Engineering Entanglement of Barut-Girardello Coherent States by Optical Beam Splitters

by

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This Dissertation is dedicated to my parents

for their endless love, support and encouragement.

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Abstract

Engineering entangled optical states is an important task in quantum-optics based quantum information processing and computation. There are many devices that can be used as an optical field state entangler. The most commonly used device in this regard is an optical beam splitter which is a passive device. For a particular scenario of path entangled optical states, the input state of a beam splitter is splitted into two parts which traverse different paths after passing through it. The output states of the beam splitter can get entangled if and only if the input states exhibit a nonclassical nature. However, it is well known fact that the outputs of a beam splitter cannot be entangled if the standard coherent states (Glauber coherent states) are taken as the input state. In this dissertation we study the entanglement generation at outputs of an optical beam splitter by considering a generalized coherent state, namely Barut-Girardello coherent state, at input.

In our study, the Barut-Girardello (BG) coherent states are constructed and their optical realization, namely Holstein-Primakoff (HP) realization, is considered. For the HP realization of BG coherent states, the photon statistics and the photon-emission correlation are investigated by means of Mandel parameter and second-order intensity correlation function, respectively. It is shown that, in contrast to the Glauber coherent states, HP realization of BG states exhibit sub-Poissonian photon statistics and exhibit anti-bunching photon correlation effect which are clearly the signatures of nonclassicality. Using BG coherent states as the inputs, the entanglement is generated between the output states of a beam splitter. The entanglement generation is analyzed by means of linear entropy. In our analysis, it is illustrated that the output states of an optical beam splitter get entangled after passing through it, if the input state exhibit a non-classical nature, otherwise, the input states appear at the outputs as separable states and no entanglement is generated. The entanglement generation of these states depend on two parameters Bargmann index k and the BG coherent states amplitude z. The variation in linear entropy by changing these parameter k and z shows the variation in degree of entanglement.

Chapter 1

Introduction and outline

1.1 Introduction

Entanglement is a characteristic phenomenon of composite quantum systems, such that, when two or more subsystems of a composite system interact with each other, they become correlated in such a way that the quantum state of each subsystem can not be described independently. The correlation of this kind is known as entanglement which do not have classical analogue. When a quantum system is composed of two subsystems, this correlation is known as bipartite entanglement. Entanglement plays a crucial role in many advanced areas, such as, quantum information processing quantum quantum computation [1], quantum teleportation [2] and quantum cryptography [3, 4, 5]. Engineering entanglement in different kinds of quantum systems is an interesting and an important task. In a particular scenario of quantum-optics based quantum information processing, the generation of entanglement is investigated through optical devices such as beam splitters and interferometers. The entanglement generation on the outputs of such an optical device depends on the nature of the optical field injected to its inputs.

In this thesis we use an optical beam splitter as an entangler with a special kind of quantum states, namely Barut-Girardello coherent states, at its inputs. A beam splitter is a passive optical device that splits an input optical field in to two parts and transmit them to the out put ports which can be entangled depending on the nature of the input optical field. In the following, we present a qualitative introduction of the main concepts involved in our work being presented in the thesis.

1.2 Coherent states and their generalization

The coherent states are very important in mathematics [6] and physics [7], such as quantum optics [8, 9] and quantum information [10, 11]. The coherent states were first introduced by Schrödinger in 1926, for harmonic oscillator. He mentioned that these states are represented by Gaussian wave packet whose centroid follows the classical trajectory of the harmonic oscillator. Moreover, these quantum mechanical states minimize the Hiesenberg uncertainty relation and therefore known as minimum uncertainty states. The minimum uncertainty states, introduced by Schrödinger, remained dormant for more than three decades till 1963. The coherent states are very popular among the mathematics and physics society after the seminal work of Glauber R. J. in 1963. Among many expressive contributions, we mention the work of Barut A.O. and Girardello L. [12] and Glauber R. J. [13, 14]. Glauber R. J. used these states in the description of coherent electromagnetic field, named as coherent states. In fact, he defined these states in a different way by making of the underlying algebra of harmonic oscillator, namely Heisenberg-Wyle algebra [15, 16]. In its classical description, the energy of a single-mode electromagnetic field is expressed by the Hamiltonian of a classical harmonic oscillator, also known as radiation oscillator. The canonical quantization of the radiation oscillators give rise to the quantized radiation energy. An eigenstate of the quantized oscillator Hamiltonian corresponds to a state of the field in which there are definite number of photons. However, the photon number states do not express the actual state of the radiation field because the expectation values of the electric field and the magnetic field vanish with respect to these number states. Glauber coherent states (standard coherent states) are perfectly suitable to represent the quantum mechanical state of the radiation field. In literature, the coherent states are define in three equivalent ways:

1. eigenstates of the annihilation operators.
2. states constructed by the action of displacement operators on the vacuum state.
3. minimum-uncertainty states or, more generally, intelligent states for position and momentum.

The coherent states are constructed using the group algebra of harmonic oscillator, i.e, Heisenberg-Weyl algebra based on the criteria (1) and (2). Initially these states are constructed for harmonic oscillator known as standard coherent states or Glauber coherent states or canonical coherent states. The coherent states are generalized for other systems using the algebra of corresponding systems [18]. Mostly the generalization of coherent states based on the first two definitions by replacing the Heisenberg-Weyl algebra with the group algebra of the corresponding system for which coherent states are being constructed [12].

The term generalized coherent state has also been used to construct coherent states for generals (Lie) groups. Using quantum group and their associated algebra of corresponding system gave the accountability to construct the generalized coherent states based on the first two definitions (1) the eigenstates of the annihilation operator (2) states constructed by the action of displacement operators on the vacuum state. Every generalization scheme has extended one of the above mentioned definition of coherent states for general systems. Most of the early generalization were made by making use of the definition based on the underlying algebra of the system [25, 26]. However, the generalization is based on definition (3) as minimum uncertainty states, for more general coherent states adapted to a local potential with at least one confined region [6]. There are two important and distinct classes of generalized coherent states associated with SU(1,1) [25, 26, 27, 28] namely Barut-Girardello coherent states [12] which are eigenstates of the lowering operator and Gilmore-Perelomov coherent states [17, 18], which can be produced by the action of a displacement operator on the vacuum state.

1.3 Barut-Girardello coherent states

In 1971, Barut and Girardello introduced new coherent states. They identified the lowering operator and found eigenstates for this operator, this lowering operator deals to quantum group. The quantum groups shows the mathematical depiction of Lie algebra, give the liability to construct new coherent states, generalized coherent states known as Barut-Girardello coherent states. The Barut-Girardello coherent state is the right eigenstate of lowering operator. In this work we present the construction of these states using their optical realization, i.e., Holstein-Primakoff (HP) realization[21, 22]of SU(1,1) algebra.

Consequently, After this exploration, due to the importance of the nonclassicality of quantum states in various theoretical and experimental fields of physics [19, 29] we naturally lead towards the entanglement generation of the Barut-Girardello coherent states which is the most important and potential application of coherent states in quantum information. It has been also required that the entangled output state from a beam splitter requires nonclassicality in the input state from Reff.[51]. Glauber-Sudarshan P-function [50] of coherent states are not positive and is more singular than delta function then these states are nonclassical. The entanglement generation in Barut-Girardello coherent states after passing through beam splitters is indication of its nonclassicality[47]. The entanglement present in a collection of states is clearly an indication of nonclassicality and a lack of it can be consider as a signature of classicality.

1.4 Nonclassicality criteria for coherent states

nonclassicality is studied through a variety of measures including squeezing [30, 31, 32], sub-Poissonian photon statistics [33], Negativity of Wigner function [44], Anti-bunching effects [8], violations of Cauchy-Schwarz inequalities [34], complementarity between particle-like and wave-like features of entangled coherent states [35], violations of a Bell's inequality [36, 37, 38, 39, 40, 41, 42] or Leggett's inequality [43] for testing local realism, and entanglement properties such as index of correlation [44], entanglement of formation [45, 46, 47] and other measures [48]. In this work, we are using the sub-Poissonian photon statistics [33] of our introduced states to show the non-classical nature of these states.

1.4.1 Sub-Poissonian statistics

Variance and mean is calculated for number operator of standard coherent states. In this case variance and mean are equal. The photon number probability distribution is Poissonian distribution for standard coherent states of harmonic oscillator. The Poissonian photon number distribution is often used as a reference in the sense that any other given quantum state is classified as having a sub-Poissonian or super-Poissonian photon number distribution. Variance is not equal to the mean for sub-Poissonian and super-Poissonian distribution. For sub-Poisson distribution variance is less than the mean and for super-Poisson distribution variance is greater than the mean. In our case for Barut-Girardello coherent states we have the sub-Poisson distribution which exhibit the non-classical nature of these states. For the sub-possonian statistics, photon number distribution is narrower than for a coherent state of same average photon number and the distribution is broader for a coherent states is said to possess super-Poissonian statistics [44]. The statistical features of these states are photon number probability distributions and Mandel's Q parameter [49] depict the sub-Poisson nature of these states which exhibits the nonclassicality in these states. The intensity correlation function [8] is also calculated to study the correlation properties of these states.

1.4.2 Negativity of Wigner function

The Wigner function is

$$W(q,p) = \frac{1}{2\Pi\hbar}\int_{-\infty}^{\infty} \langle q + \frac{1}{2}x|\hat{\rho}|q - \frac{1}{2}x\rangle e^{ipx/\hbar}dx, \tag{1.4.1}$$

negativity of Wigner-function shows that the states are nonclassical [44].

1.4.3 Quadrature squeezing

The quadrature operators are χ_1 and χ_2, for quadrature squeezing the conditions are:

$$\langle(\Delta\hat{\chi_1})^2\rangle < \frac{1}{4}, \tag{1.4.2}$$

$$\langle(\Delta\hat{\chi_2})^2\rangle < \frac{1}{4}, \tag{1.4.3}$$

for coherent states equality hold and same is for the vacuum state, but if one of the condition is holds for state, the state is said to be squeezed state.

1.4.4 Amplitude squeezing

The number phase uncertainty relation is being valid in the regime of large average photon number.

$$\Delta\phi\Delta n \geq \frac{1}{2}, \tag{1.4.4}$$

for standard coherent states of harmonic oscillator $(\Delta n)^2 = |z|^2 = \bar{n}$ and $(\Delta n) = \bar{n}^{\frac{1}{2}}$ $\Delta\phi = \frac{1}{2\bar{n}^{\frac{1}{2}}}$. Considering the definition of quadrature squeezing, if $\Delta n < \bar{n}^{\frac{1}{2}}$ the states are squeezed in number and if $\Delta\phi < \frac{1}{2\bar{n}^{\frac{1}{2}}}$ the states are squeezed in phase. The phase squeezed states are difficult to characterize the nonclassicality because it is difficult to have hermition operator representation of phase. Amplitude squeezing is a photon number squeezing, variance can be written as

$$\langle(\Delta\hat{n})^2\rangle = \langle\hat{n}\rangle + (\langle\hat{a}^{\dagger 2}\hat{a}^2\rangle - (\langle\hat{a}^\dagger\hat{a}\rangle)^2), \tag{1.4.5}$$

A state exhibiting amplitude squeezing holds the sub-possonian statistics, photon number distribution is narrower than for a coherent state of same average photon number and the distribution is broader for a coherent states is said to possess supr-Poissonian statistics.

1.4.5 Anti-bunching effects

For bunch and anti-bunch light photons are coming in pairs towards the detector. The second order correlation function $g^2(\tau)$ [8] is determining the joint probability of detecting a photon followed by other with time delayed τ. The probability of detecting a photon is 1 when they arrive independently, $g^2(\tau) = 1$. If $g^2(\tau) > g^2(0)$ we have bunching effects. If $g^2(\tau) < g^2(0)$, we have anti-bunching effects.

1.5 Entanglement measures

There exists several measures of entanglement such as the , linear entropy [66, 67], the von Neumann entropy [68, 69, 70], the concurrence [71, 72], positive partial transpose [73], the negativity [75, 76, 77] , Schmidt decomposition [74]. In this dissertation, we use the linear entropy as measure of entanglement.

1.5.1 Linear entropy

The linear entropy is use to characterize the entanglement of bipartite system. Linear entropy S_L is defined as

$$S_L = 1 - Tr(\rho_A{}^2), \tag{1.5.1}$$

where ρ_A, is the reduced density operator of the system A obtained by performing a partial trace over system B of the density operator ρ_{AB}. The linear entropy is bounded by the factor $N/(N-1)$, where N is the dimension of the state. In this work, we are interested in the spatial entanglement between the two states. If N is infinite then the linear entropy is bound to unity. The system would be completely disentangled when $S_L = 0$. On the other hand, the system would be in the maximally entangled state if $S_L = 1$.

1.5.2 von Neumann entropy

von Neumann entropy is one of the entropy entanglement measure. It is define as

$$S(\rho_A) = -Tr\{\rho_A log_2(\rho_A)\}, \tag{1.5.2}$$

Or

$$S(\rho_B) = -Tr\{\rho_B log_2(\rho_B)\}, \tag{1.5.3}$$

where ρ_A and ρ_B are reduced density operators of composite system $\rho = \rho_A \bigotimes \rho_B$. Reduced density operator ρ_A and ρ_B are define as

$$\rho_A = -Tr_B\{\rho\}, \tag{1.5.4}$$

and

$$\rho_B = -Tr_A\{\rho\}, \tag{1.5.5}$$

von Neumann entropy vanishes for pure state and it is maximum for fully mixed states. For pure state von Neumann entropy is 0 and for mixed state it is 1.

$$E(\rho) = S(\rho_A) = S(\rho_B) = 1, \tag{1.5.6}$$

the entropy of entanglement of mixed state is 1, state is maximally entangle.

1.5.3 Concurrence

Concurrence is also a measure of entanglement used in quantum information. For two-qubit pure states

$$|\Psi\rangle = \alpha|\uparrow\uparrow\rangle + \beta|\uparrow\downarrow\rangle + \gamma|\downarrow\uparrow\rangle + \delta|\downarrow\downarrow\rangle, \tag{1.5.7}$$

If $C = 2|\alpha\delta - \beta\gamma| > 0$ the state is entangle and if $C = 2|\alpha\delta - \beta\gamma| = 0$ the state is not entangle. Entanglement is maximum when concurrence $C = \frac{1}{2}$. The concurrence is used to quantify the entanglement when two-qbit systems are entangled and based on bit flip operation σ_y. For a single qbit state $|\Psi\rangle$, the overlap with the σ_y-flipped state $|\tilde{\Psi}\rangle = \sigma_y|\Psi\rangle$, define the concurrence as $c = |\langle\Psi|\tilde{\Psi}\rangle|$. We define a matrix $\tilde{\rho}$

$$\tilde{\rho}(t) = \rho(t)(\sigma y \otimes \sigma_y)\rho^*(t)(\sigma_y \otimes \sigma_y), \tag{1.5.8}$$

$\rho(t)$ is time dependent density matrix define for 2×2 systems. The square roots of eigen values of 4×4 matrix $\tilde{\rho}$ define the concurrence

$$C(t) = max[0, \Lambda(t)], \tag{1.5.9}$$

where $\Lambda(t) = \sqrt{\lambda_1(t)} - \sqrt{\lambda_2(t)} - \sqrt{\lambda_3(t)} - \sqrt{\lambda_4(t)}$, λ_i are the eigenvalues. Concurrence $C(t) = 0$ for separable states and $C(t) = 1$ for maximally entangle states.

1.5.4 Tangle

Entropy and concurrence quantify the entanglement when two-qbit system are entangled. If three-qbit system are entangle we use tangle to quantify the entanglement as measure of enta Lets consider three qbit system ABC, there exist pairwise entanglement, i.e., system A is entangle with system Bor C, but there is also three way entangled states that are not pairwise entangled. These various types of entanglement are quantified through different measures of entanglement called tangle. We take the three-qbit state starts with the avrage two tangle

$$\tau_2 = \frac{{C_{12}}^2 + {C_{23}}^2 + {C_{13}}^2}{3}, \tag{1.5.10}$$

entanglement can be measured by bipartite concurrence between one-qbit and both other qbits

$$Ci(jk) = \sqrt{2(1 - Tr({\rho_i}^2))}, \tag{1.5.11}$$

where ρ_i is density operator of qbit i, obtained by tracing over the other two j and k qbits If product state ρ is pure, ρ_i is pure state, $Tr({\rho_i}^2) = 1$, $C_{i(jk)} = 0$, ρ. If $Tr({\rho_i}^2) < 1$, $C_{i(jk)} > 0$, ρ is entangled. For maximally entangle state $Tr(\rho_i{}^2) = \frac{1}{2}$, $C_{i(jk)} = 1$. The bipartite concurrence shows the entanglement of qbit i is with the other two. It is the entanglement with one of them or with both. This is quantified through three-tangle τ_3, from bipartite concurrence we subtracts the pairwise entanglement of qbit i with j and k to obtain three-way entanglement of a three-qbit state.

$$\tau_3 = {C_{i(jk)}}^2 - ({C_{ij}}^2 + {C_{ik}}^2). \tag{1.5.12}$$

1.5.5 Positive partial transpose

The Positive Partial Transpose was introduced by Peres and by the Horodczkys. They introduced the necessary condition for joint density matrix ρ of quantum mechanical system A and system B We write the basis states$|i\rangle$ and $|j\rangle$ of system A and $|k\rangle$ and $|l\rangle$ of system B The density operator of the composite system is written as

$$\rho = \sum_{ijkl} p_{ijkl}|i\rangle|j\rangle \otimes |k\rangle|l\rangle, \tag{1.5.13}$$

the partial transposition ρ^{T_B} is interchanges the density matrix elements, e.g., system B only, as $\rho_{ijkl}\rho_{ijlk}$.

$$\rho^{T_B} = \sum_{ijkl} p_{ijlk}|i\rangle|j\rangle \otimes |l\rangle|k\rangle, \tag{1.5.14}$$

the PPT criterion states if ρ^{T_B} have negative eigen values, ρ is entangled

1.5.6 Negativity

The negativity is also a measure of entanglement based on PPT measure. PPT measure is used to quantify the entanglement in two qbit systems of high dimensions. According to it, the negative eigenvalues of the partial transposed matrix characterize the entanglement. For bipartite systems negativity is defined as

$$N(t) = -2\sum_i \lambda_i, \tag{1.5.15}$$

where λ_is are the negative values of PPT matrix, entanglement can be defined as

$$E = max[0, N(t)] \tag{1.5.16}$$

1.5.7 Schmidt decomposition

There are several measures to characterize the entanglement in bipartite systems one of them is the Schmidt decomposition. The composite state of bipartite system is written as

$$|\Psi\rangle_{AB} = \sum_i c_i|u_i\rangle_A|v_i\rangle_B, \tag{1.5.17}$$

$|u_i\rangle_A$ and $|v_i\rangle_B$ are the basis vectors defined for system Aand B respectively. The non zero eigenvalues of the reduced density matrix $\rho_A = Tr_B(\rho_{AB})$ define schmidt number d. If Schmidt number d is 1, the composite state of bipartite system is separable and system is entangled if $d > 1$.

1.6 Layout of thesis

In chapter 2, two distinct class of coherent states are discussed. Firstly, we gave a brief description of ordinary or standard coherent states of harmonic oscillator (Glauber coherent states) and their construction. The photon number probability distribution of these states are discussed. Their properties such as over completeness, non-orthognality, temporal stability etc. are discussed. The graphical representation of photon number probability distribution for these states are drawn to make the comparison of standard coherent states with the results of the new coherent states. Secondly, we present the construction of generalized coherent states known as Barut-Girardello coherent states using the optical realization of these states. The photon number probability distribution of these states are discussed. The basic properties of Barut-Girardello coherent states over-completeness, non-orthognality, temporal stability etc. are discussed.

Chapter 3 of this thesis presents, the work on optical beam splitters, engineering of beam splitters and how the input modes are related to output modes of beam splitter. If various states are the inputs of beam splitters.i.e., fock state, coherent states, Glauber coherent states etc., exhibit a very interesting phonomenon. We define a particular criteria for optical beam splitter considering the optical beam splitter as entangler. We demonstrate a condition for the output state of a optical beam splitter is entangled if input state exhibit nonclassicality. We analyze that nonclassicality in the input states of beam splitter is as a mandatory for entanglement.

Chapter 4 of this thesis presents, work on non-classical properties of Barut-Girardello coherent states. Using the statistical features of Barut-Girardello coherent states we present that these states have sub-Poisson distribution. Mandel Q parameter is calculated to investigate the sub-Poisson photon statistics of these states which is indication of nonclassicality. The second order intensity correlation function is also calculated to analyze the correlation properties of these states which exhibits the anti-bunching effects. The nonclassicality in these states leads us towards the entanglement generation of Barut-Girardello coherent states through optical device, via 50:50 beam splitter and quantification of entanglement using linear entropy as measure of entanglement. Linear entropy measures the degree of entanglement of output state. The entanglement generation of these states depend on two parameters Bargmann index k and the BG coherent states amplitude z. The variation in linear entropy by changing these parameter k and z shows the variation in degree of entanglement.

Finally chapter 5 presents the overall results, conclusion and discussion of the thesis.

Chapter 2

Optical coherent states: basic theory to Barut-Girardello generalization

2.1 Introduction

The minimum uncertainty states of a harmonic oscillator, introduced by Schrödinger, were brilliantly used by Roy Glauber to express the quantum states of coherent optical field, hence named as coherent states. In literature these states are also termed as canonical coherent states or Glauber coherent states. However, the notion of coherent states has been generalized for much more general situations than were introduced by Schrödinger and Glauber. In this chapter, we first discuss the construction of Glauber coherent states and then present a generalization of these coherent states known as Barut-Girardello coherent states.

The chapter is organized as following. In section (2.2), we briefly discuss the construction of Glauber coherent states. In order to construct these states we first discuss the Hamiltonian algebraic structure of the harmonic oscillator, that is, Hiesenberg-whyl algebra [15], and then discuss the various ways of defining coherent states. Afterwards, a set of basic characteristics of these states is presented. In Section (2.3) of this chapter we present the construction Barut-Girardello coherent states using the Holstein-Primakoff (HP) realization of su(1,1) Lie algebra [79] and their basic properties are also discussed.

2.2 Glauber coherent states

The ground breaking work of Glauber on quantum theory of optical coherences is based on a special kind of quantum states of harmonic oscillator, known as coherent states. Glauber recieved the Nobel Prize 2005 in Physics for these states. In the following we discuss the construction and properties of these states.

2.2.1 Construction

In order to construct the Glauber coherent states, first an algebraic structure of the harmonic oscillator is needed. In the following we discuss the factorization of Hamiltonian and the underlying algebraic structure of the harmonic oscillator which will be followed by the construction of coherent states.

Harmonic oscillator Hamiltonian

The harmonic oscillator (HO) is the most important model system in quantum mechanics. Its Hamiltonian is basically the sum of squares of two conjugate canonical variables that is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2, \tag{2.2.1}$$

where ω is the the angular frequency, operators $\hat{x}$ and $\hat{p}$ are, of course Hermition. Lets define two non-Hermition operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + \frac{i\hat{p}}{m\omega}), \quad \hat{a}^\dagger = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} - \frac{i\hat{p}}{m\omega}), \tag{2.2.2}$$

which are known as **annihilation** and **creation** operators respectively. The commutation relation $[\hat{a}, \hat{a}^\dagger] = I$ using canonical commutation relation

$$[\hat{a}, \hat{a}^\dagger] = (\frac{1}{2\hbar})(-i[\hat{x}, \hat{p}] + i[\hat{p}, \hat{x}]) = I. \tag{2.2.3}$$

We also define another operator as $N = \hat{a}^\dagger\hat{a}$ called number operator, which is obviously an hermition operator $N^\dagger = N$. It is straightforward to show that

$$\begin{aligned} \hat{a}^\dagger\hat{a} &= (\frac{m\omega}{2\hbar})(\hat{x}^2 + \frac{\hat{p}^2}{m^2\omega^2}) + \frac{i}{2\hbar}[\hat{x}, \hat{p}] \\ &= \frac{1}{\hbar\omega}(\frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2) - \frac{1}{2} \\ &= (\frac{\hat{H}}{\hbar\omega} - \frac{1}{2}), \end{aligned}$$

as $\hat{N} = \hat{a}^\dagger\hat{a}$, so we have an important relation between number operator and Hamiltonian operator

$$\hat{H} = \hbar\omega(\hat{N} + \frac{1}{2}), \tag{2.2.4}$$

this shows that $\hat{H}$ linearly depends on $\hat{N}$, $\hat{N}$ is diagonalize simultaneously with $\hat{H}$.

Algebraic structure of HO: Heisenberg-Weyl algebra

The annihilation operator $\hat{a}$, creation operator $\hat{a}^\dagger$ and number operator $N = \hat{a}^\dagger\hat{a}$ form Heisenberg-Weyl Algebra which satisfy the following commutation relations:

$$\begin{aligned} [\hat{a}, \hat{a}^\dagger] &= 1, \quad [\hat{a}^\dagger, \hat{a}] = -1, \\ [\hat{N}, \hat{a}^\dagger] &= \hat{a}^\dagger, \quad [\hat{N}, \hat{a}] = -\hat{a}, \\ [\hat{a}, \hat{a}] &= 0, \quad [\hat{a}^\dagger, \hat{a}^\dagger] = 0. \end{aligned} \tag{2.2.5}$$

We will proceed to construct the Glauber coherent states using the Hiesenberg-Weyl algebra of harmonic oscillator.

Fock space

The Hilbert space spanned by the number eigenstates $\{|0\rangle, |1\rangle, |2\rangle, ...|n\rangle\}$, satisfying orthonormality $\langle n|n\acute{}\rangle = \delta_{nn\acute{}}$, is known as Fock space. The number operator obeys eigenvalue equation

$$\hat{N}\,|n\rangle = n\,|n\rangle\,. \tag{2.2.6}$$

Moreover, the operators $\hat{a}^\dagger$, $\hat{a}$ act upon the number eigenstates as

$$\hat{a}^\dagger\,|n\rangle = \sqrt{n+1}\,|n+1\rangle \tag{2.2.7}$$

and

$$\hat{a}\,|n\rangle = \sqrt{n}\,|n-1\rangle\,, \tag{2.2.8}$$

where the condition

$$\hat{a}|0\rangle = 0 \tag{2.2.9}$$

defines the ground state $|0\rangle$ of the oscillator. The Fock space $\{|n\rangle\}$ may be obtained by repeated application of the creation operator $\hat{a}^\dagger$ on the vacuum state $|0\rangle$ as

$$\begin{aligned} |1\rangle &= \hat{a}^\dagger|0\rangle \\ |2\rangle &= \frac{\hat{a}^\dagger}{\sqrt{2}}|1\rangle = \frac{(\hat{a}^\dagger)^2}{\sqrt{2}}|0\rangle, \\ &. \\ &. \\ &. \\ |n\rangle &= \frac{\hat{a}^\dagger}{\sqrt{n!}}|n-1\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}|0\rangle. \end{aligned} \tag{2.2.10}$$

The Fock states thus obey completeness,

$$\sum_n |n\rangle\,\langle n| = \hat{I}. \tag{2.2.11}$$

with $\hat{I}$ being n-dimensional identity operator.

Construction of coherent states

Following the Glauber's formalism, the coherent states can be constructed by using any one of three definitions [13, 14, 17, 18, 12].

Definition 1: The coherent states $|z\rangle$ are the eigenstates of the harmonic oscillator annihilation operator $\hat{a}$, i.e.,

$$\hat{a}\,|z\rangle = z\,|z\rangle\,, \tag{2.2.12}$$

where z is a complex number.

Definition 2: They are generated by applying a displacement operator $\hat{D}\,(z)$ on the vacuum state $|0\rangle$ of the harmonic oscillator,

$$|z\rangle = \hat{D}\,(z)\,|0\rangle\,, \tag{2.2.13}$$

where the displacement operator $\hat{D}(z) = e^{z\hat{a}^\dagger - z^*\hat{a}}$, with $\hat{a}^\dagger$ being the harmonic oscillator creation operator.

Definition 3: They are the quantum states minimizing uncertainties relationship, i.e.,

$$\Delta x \Delta p = \frac{1}{2}. \tag{2.2.14}$$

This is easy to show by calculating the dispersions of position and momentum operators with respect to the coherent coherent states as

$$\begin{aligned} (\Delta x)^2 &= \langle z|\hat{x}^2|z\rangle - \langle z|\hat{x}|z\rangle^2, \\ (\Delta p)^2 &= \langle z|\hat{p}^2|z\rangle - \langle z|\hat{p}|z\rangle^2, \end{aligned}$$

where the position and momentum operators $\hat{x},\ \hat{p}$ are expressed in terms of $\hat{a}, \hat{a}^\dagger$ using equation (2.2.2).

Fock space representation of Glauber coherent states

In order to express the state of electromagnetic field, that can resemble to the corresponding classical description, it is useful to construct a linear superposition of number states or Fock states. The coherent states, defined above, can be expanded in terms of Fock basis as

$$|z\rangle = \sum_{n=0}^{\infty} c_n |n\rangle, \tag{2.2.15}$$

acting with $\hat{a}$ on each term we get

$$\hat{a}|z\rangle = \sum_{n=0}^{\infty} c_n \sqrt{n}|n-1\rangle = z \sum_{n=0}^{\infty} c_n |n\rangle, \tag{2.2.16}$$

equating coefficients of $|n\rangle$ on both sides

$$c_n = \frac{z}{\sqrt{n}} C_{n-1}, \quad n = 1, 2, 3....$$

Now

$$\begin{aligned} c_1 &= \frac{z}{\sqrt{1}} c_o, \\ c_2 &= \frac{z}{\sqrt{2}} c_1 = \frac{z^2}{\sqrt{2.1}} c_o, \\ c_3 &= \frac{z}{\sqrt{3}} c_2 = \frac{z^2}{\sqrt{3.2.1}} c_o, \\ &. \\ &. \\ c_n &= \frac{z^n}{\sqrt{n!}} c_o. \end{aligned}$$

thus we have

$$|z\rangle = c_o \sum_{n=0}^{\infty} \frac{|z|^n}{\sqrt{n!}} |n\rangle. \tag{2.2.17}$$

for normalization we require $\langle z|z\rangle = 1$

$$|c_o|^2 \sum_{n,m=0}^{\infty} \frac{|z|^n |z|^{*m}}{\sqrt{n! m!}} \langle m|n\rangle = 1 \tag{2.2.18}$$

for m=n, $\delta_{(m,n)} = 1$

$$\begin{aligned} |c_o|^2 \sum_{n=0}^{\infty} \frac{|z|^{2n}}{n!} &= 1, \\ |c_o| &= \exp\left(-\frac{|z|^2}{2}\right), \end{aligned}$$

the resulting normalized states are given by

$$|z\rangle = \exp\left(-\frac{|z|^2}{2}\right) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \tag{2.2.19}$$

the Glauber coherent states are optical coherent states which are constructed for harmonic oscillator.

2.2.2 Properties

The Glauber coherent states satisfy the following set of properties.

Over Completeness:

Coherent states form an over-complete linearly dependent set. The resolution of the identity holds in the form

$$\frac{1}{\pi} \int d^2z |z\rangle\langle z| = \sum_{n=0}^{\infty} |n\rangle\langle n| = \mathrm{I}; d^2z = d(\Re z)d(\Im z), \tag{2.2.20}$$

a state vector$|\psi\rangle$ in Hilbert space of the quantized sngle-mode field can be expressed in terms of coherent state as

$$\frac{1}{\pi} \int d^2z |z\rangle\langle z||\psi\rangle. \tag{2.2.21}$$

$|\psi\rangle$, itself is the coherent state. There are more than enough states are available to represents coherent states in terms of coherent states.

Non-orthonality

The coherent states are non-orthogonal

$$\langle z'|z\rangle = [\exp{(-\frac{|z|^2}{2})} + (-\frac{|z'|^2}{2}) + z'^* z], \tag{2.2.22}$$

here we have,

$$\langle z'|z\rangle \neq 0, \tag{2.2.23}$$

Orthognalty condition is

$$\langle z'|z\rangle = 0. \tag{2.2.24}$$

Temporal stability

The time evaluation of coherent states for a single mode free field [44] $\hat{H} = (\hat{n} + \frac{1}{2})\hbar\omega$. The time-evolved coherent state is given by

$$|z, t\rangle = \exp\left(\frac{i\hat{H}t}{\hbar}\right)|z\rangle = e^{-\frac{i\omega t}{2}} e^{-i\omega t\hat{n}}|z\rangle, \tag{2.2.25}$$

$$|z, t\rangle = e^{-\frac{i\omega t}{2}} |ze^{-i\omega t}\rangle, \tag{2.2.26}$$

the coherent state remains a coherent state under free field time-evolution.

2.3 Barut-Girardello coherent States

We can construct the generalized coherent states by replacing the algebra of harmonic oscillator with the algebra of other systems based on definition (1) and (2). In this work we are constructing the generalized coherent states namely as Barut-Girardello coherent states. The Barut-Girardello coherent states belong to SU(1,1) group. For SU(1,1) group, there are two inequivalent ways to construct coherent states: (1) eigenstates of the lowering operator and (2) as displaced vacuum state under the action of displacement operator. To construct the Barut-Girardello Coherent States, we used the criteria(1) eigenstates of the lowering operator [12]. We need su(1,1) Lie algebra because our lowering operator belongs to SU(1,1) Lie group. We are constructing the Barut-Girardello coherent states using SU(1,1) Lie algebra considering the Holstein-Primakoff (HP) realization of su(1,1) Lie algebra (in terms of a set of single mode bose annihilation and creation operators) [79].

su(1,1) algebra

We give a brief review of SU(1,1) group, there are three generators in this group K_1, K_2 and K_3. The commutation relation for the Lie algebra corresponding to the $Su(1,1)$ group:

$$[K_1, K_2] \quad = \quad -iK_3, \quad [K_2, K_3] = iK_1, \quad [K_3, K_1] = iK_2. \tag{2.3.1}$$

where the operator K_3 is considered as the generator of the geometrical rotation, while K_1 and K_2 are the Lorentz transformation[88]. The raising and lowering generators of this group are $K_\pm = K_1 \pm iK_2$, which satisfy the following commutation relations:

$$[K_-, K_+] \quad = \quad 2K_3, \qquad [K_3, K_\pm] = \pm K_\pm, \tag{2.3.2}$$

where its generators $(K_\pm, K_3)$ obey Hermicity property

$$(K_+)^\dagger = K_-, \quad (K_-)^\dagger = K_+, \quad (K_3)^\dagger = K_3. \tag{2.3.3}$$

casimir operator K^2 for any irreducible representation is

$$K^2 = (K_3)^2 - (K_1)^2 - (K_2)^2 = k(k-1), \tag{2.3.4}$$

real number k (called the Bergmann index). The state is spanned by the complete orthonormal basis of the Bargmann-Hilbert space H_B, $|k, m\rangle$, where $m = 0, 1, 2....\infty$, m is quantum number of Hilbert space. $\langle k, m'|k, m\rangle = \delta'_{m',m}$, $\sum_{m=0}^{\infty} |k, m\rangle\langle m, k| = 1$. We consider here only the representation known as the positive discrete series.

$$K^2|k,m\rangle = k(k-1)|k,m\rangle, \tag{2.3.5}$$

$$K_3|k,m\rangle = m|k,m\rangle, \tag{2.3.6}$$

$$K_\pm|k,m\rangle = \sqrt{(m\pm k)(m\mp k\pm 1)}|k,m\pm 1\rangle, \tag{2.3.7}$$

or

$$K_\pm|k,m\rangle = \sqrt{(m)(m\mp 2k\pm 1)}|k,m\pm 1\rangle, \tag{2.3.8}$$

where $k \in \{\frac{1}{2}, 1, \frac{3}{2}, ...\}$ is the Bargman index [23, 24, 25, 26, 27, 28].

Holstein-Primakoff (HP) realization of the su(1,1) Lie algebra

Bargmann (1970) gave the representations of Lie groups is constructed by realizing the operators of the Lie algebra as amplitude-squared of the boson annihilation and creation operators $\hat{a}$ and $\hat{a^\dagger}$. With these boson operators represented by operators defined over a Hilbert space of entire analytic functions, the Bargmann-Hilbert space H_B, one can also construct group representations as Hilbert spaces of entire analytic functions. One might expect that an alternative way of constructing a representation of a Lie group on a Hilbert space of analytic functions is to exploit the generalised coherent states in a fashion similar to the ordinary coherent states [79].

In the present work we concentrate on the SU(1,1) Lie group whose algebra has a number of realizations related to the quantized light field. The most known of them are the single mode realization in terms of the amplitude-squared boson operators[20, 21]. We consider here the Holstein-Primakoff (HP) single mode realization of the su(1,1) Lie algebra given below.

$$\begin{aligned} K_+(k) &= \sqrt{\hat{a}^\dagger\hat{a}+2k-1}\,\hat{a}^\dagger, \qquad \hat{a}^\dagger\hat{a}=n, \\ K_-(k) &= \hat{a}\sqrt{\hat{a}^\dagger\hat{a}+2k-1}, \\ K_3(k) &= \hat{a}^\dagger\hat{a}+k, \end{aligned}$$

Here k is the Bargmann index labeling unitary irreducible representations of the SU(1,1) Lie group. Various states associated with the (HP) SU(1,1) realization exist in the harmonic oscillator Hilbert space. These states can be conveniently treated by using general group-theoretical techniques. One can consider the generalized CS obtained as the eigenstates of the SU(1,1) lowering generator $K_-(k)$ (the so-called Barut-Girardello states)and by the action of SU(1,1) group elements on the vacuum state [123, 124, 125, 126, 127] (Perelemov coherent states).

2.3.1 Construction

The Barut-Girardello coherent states are the eigenstates of lowering operator K_-

$$K_-|z,k\rangle = z|z,k\rangle \tag{2.3.9}$$

z is a complex number. Here, we are only interested only in positive discrete series. The corresponding state is spanned by the complete orthonormal basis of the Bergman space $|k, m\rangle$ of Hilbert Space [32].

$$K_-|z,k\rangle = K_-(I|z,k\rangle), \tag{2.3.10}$$

using the completeness relation $\sum_{m=0}^{\infty} |k,m\rangle\langle m,k| = I$

$$K_-|z,k\rangle = K_-(\sum_{m=0}^{\infty} |k,m\rangle\langle m,k|z,k\rangle), \tag{2.3.11}$$

as action of lowering operator K_- on $|k,m\rangle$ from equation (2.3.9) is

$$K_-|k,m\rangle = \sqrt{(m)(m+2k-1)}|m-1,k\rangle. \tag{2.3.12}$$

$$z|z,k\rangle = \sum_{m=0}^{\infty} \sqrt{(m)(m+2k-1)}|m-1,k\rangle\langle m,k|z,k\rangle, \tag{2.3.13}$$

take inner product with $\langle n,k|$ on both sides

$$z\langle n,k|z,k\rangle = \sum_{m=0}^{\infty} \sqrt{(m)(m+2k-1)}\langle n,k|m-1,k\rangle\langle m,k|z,k\rangle, \tag{2.3.14}$$

we know that $\sum_{m=0}^{\infty} \langle n,k|m-1,k\rangle = \delta_{(n,m-1)} = 1$ If and only if $m-1=n$,

$$\begin{aligned} z\langle m-1,k|z,k\rangle &= \sqrt{(m)(m+2k-1)}\langle m,k|z,k\rangle, \\ \langle m,k|z,k\rangle &= \frac{z}{\sqrt{(m)(m+2k-1)}}\langle m-1,k|z,k\rangle, \end{aligned} \tag{2.3.15}$$

using these relations

$$(n+1)(n+2)...(n+r) = (n+r)(n+r+1)...(n+2)(n+1)\frac{\Gamma(n+1)}{\Gamma(n+1)} = \frac{\Gamma(n+r+1)}{\Gamma(n+1)},$$

$$(m+2k)(m+2k-1)(m+2k-2)... = \frac{\Gamma(m+2k)}{\Gamma(2k)}, \quad (m+2k-1)! = \Gamma(m+2k),$$

we get after re-occurence procedure

$$\langle m,k|z,k\rangle = \sum_{m=0}^{\infty} \frac{|z|^m\sqrt{\Gamma(2k)}}{\sqrt{(m)!\Gamma(m+2k)}}\langle 0,k|z,k\rangle. \tag{2.3.16}$$

$$\|\langle m,k|z,k\rangle\|^2 = \sum_{m=0}^{\infty} \frac{|z|^{2(m)}\Gamma(2k)}{(m)!\Gamma(m+2k)}\|\langle 0,k|z,k\rangle\|^2, \tag{2.3.17}$$

by normalizing to unity

$$\langle z,k|z,k\rangle = I \tag{2.3.18}$$

Using completeness relation $\sum_{m=0}^{\infty} |m,k\rangle\langle m,k| = I$ and $\sum_{m'=0}^{\infty} |m',k\rangle\langle m',k| = I$.

$$\langle z,k|z,k\rangle = I = \sum_{m=0}^{\infty}\sum_{m'=0}^{\infty} \langle z,k|m,k\rangle\langle m,k|m',k\rangle\langle m',k|z,k\rangle. \tag{2.3.19}$$

As $\sum_{m'=0}^{\infty} \langle m,k|m',k\rangle = \delta_{m,m'} = 1.$ if $m = m'$,then

$$\begin{aligned} \langle z,k|z,k\rangle &= \sum_{m=0}^{\infty} \langle z,k|m,k\rangle \sum_{m'=0}^{\infty} \langle m,k|m',k\rangle\langle m',k|z,k\rangle = 1, \\ \langle z,k|z,k\rangle &= \sum_{m=0}^{\infty} \|\langle z,k|m,k\rangle\|^2 = 1, \end{aligned}$$

using normalization condition in equation (2.3.17), we get

$$\sum_{m=0}^{\infty}\frac{|z|^{2(m)}\Gamma(2k)}{(m)!\Gamma(m+2k)}\|\langle 0,k|z,k\rangle\|^2=1, \tag{2.3.20}$$

$$\langle 0,k|z,k\rangle=\sqrt{\frac{|z|^{2k-1}}{\Gamma(2k)I_{2k-1}(2|z|)}}, \tag{2.3.21}$$

as the relation is

$$\sum_{m=0}^{\infty}\frac{|z|^{2m}}{(m)!\Gamma(m+2k)}=\frac{1}{|z|^{2k-1}}I_{2k-1}(2|z|). \tag{2.3.22}$$

$I_{(2k-1)}(2|z|)$ is the modified Bessel's function of first kind [36]. We can also write this relation in terms of Hypergeometric function [36]

$$\sum_{n=0}^{\infty}\frac{\Gamma(2k)}{n!\Gamma(n+2k)}|z|^{2n}={}_0F_1(2k;|z|^2), \tag{2.3.23}$$

using the above relation in equation (2.3.16), we have the resultant equation

$$|z,k\rangle=\sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}}\sum_{m=0}^{\infty}\frac{z^m}{\sqrt{(m)!\Gamma(m+2k)}}|m,k\rangle, \tag{2.3.24}$$

$$|z,k\rangle=\sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}}\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!\Gamma(n+2k)}}|n\rangle. \tag{2.3.25}$$

Normalization factor is

$$\aleph(|z|^2)=\sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}}, \tag{2.3.26}$$

$$|z,k\rangle=\aleph(|z|^2)\sum_{n=0}^{\infty}\frac{z^n}{\sqrt{n!\Gamma(n+2k)}}|n\rangle, \tag{2.3.27}$$

is the normalized Barut-Girardello coherent state, these states can also be written in terms of hypergeometric function as

$$|z,k\rangle=\frac{1}{\sqrt{{}_0F_1(2k;|z|^2)}}\sum_{n=0}^{\infty}\sqrt{\frac{\Gamma(2k)}{n!\Gamma(n+2k)}}z^n|n\rangle. \tag{2.3.28}$$

$0F_1(k;|z|^2)$, is Hypergeometric function usually expressed in terms of modified Bessel's function[80].

2.3.2 Properties

The Barut-Girardello coherent states satisfy the following set of properties

Over completeness:

BGCS coherent states are over complete coherent states form an over-complete linearly dependent set[26]. The resolution of the identity holds in the form

$$\int d\mu(z,k)|z,k\rangle\langle z,k|=I=\sum_{m=0}^{\infty}|k,m\rangle\langle m,k|, \tag{2.3.29}$$

$$d\mu(z,k)=\frac{2}{\Pi}K_{2k-1}(2|z|)I_{2k-1}(2|z|)d^2z;\quad d^2z=d(\Re z)d(\Im z), \tag{2.3.30}$$

The function $K_\nu(x)$ is 'ν order modified Bessel's function' of the second order. All the integrals are performed over the whole complex z-plane

$$z=r\exp(i\phi),\qquad r\epsilon[0,\infty],\qquad \phi\epsilon[0,2\Pi]. \tag{2.3.31}$$

Non-orthognality:

Barut-Girardello Coherent States are non-orthogonal

$$|z,k\rangle = \sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{(m)!\Gamma(m+2k)}}|m,k\rangle, \tag{2.3.32}$$

and

$$\langle\sigma,k| = \sqrt{\frac{|\sigma|^{2k-1}}{I_{2k-1}(2|\sigma|)}} \sum_{m'=0}^{\infty} \frac{\sigma^{*m'}}{\sqrt{m'!\Gamma(m'+2k)}}\rangle m',k|, \tag{2.3.33}$$

$$\begin{aligned}\langle\sigma,k|z,k\rangle &= (\sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}})(\sqrt{\frac{|\sigma|^{2k-1}}{I_{2k-1}(2|\sigma|)}}) \\ &\sum_{m'=0}^{\infty} \frac{|\sigma|^{m'}}{\sqrt{m'!\Gamma(m'+2k)}} \sum_{m=0}^{\infty} \frac{|z|^m}{\sqrt{m!\Gamma(m+2k)}},\end{aligned} \tag{2.3.34}$$

If $m' = m, \quad \sum_{m=k}^{\infty} \langle m\prime,k|m,k\rangle = 1,$

$$\langle\sigma,k|z,k\rangle = \sqrt{\frac{|\sigma|^{2k-1}|z|^{2k-1}}{I_{2k-1}(2|\sigma|)I_{2k-1}(2|z|)}} \overbrace{\sum_{m=0}^{\infty} \frac{|z|^m|\sigma|^{*m}}{m!\Gamma(m+2k)}}^{\frac{I_{2k-1}(2\sqrt{\sigma z})}{\sqrt{\sigma^{*2k-1}z^{2k-1}}}}, \tag{2.3.35}$$

as it is 'modified Bessels function' of first kind[92]. Finally we get,

$$\langle\sigma,k|z,k\rangle = \frac{I_{2k-1}(2\sqrt{\sigma^* z})}{\sqrt{I_{2k-1}(2|\sigma|)I_{2k-1}(2|z|)}}, \tag{2.3.36}$$

this shows that BGCS are non orthognal.

Temporal stability:

The time evalution operator of BGCS coherent states is H_k the time-evolved coherent state is given by

$$|z,k;t\rangle = \exp\left(-\frac{i}{\hbar}\hat{H}_k t\right)|z,k;0\rangle. \tag{2.3.37}$$

Considering specific system Psuedoharmonic harmonic oscillator system (PHO) [25] of SU(1,1),the Hamiltonian for PHO is H_k

$$H_k \equiv H_\alpha = \hbar\omega_o {H_\alpha}^{(red)}(y) = \hbar\omega_o(2\nu + 2K_3 - \frac{M\omega_o}{\hbar}{r_o}^2), \tag{2.3.38}$$

ν is vibrational Quantum number, Reduced Hamiltonia is

$$K_3 = \frac{1}{2}{H_\alpha}^{(red)}(y) \Longrightarrow {H_\alpha}^{(red)}(y) = 2K_3,$$

the coherent state remains a coherent state under free field time-evolution.

$$H_k \equiv H_\alpha = \hbar\omega_o {H_\alpha}^{(red)}(y) = \hbar\omega_o(2K_3 - \frac{M\omega_o}{\hbar}{r_o}^2) = (2\hbar\omega_o K_3 - M{\omega_o}^2{r_o}^2),$$

$$|z,k;t\rangle = \exp\frac{i}{\hbar}(M{\omega_o}^2{r_o}^2 - 2\hbar\omega_o K_3)t\sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}} \sum_{m=0}^{\infty} \frac{(ze^{-2i\omega_o t})^m}{\sqrt{(m)!\Gamma(m+2k)}}|m,k\rangle, \tag{2.3.39}$$

$$|z,k;t\rangle = \sqrt{\frac{|z|^{2k-1}}{I_{2k-1}(2|z|)}} \sum_{m=0}^{\infty} \frac{z^m(t)}{\sqrt{m!\Gamma(m+2k)}}|m,k\rangle, \tag{2.3.40}$$

$\nu = 0$, for ground vibrational state

$$E_o = \hbar\omega_o 2K_3 - M{\omega_o}^2{r_o}^2.$$

$$|z,t\rangle = e^{-\frac{i}{\hbar}E_o t}|z(t),k\rangle, \tag{2.3.41}$$

Hence the BGCS are the coherent state remains a coherent state under free field time-evolution.

Chapter 3

Engineering entanglement by optical beam splitters

3.1 Introduction

The optical beam splitters play a crucial role in the understanding of many phenomena concerning optics and their applications in advanced areas of research, such as, quantum optics, quantum information and quantum computation. For example in the quantum information theory, these passive optical devices are essentially used in entanglement generation, teleportation and Bell's measurements. Moreover, they are the constituent components of several other optical devices, such as, interferometers which are essentially used in precision quantum measurements by observing the interference pattern.

Engineering entanglement is an important task in the quantum information theory that provides a basis for a secure communication and a fast computation. In this chapter, we discuss the theory of beam splitters in the context of entanglement generation. In particular, we try to find the circumstances under which a beam splitter can be used as an entangler.

The chapter is organized as following. In section(3.2), we discuss the quantum mechanical description of optical beam splitters which provide the relationship between input modes and output modes. Section(3.3) is focussed on the entanglement of output states of a beam splitter by injecting different kinds of optical states through inputs. Finally, the section(3.4) deals with the description of a general criterion for using a beam splitter as an entangler.

3.2 Quantum theory of optical beam splitters

Before proceeding to the quantum theory, let us recall the action of beam splitter when we consider the classical scenario. Let us consider an optical field with complex amplitude $\mathcal{E}_1$ incident upon a beam splitter [44]. $\mathcal{E}_2$ and $\mathcal{E}_3$ are the amplitudes of the reflected and transmitted beams respectively. If R and T are the (complex) reflectance and transmittance respectively of the beam splitter, then it follows

$$\mathcal{E}_2 = R\mathcal{E}_1 \qquad and \qquad \mathcal{E}_3 = T\mathcal{E}_1, \tag{3.2.1}$$

for a 50:50 beam splitter we would have $|R| = |T| = \frac{1}{\sqrt{2}}$. However, for the sake of generality, we do not impose this condition here. Since the beam splitter is assumed lossless, the intensity of the input beam should equal the sum of the intensities of the two output beams as

$$\mathcal{E}_1 = {\mathcal{E}_2}^2 + {\mathcal{E}_3}^2, \tag{3.2.2}$$

which requires that

$$|R|^2 + |T|^2 = 1. \tag{3.2.3}$$

To treat the beam splitter quantum mechanically we might try replacing the classical complex field amplitudes $\mathcal{E}$ by a set of annihilation operators $\hat{a}_i (i = 1, 2, 3)$, in analogy with the classical case we might try setting

$$\hat{a}_2 = R\hat{a}_1 \quad and \quad \hat{a}_3 = T\hat{a}_1, \tag{3.2.4}$$

In the classical picture of the beam splitter there is an unused port which is empty there is no field on this port. However, in the quantum-mechanical picture, the unused port still contains a quantized field mode in the vacuum state and we see that fluctuations of the vacuum exhibits important physical effects. In Fig (3.1) we indicate all the modes required proper quantum description of the beam splitter. We now write the beam-splitter transformations for the field operators as

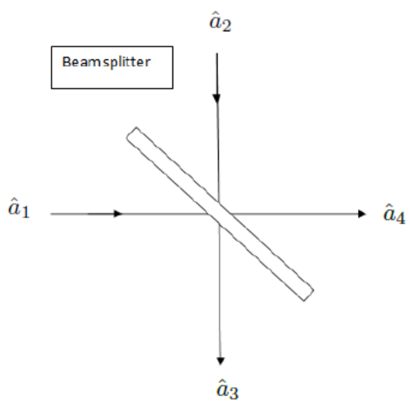

Figure 3.1: Quantum-mechanical depiction of a beam splitter.

$$\begin{aligned} \hat{a}_3 &= R\hat{a}_2 + T'\hat{a}_1, \\ \hat{a}_4 &= T\hat{a}_2 + R'\hat{a}_1, \end{aligned} \tag{3.2.5}$$

or collectively as

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} T' & R \\ R' & T \end{pmatrix} * \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix}, \tag{3.2.6}$$

where $|R|' = |R|$, $|T|' = |T|$, $|R|^2 + |T|^2 = 1$, Let us examine a couple of relevant examples. The phase shifts of the reflected and transmitted beams depend on the construction of the beam splitter [82]. If the beam splitter is constructed as a single dielectric layer, the reflected and transmitted beams will differ in phase by a factor of $exp(\pm\frac{i\pi}{2}) = \pm i$, then our device in this case is called as 50:50 beam splitter, assuming the reflected beam suffers a $\frac{\pi}{2}$ phase shift and also $|R|^2 + |T|^2 = 1$, the input and output modes are related according to

$$\hat{a}_3 = \frac{\hat{a_1} + i\hat{a_2}}{\sqrt{2}},$$
$$\hat{a}_4 = \frac{i\hat{a_1} + \hat{a_2}}{\sqrt{2}}, \tag{3.2.7}$$

the inverse of this transformation is even easier to use

$$\hat{a}_1 = \frac{\hat{a_3} - i\hat{a_4}}{\sqrt{2}},$$
$$\hat{a}_2 = \frac{-i\hat{a_3} + \hat{a_4}}{\sqrt{2}}. \tag{3.2.8}$$

3.3 Optical beam splitting and entanglement

Using the optical beam splitter we study the entanglement generation of fock states and coherent states. In this work, we focus on the analysis of the entangled states of single mode of the electromagnetic field. Optical coherent states on the optical beam splitters get entangled or not. Here, we are considering the ordinary or standard optical coherent state call as Glauber coherent states taken as one of the input on optical beam splitter and vacuum state on other port of beam splitter and check the output state whether it is entangled or not. Then we apply both Glauber coherent states on the both input ports of beam splitter and examine whether it is entangled or not.

3.3.1 Fock states as inputs of beam splitter

We are taking the fock states as inputs of optical beam splitters. Firstly we are taking the multi-photon states with vacuum state as inputs of beam splitter and generates the output state. Secondly, we are taking the multi-photon states on the both input ports of beam splitter and generates the output.

A Single photon and vacuum state:

For a given input state to the beam splitter, what is the output state. Remembering that all photon number states $|n\rangle$, hence any superposition or any statistical mixture of such states, may be constructed by the action of n powers of the creation operator on the vacuum, we may use Eq (3.2.7) to construct the output states from the action of the transformed creation operators on the vacuum states of the output modes, it being obvious that an input vacuum transforms to an output vacuum $|0\rangle_1|0\rangle_2 \rightarrow |0\rangle_3|0\rangle_4$
As an example, consider the single photon input state $|1\rangle_1|0\rangle_2$ which we may write as $\hat{a}_1^\dagger|0\rangle_1|0\rangle_2$, for the

beam splitter described by

$$\begin{aligned}\hat{a_1} &= \frac{\hat{a_2}+i\hat{a_3}}{\sqrt{2}},\\ \hat{a_1}^{\dagger} &= \frac{\hat{a_2}^{\dagger}+i\hat{a_3}^{\dagger}}{\sqrt{2}},\\ \hat{a}_1^{\dagger}|0\rangle_1|0\rangle_2 &\overset{BS}{\to} \frac{1}{\sqrt{2}}(i|1\rangle_3|0\rangle_4+|0\rangle_3|1\rangle_4),\end{aligned} \tag{3.3.1}$$

We analyzed that a single-photon incident on one of the input ports of the 50:50 beam splitter, the other port containing only the vacuum, will be either transmitted or reflected with equal probability. Of course, this is precisely as we earlier claimed and explains why no coincident counts are to be expected with photon counters placed at the outputs of the beam splitter, as confirmed by the experiment of Grangier et al. [87]. An other point needs to be made about the output state. It is an entangled state, it cannot be written as a simple product of states of the individual modes 2 and 3. For the sake of generality it is also written as

$$|out\rangle| = T|0\rangle_3|1\rangle_4 + R|1\rangle_3|0\rangle_4. \tag{3.3.2}$$

Similar considerations will allow us to obtain the output corresponding to the Fock state as input state, Input fields of same number of photons, coherent state and vacuum state as inputs and input fields of same coherent states i.e.classical-like state, Glauber coherent states and non-classical, Barut Girardello coherent states through the optical beam splitter. In order to obtain the output state as the result of a optical beam splitter on theinput fields, we introduced the effect of a beam splitter on a number state that is $|\psi\rangle_{in} = |n\rangle$ with the vacuum at the second input port. Then take the input fields of same number of photons at the input modes of beam splitter.

Multi-photon number state and vacuum state:

We firstly examine the results of beam splitter for number state. If we apply the number state (fock state) $|n\rangle$, on the one input port of beam splitter and vacuum $|0\rangle$ on the other input port of beam splitter [87, 88, 91, 92, 93].

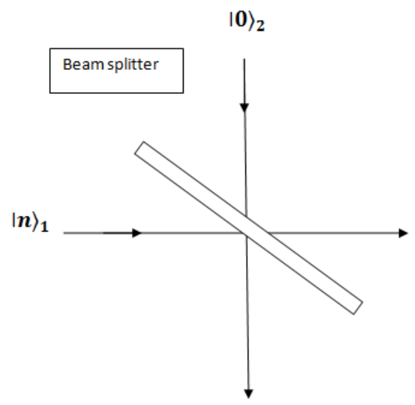

Figure 3.2: A beam splitter with a state $|n\rangle$, number state on the hone input port and a vacuum state $|0\rangle$ on the other port

$$|\psi\rangle_{in} = |n\rangle_1 |0\rangle_2, \tag{3.3.3}$$

all photon number states $|n\rangle$, hence any superposition or any statistical mixture of such states, may be constructed by the action of n powers of the creation operator on the vacuum, the action of the transformed creation operators on the vacuum states of the output modes, it being obvious that an input vacuum transforms to an output vacuum $|0\rangle_1|0\rangle_2 \rightarrow |0\rangle_3|0\rangle_4$. We construct the output state by first rewriting

$$\begin{aligned} |\psi\rangle_{in} &= \frac{(a_1{}^{\dagger})^n}{\sqrt{n!}}|0\rangle_1|0\rangle_2, \\ &= \frac{1}{\sqrt{n!}}(\frac{a_3{}^{\dagger} + ia_4{}^{\dagger}}{\sqrt{2}})^n|0\rangle_3|0\rangle_4. \end{aligned}$$

Apparently, the two or more photons emerge together such that photo-detectors placed in the output beams should not register simultaneous counts. But unlike the case of a single incident photon, the physical basis for obtaining no simultaneous counts is not a result of the particle-like nature of photons. Rather, it is caused by interference (a wave-like effect) between two possible ways of obtaining the (absent) output state the process where both photons are transmitted and the process where they are both reflected. Note the indistinguishability of the two processes for the output state. There is a simple and rather intuitive way of understanding this result. Recall Feynman's rule [94] for obtaining the probability for an outcome that can occur by several indistinguishable processes. One simply adds the probability amplitudes of all the processes and then calculates the square of the modulus. It may be attempting to interpret the result as owing to the bosonic nature of photons, a kind of clustering in the sense of Bose Einstein condensation (BEC). Indeed, in the case of fermions, such as in neutron interferometry, the output of a beam splitter for the corresponding input would find the fermions always in different beams in accordance with the Pauli exclusion principle. Of course, this behavior and that of the photons are linked to the statistical properties of the particles. Using Binomial theorem

$$(x+y)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k y^{(n-k)}, \tag{3.3.4}$$

we can expand $(\frac{a_3{}^{\dagger}+ia_4{}^{\dagger}}{\sqrt{2}})^n$ Using binomial theorem so we have

$$\begin{aligned} |\psi\rangle_{out} &= \sum_{k=0}^{n} (\frac{1}{\sqrt{n!}} \binom{n}{k} (\frac{1}{\sqrt{2}} a_3{}^{\dagger})^k (\frac{i}{\sqrt{2}} a_4{}^{\dagger})^{n-k}) |0\rangle_3 |0\rangle_4, \\ &= \sum_{k=0}^{n} (\frac{1}{\sqrt{n!}} \binom{n}{k} (\frac{1}{\sqrt{2}})^k (a_3{}^{\dagger})^k (\frac{i}{\sqrt{2}})^{(n-k)} (a_4{}^{\dagger})^{n-k}) |0\rangle_3 |0\rangle_4, \end{aligned}$$

for 50:50 beam splitter, $(\frac{1}{\sqrt{2}}) = T$ and $(\frac{i}{\sqrt{2}}) = R$

$$|\psi\rangle_{out} = \sum_{k=0}^{n} (\frac{1}{\sqrt{n!}} \binom{n}{k} T^k (a_3{}^{\dagger})^k R^{(n-k)} (a_4{}^{\dagger})^{n-k}) |0\rangle_3 |0\rangle_4, \tag{3.3.5}$$

$$= \sum_{k=0}^{n} T^k R^{(n-k)} \frac{1}{\sqrt{n!}} \binom{n}{k} (a_3{}^{\dagger})^k (a_4{}^{\dagger})^{(n-k)} |0\rangle_3 |0\rangle_4, \tag{3.3.6}$$

multiplying and dividing by $\sqrt{k!}\sqrt{(n-k)!}$

$$|\psi\rangle_{out} = \sum_{k=0}^{n} T^k R^{(n-k)} \frac{1}{\sqrt{n!}} \binom{n}{k} \frac{\sqrt{k!}\sqrt{(n-k)!}}{\sqrt{k!}\sqrt{(n-k)!}} (a_3{}^{\dagger})^k . (a_4{}^{\dagger})^{(n-k)} |0\rangle_3 |0\rangle_4,$$

aexpanding the binomial terms as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ we get

$$\begin{aligned} |\psi\rangle_{out} &= \sum_{k=0}^{n} T^k R^{(n-k)} \frac{1}{\sqrt{n!}} \frac{n!}{k!(n-k)!} . \frac{\sqrt{k!}\sqrt{(n-k)!}}{\sqrt{k!}\sqrt{(n-k)!}} (a_3{}^{\dagger})^k (a_4{}^{\dagger})^{(n-k)} |0\rangle_3 |0\rangle_4, \\ &= \sum_{k=0}^{n} T^k R^{(n-k)} \sqrt{\frac{n!}{k!(n-k)!}} \frac{(a_3{}^{\dagger})^k}{\sqrt{k!}} . \frac{(a_4{}^{\dagger})^{(n-k)}}{\sqrt{(n-k)!}} |0\rangle_3 |0\rangle_4, \\ &= \sum_{k=0}^{n} T^k R^{(n-k)} \sqrt{\frac{n!}{k!(n-k)!}} \frac{(a_3{}^{\dagger})^k}{\sqrt{k!}} |0\rangle_3 \frac{(a_4{}^{\dagger})^{(n-k)}}{\sqrt{(n-k)!}} |0\rangle_4 \end{aligned}$$

as we know that $\frac{(a_3{}^{\dagger})^k}{\sqrt{k!}}|0\rangle = |k\rangle$ and $\frac{(a_4{}^{\dagger})^{(n-k)}}{\sqrt{(n-k)!}}|0\rangle = |n-k\rangle$ hence we get the required output state

$$|\psi\rangle_{out} \overset{BS}{\rightarrow} \sum_{k=0}^{n} \sqrt{\binom{n}{k}} T^k R^{(n-k)} |k\rangle_3 |n-k\rangle_4$$

replace k with p

$$|\psi\rangle_{out} \overset{BS}{\rightarrow} \sum_{p=0}^{n} \sqrt{\binom{n}{p}} T^p R^{(n-p)} |p\rangle_3 |n-p\rangle_4,$$

we get

$$|\psi\rangle_{out} \overset{BS}{\rightarrow} \sum_{p=0}^{n} \sqrt{\binom{n}{p}} (\frac{1}{\sqrt{2}})^p (\frac{i}{\sqrt{2}})^{(n-p)} |p\rangle_3 |n-p\rangle_4, \tag{3.3.7}$$

the measure of entanglement is a convex function with its maximum for a 50:50 beam splitter, i.e., $R = T = \frac{1}{\sqrt{2}}$. In particular, when $n = 1$ the output state is $\hat{U}|1,1\rangle = \frac{1}{\sqrt{2}}|0,2\rangle| + \exp i\phi|2,0\rangle$ for a 50:50 beam splitter [95, 96], it is entangled state.

Multi-photon number states on both inputs:

If we have Multi-photon number states on both input ports of 50 : 50 beam splitter then the beam splitter transformation is

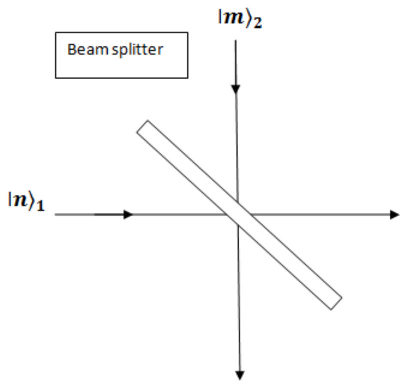

Figure 3.3: A beam splitter with a state $|n\rangle$, number state on the one input port and an other number state $|m\rangle$ on the other port

$$|\psi\rangle_{in} = |n\rangle_1 |m\rangle_2, \tag{3.3.8}$$

then we construct the output state by first rewriting

$$\begin{aligned}
|\psi\rangle_{in} &= \frac{({a_2}^{\dagger})^n}{\sqrt{n!}}.\frac{({a_2}^{\dagger})^m}{\sqrt{m!}}|0\rangle_1|0\rangle_2, \\
|\psi\rangle_{out} &= \frac{1}{\sqrt{n!}}(\frac{{a_3}^{\dagger}+i{a_4}^{\dagger}}{\sqrt{2}})^n.\frac{1}{\sqrt{m!}}(\frac{i{a_3}^{\dagger}+{a_4}^{\dagger}}{\sqrt{2}})^m|0\rangle_3|0\rangle_4,
\end{aligned}$$

using Binomial theorem $(x+y)^n = \sum\limits_{k=0}^{\infty}\binom{n}{k}x^k y^{(n-k)}$, we can expand $(\frac{{a_3}^{\dagger}+i{a_4}^{\dagger}}{\sqrt{2}})^n$ and $(\frac{i{a_3}^{\dagger}+{a_4}^{\dagger}}{\sqrt{2}})^m$

$$\begin{aligned}
|\psi\rangle_{out} &= \sum_{k=0}^{n}(\frac{1}{\sqrt{n!}}\binom{n}{k}(\frac{1}{\sqrt{2}}{a_3}^{\dagger})^k(\frac{i}{\sqrt{2}}{a_4}^{\dagger})^{n-k})\sum_{l=0}^{m}(\frac{1}{\sqrt{m!}}\binom{m}{l}(\frac{i}{\sqrt{2}}{a_3}^{\dagger})^l * (\frac{1}{\sqrt{2}}{a_4}^{\dagger})^{m-l})|0\rangle_3|0\rangle_4, \\
&= (\sum_{k=0}^{n}(\frac{1}{\sqrt{n!}}\binom{n}{k}(\frac{1}{\sqrt{2}})^k({a_3}^{\dagger})^k(\frac{i}{\sqrt{2}})^{(n-k)}({a_4}^{\dagger})^{n-k})) \\
&\quad .(\sum_{l=0}^{m}(\frac{1}{\sqrt{m!}}\binom{m}{k}(\frac{i}{\sqrt{2}})^l({a_3}^{\dagger})^l(\frac{1}{\sqrt{2}})^{(m-l)}({a_4}^{\dagger})^{m-l}))|0\rangle_3|0\rangle_4,
\end{aligned}$$

replace m with n we get

$$\begin{aligned}
|\psi\rangle_{out} &= \sum_{k=0}^{n}\sum_{l=0}^{n}\frac{1}{n!}\binom{n}{k}\binom{n}{l}(T)^{(k+n-l)}(R)^{(n-k)+l} \\
&\quad ({a_3}^{\dagger})^{k+l}({a_4}^{\dagger})^{n-k+n-l}|0\rangle_3|0\rangle_4,
\end{aligned} \tag{3.3.9}$$

put $l+k=2m$ and $l=2m-k$ as $l:0\to n$ then $k:0\to n$ $m:0\to\infty$

$$|\psi\rangle_{out} = \sum_{k=0}^{n}\sum_{m=0}^{n}\frac{1}{n!}\binom{n}{k}\binom{n}{2m-k}(T)^{(n-2m+2k)} * (R)^{(n+2m-2k)}$$

$$({a_3}^{\dagger})^{2m}({a_4}^{\dagger})^{(2n-2m)}|0\rangle_3|0\rangle_4 \tag{3.3.10}$$

multiplying and dividing by $\sqrt{2m!(2n-2m)!}$ and re-arranging the above equation, we get

$$\begin{aligned} |\psi\rangle_{out} &= \sum_{k=0}^{n}\sum_{m=0}^{n} \frac{1}{n!}\binom{n}{k}\binom{n}{2m-k}(T)^{(n-2m+2k)}(R)^{(n+2m-2k)} \\ &\quad \sqrt{2m!(2n-2m)!}\frac{(a_3{}^{\dagger})^{2m}}{\sqrt{2m!}}|0\rangle_3 \frac{(a_4{}^{\dagger})^{2n-2m}}{\sqrt{(2n-2m)!}}|0\rangle_4, \end{aligned}$$

$\frac{(a_3{}^{\dagger})^{2m}}{\sqrt{2m!}}|0\rangle = |2m\rangle$ and $\frac{(a_4{}^{\dagger})^{(2n-2m)}}{\sqrt{(2n-2m)!}}|0\rangle = |2n-2m\rangle$, so the resultant output state is

$$\begin{aligned} |\psi\rangle_{out} &= \sum_{k=0}^{n}\sum_{m=0}^{n} \binom{n}{k}\binom{n}{2m-k}(T)^{(n-2m+2k)}(R)^{(n+2m-2k)} \\ &\quad \frac{\sqrt{2m!(2n-2m)!}}{n!}|2m\rangle_3|2n-2m\rangle_4, \end{aligned} \tag{3.3.11}$$

hence the resultant output state when two number states are at the two input ports of beam splitter. Replace k with variable p and m with m' we get

$$\begin{aligned} |n\rangle|m\rangle &\overset{BS}{\to} \sum_{p=0}^{n}\sum_{m'=0}^{n} \binom{n}{p}\binom{n}{2m'-p}(T)^{(n-2m'+2p)}(R)^{(n+2m'-2p)} \\ &\quad \frac{\sqrt{2m'!(2n-2m')!}}{n!}|2m'\rangle_3|2n-2m'\rangle_4. \end{aligned} \tag{3.3.12}$$

the resultant output state is entangled [51].

3.3.2 Coherent states as inputs of beam splitter

If we take the coherent states as input of beam splitter we will examine the output states. It will get entangle or not after passing through the beam splitter.

A coherent state and vacuum state:

If we have the coherent state $|z\rangle$ on the one input port and vacuum state $|0\rangle$ at the other input port of beam splitter then the effect of beam splitter on these state is as follows

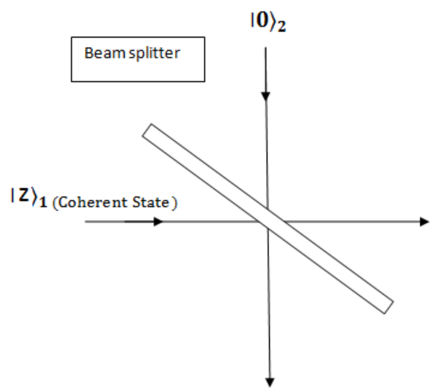

Figure 3.4: A beam splitter with $|z\rangle$, coherent state on the one input port and a vacuum state $|0\rangle$ on the other input port

$$|\psi\rangle_{in} = |z\rangle_1|0\rangle_2, \tag{3.3.13}$$

Glauber coherent state is written as

$$|z\rangle = \exp\left(-\frac{|z|^2}{2}\right)\sum_{n=0}^{\infty}\frac{|z|^n}{\sqrt{n!}}|n\rangle \tag{3.3.14}$$

comparing with this equation

$$|z\rangle = \sum_{n=0}^{\infty} C_n|n\rangle \tag{3.3.15}$$

for Glauber coherent state C_n is

$$C_n = exp\left(-\frac{|z|^2}{2}\right)\frac{|z|^n}{\sqrt{n!}},$$

now

$$|\psi\rangle_{in} = \sum_{n=0}^{\infty} C_n|n\rangle_1|0\rangle_2, \tag{3.3.16}$$

where it is evident that we have a superposition of the product states $|n\rangle_1|0\rangle_2$, because the beam splitter conserves the total number of photons, we deduce the beam-splitter transformation for the input states $|n\rangle_1|0\rangle_2$. then we construct the output state by first rewriting

$$\begin{aligned}|\psi\rangle_{in} &= \sum_{n=0}^{\infty} C_n\frac{(a_1{}^{\dagger})^n}{\sqrt{n!}}|n\rangle_1|0\rangle_2,\\ &= \sum_{n=0}^{\infty} C_n\frac{1}{\sqrt{n!}}(\frac{a_3{}^{\dagger}+ia_4{}^{\dagger}}{\sqrt{2}})^n|0\rangle_3|0\rangle_4,\end{aligned}$$

using Binomial theorem $(x+y)^n = \sum_{k=0}^{\infty}\binom{n}{k}x^k y^{(n-k)}$, we can expand $(\frac{a_3{}^{\dagger}+ia_4{}^{\dagger}}{\sqrt{2}})^n$

$$\begin{aligned}|\psi\rangle_{out} &= \sum_{n=0}^{\infty} C_n\sum_{k=0}^{n}(\frac{1}{\sqrt{n!}}\binom{n}{k}(\frac{1}{\sqrt{2}}a_3{}^{\dagger})^k(\frac{i}{\sqrt{2}}a_4{}^{\dagger})^{n-k})|0\rangle_3|0\rangle_4,\\ &= \sum_{n=0}^{\infty} C_n\sum_{k=0}^{n}(\frac{1}{\sqrt{n!}}\binom{n}{k}(\frac{1}{\sqrt{2}})^k(a_3{}^{\dagger})^k(\frac{i}{\sqrt{2}})^{(n-k)}(a_4{}^{\dagger})^{n-k})|0\rangle_3|0\rangle_4,\end{aligned}$$

$(\frac{1}{\sqrt{2}}) = T$ and $(\frac{i}{\sqrt{2}}) = R$ are the transmission and reflection coefficients for 50:50 beam splitter.

$$|\psi\rangle_{out} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_n T^k R^{(n-k)}\frac{1}{\sqrt{n!}}\binom{n}{k}(a_3{}^{\dagger})^k.(a_4{}^{\dagger})^{(n-k)}|0\rangle_3|0\rangle_4 \tag{3.3.17}$$

multiplying and dividing by $\sqrt{k!}\sqrt{(n-k)}!$

$$|\psi\rangle_{out} = \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_n T^k R^{(n-k)}\frac{1}{\sqrt{n!}}\binom{n}{k}\frac{\sqrt{k!}\sqrt{(n-k)}!}{\sqrt{k!}\sqrt{(n-k)}!}(a_3{}^{\dagger})^k * (a_4{}^{\dagger})^{(n-k)}|0\rangle_3|0\rangle_4, \tag{3.3.18}$$

as $\binom{n}{k} = \frac{n!}{k!(n-k)!}$,

$$\begin{aligned}|\psi\rangle_{out} &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_n T^k R^{(n-k)}\\ &\quad \sqrt{\frac{n!}{k!(n-k)!}} * \frac{(a_3{}^{\dagger})^k}{\sqrt{k!}}\frac{(a_4{}^{\dagger})^{(n-k)}}{\sqrt{(n-k)!}}|0\rangle_3|0\rangle_4,\end{aligned}$$

as we know that $\frac{(a_3{}^\dagger)^k}{\sqrt{k!}}|0\rangle = |k\rangle$ and $\frac{(a_4{}^\dagger)^{(n-k)}}{\sqrt{(n-k)!}}|0\rangle = |n-k\rangle$, hence the required output state

$$|z\rangle_1|0\rangle_2 \quad \overset{BS}{\rightarrow} \sum_{n=0}^{\infty}\sum_{k=0}^{n} C_n \sqrt{\binom{n}{k}} T^k R^{(n-k)} |k\rangle_3 |n-k\rangle_4, \tag{3.3.19}$$

if we replace variable k with p then the resultant output state is

$$|z\rangle_1|0\rangle_2 \quad \overset{BS}{\rightarrow} \sum_{n=0}^{\infty}\sum_{p=0}^{n} C_n \sqrt{\binom{n}{p}} T^p R^{(n-p)} |p\rangle_3 |n-p\rangle_4, \tag{3.3.20}$$

hence the resultant output state when coherent state is at the one input port of beam splitter and vacuum state is at the other input port of beam splitter. These are the general results for coherent state and vacuum state at the inputs ports of beam splitter. In this chapter taking the different examples of classical-like and non-classical states as input of optical beam splitter, we show wether the output state of optical beam splitter is entangled or not. Let us consider two more examples of beam splitting. First we consider a coherent state, a classical-like state, rather the opposite of the highly nonclassical state, incident on the beam splitter with again only the vacuum in the other input port.e.g. When Glauber coherent states and Vacuum States are at the Input Ports of Beam Splitter. Then we consider non-classical state, Barut-Girardello coherent states incident on the beam splitter with again only the vacuum in the other input port [47] which we discuss in next chapter. We are considering a coherent state, Glauber coherent state a classical-like state, rather the opposite of the highly nonclassical single-photon state, incident on the beam splitter with the vacuum in the other input port [91, 92].

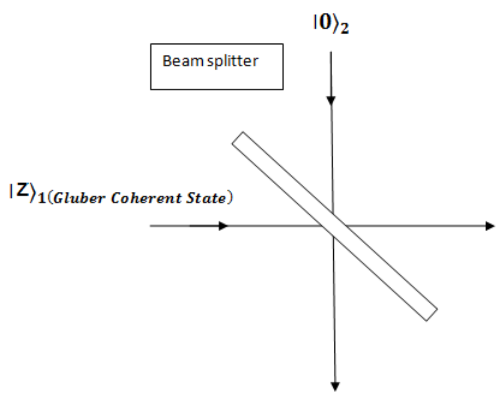

Figure 3.5: A beam splitter with a state $|z\rangle_1$, Glauber coherent state on the one input port and a vacuum state $|0\rangle$ on the other input port

for Glauber coherent state C_n is

$$C_n = exp\bigg(-\frac{|z|^2}{2}\bigg)\frac{|z|^n}{\sqrt{n!}},$$

where it is evident that we have a superposition of the product states $|n\rangle_1|0\rangle_2$. Because the beam splitter conserves the total number of photons, we deduce the beam-splitter transformation for the input states

$|z\rangle_1|0\rangle_2$. from equation (3.3.20)

$$|z\rangle_1|0\rangle_2 \quad \overset{BS}{\rightarrow} \sum_{n=0}^{\infty}\sum_{p=0}^{n} C_n \sqrt{\binom{n}{p}} T^p R^{(n-p)} |p\rangle_3 |n-p\rangle_4, \tag{3.3.21}$$

putting $C_n = exp(-\frac{|z|^2}{2})\frac{|z|^n}{\sqrt{n!}}$ we get the output state when Glauber coherent state and vacuum state is at the input ports of beam splitter given as below

$$|z\rangle_1|0\rangle \quad \overset{BS}{\rightarrow} exp(-\frac{|z|^2}{2})\sum_{n=0}^{\infty}\sum_{p=0}^{n} \sqrt{\binom{n}{p}} \frac{|z|^n}{\sqrt{n!}} T^p R^{(n-p)} |p\rangle_3 |n-p\rangle_4. \tag{3.3.22}$$

We obtain the result expected for a classical light wave where the incident intensity is evenly divided between the two output beams, e.g. half the incident average photon number $exp(-\frac{|z|^2}{2})$, emerges in each beam. We also naturally obtain the phase shift for the reflected wave $\frac{\pi}{2}$, as expected. What is about the output state of beam splitter is it entangled or separable state. If the output state is separable (or factorizable) then there is no entanglement. using output of beam splitter from equation (3.3.22)

$$\begin{aligned}
|out\rangle \quad & \overset{BS}{\rightarrow} \quad exp(-\frac{|z|^2}{2})\sum_{n=0}^{\infty}\sum_{p=0}^{n} \sqrt{\binom{n}{p}} \frac{|z|^n}{\sqrt{n!}} T^p R^{(n-p)} |p\rangle_3 |n-p\rangle_4, \\
& = \quad exp(-\frac{|z|^2}{2})\sum_{n=0}^{\infty}\sum_{p=0}^{n} \sqrt{\frac{n!}{p!(n-p)!}} \frac{|z|^n}{\sqrt{n!}} T^p R^{(n-p)} |p\rangle_3 |n-p\rangle_4, \\
& = \quad exp(-\frac{|z|^2}{2})\sum_{p=0}^{\infty} \frac{|z|^p T^p}{\sqrt{p!}} |p\rangle_3 \sum_{(n-p)=0}^{\infty} \frac{|z|^{(n-p)} R^{(n-p)}}{\sqrt{(n-p!)}} |n-p\rangle_4, \\
& = \quad exp(-\frac{|z|^2}{2})\sum_{p=0}^{\infty} \frac{|z|^p (\frac{1}{\sqrt{2}})^p}{\sqrt{p!}} |p\rangle_3 \sum_{(n-p)=0}^{\infty} \frac{|z|^{(n-p)} (\frac{1}{\sqrt{2}})^{(n-p)}}{\sqrt{(n-p!)}} |n-p\rangle_4, \\
& = \quad |\frac{z}{\sqrt{2}}\rangle_3 |\frac{iz}{\sqrt{2}}\rangle_4,
\end{aligned}$$

the output state is written as simple product of the individual states [90]. There is no entanglement generation because one of the input state on the input port of beam splitter is classical like with vacuum state on the other port of beam splitter.

Coherent states on both inputs:

If coherent states are at the both input port of beam splitter.

$$|\psi\rangle_{in} = |z\rangle_1 |z\rangle_2, \tag{3.3.23}$$

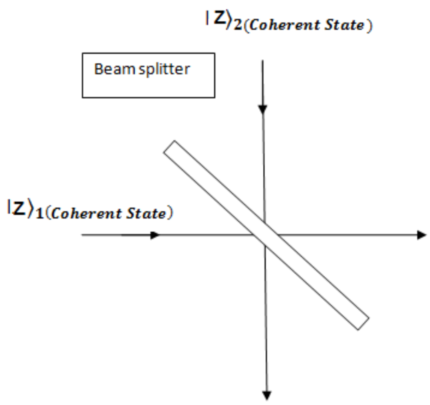

Figure 3.6: A beam splitter with a state $|z\rangle_1$, coherent state on the horizontal port and a coherent state $|z\rangle_2$ on the vertical port

$$|z\rangle_1|z\rangle_2 = \sum_{n=0}^{\infty} C_n|n\rangle \sum_{m=0}^{\infty} C_m|m\rangle, \tag{3.3.24}$$

by calculation we get

$$\begin{aligned} |n\rangle_1|m\rangle_2 \quad & \overset{BS}{\rightarrow} \quad \sum_{p=0}^{n}\sum_{m'=0}^{n} \binom{n}{p}\binom{n}{2m'-p}(T)^{(n-2m')+2p}(R)^{(n+2m'-2p)} \\ & \frac{\sqrt{2m'!(2n-2m')!}}{n!}|2m'\rangle_3|2n-2m'\rangle_4, \end{aligned} \tag{3.3.25}$$

for both coherent states as input of 50:50 beam splitter

$$\begin{aligned} |z\rangle_1|z\rangle_2 \quad & \overset{BS}{\rightarrow} \quad \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{n}\sum_{m'=0}^{n} C_nC_m\binom{n}{p}\binom{n}{2m'-p}(T)^{(n-2m'+2p)}(R)^{(n+2m'-2p)} \\ & \frac{\sqrt{2m'!(2n-2m')!}}{n!}|2m'\rangle_3|2n-2m'\rangle_4. \end{aligned} \tag{3.3.26}$$

Hence, the resultant state when there are both coherent states in general on the two input ports of beam splitter. If Glauber coherent states are on the both input port of beam splitter:

$$|\psi\rangle_{in} = |z\rangle_1|z\rangle_2, \tag{3.3.27}$$

Glauber coherent state is

$$|z\rangle_1 = \exp\left(-\frac{|z|^2}{2}\right)\sum_{n=0}^{\infty}\frac{|z|^n}{\sqrt{n!}}|n\rangle, \tag{3.3.28}$$

comparing with the equation given below

$$|z\rangle = \sum_{n=0}^{\infty} C_n|n\rangle, \tag{3.3.29}$$

we have

$$C_n = exp\left(-\frac{|z|^2}{2}\right)\frac{|z|^n}{\sqrt{n!}}.$$

And comparing with the equation given below

$$|z\rangle_2 = \exp\left(-\frac{|z|^2}{2}\right)\sum_{m=0}^{\infty}\frac{|z|^m}{\sqrt{m!}}|m\rangle \tag{3.3.30}$$

we get

$$C_m = exp(-\frac{|z|^2}{2})\frac{|z|^m}{\sqrt{m!}}, \tag{3.3.31}$$

using the results of equation (3.3.26) putting $C_n = exp(-\frac{|z|^2}{2})\frac{|z|^n}{\sqrt{n!}}$ and $C_m = exp(-\frac{|z|^2}{2})\frac{|z|^m}{\sqrt{m!}}$ we get output state

$$\begin{aligned} |z\rangle_1|z\rangle_2 \quad \overset{BS}{\to} \quad & \exp(-|z|^2)\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{n}\sum_{m'=0}^{n}\frac{|z|^n}{\sqrt{n!}} * \frac{|z|^m}{\sqrt{m!}} \\ & \binom{n}{p}\binom{n}{2m'-p}(T)^{(n-2m'+2p)} * (R)^{(n+2m'-2p)} \\ & \frac{\sqrt{2m'!(2n-2m')!}}{n!}|2m'\rangle_3|2n-2m'\rangle_4, \end{aligned} \tag{3.3.32}$$

is the required output state. The output state is not entangled becuase both states are classical-like states on the input ports of beam splitter.

3.4 Criteria for entanglement generation

There is some criteria for entanglement generation. The output state of beam splitter is entangled if input state exhibiting nonclassicality in it. We illustrate this by beam splitter entangler theorem.

3.4.1 Theorem for the beam-splitter entangler

The Optical beam splitter is an optical device which is also act as entangler. The entangler properties of a optical beam splitter have been studied in the past [89]. In particular, Kim et al. [45] studied the entangler properties with many different input states, such as a Fock (number) state, a coherent state, a squeezed state, and mixed states in Gaussian form [45]. We consider in terms of density matrix, ρ_{in} and ρ_{out} are the density operators for the input and output states, respectively. Both of them are two-mode states including mode a and mode b.

$$\rho_{out} = \hat{U}\rho_{in}\hat{U}^\dagger, \tag{3.4.1}$$

$$\hat{U}^\dagger = \hat{U}^{-1}, \tag{3.4.2}$$

$$\rho_{out} = \hat{U}\rho_{in}\hat{U}^{-1}, \tag{3.4.3}$$

now action of beam-splitter operator on (two mode) vacuum state for both mode a and b.

$$\hat{U}|00\rangle = |00\rangle, \tag{3.4.4}$$

equation is due to the simple fact of no input, no output. Without any loss of generality, we can express ρ_{in} in the P-representation in the following form:

$$\rho_{out} = \int_{-\infty}^{\infty} P(z_a, z_b, {z_a}^*, {z_b}^*)|z_a, z_b\rangle\langle z_a, z_b|d^2z_a d^2z_b, \tag{3.4.5}$$

where $|z_a, z_b\rangle$ is a coherent state in two-mode Fock space, i.e.

$$|z_a, z_b\rangle = \hat{D}_{ab}(z_a, z_b)|00\rangle, \tag{3.4.6}$$

$$|z_a, z_b\rangle = \hat{D}_{ab}(z_a, z_b)|00\rangle, \tag{3.4.7}$$

$$\hat{D}_{ab}(z_a, z_b) = e^{\hat{a}^\dagger z_a - \hat{a} z_a{}^* + \hat{b}^\dagger z_b - \hat{b} z_b{}^*}, \tag{3.4.8}$$

so

$$|z_a, z_b\rangle = e^{\hat{a}^\dagger z_a - \hat{a} z_a{}^* + \hat{b}^\dagger z_b - \hat{b} z_b{}^*}|00\rangle, \tag{3.4.9}$$

$$|z_a, z_b\rangle = e^{\hat{a}^\dagger z_a - \hat{a} z_a{}^*}|0\rangle * e^{\hat{b}^\dagger z_b - \hat{b} z_b{}^*}|0\rangle, \tag{3.4.10}$$

consider the identity (the disentangling theorem)

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}} e^{\hat{B}} e^{-\frac{1}{2}[\hat{A},\hat{B}]}, \tag{3.4.11}$$

$$e^{\hat{A}+\hat{B}} = e^{\hat{B}} e^{\hat{A}} e^{\frac{1}{2}[\hat{A},\hat{B}]}, \tag{3.4.12}$$

$$[\hat{A}, \hat{B}] = |z_a|^2, \tag{3.4.13}$$

so

$$\begin{aligned} D(z_a) &= e^{\hat{a}^\dagger z_a - \hat{a} z_a{}^*} \\ &= e^{-\frac{1}{2}|z_a|^2} e^{\hat{a}^\dagger z_a} e^{\hat{a} z_a{}^*}, \end{aligned} \tag{3.4.14}$$

as we know that $e^{\hat{a} z_a{}^*}|0\rangle = 0$ and

$$e^{\hat{a}^\dagger z_a}|0\rangle = \frac{(z_a \hat{a}^\dagger)^n}{n!}|0\rangle, \tag{3.4.15}$$

$$e^{\hat{a}^\dagger z_a}|0\rangle = \sum_{n=0}^{\infty} \frac{z_a{}^n}{\sqrt{n!}}|z_a\rangle, \tag{3.4.16}$$

similarly

$$e^{\hat{b}^\dagger z_b}|0\rangle = \sum_{n=0}^{\infty} \frac{z_b{}^n}{\sqrt{n!}}|z_b\rangle, \tag{3.4.17}$$

action of displacement operator on nvacuum state

$$\begin{aligned} \hat{D}_{ab}(z_a, z_b)|00\rangle &= e^{\hat{a}^\dagger z_a - \hat{a} z_a{}^*}|0\rangle * e^{\hat{b}^\dagger z_b - \hat{b} z_b{}^*}|0\rangle \\ &= e^{-\frac{1}{2}|z_a|^2} \sum_{0}^{\infty} \frac{z_a{}^n}{\sqrt{n!}}|z_a\rangle * e^{-\frac{1}{2}|z_b|^2} \sum_{n=0}^{\infty} \frac{z_b{}^n}{\sqrt{n!}}|z_b\rangle \\ &= |z_a\rangle|z_b\rangle \\ &= |z_a, z_b\rangle, \end{aligned} \tag{3.4.18}$$

the given state $\hat{\rho}$ of a quantum system is defined with Glauber-Sudarshan, P-representation using normalization condition$\frac{1}{\pi}\int d^2z|z\rangle\langle z| = 1$ as

$$\rho_{out} = \frac{1}{\pi^2}\int d^2z \int d^2z'|z\rangle\langle z|\rho|z'\rangle\langle z', \tag{3.4.19}$$

$$\rho_{out} = \int d^2z P(z)|z\rangle\langle z|, \tag{3.4.20}$$

such that $\int d^2z P(z) = 1$, The useful concept related to the Glauber-Sudarshan P-representation is the notion of classicality and nonclassicality. Since $P(z) \geq 0$ for all classical states and $P(z) \ngeq 0$ or can be highly singular than delta function for nonclassical states [90]. If ρ_{in} is a classical state, the distribution function $P(z_a, z_b, {z_a}^*, {z_b}^*)$ must be non-negative definite in the whole complex plane. In such a case, the ouput state is

$$\rho_{out} = \int_{-\infty}^{\infty} P(z_a, z_b, {z_a}^*, {z_b}^*) \times \hat{U}|z_a, z_b\rangle\langle z_a, z_b|\hat{U}^{-1} d^2z_a d^2z_b, \tag{3.4.21}$$

which is equivalent to

$$\rho_{out} = \int_{-\infty}^{\infty} P(z_a, z_b, {z_a}^*, {z_b}^*) \times \hat{U}\hat{D}_{ab}(z_a, z_b)\hat{U}^{-1}\hat{U}|00\rangle\langle 00|\hat{U}^{-1}\hat{U}\hat{D}_{ab}(z_a, z_b)\hat{U}^{\dagger}, \tag{3.4.22}$$

using relation $\hat{U}|00\rangle\langle 00|\hat{U}^{-1} = |00\rangle\langle 00|$ we can see that

$$\hat{U}\hat{D}_{ab}(z_a, z_b)\hat{U}^{-1} = \hat{D}_{ab}({z_a}', {z_b}'), \tag{3.4.23}$$

and

$$({z_a}', {z_b}') = M_U(z_a, z_b), \tag{3.4.24}$$

in short, the following equation can easily be obtained from

$$\hat{U}|z_a, z_b\rangle\langle z_a, z_b|\hat{U}^{-1} = |{z_a}', {z_b}'\rangle\langle {z_a}', {z_b}'|, \tag{3.4.25}$$

since det $M_U = 1$, we have the following formula for the output state

$$\rho_{out} = \int_{-\infty}^{\infty} P(z_a, z_b, {z_a}^*, {z_b}^*)|{z_a}', {z_b}'\rangle\langle {z_a}', {z_b}'| d^2{z_a}' d^2{z_b}', \tag{3.4.26}$$

this is equivalent to

$$\rho_{out} = \int_{-\infty}^{\infty} P'(z_a, z_b, {z_a}^*, {z_b}^*)|z_a, z_b\rangle\langle z_a, z_b| d^2z_a d^2z_b, \tag{3.4.27}$$

and

$$P'(z_a, z_b, {z_a}^*, {z_b}^*) = P({z_a}'', {z_b}'', {z_a}''^*, {z_b}''^*), \tag{3.4.28}$$

$$({z_a}'', {z_b}'') = M^{-1}({z_a}'', {z_b}'') \tag{3.4.29}$$

since $P(z_a, z_b, {z_a}^*, {z_b}^*) \geq 0$, the function $P'(z_a, z_b, {z_a}^*, {z_b}^*)$ must also be non-negative. By the definition of separability, the state at the output is must be separable. If such a representation can not found, i.e, $P(z_a, z_b, {z_a}^*, {z_b}^*) \ngeq 0$ the state is entangled. It is realized by an optical beam splitter, is identified to convert nonclassicality of a single-mode radiation field into bipartite entanglement. We show that the amount of nonclassicality of a single-mode radiation field is strictly transformed into the same amount of bipartite entanglement [45, 83, 84, 85, 86].

Chapter 4

Entanglement of Barut-Girardello coherent states

4.1 Introduction

Entanglement is very dominating phenomenon in quantum optics and of quantum information such as quantum computation and quantum teleportation [97], super-dense coding [98], cloning [99], quantum cryptography [100, 101, 102] and quantum metrology [103]. Entanglement plays a crucial role in secure communication. In quantum optics, fields are entangled but in quantum computation, qubits are entangled. To test whether a given quantum state is entangled it is known as quantification of entanglement, for the quantification of entanglement there are several measures of entanglement [113, 114, 115, 116, 117, 118, 119], such as, linear entropy [104, 105], concurrence [106, 107], von-Neuman entropy [108, 109, 110] and negativity [111, 112]. We choose the linear entropy as best measure of entanglement [120, 121] in this dissertation.

This chapter is organized as: section (4.2) is based on nonclassical properties of Barut-Girardello coherent states. The photon number distribution of these states are drawn on photon number probability distribution for a Glauber coherent state on the same average photon number which depicts the sub-Poissonian photon statistics for BG coherent states. Using Sub-Poissonian photon statistics as criterion to characterize the nonclassicality of BG coherent states. To gauge the nature of these states Mandel Q parameter is calculated which depicts the sub-Poissonian nature for these states which exhibited the nonclassiclity in these states. Also the second order intensity correlation function is calculated which represents the anti-bunching effects of these states. Section (4.3) presents the entanglement generation of Barut-Girardello coherent states by optical beam splitter. Section (4.4) presents the quantification of entanglement using linear entropy as a best measure of entanglement .

4.2 Nonclassical Properties of the Barut-Girardello coherent states

Nonclassical light has attracted great attention in recent years in various areas of quantum optics and quantum information. There are many criteria characterizing a nonclassical state put forth. In this section, we discuss some of the criteria of nonclassicality which are usually used, and will be helpful for investigating the nonclassicality exhibition of our introduced states leads us towards other aspects of research. To achieve this aim, we refer to sub-Poissonian statistics . To mention the common feature of this criteria we should indicate the quantum statistics of introduce state is sub-Poissonian. Using the quantum statistics of these states we analyzed the Mandel Q parameter which depicts the sub-Poissonian statistics which exhibits the signature of nonclassicality. We also calculated the intensity correlation function to depict the nonclassical nature of these states.

4.2.1 Statistical properties

To check the nonclassicality sign in Barut-Girardello coherent states we used the quantum statistical properties. The expectation values of number operator $\hat{N}$ and $\hat{N}^2$ for these states are

$$\begin{aligned} \langle z,k|\hat{N}|z,k\rangle &= \frac{|z|^2\ {}_0F_1(2k+1,|z|^2)}{2k\ {}_0F_1(2k,|z|^2)}, \\ \langle N\rangle &= \frac{|z|^2\ {}_0F_1(2k+1,|z|^2)}{2k\ {}_0F_1(2k,|z|^2)}, \end{aligned} \tag{4.2.1}$$

and

$$\begin{aligned} \langle z,k|\hat{N}^2|z,k\rangle &= \frac{|z|^4\ {}_0F_1(2k+2,|z|^2)}{2k(2k+1)\ {}_0F_1(2k,|z|^2)} + \frac{|z|^2\ {}_0F_1(2k+1,|z|^2)}{2k\ {}_0F_1(2k,|z|^2)}, \\ \langle N^2\rangle &= \frac{|z|^4\ {}_0F_1(2k+2,|z|^2)}{2k(2k+1)\ {}_0F_1(2k,|z|^2)} + \frac{|z|^2\ {}_0F_1(2k+1,|z|^2)}{2k\ {}_0F_1(2k,|z|^2)}, \end{aligned} \tag{4.2.2}$$

using these expectation values we calculate the variance of number operator given as

$$\begin{aligned} (\Delta\hat{N})^2 &= \langle N^2\rangle - \langle N\rangle^2 = \frac{|z|^4\ {}_0F_1(2k+2,|z|^2)}{2k(2k+1)\ {}_0F_1(2k,|z|^2)} + \frac{|z|^2\ {}_0F_1(2k+1,|z|^2)}{2k\ {}_0F_1(2k,|z|^2)} \\ &- \frac{|z|^2\ {}_0F_1(2k+1,|z|^2)}{2k\ {}_0F_1(2k,|z|^2)}, \end{aligned} \tag{4.2.3}$$

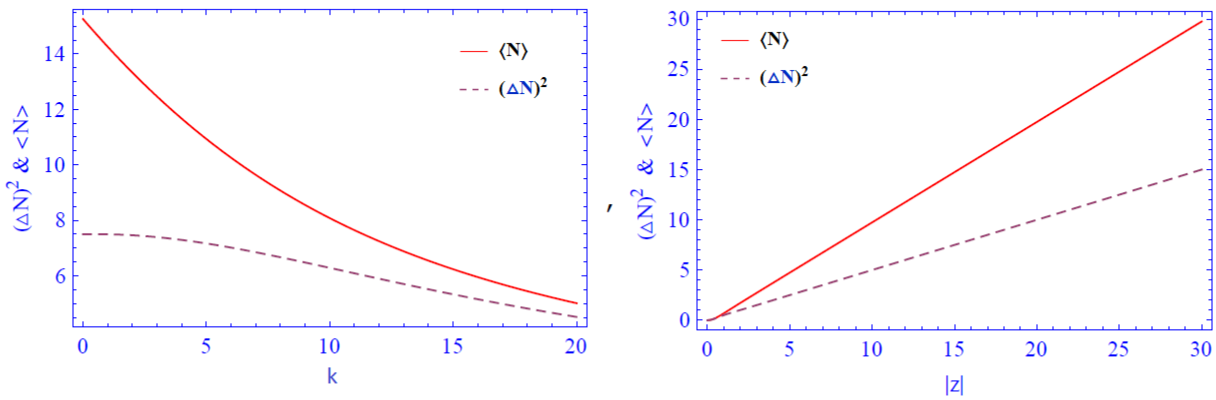

Figure 4.1: The mean $\langle \hat{N} \rangle$ (solid curve) and the variance $(\Delta \hat{N})^2$ (dashed curve) as a function of the coherent-state parameter $|z|$ and k.

In the present work, knowing the mean and the variance are necessary for the probability distribution of the coherent state. For example, in the case of standard coherent state of a harmonic oscillator, the mean $\langle N \rangle$ and the variance $(\Delta \hat{N})^2$ are equal which is a characteristic of the Poissonian distribution. However, in the present case of Barut-Girardello coherent states (BGCS) the variance $(\Delta \hat{N})^2$ is always less than the mean $\langle N \rangle$, as shown in Fig. (4.1) This indicates that the Probability distribution is sub-Poissonian in the present case.

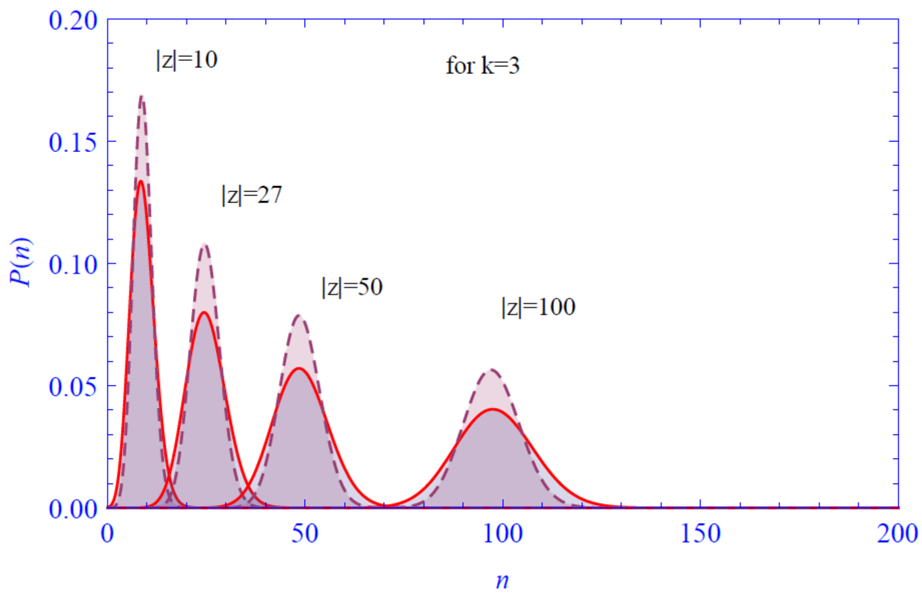

Figure 4.2: Photon number probability distribution of Barut Girardello coherent states (BGCS)(dashed line) and Glauber coherent states (red line) for $|z|$. $k = 3$, (a) $|z| = 10$,(b) $|z| = 27$,(c) $|z| = 50$, (d) $|z| = 100$

It is obvious from Fig. (4.2) that, for a particular value of n, the photon number probability distribution for Barut-Girardello coherent states (BGCS) is narrower than the Poissonian distribution is sub-Poissonian. We analyzed that as we increase coherent states amplitude $|z|$, the width of sub-Poissonian distribution is increased.

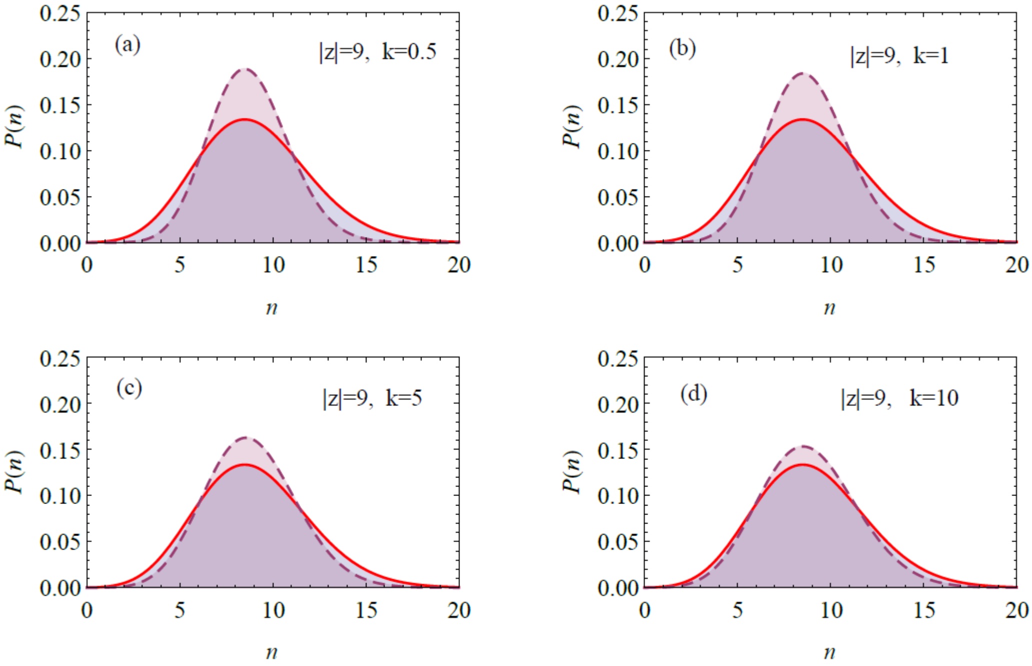

Figure 4.3: Photon number probability distribution of Barut Girardello coherent states (BGCS)(dashed line) and Glauber coherent states (red line) for $|z| = 9$, (a) $k = 0.5$,(b) $k = 1$,(c) $k = 5$, (d) $k = 10$

The photon number probability distribution of Barut-Girardello coherent states (BGCS) is decreases as we increases the parameter (Bargmann index) k, taking the value of z as constant. We increasing Bargmann index k, a limit came where sub-Possonian distribution approaches to Poissonian as shown in Fig. (4.3) of (d).

4.2.2 Mandel Q parameter:

Using the quantum statistics of Barut-Girardello coherent states (BGCS), we calculate the Mandel Q parameter [49], which can be generalized as:

$$Q = \frac{\langle N^2 \rangle - \langle N \rangle^2 - \langle N \rangle}{\langle N \rangle}, \tag{4.2.4}$$

$$Q = \frac{\langle N^2 \rangle - \langle N \rangle^2}{\langle N \rangle} - 1. \tag{4.2.5}$$

which is also written as

$$Q = \frac{\sigma^2}{\langle N \rangle} - 1. \tag{4.2.6}$$

The quantum statistics of a state is Poissonian if $Q = 0$, super-Poissonian if $Q > 0$, and sub-Poissonian if $Q < 0$ [47]. The coherent states for which Q parameter is zero, $Q = 0$ follows Poissonian statistics (standard coherent states of harmonic oscillator), e.g. Glauber coherent states. We plot Mandel Q parameter as function of $|z|$ and k shown in Fig. (4.4) Mandel Q parameter gets the lowest possible negative value, in this case $Q < 0$, Barut-Girardello coherent states have sub-Poisson photon statistics and are non-classical states.

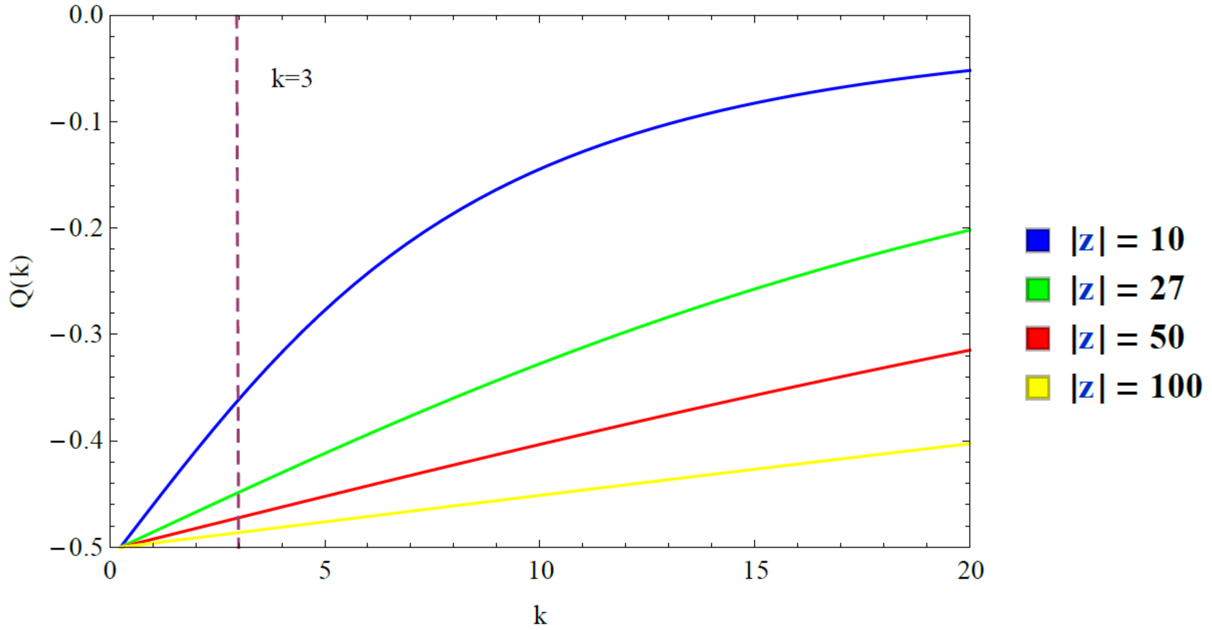

Figure 4.4: Mandel Q parameter for different $|z| = 10$ (blue), $|z| = 27$ (green), $|z| = 50$ (red), $|z| = 100$ (yellow)

Since we analyzed the results of photon number probability distribution for $k = 3$ shows the sub-Poissonian photon statistics for these states for different values of $|z|$. In Q parameter we analyzed that as we increases $|z|$, Q parameter get more negative values which reflects the more sub-Poissonian photon statistics for these states. If we increase parameter k, and fixed the parameter $|z| = 9$ for different values of k the Q parameter approaches to zero. From this we analyzed that photon statistics of (BGCS) are no more sub-Poissonian for large limit of k.

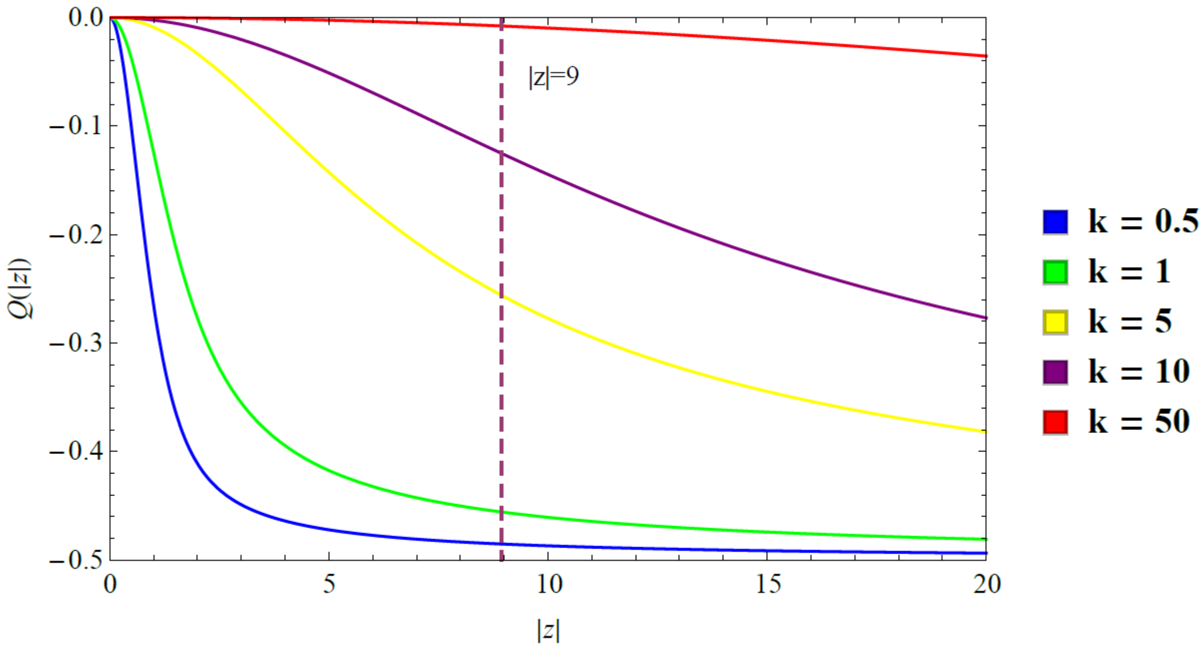

Figure 4.5: Mandel Q parameter for different k=0.5 (blue), k=1 (green), k=5 (yellow), k=10 (purple), k=50 (red)

4.2.3 Second order intensity correlation function:

The expectation value for the number operator $\hat{N}$ is defined as $N|m,k\rangle = m|m,k\rangle$

$$g^{(2)}_{(z,k)}(0) = \frac{\langle N^2\rangle_{(z,k)} - \langle N\rangle_{(z,k)}}{\langle N\rangle^2_{(z,k)}} = \frac{I_{2k-1}(2|z|)I_{2k+1}(2|z|)}{[I_{2k}(2|z|)]^2}. \tag{4.2.7}$$

it is also written as

$$g^{(2)}_{(z,k)}(0) = \frac{Q}{< N >_{(z,k)}} + 1 \tag{4.2.8}$$

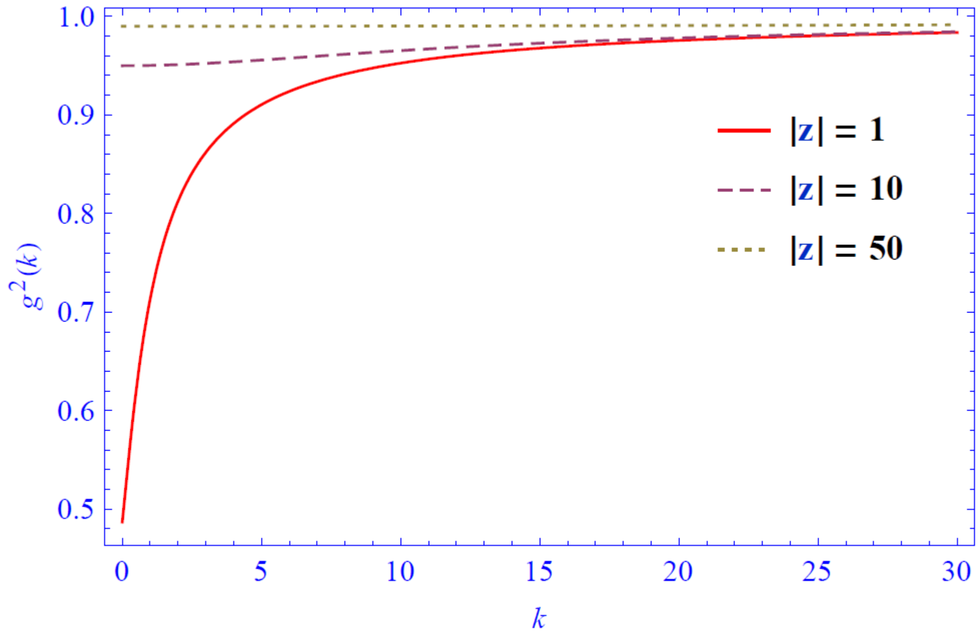

Figure 4.6: Second order correlation function $g^{(2)}_{(z,k)}$ vs k, for $|z| = 1$, (solid line), $|z| = 10$ (dashed line), $|z| = 50$ (dotted line)

If a state has sub-Poissonian or super-Poisonian distributions then the bunching and anti-bunching effects [8] are involved. As $g^{(2)}_{(z,k)}(0) > 1$, corresponds to bunching effects and $g^{(2)}_{(z,k)}(0) < 1$, corresponds to the anti-bunching effects. In our case $g^{(2)}_{(z,k)}(0) < 1$, which depicts the anti-bunching effects of these states.

4.3 Entanglement of Barut-Girardello coherent states

If Barut-Girardello coherent state is on the one input port and a vacuum state is on the other input port of beam splitter.

$$|\psi\rangle_{in} = |z\rangle_1 |0\rangle_2 \tag{4.3.1}$$

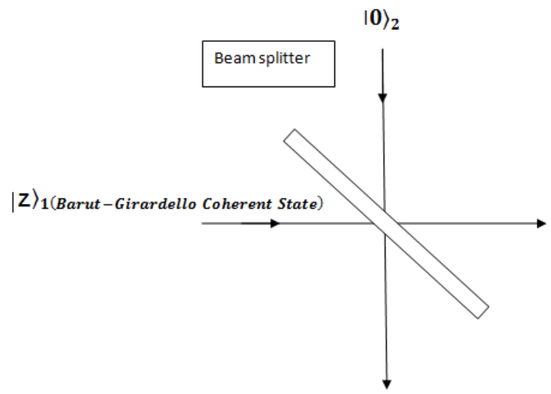

Figure 4.7: A state $|z\rangle_1$, Barut-Girardello Coherent State on the horizontal port and a vacuum state $|0\rangle_2$ on the vertical port of $50:50$ beam splitter

Barut-Girardello coherent state is

$$|z,k\rangle_{BG} = \frac{1}{\sqrt{{}_0F_1(2k;|z|^2)(2k;|z|^2)}} \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k)}{n!\Gamma(n+2k)}} z^n |n\rangle, \tag{4.3.2}$$

comparing with this equation

$$|z\rangle = \sum_{n=0}^{\infty} C_n |n\rangle, \tag{4.3.3}$$

for Barut-Girardello coherent states C_n is

$$C_n = \frac{1}{\sqrt{{}_0F_1(2k;|z|^2)}} \sqrt{\frac{\Gamma(2k)}{n!\Gamma(n+2k)}} z^n,$$

because the beam splitter conserves the total number of photons ,we deduce the beam-splitter transformation for the input states $|z\rangle_1|0\rangle_2$. Using the result from equation (3.3.20)

$$|z\rangle_{BG1}|0\rangle_2 \quad \overset{BS}{\rightarrow} \sum_{n=0}^{\infty}\sum_{p=0}^{n} C_n \sqrt{\binom{n}{p}} T^p R^{(n-p)} |p\rangle_3 |n-p\rangle_4 \tag{4.3.4}$$

putting $C_n = \frac{1}{\sqrt{{}_0F_1(2k;|z|^2)}} \sqrt{\frac{\Gamma(2k)}{n!\Gamma(n+2k)}} z^n$ the output is

$$|z\rangle_{BG1}|0\rangle_2 \quad \overset{BS}{\rightarrow} \frac{1}{\sqrt{{}_0F_1(2k;|z|^2)}} \sum_{n=0}^{\infty}\sum_{p=0}^{n} \sqrt{\frac{\Gamma(2k)}{n!\Gamma(n+2k)}} z^n \sqrt{\binom{n}{p}} T^p R^{(n-p)} |p\rangle_3 |n-p\rangle_4 \tag{4.3.5}$$

the output state become entangled, it cannot be written as a simple product of states of the individual modes 3 and 4. We write the density operatorρ for both modes.

4.4 Quantification of entanglement

There exists several measurements of entanglement such as the concurrence [105, 106], the von-Neumann entropy [107, 108, 109] or the negativity [110, 111]. The linear entropy and the von-Neumann entropy are frequently used to quantify entanglement in the quantum systems. These relations provide typical information on the entanglement. Linear entropy is easier to compute, but gives a good indication on the degree of entanglement. In this dissertation, we use the linear entropy [119, 120] to quantify entanglement.

Linear entropy

In order to measure the degree of entanglement, we use the linear entropy [119, 120]. Starting with the density operator ρ_{34} of a given output state as introduced before, the linear entropy S_L is defined as

$$S_L \quad = \quad 1 - Tr({\rho_3}^2), \tag{4.4.1}$$

where ρ_3, is the reduced density operator of the system 3 obtained by performing a partial trace over system 4 of the density operator ρ_{34}. For an output state $|Out\rangle_{3,4}$ created with a Barut-Girardello coherent state as an input through a beam splitter,

$$|Out\rangle_{3,4} = \frac{1}{\sqrt{{}_0F_1(2k;|z|^2)}}\sum_{n=0}^{\infty}\sum_{p=0}^{n}\sqrt{\frac{\Gamma(2k)}{n!\Gamma(n+2k)}}z^n\sqrt{\binom{n}{p}}T^pR^{(n-p)}|p\rangle_3|n-p\rangle_4, \tag{4.4.2}$$

the density matrix of the output state is

$$\rho_{34} = |Out\rangle_{3,4}\langle Out|_{3,4}, \tag{4.4.3}$$

as $\frac{1}{\sqrt{{}_0F_1(2k;|z|^2)}} = (\aleph(\mid z\mid^2)$

$$\begin{aligned}\rho_{34} \quad = \quad & (\aleph(\mid z\mid^2)^2\sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{n\prime=0}^{\infty}\sum_{p\prime=0}^{n\prime}\sqrt{\frac{\Gamma(2k)}{n!\Gamma(n+2k)}}\sqrt{\frac{\Gamma(2k)}{n\prime!\Gamma(n\prime+2k)}}\\ & z^nz^{*n\prime}\sqrt{\binom{n}{p}}\sqrt{\binom{n\prime}{p\prime}}T^pR^{(n-p)}T^{*p\prime}R^{*(n\prime-p\prime)}\\ & |p\rangle_3|n-p\rangle_4\langle p\prime|_3\langle n\prime-p\prime|_4,\end{aligned} \tag{4.4.4}$$

we make no measurement of say mode 4, mode 3 is then described by the reduced density matrix. This reduced density matrix can be calculated taking the trace of measured mode with un-measured mode. We are taking the inner product with $\sum\limits_{n''=0}^{\infty}\sum\limits_{p''=0}^{n\prime\prime}\langle n\prime\prime-p\prime\prime|_4|n\prime\prime-p\prime\prime\rangle_4$ from left and right we get reduced density matrixρ_3

$$\begin{aligned}Tr_4\{\rho_{34}\} \quad = \quad & (\aleph(\mid z\mid^2)^2\sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{n\prime=0}^{\infty}\sum_{p\prime=0}^{n\prime}\sqrt{\frac{\Gamma(2k)}{n!\Gamma(n+2k)}}\sqrt{\frac{\Gamma(2k)}{n\prime!\Gamma(n\prime+2k)}}\\ & z^nz^{*n\prime}\sqrt{\binom{n}{p}}\sqrt{\binom{n\prime}{p\prime}}T^pR^{(n-p)}T^{*p\prime}R^{*(n\prime-p\prime)}\\ & \sum_{n\prime\prime=0}^{\infty}\sum_{p''=0}^{n\prime\prime}\langle n\prime\prime-p\prime\prime|_4(|p\rangle_3|n-p\rangle_4\langle p\prime|_3\langle n\prime-p\prime|_4)|n\prime\prime-p\prime\prime\rangle_4,\end{aligned} \tag{4.4.5}$$

ρ_3 is the reduced density matrix of the system 3 obtained by performing a partial trace over system 4 of the density operator ρ_{34}

$$\begin{aligned}\rho_3 \quad = \quad & (\aleph(\mid z\mid^2)^2\sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{n\prime=0}^{\infty}\sum_{p\prime=0}^{n\prime}\sqrt{\frac{\Gamma(2k)}{n!\Gamma(n+2k)}}\sqrt{\frac{\Gamma(2k)}{n\prime!\Gamma(n\prime+2k)}}\\ & z^nz^{*n\prime}\sqrt{\binom{n}{p}}\sqrt{\binom{n\prime}{p\prime}}T^pR^{(n-p)}T^{*p\prime}R^{*(n\prime-p\prime)}\\ & \sum_{n''=0}^{\infty}\sum_{p''=0}^{n''}\delta_{(n''-p''),(n-p)}|p\rangle\langle p'|\delta_{(n''-p''),(n'-p'))},\end{aligned} \tag{4.4.6}$$

if $n'' - p'' = n - p$ and $n'' - p'' = n\prime - p\prime$, $n'' = (n-p) + p'$ *and* $n'' = (n' - p') + p''$, $n - p + p'' = n' - p' + p''$, $n - p = n' - p'$, $n = n'$ and $p = p'$, using kronecker delta we get

$$\begin{aligned}\rho_3 &= (\aleph(\mid z \mid^2)^2 \sum_{n=0}^{\infty}\sum_{p=0}^{n}\sum_{p\prime=0}^{n}\sqrt{\frac{n!\Gamma(2k)}{n!\Gamma(n+2k)}}\sqrt{\frac{n!\Gamma(2k)}{n!\Gamma(n+2k)}}\\ &|z|^{2n}\frac{T^p}{\sqrt{p!}}\frac{R^{(n-p)}}{\sqrt{(n-p)!}}\frac{T^{*p\prime}}{\sqrt{p\prime!}}\frac{R^{*(n-p\prime)}}{\sqrt{(n-p\prime)!}}|p\rangle\langle p'|, \end{aligned} \tag{4.4.7}$$

replacing $n - p = m$, $n - p = m\prime$, $n = m + p$, $n' = m + p$, as $n : 0 \to \infty$, $m : 0 \to \infty$, $m' : 0 \to \infty$, $p : 0 \to \infty$, $p' : 0 \to \infty$,

$$\begin{aligned}\rho_3 &= (\aleph(\mid z \mid^2)^2 \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{p\prime=0}^{\infty}\sqrt{\frac{(m+p)!\Gamma(2k)}{(m+p)!\Gamma(m+p+2k)}}\sqrt{\frac{(m+p\prime)!\Gamma(2k)}{(m+p\prime)!\Gamma(m+p\prime+2k)}}\\ &|z|^{2(m+p)}\frac{T^p}{\sqrt{p!}}\frac{T^{*p\prime}}{\sqrt{p\prime!}}\frac{|R|^{2m}}{m!}|p\rangle\langle p'|, \end{aligned} \tag{4.4.8}$$

taking square of ρ_3 we get

$$\begin{aligned}{\rho_3}^2 &= (\aleph(\mid z \mid^2)^4 \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{p\prime=0}^{\infty}\sum_{m\prime=0}^{\infty}\\ &\sqrt{\frac{(m+p)!\Gamma(2k)}{(m+p)!\Gamma(m+p+2k)}}\sqrt{\frac{(m+p\prime)!\Gamma(2k)}{(m+p\prime)!\Gamma(m+p\prime+2k)}}\\ &\sqrt{\frac{(m\prime+p)!\Gamma(2k)}{(m\prime+p\prime)!\Gamma(m\prime+p\prime+2k)}}\sqrt{\frac{(m\prime+p\prime\prime)!\Gamma(2k)}{(m\prime+p\prime\prime)!\Gamma(m\prime+p\prime\prime+2k)}}\\ &|z|^{2(m+p+m\prime+p\prime)}\frac{T^p}{\sqrt{p!}}\frac{T^{*p\prime}}{\sqrt{p\prime!}}\frac{|R|^{2m}}{m!}\frac{T^{p\prime}}{\sqrt{p\prime!}}\frac{T^{*p\prime\prime}}{\sqrt{p\prime\prime!}}\frac{|R|^{2m\prime}}{m\prime!}\\ &\sum_{p\prime=0}^{\infty}\sum_{p\prime\prime=0}^{\infty}(|p,p\prime\rangle\langle p',p\prime\prime|), \end{aligned} \tag{4.4.9}$$

taking the trace of ${\rho_3}^2$

$$\begin{aligned}Tr({\rho_3}^2) &= (\aleph(\mid z \mid^2)^4 \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{p\prime=0}^{\infty}\sum_{m\prime=0}^{\infty}\\ &\sqrt{\frac{(m+p)!\Gamma(2k)}{(m+p)!\Gamma(m+p+2k)}}\sqrt{\frac{(m+p\prime)!\Gamma(2k)}{(m+p\prime)!\Gamma(m+p\prime+2k)}}\\ &\sqrt{\frac{(m\prime+p)!\Gamma(2k)}{(m\prime+p\prime)!\Gamma(m\prime+p\prime+2k)}}\sqrt{\frac{(m\prime+p\prime\prime)!\Gamma(2k)}{(m\prime+p\prime\prime)!\Gamma(m\prime+p\prime\prime+2k)}}\\ &|z|^{2(m+p+m\prime+p\prime)}\frac{T^p}{\sqrt{p!}}\frac{T^{*p\prime}}{\sqrt{p\prime!}}\frac{|R|^{2m}}{m!}\frac{T^{p\prime}}{\sqrt{p\prime!}}\frac{T^{*p\prime\prime}}{\sqrt{p\prime\prime!}}\frac{|R|^{2m\prime}}{m\prime!}\\ &\sum_{p\prime=0}^{\infty}\sum_{p\prime\prime=0}^{\infty}\langle p,p\prime|p\prime,p\prime\prime\rangle, \end{aligned} \tag{4.4.10}$$

If $p = p\prime$ and $p = p\prime\prime$, using kronecker delta, $\sum_{p\prime=0}^{\infty}\sum_{p\prime\prime=0}^{\infty}\langle p, p\prime|p, p\prime\prime\rangle = \delta_{(p,p\prime)}\delta_{(p\prime,p\prime\prime)} = 1$, we get

$$\begin{aligned} Tr({\rho_3}^2) &= (\aleph(\mid z\mid^2)^4 \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{p\prime=0}^{\infty}\sum_{m\prime=0}^{\infty} \\ &\sqrt{\frac{(m+p)!\Gamma(2k)}{(m+p)!\Gamma(m+p+2k)}}\sqrt{\frac{(m+p\prime)!\Gamma(2k)}{(m+p\prime)!\Gamma(m+p\prime+2k)}} \\ &\sqrt{\frac{(m\prime+p)!\Gamma(2k)}{(m\prime+p)!\Gamma(m\prime+p+2k)}}\sqrt{\frac{(m\prime+p\prime)!\Gamma(2k)}{(m\prime+p\prime)!\Gamma(m\prime+p\prime+2k)}} \\ &|z|^{2(m+p+m\prime+p\prime)}\frac{|T|^{2(p+p\prime)}}{p!p\prime!}\frac{|R|^{2(m+m\prime)}}{m!m\prime!}, \end{aligned} \tag{4.4.11}$$

we calculate the linear entropy of output state at mode 3 is

$$S_L = 1 - Tr({\rho_3}^2), \tag{4.4.12}$$

$$\begin{aligned} S_L &= 1 - (\aleph(\mid z\mid^2)^4 \sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\sum_{p\prime=0}^{\infty}\sum_{m\prime=0}^{\infty} \\ &\sqrt{\frac{(m+p)!\Gamma(2k)}{(m+p)!\Gamma(m+p+2k)}}\sqrt{\frac{(m+p\prime)!\Gamma(2k)}{(m+p\prime)!\Gamma(m+p\prime+2k)}} \\ &\sqrt{\frac{(m\prime+p)!\Gamma(2k)}{(m\prime+p)!\Gamma(m\prime+p+2k)}}\sqrt{\frac{(m\prime+p\prime)!\Gamma(2k)}{(m\prime+p\prime)!\Gamma(m\prime+p\prime+2k)}} \\ &|z|^{2(m+p+m\prime+p\prime)}\frac{|T|^{2(p+p\prime)}}{p!p\prime!}\frac{|R|^{2(m+m\prime)}}{m!m\prime!}. \end{aligned} \tag{4.4.13}$$

similarly we calculate the output on mode 4 and linear entropy for mode 4 which is same as calculated for mode 3. In general, the degree of entanglement of the output state is highly dependent on the values of the amplitude $|z|$ and parameter k.

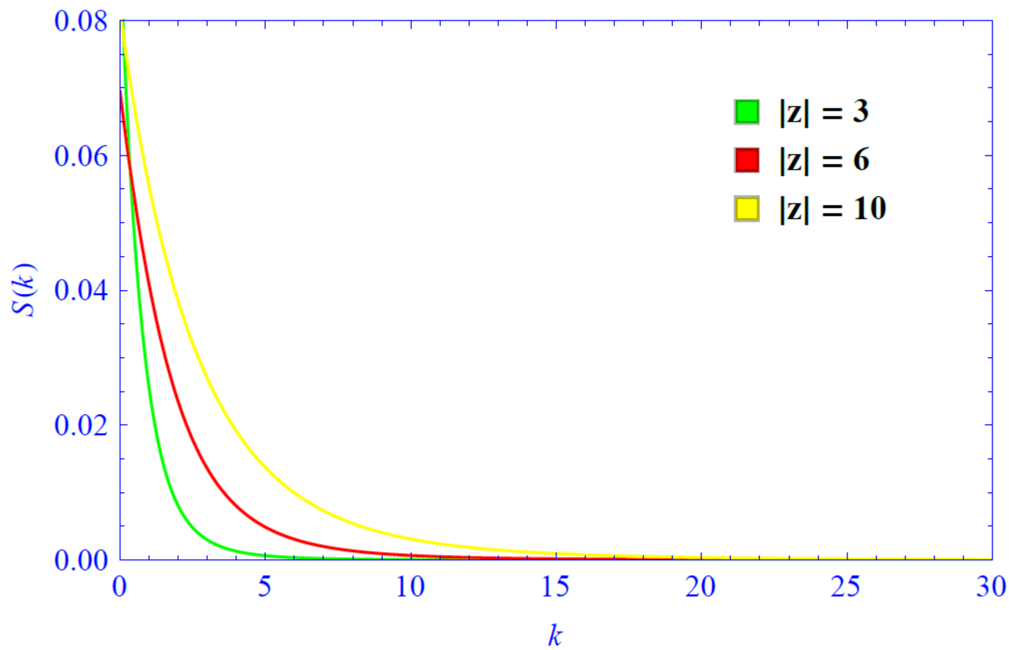

Figure 4.8: Linear entropy $S(k)$ vs k for the Barut-Girardello coherent state as a function of k for $|z| = 3$, $|z| = 6$, $|z| = 10$

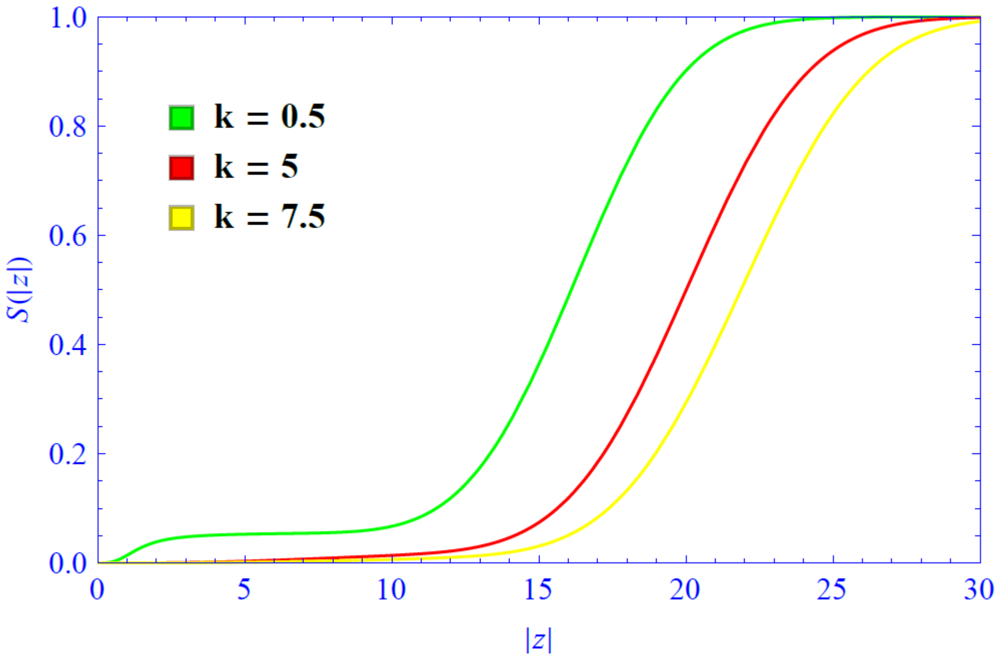

Figure 4.9: Linear entropy $S(|z|)$ vs $|z|$ for the Barut-Girardello coherent state as a function of k for $k = 0.5$, $k = 5$, $k = 7.5$

We analyzed that linear entropy increases at the start for small values of k (Bargmann index) and decreases for large values of k and get zero for the larger values of k. Since we also investigate that photon number probability distribution is not more sub-Poissonian for large values of k as shown in Fig (4.3) part (d), therefore the entanglement of Barut-Girardello coherent states approaches to zero for larger values of k. Furthermore as we increases the coherent state amplitude $|z|$, the degree of entanglement increases. It is investigated that the amount of nonclassicality of Barut-Girardello coherent states is converted into same amount of entanglement by realizing the optical beam splitter transformation.

Chapter 5

Summary and conclusion

Barut-Girardello coherent states are constructed using the (HP) realization of SU(1,1) Lie algebra and their properties are discussed. The non-classical properties of the Barut-Girardello coherent states are investigated in this dissertation. Using the quantum statistical characteristics of the Barut-Girardello coherent states we analyzed the sign of nonclassicality in these state. The mean and variance of number operator are calculated for these states which showed a significant difference of these states from the standard coherent states of harmonic oscillator (Glauber coherent states). For standard coherent states of harmonic oscillator the mean and variance are equal which exhibits the Poissonian distribution but for the Barut-Girardello coherent states variance is less than the mean which exhibits the sub-Poissonian distribution, depicted the nonclassicality in these states. We calculated the photon number probability distribution of these states and presented these results graphically in contrast to standard coherent states (Glauber coherent states) of harmonic oscillator which distinguished these states from the standard coherent states of harmonic oscillator. We investigated that the photon number probability distribution is Poissonian distribution for standard coherent states. It is found that the graphical representation of photon number probability distribution for both states, standard coherent states (Glauber coherent states) and the Barut-Girardello coherent states indicated the significant difference in both distributions. The distribution is narrower than the Poissonian distribution is the sub-Poissonian for Barut-Girardello coherent states which is the cue of nonclassicality. Mandel Q parameter is calculated, it got the negative values depicted the sub-Poissonian nature of these states which exhibited the sign of nonclassicality in these states. Also the second order intensity correlation function is calculated which depicted the anti-bunching effects (correlated photons) of these states.

Nonclassicality is the speciality of Barut-Girardello coherent states which leads us to explore an important application of these states such as entanglement generation through optical beam splitters in quantum information. We used the optical beam splitter to generate entanglement of these states. Taking different examples as input states, i.e. fock states, Glauber coherent states and Barut-Girardello coherent states on optical beam splitter we investigated wether these states are entangled or not. The main idea of this dissertation is that taking the beam splitter as entangler we illustrated output state of beam splitter is entangled if and only if input state exhibited nonclassicality in it. We analyzed that one of the input state is classical-like state with vacuum, or both of input states are classical-like there is no entanglement in the output state. Output state of the optical beam splitter is written as simple product of the individual states.

We analyzed that if one of the input state on beam splitter is exhibited noclassicality after passing through beam splitter it is entangled. It is clearly showed that fock states are entangled after passing through optical beam splitters so fock states are highly non-classical states and single state is maximally entangled. The Barut-Girardello coherent states are non-classical state, entangled after passing through optical beam splitter. It is realized by an optical beam splitter, is identified to convert nonclassicality of a single-mode radiation field i.e., Barut-Girardello coherent states into bipartite entanglement. We analyzed that the amount of nonclassicality of a single-mode radiation field is strictly transformed into the same amount of bipartite entanglement. Entanglement generation of the Barut-Girardello coherent states are measured using the linear entropy as measure of entanglement. We investigated that entanglement of Barut-Girardello coherent states depended on two parameters, Bargmann index k and coherent states amplitude z. We analyzed that entanglement goes to disappear for larger values of k. From the graphical representation of probability distribution we also showed that for larger values of k, a limit came where only the vacuum state is populated thats why entanglement tends to zero for larger values of k. Also the degree of entanglement increases as we increases the Barut-Girardello coherent state amplitude z.

There are many more aspects of further research of the Barut-Girardello coherent states. If we contracts the Lie algebra of these states on Heisenberg-Weyl algebra in the large limit of Bargmann index k, the Barut-Girardello coherent states goes to classical-like standard coherent states (Glauber coherent states) of harmonic oscillator. It is an other aspect of further research. If Lie algebras of the Barut-Girardello coherent states are converted into deformed Lie algebras then these states are called the deformed Barut-Girardello coherent states. The study on entanglement generation of deformed Barut-Girardello coherent states is also an other aspect of further research.

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