A Qualitative Study of Hermite--Hadamard Inequality

By

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Name: **Muhammad Raees**

To

My Father Muhammad Idrees

My Mother Zohra Bano

My Wife Rabia

and

My beloved Daughters Fatima, Omama

Rukayyah and Romaisa

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'In the name of Allah, the Most Gracious and the Most Merciful'

"Surely, Allah is with those who are As-Saabiroon (the patient)".

[Al Quran 6:46]

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Muhammad Raees

March, 2021

Abstract

This investigation centers to bring into the spotlight some new and generalized Hermite–Hadamard inequalities in the casings of old classical calculus, fractional calculus, quantum calculus and furthermore a few ramifications in local calculus.

This dissertation begins with the prologue to the hypothesis of convexity, some essential and generalized fractional integrals, q-differentiation and integration and a few fundamentals about local calculus. Hermite–Hadamard inequalities through approximately convex functions and its including classes are created. An exceptionally broad vital portrayal for Hermite–Hadamard inequality and its weighted partner Fejér–Hermite–Hadamard inequality are set up. The error assessment type results are likewise demonstrated. Hermite–Hadamard inequalities for (p_1, h_1)-(p_2, h_2)-convex functions on the coordinates on the rectangle in the plane are additionally settled. Some inequalities for twice differentiable m-convex functions are being developed for quantum integrals. Besides, by applying the thought of local calculus and generalized strongly m-convexity an inequality of Ostrowski type is created and its applications for numerical methods and error formula are talked about.

List of publications

Following papers have been extracted from this thesis.

1. **Muhammad Raees** and Matloob Anwar, On Hermite-Hadamard type inequalities of co-ordinated (p_1, h_1)-(p_2, h_2)-convex functions via Katugampola Fractional Integrals, Filomat 33:15 (2019), 4785–4802.

2. Artion Kashuri, **Muhammad Raees** and Matloob Anwar, Some integral inequalities for approximately h-convex functions and their applications, Proyecciones Journal of Mathematics, 40 (2) (2021), 481–504.

3. Ghulam Farid, Matloob Anwar and **Muhammad Raees**, Bounds associated to Hadamard inequality via generalized integral operators and applications for conformable and fractional integrals, Journal of fractional calculus and Applications, 11 (2) (2020), 238–251.

4. **Muhammad Raees**, Matloob Anwar and Ghulam Farid, Error bounds associated with different versions of Hadamard inequalities of mid-point type, J. Math. Computer Sci. 23 (2021), 213–229.

5. M. J. V.–Cortez, A. Kashuri, **Muhammad Raees**, Matloob Anwar , New quantum integral inequalities via m–convex functions over finite interval, (submitted).

6. Artion Kashuri, **Muhammad Raees** Matloob Anwar and Ghulam Farid, On some ostrowski type inequalities on fractal sets via generalized strongly m-convex mappings, Analele Universitatii Oradea, Fasc. Matematica, Tom XXVII, 2 (2020) 81–100.

List of Symbols

$\mathbb{R}$	The set of Reals
$[a_1, b_1]$	The real interval
$L_1[a_1, b_1]$	The space of integrable functions on $[a_1, b_1]$
P	The partition of $[a_1, b_1]$
$T(\zeta, P)$	Trapezoidal rule
$E(\zeta, P)$	Approximate Error
P	The partition of $[a_1, b_1]$
Γ	The gamma function
Γ_k	The k-gamma function
$\mathscr{E}_\beta$	The one parameter Mittag-Leffler function
$\mathscr{B}(x, y)$	Beta function
$\mathscr{B}_{q^*}(x, y)$	Generalized Beta function
${}^{\vartheta}\mathscr{J}_{a_1^+}\zeta$	The left sided Riemann- Liouville fractional integral of ζ of order $\vartheta > 0$
${}^{\vartheta}\mathscr{H}_{a_1^+}\zeta$	The left sided Hadamard fractional integral of ζ of order $\vartheta > 0$
${}^{\vartheta}\mathscr{J}^{\rho}_{a_1^+}\zeta$	The left sided Katugampola fractional integral of ζ of order $\vartheta > 0$
${}^{\vartheta}\mathscr{J}^{k}_{a_1^+}\zeta$	The left sided k-fractional integral of ζ of order $\vartheta > 0$
${}^{\vartheta}_{r}\mathscr{J}^{\gamma}_{a_1^+}\zeta$	The left sided generalized conformable fractional integral of ζ of order $\vartheta > 0$
${}^{\vartheta}_{\Phi}\mathscr{J}_{a_1^+}\zeta$	The left sided fractional integral of ζ with respect to Φ of order $\vartheta > 0$
$\mathbf{D}_q\zeta$	The q-derivative of ζ
$\mathbb{R}^\gamma$	The α-type set of real line number of fractal dimension $0 < \gamma \leq 1$

$C_\gamma[a_1, b_1]$	The space of local continuous functions defined on $[a_1, b_1]$
$\frac{\partial^\gamma \zeta}{\partial u^\gamma}$	Local fractional derivative of ζ
${}_{a_1}\mathcal{J}_{b_1}^{(\gamma)}$	The local fractional integral of ζ

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Chapter 1

Introduction

Classification of functions of real variable is concerned about different unique properties, like continuity, convexity, monotonicity and differentiability. It is known that, convexity assumes a critical part in the improvement of a few parts of science. In Section 1.1, we start with formal depiction of the thought of convexity followed by various helpful classes specifically s-convexity, m-convexity, h-convexity, p-convexity and (p,h)-convexity. By methods for these convexities, we examine here a number of properties and a few outcomes identified with the convexity. In Section 1.2, we present the Hölder's inequality and its improved forms. In Section 1.3, we talk about some special functions. In Section 1.4, we re-produce some notable fractional and conformable integrals. In Section 1.5, we give the coordinated convexity alongside its generalized classes specifically coordinated s-convexity, coordinated pq-convexity and coordinated (p_1,h_1)-(p_2,h_2)-convexity. In Section 1.6, we diagram a few analogs of Hermite–Hadamard inequality by means of traditional and fractional integrals. In Section 1.7, we layout the quantum derivatives and quantum integrals. In Section 1.8, we momentarily talk about some fundamental wordings related with the local calculus. In Section 1.9, we present the arrangement of the thesis.

1.1 Basic Concepts

1.1.1 Convex function

Recall that a set $\mathcal{Q}$ is convex set, if for any $u, v \in \mathcal{Q}, \mu \in [0,1]$,

$$\mu u + (1-\mu)v \in \mathcal{Q}. \tag{1.1}$$

Now, let $\mathcal{Q}$ be an interval in $\mathbb{R}$. A function $\zeta : \mathcal{Q} \to \mathbb{R}$ is termed as convex if for all $u, v \in \mathcal{Q}$, and $\mu \in (0,1)$, the inequality

$$\zeta(\mu u + (1-\mu)v) \leq \mu\zeta(u) + (1-\mu)\zeta(v), \tag{1.2}$$

holds. If the inequality (1.2) is strict for all $u \neq v, \mu \in (0,1)$, then ζ is known as strictly convex. The function ζ is called concave, if the inequality (1.2) holds in the reverse direction. If the inequality (1.2) is strict for all $u \neq v, \mu \in (0,1)$ in the opposite direction, then ζ is known as strictly concave (see [152]).

An understandable graphical explanation of (1.2) reveals that for any A, B and C on graph of ζ, such that B lies between A and C, then B is on or below the chord AC. Equivalently, for distinct $x_1, y_1, z_1 \in \mathcal{Q}$ such that $x_1 < y_1 < z_1$, the inequality

$$\zeta(y_1)(z_1 - x_1) \leq \zeta(x_1)(z_1 - y_1) \leq \zeta(z_1)(y_1 - x_1) \tag{1.3}$$

holds. Equivalently, we can write the inequality (1.3):

$$\frac{\zeta(y_1) - \zeta(x_1)}{y_1 - x_1} \leq \frac{\zeta(z_1) - \zeta(x_1)}{z_1 - x_1}. \tag{1.4}$$

Furthermore, if ζ defined on $[u, v]$, is convex (respectively concave) on $[u, v]$ and differentiable at u_0, then for $w \in (u, v)$ we have

$$\zeta(w) - \zeta(u_0) \geq (\text{respectively } \leq)\zeta'(u_0)(w - u_0). \tag{1.5}$$

A differentiable function ζ on (u, v) is convex (respectively concave) if and only if the inequality (1.5) holds for all $w, u_0 \in (u, v)$. Following theorem also gives a characterization of convexity.

Theorem 1.1.1. *([152]) Suppose that ζ'' exists on (u, v). Then ζ is convex (strictly convex) if and only if $\zeta''(w) \geq (>) 0$, for all $w \in (u, v)$.*

Monotonicity is a huge property for real valued functions characterized on a subset of $\mathbb{R}$ that compares to its chart being expanding or diminishing. A monotonic function or on the other hand monotonically expanding (diminishing) is only a function ζ which protects the order, i.e., for $\mathcal{Q} \subseteq \mathbb{R}$ and $u, v \in \mathcal{Q}$ with $u \leq v \, (u \geq v)$, we have $\zeta(u) \leq \zeta(v)$. The accompanying two theorems concerning the relation between the monotonicity and the derivatives of convex functions.

Theorem 1.1.2. *([152]) If $\zeta : \mathcal{Q} \to \mathbb{R}$ is convex (strictly convex), then $\zeta'_-(u)$ and $\zeta'_+(u)$ exists and are increasing (strictly increasing) on $\mathcal{Q}^o$.*

Theorem 1.1.3. *([152]) Suppose that ζ is differentiable on (u, v). Then ζ is convex (strictly convex) if and only if ζ' increasing (strictly increasing).*

The geometric characterization depends upon the idea of a support line.

Theorem 1.1.4. *([159]) A function $\zeta : (u, v) \to \mathbb{R}$ is convex if there is at least one line of support for ζ at each $u_0 \in (u, v)$.*

The following result is not direct characterization of a convex function but it is closely related to Theorem 1.1.4.

Theorem 1.1.5. *([159]) A function $\zeta : (u, v) \to \mathbb{R}$ is convex. Then ζ is differentiable at u_0 if the line of support for ζ at u_0 is unique. And in this case, $A(u) = \zeta(u_0) + \zeta'(u_0)(u - u_0)$.*

1.1.2 Breckner-s-convex functions

Due to Hudzik and Maligranda [72], two definitions of s-convexity ($0 < s < 1$) of real-valued functions are known in the literature, and given below:

Definition 1.1. A function $\zeta : \mathbb{R}_+ \to \mathbb{R}$ where $\mathbb{R}_+ = [0, \infty)$, is said to be s-convex in the first sense if

$$\zeta(\mu u + \nu v) \leq \mu^s \zeta(u) + \nu^s \zeta(v), \tag{1.6}$$

for all $u, v \in [0, \infty), \mu, \nu \geq 0$ with $\mu^s + \nu^s = 1$ and for some fixed $s \in (0, 1]$. This class of functions is denoted by K_s^1.

In 2007, Pinheiro [149], guaranteed that the class K_s^1 has numerous issues. Pinheiro considered this class of s-convex functions and clarified why the first s-convexity sense was deserted by the writing in the field. Pinheiro amended the class of s-convexity in the first sense and proposed a mathematical understanding for functions in K_s^1 with a few related outcomes. For additional outcomes concerning s-convexity in the first sense see [150].

Breckner [19] considered the following meaning of s-convexity:

Definition 1.2. ([19]) A function $\zeta : \mathbb{R}_+ \to \mathbb{R}$ where $\mathbb{R}_+ = [0, \infty)$, is said to be s-convex in the second sense if

$$\zeta(\mu u + \nu v) \leq \mu^s \zeta(u) + \nu^s \zeta(v), \tag{1.7}$$

for all $u, v \in [0, \infty), \mu, \nu \geq 0$ with $\mu + \nu = 1$ and for some fixed $s \in (0, 1]$. This class of functions is denoted by K_s^2.

In the accompanying, we think about certain results associated with s-convex functions in the second sense.

Theorem 1.1.6. *([72]) If $\zeta \in K_s^2$, then ζ is nonnegative on* $[0,1)$.

Example 1.3. ([72]) Let $0 < s < 1$ and $a_1, a_2, a_3 \in \mathbb{R}$. Defining for $v \in \mathbb{R}_+$,

$$\zeta(u) = \begin{cases} a_1, & \text{if } v = 0, \\ a_2 v^s + a_3, & \text{if } v > 0; \end{cases}$$

we have the following

1. $a_2 \geq 0$ and $0 \leq a_3 \leq a_1$, then $\zeta \in K_s^2$.

2. $a_2 > 0$ and $a_3 < 0$, then $\zeta \notin K_s^2$.

Theorem 1.1.7. *([72]) If $\zeta \in K_s^2$, then the inequality* (1.7) *holds for all $u, v \in \mathbb{R}_+$ and $\mu, \nu \geq 0$ with $\mu + \nu \leq 1$, if and only if $\zeta(0) = 0$.*

Theorem 1.1.8. *([72]) If $0 < s_1 \leq s_2 < 1$. If $\zeta \in K_{s_2}^2$ an $\zeta(0) = 0$, then $\zeta \in K_{s_1}^2$.*

Theorem 1.1.9. *([72]) Let ζ be a nondecreasing function in K_s^2 and φ be a nonnegative convex function on* $[0, \infty)$*. Then the composition $\zeta o \varphi$ of ζ with φ belongs to K_s^2.*

1.1.3 m-convex function

In [186], G. Toader, defined m-convexity, a fabulous generalization of usual convexity as follows:

Definition 1.4. Let $m \in [0,1]$. A function $\zeta : [0,b] \to \mathbb{R}, b > 0$ is called m-convex, if the inequality

$$\zeta(\mu a_1 + m(1-\mu)b_1) \leq \mu\zeta(a_1) + m(1-\mu)\zeta(b_1), \tag{1.8}$$

holds for all $a_1, b_1 \in [0,b]$ and $\mu \in [0,1]$. If $-\zeta$ is m-convex, then ζ is m-concave.

The space of all m-convex functions on $[0,b]$ for which $\zeta(0) < 0$ is denoted by $K_m(b)$. Obviously, for $m = 1$, m-convexity is the standard convexity of functions on $[0,b]$.

The following results about m-convexity are available in literature [186].

Lemma 1.1.1. *If ζ is in the space $K_m(b)$ and $0 < n < m \leq 1$, then ζ is in the space $K_n(b)$.*

Theorem 1.1.10. *([28]) For each $m \in (0,1)$, there is an m-convex function ζ such that ζ is not n-convex for any $m < n \leq 1$.*

Lemma 1.1.2. *([28]) If ζ is differentiable, then ζ is m-convex on $[0,b]$ if and only if*

$$\zeta(x) \leq m\zeta(y) + \zeta'(x)(x - my), \tag{1.9}$$

for all $x, y \in [0,b]$.

1.1.4 h-convex functions

Let $\mathcal{Q}$ be an interval in $\mathbb{R}$, then the space of Godunova-Levin functions is defined as follows.

Definition 1.5. A function $\zeta : \mathcal{Q} \to \mathbb{R}$ is stated to belong to the space of Godunova–Levin functions if ζ is nonnegative and for all $u, v \in \mathcal{Q}$ and $0 < \mu < 1$ we have

$$\zeta\left(\mu u + (1-\mu)v\right) \leq \frac{\zeta(u)}{\mu} + \frac{\zeta(v)}{1-\mu}. \tag{1.10}$$

The space of Godunova-Levin functions is denoted by $Q(\mathcal{Q})$ and was initially described in [66] by Godunova and Levin. Some further properties of it are given in [120], and [121]. It is to be noted that nonnegative monotone and nonnegative convex functions belong to this space of functions. The following definition is about the space of P-functions.

Definition 1.6. A function $\zeta : \mathcal{Q} \to \mathbb{R}$ is called P-function or belong to the space $P(\mathcal{Q})$, if ζ is nonnegative and for all $u, v \in \mathcal{Q}$ and $0 < \mu < 1$ we have

$$\zeta\left(\mu u + (1-\mu)v\right) \leq \zeta(u) + \zeta(v). \tag{1.11}$$

For further results about the space $P(\mathcal{Q})$ we refer the readers to [153]. In the paper [193] a larger space of nonnegative functions which contains nonnegative convex functions, s-convex in the second sense, Godunova-Levin functions and P-functions was considered. This space of functions is known as space of h-convex functions.

Definition 1.7. Let $\mathcal{Q}$ and J be intervals in $\mathbb{R}$ such that $(0,1) \subset J$ and let $h : J \to \mathbb{R}$ be a nonnegative function, $h \neq 0$. A nonnegative function $\zeta : \mathcal{Q} \to \mathbb{R}$ is said to be an h-convex function, or that ζ belongs to the space $SX(h;\mathcal{Q})$, if for all $u, v \in \mathcal{Q}, 0 < \mu < 1$ we have

$$\zeta\left(\mu u + (1-\mu)v\right) \leq h(\mu)\zeta(u) + h(1-\mu)\zeta(v). \tag{1.12}$$

A nonnegative function $\zeta : \mathcal{Q} \to \mathbb{R}$ is said to be an h-concave function, or that ζ belongs to the space $SV(h;\mathcal{Q})$ if the inequality (1.12) is reversed.

The following is an example for the existence of h-convex functions which are not convex.

Example 1.8. Let $h_k(u) = u^k$ where $k < 0, u > 0$ and let the function $\zeta : \mathcal{Q} = [a_1, b_1] \to \mathbb{R}$ be defined as

$$\zeta(u) = \begin{cases} 1, & \text{if } u \neq \frac{a_1+b_1}{2}, \\ 2^{1-k}, & \text{if } u = \frac{a_1+b_1}{2}. \end{cases}$$

Then ζ is an h_k-convex but not convex function.

The following results hold for h-convex functions.

Proposition 1. *Let h_1, h_2 be nonnegative functions defined on an interval $\mathcal{Q}$ with property $h_2(u) \leq h_1(u), u \in (0,1)$. If $\zeta \in SX(h_2;\mathcal{Q})$, then $\zeta \in SX(h_1;\mathcal{Q})$. If $\zeta \in SV(h_1;\mathcal{Q})$, then $\zeta \in SV(h_2;\mathcal{Q})$.*

Proposition 2. *If $\zeta, \xi \in SX(h_2; \mathcal{Q})$ and $\mu > 0$, then $\zeta + \xi, \mu\zeta \in SX(h_1; \mathcal{Q})$. If $\zeta, \xi \in SV(h_1; \mathcal{Q})$ and $\mu > 0$, then $\zeta + \xi, \mu\zeta \in SV(h_2; \mathcal{Q})$.*

For more properties on h-convex functions and related results see ([31,178,180,193]) and the references therein. Some further generalizations of convexity are stated in the next definition as p-convexity, (p, h)-convexity.

1.1.5 p-convex and (p, h)-convex functions

Definition 1.9. ([204]) A set $\mathcal{Q}_p$ is p-convex set, if for any $u, v \in \mathcal{Q}_p, \mu \in [0, 1]$, we have

$$[\mu u^p + (1-\mu)v^p]^{\frac{1}{p}} \in \mathcal{Q}_p, \tag{1.13}$$

where $p = 2r_1 + 1$ or $p = \frac{m_1}{n_1}, m_1 = 2s_1, n_1 = 2k_1 + 1$ and $r_1, s_1, k_1 \in \mathbb{N}$.

Definition 1.10. ([204]) A function ζ defined from a p-convex set $\mathcal{Q}_p$ to $\mathbb{R}$ is said to be p-convex function, if

$$\zeta([\mu u^p + (1-\mu)v^p]^{\frac{1}{p}}) \leq \mu\zeta(u) + (1-\mu)\zeta(v) \tag{1.14}$$

for all $u, v \in \mathcal{Q}_p, \mu \in [0, 1]$.

Definition 1.11. ([63]) Let $h : J \subseteq \mathbb{R} \to \mathbb{R}$ be a nonnegative and non-zero function. We say that $\zeta : \mathcal{Q}_p \to \mathbb{R}$ is a (p, h)-convex function or that ζ belongs to the class $ghx(h, p, \mathcal{Q}_p)$, if ζ is nonnegative and

$$\zeta([\mu u^p + (1-\mu)v^p]^{\frac{1}{p}}) \leq h(\mu)\zeta(u) + h(1-\mu)\zeta(v) \tag{1.15}$$

for all $u, v \in \mathcal{Q}_p, \mu \in [0, 1]$. Similarly, if the inequality sign in (1.15) is reversed, then ζ is said to be a (p, h)-concave function or belong to the class $ghv(h, p, \mathcal{Q}_p)$.

Example 1.12. ([63]) Let $h_k(\mu) = \mu^k$, where $k \leq 1$ and $\mu > 0$. If ζ is a function defined as $\zeta(u) = u^p$, where p is an odd number and $u \geq 0$, then ζ belongs to $ghx(h_k, p, \mathcal{Q}_p)$.

1.2 Hölder's Inequality

In mathematical analysis, "Hölder's inequality is one of the most influential inequality concerning integrals and is an inevitable tool to study the L_p spaces. This fabulous inequality is named after the renowned mathematician Otto Hölder".

Theorem 1.2.1. *([70]) If ζ_1 and ζ_2 are real valued functions defined on $[a_1, b_1]$ and if $|\zeta_1|^p$ and $|\zeta_2|^q$ are integrable on $[a_1, b_1]$, then*

$$\int_{a_1}^{b_1} |\zeta_1(\mathscr{S})\zeta_2(\mathscr{S})|\, d\mathscr{S} \leq \left(\int_{a_1}^{b_1} |\zeta_1(\mathscr{S})|^p\, d\mathscr{S}\right)^{\frac{1}{p}} \left(\int_{a_1}^{b_1} |\zeta_2(\mathscr{S})|^q\, d\mathscr{S}\right)^{\frac{1}{q}}, \tag{1.16}$$

where equality holds if and only if $A_1 |\zeta_1(x)|^p = B_1 |\zeta_2(x)|^q$ almost everywhere, where A_1 and B_1 are constants.

Another form of Hölder's inequality is popular among the researchers called power mean inequality.

Theorem 1.2.2. *Let $k \geq 1$. If ζ_1 and ζ_2 are real valued functions on $[a_1, b_1]$ and if $|\zeta_1|$ and $|\zeta_2| |\Psi|^q$ are integrable on $[a_1, b_1]$, then*

$$\int_{a_1}^{b_1} |\zeta_1(\mathscr{S})\zeta_2(\mathscr{S})|\, d\mathscr{S} \leq \left(\int_{a_1}^{b_1} |\zeta_1(\mathscr{S})|\, d\mathscr{S}\right)^{1-\frac{1}{k}} \left(\int_{a_1}^{b_1} |\zeta_1(\mathscr{S})|\, |\zeta_2(\mathscr{S})|^q\, d\mathscr{S}\right)^{\frac{1}{k}}, \tag{1.17}$$

Recently, Isçan [74] proved a refinement of the Hölder's integral inequality known as Hölder–Isçan inequality.

Theorem 1.2.3. *Let $k_1 > 1$ and $\frac{1}{k_1} + \frac{1}{k_2} = 1$. If ζ_1 and ζ_2 are real valued functions on $[a_1, b_1]$ and if $|\zeta_1|^{k_1}$ and $|\zeta_2|^{k_2}$ are integrable on $[a_1, b_1]$, then*

$$\begin{aligned}
&\int_{a_1}^{b_1} |\zeta_1(\mathscr{S})\zeta_2(\mathscr{S})|\, d\mathscr{S} \\
\leq\ & \frac{1}{b_1 - a_1}\Bigg\{\left(\int_{a_1}^{b_1} (b_1 - \mathscr{S})\, |\zeta_1(\mathscr{S})|^{k_1}\, d\mathscr{S}\right)^{\frac{1}{k_1}} \left(\int_{a_1}^{b_1} (b_1 - \mathscr{S})\, |\zeta_2(\mathscr{S})|^{k_2}\, d\mathscr{S}\right)^{\frac{1}{k_2}} \\
& + \left(\int_{a_1}^{b_1} (\mathscr{S} - a_1)\, |\zeta_1(\mathscr{S})|^{k_1}\, d\mathscr{S}\right)^{\frac{1}{k_1}} \left(\int_{a_1}^{b_1} (\mathscr{S} - a_1)\, |\zeta_2(\mathscr{S})|^{k_2}\, d\mathscr{S}\right)^{\frac{1}{k_2}}\Bigg\} \\
\leq\ & \left(\int_{a_1}^{b_1} |\zeta_1(\mathscr{S})|^{k_1}\, d\mathscr{S}\right)^{\frac{1}{k_1}} \left(\int_{a_1}^{b_1} |\zeta_2(\mathscr{S})|^{k_2}\, d\mathscr{S}\right)^{\frac{1}{k_2}}, \qquad (1.18)
\end{aligned}$$

where equality holds if and only if $A_1 |\Phi(\mathscr{S})|^p = B_1 |\Psi(\mathscr{S})|^q$ almost everywhere, where A_1 and B_1 are constants.

A different version named as improved power–mean integral inequality can be given by:

Theorem 1.2.4. *Let $k \geq 1$. If ζ_1 and ζ_2 are real valued functions on $[a_1, b_1]$ and if $|\zeta_1|$ and $|\zeta_2|\, |\Psi|^k$ are integrable on $[a_1, b_1]$, then*

$$\begin{aligned}
&\int_{a_1}^{b_1} |\zeta_1(\mathscr{S})\zeta_2(\mathscr{S})|\, d\mathscr{S} \\
\leq\ & \frac{1}{b_1 - a_1}\Bigg\{\left(\int_{a_1}^{b_1} (b_1 - \mathscr{S})\, |\zeta_1(\mathscr{S})|\, d\mathscr{S}\right)^{1-\frac{1}{k}} \left(\int_{a_1}^{b_1} (b_1 - \mathscr{S})\, |\zeta_1(\mathscr{S})|\, |\zeta_2(\mathscr{S})|^{k}\, d\mathscr{S}\right)^{\frac{1}{k}} \\
& + \left(\int_{a_1}^{b_1} (\mathscr{S} - a_1)\, |\zeta_1(\mathscr{S})|\, d\mathscr{S}\right)^{1-\frac{1}{k}} \left(\int_{a_1}^{b_1} (\mathscr{S} - a_1)\, |\zeta_1(\mathscr{S})|\, |\zeta_2(\mathscr{S})|^{k}\, d\mathscr{S}\right)^{\frac{1}{k}}\Bigg\} \\
\leq\ & \left(\int_{a_1}^{b_1} |\zeta_1(\mathscr{S})|\, d\mathscr{S}\right)^{1-\frac{1}{k}} \left(\int_{a_1}^{b_1} |\zeta_1(\mathscr{S})|\, |\zeta_2(\mathscr{S})|^{k}\, d\mathscr{S}\right)^{\frac{1}{k}}. \qquad (1.19)
\end{aligned}$$

1.3 Some useful Special Functions

Special functions are particular mathematical functions that, due to their significance in functional analysis, mathematical analysis, physics and other applications, have more or less established names and notations. To proceed further in this work, we first provide necessary information about Euler's gamma function and Mittag–Leffler function. These functions play a significant role in the theory of fractional calculus.

The gamma function

In 1720, Daniel Bernoulli and Christian Goldbach raised a question, how one can extend factorial function for non-integer values? Which was answered by Leonhard Euler in 1729, in his famous paper [48] "De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt". Subsequently, many methods have emerged that contribute to the concept of gamma function. Among those one representation is given by Karl Weierstras, which is now known as the gamma function.

Definition 1.13. The gamma function $\Gamma : (0, \infty) \to \mathbb{R}$ is defined as

$$\Gamma(u) = \int_0^\infty \mathscr{S}^{u-1} e^{-\mathscr{S}} d\mathscr{S}, \ u > 0. \tag{1.20}$$

By parts integration of (1.20) yields the fundamental equation

$$\Gamma(u+1) = u\Gamma(u), \ u > -1, \ u \neq 0. \tag{1.21}$$

The integral defining gamma function is uniformly convergent for all $u \in [a_1, b_1]$, where $0 < a_1 \leq b_1 < \infty$, and hence Γ is a continuous function for all $u > 0$.

The k-Gamma function

In [46], Diaz and Pariguan defined k-gamma function: $\Gamma_k(\mathscr{S}) = \lim_{n\to\infty} \frac{n! k^n (nk)^{\frac{\mathscr{S}}{k}-1}}{(\mathscr{S})_{n,k}}, k > 0$; 'where'

$(\mathscr{S})_{n,k}$ is the Pochhammer k-symbol. The integral representation of the k-gamma function is given by

$$\Gamma_k(u) = \int_0^\infty \mathscr{S}^{u-1} e^{-\frac{\mathscr{S}^k}{k}} d\mathscr{S}, \ u > 0. \tag{1.22}$$

Clearly, $\lim_{k\to 1} \Gamma_k(u) = \Gamma(u)$, $\Gamma_k(u) = k^{\frac{u}{k}-1}\Gamma\left(\frac{u}{k}\right)$ and $\Gamma_k(u+k) = u\Gamma_k(u)$.

Mittag–Leffler function and its generalized versions

In [106], Gosta Mittag Leffler (1846-1927), the swedish mathematician introduced a function as a generalization of exponential function, known as Mittag-Leffler function. The one parameter Mittag-Leffler function is defined by:

$$\mathscr{E}_\beta(\mathscr{S}) = \sum_{m=0}^{\infty} \frac{\mathscr{S}^m}{\Gamma(\beta m + 1)}, \tag{1.23}$$

where $\beta \in \mathbb{C}, \Gamma(\xi)$ is the Gamma function. Mittag-Leffler function due to its natural occurrence in the fractional order differential and integral equations attracted the attention of researchers working in the area of fractional calculus. Many generalizations of this function are then made.

In [194], Wiman first extended Mittag-Leffler function by defining

$$\mathscr{E}_{\beta,\gamma}(\mathscr{S}) = \sum_{m=0}^{\infty} \frac{\mathscr{S}^m}{\Gamma(\beta m + \gamma)}, \tag{1.24}$$

where $\beta, \gamma \in \mathbb{C}$ and $\Re(\beta), \Re(\gamma) > 0$.

In [148], Parbharker gave the following generalization:

$$\mathscr{E}^\delta_{\beta,\gamma}(\mathscr{S}) = \sum_{m=0}^{\infty} \frac{(\delta)_m \mathscr{S}^m}{\Gamma(\beta m + \gamma) m!}, \tag{1.25}$$

where $\beta, \gamma, \delta \in \mathbb{C}$ and $\Re(\beta), \Re(\gamma), \Re(\delta) > 0$.

In [176], Shukla and Prajapati added another parameter to the Mittag-Leffler function as follows:

$$\mathscr{E}_{\beta,\gamma}^{\delta,r}(\mathscr{S}) = \sum_{m=0}^{\infty} \frac{(\delta)_{rm}\mathscr{S}^m}{\Gamma(\beta m+\gamma)m!}, \tag{1.26}$$

where $\beta, \gamma, \delta \in \mathbb{C}$ and $\Re(\beta), \Re(\gamma), \Re(\delta) > 0, r \in (0,1)\cup\mathbb{N}$ and $(\delta)_{rm} = \frac{\Gamma(\delta+rm)}{\Gamma(\delta)}$ denotes generalized Pochhammer symbol.

In [160], Salim generalized the notion as follows:

$$\mathscr{E}_{\beta,\gamma}^{\delta,\lambda}(\mathscr{S}) = \sum_{m=0}^{\infty} \frac{(\delta)_{m}\mathscr{S}^m}{\Gamma(\beta m+\gamma)(\lambda)_m}, \tag{1.27}$$

where $\beta, \gamma, \delta, \lambda \in \mathbb{C}$ and $\Re(\beta), \Re(\gamma), \Re(\delta), \Re(\lambda) > 0$.

Later on, in [161] Salim and Faraj further extended it as following:

$$\mathscr{E}_{\beta,\gamma,r}^{\delta,\lambda,s}(\mathscr{S}) = \sum_{m=0}^{\infty} \frac{(\delta)_{sm}\mathscr{S}^m}{\Gamma(\beta m+\gamma)(\lambda)_{rm}}, \tag{1.28}$$

where $\beta, \gamma, \delta, \lambda \in \mathbb{C}$ and $\Re(\beta), \Re(\gamma), \Re(\delta), \Re(\lambda) > 0; r, s > 0$ and $s \leq \Re(\beta) + r$.

Recently, in [89] Kang et al. generalized the extending Mittag-Leffler function as follows:

$$\mathscr{E}_{\beta,\gamma,\nu}^{\delta,r,s,c}(\mathscr{S}, q^*) = \sum_{m=0}^{\infty} \frac{B_{q^*}(\delta+ms, c-\delta)(c)_{sm}\mathscr{S}^m}{B(\delta, c-\delta)\Gamma(\beta m+\gamma)(\nu)_{rm}}, \tag{1.29}$$

where $\mathscr{B}_{q^*}(x,y)$ is the generalized beta function while $\mathscr{B}(x,y)$ is classical beta function and are defined by

$$\mathscr{B}_{q^*}(\theta_1, \theta_2) = \int_0^1 \mathscr{S}^{\theta_1-1}(1-\mathscr{S})^{\theta_2-1} e^{\frac{-q^*}{\mathscr{S}(1-\mathscr{S})}} d\mathscr{S}, \tag{1.30}$$

$$\mathscr{B}(\theta_1, \theta_2) = \int_0^1 \mathscr{S}^{\theta_1-1}(1-\mathscr{S})^{\theta_2-1} d\mathscr{S}. \tag{1.31}$$

1.4 Summary of Fractional and Conformable integrals

The study of fractional order derivatives and integrals received more attention after the formulation of electrochemical problems. In recent years the subject is studied extensively due to its applications in different areas of natural sciences such as: quantum mechanical calculations, chemical analysis of aqueous solutions, design of heat flux meters, transmission line theory etc. see [145]. For the detailed mathematical study of fractional integral and derivative operators, see ([92, 93]). Now we outline some fractional and conformable integral operators.

Definition 1.14. ([92, 93]) Let $\zeta \in L_1[a_1, b_1]$. Then left and right-sided Riemann-Liouville fractional integrals of order $\vartheta > 0$ with $a_1 \geq 0$ are defined as follows:

$$^{\vartheta}\mathscr{I}_{a_1^+}\zeta(\theta) = \frac{1}{\Gamma(\vartheta)}\int_{a_1}^{\theta}\frac{\zeta(\mathscr{S})}{(\theta-\mathscr{S})^{1-\vartheta}}d\mathscr{S}, \quad \theta > a_1, \tag{1.32}$$

$$^{\vartheta}\mathscr{I}_{b_1^-}\zeta(\theta) = \frac{1}{\Gamma(\vartheta)}\int_{\theta}^{b_1}\frac{\zeta(\mathscr{S})}{(\mathscr{S}-\theta)^{1-\vartheta}}d\mathscr{S}, \quad \theta < b_1. \tag{1.33}$$

Definition 1.15. ([92, 93]) Let $\zeta \in L_1[a_1, b_1]$. Then left and right-sided Hadamard fractional integrals of order $\vartheta > 0$ with $a_1 \geq 0$ are defined as follows:

$$^{\vartheta}\mathscr{H}_{a_1^+}\zeta(\theta) = \frac{1}{\Gamma(\vartheta)}\int_{a_1}^{\theta}\frac{1}{\left(\ln\frac{\theta}{\mathscr{S}}\right)^{1-\vartheta}}\frac{\zeta(\mathscr{S})}{\mathscr{S}}d\mathscr{S}, \quad \theta > a_1, \tag{1.34}$$

$$^{\vartheta}\mathscr{H}_{b_1^-}\zeta(\theta) = \frac{1}{\Gamma(\vartheta)}\int_{\theta}^{b_1}\frac{1}{\left(\ln\frac{\mathscr{S}}{\theta}\right)^{1-\vartheta}}\frac{\zeta(\mathscr{S})}{\mathscr{S}}d\mathscr{S}, \quad \theta < b_1. \tag{1.35}$$

Definition 1.16. ([86]) $X_c^p(a_1, b_1)(c \in \mathbb{R}, 1 \leq p \leq \infty)$ is the set of those complex valued Labesgue measurable functions ζ defined on $[a_1, b_1]$ for which $\|\zeta\|_{X_c^p} < \infty$, where the norm is defined by $\|\zeta\|_{X_c^p} = \left(\int_{a_1}^{b_1} |\mathscr{S}^c\zeta(\mathscr{S})|^p \frac{d\mathscr{S}}{\mathscr{S}}\right)^{\frac{1}{p}} < \infty$ for $1 \leq p < \infty, c \in \mathbb{R}$ and for the case $p = \infty$, $\|\zeta\|_{X_c^p} = ess \sup_{a_1 \leq \mathscr{S} \leq b_1} [\mathscr{S}^c |\zeta(\mathscr{S})|], c \in \mathbb{R}$.

Definition 1.17. ([83–85]) Let $[a_1, b_1] \subseteq \mathbb{R}$ be a finite interval. Then left and right-sided Katugampola fractional integrals of order $\vartheta(> 0)$ of $\zeta \in X_c^p(a_1, b_1)$ with $a_1 \geq 0$ are defined by:

$$
{}^{\vartheta}\mathcal{J}^{\rho}_{a_1^+}\zeta(\theta) = \frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)} \int_{a_1}^{\theta} \frac{\mathscr{S}^{\vartheta-1}}{(\theta^{\rho} - \mathscr{S}^{\rho})^{1-\vartheta}} \zeta(\mathscr{S})\, d\mathscr{S}, \quad \theta > a_1 \tag{1.36}
$$

and

$$
{}^{\vartheta}\mathcal{J}^{\rho}_{b_1^-}\zeta(\theta) = \frac{\rho^{1-\vartheta}}{\Gamma(\vartheta)} \int_{\theta}^{b_1} \frac{\mathscr{S}^{\vartheta-1}}{(\mathscr{S}^{\rho} - \theta^{\rho})^{1-\vartheta}} \zeta(\mathscr{S})\, d\mathscr{S}, \quad \theta < b_1, \tag{1.37}
$$

with $\rho > 0$, provided the integrals exist.

Definition 1.18. ([161]) Let $\zeta \in L_1[a_1, b_1]$. Then left and right-sided generalized fractional integrals of order $\vartheta > 0$ with $a_1 \geq 0$ are defined as follows:

$$
{}^{\vartheta}\mathcal{J}^{\delta,\lambda,s}_{\beta,r,a_1^+}\zeta(\theta; w) = \int_{a_1}^{\theta} \frac{\mathcal{E}^{\delta,\lambda,s}_{\beta,\vartheta,r}(w(\theta - \mathscr{S})^{\beta})\zeta(\mathscr{S})}{(\theta - \mathscr{S})^{1-\vartheta}} d\mathscr{S}, \quad \theta > a_1, \tag{1.38}
$$

$$
{}^{\vartheta}\mathcal{J}^{\delta,\lambda,s}_{\beta,r,b_1^-}\zeta(\theta; w) = \int_{\theta}^{b_1} \frac{\mathcal{E}^{\delta,\lambda,s}_{\beta,\vartheta,r}(w(\mathscr{S} - \theta)^{\beta})\zeta(\mathscr{S})}{(\mathscr{S} - \theta)^{1-\vartheta}} d\mathscr{S}, \quad \theta < b_1, \tag{1.39}
$$

where $\mathcal{E}^{\delta,\lambda,s}_{\beta,\vartheta,r}(\mathscr{S})$ is defined in (1.28).

Definition 1.19. ([118]) Let $\zeta \in L_1[a_1, b_1]$. Then left and right-sided k-fractional integrals of order $\vartheta, k > 0$ with $a_1 \geq 0$ are defined as follows:

$$
{}^{\vartheta}\mathcal{J}^{k}_{a_1^+}\zeta(\theta) = \frac{1}{k\Gamma_k(\vartheta)} \int_{a_1}^{\theta} \frac{\zeta(\mathscr{S})}{(\theta - \mathscr{S})^{1-\frac{\vartheta}{k}}} d\mathscr{S}, \quad \theta > a_1, \tag{1.40}
$$

$$
{}^{\vartheta}\mathcal{J}^{k}_{b_1^-}\zeta(\theta) = \frac{1}{k\Gamma_k(\vartheta)} \int_{\theta}^{b_1} \frac{\zeta(\mathscr{S})}{(\mathscr{S} - \theta)^{1-\frac{\vartheta}{k}}} d\mathscr{S}, \quad \theta < b_1. \tag{1.41}
$$

Definition 1.20. ([171]) Let $\zeta \in L_1[a_1, b_1]$. Then left and right-sided (k, s)-fractional integrals of order $\vartheta > 0$ with $a_1 \geq 0$ are defined by:

$$
{}^{\vartheta}_{s}\mathcal{J}^{k}_{a_1^+}\zeta(\theta) = \frac{(1+s)^{1-\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta)} \int_{a_1}^{\theta} \frac{\mathscr{S}^{s}\zeta(\mathscr{S})}{(\theta^{1+s} - \mathscr{S}^{1+s})^{1-\frac{\vartheta}{k}}} d\mathscr{S}, \quad \theta > a_1, \tag{1.42}
$$

$$
{}^{\vartheta}_{s}\mathcal{J}^{k}_{b_1^-}\zeta(\theta) = \frac{(1+s)^{1-\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta)} \int_{\theta}^{b_1} \frac{\mathscr{S}^{s}\zeta(\mathscr{S})}{\left(\mathscr{S}^{1+s}-\theta^{1+s}\right)^{1-\frac{\vartheta}{k}}} d\mathscr{S}, \quad \theta < b_1, \tag{1.43}
$$

where $k > 0$, $s \in \mathbb{R} - \{-1\}$.

Definition 1.21. ([91]) Let ζ be a conformable integrable function on the interval $[a_1, b_1] \subseteq [0, \infty)$. The left and right-sided generalized conformable fractional integrals of order $\vartheta > 0$ with $r \in \mathbb{R}$, $\gamma \in (0, 1]$, $r + \gamma \neq 0$ are defined by

$$
{}^{\vartheta}_{r}\mathcal{J}^{\gamma}_{a_1^+}\zeta(\theta) = \frac{(r+\gamma)^{1-\vartheta}}{\Gamma(\vartheta)} \int_{a_1}^{\theta} \frac{\mathscr{S}^{r}\zeta(\mathscr{S})}{\left(\theta^{r+\gamma}-\mathscr{S}^{r+\gamma}\right)^{\vartheta-1}} d_{\gamma}\mathscr{S}, \quad \theta > a_1, \tag{1.44}
$$

$$
{}^{\vartheta}_{r}\mathcal{J}^{\gamma}_{b_1^-}\zeta(\theta) = \frac{(r+\gamma)^{1-\vartheta}}{\Gamma(\vartheta)} \int_{\theta}^{b_1} \frac{\mathscr{S}^{r}\zeta(\mathscr{S})}{\left(\mathscr{S}^{r+\gamma}-\theta^{r+\gamma}\right)^{\vartheta-1}} d_{\gamma}\mathscr{S}, \quad \theta < b_1. \tag{1.45}
$$

Definition 1.22. ([89]) Let $\zeta \in L_1[a_1, b_1]$. Then left and right-sided generalized fractional integrals of order $\vartheta > 0$ with $a_1 \geq 0$ are defined by:

$$
{}^{\vartheta}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,a_1^+}\zeta(\theta; q^*, w) = \int_{a_1}^{\theta} \frac{\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}(w(\theta-\mathscr{S})^{\beta}; q^*)\zeta(\mathscr{S})}{(\theta-\mathscr{S})^{1-\vartheta}} d\mathscr{S}, \quad \theta > a_1, \tag{1.46}
$$

$$
{}^{\vartheta}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,b_1^-}\zeta(\theta; q^*, w) = \int_{\theta}^{b_1} \frac{\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}(w(\theta-\mathscr{S})^{\beta}; q^*)\zeta(\mathscr{S})}{(\mathscr{S}-\theta)^{1-\vartheta}} d\mathscr{S}, \quad \theta < b_1, \tag{1.47}
$$

where the extended generalized Mittag-Leffler function $\mathcal{E}^{\delta,r,s,c}_{\beta,\gamma,\nu}(\mathscr{S}, q^*)$ is defined in (1.29).

Definition 1.23. ([92, 93]) Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be an integrable function. Also let Φ be an increasing and positive function on $(a_1, b_1]$, having continuous derivative Φ' on (a_1, b_1). The left and right-sided fractional integrals of a function ζ with respect to another function Φ on $[a_1, b_1]$ of order $\vartheta > 0$ are defined as:

$$
{}^{\vartheta}_{\Phi}\mathcal{J}_{a_1^+}\zeta(\theta) = \frac{1}{\Gamma(\vartheta)} \int_{a_1}^{\theta} \frac{\Phi'(\mathscr{S})\zeta(\mathscr{S})}{\left[\Phi(\theta)-\Phi(\mathscr{S})\right]^{1-\vartheta}} d\mathscr{S}, \ \theta > a_1, \tag{1.48}
$$

$$
{}^{\vartheta}_{\Phi}\mathcal{J}_{b_1^-}\zeta(\theta) = \frac{1}{\Gamma(\vartheta)} \int_{\theta}^{b_1} \frac{\Phi'(\mathscr{S})\zeta(\mathscr{S})}{[\Phi(\mathscr{S}) - \Phi(\theta)]^{1-\vartheta}} d\mathscr{S}, \ \theta < b_1. \tag{1.49}
$$

Definition 1.24. ([90]) Let $\Phi : [a_1, b_1] \to \mathbb{R}$ be an increasing and positive monotone function on $[a_1, b_1]$, having a continuous derivative Φ' on (a_1, b_1). The left and right-sided fractional integrals of a function ζ with respect to another function Φ on $[a_1, b_1]$ of order $\vartheta, k > 0$ are defined by:

$$
{}^{\vartheta}_{\Phi}\mathcal{J}^{k}_{a_1^+}\zeta(\theta) = \frac{1}{k\Gamma_k(\vartheta)} \int_{a_1}^{\theta} \frac{\Phi'(\mathscr{S})\zeta(\mathscr{S})}{[\Phi(\theta) - \Phi(\mathscr{S})]^{1-\frac{\vartheta}{k}}} d\mathscr{S}, \ \theta > a_1, \tag{1.50}
$$

$$
{}^{\vartheta}_{\Phi}\mathcal{J}^{k}_{b_1^-}\zeta(\theta) = \frac{1}{k\Gamma_k(\vartheta)} \int_{\theta}^{b_1} \frac{\Phi'(\mathscr{S})\zeta(\mathscr{S})}{[\Phi(\mathscr{S}) - \Phi(\theta)]^{1-\frac{\vartheta}{k}}} d\mathscr{S}, \ \theta < b_1, \tag{1.51}
$$

where $\Gamma_k(.)$ is the k-gamma function.

Definition 1.25. ([154]) Let $\zeta \in L_1[a_1, b_1]$. The left and right-sided integrals with special functions are denoted and defined by

$$
{}^{\sigma}_{\rho}\mathcal{J}_{\vartheta, a_1^+; w}\zeta(\theta) = \int_{a_1}^{\theta} \frac{\mathcal{F}^{\sigma}_{\rho,\vartheta}(w(\theta - \mathscr{S})^{\rho})}{(\theta - \mathscr{S})^{1-\vartheta}} \zeta(\mathscr{S}) d\mathscr{S}, \ \theta > a_1, \tag{1.52}
$$

$$
{}^{\sigma}_{\rho}\mathcal{J}_{\vartheta, b_1^-; w}\zeta(\theta) = \int_{\theta}^{b_1} \frac{\mathcal{F}^{\sigma}_{\rho,\vartheta}(w(\mathscr{S} - \theta)^{\rho})}{(\mathscr{S} - \theta)^{1-\vartheta}} \zeta(\mathscr{S}) d\mathscr{S}, \ \theta < b_1, \tag{1.53}
$$

where $\rho, \vartheta > 0$, coefficients $\sigma(j)$ generate a bounded sequence of positive real numbers and

$$
\mathcal{F}^{\sigma}_{\rho,\vartheta}(\theta) = \sum_{j=0}^{\infty} \frac{\sigma(j)}{\Gamma(\rho j + \vartheta)} \theta^j, \quad |\theta| < R, \text{ with } R > 0. \tag{1.54}
$$

Definition 1.26. ([190]) For $k > 0$, let $\Phi : [a_1, b_1] \to \mathbb{R}$ be an increasing and positive monotone function having a continuous derivative Φ' on (a_1, b_1). The left and right-sided generalized

k-fractional integrals of ζ with respect to the function Φ on $[a_1, b_1]$ are respectively defined as follows:

$$ {}^{\vartheta}_{\Phi}\mathcal{J}^{k,\rho}_{\sigma,a_1^+;w}\zeta(\theta) = \int_{a_1}^{\theta} \frac{\mathcal{F}^{\sigma,k}_{\rho,\vartheta}\left(w\left(\Phi(\theta)-\Phi(\mathscr{S})\right)^{\rho}\right)}{\left(\Phi(\theta)-\Phi(\mathscr{S})\right)^{1-\frac{\vartheta}{k}}}\Phi'(\mathscr{S})\zeta(\mathscr{S})d\mathscr{S},\ \theta > a_1, \tag{1.55} $$

$$ {}^{\vartheta}_{\Phi}\mathcal{J}^{k,\rho}_{\sigma,b_1^-;w}\zeta(\theta) = \int_{\theta}^{b_1} \frac{\mathcal{F}^{\sigma,k}_{\rho,\vartheta}\left(w\left(\Phi(\mathscr{S})-\Phi(\theta)\right)^{\rho}\right)}{\left(\Phi(\mathscr{S})-\Phi(\theta)\right)^{1-\frac{\vartheta}{k}}}\Phi'(\mathscr{S})\zeta(\mathscr{S})d\mathscr{S},\ \theta < b_1, \tag{1.56} $$

with the coefficients $\sigma(j)$ $(j \in \mathbb{N}\cup\{0\})$ form a bounded sequence of positive real numbers and

$$ \mathcal{F}^{\sigma,k}_{\rho,\vartheta}(\theta) := \sum_{j=0}^{\infty} \frac{\sigma(n)}{k\Gamma_k(\rho k j+\vartheta)}\theta^j, \quad (\rho,\vartheta > 0; |\theta| < R) \text{ with } R > 0. \tag{1.57} $$

1.5 Convex functions on the coordinates on a rectangle from the plane $\mathbb{R}^2$

In this section we outline the notion of coordinated convex functions and some generalized classes of coordinated convex functions.

Definition 1.27. ([40]) Let $\mathscr{D} = [a_1, b_1] \times [a_2, b_2]$ be a bi-dimensional interval in $\mathbb{R}^2$ such that $a_1 < b_1$ and $a_2 < b_2$. Then a function $\zeta : \mathscr{D} \to \mathbb{R}$ is called convex on the coordinates on $\mathscr{D}$, if the partial mappings:

$$ \zeta_{y_1} : [a_1, b_1] \to \mathbb{R},\ \zeta_{y_1}(u_1) = \zeta(u_1, y_1) \text{ and } \zeta_{x_1} : [a_2, b_2] \to \mathbb{R},\ \zeta_{x_1}(v_1) = \zeta(x_1, v_1), $$

are convex for all $x_1 \in [a_1, b_1]$ and $y_1 \in [a_2, b_2]$.

Formally, a function $\zeta : \mathscr{D} \to \mathbb{R}$ is called convex on the coordinates on $\mathscr{D}$, if the following inequality holds:

$$ \begin{aligned} &\zeta(\lambda x_1 + (1-\lambda)x_2, \mu y_1 + (1-\mu)y_2) \\ &\le \lambda\mu\zeta(x_1, y_1) + \lambda(1-\mu)\zeta(x_1, y_2) + (1-\lambda)\mu\zeta(x_2, y_1) \\ &\quad + (1-\lambda)(1-\mu)\zeta(x_2, y_2), \end{aligned} \tag{1.58} $$

for all $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2) \in \mathscr{D}$ and $\lambda, \mu \in [0, 1]$.

On the other side, a function $\zeta : \mathscr{D} \to \mathbb{R}$ is called convex on $\mathscr{D}$, if the following inequality holds:

$$\zeta(\lambda x_1 + (1-\lambda)x_2, \lambda y_1 + (1-\lambda)y_2) \le \lambda\zeta(x_1, y_1) + (1-\lambda)\zeta(x_2, y_2), \tag{1.59}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathscr{D}$ and $\lambda \in [0, 1]$. It has been proved that every coordinated convex function $\zeta : \mathscr{D} \to \mathbb{R}$ on the rectangle $\mathscr{D}$ is convex on $\mathscr{D}$ but the converse is not true in general, see [40].

The Definition 1.27 of coordinated convex functions is generalized as coordinated s-convex functions in both the sense as follows.

Definition 1.28. ([8,9]) Consider the bi-dimensional interval $\mathscr{D} = [a_1, b_1] \times [a_2, b_2]$ in $\in [0, \infty)^2$ with $a_1 < b_1$ and $a_2 < b_2$. The mapping $\zeta : \mathscr{D} \to \mathbb{R}$ is s-convex in the first sense (in the second sense) on $\mathscr{D}$ if

$$\zeta(\lambda x_1 + \mu x_2, \lambda y_1 + \mu y_2) \le \lambda^s \zeta(x_1, y_1) + \mu^s \zeta(x_2, y_2), \tag{1.60}$$

for all $(x_1, y_1), (x_2, y_2) \in \mathscr{D}$ and $\lambda, \mu \ge 0$ with $\lambda^s + \mu^s = 1 (\lambda + \mu = 1)$ and for some fixed $s \in (0, 1]$. We write $\zeta \in K_s^i (i = 1, 2)$ which means that ζ is s-convex in the first sense when $i = 1$, (in the second sense when $i = 2$).

A function $\zeta : \mathscr{D} \to \mathbb{R}$ is called s-convex in first sense (in the second sense) on the coordinates $\mathscr{D}$, if the partial mappings:

$$\zeta_{y_1} : [a_1, b_1] \to \mathbb{R},\ \zeta_{y_1}(u_1) = \zeta(u_1, y_1) \text{ and } \zeta_{x_1} : [a_2, b_2] \to \mathbb{R},\ \zeta_{x_1}(v_1) = \Psi(x_1, v_1),$$

are s-convex in first sense (in the second sense) for all $x_1 \in [a_1, b_1]$ and $y_1 \in [a_2, b_2]$.

The notion of coordinated convex functions is further extended as coordinated pq-convex functions.

Definition 1.29. ([134]) A function $\zeta : \mathscr{D} \to \mathbb{R}$ is called two dimensional pq-convex function on $\mathscr{D}$, if the inequality

$$\begin{aligned} &\zeta\left((\lambda x_1^p + (1-\lambda)x_2^p)^{\frac{1}{p}}, (\mu y_1^q + (1-\mu)y_2^q)^{\frac{1}{q}}\right) \\ &\leq \lambda\mu\zeta(x_1, y_1) + \lambda(1-\mu)\zeta(x_1, y_2) + (1-\lambda)\mu\zeta(x_2, y_1) + (1-\lambda)(1-\mu)\zeta(x_2, y_2), \end{aligned} \tag{1.61}$$

holds for all $(x_1, y_1), (x_1, y_2), (x_2, y_1), (x_2, y_2) \in \mathscr{D}$ and $\lambda, \mu \in [0, 1]$.

The following definition is about the notion of coordinated (p_1, h_1)-(p_2, h_2)-convex function.

Definition 1.30. ([197]) Let $h_1, h_2 : J \to \mathbb{R}$ be two nonnegative and non-zero functions. A mapping $\zeta : \mathscr{D} \to \mathbb{R}$ is said to be (p_1, h_1)-(p_2, h_2)-convex function on the coordinates on $\mathscr{D}$, if the partial mappings $\zeta_{y_1} : [a_1, b_1] \to \mathbb{R}, \zeta_{y_1}(u_1) = \zeta(u_1, y_1)$ and $\zeta_{x_1} : [a_2, b_2] \to \mathbb{R}, \zeta_{x_1}(v_1) = \zeta(x_1, v_1)$ are (p_1, h_1)-convex with respect to u_1 on $[a_1, b_1]$ and (p_2, h_2)-convex with respect to v_1 on $[a_2, b_2]$ respectively for all $y_1 \in [a_2, b_2]$ and $x_1 \in [a_1, b_1]$.

Consequent upon the definition, we say if ζ is coordinated (p_1, h_1)-(p_2, h_2)-convex function, then

$$\begin{aligned} &\zeta\left([\lambda x_1^{p_1} + (1-\lambda)x_2^{p_1}]^{\frac{1}{p_1}}, [\mu y_1^{p_2} + (1-\mu)y_2^{p_2}]^{\frac{1}{p_2}}\right) \\ &\leq h_1(\lambda)h_2(\mu)\zeta(x_1, y_1) + h_1(\lambda)h_2(1-\mu)\zeta(x_1, y_2) + h_1(1-\lambda)h_2(\mu)\zeta(x_2, y_1) \\ &\quad + h_1(1-\lambda)h_2(1-\mu)\zeta(x_2, y_2). \end{aligned} \tag{1.62}$$

Remark 1.31. If $p_1 = p_2 = 1$, then the function ζ defined in inequality (1.62) will be reduced to coordinated (h_1, h_2)-convex function.

Remark 1.32. If $h_1(\mu) = \mu^{s_1}$ and $h_2(\mu) = \mu^{s_2}$, then the function ζ defined in inequality (1.62) is coordinated (p_1, s_1)-(p_2, s_2)-convex function.

Remark 1.33. If $h_1(\mu) = \mu^{s_1}, h_2(\mu) = \mu^{s_2}$ and $p_1 = p_2 = 1$, then the function ζ defined in inequality (1.62) is called coordinated (s_1, s_2)-convex function.

1.6 Different analogues of Hermite–Hadamard and Fejér–Hermite–Hadamard inequalities via classical and fractional integrals

In the previous quite a long while, mathematical inequalities arose as a successful instrument for the advancement of numerous parts of math and other present day sciences. In twentieth century, mathematicians has perceived the capability of mathematical inequalities which has offered ascend to an assortment of new outcomes that lead to new territories of science. The particular word Inequalities by Hardy, littlewood and Pólya [71] showed up in 1934 and acquired its place as an essential reference for mathematicians. This book is a finished reference to the subject and is a valuable manual for this normally energizing field. This reference has been went with "Inequalities" by Beckenbach [17] and Analytic Inequalities by Mitrinovic [128] and "Means and their inequalities" by Bullen [26], which made extensive commitments to this field. These books end up being helpful references to investigate this point in complete profundity. The subject has applications outside of math, for example, " numerical analysis, game theory, numerical programming, control theory, variational methods, operation research, probability and statistics ". The subject has been perceived as one of the focal regions of numerical investigation all through the last century and it is a quickly developing discipline, with always expanding applications in numerous logical fields. As per American Mathematical Society, there are above $63,000$ references and uses of inequalities. This development brought about the presence of the inequalities as a free area of mathematical investigation. It is perceived that by and large some particular inequalities give a helpful and significant gadget in the improvement of various parts of science.

The historical backdrop of the word convexity is long. The roots of the hypothesis of convexity can be found in the key commitments of Hölder, Stolz and Hadamard ([69, 70, 163]). Toward the start of the last century Jensen [77] first grasped the prominence and started the symmetric examination of the convexity. In years from there on this examination brought about the presence of the convexity as an autonomous space of mathematical analysis. Lately, convexity has been a subject of broad examination and countless inequalities for convex functions have showed up.

Most common and significant inequalities are Hölder's inequality, Minkowski's inequality, power mean inequality, Jensen's inequality, Hermite–Hadamard inequality, Fejér–Hermite–Hadamard inequality, Hardy inequality, Ostrowski inequality and Gauss inequality etc.

Our interest is to study the Hermite–Hadamard and Fejér–Hermite–Hadamard inequalities for convex functions and their generalizations. The Hermite-Hadamard inequality for a convex function $\zeta : \mathcal{Q} \to \mathbb{R}$ on an interval $\mathcal{Q}$ is defined as:

$$\zeta\left(\frac{a_1 + b_2}{2}\right) \leq \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} \zeta(u)du \leq \frac{\zeta(a_1) + \zeta(b_1)}{2}, \tag{1.63}$$

for all $a_1, b_1 \in \mathcal{Q}$ with $a_1 < b_1$. The twofold inequality (1.63) was really found by Ch. Hermite [68] in 1881 in the diary Mathesis yet was no place referenced in the numerical writing and was not generally known as his outcome. In [20], Beckenbach, a main expert on the history and the theory of convexity, composed that this inequality was demonstrated by Hadamard [69] in 1893. Later on, in [119], Mitrinović discovered Hermite's note in Mathesis. This is why, the inequality (1.63) is currently normally alluded as the Hermite–Hadamard inequality. Since the revelation, Hermite–Hadamard inequality has turned into the most alluring inequality in present day math. Some of the special inequalities for means can be gotten from (1.63) for specific decisions of the function ζ. Additionally, (1.63) gives an important and adequate condition for a function ζ to be convex in (a, b).

In [51], Fejér gave a weighted partner of (1.63) as follows:

$$\zeta\left(\frac{a_1+b_1}{2}\right)\int\limits_{a_1}^{b_1}\phi(u)du \leq \frac{1}{b_1-a_1}\int\limits_{a_1}^{b_1}\zeta(u)\phi(u)du \leq \frac{\zeta(a_1)+\zeta(b_1)}{2}\int\limits_{a_1}^{b_1}\phi(u)du, \tag{1.64}$$

where $\phi : [a_1, b_1] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $(a_1 + b_1)/2$. After these results, many generalized versions of (1.63) and (1.64) are developed. We write down some important models as below.

In [181], Sarikaya et al. proved the first most important fractional version of Hermite–Hadamard.

Theorem 1.6.1. *Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a positive function with $0 \leq a_1 < b_1$ and $\zeta \in L_1[a_1, b_1]$. If ζ is a convex function on $[a_1, b_1]$, then the following inequalities holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right) \leq \frac{\Gamma(\vartheta+1)}{2(b_1-a_1)^{\vartheta}}\left[{}^{\vartheta}\mathcal{J}_{a_1^+}\zeta(b_1) + {}^{\vartheta}\mathcal{J}_{b_1^-}\zeta(a_1)\right] \leq \frac{\zeta(a_1)+\zeta(b_1)}{2}, \tag{1.65}$$

with $\vartheta > 0$.

In [76], Işcan proved a fractional counter part of Fejér–Hermite–Hadamard inequality.

Theorem 1.6.2. *Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a convex function with $0 \leq a_1 < b_1$ and $\zeta \in L_1[a_1, b_1]$. If $\phi : [a_1, b_1] \to \mathbb{R}$ is a nonnegative integrable and symmetric to $\frac{a_1+b_1}{2}$, then the following inequalities holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right)\left[{}^{\vartheta}\mathcal{J}_{a_1^+}\phi(b_1) + {}^{\vartheta}\mathcal{J}_{b_1^-}\phi(a_1)\right] \leq \left[{}^{\vartheta}\mathcal{J}_{a_1^+}(\zeta\phi)(b_1) + {}^{\vartheta}\mathcal{J}_{b_1^-}(\zeta\phi)(a_1)\right]$$

$$\leq \frac{\zeta(a_1)+\zeta(b_1)}{2}\left[{}^{\vartheta}\mathscr{I}_{a_1^+}\phi(b_1) + {}^{\vartheta}\mathscr{I}_{b_1^-}\phi(a_1)\right], \quad (1.66)$$

with $\vartheta > 0$.

In [81], Jalili and Samet proved the Hermite–Hadamard inequality for Φ-Riemann–Liouville fractional integrals.

Theorem 1.6.3. *Let* $\Phi : [a_1, b_1] \to \mathbb{R}$ *be an increasing and positive monotone function on* $[a_1, b_1]$, *having continuous derivative* $\Phi'(\mathscr{S})$ *on* (a_1, b_1) *and let* $\vartheta > 0$. *If* ζ *is a convex function on* $[a_1, b_1,]$ *then*

$$\zeta\left(\frac{a_1+b_1}{2}\right) \leq \frac{\Gamma(\vartheta+1)}{4(\Phi(b_1)-\Phi(a_1))^{\vartheta}}\left[{}^{\vartheta}_{\Phi}\mathscr{I}_{a_1^+}\widetilde{\zeta}(b_1) + {}^{\vartheta}_{\Phi}\mathscr{I}_{b_1^-}\widetilde{\zeta}(a_1)\right] \leq \frac{\zeta(a_1)+\zeta(b_1)}{2}, \quad (1.67)$$

where $\widetilde{\zeta}(\mathscr{S}) = \zeta(\mathscr{S}) + \zeta(a_1+b_1-\mathscr{S})$ *for* $\mathscr{S} \in [a_1, b_1]$.

In [18], Budak proved the weighted version of (1.67).

Theorem 1.6.4. *Let* $\vartheta > 0$. $\Phi : [a_1, b_1] \to \mathbb{R}$ *be a an increasing and positive monotone function on* $(a_1, b_1]$ *having a continuous derivative* $\Phi'(\mathscr{S})$ *on* (a_1, b_1) *and let* $\Psi : [a_1, b_1] \to \mathbb{R}$ *nonnegative integrable. If* ζ *is a convex function on* $[a_1, b_1]$, *then*

$$\zeta\left(\frac{a_1+b_1}{2}\right)\left[{}^{\vartheta}_{\Phi}\mathscr{I}_{a_1^+}\Psi(b_1) + {}^{\vartheta}_{\Phi}\mathscr{I}_{b_1^-}\Psi(a_1)\right] \leq \left[{}^{\vartheta}_{\Phi}\mathscr{I}_{a_1^+}(\zeta\Psi)(b_1) + {}^{\vartheta}_{\Phi}\mathscr{I}_{b_1^-}(\zeta\Psi)(a_1)\right]$$
$$\leq \frac{\zeta(a_1)+\zeta(b_1)}{2}\left[{}^{\vartheta}_{\Phi}\mathscr{I}_{a_1^+}\Psi(b_1) + {}^{\vartheta}_{\Phi}\mathscr{I}_{b_1^-}\Psi(a_1)\right], \quad (1.68)$$

with $\vartheta > 0$.

In [40], Dragomir proved the remarkable result for coordinated convex functions generalizing the one dimensional (1.63) as follows:

Theorem 1.6.5. *Let $\zeta : \mathcal{D} \subseteq \mathbb{R}^2 \to \mathbb{R}$ be convex on the coordinates on $\mathcal{D}$. Then*

$$\begin{aligned}
&\zeta\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right) \\
&\leq \frac{1}{2}\left[\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta\left(\lambda,\frac{a_2+b_2}{2}\right)d\lambda+\frac{1}{b_2-a_2}\int_{a_2}^{b_2}\zeta\left(\frac{a_1+b_1}{2},\mu\right)d\mu\right] \\
&\leq \frac{1}{(b_1-a_1)(b_2-a_2)}\int_{a_2}^{b_2}\int_{a_1}^{b_1}\zeta(\lambda,\mu)d\lambda d\mu \\
&\leq \frac{1}{4}\Bigg[\frac{1}{b_1-a_1}\left(\int_{a_1}^{b_1}\zeta(\lambda,a_2)d\lambda+\int_{a_1}^{b_1}\zeta(\lambda,b_2)d\lambda\right) \\
&\quad+\frac{1}{b_2-a_2}\left(\int_{a_2}^{b_2}\zeta(a_1,\mu)d\mu+\int_{a_2}^{b_2}\zeta(b_1,\mu)d\mu\right)\Bigg] \\
&\leq \frac{\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)}{4}.
\end{aligned} \tag{1.69}$$

In [164], Sarikaya established the following fractional versions of (1.69).

Theorem 1.6.6. *Let $\zeta : \mathcal{D} \subseteq \mathbb{R}^2 \to \mathbb{R}$ be a coordinated convex function on $\mathcal{D} := [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$ with $0 \leq a_1 < b_1$, $0 \leq a_2 < b_2$ and $\zeta \in L_1(\mathcal{D})$. Then*

$$\begin{aligned}
\zeta\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right) &\leq \frac{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4(b_1-a_1)^{\vartheta_1}(b_2-a_2)^{\vartheta_2}}\Big[{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,b_1^+}\zeta(b_1,b_2)+{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,b_2^-}\zeta(b_1,a_2) \\
&\quad+{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,b_1^+}\zeta(a_1,b_2)+{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,b_2^-}\zeta(a_1,a_2)\Big] \\
&\leq \frac{\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)}{4}.
\end{aligned} \tag{1.70}$$

Theorem 1.6.7. *Let $\zeta : \mathcal{D} \subset \mathbb{R}^2 \to \mathbb{R}$ be a coordinated convex function on $\mathcal{D} := [a_1, b_1] \times [a_2, b_2]$ in $\mathbb{R}^2$ with $0 \leq a_1 < b_1$, $0 \leq a_2 < b_2$ and $\zeta \in L_1(\mathcal{D})$. Then*

$$\begin{aligned}
&\zeta\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right) \\
&\leq \frac{\Gamma(\vartheta_1+1)}{4(b_1-a_1)^{\vartheta_1}}\left[{}^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta\left(b_1,\frac{a_2+b_2}{2}\right)+{}^{\vartheta_1}\mathcal{J}_{b_1^-}\zeta\left(a_1,\frac{a_2+b_2}{2}\right)\right] \\
&\quad+\frac{\Gamma(\vartheta_2+1)}{4(b_2-a_2)^{\vartheta_2}}\left[{}^{\vartheta_2}\mathcal{J}_{b_1^+}\zeta\left(\frac{a_1+b_1}{2},b_2\right)+{}^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta\left(\frac{a_1+b_1}{2},a_2\right)\right] \\
&\leq \frac{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4(b_1-a_1)^{\vartheta_1}(b_2-a_2)^{\vartheta_2}}\Big[{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,b_1^+}\zeta(b_1,b_2)+{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,b_2^-}\zeta(b_1,a_2)
\end{aligned}$$

$$+^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,b_1^+}\zeta(a_1,b_2)+^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,b_2^-}\zeta(a_1,a_2)\Big]$$
$$\leq\frac{\Gamma(\vartheta_1+1)}{4(b_1-a_1)^{\vartheta_1}}\left[^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta(b_1,a_2)+^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta(b_1,b_2)+^{\vartheta_1}\mathcal{J}_{b_1^-}\zeta(a_1,a_2)+^{\vartheta_1}\mathcal{J}_{b_1^-}\zeta(a_1,b_2)\right]$$
$$+\frac{\Gamma(\vartheta_2+1)}{4(b_2-a_2)^{\vartheta_2}}\left[^{\vartheta_2}\mathcal{J}_{b_1^+}\zeta(a_1,b_2)+^{\vartheta_2}\mathcal{J}_{b_1^+}\zeta(b_1,b_2)+^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta(a_1,a_2)+^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta(b_1,a_2)\right]$$
$$\leq\frac{\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)}{4}. \tag{1.71}$$

In [134], Noor et. al. developed the following result for coordinated pq-convex functions.

Theorem 1.6.8. *Let $\zeta:\mathcal{D}\subset\mathbb{R}^2\to\mathbb{R}$ be a p_1p_2-convex function on the coordinates on $\mathcal{D}$, then following inequalities hold:*

$$\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)$$
$$\leq\frac{p_1p_2}{(b_1^{p_1}-a_1^{p_1})(b2^{p_2}-b_1^{p_2})}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\lambda^{p_1-1}\mu^{p_2-1}\zeta(\lambda,\mu)\,d\mu d\lambda$$
$$\leq\frac{\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)}{4}. \tag{1.72}$$

In [197], Yang prove following two inequalities along with many other results.

Theorem 1.6.9. *Let $\zeta:\mathcal{D}\to\mathbb{R}$ be a (p_1,h_1)-(p_2,h_2)-convex function on the coordinates on $\mathcal{D}$. Then*

$$\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},\left[\frac{b_1^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)$$
$$\leq\frac{p_1p_2}{(b_1^{p_1}-a_1^{p_1})(b_2^{p_2}-b_1^{p_2})}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\lambda^{p_1-1}\mu^{p_2-1}\zeta(\lambda,\mu)d\mu d\lambda$$
$$\leq[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)]\int_0^1 h_1(\mathscr{S})d\mathscr{S}\int_0^1 h_2(\mathscr{S})d\mathscr{S}. \tag{1.73}$$

Theorem 1.6.10. *Let $\zeta:\mathcal{D}\to\mathbb{R}$ be a (p_1,h_1)-(p_2,h_2)-convex function on the coordinates on $\mathcal{D}$. Then one has the inequalities:*

$$
\begin{aligned}
&\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},\left[\frac{b_1^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)\\
&\leq \frac{p_1}{4h_1\left(\frac{1}{2}\right)\left(b_1^{p_1}-a_1^{p_1}\right)}\int_{a_1}^{b_1}\lambda^{p_1-1}\zeta\left(\lambda,\left[\frac{b_1^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)d\lambda\\
&+\frac{p_2}{4h_2\left(\frac{1}{2}\right)\left(b_2^{p_2}-b_1^{p_2}\right)}\int_{a_2}^{b_2}\mu^{p_2-1}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},\mu\right)d\mu\\
&\leq \frac{p_1p_2}{\left(b_1^{p_1}-a_1^{p_1}\right)\left(b_2^{p_2}-b_1^{p_2}\right)}\int_{a_1}^{b_1}\int_{a_2}^{b_2}\lambda^{p_1-1}\mu^{p_2-1}\zeta(\lambda,\mu)d\mu d\lambda\\
&\leq \frac{p_1}{2\left(b_1^{p_1}-a_1^{p_1}\right)}\left[\int_{a_1}^{b_1}x^{p_1-1}\zeta\left(\lambda,a_2\right)d\lambda+\int_{a_1}^{b_1}\lambda^{p_1-1}\zeta\left(\lambda,b_2\right)d\lambda\right]\int_0^1 h_2(t)dt\\
&+\frac{p_2}{2\left(b_2^{p_2}-b_1^{p_2}\right)}\left[\int_{a_2}^{b_2}\mu^{p_2-1}\zeta\left(a_1,\mu\right)d\mu+\int_{a_2}^{b_2}\mu^{p_2-1}\zeta\left(b_1,\mu\right)d\mu\right]\int_0^1 h_1(t)dt\\
&\leq \left[\zeta\left(a_1,a_2\right)+\zeta\left(a_1,b_2\right)+\zeta\left(b_1,a_2\right)+\zeta\left(b_1,b_2\right)\right]\int_0^1 h_1(\mathscr{S})d\mathscr{S}\int_0^1 h_2(\mathscr{S})d\mathscr{S}. \quad (1.74)
\end{aligned}
$$

The inequalities (1.63)–(1.74) are then further generalized by different authors. For instance, see ([9, 10, 29, 32, 40, 134, 164, 197]) and the references therein.

1.7 q-Derivatives and q-Integrals

The investigation of quantum calculus (or q-calculus) is the calculus without any limit. The theory has its origin in the 18th century. Popular mathematician Euler was liable for presenting q as a parameter in Newton's work of limitless arrangement. Jackson dealt with the traditional meaning of the differentiation of a function and demonstrated numerous outcomes in the area of q-calculus. Jackson got the credit for methodical advancement of q-calculus. The referenced branch is considered as an in-corporative subject among material science and math. For some current headway in this field, we allude to ([33, 36, 49, 64, 79, 80, 87, 97, 162]).

For a real function ζ, the q–derivative is portrayed by

$$\mathbf{D}_q\zeta(\omega) = \frac{\zeta(q\omega) - \zeta(\omega)}{q\omega - \omega}, \tag{1.75}$$

where $q \in (0,1)$. The Jackson integral of a real function ζ is defined by the following series expansion

$$\int_0^k \zeta(\omega) d_q\omega = (1-q)k \sum_{h=0}^{\infty} q^h \zeta(q^h k). \tag{1.76}$$

The authors in ([185, 192]), studied the notion of q–derivatives and q–integrals over the finite real interval $[a_1, b_1]$ and defined the q_{a_1}–derivative and q_{a_1}–integrals. For a continuous function $\zeta : [a_1, b_1] \to \mathbb{R}$ and $q \in (0,1)$, the q_{a_1}–derivative and q_{a_1}–integrals of ζ at $\omega \in [a_1, b_1]$ are respectively defined and denoted by

$$_{a_1}\mathbf{D}_q\zeta(\omega) = \frac{\zeta(\omega) - \zeta(q\omega + (1-q)a_1)}{(1-q)(\omega - a_1)}, \quad \omega \neq a_1, \tag{1.77}$$

and

$$\int_{a_1}^k \zeta(\omega)_{a_1} d_q\omega = (1-q)(k-a_1) \sum_{h=0}^{\infty} q^h \zeta(q^h k + (1-q^h)a_1), \quad k \in [a_1, b_1]. \tag{1.78}$$

If $a_1 = 0$, then

$$\int_0^k \zeta(\omega)_0 d_q\omega = (1-q)k \sum_{h=0}^{\infty} q^h \zeta(q^h k) = \int_0^k \zeta(\omega) d_q\omega. \tag{1.79}$$

which is the classical q-integral (1.76).

The authors in ([16, 25, 104, 133, 175]), have contributed to the ongoing research in q–calculus and have established some interesting results.

The authors in [25], discussed the notion of q–derivatives and q–integrals over the finite real interval $[a_1, b_1]$ and defined the q^{b_1}–derivative and q^{b_1}–integrals. For a continuous function

$\zeta : [a_1, b_1] \to \mathbb{R}$ and $q \in (0,1)$, the q^{b_1}–derivative and q^{b_1}–integrals of ζ at $\omega \in [a_1, b_1]$ are respectively defined and denoted by

$$ {}^{b_1}\mathbf{D}_q\zeta(\omega) = \frac{\zeta(\omega) - \zeta\left(q\omega + (1-q)b_1\right)}{(1-q)(\omega - b_1)}, \quad \omega \neq b_1, \tag{1.80}$$

and

$$\int_k^{b_1} \zeta(\omega)\ {}^{b_1}d_q\omega = (1-q)(b_1 - k)\sum_{h=0}^{\infty} q^h \zeta(q^h k + (1-q^h)b_1), \quad k \in [a_1, b_1]. \tag{1.81}$$

If $b_1 = 0$, then we have a relation with classical Jackson integral (1.76) as follows:

$$\int_k^{0} \zeta(\omega)\ {}^{0}d_q\omega = -(1-q)k\sum_{h=0}^{\infty} q^h \zeta(q^h k) = -\int_0^{k} \zeta(\omega) d_q\omega. \tag{1.82}$$

Remark 1.34. Here we point out some useful features of q–derivatives and q–integrals.

(i) From (1.75), (1.77) and (1.80), we found that

$$ {}_0d_q\omega = d_q\omega = {}^0d_q\omega. \tag{1.83}$$

(ii) The derivatives ${}^{b_1}\mathbf{D}_q\zeta(\omega)$ and ${}_{a_1}\mathbf{D}_q\zeta(\omega)$ are not same for general functions. Indeed, if $\zeta(\omega) = \omega^2$, then

$$ {}^{b_1}\mathbf{D}_q\zeta(\omega) = (1+q)\omega + (1-q)b_1$$

and

$$ {}_{a_1}\mathbf{D}_q\zeta(\omega) = (1+q)\omega + (1-q)a_1.$$

However,

$$ {}^{b_1}\mathbf{D}_q\zeta(\omega) = \zeta'(\omega) = {}_{a_1}\mathbf{D}_q\zeta(\omega)$$

provided that $q \to 1^-$.

(iii) The q–integrals $\int_{a_1}^{b_1} \zeta(\omega)\ {}^{b_1}d_q\omega$ and $\int_{a_1}^{b_1} \zeta(\omega)\ {}_{a_1}d_q\omega$ are different for general functions. For instance,

$$\int_{a_1}^{b_1} \omega\ {}^{b_1}d_q\omega = \frac{b_1 - a_1}{1+q}[a_1 + b_1 q]$$

and

$$\int_{a_1}^{b_1} \omega\ {}_{a_1}d_q\omega = \frac{b_1 - a_1}{1+q}[a_1 q + b_1].$$

Furthermore,

$$\int_{a_1}^{b_1} \omega\ {}^{b_1}d_q\omega = \frac{{b_1}^2 - {a_1}^2}{2} = \int_{a_1}^{b_1} \omega\ {}_{a_1}d_q\omega$$

subject to the condition that $q \to 1^-$.

1.8 Fractal number line system

Let $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$ and $\mathbb{R}^+$ be the the sets of natural, integer, rational, real and positive real numbers respectively. Also

$$\mathbb{J} := \mathbb{R} \setminus \mathbb{Q} \quad \text{and} \quad \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Yang [200], recently introduced his concept of fractal sets. In order to reload the definition of the local fractional derivative and local fractional integral, we follow ([38,189,200,203]). In this investigation we are also motivated by ([1]– [3]). We in this discussion use $0 < \gamma \leq 1$, to define the α-type set given in [200] of elementary sets:

1. The α-type set of integers is defined for $0 < \gamma \leq 1$, by

$$\mathbb{Z}^\gamma := \{0^\gamma\} \cup \{\pm n^\gamma : \ n \in \mathbb{N}\};$$

2. The α-type set of rational numbers is defined for $0 < \gamma \leq 1$, by

$$\mathbb{Q}^\gamma := \left\{q^\gamma : \ q \in \mathbb{Q}\right\} = \left\{q^\gamma = \left(\frac{r}{s}\right)^\gamma : \ r \in \mathbb{Z}, \ s \in \mathbb{N}\right\};$$

3. The α-type set of irrational numbers is defined for $0 < \gamma \leq 1$, by

$$\mathbb{I}^\gamma := \left\{i^\gamma : \ i \in \mathbb{I}\right\} = \left\{i^\gamma \neq \left(\frac{r}{s}\right)^\gamma : \ r \in \mathbb{Z}, \ s \in \mathbb{N}\right\};$$

4. The α-type set of real line numbers is defined for $0 < \gamma \leq 1$, by $\mathbb{R}^\gamma := \mathbb{Q}^\gamma \cup \mathbb{I}^\gamma$.

Throughout this study, whenever the α-type set are talked about, we use γ for the generation of $\mathbb{R}^\gamma$ of real line numbers with the condition $0 < \gamma \leq 1$. The two binary operations the addition $+$ and the multiplication $\cdot$ (which is conventionally omitted) on the set $\mathbb{R}^\gamma$ of real line numbers are defined as follows.

For $\mathscr{S}_1^\gamma, \mathscr{S}_2^\gamma \in \mathbb{R}^\gamma$,

$$\mathscr{S}_1^\gamma + \mathscr{S}_2^\gamma := (\mathscr{S}_1 + \mathscr{S}_2)^\gamma \quad \text{and} \quad \mathscr{S}_1^\gamma \cdot \mathscr{S}_2^\gamma = \mathscr{S}_1^\gamma \mathscr{S}_2^\gamma := (\mathscr{S}_1 \mathscr{S}_2)^\gamma.$$

Under these operations, we found that

- $(\mathbb{R}^\gamma, +)$ is a commutative group. For $\mathscr{S}_1^\gamma, \mathscr{S}_2^\gamma, \mathscr{S}_3^\gamma \in \mathbb{R}^\gamma$ the following holds:

1. $\mathscr{S}_1^\gamma + \mathscr{S}_2^\gamma \in \mathbb{R}^\gamma$;

2. $\mathscr{S}_1^\gamma + \mathscr{S}_2^\gamma = \mathscr{S}_2^\gamma + \mathscr{S}_1^\gamma$;

3. $\mathscr{S}_1^\gamma + (\mathscr{S}_2^\gamma + \mathscr{S}_3^\gamma) = (\mathscr{S}_1^\gamma + \mathscr{S}_2^\gamma) + \mathscr{S}_3^\gamma$;

4. 0^γ is the identity for $(\mathbb{R}^\gamma, +)$. For any $\mathscr{S}_1^\gamma \in \mathbb{R}^\gamma, \mathscr{S}_1^\gamma + 0^\gamma = 0^\gamma + \mathscr{S}_1^\gamma = \mathscr{S}_1^\gamma$;

5. For each $\mathscr{S}_1^\gamma \in \mathbb{R}^\gamma, (-\mathscr{S}_1)^\gamma$ is the inverse element of $\mathscr{S}_1^\gamma$ for $(\mathbb{R}^\gamma, +)$, so we have

$$\mathscr{S}_1^\gamma + (-\mathscr{S}_1)^\gamma = (\mathscr{S}_1 + (-\mathscr{S}_1))^\gamma = 0^\gamma.$$

- $(\mathbb{R}^\gamma \setminus \{0^\gamma\}, \cdot)$ is a commutative group. For $\mathscr{S}_1^\gamma, \mathscr{S}_2^\gamma, \mathscr{S}_3^\gamma \in \mathbb{R}^\gamma$ the following holds:

1. $\mathscr{S}_1^\gamma \cdot \mathscr{S}_2^\gamma \in \mathbb{R}^\gamma$;

2. $\mathscr{S}_1^\gamma \cdot \mathscr{S}_2^\gamma = \mathscr{S}_2^\gamma \cdot \mathscr{S}_1^\gamma$;

3. $\mathscr{S}_1^\gamma \cdot (\mathscr{S}_2^\gamma \cdot \mathscr{S}_3^\gamma) = (\mathscr{S}_1^\gamma \cdot \mathscr{S}_2^\gamma) \cdot \mathscr{S}_3^\gamma$;

4. Since $\mathscr{S}_1^\gamma \cdot 1^\gamma = 1^\gamma \cdot \mathscr{S}_1^\gamma = \mathscr{S}_1^\gamma \quad \forall \mathscr{S}_1^\gamma \in \mathbb{R}^\gamma$, so 1^γ is the identity for $(\mathbb{R}^\gamma, \cdot)$.

5. As for each $\mathscr{S}_1^\gamma \in \mathbb{R}^\gamma \setminus \{0^\gamma\}, \left(\frac{1}{\mathscr{S}_1}\right)^\gamma \in \mathbb{R}^\gamma \setminus \{0^\gamma\}$ such that

$$\mathscr{S}_1^\gamma \cdot \left(\frac{1}{\mathscr{S}_1}\right)^\gamma = \left(\mathscr{S}_1 \cdot \left(\frac{1}{\mathscr{S}_1}\right)\right)^\gamma = 1^\gamma,$$

so inverses exists.

- Distributive law trivially holds: $\mathscr{S}_1^\gamma \cdot (\mathscr{S}_2^\gamma + \mathscr{S}_3^\gamma) = \mathscr{S}_1^\gamma \cdot \mathscr{S}_2^\gamma + \mathscr{S}_1^\gamma \cdot \mathscr{S}_3^\gamma$.

Following are some additional properties for $(\mathbb{R}^\gamma, +, \cdot)$.

Proposition 3. *([38]) Following statements holds true for* $(\mathbb{R}^\gamma, +, \cdot)$*:*

(a) *Like the fields of real numbers,* $(\mathbb{R}^\gamma, +, \cdot)$ *is a field;*

(b) $0^\gamma \neq 1^\gamma$*;*

(c) *The elements* $(-\mathscr{S}_1)^\gamma$ *and* $\left(\frac{1}{\mathscr{S}_1}\right)^\gamma$ *are respectively additive and multiplicatives inverses for the element* $\mathscr{S}_1^\gamma$ *and are unique;*

(d) *For each* $\mathscr{S}_1^\gamma \in \mathbb{R}^\gamma$, *its additive inverse* $(-\mathscr{S}_1)^\gamma$ *may be written as* $-\mathscr{S}_1^\gamma$. *For each* $\mathscr{S}_2^\gamma \in \mathbb{R}^\gamma \setminus \{0^\gamma\}$, *its multiplicative inverse* $\left(\frac{1}{\mathscr{S}_2}\right)^\gamma$ *may be written as* $\frac{1^\gamma}{\mathscr{S}_2^\gamma}$ *but not as* $\frac{1}{\mathscr{S}_2^\gamma}$;

(e) $(\mathbb{R}^\gamma, +, \cdot, <)$ *is an ordered field like* $(\mathbb{R}, +, \cdot, <)$, *if the order* $<$ *on* $(\mathbb{R}^\gamma, +, \cdot)$ *is defined as follows:* $\mathscr{S}_1^\gamma < \mathscr{S}_2^\gamma$ *in* $\mathbb{R}^\gamma$ *if and only if* $\mathscr{S}_1 < \mathscr{S}_2$ *in* $\mathbb{R}$.

Now we give some preliminaries for local fractional calculus on $\mathbb{R}^\gamma$.

Definition 1.35. A non-differentiable function $\zeta : \mathbb{R} \longrightarrow \mathbb{R}^\gamma$, $\mathscr{S} \longrightarrow \zeta(\mathscr{S})$, is said to be local fractional continuous at $\mathscr{S}_0$ if for any $\varepsilon \in \mathbb{R}^+$, there exists $\delta \in \mathbb{R}^+$ such that

$$\left|\zeta(\mathscr{S}) - \zeta(\mathscr{S}_0)\right| < \varepsilon^\gamma$$

holds for $|\mathscr{S} - \mathscr{S}_0| < \delta$. If a function ζ is local continuous on the interval (a_1, b_1), we denote $\zeta \in C_\gamma(a_1, b_1)$.

Among several attempts to have defined local fractional derivative and local fractional integral, (see [199], Section 2.1), we choose to recall the following definitions of local fractional calculus, see ([199, 200]).

Definition 1.36. The local fractional derivative of $\zeta(\mathscr{S})$ of order γ at $\mathscr{S} = \mathscr{S}_0$ is defined by

$$\zeta^{(\gamma)}(\mathscr{S}_0) = {}_{\mathscr{S}_0}D^\gamma_{\mathscr{S}}\zeta(\mathscr{S}) = \left.\frac{d^\gamma \zeta(\mathscr{S})}{d\mathscr{S}^\gamma}\right|_{\mathscr{S}=\mathscr{S}_0} = \lim_{\mathscr{S} \longrightarrow \mathscr{S}_0} \frac{\Delta^\gamma(\zeta(\mathscr{S}) - \zeta(\mathscr{S}_0))}{(\mathscr{S} - \mathscr{S}_0)^\gamma},$$

where $\Delta^\gamma(\zeta(\mathscr{S}) - \zeta(\mathscr{S}_0)) \cong \Gamma(1+\gamma)(\zeta(\mathscr{S}) - \zeta(\mathscr{S}_0))$ and Γ is the gamma function, (see [169], Section 1.1).

Let $\zeta^{(\gamma)}(\mathscr{S}) = D^\gamma_{\mathscr{S}}\zeta(\mathscr{S})$. If there exists $\zeta^{(l+1)\gamma}(\mathscr{S}) = \overbrace{D^\gamma_{\mathscr{S}} \dots D^\gamma_{\mathscr{S}}}^{l+1 \, times}\zeta(\mathscr{S})$ for any $\mathscr{S} \in \mathcal{Q} \subseteq \mathbb{R}$ then we denote $\zeta \in D_{(l+1)\gamma}(\mathcal{Q})$, where $l \in \mathbb{N}_0$.

Remark 1.37. In this expression, γ is precisely the Hölder exponent of function defined Cantor's set. That is to say, $\left[d(\mathscr{S}-\mathscr{S}_0)\right]^\gamma$ which is a fractal span, is a fractal geometrical meaning (see [200] and references therein).

Definition 1.38. Let $\zeta \in C_\gamma[a_1, b_1]$. Assume that $P = \{a_1 = \mathscr{S}_0, \dots, \mathscr{S}_M\}$, $(M \in \mathbb{N})$ be a partition of the interval $[a_1, b_1]$ which satisfies $a_1 = \mathscr{S}_0 < \mathscr{S}_1 < \cdots < \mathscr{S}_{M-1} < \mathscr{S}_M = b_1$. For this partition P, let $\Delta\mathscr{S} := \max_{0\leq j\leq M-1} \Delta\mathscr{S}_j$, where $\Delta\mathscr{S}_j := \mathscr{S}_{j+1} - \mathscr{S}_j$ and $j = 0, \dots, M-1$. Then the local fractional integral ${}_{a_1}\mathcal{I}_{b_1}^{(\gamma)}\zeta$ of ζ of order γ is:

$$ {}_{a_1}\mathcal{I}_{b_1}^{(\gamma)}\zeta(\mathscr{S}) = \frac{1}{\Gamma(1+\gamma)}\int_{a_1}^{b_1} \zeta(\mathscr{S})(d\mathscr{S})^\gamma := \frac{1}{\Gamma(1+\gamma)} \lim_{\Delta\mathscr{S}\longrightarrow 0} \sum_{j=0}^{N-1} \zeta(\mathscr{S}_j)(\Delta\mathscr{S}_j)^\gamma, $$

such that the limit exists (in fact, this limit exists if $\zeta \in C_\gamma[a_1, b_1]$).

Observe that ${}_{a_1}\mathcal{I}_{b_1}^{(\gamma)}\zeta = 0$ if $a_1 = b_1$ and ${}_{a_1}\mathcal{I}_{b_1}^{(\gamma)}\zeta = -{}_{b_1}\mathcal{I}_{a_1}^{(\gamma)}\zeta$ if $a_1 < b_1$. If for $y \in [a_1, b_1]$ and $\varrho : [a_1, b_1] \longrightarrow \mathbb{R}^\gamma$, ${}_{a_1}\mathcal{I}_{y}^{(\gamma)}\varrho$ exists, then we write $\varrho \in \mathcal{I}_{y}^{(\gamma)}[a_1, b_1]$.

Lemma 1.8.1. *([200]) The following identities are satisfied:*

1. *(Local fractional derivative of $v^{k\gamma}$).*

$$ \frac{d^\gamma \mathscr{S}^{l\gamma}}{d\mathscr{S}^\gamma} = \frac{\Gamma(1+l\gamma)}{\Gamma(1+(l-1)\gamma)}\mathscr{S}^{(l-1)\gamma}. $$

2. *(Local fractional integration is anti-differentiation). Let $\zeta(v) = \varrho^{(\gamma)}(v) \in C_\gamma[a_1, b_1]$. Then we have*

$$ {}_{a_1}\mathcal{I}_{b_1}^{(\gamma)}\zeta(v) = \varrho(b_1) - \varrho(a_1). $$

3. *(Local fractional integration by parts). Suppose that $\lambda(v), \mu(v) \in D_\gamma[a_1, b_1]$ and $\lambda^{(\gamma)}(v)$, $\mu^{(\gamma)}(v) \in C_\gamma[a_1, b_1]$. Then we have*

$$ {}_{a_1}\mathcal{I}_{b_1}^{(\gamma)}\lambda(v)\mu^{(\gamma)}(v) = \lambda(v)\mu(v)\Big|_{a_1}^{b_1} - {}_{a_1}\mathcal{I}_{b_1}^{(\gamma)}\lambda^{(\gamma)}(v)\mu(v). $$

4. (Local fractional definite integrals of $\mathscr{S}^{k\gamma}$).

$$\frac{1}{\Gamma(1+\gamma)}\int_{a_1}^{b_1}\mathscr{S}^{l\gamma}(d\mathscr{S})^{\gamma}=\frac{\Gamma(1+l\gamma)}{\Gamma(1+(l+1)\gamma)}\left(b_1^{(l+1)\gamma}-a_1^{(l+1)\gamma}\right),\quad l\in\mathbb{R}.$$

For a detailed study about local fractional calculus, one should follow, ([198]– [202]).

1.9 Thesis outline

Aim of the thesis is to extend the inequalities (1.63),(1.64) and (1.69) via various generalized convex functions that have already been presented in the literature and also in terms of new fractional and conformable integrals namely Riemann-Liouville fractional integrals, k-Riemann Liouville fractional integrals, (k,s)- fractional integrals, generalized conformable fractional integrals, generalized k-fractional integral, fractional integrals with exponential kernel, fractional integral with extended generalized mittag-leffler kernel and Katugampola fractional integrals etc..

Plan of the present thesis is as follows:

Chapter 2 deals with the Hermite–Hadamard and trapezoid type inequalities for modified (h,d)-convex functions and its application for trapezoidal rule.

Chapter 3 discuss a general formulation for Hermite–Hadamard and Fejér–Hermite–Hadamard inequality. It also discuss various special cases for fractional and conformable integrals.

Chapter 4 present a generalized error estimation model for existing and new Hermite–Hadamard inequalities via conformable and fractional integrals.

Chapter 5 deals with the error bounds of Hermite–Hadamard type inequalities at mid-point and their examples in terms of moments of the random variable.

Chapter 6 annunciate two dimensional forms of Hermite–Hadamard inequalities via Katugampola fractional integral by keeping (p_1, h_1)-(p_2, h_2)-convex functions on the coordinated on the rectangle in the plane $\mathbb{R}^2$. It also contain interesting special cases for including classes of convex functions. Some comparison with the existing literature is also given.

Chapter 7 gives Hermite–Hadamard type inequalities via q-integrals utilizing m-convex functions.

Chapter 8 deals an ostrowski type inequality for generalized strongly m-convex functions. Application for local means and error formula are also discussed.

Chapter 9, summed up the current findings.

Chapter 2

Approximately h-convex functions and Hermite–Hadamard inequalities

Researchers of the last few decades established a lot of new inequalities of Hermite–Hadamard type with the help of different kernels of Peano type kernel and introducing different types of convex functions, see ([4, 11, 12, 15, 24, 35, 39, 43, 45, 73–75, 141, 147, 174, 184, 196]) and the references therein. This section is given to a recently characterized sort of convex functions called (h, d)-convex functions.

We initially change the idea by presenting the thought of modified (h, d)-convex functions. At that point by applying improved type of Hölder's basic inequality called Hölder–Isçan's inequality a few inequalities of Hermite–Hadamard type for modified (h, d)-convex functions have been demonstrated. Different exceptional cases including classes for example, h-convex, s-convex functions of Breckner and Godunova–Levin–Dragomir and solid forms of the previously mentioned kinds of convexity have been concluded.

The consequences of this examination gives either refinements of the trapezoidal kind inequalities given in past writing or contains them as unique cases. We show uses of our outcomes to the trapezoidal equation.

The consequences of this section has been acknowledged for publication. We partition the chapter into three sections.

Section 2.1 reload some prior writing and present the idea of modified (h,d)-convex functions. Section 2.2 manages the Hermite–Hadamard type inequalities by means of modified (h,d)-convexity alongside its exceptional cases. Section 2.3, annunciate the applications to the trapezoid formula for s-convexity.

2.1 Historical Background

The stability theory for functional inequalities has its origin in the paper of Hyers and Ulam [73], in which the notion of δ-convex functions is introduced. A function $\zeta : \Lambda \to \mathbb{R}$ is known as δ-convex function, if

$$\zeta(\wp x_1 + (1-\wp)y_1) \leq \wp\zeta(x_1) + (1-\wp)\zeta(y_1) + \delta \tag{2.1}$$

for all $x_1, y_1 \in \Lambda, \wp \in [0,1]$, where Λ is a convex subset of a real linear space X_1 and δ is a nonnegative real number. Roughly speaking, the Hyers–Ulem theorem says that a function ζ is indeed a perturbation of a convex function by a bounded function if and only if there exists a non negative real number δ which satisfies (2.1). The error term is now refer to a bounded perturbation. Páles in [147], extended the notion to (ε,δ)-convex functions while investigating the stability constants as follows:

A real valued function ζ defined on a real interval $\mathfrak{Q}$ is called (ε,δ)-convex function if it satisfies the following inequality:

$$\zeta(\wp x_1 + (1-\wp)y_1) \leq \wp\zeta(x_1) + (1-\wp)\zeta(y_1) + \varepsilon\wp(1-\wp)\left\|x_1 - y_1\right\| + \delta, \tag{2.2}$$

$\forall\, x_1, y_1 \in I$ and $\wp \in [0,1]$. In (2.2), there are two error terms. Páles in [147], shown that the error terms $\varepsilon\wp(1-\wp)\,\|x_1 - y_1\|$ and δ, respectively, corresponds to perturbation of a convex function by a Lipschitz function and by a bounded function.

Burai and Hazy [24] considered a function $d : X_1 \times X_1 \to \mathbb{R}$ a function on normed space X_1 and Λ a non-empty open, convex subset of X_1. They generalized the notion of convex function by introducing (h,d)-convex function as follows:

Definition 2.1. A function $\zeta : \Lambda \to \mathbb{R}$, is called (h,d)-convex function, if

$$\zeta(\wp x_1 + (1-\wp)y_1) \leq h(\wp)\zeta(x_1) + h(1-\wp)\zeta(y_1) + d(x_1, y_1), \quad \forall x_1, y_1 \in \Lambda,\ \wp \in (0,1). \tag{2.3}$$

The class of (h,d)-convex functions includes the class of classical convex functions, δ-convex functions, (ε,δ)-convex functions, h-convex functions, s-convex functions of Breckner type, s-convex functions of Godunova-Levin-Dragomir type etc., and classes of strongly convex versions of aforementioned convex functions.

It is worth mentioning that the convexity of the function ζ depends on the value of the function $d(.\,,.)$ and the function $d(.\,,.)$ entirely depends on the convex combination. Therefore it will be convenient to modify this definition. We know propose a modification to (h,d)-convex function.

Definition 2.2. A function $\zeta : \Lambda \to \mathbb{R}$, is called modified (h,d)-convex function, if

$$\zeta(\wp x_1 + (1-\wp)y_1) \leq h(\wp)\zeta(x_1) + h(1-\wp)\zeta(y_1) + \wp(1-\wp)d(x_1, y_1), \quad \forall x_1, y_1 \in \Lambda,\ \wp \in (0,1). \tag{2.4}$$

Now we discuss some special cases, by restricting ourselves to the only well studied notions of convex functions by assigning different values to the function d.

I. If $d(x_1, y_1) = \frac{\delta}{\wp(1-\wp)}$, then we have following extension of well known δ-convex functions given by inequality (2.1).

Definition 2.3. A function $\zeta : \Lambda \to \mathbb{R}$ is called δ-h-convex function, if ζ satisfies

$$\zeta(\wp x_1 + (1-\wp)y_1) \leq h(\wp)\zeta(x_1) + h(1-\wp)\zeta(y_1) + \delta, \quad \forall\, x_1, y_1 \in \Lambda,\ \wp \in (0,1). \tag{2.5}$$

Moreover we have class of δ-convex functions, δ-Godunova-Levin functions, δ-P-functions, δ-tgs-functions, δ-MT-convex functions, Breckner δ-s-convex functions and δ-s-Godunova-Levin functions by using $h(\wp) = \wp, \wp^{-1}, 1, \wp(1-\wp), \frac{\sqrt{\wp}}{2\sqrt{1-\wp}}, \wp^s$ and $h(\wp) = \wp^{-s}$ respectively.

II. If $d(x_1, y_1) = \varepsilon \left\| x_1 - y_1 \right\| + \frac{\delta}{\wp(1-\wp)}$, then we have following new type of convex functions.

Definition 2.4. A function $\zeta : \Lambda \to \mathbb{R}$ is called (ε, δ)-h-convex function if

$$\zeta(\wp x_1 + (1-\wp)y_1) \leq h(\wp)\zeta(x_1) + h(1-\wp)\zeta(y_1) + \varepsilon\wp(1-\wp)\left\| x_1 - y_1 \right\| + \delta, \tag{2.6}$$

$\forall\, x_1, y_1 \in \Lambda$ and $\wp \in (0,1)$.

Moreover we havee class of (ε, δ)-convex functions, (ε, δ)-Godunova-Levin functions, (ε, δ)-P-functions, (ε, δ)-tgs-functions, (ε, δ)-MT-convex functions, Breckner (ε, δ)-s-convex functions and (ε, δ)-s-Godunova-Levin functions by using $h(\wp) = \wp, \wp^{-1}, 1, \wp(1-\wp), \frac{\sqrt{\wp}}{2\sqrt{1-\wp}}, \wp^s$ and $h(\wp) = \wp^{-s}$ respectively.

III. If $d(x_1, y_1) = -\epsilon \left\| x_1 - y_1 \right\|^2$, then we have following new type of convex functions.

Definition 2.5. A function $\zeta : \Lambda \to \mathbb{R}$ is called strongly h-convex function, if

$$\zeta(\wp x_1 + (1-\wp)y_1) \leq h(\wp)\zeta(x_1) + h(1-\wp)\zeta(y_1) - \varepsilon\wp(1-\wp)\left\| x_1 - y_1 \right\|^2, \tag{2.7}$$

$\forall\, x_1, y_1 \in \Lambda$ and $\wp \in (0,1)$.

Moreover, classes of strongly convex functions, strongly Godunova-Levin functions, strongly P-functions, strongly tgs-functions, strongly MT-convex functions, Breckner strongly s-convex functions and strongly s-Godunova-Levin functions are obtained by using $h(\wp) = \wp, \wp^{-1}, 1, \wp(1-\wp), \frac{\sqrt{\wp}}{2\sqrt{1-\wp}}, \wp^s$ and $h(\wp) = \wp^{-s}$ respectively.

Now we give some preliminaries which will be helpful for further study.

Sarikaya et al. [180] prove the following Hadamard inequality for h-convex function.

Theorem 2.1.1. *Let $\zeta : \mathcal{Q} \to \mathbb{R}$ be an h-convex function, $a_1, b_1 \in \mathcal{Q}$ with $a_1 < b_1$ and $\zeta \in L_1[a_1, b_1]$. Then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\zeta\left(\frac{a_1+b_1}{2}\right) \leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S} \leq [\zeta(a_1)+\zeta(b_1)]\int_0^1 h(r)dr. \tag{2.8}$$

In [43], following identity for differentiable mappings is proved.

Lemma 2.1.1. *Let $\zeta : \mathcal{Q} \subset \mathbb{R} \to \mathbb{R}$ be a differentiable function on $\mathcal{Q}^o$ where $a_1, b_1 \in \mathcal{Q}$ with $a_1 < b_1$. If $\zeta' \in L_1[a_1, b_1]$, then the following equality holds:*

$$\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S} = \left(\frac{b_1-a_1}{2}\right)\int_0^1(1-2\wp)\zeta'(\wp a_1+(1-\wp)b_1)d\wp. \tag{2.9}$$

2.2 Hermite–Hadamard inequality for modified (h, d)-convex functions and error estimates

We have the following results.

Theorem 2.2.1. *Let ζ be an integrable function defined on $[a_1, b_1]$. If ζ is a modified (h, d)-convex function on $[a_1, b_1]$, then the following inequality holds:*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\left[\zeta\left(\frac{a_1+b_1}{2}\right)-\frac{1}{4}\int\limits_0^1 d\left(\mathscr{S}a_1+(1-\mathscr{S})b_1,(1-\mathscr{S})a_1+\mathscr{S}b_1\right)d\mathscr{S}\right]$$
$$\leq \int\limits_0^1 \zeta(\mathscr{S}a_1+(1-\mathscr{S})b_1)d\mathscr{S}$$
$$\leq [\zeta(a_1)+\zeta(b_1)]\int\limits_0^1 h(\mathscr{S})d\mathscr{S}+\frac{1}{6}d(a_1,b_1). \tag{2.10}$$

Proof. By the notion of modified (h,d)-convex functions, we have

$$\zeta\left(\frac{u_1+u_2}{2}\right)\leq h\left(\frac{1}{2}\right)[\zeta(u_1)+\zeta(u_2)]+\frac{1}{4}d(u_1,u_2) \tag{2.11}$$

for all $u_1,u_2\in[a_1,b_1]$. Now, by taking $u_1=\mathscr{S}a_1+(1-\mathscr{S})b_1$ and $u_2=\mathscr{S}b_1+(1-\mathscr{S})a_1$, $\mathscr{S}\in[0,1]$, from (2.11), we get

$$\frac{1}{h\left(\frac{1}{2}\right)}\left[\zeta\left(\frac{a_1+b_1}{2}\right)-\frac{1}{4}d\left(\mathscr{S}a_1+(1-\mathscr{S})b_1,(1-\mathscr{S})a_1+\mathscr{S}b_1\right)\right]$$
$$\leq \zeta(\mathscr{S}a_1+(1-\mathscr{S})b_1)+\zeta((1-\mathscr{S})a_1+\mathscr{S}b_1). \tag{2.12}$$

Now integrating (2.12) over $(0,1)$ with respect to $\mathscr{S}$, we obtain

$$\frac{1}{2h\left(\frac{1}{2}\right)}\left[\zeta\left(\frac{a_1+b_1}{2}\right)-\frac{1}{4}\int\limits_0^1 d\left(\mathscr{S}a_1+(1-\mathscr{S})b_1,(1-\mathscr{S})a_1+\mathscr{S}b_1\right)d\mathscr{S}\right]$$
$$\leq \int\limits_0^1 \zeta(\mathscr{S}a_1+(1-\mathscr{S})b_1)d\mathscr{S}. \tag{2.13}$$

On the other hand, since ζ is a modified (h,d)-convex function, so

$$\int\limits_0^1 \zeta(\mathscr{S}a_1+(1-\mathscr{S})b_1)d\mathscr{S} \leq \zeta(a_1)\int\limits_0^1 h(\mathscr{S})d\mathscr{S}+\zeta(b_1)\int\limits_0^1 h(1-\mathscr{S})d\mathscr{S}$$
$$+d(a_1,b_1)\int\limits_0^1 \mathscr{S}(1-\mathscr{S})d\mathscr{S}. \tag{2.14}$$

Required double inequality (2.10) is obtained by combining (2.13) and (2.14). □

Remark 2.6. If $d(x_1, y_1) = 0$, then we obtain [180, Theorem 6].

Corollary 2.7. *If ζ is δ-h-convex function on $[a_1, b_1]$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\left[\zeta\left(\frac{a_1+b_1}{2}\right)-\delta\right] \leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S} \leq [\zeta(a_1)+\zeta(b_1)]\int_{0}^{1}h(\mathscr{S})d\mathscr{S}+\delta.$$

Corollary 2.8. *If ζ is (ε, δ)-h-convex function on $[a_1, b_1]$, then*

$$\begin{aligned}\frac{1}{2h\left(\frac{1}{2}\right)}\left[\zeta\left(\frac{a_1+b_1}{2}\right)-\delta\right] &\leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S} \\ &\leq [\zeta(a_1)+\zeta(b_1)]\int_{0}^{1}h(\mathscr{S})d\mathscr{S}+\frac{\varepsilon}{6}(b_1-a_1)+\delta.\end{aligned}$$

Corollary 2.9. *If ζ is (ε, δ)-s-convex function of Breckner type on $[a_1, b_1]$, then*

$$\begin{aligned}2^{s-1}\left[\zeta\left(\frac{a_1+b_1}{2}\right)-\delta\right] &\leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S} \\ &\leq \frac{\zeta(a_1)+\zeta(b_1)}{s+1}+\frac{\varepsilon}{6}(b_1-a_1)+\delta.\end{aligned}$$

Corollary 2.10. *If ζ is (ε, δ)-s-convex function of Godunova-Levin-Dragomir type on $[a_1, b_1]$, then*

$$\begin{aligned}2^{-(s+1)}\left[\zeta\left(\frac{a_1+b_1}{2}\right)-\delta\right] &\leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S} \\ &\leq \frac{\zeta(a_1)+\zeta(b_1)}{1-s}+\frac{\varepsilon}{6}(b_1-a_1)+\delta.\end{aligned}$$

Corollary 2.11. *If ζ is (ε, δ)-P-convex function on $[a_1, b_1]$, then*

$$\begin{aligned}\frac{1}{2}\left[\zeta\left(\frac{a_1+b_1}{2}\right)-\delta\right] &\leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S} \\ &\leq [\zeta(a_1)+\zeta(b_1)]+\frac{\varepsilon}{6}(b_1-a_1)+\delta.\end{aligned}$$

Corollary 2.12. *If ζ is strongly-h-convex function on $[a_1, b_1]$, then*

$$\frac{1}{2h\left(\frac{1}{2}\right)}\left[\zeta\left(\frac{a_1+b_1}{2}\right)+\frac{\varepsilon}{12}(b_1-a_1)^2\right]$$
$$\leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}$$
$$\leq [\zeta(a_1)+\zeta(b_1)]\int_0^1 h(s)ds-\frac{\epsilon}{6}(b_1-a_1)^2,$$

which was proved in [12].

Corollary 2.13. *If ζ is strongly s-convex function of Breckner type on $[a_1, b_1]$, then*

$$2^{s-1}\left[\zeta\left(\frac{a_1+b_1}{2}\right)+\frac{\varepsilon}{12}(b_1-a_1)^2\right]\leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\leq \frac{\zeta(a_1)+\zeta(b_1)}{s+1}+\frac{\varepsilon}{6}(b_1-a_1)^2.$$

Corollary 2.14. *If ζ is strongly δ-s-convex function of Godunova-Levin-Dragomir type on $[a_1, b_1]$, then*

$$2^{-(s+1)}\left[\zeta\left(\frac{a_1+b_1}{2}\right)+\frac{\varepsilon}{12}(b_1-a_1)^2\right]\leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\leq \frac{\zeta(a_1)+\zeta(b_1)}{1-s}+\frac{\varepsilon}{6}(b_1-a_1)^2.$$

Corollary 2.15. *If ζ is strongly P-convex function on $[a_1, b_1]$, then*

$$\frac{1}{2}\left[\zeta\left(\frac{a_1+b_1}{2}\right)+\frac{\varepsilon}{12}(b_1-a_1)^2\right]\leq \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\leq [\zeta(a_1)+\zeta(b_1)]+\frac{\varepsilon}{6}(b_1-a_1)^2.$$

Theorem 2.2.2. *Let ζ be a differentiable function defined on $[a_1, b_1]$. If $|\zeta'|^m$ is a modified (h, d)-convex function on $[a_1, b_1]$, where $m > 1$ and $\frac{1}{r}+\frac{1}{m}=1$, then the following inequality holds:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right|$$
$$\leq \frac{(b_1-a_1)}{2\sqrt[r]{2(r+1)}}\left\{\left[K_1|\zeta'(a_1)|^m+K_2|\zeta'(b_1)|^m+\frac{1}{12}d(a_1,b_1)\right]^{\frac{1}{l}}\right.$$
$$\left.+\left[K_1|\zeta'(b_1)|^m+K_2|\zeta'(a_1)|^m+\frac{1}{12}d(a_1,b_1)\right]^{\frac{1}{m}}\right\}, \tag{2.15}$$

where

$$K_1 = \int_0^1 \wp h(\wp) d\wp, \quad K_2 = \int_0^1 \wp h(1-\wp) d\wp.$$

Proof. Taking the modulus on both sides of identity (2.9) and using Hölder–Isçan integral inequality, we have

$$\left| \frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1} \int_{a_1}^{b_1} \zeta(\mathscr{S}) d\mathscr{S} \right| \leq \left(\frac{b_1-a_1}{2}\right) \int_0^1 |1-2\wp| \left|\zeta'(\wp a_1 + (1-\wp)b_1)\right| d\wp$$

$$\leq \left(\frac{b_1-a_1}{2}\right) \left\{ \left(\int_0^1 (1-\wp) |1-2\wp|^r d\wp \right)^{\frac{1}{r}} \left(\int_0^1 (1-\wp) \left|\zeta'(\wp a_1 + (1-\wp)b_1)\right|^m d\wp \right)^{\frac{1}{m}} \right.$$

$$\left. + \left(\int_0^1 \wp |1-2\wp|^r d\wp \right)^{\frac{1}{r}} \left(\int_0^1 \wp \left|\zeta'(\wp a_1 + (1-\wp)b_1)\right|^m d\wp \right)^{\frac{1}{m}} \right\}. \tag{2.16}$$

Since $|\zeta'|^m$ is modified (h,d)-convex function on $[a_1,b_1]$, we get

$$\int_0^1 \wp \left|\zeta'(\wp a_1 + (1-\wp)b_1)\right|^m d\wp$$

$$\leq \int_0^1 \wp[h(\wp) |\zeta'(a_1)|^m + h(1-\wp) |\zeta'(b_1)|^m + \wp(1-\wp) d(a_1,b_1)] d\wp$$

$$= \int_0^1 \wp h(\wp) |\zeta'(a_1)|^m d\wp + \int_0^1 \wp h(1-\wp) |\zeta'(b_1)|^m d\wp + \frac{1}{12} d(a_1,b_1) \tag{2.17}$$

and

$$\int_0^1 (1-\wp) \left|\zeta'(\wp a_1 + (1-\wp)b_1)\right|^m d\wp$$

$$\leq \int_0^1 \wp h(1-\wp) |\zeta'(a_1)|^m d\wp + \int_0^1 \wp h(\wp) |\zeta'(b_1)|^m d\wp + \frac{1}{12} d(a_1,b_1). \tag{2.18}$$

On the other hand,

$$\int_0^1 \wp |1-2\wp|^r d\wp = \int_0^1 (1-\wp) |1-2\wp|^r d\wp = \frac{1}{2(r+1)}. \tag{2.19}$$

Required inequality (2.15) is obtained by using (2.17), (2.18) and (2.19) in (2.16). □

Corollary 2.16. *If ζ is h-convex function on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int\limits_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right|$$
$$\leq \frac{(b_1-a_1)}{2\sqrt[r]{2(r+1)}}\left\{\left[K_1\left|\zeta'(a_1)\right|^m+K_2\left|\zeta'(b_1)\right|^m\right]^{\frac{1}{m}}+\left[K_1\left|\varrho'(b_1)\right|^m+K_2\left|\zeta'(a_1)\right|^m\right]^{\frac{1}{m}}\right\}.$$

Remark 2.17. If ζ is s-convex function of Breckner type, then we get [75, Theorem 2.2].

Corollary 2.18. *If ζ is s-convex function of Godunova-Levin-Dragomir type on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int\limits_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \leq \frac{(b_1-a_1)}{2\sqrt[r]{2(r+1)}}\left[\left(\frac{(1-s)\left|\zeta'(a_1)\right|^m+\left|\zeta'(b_1)\right|^m}{(1-s)(2-s)}\right)^{\frac{1}{m}}\right.$$
$$\left.+\left(\frac{(1-s)\left|\zeta'(b_1)\right|^m+\left|\zeta'(a_1)\right|^m}{(1-s)(2-s)}\right)^{\frac{1}{m}}\right].$$

Corollary 2.19. *If ζ is P-convex function on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\varrho(b_1)}{2}-\frac{1}{b_1-a_1}\int\limits_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \leq \frac{(b_1-a_1)}{2\sqrt[r]{r+1}}\left[\left|\zeta'(a_1)\right|^m+\left|\zeta'(b_1)\right|^m\right].$$

Remark 2.20. We skip the cases of δ-h-convex functions as they gives similar estimates to h-convex functions as discussed earlier.

Corollary 2.21. *If ζ is (ε, δ)-h-convex function on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int\limits_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right|$$
$$\leq \frac{(b_1-a_1)}{2\sqrt[r]{2(r+1)}}\left\{\left[K_1\left|\zeta'(a_1)\right|^m+K_2\left|\zeta'(b_1)\right|^m+\frac{1}{2}\left[\frac{\varepsilon}{6}(b_1-a_1)+\delta\right]\right]^{\frac{1}{m}}\right.$$
$$\left.+\left[K_1\left|\zeta'(b_1)\right|^m+K_2\left|\zeta'(a_1)\right|^m+\frac{1}{2}\left[\frac{\varepsilon}{6}(b_1-a_1)+\delta\right]\right]^{\frac{1}{m}}\right\}. \tag{2.20}$$

Corollary 2.22. *If ζ is (ε, δ)-s-convex function of Breckner type on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int\limits_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \leq \frac{(b_1-a_1)}{2\sqrt[r]{2(r+1)}}$$

$$\times\left[\left(\frac{(1+s)\left|\zeta'(a_1)\right|^m+\left|\zeta'(b_1)\right|^m}{(1+s)(2+s)}+\frac{1}{2}\left[\frac{\varepsilon}{6}(b_1-a_1)+\delta\right]\right)^{\frac{1}{m}} + \left(\frac{(1+s)\left|\zeta'(b_1)\right|^m+\left|\zeta'(a_1)\right|^m}{(1+s)(2+s)}+\frac{1}{2}\left[\frac{\varepsilon}{6}(b_1-a_1)+\delta\right]\right)^{\frac{1}{m}}\right]. \tag{2.21}$$

Corollary 2.23. *If ζ is (ε,δ)-s-convex function of Godunova-Levin-Dragomir type on $[a_1,b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \leq \frac{(b_1-a_1)}{2\sqrt[p]{2(p+1)}}\left[\left(\frac{(1-s)\left|\varrho'(a_1)\right|^q+\left|\varrho'(b_1)\right|^q}{(1-s)(2-s)}+\frac{1}{2}\left[\frac{\epsilon}{6}(b_1-a_1)+\delta\right]\right)^{\frac{1}{q}} + \left(\frac{(1-s)\left|\zeta'(b_1)\right|^q+\left|\zeta'(a_1)\right|^q}{(1-s)(2-s)}+\frac{1}{2}\left[\frac{\varepsilon}{6}(b_1-a_1)+\delta\right]\right)^{\frac{1}{q}}\right]. \tag{2.22}$$

Corollary 2.24. *If ζ is (ε,δ)-P-convex function on $[a_1,b_1]$, then following inequality hold:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \leq \frac{(b_1-a_1)}{2\sqrt[r]{r+1}}\left[\left|\zeta'(a_1)\right|^m+\left|\zeta'(b_1)\right|^m+\left[\frac{\epsilon}{6}(b_1-a_1)+\delta\right]\right]^{\frac{1}{m}}.$$

Corollary 2.25. *If ζ is strongly h-convex function on $[a_1,b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \leq \frac{(b_1-a_1)}{2\sqrt[r]{2(r+1)}}\left[\left(K_1\left|\zeta'(a_1)\right|^m+K_2\left|\zeta'(b_1)\right|^m-\frac{\varepsilon}{12}(b_1-a_1)^2\right)^{\frac{1}{m}} + \left(K_1\left|\zeta'(b_1)\right|^m+K_2\left|\zeta'(a_1)\right|^m-\frac{\varepsilon}{12}(b_1-a_1)^2\right)^{\frac{1}{m}}\right]. \tag{2.23}$$

Corollary 2.26. *If ζ is strongly s-convex function of Breckner type on $[a_1,b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \leq \frac{(b_1-a_1)}{2\sqrt[r]{2(r+1)}}\left[\left(\frac{(1+s)\left|\zeta'(a_1)\right|^m+\left|\zeta'(b_1)\right|^m}{(1+s)(2+s)}-\frac{\varepsilon}{12}(b_1-a_1)^2\right)^{\frac{1}{m}} + \left(\frac{(1+s)\left|\zeta'(b_1)\right|^m+\left|\zeta'(a_1)\right|^m}{(1+s)(2+s)}-\frac{\varepsilon}{12}(b_1-a_1)^2\right)^{\frac{1}{m}}\right]. \tag{2.24}$$

Corollary 2.27. *If ζ is strongly s-convex function of Godunova-Levin-Dragomir type on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right|$$
$$\leq \frac{(b_1-a_1)}{2\sqrt[r]{2(r+1)}}\left[\left(\frac{(1-s)\left|\zeta'(a_1)\right|^m+\left|\zeta'(b_1)\right|^m}{(1-s)(2-s)} - \frac{\varepsilon}{12}(b_1-a_1)^2\right)^{\frac{1}{m}}\right.$$
$$\left.+\left(\frac{(1-s)\left|\zeta'(b_1)\right|^m+\left|\zeta'(a_1)\right|^m}{(1-s)(2-s)} - \frac{\varepsilon}{12}(b_1-a_1)^2\right)^{\frac{1}{m}}\right]. \tag{2.25}$$

Corollary 2.28. *If ζ is strongly P-convex function on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \leq \frac{(b_1-a_1)}{2\sqrt[r]{r+1}}\left[\left|\zeta'(a_1)\right|^m+\left|\zeta'(b_1)\right|^m - \frac{\varepsilon}{6}(b_1-a_1)\right]^{\frac{1}{m}}.$$

Theorem 2.2.3. *Let ζ be a differentiable function defined on $[a_1, b_1]$. If $|\zeta'|^m$ for $m \geq 1$, is a modified (h, d)-convex function on $[a_1, b_1]$, then the following inequality holds:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right|$$
$$\leq \frac{(b_1-a_1)}{2\sqrt[r]{4}}\left[\left(K_3\left|\zeta'(a_1)\right|^m+K_4\left|\zeta'(b_1)\right|^m+\frac{1}{32}d(a_1,b_1)\right)^{\frac{1}{m}}\right.$$
$$\left.+\left(K_3\left|\zeta'(b_1)\right|^m+K_4\left|\zeta'(a_1)\right|^m+\frac{1}{32}d(a_1,b_1)\right)^{\frac{1}{m}}\right], \tag{2.26}$$

where

$$K_3 = \int_0^1 \wp\left|1-2\wp\right| h(\wp)d\wp, \quad K_4 = \int_0^1 \wp\left|1-2\wp\right| h(1-\wp)d\wp,$$

and $r = \frac{m}{m-1}$.

Proof. Taking the modulus on both sides of identity (2.9) and applying the improved power–mean integral inequality, we have

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \leq \left(\frac{b_1-a_1}{2}\right)\int_0^1 \left|1-2\wp\right|\left|\zeta'(\wp a_1+(1-\wp)b_1)\right| d\wp$$

$$\times \leq \left(\frac{b_1-a_1}{2}\right)$$
$$\times \left\{ \left(\int_0^1 (1-\wp)\,|1-2\wp|\,d\wp\right)^{1-\frac{1}{m}} \left(\int_0^1 (1-\wp)\,|1-2\wp|\,|\zeta'(\wp a_1+(1-\wp)b_1)|^m\,d\wp\right)^{\frac{1}{m}}\right.$$
$$\left. + \left(\int_0^1 \wp\,|1-2\wp|\,d\wp\right)^{1-\frac{1}{m}} \left(\int_0^1 \wp\,|1-2\wp|\,|\zeta'(\wp a_1+(1-\wp)b_1)|^m\,d\wp\right)^{\frac{1}{m}}\right\}. \tag{2.27}$$

Since $|\zeta'|^m$ is a modified (h,d)-convex function on $[a_1,b_1]$, we get

$$\int_0^1 \wp\,|1-2\wp|\,|\zeta'(\wp a_1+(1-\wp)b_1)|^m\,d\wp$$
$$\leq \int_0^1 \wp\,|1-2\wp|\,h(\wp)\,|\zeta'(a_1)|^m\,d\wp + \int_0^1 \wp\,|1-2\wp|\,h(1-\wp)\,|\zeta'(b_1)|^m\,d\wp + \frac{1}{32}d(a_1,b_1) \tag{2.28}$$

and

$$\int_0^1 (1-\wp)\,|1-2\wp|\,|\zeta'(\wp a_1+(1-\wp)b_1)|^m\,d\wp$$
$$\leq \int_0^1 \wp\,|1-2\wp|\,h(1-\wp)\,|\zeta'(a_1)|^m\,d\wp + \int_0^1 \wp\,|1-2\wp|\,h(\wp)\,|\zeta'(b_1)|^m\,d\wp + \frac{1}{32}d(a_1,b_1). \tag{2.29}$$

On the other hand,

$$\int_0^1 \wp\,|1-2\wp|\,d\wp = \int_0^1 (1-\wp)\,|1-2\wp|\,d\wp = \frac{1}{4}. \tag{2.30}$$

Required inequality (2.26) is obtained by using (2.28), (2.29) and (2.30) in (2.27). □

Corollary 2.29. *If ζ is h-convex function on $[a_1,b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1} \zeta(\mathscr{S})d\mathscr{S}\right|$$
$$\leq \frac{(b_1-a_1)}{2\sqrt[r]{4}}\left[\left(K_3\,|\zeta'(a_1)|^q + K_4\,|\zeta'(b_1)|^m\right)^{\frac{1}{m}} + \left(K_3\,|\zeta'(b_1)|^m + K_4\,|\zeta'(a_1)|^m\right)^{\frac{1}{m}}\right]. \tag{2.31}$$

Remark 2.30. If ζ is s-convex function of Breckner type, then we get [75, Theorem 2.4].

Corollary 2.31. *If ζ is s-convex function of Godunova-Levin-Dragomir type on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int\limits_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \leq \frac{(b_1-a_1)}{2\sqrt[r]{4}}$$
$$\times\left\{\left[\frac{(1-s)(1-s+2^{s-1})\left|\zeta'(a_1)\right|^m + ((5-s)2^{s-1}-s-1)\left|\zeta'(b_1)\right|^m}{(1-s)(2-s)(3-s)}\right]^{\frac{1}{m}}\right.$$
$$\left.+\left[\frac{(1-s)(1-s+2^{s-1})\left|\zeta'(b_1)\right|^m + ((5-s)2^{s-1}-s-1)\left|\zeta'(a_1)\right|^m}{(1-s)(2-s)(3-s)}\right]^{\frac{1}{m}}\right\}. \tag{2.32}$$

Corollary 2.32. *If ζ is P-convex function on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int\limits_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \leq \frac{(b_1-a_1)}{4}\left[\left|\zeta'(a_1)\right|^m + \left|\zeta'(b_1)\right|^m\right]^{\frac{1}{m}}.$$

Remark 2.33. We skip the cases of δ-h-convex functions as they gives similar estimates to h-convex functions as discussed earlier.

Corollary 2.34. *If ζ is (ε, δ)-h-convex function on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int\limits_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right|$$
$$\leq \frac{(b_1-a_1)}{2\sqrt[r]{4}}\left\{\left[K_3\left|\zeta'(a_1)\right|^m + K_4\left|\zeta'(b_1)\right|^m + \frac{1}{4}\left[\frac{\varepsilon}{8}(b_1-a_1)+\delta\right]\right]^{\frac{1}{m}}\right.$$
$$\left.+\left[K_3\left|\zeta'(b_1)\right|^m + K_4\left|\zeta'(a_1)\right|^m + \frac{1}{4}\left[\frac{\varepsilon}{8}(b_1-a_1)+\delta\right]\right]^{\frac{1}{m}}\right\}. \tag{2.33}$$

Corollary 2.35. *If ζ is (ε, δ)-s-convex function of Breckner type on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int\limits_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right|$$
$$\leq \frac{(b_1-a_1)}{2\sqrt[r]{4}}\left\{\left[\frac{(s+1)(s+1+2^{-s-1})\left|\zeta'(a_1)\right|^m + ((5+s)2^{-s-1}+s-1)\left|\zeta'(b_1)\right|^m}{(s+1)(s+2)(s+3)}\right.\right.$$
$$\left.+\frac{1}{4}\left[\frac{\varepsilon}{8}(b_1-a_1)+\delta\right]\right]^{\frac{1}{m}}$$
$$+\left[\frac{(s+1)(s+1+2^{-s-1})\left|\zeta'(b_1)\right|^m + ((5+s)2^{-s-1}+s-1)\left|\zeta'(a_1)\right|^q}{(s+1)(s+2)(s+3)}\right.$$
$$\left.\left.+\frac{1}{4}\left[\frac{\varepsilon}{8}(b_1-a_1)+\delta\right]\right]^{\frac{1}{m}}\right\}. \tag{2.34}$$

Corollary 2.36. *If ζ is (ε,δ)-s-convex function of Godunova-Levin-Dragomir type on $[a_1,b_1]$, then*

$$\begin{aligned}
&\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \\
&\leq \frac{(b_1-a_1)}{2\sqrt[r]{4}}\left\{\left[\frac{(1-s)(1-s+2^{s-1})\left|\zeta'(a_1)\right|^m+((5-s)2^{s-1}-s-1)\left|\zeta'(b_1)\right|^m}{(1-s)(2-s)(3-s)}\right.\right. \\
&\quad \left.+\frac{1}{4}\left[\frac{\varepsilon}{8}(b_1-a_1)+\delta\right]\right]^{\frac{1}{q}} \\
&\quad +\left[\frac{(1-s)(1-s+2^{s-1})\left|\zeta'(b_1)\right|^m+((5-s)2^{s-1}-s-1)\left|\zeta'(a_1)\right|^m}{(1-s)(2-s)(3-s)}\right. \\
&\quad \left.\left.+\frac{1}{4}\left[\frac{\varepsilon}{8}(b_1-a_1)+\delta\right]\right]^{\frac{1}{m}}\right\}. \qquad (2.35)
\end{aligned}$$

Corollary 2.37. *If ζ is (ε,δ)-P-convex function on $[a_1,b_1]$, then*

$$\begin{aligned}
&\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \\
&\leq \frac{(b_1-a_1)}{4}\left[\left|\zeta'(a_1)\right|^m+\left|\zeta'(b_1)\right|^m+\left[\frac{\varepsilon}{8}(b_1-a_1)+\delta\right]\right]^{\frac{1}{m}}.
\end{aligned}$$

Corollary 2.38. *If ζ is strongly h-convex function on $[a_1,b_1]$, then*

$$\begin{aligned}
&\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \\
&\leq \frac{(b_1-a_1)}{2\sqrt[r]{4}}\left\{\left[K_3\left|\zeta'(a_1)\right|^m+K_4\left|\zeta'(b_1)\right|^m-\frac{\epsilon}{32}(b_1-a_1)^2\right]^{\frac{1}{m}}\right. \\
&\quad \left.+\left[K_3\left|\zeta'(b_1)\right|^m+K_4\left|\zeta'(a_1)\right|^m-\frac{\varepsilon}{32}(b_1-a_1)^2\right]^{\frac{1}{m}}\right\}. \qquad (2.36)
\end{aligned}$$

Corollary 2.39. *If ζ is strongly s-convex function of Breckner type on $[a_1,b_1]$, then*

$$\begin{aligned}
&\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\mathscr{S})d\mathscr{S}\right| \\
&\leq \frac{(b_1-a_1)}{2\sqrt[r]{4}}\left\{\left[\frac{(s+1)(s+1+2^{-s-1})\left|\zeta'(a_1)\right|^m+((5+s)2^{-s-1}+s-1)\left|\zeta'(b_1)\right|^m}{(s+1)(s+2)(s+3)}\right.\right. \\
&\quad \left.-\frac{\epsilon}{32}(b_1-a_1)^2\right]^{\frac{1}{m}}+\left[\frac{(s+1)(1+s+2^{-s-1})\left|\zeta'(b_1)\right|^m+((5+s)2^{-s-1}+s-1)\left|\zeta'(a_1)\right|^m}{(s+1)(s+2)(s+3)}\right. \\
&\quad \left.\left.-\frac{\varepsilon}{32}(b_1-a_1)^2\right]^{\frac{1}{m}}\right\}. \qquad (2.37)
\end{aligned}$$

Corollary 2.40. *If ζ is strongly s-convex function of Godunova-Levin-Dragomir type on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{a_1 - a_1}\int_{a_1}^{b_1} \zeta(\mathscr{S})d\mathscr{S}\right|$$
$$\leq \frac{(b_1 - a_1)}{2\sqrt[r]{4}}\left\{\left[\frac{(1-s)(1-s+2^{s-1})\left|\zeta'(a_1)\right|^m + ((5-s)2^{s-1}-s-1)\left|\zeta'(b_1)\right|^m}{(1-s)(2-s)(3-s)}\right.\right.$$
$$\left. - \frac{\varepsilon}{32}(b_1 - a_1)^2\right]^{\frac{1}{m}}$$
$$+ \left[\frac{(1-s)(1-s+2^{s-1})\left|\zeta'(b_1)\right|^m + ((5-s)2^{s-1}-s-1)\left|\zeta'(a_1)\right|^m}{(1-s)(2-s)(3-s)}\right.$$
$$\left.\left. - \frac{\varepsilon}{32}(b_1 - a_1)^2\right]^{\frac{1}{m}}\right\}. \tag{2.38}$$

Corollary 2.41. *If ζ is strongly P-convex function on $[a_1, b_1]$, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1 - a_1}\int_{a_1}^{b_1} \zeta(\mathscr{S})d\mathscr{S}\right| \leq \frac{(b_1 - a_1)}{4}\left[\left|\zeta'(a_1)\right|^m + \left|\zeta'(b_1)\right|^m - \frac{\epsilon}{8}(b_1 - a_1)^2\right]^{\frac{1}{m}}.$$

2.3 Applications towards the error formula

In this section, we give some new error estimates for the trapezoidal quadrature formula by using inequalities developed in the Section 2.2. Assume that P is a partition of the interval $[a_1, b_1]$, i.e.,

$$P : a_1 = u_0 < u_1 < u_2 \ldots < u_{n-1} < u_n = b_1.$$

Trapezoidal quadrature formula is defined by

$$\int_{a_1}^{b_1} \zeta(\mathscr{S})d\mathscr{S} = T(\zeta, P) + E(\zeta, P),$$

where

$$T(\zeta, P) = \sum_{i=0}^{n-1}\left[\frac{\zeta(u_i)+\zeta(u_{i+1})}{2}\right] h_i \tag{2.39}$$

is the trapezoidal version for $h_i = (u_{i+1} - u_i)$ and $i = 0, 1, \ldots, n-1$, and $E(\zeta, P)$ is denote their associated approximation error.

Proposition 4. *Let $\zeta : [a_1, b_1] \longrightarrow \mathbb{R}$ be a differentiable function on (a_1, b_1), where $a_1 < b_1$. If $|\zeta'|^m$ is strongly s-convex of Godunova-Levin-Dragomir type on $[a_1, b_1]$ for $m > 1$ and $\frac{1}{r} + \frac{1}{m} = 1$, then the following inequality holds:*

$$\begin{aligned} \left|E(\zeta, P)\right| \quad &\leq \frac{1}{2\sqrt[r]{2(r+1)}} \sum_{i=0}^{n-1} h_i^2 \left[\left(\frac{(1-s)\,|\zeta'(u_i)|^m + |\zeta'(u_{i+1})|^m}{(1-s)(2-s)} - \frac{\varepsilon}{12} h_i^2 \right)^{\frac{1}{m}} \right. \\ &\left. + \left(\frac{(1-s)\,|\zeta'(u_{i+1})|^m + |\zeta'(u_i)|^m}{(1-s)(2-s)} - \frac{\varepsilon}{12} h_i^2 \right)^{\frac{1}{m}} \right]. \end{aligned} \tag{2.40}$$

Proof. Applying inequality (2.25) in Corollary 2.27 on the subintervals $[x_i, x_{i+1}]$ $(i = 0, \ldots, n-1)$ of the partition P, we have

$$\begin{aligned} &\left| \frac{\zeta(u_i) + \zeta(u_{i+1})}{2} - \frac{1}{u_{i+1} - u_i} \int_{u_i}^{u_{i+1}} \zeta(\mathscr{S}) d\mathscr{S} \right| \\ &\leq \frac{(u_{i+1} - u_i)}{2\sqrt[r]{2(r+1)}} \left[\left(\frac{(1-s)\,|\zeta'(u_i)|^m + |\zeta'(u_{i+1})|^m}{(1-s)(2-s)} - \frac{\varepsilon}{12} h_i^2 \right)^{\frac{1}{m}} \right. \\ &\left. + \left(\frac{(1-s)\,|\zeta'(u_{i+1})|^m + |\zeta'(u_i)|^m}{(1-s)(2-s)} - \frac{\varepsilon}{12} h_i^2 \right)^{\frac{1}{m}} \right]. \end{aligned} \tag{2.41}$$

Hence from (2.41), we get

$$\begin{aligned} \left|E(\zeta, P)\right| &= \left| \int_{a_1}^{b_1} \zeta(\mathscr{S}) d\mathscr{S} - T(\zeta, P) \right| \\ &\leq \left| \sum_{i=0}^{n-1} \left\{ \int_{u_i}^{u_{i+1}} \zeta(\mathscr{S}) d\mathscr{S} - \frac{\zeta(u_i) + \zeta(u_{i+1})}{2} h_i \right\} \right| \\ &\leq \sum_{i=0}^{n-1} \left| \left\{ \int_{u_i}^{u_{i+1}} \zeta(\mathscr{S}) d\mathscr{S} - \frac{\zeta(u_i) + \zeta(u_{i+1})}{2} h_i \right\} \right| \\ &\leq \frac{1}{2\sqrt[r]{2(r+1)}} \sum_{i=0}^{n-1} h_i^2 \left[\left(\frac{(1-s)\,|\zeta'(u_i)|^m + |\zeta'(u_{i+1})|^m}{(1-s)(2-s)} - \frac{\varepsilon}{12} h_i^2 \right)^{\frac{1}{m}} \right. \\ &\left. + \left(\frac{(1-s)\,|\zeta'(u_{i+1})|^m + |\zeta'(u_i)|^m}{(1-s)(2-s)} - \frac{\varepsilon}{12} h_i^2 \right)^{\frac{1}{m}} \right]. \end{aligned}$$

The proof of Proposition 4 is completed. □

Proposition 5. *Let $\zeta : [a_1, b_1] \longrightarrow \mathbb{R}$ be a differentiable function on (a_1, b_1), where $a_1 < b_1$. If $|\zeta'|^m$ is strongly s-convex of Godunova-Levin-Dragomir type on $[a_1, b_1]$ for $m \geq 1$, then the following inequality holds:*

$$\begin{aligned} \big|E(\zeta, P)\big| &\leq \frac{1}{2\sqrt[r]{4}} \sum_{i=0}^{n-1} h_i^2 \\ &\times \left\{ \left[\frac{(1-s)(1-s+2^{s-1}) |\zeta'(u_i)|^m + ((5-s)2^{s-1} - s - 1) |\zeta'(u_{i+1})|^m}{(1-s)(2-s)(3-s)} - \frac{\varepsilon}{32} h_i^2 \right]^{\frac{1}{m}} \right. \\ &+ \left. \left[\frac{(1-s)(1-s+2^{s-1}) |\zeta'(u_{i+1})|^m + ((5-s)2^{s-1} - s - 1) |\zeta'(u_i)|^m}{(1-s)(2-s)(3-s)} - \frac{\varepsilon}{32} h_i^2 \right]^{\frac{1}{m}} \right\}. \end{aligned} \tag{2.42}$$

Proof. The proof is analogous as to that of Proposition 4 but use inequality (2.38) in Corollary 2.40. □

Remark 2.42. By using Theorem 2.2.1, for appropriate choices of function ζ and function $h(\wp) = \wp, \wp^{-1}, \wp^s, \wp^{-s}, \wp(1-\wp), \frac{\sqrt{\wp}}{2\sqrt{1-\wp}}$ and $h(\wp) = 1$, we can obtain some new integral inequalities using special means, such as arithmetic, geometric, logarithmic etc. Also, applying our Theorems 2.2.2 and 2.2.3 for above suitable functions h, we can deduce some new integral inequalities using ideas and techniques of Propositions 4 and 5. We omit their proofs and the details are left to the interested reader.

Chapter 3

Generalized Hermite–Hadamard and Fejér–Hermite–Hadamard inequalities with associated conformable and fractional integral inequalities

Due to the natural occurrence of half order derivatives and integrals in physical problems, interest of the researcher has been stimulated in this special field of mathematics. Its applications are discovered in different areas of natural sciences such as: quantum mechanical calculations, chemical analysis of aqueous solutions, design of heat flux meters, transmission line theory etc. see ([108, 109, 112, 137–139, 143–145]). For the detailed mathematical study of fractional integral and derivative operators, see ([92, 93]). In the recent past, researchers have used the notion of fractional order derivatives and integrals for the establishment of some fractional integrals versions of Hadamard inequality, see for example ([7, 13, 18, 50, 59, 62, 63, 76, 81, 88–90, 102, 103, 110, 140, 174, 181, 190]), etc. and the references therein).

This section dedicated to the generalized form of Hermite–Hadamard inequality to not just bind together the diverse fractional and classical forms yet to acquire some very new partner by using a generalized integral operator given in Definition 3.1.2. We derive some new Hermite–Hadamard type inequalities for various fractional integrals including generalized Riemann-Liouville fractional integral, generalized k-fractional integral of a function with respect to another function, generalized fractional integral with Mittag-Leffler function as its kernel, and so on. We have likewise acquired a generalized weighted adaptation of Hermite–Hadamard inequality. Results because of this examination keep on holding for recently known outcomes. Besides, a decent number of new outcomes are found for various fractional and conformable integrals. The inequality (3.19) and (3.34) can be use as a wellspring of Hermite–Hadamard and Fejer–Hermite–Hadamard inequalities. We put together this chapter as follows:

In Section 3.1, we sum up the various types of fractional and conformable integrals arising out of a generalized integral operator presented by Farid [52]. In Section 3.2, we build up a generalized Hermite–Hadamard inequality by utilizing the generalized integral operator. In Section 3.3, we reason various forms of Hermite–Hadamard inequalities for fractional and conformable integrals. In Section 3.4, we demonstrate a weighted generalized Hermite–Hadamard inequality. In Section 3.5, we acquire relating weighted adaptations for fractional and similar conformable integrals.

3.1 Generalized integral operator and its special cases

Recently, Sarikaya and Ertuğral [172] introduce a generalized fractional integral. The operator by Sarikaya and Ertuğral gave a general formulation for different kinds of singular Kernel that appeared in various kinds of fractional integrals.

Definition 3.1.1. *The left and right-sided fractional integral of a function ζ are defined by*

$$ {}^{\varphi}S_{a_1^+}\zeta\left(\theta\right)=\int\limits_{a_1}^{\theta}\frac{\varphi\left(\theta-\tau\right)}{\theta-\tau}\zeta\left(\tau\right)d\tau,\ \ \theta>a_1, \tag{3.1}$$

$$
{}^{\varphi}S_{b_1^-}\zeta(\theta) = \int_{\theta}^{b_1} \frac{\varphi(\tau-\theta)}{\tau-\theta}\zeta(\tau)\,d\tau,\ \ \theta > a_1, \tag{3.2}
$$

where $\varphi : [0,\infty) \to [0,\infty)$ *satisfying the condition*

$$
\int_0^1 \frac{\varphi(\tau)}{\tau} d\tau < \infty.
$$

The operator includes Riemann-Liouville, k-Riemann-Liouville, (k,s)-fractional integral, Hadamard, conformable fractional integrals etc.. For the detail see [172].

Recently, in [52], Farid formulated a generalized integral model similar to the model of Sarikaya and Ertuğral by using a monotone increasing function in the kernel to obtain more fractional integrals as a special case.

Definition 3.1.2. *Let* $\zeta, \Phi : [a_1, b_1] \to \mathbb{R}$, $0 < a_1 < b_1$, *be the functions such that* ζ *be positive and* $\zeta \in L_1[a_1, b_1]$ *and* Φ *be differentiable and increasing. Also let* φ *be a positive function such that* $\frac{\varphi}{\tau}$ *is increasing on* $[a_1,\infty)$. *Then for* $\tau \in [a_1, b_1]$ *the left and right-sided integral operators is defined by*

$$
{}^{\varphi}_{\Phi}\mathcal{G}_{a_1^+}\zeta(\theta) = \int_{a_1}^{\theta} \frac{\Phi'(\tau)\varphi(\Phi(\theta)-\Phi(\tau))\zeta(\tau)}{\Phi(\theta)-\Phi(\tau)} d\tau, \quad a_1 > \theta, \tag{3.3}
$$

$$
{}^{\varphi}_{\Phi}\mathcal{G}_{b_1^-}\zeta(\theta) = \int_{\theta}^{b_1} \frac{\Phi'(\tau)\varphi(\Phi(\tau)-\Phi(\theta))\zeta(\tau)}{\Phi(\tau)-\Phi(\theta)} d\tau, \quad b_1 < \theta, \tag{3.4}
$$

Remark 3.1. It is worth mentioning that we can deduce aforementioned fractional integrals by some suitable settings of functions φ and Φ. For instance,

(i) If $\varphi(\tau) = \frac{1}{k\Gamma_k(\vartheta)}\tau^{\frac{\vartheta}{k}}$, then we have k-analogue of generalized Riemann-Liouville fractional integrals (1.50) and (1.51). For $k = 1$, we will get generalized Riemann-liouville fractional integrals (1.48) and (1.49). Furthermore if Φ is identity function then we have k-Riemann

Liouville fractional integral (1.40) and (1.41).

(ii) If $\varphi(\tau) = \frac{1}{\Gamma(\vartheta)}\tau^{\vartheta}, \Phi(\tau) = \frac{\tau^{r+\gamma}}{r+\gamma}, r + \gamma \neq 0$, then we have generalized conformable fractional integral (1.44) and (1.45).

(iii) For $\varphi(\tau) = \frac{1}{\Gamma(\vartheta)}\tau^{\vartheta}$, $\Phi(\tau) = \frac{\tau^{\rho}}{\rho}$, $\rho > 0$ and $k = 1$ in (3.3) and (3.4), then we obtain Katugampola fractional integral (1.36) and (1.37).

(iv) If we take $\varphi(\tau) = \frac{1}{k\Gamma_k(\vartheta)}\tau^{\frac{\vartheta}{k}}, \Phi(\tau) = \frac{\tau^{1+s}}{1+s}$, then (3.3) and (3.4) reduces to (k, s)-fractional integrals (1.42) and (1.43).

(v) If we take $\varphi(\tau) = \frac{1}{\vartheta}\exp\left(-\frac{1-\vartheta}{\vartheta}\tau\right), \vartheta \in (0, 1)$, then we get generalized exponential fractional integral [41] as following:

$$ {}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{a_1^+}\zeta(\theta) = \frac{1}{\vartheta}\int_{a_1}^{\theta} \exp\left(-\frac{1-\vartheta}{\vartheta}(\Phi(\theta) - \Phi(\tau))\right)\Phi'(\tau)\zeta(\tau)\,d\tau, \quad \theta > a_1, \tag{3.5} $$

$$ {}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{b_1^-}\zeta(\theta) = \frac{1}{\vartheta}\int_{\theta}^{b_1} \exp\left(-\frac{1-\vartheta}{\vartheta}(\Phi(\tau) - \Phi(\theta))\right)\Phi'(\tau)\zeta(\tau)\,d\tau, \quad \theta < b_1. \tag{3.6} $$

It reduces to the operator of Kirane and Torebik if g is identity function [7].

(vi) If $\varphi(\tau) = \tau^{\frac{\vartheta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\vartheta}(w(\tau)^{\rho})$ in (3.3) and (3.4), then we have generalized k-fractional integral operator (1.55) and (1.56).

(vii) If we choose $\varphi(\tau) = \tau^{\vartheta}\mathcal{E}^{\delta,\lambda,s}_{\beta,\vartheta,r}(w(\tau)^{\beta})$, then we get a natural extension of fractional integral operator (1.38) and (1.39) as follows:

$$ {}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,\lambda,s}_{\beta,r,a_1^+}\zeta(\theta; w) = \int_{a_1}^{\theta} \frac{\Phi'(\tau)\mathcal{E}^{\delta,\lambda,s}_{\beta,\vartheta,r}\left(w\left(\Phi(\theta) - \Phi(\tau)\right)^{\beta}\right)\zeta(\tau)}{(\Phi(\theta) - \Phi(\tau))^{1-\vartheta}}d\tau, \quad a_1 > \theta. \tag{3.7} $$

$$ {}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,\lambda,s}_{\beta,r,b_1^-}\zeta(\theta; w) = \int_{\theta}^{b_1} \frac{\Phi'(\tau)\mathcal{E}^{\delta,\lambda,s}_{\beta,\vartheta,r}\left(w\left(\Phi(\tau) - \Phi(\theta)\right)^{\beta}\right)f(\tau)}{(\Phi(\tau) - \Phi(\theta))^{1-\vartheta}}d\tau, \quad \theta < b_1. \tag{3.8} $$

(viii) If we choose $\varphi(\tau) = \tau^{\vartheta}\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}(w(\tau)^{\beta}; q^*)$, then we get a natural extension of fractional integral operator (1.46) and (1.47) as follows:

$$ {}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,a_1^+}\zeta\left(\theta;q^*,w\right)=\int\limits_{a_1}^{\theta}\frac{\Phi'(\tau)\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}\left(w\left(\Phi(\theta)-\Phi(\tau)\right)^{\beta};q^*\right)\zeta(\tau)}{(\Phi(\theta)-\Phi(\tau))^{1-\vartheta}}d\tau,\quad \theta>a_1. \tag{3.9} $$

$$ {}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,b_1^-}\zeta\left(\theta;q^*,w\right)=\int\limits_{\theta}^{b_1}\frac{\Phi'(\tau)\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}\left(w\left(\Phi(\tau)-\Phi(\theta)\right)^{\beta};q^*\right)\zeta(\tau)}{\left(\Phi(\tau)-\Phi(\theta)\right)^{1-\vartheta}}d\tau,\quad \theta<b_1. \tag{3.10} $$

(ix) If $\varphi(x)=\frac{1}{\Gamma(\vartheta)}x^{\vartheta}$, and $\Phi(x)=-x^{-1}$, then Harmonic fractional integral operators will be obtained defined in [41] as follows:

$$ {}^{\vartheta}H_{a_1^+}\zeta(\theta)=\frac{\theta^{1-\vartheta}}{\Gamma(\vartheta)}\int\limits_{a_1}^{\theta}\left(\theta-\tau\right)^{\vartheta-1}\frac{\zeta(\tau)}{\tau^{\vartheta+1}}d\tau,\ \theta>a_1, \tag{3.11} $$

$$ {}^{\vartheta}H_{b_1^+}\zeta(\theta)=\frac{\theta^{1-\vartheta}}{\Gamma(\vartheta)}\int\limits_{\theta}^{b_1}\left(\tau-\theta\right)^{\vartheta-1}\frac{\zeta(\tau)}{\tau^{\vartheta+1}}d\tau,\ \theta<b_1. \tag{3.12} $$

(x) If $\varphi(x)=\frac{1}{\Gamma(\vartheta)}x^{\vartheta}$, and $\Phi(x)=\exp(\beta x)$,then β-Exponential fractional integral operators with order $\vartheta>0$ will be obtained [41]:

$$ {}^{\vartheta}_{\beta}\mathcal{J}_{c^+}\zeta(\theta)=\frac{\beta}{\Gamma(\vartheta)}\int\limits_{a_1}^{\theta}\left(\exp(\beta\theta)-\exp(\beta\tau)\right)^{\vartheta-1}\exp(\beta\tau)\zeta(\tau)d\tau,\ \theta>a_1, \tag{3.13} $$

$$ {}^{\vartheta}_{\beta}\mathcal{J}_{c^+}\zeta(\theta)=\frac{\beta}{\Gamma(\vartheta)}\int\limits_{a_1}^{\theta}\left(\exp(\beta\theta)-\exp(\beta\tau)\right)^{\vartheta-1}\exp(\beta\tau)\zeta(\tau)d\tau,\ \theta<b_1. \tag{3.14} $$

(xi) If $\varphi(x)=x^{\beta}\ln x$, then left and right-sided logarithmic fractional integrals which were introduced in [41] is obtained:

$$ {}^{\vartheta}_{\Phi}\mathcal{L}_{a_1^+}\zeta(\theta)=\int\limits_{a_1}^{\theta}\frac{\ln(\Phi(\theta)-\Phi(\tau))\Phi'(\tau)\zeta(\tau)}{(\Phi(\theta)-\Phi(\tau))^{1-\vartheta}}d\tau,\quad \theta>a_1, $$

$$ {}^{\vartheta}_{\Phi}\mathcal{L}_{b_1^-}\zeta(\theta)=\int\limits_{\theta}^{b_1}\frac{\ln(\Phi(\tau)-\Phi(\theta))\Phi'(\tau)\zeta(\tau)}{(\Phi(\tau)-\Phi(\theta))^{1-\vartheta}}d\tau,\quad \theta<b_1. $$

(xii) If $\varphi(x)=\frac{1}{\Gamma(\vartheta)}x^{\vartheta}$ and $\Phi(x)=\ln x$, then Hadamard fractional integral operators (1.34) and (1.35) are recaptured.

3.2 A Generalized Hermite–Hadamard inequality

This section is reserved for a generalized version of Hermite–Hadamard inequality. In the upcoming results, the following notations will be used frequently.

$$\begin{aligned}\widetilde{\zeta}(\tau) &= \zeta(\tau)+\zeta(a_1+b_1-\tau),\ \tau\in[a_1,b_1],\\ \Delta_{\varphi,\Phi}(\theta;\omega) &= \int_0^\theta \Im(\varphi,\Phi,\omega;s)ds, \end{aligned}\tag{3.15}$$

where

$$\begin{aligned}\Im(\varphi,\Phi,\omega;s) = &\frac{\varphi\left(\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)\right)\omega\left(sb_1+(1-s)a_1\right)}{\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)}\Phi'\left(sb_1+(1-s)a_1\right)\\ &+\frac{\varphi\left(\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)\right)\omega\left(sa_1+(1-s)b_1\right)}{\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)}\Phi'\left(sa_1+(1-s)b_1\right).\end{aligned}\tag{3.16}$$

Also note that for $\omega=1$, above conventions will take the form

$$\Delta_{\varphi,\Phi}(\theta) = \int_0^\theta \Im(\varphi,\Phi;s)ds, \tag{3.17}$$

where

$$\begin{aligned}\Im(\varphi,\Phi;s) = &\frac{\varphi\left(\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)\right)}{\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)}\Phi'\left(sb_1+(1-s)a_1\right)\\ &+\frac{\varphi\left(\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)\right)}{\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)}\Phi'\left(sa_1+(1-s)b_1\right).\end{aligned}\tag{3.18}$$

Now we have the following generalized Hermite–Hadamard inequality.

Theorem 3.2. *Let $\zeta,\Phi:[a_1,b_1]\to\mathbb{R}$, $0<a_1<b_1$, be the functions such that ζ be positive and $\zeta\in L_1[a_1,b_1]$ and Φ be differentiable and increasing. Also let φ be a positive function such that $\frac{\varphi}{\tau}$ is increasing on $[a_1,\infty)$. If ζ is convex, then following inequality holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right)\Delta_{\varphi,\Phi}(1)\le\frac{1}{2(b_1-a_1)}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{a_1^+}\widetilde{\zeta}(b_1)+{}^{\varphi}_{\Phi}\,\mathcal{G}_{b_1^-}\widetilde{\zeta}(a_1)\right]\le\frac{\zeta(a_1)+\zeta(b_1)}{2}\Delta_{\varphi,\Phi}(1). \tag{3.19}$$

Proof. By the notion of convexity of ζ, we have for all $u, v \in [a_1, b_1]$,

$$\zeta\left(\frac{u+v}{2}\right) \leq \frac{1}{2}\zeta(u) + \frac{1}{2}\zeta(v). \tag{3.20}$$

Fixing $u = sa_1 + (1-s)b_1$ and $v = (1-s)a_1 + sb_1, s \in [0,1]$, we get

$$2\zeta\left(\frac{a_1+b_1}{2}\right) \leq \zeta(sa_1 + (1-s)b_1) + \zeta((1-s)a_1 + sb_1). \tag{3.21}$$

Multiplying both sides of (3.21) by $\Im(\varphi, \Phi; s)$ and integrating over $(0,1)$ with respect to s, we get

$$\begin{aligned}
&2\zeta\left(\frac{a_1+b_1}{2}\right) \int_0^1 \left[\frac{\varphi\left(\Phi\left(sb_1 + (1-s)a_1\right) - \Phi(a_1)\right)}{\Phi\left(sb_1 + (1-s)a_1\right) - \Phi(a_1)} \Phi'\left(sb_1 + (1-s)a_1\right)\right. \\
&+ \left.\frac{\varphi\left(\Phi(b_1) - \Phi\left(sa_1 + (1-s)b_1\right)\right)}{\Phi(b_1) - \Phi\left(sa_1 + (1-s)b_1\right)} \Phi'\left(sa_1 + (1-s)b_1\right)\right] ds \\
&\leq \int_0^1 \left[\frac{\varphi\left(\Phi\left(sb_1 + (1-s)a_1\right) - \Phi(a_1)\right)}{\Phi\left(sb_1 + (1-s)a_1\right) - \Phi(a_1)} \Phi'\left(sb_1 + (1-s)a_1\right)\right. \\
&+ \left.\frac{\varphi\left(\Phi(b_1) - \Phi\left(sa_1 + (1-s)b_1\right)\right)}{\Phi(b_1) - \Phi\left(sa_1 + (1-s)b_1\right)} \Phi'\left(sa_1 + (1-s)b_1\right)\right] \zeta(sa_1 + (1-s)b_1) ds \\
&+ \int_0^1 \left[\frac{\varphi\left(\Phi\left(sb_1 + (1-s)a_1\right) - \Phi(a_1)\right)}{\Phi\left(sb_1 + (1-s)a_1\right) - \Phi(a_1)} \Phi'\left(sb_1 + (1-s)a_1\right)\right. \\
&+ \left.\frac{\varphi\left(\Phi(b_1) - \Phi\left(sa_1 + (1-s)b_1\right)\right)}{\Phi(b_1) - \Phi\left(sa_1 + (1-s)b_1\right)} \Phi'\left(sa_1 + (1-s)b_1\right)\right] \zeta((1-s)a_1 + sb_1) ds.
\end{aligned} \tag{3.22}$$

Now,

$$\begin{aligned}
&\int_0^1 \left[\frac{\varphi\left(\Phi\left(sb_1 + (1-s)a_1\right) - \Phi(a_1)\right)}{\Phi\left(sb_1 + (1-s)a_1\right) - \Phi(a_1)} \Phi'\left(sb_1 + (1-s)a_1\right)\right. \\
&+ \left.\frac{\varphi\left(\Phi(b_1) - \Phi\left(sa_1 + (1-s)b_1\right)\right)}{\Phi(b_1) - \Phi\left(sa_1 + (1-s)b_1\right)} \Phi'\left(sa_1 + (1-s)b_1\right)\right] \zeta((1-s)a_1 + sb_1) ds \\
&= \int_0^1 \frac{\varphi\left(\Phi\left(sb_1 + (1-s)a_1\right) - \Phi(a_1)\right)}{\Phi\left(sb_1 + (1-s)a_1\right) - \Phi(a_1)} \Phi'\left(sb_1 + (1-s)a_1\right) \zeta((1-s)a_1 + sb_1) ds \\
&+ \int_0^1 \frac{\varphi\left(\Phi(b_1) - \Phi\left(\tau a_1 + (1-\tau)b_1\right)\right)}{\Phi(b_1) - \Phi\left(\tau a_1 + (1-\tau)b_1\right)} \Phi'\left(\tau a_1 + (1-\tau)b_1\right) \zeta((1-\tau)a_1 + \tau b_1) d\tau
\end{aligned}$$

On replacing s by $\dfrac{u-a_1}{b_1-a_1}$ and τ by $\dfrac{b_1-v}{b_1-a_1}$, we have

$$=\int_{a_1}^{b_1}\frac{\varphi\left(\Phi(u)-\Phi(a_1)\right)\Phi'(u)}{\Phi(u)-\Phi(a_1)}\frac{\zeta(u)}{b_1-a_1}du+\int_{a_1}^{b_1}\frac{\varphi\left(\Phi(b_1)-\Phi(v)\right)\Phi'(v)}{\Phi(b_1)-\Phi(v)}\frac{\zeta(a_1+b_1-v)}{b_1-a_1}dv. \quad (3.23)$$

Similarly,

$$\begin{aligned}&\int_0^1\left[\frac{\varphi\left(\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)\right)}{\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)}\Phi'\left(sb_1+(1-s)a_1\right)\right.\\&+\left.\frac{\varphi\left(\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)\right)}{\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)}\Phi'\left(sa_1+(1-s)b_1\right)\right]\zeta(sa_1+(1-s)b_1)ds\\&=\int_{a_1}^{b_1}\frac{\varphi\left(\Phi(u)-\Phi(a_1)\right)\Phi'(u)}{\Phi(u)-\Phi(a_1)}\frac{\zeta(a_1+b_1-u)}{b_1-a_1}du+\int_{a_1}^{b_1}\frac{\varphi\left(\Phi(b_1)-\Phi(v)\right)\Phi'(v)}{\Phi(b_1)-\Phi(v)}\frac{\zeta(v)}{b_1-a_1}dv. \quad (3.24)\end{aligned}$$

Using (3.23) and (3.24) in (3.22) and applying the Definition 3.1.2 of generalized integral operator, we get the first inequality of (3.19).

By the application of the convexity of ζ, we have

$$\zeta(sa_1+(1-s)b_1)+\zeta((1-s)a_1+sb_1)\leq\zeta(a_1)+\zeta(b_1). \quad (3.25)$$

Multiplying both sides of (3.25) by $\Im(\varphi,\Phi;s)$ and integrating over $(0,1)$ with respect to s, we get

$$\begin{aligned}&\leq\int_0^1\left[\frac{\varphi\left(\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)\right)}{\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)}\Phi'\left(sb_1+(1-s)a_1\right)\right.\\&+\left.\frac{\varphi\left(\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)\right)}{\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)}\Phi'\left(sa_1+(1-s)b_1\right)\right]\zeta(sa_1+(1-s)b_1)ds\\&+\int_0^1\left[\frac{\varphi\left(\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)\right)}{\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)}\Phi'\left(sb_1+(1-s)a_1\right)\right.\\&+\left.\frac{\varphi\left(\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)\right)}{\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)}\Phi'\left(sa_1+(1-s)b_1\right)\right]\zeta((1-s)a_1+sb_1)ds\\&\leq[\zeta(a_1)+\zeta(b_1)]\int_0^1\left[\frac{\varphi\left(\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)\right)}{\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)}\Phi'\left(sb_1+(1-s)a_1\right)\right.\\&+\left.\frac{\varphi\left(\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)\right)}{\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)}\Phi'\left(sa_1+(1-s)b_1\right)\right]ds. \quad (3.26)\end{aligned}$$

Again by suitable change of variable and by applying Definition 3.1.2 of generalized integral operator, in (3.26), we have the second inequality of (3.19), which then completes the proof. □

Remark 1. *We have found some connections of Theorem 3.2 with existing results.*

(a) If $\varphi(u) = \frac{1}{\Gamma(\vartheta)} u^{\vartheta}, \vartheta > 0$*, then Theorem 3.2 coincide with Theorem* 2.1 *of [81].*

(b) If $\varphi(u) = \frac{1}{k\Gamma_k(\vartheta)} u^{\frac{\vartheta}{k}}, \vartheta, k > 0$ *and* Φ *is an identity function, then Theorem 3.2 reduces to Theorem* 2.1 *of [62].*

(c) If $\varphi(u) = \frac{1}{\Gamma(\vartheta)} u^{\vartheta}, \vartheta > 0$ *and* Φ *is identity function, then Theorem 3.2 gives Theorem* 2 *of [181].*

(d) If $\varphi(u) = u^{\frac{\vartheta}{k}} \mathcal{F}^{\sigma,k}_{\rho,\vartheta}(w(u)^{\rho})$*, then we have Theorem* 1 *of [190].*

3.3 Associated fractional Hadamard inequalities

In this section, we give some new results which can be deduce from Theorem 3.2.

Corollary 3.3.1. *Let* $\Phi : [a_1, b_1] \to \mathbb{R}$ *be a monotone increasing function with continuous derivative* Φ' *on* (a_1, b_1)*. If* ζ *is a convex function on* $[a_1, b_1]$*, then following inequality for integral operators* (3.7) *and* (3.8) *holds:*

$$\begin{aligned}
&\zeta\left(\frac{a_1+b_1}{2}\right) \\
&\leq \frac{1}{4\left[\Phi(b_1)-\Phi(a_1)\right]^{\vartheta} \mathcal{E}^{\delta,\lambda,s}_{\beta,\vartheta,r}\left(w[\Phi(b_1)-\Phi(a_1)]^{\vartheta}\right)} \left[{}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,\lambda,s}_{\beta,r,a_1^+}\widetilde{\zeta}\left(b_1;w\right) + {}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,\lambda,s}_{\beta,r,b_1^-}\widetilde{\zeta}\left(a_1;w\right)\right] \\
&\leq \frac{\zeta(a_1)+\zeta(b_1)}{2}.
\end{aligned} \tag{3.27}$$

Proof. If $\varphi(u) = u^{\vartheta}\mathcal{E}^{\delta,\lambda,s}_{\beta,\vartheta,r}(w(u)^{\beta})$ in inequality (3.19), then simple calculations will leads to the inequality (3.27). □

Corollary 3.3.2. *Let $\Phi : [a_1, b_1] \to \mathbb{R}$ be a monotone increasing function with continuous derivative Φ' on (a_1, b_1). If ζ is a convex function on $[a_1, b_1]$, then following inequality for fractional integral operators* (1.46) *and* (1.47) *holds:*

$$\begin{aligned}
&\zeta\left(\frac{a_1+b_1}{2}\right) \\
&\leq \frac{1}{4\left[\Phi(b_1)-\Phi(a_1)\right]^{\vartheta}\mathcal{E}^{\delta,\lambda,s}_{\beta,\vartheta,r}\left(w[\Phi(b_1)-\Phi(a_1)]^{\vartheta}\right)}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,a_1^+}\widetilde{\zeta}\,(b_1;q^*,w)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,b_1^-}\widetilde{\zeta}\,(a_1;q^*,w)\right] \\
&\leq \frac{\zeta(a_1)+\zeta(b_1)}{2}. \qquad (3.28)
\end{aligned}$$

Proof. If $\varphi(u) = u^{\vartheta}\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}(w(u)^{\beta};q^*)$, then inequality (3.19) reduces to (3.28), which is the required inequality. □

Remark 2. *If Φ is identity function, then we have [89, Theorem 2.1].*

Remark 3. *If $\varphi(\tau) = \tau^{\frac{\vartheta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\vartheta}(w(\tau)^{\rho})$, then we have*

$$\begin{aligned}
&\zeta\left(\frac{a_1+b_1}{2}\right) \\
&\leq \frac{1}{4k\left(\Phi(b_1)-\Phi(a_1)\right)^{\frac{\vartheta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\vartheta+k}[w\left(\Phi(b_1)-\Phi(a_1)\right)^{\rho}]}\left[{}^{\sigma}_{\rho}\mathcal{J}^{k,\psi}_{\vartheta,a_1^+;w}\widetilde{\zeta}\,(b_1)+{}^{\sigma}_{\rho}\,\mathcal{J}^{k,\psi}_{\vartheta,b_1^-;w}\widetilde{\zeta}\,(a_1)\right] \\
&\leq \frac{\zeta(a_1)+\zeta(b_1)}{2}, \qquad (3.29)
\end{aligned}$$

which is proved in [190]

Corollary 3.3.3. *Let $\vartheta, k > 0$ and Φ be a monotone increasing function with continuous derivative Φ' on (a_1, b_1). If ζ is a convex function on $[a_1, b_1]$, then following inequality for integral operators* (1.50) *and* (1.51) *holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right) \leq \frac{\Gamma_k(\vartheta+k)}{4\left(\Phi(b_1)-\Phi(a_1)\right)^{\frac{\vartheta}{k}}}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{k}_{a_1^+}\widetilde{\zeta}\,(b_1)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{k}_{b_1^-}\widetilde{\zeta}\,(a_1)\right] \leq \frac{\zeta(a_1)+\zeta(b_1)}{2}. \qquad (3.30)$$

Proof. If $\varphi(u) = \frac{u^{\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta)}$ in inequality (3.19), then simple calculations will leads to the inequality (3.30). □

Remark 4. *Theorem 3.3.3 coincide to Theorem* 2.1 *of [81] subject to the condition that* $k = 1$.

Corollary 3.3.4. *Let* $\zeta : [a_1, b_1] \to \mathbb{R}$ *with* $0 \leq a_1 < b_1$. *If* ζ *is a convex function on* $[a_1, b_1]$, *then following version of Hadamard inequality for generalized conformable integral* (1.44) *and* (1.45) *holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right) \leq \frac{(\gamma+r)^{\vartheta}\Gamma(\vartheta+1)}{4\left(b_1^{\gamma+r}-a_1^{\gamma+r}\right)^{\vartheta}}\left[{}_r^{\vartheta}\mathcal{J}_{a_1^+}^{\gamma}\widetilde{\zeta}\,(b_1) + {}_r^{\vartheta}\,\mathcal{J}_{b_1^-}^{\gamma}\widetilde{\zeta}\,(a_1)\right] \leq \frac{\zeta(a_1)+\zeta(b_1)}{2}. \tag{3.31}$$

Proof. Choose $\Phi(u) = \frac{u^{\gamma+r}}{\gamma+r}$, where $\gamma \in (0,1)$, $r \in \mathbb{R}$ and $\gamma + r \neq 0$, then inequality (3.30) will leads to the required inequality (3.31). □

Corollary 3.3.5. *Let* $\zeta : [a_1, b_1] \to \mathbb{R}$ *with* $0 \leq a_1 < b_1$. *If* ζ *is a convex function on* $[a_1, b_1]$, *then following version of Hadamard inequality for fractional integral* (1.42) *and* (1.43) *holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right) \leq \frac{(s+1)^{\frac{\vartheta}{k}}\Gamma_k(\vartheta+k)}{4\left(b_1^{s+1}-a_1^{s+1}\right)^{\frac{\vartheta}{k}}}\left[{}_s^{\vartheta}\mathcal{J}_{a_1^+}^{k}\widetilde{\zeta}(b_1) + {}_s^{\vartheta}\,\mathcal{J}_{b_1^-}^{k}\widetilde{\zeta}(a_1)\right] \leq \frac{\zeta(a_1)+\zeta(a_2)}{2}. \tag{3.32}$$

Proof. If we choose $\Phi(u) = \frac{u^{s+1}}{s+1}$, $s \neq -1$, then inequality (3.30) will provide inequality (3.32). □

Corollary 3.3.6. *Let* $\Phi : [a_1, b_1] \to \mathbb{R}$ *be a monotone increasing function with continuous derivative* Φ' *on* (a_1, b_1). *If* ζ *is a convex function on* $[a_1, b_1]$, *then following inequality for fractional integral operators* (3.5) *and* (3.6) *holds:*

$$\begin{aligned}\zeta\left(\frac{a_1+b_1}{2}\right) &\leq \frac{1-\vartheta}{4\left[1-\exp\left(-A\left(\Phi(b_1)-\Phi(a_1)\right)\right)\right]}\left[{}_{\Phi}^{\vartheta}\mathcal{J}_{a_1^+}^{e}\widetilde{\zeta}\,(b_1) + {}_{\Phi}^{\vartheta}\,\mathcal{J}_{b_1^-}^{e}\widetilde{\zeta}\,(a_1)\right] \\ &\leq \frac{\zeta(a_1)+\zeta(b_1)}{2},\end{aligned} \tag{3.33}$$

where $A = \frac{1-\vartheta}{\vartheta}, \vartheta \in (0,1)$.

Proof. If $\varphi(u) = \frac{u}{\vartheta}\exp(-Au)$, where $A = \frac{1-\vartheta}{\vartheta}$, $\vartheta \in (0,1)$ in (3.19), then we have required inequality (3.33). □

Remark 5. *If* $\Phi(u) = u$, *then we have Theorem* 3.1 *of [7].*

3.4 Generalized Fejér–Hermite–Hadamard inequality

Theorem 3.3. *Let $\zeta, \Phi : [a_1, b_1] \to \mathbb{R}$, $0 < a_1 < b_1$, be the functions such that ζ be positive and $\zeta \in L_1[a_1, b_1]$ and Φ be differentiable and increasing. Also let φ be a positive function such that $\frac{\varphi}{\tau}$ is increasing on $[a_1, \infty)$. Let $\omega : [a_1, b_1] \to \mathbb{R}$ be a non-negative and integrable function on $[a_1, b_1]$. If ζ is convex, then the following Fejér–Hermite–Hadamard inequality holds:*

$$\begin{aligned}\zeta\left(\frac{a_1+b_1}{2}\right)\left[{}^{\varphi}_{\Phi}\mathcal{G}_{a_1^+}\omega(b_1) + {}^{\varphi}_{\Phi}\,\mathcal{G}_{b_1^-}\omega(a_1)\right] &\leq \frac{1}{2}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{a_1^+}(\omega\widetilde{\zeta})(b_1) + {}^{\varphi}_{\Phi}\,\mathcal{G}_{b_1^-}(\omega\widetilde{\zeta})(a_1)\right] \\ &\leq \frac{\zeta(a_1)+\zeta(b_1)}{2}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{a_1^+}\omega(b_1) + {}^{\varphi}_{\Phi}\,\mathcal{G}_{b_1^-}\omega(a_1)\right].\end{aligned} \tag{3.34}$$

Proof. By the notion of convexity of ζ, we have for all $\xi_1, \xi_2 \in [a_1, b_1]$,

$$\zeta\left(\frac{\xi_1+\xi_2}{2}\right) \leq \frac{1}{2}\zeta(\xi_1) + \frac{1}{2}\zeta(\xi_2). \tag{3.35}$$

Choosing $\xi_1 = sa_1 + (1-s)b_1$ and $\xi_2 = (1-s)a_1 + sb_1$, $s \in [0,1]$, we have

$$\zeta\left(\frac{a_1+b_1}{2}\right) \leq \frac{1}{2}\zeta(sa_1 + (1-s)b_1) + \frac{1}{2}\zeta((1-s)a_1 + sb_1). \tag{3.36}$$

Multiplying both sides of (3.36) by $\Im(\varphi, \Phi, \omega; s)$ and integrating over $(0,1)$ with respect to s, we get

$$\begin{aligned}\zeta\left(\frac{a_1+b_1}{2}\right)\int_0^1 \Im(\varphi, \Phi, \omega; s)ds &\leq \frac{1}{2}\int_0^1 \Im(\varphi, \Phi, \omega; s)\zeta(sa_1 + (1-s)b_1)ds \\ &+ \frac{1}{2}\int_0^1 \Im(\varphi, \Phi, \omega; s)\zeta(sb_1 + (1-s)a_1)ds.\end{aligned} \tag{3.37}$$

Using the substitutions $s = \frac{u-a_1}{b_1-a_1}$ and $\tau = \frac{b_1-v}{b_1-a_1}$, we have,

$$
\begin{aligned}
&\int_0^1 \Im(\varphi,\Phi,\omega;s)ds\\
&=\int_0^1 \frac{\varphi\left(\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)\right)}{\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)}\Phi'\left(sb_1+(1-s)a_1\right)\omega(sb_1+(1-s)a_1)ds\\
&+\frac{\varphi\left(\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)\right)}{\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)}\Phi'\left(sa_1+(1-s)b_1\right)\omega(sa_1+(1-s)b_1)ds\\
&=\int_0^1 \frac{\varphi\left(\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)\right)}{\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)}\Phi'\left(sb_1+(1-s)a_1\right)\omega(sb_1+(1-s)a_1)ds\\
&+\int_0^1 \frac{\varphi\left(\Phi(b_1)-\Phi\left(\tau a_1+(1-\tau)b_1\right)\right)}{\Phi(b_1)-\Phi\left(\tau a_1+(1-\tau)b_1\right)}\Phi'\left(\tau a_1+(1-\tau)b_1\right)\omega(\tau a_1+(1-\tau)b_1)d\tau\\
&=\int_{a_1}^{b_1} \frac{\varphi\left(\Phi(u)-\Phi(a_1)\right)\Phi'(u)}{\Phi(u)-\Phi(a_1)}\frac{\omega(u)}{b_1-a_1}du+\int_{a_1}^{b_1} \frac{\varphi\left(\Phi(b_1)-\Phi(v)\right)\Phi'(u)}{\Phi(b_1)-\Phi(v)}\frac{\omega(v)}{b_1-a_1}dv. \qquad (3.38)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\int_0^1 \Im(\varphi,\Phi,\omega;s)\zeta((1-s)a_1+sb_1)ds\\
&=\int_0^1 \frac{\varphi\left(\Phi(sb_1+(1-s)a_1)-\Phi(a_1)\right)\Phi'(sb_1+(1-s)a_1)\times\omega(sb_1+(1-s)a_1)\zeta(sb_1+(1-s)a_1)}{\Phi(sb_1+(1-s)a_1)-\Phi(a_1)}ds\\
&\quad+\int_0^1 \frac{\varphi\left(\Phi(b_1)-\Phi(\tau a_1+(1-\tau)b_1)\right)\Phi'(\tau a_1+(1-\tau)b_1)\times\omega(\tau a_1+(1-\tau)b_1)\zeta(\tau b_1+(1-\tau)a_)}{\Phi(b_1)-\Phi(\tau a_1+(1-\tau)b_1)}d\tau\\
&=\int_{a_1}^{b_1} \frac{\varphi\left(\Phi(u)-\Phi(a_1)\right)\Phi'(u)}{\Phi(u)-\Phi(a_1)}\frac{\omega(u)\zeta(u)}{b_1-a_1}du\\
&\quad+\int_{a_1}^{b_1} \frac{\varphi\left(\Phi(b_1)-\Phi(v)\right)\Phi'(v)}{\Phi(b_1)-\Phi(v)}\frac{\omega(u)\zeta(a_1+b_1-v)}{b_1-a_1}dv, \qquad (3.39)
\end{aligned}
$$

and

$$
\begin{aligned}
&\int_0^1 \Im(\varphi,\Phi,\omega;s)\zeta(sa_1+(1-s)b_1)ds\\
&=\int_0^1 \frac{\varphi\left(\Phi(sb_1+(1-s)a_1)-\Phi(a_1)\right)\Phi'(sb_1+(1-s)a_1)\times\omega(sb_1+(1-s)a_1)\zeta(sa_1+(1-s)b_1)}{\Phi(sb_1+(1-s)a_1)-\Phi(a_1)}ds
\end{aligned}
$$

$$
+\int_0^1 \frac{\begin{array}{c}\varphi\left(\Phi(b_1)-\Phi(\tau a_1+(1-\tau)b_1)\right)\Phi'(\tau a_1+(1-\tau)b_1)\\ \times\omega(\tau a_1+(1-\tau)b_1)\zeta(\tau a_1+(1-\tau)b_1)\end{array}}{\Phi(b_1)-\Phi(\tau a_1+(1-\tau)b_1)}d\tau
$$

$$
=\int_{a_1}^{b_1} \frac{\varphi\left(\Phi(u)-\Phi(a_1)\right)\Phi'(u)}{\Phi(u)-\Phi(a_1)}\frac{\omega(u)\zeta(a_1+b_1-u)}{b_1-a_1}du
$$

$$
+\int_{a_1}^{b_1} \frac{\varphi\left(\Phi(b_1)-\Phi(v)\right)\Phi'(v)}{\Phi(b_1)-\Phi(v)}\frac{\omega(u)\zeta(v)}{b_1-a_1}dv. \tag{3.40}
$$

Using equations (3.38)–(3.40) in (3.37) and applying the Definition 3.1.2, we have the first inequality of (3.34). For the second inequality we note by the convexity of ζ, that

$$
\zeta(sa_1+(1-s)b_1)+\zeta((1-s)a_1+sb_1)\leq \zeta(a_1)+\zeta(b_1). \tag{3.41}
$$

Multiplying both sides of (3.39) by $\Im(\varphi,\Phi,\omega;s)$ and integrating over $(0,1)$ with respect to s, we get

$$
\frac{1}{2}\int_0^1 \Im(\varphi,\Phi,\omega;s)\zeta(sa_1+(1-s)b_1)ds+\frac{1}{2}\int_0^1 \Im(\varphi,\Phi,\omega;s)\zeta(sb_1+(1-s)a_1)ds.
$$

$$
\leq \frac{\zeta(a_1)+\zeta(b_1)}{2}\int_0^1 \Im(\varphi,\Phi,\omega;s)ds, \tag{3.42}
$$

Using equations (3.38)–(3.40) in (3.42) and applying the Definition 3.1.2, we have the second inequality of (3.34), which completes the proof. □

Remark 6. *(a) If $\varphi(u)=\frac{1}{\Gamma(\vartheta)}u^{\vartheta}$, then Theorem 3.3 coincide with Theorem* 6 *of [18].*

(b) If $\varphi(u)=\frac{1}{\Gamma(\vartheta)}u^{\vartheta}$ and Φ is identity function, then Theorem 3.3 reduces to Theorem 2.2 *of [76] subject to the condition that ω is symmetric with respect to $\frac{a_1+b_1}{2}$.*

3.5 Fejér–Hermite–Hadamard inequalities for fractional and conformable integrals

In this section, we give some new results which can be deduce from Theorem 3.3.

Corollary 3.5.1. *Let $\Phi : [a_1, b_1] \to \mathbb{R}$ be a monotone increasing function with continuous derivative $\Phi' \in (a_1, b_1)$. Let $\omega : [a_1, b_1] \to \mathbb{R}$ be a non-negative and integrable function on $[a_1, b_1]$. If ζ is a convex function on $[a_1, b_1]$, then the following Fejér–Hermite–Hadamard inequality holds:*

$$\begin{aligned}
&\zeta\left(\frac{a_1+b_1}{2}\right)\left[{}_{\Phi}^{\vartheta}\mathcal{J}_{\beta,r,a_1^+}^{\delta,\lambda,s}\omega\left(b_1;w\right)+{}_{\Phi}^{\vartheta}\,\mathcal{J}_{\beta,r,b_1^-}^{\delta,\lambda,s}\omega\left(a_1;w\right)\right]\\
&\leq \frac{1}{2}\left[{}_{\Phi}^{\vartheta}\mathcal{J}_{\beta,r,a_1^+}^{\delta,\lambda,s}(\omega\widetilde{\zeta})\left(b_1;w\right)+{}_{\Phi}^{\vartheta}\,\mathcal{J}_{\beta,r,b_1^-}^{\delta,\lambda,s}(\omega\widetilde{\zeta})\left(a_1;w\right)\right]\\
&\leq \frac{\zeta(a_1)+\zeta(b_1)}{2}\left[{}_{\Phi}^{\vartheta}\mathcal{J}_{\beta,r,a_1^+}^{\delta,\lambda,s}\omega\left(b_1;w\right)+{}_{\Phi}^{\vartheta}\,\mathcal{J}_{\beta,r,b_1^-}^{\delta,\lambda,s}\omega\left(a_1;w\right)\right]. \qquad (3.43)
\end{aligned}$$

Proof. If $\varphi(u) = u^{\vartheta}\mathcal{E}_{\beta,\vartheta,r}^{\delta,\lambda,s}(w(u)^{\beta})$ in inequality (3.34), then simple calculations will leads to the Inequality (3.43). □

Corollary 3.5.2. *Let $\Phi : [a_1, b_1] \to \mathbb{R}$ be a monotone increasing function with continuous derivative $\Phi' \in (a_1, b_1)$. Let $\omega : [a_1, b_1] \to \mathbb{R}$ be a non-negative and integrable function on $[a_1, b_1]$. If ζ is a convex function on $[a_1, b_1]$, then following Fejér–Hermite–Hadamard inequality for integral operators* (1.46) *and* (1.47) *holds:*

$$\begin{aligned}
&\zeta\left(\frac{a_1+b_1}{2}\right)\left[{}_{\Phi}^{\vartheta}\mathcal{J}_{\beta,\nu,a_1^+}^{\delta,r,s,c}\omega\left(b_1;q^*,w\right)+{}_{\Phi}^{\vartheta}\,\mathcal{J}_{\beta,\nu,b_1^-}^{\delta,r,s,c}\omega\left(a_1;q^*,w\right)\right]\\
&\leq \frac{1}{2}\left[{}_{\Phi}^{\vartheta}\mathcal{J}_{\beta,\nu,a_1^+}^{\delta,r,s,c}(\omega\widetilde{\zeta})\left(b_1;q^*,w\right)+{}_{\Phi}^{\vartheta}\,\mathcal{J}_{\beta,\nu,b_1^-}^{\delta,r,s,c}(\omega\widetilde{\zeta})\left(a_1;q^*,w\right)\right]\\
&\leq \frac{\zeta(a_1)+\zeta(b_1)}{2}\left[{}_{\Phi}^{\vartheta}\mathcal{J}_{\beta,\nu,a_1^+}^{\delta,r,s,c}\omega\left(b_1;q^*,w\right)+{}_{\Phi}^{\vartheta}\,\mathcal{J}_{\beta,\nu,b_1^-}^{\delta,r,s,c}\omega\left(a_1;q^*,w\right)\right]. \qquad (3.44)
\end{aligned}$$

Proof. If $\varphi(u) = u^{\vartheta}\mathcal{E}_{\beta,\vartheta,\nu}^{\delta,r,s,c}(w(u)^{\beta};q^*)$, then inequality (3.34) reduces to the desired inequality (6.2). □

Corollary 3.5.3. *Let $\vartheta, k > 0$ and Φ be a monotone increasing function with continuous derivative Φ' on (a_1, b_1). Let ω be a non-negative integrable function on $[a_1, b_1]$. If ζ is a convex* *function, then following Fejér–Hermite–Hadamard inequality holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right)\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{k}_{a_1^+}\omega\,(b_1)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{k}_{b_1^-}\omega\,(a_1)\right]\leq\frac{1}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{k}_{a_1^+}(\omega\widetilde{\zeta})\,(b_1)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{k}_{b_1^-}(\omega\widetilde{\zeta})\,(a_1)\right]$$
$$\leq\frac{\zeta(a_1)+\zeta(b_1)}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{k}_{a_1^+}\omega\,(b_1)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{k}_{b_1^-}\omega\,(a_1)\right]. \tag{3.45}$$

Proof. If $\varphi(u)=\frac{u^{\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta)}$, then inequality (3.34) produces the required inequality (3.45). □

Corollary 3.5.4. *Let ω be a non-negative integrable function on $[a_1,b_1]$. If ζ is a convex function, then following Fejér–Hermite–Hadamard inequality for generalized conformable integrals* (1.44) *and* (1.45) *holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right)\left[{}^{\vartheta}_{r}\mathcal{J}^{\gamma}_{a_1^+}\omega\,(b_1)+{}^{\vartheta}_{r}\,\mathcal{J}^{\gamma}_{b_1^-}\omega\,(a_1)\right]\leq\frac{1}{2}\left[{}^{\vartheta}_{r}\mathcal{J}^{\gamma}_{a_1^+}(\omega\widetilde{\zeta})\,(b_1)+{}^{\vartheta}_{r}\,\mathcal{J}^{\gamma}_{b_1^-}(\omega\widetilde{\zeta})\,(a_1)\right]$$
$$\leq\frac{\zeta(a_1)+\zeta(b_1)}{2}\left[{}^{\vartheta}_{r}\mathcal{J}^{\gamma}_{a_1^+}\omega\,(b_1)+{}^{\vartheta}_{r}\,\mathcal{J}^{\gamma}_{b_1^-}\omega\,(a_1)\right]. \tag{3.46}$$

Proof. Choose $\Phi(u)=\frac{u^{\gamma+r}}{\gamma+r}$, where $\gamma\in(0,1)$, $r\in\mathbb{R}$ and $\gamma+r\neq0$, then inequality (3.45) will reduced to the required inequality (3.46). □

Corollary 3.5.5. *Let ω be a non-negative integrable function on $[a_1,b_1]$. If ζ is a convex function, then following Fejér–Hermite–Hadamard inequality for (k,s)-fractional integrals* (1.42) *and* (1.43) *holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right)\left[{}^{\vartheta}_{s}\mathcal{J}^{k}_{a_1^+}\omega(b_1)+{}^{\vartheta}_{s}\,\mathcal{J}^{k}_{b_1^-}\omega(a_1)\right]\leq\frac{1}{2}\left[{}^{\vartheta}_{s}\mathcal{J}^{k}_{a_1^+}(\omega\widetilde{\zeta})(b_1)+{}^{\vartheta}_{s}\,\mathcal{J}^{k}_{b_1^-}(\omega\widetilde{\zeta})(a_1)\right]$$
$$\leq\frac{\zeta(a_1)+\zeta(b_1)}{2}\left[{}^{\vartheta}_{s}\mathcal{J}^{k}_{a_1^+}\omega(b_1)+{}^{\vartheta}_{s}\,\mathcal{J}^{k}_{b_1^-}\omega(a_1)\right]. \tag{3.47}$$

Proof. If we choose $\Phi(u)=\frac{u^{s+1}}{s+1}$, $s\neq-1$, then inequality (3.45) will provide inequality (3.47). □

Corollary 3.5.6. *Let $\vartheta,k>0$ and Φ be a monotone increasing function with continuous derivative Φ' on (a_1,b_1). Let ω be a non-negative integrable function on $[a_1,b_1]$. If ζ is a convex* *function, then following Fejér–Hermite–Hadamard inequality for the generalized k-fractional integral holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right)\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{k,\rho}_{\sigma,a_1^+;w}\omega\left(b_1\right)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{k,\rho}_{\sigma,b_1^-;w}\omega\left(a_1\right)\right]\leq\frac{1}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{k,\rho}_{\sigma,a_1^+;w}(\omega\widetilde{\zeta})\left(b_1\right)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{k,\rho}_{\sigma,b_1^-;w}(\omega\widetilde{\zeta})\left(a_1\right)\right]$$
$$\leq\frac{\zeta(a_1)+\zeta(b_1)}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{k,\rho}_{\sigma,a_1^+;w}\omega\left(b_1\right)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{k,\rho}_{\sigma,b_1^-;w}\omega\left(a_1\right)\right]. \tag{3.48}$$

Proof. Choose $\varphi(u)=u^{\frac{\vartheta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\vartheta}(w(u)^{\rho})$, then inequality (3.34) leads to the required inequality. □

Corollary 3.5.7. *Let $\vartheta\in(0,1), A=\frac{1-\vartheta}{\vartheta}$ and Φ be a monotone increasing function with continuous derivative Φ' on (a_1,b_1). Let ω be a non-negative integrable function on $[a_1,b_1]$. If ζ is a convex function, then following Fejér–Hermite–Hadamard inequality for the generalized exponential fractional integral* (3.5) *holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right)\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{a_1^+}\omega(b_1)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{e}_{b_1^-}\omega(a_1)\right]\leq\frac{1}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{a_1^+}(\omega\widetilde{\zeta})(b_1)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{e}_{b_1^-}(\omega\widetilde{\zeta})(a_1)\right]$$
$$\leq\frac{\zeta(a_1)+\zeta(b_1)}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{a_1^+}\omega(b_1)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{e}_{b_1^-}\omega(a_1)\right]. \tag{3.49}$$

Proof. If $\varphi(u)=\frac{u}{\vartheta}\exp(-Au)$, where $A=\frac{1-\vartheta}{\vartheta}$, $\vartheta\in(0,1)$ in (3.34), then we have required inequality (3.49). □

Chapter 4

Bounds associated to Hadamard inequality via generalized integral operators and applications for conformable and fractional integrals

Amid the previous decades, many investigation have been centered around the error bounds of different versions of Hermite–Hadamard inequality. For these works, the inspiration comes from the applications of error bounds. See for the reference ([23, 43, 45, 57, 75, 81, 90, 103, 111, 115–117, 124, 125, 170, 171, 181]) etc. and the references therein.

This examination brings into spotlight an extremely generalized error assessment detailing for the distinction of first and second term of the generalized version of Hermite–Hadamard disparity (3.19). An examination with late published outcomes has been done and found in complete arrangement. This chapter follows the clear configuration.

Section 4.1 manages the generalized identity and its uncommon cases. The error bound for the generalized Hermite–Hadamard inequality (3.19) is set up. A few associations are likewise given with past discoveries. Section 4.2 annunciating the critical discoveries in regards to the error estimation for different uncommon kinds of Hermite–Hadamard inequality through fractional and conformable integrals.

4.1 Generalized Upper bound error assessment inequality

In this section we utilize the following conventions oftentimes:

$$\widetilde{\zeta}(x) \quad := \quad \zeta(x)+\zeta(a_1+b_1-x),$$

$$\Delta_1(t)=\int_0^t \frac{\varphi\left(\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)\right)}{\Phi\left(sb_1+(1-s)a_1\right)-\Phi(a_1)}\Phi'\left(sb_1+(1-s)a_1\right)ds,$$

and

$$\Delta_2(t)=\int_0^t \frac{\varphi\left(\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)\right)}{\Phi(b_1)-\Phi\left(sa_1+(1-s)b_1\right)}\Phi'\left(sa_1+(1-s)b_1\right)ds.$$

Then $\Delta_{\varphi,\Phi}(t)=\Delta_1(t)+\Delta_2(t)$, where $\Delta_{\varphi,\Phi}(t)$ is defined by (3.17).

The following lemma is useful to establish the bounds of the Hadamard inequality for integral operators (3.3) and (3.4).

Lemma 4.1.1. *Let $\zeta,\Phi:[a_1,b_1]\to\mathbb{R}$, $0<a_1<b_1$, be the functions such that ζ be positive and $\zeta\in L_1[a_1,b_1]$ and Φ be differentiable and increasing. Also let φ be a positive function such that $\frac{\varphi}{\tau}$ is increasing on $[a_1,\infty)$. Then following inequality holds. Let $\zeta:[a_1,b_1]\to\mathbb{R}$ be a differentiable mapping on (a_1,b_1) with $a_1<b_1$. If $\zeta'\in L_1[a_1,b_1]$, then the following equalities for integral operators* (3.3) *and* (3.4) *hold:*

$$\begin{aligned}
&\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{2(b_1-a_1)\Delta_{\varphi,\Phi}(1)}\left[{}_{\Phi}^{\varphi}\mathcal{G}_{a_1^+}\widetilde{\zeta}(b_1)+{}_{\Phi}^{\varphi}\,\mathcal{G}_{b_1^-}\widetilde{\zeta}(a_1)\right]\\
=\;&\frac{b_1-a_1}{2\Delta_{\varphi,\Phi}(1)}\int_0^1 \Upsilon_1(\tau)\zeta'(\tau a_1-t)b_1)dt\\
=\;&\frac{b_1-a_1}{2\Delta_{\varphi,\Phi}(1)}\int_0^1 \Upsilon_2(\tau)\widetilde{\zeta}'(\tau b_1+(1-\tau)a_1)d\tau,
\end{aligned}$$

where

$$\Upsilon_1(t) = \Delta_1(1-\tau) - \Delta_1(\tau) + \Delta_2(1-\tau) - \Delta_2(\tau),$$

$$\Upsilon_2(\tau) = \Delta_1(\tau) + \Delta_2(\tau) = \Delta_{\varphi,\Phi}(\tau).$$

Proof. It is easy to see that,

$$\begin{aligned}
&\int_0^1 (\Delta_1(1-\tau) - \Delta_1(\tau))\, \zeta'(\tau a_1 + (1-\tau)b_1)d\tau \\
= &\int_0^1 \Delta_1(1-u)\zeta'(ua_1 + (1-u)b_1)du - \int_0^1 \Delta_1(\tau)\zeta'(\tau a_1 + (1-\tau)b_1)d\tau \\
= &\int_0^1 \Delta_1(\tau)\zeta'(\tau b_1 + (1-\tau)b_1)d\tau - \int_0^1 \Delta_1(\tau)\zeta'(\tau a_1 + (1-\tau)b_1)d\tau \\
= &\int_0^1 \Delta_1(\tau)\left[\zeta'(\tau b_1 + (1-\tau)b_1) - \zeta'(\tau a_1 + (1-\tau)b_1)\right] d\tau. \qquad (4.1)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 (\Delta_2(1-\tau) - \Delta_2(\tau))\, \zeta'(\tau a_1 + (1-\tau)b_1)d\tau \\
= &\int_0^1 \Delta_2(\tau)\left[\zeta'(\tau b_1 + (1-\tau)b_1) - \zeta'(\tau a_1 + (1-\tau)b_1)\right] d\tau. \qquad (4.2)
\end{aligned}$$

As

$$\begin{aligned}
\int_0^1 \Upsilon_2(\tau)\widetilde{\zeta}'(tb_1 + (1-t)a_1)dt &= \int_0^1 \Delta_1(\tau)\widetilde{\zeta}'(\tau b_1 + (1-\tau)a_1)d\tau \\
&+ \int_0^1 \Delta_2(\tau)\widetilde{\zeta}'(\tau b_1 + (1-\tau)a_1)dt. \qquad (4.3)
\end{aligned}$$

Let

$$I_{a_1} = \int_0^1 \Delta_1(\tau)\widetilde{\zeta}'(\tau b_1 + (1-\tau)a_1)d\tau \qquad (4.4)$$

and

$$I_{b_1} = \int_0^1 \Delta_2(\tau)\widetilde{\zeta}'(\tau b_1 + (1-\tau)a_1)d\tau. \tag{4.5}$$

Then one can have by integration by parts,

$$\begin{aligned}
&(b_1 - a_1)I_{a_1} \\
&= \Delta_1(1)\widetilde{\zeta}(b_1) \\
&- \int_0^1 \frac{\varphi\left(\Phi\left(\tau b_1 + (1-\tau)a_1\right) - \Phi(a_1)\right)}{\Phi\left(\tau b_1 + (1-\tau)a_1\right) - \Phi(a_1)} \Phi'\left(\tau b_1 + (1-\tau)a_1\right)\widetilde{\zeta}(\tau b_1 + (1-\tau)a_1)d\tau.
\end{aligned}$$

By taking $\tau = \frac{x-a_1}{b_1-a_1}$, we have

$$(b_1 - a_1)I_{a_1} = \Delta_1(1)\widetilde{\zeta}(b_1) - \frac{1}{b_1 - a_1}\int_{a_1}^{b_1} \frac{\varphi\left(\Phi\left(x\right) - \Phi(a_1)\right)}{\Phi\left(x\right) - \Phi(a_1)} \Phi'\left(x\right)\widetilde{\zeta}(x)dx.$$

By applying (3.3) and (3.4) of generalized integral operators, we have

$$(b_1 - a_1)I_{a_1} = \Delta_1(1)[\zeta(a_1) + \zeta(b_1)] - \frac{1}{(b_1 - a_1)} {}_{\Phi}^{\varphi}\mathcal{G}_{b_1^-}\widetilde{\zeta}(a_1). \tag{4.6}$$

Similarly,

$$(b_1 - a_1)I_{b_1} = \Delta_1(1)[\zeta(a_1) + \zeta(b_1)] - \frac{1}{(b_1 - a_1)} {}_{\Phi}^{\varphi}\mathcal{G}_{a_1^+}\widetilde{\zeta}(b_1). \tag{4.7}$$

From (4.1), (4.2), (4.3), (4.6) and (4.7), the required equalities can be acheived. □

Remark 4.1. The aforementioned lemma holds for all kinds of integral operators comprises in Remark 19. In particular one can obtain [43, Lemma 2.1], [81, Lemma 2.4], [62, Lemma 2.3], [7, Theorem 4.1], [172, Lemma 5], [181, Lemma 2] and [190, Lemma 1] etc. Furthermore, some new equalities can be obtain for the aforementioned fractional integral operators by using appropriate settings of φ and Φ as given in Remark 3.1.

Theorem 4.1.1. *Let $\Phi : [a_1, b_1] \to \mathbb{R}$ be a positive monotone increasing function on $(a_1, b_1]$, having continuous derivatives Φ' on (a_1, b_1). Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a differentiable mapping on (a_1, b_1). If $|\zeta'|$ is convex on $[a_1, b_1]$, then the following inequality for generalized fractional integrals hold:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)\Delta_{\varphi,\Phi}(1)}\left[{}_{\Phi}^{\varphi}\mathcal{G}_{a_1^+}\widetilde{\zeta}(b_1) + {}_{\Phi}^{\varphi}\,\mathcal{G}_{b_1^-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{b_1-a_1}{\Delta_{\varphi,\Phi}(1)}\left(\frac{|\zeta'(a_1)|+|\zeta'(b_1)|}{2}\right)\int_0^1 \tau\,|\Upsilon_1(\tau)|\,d\tau, \tag{4.8}$$

where $\Upsilon_1(\tau)$ is same as defined in Lemma 4.1.1 and $\Delta_{\varphi,\Phi}(t)$ is defined in (3.17).

Proof. By Lemma 4.1.1, and property of modulus, we have

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)\Delta_{\varphi,\Phi}(1)}\left[{}_{\Phi}^{\varphi}\mathcal{G}_{a_1^+}\widetilde{\zeta}(b_1) + {}_{\Phi}^{\varphi}\,\mathcal{G}_{b_1^-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{b_1-a_1}{2\Delta_{\varphi,\Phi}(1)}\int_0^1 |\Upsilon_1(\tau)|\,|\zeta'(\tau a_1 + (1-\tau)b_1)|\,d\tau.$$

By convexity of $|\zeta'|$, we get

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)\Delta_{\varphi,\Phi}(1)}\left[{}_{\Phi}^{\varphi}\mathcal{G}_{a_1^+}\widetilde{\zeta}(b_1) + {}_{\Phi}^{\varphi}\,\mathcal{G}_{b_1^-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{b_1-a_1}{2\Delta_{\varphi,\Phi}(1)}\left[|\zeta'(a_1)|\int_0^1 \tau\,|\Upsilon_1(\tau)|\,d\tau + |\zeta'(b_1)|\int_0^1 (1-\tau)\,|\Upsilon_1(\tau)|\,d\tau\right]$$
$$= \frac{b_1-a_1}{2\Delta_{\varphi,\Phi}(1)}\left[|\zeta'(a_1)|\int_0^1 \tau\,|\Upsilon_1(\tau)|\,d\tau + |\zeta'(b_1)|\int_0^1 \tau\,|\Upsilon_1(1-\tau)|\,d\tau\right].$$

Note that

$$\Upsilon_1(t) = \Delta_1(1-\tau) - \Delta_1(\tau) + \Delta_2(1-\tau) - \Delta_2(\tau).$$

So,

$$\Upsilon_1(1-\tau) \;=\; -\Upsilon_1(\tau).$$

Using value of $\Upsilon_1(1-\tau)$ in above inequality, we get required inequality (4.8). □

Remark 4.2. The aforementioned inequality gives error bounds of Hadamard inequalities of all kinds of integral operators comprises in Remark 3.1. In particular by using suitable settings of φ and Φ as given in Remark 3.1, one can obtain [43, Theorem 2.2], [81, Theorem 2.5], [62, Theorem 2.4], [7, Theorem 4.1], [172, Theorem 6], [172, Corollary 5], [181, Theorem 3] and [190, Theorem 2].

4.2 Error bounds associated to Hadamard inequalities via conformable and fractional integrals

In this section we construct error bounds of the Hadamard inequalities for various kinds of fractional and conformable integral operators.

Theorem 4.2.1. *Let $\Phi : [a_1, b_1] \to \mathbb{R}$ be a positive monotone increasing function on $(a_1, b_1]$, having continuous derivatives $\Phi'(x)$ on (a_1, b_1). Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a differentiable mapping on (c, d). If $|\zeta'|$ is convex on $[a_1, b_1]$, then the following inequality for operators* (1.50) *and* (1.51) *holds:*

$$\begin{aligned}&\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{\Gamma_k(\vartheta+k)}{4\left[\Phi(b_1)-\Phi(a_1)\right]^{\frac{\beta}{k}}}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{k}_{a_1^+}\widetilde{\zeta}(b_1) + {}^{\vartheta}_{\Phi}\mathcal{J}^{k}_{b_1^-}\widetilde{\zeta}(a_1)\right]\right| \\ &\leq \frac{{}^{\vartheta}_{k}A^{\Phi}\left(a_1, b_1\right)}{4\left[\Phi(b_1)-\Phi(a_1)\right]^{\frac{\vartheta}{k}}(b_1-a_1)}\left(|\zeta'(a_1)|+|\zeta'(b_1)|\right),\end{aligned} \tag{4.9}$$

where,

$${}^{\vartheta}_{k}A^{\Phi}\left(a_1, b_1\right) = {}^{\vartheta}_{k}\chi^{\Phi}(b_1, b_1) + {}^{\vartheta}_{k}\chi^{\Phi}(a_1, b_1) - {}^{\vartheta}_{k}\chi^{\Phi}(b_1, a_1) - {}^{\vartheta}_{k}\chi^{\Phi}(a_1, a_1), \tag{4.10}$$

and

$${}^{\vartheta}_{k}\chi^{\Phi}(x, y) := \int_{a_1}^{\frac{a_1+b_1}{2}} |x-\tau|\,|\Phi(y)-\Phi(\tau)|^{\frac{\vartheta}{k}}\,dt - \int_{\frac{a_1+b_1}{2}}^{b_1} |x-\tau|\,|\Phi(y)-\Phi(\tau)|^{\frac{\vartheta}{k}}\,dt, \tag{4.11}$$

for all $x, y \in [c, d]$.

Proof. Let us define the function φ by $\varphi(\tau) = \frac{\tau^{\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta)}$. Then we have

$$\begin{aligned} \Upsilon_1(\tau) &= \frac{1}{(b_1 - a_1)\Gamma_k(\vartheta + k)} \\ &\quad \times \Big[(\Phi(\tau a_1 + (1-\tau)b_1) - \Phi(a_1))^{\frac{\vartheta}{k}} - (\Phi(\tau a_1 + (1-\tau)b_1) - \Phi(a_1))^{\frac{\vartheta}{k}} \\ &\quad + (\Phi(\tau a_1 + (1-\tau)b_1) - \Phi(a_1))^{\frac{\vartheta}{k}} - (\Phi(\tau a_1 + (1-\tau)b_1) - \Phi(a_1))^{\frac{\vartheta}{k}} \Big] \end{aligned}$$

and

$$\Delta_1(\tau) + \Delta_1(\tau) = \Delta_{\varphi,\Phi}(1) = \frac{2}{(b_1 - a_1)\Gamma_k(\vartheta + k)} [(\Phi(b_1) - \Phi(a_1))^{\frac{\vartheta}{k}}]. \tag{4.12}$$

Also by change of variables we have

$$\int_0^1 \tau \left|\Upsilon_1(\tau)\right| d\tau = \frac{1}{(b_1 - a_1)^3 \Gamma_k(\vartheta + k)} \int_{a_1}^{b_1} (b_1 - u) \left|\psi(u)\right| du, \tag{4.13}$$

where

$$\begin{aligned} \psi(u) &= (\Phi(u) - \Phi(a_1))^{\frac{\vartheta}{k}} - (\Phi(a_1 + b_1 - u) - \Phi(a_1))^{\frac{\vartheta}{k}} - (\Phi(b_1) - \Phi(a_1 + b_1 - u))^{\frac{\vartheta}{k}} \\ &\quad - (\Phi(b_1) - \Phi(u))^{\frac{\vartheta}{k}}. \end{aligned}$$

Observe that ψ is a non-decreasing function on $[a_1, b_1]$. We have indeed,

$$\psi(a_1) = 2\left(\Phi(a_1)\right)^{\vartheta} - 2\left(\Phi(b_1)\right)^{\vartheta} < 0,$$

$$\psi(\frac{a_1 + b_1}{2}) = 0$$

and

$$\psi(b_1) = 2\left(\Phi(b_1)\right)^{\vartheta} - 2\left(\Phi(a_1)\right)^{\vartheta} > 0.$$

Hence we have,

$$\int_{a_1}^{b_1} (b_1 - u) |\psi(u)| \, du = I_1 + I_2 + I_3 + I_4, \tag{4.14}$$

where

$$I_1 = \int_{a_1}^{\frac{a_1+b_1}{2}} (b_1 - u)[\Phi(a_1 + b_1 - u) - \Phi(a_1)]^{\frac{\vartheta}{k}} du - \int_{\frac{a_1+b_1}{2}}^{b_1} (b_1 - u)[\Phi(a_1 + b_1 - u) - \Phi(a_1)]^{\frac{\vartheta}{k}} du,$$

$$I_2 = \int_{a_1}^{\frac{a_1+b_1}{2}} (b_1 - u)[\Phi(b_1) - \Phi(u)]^{\frac{\vartheta}{k}} du - \int_{\frac{a_1+b_1}{2}}^{b_1} (b_1 - u)[\Phi(b_1) - \Phi(u)]^{\frac{\vartheta}{k}} du,$$

$$I_3 = -\int_{a_1}^{\frac{a_1+b_1}{2}} (b_1 - u)[\Phi(b_1) - \Phi(a_1 + b_1 - u)]^{\frac{\vartheta}{k}} du + \int_{\frac{a_1+b_1}{2}}^{b_1} (b_1 - u)[\Phi(b_1) - \Phi(a_1 + b_1 - u)]^{\frac{\vartheta}{k}} du,$$

$$I_4 = -\int_{a_1}^{\frac{a_1+b_1}{2}} (b_1 - u)[\Phi(u) - \Phi(a_1)]^{\frac{\vartheta}{k}} du + \int_{\frac{a_1+b_1}{2}}^{b_1} (b_1 - u)[\Phi(u) - \Phi(a_1)]^{\frac{\vartheta}{k}}.$$

By (4.11),

$$I_2 =^{\vartheta}_{k} \chi^{\Phi}(b_1, b_1), I_4 = -^{\vartheta}_{k}\chi^{\Phi}(b_1, a_1) \tag{4.15}$$

and by suitable change of variables,

$$I_1 = -^{\vartheta}_{k}\chi^{\Phi}(a_1, a_1), I_3 =^{\vartheta}_{k} \chi^{\Phi}(a_1, b_1). \tag{4.16}$$

Using (4.10), (4.14), (4.15) and (4.16) in (4.13), one have

$$\int_0^1 \tau |\Upsilon_1(\tau)| \, d\tau = \frac{^{\vartheta}_{k}A^{\Phi}(a_1, b_1)}{(b_1 - a_1)^3 \Gamma_k(\vartheta + k)}. \tag{4.17}$$

Thus inequality (4.8) along with (6.13) and (4.17) reduced to the required inequality (4.9). □

Corollary 4.3. *Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a differentiable mapping on (a_1, b_1). If $|\zeta'|$ is convex on $[a_1, b_1]$, then the following inequality for operators* (1.44) *and* (1.45) *hold:*

$$\left| \frac{\zeta(a_1) + \zeta(b_1)}{2} - \frac{(r+\gamma)^{\vartheta}\Gamma(\vartheta+1)}{4\left[b_1^{r+\gamma} - a_1^{r+\gamma}\right]^{\beta}} \left[^{\vartheta}_{r}\mathcal{J}^{\gamma}_{a_1^+}\widetilde{\zeta}(b_1) +^{\vartheta}_{r} \mathcal{J}^{\gamma}_{b_1^-}\widetilde{\zeta}(a_1)\right] \right|$$

$$\leq \frac{{}_{r}^{\vartheta}B^{\gamma}\left(a_{1},b_{1}\right)}{4\left[b_{1}^{r+\gamma}-a_{1}^{r+\gamma}\right]^{\vartheta}(d-c)}(|u'(c)|+|u'(d)|), \tag{4.18}$$

where

$${}_{r}^{\vartheta}B^{\gamma}(a_{1},b_{1}) ={}_{r}^{\vartheta} \varrho^{\gamma}(b_{1},b_{1}) +{}_{r}^{\vartheta} \varrho^{\gamma}(a_{1},b_{1}) -{}_{r}^{\vartheta} \varrho^{\gamma}(b_{1},a_{1}) -{}_{r}^{\vartheta} \varrho^{\gamma}(a_{1},a_{1})$$

and

$${}_{r}^{\vartheta}\varrho^{\gamma}(x,y) := \int_{a_{1}}^{\frac{c+d}{2}} |x-\tau| \left|y^{r+\gamma}-\tau^{r+\gamma}\right|^{\gamma} d\tau - \int_{\frac{a_{1}+b_{1}}{2}}^{b_{1}} |x-\tau| \left|y^{r+\gamma}-\tau^{r+\gamma}\right|^{\vartheta} d\tau,$$

for all $x, y \in [a_1, b_1]$.

Proof. Considering $\Phi(\tau) = \frac{\tau^{r+\gamma}}{r+\gamma}$, $r+\gamma \neq 0, \gamma \in (0,1)$ and $k = 1$, in inequality (4.9), one gets the required result. □

Corollary 4.4. *Let* $\zeta : [a_1, b_1] \to \mathbb{R}$ *be a differentiable mapping on* (a_1, b_1). *If* $|\zeta'|$ *is convex on* $[a_1, b_1]$, *then the following inequality for operators* (1.42) *and* (1.43) *hold:*

$$\left|\frac{\zeta(a_{1})+\zeta(b_{1})}{2} - \frac{(1+s)^{\frac{\vartheta}{k}}\Gamma(\vartheta+1)}{4\left[b_{1}^{1+s}-a_{1}^{1+s}\right]^{\beta}}\left[{}_{s}^{\vartheta}\mathcal{J}_{a_{1}^{+}}^{k}\widetilde{\zeta}(b_{1}) +{}_{s}^{\vartheta} \mathcal{J}_{b_{1}^{-}}^{k}\widetilde{\zeta}(a_{1})\right]\right|$$
$$\leq \frac{{}_{s}^{\vartheta}C^{k}\left(a_{1},b_{1}\right)}{4\left[b_{1}^{1+s}-a_{1}^{1+s}\right]^{\frac{\beta}{k}}(b_{1}-a_{1})}(|\zeta'(a_{1})|+|\zeta'(b_{1})|), \tag{4.19}$$

where

$${}_{s}^{\vartheta}C^{k}(a_{1},b_{1}) ={}_{s}^{\vartheta} \omega^{k}(b_{1},b_{1}) +{}_{s}^{\vartheta} \omega^{k}(a_{1},b_{1}) -{}_{s}^{\vartheta} \omega^{k}(b_{1},a_{1}) -{}_{s}^{\vartheta} \omega^{k}(a_{1},a_{1})$$

and

$${}_{s}^{\vartheta}\omega^{k}(x,y) := \int_{c}^{\frac{c+d}{2}} |x-\tau| \left|y^{1+s}-\tau^{1+s}\right|^{\frac{\vartheta}{k}} d\tau - \int_{\frac{a_{1}+b_{1}}{2}}^{b_{1}} |x-\tau| \left|y^{1+s}-\tau^{1+s}\right|^{\frac{\vartheta}{k}} d\tau,$$

for all $x, y \in [a_1, b_1]$.

Proof. By using $\Phi(\tau) = \frac{\tau^{1+s}}{1+s}$, $s \in \mathbb{R} - \{-1\}$, in inequality (4.9), one gets the required inequality.

□

Corollary 4.5. *Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a differentiable mapping on (a_1, b_1). If $|\zeta'|$ is convex on $[a_1, b_1]$, then the following inequality for β-exponential fractional integrals (3.13) and (3.14) of order ϑ hold:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{\Gamma(\vartheta+1)}{4\left[\exp(\beta b_1)-\exp(\beta a_1)\right]^{\vartheta}}\left[{}_{\beta}^{\vartheta}\mathcal{J}_{a_1^+}\widetilde{\zeta}(b_1) + {}_{\beta}^{\vartheta}\mathcal{J}_{b_1^-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{{}_{\vartheta}^{\beta}N\left(a_1, b_1\right)}{4\left[\exp(\beta b_1)-\exp(\beta a_1)\right]^{\vartheta}(b_1-a_1)}(|\zeta'(a_1)| + |\zeta'(b_1)|), \tag{4.20}$$

where

$${}_{\vartheta}^{\beta}N(a_1, b_1) = {}_{\vartheta}^{\beta}\Lambda(b_1, b_1) + {}_{\vartheta}^{\beta}\Lambda(a_1, b_1) - {}_{\vartheta}^{\beta}\Lambda(b_1, a_1) - {}_{\vartheta}^{\beta}\Lambda(a_1, a_1)$$

and

$${}_{\vartheta}^{\beta}\Lambda(x, y) := \int_{a_1}^{\frac{a_1+b_1}{2}} |x-\tau|\left|\exp(\beta y)-\exp(\beta\tau)\right|^{\vartheta} d\tau - \int_{\frac{a_1+b_1}{2}}^{b_1} |x-\tau|\left|\exp(\beta y)-\exp(\beta\tau)\right|^{\vartheta} d\tau,$$

for all $x, y \in [a_1, b_1]$.

Proof. Using $\Phi(\tau) = \exp(\beta\tau)$, $\beta > 0, k = 1$, in inequality (4.9), one gets the required inequality.

□

Corollary 4.6. *Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a differentiable mapping on (a_1, b_1). If $|u'|$ is convex on $[a_1, b_1]$, then the following inequality for operators (3.11) and (3.12) holds:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{(a_1 b_1)^{\beta}\Gamma(\vartheta+1)}{4(b_1-a_1)^{\vartheta}}\left[{}^{\vartheta}H_{a_1^+}\widetilde{\zeta}(b_1) + {}^{\vartheta}H_{b_1^-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{{}^{\vartheta}L\left(a_1, b_1\right)}{4(b_1-a_1)^{\vartheta+1}}(|\zeta'(a_1)| + |\zeta'(b_1)|), \tag{4.21}$$

where

$${}_{\nu}^{\vartheta}L(a_1, b_1) = a_1^{\vartheta}\left[{}^{\vartheta}\Psi(b_1, b_1) + {}^{\vartheta}\Psi(a_1, b_1)\right] - b_1^{\vartheta}\left[{}^{\beta}\Psi(a_1, a_1) - {}^{\vartheta}\Psi(b_1, a_1)\right]$$

and

$$^{\vartheta}\Psi(x,y) := \int\limits_{a_1}^{\frac{a_1+b_1}{2}} |x-\tau| \frac{|y-\tau|^{\vartheta}}{\tau^{\vartheta}} d\tau - \int\limits_{\frac{a_1+b_1}{2}}^{b_1} |x-\tau| \frac{|y-\tau|^{\vartheta}}{\tau^{\vartheta}} d\tau,$$

for all $x, y \in [a_1, b_1]$.

Proof. Inequality (4.9), reduced to the required inequality subject to the condition that $\Phi(\tau) = -\tau^{-1}$, $t > 0, k = 1$. □

Corollary 4.7. *Let* $\zeta : [a_1, b_1] \to \mathbb{R}$ *be a differentiable mapping on* (a_1, b_1). *If* $|\zeta'|$ *is convex on* $[a_1, b_1]$, *then the following inequality for Riemann integrals hold:*

$$\begin{aligned} &\left| \frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(\Phi(b_1)-\Phi(a_1))} \int\limits_{a_1}^{b_1} \Phi'(\tau)\widetilde{\zeta}(\tau)d\tau \right| \\ &\leq \frac{1}{(b_1-a_1)(\Phi(b_1)-\Phi(a_1))} \left(\frac{|\zeta'(a_1)|+|\zeta'(b_1)|}{2} \right) \left[\Theta^{a_1,\Phi}(x) - \Theta^{b_1,\Phi}(x) \right], \end{aligned} \tag{4.22}$$

where

$$\Theta^{y,\Phi}(x) = \int\limits_{a_1}^{\frac{a_1+b_1}{2}} |y-x| \, \Phi(x)dx - \int\limits_{\frac{a_1+b_1}{2}}^{b_1} |y-x| \, \Phi(x)dx.$$

Proof. By taking φ as identity function in inequality (4.8) and using the same lines as adopted in Theorem (4.2.1), we come to the desired inequality. □

Theorem 4.2.2. *Let* $\Phi : [a_1, b_1] \to \mathbb{R}$ *be a positive monotone increasing function on* $(a_1, b_1]$, *having continuous derivatives* $\Phi'(x)$ *on* (a_1, b_1). *Let* $\zeta : [a_1, b_1] \to \mathbb{R}$ *be a differentiable mapping on* (a_1, b_1). *If* $|\zeta'|$ *is convex on* $[a_1, b_1]$, *then the following inequality for operators 3.5 and 3.6 holds:*

$$\left| \frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1-\vartheta}{4(1-\exp(-B))} \left[{}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{a_1^+}\widetilde{\zeta}(b_1) + {}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{b_1^-}\widetilde{\zeta}(a_1) \right] \right| \leq \frac{1}{4(1-\exp(-B))} \frac{N(a_1,b_1)}{(b_1-a_1)}, \tag{4.23}$$

where

$$N(a_1,b_1)=\xi(b_1,a_1)-\xi(a_1,b_1)+\xi(a_1,a_1)-\xi(b_1,b_1),$$

$$\begin{aligned}\xi(x,y) \quad : \quad = & \int_{a_1}^{\frac{a_1+b_1}{2}} |x-u| \exp\left(-A\left(|\Phi(u)-\Phi(y)|\right)\right) du \\ & - \int_{\frac{a_1+b_1}{2}}^{b_1} |x-u| \exp\left(-A\left(|\Phi(u)-\Phi(y)|\right)\right) du,\end{aligned}$$

$A=\frac{1-\vartheta}{\vartheta}$ *and* $B=A(\Phi(b_1)-\Phi(a_1))$.

Proof. If we use $\varphi(u)=\frac{u}{\vartheta}\exp(-Au)$, where $A=\frac{1-\vartheta}{\vartheta}$, $\vartheta\in(0,1)$, then by same lines as followed in the proof of Theorem 4.2.1, required inequality (4.23) is obtained. □

Chapter 5

Error bounds associated with different versions of Hermite–Hadamard inequalities of mid-point type

This chapter analyzing the distinctive generalized error bound for the Hermite–Hadamard inequality and brings into the light an overall numerical definition for error formula of first and second term of generalized Hermite–Hadamard inequality at mid-point. As of late, utilizations of error assessments are distributed as far as central moments of random variables, see ([21–23, 34, 82]) and so forth and the references in that. This exploration build up some new outcomes for various recently presented fractional integrals and afterward present the applications regarding focal snapshot of the random variable. Examination of current examination are settled on and found in incredible concurrence with the published outcomes. The outcomes have been published. This section follows the positive configuration.

Section 5.1 manages the two generalized identities for generalized exponential fractional integral and generalized fractional integral with mittag–leffler function in the kernel. Section 5.2 annunciate the critical findings of the whole chapter. Section 5.3 gives a few models including focal snapshot of a random variable for its central moment.

5.1 Some Fractional Hadamard inequalities at midpoint

In this section, we give some error estimates for Hadamard inequalities associated to fractional integral operators namely generalized k-fractional integrals and generalized exponential fractional integrals. Through out the investigation, We consider $\widetilde{\zeta}(\tau) = \zeta(u) + \zeta(a_1 + b_1 - \tau)$ for $\tau \in [a_1, b_1]$. We start with the following identity:

Lemma 5.1.1. *Let $\Phi : [a_1, b_1] \to \mathbb{R}$ be a positive monotone increasing function with continuous derivative Φ' on (a_1, b_1). Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a differentiable function on (a_1, b_1) with $0 \le a_1 < b_1$. If $\zeta' \in L_1[a_1, b_1]$, then the following identity holds for operator* (3.9) *and* (3.10)*:*

$$
\begin{aligned}
&{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{1}{b_1-a_1}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\left(\frac{a_1+b_1}{2}\right)^+}\widetilde{\zeta}(b_1;q^*,w) + {}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\left(\frac{a_1+b_1}{2}\right)^-}\widetilde{\zeta}(a_1;q^*,w)\right] \\
&= \frac{b_1-a_1}{4}\int_0^1 {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\left[\zeta'\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right) - \zeta'\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right]dr,
\end{aligned}
\tag{5.1}
$$

where

$$
\begin{aligned}
&{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r) \\
&= \left[\Phi(b_1) - \Phi\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right] {}^{\vartheta}\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta+1,\nu}\left(\left[\Phi(b_1) - \Phi\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right]^{\beta}; q^*\right) \\
&\quad + \left[\Phi\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right) - \Phi(c)\right] {}^{\vartheta}\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta+1,\nu}\left(\left[\Phi\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right) - \Phi(a_1)\right]^{\beta}; q^*\right).
\end{aligned}
\tag{5.2}
$$

Proof. We have

$$
\begin{aligned}
&\int_0^1 {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\left[\zeta'\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right) - \zeta'\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right]dr \\
&= \int_0^1 {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\zeta'\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right)dr - \int_0^1 {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\zeta'\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)dr.
\end{aligned}
\tag{5.3}
$$

Integrating by parts, we have

$$\int_0^1 {}_{\Phi}R_{\beta,\vartheta,\nu}^{\delta,r,s,c}(r)\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)dr = \frac{2}{b_1-a_1}{}_{\Phi}R_{\beta,\vartheta,\nu}^{\delta,r,s,c}(r)\zeta\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)\Big|_0^1$$
$$-\frac{2}{b_1-a_1}\sum_{j=0}^{\infty}\frac{B_{q^*}(\delta+js,c-\delta)(c)_{sj}}{B(\delta,c-\delta)\Gamma(\beta j+\vartheta)(\nu)_{rj}}\int_0^1\frac{\Phi'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\zeta\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)}{\left[\Phi(b_1)-\Phi\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]^{1-\vartheta-\beta j}}dr$$
$$-\frac{2}{d-c}\sum_{j=0}^{\infty}\frac{B_{q^*}(\delta+js,c-\delta)(c)_{sj}}{B(\delta,c-\delta)\Gamma(\beta j+\vartheta)(\nu)_{rj}}\int_0^1\frac{\Phi'\left(\frac{r}{2}b_1+\frac{2-r}{2}c\right)\zeta\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)}{\left[\Phi\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)-\Phi(a_1)\right]^{1-\vartheta-\beta j}}dr$$
$$=\frac{2}{d-c}{}_{\Phi}R_{\beta,\vartheta,\nu}^{\delta,r,s,c}(1)\zeta\left(\frac{a_1+b_1}{2}\right)$$
$$-\frac{2}{b_1-a_1}\int_0^1\frac{\Phi'\left(\frac{u}{2}a_1+\frac{2-u}{2}b_1\right)\mathcal{E}_{\beta,\vartheta,\nu}^{\delta,r,s,c}\left(\left[\Phi(b_1)-\Phi\left(\frac{u}{2}a_1+\frac{2-u}{2}b_1\right)\right]^{\beta};q^*\right)\times\zeta\left(\frac{u}{2}b_1+\frac{2-u}{2}a_1\right)}{\left[\Phi(b_1)-\Phi\left(\frac{u}{2}a_1+\frac{2-u}{2}b_1\right)\right]^{1-\vartheta}}du$$
$$-\frac{2}{b_1-a_1}\int_0^1\frac{\Phi'\left(\frac{v}{2}b_1+\frac{2-v}{2}a_1\right)\mathcal{E}_{\beta,\vartheta,\nu}^{\delta,r,s,c}\left(\left[\Phi\left(\frac{v}{2}b_1+\frac{2-v}{2}a_1\right)-\Phi(a_1)\right]^{\beta};q^*\right)\times\zeta\left(\frac{v}{2}b_1+\frac{2-v}{2}a_1\right)}{\left[\Phi\left(\frac{v}{2}b_1+\frac{2-v}{2}a_1\right)-\Phi(a_1)\right]^{1-\vartheta}}dv. \tag{5.4}$$

Let $u=\frac{2(b_1-u_1)}{b_1-a_1}$ and $v=\frac{2(v_1-a_1)}{b_1-a_1}$, then

$$\int_0^1 {}_{\Phi}R_{\beta,\vartheta,\nu}^{\delta,r,s,c}(r)\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)dr$$
$$=\frac{2}{d-c}{}_{\Phi}R_{\beta,\vartheta,\nu}^{\delta,r,s,c}(1)\zeta\left(\frac{a_1+b_1}{2}\right)$$
$$-\frac{4}{(b_1-a_1)^2}\int_{\frac{a_1+b_1}{2}}^{b_1}\frac{\Phi'(u_1)\mathcal{E}_{\beta,\vartheta,\nu}^{\delta,r,s,c}\left([\Phi(b_1)-\Phi(u_1)]^{\beta};q^*\right)\zeta(a_1+b_1-u_1)}{[\Phi(b_1)-\Phi(u_1)]^{1-\vartheta}}du_1$$
$$-\frac{4}{(b_1-a_1)^2}\int_{a_1}^{\frac{a_1+b_1}{2}}\frac{\Phi'(v_1)\mathcal{E}_{\beta,\vartheta,\nu}^{\delta,r,s,c}\left([\Phi(v_1)-\Phi(a_1)]^{\beta};q^*\right)\zeta(v_1)}{[\Phi(v_1)-\Phi(a_1)]^{1-\vartheta}}dv_1. \tag{5.5}$$

Similarly,

$$\int_0^1 {}_{\Phi}R_{\beta,\vartheta,\nu}^{\delta,r,s,c}(r)\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)dr$$
$$=-\frac{2}{b_1-a_1}{}_{\Phi}R_{\beta,\vartheta,\nu}^{\delta,r,s,c}(1)\zeta\left(\frac{a_1+b_1}{2}\right)$$
$$+\frac{4}{(b_1-a_1)^2}\int_{\frac{a_1+b_1}{2}}^{b_1}\frac{\Phi'(u_1)\mathcal{E}_{\beta,\vartheta,\nu}^{\delta,r,s,c}\left([\Phi(b_1)-\Phi(u_1)]^{\beta};q^*\right)\zeta(u_1)}{[\Phi(b_1)-\Phi(u_1)]^{1-\vartheta}}du_1$$

$$+\frac{4}{(b_1-a_1)^2}\int\limits_{a_1}^{\frac{a_1+b_1}{2}}\frac{\Phi'(v_1)\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}\left([\Phi(v_1)-\Phi(a_1)]^{\beta};q^*\right)\zeta(a_1+b_1-v_1)}{[\Phi(v_1)-\Phi(a_1)]^{1-\vartheta}}dv_1. \tag{5.6}$$

Equation (5.3) together with (5.5) and (5.6), leads to

$$\begin{aligned}
&\int\limits_0^1 {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\left[\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)-\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]dr\\
&=\frac{4}{b_1-a_1}{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(1)\zeta\left(\frac{a_1+b_1}{2}\right)\\
&\quad-\frac{4}{(b_1-a_1)^2}\int\limits_{\frac{a_1+b_1}{2}}^{b_1}\frac{\Phi'(u_1)\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}\left([\Phi(b_1)-\Phi(u_1)]^{\beta};q^*\right)[\zeta(u_1)+\zeta(a_1+b_1-u_1)]}{[\Phi(b_1)-\Phi(u_1)]^{1-\vartheta}}du_1\\
&\quad-\frac{4}{(b_1-a_1)^2}\int\limits_{a_1}^{\frac{a_1+b_1}{2}}\frac{\Phi'(v_1)\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}\left([\Phi(v_1)-\Phi(a_1)]^{\beta};q^*\right)[\zeta(v_1)+\zeta(a_1+b_1-v_1)]}{[\Phi(v_1)-\Phi(a_1)]^{1-\vartheta}}dv_1.
\end{aligned} \tag{5.7}$$

Applying (3.9) and (3.10), we finally have

$$\begin{aligned}
&\int\limits_0^1 {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\left[\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)-\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]dr\\
&=\frac{4}{b_1-a_1}{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(1)\zeta\left(\frac{a_1+b_1}{2}\right)-\frac{4}{(b_1-a_1)^2}{}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\left(\frac{a_1+b_1}{2}\right)^+}\widetilde{\zeta}(b_1)\\
&\quad-\frac{4}{(b_1-a_1)^2}{}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\left(\frac{a_1+b_1}{2}\right)^-}\widetilde{\zeta}(a_1).
\end{aligned} \tag{5.8}$$

Multiplying (5.8) by $\frac{b_1-a_1}{4}$, we have the required identity. □

Lemma 5.1.2. *Let $\Phi:[a_1,b_1]\to\mathbb{R}$ be a positive and monotone increasing function with continuous derivative Φ' on (a_1,b_1). Let $\zeta:[a_1,b_1]\to\mathbb{R}$ be a differentiable function on (a_1,b_1) with $0\le a_1<b_1$. If $\zeta'\in L_1[a_1,b_1]$, then the following identity holds for the fractional integral operator (3.5) and (3.6):*

$$\begin{aligned}
&N^{\Phi}_{\vartheta}(1)\zeta\left(\frac{a_1+b_1}{2}\right)-\frac{1-\vartheta}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{\left(\frac{a_1+b_1}{2}\right)^+}\widetilde{\zeta}(b_1)+{}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{\left(\frac{a_1+b_1}{2}\right)^-}\widetilde{\zeta}(a_1)\right]\\
&=\frac{b_1-a_1}{4}\int\limits_0^1 N^{\Phi}_{\vartheta}(r)\left[\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)-\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]dr,
\end{aligned} \tag{5.9}$$

where

$$N_{\vartheta}^{\Phi}(r): \quad = \quad 2-\exp\left(-A\left(\left(\Phi(b_1)-\Phi\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right)\right)\right)$$
$$-\exp\left(-A\left(\left(\Phi\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)-\Phi(a_1)\right)\right)\right), \tag{5.10}$$

and $A=\frac{1-\vartheta}{\vartheta}, \vartheta\in(0,1)$.

Proof. Integrating by parts we have,

$$\begin{aligned}
&\int_0^1 N_{\vartheta}^{\Phi}(r)\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)dr = \frac{2}{b_1-a_1}N_{\vartheta}^{\Phi}(r)\zeta\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)\Big|_0^1 \\
&\quad - A\int_0^1 \begin{matrix}\exp\left(-A\left(\Phi(b_1)-\Phi\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right)\right)\Phi'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\\ \times g\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)\end{matrix}\, dr \\
&\quad - A\int_0^1 \begin{matrix}\exp\left(-A\left(\Phi\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)-\Phi(a_1)\right)\right)\Phi'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)\\ \times\zeta\left(\frac{r}{2}b_1+\frac{2-r}{2}c\right)\end{matrix}\, dr \\
&= \frac{2}{b_1-a_1}N_{\vartheta}^{\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right) \\
&\quad - \frac{2A}{b_1-a_1}\int_{\frac{a_1+b_1}{2}}^{b_1} \exp\left(-A\left(\Phi(b_1)-\Phi(x)\right)\right)\Phi'(x)\zeta(a_1+b_1-x)dx \\
&\quad - \frac{2A}{b_1-a_1}\int_{a_1}^{\frac{a_1+b_1}{2}} \exp\left(-A\left(\Phi(x)-\Phi(c)\right)\right)\Phi'(x)\zeta(x)dx.
\end{aligned} \tag{5.11}$$

Similarly,

$$\begin{aligned}
&\int_0^1 N_{\vartheta}^{\Phi}(r)\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)dr \\
&= -\frac{2}{(b_1-a_1)}N_{\vartheta}^{\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right) \\
&\quad + \frac{2A}{b_1-a_1}\int_{\frac{a_1+b_1}{2}}^{b_1} \exp\left(-A\left(\Phi(b_1)-\Phi(x)\right)\right)\Phi'(x)\zeta(x)dx \\
&\quad + \frac{2A}{b_1-a_1}\int_{a_1}^{\frac{a_1+b_1}{2}} \exp\left(-A\left(\Phi(x)-\Phi(a_1)\right)\right)\Phi'(x)\zeta(a_1+b_1-x)dx.
\end{aligned} \tag{5.12}$$

By applying (3.5) and (3.6) to (5.11) and (5.12), we have

$$\int_0^1 N_{\vartheta}^{\Phi}(r)\left[\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)-\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]dr$$
$$=\frac{4}{(b_1-a_1)}N_{\vartheta}^{\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right)-\frac{2(1-\vartheta)}{b_1-a_1}{}^{\vartheta}_{\Phi}\mathcal{G}^{e}_{\left(\frac{a_1+b_1}{2}\right)^+}\widetilde{\zeta}(b_1)-\frac{2(1-\gamma)}{b_1-a_1}{}^{\vartheta}_{\Phi}\mathcal{G}^{e}_{\left(\frac{a_1+b_1}{2}\right)^-}\widetilde{\zeta}(a_1). \quad (5.13)$$

Equation (5.13) leads to the required identity. □

Now, we give some error estimates for Hermite–Hadamard inequalities via fractional integrals generalized k fractional integrals and generalized exponential fractional integrals.

Theorem 5.1.1. *Let Φ be a positive and montone increasing function on $[a_1, b_1]$ with continuous derivative Φ' on (a_1, b_1). If $|\zeta'|$ is convex, then the following inequality holds for the fractional integral* (3.9) *and* (3.10)*:*

$$\left|{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(1)\zeta\left(\frac{a_1+b_1}{2}\right)-\frac{1}{b_1-a_1}\left[{}^{\vartheta}_{\Phi}\mathcal{G}^{\delta,r,s,c}_{\beta,\nu,\frac{a_1+b_1}{2}^+}\widetilde{\zeta}(b_1)+{}^{\vartheta}_{\Phi}\mathcal{G}^{\delta,r,s,c}_{\beta,\nu,\frac{a_1+b_1}{2}^-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq\frac{(b_1-a_1)\left[|\zeta'(a_1)|+|\zeta'(b_1)|\right]}{4}\int_0^1\left|{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\right|dr. \quad (5.14)$$

Proof. By Lemma 5.1.1, and property of modulus, we have

$$\left|{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(1)\zeta\left(\frac{a_1+b_1}{2}\right)-\frac{1}{b_1-a_1}\left[{}^{\vartheta}_{\Phi}\mathcal{G}^{\delta,r,s,c}_{\beta,\nu,\frac{a_1+b_1}{2}^+}\widetilde{\zeta}(b_1)+{}^{\vartheta}_{\Phi}\mathcal{G}^{\delta,r,s,c}_{\beta,\nu,\frac{a_1+b_1}{2}^-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq\frac{(b_1-a_1)}{4}\int_0^1\left|{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\right|\left[\left|\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)\right|+\left|\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right|\right]dr. \quad (5.15)$$

By convexity of $|\zeta'|$, we get

$$\left|\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)\right|+\left|\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right|$$
$$\leq\frac{r}{2}|\zeta'(b_1)|+\frac{2-r}{2}|\zeta'(a_1)|+\frac{r}{2}|\zeta'(a_1)|+\frac{2-r}{2}|\zeta'(b_1)|$$
$$=|\zeta'(a_1)|+|\zeta'(b_1)|. \quad (5.16)$$

Using (5.16) in (5.15), we obtain the required inequality. □

Theorem 5.1.2. *Let Φ be a positive and montone increasing function on $[a_1, b_1]$ with continuous derivative Φ' on (a_1, b_1). If $|\zeta'|$ is convex, then the following inequality holds for the fractional integral operator* (3.5) *and* (3.6)*:*

$$\left| N_{\vartheta}^{\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{1-\vartheta}{2}\left[{}_{\Phi}^{\vartheta}\mathcal{J}^{e}_{\frac{a_1+b_1}{2}+}\widetilde{\zeta}(b_1) + {}_{\Phi}^{\vartheta}\,\mathcal{J}^{e}_{\frac{a_1+b_1}{2}-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{(b_1-a_1)\left[|\zeta'(a_1)| + |\zeta'(b_1)|\right]}{4}\int_0^1 \left|N_{\vartheta}^{\Phi}(r)\right| dr. \tag{5.17}$$

Proof. By Lemma 5.1.2 and following the steps in the proof of Theorem 3.2, we obtain the required inequality. □

Theorem 5.1.3. *Let Φ be a positive and montone increasing function on $[a_1, b_1]$ with continuous derivative Φ' on (a_1, b_1). If $|\zeta'|^m, m > 1$ is convex, then the following inequality holds for the fractional integral* (3.9) *and* (3.10)*:*

$$\left| {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{1}{b_1-a_1}\left[{}_{\Phi}^{\vartheta}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\frac{a_1+b_1}{2}+}\widetilde{\zeta}(b_1) + {}_{\Phi}^{\vartheta}\,\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\frac{a_1+b_1}{2}-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{(b_1-a_1)}{4}\left(\int_0^1 \left|{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\right|^p dr\right)^{\frac{1}{p}}$$
$$\times\left\{\left[\frac{|\zeta'(b_1)|^m + 3|\zeta'(a_1)|^m}{4}\right]^{\frac{1}{m}} + \left[\frac{|\zeta'(a_1)|^m + 3|\zeta'(b_1)|^m}{4}\right]^{\frac{1}{m}}\right\}, \tag{5.18}$$

where $\frac{1}{p} + \frac{1}{m} = 1$.

Proof. From Lemma 5.1.1, we have

$$\left| {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{1}{b_1-a_1}\left[{}_{\Phi}^{\vartheta}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\frac{a_1+b_1}{2}+}\widetilde{\zeta}(b_1) + {}_{\Phi}^{\vartheta}\,\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\frac{a_1+b_1}{2}-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{(d-c)}{4}\left\{\int_0^1 \left|{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\right|\left|\zeta'\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right)\right| dr\right.$$
$$\left.+\int_0^1 \left|{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\right|\left|\zeta'\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right| dr\right\}. \tag{5.19}$$

Now by the Hölder's inequality, we have

$$\begin{aligned}
&\int_0^1 \left| {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r) \right| \left| \zeta'\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right) \right| dr \\
&\leq \left(\int_0^1 \left| {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r) \right|^p dr \right)^{\frac{1}{p}} \left(\int_0^1 \left| \zeta'\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right) \right|^m dr \right)^{\frac{1}{m}}. \qquad (5.20)
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^1 \left| {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r) \right| \left| \zeta'\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right) \right| dr \\
&\leq \left(\int_0^1 \left| {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r) \right|^p dr \right)^{\frac{1}{p}} \left(\int_0^1 \left| \zeta'\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right) \right|^m dr \right)^{\frac{1}{m}}. \qquad (5.21)
\end{aligned}$$

Since $|\zeta'|^m, m > 1$, is convex, so we have

$$\int_0^1 \left| \zeta'\left(\frac{r}{2}b_1 + \left(\frac{2-r}{2}\right)a_1\right) \right|^m dr \leq \frac{|\zeta'(b_1)|^m + 3|\zeta'(a_1)|^m}{4}, \qquad (5.22)$$

and

$$\int_0^1 \left| \zeta'\left(\frac{r}{2}a_1 + \left(\frac{2-r}{2}\right)b_1\right) \right|^m dr \leq \frac{|\zeta'(a_1)|^m + 3|\zeta'(b_1)|^m}{4}. \qquad (5.23)$$

Utilizing (5.20), (5.21), (5.22) and (5.23) in inequality (5.19), we obtain the desired inequality (5.18). □

Theorem 5.1.4. *Let Φ be a positive and montone increasing function on $[a_1, b_1]$ with continuous derivative Φ' on (a_1, b_1). If $|\zeta'|^m, m > 1$ is convex, then the following inequality for the fractional integral operator (3.5) and (3.6) holds:*

$$\begin{aligned}
&\left| N^{\Phi}_{\vartheta}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{1-\vartheta}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{\frac{a_1+b_1}{2}+}\widetilde{\zeta}(b_1) + {}^{\vartheta}_{\Phi}\,\mathcal{J}^{e}_{\frac{a_1+b_1}{2}-}\widetilde{\zeta}(a_1) \right] \right| \\
&\leq \frac{(b_1-a_1)}{4}\left(\int_0^1 \left| N^{\Phi}_{\vartheta}(r) \right|^p dr \right)^{\frac{1}{p}} \left\{ \left[\frac{|\zeta'(b_1)|^m + 3|\zeta'(a_1)|^m}{4} \right]^{\frac{1}{m}} + \left[\frac{|\zeta'(a_1)|^m + 3|\zeta'(b_1)|^m}{4} \right]^{\frac{1}{m}} \right\}, \qquad (5.24)
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{m} = 1$.

Proof. The required inequality is obtained by Lemma 5.1.2 and application of the Hölder's inequality. □

5.2 Mid-point type generalized integral identity and error estimation

Now and onward we consider,

$$\Delta^*_{\varphi,\Phi}(r) = \int_0^r \phi_{\varphi,\mu}(s)dr,$$

where

$$\begin{aligned}\phi_{\varphi,\Phi}(s) = &\frac{\varphi\left(\Phi\left(\frac{s}{2}b_1 + \frac{2-s}{2}a_1\right) - \Phi(a_1)\right)}{\Phi\left(\frac{s}{2}b_1 + \frac{2-s}{2}a_1\right) - \Phi(a_1)}\Phi'\left(\frac{s}{2}b_1 + \frac{2-s}{2}a_1\right) \\ &+ \frac{\varphi\left(\Phi(b_1) - \Phi\left(\frac{s}{2}a_1 + \frac{2-s}{2}b_1\right)\right)}{\Phi(b_1) - \Phi\left(\frac{s}{2}a_1 + \frac{2-s}{2}b_1\right)}\Phi'\left(\frac{s}{2}a_1 + \frac{2-s}{2}b_1\right). \end{aligned} \tag{5.25}$$

and $\widetilde{\zeta}(x) = \zeta(x) + \widehat{\zeta}(x)$ with $\widehat{\zeta}(x) = \zeta(a_1 + b_1 - x)$ for all $x \in [a_1, b_1]$.

Lemma 5.2.1. *Let $\zeta, \Phi : [a_1, b_1] \to \mathbb{R}$, $0 < a_1 < b_1$, be the functions such that ζ be positive and $\zeta \in L_1[a_1, b_1]$ and Φ be differentiable and increasing. Also let φ be a positive function such that $\frac{\varphi}{x}$ is increasing on $[a_1, \infty)$. If $\zeta' \in L_1[a_1, b_1]$, then the following identity holds for the generalized integral operator* (3.3) *and* (3.4)*:*

$$\begin{aligned}&\Delta^*_{\varphi,\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{1}{b_1-a_1}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^+}\widetilde{\zeta}(b_1) + {}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^-}\widetilde{\zeta}(a_1)\right] \\ &= \frac{b_1-a_1}{4}\int_0^1 \Delta^*_{\varphi,\Phi}(r)\left[\zeta'\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right) - \zeta'\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right]dr. \end{aligned} \tag{5.26}$$

Proof. Here,

$$\int_0^1 \Delta^*_{\varphi,\Phi}(r)\left[\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)-\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]dr$$
$$=\int_0^1 \Delta^*_{\varphi,\Phi}(r)\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)dr-\int_0^1 \Delta^*_{\varphi,\Phi}(r)\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)dr.$$

By integrating by parts, we have

$$\int_0^1 \Delta^*_{\varphi,\Phi}(r)\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)dr$$
$$=\frac{2}{b_1-a_1}\Delta^*_{\varphi,\Phi}(r)\zeta\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)\Big|_0^1-\frac{2}{b_1-a_1}\int_0^1 \phi_{\varphi,\Phi}(r)\zeta\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)dr$$
$$=\frac{2}{b_1-a_1}\Delta^*_{\varphi,\Phi}(1)g\left(\frac{a_1+b_1}{2}\right)-\frac{4}{(b_1-a_1)^2}\int_{a_1}^{\frac{a_1+b_1}{2}}\frac{\varphi\left(\Phi(x)-\Phi(a_1)\right)\Phi'(x)\zeta(x)}{\Phi(x)-\Phi(a_1)}dx$$
$$-\frac{4}{(b_1-a_1)^2}\int_{\frac{a_1+b_1}{2}}^{b_1}\frac{\varphi\left(\Phi(b_1)-\Phi(x)\right)\Phi'(x)\widetilde{\zeta}(x)}{\Phi(b_1)-\Phi(x)}dx. \tag{5.27}$$

Similarly,

$$\int_0^1 \Delta^*_{\varphi,\Phi}(r)\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)dr$$
$$=-\frac{2}{b_1-a_1}\Delta^*_{\varphi,\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right)+\frac{4}{(b_1-a_1)^2}\int_{a_1}^{\frac{a_1+b_1}{2}}\frac{\varphi\left(\Phi(x)-\Phi(a_1)\right)\Phi'(x)\widetilde{\zeta}(x)}{\Phi(x)-\Phi(a_1)}dx$$
$$+\frac{4}{(b_1-a_1)^2}\int_{\frac{a_1+b_1}{2}}^{b_1}\frac{\varphi\left(\Phi(b_1)-\Phi(x)\right)\Phi'(x)\zeta(x)}{\Phi(b_1)-\Phi(x)}dx. \tag{5.28}$$

Applying Definition 3.1.2 in (5.27) and (5.28), we have respectively,

$$\int_0^1 \Delta^*_{\varphi,\Phi}(r)\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)dr$$
$$=\frac{2}{b_1-a_1}\Delta^*_{\varphi,\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right)-\frac{4}{\left(b_1-a_1\right)^2}\,{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^-}\zeta(a_1)$$
$$-\frac{4}{\left(b_1-a_1\right)^2}\,{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^+}\widehat{\zeta}(b_1), \tag{5.29}$$

and

$$\int_0^1 \Delta^*_{\varphi,\Phi}(r)\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)dr$$
$$= -\frac{2}{b_1-a_1}\Delta^*_{\varphi,\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right)+\frac{4}{(b_1-a_1)^2}\,{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^-}\widehat{\zeta}(a_1)$$
$$+\frac{4}{(b_1-a_1)^2}\,{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^+}\zeta(b_1). \tag{5.30}$$

Equations (5.29) and (5.30) lead to the required equality (5.26). □

Remark 5.1. We can deduce associated identities for aforementioned fractional and conformable integrals from Lemma 5.2.1.

(i) If Φ is the identity function, then we have [23, Lemma 3].

(ii) If $\varphi(u)=\frac{u^{\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta)}$, then we have,

$$A_1^{\vartheta,k}(1)\zeta\left(\frac{a_1+b_1}{2}\right)-\frac{\Gamma_k(\vartheta+k)}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^k_{\left(\frac{a_1+b_1}{2}\right)^+}\widetilde{\zeta}(b_1)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^k_{\left(\frac{a_1+b_1}{2}\right)^-}\widetilde{\zeta}(a_1)\right]$$
$$=\frac{b_1-a_1}{4}\int_0^1 A_1^{\vartheta,k}(r)\left(\zeta'\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)-\zeta'\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right)dr,$$

where

$$A_1^{\vartheta,k}(r)=\left[\Phi\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)-\Phi(a_1)\right]^{\frac{\vartheta}{k}}+\left[\Phi(b_1)-\Phi\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]^{\frac{\vartheta}{k}}.$$

Moreover, if $k=1$, then it coincides to [170, Corollary 4].

(iii) If $\varphi(u)=u^{\frac{\vartheta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\vartheta}\left(w(x)^{\rho}\right)$, then we get [170, Lemma 2]. Furthermore, if $k=1$, then we obtain [170, Corollary 3].

(iv) If we choose $\varphi(u)=u^{\vartheta}\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}(w(u)^{\beta};q^*)$, then we obtain Lemma 5.1.1.

(v) If $\varphi(u)=\frac{u}{\vartheta}\exp(-Au)$, where $A=\frac{1-\vartheta}{\vartheta}$, $\vartheta\in(0,1)$, then we have Lemma 5.1.2.

Theorem 5.2.1. *Let $\zeta, \Phi : [a_1, b_1] \to \mathbb{R}$, $0 < a_1 < b_1$, be the functions such that ζ be positive and $\zeta \in L_1[a_1, b_1]$ and Φ be differentiable and increasing. Also let φ be a positive function such that $\frac{\varphi}{x}$ is increasing on $[a_1, \infty)$. If $|\zeta'|$ is convex on $[a_1, b_1]$, then the following inequality for the generalized integral operator* (3.3) *and* (3.4)*holds:*

$$\left|\Delta^*_{\varphi,\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{1}{b_1-a_1}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^+}\widetilde{\zeta}(b_1) + {}^{\varphi}_{\Phi}\,\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{(b_1-a_1)\left[|\zeta'(a_1)| + |\zeta'(b_1)|\right]}{4}\int_0^1 \left|\Delta^*_{\varphi,\Phi}(r)\right| dr.$$

Proof. From Lemma 5.2.1, we have

$$\left|\Delta^*_{\varphi,\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{1}{b_1-a_1}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^+}\widetilde{\zeta}(b_1) + {}^{\varphi}_{\Phi}\,\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{b_1-a_1}{4}\int_0^1 \left|\Delta^*_{\varphi,\Phi}(r)\right|\left[\left|\zeta'\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right)\right| + \left|\zeta'\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right|\right] dr. \tag{5.31}$$

Now utilizing (5.16) in (5.31), we get the required inequality. □

Remark 5.2. We can deduce associated inequalities for various fractional and conformable integrals from Theorem 5.2.1.

(i) If Φ is identity function, then we have [23, Theorem 4].

(ii) If $\varphi(u) = \frac{u^{\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta)}$, then we have,

$$\left|A_1^{\vartheta,k}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{\Gamma_k(\vartheta+k)}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^k_{\left(\frac{a_1+b_1}{2}\right)^+}\widetilde{\zeta}(b_1) + {}^{\vartheta}_{\Phi}\,\mathcal{J}^k_{\left(\frac{a_1+b_1}{2}\right)^-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{(b_1-a_1)[|\zeta'(a_1)| + |\zeta'(b_1)|]}{4}\int_0^1 \left|A_1^{\vartheta,k}(r)\right| dr,$$

where $A_1^{\vartheta,k}(r)$ is same as in Remark 5.1. Moreover, if $k = 1$, then it coincides to [170, Corollary 5].

(iii) If $\varphi(u) = u^{\frac{\vartheta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\vartheta}\left(w(x)^{\rho}\right)$, then we obtain [170, Theorem 3].

(iv) If we choose $\varphi(u) = u^{\vartheta}\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}(w(u)^{\beta}; q^{*})$, then we get Theorem 5.1.1.

(v) If $\varphi(u) = \frac{u}{\vartheta}\exp(-Au)$, where $A = \frac{1-\vartheta}{\vartheta}$, $\vartheta \in (0,1)$, then we have Theorem 5.1.2.

Theorem 5.2.2. *Let $\zeta, \Phi : [a_1, b_1] \to \mathbb{R}$, $0 < a_1 < b_1$, be the functions such that ζ be positive and $\zeta \in L_1[a_1, b_1]$ and Φ be differentiable and increasing. Also let φ be a positive function such that $\frac{\varphi}{x}$ is increasing on $[a_1, \infty)$. If $|\zeta'|^m, m > 1$ is convex on $[a_1, b_1]$, then the following inequality for the generalized integral operator* (3.3) *and* (3.4)*holds:*

$$\begin{aligned}
&\left|\Delta^{*}_{\varphi,\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{1}{b_1-a_1}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^{+}}\widetilde{\zeta}(b_1) + {}^{\varphi}_{\Phi}\,\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^{-}}\widetilde{\zeta}(a_1)\right]\right| \\
&\leq \frac{(b_1-a_1)}{4}\left(\int\limits_0^1 \left|\Delta^{*}_{\varphi,\Phi}(r)\right|^p dr\right)^{\frac{1}{p}} \\
&\quad\times\left[\left(\frac{|\zeta'(b_1)|^m + 3\,|\zeta'(a_1)|^m}{4}\right)^{\frac{1}{m}} + \left(\frac{|\zeta'(a_1)|^m + 3\,|\zeta'(b_1)|^m}{4}\right)^{\frac{1}{m}}\right], \qquad (5.32)
\end{aligned}$$

where $\frac{1}{p} + \frac{1}{m} = 1$.

Proof. By using the property of modulus in Lemma 5.2.1, we have

$$\begin{aligned}
&\left|\Delta^{*}_{\varphi,\Phi}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{1}{b_1-a_1}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^{+}}\widetilde{\zeta}(b_1) + {}^{\varphi}_{\Phi}\,\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^{-}}\widetilde{\zeta}(a_1)\right]\right| \\
&\leq \frac{(b_1-a_1)}{4} \\
&\times\left\{\int\limits_0^1 \left|\Delta^{*}_{\varrho,\Phi}(r)\right|\left|\zeta'\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right)\right| dr + \int\limits_0^1 \left|\Delta^{*}_{\varphi,\Phi}(r)\right|\left|\zeta'\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right| dr\right\}. \qquad (5.33)
\end{aligned}$$

By applying the Hölder's inequality, we get

$$\begin{aligned}
&\int\limits_0^1 \left|\Delta^{*}_{\varphi,\Phi}(r)\right|\left|\zeta'\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right)\right| dr \\
&\leq \left(\int\limits_0^1 \left|\Delta^{*}_{\varphi,\Phi}(r)\right|^p dr\right)^{\frac{1}{p}}\left(\int\limits_0^1 \left|\zeta'\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right)\right|^m dr\right)^{\frac{1}{m}}, \qquad (5.34)
\end{aligned}$$

and

$$\int_0^1 \left|\Delta^*_{\varphi,\Phi}(r)\right| \left|\zeta'\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right| dr$$
$$\leq \left(\int_0^1 \left|\Delta^*_{\varphi,\Phi}(r)\right|^p dr\right)^{\frac{1}{p}} \left(\int_0^1 \left|\zeta'\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right|^m dr\right)^{\frac{1}{m}}. \tag{5.35}$$

Inequality (5.33) together with (5.22), (5.23), (5.34) and (5.35) leads to the required inequality (5.32). □

Remark 5.3. We deduce some inequalities for aforementioned fractional and conformable integrals from Theorem 5.2.2.

(i) If Φ is identity function, then we have [23, Theorem 5].

(ii) If $\varphi(u) = \frac{u^{\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta)}$, then we obtain,

$$\left|A_1^{\vartheta,k}(1)\zeta\left(\frac{a_1+b_1}{2}\right) - \frac{\Gamma_k(\vartheta+k)}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^k_{\left(\frac{a_1+b_1}{2}\right)^+}\widetilde{\zeta}(b_1) + {}^{\vartheta}_{\Phi}\mathcal{J}^k_{\left(\frac{a_1+b_1}{2}\right)^-}\widetilde{\zeta}(a_1)\right]\right|$$
$$\leq \frac{(b_1-a_1)}{4}\left(\int_0^1 \left|A_1^{\vartheta,k}(r)\right|^p d\xi\right)^{\frac{1}{p}}$$
$$\times\left[\left(\frac{|\zeta'(b_1)|^m + 3|\zeta'(a_1)|^m}{4}\right)^{\frac{1}{m}} + \left(\frac{|\zeta'(a_1)|^m + 3|\zeta'(b_1)|^m}{4}\right)^{\frac{1}{m}}\right],$$

where $A_1^{\vartheta,k}(r)$ is same as in Remark 5.1. Moreover, if $k=1$, then it will gives [170, Corollary 6].

(iii) If $\varphi(u) = u^{\frac{\vartheta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\vartheta}(w(x)^\rho)$, then we get [170, Theorem 4].

(iv) If we choose $\varphi(u) = u^\vartheta \mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}(w(u)^\beta; q^*)$, then we obtain Theorem 5.1.3.

(v) If $\varphi(u) = \frac{u}{\vartheta}\exp(-Au)$, where $A = \frac{1-\vartheta}{\vartheta}$, $\vartheta \in (0,1)$, then we have Theorem 5.1.4.

5.3 Applications

Density and distribution functions give total depictions of the distribution of probability for a random variable. They don't allow us to make examinations between two distinct distributions without any problem. In any case, the set of moments are helpful in making examination under sensible conditions. Moments can be resolved remarkably by the probability distribution functions. Applying numerical inequalities, a few assessments were made for the snapshots of random variables by numerous researchers, for additional subtleties see ([21, 22, 34, 82, 124]).

Leave X alone a random variable whose probability function is $\psi : \mathcal{Q} \subset \mathbb{R} \to \mathbb{R}^+$. The m^{th} moment about any arbitrary point x of the random variable X is indicated and characterized as follows:

$$M^m(x) = \int_{a_1}^{b_1} (t-x)^m \psi(t)dt, \quad \text{m=0,1,2,3...} \tag{5.36}$$

Now we give some applications of our results for central moment. Now and onward we will use $\widehat{M^m}(u) := M^m(u) + M^m(a_1 + b_1 - u)$ for all $u \in [a_1, b_1]$ and $m \geq 2s+1, s = 0, 1, 2, 3, ...$

Proposition 6. *Let X be a random variable whose probability function is $\psi : \mathcal{Q} \subset \mathbb{R} \to \mathbb{R}^+$, where ψ is a convex function on the interval of real numbers I such that $a_1, b_1 \in \mathcal{Q}$ with $a_1 < b_1$, then the following identity holds for the generalized integral operator* (3.3) *and* (3.4)*:*

$$\begin{aligned}
&\Delta^*_{\varphi,\Phi}(1) M^m\left(\frac{a_1+b_1}{2}\right) - \frac{1}{b_1-a_1}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1) + {}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1)\right] \\
&= \frac{m(b_1-a_1)}{4}\int_0^1 \Delta^*_{\varphi,\Phi}(r)\left[M^{m-1}\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right) - M^{m-1}\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right] dr.
\end{aligned} \tag{5.37}$$

Proof. Required identity is obtained by setting $\zeta(u) = M^m(u)$ in Lemma 5.2.1. □

Remark 5.4. We can deduce associated identities for aforementioned fractional and conformable integrals from (5.37).

(i) If $\varphi(u) = \frac{u^{\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta)}$, then we have,

$$A_1^{\vartheta,k}(1)M^m\left(\frac{a_1+b_1}{2}\right) - \frac{\Gamma_k(\vartheta+k)}{2}\left[{}_{\Phi}^{\vartheta}\mathcal{J}^k_{\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1) + {}_{\Phi}^{\vartheta}\mathcal{J}^k_{\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1)\right]$$
$$= \frac{m(b_1-a_1)}{4}\int_0^1 A_1^{\vartheta,k}(r)\left[M^{m-1}\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right) - M^{m-1}\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]dr. \quad (5.38)$$

where $A_1^{\vartheta,k}(r)$ is same as in Remark 5.1.

(ii) If $\varphi(u) = u^{\frac{\vartheta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\vartheta}(w(u)^\rho)$, then we get

$$^{\Phi}C^{\vartheta,\sigma}_{\rho,k}(1)M^m\left(\frac{a_1+b_1}{2}\right) - \frac{1}{2k}\left[{}_{\Phi}^{\vartheta}\mathcal{J}^{k,\rho}_{\sigma,\left(\frac{a_1+b_1}{2}\right)^+;w}\widehat{M^m}(b_1) + {}_{\Phi}^{\vartheta}\mathcal{J}^{k,\rho}_{\sigma,\left(\frac{a_1+b_1}{2}\right)^-;w}\widehat{M^m}(a_1)\right]$$
$$= \frac{m(b_1-a_1)}{4}\int_0^1 {}^{\Phi}C^{\vartheta,\sigma}_{\rho,k}(r)\left[M^{m-1}\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right) - M^{m-1}\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]dr, \quad (5.39)$$

where

$$^{\Phi}C^{\vartheta,\sigma}_{\rho,k}(r)$$
$$= \left[\Phi(b_1) - \Phi\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]^{\frac{\vartheta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\vartheta+k}\left(w\left[\Phi(b_1) - \Phi\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]^\rho\right)$$
$$+ \left[\Phi\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right) - \Phi(a_1)\right]^{\frac{\vartheta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\vartheta+k}\left(w\left[\Phi\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right) - \Phi(a_1)\right]^\rho\right). \quad (5.40)$$

(iii) If we choose $\varphi(u) = u^\vartheta\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}(w(u)^\beta; q^*)$, then we obtain

$${}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(1)M^m\left(\frac{a_1+b_1}{2}\right)$$
$$- \frac{1}{b_1-a_1}\left[{}_{\Phi}^{\vartheta}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1;q^*,w) + {}_{\Phi}^{\vartheta}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1;q^*,w)\right]$$
$$= \frac{m(b_1-a_1)}{4}\int_0^1 {}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\left[M^{m-1}\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right) - M^{m-1}\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]dr. \quad (5.41)$$

(iv) If $\varphi(u) = \frac{u}{\vartheta}\exp(-Au)$, where $A = \frac{1-\vartheta}{\vartheta}$, $\vartheta \in (0,1)$, then we have

$$N^{\Phi}_{\vartheta}(1)M^m\left(\frac{a_1+b_1}{2}\right) - \frac{1-\vartheta}{2}\left[{}_{\Phi}^{\vartheta}\mathcal{J}^e_{\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1) + {}_{\Phi}^{\vartheta}\mathcal{J}^e_{\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1)\right]$$
$$= \frac{m(b_1-a_1)}{4}\int_0^1 N^{\Phi}_{\vartheta}(r)\left[M^{m-1}\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right) - M^{m-1}\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right]dr. \quad (5.42)$$

Proposition 7. *Let X be a random variable whose probability function is $\psi : \mathcal{Q} \subset \mathbb{R} \to \mathbb{R}^+$, where ψ is a convex function on the interval of real numbers $\mathcal{Q}$ such that $a_1, b_1 \in \mathcal{Q}$ with $a_1 < b_1$. If $|\psi|$ is bounded, then the following inequality holds for the generalized integral operators* (3.3) *and* (3.4)*:*

$$\left|\Delta^*_{\varphi,\Phi}(1)M^m\left(\frac{a_1+b_1}{2}\right) - \frac{1}{b_1-a_1}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1) + {}^{\varphi}_{\Phi}\,\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1)\right]\right|$$
$$\leq \frac{(1-(-1)^m)(b_1-a_1)^{m+1}\|\psi\|_\infty}{4}\int\limits_0^1 \left|\Delta^*_{\varphi,\Phi}(r)\right|\left[\left(\frac{r}{2}\right)^m + \left(\frac{2-r}{2}\right)^m\right] dr. \tag{5.43}$$

Proof. From Proposition 6, we have

$$\left|\Delta^*_{\varphi,\Phi}(1)M^m\left(\frac{a_1+b_1}{2}\right) - \frac{1}{b_1-a_1}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1) + {}^{\varphi}_{\Phi}\,\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1)\right]\right|$$
$$\leq \frac{m(b_1-a_1)}{4}\int\limits_0^1 |\Delta_{\varrho,\mu}(r)|\left[\left|M^{m-1}\left(\frac{r}{2}b_1 + \frac{2-r}{2}a_1\right)\right| + \left|M^{m-1}\left(\frac{r}{2}a_1 + \frac{2-r}{2}b_1\right)\right|\right] dr. \tag{5.44}$$

The required estimate is obtained by evalauting the integrals on the right side of the inequality.

□

Remark 5.5. We now deduce new inequalities involving central moment of a random variable for different fractional and conformable integrals from Proposition 7.

(i) If $\varphi(u) = \frac{u^{\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta)}$, then we have,

$$\left|A_1^{\vartheta,k}(1)M^m\left(\frac{a_1+b_1}{2}\right) - \frac{\Gamma_k(\vartheta+k)}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{I}^k_{\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1) + {}^{\vartheta}_{\Phi}\,\mathcal{I}^k_{\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1)\right]\right|$$
$$\leq \frac{(1-(-1)^m)(b_1-a_1)^{m+1}\|\psi\|_\infty}{4}\int\limits_0^1 \left|A_1^{\vartheta,k}(r)\right|\left[\left(\frac{r}{2}\right)^m + \left(\frac{2-r}{2}\right)^m\right] dr, \tag{5.45}$$

where $A_1^{\vartheta,k}(r)$ is same as in Remark 5.1.

(ii) If $\varphi(u) = u^{\frac{\vartheta}{k}}\mathcal{F}^{\sigma,k}_{\rho,\vartheta}(w(u)^\rho)$, then we get

$$\left|{}^{\Phi}C^{\vartheta,\sigma}_{\rho,k}(1)M^m\left(\frac{a_1+b_1}{2}\right)-\frac{1}{2k}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{k,\rho}_{\sigma,\left(\frac{a_1+b_1}{2}\right)^+;w}\widehat{M^m}(b_1)+{}^{\vartheta}_{\Phi}\mathcal{J}^{k,\rho}_{\sigma,\left(\frac{a_1+b_1}{2}\right)^-;w}\widehat{M^m}(a_1)\right]\right|$$
$$\leq\frac{(1-(-1)^m)(b_1-a_1)^{m+1}\|\psi\|_\infty}{4}\int\limits_0^1\left|{}^{\Phi}C^{\vartheta,\sigma}_{\rho,k}(r)\right|\left[\left(\frac{r}{2}\right)^m+\left(\frac{2-r}{2}\right)^m\right]dr, \tag{5.46}$$

where ${}^{\Phi}C^{\vartheta,\sigma}_{\rho,k}(r)$ is given in (5.40).

(iii) If we choose $\varphi(u)=u^{\vartheta}\mathcal{E}^{\delta,r,s,c}_{\beta,\vartheta,\nu}(w(u)^{\beta};q^*)$, then we obtain

$$\left|{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(1)M^m\left(\frac{a_1+b_1}{2}\right)\right.$$
$$\left.-\frac{1}{b_1-a_1}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1;q^*,w)+{}^{\vartheta}_{\Phi}\mathcal{J}^{\delta,r,s,c}_{\beta,\nu,\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1;q^*,w)\right]\right|$$
$$\leq\frac{(1-(-1)^m)(b_1-a_1)^{m+1}\|\psi\|_\infty}{4}\int\limits_0^1\left|{}_{\Phi}R^{\delta,r,s,c}_{\beta,\vartheta,\nu}(r)\right|\left[\left(\frac{r}{2}\right)^m+\left(\frac{2-r}{2}\right)^m\right]dr. \tag{5.47}$$

(iv) If $\varphi(u)=\frac{u}{\vartheta}\exp(-Au)$, where $A=\frac{1-\vartheta}{\vartheta}$, $\vartheta\in(0,1)$, then we have

$$\left|N^{\Phi}_{\vartheta}(1)M^m\left(\frac{a_1+b_1}{2}\right)-\frac{1-\vartheta}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1)+{}^{\vartheta}_{\Phi}\mathcal{J}^{e}_{\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1)\right]\right|$$
$$\leq\frac{(1-(-1)^m)(b_1-a_1)^{m+1}\|\psi\|_\infty}{4}\int\limits_0^1\left|N^{\Phi}_{\vartheta}(r)\right|\left[\left(\frac{r}{2}\right)^m+\left(\frac{2-r}{2}\right)^m\right]dr. \tag{5.48}$$

Proposition 8. *Let X be a random variable whose probability function is $\psi:\mathcal{Q}\subset\mathbb{R}\to\mathbb{R}^+$, where ψ is a convex function on the interval of real numbers $\mathcal{Q}$ such that $a_1,b_1\in\mathcal{Q}$ with $a_1<b_1$ and $\psi\in L_p[a_1,b_1], p>1$. If $|\psi|$ is bounded, then*

$$\left|\Delta^*_{\varphi,\Phi}(1)M^m\left(\frac{a_1+b_1}{2}\right)-\frac{1}{b_1-a_1}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1)+{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1)\right]\right|$$
$$\leq\frac{m(1-(-1)^{(m-1)q+1})(b_1-a_1)^{(m-1)q+2}\|\psi\|_p}{4((m-1)q+1)}$$
$$\times\int\limits_0^1\left|\Delta^*_{\varphi,\Phi}(r)\right|\left[\left(\frac{r}{2}\right)^{(m-1)q+1}+\left(\frac{2-r}{2}\right)^{(m-1)q+1}\right]dr, \tag{5.49}$$

where $\frac{1}{p}+\frac{1}{q}=1$.

Proof. From Proposition 6, we get

$$\left|\Delta^*_{\varphi,\Phi}(1)M^m\left(\frac{a_1+b_1}{2}\right)-\frac{1}{b_1-a_1}\left[{}^{\varphi}_{\Phi}\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1)+{}^{\varphi}_{\Phi}\,\mathcal{G}_{\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1)\right]\right|$$
$$\leq\frac{m(b_1-a_1)}{4}\int_0^1|\Delta_{\varrho,\mu}(r)|\left[\left|M^{m-1}\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)\right|+\left|M^{m-1}\left(\frac{r}{2}a_1+\frac{2-r}{2}b_1\right)\right|\right]dr. \quad (5.50)$$

By the Hölder's inequality, we have

$$\left|M^{m-1}\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)\right|\leq\int_{a_1}^{b_1}\left(x-\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)\right)^{m-1}|\psi(x)|\,dx$$
$$\leq\left(\int_{a_1}^{b_1}|\psi(x)|^p\,dx\right)^{\frac{1}{p}}\left(\int_{a_1}^{b_1}\left(x-\left(\frac{r}{2}b_1+\frac{2-r}{2}a_1\right)\right)^{(m-1)q}dx\right)^{\frac{1}{q}}. \quad (5.51)$$

Inequalities (5.50) and (5.51) give the required estimate. □

Remark 5.6. We deduce associated inequalities from Proposition 8 for central moment of a random variable via different fractional and conformable integrals.

(i) If $\varphi(u)=\frac{u^{\frac{\vartheta}{k}}}{k\Gamma_k(\vartheta)}$, then we have

$$\left|A_1^{\vartheta,k}(1)M^m\left(\frac{a_1+b_1}{2}\right)-\frac{\Gamma_k(\vartheta+k)}{2}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^k_{\left(\frac{a_1+b_1}{2}\right)^+}\widehat{M^m}(b_1)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^k_{\left(\frac{a_1+b_1}{2}\right)^-}\widehat{M^m}(a_1)\right]\right|$$
$$\leq\frac{m(1-(-1)^{(m-1)q+1})(b_1-a_1)^{(m-1)q+2}\|\psi\|_p}{4((m-1)q+1)}$$
$$\times\int_0^1\left|A_1^{\vartheta,k}(r)\right|\left[\left(\frac{r}{2}\right)^{(m-1)q+1}+\left(\frac{2-r}{2}\right)^{(m-1)q+1}\right]dr, \quad (5.52)$$

where $A_1^{\alpha,k}(r)$ is same as in Remark 5.1.

(ii) If $\varrho(u)=u^{\frac{\gamma}{k}}\mathcal{F}^{\sigma,k}_{\rho,\gamma}(w(u)^\rho)$, then following inequality holds:

$$\left|{}^{\Phi}C^{\vartheta,\sigma}_{\rho,k}(1)M^m\left(\frac{a_1+b_1}{2}\right)-\frac{1}{2k}\left[{}^{\vartheta}_{\Phi}\mathcal{J}^{k,\rho}_{\sigma,\left(\frac{a_1+b_1}{2}\right)^+;w}\widehat{M^m}(b_1)+{}^{\vartheta}_{\Phi}\,\mathcal{J}^{k,\rho}_{\sigma,\left(\frac{a_1+b_1}{2}\right)^-;w}\widehat{M^m}(a_1)\right]\right|$$
$$\leq\frac{m(1-(-1)^{(m-1)q+1})(b_1-a_1)^{(m-1)q+2}\|\psi\|_p}{4((m-1)q+1)}$$
$$\times\int_0^1\left|{}^{\Phi}C^{\vartheta,\sigma}_{\rho,k}(r)\right|\left[\left(\frac{r}{2}\right)^{(m-1)q+1}+\left(\frac{2-r}{2}\right)^{(m-1)q+1}\right]dr, \quad (5.53)$$

where ${}^{\Phi}C_{\rho,k}^{\vartheta,\sigma}(r)$ is same as in (5.40).

(iii) If we choose $\varphi(u) = u^{\vartheta}\mathcal{E}_{\beta,\vartheta,\nu}^{\delta,r,s,c}(w(u)^{\beta}; q^*)$, then following inequality holds:

$$\begin{aligned}
&\Bigg|{}_{\Phi}R_{\beta,\vartheta,\nu}^{\delta,r,s,c}(1)M^m\left(\frac{a_1+b_1}{2}\right) \\
&- \frac{1}{b_1-a_1}\left[{}_{\Phi}^{\vartheta}\mathcal{J}_{\beta,\nu,\left(\frac{a_1+b_1}{2}\right)^+}^{\delta,r,s,c}\widehat{M^m}\,(b_1; q^*, w) + {}_{\Phi}^{\vartheta}\mathcal{J}_{\beta,\nu,\left(\frac{a_1+b_1}{2}\right)^-}^{\delta,r,s,c}\widehat{M^m}\,(a_1; q^*, w)\right]\Bigg| \\
&\leq \frac{m(1-(-1)^{(m-1)q+1})(b_1-a_1)^{(m-1)q+2}\|\psi\|_p}{4((m-1)q+1)} \\
&\quad\times \int_0^1 \left|{}_{\Phi}R_{\beta,\vartheta,\nu}^{\delta,r,s,c}(r)\right| \left[\left(\frac{r}{2}\right)^{(m-1)q+1} + \left(\frac{2-r}{2}\right)^{(m-1)q+1}\right] dr. \qquad (5.54)
\end{aligned}$$

(iv) If $\varphi(u) = \frac{u}{\vartheta}\exp(-Au)$, where $A = \frac{1-\vartheta}{\vartheta}$, $\vartheta \in (0,1)$, then following inequality holds:

$$\begin{aligned}
&\left|N_{\vartheta}^{\Phi}(1)M^m\left(\frac{a_1+b_1}{2}\right) - \frac{1-\vartheta}{2}\left[{}_{\Phi}^{\vartheta}\mathcal{J}_{\left(\frac{a_1+b_1}{2}\right)^+}^{e}\widehat{M^m}(b_1) + {}_{\Phi}^{\vartheta}\mathcal{J}_{\left(\frac{a_1+b_1}{2}\right)^-}^{e}\widehat{M^m}(a_1)\right]\right| \\
&\leq \frac{m(1-(-1)^{(m-1)q+1})(b_1-a_1)^{(m-1)q+2}\|\psi\|_p}{4((m-1)q+1)} \\
&\quad\times \int_0^1 \left|N_{\gamma}^{\mu}(r)\right| \left[\left(\frac{r}{2}\right)^{(m-1)q+1} + \left(\frac{2-r}{2}\right)^{(m-1)q+1}\right] dr. \qquad (5.55)
\end{aligned}$$

Chapter 6

On Hermite–Hadamard type inequalities of coordinated (p_1, h_1)-(p_2, h_2)-convex functions via Katugampola fractional integrals

In the recent past coordinated convexity has been utilized all the more often for the foundation of Hermite–Hadamard type inequality. Noor et al. [134] characterized another class of convex functions named as two dimensional(coordinated) pq-convex functions. They sum up certain inequality for the noted family of functions. Yang [197] broaden this thought for a more broad class coined as coordinated (p_1, h_1)-(p_2, h_2)-convex functions. This is an exceptionally enormous class as it incorporates the families called coordinated (p_1, s_1)-(p_2, s_2)-convex, pq-convex, (h_1, h_2)-convex, (s_1, s_2)-convex, h-convex, s-convex and coordinated convex functions as special cases. Yang set up inequalities of Hermite–Hadamard type for this class of convexity and associated the outcomes to the recently known outcomes for traditional integrals. Sarikaya [164] settled the fractional Hermite–Hadamard type inequalities for coordinated convex functions. Chen [32] set up fractional counterpart of Hermite–Hadamard type inequalities for coordinated s-convex functions. Set et al. [179] considered coordinated h-convex functions and generalized certain inequalities of Hermite–Hadamard type.

This chapter build up the Hermite–Hadamard type inequalities for coordinated (p_1, h_1)-(p_2, h_2)-convex functions through Katugampola fractional integrals. We introduced our outcomes for a few exceptional cases as mentioned earlier. Results demonstrated in this examination are keep on holding for set up outcomes, see [32, 164, 179]. The results of this section has been published. This chapter follows the definite succession.

Section 6.1 reload a few definitions and twofold type of Katugampola fractional integrals. Section 6.2 annunciate some new Hermite–Hadamard type inequalities of (p_1, h_1)-(p_2, h_2)-convex functions through Katugampola fractional integrals and conclude some fascinating inequalities for including classes of coordinated convexity.

6.1 Katugampola fractional integral and its useful identities

This section recall some useful definitions.

Definition 6.1. [86] $X_c^p(a_1, b_1)(c \in \mathbb{R}, 1 \leq p \leq \infty)$ is the set of those complex valued Lebesgue measurable functions ζ defined on $[a_1, b_1]$ for which $\|\zeta\|_{X_c^p} < \infty$, where the norm is defined by $\|\zeta\|_{X_c^p} = \left(\int\limits_{a_1}^{b_1} |\tau^c \zeta(\tau)|^p \frac{d\tau}{\tau}\right)^{\frac{1}{p}} < \infty$ for $1 \leq p < \infty, c \in \mathbb{R}$ and for the case $p = \infty$, $\|\zeta\|_{X_c^p} = ess \sup\limits_{a_1 \leq \tau \leq b_1} [\tau^c |\zeta(\tau)|], c \in \mathbb{R}$.

Katugampla fractional integrals into two dimensional case may be given as follows:

Definition 6.2. Let $\zeta \in X_c^p(\Delta)$. The Katugampola fractional integrals ${}^{\vartheta_1,\vartheta_2}\mathcal{I}^{p_1,p_2}_{a_1^+,a_2^+}$, ${}^{\vartheta_1,\vartheta_2}\mathcal{I}^{p_1,p_2}_{a_1^+,b_2^-}$,

${}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{b_1^-,a_2^+}$ and ${}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{b_1^-,b_2^-}$ of order $\vartheta_1, \vartheta_2 > 0$ with $a_1, a_2 \geq 0$ are defined by

$$ {}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{a_1^+,a_2^+}\zeta(x,y) = \frac{p_1^{1-\vartheta_1}\ p_2^{1-\vartheta_2}}{\Gamma(\vartheta_1)\Gamma(\vartheta_2)} \int_{a_1}^{x}\int_{a_2}^{y} \frac{t^{p_1-1}s^{p_2-1}}{(x^{p_1}-t^{p_1})^{1-\vartheta_1}(y^{p_2}-s^{p_2})^{1-\vartheta_2}}\zeta(t,s)\,dsdt,\ x > a_1, y > c_1, $$

$$ {}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{a_1^+,b_2^-}\zeta(x,y) = \frac{p_1^{1-\vartheta_1}\ p_2^{1-\vartheta_2}}{\Gamma(\vartheta_1)\Gamma(\vartheta_2)} \int_{a_1}^{x}\int_{y}^{b_2} \frac{t^{p_1-1}s^{p_2-1}}{(x^{p_1}-t^{p_1})^{1-\vartheta_1}(s^{p_2}-y^{p_2})^{1-\vartheta_2}}\zeta(t,s)\,dsdt,\ x > a_1, y < b_2, $$

$$ {}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{b_1^-,a_2^+}\zeta(x,y) = \frac{p_1^{1-\vartheta_1}\ p_2^{1-\vartheta_2}}{\Gamma(\vartheta_1)\Gamma(\vartheta_2)} \int_{x}^{b_1}\int_{a_2}^{y} \frac{t^{p_1-1}s^{p_2-1}}{(t^{p_1}-x^{p_1})^{1-\vartheta_1}(y^{p_2}-s^{p_2})^{1-\vartheta_2}}\zeta(t,s)\,dsdt,\ x < b_1, y > a_2 $$

and

$$ {}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{b_1^-,b_2^-}\zeta(x,y) = \frac{p_1^{1-\vartheta_1}\ p_2^{1-\vartheta_2}}{\Gamma(\vartheta_1)\Gamma(\vartheta_2)} \int_{x}^{b_1}\int_{y}^{b_2} \frac{t^{p_1-1}s^{p_2-1}}{(t^{p_1}-x^{p_1})^{1-\vartheta_1}(s^{p_2}-y^{p_2})^{1-\vartheta_2}}\zeta(t,s)\,dsdt,\ x < b_1, y < b_2 $$

respectively and $p_1, p_2 > 0$. Here Γ is the Gamma function. Moreover,

$$ {}^{0,0}\mathcal{J}^{p_1,p_2}_{a_1^+,a_2^+}\zeta(x,y) = {}^{0,0}\mathcal{J}^{p_1,p_2}_{a_1^+,b_2^-}\zeta(x,y) = {}^{0,0}\mathcal{J}^{p_1,p_2}_{b_1^-,a_2^+}\zeta(x,y) = {}^{0,0}\mathcal{J}^{p_1,p_2}_{b_1^-,b_2^-}\zeta(x,y) = \zeta(x,y) $$

and

$$ {}^{1,1}\mathcal{J}^{p_1,p_2}_{a_1^+,b_2^-}\zeta(x,y) = \int_{a_1}^{x}\int_{y}^{b_2} t^{p_1-1}s^{p_2-1}\zeta(t,s)\,dsdt. $$

Similar to one dimensional form given in Definition 1.17, we introduce the following fractional integrals:

$$ {}^{\vartheta_1}\mathcal{J}^{p_1}_{a_1^+}\zeta\left(x, \left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) = \frac{p_1^{1-\vartheta_1}}{\Gamma(\vartheta_1)} \int_{a_1}^{x} \frac{t^{p_1-1}}{(x^{p_1}-t^{p_1})^{1-\vartheta_1}}\zeta\left(t, \left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) dt,\ x > a_1, $$

$$ {}^{\vartheta_1}\mathcal{J}^{p_1}_{b_1^-}\zeta\left(x, \left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) = \frac{p_1^{1-\vartheta_1}}{\Gamma(\vartheta_1)} \int_{x}^{b_1} \frac{t^{p_1-1}}{(t^{p_1}-x^{p_1})^{1-\vartheta_1}}\zeta\left(t, \left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) dt,\ x < b_1, $$

$$ {}^{\vartheta_2}\mathcal{J}^{p_1}_{a_2^+}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, y\right) = \frac{p_2^{1-\vartheta_2}}{\Gamma(\vartheta_2)} \int_{a_2}^{y} \frac{s^{p_2-1}}{(y^{p_2}-s^{p_2})^{1-\vartheta_2}}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, s\right) ds,\ y > a_2 $$

$$ {}^{\vartheta_2}\mathcal{J}^{p_1}_{b_2^-}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, y\right) = \frac{p_2^{1-\vartheta_2}}{\Gamma(\vartheta_2)} \int_{y}^{b_2} \frac{s^{p_2-1}}{(s^{p_2}-y^{p_2})^{1-\vartheta_2}}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, s\right) ds,\ y < b_2 $$

6.2 Hermite–Hadamard type inequalities of coordinated (p_1, h_1)-(p_2, h_2)-convex functions via Katugampola fractional integrals

In this section we give the Hadamard type inequalities by using (p_1, h_1)-(p_2, h_2)-convex functions of two variables on $\mathscr{D} = [a_1, b_1] \times [a_2, b_2]$.

Theorem 6.2.1. *Suppose that $\zeta : \mathscr{D} \to \mathbb{R}$ is a (p_1, h_1)-(p_2, h_2)-convex function on the coordinates on $\mathscr{D}$ and $\zeta \in L_1(\mathscr{D})$. Then one has the inequalities:*

$$\begin{aligned}
&\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\
&\leq \frac{p_1^{\vartheta_1}p_2^{\vartheta_2}\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\Big[\, {}^{\vartheta_1,\vartheta_2}\mathscr{I}^{p_1,p_2}_{a_1^+,a_2^+}\zeta(b_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathscr{I}^{p_1,p_2}_{a_1^+,b_2^-}\zeta(b_1,a_2) \\
&\quad + {}^{\vartheta_1,\vartheta_2}\mathscr{I}^{p_1,p_2}_{b_1^-,a_2^+}\zeta(a_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathscr{I}^{p_1,p_2}_{b_1^-,b_2^-}\zeta(a_1,a_2)\Big] \\
&\leq \frac{\vartheta_1\vartheta_2}{4}\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right] \\
&\quad \times \int_0^1\int_0^1 t_1^{\vartheta_1-1}t_2^{\vartheta_2-1}\left[h_1(t_1)+h_1(1-t_1)\right]\left[h_2(t_2)+h_2(1-t_2)\right]dt_1dt_2. \qquad (6.1)
\end{aligned}$$

Proof. Let $\xi_1^{p_1} = t_1 a_1^{p_1} + (1-t_1) b_1^{p_1}$, $\xi_2^{p_1} = (1-t_1) a_1^{p_1} + t_1 b_1^{p_1}$ and $\mu_1^{p_2} = t_2 a_2^{p_2} + (1-t_2) b_2^{p_2}$, $\mu_2^{p_2} = (1-t_2) a_2^{p_2} + t_2 b_2^{p_2}$, then by coordinated (p_1, h_1)-(p_2, h_2)-convexity of ζ, we have,

$$\begin{aligned}
&\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) = \zeta\left(\left[\frac{\xi_1^{p_1}+\xi_2^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{\mu_1^{p_2}+\mu_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\
&\leq h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)\left[\zeta(\xi_1,\mu_1)+\zeta(\xi_1,\mu_2)+\zeta(\xi_2,\mu_1)+\zeta(\xi_2,\mu_2)\right]. \qquad (6.2)
\end{aligned}$$

An integration over $([0,1]\times[0,1])$, after multiplying with $\frac{\vartheta_1\vartheta_2}{4}t_1^{\alpha-1}t_2^{\beta-1}$, we obtain

$$\begin{aligned}
&\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\
&\leq \frac{\vartheta_1\vartheta_2}{4}\int_0^1\int_0^1 t_1^{\vartheta_1-1}t_2^{\vartheta_2-1}\left[\zeta(\xi_1,\mu_1)+\zeta(\xi_1,\mu_2)+\zeta(\xi_2,\mu_1)+\zeta(\xi_2,\mu_2)\right]dt_1dt_2. \qquad (6.3)
\end{aligned}$$

Note that by the change of variable, we have on the right-hand side of the inequality (6.3):

$$\begin{aligned}
&\int_0^1\int_0^1 t_1^{\vartheta_1-1}t_2^{\vartheta_2-1}\left[\zeta\left(\left[t_1a_1^{p_1}+(1-t_1)\,b_1^{p_1}\right]^{\frac{1}{p_1}},\left[t_2a_2^{p_2}+(1-t_2)\,b_2^{p_2}\right]^{\frac{1}{p_2}}\right)\right.\\
&+\zeta\left(\left[t_1a_1^{p_1}+(1-t_1)\,b_1^{p_1}\right]^{\frac{1}{p_1}},\left[(1-t_2)\,a_2^{p_2}+t_2b_2^{p_2}\right]^{\frac{1}{p_2}}\right)\\
&+\zeta\left(\left[(1-t_1)\,a_1^{p_1}+t_1b_1^{p_1}\right]^{\frac{1}{p_1}},\left[t_2a_2^{p_2}+(1-t_2)\,b_2^{p_2}\right]^{\frac{1}{p_2}}\right)\\
&\left.+\,\zeta\left(\left[(1-t_1)\,a_1^{p_1}+t_1b_1^{p_1}\right]^{\frac{1}{p_1}},\left[(1-t_2)\,a_2^{p_2}+t_2b_2^{p_2}\right]^{\frac{1}{p_2}}\right)\right]dt_1dt_2\\
=\;&\frac{p_1p_2}{\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\left[\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{x^{p_1-1}y^{p_2-1}}{\left(b_1^{p_1}-x^{p_1}\right)^{1-\vartheta_1}\left(b_2^{p_2}-y^{p_2}\right)^{1-\vartheta_2}}\zeta(x,y)\,dydx\right.\\
&+\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{x^{p_1-1}y^{p_2-1}}{\left(b_1^{p_1}-x^{p_1}\right)^{1-\vartheta_1}\left(y^{p_2}-a_2^{p_2}\right)^{1-\vartheta_2}}\zeta(x,y)\,dydx\\
&+\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{x^{p_1-1}y^{p_2-1}}{\left(x^{p_1}-a_1^{p_1}\right)^{1-\vartheta_1}\left(b_2^{p_2}-y^{p_2}\right)^{1-\vartheta_2}}\zeta(x,y)\,dydx\\
&\left.+\int_a^b\int_c^d\frac{x^{p_1-1}y^{p_2-1}}{\left(x^{p_1}-a_1^{p_1}\right)^{1-\alpha}\left(y^{p_2}-a_2^{p_2}\right)^{1-\beta}}\zeta(x,y)\,dydx\right].
\end{aligned}$$

The application of the Definition 6.2 leads to the firs inequality of (6.1).

We use the coordinated (p_1,h_1)-(p_2,h_2)-convexity of ζ to obtain the second inequality on the right hand side of (6.1):

$$\begin{aligned}
\zeta(x,u) &= \zeta\left(\left[t_1a_1^{p_1}+(1-t_1)\,b_1^{p_1}\right]^{\frac{1}{p_1}},\left[t_2a_2^{p_2}+(1-t_2)\,b_2^{p_2}\right]^{\frac{1}{p_2}}\right)\\
&\le h_1(t_1)\,h_2(t_2)\,\zeta(a_1,a_2)+h_1(t_1)\,h_2(1-t_2)\,\zeta(a_1,b_2) \qquad (6.4)\\
&\quad +h_1(1-t_1)\,h_2(t_2)\,\zeta(b_1,a_2)+h_1(1-t_1)\,h_2(1-t_2)\,\zeta(b_1,b_2),
\end{aligned}$$

$$\begin{aligned}
\zeta(x,v) &= \zeta\left(\left[t_1a_1^{p_1}+(1-t_1)\,b_1^{p_1}\right]^{\frac{1}{p_1}},\left[(1-t_2)\,a_2^{p_2}+t_2b_2^{p_2}\right]^{\frac{1}{p_2}}\right)\\
&\le h_1(t_1)\,h_2(1-t_2)\,\zeta(a_1,a_2)+h_1(t_1)\,h_2(t_2)\,\zeta(a_1,b_2) \qquad (6.5)\\
&\quad +h_1(1-t_1)\,h_2(1-t_2)\,\zeta(b_1,a_2)+h_1(1-t_1)\,h_2(t_2)\,\zeta(b_1,b_2),
\end{aligned}$$

$$\begin{aligned}\zeta(y,u) &= \zeta\left(\left[(1-t_1)a_1^{p_1}+t_1b_1^{p_1}\right]^{\frac{1}{p_1}},\left[t_2a_2^{p_2}+(1-t_2)b_2^{p_2}\right]^{\frac{1}{p_2}}\right)\\ &\le h_1(1-t_1)h_2(t_2)\zeta(a_1,a_2)+h_1(1-t_1)h_2(1-t_2)\zeta(a_1,b_2)\\ &\quad +h_1(t_1)h_2(t_2)\zeta(b_1,a_2)+h_1(t_1)h_2(1-t_2)\zeta(b_1,b_2) \qquad (6.6)\end{aligned}$$

and

$$\begin{aligned}\zeta(y,v) &= \zeta\left(\left[(1-t_1)a_1^{p_1}+t_1b_1^{p_1}\right]^{\frac{1}{p_1}},\left[(1-t_2)a_2^{p_2}+t_2b_2^{p_2}\right]^{\frac{1}{p_2}}\right)\\ &\le h_1(1-t_1)h_2(1-t_2)\zeta(a_1,a_2)+h_1(1-t_1)h_2(t_2)\zeta(a_1,b_2)\\ &\quad +h_1(t_1)h_2(1-t_2)\zeta(b_1,a_2)+h_1(t_1)h_2(t_2)\zeta(b_1,b_2). \qquad (6.7)\end{aligned}$$

Adding inequalities (6.4), (6.5), (6.6), and (6.7), we come to the result:

$$\begin{aligned}&\zeta(x,u)+\zeta(x,v)+\zeta(y,u)+\zeta(y,v)\\ &\le\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right]\{h_1(t_1)h_2(t_2)+h_1(t_1)h_2(1-t_2)\\ &+h_1(1-t_1)h_2(t_2)+h_1(1-t_1)h_2(1-t_2)\}. \qquad (6.8)\end{aligned}$$

An integration over $([0,1]\times[0,1])$, after multiplying (6.8) by $\frac{\vartheta_1\vartheta_2}{4}t_1^{\vartheta_1-1}t_2^{\vartheta_2-1}$ we get the second inequality of (6.1) by applying Definition 6.2, which then completes the proof. □

Remark 6.3. If $\vartheta_1=1=\vartheta_2$, then above result become Theorem 1.6.9 which was proved in [197].

Remark 6.4. If $h_1(t)=t=h_2(t)$ and $\vartheta_1=\vartheta_2=1$, then above result coincide to Theorem 1.6.8 which was proved in [134].

Remark 6.5. If $p_1=p_2=1$, then inequality (6.1) reduced to:

$$
\begin{aligned}
&\frac{1}{4h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\zeta\left(\frac{a_1+b_1}{2},\frac{a_2+b_2}{2}\right)\\
\leq\ &\frac{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4(b_1-a_1)^{\vartheta_1}(b_2-a_2)^{\vartheta_2}}\Big[\ {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,a_2^+}\zeta(b_1,b_2)+{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,b_2^-}\zeta(b_1,a_2)\\
&+{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,a_2^+}\zeta(a_1,b_2)+{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,b_2^-}\zeta(a_1,a_2)\Big]\\
\leq\ &\frac{\vartheta_1\vartheta_2}{4}\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right]\\
&\quad\times\int_0^1\int_0^1 t_1^{\vartheta_1-1}t_2^{\vartheta_2-1}\left[h_1(t_1)+h_1(1-t_1)\right]\left[h_2(t_2)+h_2(1-t_2)\right]dt_1dt_2.
\end{aligned}
$$

This result broaden the scope of Theorem 2.1 of [179]. It coincide with Theorem 2.1 of [179], if $h_1(\theta)=h_2(\theta)=h(\theta)$. Furthermore, if $\vartheta_1=\vartheta_2=1$, it reduced to Theorem 7 of [98].

Remark 6.6. If $h_1(t)=h_2(t)=t$ and $p_1=p_2=1$, then our result coincide with Theorem 1.6.6.

Corollary 6.7. *Suppose that $\zeta:\mathcal{D}\to\mathbb{R}$ is (p_1,s_1)-(p_2,s_2)-convex function on the coordinates on $\mathcal{D}$ and $\zeta\in L_1(\mathcal{D})$. Then one has the inequalities:*

$$
\begin{aligned}
&2^{s_1+s_2-2}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)\\
&\leq\frac{p_1^{\vartheta_1}p_2^{\vartheta_2}\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\Big[\ {}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{a_1^+,a_2^+}\zeta(b_1,b_2)+{}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{a_1^+,b_2^-}\zeta(b_1,a_2)\\
&\quad+{}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{b_1^-,a_2^+}\zeta(a_1,b_2)+{}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{b_1^-,b_2^-}\zeta(a_1,a_2)\Big]\\
&\leq\frac{\vartheta_1\vartheta_2}{4}\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right]\Bigg\{\frac{1}{(\vartheta_1+s_1)(\vartheta_2+s_2)}\\
&\quad+\frac{B(\vartheta_2,s_2+1)}{(\vartheta_1+s_1)}+\frac{B(\vartheta_1,s_1+1)}{(\vartheta_2+s_2)}+B(\vartheta_2,s_2+1)B(\vartheta_1,s_1+1)\Bigg\},
\end{aligned}
$$

where $\mathcal{B}(.,.)$ is the Beta function defined by (1.31).

Proof. If one chooses $h_1(t)=t^{s_1},h_2(t)=t^{s_2}$ for $s_1,s_2\in(0,1]$, then calculation of integrals involved in inequality (6.1) leads to the required result. □

Corollary 6.8. *Suppose that $\zeta : \mathscr{D} \to \mathbb{R}$ is an (s_1, s_2)-convex function on the coordinates on $\mathscr{D}$ and $\zeta \in L_1(\mathscr{D})$. Then one has the inequalities:*

$$
\begin{aligned}
& 2^{s_1+s_2-2}\zeta\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) \\
\leq \; & \frac{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4(b_1-a_1)^{\vartheta_1}(b_2-a_2)^{\vartheta_2}}\Big[{}^{\vartheta_1,\vartheta_2}\mathscr{J}_{a_1^+,a_2^+}\zeta(b_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathscr{J}_{a_1^+,b_2^-}\zeta(b_1,a_2) \\
& \quad + {}^{\vartheta_1,\vartheta_2}\mathscr{J}_{b_1^-,a_2^+}\zeta(a_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathscr{J}_{b_1^-,b_2^-}\zeta(a_1,a_2)\Big] \\
\leq \; & \frac{\vartheta_1\vartheta_2}{4}\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right]\Bigg\{\frac{1}{(\vartheta_1+s_1)(\vartheta_2+s_2)} \\
& \quad + \frac{B(\vartheta_2,s_2+1)}{(\vartheta_1+s_1)} + \frac{B(\vartheta_1,s_1+1)}{(\vartheta_2+s_2)} + B(\vartheta_2,s_2+1)B(\vartheta_1,s_1+1)\Bigg\},
\end{aligned}
$$

where $\mathscr{B}(.,.)$ is the Beta function defined by (1.31). Above result extends Theorem 10 of [32].

Corollary 6.9. *Suppose that $\zeta : \mathscr{D} \to \mathbb{R}$ is a (p_1, s)-(p_2, s)-convex function on the coordinates on $\mathscr{D}$ and $\zeta \in L_1(\mathscr{D})$. Then one has the inequalities:*

$$
\begin{aligned}
& 4^{s-1}\zeta\left(\left[\frac{a^p+b^p}{2}\right]^{\frac{1}{p}}, \left[\frac{c^p+d^p}{2}\right]^{\frac{1}{p}}\right) \\
\leq & \frac{p_1^{\vartheta_1}p_2^{\vartheta_2}\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4(b_1^{p_1}-a_1^{p_1})^{\vartheta_1}(b_2^{p_2}-a_2^{p_2})^{\vartheta_2}}\Big[{}^{\vartheta_1,\vartheta_2}\mathscr{J}^{p_1,p_2}_{a_1^+,a_2^+}\zeta(b_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathscr{J}^{p_1,p_2}_{a_1^+,b_2^-}\zeta(b_1,a_2) \\
& + {}^{\vartheta_1,\vartheta_2}\mathscr{J}^{p_1,p_2}_{b_1^-,a_2^+}\zeta(a_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathscr{J}^{p_1,p_2}_{b_1^-,b_2^-}\zeta(a_1,a_2)\Big] \\
\leq & \frac{\vartheta_1\vartheta_2}{4}\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right]\Bigg\{\frac{1}{(\vartheta_1+s_1)(\vartheta_2+s_2)} \\
& + \frac{B(\vartheta_2,s+1)}{(\vartheta_1+s)} + \frac{B(\vartheta_1,s+1)}{(\vartheta_2+s)} + B(\vartheta_2,s+1)B(\vartheta_1,s+1)\Bigg\}.
\end{aligned}
$$

Corollary 6.10. *Suppose that $\zeta : \mathscr{D} \to \mathbb{R}$ is (p, s)-convex function on the coordinates on $\mathscr{D}$ and $\zeta \in L_1(\mathscr{D})$. Then one has the inequalities:*

$$
\begin{aligned}
& 4^{s-1}\zeta\left(\left[\frac{a_1^p+b_1^p}{2}\right]^{\frac{1}{p}}, \left[\frac{a_2^p+b_2^p}{2}\right]^{\frac{1}{p}}\right) \\
\leq & \frac{p^{\vartheta_1+\vartheta_2}\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4(b_1^p-a_1^p)^{\vartheta_1}(b_2^p-a_2^p)^{\vartheta_2}}\Big[{}^{\vartheta_1,\vartheta_2}\mathscr{J}^{p,p}_{a_1^+,a_2^+}\zeta(b_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathscr{J}^{p,p}_{a_1^+,b_2^-}\zeta(b_1,a_2) \\
& + {}^{\vartheta_1,\vartheta_2}\mathscr{J}^{p,p}_{b_1^-,a_2^+}\zeta(a_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathscr{J}^{p,p}_{b_1^-,b_2^-}\zeta(a_1,a_2)\Big]
\end{aligned}
$$

$$\leq \frac{\vartheta_1\vartheta_2}{4}\left[\zeta\left(a_1,a_2\right)+\zeta\left(a_1,b_2\right)+\zeta\left(b_1,a_2\right)+\zeta\left(b_1,b_2\right)\right]\left\{\frac{1}{(\vartheta_1+s_1)(\vartheta_2+s_2)}\right.$$
$$\left.+\frac{B(\vartheta_2,s+1)}{(\vartheta_1+s)}+\frac{B(\vartheta_1,s+1)}{(\vartheta_2+s)}+B(\vartheta_2,s+1)B(\vartheta_1,s+1)\right\}.$$

This result also gives a generalization of Theorem 10 of [32].

For the next result, we derive following result for one dimensional case.

Proposition 9. *Let $\zeta : I=[a_1,b_1]\subseteq(0,\infty)\to\mathbb{R}$ be a (p,h)-convex function and $\zeta\in L_1[a_1,b_1]$. Then following double inequality holds:*

$$\begin{aligned}\frac{1}{h\left(\frac{1}{2}\right)}\zeta\left(\left[\frac{a_1^p+b_1^p}{2}\right]^{\frac{1}{p}}\right) &\leq \frac{p^{\vartheta}\Gamma\left(\vartheta+1\right)}{\left(b_1^p-a_1^p\right)^{\vartheta}}\left[{}^{\vartheta}\mathcal{J}^p_{a_1^+}\zeta\left(b_1\right)+{}^{\vartheta}\mathcal{J}^p_{b_1^-}\zeta\left(a_1\right)\right]\\ &\leq \vartheta\left[\zeta\left(a_1\right)+\zeta\left(b_1\right)\right]\int_0^1\lambda^{\vartheta-1}\left[h\left(\lambda\right)+h\left(1-\lambda\right)\right]d\lambda. \qquad (6.9)\end{aligned}$$

Proof. Since ζ is a (p,h)-convex function on $[a_1,b_1]$, so by taking $\xi_1^p=\lambda a_1^p+(1-\lambda)\,b_1^p,\xi_2^p=(1-\lambda)\,a_1^p+\lambda b_1^p$ and for all $\lambda\in[0,1]$,

$$\frac{1}{h\left(\frac{1}{2}\right)}\zeta\left(\left[\frac{a_1^p+b_1^p}{2}\right]^{\frac{1}{p}}\right)\leq\zeta\left([\lambda a_1^p+(1-\lambda)b_1^p]^{\frac{1}{p}}\right)+\zeta\left([(1-\lambda)a_1^p+tb_1^p]^{\frac{1}{p}}\right). \qquad (6.10)$$

Multiplying both sides of (6.10) by $\lambda^{\vartheta-1}$ and integrating w.r.t. λ over $[0,1]$,

$$\begin{aligned}\frac{1}{\vartheta h\left(\frac{1}{2}\right)}\zeta\left(\left[\frac{a_1^p+b_1^p}{2}\right]^{\frac{1}{p}}\right) &\leq \int_0^1\lambda^{\vartheta-1}\zeta\left(\left[\lambda a_1^p+(1-\lambda)\,b_1^p\right]^{\frac{1}{p}}\right)d\lambda\\ &+\int_0^1\lambda^{\vartheta-1}\zeta\left(\left[(1-\lambda)\,a_1^p+\lambda b_1^p\right]^{\frac{1}{p}}\right)d\lambda. \qquad (6.11)\end{aligned}$$

By change of variable in (6.11), we have

$$\begin{aligned}&\frac{1}{\vartheta h\left(\frac{1}{2}\right)}\zeta\left(\left[\frac{a_1^p+b_1^p}{2}\right]^{\frac{1}{p}}\right)\\ &\leq \frac{p}{\left(b_1^p-a_1^p\right)^{\vartheta}}\left[\int_{a_1}^{b_1}\frac{\theta^{p-1}}{\left(b_1^p-\theta^p\right)^{\vartheta}}\zeta\left(\theta\right)d\theta+\int_{a_1}^{b_1}\frac{\theta^{p-1}}{\left(\theta^p-a_1^p\right)^{\vartheta}}\zeta\left(\theta\right)d\theta\right].\end{aligned}$$

Application of Definition 1.17 leads to the first inequality of (6.9).

For the next inequality, we apply the (p,h)-convexity of ζ,

$$\begin{aligned}\zeta(\xi_1)+\zeta(\xi_2) &= \zeta\left(\left[\lambda a_1^p+(1-\lambda)b_1^p\right]^{\frac{1}{p}}\right)+\zeta\left(\left[(1-\lambda)a_1^p+tb_1^p\right]^{\frac{1}{p}}\right)\\ &\leq \left[\zeta(a_1)+\zeta(b_1)\right](h(\lambda)+h(1-\lambda)).\end{aligned}$$

Multiplying by $\lambda^{\vartheta-1}$ on both sides and integrating over $[0,1]$, we obtained the second inequality of (6.9). □

Remark 6.11. If $\vartheta=1$, then above result coincide to Theorem 5 of [63].

Remark 6.12. If $p=1$ and $h(\lambda)=\lambda$, then we get Theorem 2 of [181].

Now we give our next main result.

Theorem 6.2.2. *Let $\zeta:\mathcal{D}\to\mathbb{R}$ be a coordinated (p_1,h_1)-(p_2,h_2)-convex function and $\zeta\in L_1(\mathcal{D})$. Then one has the inequalities:*

$$\begin{aligned}
&\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)\\
\leq\ & \frac{p_1^{\vartheta_1}\Gamma(\vartheta_1+1)}{2h_2\left(\frac{1}{2}\right)\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}\\
&\times\left[{}^{\vartheta_1}\mathcal{J}_{a_1^+}^{p_1}\zeta\left(b_1,\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)+{}^{\vartheta_1}\mathcal{J}_{b_1^-}^{p_1}\zeta\left(a_1,\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)\right]\\
&+\frac{p_2^{\vartheta_2}\Gamma(\vartheta_2+1)}{2h_1\left(\frac{1}{2}\right)\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\\
&\times\left[{}^{\vartheta_2}\mathcal{J}_{a_2^+}^{p_2}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},b_2\right)+{}^{\vartheta_2}\mathcal{J}_{b_2^-}^{p_2}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},a_2\right)\right]\\
\leq\ & \frac{p_1^{\vartheta_1}p_2^{\vartheta_2}\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\left[{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,a_2^+}^{p_1,p_2}\zeta(b_1,b_2)+{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,b_2^-}^{p_1,p_2}\zeta(b_1,a_2)\right.\\
&\left.+{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,a_2^+}^{p_1,p_2}\zeta(a_1,b_2)+{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,b_2^-}^{p_1,p_2}\zeta(a_1,a_2)\right]\\
\leq\ & \frac{\vartheta_2 p_1^{\vartheta_1}\Gamma(\vartheta_1+1)}{2\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}\left[{}^{\vartheta_1}\mathcal{J}_{a_1^+}^{p_1}\zeta(b_1,a_2)+{}^{\vartheta_1}\mathcal{J}_{a_1^+}^{p_1}\zeta(b_1,b_2)+{}^{\vartheta_1}\mathcal{J}_{b_1^-}^{p_1}\zeta(a_1,a_2)\right.
\end{aligned}$$

$$+ \ {}^{\vartheta_1}\mathcal{J}^{p_1}_{b_1^-}\zeta(a_1,b_2)\Big]\int_0^1 t_2^{\vartheta_2-1}\left[h_2(t_2)+h_2(1-t_2)\right]dt_2$$

$$+\frac{\vartheta_1 p_2^{\vartheta_2}\Gamma(\vartheta_2+1)}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\Big[\ {}^{\vartheta_2}\mathcal{J}^{p_2}_{a_2^+}\zeta(a_1,b_2)+\ {}^{\vartheta_2}\mathcal{J}^{p_2}_{a_2^+}\zeta(b_1,b_2)+\ {}^{\vartheta_2}\mathcal{J}^{p_2}_{b_2^-}\zeta(a_1,a_2)$$

$$+\ {}^{\vartheta_2}\mathcal{J}^{p_2}_{b_2^-}\zeta(b_1,a_2)\Big]\int_0^1 t_1^{\vartheta_1-1}\left[h_1(t_1)+h_1(1-t_1)\right]dt_1$$

$$\leq \vartheta_1\vartheta_2\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right]$$

$$\times\int_0^1\int_0^1 t_1^{\vartheta_1-1}t_2^{\vartheta_2-1}\left[h_2(t_2)+h_2(1-t_2)\right]\left[h_1(t_1)+h_1(1-t_1)\right]dt_2dt_1. \tag{6.12}$$

Proof. Since $\zeta:\mathcal{D}\to\mathbb{R}$ is a (p_1,h_1)-(p_2,h_2)-convex function, so the partial mapping $\zeta_x:[a_2,b_2]\to\mathbb{R}$ defined by $\zeta_x(v)=\zeta(x,v)$ for all $x\in[a_1,b_1]$ is (p_2,h_2)-convex on $[a_2,b_2]$. Similarly $\zeta_y:[a_1,b_1]\to\mathbb{R}$ defined by $\zeta_y(u)=\zeta(u,y)$ for all $y\in[a_2,b_2]$ is (p_1,h_1)-convex on $[a_1,b_1]$. Then by Proposition 9 and applying the (p_2,h_2)-convexity of ζ_x, we have

$$\frac{1}{h_2\left(\frac{1}{2}\right)}\zeta_x\left(\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \leq \frac{p_2^{\vartheta_2}\Gamma(\vartheta_2+1)}{\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\left[{}^{\vartheta_2}\mathcal{J}^{p_2}_{a_2^+}\zeta_x(b_2)+{}^{\vartheta_2}\mathcal{J}^{p_2}_{b_2^-}\zeta_x(a_2)\right]$$
$$\leq \vartheta_2\left[\zeta_x(a_2)+\zeta_x(b_2)\right]\int_0^1 t_2^{\vartheta_2-1}\left[h_2(t_2)+h_2(1-t_2)\right]dt_2.$$

Or

$$\frac{1}{h_2\left(\frac{1}{2}\right)}\zeta\left(x,\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)$$
$$\leq \frac{p_2^{\vartheta_2}\Gamma(\vartheta_2+1)}{\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\left[{}^{\vartheta_2}\mathcal{J}^{p_2}_{a_2^+}\zeta(x,b_2)+{}^{\vartheta_2}\mathcal{J}^{p_2}_{b_2^-}\zeta(x,a_2)\right]$$
$$\leq \vartheta_2\left[\zeta(x,a_2)+\zeta(x,b_2)\right]\int_0^1 t_2^{\vartheta_2-1}\left[h_2(t_2)+h_2(1-t_2)\right]dt_2. \tag{6.13}$$

Integrating (6.13) w.r.t. x over $[a_1,b_1]$ after multiplying by $\frac{\vartheta_1 p_1 x^{p_1-1}}{2\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}\left(b_1^{p_1}-x^{p_1}\right)^{1-\vartheta_1}}$ and $\frac{\vartheta_1 p_1 x^{p_1-1}}{2\left(b_1^{p_1}-a_1^{p_1}\right)^{\alpha}\left(x^{p_1}-a_1^{p_1}\right)^{1-\vartheta_1}}$,

$$\frac{\vartheta_1 p_1}{2h_2\left(\frac{1}{2}\right)\left(b_1^{p_1}-a_1^{p_1}\right)^{\alpha}}\int_{a_1}^{b_1}\frac{x^{p_1-1}}{\left(b_1^{p_1}-x^{p_1}\right)^{1-\vartheta_1}}\zeta\left(x,\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)dx$$

$$\leq \frac{\vartheta_1\vartheta_2 p_1 p_2}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}\left(b_2^{p_1}-a_2^{p_1}\right)^{\vartheta_1}}\left[\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{x^{p_1-1}y^{p_2-1}\zeta(x,y)}{\left(b_2^{p_1}-x^{p_1}\right)^{1-\vartheta_1}\left(b_2^{p_2}-y^{p_2}\right)^{1-\vartheta_2}}dydx\right.$$
$$\left.+\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{x^{p_1-1}y^{p_2-1}\zeta(x,y)}{\left(b_1^{p_1}-x^{p_1}\right)^{1-\vartheta_1}\left(y^{p_2}-a_2^{p_2}\right)^{1-\vartheta_2}}dydx\right]$$
$$\leq \frac{\vartheta_1\vartheta_2 p_1}{2\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}\left[\int_{a_1}^{b_1}\frac{x^{p_1-1}\zeta(x,a_2)}{\left(b_1^{p_1}-x^{p_1}\right)^{1-\vartheta_1}}dx+\int_{a_1}^{b_1}\frac{x^{p_1-1}\zeta(x,b_2)}{\left(b_1^{p_1}-x^{p_1}\right)^{1-\vartheta_1}}dx\right]$$
$$\times\int_0^1 t_2^{\vartheta_2-1}\left[h_2(t_2)+h_2(1-t_2)\right]dt_2, \tag{6.14}$$

and

$$\frac{\vartheta_1 p_1}{2h_2\left(\frac{1}{2}\right)\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}\int_{a_1}^{b_1}\frac{x^{p_1-1}}{\left(x^{p_1}-a_1^{p_1}\right)^{1-\vartheta_1}}\zeta\left(x,\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)dx$$
$$\leq \frac{\vartheta_1\vartheta_2 p_1 p_2}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}\left[\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{x^{p_1-1}y^{p_2-1}\zeta(x,y)}{\left(x^{p_1}-a_1^{p_1}\right)^{1-\vartheta_1}\left(b_2^{p_2}-y^{p_2}\right)^{1-\vartheta_2}}dydx\right.$$
$$\left.+\int_{a_1}^{b_1}\int_{a_2}^{b_2}\frac{x^{p_1-1}y^{p_2-1}\zeta(x,y)}{\left(x^{p_1}-a_1^{p_1}\right)^{1-\vartheta_1}\left(y^{p_2}-a_2^{p_2}\right)^{1-\vartheta_2}}dydx\right]$$
$$\leq \frac{\vartheta_1\vartheta_2 p_1}{2\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}\left[\int_{a_1}^{b_1}\frac{x^{p_1-1}\zeta(x,a_2)}{\left(x^{p_1}-a_1^{p_1}\right)^{1-\vartheta_1}}dx+\int_{a_1}^{b_1}\frac{x^{p_1-1}\zeta(x,b_2)}{\left(x^{p_1}-a_1^{p_1}\right)^{1-\vartheta_1}}dx\right]$$
$$\times\int_0^1 t_2^{\vartheta_2-1}\left[h_2(t_2)+h_2(1-t_2)\right]dt_2. \tag{6.15}$$

Now again by Proposition 9 and applying (p_1, h_1)-convexity of ζ_y, we have

$$\frac{1}{h_1\left(\frac{1}{2}\right)}\zeta_y\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}\right)$$
$$\leq \frac{p_1^{\vartheta_1}\Gamma(\vartheta_1+1)}{\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}\left[{}^{\vartheta_1}\mathcal{J}_{a_1^+}^{p_1}\zeta_y(b_1)+{}^{\vartheta_1}\ \mathcal{J}_{b_1^-}^{p_1}\zeta_y(a_1)\right]$$
$$\leq \vartheta_1\left[\zeta_y(a_1)+\zeta_y(b_1)\right]\int_0^1 t_1^{\vartheta_1-1}\left[h_1(t_1)+h_1(1-t_1)\right]dt_1. \tag{6.16}$$

Integrating (6.16) w.r.t. y over $[a_2, b_2]$ after multiplying by $\frac{\vartheta_2 p_2 y^{p_2-1}}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}\left(b_2^{p_2}-y^{p_2}\right)^{1-\vartheta_2}}$ and

$\frac{\vartheta_2 p_2 y^{p_2-1}}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}\left(y^{p_2}-a_2^{p_2}\right)^{1-\vartheta_2}}$, we have

$$\begin{aligned}
& \frac{\vartheta_2 p_2}{2h_1\left(\frac{1}{2}\right)\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}} \int_{a_2}^{b_2} \frac{y^{p_2-1}}{\left(b_2^{p_2}-y^{p_2}\right)^{1-\vartheta_2}} \zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, y\right) dy \\
\leq \; & \frac{\vartheta_1 \vartheta_2 p_1 p_2}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}} \left[\int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{x^{p_1-1} y^{p_2-1} \zeta(x,y)}{\left(b_1^{p_1}-x^{p_1}\right)^{1-\vartheta_1}\left(b_2^{p_2}-y^{p_2}\right)^{1-\vartheta_2}} dy dx \right. \\
& \left. + \int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{x^{p_1-1} y^{p_2-1} \zeta(x,y)}{\left(x^{p_1}-a_1^{p_1}\right)^{1-\vartheta_1}\left(b_2^{p_2}-y^{p_2}\right)^{1-\vartheta_2}} dy dx \right] \qquad (6.17) \\
\leq \; & \frac{\vartheta_1 \vartheta_2 p_2}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}} \left[\int_{a_2}^{b_2} \frac{y^{p_2-1}\zeta(a_1,y)}{\left(b_2^{p_2}-y^{p_2}\right)^{1-\vartheta_2}} dy + \int_{a_2}^{b_2} \frac{y^{p_2-1}\zeta(b_1,y)}{\left(b_2^{p_2}-y^{p_2}\right)^{1-\vartheta_2}} dy\right] \\
& \times \int_0^1 t_1^{\vartheta_1-1}\left[h_1(t_1)+h_2(1-t_1)\right] dt_1 \qquad (6.18)
\end{aligned}$$

and

$$\begin{aligned}
& \frac{\vartheta_2 p_2}{2h_1\left(\frac{1}{2}\right)\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}} \int_{a_2}^{b_2} \frac{y^{p_2-1}}{\left(y^{p_2}-a_2^{p_2}\right)^{1-\beta}} \zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, y\right) dy \\
\leq \; & \frac{\vartheta_1 \vartheta_2 p_1 p_2}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}} \left[\int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{x^{p_1-1} y^{p_2-1} \zeta(x,y)}{\left(b_1^{p_1}-x^{p_1}\right)^{1-\vartheta_1}\left(y^{p_2}-a_2^{p_2}\right)^{1-\vartheta_2}} dy dx \right. \\
& \left. + \int_{a_1}^{b_1}\int_{a_2}^{b_2} \frac{x^{p_1-1} y^{p_2-1} \zeta(x,y)}{\left(x^{p_1}-a_1^{p_1}\right)^{1-\vartheta_1}\left(y^{p_2}-a_2^{p_2}\right)^{1-\vartheta_2}} dy dx \right] \\
\leq \; & \frac{\vartheta_1 \vartheta_2 p_2}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}} \left[\int_{a_2}^{b_2} \frac{y^{p_2-1}\zeta(a_1,y)}{\left(y^{p_2}-a_2^{p_2}\right)^{1-\vartheta_2}} dy + \int_{a_2}^{b_2} \frac{y^{p_2-1}\zeta(b_1,y)}{\left(y^{p_2}-a_2^{p_2}\right)^{1-\vartheta_2}} dy\right] \\
& \times \int_0^1 t_1^{\vartheta_1-1}\left[h_1(t_1)+h_2(1-t_1)\right] dt_1. \qquad (6.19)
\end{aligned}$$

Adding inequalities (6.14),(6.15),(6.17),(6.19) and applying Definition 6.2, one obtained

$$\begin{aligned}
\leq \; & \frac{p_1^{\vartheta_1} p_2^{\vartheta_2}\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}} \left[{}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{a_1^+,a_2^+}\zeta(b_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{a_1^+,b_2^-}\zeta(b_1,a_2) \right. \\
& \left. + {}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{b_1^-,a_2^+}\zeta(a_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathcal{J}^{p_1,p_2}_{b_1^-,b_2^-}\zeta(a_1,a_2)\right] \\
\leq \; & \frac{\vartheta_2 p_1^{\vartheta_1}\Gamma(\vartheta_1+1)}{2\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}} \left[{}^{\vartheta_1}\mathcal{J}^{p_1}_{a_1^+}\zeta(b_1,a_2) + {}^{\vartheta_1}\mathcal{J}^{p_1}_{a_1^+}\zeta(b_1,b_2) + {}^{\vartheta_1}\mathcal{J}^{p_1}_{b_1^-}\zeta(a_1,a_2) \right.
\end{aligned}$$

$$+ \ {}^{\vartheta_1}\mathcal{J}^{p_1}_{b_1^-}\zeta(a_1,b_2)\Big]\int_0^1 t_2^{\vartheta_2-1}\left[h_2(t_2)+h_2(1-t_2)\right]dt_2$$
$$+\frac{\vartheta_1 p_2^{\vartheta_2}\Gamma(\vartheta_2+1)}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\Big[\ {}^{\vartheta_2}\mathcal{J}^{p_2}_{a_2^+}\zeta(a_1,b_2)+\ {}^{\vartheta_2}\mathcal{J}^{p_2}_{a_2^+}\zeta(b_1,b_2)+\ {}^{\vartheta_2}\mathcal{J}^{p_2}_{b_2^-}\zeta(a_1,a_2)$$
$$+ \ {}^{\vartheta_2}\mathcal{J}^{p_2}_{b_2^-}\zeta(b_1,a_2)\Big]\int_0^1 t_1^{\vartheta_1-1}\left[h_1(t_1)+h_1(1-t_1)\right]dt_1.$$

Which are the second and third inequalities of (6.12).

For the last inequality of (6.12), applying Proposition 9 to the last part of above inequality, we have

$$\frac{\vartheta_2 p_1^{\vartheta_1}\Gamma(\vartheta_1+1)}{2\left(b^{p_1}-a^{p_1}\right)^{\vartheta_1}}\Big[\ {}^{\vartheta_1}\mathcal{J}^{p_1}_{a_1^+}\zeta(b_1,a_2)+\ {}^{\vartheta_1}\mathcal{J}^{p_1}_{a_1^+}\zeta(b_1,b_2)+\ {}^{\vartheta_1}\mathcal{J}^{p_1}_{b_1^-}\zeta(a_1,a_2)$$
$$+ \ {}^{\vartheta_1}\mathcal{J}^{p_1}_{b_1^-}\zeta(a_1,b_2)\Big]\int_0^1 t_2^{\vartheta_2-1}\left[h_2(t_2)+h_2(1-t_2)\right]dt_2$$
$$+\frac{\vartheta_1 p_2^{\vartheta_2}\Gamma(\vartheta_2+1)}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\Big[\ {}^{\vartheta_2}\mathcal{J}^{p_2}_{a_2^+}\zeta(a_1,b_2)+\ {}^{\vartheta_2}\mathcal{J}^{p_2}_{a_2^+}\zeta(b_1,b_2)+\ {}^{\vartheta_2}\mathcal{J}^{p_2}_{b_2^-}\zeta(a_1,a_2)$$
$$+ \ {}^{\vartheta_2}\mathcal{J}^{p_2}_{b_2^-}\zeta(b_1,a_2)\Big]\int_0^1 t_1^{\vartheta_1-1}\left[h_1(t_1)+h_1(1-t_1)\right]dt_1$$
$$\leq \vartheta_1\vartheta_2\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right]$$
$$\times\int_0^1\int_0^1 t_1^{\vartheta_1-1}t_2^{\vartheta_2-1}\left[h_2(t_2)+h_2(1-t_2)\right]\left[h_1(t_1)+h_1(1-t_1)\right]dt_2dt_1.$$

For the first inequality of (6.12), we again use Proposition 9 and get

$$\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)$$
$$\leq\frac{h_1\left(\frac{1}{2}\right)p_1^{\vartheta_1}\Gamma(\vartheta_1+1)}{\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}$$
$$\times\left[\ {}^{\vartheta_1}\mathcal{J}^{p_1}_{a_1^+}\zeta\left(b_1,\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)+{}^{\vartheta_1}\mathcal{J}^{p_1}_{b_1^-}\zeta\left(a_1,\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)\right]. \tag{6.20}$$

Similarly,

$$\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)$$

$$\leq \frac{h_2\left(\frac{1}{2}\right) p_2^{\vartheta_2}\Gamma(\vartheta_2+1)}{\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}$$
$$\times\left[{}^{\vartheta_2}\mathcal{J}_{a_2^+}^{p_2}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, b_2\right) + {}^{\vartheta_2}\mathcal{J}_{b_2^-}^{p_2}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, a_2\right)\right]. \tag{6.21}$$

Adding (6.20) and (6.21), then dividing by $2h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)$, one has the first inequality of (6.12), which then completes the proof. □

Remark 6.13. If $\vartheta_1 = 1 = \vartheta_2$, then above result gives Theorem 1.6.10, which was proved in [197].

Remark 6.14. If $p_1 = p_2 = 1$ and $h_1(t) = h_2(t) = t$, our result reduced to Theorem 4.3, which was proved in [164].

Corollary 6.15. *Let ζ be an (h_1, h_2)-convex function on the coordinates on $\mathcal{D}$ and $\zeta \in L_1(\mathcal{D})$, then one has following inequalities:*

$$\frac{1}{h_1\left(\frac{1}{2}\right)h_2\left(\frac{1}{2}\right)}\zeta\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right)$$
$$\leq \frac{\Gamma(\vartheta_1+1)}{2h_2\left(\frac{1}{2}\right)(b_1-a_1)^{\vartheta_1}}$$
$$\times\left[{}^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta\left(b_1, \frac{a_2+b_2}{2}\right) + {}^{\vartheta_1}\mathcal{J}_{b_1^-}\zeta\left(a_1, \frac{a_2+b_2}{2}\right)\right]$$
$$+\frac{\Gamma(\vartheta_2+1)}{2h_1\left(\frac{1}{2}\right)(b_2-a_2)^{\vartheta_2}}$$
$$\times\left[{}^{\vartheta_2}\mathcal{J}_{a_2^+}\zeta\left(\frac{a_1+b_1}{2}, b_2\right) + {}^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta\left(\frac{a_1+b_1}{2}, a_2\right)\right]$$
$$\leq \frac{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4(b_1-a_1)^{\vartheta_1}(b_2-a_2)^{\vartheta_2}}\left[{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,a_2^+}\zeta(b_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,b_2^-}\zeta(b_1,a_2)\right.$$
$$\left.+ {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,a_2^+}\zeta(a_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,b_2^-}\zeta(a_1,a_2)\right]$$
$$\leq \frac{\vartheta_2\Gamma(\vartheta_1+1)}{2(b_1-a_1)^{\vartheta_1}}\left[{}^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta(b_1,a_2) + {}^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta(b_1,b_2) + {}^{\vartheta_1}\mathcal{J}_{b_1^-}\zeta(a_1,a_2)\right.$$
$$\left.+ {}^{\vartheta_1}\mathcal{J}_{b_1^-}\zeta(a_1,b_2)\right]\int_0^1 t_2^{\vartheta_2-1}\left[h_2(t_2)+h_2(1-t_2)\right]dt_2$$
$$+\frac{\vartheta_1\Gamma(\vartheta_2+1)}{2(b_2-a_2)^{\vartheta_2}}\left[{}^{\vartheta_2}\mathcal{J}_{a_2^+}\zeta(a_1,b_2) + {}^{\vartheta_2}\mathcal{J}_{a_2^+}\zeta(b_1,b_2) + {}^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta(a_1,a_2)\right.$$

$$
\begin{aligned}
&+ {}^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta\left(b_1, a_2\right)\Big] \int_0^1 t_1^{\vartheta_1-1}\left[h_1\left(t_1\right)+h_1\left(1-t_1\right)\right] dt_1 \\
\leq\ & \vartheta_1\vartheta_2\left[\zeta\left(a_1, a_2\right)+\zeta\left(a_1, b_2\right)+\zeta\left(b_1, a_2\right)+\zeta\left(b_1, b_2\right)\right] \\
&\times \int_0^1\int_0^1 t_1^{\vartheta_1-1}t_2^{\vartheta_2-1}\left[h_2\left(t_2\right)+h_2\left(1-t_2\right)\right]\left[h_1\left(t_1\right)+h_1\left(1-t_1\right)\right] dt_2 dt_1.
\end{aligned}
$$

Remark 6.16. Corollary 6.15 gives the classical version of Hadamard type inequalities for coordinated (h_1, h_2)-convex functions, if $\vartheta_1 = \vartheta_2 = 1$.

Corollary 6.17. *Let $\zeta : \mathcal{D} \to \mathbb{R}$ be a (p_1, s_1)-(p_2, s_2)-convex function on the coordinates on $\mathcal{D}$ and $\zeta \in L_1(\mathcal{D})$. Then one has the inequalities:*

$$
\begin{aligned}
& 2^{s_1+s_2}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, \left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\
\leq\ & \frac{2^{s_2-1}p_1^{\vartheta_1}\Gamma\left(\vartheta_1+1\right)}{\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}\left[{}^{\vartheta_1}\mathcal{J}_{a_1^+}^{p_1}\zeta\left(b_1, \left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) + {}^{\vartheta_1}\mathcal{J}_{b_1^-}^{p_1}\zeta\left(a_1, \left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)\right] \\
& + \frac{2^{s_1-1}p_2^{\vartheta_2}\Gamma\left(\vartheta_2+1\right)}{\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}} \\
& \times \left[{}^{\vartheta_2}\mathcal{J}_{a_2^+}^{p_2}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, b_2\right) + {}^{\vartheta_2}\mathcal{J}_{b_2^-}^{p_2}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}}, a_2\right)\right] \\
& \leq \frac{p_1^{\vartheta_1}p_2^{\vartheta_2}\Gamma\left(\vartheta_1+1\right)\Gamma\left(\vartheta_2+1\right)}{4\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\Big[{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,a_2^+}^{p_1,p_2}\zeta\left(b_1, b_2\right) + {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,b_2^-}^{p_1,p_2}\zeta\left(b_1, a_2\right) \\
& \quad + {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,a_2^+}^{p_1,p_2}\zeta\left(a_1, b_2\right) + {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,b_2^-}^{p_1,p_2}\zeta\left(a_1, a_2\right)\Big] \\
& \leq \frac{\vartheta_2 p_1^{\vartheta_1}\Gamma\left(\vartheta_1+1\right)}{2\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}\Big[{}^{\vartheta_1}\mathcal{J}_{a_1^+}^{p_1}\zeta\left(b_1, a_2\right) + {}^{\vartheta_1}\mathcal{J}_{a_1^+}^{p_1}\zeta\left(b_1, b_2\right) + {}^{\vartheta_1}\mathcal{J}_{b_1^-}^{p_1}\zeta\left(a_1, a_2\right) \\
& + {}^{\vartheta_1}\mathcal{J}_{b_1^-}^{p_1}\zeta\left(a_1, b_2\right)\Big]\left\{\frac{1}{\vartheta_2+s_2}+B(\vartheta_2, s_2+1)\right\} \\
& + \frac{\vartheta_1 p_2^{\vartheta_2}\Gamma\left(\vartheta_2+1\right)}{2\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\Big[{}^{\vartheta_2}\mathcal{J}_{a_2^+}^{p_2}\zeta\left(a_1, b_2\right) + {}^{\vartheta_2}\mathcal{J}_{a_2^+}^{p_2}\zeta\left(b_1, b_2\right) + {}^{\vartheta_2}\mathcal{J}_{b_2^-}^{p_2}\zeta\left(a_1, a_2\right) \\
& + {}^{\vartheta_2}\mathcal{J}_{b_2^-}^{p_2}\zeta\left(b_1, a_2\right)\Big]\left\{\frac{1}{\vartheta_1+s_1}+B(\vartheta_1, s_1+1)\right\} \\
\leq\ & \vartheta_1\vartheta_2\left[\zeta\left(a_1, a_2\right)+\zeta\left(a_1, b_2\right)+\zeta\left(b_1, a_2\right)+\zeta\left(b_1, b_2\right)\right]\left\{\frac{1}{(\vartheta_1+s_1)(\vartheta_2+s_2)}\right. \\
& \left. + \frac{B(\vartheta_1, s_1+1)}{\vartheta_2+s_2} + \frac{B(\vartheta_2, s_2+1)}{\vartheta_1+s_1} + B(\vartheta_1, s_1+1)B(\vartheta_2, s_2+1)\right\}.
\end{aligned}
$$

Remark 6.18. If $\vartheta_1 = 1 = \vartheta_2$, then Corollary 6.17 reduces to a new result for (p_1, s_1)-(p_2, s_2)-

convex functions on the coordinates on $\mathscr{D}$ via classical integrals as follows:

$$\begin{aligned}
& 2^{s_1+s_2}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\
\leq\ & \frac{2^{s_2}p_1}{\left(b_1^{p_1}-a_1^{p_1}\right)}\int_{a_1}^{b_1} x^{p_1-1} f\left(x,\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)dx+\frac{2^{s_1}p_2}{\left(b_2^{p_2}-a_2^{p_2}\right)} \\
& \times\int_{a_2}^{b_2} y^{p_2-1}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},y\right)dy \\
\leq\ & \frac{4p_1p_2}{\left(b_1^{p_1}-a_1^{p_1}\right)\left(b_2^{p_2}-a_2^{p_2}\right)}\int_{a_1}^{b_1}\int_{a_2}^{b_2} x^{p_1-1}y^{p_2-1}f(x,y)dydx \\
\leq\ & \frac{2p_1}{\left(b_1^{p_1}-a_1^{p_1}\right)(s_2+1)}\int_{a_1}^{b} x^{p_1-1}\left[\zeta(x,a_2)+\zeta(x,b_2)\right]dx+\frac{2p_2}{\left(b_2^{p_2}-a_2^{p_2}\right)(s_1+1)} \\
& \times\int_{a_2}^{b_2} y^{p_2-1}\left[\zeta(a_1,y)+\zeta(b_1,y)\right]dy \\
\leq\ & 4\frac{\left[\zeta\left(a_1,a_2\right)+\zeta\left(a_1,b_2\right)+\zeta\left(b_1,a_2\right)+\zeta\left(b_1,b_2\right)\right]}{(s_1+1)(s_2+1)}.
\end{aligned}$$

Corollary 6.19. *Let $\zeta : \mathscr{D} \to \mathbb{R}$ be a (p_1, s)-(p_2, s)-convex function on the coordinates on $\mathscr{D}$ and $\zeta \in L_1(\mathscr{D})$. Then one has the inequalities:*

$$\begin{aligned}
& 4^{s}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right) \\
\leq\ & \frac{2^{s-1}p_1^{\vartheta_1}\Gamma\left(\vartheta_1+1\right)}{\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}\left[{}^{\vartheta_1}\mathscr{J}_{a_1^+}^{p_1}\zeta\left(b_1,\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)+{}^{\vartheta_1}\mathscr{J}_{b_1^-}^{p_1}\zeta\left(a_1,\left[\frac{a_2^{p_2}+b_2^{p_2}}{2}\right]^{\frac{1}{p_2}}\right)\right] \\
& +\frac{2^{s-1}p_2^{\vartheta_2}\Gamma\left(\vartheta_2+1\right)}{\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}} \\
& \times\left[{}^{\vartheta_2}\mathscr{J}_{a_2^+}^{p_2}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},b_2\right)+{}^{\vartheta_2}\mathscr{J}_{b_2^-}^{p_2}\zeta\left(\left[\frac{a_1^{p_1}+b_1^{p_1}}{2}\right]^{\frac{1}{p_1}},a_2\right)\right] \\
\leq\ & \frac{p_1^{\vartheta_1}p_2^{\vartheta_2}\Gamma\left(\vartheta_1+1\right)\Gamma\left(\vartheta_2+1\right)}{4\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}\left(b_2^{p_2}-a_2^{p_2}\right)^{\vartheta_2}}\left[{}^{\vartheta_1,\vartheta_2}\mathscr{J}_{a_1^+,a_2^+}^{p_1,p_2}\zeta\left(b_1,b_2\right)+{}^{\vartheta_1,\vartheta_2}\mathscr{J}_{a_1^+,b_2^-}^{p_1,p_2}\zeta\left(b_1,a_2\right)\right. \\
& \left.+{}^{\vartheta_1,\vartheta_2}\mathscr{J}_{b_1^-,a_2^+}^{p_1,p_2}\zeta\left(a_1,b_2\right)+{}^{\vartheta_1,\vartheta_2}\mathscr{J}_{b_1^-,b_2^-}^{p_1,p_2}\zeta\left(a_1,a_2\right)\right] \\
\leq\ & \frac{\vartheta_2p_1^{\vartheta_1}\Gamma\left(\vartheta_1+1\right)}{2\left(b_1^{p_1}-a_1^{p_1}\right)^{\vartheta_1}}\left[{}^{\vartheta_1}\mathscr{J}_{a_1^+}^{p_1}\zeta\left(b_1,a_2\right)+{}^{\vartheta_1}\mathscr{J}_{a_1^+}^{p_1}\zeta\left(b_1,b_2\right)+{}^{\vartheta_1}\mathscr{J}_{b_1^-}^{p_1}\zeta\left(a_1,a_2\right)\right. \\
& \left.+{}^{\vartheta_1}\mathscr{J}_{b_1^-}^{p_1}\zeta\left(a_1,b_2\right)\right]\left\{\frac{1}{\vartheta_2+s}+B(\vartheta_2,s+1)\right\}
\end{aligned}$$

$$+\frac{\vartheta_1 p_2^{\vartheta_2}\Gamma(\vartheta_2+1)}{2(b_2^{p_2}-a_2^{p_2})^{\vartheta_2}}\Big[\ {}^{\vartheta_2}\mathcal{J}_{a_2^+}^{p_2}\zeta(a_1,b_2)+\ {}^{\vartheta_2}\mathcal{J}_{a_2^+}^{p_2}\zeta(b_1,b_2)+\ {}^{\vartheta_2}\mathcal{J}_{b_2^-}^{p_2}\zeta(a_1,a_2)$$

$$+\ {}^{\vartheta_2}\mathcal{J}_{b_2^-}^{p_2}\zeta(b_1,a_2)\Big]\left\{\frac{1}{\vartheta_1+s}+B(\vartheta_1,s+1)\right\}$$

$$\leq \vartheta_1\vartheta_2\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right]\left\{\frac{1}{(\vartheta_1+s)(\vartheta_2+s)}\right.$$

$$\left.+\frac{B(\vartheta_1,s+1)}{\vartheta_2+s}+\frac{B(\vartheta_2,s+1)}{\vartheta_1+s}+B(\vartheta_1,s+1)B(\vartheta_2,s+1)\right\}.$$

Remark 6.20. If one use $\vartheta_1=1=\vartheta_2$, then inequalities in Corollary 6.19 will present the classical version of Hadamard type inequalities for coordinated (p_1,s)-(p_2,s)-convex functions.

Corollary 6.21. *Let $\zeta:\mathcal{D}\to\mathbb{R}$ be a (p,s)-convex function on the coordinates on $\mathcal{D}$ and $\zeta\in L_1(\mathcal{D})$. Then one has the inequalities:*

$$4^s\zeta\left(\left[\frac{a_1^p+b_1^p}{2}\right]^{\frac{1}{p}},\left[\frac{a_2^p+b_2^p}{2}\right]^{\frac{1}{p}}\right)$$

$$\leq \frac{2^{s-1}p^{\vartheta_1}\Gamma(\vartheta_1+1)}{(b_1^p-a_1^p)^{\vartheta_1}}\left[\ {}^{\vartheta_1}\mathcal{J}_{a_1^+}^{p}\zeta\left(b_1,\left[\frac{a_2^p+b_2^p}{2}\right]^{\frac{1}{p}}\right)+{}^{\vartheta_1}\ \mathcal{J}_{b_1^-}^{p}\zeta\left(a_1,\left[\frac{a_2^p+b_2^p}{2}\right]^{\frac{1}{p}}\right)\right]$$

$$+\frac{2^{s-1}p^{\vartheta_2}\Gamma(\vartheta_2+1)}{(b_2^p-a_2^p)^{\vartheta_2}}$$

$$\times\left[\ {}^{\vartheta_2}\mathcal{J}_{a_2^+}^{p}\zeta\left(\left[\frac{a_1^p+b_1^p}{2}\right]^{\frac{1}{p}},b_2\right)+\ {}^{\vartheta_2}\mathcal{J}_{b_2^-}^{p}\zeta\left(\left[\frac{a_1^p+b_1^p}{2}\right]^{\frac{1}{p}},a_2\right)\right]$$

$$\leq \frac{p^{\vartheta_1+\vartheta_2}\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4(b_1^p-a_1^p)^{\vartheta_1}(b_2^p-a_2^p)^{\vartheta_2}}\Big[\ {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,a_2^+}^{p,p}\zeta(b_1,b_2)+\ {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,b_2^-}^{p,p}\zeta(b_1,a_2)$$

$$+\ {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,a_2^+}^{p,p}\zeta(a_1,b_2)+\ {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,b_2^-}^{p,p}\zeta(a_1,a_2)\Big]$$

$$\leq \frac{\vartheta_2 p^{\vartheta_1}\Gamma(\vartheta_1+1)}{2(b_1^p-a_1^p)^{\vartheta_1}}\Big[\ {}^{\vartheta_1}\mathcal{J}_{a_1^+}^{p}\zeta(b_1,a_2)+\ {}^{\vartheta_1}\mathcal{J}_{a_1^+}^{p}\zeta(b_1,b_2)+\ {}^{\vartheta_1}\mathcal{J}_{b_1^-}^{p}\zeta(a_1,a_2)$$

$$+\ {}^{\vartheta_1}\mathcal{J}_{b_1^-}^{p}\zeta(a_1,b_2)\Big]\left\{\frac{1}{\vartheta_2+s}+B(\vartheta_2,s+1)\right\}$$

$$+\frac{\vartheta_1 p_2^{\vartheta_2}\Gamma(\vartheta_2+1)}{2(b_2^p-a_2^p)^{\vartheta_2}}\Big[\ {}^{\vartheta_2}\mathcal{J}_{a_2^+}^{p}\zeta(a_1,b_2)+\ {}^{\vartheta_2}\mathcal{J}_{a_2^+}^{p}\zeta(b_1,b_2)+\ {}^{\vartheta_2}\mathcal{J}_{b_2^-}^{p}\zeta(a_1,a_2)$$

$$+\ {}^{\vartheta_2}\mathcal{J}_{b_2^-}^{p}\zeta(b_1,a_2)\Big]\left\{\frac{1}{\vartheta_1+s}+B(\vartheta_1,s+1)\right\}$$

$$\leq \vartheta_1\vartheta_2\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right]\left\{\frac{1}{(\vartheta_1+s)(\vartheta_2+s)}\right.$$

$$\left.+\frac{B(\vartheta_1,s+1)}{\vartheta_2+s}+\frac{B(\vartheta_2,s+1)}{\vartheta_1+s}+B(\vartheta_1,s+1)B(\vartheta_2,s+1)\right\}.$$

Corollary 6.22. *Let $\zeta : \mathcal{D} \to \mathbb{R}$ be an (s_1, s_2)-convex function on the coordinates on $\mathcal{D}$ and $\zeta \in L_1(\mathcal{D})$. Then one has the inequalities:*

$$
\begin{aligned}
& 2^{s_1+s_2}\zeta\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) \\
\leq\ & \frac{2^{s_2-1}\Gamma(\vartheta_1+1)}{(b_1^{p_1}-a_1^{p_1})^{\vartheta_1}}\left[\,^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta\left(b_1, \frac{a_2+b_2}{2}\right) +^{\vartheta_1} \mathcal{J}_{b_1^-}\zeta\left(a_1, \frac{a_2+b_2}{2}\right)\right] \\
& +\frac{2^{s_1-1}\Gamma(\vartheta_2+1)}{(b_2-a_2)^{\vartheta_2}} \\
& \times\left[\,^{\vartheta_2}\mathcal{J}_{a_2^+}\zeta\left(\frac{a_1+b_1}{2}, b_2\right) + \,^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta\left(\frac{a_1+b_1}{2}, a_2\right)\right] \\
\leq\ & \frac{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4(b_1-a_1)^{\vartheta_1}(b_2-a_2)^{\vartheta_2}}\left[\,^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,a_2^+}\zeta(b_1,b_2) + \,^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,b_2^-}\zeta(b_1,a_2)\right. \\
& \left. + \,^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,a_2^+}\zeta(a_1,b_2) + \,^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,b_2^-}\zeta(a_1,a_2)\right] \\
\leq\ & \frac{\vartheta_2\Gamma(\vartheta_1+1)}{2(b_1-a_1)^{\vartheta_1}}\left[\,^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta(b_1,a_2) + \,^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta(b_1,b_2) + \,^{\vartheta_1}\mathcal{J}_{b_1^-}\zeta(a_1,a_2)\right. \\
& \left. + \,^{\vartheta_1}\mathcal{J}_{b_1^-}\zeta(a_1,b_2)\right]\left\{\frac{1}{\vartheta_2+s_2} + B(\vartheta_2, s_2+1)\right\} \\
& +\frac{\vartheta_1 p_2^{\vartheta_2}\Gamma(\vartheta_2+1)}{2(b_2-a_2)^{\vartheta_2}}\left[\,^{\vartheta_2}\mathcal{J}_{a_2^+}\zeta(a_1,b_2) + \,^{\vartheta_2}\mathcal{J}_{a_2^+}\zeta(b_1,b_2) + \,^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta(a_1,a_2)\right. \\
& \left. + \,^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta(b_1,a_2)\right]\left\{\frac{1}{\vartheta_1+s_1} + B(\vartheta_1, s_1+1)\right\} \\
\leq\ & \vartheta_1\vartheta_2\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right]\left\{\frac{1}{(\vartheta_1+s_1)(\vartheta_2+s_2)}\right. \\
& \left. + \frac{B(\vartheta_1,s_1+1)}{\vartheta_2+s_2} + \frac{B(\vartheta_2,s_2+1)}{\vartheta_1+s_1} + B(\vartheta_1,s_1+1)B(\vartheta_2,s_2+1)\right\}.
\end{aligned}
$$

Remark 6.23. If $\vartheta_1 = 1 = \vartheta_2$, then the inequalities in Corollary 6.22 extends Theorem 2.1 of [9]. It will coincide to the Theorem 2.1 of [9] if $s_1 = s_2 = s$.

Corollary 6.24. *Let $\zeta : \mathcal{D} \to \mathbb{R}$ be an s-convex function on the coordinates on $\mathcal{D}$ and $\zeta \in L_1(\mathcal{D})$. Then one has the inequalities:*

$$
\begin{aligned}
& 4^{s}\zeta\left(\frac{a_1+b_1}{2}, \frac{a_2+b_2}{2}\right) \\
\leq\ & \frac{2^{s-1}\Gamma(\vartheta_1+1)}{(b_1^{p_1}-a_1^{p_1})^{\vartheta_1}}\left[\,^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta\left(b_1, \frac{a_2+b_2}{2}\right) +^{\vartheta_1} \mathcal{J}_{b_1^-}\zeta\left(a_1, \frac{a_2+b_2}{2}\right)\right] \\
& +\frac{2^{s-1}\Gamma(\vartheta_2+1)}{(b_2-a_2)^{\vartheta_2}}
\end{aligned}
$$

$$
\begin{aligned}
&\times\left[{}^{\vartheta_2}\mathcal{J}_{a_2^+}\zeta\left(\frac{a_1+b_1}{2},b_2\right) + {}^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta\left(\frac{a_1+b_1}{2},a_2\right)\right] \\
\leq\ & \frac{\Gamma(\vartheta_1+1)\Gamma(\vartheta_2+1)}{4(b_1-a_1)^{\vartheta_1}(b_2-a_2)^{\vartheta_2}}\left[{}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,a_2^+}\zeta(b_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{a_1^+,b_2^-}\zeta(b_1,a_2)\right. \\
&\left. + {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,a_2^+}\zeta(a_1,b_2) + {}^{\vartheta_1,\vartheta_2}\mathcal{J}_{b_1^-,b_2^-}\zeta(a_1,a_2)\right] \\
\leq\ & \frac{\vartheta_2\Gamma(\vartheta_1+1)}{2(b_1-a_1)^{\vartheta_1}}\left[{}^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta(b_1,a_2) + {}^{\vartheta_1}\mathcal{J}_{a_1^+}\zeta(b_1,b_2) + {}^{\vartheta_1}\mathcal{J}_{b_1^-}\zeta(a_1,a_2)\right. \\
&\left. + {}^{\vartheta_1}\mathcal{J}_{b_1^-}\zeta(a_1,b_2)\right]\left\{\frac{1}{\vartheta_2+s} + B(\vartheta_2,s+1)\right\} \\
&+\frac{\vartheta_1 p_2^{\vartheta_2}\Gamma(\vartheta_2+1)}{2(b_2-a_2)^{\vartheta_2}}\left[{}^{\vartheta_2}\mathcal{J}_{a_2^+}\zeta(a_1,b_2) + {}^{\vartheta_2}\mathcal{J}_{a_2^+}\zeta(b_1,b_2) + {}^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta(a_1,a_2)\right. \\
&\left. + {}^{\vartheta_2}\mathcal{J}_{b_2^-}\zeta(b_1,a_2)\right]\left\{\frac{1}{\vartheta_1+s} + B(\vartheta_1,s+1)\right\} \\
\leq\ & \vartheta_1\vartheta_2\left[\zeta(a_1,a_2)+\zeta(a_1,b_2)+\zeta(b_1,a_2)+\zeta(b_1,b_2)\right]\left\{\frac{1}{(\vartheta_1+s)(\vartheta_2+s)}\right. \\
&\left. + \frac{B(\vartheta_1,s+1)}{\vartheta_2+s} + \frac{B(\vartheta_2,s+1)}{\vartheta_1+s} + B(\vartheta_1,s+1)B(\vartheta_2,s+1)\right\}.
\end{aligned}
$$

Remark 6.25. Inequalities in Corollary 6.24, will be reduced to special case of classical integrals if $\vartheta_1 = 1 = \vartheta_2$. In that case it will coincide to Theorem 2.1 of [9].

Chapter 7

New quantum estimates of Hermite–Hadamard type inequalities for twice differentiable m-convex functions

The calculus without limit is known as quantum calculus. It supplant the old style thought of differentiation by a difference operator, which than permits us to manage non-differentiable functions. The acquired notion locate an intriguing job because of uses in a few numerical zones, for example, "orthogonal polynomials, essential hyper geometric functions, combinatorics, the calculus of variations, mechanics and the theory of relativity".

As of late, specialists researched a few intriguing new inequalities with regards to the premises of quantum math. New discoveries opened another and intriguing space for the analysts working nearby inequalities. We allude to ([16, 25, 104, 133, 175]). This section talk about some Hermite–Hadamard type inequalities for twice differentiable m-convex functions. The chapter follows the clear arrangement.

Section 7.1 present two new identities out of a current one. Section 7.2 annunciate some new Hermite–Hadamard type inequalities for twice differentiable m-convex functions.

7.1 New identities for twice q–differentiable functions

Let recall some useful results and conventions used in the indicated domain.

Definition 7.1. [87] For a fixed real number p,

$$[p] = \frac{1-q^p}{1-q} \tag{7.1}$$

is called q–analogue of p. In particular,

$$[s] = \frac{1-q^s}{1-q} = q^{s-1} + \ldots + q + 1, \tag{7.2}$$

provided that s is a positive integer.

Definition 7.2. [87] For any integer s, the q–analogue of $(w-c)^s$ is the polynomial

$$(w-c)_q^s = \begin{cases} 1, & \text{if} \quad s = 0, \\ (w-c)(w-qc)\cdots(w-q^{s-1}c), & \text{if} \quad s \geq 1. \end{cases} \tag{7.3}$$

Definition 7.3. [87] For any $\gamma, \delta > 0$,

$$B_q(\gamma, \delta) = \int_0^1 \omega^{\gamma-1}(1-q\omega)_q^{\delta-1} d_q\omega, \tag{7.4}$$

is called the q–Beta function. Notice that,

$$B_q(\gamma, 1) = \int_0^1 \omega^{\gamma-1} d_q\omega = \frac{1}{[\gamma]}, \tag{7.5}$$

where $[\gamma]$ is the q–analogue of γ. Furthermore, if $q \to 1^-$, then we have the following classical Beta function

$$B(\gamma, \delta) = \int_0^1 \omega^{\gamma-1}(1-\omega)^{\delta-1} d\omega. \tag{7.6}$$

In [104], the authors introduced some new inequalities of Hadamard type by proving the following identity.

Lemma 7.1.1. *Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a twice differentiable function on $\mathcal{Q}^\circ$ (the interior of $\mathcal{Q}$) with ${}_{a_1}\mathbf{D}_q^2\zeta$ be continuous and integrable on $\mathcal{Q}$, where $0 < q < 1$. Then the following identity holds:*

$$\frac{q\zeta(a_1)+\zeta(b_1)}{1+q} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1} \zeta(\omega)\ {}_{a_1}d_q\omega = \frac{q^2(b_1-a_1)^2}{1+q} \times \int_0^1 \mu(1-q\mu)_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\ {}_0d_q\mu. \tag{7.7}$$

Now and onward, we adopt the convention $d_q\omega$ for ${}_0d_q\omega$ and ${}^0d_q\omega$ if the lower limit is 0 due to the fact given in Remark 1.34. Some very similar calculations leads to the following counterpart of Lemma 7.1.1.

Lemma 7.1.2. *Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a twice differentiable function on $\mathcal{Q}^\circ$ (the interior of $\mathcal{Q}$) with ${}^{a_1}\mathbf{D}_q^2\zeta$ be continuous and integrable on $\mathcal{Q}$, where $0 < q < 1$. Then the following identity holds:*

$$\frac{\zeta(a_1)+q\zeta(b_1)}{1+q} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1} \zeta(\omega)\ {}^{b_1}d_q\omega = \frac{q^2(b_1-a_1)^2}{1+q} \times \int_0^1 \mu(1-q\mu)^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\ d_q\mu. \tag{7.8}$$

Now, we have the following new identity from Lema 7.1.1 and Lema 7.1.2.

Lemma 7.1.3. *Let $\zeta : [a_1, b_1] \to \mathbb{R}$ be a twice differentiable function on $\mathcal{Q}^\circ$ (the interior of $\mathcal{Q}$) with ${}_{a_1}\mathbf{D}_q^2\zeta$ and ${}^{b_1}\mathbf{D}_q^2\zeta$ are continuous and integrable on $\mathcal{Q}$, where $0 < q < 1$. Then the following identity holds:*

$$\begin{aligned} &\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1} \zeta(\omega)\ {}_{a_1}d_q\omega + \int_{a_1}^{b_1} \zeta(\omega)\ {}^{b_1}d_q\omega\right] \\ &= \frac{q^2(b_1-a_1)^2}{2(1+q)}\int_0^1 \mu(1-q\mu)\left[\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1) + {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\right] d_q\mu. \end{aligned} \tag{7.9}$$

7.2 New quantum integral inequalities via m–convex functions

In this section, before we present our main results, let denote, respectively, $\mathcal{Q} = [a_1, b_1]$ and $\mathcal{Q}^\circ = (a_1, b_1)$.

Theorem 7.2.1. *Let $\zeta : \mathcal{Q} \to \mathbb{R}$ be a twice differentiable convex function on $\mathcal{Q}^\circ$ with ${}_{a_1}\mathbf{D}_q^2\zeta$ and ${}^{b_1}\mathbf{D}_q^2\zeta$ are continuous and integrable on $\mathcal{Q}$, where $0 < q < 1$. If $\left|{}_{a_1}\mathbf{D}_q^2\zeta\right|^r$ and $\left|{}^{b_1}\mathbf{D}_q^2\zeta\right|^r$ are m–convex functions on $\mathcal{Q}$ for $r \geq 1$, where $m \in (0,1]$ is fixed, then*

$$\begin{aligned}
&\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1} \zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1} \zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right| \\
&\leq \frac{q^2(b_1-a_1)^2}{2(1+q)^{2-\frac{1}{r}}}\left\{\left((1-q)\sum_{p=0}^{\infty} q^{3p}(1-q^{p+1})^r \left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r\right.\right. \\
&\left. + m(1-q)\sum_{p=0}^{\infty}(q^{2p}-q^{3p})(1-q^{p+1})^r \left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r d_q\mu\right)^{\frac{1}{r}} \\
&+\left((1-q)\sum_{p=0}^{\infty} q^{3p}(1-q^{p+1})^r \left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r\right. \\
&\left.\left. + m(1-q)\sum_{p=0}^{\infty}(q^{2p}-q^{3p})(1-q^{p+1})^r \left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r d_q\mu\right)^{\frac{1}{r}}\right\}. \qquad (7.10)
\end{aligned}$$

Proof. By the Lemma 7.1.3 and properties of the modulus, we have

$$\begin{aligned}
&\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1} \zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1} \zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right| \\
&= \frac{q^2(b_1-a_1)^2}{2(1+q)}\int_0^1 \mu(1-q\mu)\left|\left[\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1) + \ {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\right]\right|\ d_q\mu \\
&\leq \frac{q^2(b_1-a_1)^2}{2(1+q)}\left[\int_0^1 \mu(1-q\mu)\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|\ d_q\mu\right. \\
&\left. + \int_0^1 \mu(1-q\mu)\left|\ {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\right|\ d_q\mu\right]. \qquad (7.11)
\end{aligned}$$

By m–convexity, application of q–integral (1.79) and power mean inequality, we get

$$\int_0^1 \mu(1-q\mu)\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|\ d_q\mu$$

$$\leq \left(\int_0^1 \mu d_q\mu\right)^{1-\frac{1}{r}} \left(\int_0^1 \mu(1-q\mu)^r \left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|^r\ d_q\mu\right)^{\frac{1}{r}}$$

$$\leq \left(\int_0^1 \mu d_q\mu\right)^{1-\frac{1}{r}} \left(\int_0^1 \mu(1-q\mu)^r \left[\mu\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + m(1-\mu)\left|{}_{a_1}D_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\right] d_q\mu\right)^{\frac{1}{r}}$$

$$= \left(\int_0^1 \mu d_q\mu\right)^{1-\frac{1}{r}}$$

$$\times\left(\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r \int_0^1 \mu^2(1-q\mu)^r d\mu + m\left|{}_{a_1}D_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r \int_0^1 \mu(1-\mu)(1-q\mu)^r d_q\mu\right)^{\frac{1}{r}}$$

$$= \left(\frac{1}{1+q}\right)^{1-\frac{1}{r}} \left((1-q)\sum_{p=0}^{\infty} q^{3p}(1-q^{p+1})^r \left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r\right.$$

$$\left. + m(1-q)\sum_{p=0}^{\infty}(q^{2p}-q^{3p})(1-q^{p+1})^r \left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\right)^{\frac{1}{r}}. \tag{7.12}$$

Similarly,

$$\int_0^1 \mu(1-q\mu)\left|\ {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\right|\ d_q\mu$$

$$\leq \left(\frac{1}{1+q}\right)^{1-\frac{1}{r}} \left((1-q)\sum_{p=0}^{\infty} q^{3p}(1-q^{p+1})^r \left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r\right.$$

$$\left. + m(1-q)\sum_{p=0}^{\infty}(q^{2p}-q^{3p})(1-q^{p+1})^r \left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r\right)^{\frac{1}{r}}. \tag{7.13}$$

Inequality (7.11) leads to the required inequality (7.10) by utilizing inequalities (7.12) and (7.13). □

Corollary 7.4. *If r is a positive integer, then using the facts that*

$$(1-q\mu)^r \leq (1-q\mu)_q^r, \quad (1-\mu)(1-q\mu)^r \leq (1-q\mu)_q^{r+1}, \tag{7.14}$$

we obtain

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma+\int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right| \tag{7.15}$$

$$\leq \frac{q^2(b_1-a_1)^2}{2(1+q)^{2-\frac{1}{r}}}\left\{\left(B_q(3,r+1)\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r+mB_q(2,r+2)\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right.$$

$$\left.+\left(B_q(3,r+1)\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r+mB_q(2,r+2)\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{7.16}$$

Remark 7. *If $q\to 1^-$, then we have the following inequality for m–convex functions:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\varsigma)\ d\varsigma\right|\leq\frac{(b_1-a_1)^2}{2^{3-\frac{1}{r}}}$$

$$\times\left\{\left(\frac{2\left|\zeta''(b_1)\right|^r+m(r+1)\left|\zeta''\left(\frac{a_1}{m}\right)\right|^r}{(r+1)(r+2)(r+3)}\right)^{\frac{1}{r}}+\left(\frac{2\left|\zeta''(a_1)\right|^r+m(r+1)\left|\zeta''\left(\frac{b_1}{m}\right)\right|^r}{(r+1)(r+2)(r+3)}\right)^{\frac{1}{r}}\right\}. \tag{7.17}$$

Theorem 7.2.2. *Let $\zeta:\mathcal{Q}\to\mathbb{R}$ be a twice differentiable convex function on $\mathcal{Q}^\circ$ with ${}_{a_1}\mathbf{D}_q^2\zeta$ and ${}^{b_1}\mathbf{D}_q^2\zeta$ are continuous and integrable on $\mathcal{Q}$, where $0<q<1$. If $\left|{}_{a_1}\mathbf{D}_q^2\zeta\right|^r$ and $\left|{}^{b_1}\mathbf{D}_q^2\zeta\right|^r$ are m–convex functions on $\mathcal{Q}$, where $r,u>1$, $\frac{1}{u}+\frac{1}{r}=1$ and $m\in(0,1]$ is fixed, then the following inequality holds:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma+\int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right|$$

$$\leq\frac{q^2(b_1-a_1)^2}{2(1+q)}\left((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^u\right)^{\frac{1}{u}}\left\{\left(\frac{(1+q)\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r+mq^2\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{(q+1)(1+q+q^2)}\right)^{\frac{1}{r}}\right.$$

$$\left.+\left(\frac{(1+q)\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r+mq^2\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{(q+1)(1+q+q^2)}\right)^{\frac{1}{r}}\right\}. \tag{7.18}$$

Proof. The Hölder's inequality, m–convexity and the application of q–integral leads to

$$\int_0^1\mu(1-q\mu)\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|\ d_q\mu$$

$$\leq\left(\int_0^1\mu(1-q\mu)^u d_q\mu\right)^{\frac{1}{u}}\left(\int_0^1\mu\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|^r\ d_q\mu\right)^{\frac{1}{r}}$$

$$\leq\left((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^{u}\right)^{\frac{1}{u}}\left(\int_{0}^{1}\mu\left[\mu\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r+m(1-\mu)\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\right]d_q\mu\right)^{\frac{1}{r}}$$

$$=\left((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^{u}\right)^{\frac{1}{u}}\left(\frac{(1+q)\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r+mq^2\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{(q+1)(1+q+q^2)}\right)^{\frac{1}{r}}. \tag{7.19}$$

Similarly,

$$\int_{0}^{1}\mu(1-q\mu)\left|{}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\right|\ d_q\mu$$

$$\leq\left((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^{u}\right)^{\frac{1}{u}}\left(\frac{(1+q)\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r+mq^2\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{(q+1)(1+q+q^2)}\right)^{\frac{1}{r}}. \tag{7.20}$$

Inequality (7.18) is then obtained by inequalities (7.11), (7.19) and (7.20). □

Corollary 7.5. *If u is a positive integer, then using the facts that*

$$(1-q\mu)^u\leq(1-q\mu)_q^u, \tag{7.21}$$

we have

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma+\int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right|$$

$$\leq\frac{q^2(b_1-a_1)^2}{2(1+q)}\left(B_q(2,u+1)\right)^{\frac{1}{u}}\left\{\left(\frac{(1+q)\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r+mq^2\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{(1+q+q^2)(1+q)}\right)^{\frac{1}{r}}\right.$$

$$\left.+\left(\frac{(1+q)\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r+mq^2\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{(1+q+q^2)(1+q)}\right)^{\frac{1}{r}}\right\}. \tag{7.22}$$

Remark 8. *If $q\to1^-$, then we get the following inequality for m–convex functions:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\varsigma)\ d\varsigma\right|$$

$$\leq\frac{(b_1-a_1)^2}{4}\left(\frac{1}{(u+1)(u+2)}\right)^{\frac{1}{u}}$$

$$\times\left\{\left(\frac{2\left|\zeta''(b_1)\right|^r+m\left|\zeta''\left(\frac{a_1}{m}\right)\right|^r}{6}\right)^{\frac{1}{r}}+\left(\frac{2\left|\zeta''(a_1)\right|^r+m\left|\zeta''\left(\frac{b_1}{m}\right)\right|^r}{6}\right)^{\frac{1}{r}}\right\}. \tag{7.23}$$

Theorem 7.2.3. *Let $\zeta : \mathcal{Q} \to \mathbb{R}$ be a twice differentiable convex function on $\mathcal{Q}^\circ$ with ${}_{a_1}\mathbf{D}_q^2\zeta$ and ${}^{b_1}\mathbf{D}_q^2\zeta$ are continuous and integrable on $\mathcal{Q}$, where $0 < q < 1$. If $\left|{}_{a_1}\mathbf{D}_q^2\zeta\right|^r$ and $\left|{}^{b_1}\mathbf{D}_q^2\zeta\right|^r$ are m–convex functions on $\mathcal{Q}$ for $r \geq 1$, where $m \in (0,1]$ is fixed, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right|$$
$$\leq \frac{q^2(b_1-a_1)^2}{2(1+q)}\left\{\left((1-q)\sum_{s=0}^{\infty}(q^s)^{r+2}(1-q^{s+1})^r\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r\right.\right.$$
$$\left. + m(1-q)\sum_{s=0}^{\infty}(q^s)^{r+1}(1-q^{s+1})^r(1-q^s)\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\right)^{\frac{1}{r}}$$
$$+\left((1-q)\sum_{s=0}^{\infty}(q^s)^{r+2}(1-q^{s+1})^r\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r\right.$$
$$\left.\left. + m(1-q)\sum_{s=0}^{\infty}(q^s)^{r+1}(1-q^{s+1})^r(1-q^s)\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{7.24}$$

Proof. Applying the power mean inequality, m–convexity and the q–integral (1.79), we get

$$\int_0^1 \mu(1-q\mu)\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|\ d_q\mu$$
$$\leq \left(\int_0^1 1\ d_q\mu\right)^{1-\frac{1}{r}}\left(\int_0^1 \mu^r(1-qt)^r\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|^r\ d_q\mu\right)^{\frac{1}{r}}$$
$$\leq \left(\int_0^1 \mu^{r+1}(1-qt)^r\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r d_qt + m\int_0^1 \mu^r(1-qt)^r(1-\mu)\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r d_q\mu\right)^{\frac{1}{r}}$$
$$= \left((1-q)\sum_{s=0}^{\infty}(q^s)^{r+2}(1-q^{s+1})^r\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r\right.$$
$$\left. + m(1-q)\sum_{s=0}^{\infty}(q^s)^{r+1}(1-q^{s+1})^r(1-q^s)\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\right)^{\frac{1}{r}}. \tag{7.25}$$

Similarly,

$$\int_0^1 \mu(1-q\mu)\left|\ {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\right|\ d_q\mu \leq \left((1-q)\sum_{s=0}^{\infty}(q^s)^{r+2}(1-q^{s+1})^r\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r\right.$$
$$\left. + m(1-q)\sum_{s=0}^{\infty}(q^s)^{r+1}(1-q^{s+1})^r(1-q^s)\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r\right)^{\frac{1}{r}}. \tag{7.26}$$

Inequality (7.24) is then obtained by inequalities (7.11), (7.25) and (7.26). □

Corollary 7.6. *If r is a positive integer, then using the facts that*

$$(1-q\mu)^r \le (1-q\mu)_q^r, \quad (1-\mu)(1-q\mu)^r \le (1-q\mu)_q^{r+1}, \tag{7.27}$$

we obtain

$$\begin{aligned}
&\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right| \le \frac{q^2(b_1-a_1)^2}{2(1+q)} \\
&\times\left\{\left(B_q(r+2,r+1)\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mB_q(r+1,r+2)\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right. \\
&\left.+\left(B_q(r+2,r+1)\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + mB_q(r+1,r+2)\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}.
\end{aligned} \tag{7.28}$$

Remark 9. *If $q \to 1^-$, then we have the following inequality for m–convex functions:*

$$\begin{aligned}
&\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\varsigma)\ d\varsigma\right| \le \frac{(b_1-a_1)^2}{4} \\
&\times\left\{\left(B(r+2,r+1)\left|\zeta''(b_1)\right|^r + mB(r+1,r+2)\left|\zeta''\left(\frac{a_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right. \\
&\left.+\left(B(r+2,r+1)\left|\zeta''(a_1)\right|^r + mB(r+1,r+2)\left|\zeta''\left(\frac{b_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}.
\end{aligned} \tag{7.29}$$

Theorem 7.2.4. *Let $\zeta : \mathcal{Q} \to \mathbb{R}$ be a twice differentiable convex function on $\mathcal{Q}^\circ$ with ${}_{a_1}\mathbf{D}_q^2\zeta$ and ${}^{b_1}\mathbf{D}_q^2\zeta$ are continuous and integrable on $\mathcal{Q}$, where $0<q<1$. If $\left|{}_{a_1}\mathbf{D}_q^2\zeta\right|^r$ and $\left|{}^{b_1}\mathbf{D}_q^2\zeta\right|^r$ are m–convex functions on $\mathcal{Q}$, where $r,u>1$, $\frac{1}{u}+\frac{1}{r}=1$ and $m\in(0,1]$ is fixed, then*

$$\begin{aligned}
&\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right| \le \frac{q^2(b_1-a_1)^2}{2(1+q)} \\
&\times\left((1-q)\sum_{s=0}^{\infty}(q^s)^{u+1}(1-q^{s+1})^u\right)^{\frac{1}{u}}\left\{\left(\frac{\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mq\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}}\right. \\
&\left.+\left(\frac{\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + mq\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}}\right\}.
\end{aligned} \tag{7.30}$$

Proof. By the Hölder's inequality, m–convexity and the q–integral (1.79), we have

$$\begin{aligned}
&\int_0^1 \mu(1-q\mu)\left| {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right| \, d_q\mu \\
&\le \left(\int_0^1 \mu^u(1-q\mu)^u d_q\mu\right)^{\frac{1}{u}} \left(\int_0^1 \left| {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|^r \, d_q\mu\right)^{\frac{1}{r}} \\
&\le \left(\int_0^1 \mu^u(1-q\mu)^u d_q\mu\right)^{\frac{1}{u}} \left(\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r \int_0^1 \mu d_q\mu + m\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r \int_0^1 (1-\mu)d_q\mu\right)^{\frac{1}{r}} \\
&= \left((1-q)\sum_{s=0}^{\infty}(q^s)^{u+1}(1-q^{s+1})^u\right)^{\frac{1}{u}} \left(\frac{\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mq\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}}. \qquad (7.31)
\end{aligned}$$

Similarly,

$$\begin{aligned}
&\int_0^1 \mu(1-q\mu)\left| {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\right| \, d_q\mu \\
&\le \left((1-q)\sum_{s=0}^{\infty}(q^s)^{u+1}(1-q^{s+1})^u\right)^{\frac{1}{u}} \left(\frac{\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + mq\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}}. \qquad (7.32)
\end{aligned}$$

Inequality (7.30) is then obtained by inequalities (7.11), (7.31) and (7.32). □

Corollary 7.7. *If u is a positive integer, then using the facts that*

$$(1-q\mu)^u \le (1-q\mu)_q^u, \qquad (7.33)$$

we get

$$\begin{aligned}
&\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right| \\
&\le \frac{q^2(b_1-a_1)^2}{2(1+q)}\left(B_q(u+1,u+1)\right)^{\frac{1}{u}}\left\{\left(\frac{\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mq\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}}\right. \\
&\left.+\left(\frac{\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + mq\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{1+q}\right)^{\frac{1}{r}}\right\}. \qquad (7.34)
\end{aligned}$$

Remark 10. *Considering the integral,*

$$B(u+1,u+1)=\int_0^1 \mu^u(1-\mu)^u d\mu, \tag{7.35}$$

and taking the following facts into account that,

$$2^{2\mu-1}\Gamma(\mu)\Gamma\left(\mu+\frac{1}{2}\right)=\Gamma\left(\frac{1}{2}\right)\Gamma(2\mu),$$

and

$$B(\mu,\lambda)=\frac{\Gamma(\mu)\Gamma(\lambda)}{\Gamma(\mu+\lambda)},$$

then

$$B(u+1,u+1)=\frac{2^{-1-2u}\Gamma(u+1)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(u+\frac{3}{2}\right)},$$

where $\Gamma(\cdot)$ *is the Gamma function.*

If $q\to 1^-$, *then we have the following inequality for* m*–convex functions from inequality (7.30):*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\varsigma)\ d\varsigma\right|\leq\frac{(b_1-a_1)^2}{32}\left(\frac{\sqrt{\pi}\Gamma(u+1)}{\Gamma\left(u+\frac{3}{2}\right)}\right)^{\frac{1}{u}}$$
$$\times\left\{\left(|\zeta''(b_1)|^r+m\left|\zeta''\left(\frac{a_1}{m}\right)\right|^r\right)^{\frac{1}{r}}+\left(|\zeta''(a_1)|^r+m\left|\zeta''\left(\frac{b_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{7.36}$$

Theorem 7.2.5. *Let* $\zeta:\mathcal{Q}\to\mathbb{R}$ *be a twice differentiable convex function on* $\mathcal{Q}^\circ$ *with* ${}_{a_1}\mathbf{D}_q^2\zeta$ *and* ${}^{b_1}\mathbf{D}_q^2\zeta$ *are continuous and integrable on* $\mathcal{Q}$, *where* $0<q<1$. *If* $\left|{}_{a_1}\mathbf{D}_q^2\zeta\right|^r$ *and* $\left|{}^{b_1}\mathbf{D}_q^2\zeta\right|^r$ *are* m*–convex functions on* $\mathcal{Q}$, *where* $r,u>1$, $\frac{1}{u}+\frac{1}{r}=1$ *and* $m\in(0,1]$ *is fixed, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma+\int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right|$$
$$\leq\frac{q^2(b_1-a_1)^2}{2(1+q)}\left(\frac{1}{[u+1]}\right)^{\frac{1}{u}}\left\{\left((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^r\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r\right.\right.$$

$$+ m(1-q)\sum_{s=0}^{\infty}(q^s - q^{2s})(1-q^{s+1})^r \left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\Bigg)^{\frac{1}{r}}$$
$$+ \Bigg((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^r \left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r$$
$$+ m(1-q)\sum_{s=0}^{\infty}(q^s - q^{2s})(1-q^{s+1})^r \left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r\Bigg)^{\frac{1}{r}}\Bigg\}. \tag{7.37}$$

Proof. By using Hölder's inequality, m–convexity and the application of q–integral, we have

$$\int_0^1 \mu(1-q\mu)\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1 + (1-\mu)a_1)\right|\ d_q\mu$$
$$\leq \left(\int_0^1 \mu^u d_q\mu\right)^{\frac{1}{u}}\left(\int_0^1 (1-q\mu)^r \left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1 + (1-\mu)a_1)\right|^r\ d_q\mu\right)^{\frac{1}{r}}$$
$$\leq \left(\frac{1-q}{1-q^{u+1}}\right)^{\frac{1}{u}}$$
$$\times\left(\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r\int_0^1 \mu(1-q\mu)^r d_q\mu + m\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\int_0^1 (1-\mu)(1-q\mu)^r d_q\mu\right)^{\frac{1}{r}}$$
$$= \left(\frac{1}{[u+1]}\right)^{\frac{1}{u}}\Bigg((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^r\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r$$
$$+ m(1-q)\sum_{s=0}^{\infty}(q^s - q^{2s})(1-q^{s+1})^r\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\Bigg)^{\frac{1}{r}}. \tag{7.38}$$

Similarly,

$$\int_0^1 \mu(1-q\mu)\left|\ {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1 + (1-\mu)b_1)\right|\ d_q\mu$$
$$\leq \left(\frac{1}{[u+1]}\right)^{\frac{1}{u}}\Bigg((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^r\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r$$
$$+ m(1-q)\sum_{s=0}^{\infty}(q^s - q^{2s})(1-q^{s+1})^r\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r\Bigg)^{\frac{1}{r}}. \tag{7.39}$$

Inequality (7.37) is then obtained by inequalities (7.11), (7.38) and (7.39). □

Corollary 7.8. *If r is a positive integer, then using the facts that*

$$(1-\mu)^{r+1} \leq (1-q\mu)_q^{r+1}, \quad (1-\mu)^r \leq (1-q\mu)_q^r, \tag{7.40}$$

we get

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1} \zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1} \zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right| \leq \frac{q^2(b_1-a_1)^2}{2(1+q)}\left(\frac{1}{[u+1]}\right)^{\frac{1}{u}}$$
$$\times\left\{\left(B_q(2,r+1)\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mB_q(1,r+2)\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right.$$
$$\left. + \left(B_q(2,r+1)\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + mB_q(1,r+2)\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{7.41}$$

Remark 11. *If $q \to 1^-$, then we obtain the following inequality for m–convex functions:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1} \zeta(\varsigma)\ d\varsigma\right| \leq \frac{(b_1-a_1)^2}{4}\left(\frac{1}{u+1}\right)^{\frac{1}{u}}$$
$$\times\left\{\left(\frac{|\zeta''(b_1)|^r + m(r+1)\left|\zeta''\left(\frac{a_1}{m}\right)\right|^r}{(r+1)(r+2)}\right)^{\frac{1}{r}} + \left(\frac{|\zeta''(a_1)|^r + m(r+1)\left|\zeta''\left(\frac{b_1}{m}\right)\right|^r}{(r+1)(r+2)}\right)^{\frac{1}{r}}\right\}. \tag{7.42}$$

Theorem 7.2.6. *Let $\zeta : \mathcal{Q} \to \mathbb{R}$ be a twice differentiable convex function on $\mathcal{Q}^\circ$ with ${}_{a_1}\mathbf{D}_q^2\zeta$ and ${}^{b_1}\mathbf{D}_q^2\zeta$ are continuous and integrable on $\mathcal{Q}$, where $0 < q < 1$. If $\left|{}_{a_1}\mathbf{D}_q^2\zeta\right|^r$ and $\left|{}^{b_1}\mathbf{D}_q^2\zeta\right|^r$ are m–convex functions on $\mathcal{Q}$, where $r, u > 1, \frac{1}{u}+\frac{1}{r} = 1$ and $m \in (0,1]$ is fixed, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1} \zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1} \zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right| \leq \frac{q^2(b_1-a_1)^2}{2(1+q)}$$
$$\times\left((1-q)\sum_{s=0}^{\infty} q^s(1-q^{s+1})^u\right)^{\frac{1}{u}}\left\{\left(\frac{[r+1]\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mq^{r+1}\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}}\right.$$
$$\left. + \left(\frac{[r+1]\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + mq^{r+1}\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}}\right\}. \tag{7.43}$$

Proof. By utilizing m–convexity, the Hölder's inequality and the q–integral leads to

$$\int_0^1 \mu(1-q\mu)\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|\ d_q\mu$$
$$\leq \left(\int_0^1 (1-\mu)^u d_q\mu\right)^{\frac{1}{u}} \left(\int_0^1 \mu^r \left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|^r\ d_q\mu\right)^{\frac{1}{r}}$$
$$\leq \left((1-q)\sum_{s=0}^{\infty} q^s(1-q^{s+1})^u\right)^{\frac{1}{u}}$$
$$\times\left(\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r \int_0^1 \mu^{r+1}d_q\mu + m\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r \int_0^1 \mu^r(1-\mu)d_q\mu\right)^{\frac{1}{r}}$$
$$= \left((1-q)\sum_{s=0}^{\infty} q^s(1-q^{s+1})^u\right)^{\frac{1}{u}}$$
$$\times\left(\frac{[r+1]\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mq^{r+1}\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}}. \tag{7.44}$$

Similarly,

$$\int_0^1 \mu(1-q\mu)\left|\ {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\right|\ d_q\mu$$
$$\leq \left((1-q)\sum_{s=0}^{\infty} q^s(1-q^{s+1})^u\right)^{\frac{1}{u}} \left(\frac{[r+1]\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + mq^{r+1}\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}}. \tag{7.45}$$

Inequality (7.43) is then obtained by inequalities (7.11), (7.44) and (7.45). □

Corollary 7.9. *If u is a positive integer, then using the facts that*

$$(1-q\mu)^u \leq (1-q\mu)_q^u, \tag{7.46}$$

we have

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right|$$
$$\leq \frac{q^2(b_1-a_1)^2}{2(1+q)}\left(B_q(1,u+1)\right)^{\frac{1}{u}}\left\{\left(\frac{[r+1]\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mq^{r+1}\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}}\right.$$
$$\left. + \left(\frac{[r+1]\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + mq^{r+1}\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{[r+1][r+2]}\right)^{\frac{1}{r}}\right\}. \tag{7.47}$$

Remark 12. *If $q \to 1^-$, then we get the following inequality for m–convex functions:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1} \zeta(\varsigma)\ d\varsigma\right| \leq \frac{(b_1-a_1)^2}{4}\left(\frac{1}{u+1}\right)^{\frac{1}{u}}$$
$$\times \left\{ \left(\frac{(r+1)\left|\zeta''(b_1)\right|^r + m\left|\zeta''\left(\frac{a_1}{m}\right)\right|^r}{(r+1)(r+2)}\right)^{\frac{1}{r}} + \left(\frac{(r+1)\left|\zeta''(a_1)\right|^r + m\left|\zeta''\left(\frac{b_1}{m}\right)\right|^r}{(r+1)(r+2)}\right)^{\frac{1}{r}} \right\}. \quad (7.48)$$

Theorem 7.2.7. *Let $\zeta : \mathcal{Q} \to \mathbb{R}$ be a twice differentiable convex function on $\mathcal{Q}^\circ$ with ${}_{a_1}\mathbf{D}_q^2\zeta$ and ${}^{b_1}\mathbf{D}_q^2\zeta$ are continuous and integrable on $\mathcal{Q}$, where $0 < q < 1$. If $\left|{}_{a_1}\mathbf{D}_q^2\zeta\right|$ and $\left|{}^{b_1}\mathbf{D}_q^2\zeta\right|$ are m–convex functions on $\mathcal{Q}$, where $m \in (0,1]$ is fixed, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1} \zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1} \zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right| \leq \frac{q^2(b_1-a_1)^2}{2(1+q)}$$
$$\times \left\{ \frac{\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + \left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right| + mq^2\left[\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right| + \left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|\right]}{(1+q+q^2)(1+q+q^2+q^3)} \right\}. \quad (7.49)$$

Proof. By m–convexity and the application of q–integral, we get

$$\int_0^1 \mu(1-q\mu)\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1 + (1-\mu)a_1)\right|\ d_q\mu$$
$$\leq \left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right| \int_0^1 \mu^2(1-q\mu)d_q\mu + m\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right| \int_0^1 \mu(1-\mu)(1-q\mu)d_q\mu$$
$$= \frac{\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right| + mq^2\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|}{(1+q+q^2)(1+q+q^2+q^3)}. \quad (7.50)$$

Similarly,

$$\int_0^1 \mu(1-q\mu)\left|\ {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1 + (1-\mu)b_1)\right|\ d_q\mu \leq \frac{\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right| + mq^2\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|}{(1+q+q^2)(1+q+q^2+q^3)}. \quad (7.51)$$

Inequality (7.49) is then obtained by inequalities (7.11), (7.50) and (7.51). □

Remark 13. *If $q \to 1^-$, then we have the following inequality for m–convex functions:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1} \zeta(\varsigma)\ d\varsigma\right| \leq \frac{(b_1-a_1)^2}{48}$$
$$\times \left\{ \left|\zeta''(b_1)\right| + \left|\zeta''(a_1)\right| + m\left[\left|\zeta''\left(\frac{a_1}{m}\right)\right| + \left|\zeta''\left(\frac{b_1}{m}\right)\right|\right] \right\}. \quad (7.52)$$

Theorem 7.2.8. *Let $\zeta : \mathcal{Q} \to \mathbb{R}$ be a twice differentiable convex function on $\mathcal{Q}^\circ$ with ${}_{a_1}\mathbf{D}_q^2\zeta$ and ${}^{b_1}\mathbf{D}_q^2\zeta$ are continuous and integrable on $\mathcal{Q}$, where $0 < q < 1$. If $\left|{}_{a_1}\mathbf{D}_q^2\zeta\right|^r$ and $\left|{}^{b_1}\mathbf{D}_q^2\zeta\right|^r$ are m–convex functions on $\mathcal{Q}$ for $r \geq 1$, where $m \in (0,1]$ is fixed, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right| \leq \frac{q^2(b_1-a_1)^2}{2(1+q)^{2-\frac{1}{r}}(1+q+q^2)}$$
$$\times\left\{\left(\frac{\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mq^2\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{1+q+q^2+q^3}\right)^{\frac{1}{r}} + \left(\frac{\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + mq^2\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{1+q+q^2+q^3}\right)^{\frac{1}{r}}\right\}. \quad (7.53)$$

Proof. By using power mean inequality, m–convexity and the application of q–integral, we have

$$\begin{aligned}
&\int_0^1 \mu(1-q\mu)\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|\ d_q\mu \\
&\leq \left(\int_0^1 \mu(1-q\mu)d_q\mu\right)^{1-\frac{1}{r}}\left(\int_0^1 \mu(1-q\mu)\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|^r\ d_q\mu\right)^{\frac{1}{r}} \\
&\leq \left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}} \\
&\times\left(\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r\int_0^1 \mu^2(1-q\mu)d_q\mu + m\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\int_0^1 \mu(1-\mu)(1-q\mu)d_q\mu\right)^{\frac{1}{r}} \\
&= \left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}}\left(\frac{\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mq^2\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{(1+q+q^2)(1+q+q^2+q^3)}\right)^{\frac{1}{r}}.
\end{aligned} \quad (7.54)$$

Similarly,

$$\begin{aligned}
&\int_0^1 \mu(1-q\mu)\left|\ {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\right|\ d_q\mu \\
&\leq \left(\frac{1}{(1+q)(1+q+q^2)}\right)^{1-\frac{1}{r}}\left(\frac{\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + mq^2\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{(1+q+q^2)(1+q+q^2+q^3)}\right)^{\frac{1}{r}}.
\end{aligned} \quad (7.55)$$

Inequality (7.53) is then obtained by inequalities (7.11), (7.54) and (7.55). ∎

Remark 14. *If $q \to 1^-$, then we have the following inequality for m–convex functions:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\varsigma)\ d\varsigma\right| \le \frac{(b_1-a_1)^2}{3\cdot 2^{3+\frac{1}{r}}}$$
$$\times\left\{\left(|\zeta''(b_1)|^r + m\left|\zeta''\left(\frac{a_1}{m}\right)\right|^r\right)^{\frac{1}{r}} + \left(|\zeta''(a_1)|^r + m\left|\zeta''\left(\frac{b_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{7.56}$$

Theorem 7.2.9. *Let $\zeta : \mathcal{Q} \to \mathbb{R}$ be a twice differentiable convex function on $\mathcal{Q}^\circ$ with ${}_{a_1}\mathbf{D}_q^2\zeta$ and ${}^{b_1}\mathbf{D}_q^2\zeta$ are continuous and integrable on $\mathcal{Q}$, where $0<q<1$. If $\left|{}_{a_1}\mathbf{D}_q^2\zeta\right|^r$ and $\left|{}^{b_1}\mathbf{D}_q^2\zeta\right|^r$ are m–convex functions on $\mathcal{Q}$ for $r\ge 1$, where $m\in(0,1]$ is fixed, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma + \int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right| \le \frac{q^2(b_1-a_1)^2}{2(1+q)}\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}$$
$$\times\left\{\left(\frac{[r+1]\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mq^{r+1}(1+q)\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{[r+1][r+2][r+3]}\right)^{\frac{1}{r}}\right.$$
$$\left.+\left(\frac{[r+1]\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + mq^{r+1}(1+q)\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{[r+1][r+2][r+3]}\right)^{\frac{1}{r}}\right\}. \tag{7.57}$$

Proof. Applying the power mean inequality, m–convexity and the application of q–integral leads to

$$\int_0^1 \mu(1-q\mu)\left|\ {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|\ d_q\mu$$
$$\le\left(\int_0^1(1-q\mu)d_q\mu\right)^{1-\frac{1}{r}}\left(\int_0^1\mu^r(1-q\mu)\left|\ {}_{a}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|^r\ d_q\mu\right)^{\frac{1}{r}}$$
$$\le\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}$$
$$\times\left(\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r\int_0^1\mu^{r+1}(1-q\mu)d_q\mu + m\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\int_0^1\mu^r(1-\mu)(1-q\mu)^r d_q\mu\right)^{\frac{1}{r}}$$
$$=\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left(\frac{[r+1]\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + mq^{r+1}(1+q)\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{[r+1][r+2][r+3]}\right)^{\frac{1}{r}}. \tag{7.58}$$

Similarly,

$$\int_0^1 \mu(1-q\mu)\left|\ {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\right|\ d_q\mu$$
$$\leq\left(\frac{1}{1+q}\right)^{1-\frac{1}{r}}\left(\frac{[r+1]\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r+mq^{r+1}(1+q)\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{[r+1][r+2][r+3]}\right)^{\frac{1}{r}}. \tag{7.59}$$

Inequality (7.57) is then obtained by inequalities (7.11), (7.58) and (7.59). □

Remark 15. *If $q\to 1^-$, then we have the following inequality for m–convex functions:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{b_1-a_1}\int_{a_1}^{b_1}\zeta(\varsigma)\ d\varsigma\right|\leq\frac{(b_1-a_1)^2}{8}\left(\frac{2}{(r+1)(r+2)(r+3)}\right)^{\frac{1}{r}}$$
$$\times\left\{\left((r+1)\left|\zeta''(b_1)\right|^r+2m\left|\zeta''\left(\frac{a_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right.$$
$$\left.+\left((r+1)\left|\zeta''(a_1)\right|^r+2m\left|\zeta''\left(\frac{b_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{7.60}$$

Theorem 7.2.10. *Let $\zeta:\mathcal{Q}\to\mathbb{R}$ be a twice differentiable convex function on $\mathcal{Q}^\circ$ with ${}_{a_1}\mathbf{D}_q^2\zeta$ and ${}^{b_1}\mathbf{D}_q^2\zeta$ are continuous and integrable on $\mathcal{Q}$, where $0<q<1$. If $\left|{}_{a_1}\mathbf{D}_q^2\zeta\right|^r$ and $\left|{}^{b_1}\mathbf{D}_q^2\zeta\right|^r$ are m–convex functions on $\mathcal{Q}$, where $r,u>1$, $\frac{1}{u}+\frac{1}{r}=1$ and $m\in(0,1]$ is fixed, then*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2}-\frac{1}{2(b_1-a_1)}\left[\int_{a_1}^{b_1}\zeta(\varsigma)\ {}_{a_1}d_q\varsigma+\int_{a_1}^{b_1}\zeta(\varsigma)\ {}^{b_1}d_q\varsigma\right]\right|$$
$$\leq\frac{q^2(b_1-a_1)^2}{2(1+q)}\left(\frac{1}{[u+1]}\right)^{\frac{1}{u}}\left\{\left((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^r\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r\right.\right.$$
$$\left.+m(1-q)\sum_{s=0}^{\infty}(q^s-q^{2s})(1-q^{s+1})^r\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r\right)^{\frac{1}{r}}$$
$$+\left((1-q)\sum_{s=0}^{\infty}q^{2s}(1-q^{s+1})^r\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r\right.$$
$$\left.\left.+\ m(1-q)\sum_{s=0}^{\infty}(q^s-q^{2s})(1-q^{s+1})^r\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r\right)^{\frac{1}{r}}\right\}. \tag{7.61}$$

Proof. Applying Hölder's inequality, m–convexity and the application of q–integral leads to

$$\int_0^1 \mu(1-q\mu)\left|\, {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right| \, d_q\mu$$

$$\leq \left(\int_0^1 \mu^u(1-q\mu)d_q\mu\right)^{\frac{1}{u}} \left(\int_0^1 (1-q\mu)\left|\, {}_{a_1}\mathbf{D}_q^2\zeta(\mu b_1+(1-\mu)a_1)\right|^r \, d_q\mu\right)^{\frac{1}{r}}$$

$$\leq \left(\int_0^1 \mu^u(1-q\mu)_q^1 \, d_q\mu\right)^{\frac{1}{u}}$$

$$\times \left(\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r \int_0^1 \mu(1-q\mu)d_q\mu + m\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r \int_0^1 (1-\mu)(1-q\mu)d_q\mu\right)^{\frac{1}{r}}$$

$$= (B_q(u+1,2))^{\frac{1}{u}} \left(\frac{\left|{}_{a_1}\mathbf{D}_q^2\zeta(b_1)\right|^r + m(q+q^2)\left|{}_{a_1}\mathbf{D}_q^2\zeta\left(\frac{a_1}{m}\right)\right|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}}. \tag{7.62}$$

Similarly,

$$\int_0^1 \mu(1-q\mu)\left|\, {}^{b_1}\mathbf{D}_q^2\zeta(\mu a_1+(1-\mu)b_1)\right| \, d_q\mu$$

$$\leq (B_q(u+1,2))^{\frac{1}{u}} \left(\frac{\left|{}^{b_1}\mathbf{D}_q^2\zeta(a_1)\right|^r + m(q+q^2)\left|{}^{b_1}\mathbf{D}_q^2\zeta\left(\frac{b_1}{m}\right)\right|^r}{(1+q)(1+q+q^2)}\right)^{\frac{1}{r}}. \tag{7.63}$$

Inequality (7.61) is then obtained by inequalities (7.11), (7.62) and (7.63). □

Remark 16. *If $q \to 1^-$, then we have the following inequality for m–convex functions:*

$$\left|\frac{\zeta(a_1)+\zeta(b_1)}{2} - \frac{1}{b_1-a_1}\int_{a_1}^{b_1} \zeta(\varsigma)\, d\varsigma\right| \leq \frac{(b_1-a_1)^2}{4}\left(\frac{1}{(u+1)(u+2)}\right)^{\frac{1}{u}}$$

$$\times\left\{\left(\frac{|\zeta''(b_1)|^r + 2m\left|\zeta''\left(\frac{a_1}{m}\right)\right|^r}{6}\right)^{\frac{1}{r}} + \left(\frac{|\zeta''(a_1)|^r + 2m\left|\zeta''\left(\frac{b_1}{m}\right)\right|^r}{6}\right)^{\frac{1}{r}}\right\}. \tag{7.64}$$

Remark 17. *Taking $m = 1$ in our above theorems, we can obtain some interesting results for convex functions. We omit their proofs and the details are left to the interested reader.*

Chapter 8

On some Ostrowski type inequalities on factal sets via generalized strongly m–convex mappings

As of late, the hypothesis of fractal sets has become the hot zone of examination around the field of inequalities. This hypothesis has its begun by X.- J. Yang [200]. For additional subtleties on local calculus, one may allude to ([198]– [202]). The first and most fundamental work about Hermite–Hadamard inequality in the casing of fractal sets is because of [129]. The most closely related inequality to that of Hermite–Hadamard inequality is the renowned Ostrowski inequality. In recent past, Sarikaya and Budak [167] published a paper on the local sort of renowned Ostrowski inequality.

After this acknowledgment by the most elevated gathering of mathematician, a great deal of exploration has been done toward this path. This part is given to investigate an Ostrowski type inequality for twice differentiable convex functions keeping in to account the idea of generalized strongly m-convexity. The result of this examination has been published. This section follows the distinct succession.

In Section 8.1, we present some essential writing. In Section 8.2, we first give an identity for twice locally differentiable functions. By using the Lemma 8.1.2, we build up certain outcomes for Ostrowski type inequalities on fractal sets keeping generalized strongly m-convexity in to account. We talk about some new special cases which can be reasoned from our fundamental outcomes. In Section 8.3, we will consider uses of the indispensable inequalities including inequalities created in the Section 8.2. In Section 8.4, a few uses of these inequalities for generalized means are introduced.

8.1 Basic concepts

Definition 8.1. [151] We call function $\zeta : \mathcal{Q} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ a strongly convex with modulus $\varepsilon \in \mathbb{R}^+$, if

$$\zeta(\nu\xi_1 + (1-\nu)\xi_2) \leq v\zeta(\xi_1) + (1-\nu)\zeta(\xi_2) - \varepsilon\nu(1-\nu)(\xi_1 - \xi_2)^2$$

for all $\xi_1, \xi_2 \in \mathcal{Q}$ and $\nu \in [0,1]$.

Definition 8.2. [100,101] We call function $\zeta : \mathcal{Q} \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ a strongly m-convex with modulus $\varepsilon \in \mathbb{R}^+$, if

$$\zeta(\nu\xi_1 + m(1-\nu)\xi_2) \leq \nu\zeta(\xi_1) + m(1-\nu)\zeta(\xi_2) - \varepsilon\nu(1-\nu)(m\xi_2 - \xi_1)^2$$

holds for every $\xi_1, \xi_2 \in \mathcal{Q}$ and $\nu, m \in [0,1]$.

Remark 18. *Any strongly m-convex function is particularly m-convex but m-convex functions are not strongly m-convex with modulus c (see [100], Example 1.8).*

Definition 8.3. [129] We call $\zeta : \mathcal{Q} \subseteq \mathbb{R} \longrightarrow \mathbb{R}^\gamma$ a generalized convex if for $\xi_1, \xi_2 \in \mathcal{Q}$ and $\nu \in [0,1]$,

$$\zeta(\nu\xi_1 + (1-\nu)\xi_2) \leq \nu^\gamma\zeta(\xi_1) + (1-\nu)^\gamma\zeta(\xi_2).$$

Following are examples of generalized convex functions:

1. $h_1(v) = v^{\gamma p}$, where $v \geq 0$ and $p > 1$;
2. $h_2(v) = E_\gamma(v^\gamma)$, $v \in \mathbb{R}$, where $E_\gamma(v^\gamma) := \sum_{l=0}^{\infty} \frac{v^{\gamma l}}{\Gamma(1+l\gamma)}$ is the Mittag-Leffer function.

For a recent advancement, see ([4, 14, 38, 47, 78, 100, 101, 126, 129, 182, 189]) and ([198]– [203]).

Theorem 8.1.1. *[129] Let $\zeta : [a_1, b_1] \longrightarrow \mathbb{R}^\gamma$ be a generalized convex function with $a_1 < b_1$. Then, for all $x \in [a_1, b_1]$, the following inequality holds:*

$$\zeta\left(\frac{a_1+b_1}{2}\right) \leq \frac{\Gamma(1+\gamma)}{(b_1-a_1)^\gamma} {}_{a_1}I_{b_1}^{(\gamma)}\zeta(v) \leq \frac{\zeta(a_1)+\zeta(b_1)}{2^\gamma}. \tag{8.1}$$

The generalized Hölder's inequality for fractal set $\mathbb{R}^\gamma$ is given by the next lemma.

Lemma 8.1.1. *[200] Let $\varphi, \psi \in C_\gamma[a_1, a_2]$ with $p_1^{-1} + q_1^{-1} = 1$, where $p_1, q_1 > 1$. Then we have*

$$\frac{1}{\Gamma(1+\gamma)} \int_{a_1}^{a_2} \left|\varphi(v)\psi(v)\right|(dv)^\gamma \leq \left(\frac{1}{\Gamma(1+\gamma)} \int_{a_1}^{a_2} \left|\varphi(v)\right|^{p_1}(dv)^\gamma\right)^{\frac{1}{p_1}} \left(\frac{1}{\Gamma(1+\gamma)} \int_{a_1}^{a_2} \left|\psi(v)\right|^{q_1}(dv)^\gamma\right)^{\frac{1}{q_1}}.$$

Theorem 8.1.2. *(Generalized Ostrowski inequality) Let $\mathcal{Q} \subseteq \mathbb{R}$ be an interval, $\zeta : \mathcal{Q}^0 \subseteq \mathbb{R} \longrightarrow \mathbb{R}^\gamma$ ($\mathcal{Q}^0$ is the interior of $\mathcal{Q}$) such that $\zeta \in D_\gamma(\mathcal{Q}^0)$, and $\zeta^{(\gamma)} \in C_\gamma[a_1, b_1]$ for $a_1, b_1 \in \mathcal{Q}^0$ with $a_1 < b_1$. Then, for all $v \in [a_1, b_1]$, the following inequality holds:*

$$\left|\zeta(v) - \frac{\Gamma(1+\gamma)}{(b_1-a_1)^\gamma} {}_{a_1}\mathcal{I}_{b_1}^{(\gamma)}\zeta(t)\right| \leq 2^\gamma \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left[\frac{1}{4^\gamma} + \left(\frac{x - \frac{a_1+b_1}{2}}{b_1-a_1}\right)^{2\gamma}\right](b_1-a_1)^\gamma \|\zeta^{(\gamma)}\|_\infty. \tag{8.2}$$

Choi et al. in [38], proved the following identity for twice local fractional differentiable functions.

Lemma 8.1.2. *Let $\mathcal{Q} \subseteq \mathbb{R}$ be an interval, $\zeta : \mathcal{Q}^0 \subseteq \mathbb{R} \to \mathbb{R}^\gamma$ ($\mathcal{Q}^0$ is the interior of $\mathcal{Q}$) such that $\zeta, \zeta^{(\gamma)} \in D_\gamma(\mathcal{Q}^0)$ and $\zeta^{(2\gamma)} \in C_\gamma[a_1, b_1]$ for $a_1, b_1 \in \mathcal{Q}^0$ with $a_1 < b_1$. Then the following equality* *holds true: For any* $v \in \left[\frac{a_1+b_1}{2}, b_1\right]$,

$$\Lambda(a_1,b_1;x) = \frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+\gamma)\Gamma(1+2\gamma)} \int_0^1 h(v)\zeta^{(2\gamma)}(va_1+(1-v)b_1)(dv)^{\gamma}, \tag{8.3}$$

where

$$h(v) = \begin{cases} v^{2\gamma}, & \text{if } \; 0 \le v < \frac{b_1-v}{b_1-a_1}; \\ \left(v-\frac{1}{2}\right)^{2\gamma}, & \text{if } \; \frac{b_1-v}{b_1-a_1} \le v < \frac{v-a_1}{b_1-a_1}; \\ (v-1)^{2\gamma}, & \text{if } \; \frac{v-a_1}{b_1-a_1} \le v < 1, \end{cases}$$

and

$$\begin{aligned} \Lambda(a_1,b_1;v) \;\; = \;\; & \frac{1}{(b_1-a_1)^{\gamma}} \, {}_{a_1}\mathcal{I}_{b_1}^{(\gamma)}\zeta - \frac{1}{\Gamma(1+\gamma)\Gamma(1+2\gamma)}[\zeta(v)-\zeta(a_1+b_1-v)] \\ & + \frac{1}{2^{\gamma}\Gamma(1+\gamma)}\left(v-\frac{a_1+3b_1}{4}\right)^{\gamma}[\zeta^{(\gamma)}(v)-\zeta^{(\gamma)}(a_1+b_1-v)]. \end{aligned}$$

Definition 8.4. [47] A function $\zeta : \mathcal{Q} \subseteq \mathbb{R} \longrightarrow \mathbb{R}^{\gamma}$ is called generalized strongly m-convex with $m \in [0,1]$, if

$$\zeta(\nu\xi_1 + m(1-\nu)\xi_2) \le \nu^{\gamma}\zeta(\xi_1) + m^{\gamma}(1-\nu)^{\gamma}\zeta(\xi_2) \tag{8.4}$$

holds for any $\xi_1, \xi_2 \in \mathcal{Q}$ and $\nu \in [0,1]$.

Definition 8.5. [14] A function $\zeta : \mathcal{Q} \subseteq \mathbb{R} \longrightarrow \mathbb{R}^{\gamma}$ is called generalized strongly m-convex with $m \in [0,1]$ and modulus $c \in \mathbb{R}^{+}$, if

$$\zeta(\nu\xi_1 + m(1-\nu)\xi_2) \le v^{\gamma}\zeta(\xi_1) + m^{\gamma}(1-\nu)^{\gamma}\zeta(\xi_2) - (cm)^{\gamma}\nu^{\gamma}(1-\nu)^{\gamma}(\xi_1-\xi_2)^{2\gamma} \tag{8.5}$$

holds for any $\xi_1, \xi_2 \in \mathcal{Q}$ and $v \in [0,1]$.

Remark 19. *From the Definition 8.5, it is obvious that every generalized strongly* m*-convex with modulus* c*, is indeed a generalized* m*-convex functions but the converse is not true which is proved in the upcoming example.*

Example 8.1.3. *The mapping $\zeta : [0,\infty) \to \mathbb{R}^{\gamma}$ given by*

$$\zeta(v) = \left(\frac{1}{12}v^4\right)^{\gamma} - \left(\frac{5}{12}v^3\right)^{\gamma} + \left(\frac{3}{4}v^2\right)^{\gamma} - \left(\frac{5}{12}v\right)^{\gamma} \tag{8.6}$$

is generalized $\frac{16}{17}$-convex function (see [47, Example 2.1]) and hence is generalized $\frac{1}{2}$-convex function (see [47, Proposition 2.1]). But it is not generalized strongly $\frac{1}{2}$-convex. Indeed for $c < \frac{1}{3}$, if above function is generalized strongly $\frac{1}{2}$-convex, then

$$\zeta\left(xv + \frac{1}{2}(1-v)y\right) \;\; \leq \;\; x^{\gamma}\zeta(v) + \left(\frac{1}{2}\right)^{\gamma}(1-v)^{\gamma}\zeta(y) - \left(c\frac{1}{2}\right)^{\gamma}(x-y)^{2\gamma}. \tag{8.7}$$

If $x = 1, y = 2, v = \frac{1}{2}$, then

$$0^{\gamma} = \zeta(1) \;\; \leq \;\; \left(\frac{1}{24}\right)^{\gamma} - \left(c\frac{1}{8}\right)^{\gamma}, \tag{8.8}$$

which contradicting the fact that $c < \frac{1}{3}$. So generalized m-convex functions need not be generalized strongly m-convex function with modulus c.

8.2 Main results

First, we need the following motivation.

Lemma 8.2.1. *Let $\mathcal{Q} \subseteq \mathbb{R}$ be an interval, $\zeta : \mathcal{Q}^0 \subseteq \mathbb{R} \to \mathbb{R}^{\gamma}$ ($\mathcal{Q}^0$ is the interior of $\mathcal{Q}$) such that $\zeta, \zeta^{(\gamma)} \in D_{\gamma}(\mathcal{Q}^0)$ and $\zeta^{(2\gamma)} \in C_{\gamma}[a_1, b_1]$ for $a, b \in \mathcal{Q}^0$ with $a_1 < b_1$. Then, for any $x \in [\frac{a_1+b_1}{2}, b_1]$ the following equality holds true:*

$$\Lambda(a_1, b_1; v) = \frac{(b_1 - a_1)^{2\gamma}}{\Gamma(1+\gamma)\Gamma(1+2\gamma)} \int_0^1 h(\wp)\zeta^{(2\gamma)}(\wp b_1 + (1-\wp)a_1)(d\wp)^{\gamma}, \tag{8.9}$$

where $\Lambda(a_1, b_1; v)$ and $h(\wp)$ are defined in Lemma 8.1.2. This identity can be prove analogously as the one in Lemma 8.1.2.

Theorem 8.2.1. *Let $\mathcal{Q} \subseteq \mathbb{R}$ be an interval, $\zeta : \mathcal{Q}^0 \subseteq \mathbb{R} \to \mathbb{R}^\gamma$ ($\mathcal{Q}^0$ is the interior of $\mathcal{Q}$) such that $\zeta, \zeta^{(\gamma)} \in D_\gamma(\mathcal{Q}^0)$ and $\zeta^{(2\gamma)} \in C_\gamma[a_1, b_1]$ for $a_1, b_1 \in \mathcal{Q}^0$ with $a_1 < b_1$. If $\left|\zeta^{(2\gamma)}\right|$ is generalized strongly m-convex, then for any $x \in [\frac{a_1+b_1}{2}, b_1]$, the following inequality holds true:*

$$|\Lambda(a_1, b_1; v)| \leq \frac{(b_1 - a_1)^{2\gamma}}{\Gamma(1 + 2\gamma)} \min\{M_1, M_2\}, \tag{8.10}$$

where

$$M_1 = \Omega_1(a_1, b_1; x) \left|\zeta^{(2\gamma)}(a_1)\right| + m^\gamma \Omega_2(a_1, b_1; x) \left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right| - \left(\frac{c}{m}\right)^\gamma \Omega_3(a_1, b_1; x)(ma_1 - b_1)^{2\gamma},$$

$$M_2 = \Omega_1(a_1, b_1; x) \left|\zeta^{(2\gamma)}(b_1)\right| + m^\gamma \Omega_2(a_1, b_1; x) \left|\zeta^{(2\gamma)}\left(\frac{a_1}{m}\right)\right| - \left(\frac{c}{m}\right)^\gamma \Omega_3(a_1, b_1; x)(mb_1 - a_1)^{2\gamma},$$

and

$$\begin{aligned} &\Omega_1(a_1, b_1; v) \\ = {} & \frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)} \left(\frac{(v-a_1)^{4\gamma} - (b_1 - v)^{4\gamma}}{(b_1 - a_1)^{4\gamma}}\right) - \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} \left(\frac{(v-a_1)^{3\gamma} - 2^\gamma (b_1 - v)^{3\gamma}}{(b_1 - a_1)^{3\gamma}}\right) \\ & + \left(\frac{1}{4}\right)^\gamma \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left(\frac{(v-a_1)^{2\gamma} - (b_1 - v)^{2\gamma}}{(b_1 - a_1)^{2\gamma}}\right), \end{aligned}$$

$$\begin{aligned} &\Omega_2(a_1, b_1; v) \\ = {} & \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} \left(\frac{2^\gamma (v-a_1)^{3\gamma} - (b_1 - v)^{3\gamma}}{(b_1 - a_1)^{3\gamma}}\right) - \frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)} \left(\frac{(v-a_1)^{4\gamma} - (b_1 - v)^{4\gamma}}{(b_1 - a_1)^{4\gamma}}\right) \\ & - \left(\frac{5}{4}\right)^\gamma \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left(\frac{(v-a_1)^{2\gamma} - (b_1 - v)^{2\gamma}}{(b_1 - a_1)^{2\gamma}}\right) + \left(\frac{1}{4}\right)^\gamma \frac{1}{\Gamma(1+\gamma)} \left(\frac{2v - a_1 - b_1}{b_1 - a_1}\right)^\gamma, \end{aligned}$$

$$\begin{aligned} \Omega_3(a_1, b_1; v) = {} & 2^\gamma \frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)} \left(\frac{v-a_1}{b_1 - a_1}\right)^{4\gamma} - \frac{\Gamma(1+4\gamma)}{\Gamma(1+5\gamma)} \left(\frac{(v-a_1)^{5\gamma} + (b_1 - v)^{5\gamma}}{(b_1 - a_1)^{5\gamma}}\right) \\ & + \left(\frac{1}{4}\right)^\gamma \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left(\frac{(v-a_1)^{2\gamma} - (b_1 - v)^{2\gamma}}{(b_1 - a_1)^{2\gamma}}\right) \\ & - \left(\frac{5}{4}\right)^\gamma \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} \left(\frac{(v-a_1)^{3\gamma} - (b_1 - v)^{3\gamma}}{(b_1 - a_1)^{3\gamma}}\right). \end{aligned}$$

Proof. Taking modulus in the identity (8.3) of Lemma 8.1.2, we have

$$\begin{aligned} |\Lambda(a_1,b_1;v)| &\leq \frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)}\frac{1}{\Gamma(1+\gamma)}\int_0^1 |h(\wp)|\left|\zeta^{(2\gamma)}(\wp a_1+(1-\wp)b_1)\right|(d\wp)^\gamma \\ &\leq \frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)}[A_1+A_2+A_3], \end{aligned} \tag{8.11}$$

where

$$\begin{aligned} A_1 &= \frac{1}{\Gamma(1+\gamma)}\int_0^{\frac{b_1-v}{b_1-a_1}} \wp^{2\gamma}\left|\zeta^{(2\gamma)}(\wp a_1+(1-\wp)b_1)\right|(d\wp)^\gamma, \\ A_2 &= \frac{1}{\Gamma(1+\gamma)}\int_{\frac{b_1-v}{b_1-a_1}}^{\frac{v-a_1}{b_1-a_1}} \left(\wp-\frac{1}{2}\right)^{2\gamma}\left|\zeta^{(2\gamma)}(\wp a_1+(1-\wp)b_1)\right|(d\wp)^\gamma \end{aligned}$$

and

$$A_3=\frac{1}{\Gamma(1+\gamma)}\int_{\frac{v-a_1}{b_1-a_1}}^1 (\wp-1)^{2\gamma}\left|\zeta^{(2\gamma)}(\wp a_1+(1-\wp)b_1)\right|(d\wp)^\gamma.$$

By using generalized strongly m-convexity of $\left|\zeta^{(2\gamma)}\right|$, we have

$$\begin{aligned} A_1 &\leq \frac{1}{\Gamma(1+\gamma)}\int_0^{\frac{b_1-v}{b_1-a_1}} \left[\wp^{3\gamma}\left|\zeta^{(2\gamma)}(a_1)\right|+m^\gamma\wp^{2\gamma}(1-\wp)^\gamma\left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right|\right. \\ &\quad \left.-\left(\frac{c}{m}\right)^\gamma\wp^{3\gamma}(1-\wp)^\gamma(ma_1-b_1)^{2\gamma}\right](d\wp)^\gamma \\ &= \frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{4\gamma}\left|\zeta^{(2\gamma)}(a_1)\right| \\ &\quad +m^\gamma\left[\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{3\gamma}-\frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{4\gamma}\right]\left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right| \\ &\quad -\left(\frac{c}{m}\right)^\gamma(ma_1-b_1)^{2\gamma}\left[\frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{4\gamma}-\frac{\Gamma(1+4\gamma)}{\Gamma(1+5\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{5\gamma}\right]. \end{aligned}$$

Similarly,

$$A_2\leq\frac{1}{\Gamma(1+\gamma)}\int_{\frac{b_1-v}{b_1-a_1}}^{\frac{v-a_1}{b_1-a_1}} \left[\left(\wp-\frac{1}{2}\right)^{2\gamma}\wp^\gamma\left|\zeta^{(2\gamma)}(a_1)\right|+m^\gamma\left(\wp-\frac{1}{2}\right)^{2\gamma}(1-\wp)^\gamma\left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right|\right.$$

$$
\begin{aligned}
& - \left(\frac{c}{m}\right)^{\gamma} \left(\wp - \frac{1}{2}\right)^{2\gamma} \wp^{\gamma}(1-\wp)^{\gamma} (ma_1 - b_1)^{2\gamma}\Bigg] (d\wp)^{\gamma} \\
& = \left[\frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)} \left(\frac{(v-a_1)^{4\gamma} - (b_1-v)^{4\gamma}}{(b_1-a_1)^{4\gamma}}\right) - \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} \left(\frac{(v-a_1)^{3\gamma} - (b_1-v)^{3\gamma}}{(b_1-a_1)^{3\gamma}}\right)\right. \\
& \left. + \left(\frac{1}{4}\right)^{\gamma} \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left(\frac{(v-a_1)^{2\gamma} - (b_1-v)^{2\gamma}}{(b_1-a_1)^{2\gamma}}\right)\right] \left|\zeta^{(2\gamma)}(a_1)\right| \\
& + m^{\gamma} \left[2^{\gamma} \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} \left(\frac{(v-a_1)^{3\gamma} - (b_1-v)^{3\gamma}}{(b_1-a_1)^{3\gamma}}\right) + \left(\frac{1}{4}\right)^{\gamma} \frac{1}{\Gamma(1+\gamma)} \left(\frac{(2v-a_1-b_1)^{\gamma}}{(b_1-a_1)^{\gamma}}\right)\right. \\
& - \frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)} \left(\frac{(v-a_1)^{4\gamma} - (b_1-v)^{4\gamma}}{(b_1-a_1)^{4\gamma}}\right) \\
& \left. - \left(\frac{5}{4}\right)^{\gamma} \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left(\frac{(v-a_1)^{2\gamma} - (b_1-v)^{2\gamma}}{(b_1-a_1)^{2\gamma}}\right)\right] \left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right| \\
& - \left(\frac{c}{m}\right)^{\gamma} (ma_1 - b_1)^{2\gamma} \left[2^{\gamma} \frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)} \left(\frac{(v-a_1)^{4\gamma} - (b_1-v)^{4\gamma}}{(b_1-a_1)^{4\gamma}}\right)\right. \\
& + \left(\frac{1}{4}\right)^{\gamma} \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left(\frac{(v-a_1)^{2\gamma} - (b_1-v)^{2\gamma}}{(b_1-a_1)^{2\gamma}}\right) \\
& \left. - \frac{\Gamma(1+4\gamma)}{\Gamma(1+5\gamma)} \left(\frac{(v-a_1)^{5\gamma} - (b_1-v)^{5\gamma}}{(b_1-a_1)^{5\gamma}}\right) - \left(\frac{5}{4}\right)^{\gamma} \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} \left(\frac{(v-a_1)^{3\gamma} - (b_1-v)^{3\gamma}}{(b_1-a_1)^{3\gamma}}\right)\right]
\end{aligned}
$$

and

$$
\begin{aligned}
A_3 \leq & \frac{1}{\Gamma(1+\gamma)} \int_{\frac{v-a_1}{b_1-a}}^{1} \left[(\wp-1)^{2\gamma} \wp^{\gamma} \left|\zeta^{(2\gamma)}(a_1)\right| + m^{\gamma} (\wp-1)^{2\gamma} (1-\wp)^{\gamma} \left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right|\right. \\
& \left. - \left(\frac{c}{m}\right)^{\gamma} (\wp-1)^{2\gamma} t^{\gamma}(1-\wp)^{\gamma} (ma_1 - b_1)^{2\gamma}\right] (d\wp)^{\gamma} \\
= & \left[\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} \left(\frac{b_1-v}{b_1-a_1}\right)^{3\gamma} - \frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)} \left(\frac{b_1-v}{b_1-a_1}\right)^{4\gamma}\right] \left|\zeta^{(2\gamma)}(a_1)\right| \\
& + m^{\gamma} \frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)} \left(\frac{b_1-v}{b_1-a_1}\right)^{4\gamma} \left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right| \\
& - \left(\frac{c}{m}\right)^{\gamma} (ma_1 - b_1)^{2\gamma} \left[\frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)} \left(\frac{b_1-v}{b_1-a_1}\right)^{4\gamma} - \frac{\Gamma(1+4\gamma)}{\Gamma(1+5\gamma)} \left(\frac{b_1-v}{b_1-a_1}\right)^{5\gamma}\right].
\end{aligned}
$$

Clearly, with these values, we obtain from (8.11);

$$
|\Lambda(a_1, b_1; v)| \leq \frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)} M_1.
$$

Analogously one can obtain from the identity (8.9) of Lemma 8.2.1,

$$
|\Lambda(a_1, b_1; v)| \leq \frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)} M_2,
$$

which then completes the proof. $\square$

Corollary 8.2.2. *If in addition to the conditions of Theorem 8.2.1, ζ is generalized m-convex function, then following inequality holds:*

$$|\Lambda(a_1, b_1; v)| \leq \frac{(b_1 - a_1)^{2\gamma}}{\Gamma(1+2\gamma)} \min\{M_3, M_4\}, \tag{8.12}$$

where

$$M_3 = \Omega_1(a_1, b_1; v)\left|\zeta^{(2\gamma)}(a_1)\right| + m^\gamma \Omega_2(a_1, b_1; v)\left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right|,$$

$$M_4 = \Omega_1(a_1, b_1; v)\left|\zeta^{(2\gamma)}(b_1)\right| + m^\gamma \Omega_2(a_1, b_1; v)\left|\zeta^{(2\gamma)}\left(\frac{a_1}{m}\right)\right|.$$

where $\Omega_i(a_1, b_1; x), i = 1, 2, 3$ are same as given in Theorem 8.2.1.

Corollary 8.2.3. *If in addition to the conditions of Theorem 8.2.1, ζ is generalized strongly convex function, then following inequality holds:*

$$|\Lambda(a_1, b_1; v)| \leq \frac{(b_1 - a_1)^{2\gamma}}{\Gamma(1+2\gamma)} M^*, \tag{8.13}$$

where

$$M^* = \Omega_1(a_1, b_1; v)\left|\zeta^{(2\gamma)}(a_1)\right| + \Omega_2(a_1, b_1; v)\left|\zeta^{(2\gamma)}(b_1)\right| - c^\gamma \Omega_3(a_1, b_1; v)(a_1 - b_1)^{2\gamma},$$

Remark 20. *If $m = 1, c = 0$ in the inequality of Theorem 8.2.1, then we have [38, Theorem 10].*

Theorem 8.2.2. *Let $\mathcal{Q} \subseteq \mathbb{R}$ be an interval, $\zeta : \mathcal{Q}^\circ \subseteq \mathbb{R} \to \mathbb{R}^\gamma$ ($\mathcal{Q}^0$ is the interval of $\mathcal{Q}$) such that $\zeta, \zeta^{(\gamma)} \in D_\gamma(\mathcal{Q}^\circ)$ and $\zeta(2\gamma) \in C_\gamma[a_1, b_1]$ for $a_1, b_1 \in \mathcal{Q}^\circ$ with $a_1 < b_1$. Also let $p, q > 1$ and*

$\frac{1}{p}+\frac{1}{q}=1$. *If* $\left|\zeta^{(2\gamma)}\right|^q$ *is generalized strongly m-convex, then for any* $v\in[\frac{a_1+b_1}{2},b_1]$, *the following inequality holds true:*

$$|\Lambda(a_1,b_1;v)|\leq\frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)}\left(\frac{\Gamma(1+2\gamma p)}{\Gamma(1+(2p+1)\gamma)}\right)^{\frac{1}{p}}\left(\frac{b_1-v}{b_1-a_1}\right)^{(2+\frac{1}{p})\gamma}\min\left\{\sum_{i=1}^{3}J_i^\gamma,\ \sum_{i=1}^{3}K_i^\gamma\right\},\tag{8.14}$$

where

$$J_1^\gamma = \left[B_1(v)\left[\left|\zeta^{(2\gamma)}(a_1)\right|^q+m^\gamma\left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right|^q\right]-\left(\frac{c}{m}\right)^\gamma(ma_1-b_1)^{2\gamma}B_2(v)\right]^{\frac{1}{q}},$$

$$\begin{aligned}J_2^\gamma &= \left[\left(\frac{2v-a_1-b_1}{2(b_1-a_1)}\right)^{(2p+1)\gamma}-\left(\frac{a_1+b_1-2v}{2(b_1-a_1)}\right)^{(2p+1)\gamma}\right]^{\frac{1}{p}}\\ &\quad\times\left[B_3(v)\left[\left|\zeta^{(2\gamma)}(a_1)\right|^q+m^\gamma\left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right|^q\right]-\left(\frac{c}{m}\right)^\gamma(ma_1-b_1)^{2\gamma}B_4(v)\right]^{\frac{1}{q}},\end{aligned}$$

$$J_3^\gamma = \left[B_5(v)\left|\zeta^{(2\gamma)}(a_1)\right|^q+m^\gamma B_6(v)\left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right|^q-\left(\frac{c}{m}\right)^\gamma(ma_1-b_1)^{\gamma}B_7(v)\right]^{\frac{1}{q}},$$

$$K_1^\gamma = \left[B_1(v)\left[\left|\zeta^{(2\gamma)}(b_1)\right|^q+m^\gamma\left|\zeta^{(2\gamma)}\left(\frac{a_1}{m}\right)\right|^q\right]-\left(\frac{c}{m}\right)^\gamma(mb_1-a_1)^{2\gamma}B_2(v)\right]^{\frac{1}{q}},$$

$$\begin{aligned}K_2^\gamma &= \left[\left(\frac{2v-a_1-b_1}{2(b_1-a_1)}\right)^{(2p+1)\gamma}-\left(\frac{a_1+b_1-2v}{2(b_1-a_1)}\right)^{(2p+1)\gamma}\right]^{\frac{1}{p}}\\ &\quad\times\left[B_3(v)\left[\left|\zeta^{(2\gamma)}(b_1)\right|^q+m^\gamma\left|\zeta^{(2\gamma)}\left(\frac{a_1}{m}\right)\right|^q\right]-\left(\frac{c}{m}\right)^\gamma(mb_1-a_1)^{2\gamma}B_4(v)\right]^{\frac{1}{q}},\end{aligned}$$

$$K_3^\gamma = \left[B_5(v)\left|\zeta^{(2\gamma)}(b_1)\right|^q+m^\gamma B_6(v)\left|\zeta^{(2\gamma)}\left(\frac{a_1}{m}\right)\right|^q-\left(\frac{c}{m}\right)^\gamma(mb_1-a_1)^{\gamma}B_7(v)\right]^{\frac{1}{q}}$$

and

$$B_1(v)=\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left(\frac{b_1-v}{b_1-a}\right)^{2\gamma},\quad B_2(v)=\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{2\gamma}-\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{3\gamma},$$

$$B_3(v)=\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left[\frac{(v-a_1)^{2\gamma}-(b_1-v)^{2\gamma}}{(b_1-a_1)^{2\gamma}}\right],$$

$$B_4(v)=\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left[\frac{(v-a_1)^{2\gamma}-(b_1-v)^{2\gamma}}{(b_1-a_1)^{2\gamma}}\right]-\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\left[\frac{(v-a_1)^{3\gamma}-(b_1-v)^{3\gamma}}{(b_1-a_1)^{3\gamma}}\right],$$

$$B_5(v)=\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left[\frac{(b_1-a_1)^{2\gamma}-(b_1-v)^{2\gamma}}{(b_1-a_1)^{2\gamma}}\right],\quad B_6(v)=\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left(\frac{b-v}{b_1-a_1}\right)^{2\gamma}$$

and

$$B_7(v)=\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left[\frac{(v-a_1)^{2\gamma}-(b_1-v)^{2\gamma}}{(b_1-a_1)^{2\gamma}}\right]-\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\left[\frac{(v-a_1)^{3\gamma}-(b_1-v)^{3\gamma}}{(b_1-a_1)^{3\gamma}}\right].$$

Proof. Taking modulus on both sides of equality in Lemma 8.1.2, we have

$$\begin{aligned}|\Lambda(a_1,b_1;v)| &\le \frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)}\frac{1}{\Gamma(1+\gamma)}\int_0^1 |h(\wp)|\left|\zeta^{(2\gamma)}(\wp a_1+(1-\wp)b_1)\right|(d\wp)^\gamma\\ &\le \frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)}[I_1+I_2+I_3].\end{aligned}\tag{8.15}$$

By Hölder's inequality for local fractional integrals, we obtain

$$I_1\le\left(\frac{1}{\Gamma(1+\gamma)}\int_0^{\frac{b_1-v}{b_1-a_1}}\wp^{2\gamma p}(dt)^\gamma\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\gamma)}\int_0^{\frac{b_1-v}{b_1-a_1}}\left|\zeta^{(2\gamma)}(\wp a_1+(1-\wp)b_1)\right|^q(dt)^\gamma\right)^{\frac{1}{q}},$$

$$I_2\le\left(\frac{1}{\Gamma(1+\gamma)}\int_{\frac{b_1-v}{b_1-a_1}}^{\frac{v-a_1}{b_1-a_1}}\left(\wp-\frac{1}{2}\right)^{2\gamma p}(d\wp)^\gamma\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\gamma)}\int_{\frac{b_1-v}{b_1-a_1}}^{\frac{v-a_1}{b_1-a_1}}\left|\zeta^{(2\gamma)}(\wp a_1+(1-\wp)b_1)\right|^q(d\wp)^\gamma\right)^{\frac{1}{q}}$$

and

$$I_3\le\left(\frac{1}{\Gamma(1+\gamma)}\int_{\frac{v-a_1}{b_1-a_1}}^{1}(\wp-1)^{2\gamma p}(d\wp)^\gamma\right)^{\frac{1}{p}}\left(\frac{1}{\Gamma(1+\gamma)}\int_{\frac{v-a_1}{b_1-a_1}}^{1}\left|\zeta^{(2\gamma)}(\wp a_1+(1-\wp)b_1)\right|^q(d\wp)^\gamma\right)^{\frac{1}{q}}.$$

Here, using Lemma 1.8.1, we get

$$\frac{1}{\Gamma(1+\gamma)}\int_0^{\frac{b_1-v}{b_1-a_1}}\wp^{2\gamma p}(d\wp)^\gamma=\frac{\Gamma(1+2\gamma p)}{\Gamma(1+(2p+1)\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{(2p+1)\gamma},$$

$$\frac{1}{\Gamma(1+\gamma)}\int_{\frac{b_1-v}{b_1-a_1}}^{\frac{v-a_1}{b_1-a_1}}\left(\wp-\frac{1}{2}\right)^{2\gamma p}(d\wp)^{\gamma}=\frac{\Gamma(1+2\gamma p)}{\Gamma(1+(2p+1)\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{(2p+1)\gamma}$$
$$\times\left[\left(\frac{2v-a_1-b_1}{2(b_1-a_1)}\right)^{(2p+1)\gamma}-\left(\frac{a_1+b_1-2v}{2(b_1-a_1)}\right)^{(2p+1)\gamma}\right]$$

and

$$\frac{1}{\Gamma(1+\gamma)}\int_{\frac{v-a_1}{b_1-a_1}}^{1}(\wp-1)^{2\gamma p}(d\wp)^{\gamma}=\frac{\Gamma(1+2\gamma p)}{\Gamma(1+(2p+1)\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{(2p+1)\gamma}.$$

Now applying the generalized strongly m-convexity of $\left|\zeta^{(2\gamma)}\right|^q$, one has

$$\int_{0}^{\frac{b_1-v}{b_1-a_1}}\left|\zeta^{(2\gamma)}(\wp a_1+(1-\wp)b_1)\right|^q(d\wp)^{\gamma}$$
$$\leq\int_{0}^{\frac{b_1-v}{b_1-a_1}}\left[\wp^{\gamma}\left|\zeta^{(2\gamma)}(a_1)\right|^q+m^{\gamma}(1-\wp)^{\gamma}\left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right|^q-\left(\frac{c}{m}\right)^{\gamma}\wp^{\gamma}(1-\wp)^{\gamma}(ma_1-b_1)^{2\gamma}\right](d\wp)^{\gamma}$$
$$=\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{2\gamma}\left[\left|\zeta^{(2\gamma)}(a_1)\right|^q+m^{\gamma}\left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right|^q\right]$$
$$-\left(\frac{c}{m}\right)^{\gamma}(ma_1-b_1)^{2\gamma}\left[\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{2\gamma}-\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\left(\frac{b_1-v}{b_1-a_1}\right)^{3\gamma}\right].$$

Similarly,

$$\frac{1}{\Gamma(1+\gamma)}\int_{\frac{b_1-v}{b_1-a_1}}^{\frac{v-a_1}{b_1-a_1}}\left|\zeta^{(2\gamma)}(\wp a_1+(1-\wp)b_1)\right|^q(d\wp)^{\gamma}$$
$$\leq\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left\{\frac{(v-a_1)^{2\gamma}-(b_1-v)^{2\gamma}}{(b_1-a_1)^{2\gamma}}\right\}\left[\left|\zeta^{(2\gamma)}(a_1)\right|^q+m^{\gamma}\left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right|^q\right]$$
$$-\left(\frac{c}{m}\right)^{\gamma}(ma_1-b_1)^{2\gamma}\left[\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left\{\frac{(v-a_1)^{2\gamma}-(b_1-v)^{2\gamma}}{(b_1-a_1)^{2\gamma}}\right\}\right.$$
$$\left.-\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\left\{\frac{(v-a_1)^{3\gamma}-(b_1-v)^{3\gamma}}{(b_1-a_1)^{3\gamma}}\right\}\right]$$

and

$$\frac{1}{\Gamma(1+\gamma)}\int_{\frac{v-a_1}{b_1-a_1}}^{1}\left|\zeta^{(2\gamma)}(\wp a_1+(1-\wp)b_1)\right|^q(d\wp)^{\gamma}$$

$$
\begin{aligned}
&\le \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left\{ \frac{(b_1-a_1)^{2\gamma}-(b_1-v)^{2\gamma}}{(b_1-a_1)^{2\gamma}} \right\} \left|\zeta^{(2\gamma)}(a_1)\right|^q \\
&+ m^\gamma \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left(\frac{b_1-v}{b_1-a_1} \right)^{2\gamma} \left| \zeta^{(2\gamma)}\left(\frac{b_1}{m}\right) \right|^q \\
&- \left(\frac{c}{m}\right)^\gamma (ma_1-b_1)^\gamma \left[\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left\{ \frac{(v-a_1)^{2\gamma}-(b_1-v)^{2\gamma}}{(b_1-a_1)^{2\gamma}} \right\} \right. \\
&\left. - \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} \left\{ \frac{(v-a_1)^{3\gamma}-(b_1-v)^{3\gamma}}{(b_1-a_1)^{3\gamma}} \right\} \right].
\end{aligned}
$$

Using (8.15), we have

$$
|\Lambda(a_1,b_1;v)| \le \frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)} \left(\frac{\Gamma(1+2\gamma p)}{\Gamma(1+(2p+1)\gamma)} \right)^{\frac{1}{p}} \left(\frac{b_1-v}{b_1-a_1} \right)^{(2+\frac{1}{p})\gamma} \sum_{i=1}^{3} J_i^\gamma.
$$

Analogously from identity of Lemma 8.2.1, we obtain

$$
|\Lambda(a_1,b_1;v)| \le \frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)} \left(\frac{\Gamma(1+2\gamma p)}{\Gamma(1+(2p+1)\gamma)} \right)^{\frac{1}{p}} \left(\frac{b_1-v}{b_1-a_1} \right)^{(2+\frac{1}{p})\gamma} \sum_{i=1}^{3} K_i^\gamma,
$$

which then completes the proof. □

Corollary 8.2.4. *If in addition to the conditions of Theorem 8.2.2, ζ is generalized m-convex function, then following inequality holds:*

$$
\begin{aligned}
&|\Lambda(a_1,b_1;v)| \\
&\le \frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)} \left(\frac{\Gamma(1+2\gamma p)}{\Gamma(1+(2p+1)\gamma)} \right)^{\frac{1}{p}} \left(\frac{b_1-v}{b_1-a_1} \right)^{(2+\frac{1}{p})\gamma} \min\left\{ \sum_{i=4}^{6} J_i^\gamma,\ \sum_{i=4}^{6} K_i^\gamma \right\}, \qquad (8.16)
\end{aligned}
$$

where

$$
J_4^\gamma = \left[B_1(v) \left[\left|\zeta^{(2\gamma)}(a_1)\right|^q + m^\gamma \left| \zeta^{(2\gamma)}\left(\frac{b_1}{m}\right) \right|^q \right] \right]^{\frac{1}{q}},
$$

$$
\begin{aligned}
J_5^\gamma &= \left[\left(\frac{2v-a_1-b_1}{2(b_1-a_1)} \right)^{(2p+1)\gamma} - \left(\frac{a_1+b_1-2v}{2(b_1-a_1)} \right)^{(2p+1)\gamma} \right]^{\frac{1}{p}} \\
&\quad \times \left[B_3(v) \left[\left|\zeta^{(2\gamma)}(a_1)\right|^q + m^\gamma \left| \zeta^{(2\gamma)}\left(\frac{b_1}{m}\right) \right|^q \right] \right]^{\frac{1}{q}},
\end{aligned}
$$

$$J_6^\gamma = \left[B_5(v)\left|\zeta^{(2\gamma)}(a_1)\right|^q + m^\gamma B_6(v)\left|\zeta^{(2\gamma)}\left(\frac{b_1}{m}\right)\right|^q\right]^{\frac{1}{q}},$$

$$K_4^\gamma = \left[B_1(v)\left[\left|\zeta^{(2\gamma)}(b_1)\right|^q + m^\gamma\left|\zeta^{(2\gamma)}\left(\frac{a_1}{m}\right)\right|^q\right]\right]^{\frac{1}{q}},$$

$$K_5^\gamma = \left[\left(\frac{2v-a_1-b_1}{2(b_1-a_1)}\right)^{(2p+1)\gamma} - \left(\frac{a_1+b_1-2v}{2(b_1-a_1)}\right)^{(2p+1)\gamma}\right]^{\frac{1}{p}}$$
$$\times\left[B_3(v)\left[\left|\zeta^{(2\gamma)}(b_1)\right|^q + m^\gamma\left|\zeta^{(2\gamma)}\left(\frac{a_1}{m}\right)\right|^q\right]\right]^{\frac{1}{q}},$$

$$K_6^\gamma = \left[B_5(v)\left|\zeta^{(2\gamma)}(b_1)\right|^q + m^\gamma B_6(v)\left|\zeta^{(2\gamma)}\left(\frac{a_1}{m}\right)\right|^q\right]^{\frac{1}{q}}$$

and $B_i(x), i = 1,2,3,4,5,6$ *are same as given in Theorem 8.2.2.*

Corollary 8.2.5. *If in addition to the conditions of Theorem 8.2.2,* ζ *is generalized strongly convex function, then following inequality holds:*

$$|\Lambda(a_1,b_1;v)| \le \frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)}\left(\frac{\Gamma(1+2\gamma p)}{\Gamma(1+(2p+1)\gamma)}\right)^{\frac{1}{p}}\left(\frac{b_1-v}{b_1-a_1}\right)^{(2+\frac{1}{p})\gamma}\sum_{i=1}^{3} J_i^\gamma, \tag{8.17}$$

where

$$J_1^\gamma = \left[B_1(v)\left[\left|\zeta^{(2\gamma)}(a_1)\right|^q + \left|\zeta^{(2\gamma)}(b_1)\right|^q\right] - c^\gamma (a_1-b_1)^{2\gamma} B_2(v)\right]^{\frac{1}{q}},$$

$$J_2^\gamma = \left[\left(\frac{2v-a_1-b_1}{2(b_1-a_1)}\right)^{(2p+1)\gamma} - \left(\frac{a_1+b_1-2v}{2(b_1-a_1)}\right)^{(2p+1)\gamma}\right]^{\frac{1}{p}}$$
$$\times\left[B_3(v)\left[\left|\zeta^{(2\gamma)}(a_1)\right|^q + \left|\zeta^{(2\gamma)}(b_1)\right|^q\right] - c^\gamma (a_1-b_1)^{2\gamma} B_4(v)\right]^{\frac{1}{q}},$$

$$J_3^\gamma = \left[B_5(v)\left|\zeta^{(2\gamma)}(a_1)\right|^q + B_6(v)\left|\zeta^{(2\gamma)}(b_1)\right|^q - c^\gamma(a_1-b_1)^\gamma B_7(v)\right]^{\frac{1}{q}},$$

and $B_i(v), i = 1,2,3,4,5,6,7$ *are same as given in Theorem 8.2.2.*

Remark 21. *If* $m = 1, c = 0$ *in the inequality of Theorem 8.2.1, then we have [38, Theorem 11].*

8.3 Applications to numerical integration

In this section, let us consider some applications of the integral inequalities involving local fractional integral developed in section 8.2.

Proposition 10. *Let $\mathscr{Q} \subseteq \mathbb{R}$ be an interval, $\zeta : \mathscr{Q}^\circ \subseteq \mathbb{R} \to \mathbb{R}^\gamma$ ($\mathscr{Q}^\circ$ is the interval of $\mathscr{Q}$) such that $\zeta, \zeta^{(\gamma)} \in D_\gamma(\mathscr{Q}^\circ)$ and $\zeta^{(2\gamma)} \in C_\gamma[a_1, b_1]$ for $a_1, b_1 \in \mathscr{Q}^\circ$ with $a_1 < b_1$. If $\left|\zeta^{(2\gamma)}\right|$ is generalized strongly m-convex and $P_n : a_1 = \mathscr{S}_0 < \mathscr{S}_1 < \ldots < \mathscr{S}_{n-1} < \mathscr{S}_n = b_1$ is a partition of $[a_1, b_1]$, where $h_i = \mathscr{S}_{i+1} - \mathscr{S}_i$, $i = 0, 1, \ldots, n-1$, then for any $\mathscr{S} \in [\frac{a_1+b_1}{2}, b_1]$, we have*

$$\frac{1}{\Gamma(1+\gamma)} \int_{a_1}^{b_1} \zeta(\mathscr{S})(d\mathscr{S})^\gamma = W(P_n, \zeta) + R(P_n, \zeta),$$

where

$$\begin{aligned} W(P_n, \zeta) &= -\frac{1}{[\Gamma(1+\gamma)]^2 \Gamma(1+2\gamma)} \sum_{i=0}^{n-1} h_i^\gamma \left[\zeta(\mathscr{S}_i^\star) - \zeta(\mathscr{S}_i + \mathscr{S}_{i+1} - \mathscr{S}_i^\star)\right] \\ &+ \frac{1}{2^\gamma [\Gamma(1+\gamma)]^2} \sum_{i=0}^{n-1} h_i^\gamma \left(\mathscr{S}_i^\star - \frac{\mathscr{S}_i + 3\mathscr{S}_{i+1}}{4}\right)^\gamma \left[\zeta^{(\gamma)}(\mathscr{S}_i^\star) - \zeta^{(\gamma)}(\mathscr{S}_i + \mathscr{S}_{i+1} - \mathscr{S}_i^\star)\right] \end{aligned}$$

and the remainder term satisfies the estimation:

$$|R(P_n, \zeta)| \leq \frac{1}{\Gamma(1+\gamma)\Gamma(1+2\gamma)} \sum_{i=0}^{n-1} h_i^{3\gamma} \min\{E_{i,1}, E_{i,2}\}, \tag{8.18}$$

with

$$\begin{aligned} E_{i,1} &= \Psi_1(\mathscr{S}_i, \mathscr{S}_{i+1}; \mathscr{S}_i^\star) \left|\zeta^{(2\gamma)}(\mathscr{S}_i)\right| + m^\gamma \Psi_2(\mathscr{S}_i, \mathscr{S}_{i+1}; \mathscr{S}_i^\star) \left|\zeta^{(2\gamma)}\left(\frac{\mathscr{S}_{i+1}}{m}\right)\right| \\ &- \left(\frac{c}{m}\right)^\gamma \Psi_3(\mathscr{S}_i, \mathscr{S}_{i+1}; \mathscr{S}_i^\star)(m\mathscr{S}_i - \mathscr{S}_{i+1})^{2\gamma}, \end{aligned}$$

$$\begin{aligned} E_{i,2} &= \Psi_1(\mathscr{S}_i, \mathscr{S}_{i+1}; \mathscr{S}_i^\star) \left|\zeta^{(2\gamma)}(\mathscr{S}_{i+1})\right| + m^\gamma \Psi_2(\mathscr{S}_i, \mathscr{S}_{i+1}; \mathscr{S}_i^\star) \left|\zeta^{(2\gamma)}\left(\frac{\mathscr{S}_i}{m}\right)\right| \\ &- \left(\frac{c}{m}\right)^\gamma \Psi_3(\mathscr{S}_i, \mathscr{S}_{i+1}; \mathscr{S}_i^\star)(m\mathscr{S}_{i+1} - \mathscr{S}_i)^{2\gamma} \end{aligned}$$

and

$$\begin{aligned}\Psi_1(\mathscr{S}_i,\mathscr{S}_{i+1};\mathscr{S}_i^\star) &= \frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)}\left(\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{4\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{4\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{4\gamma}}\right)\\ &- \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\left(\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{3\gamma}-2^\gamma(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{3\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{3\gamma}}\right)\\ &\quad+\left(\frac{1}{4}\right)^\gamma\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left(\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{2\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{2\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{2\gamma}}\right),\end{aligned}$$

$$\begin{aligned}\Psi_2(\mathscr{S}_i,\mathscr{S}_{i+1};\mathscr{S}_i^\star) &= \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\left(\frac{2^\gamma(\mathscr{S}_i^\star-\mathscr{S}_i)^{3\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{3\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{3\gamma}}\right)\\ &- \frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)}\left(\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{4\gamma}-(\mathscr{S}_{i+1}-x)^{4\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{4\gamma}}\right)\\ &\quad-\left(\frac{5}{4}\right)^\gamma\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left(\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{2\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{2\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{2\gamma}}\right)\\ &\quad+\left(\frac{1}{4}\right)^\gamma\frac{1}{\Gamma(1+\gamma)}\left(\frac{2\mathscr{S}_i^\star-\mathscr{S}_i-\mathscr{S}_{i+1}}{\mathscr{S}_{i+1}-\mathscr{S}_i}\right)^\gamma,\end{aligned}$$

$$\begin{aligned}\Psi_3(\mathscr{S}_i,\mathscr{S}_{i+1};\mathscr{S}_i^\star) &= 2^\gamma\frac{\Gamma(1+3\gamma)}{\Gamma(1+4\gamma)}\left(\frac{\mathscr{S}_i^\star-\mathscr{S}_i}{\mathscr{S}_{i+1}-\mathscr{S}_i}\right)^{4\gamma}-\frac{\Gamma(1+4\gamma)}{\Gamma(1+5\gamma)}\left(\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{5\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{5\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{5\gamma}}\right)\\ &+\left(\frac{1}{4}\right)^\gamma\frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)}\left(\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{2\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{2\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{2\gamma}}\right)\\ &-\left(\frac{5}{4}\right)^\gamma\frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)}\left(\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{3\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{3\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{3\gamma}}\right),\end{aligned}$$

for any $\mathscr{S}_i^\star \in \left[\frac{\mathscr{S}_i+\mathscr{S}_{i+1}}{2},\mathscr{S}_{i+1}\right]$.

Proof. Applying Theorem 8.2.1 on the interval $[\mathscr{S}_i,\mathscr{S}_{i+1}]$ for all $i=0,1,\ldots,n-1$, we get

$$\Bigg|\, {}_{\mathscr{S}_i}I^{(\gamma)}_{\mathscr{S}_{i+1}}\zeta-\frac{1}{[\Gamma(1+\gamma)]^2\Gamma(1+2\gamma)}\sum_{i=0}^{n-1}h_i^\gamma\left[\zeta(\mathscr{S}_i^\star)-\zeta(\mathscr{S}_i+\mathscr{S}_{i+1}-\mathscr{S}_i^\star)\right]$$

$$+\frac{1}{2^\gamma[\Gamma(1+\gamma)]^2}\sum_{i=0}^{n-1}h_i^\gamma\left(\mathscr{S}_i^\star-\frac{\mathscr{S}_i+3\mathscr{S}_{i+1}}{4}\right)^\gamma\left[\zeta^{(\gamma)}(\mathscr{S}_i^\star)-\zeta^{(\gamma)}(\mathscr{S}_i+\mathscr{S}_{i+1}-\mathscr{S}_i^\star)\right]\Bigg|$$

$$\leq\frac{h_i^{3\gamma}}{\Gamma(1+\gamma)\Gamma(1+2\gamma)}\min\left\{E_{i,1},E_{i,2}\right\}.$$

Summing over i from 0 to $n-1$ and using the triangle inequality we obtain the estimation (8.18). □

Proposition 11. *Let $\mathcal{Q} \subseteq \mathbb{R}$ be an interval, $\zeta : \mathcal{Q}^{\circ} \subseteq \mathbb{R} \to \mathbb{R}^{\gamma}$ ($\mathcal{Q}^{\circ}$ is the interval of $\mathcal{Q}$) such that $\zeta, \zeta^{(\gamma)} \in D_{\gamma}(\mathcal{Q}^{\circ})$ and $\zeta^{(2\gamma)} \in C_{\gamma}[a_1, b_1]$ for $a_1, b_1 \in \mathcal{Q}^{\circ}$ with $a_1 < b_1$. Also let $p, q > 1$ and $\frac{1}{p}+\frac{1}{q} = 1$. If $\left|\zeta^{(2\gamma)}\right|$ is generalized strongly m-convex and $P_n : a_1 = \mathscr{S}_0 < \mathscr{S}_1 < \ldots < \mathscr{S}_{n-1} < \mathscr{S}_n = b_1$ is a partition of $[a_1, b_1]$, where $h_i = \mathscr{S}_{i+1} - \mathscr{S}_i$, $i = 0, 1, \ldots, n-1$,*

$$|R(P_n, \zeta)| \leq \frac{1}{\Gamma(1+\gamma)\Gamma(1+2\gamma)} \left(\frac{\Gamma(1+2\gamma p)}{\Gamma(1+(2p+1)\gamma)}\right)^{\frac{1}{p}} \times \sum_{i=1}^{3} (\mathscr{S}_{i+1} - \mathscr{S}_i^{\star})^{(2+\frac{1}{p})\gamma} h_i^{\frac{\gamma}{q}} \min\left\{U_i^{\gamma},\ V_i^{\gamma}\right\}, \tag{8.19}$$

where

$$U_1^{\gamma} = \left[B_1(\mathscr{S}_i^{\star})\left(\left|\zeta^{(2\gamma)}(\mathscr{S}_i)\right|^q + m^{\gamma}\left|\zeta^{(2\gamma)}\left(\frac{\mathscr{S}_{i+1}}{m}\right)\right|^q\right) - \left(\frac{c}{m}\right)^{\gamma}(m\mathscr{S}_i - \mathscr{S}_{i+1})^{2\gamma} B_2(\mathscr{S}_i^{\star})\right]^{\frac{1}{q}},$$

$$U_2^{\gamma} = \left[\left(\frac{2\mathscr{S}_i^{\star} - \mathscr{S}_i - \mathscr{S}_{i+1}}{2(\mathscr{S}_{i+1} - \mathscr{S}_i)}\right)^{(2p+1)\gamma} - \left(\frac{\mathscr{S}_i + \mathscr{S}_{i+1} - 2\mathscr{S}_i^{\star}}{2(\mathscr{S}_{i+1} - \mathscr{S}_i)}\right)^{(2p+1)\gamma}\right]^{\frac{1}{p}} \times \left[B_3(\mathscr{S}_i^{\star})\left(\left|\zeta^{(2\gamma)}(\mathscr{S}_i)\right|^q + m^{\gamma}\left|\zeta^{(2\gamma)}\left(\frac{\mathscr{S}_{i+1}}{m}\right)\right|^q\right) - \left(\frac{c}{m}\right)^{\gamma}(m\mathscr{S}_i - \mathscr{S}_{i+1})^{2\gamma} B_4(\mathscr{S}_i^{\star})\right]^{\frac{1}{q}},$$

$$U_3^{\gamma} = \left[B_5(\mathscr{S}_i^{\star})\left|\zeta^{(2\gamma)}(\mathscr{S}_i)\right|^q + m^{\gamma} B_6(\mathscr{S}_i^{\star})\left|\zeta^{(2\gamma)}\left(\frac{\mathscr{S}_{i+1}}{m}\right)\right|^q - \left(\frac{c}{m}\right)^{\gamma}(m\mathscr{S}_i - \mathscr{S}_{i+1})^{\gamma} B_7(\mathscr{S}_i^{\star})\right]^{\frac{1}{q}},$$

$$V_1^{\gamma} = \left[B_1(\mathscr{S}_i^{\star})\left(\left|\zeta^{(2\gamma)}(\mathscr{S}_{i+1})\right|^q + m^{\gamma}\left|\zeta^{(2\gamma)}\left(\frac{\mathscr{S}_i}{m}\right)\right|^q\right) - \left(\frac{c}{m}\right)^{\gamma}(m\mathscr{S}_{i+1} - \mathscr{S}_i)^{2\gamma} B_2(\mathscr{S}_i^{\star})\right]^{\frac{1}{q}},$$

$$V_2^{\gamma} = \left[\left(\frac{2\mathscr{S}_i^{\star} - \mathscr{S}_i - \mathscr{S}_{i+1}}{2(\mathscr{S}_{i+1} - \mathscr{S}_i)}\right)^{(2p+1)\gamma} - \left(\frac{\mathscr{S}_i + \mathscr{S}_{i+1} - 2\mathscr{S}_i^{\star}}{2(\mathscr{S}_{i+1} - \mathscr{S}_i)}\right)^{(2p+1)\gamma}\right]^{\frac{1}{p}} \times \left[B_3(\mathscr{S}_i^{\star})\left(\left|\zeta^{(2\gamma)}(\mathscr{S}_{i+1})\right|^q + m^{\gamma}\left|\zeta^{(2\gamma)}\left(\frac{\mathscr{S}_i}{m}\right)\right|^q\right) - \left(\frac{c}{m}\right)^{\gamma}(m\mathscr{S}_{i+1} - \mathscr{S}_i)^{2\gamma} B_4(\mathscr{S}_i^{\star})\right]^{\frac{1}{q}},$$

$$V_3^{\gamma} = \left[B_5(\mathscr{S}_i^{\star})\left|\zeta^{(2\gamma)}(\mathscr{S}_{i+1})\right|^q + m^{\gamma} B_6(\mathscr{S}_i^{\star})\left|\zeta^{(2\gamma)}\left(\frac{\mathscr{S}_i}{m}\right)\right|^q - \left(\frac{c}{m}\right)^{\gamma}(m\mathscr{S}_{i+1} - \mathscr{S}_i)^{\gamma} B_7(\mathscr{S}_i^{\star})\right]^{\frac{1}{q}},$$

$$B_1(\mathscr{S}_i^\star) = \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left(\frac{\mathscr{S}_{i+1}-\mathscr{S}_i^\star}{\mathscr{S}_{i+1}-\mathscr{S}_i}\right)^{2\gamma},$$

$$B_2(\mathscr{S}_i^\star) = \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left(\frac{\mathscr{S}_{i+1}-\mathscr{S}_i^\star}{\mathscr{S}_{i+1}-\mathscr{S}_i}\right)^{2\gamma} - \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} \left(\frac{\mathscr{S}_{i+1}-\mathscr{S}_i^\star}{\mathscr{S}_{i+1}-\mathscr{S}_i}\right)^{3\gamma},$$

$$B_3(\mathscr{S}_i^\star) = \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left[\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{2\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{2\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{2\gamma}}\right],$$

$$B_4(\mathscr{S}_i^\star) = \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left[\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{2\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{2\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{2\gamma}}\right] - \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} \left[\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{3\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{3\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{3\gamma}}\right],$$

$$B_5(\mathscr{S}_i^\star) = \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left[\frac{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{2\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{2\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{2\gamma}}\right], \quad B_6(\mathscr{S}_i^\star) = \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left(\frac{\mathscr{S}_{i+1}-\mathscr{S}_i^\star}{\mathscr{S}_{i+1}-\mathscr{S}_i}\right)^{2\gamma}$$

and

$$B_7(\mathscr{S}_i^\star) = \frac{\Gamma(1+\gamma)}{\Gamma(1+2\gamma)} \left[\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{2\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{2\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{2\gamma}}\right] - \frac{\Gamma(1+2\gamma)}{\Gamma(1+3\gamma)} \left[\frac{(\mathscr{S}_i^\star-\mathscr{S}_i)^{3\gamma}-(\mathscr{S}_{i+1}-\mathscr{S}_i^\star)^{3\gamma}}{(\mathscr{S}_{i+1}-\mathscr{S}_i)^{3\gamma}}\right],$$

for any $\mathscr{S}_i^\star \in \left[\frac{\mathscr{S}_i+\mathscr{S}_{i+1}}{2}, \mathscr{S}_{i+1}\right]$.

Proof. The proof is similarly as Proposition 10 but using Theorem 8.2.2, so we omit it. □

Remark 22. *One can have applications for numerical integration subject to generalized m-convex functions and generalized strongly convex function by respectively considering* $c = 0$ *and* $m = 1$ *in above propositions.*

8.4 Applications to some special means

Let us recall some generalized means:

$$\mathscr{A}(a_1, b_1) = \frac{{a_1}^\gamma + {b_1}^\gamma}{2^\gamma},$$

$$\mathscr{L}_l(a_1, b_1) = \left[\frac{\Gamma(1+l\gamma)}{\Gamma(1+(l+1)\gamma)} \left(\frac{{b_1}^{(l+1)\gamma} - {a_1}^{(l+1)\gamma}}{(b_1-a_1)^\gamma}\right)\right]^{\frac{1}{l}},$$

where $n \in \mathbb{Z} - \{-1, 0\}$, $a_1, b_1 \in \mathbb{R}$ and $a_1 \neq b_1$. Now by considering the function $\zeta : (0, +\infty) \to \mathbb{R}^\gamma$, defined by $\zeta(x) = x^{l\gamma}$, $l \in \mathbb{Z} - \{-1, 0\}$ and $0 < a_1 < b_1$, we have $\mathscr{L}_l^l(a_1, b_1) = \frac{1}{(b_1-a_1)^\gamma} {}_{a_1}\mathscr{I}_{a_1}^\gamma$.

Using Lemma 1.8.1, we get $\zeta^{(2\gamma)}(u) = \frac{\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)} u^{(l-2)\gamma}$. Then we have the following applications from our results:

Proposition 12. *Let $0 < a_1 < b_1,\ l > 1$ and $\gamma \in (0, 1]$. Then the following inequality for generalized means holds:*

$$\begin{aligned} &\left| \mathcal{L}_l^l(a_1, b_1) - \frac{(2v - a_1 - b_1)^\gamma \Gamma(1 + l\gamma)}{\Gamma(1+\gamma)\Gamma(1+2\gamma)\Gamma(1+(l-1)\gamma)} \mathcal{L}_{l-1}^{l-1}(a_1 + b_1 - v, v) \right. \\ &\left. + \frac{1}{\Gamma(1+\gamma)} \left(v - \frac{a_1 + 3b_1}{4} \right)^\gamma \mathcal{A}(v^{l-1}, (a_1 + b_1 - v)^{l-1}) \right| \\ \leq\ & \frac{(b_1 - a_1)^{2\gamma}}{\Gamma(1+2\gamma)} \min\{M_3, M_4\}, \end{aligned}$$

where

$$\begin{aligned} M_3 &= \left(\frac{\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)} \right) \Omega_1(a_1, b_1; v) {a_1}^{(l-2)\gamma} \\ &+ m^\gamma \left(\frac{\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)} \right) \Omega_2(a_1, b_1; v) \left(\frac{b_1}{m} \right)^{(l-2)\gamma} - \left(\frac{c}{m} \right)^\gamma \Omega_3(a_1, b_1; v)(ma_1 - b_1)^{2\gamma}, \end{aligned}$$

$$\begin{aligned} M_4 &= \left(\frac{\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)} \right) \Omega_1(a_1, b_1; v) \left(\frac{b_1}{m} \right)^{(l-2)\gamma} \\ &+ m^\gamma \left(\frac{\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)} \right) \Omega_2(a_1, b_1; v) {a_1}^{(l-2)\gamma} - \left(\frac{c}{m} \right)^\gamma \Omega_3(a_1, b_1; v)(mb_1 - a_1)^{2\gamma} \end{aligned}$$

and $\Omega_i(a_1, b_1; v)$ for $i = 1, 2, 3$ are defined in Theorem 8.2.1.

Proof. By considering $\zeta(u) = u^{l\gamma}$, $l \in \mathbb{Z} - \{-1, 0\}$ and $0 < a_1 < b_1$, required inequality follows from the inequality (8.10). □

Proposition 13. *Let $0 < a_1 < b_1,\ l > 1$ and $\gamma \in (0, 1]$. Then the following inequality for generalized means holds:*

$$
\begin{aligned}
&\Bigg|\mathcal{L}_l^l(a_1,b_1)-\frac{(2v-a_1-b_1)^\gamma\Gamma(1+l\gamma)}{\Gamma(1+\gamma)\Gamma(1+2\gamma)\Gamma(1+(l-1)\gamma)}\mathcal{L}_{l-1}^{l-1}(a_1+b_1-v,v)\\
&+\frac{1}{\Gamma(1+\gamma)}\left(v-\frac{a_1+3b_1}{4}\right)^\gamma\mathcal{A}(v^{l-1},(a_1+b_1-v)^{n-1})\Bigg|\\
\leq\ &\frac{(b_1-a_1)^{2\gamma}}{\Gamma(1+2\gamma)}\left(\frac{\Gamma(1+2\gamma p)}{\Gamma(1+(2p+1)\gamma)}\right)^{\frac{1}{p}}\left(\frac{b_1-v}{b_1-a_1}\right)^{(2+\frac{1}{p})\gamma}\min\left\{\sum_{i=4}^{6}J_i^\gamma,\ \sum_{i=4}^{6}K_i^\gamma\right\},
\end{aligned}
$$

where

$$
J_4^\gamma = \left[\frac{B_1(v)\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)}\left({a_1}^{q(l-2)\gamma}+m^\gamma\left(\frac{b_1}{m}\right)^{q(l-2)\gamma}\right)-\left(\frac{c}{m}\right)^\gamma(ma_1-b_1)^{2\gamma}B_2(v)\right]^{\frac{1}{q}},
$$

$$
\begin{aligned}
J_5^\gamma = &\left[\left(\frac{2v-a_1-b_1}{2(b_1-a_1)}\right)^{(2p+1)\gamma}-\left(\frac{a_1+b_1-2v}{2(b_1-a_1)}\right)^{(2p+1)\gamma}\right]^{\frac{1}{p}}\\
&\times\left[\frac{B_3(v)\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)}\left({a_1}^{q(l-2)\gamma}+m^\gamma\left(\frac{b_1}{m}\right)^{q(l-2)\gamma}\right)-\left(\frac{c}{m}\right)^\gamma(ma_1-b_1)^{2\gamma}B_4(v)\right]^{\frac{1}{q}},
\end{aligned}
$$

$$
\begin{aligned}
J_6^\gamma = &\left[\frac{B_5(v)\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)}{a_1}^{q(l-2)\gamma}+m^\gamma\frac{B_6(v)\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)}\left(\frac{b_1}{m}\right)^{q(l-2)\gamma}\right.\\
&\left.-\left(\frac{c}{m}\right)^\gamma(ma_1-b_1)^\gamma B_7(v)\right]^{\frac{1}{q}},
\end{aligned}
$$

$$
K_4^\gamma = \left[\frac{B_1(v)\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)}\left({b_1}^{q(l-2)\gamma}+m^\gamma\left(\frac{a_1}{m}\right)^{q(l-2)\gamma}\right)-\left(\frac{c}{m}\right)^\gamma(mb_1-a_1)^{2\gamma}B_2(v)\right]^{\frac{1}{q}},
$$

$$
\begin{aligned}
K_5^\gamma = &\left[\left(\frac{2v-a_1-b_1}{2(b_1-a_1)}\right)^{(2p+1)\gamma}-\left(\frac{a_1+b_1-2v}{2(b_1-a_1)}\right)^{(2p+1)\gamma}\right]^{\frac{1}{p}}\\
&\times\left[\frac{B_3(v)\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)}\left({b_1}^{q(l-2)\gamma}+m^\gamma\left(\frac{a_1}{m}\right)^{q(l-2)\gamma}\right)-\left(\frac{c}{m}\right)^\gamma(mb_1-a_1)^{2\gamma}B_4(v)\right]^{\frac{1}{q}}
\end{aligned}
$$

and

$$
\begin{aligned}
K_6^\gamma = &\left[\frac{B_5(v)\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)}{b_1}^{q(l-2)\gamma}+m^\gamma\frac{B_6(v)\Gamma(1+l\gamma)}{\Gamma(1+(l-2)\gamma)}\left(\frac{a_1}{m}\right)^{q(l-2)\gamma}\right.\\
&\left.-\left(\frac{c}{m}\right)^\gamma(mb_1-a_1)^\gamma B_7(v)\right]^{\frac{1}{q}}.
\end{aligned}
$$

Proof. By considering $\zeta(u) = u^{l\gamma}$, $l \in \mathbb{Z} - \{-1, 0\}$ and $0 < a_1 < b_1$, required inequality follows from the inequality (8.16). $\square$

Chapter 9

Conclusion and Future Work

In this chapter, we conclude the main findings and also talk about new ideas for future.

9.1 Conclusion

In the primary Chapter 1, we give a concise prologue to the hypothesis of convexity, fractional calculus, quantum calculus and local calculus alongside certain outcomes which were the fundamental focal point of this investigation. Section 1.1 of section 1 presents a few primers about convex functions and its various types. Section 1.2 gives the Hölder's inequality and its improved forms. Section 1.3 dedicated for the presentation of Gamma functions and its k-analogue. It likewise contain a short review of Mittag-Leffler function and its generalized forms. Section 1.4 present different fractional integrals. Section 1.5 draws out the writing about the idea of coordinated convexity and its generalized classes. Section 1.6 talk about the partners of celebrated Hermite-Hadamard inequality including different fractional integrals. Section 1.7 contains the idea of q-differentiation and q-integration. Section 1.8 talk about some basic ideas of local calculus.

Chapter 2 present the idea of modified (h, d)-convex functions and set up the Hermite–Hadamard type inequality. We additionally broaden the current inequalities for twice differentiable convex functions using improved Işcan-Hölder's inequality. Toward the finish of the part, we give the uses of our outcomes for trapezoid formula.

In Chapter 3, we demonstrated a generalized model for the classical and fractional integral forms of Hermite–Hadamard inequality with the assistance of generalized integral operator by Farid [52]. We examine different set up outcomes which can be derive from our inequality by various settings of the functions. We likewise get the weighted version called Fejér–Hermite–Hadamard inequality. We at that point present some new and existing outcomes that are uncommon instances of our inequality. The consequences of the Chapter 3 are in fantastic concurrence with those of published one.

In Chapter 4, we created error estimates of the generalized inequality introduced in Chapter 3. The obtained inequality has the significant component that a Peano type kernel can be develop by various settings of the functions, which at that point convert the outcome into a fractional integral inequalities of our choice. We have reasoned a few outcomes to investigate this element of our focused inequality. The outcomes found are in astounding concurrence with the published literature.

In Chapter 5, we used the mid-point approach o examine the error estimates for the lower bound of the generalized inequality. We in the section 5.1 create two identities for fractional integrals containing extended generalized mittag-leffler kernel and generalized exponential kernel. We at that point set up the error estimates for the indicated fractional integrals. In Section 5.2, we build up a generalized identity to set up the generalized error bound inequality. We find a few uncommon cases to make the correlation with existing results. In Section 5.3, we present our outcomes for the central moment of the random variables. We present various cases for fractional and conformable integrals.

Chapter 6 talks about some new Hermite–Hadamard type inequalities by keeping into account the bigger class of coordinated convexity, specifically coordinated (p_1, h_1)-(p_2, h_2)-convex functions. Section 6.1 present some fundamental results. Section 6.2 presents fundamental discoveries of this exploration. The outcomes of this examination gives a detailed study of the notable outcomes for various types of coordinated convex functions, for example, coordinated s-convex functions, coordinated pq-convex functions,coordinated (p, s)-convex functions, coordinated h-convex functions etc..

In Chapter 7, we change the classical results through quantum integrals. Section 7.1 presents some valuable outcomes and a new identity. Section 7.2 presents the new Hermite–Hadamard type inequalities for twice differentiable m-convex functions. Some new outcomes for classical integrals are derived by setting $q \to 1^-$. Results are again found in concurrence with the recently published one.

In Chapter 8, we consider the fractal number line framework and use the generalized strongly m-convex functions. An Ostrowski inequality is then settled for fractal sets. In Section 8.1, some writing for helpful results is given. Section 8.2 build up the nearby form of Ostrowski inequality and in Section 8.3, we used the outcomes of Section 8.2 and demonstrate error formula for local fractional integrals. In Section 8.4 of this part we present the outcomes for generalized special means.

9.2 Future Work

After the emergence of the notion of coordinated convex function and the appearence of the famous Hadamard inequality (1.69), several inequalities for the improvement and extensions are appeared in the literature. In 2010, Sarikaya et al. [177] obtained the error estimates for the third and fourth inequality in (1.69). In 2012, in similar passion, Latif and Dragomir [99] developed error estimates for the the first and second inequality in (1.69).

In 2020, Alp and Sarikaya [16] established the q-Hadamard integral inequalities via coordinated convex functions.

We feel that the error estimates of the inequalities presented in [16] can be achieved on the similar lines as have been followed in [99, 177]. In future we will try to obtain these results. We will further discuss some simpson's type inequalities in 4 and six panels to check the suitability of the algorithm in quantum frame work of calculus. Finally we aim to obtain the same results for post-quantum calculus.

We recommend other researchers to go deep in q-calculus to develop these and other types of inequalities and so that to reach some new applications in the numerical integrations.

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