Accretion of Phantom Energy onto Rotating Banados-Teitelboim-Zanelli (BTZ) Black Hole

by

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Dedicated

to

My parents, brothers and sisters

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Abstract

In this dissertation, we have considered the phenomena of phantom energy accretion onto (2+1)-dimensional BTZ black hole. The first two chapters are devoted to introductions to General Relativity and black hole solutions. In the first chapter, we discuss basic equations of General Relativity and also focus on different types of energies. In the second chapter, we briefly discuss some black hole solutions: namely Schwarzschild solution and (2+1)-dimensional BTZ black hole solution.

In chapter three, we study the accretion of phantom energy onto a (2+1)-dimensional non-rotating BTZ black hole as well as rotating BTZ black hole. An interesting finding is that the rate of change of mass of the non-rotating BTZ black hole is independent of its mass and depends only on energy density and pressure of the phantom energy. Similarly the rate of change of mass of the rotating BTZ black hole depends not only on the energy density and pressure of the phantom energy but also on the mass of the rotating BTZ black hole. In the case of rotating BTZ black hole, we find that the mass will decrease only if, $\frac{\sqrt{M^2-\frac{J^2}{l^2}}+M}{2\sqrt{M^2-\frac{J^2}{l^2}}} = 2n$, $M > \frac{r^2}{l^2}$ and $M > \frac{J}{l}$ where $n = 0, \pm 1, \pm 2, \cdots$. Finally we conclude the dissertation and discuss some further lines of work.

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Chapter 1

Introduction

In Astrophysics, accretion is defined as "a process by which matter is collected around a central object" [1]. In binary systems accretion, one star is tidally deformed and matter flows out from it to the compact companion. When one deals with an isolated object, it may accrete from the interstellar medium at a very low rate. In many of the galactic centers, there is evidence of supermassive black holes [1]. There are no companions, but matter is accreted from winds of the surrounding stars. In these cases, stars may also be tidally disrupted if they come very close to the black hole and the matter would be accreted from the disrupted star to the central black hole.

Though the subject began with the study of accretion onto ordinary stars, very quickly a similar method was found to be useful for the study of accretion onto compact objects, such as white dwarfs, neutron stars and black holes.

The recent observational evidence obtained from Wilkinson Microwave Anisotropy Probe (WMAP) strongly suggests that the current expansion of the Universe is accelerating [2]. This positive accelerated expansion of the Universe is explained by the dominance of dark energy with a negative pressure in the Einstein theory of gravity [3]. A peculiar property of cosmological models with dark energy is the possibility of a Big Rip [4]: an infinite increase in the scale factor of the Universe in a finite time. The Big Rip scenario is realized in the case of phantom energy (for which $(\rho + p) < 0$, where ρ is the energy density and p is the pressur of the phantom energy). In the Big Rip scenario the phantom energy tends to infinity and all the bound objects are torn apart upto subatomic scales. It should be noted, however, that the condition $(\rho + p) < 0$ alone is not enough for the Big Rip scenario to be realized [5].

The history of research on the accretion of an ideal fluid onto a compact object began with Hoyle and Lyttleton's paper in 1939 [1]. The problem was to study how much matter would accrete on a star moving through an interstellar medium. The work was not satisfactory as pressure effects were ignored. In 1952, the classic paper of Bondi [6] was published. There he computed the mass accretion rate on a star 'in rest in an infinite cloud of gas' by including the pressure effects. According to Bondi, the compressional heat may be lost and pressure at the inner edge may be diminished causing much larger inflow rate. His work was not suitable to discuss the accretion onto objects like black holes, neutron stars and white dwarfs, etc. A few years later, Michel extended the Bondi's work by studying the accretion phenomena near the Schwarzschild black hole [7]. Michel showed that high energy X-rays and gamma rays will be emitted in the accretion process. Carr and Hawking consider the accretion of dust and radiation onto a black hole by solving the complete system of the Einstein equations and taking into account back reaction of the surrounding matter [8].

Babichev et al, discuss the effect of dark energy accretion onto Schwarzschild black hole. They obtain a solution for the stationary accretion of a test relativistic ideal fluid with an arbitrary equation of state $p = p(\rho)$ onto the Schwarzschild black hole. They found that

$$dM = 4\pi A M^2 (\rho + p)\, dt, \tag{1.0.1}$$

where it is clear that the mass of the black hole increases as it accretes the gas of particles when $p > 0$, but decreases as it accretes the phantom energy. In particular this implies that the black hole mass in the Universe filled with phantom energy must decrease [9].

Babichev et al, showed that accretion of phantom energy will induce the mass of black hole to decrease. They conclude that the mass of all black holes will vanish before the Big Rip is reached.

The accretion of phantom energy onto a (2+1)-dimensional non-rotating Banados-Teitelboim-Zanelli (BTZ) black hole was studied by M. Jamil and M. Akbar [11]. In their work they showed that the mass of a (2+1)-dimensional BTZ black hole decreases due to the accretion of phantom energy. They showed that the change in BTZ black hole mass is

$$dM = 2\pi A_1(\rho + p)dt. \tag{1.0.2}$$

Note that for phantom energy $(\rho + p) < 0$ which leads to the decrease in the mass of the BTZ black hole. We extend this work to rotating BTZ black hole and find that the mass of rotating BTZ black hole will decrease only if, $\frac{\sqrt{M^2 - \frac{J^2}{l^2}} + M}{2\sqrt{M^2 - \frac{J^2}{l^2}}} = 2n$, $M > \frac{r^2}{l^2}$ and $M > \frac{J}{l}$ where $n = 0, \pm 1, \pm 2, ...$

Most of the energy of the Universe is in the form of dark energy and dark matter. It has been found that 97 percent energy of the Universe consist of dark energy and dark matter (73 percent dark energy and 24 percent dark matter) [12]. This dark energy has some remarkable effects on the Universe. In fact the effect of dark energy on the black hole produces a huge change in the physics of the black hole. In this dissertation, we will consider the effect of phantom energy on the physics of a (2+1)-dimensional BTZ black hole.

Einstein's theory of General Relativity (GR) is the most beautiful and elegant of physical theories. It is the foundation of cosmology, the subject that discovers the evolution of the Universe from its first intensely hot and dense beginning to its possible features. GR is also the foundation for our understanding of compact stars. Neutron stars and black holes can be understood correctly only in GR as formulated by Einstein.

In this chapter, we discuss some basic definitions and known results which will provide us a background for our research work in subsequent chapters. We begin by giving some ideas about vectors and tensors. Our next and most important task will be to discuss curvature tensors and Einstein's field equations, which relate the geometry of spacetime to the distribution of matter in the Universe. Finally we will discuss dark energy and phantom energy.

In this dissertation, we use the following conventions and units: the signature of the underlying metric will be $(-,+,+,+)$, unless otherwise mentioned, we use the natural units in which $c = G = 1$, also the Planck's units in which $c = \hbar\ (= \frac{h}{2\pi}) = 8G = 1$. Here c is the speed of light, G is Newton's gravitational constant and h is the Planck's constant.

1.1 Definitions

1.1.1 Vectors

A vector is defined as a geometric object that has both magnitude and direction. Usually a vector is denoted by a letter with an arrow above it, for example $\vec{u}$, or by boldface letter, for example $\mathbf{u}$. We shall use the latter notation.

Vectors can also be defined as quantities satisfying certain axioms:

1. If $\mathbf{u}$ and $\mathbf{v}$ are vectors then $\mathbf{u}+\mathbf{v}$ is also a vector,

2. $a\mathbf{u}$ is a vector, for every vector $\mathbf{u}$ and real number a.

An expression of the form $\alpha^\nu \mathbf{e}_\nu$ where α^ν (with $\nu \in \{1, ...n\}$) are real numbers, is called a linear combination of the vectors $\mathbf{e}_\nu$. Here we use the Einstein summation convention: according to this convention, when an index variable appears twice in a single term, one subscript and one superscript, then we are summing over all of its possible values, i.e., $\sum_{\nu=1}^{n} \alpha^\nu \mathbf{e}_\nu = \alpha^\nu \mathbf{e}_\nu$. The vectors $\mathbf{e}_1, ..\mathbf{e}_n$ are said to be linearly independent if no real numbers $\alpha^\nu \neq 0$ exist so that $\alpha^\nu \mathbf{e}_\nu = 0$.

A set of linearly independent vectors $\{\mathbf{e}_\nu\}$ is said to be maximally linearly independent if for all vectors $\mathbf{u}$ the set of vectors $\{\mathbf{e}_\nu, \mathbf{u}\}$ is linearly dependent. Then there exist non-zero real numbers α^ν, such that

$$\alpha^\nu \mathbf{e}_\nu + \mathbf{u} = 0 \tag{1.1.1}$$

A vector basis for a space V is defined as a set of vectors in V that are maximally linearly independent. The number of vectors in the basis is called the dimension of V.

Let the vector set $\{\mathbf{e}_1, ..., \mathbf{e}_n\}$ be a basis in an n-dimensional space. Setting $\alpha^\nu = -u^\nu$ in equation (1.3), we get

$$\mathbf{u} = u^\nu \mathbf{e}_\nu. \tag{1.1.2}$$

The numbers u^ν are called the components of $\mathbf{u}$ relative to the basis $\{\mathbf{e}_\nu\}$.

1.1.2 One-forms

Let the set of real numbers be denoted by $\mathbb{R}$ and let V be a vector space. A function f is said to be linear if

$$f(a\mathbf{u} + b\mathbf{v}) = af(\mathbf{u}) + bf(\mathbf{v}), \tag{1.1.3}$$

where $a, b \in \mathbb{R}$ and $\mathbf{u}, \mathbf{v} \in V$.

A one-form, $\mathbf{a}$ is defined as a linear function from V into $\mathbb{R}$; i.e., $\mathbf{a} : V \mapsto \mathbb{R}$. In other words, a one-form, $\mathbf{a}$, acts on a vector, $\mathbf{v}$, and gives out a real number, $\mathbf{a}(\mathbf{v})$. In order to be able to write a form in component-form, we have to define a one-form basis $\{\boldsymbol{\omega}^\mu\}$. The basis is defined by

$$\boldsymbol{\omega}^\mu(\mathbf{e}_\nu) = \delta^\mu_\nu, \tag{1.1.4}$$

where δ^μ_ν is the Kronecker-symbol. We can now write a one-form as a linear combination of the basis-forms

$$\mathbf{a} = a_\mu \boldsymbol{\omega}^\mu. \tag{1.1.5}$$

The numbers a_μ are called the components of $\mathbf{a}$ relative to the basis $\{\boldsymbol{\omega}^\mu\}$. By means of equation (1.6) and (1.7), we have

$$\mathbf{a}(\mathbf{e}_\mu) = a_\nu \boldsymbol{\omega}^\nu(\mathbf{e}_\mu) = a_\nu \delta^\nu_\mu = a_\mu. \tag{1.1.6}$$

The number $\mathbf{a}(\mathbf{v})$ is called the contraction or interior product of $\mathbf{a}$ with $\mathbf{v}$ which may be expressed by the components of $\mathbf{a}$ and $\mathbf{v}$ as

$$\mathbf{a}.\mathbf{v} = v^\mu a_\mu. \tag{1.1.7}$$

This is just the same number that is obtained by taking the scalar product of two vectors $\mathbf{v}$ and $\mathbf{a}$. One-forms correspond to vectors and the contraction of a one-form by a vector to the scalar-product of two vectors.

Just like forms, vectors can also be perceived as linear functions. If a vector $\mathbf{v}$ acts on a form $\mathbf{a}$, it gives out the number $\mathbf{v}(\mathbf{a}) = a_\mu v^\mu$. Since this is equal to $v^\mu a_\mu$ we have

$\mathbf{v}(\mathbf{a}) = \mathbf{a}(\mathbf{v})$, which corresponds to the symmetry of the scalar-product of two vectors. It follows that the vector components v^μ can be expressed as

$$v^\mu = \mathbf{v}(\boldsymbol{\omega}^\mu). \tag{1.1.8}$$

The components of a vector are the contractions of the vector with the basis forms.

Like the vectors, one-forms satisfy the axioms of a vector space. Therefore, one-forms are sometimes referred to as dual vectors. In Dirac's bra-ket notation in quantum mechanics, the kets $|\psi\rangle$ are the vectors and the bras $\langle\psi|$ are the forms.

1.1.3 Tensors

GR is formulated in the language of tensors. A tensor is a quantity which remains invariant under coordinate transformation [13]. The component of a tensor can be written either in *covariant* $\xi_{\mu\nu}$, *contravariant* $\xi^{\mu\nu}$ or *mixed* ξ^μ_ν form. We can also define the rank of a tensor as "the total number of free indices". For example a vector has rank 1, a scalar has rank 0. Similarly $\xi_{\mu\nu}$, $\xi^{\mu\nu}$ and ξ^μ_ν all have rank 2 and so on. The transformation rule for the components of a *contravariant* vector is given by

$$\xi'^\mu = \frac{\partial x'^\mu}{\partial x^\nu}\xi^\nu. \tag{1.1.9}$$

Similarly the transformation rule for the components of a *covariant* vector is given by

$$\xi'_\mu = \frac{\partial x^\nu}{\partial x'^\mu}\xi_\nu. \tag{1.1.10}$$

The transformation rules for *contavariant*, *covariant* and *mixed* tensors of rank 2 are written respectively as

$$\xi'^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x'^\nu}{\partial x^\delta}\xi^{\lambda\delta}, \tag{1.1.11}$$

$$\xi'_{\mu\nu} = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x^\delta}{\partial x'^\nu}\xi_{\lambda\delta}, \tag{1.1.12}$$

and

$$\xi'^\mu_\nu = \frac{\partial x'^\mu}{\partial x^\lambda}\frac{\partial x^\delta}{\partial x'^\nu}\xi^\lambda_\delta. \tag{1.1.13}$$

1.1.4 Metric Tensor

This is the most important tensor in GR and is a symmetric tensor. A metric basically defines the distance or the length of a vector. This distance ds between two neighboring points in space is defined by

$$ds^2 := g_{\mu\nu} dx^\mu dx^\nu, \tag{1.1.14}$$

where $g_{\mu\nu}$ is the metric tensor. In flat spacetime $g_{\mu\nu} = \eta_{\mu\nu}$, where $\eta_{\mu\nu} = \text{diag}(-1, 1, 1, 1)$ and is called the Minkowski metric, so in flat spacetime

$$ds^2 = -c^2 dt^2 + dx^2 + dy^2 + dz^2. \tag{1.1.15}$$

Depending upon the sign of the metric tensor, we have the following three kinds of separation

$ds^2 > 0$	spacelike separated,
$ds^2 = 0$	null separated,
$ds^2 < 0$	timelike separated.

1.1.5 4-Vectors

Spacetime is four-dimensional. Every point of spacetime can be characterized by four linearly independent basis vectors $\mathbf{e}_\nu$. Thus, a vector in spacetime has four components. Such vectors are called four-vectors. The basis vectors in this system are denoted by $\{\mathbf{e}_t, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$. They are mutually orthogonal unit vectors. Since they are unit vectors, so they form an orthonormal basis.

According to the Galilean and Newtonian kinematics, all particles move in a three-dimensional space, and the velocity $\mathbf{u}$ is a vector. The ordinary velocity of a particle is

$$\mathbf{u} = u^x \mathbf{e}_x + u^y \mathbf{e}_y + u^z \mathbf{e}_z = \frac{dx}{dt}\mathbf{e}_x + \frac{dy}{dt}\mathbf{e}_y + \frac{dz}{dt}\mathbf{e}_z. \tag{1.1.16}$$

According to the relativistic description, however, particles exist in a four dimensional spacetime. In this description the ordinary velocity of a particle is not a vector. Instead one defines a four-velocity

$$\mathbf{U} = c\frac{dt}{d\tau}\mathbf{e}_t + \frac{dx}{d\tau}\mathbf{e}_x + \frac{dy}{d\tau}\mathbf{e}_y + \frac{dz}{d\tau}\mathbf{e}_z, \tag{1.1.17}$$

where τ is the proper time of the particle, i.e., the time measured by a standard clock carried by the particle. Using Einsteins summation convention we may write

$$\mathbf{U} = u^\mu \mathbf{e}_\mu = \frac{dx^\mu}{d\tau}\mathbf{e}_\mu, \quad x^\mu \in \{x^o, x^1, x^2, x^3\}, \tag{1.1.18}$$

where $x^o = ct$, $x^1 = x$, $x^2 = y$ and $x^3 = z$. Since $dt/d\tau = \gamma$, so the components of the four-velocity are given in terms of the components of ordinary velocity as

$$\mathbf{U} = \gamma(c, u^x, u^y, u^z) = \gamma(c, \mathbf{u}). \tag{1.1.19}$$

In the rest frame of the particle, $\mathbf{u} = 0$ and $\gamma = 1$. Hence, the four-velocity reduces to

$$\mathbf{U} = c\mathbf{e}_t. \tag{1.1.20}$$

In this frame the particle moves in the time direction with the speed of light. One often uses units such that $c = 1$. In these units, both time and space are measured in units of length. In such geometrical units of measurement the particle moves with unit velocity in the time-direction in its own rest frame. The four-momentum, $\mathbf{P}$, of a particle with rest-mass m_0 is defined by

$$\mathbf{P} = m_0\mathbf{U}. \tag{1.1.21}$$

Referring to the rest-frame of the particle and using units such that $c = 1$, we see that the magnitude of the four-momentum is equal to the rest-mass of the particle. The ordinary (three-dimensional) relativistic momentum of the particle is

$$\mathbf{p} = m\mathbf{u} = \gamma m_0 \mathbf{u}. \tag{1.1.22}$$

1.1.6 Covariant Derivative

The covariant derivative is a way to specify a derivative along the tangent vector to a manifold. This is an extension of the gradient operator to tensors and is denoted by ∇_μ. The covariant derivative of a scalar field, φ, is simply the partial derivative and is defined as

$$\nabla_\mu \varphi := \varphi_{,\mu}. \tag{1.1.23}$$

The covariant derivative of *covariant* and *contravariant* vectors are, respectively, defined as

$$\nabla_\nu \xi_\mu := \xi_{\mu;\nu} = \xi_{\mu,\nu} - \Gamma^{\rho}{}_{\mu\nu}\xi_\rho \tag{1.1.24}$$

and

$$\nabla_\nu \xi^\mu := \xi^{\mu}{}_{;\nu} = \xi^{\mu}{}_{,\nu} + \Gamma^{\mu}{}_{\rho\nu}\xi^\rho, \tag{1.1.25}$$

where $\Gamma^{\mu}{}_{\rho\nu}$ is the Christoffel symbol defined by

$$\Gamma^{\mu}{}_{\rho\nu} := \frac{1}{2} g^{\mu\lambda}(g_{\rho\lambda,\nu} + g_{\lambda\nu,\rho} - g_{\rho\nu,\lambda}). \tag{1.1.26}$$

Similarly the the expression for the covariant derivative of a *mixed* tensor is

$$\nabla_\nu \xi^{\mu}{}_{\rho} = \xi^{\mu}{}_{\rho;\nu} = \xi^{\mu}{}_{\rho,\nu} + \Gamma^{\mu}{}_{\alpha\nu}\xi^{\alpha}{}_{\rho} - \Gamma^{\alpha}{}_{\rho\nu}\xi^{\mu}{}_{\alpha}. \tag{1.1.27}$$

In general the covariant derivative of a higher rank *mixed* tensor is

$$\xi^{\alpha\beta..}{}_{\gamma\lambda..;\nu} = \xi^{\alpha\beta..}{}_{\gamma\lambda..,\nu} + \Gamma^{\alpha}{}_{\mu\nu}\xi^{\mu\beta..}{}_{\gamma\lambda..} + \Gamma^{\beta}{}_{\mu\nu}\xi^{\alpha\mu..}{}_{\gamma\lambda..} .. - \Gamma^{\mu}{}_{\gamma\nu}\xi^{\alpha\beta..}{}_{\mu\lambda..} - \Gamma^{\mu}{}_{\lambda\nu}\xi^{\alpha\beta..}{}_{\gamma\mu..} ... \tag{1.1.28}$$

1.2 Riemann Tensor

The Riemann tensor, also known as Riemann curvature tensor, measures the deviation of spacetime from Minkowski spacetime [13]. It is given by

$$R^{\mu}{}_{\nu\lambda\delta} = \Gamma^{\mu}{}_{\nu\delta,\lambda} - \Gamma^{\mu}{}_{\nu\lambda,\delta} + \Gamma^{\rho}{}_{\nu\delta}\Gamma^{\mu}{}_{\rho\lambda} - \Gamma^{\rho}{}_{\nu\lambda}\Gamma^{\mu}{}_{\rho\delta}. \tag{1.2.1}$$

It is useful to look at the components of Riemann curvature tensor $\mathbf{R}$ in a locally inertial frame at a point $\mathfrak{p}$. In a locally inertial frame $\Gamma^{\alpha}{}_{\mu\nu} = 0$ but $\Gamma^{\alpha}{}_{\mu\nu,\lambda} \neq 0$, because the second derivative of $g_{\mu\nu}$ does not vanish. Thus from equation (1.28), we have

$$\Gamma^{\mu}{}_{\nu\delta,\lambda} = \frac{1}{2} g^{\mu\sigma}(g_{\nu\sigma,\delta\lambda} + g_{\sigma\delta,\nu\lambda} - g_{\nu\delta,\sigma\lambda}). \tag{1.2.2}$$

Using equation (1.32) in (1.31), the components of the Riemann curvature tensor reduce to

$$R^{\mu}{}_{\nu\lambda\delta} = \frac{1}{2} g^{\mu\sigma}(g_{\nu\sigma,\delta\lambda} + g_{\sigma\delta,\nu\lambda} - g_{\nu\delta,\sigma\lambda} - g_{\sigma\nu,\lambda\delta} - g_{\sigma\lambda,\nu\delta} + g_{\nu\lambda,\sigma\delta}). \tag{1.2.3}$$

Using the symmetry of $g_{\alpha\beta}$ and the fact that

$$g_{\alpha\beta,\mu\nu} = g_{\alpha\beta,\nu\mu}, \tag{1.2.4}$$

because partial derivatives commute, we find

$$R^{\mu}{}_{\nu\lambda\delta} = \frac{1}{2} g^{\mu\sigma}(g_{\sigma\delta,\nu\lambda} - g_{\sigma\lambda,\nu\delta} + g_{\nu\lambda,\sigma\delta} - g_{\nu\delta,\sigma\lambda}). \tag{1.2.5}$$

Lowering the indices μ by multiplying equation (1.35) with $g_{\mu\gamma}$, we have

$$R_{\mu\nu\lambda\delta} = g_{\mu\gamma} R^{\gamma}{}_{\nu\lambda\delta} = \frac{1}{2}(g_{\mu\delta,\nu\lambda} - g_{\mu\lambda,\nu\delta} + g_{\nu\lambda,\mu\delta} - g_{\nu\delta,\mu\lambda}). \tag{1.2.6}$$

In this form it is easy to verify the following identities:

$$R_{\mu\nu\lambda\delta} = -R_{\nu\mu\lambda\delta} = -R_{\mu\nu\delta\lambda} = R_{\lambda\delta\mu\nu}, \tag{1.2.7}$$

and

$$R_{\mu\nu\lambda\delta} + R_{\mu\delta\nu\lambda} + R_{\mu\lambda\delta\nu} = 0. \tag{1.2.8}$$

From equation (1.37) it is clear that $R_{\mu\nu\lambda\delta}$ is antisymmetric in both the first and the second pair of indices and symmetric in exchange of the two pairs. Equations (1.37) and (1.38) are tensor equations. Therefore if they are true in one coordinate system, they are true in all coordinate systems.

1.3 Bianchi Identities: Ricci and Einstein Tensors

Bianchi Identities. Differentiating equation (1.31) with respect to x^{α} and evaluating the result in locally inertial coordinates, we have

$$R_{\mu\nu\lambda\delta,\alpha} = \frac{1}{2}(g_{\mu\delta,\nu\lambda\alpha} - g_{\mu\lambda,\nu\delta\alpha} + g_{\nu\lambda,\mu\delta\alpha} - g_{\nu\delta,\mu\lambda\alpha}). \tag{1.3.1}$$

Using the symmetry of $g_{\mu\nu}$ and the fact that partial derivative commute, it is easy to show that

$$R_{\mu\nu\lambda\delta,\alpha} + R_{\mu\nu\alpha\lambda,\delta} + R_{\mu\nu\delta\alpha,\lambda} = 0. \tag{1.3.2}$$

Since equation (1.40) is a valid tensor equation and is true in one coordinate system, it is true in all coordinate systems, i.e.,

$$R_{\mu\nu\lambda\delta;\alpha} + R_{\mu\nu\alpha\lambda;\delta} + R_{\mu\nu\delta\alpha;\lambda} = 0. \tag{1.3.3}$$

the results in equation (1.41) are called the Bianchi identities [13].

The Ricci tensor. In order to discuss the consequences of the Bianchi identities, we first need to define the Ricci tensor $R_{\mu\nu}$:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu} = R_{\nu\mu}. \tag{1.3.4}$$

It is the contraction of $R^{\lambda}{}_{\mu\delta\nu}$ on the first and third indices. Other contractions would in principle also be possible: on the first and second, the first and fourth, etc, but because $R_{\mu\nu\delta\lambda}$ is antisymmetric in μ and ν and in δ and λ, all these contractions either vanish identically or reduce to $\pm R_{\mu\nu}$. Therefore the Ricci tensor is the only independent contraction of the Riemann tensor.

Ricci scalar. Contracting $R_{\mu\nu}$ with $g^{\mu\nu}$, we have

$$R = g^{\mu\nu} R_{\mu\nu}. \tag{1.3.5}$$

This scalar R is known as the Ricci scalar. Notice that $R^{\mu}{}_{\nu\lambda\delta} = 0$ implies that $R_{\mu\nu} = 0$ yielding $R = 0$, but the inverse may not be true in general.

The Einstein tensor. Contracting the Bianchi identities given in equation (1.41), we have

$$g^{\mu\lambda}(R_{\mu\nu\lambda\delta;\alpha} + R_{\mu\nu\alpha\lambda;\delta} + R_{\mu\nu\delta\alpha;\lambda}) = 0,$$

or

$$R_{\nu\delta;\alpha} - R_{\nu\alpha;\delta} + R^{\lambda}{}_{\nu\delta\alpha;\lambda} = 0. \tag{1.3.6}$$

To derive this result we use the following facts. First $g_{\delta\lambda;\mu} = 0$, since $g^{\delta\lambda}$ is a function only of $g_{\delta\lambda}$, it follows that $g^{\delta\lambda}{}_{;\mu} = 0$. Therefore $g^{\delta\mu}$ and $g_{\lambda\nu}$ can be taken in and out of covariant derivatives. Equation (1.44) is called the contracted Bianchi identities. A more useful equation is obtaine by contracting again on the indices ν and δ, we have

$$g^{\nu\delta}[R_{\nu\delta;\alpha} - R_{\nu\alpha;\delta} + R^{\lambda}{}_{\nu\delta\alpha;\lambda}] = 0,$$

or

$$R_{;\alpha} - R^{\lambda}{}_{\alpha;\lambda} - R^{\lambda}{}_{\alpha;\lambda} = 0. \tag{1.3.7}$$

Again the antisymmetry of $\mathbf{R}$ has been used to get the correct sign in the last term. Now equation (1.45) can be written in the form

$$\begin{aligned}(R^{\lambda}{}_{\alpha} - \frac{1}{2}\delta^{\lambda}{}_{\alpha}R)_{;\lambda} &= 0,\\ (g^{\delta\alpha}R^{\lambda}{}_{\alpha} - \frac{1}{2}g^{\delta\alpha}\delta^{\lambda}{}_{\alpha}R)_{;\lambda} &= 0,\\ (R^{\delta\lambda} - \frac{1}{2}g^{\delta\lambda}R)_{;\lambda} &= 0,\end{aligned} \tag{1.3.8}$$

These are the twice contracted Bianchi identities. If we define the symmetric tensor

$$G^{\alpha\beta} = R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R, \tag{1.3.9}$$

then we see that equation (1.46) is equivalent to

$$G^{\alpha\beta}{}_{;\beta} = 0. \tag{1.3.10}$$

The tensor $G^{\alpha\beta}$ is obtained from the Riemann tensor and the metric, and is automatically divergence free as an identity. It is called the Einstein tensor and its importance for gravity was first understood by Einstein.

1.4 Energy-Momentum Tensor

The energy momentum tensor, $T^{\mu\nu}$, is a tensor which describes the matter and energy contents of a spacetime. The components of energy-momentum tensor can be described as follows [13]:

1. T^{00} represents the energy or mass density

2. T^{0i} is the energy flux

3. T^{i0} is the momentum density

4. T^{ij} is the momentum flux

5. T^{ii} represent the pressure.

The conservation of energy momentum tensor yields

$$\nabla_\nu T^{\mu\nu} = 0. \tag{1.4.1}$$

In a locally inertial frame, this equation reduces to

$$T^{\mu\nu}_{\ \ ,\nu} = 0. \tag{1.4.2}$$

1.5 The Einstein Field Equations

The Einstein field equations (EFEs) are basically a relationship between the geometry of spacetime and the distribution of matter. These equations are obtained [13] by varying the Einstein-Hilbert action with respect to the metric tensor and are given as

$$G_{\mu\nu} = \kappa T_{\mu\nu}, \tag{1.5.1}$$

where $G_{\mu\nu}$ is the Einstein tensor which is symmetric i.e., $G_{\mu\nu} = G_{\nu\mu}$, $T_{\mu\nu}$ is the energy-momentum tensor which can be thought of as the source for the gravitational field. It is a divergenceless tensor due to Bianchi Identities. The coupling constant κ, written in arbitrary units is $8\pi G/c^4$, while it reduces to 8π in natural units ($c = G = 1$). The complete form of Einstein field equations contain an extra term called the cosmological constant denoted by Λ. Its value was found to be extremely small. This constant Λ is responsible for the present accelerated expansion of the Universe. Thus the Einstein field equations with cosmological constant, Λ, can be written as

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{1.5.2}$$

The right hand side of equation (1.52) gives information about the matter distribution while the left hand side represents the geometry of the spacetime. The Einstein field equations are a system of second order, non-linear partial differential equations in the metric tensor.

The significance of the cosmological constant appears mostly in the context of cosmology in which one studies the fate of the Universe. The cosmological constant will appear again when we discuss the BTZ black hole in chapter 2.

1.6 Energy Momentum Conservation

The law of conservation of energy-momentum states that the total flux of energy momentum into a four dimensional region Ω is zero, i.e.,

$$\int_{\partial\Omega} T^{\mu\nu} n_\nu d\sigma = 0, \tag{1.6.1}$$

where $\partial\Omega$ is the boundary of Ω and n_ν is the outward normal vector to $\partial\Omega$. From Gauss's divergence theorem we obtain

$$\int_{\Omega} T^{\mu\nu}{}_{;\nu} \sqrt{-g} d^4x = 0, \tag{1.6.2}$$

for an arbitrary region Ω. Hence, the local formulation of the law of energy momentum conservation has the form

$$T^{\mu\nu}{}_{;\nu} = 0. \tag{1.6.3}$$

Here the time component describes conservation of energy and the space components represent the conservation of momentum.

1.7 Perfect Fluid

A fluid is defined as [13] "a special kind of continuum in which the collection of particles is so large that the dynamics of individual particle can not be followed, leaving only the description of the collection interms of average quantities". The average quantities are the number of particles per unit volume, density of momentum, density of pressure, temperature, pressure, etc. The behavior of a lake of water does not depend upon a single particle, but it mainly depends on the average properties of a large collection of particles.

The above mentioned properties change from point to point in a lake of water. For example, the pressure at the bottom of a lake is greater than at the top. The temperature, pressure, etc may also change from point to point in the lake. A question arises that, how large a collection of particles to average over, this collection must be clearly large enough so that the single particles do not matter, but it must be small enough so that it is relatively homogeneous: the kinetic energy, interparticle spacing and the average velocity must be the same every where in the collection and this collection is called an element.

A fluid can also be defined as “a continuum that flows”. As we know that most of the solids flow under high temperature and pressure, so this definition is not very precise, because the division between fluid and solid is not very well defined. What makes a substance rigid? The rigidity of a substance comes from the forces parallel to the interface between two adjacent elements.

A perfect fluid is defined as “a fluid that has no viscosity and no heat conduction”. It is the generalization of the ideal gas of ordinary thermodynamics. The two restrictions (no viscosity and no heat conduction) in the definition of perfect fluid has many properties which we will discuss in detail. Viscosity is a force which is parallel to the interface between particles, the absence of this force means that the force should always be perpendicular to the interface. This condition makes the perfect fluid a continuum that can flow. Another interesting property of a perfect fluid which makes it distinguished from other fluids is the ‘no heat conduction’. A fluid is able to exchange energy with its surrounding in only two ways: by heat conduction and by work (doing an amount of work). If there is no heat conduction then the perfect fluid can only exchange energy with its surrounding by doing work. The energy momentum tensor for a perfect fluid is

$$T^{\mu\nu} = (\rho + p)u^{\mu}u^{\nu} + pg^{\mu\nu}, \tag{1.7.1}$$

where ρ is the energy density, p is the pressure and u^{μ} is the four velocity vector of the fluid.

1.8 Dark Energy

Dark energy is the most popular way to explain recent observations that the Universe appears to be expanding at an accelerated rate. The exact nature of dark energy is a matter of speculation. It is known to be homogeneous, not very dense and is not known to interact through any of the fundamental forces other than gravity. Since it is not very dense, roughly 10^{-29} grams per cubic centimeter, it is hard to imagine experiments to detect it in the laboratory. Dark energy is supposed to have a strong negative pressure in order to explain the observed acceleration in the expansion rate of the Universe.

The theory of GR has resolved many problems since its birth. There are still issues which need attention for their solution. For example, there are some problems in astrophysics and cosmology like the issue of dark energy and dark matter where we experience several theoretical difficulties. It has been found that 97 percent energy of the Universe consist of dark energy and dark matter (73 percent dark energy and 24 percent dark matter) [12].

In the frame work of GR, the positive accelerated expansion of the Universe means that, at present our Universe is dominated by a mysterious component called the dark energy [14], the component with a positive density $\rho > 0$ and with a negative pressure $p < -\frac{1}{3}\rho$. This dark energy may be in the form of vacuum energy (cosmological constant Λ) with $p = -\rho$ or a dynamical evolving scalar field with a negative pressure. One of the peculiar features of the cosmological dark energy is a possibility of the Big Rip [4], the infinite expansion of the Universe in a finite time. The Big Rip scenario is realized if a dark energy is in the form of the phantom energy with $(\rho + p) < 0$. In this case the cosmological phantom energy density grows at large times and disrupts finally all object upto subatomic scale.

1.9 Phantom Energy

Phantom energy is a hypothetical form of dark energy which possesses strong negative pressure. We shall consider the simplest case where the energy is given by a real scalar field φ with Lagrangian density [15]

$$\mathcal{L} = \frac{-\chi}{2}\frac{\partial\varphi}{\partial x^{\mu}}\frac{\partial\varphi}{\partial x_{\mu}} - V(\varphi). \tag{1.9.1}$$

In the above equation the first term represents the kinetic energy and the second term $V(\varphi)$, represents the potential energy. $\chi = -1$ corresponds to the phantom energy while $\chi = +1$ represent the standard scalar field called quintessence field. Considering the perfect fluid energy momentum tensor, the above Lagrangian gives the following expressions for the energy density and the pressure respectively:

$$\rho = \frac{l}{2}\dot{\varphi}^2 + V(\varphi), \tag{1.9.2}$$

and

$$p = \frac{l}{2}\dot{\varphi}^2 - V(\varphi). \tag{1.9.3}$$

From equations (1.58) and (1.59), we have

$$\rho + p = l\dot{\varphi}^2. \tag{1.9.4}$$

It is clear from equation (1.60) that, for $\chi = -1$, $\rho + p < 0$, so it results in the violation of the Null energy condition $(\rho + p) > 0$. The most striking property of phantom energy is that its energy density varies as some power of the scale factor $a(t)$ [16] and hence grows as the Universe expands, i.e.,

$$\rho \propto a^{3|1+\omega|}, \qquad \omega < -1, \tag{1.9.5}$$

which is quite unlike the behavior of normal matter whose density decreases with the growth of the Universe. Thus phantom energy causes a future singularity commonly called the 'Big Rip'. At this singularity, the energy density of the phantom energy will become infinite and the scale factor reaches infinity in a finite time [17]. Near this singularity, everything is pulled apart and converted into the subparticles. The phantom energy will destroy first the galactic clusters, then galaxies, solar system, atoms and nuclei. Eventually at the Big Rip, even the very fabric of spacetime will be pulled apart.

Chapter 2

Some Black Hole Solutions

2.1 Introduction

The idea of a black hole is fairly old. John Michell and Laplace at the end of the eighteenth century showed that light cannot escape from an object more compact than a radius less than $2GM$ (with $c = 1$). This radius is now called the Schwarzschild radius R_s. A particle of mass m moving with velocity v has its kinetic energy $T = mv^2/2$. This particle cannot escape from an object with mass M only if its kinetic energy $mv^2/2$ is less than the absolute value of the potential energy GMm/R. If we take $v = c$, than one can get easily from this condition the limit on the size of the object M from which nothing can escape. Later in 1916, Carl Schwarzschild was able to solve the Einstein field equation in the vacuum for uncharged spherical systems. His solution known as the Schwarzschild solution, represent the simplest black hole type called the Schwarzschild black hole which is determined by a single parameter only, namely its mass M.

The physics of black holes is largely based on Einstein's general theory of relativity, which is a theory of gravitation. Relativity is a geometrical theory because the mathematical study of spacetime, whether curved or flat, is geometry (it can simply be said that in Special Relativity (SR) we deal with the Euclidean geometry, which is a geometry of flat spacetime, whereas in GR we deal with the non-Euclidean geometry which is the Riemannian spacetime). There is no doubt that GR is one of the most successful physical theories (of course it cannot tell everything about the universe) owing to its predictions that have been confirmed by a number of experimental tests [18].

2.2 Black Hole

The pressure supporting a star comes from the heat produced by fusion of light nuclei into heavier ones. When nuclear fuel is used up, the temperature at the core of the star decreases and the star begins to collapse under the influence of gravity. This collapse may suddenly be stopped due to Fermi degeneracy pressure: the electrons are bring so close to each other that they resist further collapsing. A star supported by Fermi degeneracy pressure is called a white dwarf. White dwarfs are found throughout the Universe and are the end state for most stars

If the total mass of a star is sufficiently high, it will reach the Chandrasekhar limit ($1.5M_s$, where $M_s = 1.2 \times 10^{33}g$ is the mass of the Sun), where even the Fermi degeneracy pressure is not high enough to resist the gravitational pull. At this limit, the star is forced to collapse into a smaller radius and the electrons combine with protons to produce neutrons and neutrinos. The neutrinos simply fly away and the core of the star becomes rich with neutrons. Such a type of star is called a neutron star [19].

The conditions at the core of a neutron star are very different from those of the earth. The massive neutron stars will itself be unable to resist the gravitational pull and will continue to collapse; current estimates of the maximum possible neutron star mass are around $3 - 4M_s$. Since a fluid of neutrons is the densest material we know about, it is believed that the outcome of such a star is a black hole.

A black hole is a region of spacetime, where gravity is so strong that nothing including light can escape from it. A black hole is formed when a star of mass M contracts to a size less then $r = \frac{2GM}{c^2}$. A star with mass at least 3.2 times the mass of our sun, reduces to such a small size when it runs out of its hydrogen and other nuclear fuel. It gets cold and will shrink to an infinitely small, infinitely dense, hypothetical point which is called a singularity. Its surface gravity becomes so strong that it sucks everything into it that is in the range of $r = \frac{2GM}{c^2}$, including light.

How would we know if there was a black hole? Black holes have strong gravitational fields, so one can detect them indirectly by observing matter being influenced by these fields. When matter falls into a black hole, it will accelerate and heat up due to which the matter emits X-rays, which one can easily detect with satellite observatories. A large number of black hole candidates have been detected by this method and the probability of the existence of real black holes in our universe is very large [20]. The large majority of candidates fall into one of the two classes. There are black holes having masses of the order of solar mass or somewhat higher. The other category describes suppermassive black holes, with masses between 10^6 and 10^9 solar masses, which are found at the centers of galaxies. Our own Milky Way galaxy contains an object that is believed to be a black hole of at least $2 \times 10^6 M_s$.

2.3 The Schwarzschild Black Hole

Einstein in 1915 published his theory of GR, a new theory of gravitation that made fundamental predictions on the effect of gravity on light. A few months after the publication of Einstein's GR, Carl Schwarzschild solved the Einstein field equations by assuming a static and spherically symmetric geometry, obtaining what is now called the Schwarzschild solution.

The Newtonian potential around a point mass situated at the origin in its own rest frame is spherically symmetric. Also for objects like stars and planets the same is true to lowest order. Exterior to such objects there is a static, spherically symmetric space. Motivated by this we study spherically symmetric solutions to the Einstein field equation for empty space.

The geometry of a spherically symmetric vacuum, i.e., vacuum spacetime outside the spherical black hole is the Schwarzschild geometry describable in terms of the Schwarzschild metric

$$ds^2 = -(1 - 2GM/r)dt^2 + (1 - 2GM/r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (2.3.1)$$

The Schwarzschild metric is asymptotically flat, that is, for large r

$$ds^2 \approx -(1 - 2GM/r)dt^2 + (1 + 2GM/r)dr^2 + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \quad (2.3.2)$$

It is clear from equation (2.1), that the metric coefficients g_{tt} and g_{rr} become infinite at $r = 0$ and $r = 2GM$ respectively. These are called singularities: the values of coordinates at a given point where the matric ds^2 become undefined [19]. What type of singularities these are? In order to check whether a singularity is a physical singularity or a coordinate singularity, we check a simple criterion. As we see that the metric coefficients are coor-

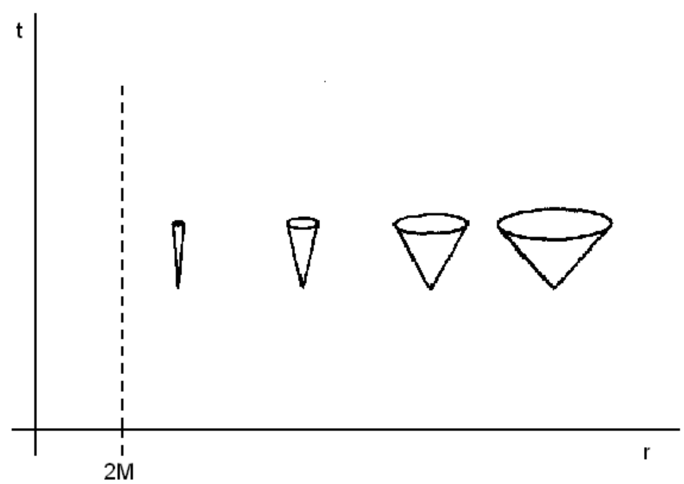

Figure 2.1: In Schwarzschild coordinates, light cones appear to close up as we approach $r = 2GM$. This trajectory is given by solving the Schwarzschild metric $ds^2 = 0$ (null geodesic) with $d\theta^2 + \sin^2\theta d\varphi^2 = 0$.

dinate dependent, the curvature is measured by the Riemann curvature tensor. Thus it is difficult to say when this tensor becomes infinite, since its components are coordinate dependent. But from Riemann curvature tensor we can construct various scalar quantities and since scalars are coordinate independent it is meaningful to say that they become infinite. The simplest such scalars are $R = g^{\lambda\delta}R_{\lambda\delta}$, $R^{\lambda\delta}R_{\lambda\delta}$, $R^{\lambda\delta\mu\nu}R_{\lambda\delta\mu\nu}$ and so on. If any of these scalars goes to infinity at a given point, we say that the point is a singularity of the curvature otherwise not. In the case of Schwarzschild metric (2.1), direct calculation of 'Kretschmann's curvature scalar' reveals that

$$R^{\lambda\delta\mu\nu}R_{\lambda\delta\mu\nu} = \frac{48G^2M^2}{r^6}. \tag{2.3.3}$$

The curvature scalar (2.3) become infinite at $r = 0$, so $r = 0$ is a physical (curvature) singularity, whereas $r = 2GM$ is a coordinate singularity, i.e., this singularity appears due to the bad selection of coordinate. The singularity at $r = 2GM$ is however removable by choosing appropriate coordinates like the Kruskal-Szekeres coordinates found in 1960, which represents the spacetime more appropriately. The Kruskal-Szekeres coordinates are

$$v = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \sinh\left(\frac{t}{4GM}\right), \tag{2.3.4}$$

and

$$u = \left(\frac{r}{2GM} - 1\right)^{1/2} e^{r/4GM} \cosh\left(\frac{t}{4GM}\right). \tag{2.3.5}$$

Using these coordinates, the metric in equation (2.1) becomes

$$ds^2 = \frac{32G^3M^3}{r} e^{-r/2GM}(-dv^2 + du^2) + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{2.3.6}$$

where, now, r is not to be regarded as a coordinate but as a function of u and v, given implicitly by the inverse of equations (2.4) and (2.5):

$$\left(\frac{r}{2GM} - 1\right) e^{r/2GM} = u^2 - v^2. \tag{2.3.7}$$

From equation (2.6), it is clear that, there is nothing singular about any metric coefficient at $r = 2GM$. There is, however, a singularity at $r = 0$, where we expect it. Hence the coordinate singularity at $r = 2GM$ is removed by using Kruskal-Szekeres coordinates.

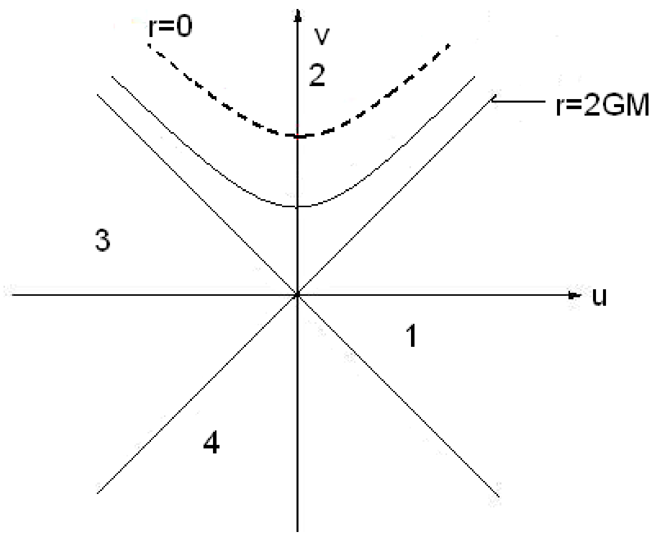

Figure 2.2: The Schwarzschild black hole in the Kruskal-Szekeres coordinates have no "coordinate singularity", hence represents the real spacetime where $r = 0$ is the singularity which is represented by the broken thick line in the figure.

2.4 General Relativity in (2+1)-Dimensions

For many purposes especially in the field of quantum gravity, a particularly useful model is general relativity in three (spacetime) dimensions. Work on (2+1)-dimensional gravity dates back at least to 1963, and occasional articles appeared over the next twenty years [21]. But credit for the recent growth of interest should probably go to Deser, Jackiw, and t' Hooft [22], who examined the classical and quantum dynamics of point sources.

Let us begin by examining the reasons for the simplicity of general relativity in 2+1 dimensions. The Einstein-Hilbert action in three spacetime dimensions becomes

$$I = \frac{1}{2\pi} \int_S d^3x \sqrt{-g}(R - 2\Lambda) + I_m, \tag{2.4.1}$$

where I_m represents the matter part of the action, the units are chosen such that $8G = 1$. The resulting Euler-Lagrange equations are the standard Einstein field equations in (2+1)-dimensions

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \pi T_{\mu\nu}. \tag{2.4.2}$$

In any spacetime, the Riemann curvature tensor may be decomposed into a curvature scalar R, a Ricci tensor $R_{\mu\nu}$, and a remaining trace-free Weyl tensor $C^{\sigma}{}_{\mu\nu\rho}$, i.e., in n-dimensions with $n \geq 3$, the Riemann curvature tensor can be written as follows

$$R_{\mu\nu\rho\sigma} = C_{\mu\nu\rho\sigma} + \frac{2}{n-2}(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}) - \frac{2}{(n-1)(n-2)}Rg_{\mu[\rho}g_{\sigma]\nu}. \qquad (2.4.3)$$

In 2+1 dimensions, however, the Weyl tensor vanishes identically [23], and the full curvature tensor is determined algebraically by the curvature scalar and the Ricci tensor

$$R_{\mu\nu\rho\sigma} = g_{\mu\rho}R_{\nu\sigma} + g_{\nu\sigma}R_{\mu\rho} - g_{\nu\rho}R_{\mu\sigma} - g_{\mu\sigma}R_{\nu\rho} - \frac{1}{2}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})R. \qquad (2.4.4)$$

There is no surprise since the number of independent components of $R_{\mu\nu\rho\sigma}$ and $R_{\mu\nu}$ are, respectively $\frac{n^2(n^2-1)}{12}$ and $\frac{n(n+1)}{2}$ in n-dimensions and are both equal to 6 when $n = 3$. Thus the fundamental physical difference between general relativity in 2+1 and 3+1 dimensions originates from the fact that the curvature tensor in 2+1 dimensions depends linearly on the Ricci tensor. In particular, this implies that any solution of the vacuum Einstein field equations is flat, and that any solution of the field equations with a cosmological constant,

$$R_{\mu\nu} = 2\Lambda g_{\mu\nu}, \qquad (2.4.5)$$

has constant curvature.

2.5 The (2+1)-Dimensional BTZ Black Hole

Maximo Banados, Claudio Teitelboim and Jorge Zanelli discovered, in 1992, the black hole solution of Einstein's equation with a negative cosmological constant, in 2+1 dimensions [24]. This discovery was rather surprising as there was no speculation that there would exist a black hole solution in 2+1 dimensions at that time. The BTZ black hole is known as a simple toy model for a number of studies including the string and supergravity theory. The cosmological constant, Λ is written as $-1/l^2$ in the BTZ black hole's metric which reads:

$$ds^2 = -N(r)dt^2 + N^{-1}(r)dr^2 + r^2(d\varphi - \frac{J}{2r^2}dt)^2, \qquad (2.5.1)$$

where the lapse function is given by

$$N(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} \tag{2.5.2}$$

with $-\infty < t < +\infty$, $0 < r < +\infty$ and $0 \leq \varphi \leq 2\pi$. In equation (2.13), M is the mass and J is the angular momentum of the BTZ black hole. The lapse function $N(r)$ vanishes for two values of the radial coordinate r given by

$$r_{\pm}^2 = \frac{l^2}{2}\left[M \pm \sqrt{M^2 - \frac{J^2}{l^2}}\right]. \tag{2.5.3}$$

The larger root, r_+, is the event horizon of the BTZ black hole. It is evident that in order for the horizon to exist one must have

$$M > 0, \quad |J| \leq Ml. \tag{2.5.4}$$

Therefore, negative black hole masses are excluded from the physical spectrum. There is, however, an important exceptional case. When one sets $M = -1$ and $J = 0$, the singularity, i.e., $r = 0$, disappears. There is neither a horizon nor a singularity to hide.

For the special case of spinless ($J = 0$) BTZ black hole, the line element (2.13) takes the simple form

$$ds^2 = -N(r)dt^2 + N^{-1}(r)dr^2 + r^2 d\varphi^2, \tag{2.5.5}$$

where the lapse function is now given by

$$N(r) = -M + \frac{r^2}{l^2}. \tag{2.5.6}$$

In this specific case, the lapse function also vanishes for two values of r given by

$$r_{\pm} = \pm l\sqrt{M}. \tag{2.5.7}$$

The rotating BTZ black hole with the incorporation of the charge Q, i.e., charged rotating BTZ (CR-BTZ) black hole solution is given by [25]

$$ds^2 = -N(r)dt^2 + N^{-1}(r)dr^2 + r^2(d\varphi - \frac{J}{2r^2}dt)^2, \tag{2.5.8}$$

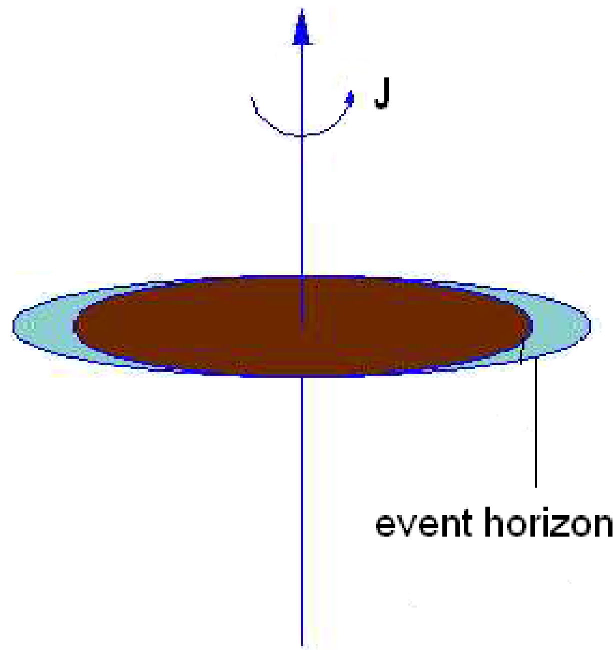

Figure 2.3: The 2+1 dimensional BTZ black hole. This black hole can be visualized as a circular disc with spin J. Its event horizon depends on its mass.

with lapse function given by

$$N(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2} - \frac{\pi}{2}Q^2 \ln r. \tag{2.5.9}$$

Chapter 3

Phantom Energy Accreting BTZ Black Hole

3.1 Introduction

Over the past decade, (2+1)-dimensional gravity has become an active field of research, drawing insights from general relativity, differential geometry and topology, high energy particle physics, field theory and string theory. Interest in (2+1)-dimensional gravity - general relativity in two spatial dimensions plus time - dates back at least to 1963, when Staruszkiewicz first showed that point particles in a (2+1)-dimensional spacetime could be given a simple and elegant geometrical description [26]. Over the next twenty years, occasional papers on classical [27] and quantum mechanical [28] aspects appeared, but until recently the subject remained largely a curiosity.

In this chapter, we first review the accretion of phantom energy onto a static non-rotating (2+1)-dimensional BTZ black hole discussed in [11]. After this review, we will discuss the accretion of phantom energy onto a rotating (2+1)-dimensional BTZ black hole. We investigate the accretion of phantom energy onto a static non-rotating (2+1)-dimensional BTZ black hole. It is obvious that the usual spacetime has three spatial dimensions, so the BTZ black hole is merely a mathematical construct. In the present chapter, we are interested in understanding how the accretion of phantom energy will effect a lower dimensional black hole. The expression for the evolution of non-rotating BTZ black hole mass is independent of its mass and depends only on the energy density and pressure of the phantom energy, although the mass decreases due to accretion. We are also interested in understanding how the accretion of phantom energy will effect the rotating BTZ black hole. We will show that the expression of the evolution of rotating BTZ black hole mass depends not only on the energy density and pressure of the phantom energy but also on the mass of the rotating BTZ black hole.

3.2 Model of Accretion for Non-Rotating BTZ Black Hole

In this section, we review the model of accretion for a (2+1)-dimensional non-rotating BTZ black hole discussed in [11]. The Einstein field equations for a (2+1)-dimensional spacetime with a negative cosmological constant, $\Lambda = -\frac{1}{l^2}$, is

$$R_{ab} - \frac{1}{2}Rg_{ab} - \frac{1}{l^2}g_{ab} = \pi T_{ab}, \quad a, b = 0, 1, 2, \tag{3.2.1}$$

where R_{ab} is the Ricci tensor in (2+1)-dimensions, R is the Ricci scalar and T_{ab} is the stress energy tensor of the matter field. The units are chosen such that $c = 1$ and $8G = 1$. Consider the vacuum stress energy tensor for which the Einstein field equations has the solution of a (2+1)-dimensional BTZ black hole metric [24]

$$ds^2 = -N(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2 d\phi^2, \tag{3.2.2}$$

where $N(r) = -M + r^2/l^2$ is the lapse function, M is the dimensionless mass of the BTZ black hole and $l^2 = -1/\Lambda$ is a positive constant. The event horizon of the BTZ black hole is obtained by setting $N(r) = 0$, which yields

$$r_{\pm} = \pm l\sqrt{M}, \tag{3.2.3}$$

where $r_+ = l\sqrt{M}$ is the outer horizon and $r_- = -l\sqrt{M}$ is the inner horizon of the (2+1)-dimensional non-rotating BTZ black hole. Now the metric tensor and its inverse have the components

$$g_{ab} = \begin{pmatrix} -N(r) & 0 & 0 \\ 0 & \frac{1}{N(r)} & 0 \\ 0 & 0 & r^2 \end{pmatrix}, \tag{3.2.4}$$

$$g^{ab} = \begin{pmatrix} -\frac{1}{N(r)} & 0 & 0 \\ 0 & N(r) & 0 \\ 0 & 0 & r^{-2} \end{pmatrix}. \tag{3.2.5}$$

Also we have $\sqrt{|g|} = r$, where g is the determinant of the metric g_{ab}. Considering the phantom energy to be a perfect fluid with the energy momentum tensor

$$T^{ab} = (\rho + p)u^a u^b + pg^{ab}, \tag{3.2.6}$$

where ρ is the energy density and p is the pressure of the phantom energy while $u^a = (u^0, u^1, 0)$ is the three vector velocity of the fluid flow. Considering $u^1 = u$, which is the radial component of velocity of the flow while the third component of the three vector velocity is zero because of the spherical symmetry of the BTZ black hole. The normalization condition of the three vector velocity is

$$g_{ab}u^a u^b = -1,$$

or

$$g_{00}(u^0)^2 + g_{11}u^2 = -1. \tag{3.2.7}$$

The solution of equation (3.7) is

$$u^0 = \pm\frac{\sqrt{N(r) + u^2}}{N(r)}. \tag{3.2.8}$$

The metric g_{ab} can be used to raise and lower the indices, i.e., $u_0 = g_{00}u^0$, which yields

$$u_0 = \mp\sqrt{N(r) + u^2}. \tag{3.2.9}$$

In order to discuss the accretion dynamics of phantom energy onto a (2+1)-dimensional non-rotating BTZ black hole one can employ the formalism from the work of Babichev et al. [9]. The accretion phenomenon mainly depends on two important conservation equations, one which controls the energy flux $T^{0a}{}_{;a} = 0$ across the event horizon and the other that controls the conservation of mass flux $J^{a}{}_{;a} = 0$, where J^a is the current density whose components are $(\rho u^0, \rho u, 0)$. Since the black hole is stationary, the only component of stress energy tensor of interest is T^{01}. Thus from the equation of energy conservation $T^{0a}{}_{;a} = 0$, one gets

$$\frac{T^{01}{}_{,1}}{T^{01}} + \frac{N'(r)}{N(r)} + \frac{1}{r} = 0, \tag{3.2.10}$$

where ", 1" represent partial derivative with respect to the radial coordinate. Integrating equation (3.10), gives

$$\ln|T^{01}| + \ln|N(r)| + \ln|r| = C_0,$$

or

$$|rN(r)T^{01}| = e^{C_0},$$

or

$$rN(r)(\rho + p)u^0 u = \pm C_1, \tag{3.2.11}$$

where $C_1 = e^{C_0}$ is a positive constant of integration. Substituting equation (3.8) in (3.11), gives

$$\pm\left(ur(\rho + p)\sqrt{N(r) + u^2}\right) = \pm C_1. \tag{3.2.12}$$

Since for the inward flow $u < 0$. Also for phantom energy $(\rho + p) < 0$. On the left hand side of equation (3.12), the quantity inside the brackets is positive. Hence we take $u^0 = \frac{\sqrt{N(r)+u^2}}{N(r)}$ and $u_0 = -\sqrt{N(r) + u^2}$, therefore

$$ur(\rho + p)\sqrt{N(r) + u^2} = C_1. \tag{3.2.13}$$

In order to find the second integral of motion one can use the energy momentum conservation along the velocity three vector (the energy flux equation)

$$u_a T^{ab}{}_{;b} = 0. \tag{3.2.14}$$

Assume phantom energy to be a perfect fluid for which the conservation law becomes [?].

$$u^b \rho_{,b} + (\rho + p) u^b_{;b} = 0. \tag{3.2.15}$$

Simplifying equation (3.15), gives

$$\frac{\rho_{,1}}{\rho + p} + \frac{u_{,1}}{u} + \frac{1}{r} = 0.$$

Integrating this equation, yields

$$\int_{\rho_\infty}^{\rho_h} \frac{d\rho}{\rho + p} + \ln|ur| = A$$

or

$$ur\ exp\left[\int_{\rho_\infty}^{\rho_h} \frac{d\rho}{\rho + p}\right] = \pm e^A, \tag{3.2.16}$$

since $u < 0$, while all other quantities on the left hand side of equation (3.16) are positive, this equation holds only if one take negative sign on the right hand side, i.e.,

$$ur\ exp\left[\int_{\rho_\infty}^{\rho_h} \frac{d\rho}{\rho + p}\right] = -A_1, \tag{3.2.17}$$

where $A_1 = e^A$ is the positive constant of integration. Also ρ_h and ρ_∞ are the energy densities of the phantom energy at the BTZ black hole horizon and at infinity respectively. Using equation (3.17) in equation (3.13), gives

$$(\rho + p)\sqrt{N(r) + u^2}\ exp\left[-\int_{\rho_\infty}^{\rho_h} \frac{d\rho}{\rho + p}\right] = C_2, \tag{3.2.18}$$

where $C_2 = -C_1/A_1 = \rho_\infty + p_\infty$. In order to calculate the rate of change of mass of BTZ black hole $\dot{M}$, we integrate the flux of phantom energy over the entire BTZ black hole horizon to get

$$\dot{M} = \oint T^1{}_0 dS. \tag{3.2.19}$$

Here T^1_0 determines the energy flux in the radial direction only and $dS = \sqrt{-g}d\varphi$ is the infinitesimal surface element of the BTZ black hole horizon. Equation (3.19) gives

$$\dot{M} = 2\pi r T^1_0. \tag{3.2.20}$$

Since $T^1_0 = -(\rho + p)u\sqrt{N(r) + u^2}$, thus equation (3.20), becomes

$$\dot{M} = -2\pi ur(\rho + p)\sqrt{N(r) + u^2}. \tag{3.2.21}$$

Substituting equations (3.17) and (3.18) in equation (3.21), we have

$$dM = 2\pi A_1(\rho_\infty + p_\infty)dt. \tag{3.2.22}$$

It follows from equation (3.22) that the mass of a (2+1)-dimensional non-rotating BTZ black hole decreases as it accretes the phantom energy (because all the parameters are positive except $(\rho + p)$ which is negative). In particular, this implies that the BTZ black hole mass in the Universe filled with phantom energy must decrease. Moreover, equation (3.22) is independent of mass contrary to the case of Schwarzschild black hole [9]. The physical reason of the decrease in the black hole mass is as follows: The phantom energy falls on the black hole, but the energy flux associated with this fall is directed away from the black hole. Further, equation (3.22) is valid for any general ρ and p violating the null energy condition, thus one can write

$$dM = 2\pi A_1(\rho + p)dt. \tag{3.2.23}$$

3.2.1 Critical Accretion for Non-Rotating BTZ Black Hole

In this section, we review critical accretion for non-rotating BTZ black hole discussed in [11]. In order to evaluate the critical point of accretion, one can find out those solutions which pass through the critical point. Such solution correspond to the material falling into the black hole with monotonically increasing speed. Near the critical point of accretion, the falling fluid can exhibit a variety of behaviors, close to the compact object. The continuity equation or the equation of mass flux is

$$J^a{}_{;a} = 0, \tag{3.2.24}$$

where J^a is called the current density and has components $J^a = (\rho u^0, \rho u, 0)$. Equation (3.24) after solving gives

$$\rho u r = K_1, \tag{3.2.25}$$

where K_1 is a constant of integration. Using equation (3.25) in equation (3.13) and squaring, we get

$$\left(\frac{\rho + p}{\rho}\right)^2 \left(-M + \frac{r^2}{l^2} + u^2\right) = C_3, \tag{3.2.26}$$

where $C_3 = (C_1/K_1)^2$. Taking differential of equation (3.25) and (3.26), we get

$$\frac{d\rho}{\rho} = -\frac{du}{u} - \frac{dr}{r}, \tag{3.2.27}$$

and

$$v^2\frac{d\rho}{\rho} + \frac{r^2/l^2}{N(r)+u^2}\frac{dr}{r} + \frac{u^2}{N(r)+u^2}\frac{du}{u} = 0, \tag{3.2.28}$$

respectively, where

$$v^2 = \frac{d\ln(\rho+p)}{d\rho} - 1. \tag{3.2.29}$$

Now eliminating $d\rho/\rho$ from equations (3.27) and (3.28), one gets

$$\left[-v^2 + \frac{u^2}{N(r)+u^2}\right]\frac{du}{u} + \left[-v^2 + \frac{r^2/l^2}{N(r)+u^2}\right]\frac{dr}{r} = 0. \tag{3.2.30}$$

Equation (3.30) is the general equation which can be used to get the critical speed of flow for the phantom energy. It is clear from equation (3.30) that if one or the other bracket factor is zero , one gets a turn-around point corresponding to double valued solution in either r or u. We are interested only in those solutions that passes through the critical point, such solutions will correspond to material falling into the black hole. In order to get the critical point we take both the bracket factors in equation (3.30) to be equal to zero, this will gives the critical points of accretion

$$v_c^2 = \frac{r_c^2/l^2}{N(r_c)+u_c^2}, \tag{3.2.31}$$

and

$$v_c^2 = \frac{u_c^2}{N(r_c) + u_c^2}, \tag{3.2.32}$$

here the subscript c refer to the critical quantities. Comparing equations (3.31) and (3.32), one can easily get

$$u_c^2 = \frac{r_c^2}{l^2}, \tag{3.2.33}$$

and

$$v_c^2 = \frac{u_c^2}{-M + 2u_c^2}. \tag{3.2.34}$$

Here the quantity u_c represents the critical speed of flow at the critical points which are to be determined below. For physically acceptable solution, we require $v_c^2 > 0$, hence we get the following restriction on speed and location of critical points

$$u_c^2 > \frac{M}{2}, \tag{3.2.35}$$

and

$$r_c^2 > \frac{r_+^2}{2}. \tag{3.2.36}$$

3.3 Model of Accretion for Rotating BTZ Black Hole

The accretion would be much more interesting when an additional parameter like angular momentum is also incorporated in the BTZ black hole spacetime. In this section, we extended the work of [11], discussed in section (3.2) for rotating BTZ black hole, by studying the accretion of phantom energy onto a rotating BTZ black hole.

The axial symmetric rotating solution to the (2+1)-dimensional Einstein field equations (3.1), is represented by the rotating BTZ black hole metric [24].

$$ds^2 = -N(r)dt^2 + \frac{1}{N(r)}dr^2 + r^2\left(d\varphi - \frac{J}{2r^2}dt\right)^2, \tag{3.3.1}$$

where $N(r) = -M + \frac{r^2}{l^2} + \frac{J^2}{4r^2}$ is the lapse function and M is the dimensionless mass, while J is the angular momentum of the rotating BTZ black hole. From equation (3.35), we can write

$$g_{ab} = \begin{pmatrix} -N(r) + \frac{J^2}{4r^2} & 0 & -\frac{1}{2}J \\ \\ 0 & \frac{1}{N(r)} & 0 \\ \\ -\frac{1}{2}J & 0 & r^2 \end{pmatrix}, \tag{3.3.2}$$

and

$$g^{ab} = \begin{pmatrix} \frac{-1}{N(r)} & 0 & \frac{-J}{2r^2 N(r)} \\ \\ 0 & N(r) & 0 \\ \\ \frac{-J}{2r^2 N(r)} & 0 & \frac{4r^2 N(r) - J^2}{4r^4 N(r)} \end{pmatrix}. \tag{3.3.3}$$

The outer event horizon of the rotating BTZ black hole is obtained by setting $N(r) = 0$, which gives

$$r_e = l \left[\frac{M}{2} \left\{ 1 + \left(1 - \left(\frac{J}{Ml} \right)^2 \right)^{1/2} \right\} \right]^{1/2}. \tag{3.3.4}$$

Note that $\sqrt{|g|} = r$, where g is the determinant of g_{ab}. In order to discuss the accretion dynamics of phantom energy onto the rotating BTZ black hole we follow the formalism from the work of M. Jamil and M. Akbar [11]. We consider the phantom energy to be a perfect fluid for which the energy momentum tensor is

$$T^{ab} = (\rho + p)u^a u^b + p g^{ab}, \tag{3.3.5}$$

where ρ and p are the energy density and pressure of the phantom energy respectively, while $u^a = (u^0, u^1, 0)$ is the three vector velocity of the fluid with pressure $p = p(r)$ and energy density $\rho = \rho(r)$. The normalization condition of the fluid velocity implies

$$g_{ab} u^a u^b = -1. \tag{3.3.6}$$

We can write equation (3.42) in components form as

$$g_{00}(u^0)^2 + g_{11}(u^1)^2 = -1. \tag{3.3.7}$$

Here we consider $u^1 = u$, which is the radial component of velocity of the fluid flow while, the third component of the three vector velocity u^2 is zero because of the spherical symmetry of the rotating BTZ black hole. We solve equation (3.43) for u^0, which after substitution of g_{00} and g_{11} from the metric (3.38), gives

$$u^0 = \sqrt{\frac{N(r)+u^2}{N(r)\left(N(r)-\frac{J^2}{4r^2}\right)}}, \tag{3.3.8}$$

and lowering the indices of the velocity component by multiplying equation (3.44) by g_{00}, we have

$$u_0 = -\sqrt{N(r)-\frac{J^2}{4r^2}}\sqrt{\frac{N(r)+u^2}{N(r)}}. \tag{3.3.9}$$

There are two important equations of conservation on which the accretion process mainly depends, among which one is the energy flux equation $T^{0a}{}_{;a}$ and the other is the conservation of mass flux equation $J^a{}_{;a}$, where J^a is the current density of the fluid. Since the black hole is stationary, from the energy flux conservation equation $T^{0a}{}_{;a}$, one gets the equation of conservation

$$\frac{T^{01}{}_{,1}}{T^{01}} + \frac{N'(r)}{N(r)} + \frac{1}{r} + \frac{J^2}{2r^3N(r)} = 0, \tag{3.3.10}$$

where ", 1" represent the partial derivative with respect to the radial coordinate r. Integrating equation (3.46), we have

$$\ln|T^{01}rN(r)| + \frac{J^2}{2}\int\frac{dr}{r^3N(r)} = \ln C. \tag{3.3.11}$$

Using $T^{01} = (\rho+p)uu^0$ and the value of u^0 and $N(r)$ in equation (3.47), we have

$$\ln\left|(\rho+p)ur\sqrt{\frac{N(r)(N(r)+u^2)}{N(r)-\frac{J^2}{4r^2}}}\right| + \frac{J^2}{2}\int\frac{dr}{r\left(\frac{r^4}{l^2}-Mr^2+\frac{J^2}{4}\right)} = \ln C. \tag{3.3.12}$$

We first solve the integral on the left hand side of equation (3.48), which after simplification gives

$$\begin{aligned}
&\int\frac{dr}{r(\frac{r^4}{l}-Mr^2+\frac{J^2}{4})} \\
&= \frac{4}{J^2}\ln|r| + \left(\frac{M-\sqrt{M^2-\frac{J^2}{l^2}}}{J^2\sqrt{M^2-\frac{J^2}{l^2}}}\right)\ln\left|l^2\left(M+\sqrt{M^2-\frac{J^2}{l^2}}-\frac{2r^2}{l^2}\right)\right| \\
&- \left(\frac{M+\sqrt{M^2-\frac{J^2}{l^2}}}{J^2\sqrt{M^2-\frac{J^2}{l^2}}}\right)\ln\left|l^2\left(\frac{2r^2}{l^2}-M+\sqrt{M^2-\frac{J^2}{l^2}}\right)\right|.
\end{aligned} \tag{3.3.13}$$

Using equation (3.49) in (3.48) and simplifying, we get

$$(\rho+p)ur^3\sqrt{\frac{N(r)\,(N(r)+u^2)}{N(r)-\frac{J^2}{4r^2}}}\left(\frac{\sqrt{M^2-\frac{J^2}{l^2}}+(M-\frac{2r^2}{l^2})}{\sqrt{M^2-\frac{J^2}{l^2}}-(M-\frac{2r^2}{l^2})}\right)^{\frac{M}{2\sqrt{M^2-\frac{J^2}{l^2}}}}\times \frac{1}{\sqrt{r^2l^2(M-\frac{r^2}{l^2})-\frac{J^2l^2}{4}}}=C_1, \tag{3.3.14}$$

where $C_1 = 2e^C$ is the constant of integration. In order to find the second integral of motion we use the energy momentum conservation along the three-vector velocity (the energy flux equation)

$$u_aT^{ab}{}_{;b}=0. \tag{3.3.15}$$

We consider the phantom energy to be a perfect fluid for which the conservation law becomes

$$\rho_{,b}u^b+(\rho+p)u^b{}_{;b}=0, \tag{3.3.16}$$

The solution of equation (3.52) is

$$ur\ exp\left(\int_{\rho_\infty}^{\rho_h}\frac{d\rho}{\rho+p}\right)=-A_1, \tag{3.3.17}$$

where $A_1 = e^A$ is a positive constant of integration. Also ρ_h and ρ_∞ are the energy density of the phantom energy at BTZ black hole horizon and at infinity respectively. Substituting equation (3.53) in (3.50), we have

$$(\rho+p)r^2\ exp\left[-\int_{\rho_\infty}^{\rho_h}\frac{d\rho}{\rho+p}\right]\sqrt{\frac{N(r)(N(r)+u^2)}{N(r)-\frac{J^2}{4r^2}}}\times \left(\frac{\sqrt{M^2-\frac{J^2}{l^2}}+(M-\frac{2r^2}{l^2})}{\sqrt{M^2-\frac{J^2}{l^2}}-(M-\frac{2r^2}{l^2})}\right)^{\frac{M}{2\sqrt{M^2-\frac{J^2}{l^2}}}}\frac{1}{\sqrt{r^2l^2(M-\frac{r^2}{l^2})-\frac{J^2l^2}{4}}}=C_2, \tag{3.3.18}$$

where

$$C_2=\frac{-C_1}{A_1}=\rho_\infty+p_\infty. \tag{3.3.19}$$

The rate of change in the mass of rotating BTZ black hole is

$$\dot{M}=2\pi rT_0^1. \tag{3.3.20}$$

The value of T_0^1 from the stress energy tensor is

$$T_0^1 = (\rho + p)u\sqrt{N(r) + u^2}\sqrt{\frac{N(r) - \frac{J^2}{4r^2}}{N(r)}}. \tag{3.3.21}$$

Using equation (3.53) along with (3.54), (3.55) and (3.57) in equation (3.56), we have

$$\dot{M} = 2\pi\frac{l}{r}A_1(\rho_\infty + p_\infty)\left(\frac{M - \frac{r^2}{l^2}}{[-N(r)]^{\frac{\sqrt{M^2-\frac{J^2}{l^2}}+M}{2\sqrt{M^2-\frac{J^2}{l^2}}}}}\right)\left(\frac{\left(\sqrt{M^2 - \frac{J^2}{l^2}} - (M - \frac{2r^2}{l^2})\right)^2}{\frac{4r^2}{l^2}}\right)^{\frac{M}{2\sqrt{M^2-\frac{J^2}{l^2}}}}. \tag{3.3.22}$$

By taking $J = 0$ in equation (3.58) one gets the result of the non-rotating BTZ black hole (3.22). In equation (3.58), $(\rho_\infty + p_\infty) < 0$, so the mass of the rotating BTZ black hole must decrease only if, $\frac{\sqrt{M^2-\frac{J^2}{l^2}}+M}{2\sqrt{M^2-\frac{J^2}{l^2}}} = 2n$, $M > \frac{r^2}{l^2}$ and $M > \frac{J}{l}$ where $n = 0, \pm1, \pm2, ...$ Further the last equation is valid for any general ρ and p violating the null energy condition, thus we have

$$dM = 2\pi\frac{l}{r}A_1(\rho + p)\left(\frac{M - \frac{r^2}{l^2}}{[-N(r)]^{\frac{\sqrt{M^2-\frac{J^2}{l^2}}+M}{2\sqrt{M^2-\frac{J^2}{l^2}}}}}\right)\left(\frac{\left(\sqrt{M^2 - \frac{J^2}{l^2}} - (M - \frac{2r^2}{l^2})\right)^2}{\frac{4r^2}{l^2}}\right)^{\frac{M}{2\sqrt{M^2-\frac{J^2}{l^2}}}} dt. \tag{3.3.23}$$

3.3.1 Critical Accretion for Rotating BTZ Black Hole

Now we are in a position to evaluate the critical point of accretion. At the critical point of accretion the phantom energy shows a variety of changes in their behaviors close to the compact object. In order to observe these changes in the behavior of the phantom energy we find out those solutions that pass through the critical point as these correspond to the material falling into the black hole with monotonically increasing speed. The equation of mass flux or the continuity equation is

$$J^a_{;a} = 0, \tag{3.3.24}$$

where J is the current density. Simplifying equation (3.60), we have

$$\rho u r = C_3, \tag{3.3.25}$$

where C_3 is the constant of integration. Putting equation (3.61) in equation (3.50), we have

$$\left(\frac{\rho+p}{\rho}\right)^2 r^4 \frac{N(r)\left(N(r)+u^2\right)}{N(r)-\frac{J^2}{4r^2}} \left(\frac{\sqrt{M^2-\frac{J^2}{l^2}}+(M-\frac{2r^2}{l^2})}{\sqrt{M^2-\frac{J^2}{l^2}}-(M-\frac{2r^2}{l^2})}\right)^{\frac{M}{\sqrt{M^2-\frac{J^2}{l^2}}}} \times \frac{1}{r^2l^2(M-\frac{r^2}{l^2})-\frac{J^2l^2}{4}} = C_4, \tag{3.3.26}$$

where $C_4 = (\frac{C_1}{C_3})^2$. Taking the differential of equation (3.61) and (3.62) and eliminating $\frac{d\rho}{\rho}$, we get

$$\left[-v^2+\frac{u^2}{N(r)+u^2}\right]\frac{du}{u} + \left[-v^2+\frac{\frac{r^2}{l^2}-\frac{J^2}{4r^2}}{N(r)+u^2}-\frac{M}{-M+\frac{r^2}{l^2}}+\frac{M}{f(r)}\right]\frac{dr}{r}=0, \tag{3.3.27}$$

where

$$v^2 = \frac{d\ln(\rho+p)}{d\rho}-1. \tag{3.3.28}$$

It is clear from equation (3.63) that if one or the other bracket factor vanishes, we get a turn-around point corresponding to double valued solution in either r or u. The only solution that passes through the critical point of accretion is feasible. The feasible solutions will correspond to the material falling into the black hole with monotonically increasing velocity. In order to get the critical point of accretion we take both the bracket factors in equation (3.63) equal to zero, this will give us the critical point of accretion as

$$v_c^2 = \frac{\frac{r_c^2}{l^2}-\frac{J^2}{4r_c^2}}{N(r_c)+u^2}-\frac{M}{-M+\frac{r_c^2}{l^2}}+\frac{M}{N(r_c)}, \tag{3.3.29}$$

and

$$v_c^2 = \frac{u_c^2}{N(r_c)+u_c^2}, \tag{3.3.30}$$

where the subscript c refers to the critical quantities. Comparing equation (3.65) and (3.66), we have

$$u_c^2 = \frac{r_c^2}{l^2}\left(\frac{(-M+\frac{r_c^2}{l^2})^2-(\frac{J^2}{4r_c^2})^2}{(-M+\frac{r_c^2}{l^2})^2+\frac{J^2}{4l^2}}\right), \tag{3.3.31}$$

In equation (3.67) the quantity u_c represents the critical speed of flow at the critical point. Furthermore by taking $J = 0$ in equation (3.66) and (3.67), one gets the corresponding results of the non-rotating BTZ black hole equations (3.34) and (3.33) respectively. For physically acceptable solution, we require $v_c^2 > 0$. Hence we get the following restriction on speed and location of critical points

$$u_c^2 > M - \frac{r_c^2}{l^2} - \frac{J^2}{4r_c^2}, \tag{3.3.32}$$

and

$$r_c^2 > \frac{l^2 M}{2}. \tag{3.3.33}$$

Conditions (3.68) and (3.69) reduce to conditions (3.35) and (3.36) by taking $J = 0$.

3.4 Conclusion

The main objective of this dissertation was to study the accretion of phantom energy onto a (2+1)-dimensional BTZ black hole. The motivation behind this work is to study the accretion dynamics in low dimensional gravity.

In chapter 1, we discussed some definitions and basic equations. We also made the background by discussing perfect fluid, dark energy, and phantom energy.

In the second chapter we discussed some black hole solutions, such as Schwarzschild solution, (2+1)-dimensional BTZ black hole solution and the properties of these black hole solutions.

Our main focus is chapter three, which consists of two major sections. In the first section, we consider the effect of the phantom energy accretion onto a (2+1)-dimensional non-rotating BTZ black hole, while in second section we discuss the effect of phantom energy accretion onto a (2+1)-dimensional rotating BTZ black hole. In first section our analysis has shown that the evolution of the non-rotating BTZ black hole mass dependent only on the energy density and pressure of the phantom energy and is independent of mass of the non-rotating BTZ black hole, the black hole mass decreases due to the accretion of phantom energy. The physical cause for the decrease in the black hole mass is as follows: the phantom energy falls to the black hole, but the energy flux associated with this fall is directed away from the back hole. We also discuss the critical accretion for phantom energy and found the critical speed of flow.

The accretion would be much more interesting when additional parameters like charge and angular momentum are also incorporated in the BTZ spacetime. So in second section of chapter three we discuss the accretion of phantom energy by introducing the angular momentum in the BTZ spacetime. We follow the same procedure as followed in the non-rotating BTZ black hole case. By taking $J = 0$ all the results of the rotating BTZ black hole convert to the results of non-rotating BTZ black hole. Our analysis has shown that the evolution of the rotating BTZ black hole mass depends not only on the energy density and pressure of the phantom energy but also on the mass of the rotating BTZ black hole. In case of rotating BTZ black hole the mass will decrease only if, $\frac{\sqrt{M^2-\frac{J^2}{l^2}}+M}{2\sqrt{M^2-\frac{J^2}{l^2}}} = 2n$, $M > \frac{r^2}{l^2}$ and $M > \frac{J}{l}$ where $n = 0, \pm 1, \pm 2, ...$ We also discuss the critical accretion and found the critical parameters.

3.5 Further Line of Work

In this dissertation, we focused on the accretion of phantom energy onto a non-rotating and rotating BTZ black hole. It is also important to discuss the accretion of phantom energy onto a charged rotating BTZ black hole. One can also investigate the thermodynamic features of phantom energy accretion onto rotating BTZ black holes and charged rotating BTZ black hole. It is important to verify whether the laws of thermodynamics are respected in the processes under investigation. In particular, it is evident that the second law of thermodynamics is violated, i.e., $\dot{S} < 0$ due to violation of the null energy condition. This suggests that we use the generalized second law of thermodynamics for black holes accreting phantom energy, because generalized second law of thermodynamics holds in most cases.

References

[1] S.K. Chakrabarti, Study of accretion processes on black holes: Fifty years of developments, arXiv:astro-ph/0402562v1.

[2] D.N. Spergel et. al. [WMAP Collaboration] Ap. J. Supple 148 (2003) 175; H.V. Peires et. al. [WMAP Collaboration] Ap. J. Supple 148 (2003) 213.

[3] N. Bahcall, J.P. Ostriker, S. Perlmutter and P.J. Steinhardt, Science 284 (1999) 1481.

[4] R.R. Caldwell, Phys. Lett. B 23 (2002) 545.

[5] B. Melnnes, J. High Energy Phy. 029 (2002) 0208.

[6] H. Bondi, Mon. Not. Roy. Astron. Soc. 112 (1952) 195.

[7] F.C. Michel, Ap. Sp. Sc. 15 (1972) 153; C. Armendarize-Picon, T. Damour and V. Mukhanov, Phy. Lett. B 458 (1999) 209.

[8] B.J. Carr, S.W. Hawking, Mon. Not. R. Astron. Soc. 168 (1974) 399.

[9] E. Babichev et al, Phys. Rev. Lett. 93 (2004) 021102; E. Babichev et al, arXiv:0806.0916v3 [gr-qc].

[10] C. Gao, X. Chen, V. Faraoni, Y.G. Shen, Phys. Rev. D 78 (2008) 024008.

[11] M. Jamil, M. Akbar, Gen. Rel. Grav. 43 (2011) 1061.

[12] A.G. Riess et al. (Supernova Search Team), Astron. J. 116 (1998) 1009; S. Perlmuttre et al. Astrophys. J. 517 (1999) 565; P.J.E. Peebles, and B. Ratra, Rev. Mod. Phys 75 (2003) 559; A.G. Riess et al. Ap. J. 607 (2004) 665.

[13] B.F. Schutz, A First Course in General Relativity, (Cambridge University Press, 1985).

[14] E.J. Copeland, M. Sami and S. Tsujikawa, Int. J. Mod, Phys. D 15 (2002) 1753.

[15] O. Gron, S. Hervik, Einstein's General Theory of Relativity With Modern Applications in Cosmology, (Cambridge University Press, 1985).

[16] J.A.S. Lima and J. S. Alacaniz, Phys. Lett. B 600 (2004)191; J.A.S. Lima et al, astro-ph/0807.3379.

[17] V. Johri, Phys. Rev. D 70 (2004) 041303.

[18] J.B. Hartle, Gravity. An Introduction to Einstein's General Relativity (Addison Wesley, 2003).

[19] S.M. Carroll, Spacetime and Geometry. An Introduction to General Relativity (Addison Wesley, 2004).

[20] A. Celotti, J.C. Miller and D.W. Sciama, Class. Quant. Grav. 16 (1999) A3.

[21] H. Leutwyler, Nuovo Cimento 42A (1966) 159; P. Collas, Am. J. Phys. 45 (1977) 833, G. Clement, Nucl. Phys. B 114 (1976) 437.

[22] S. Deser, R. Jackiw, and G. t' Hooft, Ann. Phys. 152 (1984) 220; S. Deser and R. Jackiw, Commun. Math. Phys. 118 (1988) 495; G. t' Hooft, Commun. Math. Phys. 117 (1988) 685.

[23] S. Carlip, Quantum Gravity in 2+1 Dimensions, (Cambridge University Press, 2003).

[24] M. Banados et al, Phys. Rev. Lett. 69 (1992) 1849; M. Banados et al, Phys. Rev. D 48 (1993) 1506.

[25] C. Martnez, C. Teitelboim and J. Zanelli, Phys. Rev. D 61 (2000) 104013.

[26] A. Staruszkiewicz, Acta. Phys. Pol. 6 (1963) 735.

[27] G. Clement, Nuc. Phys. B 114 (1976) 437.

[28] H. Leutwyler, Nuo. Cim. 42 (1966) 159.