M_v}{\partial\hat{z}^2} \qquad (56)$$

Now consider RHS of equation 54

$$\gamma_1\frac{\partial\left(\hat{\rho}\hat{h}\right)}{\partial\hat{\theta}}=\gamma_1\left[\hat{h}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+\hat{\rho}\frac{\partial\hat{h}}{\partial\hat{\theta}}\right] \qquad (57)$$

$$\gamma_2\frac{\partial\left(\hat{\rho}\hat{h}\right)}{\partial\hat{z}}=\gamma_2\left[\hat{h}\frac{\partial\hat{\rho}}{\partial\hat{z}}+\hat{\rho}\frac{\partial\hat{h}}{\partial\hat{z}}\right] \qquad (58)$$

$$\gamma\frac{\partial\left(\hat{\rho}\hat{h}\right)}{\partial\hat{t}}=\gamma\left[\hat{h}\frac{\partial\hat{\rho}}{\partial\hat{t}}+\hat{\rho}\frac{\partial\hat{h}}{\partial\hat{t}}\right] \qquad (59)$$

Putting equations (55-59) in (54)

$$\left[\begin{array}{l}-\frac{1.5}{a_2}\left[\begin{array}{l}\hat{h}^{0.5}M_v\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+0.5\hat{h}^{-0.5}\hat{\rho}M_v\frac{\partial\hat{h}}{\partial\hat{\theta}}\\+\hat{\rho}\hat{h}^{0.5}\frac{\partial M_v}{\partial\hat{\theta}}-\hat{h}^{0.5}M_v\hat{\rho}\end{array}\right]\frac{\partial\hat{h}}{\partial\hat{\theta}}-\frac{1.5\hat{h}^{0.5}M_v\hat{\rho}}{a_2}\frac{\partial^2\hat{h}}{\partial\hat{\theta}^2}\\+\frac{1}{a_2}\left[1.5\hat{h}^{0.5}\hat{\rho}\frac{\partial\hat{h}}{\partial\hat{\theta}}+\hat{h}^{1.5}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}-\hat{h}\hat{\rho}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\right]\frac{\partial M_v}{\partial\hat{\theta}}+\frac{\hat{h}^{1.5}\hat{\rho}}{a_2}\frac{\partial^2 M_v}{\partial\hat{\theta}^2}\end{array}\right]+K^2\left[\begin{array}{l}-\frac{1.5}{a_2}\left[\begin{array}{l}\hat{h}^{0.5}M_v\frac{\partial\hat{\rho}}{\partial\hat{z}}+0.5\hat{h}^{-0.5}\hat{\rho}M_v\frac{\partial\hat{h}}{\partial\hat{z}}\\+\hat{\rho}\hat{h}^{0.5}\frac{\partial M_v}{\partial\hat{z}}-\hat{h}^{0.5}M_v\hat{\rho}\end{array}\right]\frac{\partial\hat{h}}{\partial\hat{z}}-\frac{1.5\hat{h}^{0.5}M_v\hat{\rho}}{a_2}\frac{\partial^2\hat{h}}{\partial\hat{z}^2}\\+\frac{1}{a_2}\left[1.5\hat{h}^{0.5}\hat{\rho}\frac{\partial\hat{h}}{\partial\hat{z}}+\hat{h}^{1.5}\frac{\partial\hat{\rho}}{\partial\hat{z}}-\hat{h}\hat{\rho}\frac{\partial\hat{\rho}}{\partial\hat{z}}\right]\frac{\partial M_v}{\partial\hat{z}}+\frac{\hat{h}^{1.5}\hat{\rho}}{a_2}\frac{\partial^2 M_v}{\partial\hat{z}^2}\end{array}\right]$$
$$=\gamma_1\left[\hat{h}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+\hat{\rho}\frac{\partial\hat{h}}{\partial\hat{\theta}}\right]+\gamma_2\left[\hat{h}\frac{\partial\hat{\rho}}{\partial\hat{z}}+\hat{\rho}\frac{\partial\hat{h}}{\partial\hat{z}}\right]+\gamma\left[\hat{h}\frac{\partial\hat{\rho}}{\partial\hat{t}}+\hat{\rho}\frac{\partial\hat{h}}{\partial\hat{t}}\right]$$

Divide this equation by $\frac{\hat{\rho}\hat{h}^{1.5}}{a_2}$

$$\left(\frac{\partial^2 M_v}{\partial\hat{\theta}^2}+K^2\frac{\partial^2 M_v}{\partial\hat{z}^2}\right)+\frac{1}{\hat{\rho}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial M_v}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial M_v}{\partial\hat{z}}\right)-\frac{\hat{h}^{-0.5}}{a_2}\left(a_3\frac{\partial M_v}{\partial\hat{\theta}}+K^2a_4\frac{\partial M_v}{\partial\hat{z}}\right)$$

$$=M_v\left[\begin{array}{l}\frac{1.5}{\hat{\rho}\hat{h}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right)-\frac{1.5}{a_2\hat{h}}\left(a_3\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2a_4\frac{\partial\hat{h}}{\partial\hat{z}}\right)\\+\frac{0.75}{\hat{h}^2}\left(\left(\frac{\partial\hat{h}}{\partial\hat{\theta}}\right)^2+K^2\left(\frac{\partial\hat{h}}{\partial\hat{z}}\right)^2\right)+\frac{1.5}{\hat{h}}\left(\frac{\partial^2\hat{h}}{\partial\hat{\theta}^2}+K^2\frac{\partial^2\hat{h}}{\partial\hat{z}^2}\right)\end{array}\right]$$

$$+\gamma_1\left[\frac{a_2}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+\frac{a_2}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{\theta}}\right]+\gamma_2\left[\frac{a_2}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{z}}+\frac{a_2}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{z}}\right]+\gamma\left[\frac{a_2}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{t}}+\frac{a_2}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{t}}\right]$$

$$\boxed{\begin{array}{l}\left(\frac{\partial^2 M_v}{\partial\hat{\theta}^2}+K^2\frac{\partial^2 M_v}{\partial\hat{z}^2}\right)+\frac{1}{\hat{\rho}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial M_v}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial M_v}{\partial\hat{z}}\right)-\frac{\hat{h}^{-0.5}}{a_2}\left(a_3\frac{\partial M_v}{\partial\hat{\theta}}+K^2a_4\frac{\partial M_v}{\partial\hat{z}}\right)\\=M_v\left(F\right)+G\end{array}} \quad (60)$$

$$F=\frac{1.5}{\hat{\rho}\hat{h}}\left(\frac{\partial\hat{\rho}}{\partial\hat{\theta}}\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2\frac{\partial\hat{\rho}}{\partial\hat{z}}\frac{\partial\hat{h}}{\partial\hat{z}}\right)-\frac{1.5}{a_2\hat{h}}\left(a_3\frac{\partial\hat{h}}{\partial\hat{\theta}}+K^2a_4\frac{\partial\hat{h}}{\partial\hat{z}}\right)+\frac{0.75}{\hat{h}^2}\left(\left(\frac{\partial\hat{h}}{\partial\hat{\theta}}\right)^2+K^2\left(\frac{\partial\hat{h}}{\partial\hat{z}}\right)^2\right)$$

$$+\frac{1.5}{\hat{h}}\left(\frac{\partial^2\hat{h}}{\partial\hat{\theta}^2}+K^2\frac{\partial^2\hat{h}}{\partial\hat{z}^2}\right)$$

$$G=\gamma_1\left[\frac{a_2}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{\theta}}+\frac{a_2}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{\theta}}\right]+\gamma_2\left[\frac{a_2}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{z}}+\frac{a_2}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{z}}\right]+\gamma\left[\frac{a_2}{\hat{\rho}\hat{h}^{0.5}}\frac{\partial\hat{\rho}}{\partial\hat{t}}+\frac{a_2}{\hat{h}^{1.5}}\frac{\partial\hat{h}}{\partial\hat{t}}\right]$$

Where

$$a_3=\lambda\tau_{rref}\frac{\partial\hat{\tau}_{rr}}{\partial\hat{\theta}}-\frac{\partial\hat{\mu}}{\partial\hat{\theta}}$$

$$a_4=\lambda\tau_{rref}\frac{\partial\hat{\tau}_{rr}}{\partial\hat{z}}-\frac{\partial\hat{\mu}}{\partial\hat{z}}$$

Lateral velocity of the crankshaft is calculated by [41]

$$V=U\sin\gamma_m \qquad \gamma_m=\frac{PL^2}{16EJ} \quad (61)$$

The non-dimensional form of film thickness equation is given as [42]

$$\hat{h}=1+\varepsilon\cos(\hat{\theta}) \quad (62)$$

<u>CERTIFICATE OF COMPLETENESS</u>

It is hereby certified that the dissertation submitted by NS Talha Zia, Reg No. **NUST201464474MCEME35114F**, Titled: **Modeling the Viscoelastic Behavior of Engine Oil in the Crankshaft-Journal-Bearing of High Torque Low-Speed Diesel Engine** has been checked/reviewed and its contents are complete in all respects.

Supervisor's Name: **Dr. Raja Amer Azim**

Signature: ________________

Date: ________________