Entanglement Sharing in Continuous Variable with Separable State

By
Muhammad Naveed Anjam

A dissertation submitted in partial fulfillment of the
DEGREE OF MASTER OF PHILOSOPHY
IN
PHYSICS

Supervised by
Dr. Aeysha Khalique

SCHOOL OF NATURAL SCIENCES
NATIONAL UNIVERSITY OF SCIENCES AND TECHNOLOGY
ISLAMABAD, PAKISTAN

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Muhammad Naveed Anjam, Regn No. 00000117020 Titled: Entanglement Sharing in Continuous Variable with Separable State be accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: DR. SHAHID IQBAL Signature:

2. Name: DR. RIZWAN KHALID Signature:

External Examiner: DR.RAMEEZ UL ISLAM Signature:

Supervisor's Name DR. AEYSHA KHALIQUE Signature:

Head of Department

Date 27/8/2019

COUNTERSINGED

Date: 27/9/2019

Dean/Principal

THESIS ACCEPTANCE CERTIFICATE

Certified that final copy of MS thesis written by **Mr. Muhammad Naveed Anjam** (Registration No. **000001117020**), of **School of Natural Sciences** has been vetted by undersigned, found complete in all respects as per NUST statutes/regulations, is free of plagiarism, errors, and mistakes and is accepted as partial fulfillment for award of MS/M.Phil degree. It is further certified that necessary amendments as pointed out by GEC members and external examiner of the scholar have also been incorporated in the said thesis.

Signature: ____________________

Name of Supervisor: Dr. Aeysha Khalique

Date: 27/08/19

Signature (HoD): ____________________

Date: 29/8/2019

Signature (Dean/Principal): ____________________

Date: 27-8-2019

To my parents whom I love from the depth Of my heart and soul.

Acknowledgment

First of all, I humbly thank Allah Almighty, the Merciful and the Beneficent, who gave me health, thoughts and co-operative people to enable me achieve this goal. This thesis appears in its current form due to the assistance and guidance of several people. I would therefore like to offer my sincere thanks to all of them.

I express my deepest gratitude to my supervisor Dr. Aeysha Khalique for her special attention, kind behavior, patience, care and encouraging remarks, during the entire course of my work. Which made it possible to complete this work. I am gratefully obliged to Dr. Shahid Iqbal, head of Physics Department, for providing a sound research atmosphere.

I would also like to express my gratitude to my class fellows and specially my friends for their support and encouragement. Finally, I am forever indebted to my parents, sisters, and brother. I can just say thanks for everything and may Allah give them all the best in return.

Muhammad Naveed Anjam

Abstract

Entanglement is an amazing feature of quantum mechanics which is non trivial non-local correlation between the states. Discrete and continuous are two variables used to study quantum communication protocol. To study the spin of electrons and polarization of photon etc, we use discrete variables. To study the quadrature of electromagnetic field, we use continuous variables. The efficient and compact information processing is done with the help of continuous variables. It is worthwhile to study quantum information with continuous variables which is robust against decoherence. In this thesis we reviewed the distribution of entanglement between two parties using ancilla which remain have separable throughout process. In this scheme Gaussian states are used, which are easy to generate and analyzed mathematically using covariance matrix phase space formalism. We have extended the two partite entanglement distribution to three parties using two ancilla.

List of Figures

Contents

1

Introduction

20^{th} century has brought radical changes in the world of science. Max Plank, while explaining the Black body radiation proposed that energy emitted could be thought as consisting of discrete packets of energy. These are called *Quantas* [1]. His effort proved successful when Einstein employed his idea for explaining photoelectric effect and was granted the Nobel prize in 1921. He also used this idea for explaining specific heat at low temperature [2]. In 1913, Bohr postulated that electrons had confined set of orbits. He successfully explained the spectral lines for hydrogen and helium[3, 4]. Explanation of extremely small things has found many applications like laser and superconductivity.

By the end of 19th century majority of the scientists agreed that light has a wave nature and the supporting experiment was Young's double-slit experiment. Einstein's work on Photo-electric effect proved the particle nature. Compton effect in 1923 was another proof of particle nature of light [5]. Both these explanation initiated a new concept of wave particle duality. de Broglie postulated that not only light has dual behavior but all the particles should have a wave nature [6].

Devisson confirmed his hypothesis by proving the wave nature of electron of nickel crystal [7].

Proving the quantum effects led to think about the formalism through which we used to explain quantum effects. All the efforts on this proved successful after a few years and it brought out two new formalism. The first one was developed by Heisenberg and the second one developed by Schrodinger. First formalism is known as Heisenberg 's matrix mechanics which he developed with Born and Jordan [8]. This was purely mathematical approach, so not well appreciated by Most of the physicist. This is known as Schrodinger wave equation and it is well appreciated [9].

Schrodinger applied law of conservation of energy and used de Broglie's concept in his derivation. Born further added to the Schrodinger concept that the square of wave function gives the probability of wave function at a given state [10]. Now this probabilistic world has proved when Heisenberg discovered the uncertainty principle in 1927. Most of the scientists rejected the idea of probabilistic world. Most of the physicist believed that quantum mechanics should be deterministic.

In 1935 Schrodinger proposed a thought experiment known as cat's experiment. In this thought experiment he imagined a cat in a box with radioactive element in the box and a counter that initiate the radio active element having poison fumes. This poison kills the cat as the radioactive element is in the superposition of decaying and not decaying states and hence cat also is in the superposition of live and dead state. The response of the Copenhagen interpretation to this paradox was that the act of observation collapses the wave function in one of both states of the cat.

In 1935 first time Schrodinger used the term entanglement. Quantum entanglement is the phenomenon that occurs when two particles are interacted physically. In 1964 John Bell derived his famous inequality which is known as Bell's Inequality[11]. The first Bell test was tested in 1972 by Freedman[12].

All this progress in the fundamentals of quantum mechanics finally led to use entanglement as a resource in quantum information processes in 1980's. These protocols were mainly based on discrete variables which were prone to noise. In the beginning of 21st century, continuous variable quantum information flourished, which is more robust to noise. We study how to create one resource of continuous variable quantum information, which is quantum entanglement.

In the rest of the chapter we discuss the basics needed for our problem. In section 1.2, we define the qubit which plays a vital role in defining a quantum system and play their role as the carrier of information. In section 1.3, we discuss the quadrature and define the position and momentum quadrature for continuous variable system. In section 1.4, we discuss the action of beam splitter on quantum states and its role in the entanglement which differentiates the classical and quantum case.

1.1 Qubit

The basic unit of classical information is bit. It is the carrier of information. All the information is encoded in qubit. This qubit may be in the form of spin of particles or polarization of particles. In case of continuous variable qubit takes the form of position and momentum. It can have two possible states which are "0" and "1" . A qubit is simply a quantum system that contains a two level quantum state.

All the information of a quantum system is encoded in the state vector. Now if we want to extract the required information of the system, we use corresponding operator that gives the information. We realize a classical bit as a mechanical switch which has two distinct states. Mathematically the state of a qubit can be written as the superposition of $|0\rangle$ and $|1\rangle$

$$|\psi\rangle = \alpha\,|\mathbf{0}\rangle + \beta\,|\mathbf{1}\rangle \tag{1.1}$$

where α and β are the probabilities amplitudes of the states and hence the total probabilities is equal to one i.e.

$$|\alpha|^{\mathbf{2}} + |\beta|^{\mathbf{2}} = \mathbf{1} \tag{1.2}$$

The square of probability amplitude is known as probability in the corresponding state. On measurement, the state collapses into $|0\rangle$ with probability $|\alpha|^2$ and into state $|1\rangle$ with probability $|\beta|^2$.

1.2 Quadrature

Quadrature are the objects that are 90 degree apart from each other. In this context the position and momentum operator are 90 degree apart from each other in a complex plane that is why they are called quadrature operator. Position and momentum operators are continuous. These quadratures are mathematically written as

$$\begin{aligned} \hat{X} &= \tfrac{1}{\sqrt{2}}(a + a^{\dagger}) \\ \hat{P} &= \tfrac{1}{\sqrt{2}i}(a - a^{\dagger}) \end{aligned} \tag{1.3}$$

In the next section we are going to discuss the action of beam spliter and its role in the entanglement.

1.3 Beam Splitter

It is an optical device that splits a beam of light. It is the essential part of interferometer. Many quantum operation can be implemented by optical beam spliter. It is the most general SU(2) operator. It is an entangling agent between the input states. Beam splitter plays vital role in the study of many aspects of optics especially entanglement.

1.3.1 Classical View of Beam Splitter

First, I explain the lossless beam splitter on the basses of classical physics. Consider a classical light with complex amplitude incident on lossless beam splitter as shown in figure(1.1)

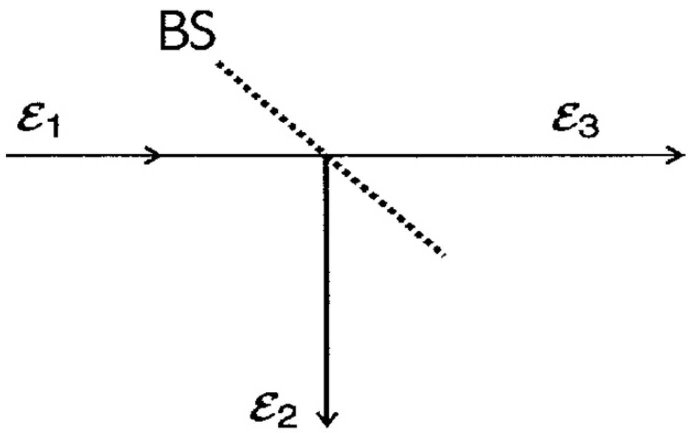

Figure 1.1: classical description of Beam Splitter having one input ε_1 and two output ε_2 and ε_3

$$\varepsilon_2 = r\varepsilon_1, \qquad \varepsilon_3 = t\varepsilon_1, \tag{1.4}$$

where r is the reflectance and t is the transmittance and ε_1 , ε_2 and ε_3 are the complex amplitude of incidence, reflected and transmitted beams. The total intensity of reflected and transmitted intensities must be equal to incident intensity.

$$|\varepsilon_1|^2 = |\varepsilon_2|^2 + |\varepsilon_3|^2 \tag{1.5}$$

which results into,

$$|r|^2 + |t|^2 = 1 \tag{1.6}$$

giving the probability of reflection and transmission to be one.

1.3.2 Quantum View of Beam Splitter

Now we use quantum mechanical model to explain this problem. In quantum approach we replace complex amplitude with annihilation operator as

$$\hat{a}_2 = r\hat{a}_1, \qquad \hat{a}_3 = t\hat{a}_1 \tag{1.7}$$

These operators must satisfy the commutation relation that are given below,

$$\left[\hat{a}_i, \hat{a}_j^\dagger\right] = \delta_{ij} \tag{1.8}$$

$$[\hat{a}_i, \hat{a}_j] = 0 = \left[\hat{a}_i^\dagger, \hat{a}_j^\dagger\right] \tag{1.9}$$

$$\left[\hat{a}_2, \hat{a}_2^\dagger\right] = |r|^2 \left[\hat{a}_1, \hat{a}_1^\dagger\right] = |r|^2 \tag{1.10}$$

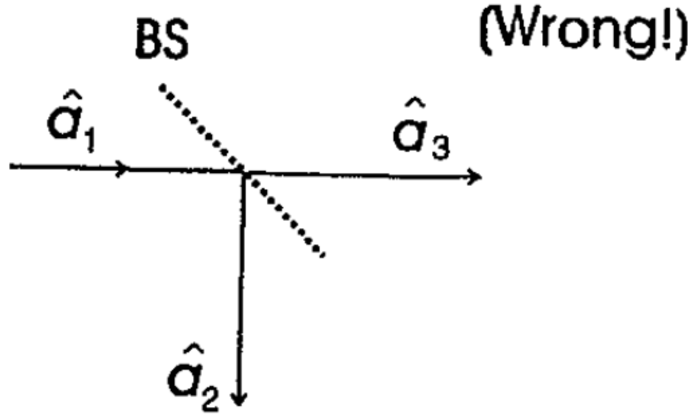

Figure 1.2: wrong description of Quantum beam splitter having one input and two output

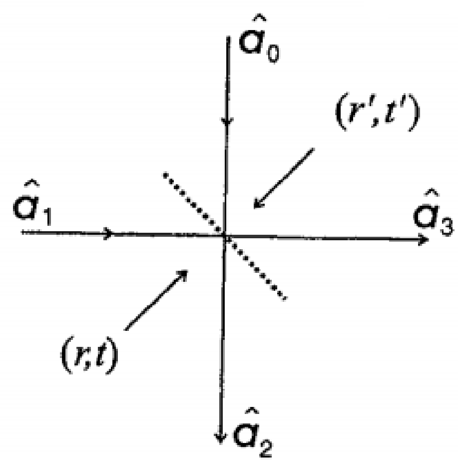

Figure 1.3: Quantum beam splitter with two input and two output

$$\left[\hat{a}_3, \hat{a}_3^\dagger\right] = |t|^2 \left[\hat{a}_1, \hat{a}_1^\dagger\right] = |t|^2 \tag{1.11}$$

$$\left[\hat{a}_2, \hat{a}_3^\dagger\right] = rt^* \neq 0 \tag{1.12}$$

These transformation with only one output do not preserve the commutation relation and hence quantum mechanical approach cannot give the correct result. The failure of this lies in that we are not using the port which is correct according to classical physics. But according to quantum physics it is not correct. In quantum mechanical picture, the output port that we are not using still contains a quantized field that is vacuum state,

$$\hat{a}_2 = t\hat{a}_1 + r^{'}\hat{a}_0 \tag{1.13}$$

$$\hat{a}_3 = t^{'}\hat{a}_1 + r\hat{a}_0 \tag{1.14}$$

$$\begin{pmatrix} \hat{a}_2 \\ \hat{a}_3 \end{pmatrix} = \begin{pmatrix} t^{'} & r \\ r^{'} & t \end{pmatrix} \begin{pmatrix} \hat{a}_0 \\ \hat{a}_1 \end{pmatrix} \tag{1.15}$$

These relations are called reciprocity relations [13].

In the next section, I will discuss entanglement and also classical and quantum view of entanglement.

1.4 Entanglement

If a system consists of more than one subsystem, then this mixture exhibits a very interesting feature which is known as entanglement. Entanglement is an important property of quantum mechanical system which plays a vital role in the field of quantum computation and quantum information. It is perhaps the main difference between classical and quantum cases. The thought experiment described by Einstein, Podolsky and Rosen argued that laws of quantum mechanics are incomplete. This thought experiment is known as EPR-Paradox[14]. Classical and quantum correlation are different from manipulation point of view.

1.4.1 Classical correlation

Classical correlation are well explained and understood by using conservation laws. For example, in pair production a photon having rest mass zero splits into a particle and its anti-particles in opposite direction. If we calculate the momentum of first particle then we can guess the antiparticle's momentum by using law of conservation of momentum.

1.4.2 Quantum correlation

Now if we have a system of two entangled state and we have a system having total spin zero that can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|1_A 0_B\rangle + |0_A 1_B\rangle), \tag{1.16}$$

When $|1\rangle$ is spin up state and $|0\rangle$ is the spin down state. Where we measure the spin state of one of the system them we find the state of second system without measuring it. The difference lies in the superposition of sates that makes the result be quite random. As we measure the system, state collapses. An immediate transfer happens from one state if the states are separated very far from each other. As the transfer occurs within no time then it is thought that perhaps this transfer violates special relativity that put a bound on the speed of traveling of information that no information can travel faster than the speed of light. But we can say that in this case useful information is traveled between two parties. This can be thought of as the entanglement is prepared between the states and when the states move apart from each other, the information is spreads over the Hilbert space and hence

during measurement no information is traveled.
In the next section, I discuss the outline of the thesis.

1.5 Thesis Outline

In this thesis we distribute entanglement between three parties Alice, Bob and Charlie using two helping bit that are separable through out the whole process. We use continuous variables that are vulnerable to decoherence. This thesis is arranged as follows.
In chapter two, I have discussed continuous variable entanglement by first explaining the continuous variables , differentiating them from discrete variable and outlining advantages over discrete case.I discuss different cases staring from number states, Gaussian states and then the squeezed states. Then I explain the covariance matrix and its importance in the entanglement and then explain how it is used to guess weather a system is entangled or separable by computing the symplectic eigenvalues of that covariance matrix with respect to different parties.
In the third chapter I review the previous technique to apply on different system to check entanglement. First I review when two vacuum states added on 50:50 beam splitter then we add a squeeze state and vacuum state on beam splitter and at last I add two squeeze vacuum state on beam splitter and discuss the results i.e. In sec 3.1 I added two vacuum state of 50:50 beam splitter. In sec 3.2, I checked the entanglement when we add one vacuum state and the second squeeze vacuum state. In sec 3.3, I discussed check the entanglement when we add two single mode squeeze vacuum states.

In fourth chapter, I review the entanglement between Alice and Bob with the help of Charlie which plays the role of helping bit and carrier of information. This chapter is the review of [15].

In chapter five, I distribute the entanglement between between three parties Alice, Bob and charlie with the help of two helping qubits D_1 and D_2.

2

Continuous Variable Quantum Information

2.1 Continuous Variable in Quantum Optics

Continuous variables are those variables whose possible values are infinite and continuous. When we are encoding the quantum information, there are two set of coordinate that are continuous and discrete variable but continuous variables have a lot of advantages over discrete variables. These variables can take any value. Variables that have finite set of states are called discrete variables. The main differences of continuous and discrete variables are

1. In Contionuous varible the set of possible states are infinite while in discrete variables the set of possile states are finite.

2. Posion amd momentum are the operators instead of lowering and raising operators.

3. An example of discrete variables is polarization states of photon while an example of continuos variable is quantized hormonic oscillator

When a transition is made from classical to quantum mechanics the observable of particles of a system can turn into non hermitian operators in the Hamiltonian of that system. The electromagnetic modes of the system correspond to quantum harmonic oscillators and the quadrature of that mode plays the role of position and momentum of the oscillators. In addition to these, continuous variable quantum information has many practical advantages over its quantum-bit quantum information as

1. Current optical sources of entangled qubits do not succeed in generating entanglement on demand. System comprising qubits are easy to manipulate but single photons are hard to produce on demand.

2. Continuous variable quantum states can be relatively easily generated [16].
3. The measurement in the basis of entangled states is not unconditional.

The generation, manipulation and measurement of entangled states makes continuous-variable quantum information even more interesting. Gaussian states and Gaussian operation are most extensively used in quantum information technology in these days. These states are manipulated by using continuous variable. Also Gaussian sates are easily understood hence it is also an advantage.
In the next section, we define and explain the importance of covariance matrix and

also explain how we calculate the covariance matrix when the density matrix are given.

2.2 Covariance Matrix

The covariance matrix is quantum state description that is an alternative to a density matrix or wave function. It contains all the information about the system just like the wave function or density matrix . The complete description of a Gaussian state is done by their first and second moments. First moments give the values of canonical variable and the second moments are collected in a real, symmetric covariance matrix [17]. The vector of quadratures is written as,

$$\hat{\chi} = (\hat{x}_1, \hat{p}_1, \hat{x}_2, \hat{p}_2, ..., \hat{x}_N, \hat{p}_N)^T \tag{2.1}$$

For a state with density matrix ρ covariance matrix is defined as, [18]

$$\gamma_{ij} := \frac{1}{2}\langle\{\Delta\hat{\chi}_i, \Delta\hat{\chi}_j\}\rangle = Tr[\hat{\rho}(\Delta\hat{\chi}_i\Delta\hat{\chi}_j + \Delta\hat{\chi}_j\Delta\hat{\chi}_i)/2], \tag{2.2}$$

where,

$$\Delta\hat{\chi}_i = \hat{\chi}_i - \langle\hat{\chi}_i\rangle \tag{2.3}$$

and for zero mean values this becomes equal to $\hat{\chi}_i$.

When we need to take partial transpose of a mode with respect to rest of modes, then covariance matrix transforms from γ to γ^{T_A} as,

$$\gamma^{(T_A)} = \Lambda_A\gamma\Lambda_A \tag{2.4}$$

where Λ_A are the diagonal matrix, that are written for the system consisting of three parties x , y and z,

$$\begin{aligned} \Lambda_x &= \sigma_z \oplus \mathbb{1} \oplus \mathbb{1} \\ \Lambda_y &= \mathbb{1} \oplus \sigma_z \oplus \mathbb{1} \\ \Lambda_z &= \mathbb{1} \oplus \mathbb{1} \oplus \sigma_z \end{aligned} \tag{2.5}$$

with σ_z being the Pauli diagonal matrix which is written as,

$$\sigma_z = diag\{1, -1\}$$

and $\mathbb{1}$ is the 2x2 identity matrix.
Now I explain covariance matrix with one example. Let's take the density matrix of simple system that is number states that is written as,

$$\rho = |n\rangle \langle n| \tag{2.6}$$

and x and p are the quadrature of this state. The first element of covariance can be briefly calculated as,

$$\begin{aligned} \gamma_{11} &= Tr(|n\rangle \langle n| \xi_1 \xi_1) \\ \gamma_{11} &= Tr(|n\rangle \langle n| XX) \end{aligned} \tag{2.7}$$

here

$$\xi = (x_1, p_1, x_2, p_2, ..., x_N, p_N) \tag{2.8}$$

are the quadrature operators defined previously and X and P quadrature are written is raising and lowering operator as,

$$\begin{aligned} X &= \tfrac{1}{\sqrt{2}}(\hat{a} + \hat{a}^{\dagger}) \\ P &= \tfrac{1}{\sqrt{2}}(\hat{a} - \hat{a}^{\dagger}) \end{aligned} \tag{2.9}$$

Now putting these values in the above equation and giving the operators in order form as,

$$\begin{aligned} \gamma_{11} &= Tr[|n\rangle \langle n| (\hat{a}^2 + \hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a} + \hat{a}^{\dagger})] \\ \gamma_{11} &= Tr[|n\rangle \langle n| (\hat{a}^2 + \hat{a}^{\dagger}\hat{a} + 1 + \hat{a}^{\dagger}\hat{a} + \hat{a}^{\dagger})] \end{aligned} \tag{2.10}$$

Now operating the lowering and raising operator, in a manner as their operation is discussed in the previous section, and tracing out the sates,

$$\gamma_{11} = 2n + 1 \tag{2.11}$$

On the same lines we can calculate the other element of the covariance matrix. γ_{12} and γ_{21} are brought to be zero and γ_{22} is calculated as $-2n - 1$. So the covariance matrix is written as,

$$\gamma = \begin{bmatrix} 2n+1 & 0 \\ 0 & -2n-1 \end{bmatrix}, \tag{2.12}$$

which is the covariance matrix of number state. In the next section, I discuss how these variables are used to manipulate the Gaussian states.

2.3 Gaussian State

Gaussian states are the states whose Wigner function is Gaussian. These states are minimum uncertainty state. These states are very useful in quantum information, as they are easily prepared [19] and also vulnerable to decoherence[20]. These states are frequently used in quantum information protocols. These states are easily displaced, rotated and measured. The ground state and thermal states of bosonic systems are Gaussian.

Vacuum state is the most frequently used Gaussian state. This state contains zero photon having eigenvalue zero of annihilation operator.

2.4 Number State

Number states are the basis for all squeezed and coherent states. The operator product $\hat{a}^\dagger\hat{a}$ has a special significance and is called the number operator, which we denote as $|n\rangle$. The eigenstates of the number operator is called the number state, which is also the energy eigenstate of the single mode field with the energy eigenvalue E_n such that

$$\hat{H}\,|n\rangle = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})\,|n\rangle \tag{2.13}$$

$$\hat{n}\,|n\rangle = n\,|n\rangle \tag{2.14}$$

$$\hat{a}\,|0\rangle = 0 \tag{2.15}$$

$$\hat{a}\,|n\rangle = \sqrt{n-1}\,|n-1\rangle \tag{2.16}$$

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{2.17}$$

$$|n\rangle = \left[\frac{(\hat{a}^{\dagger})^{n}}{\sqrt{n!}}\right]|0\rangle \tag{2.18}$$

The number states are orthogonal and complete, satisfying the following orthogonality and completeness conditions

$$\langle n|\, m\rangle = \delta_{n,m} \tag{2.19}$$

$$\sum_{n=0}^{\infty}|n\rangle\langle n| = I \tag{2.20}$$

. In the next section we explain the squeezed states.

2.5 Squeezed State

Squeezing is a process that decreases the variance of one continuous variable[18]. The minimum uncertainty states are called Squeezed States. It has a less variance in one quadrature while grater variance in the other quadrature. The squeezed states are called as such because the quadrature variance of one quadrature is smaller than $\frac{1}{4}$. This reduced or squeezed quadrature means that the variance of the other quadrature must be larger than $\frac{1}{4}$, such that the variance product of the two quadratures is still governed by the Heisenberg uncertainty relation.

$$\left\langle(\Delta\hat{x}_k)^2\right\rangle\left\langle(\Delta\hat{p}_k)^2\right\rangle \geq \frac{1}{4}|[\hat{x}_k,\hat{p}_k]|^2 = \frac{1}{16} \tag{2.21}$$

where,

$$\begin{aligned}\left\langle(\Delta\hat{x}_k)^2\right\rangle &\equiv \left\langle(\hat{x}_k - \langle\hat{x}_k\rangle)^2\right\rangle = \langle\hat{x}_k^2\rangle - \langle\hat{x}_k\rangle^2\\ \left\langle(\Delta\hat{p}_k)^2\right\rangle &\equiv \left\langle(\hat{p}_k - \langle\hat{p}_k\rangle)^2\right\rangle = \langle\hat{p}_k^2\rangle - \langle\hat{p}_k\rangle^2\end{aligned} \tag{2.22}$$

Most generally a state is called squeezed if the covariance matrix has eigenvalue smaller than one as shown in figure. The simplest single mode squeeze state can be written as [21]

$$|\zeta, 0\rangle = S(\zeta)\,|0\rangle\,, \tag{2.23}$$

where $|0\rangle$ is the vacuum.

Squeezed state is generated by squeezing operator that is written as,

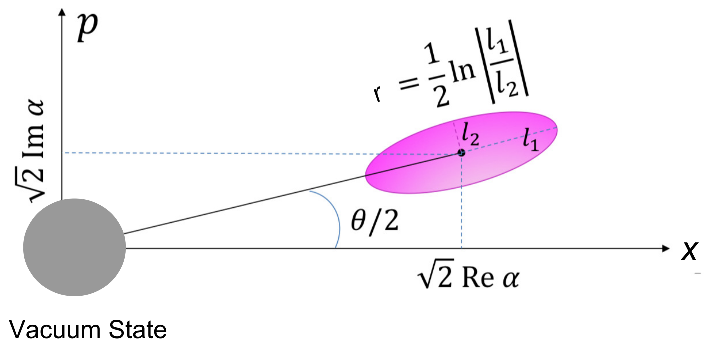

Figure 2.1: Squeezed vacuum state and vacuum state.

$$S(\zeta) = \exp(\frac{1}{2}(\zeta^* \hat{a}^2 - \zeta \hat{a}^{\dagger 2})) \tag{2.24}$$

This Squeezing operator is unitary as

$$\hat{S}(\xi)\hat{S}^{\dagger}(\xi) = \hat{S}^{\dagger}(\xi)\hat{S}(\xi) = 1 \tag{2.25}$$

The covariance matrix for squeeze vacuum state can be written as,

$$\gamma_{SV}(d) = \begin{pmatrix} e^{-d} & 0 \\ 0 & e^{d} \end{pmatrix} \gamma_0 \begin{pmatrix} e^{-d} & 0 \\ 0 & e^{d} \end{pmatrix} = \begin{pmatrix} e^{-2d} & 0 \\ 0 & e^{2d} \end{pmatrix} \tag{2.26}$$

Where e^{-2d} is the variance of position and e^{2d} is the variance of momentum. Position noise is increased when d is negative and opposite is true in case of momentum.
The squeeze state can be written in ordered form as [22],

$$S(\xi) = \frac{1}{\sqrt{\cosh(\xi)}} \times \exp(-\frac{a^{\dagger 2}}{2} e^{i\phi} \tanh(\xi)) \exp(-a^{\dagger} a(\ln\cosh(\xi))) \exp(\frac{1}{2} a^2 e^{i\phi} \tanh(\xi)) \tag{2.27}$$

2.6 Density Matrix

A density matrix is, just like a state vector, a state representation. Just like state vector, it contains all the information about the system. If we have a system that consists of more than one quantum states then statistical mixture can be represented by density matrix. It is a powerful tool to distinguish between pure and mixed states. The density matrix of a pure state is written as,

$$\rho = |\psi\rangle \langle\psi| \tag{2.28}$$

If system has mixture of states then their density matrix is written as,

$$\rho = \sum_n p_n |\psi_n\rangle \langle\psi_n| , \tag{2.29}$$

where p_n is the probability of the system to be in the state ψ_n
The density matrix has following properties [23, 24],

1. The density matrix is always non-negative as,

$$\rho \geq 0 \tag{2.30}$$

2. The expectation value of any operator $\hat{A}$ using density matrix is computed as,

$$\langle A \rangle_\rho = Tr(\rho A) \tag{2.31}$$

3. The Density matrix is normalized that can be written as,

$$Tr(\rho) = 1 \tag{2.32}$$

4. The Density matrix representing a pure state iff

$$Tr(\rho^2) = 1 \tag{2.33}$$

5. The Density matrix representing a mixed state iff

$$0 < Tr(\rho^2) < 1 \tag{2.34}$$

In the next section, I am going to discuss the covariance matrix of different states.

2.6.1 Vacuum State

It is a quantum state which has the lowest possible energy. In phase space representation the vacuum state is the vacuum states lies at the origin as shown in figure 3.1.

Its covariance matrix can be written as,

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.35}$$

2.6.2 Coherent State

These states are minimum uncertainty state. A coherent state $|\alpha\rangle$ is an eigenstate of annihilation operator $\hat{a}$.

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle \tag{2.36}$$

α is a complex number.

Coherent state is generated by the action of displacement operator on the vacuum state as,

$$|\alpha\rangle = D(\alpha)\,|0\rangle \tag{2.37}$$

$D(\alpha)$ is the displacement operator that is written as,

$$D(\alpha) = \mathrm{e}^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \tag{2.38}$$

Coherent states are not orthogonal and cannot satisfy the completeness relation. The covariance matrix of coherent state is also identity matrix.

2.6.3 Squeezed State

It is the minimum uncertainty state. The squeeze state is generated by unitary squeezing operator

$$S(\chi) = \exp[\frac{1}{2}(\chi^*\hat{a}^2 - \chi\hat{a}^{\dagger 2})] \tag{2.39}$$

where $\chi = re^{2i\theta}$, r and θ are the squeezing parameter and squeezing angle respectively.

We define squeezing parameter as,

$$|0, \chi\rangle = S(\chi)\,|0\rangle \tag{2.40}$$

An arbitrary squeezed state is obtained by squeezing the state first and then displace it,

$$|\alpha, \chi\rangle = D(\alpha)S(\chi)\,|0\rangle \tag{2.41}$$

Hence the covariance matrix for squeezed state is written as,

$$\begin{pmatrix} e^{-d} & 0 \\ 0 & e^{d} \end{pmatrix} \tag{2.42}$$

Where $d = 2r$, d is the squeezing parameter. The effect of this operator is to squeeze one of the quadrature while the uncertainty of the other quadrature is increased.

In the next section, we are going to calculate the symplectic eigenvalues and show how these eigenvalues are helpful in distinguishing between entangled and separable states. We discuss the role of symplectic eigenvalue to check whether a state is entangled or not.

2.7 Symplectic Eigenvalues of a Covariance Matrix

For Unitary operator $U|U^\dagger U = I$ in Hilbert space, there is Symplectic one $S|SJS^T = J$ in phase space, where J is an antisymmetric matrix. The unitary operator maps the density matrix to $\rho \mapsto U\rho U^\dagger$, whereas the corresponding symplectic matrix maps the covariance matrix γ to $\gamma \mapsto S\gamma S^\dagger$. Symplectic eigenvalues η of γ are defined as the positive roots of the polynomial [25]

$$|\gamma - i\eta J_N| = 0 \tag{2.43}$$

where j_N is the N-mode symplectic matrix,

$$J_N = \bigoplus_{j=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \tag{2.44}$$

The density matrix representing a physical state must be positive definite, hence its eigenvalues must be positive. This condition in the phase space translates as [25]

1. state is physical if $\eta \geq 1$.

2. state is non-physical if $\eta < 1$.

These eigenvalues play a vital role in determining whether a state is entangled or separable.

2.7.1 Separability Criteria

The Positive Partial Transpose (PPT) criteria for the separability of two states can be translated to phase space. Partial transpose is a non-unitary operation and after this operation state remains physical only if it was separable before. The state becomes unphysical if it was entangled. So the separability criteria leads to a condition on symplectic eigenvalue η of partial transposed state γ^{T_x}, i.e. positive root η of the polynomial [25].

$$\left|\gamma^{T_x} - i\eta J_N\right| = 0 \tag{2.45}$$

1. If $\eta \geq 1$, then state x is separable from rest of the parties

2. If $\eta < 1$ then state x is entangled to the rest of the parties

This criteria applies to separability of one-many parties.

Having explained covariance matrix in this chapter, in next chapter I will explain bi-bipartite entanglement with reference to covariance matrix.

3

Bipartite Entanglement

In this chapter I will explain that how two states becomes entangled. I will use symplectic eigenvalue technique to check the state that whether it is entangled or not. In section 3.1, I will add two vacuum states on beam splitter. In sec. 3.2, I add a squeeze sates and one vacuum state on beam splitter. In sec. 3.3, I add two squeeze states on beam splitter and using the symplectic eigenvalue criteria check that whether these state are entangled or not.

3.1 Addition of Two Vacuum states on Beam Splitter

The covariance matrix of vacuum in matrix form can be written as

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{3.1}$$

The combined state of two vacuum state is

$$\gamma_{AB} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.2}$$

Now if we want to join two vacuum states on a 50:50 beam splitter then as the unitary matrix describing the beam splitter action is,

$$U_{AB} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{3.3}$$

Applying the beam splitter on the two vacuum states as,

$$\gamma_2 = U_{AB}.\gamma_{AB}.{U_{AB}}^T \tag{3.4}$$

After the action of balance beams splitter, the state stays the same.

$$\gamma_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{3.5}$$

Now to check whether the state is entangled or not, we calculate the symplectic eigenvalues of the system that reads as 1 that tells that these states are separable.

In the next section we explain the action of beam splitter on two single mode squeeze vacuum states.

3.2 Addition of Two Squeeze Vacuum States on Beam Splitter

Now we are adding the two squeezed states on a balance beam splitter. Single mode squeezed vacuum state in ordered form can be written as,

$$|\psi_A\rangle = e^{-\frac{1}{2}e^{i\theta.\,\tanh(r)a^\dagger}} e^{-\frac{1}{2}(a^\dagger a+a^\dagger a)\ln(\cosh(r))}\,|0\rangle \tag{3.6}$$

Also the state of second single mode squeeze vacuum is

$$|\psi_B\rangle = e^{-\frac{1}{2}e^{i\theta.\,\tanh(r)b^\dagger}} e^{-\frac{1}{2}(b^\dagger b+b^\dagger b)\ln(\cosh(r))}\,|0\rangle \tag{3.7}$$

Now tensor product of these sates can be written as,

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle \tag{3.8}$$

Now for a and b as the input states and c and d are the output states, the action of beam splitter can be written as,

$$a = c + id \tag{3.9}$$

$$b = ic + d \tag{3.10}$$

$$a^{\dagger} = c^{\dagger} + id^{\dagger} \tag{3.11}$$

$$b^{\dagger} = ic^{\dagger} + d^{\dagger} \tag{3.12}$$

Putting the states together, final state is ,

$$|\psi_{AB}\rangle = e^{\frac{1}{2}e^{i\theta}\tanh(r)[a^{\dagger 2}+b^{\dagger 2}]}e^{-\ln(\cosh(r))\frac{1}{2}(aa^{\dagger}+a^{\dagger}a+b^{\dagger}b+bb^{\dagger})}|0\rangle_A|0\rangle_B$$

After applying beam splitter on the above state, it becomes

$$|\psi_{AB}\rangle = e^{\frac{1}{2}e^{i\theta}\tanh(r)[(\frac{c^{\dagger}+id^{\dagger}}{\sqrt{2}})^2+(\frac{ic^{\dagger}+d^{\dagger}}{\sqrt{2}})^2]}$$
$$e^{-\ln(\cosh(r))\frac{1}{2}((\frac{c+id}{\sqrt{2}})(\frac{c^{\dagger}+id^{\dagger}}{\sqrt{2}})+(\frac{c^{\dagger}+id^{\dagger}}{\sqrt{2}})(\frac{c+id}{\sqrt{2}})+(\frac{ic^{\dagger}+d^{\dagger}}{\sqrt{2}})(\frac{ic+d}{\sqrt{2}})+(\frac{ic+d}{\sqrt{2}})(\frac{ic^{\dagger}+d^{\dagger}}{\sqrt{2}}))}|0\rangle|0\rangle \tag{3.13}$$

Multiplying the operators and simplifying the above state by using the binomial formula

$$(x+y)^n = \sum_{k=0}^{n}\begin{pmatrix} n \\ k \end{pmatrix} x^{n-k}y^k, \tag{3.14}$$

We get two mode squeezed state as,

$$|\psi_{AB}\rangle = \sqrt{1-\tanh r^2}\sum_{n=0}^{\infty}(\tanh r)^n|n\rangle|n\rangle \tag{3.15}$$

The beam splitter operation graphically as shown in figure 3.1

The density matrix of the combined system is

$$\begin{aligned}\rho_{AB} &= |\psi_{AB}\rangle\langle\psi_{AB}| \\ \rho_{AB} &= (1-tanh(r)^2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[\tanh(r)]^{n+m}|n\rangle|n\rangle\langle m|\langle m|\end{aligned} \tag{3.16}$$

Now using this density matrix, we can calculate the covariance matrix that

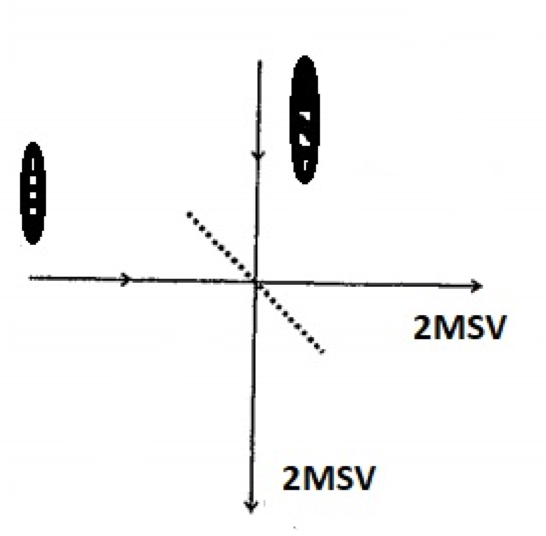

Figure 3.1: Adition of two squeeze vacuum state on beam splitter. Here the input states are squeeze vacuum state which on addition to beam splitter give rise to 2MSV states as output sate.

after some calculation comes out to be,

$$\gamma_{AB} = \begin{pmatrix} \cosh(2t) & 0 & \sinh(2t) & 0 \\ 0 & \cosh(2t) & 0 & -\sin h(2t) \\ \sinh(2t) & 0 & \cosh(2t) & 0 \\ 0 & -\sinh(2t) & 0 & \cosh(2t) \end{pmatrix} \tag{3.17}$$

Now computing the symplectic eigenvalues of the above two mode squeeze state. The lowest symplectic eigenvalue can be written as $e^{-2t} < 1$, when $t > 0$, as shown in Fig. 3.2. This confirms that both sates are entangled.

I have explained how the entanglement can be established between two parties using beam splitter and how the same can be checked using symplectic eigenvalues. In the next chapter, I will explain how this technique can be applied to the

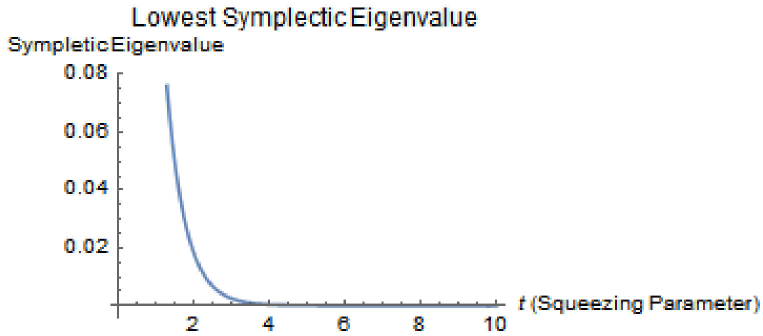

Figure 3.2: Addition of two squeeze vacuum state on beam splitter. As the lowest symplectic eigenvalue is less than one for non zero squeezing, hence the state is entangled.

distribution of entanglement between two parties using separable ancilla.

4

Distribution of Entanglement between Two Parties using Separable States

In this chapter, I review a scheme to distribute the entanglement between two parties using a helping ancilla [15]. In this scheme we have two parties A (Alice) and C(Charlie) that are in squeeze vacuum state and B (Bob) is in vacuum state. The scheme for this work is shown in Fig. 4.1. In this model we distribute the entanglement between Alice and Bob with the help of Charlie which remains separable at each step. We are using two 50:50 beam splitter that are entangling agent.

At the first step, we add Alice and Charlie on the beam splitter which gives two

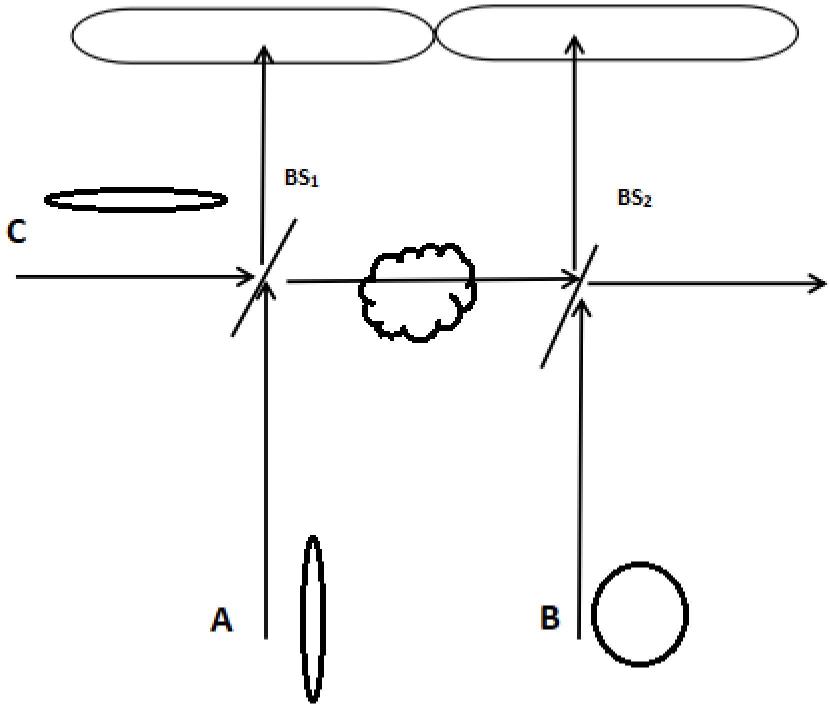

Figure 4.1: Entanglement sharing scheme between two parties using separable state. Here a mode as A and C are in squeeze vacuum sate while Bob's distant mode is in vacuum state. Mixing A and C on balanced beam splitter entangles A and BC while C remains separable after this step. Mixing the mode B and C on the second beam splitter that entangles A and B while C remains separable

mode squeeze vacuum state that is written as ,

$$\gamma_{AC} = \begin{pmatrix} \cosh[2t] & 0 & \sinh[2t] & 0 \\ 0 & \cosh[2t] & 0 & -\sinh[2t] \\ \sinh[2t] & 0 & \cosh[2t] & 0 \\ 0 & -\sinh[2t] & 0 & \cosh[2t] \end{pmatrix}, \tag{4.1}$$

where $t \geq 0$ is the squeezing parameter.

Now computing the symplectic eigenvalues of the above covariance matrix, which is shown in Fig. 4.2. As the symplectic eigenvalue is less than one when squeezing

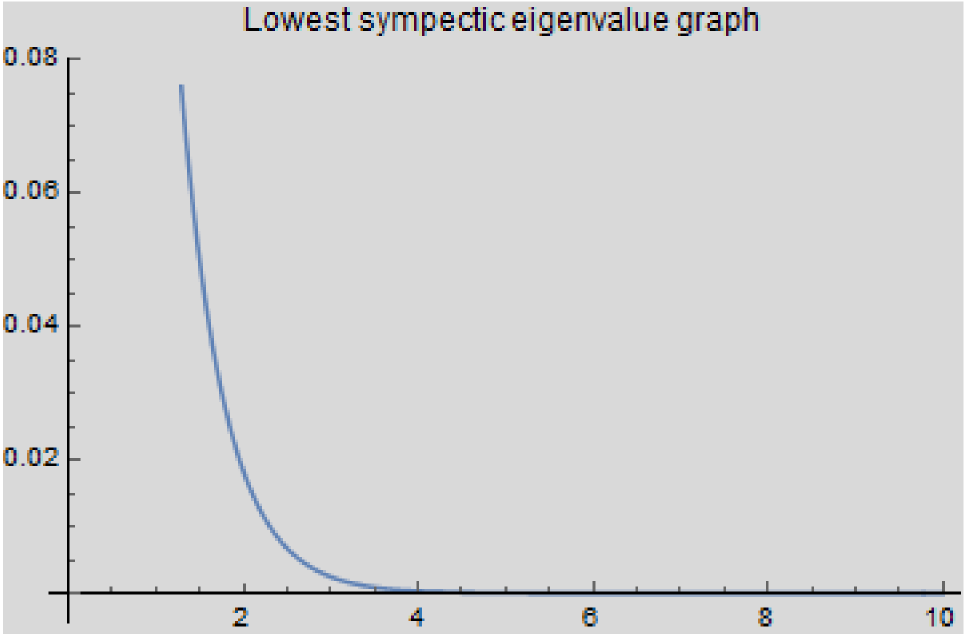

Figure 4.2: Entanglement sharing between two parties. Here the graph of lowest symplectic eigenvalue (along y-axis) and squeezing parameter (along x-axis). Lowest symplectic eigenvalue is less than one for non zero squeezing that is the indication of entanglement between A and C.

parameter is greater than zero, this shows that mode A and C are entangled.

Now adding the state B which is a vacuum state, the entire system comes out to be,

$$\gamma_{ABC} = \begin{pmatrix} \cosh[2t] & 0 & 0 & 0 & \sinh[2t] & 0 \\ 0 & cosh[2t] & 0 & 0 & 0 & -sinh[2t] \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \sinh[2t] & 0 & 0 & 0 & \cosh[2t] & 0 \\ 0 & -\sinh[2t] & 0 & 0 & 0 & \cosh[2t] \end{pmatrix} \tag{4.2}$$

Now to make the state C separable, we add a noise matrix to the above covariance matrix which can disentangle C from AB and entangle the modes A and B. We try the noise matrix,

$$Q = \begin{pmatrix} 1 & 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ -2 & 0 & 4 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 & 0 & 2 \\ -1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \end{pmatrix}, \tag{4.3}$$

This noise matrix is obtained by computing the eigenvector corresponding the negative eigenvalue of the matrix $\gamma_{AB} - i\tilde{J}$ [26]. Extending this eigenvector and sum this state. The eigenvector is written in the form as, $q_\alpha = q_1 + iq_2$. Where $q_1 = \{0, 1, 0, 1\}^T, q_2 = \{1, 0, -1, 0\}^T$. The extended vector is written as,

$$\begin{aligned} \tilde{q}_1 &= \{0, 1, 0, 2, 0, 1\}^T , \\ \tilde{q}_2 &= \{-1, 0, 2, 0, 1, 0\}^T \end{aligned} \tag{4.4}$$

So the noise matrix is finally written as,

$$Q = \tilde{q}_1\tilde{q}_1^T + \tilde{q}_2\tilde{q}_2^T \tag{4.5}$$

Now adding the noise matrix to the γ_{AB}, finally the state becomes $\gamma_{AB} + xQ$ where

$x \geq 0$ and is written as,

$$\begin{pmatrix} x+\cosh[2t] & 0 & -2x & 0 & -x+\sinh[2t] & 0 \\ 0 & x+\cosh[2t] & 0 & 2x & 0 & x-\sinh[2t] \\ -2x & 0 & 1+4x & 0 & 2x & 0 \\ 0 & 2x & 0 & 1+4x & 0 & 2x \\ -x+\sinh[2t] & 0 & 2x & 0 & x+\cosh[2t] & 0 \\ 0 & x-\sinh[2t] & 0 & 2x & 0 & x+\cosh[2t] \end{pmatrix} \tag{4.6}$$

The noise matrix destroys the initial entanglement between mode A and C. Now taking the partial transpose of the covariance matrix with respect to C. When we are dealing with covariance matrix then we take the transposition with respect to mode "A" as,

$$\gamma^{T_A} = \Lambda_A \gamma \Lambda_A \tag{4.7}$$

This partial transposition transforms the covariance matrix from γ to γ^{T_A} . Where Λ_A are the diagonal matrices,

$$\begin{aligned} \Lambda_A &= \sigma_Z \oplus 1 \oplus 1, \\ \Lambda_B &= 1 \oplus \sigma_Z \oplus 1, \\ \Lambda_C &= 1 \oplus 1 \oplus \sigma_Z. \end{aligned} \tag{4.8}$$

σ_Z is the pauli's matrix that is written as $\sigma_Z = \text{diag}(1, -1)$ and 1 is the 2x2 identity matrix .

Λ_B is the partial transposition in Bob's mode only. This transformation only effects the Bob's state while do not effect the Allice's and Charlie's state [26] . Now the main task is to separate mode C from A and B. The partial transpose is positive when the symplectic eigenvalue λ of the covariance matrix is

$$\lambda \geq 1 \tag{4.9}$$

. The lowest symplectic eigenvalues with respect to mode C is written as,

$$\frac{1}{2}\mathrm{e}^{-2t}\left(1-\mathrm{e}^{2t}\left(1+2x\right)+\mathrm{e}^{t}\sqrt{2+\mathrm{e}^{-2t}+12x+\mathrm{e}^{2t}\left(1+12x+4x^{2}\right)}\right) \tag{4.10}$$

In order to make the state C separable, the symplectic eigenvalue must be equal

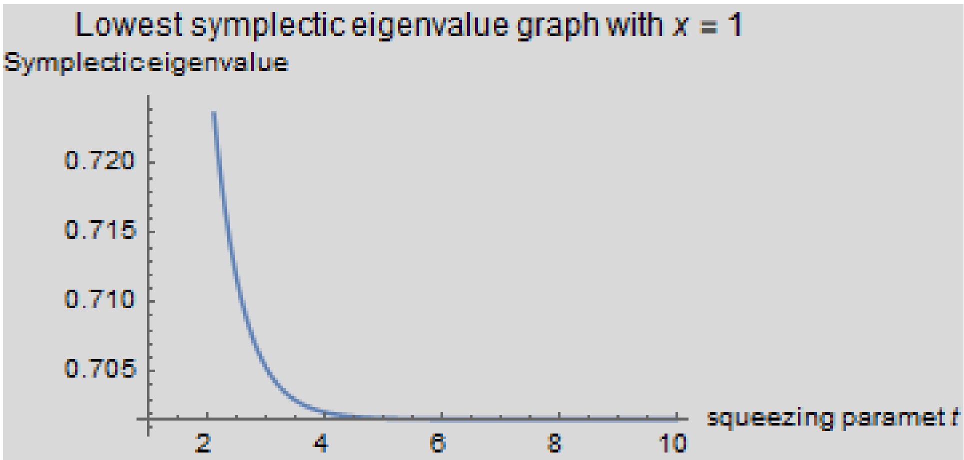

Figure 4.3: Entanglement sharing scheme between two parties using separable state. The lowest symplectic eigenvalue og mode C w.r.t mode AB. As the lowest symplectic eigenvalue is less than one for non-zero x which is the indication that C is still entangled with AC.

or greater than one. For that purpose we make the lowest symplectic eigenvalue equal to one.

$$\frac{1}{2}\mathrm{e}^{-2t}\left(1-\mathrm{e}^{2t}\left(1+2x\right)+\mathrm{e}^{t}\sqrt{2+\mathrm{e}^{-2t}+12x+\mathrm{e}^{2t}\left(1+12x+4x^{2}\right)}\right)=1 \tag{4.11}$$

When we simplify this relation the result comes out to be,

$$x_{sep} = \frac{1}{2}\left(-1 + \mathrm{e}^{2t}\right) \tag{4.12}$$

This is the separability condition. Now if x is greater than or equal to x_{sep}. then the state is separable. Now checking the entanglement between mode A and B by computing the lower symplectic eigenvalue $\gamma_2^{(T_A)}$ which does not come out to be less than 1. So although this noise makes C separable but this disentangles A and B as well.

Now we use another noise matrix as given in [15].

$$q_1 = \{0, -1, 0, 2, 0, -1\}^T, \quad q_2 = \{1, 0, 2, 0, -1, 0\}^T \tag{4.13}$$

and follow the same procedure. This noise disentangles C and entangles A and B as shown in Fig. 4.4 for the same $x_{sep} = \frac{1}{2}\left(-1 + \mathrm{e}^{2t}\right)$ as for the previous noise.

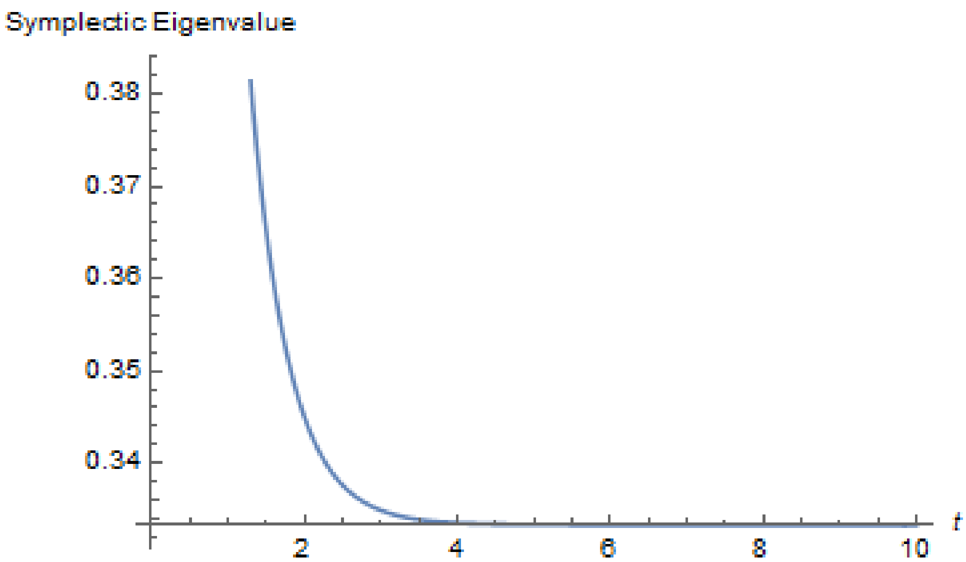

Figure 4.4: Entanglement of mode A with B.

I have explained how the entanglement can be distributed between two parties. In the next chapter I explain how this technique can be applied to the distribution of entanglement between three parties using separable.

5

Distribution of Entanglement Between Three Parties

In this chapter I explain our work on distribution of entanglement between three parties. Our scheme is based on two partite entanglement describes in chapter 4. However it is innovative and novel in extending the existing scheme to three partite case.

5.1 System and Scheme

We propose to use two ancilla D_1 and D_2 to distribute entanglement between three parties. In this case A and D_2 are in squeeze vacuum state while B , C and D_1 are in vacuum state. The scheme that we follow is shown in figure. In step 1, we add two squeeze vacuum states on the balanced beam splitter. This entangles A and D_2. B is then added and the noise term is determined that can make D_2 separable. In step-3, D_2 is separable and A is entangled with BD_2. Now at step-3,

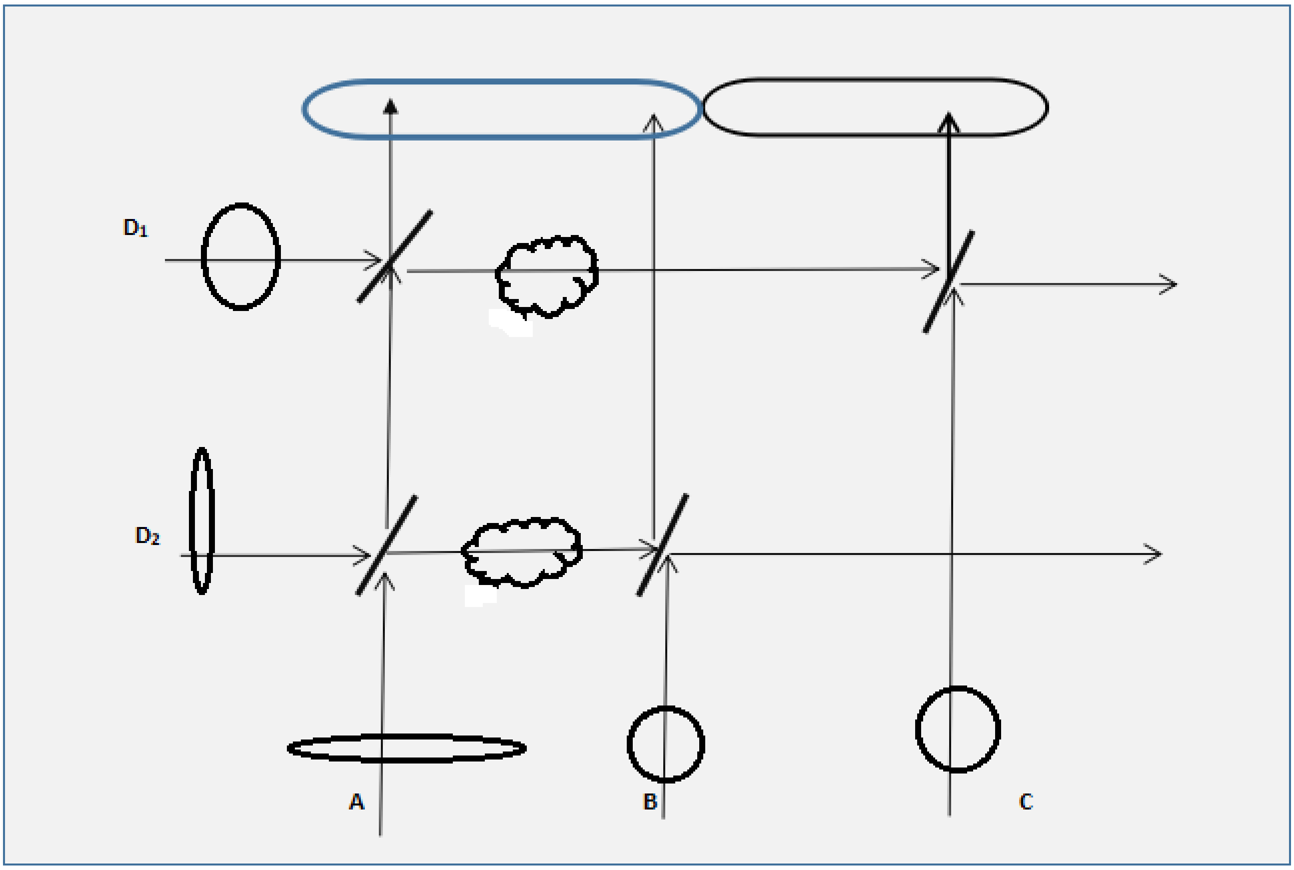

Figure 5.1: Distribution of entanglement between three parties with two helping bit. Here D_2 and A are in squeeze state while B, C and D_1 are in vacuum state.

mixing the states D_2 and B on a beam splitter entangles A and B while D_2 remains separable. As the state D_2 is separable so we can take the reduced state of AB. In step-4, mixing D_1 on the balanced beam splitter with A, entangles D_1 with AB. Adding the third party C and with the help of noise term separates D_1. Mixing C and D_1 on the balanced beam splitter entangles ABC while D_1 remains separable throughout whole step.

When we combine two squeezed states of A and D_1 on the balanced beam splitter, then the beam splitting operation results in two mode squeeze vacuum state that can be written in matrix form as

$$\gamma_{AD_2} = \begin{pmatrix} \cosh[2t] & 0 & \sinh[2t] & 0 \\ 0 & \cosh[2t] & 0 & -\sinh[2t] \\ \sinh[2t] & 0 & \cosh[2t] & 0 \\ 0 & -\sinh[2t] & 0 & \cosh[2t] \end{pmatrix}, \tag{5.1}$$

where t is the squeezing parameter and $t \geq 0$. When we check the symplectic eigenvalues [15] of the above covariance matrix, then the lowest symplectic eigen value is shown in Fig. 5.2 As the symplectic eigenvalues is less than one which

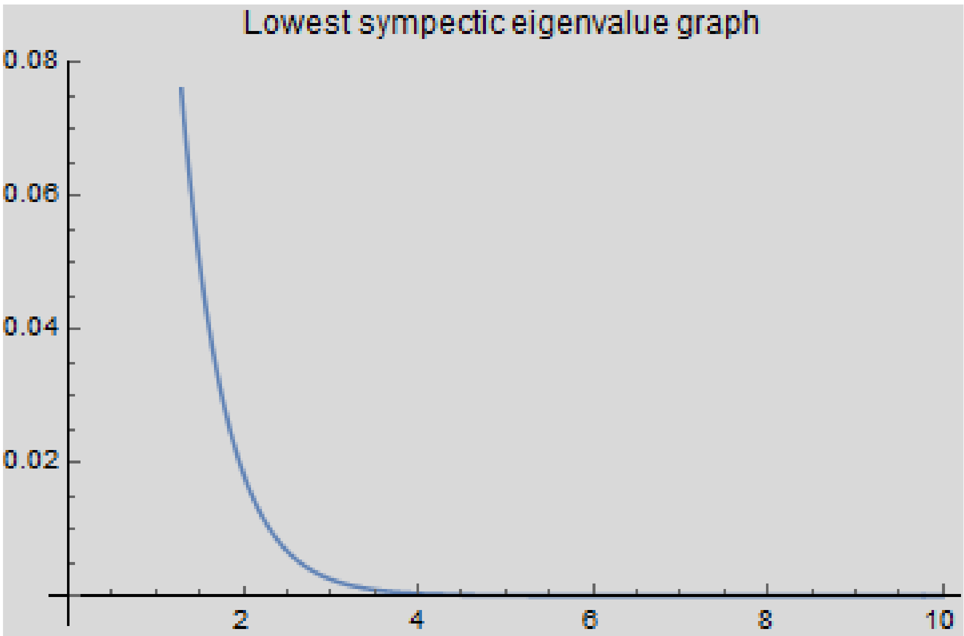

Figure 5.2: The lowest symplectic eigenvalue os A w.r.t D_2 showing the entanglement between A and D_2

indicates that there is entanglement between these two states. Now by adding the mode B in this sates which is vacuum state and the entire covariance can be written as

$$\gamma_{ABD_2} = \begin{pmatrix} \cosh[2t] & 0 & 0 & 0 & \sinh[2t] & 0 \\ 0 & \cosh[2t] & 0 & 0 & 0 & -\sinh[2t] \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ \sinh[2t] & 0 & 0 & 0 & \cosh[2t] & 0 \\ 0 & -\sinh[2t] & 0 & 0 & 0 & \cosh[2t] \end{pmatrix} \tag{5.2}$$

In order to destroy entanglement between AB and D_2 , we add a noise term to the above state. Now addition of noise term is done by taking the eigenvector corresponding to negative eigenvalue of the matrix $\gamma - i\tilde{J}_A$ [26], where $\tilde{J}_A = \Lambda_A J_N \Lambda_A$

and

$$J_N = \bigoplus_{J=1}^{N} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tag{5.3}$$

The negative eigen vector can be written as

$$Q_\lambda = Q_1 + iQ_2$$

where $q_1 = \{0, -1, 0, -1\}$ and $q_2 = \{1, 0, -1, 0\}$. as $Q = q_1{q_1}^T + iq_2{q_2}^T$. The matrix Q is positive by construction. we can make extensions of q_1 and q_2 as

$$\begin{aligned} q_1 &= (0, -1, 0, 2, 0, -1)^T \\ q_2 &= (1, 0, 2, 0, -1, 0)^T \end{aligned} \tag{5.4}$$

Now adding a sufficiently large non-negative multiple xQ to the covariance matrix destroys the initial entanglement between mode A and D_2. After adding the noise term, the covariance is

$$\gamma_{ABD_2} = \begin{pmatrix} (x+\cosh[2t])\mathbb{1} & 2x\sigma_z & (x+\sinh[2t])\sigma_z \\ 2x\sigma_z & (1+4x)\mathbb{1} & -2x\mathbb{1} \\ (x+\sinh[2t])\sigma_z & -2x1 & (x+\cosh[2t])\mathbb{1} \end{pmatrix} \tag{5.5}$$

Now taking partial transpose with respect to D_2. The corresponding covariance matrix can be written as

$$\Lambda_{D_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix} \tag{5.6}$$

Computing the lowest symplectic eigenvalue with respect to D_2, which is written as

$$-\frac{1}{2}\mathrm{e}^{-2t}\left(-1+\mathrm{e}^{2t}(1+2x)+\mathrm{e}^{t}\sqrt{2+\mathrm{e}^{-2t}+12x+\mathrm{e}^{2t}(1+12x+4x^2)}\right) \tag{5.7}$$

Now to make the state separable , the lowest symplectic eigen values must be equal or grater than one. So,

$$-\frac{1}{2}\mathrm{e}^{-2t}\left(-1+\mathrm{e}^{2t}(1+2x)+\mathrm{e}^{t}\sqrt{2+\mathrm{e}^{-2t}+12x+\mathrm{e}^{2t}(1+12x+4x^2)}\right)=1 \tag{5.8}$$

When the lowest symplectic eigenvalue is equal to one and the threshold is written as

$$x_{sep} \to \frac{1}{2}\left(-1 + \mathrm{e}^{2t}\right) \tag{5.9}$$

So the D_2 is separable when $x \geq x_{sep}$ where

$$x_{sep} = \frac{1}{2}\left(-1 + \mathrm{e}^{2t}\right) \tag{5.10}$$

Now in order to check the the entanglement between B and AD_2, we compute the lowest symplectic eigenvalue w.r.t the state D_2 , which is

$$\frac{1 + 6x + e^{-2t} - \sqrt{\left(1 + 2x - e^{-2t}\right)^2 + 32x^2}}{2} \tag{5.11}$$

and its graphical representation is shown as in Fig. 5.3. As the symplectic eigenval-

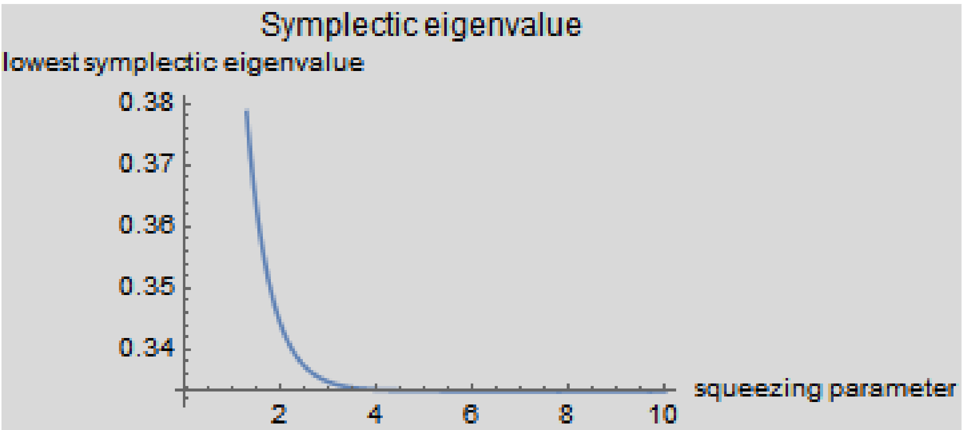

Figure 5.3: The lowest symplectic eigenvalue of mode B w.r.t AD_2 showing the entanglement between B and AD_2 .

ues are less than one for non-zero squeezing, so mode A is entangled with D_2B. Now mixing the state B with D_2 of beam splitter BS_{D_2B} that is mathematically written as,

$$U_{BD_2} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} \end{pmatrix}. \quad (5.12)$$

Now the entire 6×6 matrix can be written as

$$\begin{pmatrix} A & B & E \\ B & C & D \\ E & D & F \end{pmatrix} \quad (5.13)$$

where, $A = \mathrm{diag}(a, a)$, $B = \mathrm{diag}(b, -b)$, $C = \mathrm{diag}(c, c)$, $D = \mathrm{diag}(d, d)$, $E = \mathrm{diag}(-e, e)$ and $F = \mathrm{diag}(f, f)$ with

$$a = x + \cosh[2t] \ , \qquad b = \frac{-3x + \sinh[2t]}{\sqrt{2}} \ ,$$
$$c = \frac{1}{2}(1 + 9x + \cosh[2t]) \ , \qquad d = \frac{1}{2}(1 + 3x - \cosh[2t]) \ ,$$
$$e = \frac{x + \sinh[2t]}{\sqrt{2}} \ , \qquad f = \frac{1}{2}(1 + x + \cosh[2t]) \ .$$

As state D_2 is separable from A and B so we can reduce the state as

$$\begin{pmatrix} x + \cosh[2t] & 0 & \frac{x+\sinh[2t]}{\sqrt{2}} & 0 \\ 0 & x + \cosh[2t] & 0 & -\frac{x+\sinh[2t]}{\sqrt{2}} \\ \frac{x+\sinh[2t]}{\sqrt{2}} & 0 & x + \cosh[2t] & 0 \\ 0 & -\frac{x+\sinh[2t]}{\sqrt{2}} & 0 & x + \cosh[2t] \end{pmatrix} \quad (5.14)$$

We computed symplectic eigenvalues with respect to B and these being less than 1 prove that A and B are entangled, as shown by eigenvalue graph in Fig. 5.4 Now

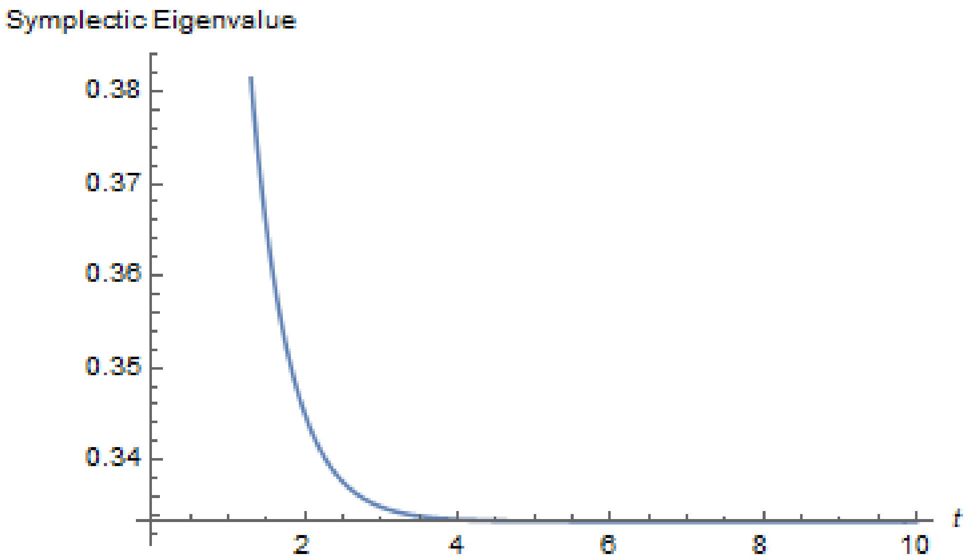

Figure 5.4: Symplectic Eigenvalue Symplectic Eigenvalue (y-axis) vs squeezing parameter t (x-axis) of the reduced state (γ_{AB}) w.r.t B showing the enatnglement between A and B

adding the ancilla D_1 in vacuum state , the covariance matrix can now become

$$\gamma_{ABD_1} = \begin{pmatrix} x+\cosh[2t] & 0 & \frac{x+\sinh[2t]}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & x+\cosh[2t] & 0 & -\frac{x+\sinh[2t]}{\sqrt{2}} & 0 & 0 \\ \frac{x+\sinh[2t]}{\sqrt{2}} & 0 & x+\cosh[2t] & 0 & 0 & 0 \\ 0 & -\frac{x+\sinh[2t]}{\sqrt{2}} & 0 & x+\cosh[2t] & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{5.15}$$

Applying a beam splitter AD_1 which is written as

$$U_{AD_1} = \begin{pmatrix} \frac{1}{\sqrt{2}}I & 0 & \frac{1}{\sqrt{2}}I \\ 0 & I & 0 \\ \frac{1}{\sqrt{2}}I & 0 & -\frac{1}{\sqrt{2}}I \end{pmatrix} \tag{5.16}$$

,

where I is an 2×2 identity matrix. Finally state become

$$\begin{pmatrix} A_1 & B_1 & C_1 \\ B_1 & D_1 & B_1 \\ C_1 & B_1 & A_1 \end{pmatrix} \tag{5.17}$$

where $A_1 = \text{diag}(a, a),\ B_1 = \text{diag}(b, -b),\ C_1 = \text{diag}(c, -c),\ D_1 = \text{diag}(d, d)$, with

$$a = \frac{1}{2} + \frac{1}{2}\left(x + \cosh\left[2t\right]\right);, \qquad b = 1/2(x + \sinh[2t]);,$$
$$c = -(1/2) + 1/2(x + \cosh[2t]);, \qquad d = x + \cosh\left[2t\right];.$$

Now computing symplectic eigenvalues with respect to state D_1, the graph of lowest symplectic eigenvalue, is shown in Fig 5.5. The next step is to entangle C with the system, which is done by combining C and D_1 on a balanced beam splitter. The same value of symplectic eigen value is obtained for A, B and C w.r.t. the rest. At this point all three parties along with D_1 are found to be entangled for the range $0 < t < 0.5$.

Now we have to make the state D_1 separable by computing the eigenvectors corresponding to the negative eigenvalues of γ_{ABD_1} after attaching system C with

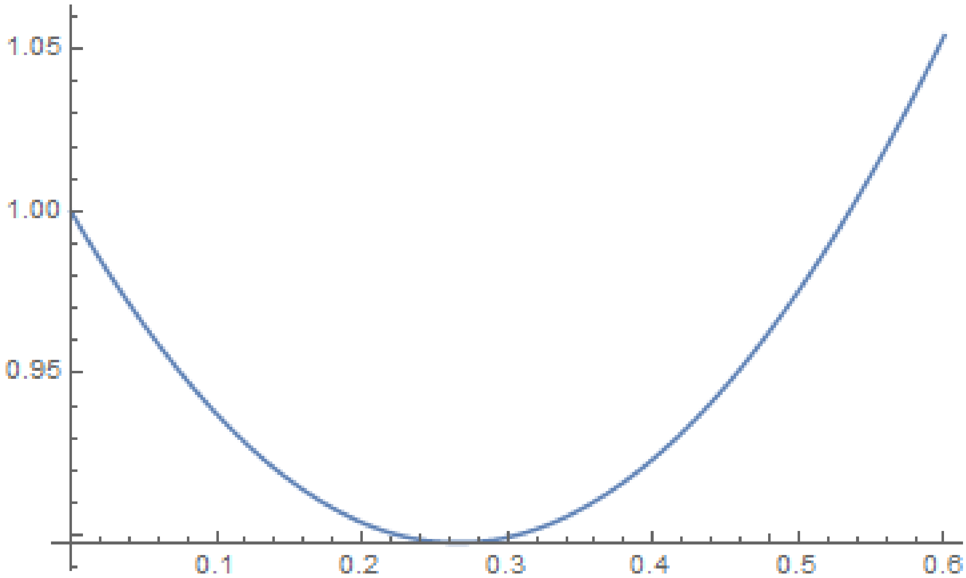

Figure 5.5: Symplectic Eigenvalue (y-axis) vs squeezing parameter t (x-axis) w.r.t. D_1 with AB
. Then same plot is achieved after entangling C and computing symplectic eigen value w.r.t. A, B and C w.r.t. the rest.

it. To compute the noise term and making D_1 separable requires calculation of a bigger system which is computationally tedious. We aim to do it numerically and compute the entanglement at different values of t.

6

Conclusion

Continuous variable quantum information is interesting as it is more robust against decoherence. Lately a lot of research has been done to use continuous variable quantum entanglement as a resource in various quantum information protocols. The continuos variables are the position and momentum quadratures, which can take continuous infinite values. We have studied the use of Gaussian states to establish entanglement between distant parties. Gaussian states are easier to prepare and their analysis is simplified in phase space using symplectic formalism. In phase space, the tensor products of unitary operators are replaced by the direct sum in symplectic operators. The covariance matrix representation of quantum states needs only to track the evolution of covariance matrix with quantum operations.

With the tools of covariance matrix approach and symplectic operations in phase space in hand, we have analyzed the distribution of continuous variable entanglement between two and three distant parties. First we have reviewed the distribution of entanglement between two parties using ancilla, which remains separable throughout the process [15]. The initial states of three parties are taken as Gaussian, the coherent or squeezed states. Entanglement is generated by beam splitting action between ancilla and the two parties to be entangled. The interesting feature of this analysis is the deliberate addition of noise to make the ancilla separable. The noise is added in such a way that it makes the ancilla separable but keeps entangles the two parties. We have found that the general recommended noise, though disentangles the ancilla but it also disentangles the two legitimate parties. We have then added the modified noise as in [15], which serves the purpose.

We have also extended the procedure to three party entanglement distribution, where we have proposed to use two ancilla to distribute the entanglement. One ancilla is first entangled with first and a bipartite entanglement is established first as in the previous case. The ancilla is then made separable by adding noise. The first party is then entangled with the second ancilla. We have found that the second ancilla gets entangled only for range of squeezing parameter $0 < t < 0.5$. The ancilla is then entangled with the third party by combining the two on beam splitter. They are entangled for the same range, which is a very positive result. Now all three parties and the second ancilla are entangled. The task to make the second ancilla separable becomes tedious as the system and hence the matrix size becomes bigger. We intend to further solve it numerically.

The three partite entanglement distribution system can be analyzed for various combinations of input states and the range of squeezing parameter for which system is entangled may be increased. We intend to further analyze the system in this respect.

Bibliography

[1] Max Planck. On the law of distribution of energy in the normal spectrum. *Annalen der physik*, 4(553):1, 1901.

[2] A Einstein. The planck theory of radiation and the theory of specific waste [adp 22, 180 (1907)]. *Annalen der Physik*, 14(S1 1):280–291, 2005.

[3] Niels Bohr. The spectra of helium and hydrogen. *Nature*, 92(2295):231, 1913.

[4] Niels Bohr. On the constitution of atoms and molecules. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*, 26(151):1–25, 1913.

[5] Arthur H Compton. A quantum theory of the scattering of x-rays by light elements. *Physical Review*, 21(5):483, 1923.

[6] Louis de Broglie. *Thèses présentées à la Faculté des sciences de l'Université de Paris, pour obtenir le grade de docteur ès sciences physiques, par Louis de Broglie. 1re Thèse. Recherches sur la théorie des quanta. 2e Thèse. Propositions données par la Faculté.* Masson et Cie, éditeurs 120, boulevard Saint-Germain, 1924.

[7] Clinton Davisson and Lester Halbert Germer. The scattering of electrons by a single crystal of nickel. *Nature*, 119(2998):558, 1927.

[8] William A Fedak and Jeffrey J Prentis. The 1925 born and jordan paper "on quantum mechanics". *American Journal of Physics*, 77(2):128–139, 2009.

[9] Erwin Schrödinger. An undulatory theory of the mechanics of atoms and molecules. *Physical review*, 28(6):1049, 1926.

[10] Max Born. Quantenmechanik der stoßvorgänge. *Zeitschrift für Physik*, 38(11-12):803–827, 1926.

[11] John S Bell. On the einstein podolsky rosen paradox. *Physics Physique Fizika*, 1(3):195, 1964.

[12] Stuart J Freedman and John F Clauser. Experimental test of local hidden-variable theories. *Physical Review Letters*, 28(14):938, 1972.

[13] Christopher Gerry and Peter L Knight. *Introductory quantum optics*. Cambridge university press, 2005.

[14] Albert Einstein, Boris Podolsky, and Nathan Rosen. Can quantum-mechanical description of physical reality be considered complete? *Physical Review*, 47(10):777, 1935.

[15] Ladislav Mišta Jr and Natalia Korolkova. Improving continuous-variable entanglement distribution by separable states. *Physical Review A*, 80(3):032310, 2009.

[16] Leuchs Gerd et al. *Quantum information with continuous variables of atoms and light*. World Scientific, 2007.

[17] Jens Eisert, Stefan Scheel, and Martin B Plenio. Distilling gaussian states with gaussian operations is impossible. *Physical Review Letters*, 89(13):137903, 2002.

[18] Christian Weedbrook, Stefano Pirandola, Raúl García-Patrón, Nicolas J Cerf, Timothy C Ralph, Jeffrey H Shapiro, and Seth Lloyd. Gaussian quantum information. *Reviews of Modern Physics*, 84(2):621, 2012.

[19] Ulrik L Andersen, Tobias Gehring, Christoph Marquardt, and Gerd Leuchs. 30 years of squeezed light generation. *Physica Scripta*, 91(5):053001, 2016.

[20] Rafał Demkowicz-Dobrzański, Konrad Banaszek, and Roman Schnabel. Fundamental quantum interferometry bound for the squeezed-light-enhanced gravitational wave detector geo 600. *Physical Review A*, 88(4):041802, 2013.

[21] Xiang-Bin Wang, Tohya Hiroshima, Akihisa Tomita, and Masahito Hayashi. Quantum information with gaussian states. *Physics reports*, 448(1-4):1–111, 2007.

[22] Stephen Barnett and Paul M Radmore. *Methods in theoretical quantum optics*, volume 15. Oxford University Press, 2002.

[23] Peter Lambropoulos and David Petrosyan. *Fundamentals of quantum optics and quantum information*, volume 23. Springer, 2007.

[24] Michael A Nielsen and Isaac Chuang. Quantum computation and quantum information, 2002.

[25] Christina E Vollmer, Daniela Schulze, Tobias Eberle, Vitus Händchen, Jaromír Fiurášek, and Roman Schnabel. Experimental entanglement distribution by separable states. *Physical review letters*, 111(23):230505, 2013.

[26] Géza Giedke, Barbara Kraus, Maciej Lewenstein, and J Ignacio Cirac. Separability properties of three-mode gaussian states. *Physical Review A*, 64(5):052303, 2001.