Motion Of a Charged Particle Around a Slowly Rotating Kerr Black Hole Immersed in a Magnetic Field

by

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This Dissertation is dedicated to my parents

for their endless love, support and encouragement.

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Abstract

In this thesis motion of a charged particle around a slowly rotating Kerr black hole immersed in a magnetic field is investigated. There is both theoretical and observational evidence that a magnetic field is present in the vicinity of a black hole. This is probably due to plasma present in the accretion disk of the black hole. Particularly our focus is on escape trajectories of the particle moving around a black hole in the innermost stable circular orbit. It is found that either the particle moves in clockwise or anticlockwise direction around the black hole, after the kick by another particle, it will escape to infinity or its motion remain bound, depends on its energy. The change in the energy of the particle is investigated and its escape velocity is also calculated. It is observed that the magnetic field in the vicinity of black hole has a very strong effect on the particle's motion. It plays the key role in the transfer mechanism of energy to the escaping particle. As the particle moves away from the black hole the effect of the magnetic field on its motion is reduced (the magnetic field becomes homogeneous far from the black hole vicinity). In general, the motion of the particle is unpredictable (chaotic).

Chapter 1

Introduction

General relativity (GR) theory is a theory of motion of macroscopic objects. It gives a satisfactory explanation of the motion of macroscopic particles as it explains the perihelion shift of Mercury, bending of light due to gravity and gravitational red shift in total agreement with the experimental results [1]. But at a microscopic level the limitations of the theory are felt. For fundamental level (microscopic level) the Quantum theory is needed. This theory also has problems in dealing with gravitational forces. Up till now there is no theory which can satisfactorily explain motion at both the macroscopic and microscopic levels [2].

GR is the only theory by which we understand the astrophysical phenomena like black hole, pulsars, quasars, the big bang and the Universe itself. It is the essential ingredient in the system of global positioning system and it also deals with the slight shift (like perihelion shift) of the orbit of planets [3]. It was presented by Einstein in 1915 not as a new force law or new theory of gravity, but as a conceptual revolution in our views of space and time. That all bodies fall with the same acceleration in a gravitational field led him to understand gravity in terms of curvature of the space-time. Mass curves the space-time in its surroundings, and paths along which all the bodies fall are the "straightest trajectories" in this curved space-time [4].

According to GR the mass of the Sun curves the surrounding space-time and the planets in its vicinity move on straight trajectories in that space-time. *Gravity is geometry.*

Some relativistic phenomena and relativistic stars are explained below. Most stars support themselves by the radiation pressure produced due to nuclear reaction inside the star against the gravitational attraction. When the nuclear fuel inside the star vanishes or is not sufficient then gravitational attraction dominates over the radiation pressure produced due to the nuclear reaction inside it and gravitational collapse occur as discussed in [5]. The core of some stars become highly compact like “white dwarfs”, “neutron stars”, or “black holes”. If the mass of the star is $m \leq 1.44$ solar masses (Chandrasekhar limit) then it will become a white dwarf after. GR also place a strict limit on the maximum mass of the neutron star which is 3.2 solar masses [6] and the mass greater then this leads to the black hole. GR predicts that black hole is formed when mass is compressed to the extend that the gravitational pull on the surface is too large, such that nothing can escape from the surface, even light. According to Newtonian gravity a body can escape from the vicinity of a star and its velocity can be calculated by using its kinetic and potential energy as

$$\frac{1}{2}mv_{esc}^2 = \frac{GmM}{R}. \tag{1.0.1}$$

According to the above expression the escape velocity is ($v_{esc} = \sqrt{\frac{2GM}{R}}$) and a body can escape from the vicinity of star if its initial velocity is greater then escape velocity ($v_{initial} > v_{esc}$). Therefore, the velocity of body can exceed from velocity of light if $2GM > c^2R$. The newtonian theory of gravity puts no limit on the speed, and it is not applicable to the relativistic situation [3, 7]. The black hole is defined by its surface, called the event horizon (boundary beyond which events cannot effect the outside observer). Things can fall through it but nothing can emerge out. Laplace was the first to suggest such objects and John Wheeler gave it the name “black hole”[8]. For a star of mass ($M = 2 \times 10^{30}$), to become a black hole its radius would be about 0.3km [1].

GR predicts gravitational waves. As mass causes space-time curvature, moving mass should produce ripples in space-time, which propagate with the speed of light. These ripples are called gravitational waves. The sources of gravitational waves are binary stars, supernova explosions, black holes and the big bang. Gravitational waves cannot be easily detected due to the weakness of the gravitational attraction. But it could enable us to observe the black hole horizon more closely and other earlier events which occur in the Universe.

The motion of a particle around a black hole is one of the most important problems of black hole astrophysics as it helps in understanding the geometrical structure of space-time near the black hole. Motion of the particle outside the black hole might be predictable. There are observational and theoretical evidence that magnetic field is present in the nearby surrounding of the black hole [7]. The origin of this magnetic field is the probable existence of plasma in the surrounding of the black hole such as the accretion disk or a charged gas cloud. The relativistic motion of particles in the conducting matter (plasma) in the accretion disk can generate a magnetic field in the vicinity of black hole. Therefore, near the black hole's event horizon, there may exists very strong magnetic fields. This field does not effect the geometry of the black hole much but its effect on the motion of a charged particle moving around it can be significant as explained in [9]. More importantly, a rotating black hole may provide sufficient energy to the particle for it to escape to spatial infinity. This physical effect appears to play a crucial role in the ejection of high energy particles from accretion disks around black holes. But it is predicted that the main role is played by the magnetic field, along with the black hole's rotation, in the transfer of energy to the particle [5,10].

Our main focus in this thesis is to investigate how a particle behaves when it interacts with another particle during its motion around the black hole. We are interested in the conditions for the particle: when it escapes from the surrounding of the black hole, or is captured by the black hole.

Modeling the motion of a particle around a black hole is a very complicated problem because during its motion around the black hole, the particle is under the influence of both gravitational and magnetic forces. Hence before dealing with a complicated problem of dynamics of a charged particle around a black hole, we start with the simpler case of a neutral particle. Generally the dynamical equations for a particle, obtained from a Lagrangian or some other methods, are not solvable analytically. In this thesis we consider a slowly rotating Kerr black hole which is surrounded by an axially-symmetric magnetic field, which is homogeneous at spacial infinity. The motion of the charged particle around a magnetized black hole was studied in [11]. Main features of their study are, there exist three asymptotic behaviour behaviour for a particle during its motion around a black hole: capture by the black hole and escape to infinity. There is an asymptotic limit on energy of the particle or the minimum energy required to escape. Same problem was studied for weakly charged rotating black holes in [12]. The main conclusion is that if the magnetic field is present then the innermost stable circular orbit is located closer to the black hole horizon. In this case, the action of the black hole rotation on a neutral particle is the same as the action of the magnetic field on a charged particle. In this thesis we use sign convention $(+,-,-,-)$ and units where $c = 1, G = 1$.

Chapter 2

General Relativity Theory: A Review

In this chapter we explain the curvature tensor and its properties which are essentially required for the derivation of the Einstein tensor and to study the space-time curvature. We derive the Einstein field equations by using a variational principle and the Lagrangian equations of motion. We solve the Einstein field equations for a point mass which is static and spherically symmetric in an empty space and also discuss its singularities and event horizon. We also present the Kerr solution, which is stationary and axially symmetric, and also study some of the aspects of this space-times.

2.1 Metric Tensor

Metric tensor is defined as a bilinear map of two vectors into the reals ($\mathbb{R}$), i. e. giving their inner product, represented as given in [13]

$$\mathbf{g}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}. \tag{2.1.1}$$

From the above definition we can see that the metric tensor is symmetric tensor and its covariant and contravariant components are respectively

$$g_{ij} = \mathbf{g}(\mathbf{e_i}, \mathbf{e_j}) = \mathbf{e_i} \cdot \mathbf{e_j} \tag{2.1.2}$$

$$g^{ij} = \mathbf{g}(\mathbf{e^i}, \mathbf{e^j}) = \mathbf{e^i} \cdot \mathbf{e^j}. \tag{2.1.3}$$

A metric tensor g_{ij} is a non singular covariant symmetric $g_{ij} = g_{ji}$ tensor-field of rank 2. In terms of its components non singular mean, its determinant is non zero. It has the unique inverse denoted by g^{ij} which guarantees that the rank of inverse metric is also 2.

The metric tensor defines the infinitesimal distance ds, between two points on a curve $x^i(\lambda)$ and $x^i(\lambda+\Delta\lambda)$. If v^i is the tangent vector field of the curve. Then

$$ds^2 = g(\mathbf{v}, \mathbf{v})d\lambda^2 = g_{ij}v^i v^j d\lambda^2 = g_{ij}dx^i dx^j, \tag{2.1.4}$$

where $v^i = \frac{dx^i}{d\lambda}$. The quantity ds is called the line element of metric tensor g_{ij}.

We now assume a function of several variables. A function is multi-linear if it is linear in all its arguments. A Tensor is a multi-linear function that maps vectors and one-forms into $\mathbb{R}$.

2.2 Riemann Curvature Tensor and Other Related Tensor

The Riemann curvature tensor describes the curvature in an invariant way and is defined as [1]

$$R^a{}_{bcd} = \Gamma^a{}_{bd,c} - \Gamma^a{}_{bc,d} + \Gamma^a{}_{ec}\Gamma^e{}_{bd} - \Gamma^a{}_{ed}\Gamma^e{}_{bc}. \tag{2.2.1}$$

Here Γ is the Christoffel symbol which is defined by

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\big(g_{dc,b} + g_{bd,c} - g_{bc,d}\big). \tag{2.2.2}$$

The Riemann tensor can also be written in covariant form as

$$R_{abcd} = g_{ae}R^e{}_{bcd}. \tag{2.2.3}$$

We can write R_{abcd} as

$$R_{abcd} = \frac{1}{2}(g_{bc,ad} - g_{ac,bd} + g_{ad,bc} - g_{bd,ac}) - g^{ef}(\Gamma_{eac}\Gamma_{fbd} - \Gamma_{ead}\Gamma_{fbc}). \tag{2.2.4}$$

From the above expression we can easily deduce the symmetry properties of curvature tensor. But by choosing the arbitrary point P on the manifold and constructing the geodesic coordinate system (Riemanian normal coordinates) in which the Christoffel symbol vanishes $\Gamma^a{}_{bc} = 0$, which is not true in general. Therefore we can reduce the expression (2.2.4) to

$$(R_{abcd})_P = \frac{1}{2}(g_{bc,ad} - g_{ac,bd} + g_{ad,bc} - g_{bd,ac})_P \tag{2.2.5}$$

It is antisymmetric with respect to first two indices and last two indices

$$R_{abcd} = -R_{bacd}, \quad R_{abcd} = -R_{abdc}. \tag{2.2.6}$$

Now it is very easy to check, it is symmetric with respect to pair of the indices

$$R_{abcd} = R_{cdab}, \tag{2.2.7}$$

and Riemann tensor satisfies the following two Bianchi identities

$$R^a{}_{bcd} + R^a{}_{cbd} + R^a{}_{bdc} = 0, \tag{2.2.8}$$

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0. \tag{2.2.9}$$

Here ";" represents the covariant derivative. The Riemann curvature tensor (2.2.3) appears to have n^4 independent components but by using the properties (2.2.6) – (2.2.8) independent components reduce to $\frac{n^2(n^2-1)}{12}$ [2].

For the flat region of space we have the coordinate in which connection symbol $\Gamma^a{}_{bc}$ and its derivative are zero, hence

$$R^e{}_{abc} = 0, \tag{2.2.10}$$

everywhere in the region for flat space. This relation must hold for every coordinate system for the flat space because it is most general as it is a tensor defined relation. conversely we can say that Riemann tensor is zero then the space must be flat.

From the symmetry properties of the curvature tensor it follows only two independent contractions. We can do these contractions on the first and last indices or on the first two indices. From (2.2.6) raising the first index and contracting it with the last index or contracting the first two indices respectively we get

$$R_{bd} = R^{a}{}_{bda} = R_{bd} = R^{a}{}_{abd}. \tag{2.2.11}$$

From the above equation it is clear that Ricci tensor is the trace of Reimann curvature tensor The expended form of it is

$$R_{bd} = \Gamma^{a}{}_{bd,a} - \Gamma^{a}{}_{ba,d} + \Gamma^{a}{}_{ea}\Gamma^{e}{}_{bd} - \Gamma^{a}{}_{ed}\Gamma^{e}{}_{ba}. \tag{2.2.12}$$

Further contraction gives curvature scalar (Ricci scalar).

$$R = g^{bd}R_{db}. \tag{2.2.13}$$

Ricci scalar is the trace of Ricci tensor.

2.2.1 The Einstein Tensor

Here we are deriving the Einstein tensor from the Bianchi identity as follow

$$R_{abcd;e} + R_{abde;c} + R_{abec;d} = 0. \tag{2.2.14}$$

Raising the index a and contracting it with d gives

$$R_{bc;e} + R^{a}{}_{bae;c} + R^{a}{}_{bec;a} = 0. \tag{2.2.15}$$

Further by using the antisymmetric property for the second term we can write

$$R_{bc;e} - R_{be;c} + R^{a}{}_{bec;a} = 0. \tag{2.2.16}$$

If we raise b and contract it with e we get

$$R^{b}{}_{c;b} - R_{;c} + R^{ab}{}_{bc;a} = 0. \tag{2.2.17}$$

By using the symmetry property we have

$$R^{ab}{}_{bc;a} = R^{ba}{}_{cb;a} = R^{a}{}_{c;a} = R^{b}{}_{b;c}. \tag{2.2.18}$$

Hence the first and the last terms are same of equation (2.2.17) and we can write it as

$$2R^{b}{}_{c;b} - R_{;c} = (R^{b}{}_{c} - \delta^{b}_{c}R)_{;b} = 0. \tag{2.2.19}$$

Here we get the Einstein tensor by lowering the index c in the last expression

$$(R^{bc} - \frac{1}{2}g^{bc}R)_{;b} = 0. \tag{2.2.20}$$

The expression in the parentheses is the Einstein tensor and is usually denoted by

$$G^{ab} \equiv R^{ab} - \frac{1}{2}g^{ab}R. \tag{2.2.21}$$

The Einstein tensor is symmetric and divergence free. This is the tensor which describes the curvature of space-time in the field equations of GR.

2.3 The Einstein Field Equations

We will derive the Einstein field equations by using variational principle,

$$\delta S_G = 0, \tag{2.3.1}$$

where S_G is the action integral for gravitation [13]. As it is geometrical, the Lagrangian is a function of the metric tensor.

$$S_G = \frac{1}{2\kappa}\int_M \mathcal{L}(g_{\mu\nu}, g_{\mu\nu,\lambda})\sqrt{-g}d^4x. \tag{2.3.2}$$

Here $\kappa = 8\pi$ is a constant and calculated by the required condition that the Einstein field equations reduce to Newton' law in the weak field limit. For the integral to be invariant under any transformation the function $\mathcal{L}[g_{\mu\nu}]$ should be scalar. So, we are using

$$\mathcal{L}(g_{\mu\nu}, g_{\mu\nu,\lambda}) = R - 2\Lambda. \tag{2.3.3}$$

Λ is the cosmological constant. After putting the above values in (2.3.2) action will reads

$$S_G = \frac{1}{2\kappa}\int (R - 2\Lambda)\sqrt{-g}d^4x. \tag{2.3.4}$$

Now we will vary the action inside a infinitesimal region V and assuming that the variation of the metric and its differentiation on the boundary of the region will vanishes. Then we deduce the Einstein field's equation by the requirement that $\delta S_G = 0$, for any variation in the metric. We can write the metric as

$$S_G = \frac{1}{2\kappa}\int (R_{\mu\nu}g^{\mu\nu}\sqrt{-g} - 2\Lambda\sqrt{-g})d^4x. \tag{2.3.5}$$

We get

$$\delta S_G = \frac{1}{2\kappa}\int (g^{\mu\nu}\sqrt{-g}\delta R_{\mu\nu} + R_{\mu\nu}\delta g^{\mu\nu}\sqrt{-g} - 2\Lambda\delta\sqrt{-g})d^4x. \tag{2.3.6}$$

As we said the variation of the metric and its differentiation on the boundary of the region will vanishes. Then the Ricci tensor is given by the equation (2.2.12) will reduce to

$$R_{\mu\nu} = \Gamma^{\lambda}_{\mu\nu,\lambda} - \Gamma^{\lambda}_{\mu\lambda,\nu}. \tag{2.3.7}$$

Thus

$$\delta R_{\mu\nu} = \delta\Gamma^{\lambda}_{\mu\nu,\lambda} - \delta\Gamma^{\lambda}_{\mu\lambda,\nu}. \tag{2.3.8}$$

As the partial derivative commute with the variation

$$\delta R_{\mu\nu} = (\delta\Gamma^{\lambda}_{\mu\nu})_{,\lambda} - (\delta\Gamma^{\lambda}_{\mu\lambda})_{,\nu}. \tag{2.3.9}$$

Partial derivative of the metric tensor will vanish at boundary of V. Then we may write the above equation as

$$g^{\mu\nu}\delta R_{\mu\nu} = (g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\nu})_{,\lambda}. \tag{2.3.10}$$

We introducing a vector A^{λ}

$$A^{\lambda} = g^{\mu\nu}\delta\Gamma^{\lambda}_{\mu\nu} - g^{\mu\lambda}\delta\Gamma^{\nu}_{\mu\nu}. \tag{2.3.11}$$

Then the above equation can be written as

$$g^{\mu\nu}\delta R_{\mu\nu} = A^{\mu}_{,\mu}\ , \tag{2.3.12}$$

which is the total divergence. As the metric tensor and its derivative vanishes at the boundaries then according to Stoke's theorem the first term will vanish and contribute nothing to δS_G

$$\int (\sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu}) d^4x = 0. \tag{2.3.13}$$

For $\delta\sqrt{-g}$:

$$\delta\sqrt{-g} = [\frac{\partial\sqrt{-g}}{\partial g_{\alpha\beta}}]\delta g_{\alpha\beta} = -\frac{1}{2\sqrt{-g}}(\frac{\partial g}{\partial g_{\alpha\beta}})\delta g_{\alpha\beta}. \tag{2.3.14}$$

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g} g^{\alpha\beta} \delta g_{\alpha\beta}. \tag{2.3.15}$$

Now for the second term

$$\delta\big[g^{\mu\nu}\sqrt{-g}\big] = \sqrt{-g}\delta g^{\mu\nu} + g^{\mu\nu}\delta\sqrt{-g} \tag{2.3.16}$$

as we know $g^{\mu\beta} g_{\alpha\beta} = \delta^{\mu}_{\beta}$. We get

$$\delta(g^{\mu\alpha} g_{\alpha\beta}) = 0 \tag{2.3.17}$$

Accordingly we can write

$$\delta g_{\alpha\beta} = -g_{\alpha\mu} g_{\beta\nu} \delta g^{\mu\nu}. \tag{2.3.18}$$

After putting the value we get

$$\delta\big[g^{\mu\nu}\sqrt{-g}\big] = \sqrt{-g}\big(\delta g^{\mu\nu} + \frac{1}{2} g^{\mu\nu} g^{\alpha\beta} \delta g_{\alpha\beta}\big) = \sqrt{-g}\big(\delta g^{\mu\nu} - \frac{1}{2} g^{\mu\nu} g_{\alpha\beta} \delta g^{\alpha\beta}\big). \tag{2.3.19}$$

After putting the values of equation (2.3.15) and (2.3.19) in (2.3.6) then the variation in the action reads

$$\delta S_G = \frac{1}{2\kappa}\int \sqrt{-g}\big(R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta}\big)\delta g^{\alpha\beta} d^4x. \tag{2.3.20}$$

It is required for the vacuum field equations of the general theory of relativity that the $\delta S_G = 0$, for any arbitrary variation in the metric as explained [13].

The variation in the action δS_G can only be zero if the integrant is zero. Accordingly we get from the above equation (2.2.12)

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = 0 \quad (2.3.21)$$

Since $R_{\alpha\beta}$ and $g_{\alpha\beta}$ are symmetric. Therefore it has only 6 independent components and there are only six field equations.

2.4 The Einstein Field Equations in the Presence of Matter

Now we derive the field equation for non vanishing energy momentum tensor from the variational principle

$$\delta(S_G + S_M) = 0. \quad (2.4.1)$$

S_M is the action integral for matter and energy.

$$S_M = \int \mathcal{L}_M(g_{\mu\nu}, g_{\mu\nu,\lambda})\sqrt{-g}d^4x. \quad (2.4.2)$$

$\mathcal{L}_M$ is the Lagrange density for matter. For the variation in the argument gives

$$\delta[\sqrt{-g}\mathcal{L}_M(g_{\mu\nu}, g_{\mu\nu,\lambda})] = \frac{\partial[\sqrt{-g}\mathcal{L}_M]}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial[\sqrt{-g}\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}\delta g^{\mu\nu}{}_{,\lambda}. \quad (2.4.3)$$

Generally the Lagrangian depends on the metric tensor and its derivative. We define a vector B^λ as

$$B^\lambda = \frac{\partial[\sqrt{-g}\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}\delta g^{\mu\nu}. \quad (2.4.4)$$

The divergence of B^λ is

$$B^\lambda{}_{,\lambda} = [\frac{\partial[\sqrt{-g}\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}]_{,\lambda}\delta g^{\mu\nu} + \frac{\partial[\sqrt{-g}\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}\delta g^{\mu\nu}{}_{,\lambda}. \quad (2.4.5)$$

From equations (2.4.5) and (2.4.3)

$$\delta[\sqrt{-g}\mathcal{L}_M] = \frac{\partial[\sqrt{-g}\mathcal{L}_M]}{\partial g^{\mu\nu}}\delta g^{\mu\nu} - [\frac{\partial[\sqrt{-g}\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}]_{,\lambda}\delta g^{\mu\nu} + B^\lambda{}_{,\lambda}. \quad (2.4.6)$$

We have assumed that the variation vanishes at the boundary so, according to Gauss integral theorem $\int B^{\lambda}{}_{,\lambda} d^4x = 0$. Finally we get

$$\delta S_M = \int \left(\frac{\partial[\sqrt{-g}\mathcal{L}_M]}{\partial g^{\mu\nu}} - \left[\frac{\partial[\sqrt{-g}\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}\right]_{,\lambda}\right)\delta g^{\mu\nu} d^4x. \tag{2.4.7}$$

The energy momentum tensor with Lagrange density is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\left(\frac{\partial[\sqrt{-g}\mathcal{L}_M]}{\partial g^{\mu\nu}} - \left[\frac{\partial[\sqrt{-g}\mathcal{L}_M]}{\partial g^{\mu\nu}{}_{,\lambda}}\right]_{,\lambda}\right). \tag{2.4.8}$$

This will give us

$$\delta S_M = -\frac{1}{2}\int T_{\mu\nu}\sqrt{-g}\delta g^{\mu\nu} d^4x. \tag{2.4.9}$$

From equation (2.3.20) and (2.4.9) we get the gravitational field equation in the presence of matter and energy for the theory of general relativity

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu}. \tag{2.4.10}$$

These are the Einstein's field equations. As like Einstein tensor the stress energy tensor is also divergence free which assures the conservation of energy and momentum.

2.5 The Schwarzschild Solution of the Einstein Field Equations

The Schwarzschild obtained the solution to Einstein's Field equations for a point mass in an empty space which is static and spherically symmetric [2]. The Einstein field equations are 10 partial differential equations of 10 independent functions ($g_{\mu\nu}$) of 4 variables (x^{μ}). They are second order and non linear in the first order. By choosing some assumptions and appropriate coordinate we can reduce the number of independent variable and unknowns. The simplest case is $T^{\mu\nu} = 0$.

Let a point gravitational source of mass M, situated at the origin in its own rest frame. This implies that the solution is spherically symmetric and, excluding the origin, there is a vacuum. The most general spherically symmetric metric is

$$ds^2 = e^{v(t,r)}dt^2 - e^{\lambda(t,r)}dr^2 - f(t,r)d\Omega^2, \tag{2.5.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$. The gravitational field due to point mass can not vary with time. Then the above equation becomes

$$ds^2 = e^{v(r)}dt^2 - e^{\lambda(r)}dr^2 - f(r)d\Omega^2. \tag{2.5.2}$$

As $T_{\mu\nu} = 0$ therefore the Einstein's field equations become

$$R_{\mu\nu} = 0. \tag{2.5.3}$$

The metric tensor is

$$g_{\mu\nu} = \begin{bmatrix} e^v & 0 & 0 & 0 \\ 0 & -e^\lambda & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} \tag{2.5.4}$$

The inverse of a diagonal matrix is simply the inverse of the diagonal entries. Therefore we have

$$g^{\mu\nu} = \begin{bmatrix} e^{-v} & 0 & 0 & 0 \\ 0 & -e^{-\lambda} & 0 & 0 \\ 0 & 0 & -\frac{1}{r^2} & 0 \\ 0 & 0 & 0 & -\frac{1}{r^2\sin^2\theta} \end{bmatrix}. \tag{2.5.5}$$

The determinant of the metric $g_{\mu\nu}$ is

$$|g| = -e^{v+\lambda}r^4\sin^2\theta. \tag{2.5.6}$$

The metric is independent of t, ϕ. The non-zero Christoffel symbols are

$$\begin{aligned} \Gamma^0_{01} = \frac{1}{2}v', \quad \Gamma^1_{00} = \frac{1}{2}v'e^{v-\lambda}, \quad \Gamma^1_{11} = \frac{1}{2}\lambda', \quad \Gamma^1_{22} = -re^{-\lambda}, \\ \Gamma^1_{33} = -r\sin^2\theta e^{-\lambda}, \quad \Gamma^2_{21} = \frac{1}{r} = \Gamma^3_{31}, \\ \Gamma^2_{33} = -\sin\theta\cos\theta, \quad \Gamma^3_{32} = \cot\theta. \end{aligned} \tag{2.5.7}$$

There are 4 equations for $R_{00}, R_{11}, R_{22}, R_{33}$ given below

$$R_{00} = \frac{1}{2}\big[v'' + \frac{1}{2}v'\lambda' - \frac{1}{2}v'\lambda' + \frac{2}{r}v'\big]e^{v-\lambda} = 0, \tag{2.5.8}$$

$$R_{11} = -\frac{1}{2}\big[v'' + \frac{1}{2}v'\lambda' - \frac{1}{2}v'\lambda' - \frac{2}{r}\lambda'\big] = 0, \tag{2.5.9}$$

Comparing equations (2.5.8) and (2.5.9) we see

$$v(r) + \lambda(r) = constant. \tag{2.5.10}$$

We can take the *constant* = 0 or may be absorbed in the units. Then we have

$$v(r) = -\lambda(r). \tag{2.5.11}$$

The equation for R_{22} is

$$R_{22} = (-re^{-\lambda})' + \big[\frac{1}{2}(v' + \lambda') + \frac{2}{r}\big](-re^{-\lambda}) - \frac{2}{r}(-re^{-\lambda}) + 1 = 0. \tag{2.5.12}$$

By using equation (2.5.11) in R_{22}

$$R_{22} = (-re^{-\lambda})' + 1 = 0. \tag{2.5.13}$$

Solving the above equation we get

$$e^{-\lambda} = e^{v} = 1 + \frac{\alpha}{r}. \tag{2.5.14}$$

Here α is a constant of integration.

$$R_{33} = \sin^2\theta R_{22} = 0. \tag{2.5.15}$$

The other components of the Ricci tensor are identically zero.

Now we have the complete solution for the vacuum Einstein's field equations for spherical symmetric gravitating object

$$ds^2 = (1 + \frac{\alpha}{r})dt^2 - \frac{1}{1 + \frac{\alpha}{r}}dr^2 - r^2 d\Omega^2. \tag{2.5.16}$$

It can be seen that the integration constant α must be representing a constant mass which produces gravitational field. Considering the weak field limit we can identify α. As we require in weak field limit

$$\frac{e^{\nu(r)}}{c^2} = 1 + 2\alpha. \tag{2.5.17}$$

Here α is the Newtonian gravitational potential. Furthermore, in the weak limit r can be identified as the radial distance. If the gravitational source is spherically symmetric having mass M then in the weak field limit we have

$$\alpha = -2M. \tag{2.5.18}$$

Thus the Schwarzschild metric for the empty space-time outside a spherically symmetric body of mass M will become

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \frac{1}{\left(1 - \frac{2M}{r}\right)}dr^2 - r^2 d\Omega^2. \tag{2.5.19}$$

This is the required metric for the spherically symmetric Schwarzschild space-time with Schwarzschild radius $r = 2M = r_g$. We can see that this metric reduces to Minkowski space-time as $M \to 0$ or $r \to \infty$. The Schwarzschild solution is also an asymptotically flat solution. We can use this metric to investigate the motion of the particle (macro or micro) in the vicinity of spherically symmetric object of mass M. This metric is valid down to $r = 2M$.

2.5.1 Singularities and Event Horizon

For large r it will approximately reduce to flat Minkowski space-time. Metric becomes singular at $r = r_g$ and $r = 0$ these two values of r have special physical importance. At $r = r_g$, g_{11} becomes infinite and at $r = 0$, g_{00} becomes infinite. The singularity $r = 2M$ is not essential singularity as it arises due to bad choice of coordinates. Hence we can remove it by the proper choice of coordinates like Eddington Finkelstein coordinate , Kruskal coordinate and Kruskal-Szekeres coordinates etc [2].

The singularity at $(r = 0)$ is the essential singularity or a physical singularity where the curvature becomes infinite or the curvature invariants diverge. We cannot remove it by any choice of coordinates. There are many interesting properties associated with $r = r_g$, such that things can fall into the region $r < r_g$ and cannot emerge from it. Due to this reason it is called event horizon and the region of space inside it is called the Schwarzschild black hole. The observer which is just outside the event horizon $r = r_g$ can see nothing beyond this surface. The Schwarzschild radius $r = r_g$, represents the boundary of the events which can be observed.

2.6 The Kerr Solution of the Einstein Field Equations

We will not derive here the complete Kerr solution. We take it as granted from the book [13]. We will discuss the singularities and event horizon and some other features of Kerr black hole. Space-time is called stationary if there is a Killing vector ξ^{μ} which is time-like at spatial infinity. If this Killing vector is orthogonal to the other three space-like vectors then the space-time is static e. g. the Schwarzschild. Here we are assuming the space-time is stationary and axi-symmetric. It has two dimensional surface which is orthogonal to Killing vectors $\xi_t = \frac{\partial}{\partial t}$ and $\xi_\phi = \frac{\partial}{\partial \phi}$.

For stationary and axi-symmetric space-time the metric components should be independent of t and ϕ; i. e.

$$g_{\mu\nu} = g(x^1, x^2) = g(r, \theta). \tag{2.6.1}$$

Besides the axially symmetric and stationary, the line element (ds^2) should be invariant under the simultaneous inversions $t \to -t$ and $\phi \to -\phi$. It means that the motion of the gravitational source must be purely rotational about the axis of symmetry. According to the above assumption we require that

$$g_{01} = g_{02} = g_{12} = g_{13} = g_{23} = 0. \tag{2.6.2}$$

Hence, the line element must have the form

$$ds^2 = g_{00}dt^2 + 2g_{03}dtd\phi + g_{33}d\phi^2 + [g_{11}dr^2 + g_{22}d\theta^2]. \tag{2.6.3}$$

Metric coefficients depend only on r and θ. Further simplification to this kind of metric can be achieved by the fact that any two dimensional (pseudo) Riemannian curvature must be flat. Hence, by using this fact and writing the result for the rotating body, the line element (2.6.3) takes the form

$$ds^2 = Adt^2 - Cdr^2 - Dd\theta^2 - B(d\phi - \omega dt)^2. \tag{2.6.4}$$

Here $(A, B, C, D$ and $\omega)$ are function of two spacelike coordinate r and θ. We can write the metric coefficients as

$$g_{00} = A - B\omega^2, \quad g_{03} = B\omega, \quad g_{33} = -B, \quad g_{11} = -C, \quad g_{22} = -D. \tag{2.6.5}$$

Finally, we have the metric of the form (2.6.4) for the spherically symmetric and stationary space-time. The metric we have (2.6.4) is the general one as it is valid for outside the rotating "extended" axi-symmetric body. To calculate Kerr metric we have to impose the condition on solution that space time geometry should transform to Minkowiski form at $r \to \infty$ then the solution is unique. In Boyer-Lindquist coordinate the line element for Kerr geometry is given by [25]

$$\begin{aligned} ds^2 &= \frac{\Delta - a^2\sin^2\theta}{\rho^2}dt^2 + \frac{4Mar\sin^2\theta}{\rho^2}d\phi dt - \frac{\rho^2}{\Delta}dr^2 - \rho^2 d\theta^2 - \frac{A\sin^2\theta}{\rho^2}d\phi^2, \\ & \Delta = r^2 - 2Mr + a^2 \\ & \rho^2 = r^2 + a^2\cos^2\theta \\ & A = (r^2 + a^2)^2 - a^2\Delta\sin^2\theta, \end{aligned} \tag{2.6.6}$$

where M is the mass of the black hole and a is the rotation of the black hole and it is interpreted as angular momentum per unit mass of black hole

$a = \frac{L}{M}$. This solution (2.6.6) is for vacuum Einstein's field equations outside the axially symmetric rotating body. The Kerr metric diverges where $\Delta = 0$, or $\rho = 0$,

$$\Delta(r) = r^2 + a^2 - 2Mr = 0. \tag{2.6.7}$$

From the above equation we get two values of r

$$r_+ = M + \sqrt{M^2 - a^2}, r = r_- = M - \sqrt{M^2 - a^2}. \tag{2.6.8}$$

$\Delta > 0$ for $r_+ < r < r_-$ and $\Delta < 0$ for $r > r_-$ and $r < r_+$. The region $r = r_+$ and $r = r_-$ are the event horizon. The Kerr metric has two event horizons and for $a = 0$, there is only one which is at $r = 2M = r_g$ like Schwarzschild metric. The intrinsic geometry of both horizon is not spherically symmetric. The singularities r_+ and r_- are only the coordinate singularity which occur on the surfaces.

The place $\rho = 0$ is the curvature singularity

$$\rho^2 = r^2 + a^2 \cos^2\theta = 0, \tag{2.6.9}$$

and we can see from the above equation that $\rho = 0$ only for $r = 0$ and $\theta = \frac{\pi}{2}$. We can say that $r = 0$ and $\theta = \frac{\pi}{2}$ represents disc of coordinate of radius a in the equatorial plane and $r = 0$ and $\theta = \frac{\pi}{2}$ is the outer edge of the disc which means the singularity is a ring like singularity.

Chapter 3

Motion of a Charged Particle Around the Schwarzschild Black Hole Immersed in a Magnetic Field

This chapter is devoted to review the motion of charged particles around the Schwarzschild black hole immersed in a magnetic field [11]. We calculate the location of the innermost stable circular orbit (ISCO) in the vicinity of the black hole which is immersed in a magnetic field. There are many astrophysical observations which show that a magnetic field is present in the vicinity of the black hole, probably due to a plasma in the accretion disc [16, 19]. Therefore the particle moves around the black hole under the influence of both gravitational and magnetic forces.

Constants of motion and the effective potential will be reviewed for the Schwarzschild space-time and discussed in detail in this chapter [11,16]. Two cases are considered: motion near the black hole when the particle is neutral; and when it is charged. The main focus is on trajectories of the particle's motion near the black hole, how it is effected by the magnetic field present in the vicinity of the black hole as we follow from the paper [10,20]. Conditions on the energy of the particle to escape to infinity as it is kicked by another particle are discussed in detail. Escape velocity of the particle moving under the influence of both gravitational and magnetic forces are calculated.

This chapter is organised as follows: In section 3.1 we discuss the motion of the neutral particle in the absence of magnetic field. In section 3.2 the expression for magnetic field present in the accretion disc of Schwarzschild black hole is given. Equations of motion and constants of motion are presented in section 3.3 and discuss their limiting cases in section 3.4. We study the behaviour of effective potential in section 3.5.

3.1 Motion of a Neutral Particle Around the Schwarzschild Black Hole

First we consider the simple case of the neutral particle in the absence of a magnetic field . In this case the Lagrangian for a particle outside a spherically symmetric body is defined as [13]

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} \dot{x}^{\mu} \dot{x}^{\nu}. \tag{3.1.1}$$

From (2.5.19), we get

$$\mathcal{L} = \frac{1}{2}\big[(1 - \frac{r_g}{r})\dot{t}^2 - \frac{\dot{r}^2}{1 - \frac{r_g}{r}} - r^2\dot{\theta}^2 - r^2 \sin^2\theta \dot{\phi}^2\big], \tag{3.1.2}$$

Note that t and ϕ are cyclic coordinates, so the canonical momenta, p_t and p_ϕ respectively are constant. There exist three constants of motion, two of them are calculated by the Lagrangian equation which is given below.

$$\frac{d}{d\tau}\big(\frac{\partial \mathcal{L}}{\partial \dot{x}^{\mu}}\big) - \frac{\partial \mathcal{L}}{\partial x^{\mu}} = 0 \tag{3.1.3}$$

The corresponding conserved quantities are the energy E and the azimuthal angular momentum L_z as measured by an observer at infinity. Using Lagrangian (3.1.2) and the Euler-Lagrange equation (3.1.3) we have

$$P_t = E = \dot{t}\big(1 - \frac{r_g}{r}\big), \tag{3.1.4}$$

where E is the energy per unit mass and

$$P_\phi = L_z = \dot{\phi} r^2 \sin^2\theta, \tag{3.1.5}$$

where L_z is the angular momentum per unit mass. The over dot represents the derivative with respect to proper time, τ. The third constant of motion is the square of the total angular momentum L^2

$$L^2 = r^4\dot{\theta}^2 + \frac{L_z^2}{\sin^2\theta}. \tag{3.1.6}$$

Introducing the symbol $v = -r\dot{\theta}_o$, then the above equation can be written as

$$L^2 = r^2 v^2 + \frac{L_z^2}{\sin^2\theta}. \tag{3.1.7}$$

Without loss of generality by appropriate choice of the z-axis we can take $\theta(\tau_o) = \frac{\pi}{2}$ and $\dot{\theta}(\tau_o) = 0$. Then the Lagrangian equation of motion for θ is

$$0 = \frac{d}{d\tau}(r^2\dot{\theta})^2 + \frac{d}{d\tau}(\frac{L_z^2}{\sin^2\theta}). \tag{3.1.8}$$

By integrating the equation (3.1.8) and taking integration constant to be zero we get

$$(r^2\dot{\theta})^2 = -L_z^2 \csc^2\theta. \tag{3.1.9}$$

Both sides have to be zero to satisfy the equation because the left side is never negative and the right side can never be positive. Hence the orbit is plane and therefore ($\dot{\theta} = 0$).

The equation of motion for the massive particle in the equatorial plane is

$$\frac{d^2u}{d\phi^2} + u = \frac{m}{L_z^2} + 3mu^2, \tag{3.1.10}$$

where $u = \frac{1}{r}$ and solution of the above equation is

$$u = \frac{m}{L_z^2} + 3mu^2. \tag{3.1.11}$$

Hence, for ISCO, as $\frac{du}{d\phi} = 0$, we get $r_o = 3r_g$. Let the particle is in the circular orbit, $r = r_o$, where r_o is the local minima of the effective potential and

$r_o \in (3r_g, \infty)$. We get the critical values for energy and azimuthal angular momentum corresponding to this local minima which are given below.

$$L_{zo} = \frac{r_o\sqrt{r_g}}{\sqrt{2r_o - 3r_g}}, \tag{3.1.12}$$

$$E_o = \frac{\sqrt{2}(r_o - r_g)}{\sqrt{r_o(2r_o - 3r_g)}}. \tag{3.1.13}$$

The ISCO for the Schwarzschild black hole is defined by $r_o = 3r_g$. which is the convolution point of the effective potential.

Now we are assuming that the particle is in the circular orbit and collides with another particle. After the collision, particle will move in a new plane. There can be three possibilities: (*i*) captured by the black hole, (*ii*) bounded motion and (*iii*) escape to infinity. It depends on the mechanism of the collision. For small change in the energy and the momentum, the orbit of the particle will be slightly perturbed and its motion will remain bound but for large change, it may escape to infinity or it might be captured by the black hole.

Generally, after the collision the particle will have new values of the constants of motion E, L_z and L^2. The problem is simplified by considering the following assumptions: (i) the azimuthal angular momentum is not changed after the collision and (ii) the radial velocity is also unchanged after collision. Then using one parameter (the energy E) after the collision we can determine the motion of the particle as explained in [11].

After the collision the azimuthal angular momentum, L_z^2, changes to L^2 and the energy of the particle changes which corresponds to the same local minima, r_o, and are given respectively as

$$L^2 = r_o^2 v^2 + L_z^2, \tag{3.1.14}$$

and

$$E = \sqrt{E_o^2 + v^2 \frac{r_o - r_g}{r_o}}. \tag{3.1.15}$$

These values of the energy and angular momentum after collision are greater than the values of energy and angular momentum before collision. Thus, immediately after the collision, the particle is still at the turning point where $\dot{r}_o = 0$ and lies in between the extreme values of the energy. We can also see from the equation (3.1.15) that as $r \to \infty$ then $E \to 1$. This implies that the particle will escapes to infinity if $E \geq 1$. We can also find the escape velocity of the particle by solving the equation (3.1.15)

$$| v | \geq \sqrt{\frac{r_o(1 - E_o^2)}{(r_o - r_g)}}, \tag{3.1.16}$$

where the above equality correspond to $E = 1$.

3.2 Magnetic Field Around the Schwarzschild Black Hole

The motion of the particle is one of the most important problem of modern astrophysics. The matter (plasma) in the vicinity of the black hole and rotation of the black hole might provide the sufficient energy to the particle moving around it to escape. But the main role in the energy transfer mechanism is played by the magnetic field [5, 10]. During the motion in the surrounding of black hole, the charged particle will radiate and stability might be lost. Here we consider that the effect of radiation energy on the motion of the particle is very small as compare to the effect of magnetic field and black hole rotation, therefore it can be ignorable.

There are theoretical and experimental evidence that magnetic field should be present in the surrounding of the black hole [14]. A regular magnetic field exists in the vicinity of the black hole due to plasma (conducting matter), e.g., if the black hole has the accretion disk. The motion in the conducting matter around the accretion disk can generate the regular magnetic field inside the disk. This type of magnetic field is trapped in the vicinity of black hole [11]. The effect of magnetic field on the geometry of black hole is very small but it effects the motion of the charged particle largely. This type of black hole is called magnetized black hole [11].

The mechanism for the extraction of the rotational energy from the black hole in the presence of magnetic field is proposed by Blandford and Znajek [16] and it is given in detail in [17]. Different features of the motion of the charged particle moving in both gravitational and magnetic field in the vicinity of the black hole are discussed here .

Magnetic Field

We are assuming the case that the particle has charge q and mass m moving around the black hole which is immersed in a magnetic field. The particle motion is effected by the magnetic field in the black hole vicinity. We have considered that there exists a magnetic field in the black hole exterior which is axi-symmetric and it is decreasing as we move away from the black hole. It has the strength B far away from the black hole. We can calculate such a magnetic field by the procedure given in [12, 18].

As the metric is asymptotically flat then for the Killing vectors, we have

$$\xi^{\mu}{}_{;\nu}{}^{;\nu} = 0, \tag{3.2.1}$$

where ξ^{μ} is a Killing vector. Eq. (3.2.1) coincides with the Maxwell equation for 4-potential A^{μ} in the Lorentz gauge $A^{\mu}{}_{;\mu} = 0$. The special choice for A^{μ} is [12]

$$A^{\mu} = \frac{B}{2}\xi^{\mu}_{(\phi)}, \tag{3.2.2}$$

where B is the magnetic field strength. The 4-potential is invariant under the symmetries which correspond to the Killing vectors, i.e.,

$$L_\xi A_\mu = A_{\mu,\nu}\xi^\nu + A_\nu \xi^\nu_{,\mu} = 0. \tag{3.2.3}$$

A magnetic field vector is defined as [11]

$$B^\mu = -\frac{1}{2} e^{\mu\nu\lambda\sigma} F_{\lambda\sigma} u_\nu, \tag{3.2.4}$$

where

$$e^{\mu\nu\lambda\sigma} = \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}, \quad \epsilon_{0123} = 1, \quad g = det(g_{\mu\nu}), \tag{3.2.5}$$

and $\epsilon^{\mu\nu\lambda\sigma}$ is the Levi Civita symbol. The Maxwell tensor is defined as

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu;\mu} - A_{\mu;\nu}. \tag{3.2.6}$$

In the schwarzschild space-time. a stationary observer has the 4-velocity like $u_0^\nu = (1 - \frac{r_g}{r})\xi_t^\nu$. From equations (3.2.2) – (3.2.6) we get the magnetic field

$$B^\mu = B(1 - \frac{r_g}{r})\frac{1}{r}\left(r\cos\theta\frac{\partial}{\partial r} - \sin\theta\frac{\partial}{\partial\theta}\right). \tag{3.2.7}$$

Magnetic field is directed upward along the z direction at the spacial infinity. We assume that the magnetic field is along the positive z direction that is why we take it positive $B > 0$.

3.3 Equations of Motion of the Charged Particle

If the particle is charged then its motion is effected by the magnetic field in the vicinity of the black hole and the generalized 4 momentum is defined as

$$P_\mu = mu_\mu + qA_\mu. \tag{3.3.1}$$

Lagrangian for the charged particle moving around the magnetized black hole is defined as [24]

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu + \frac{qA_\mu}{m}\dot{x}^\mu. \tag{3.3.2}$$

Since the Lagrangian (3.3.2) does not depend explicitly on the coordinates t and ϕ. Therefor we can calculate two conserved quantities correspond to these two coordinates. These conserved quantities are energy per unit mass and azimuthal angular momentum per unit mass as measured by an observer at infinity. From lagrangian (3.3.2) and (3.1.3) we have

$$E = \dot{t}\big(1 - \frac{r_g}{r}\big), \tag{3.3.3}$$

$$L_z = \big(\dot{\phi} + \beta\big) r^2 \sin^2\theta. \tag{3.3.4}$$

Here we denote

$$\beta = \frac{qB}{r_g}. \tag{3.3.5}$$

From the equations (3.3.3) and (3.3.4), we have

$$\dot{t} = \frac{E}{\big(1 - \frac{r_g}{r}\big)}, \tag{3.3.6}$$

$$\dot{\phi} = \frac{L_z}{r^2 \sin^2\theta} - \beta. \tag{3.3.7}$$

By putting the values of $\dot{t}$ and $\dot{\phi}$ into the Lagrangian and solving for the Lagrangian equation for r and θ we get

$$\ddot{r} = \frac{r_g}{2r^2}(2L_z\beta - 1) + \frac{(2r - 3r_g)(\dot{\theta}^2)}{2} + \frac{(2r - 3r_g)L_z^2}{2r^4 \sin^2\theta} - \frac{\beta^2(2r - 3r_g)\sin^2\theta}{2}, \tag{3.3.8}$$

$$\ddot{\theta} = \frac{L_z^2 \cot\theta}{r^4 \sin^2\theta} - 2\beta^2 \sin 2\theta - \frac{2\dot{r}\dot{\theta}}{r}. \tag{3.3.9}$$

Using the normalization condition $u^\mu u_\mu = 1$ and putting the values of $\dot{t}$ and $\dot{\phi}$ we get

$$E^2 = \big(1 - \frac{r_g}{r}\big)\left[1 + r^2 \sin^2\theta \left(\frac{L_z}{r^2 \sin^2\theta} - \beta\right)^2\right] + \dot{r}^2 + r^2\big(1 - \frac{r_g}{r}\big)\dot{\theta}^2. \tag{3.3.10}$$

From the above equation (3.3.10) we can define the effective potential for a charged particle in the vicinity of black hole as

$$V_{\text{eff}} = \big(1 - \frac{r_g}{r}\big)\left[1 + r^2 \sin^2\theta \left(\frac{L_z}{r^2 \sin^2\theta} - \beta\right)^2\right]. \tag{3.3.11}$$

Equation (3.3.11) is the effective potential by which we can predict the motion of the particle. Equation (3.3.11) is a constraint if it is satisfied initially then it will always be satisfied provided that the dynamics of r and θ are controlled by equations (3.3.8) and (3.3.9) respectively.

There are some symmetries associated with the equations (3.3.6)−(3.3.11). These equations are invariant under the following transformations

$$\phi \to -\phi, \quad L_z \to -L_z, \quad \beta \to -\beta. \tag{3.3.12}$$

For ($\beta > 0$) then we will have to study both the cases of positive L_z and negative $-L_z$ because they are physically different. If we change the sign of L_z, we are actually changing the direction of the Lorentz force on the particle [11].

The system of equations ($3.3.6 - 3.3.11$) is invariant under reflection $\theta \to \pi - \theta$. According to this transformation the initial position of the particle remain fix but the direction of its velocity will change as $v \to -v$.

The geodesic equation in the Schwarzschild space-time is already studied very well. Our focus is that how the motion of the charged particle changes due to presence of magnetic field. The solution of the problem largely depends upon the initial conditions $r(\tau_o)$, $\phi(\tau_o)$ and $t(\tau_o)$ because by these condition we may decide the initial position of the particle.

3.4 Weak Gravitational Field

3.4.1 Flat Space-time Limit

Before the study of the motion of the particle close to the black hole we discuss the motion in the weak gravitational field i.e. the gravitational field approaches to zero, $M \to 0$. Let the particle escapes to infinity after collision. The gravitational field of the black hole vanishes at infinity. Hence particle moves in the homogeneous magnetic field in the flat space-time. Such type of motion is discussed in [15]. By introducing the cylindrical coordinates

$$r = a\sin\alpha, \quad z = a\cos\alpha. \tag{3.4.1}$$

and taking the limit $|z| \to \infty$. The equations (3.3.6) – (3.3.11) become

$$E = \dot{t}, \quad \dot{\phi} = \frac{L_z - \beta r^2}{r^2}, \tag{3.4.2}$$

$$\ddot{r} = \frac{L_z^2 - \beta^2 r^4}{r^3}, \tag{3.4.3}$$

$$E^2 = 1 + \dot{z}^2 + \dot{r}^2 + \frac{(L_z - \beta r^2)^2}{r^2}. \tag{3.4.4}$$

The solution of the above equations represents a helix with the axis directed along the magnetic field, i.e. as parallel to z axis. If the components of the velocity along the z direction vanishes then the trajectory of the particle becomes a circle in a plane. Suppose $\mathbf{r_c}$ be the radius of the circle in which particle is moving then its velocity can be written as

$$\mathbf{v} = -\frac{q}{m}[\mathbf{B} \times \mathbf{r_c}]. \tag{3.4.5}$$

Here $(\mathbf{B} \times \mathbf{r})$ is the cross product of 3-vectors $\mathbf{B}$ and $\mathbf{r}$ defined in the standard way in the Euclidean space. The vector potential for a uniform magnetic field as defined by (3.2.2) become

$$\mathbf{A} = \frac{1}{2}(\mathbf{B} \times \mathbf{r}). \tag{3.4.6}$$

Therefore, generalized 3-momentum of the particle is defined by

$$\mathbf{P} = \frac{q(\mathbf{B} \times \mathbf{r})}{2} - q(\mathbf{B} \times \mathbf{r_c}), \tag{3.4.7}$$

and the corresponding angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{P}. \tag{3.4.8}$$

Angular momentum is directed along the positive z direction. Let the center of the circle of radius $(r = r_c)$ is on the z-axis then the z component of 3-angular momentum is

$$L = -\frac{qBr_c^2}{2} < 0. \tag{3.4.9}$$

By using (3.3.5) we can write the above equation as

$$L = -r_c^2\beta. \tag{3.4.10}$$

If $\dot{z} = 0$, then the solution of the equations (3.4.2) − (3.4.4) is

$$r = r_c, \quad \phi = \phi_o - \frac{2\beta t}{E}. \tag{3.4.11}$$

Here ϕ_o is the solution corresponding to $t = 0$. We can calculate the general solution of these equations by shifting the center of the circle from (r_c, ϕ_o) to some other point (r_d, ϕ_d) on the same plane where it is moving [11]. The solution of the equations (3.4.11) transforms as

$$r = \sqrt{r_d^2 + r_c^2 + 2r_c r_d \cos\big(\phi_o - \phi_d - \frac{2\beta t}{E}\big)}, \tag{3.4.12}$$

$$\phi = \phi_d + \arccos\big(\frac{r^2 - r_c^2 + r_d^2}{2r_d r}\big), \tag{3.4.13}$$

$$z = z_o + \frac{Vt}{E}. \tag{3.4.14}$$

By these transformations the angular momentum will become

$$L = \beta(r_d^2 - r_c^2). \tag{3.4.15}$$

Here we boosted the solution to a distance r_d, where r_d is the distance from the z-axis to the helix. We can see from the above equation for $(r_d > r_c)$, the z-axis is located outside the circle and for $(r_d < r_c)$ the z-axis is located inside the circle. If $(r_d = r_c)$ then the z-axis will pass through the circle.

The particle's energy does not depend on the position of the circle. It can be expressed as

$$E^2 = 1 + \dot{z}^2 + 4\beta(\beta r_d^2 - L). \tag{3.4.16}$$

The energy $E = 1$, if the particle is at rest. This corresponds to $\dot{z} = 0$, and $r_c = 0$. These conditions imply

$$L = \beta r_d^2 \geq 0. \tag{3.4.17}$$

If the particle is located on the z axis then $r_d = 0$ and $L = 0$. If $\dot{z} = 0$ and $L < 0$ then the energy can be written as

$$E^2 = 1 + 4\beta(\beta r_d^2 - L) > 1. \tag{3.4.18}$$

3.4.2 Approximation of the Weak Gravitational Field

In the presence of the black hole, symmetry of the flat space-time is broken and it also provides the strong gravitational force to the particle. Let us discuss the motion of the particle in the approximation when gravitational force exerted by the black hole is very weak and particle is moving in a constant magnetic field. Equations (3.3.6) – (3.3.8) in the limit $r >> r_g$ become

$$\dot{t} = E, \quad \dot{\phi} = \frac{L - \beta r^2}{r^2}, \tag{3.4.19}$$

$$\ddot{r} = \frac{L^2}{r^3} - \beta^2 r - \mathrm{g}, \tag{3.4.20}$$

where g is the Newtonian gravitational force and it is perpendicular to the magnetic field. We consider a point (r_o, ϕ_o) and define the Cartesian coordinate (x, y) closer to (r_o, ϕ_o) as

$$r - r_o = y, \quad \phi_o - \phi = \frac{x}{r_o}. \tag{3.4.21}$$

The value of $\mathrm{g}_o = \frac{r_g}{2r_o^2}$, is the only leading term because the other higher terms are very small. By putting the values from (3.4.21) into equations (3.4.19) – (3.4.20) and retaining the linearity in x and y we get for the zero order

$$L = \beta r_o^2. \tag{3.4.22}$$

For the first order

$$\ddot{y} = -4\beta^2 y - \mathrm{g}_o, \tag{3.4.23}$$

$$\dot{x} = 2\beta y. \tag{3.4.24}$$

The solution of the above equations are

$$y(\tau) = a\cos(2\beta\tau) - \frac{g_o}{4\beta^2}, \tag{3.4.25}$$

$$x(\tau) = a\sin(2\beta\tau) - (\frac{g_o}{2\beta})\tau. \tag{3.4.26}$$

The above solution describes the plane called trochoid. If $g_o = 0$, then the above solution describes the motion of the particle moving along a circle with frequency $\omega = 2\beta = \omega_c E$ which is the relativistic cyclotron frequency. In the presence of gravity the centre of circle moves in the negative x direction with the velocity given by

$$v = \frac{g_o}{2\beta}. \tag{3.4.27}$$

The velocity with respect to the rest frame of the observer is defined by

$$\mathbf{V} = \mathbf{v}E. \tag{3.4.28}$$

For $a = \frac{g_o}{\omega^2}$, the solution becomes

$$y(\tau) = \frac{g_o\cos(\omega\tau)}{\omega^2} - \frac{g_o}{\omega^2}, \tag{3.4.29}$$

$$x(\tau) = \frac{g_o\sin(\omega\tau)}{\omega^2} - \frac{g_o\tau}{\omega}. \tag{3.4.30}$$

The above solution represents a cycloid. In this solution if $a > \frac{g_o}{\omega^2}$, then the path is curly and if $a < \frac{g_o}{\omega^2}$, then there are no curls in the path of the particle as explained in [9].

The interpretation of this solution is, consider a frame which is moving with the velocity as given in (3.4.28). The changing magnetic field produces the electric field $\varepsilon = \gamma vB$, which is perpendicular to both velocity and the magnetic field B. This electric field applies the force $q\varepsilon$, directed along the y axis. The above velocity is defined by the condition that this electric force exactly compensates the gravitational force. Hence the motion of the charged particle under the influence of the gravitational force which is orthogonal to the magnetic field is exactly analogous to the motion of the particle under the influence of electric and magnetic field which are orthogonal to each other [15].

3.5 Charged Particle in the Schwarzschild Space-time

Limiting cases of motion in the flat space-time and weak gravitational field limit is discussed above. Now we return to our problem. First we introduce some dimensionless quantities: σ, ρ, b, and ℓ which are defined below

$$t = \tau r_g, \quad r = \rho r_g, \quad \tau = \sigma r_g, \quad L_z = r_g \ell, \quad b = \beta r_g. \tag{3.5.1}$$

Using these dimensionless quantities in (3.3.6) – (3.3.11), we have

$$\frac{d^2\rho}{d\sigma^2} = \Big(\frac{d\theta}{d\sigma}\Big)^2 \frac{(2\rho-3)}{2} + \frac{(2\rho-3)\ell^2}{2\rho^4 \sin^2\theta} + \frac{1}{2\rho^2}(2\ell b - 1) - \frac{b^2(2\rho-1)\sin^2\theta}{2}, \tag{3.5.2}$$

$$\frac{d^2\theta}{d\sigma^2} = \frac{\ell^2 \cot\theta}{\rho^4 \sin^2\theta} - \frac{2}{\rho}\frac{d\rho}{d\sigma}\frac{d\theta}{d\sigma} - 2b^2 \sin 2\theta, \tag{3.5.3}$$

$$V_{\text{eff}} = \left[1 + \frac{(\ell - b\rho^2 \sin^2\theta)}{\rho^2 \sin^2\theta}\right]\left(1 - \frac{1}{\rho}\right), \tag{3.5.4}$$

$$E^2 = \Big(\frac{d\rho}{d\sigma}\Big)^2 + (\rho^2 - \rho)\Big(\frac{d\theta}{d\sigma}\Big) + V_{eff}. \tag{3.5.5}$$

If the particle is moving around the black hole at the equatorial plane $\theta = \frac{\pi}{2}$, in the circular orbit of radius ρ_o, then the equation (3.5.4) becomes

$$V_o = E_o = \left(1 - \frac{1}{\rho_o}\right)\left[1 + \frac{(\ell - b\rho_o^2)^2}{\rho^2}\right]. \tag{3.5.6}$$

As we have assumed that for a neutral particle that during motion of the particle around black hole if it collides with another particle its azimuthal angular momentum ℓ does not change and only energy will change. This collision gives the particle the transverse velocity $v > 0$. The energy change is given by

$$E_{\text{new}} = \sqrt{\left(E_o^2 - \frac{v^2(1-\rho)}{\rho}\right)}. \tag{3.5.7}$$

For $b \geq 0$, the parameter ℓ can be positive or negative $\ell = \pm\ell$. If $\ell > 0$ then the Lorentz force on the particle is repulsive, i.e. directed away from the black hole and if $\ell < 0$ then it is attractive, i.e. directed toward black hole. To study the effective potential's characteristics we have assumed that the magnetic field strength is fixed, hence the parameter b is fixed. There are only two parameters in the effective potential ℓ and ρ which can vary. We are interested only in the black hole exterior where $\rho > 1$ and the effective potential is positive. It vanishes at the black hole horizon at $\rho = 1$ and increases at the rate $b^2\rho^2$ for $\rho > 1$.

In figure 3.1 we plot the effective potential as a function of ρ, for $\ell = 10$, and $b = 0.5$

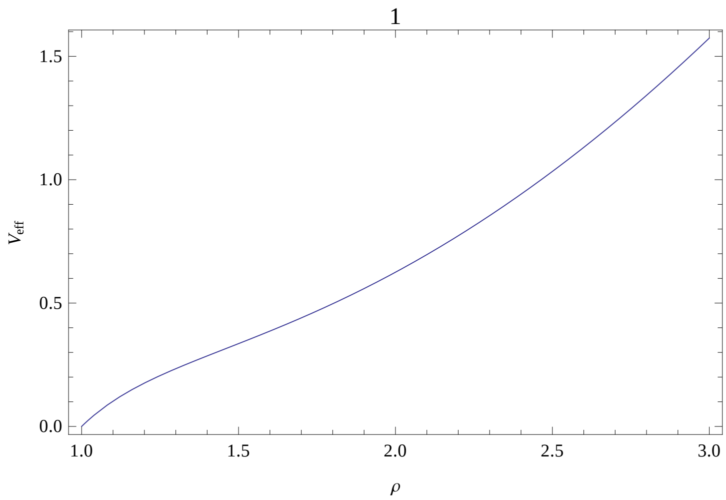

Figure 3.1: Plot of effective potential for $b = 0.5$ and $\ell = 1$. We can see from the figure that there are no circular orbits.

3.5.1 Innermost Stable Circular Orbit

Initially we assume the particle is in ISCO. To study the characteristic aspect of the effective potential we will find its the critical points (maximum, minimum) using first and second derivative test of equation (3.5.6). Using $\frac{\partial V_o}{\partial \rho} = 0$ and $\frac{\partial^2 V_o}{\partial \rho^2} = 0$ we can determine the ISCO.

$$\frac{1}{\rho^4}(b^2\rho^4(2\rho - 1) + \ell^2(3 - 2\rho) + \rho^2(1 - 2\ell b)) = 0, \tag{3.5.8}$$

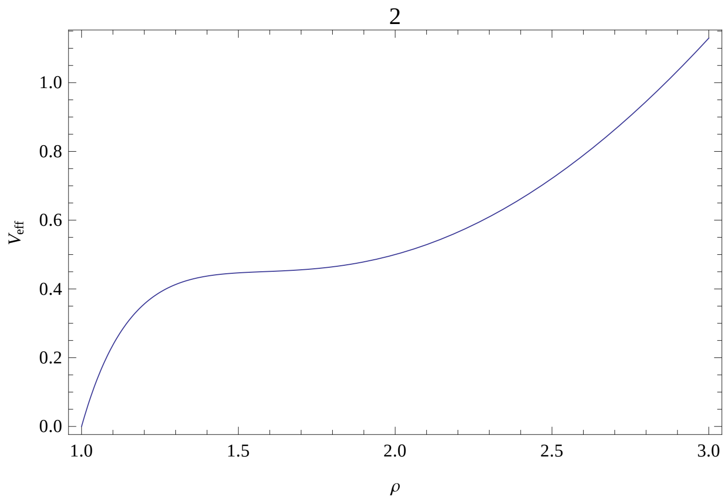

Figure 3.2: In this figure $\ell = 2$, there is an ISCO corresponding to $\rho \approx 1.60$.

$$\frac{2}{\rho^5}(\rho^2(b^2\rho^3 + 2\ell b - 1) + 3\ell^2(\rho - 2)) = 0. \tag{3.5.9}$$

The extreme values of the effective potential correspond to equation (3.5.8). From the above two equations (3.5.8) and (3.5.9) we can find ℓ and b in term of ρ. By adding (3.5.8) and (3.5.9) we get [27]

$$\ell_{\pm} = \pm\frac{b\rho_{\pm}^2\sqrt{3\rho_{\pm} - 1}}{\sqrt{-\rho_{\pm} + 3}}. \tag{3.5.10}$$

From the above equation we can see that the real solution exist only in the interval $\rho_{\pm} \in (\frac{1}{3}, 3]$. We are considering the black hole exterior, so we have $\rho_{\pm} \in (1, 3]$. By inserting the values from equation (3.5.10) into either of the equation (3.5.8) or (3.5.9) we get the equation in terms of $\rho_{\pm}$

$$\pm\sqrt{(3 - \rho_{\pm})(3\rho_{\pm} - 1)} + 4\rho_{\pm} - 9\rho_{\pm} + 3 + \frac{\rho_{\pm} - 3}{2b^2\rho_{\pm}^2} = 0. \tag{3.5.11}$$

By solving the above equation for b we get

$$b = \frac{\sqrt{2(3 - \rho_{\pm})}}{2\rho_{\pm}\sqrt{\left(4\rho_{\pm} - 9\rho_{\pm} + 3 \pm \sqrt{(3 - \rho_{\pm})(3\rho_{\pm} - 1)}\right)}}. \tag{3.5.12}$$

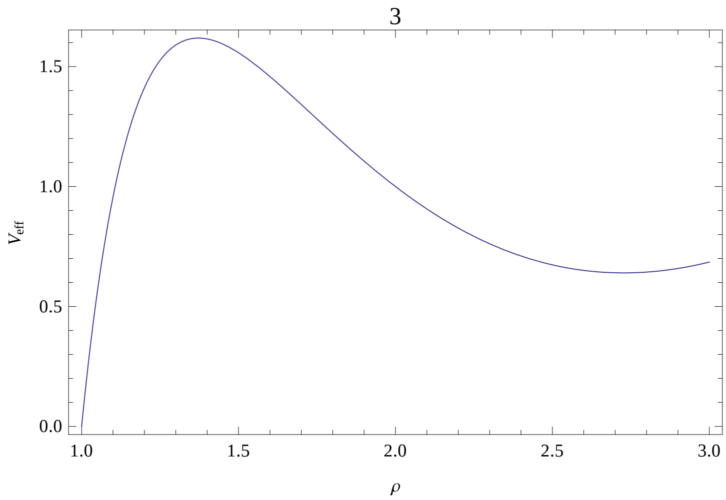

Figure 3.3: In this figure $\ell = 4$, there are both unstable and stable circular orbits defined by the minimum and maximum value of the effective potential.

For $\rho_\pm = 3$ we have $b = 0$. Inserting the values of b from equation (3.5.12) into equation (3.5.10) we obtain $\ell_\pm$ in term of $\rho_\pm$, i.e.

$$\ell_\pm = \pm \frac{\rho_\pm \sqrt{3\rho_\pm - 1}}{\sqrt{2\big(4\rho_\pm - 9\rho_\pm + 3 \pm \sqrt{(3-\rho_\pm)(3\rho_\pm - 1)}\big)}}. \tag{3.5.13}$$

For b to be real, it will remain same as we defined above $1 < \rho_+ \leq 3$. But for ρ_- it will change and impose an extra restriction which is $\frac{\sqrt{13}+5}{4} < \rho_- \leq 3$. The location of ISCO around the Schwarzschild black hole corresponds to $\rho_\pm = 3$, $\ell_\pm = \pm\sqrt{3}$ and $b = 0$.

Figure 3.4 shows that as the magnetic field increases then the ρ will decrease. It means that the magnetic field is stronger in the accretion disc of the black hole and it decreases far away from it. For the repulsive Lorentz force and the strong magnetic field, the innermost stable orbit may be very close to the black hole horizon as explained in [12] and [26].

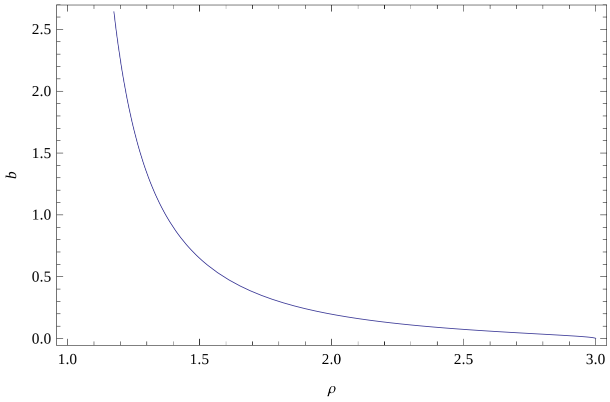

Figure 3.4: Dependance of the stable orbit on the magnetic field b. As ρ increases magnetic field decreases this may shift the ISCO near to horizon.

From (3.5.6) and (3.5.13) we have

$$E_o = 1 + \frac{(\rho_o - 3.463)0.115}{(1.85 - \rho_o)(3.433 - \rho_o)}, \tag{3.5.14}$$

and

$$E_o = 1 + \frac{(3.19 - \rho_o)0.439}{(2.10 - \rho_o)(\rho_o - 3.667)}. \tag{3.5.15}$$

We have plotted equations (3.5.14) and (3.5.15) as shown in figure 3.5 and 3.6.

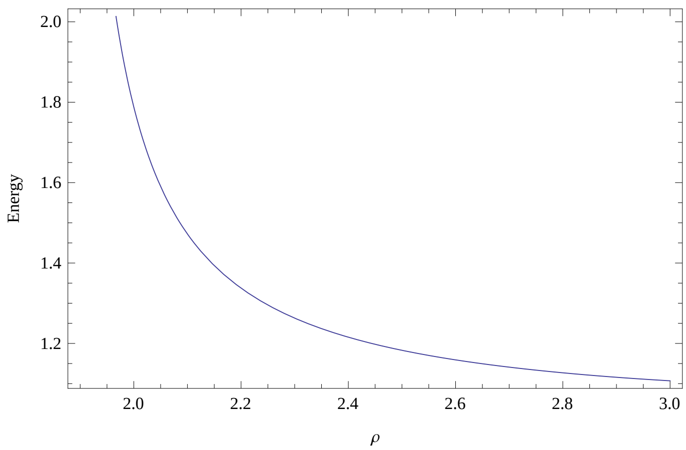

Figure 3.5: Here we plot the energy as a function of ρ for $b = 0.5$ and $\ell = 0.3$.

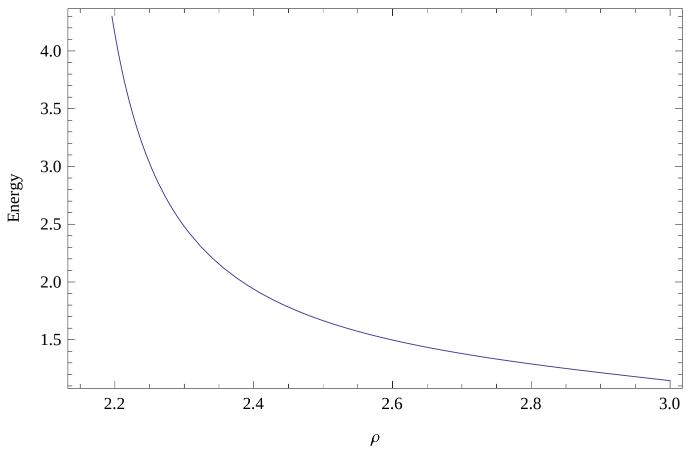

Figure 3.6: Here we plot the energy as a function of ρ for $b = 0.5$ and $\ell = -0.3$.

For escape velocity putting the values of equations (3.5.6) and (3.5.14) in equation (3.5.7) for $\ell > 0$ we get

$$v = \frac{\sqrt{\left(1 + \frac{(\rho_o - 3.463)0.115}{(1.85 - \rho_o)(3.433 - \rho_o)}\right)^2 \rho_o - (\rho_o - 1)\left(1 + \frac{(1.22 - 0.5\rho_o^2)^2}{\rho_o^2}\right)}}{\sqrt{\rho_o - 1}}, \quad (3.5.16)$$

for $\ell < 0$ from equations (3.5.6) and (3.5.15)

$$v = \frac{\sqrt{\left(1 + \frac{(3.19 - \rho_o)0.439}{(2.10 - \rho_o)(\rho_o - 3.667)}\right)^2 \rho_o - (\rho_o - 1)\left(1 + \frac{(1.22 - 0.5\rho_o^2)^2}{\rho_o^2}\right)}}{\sqrt{\rho_o - 1}}. \quad (3.5.17)$$

We have plotted equations (3.5.16) and (3.5.17) for escape velocity for both cases $\ell > 0$ and $\ell < 0$ in figures 3.7 and 3.8.

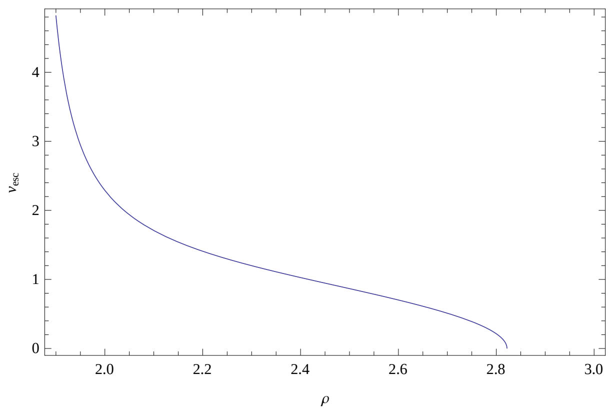

Figure 3.7: Here we plot the escape velocity against ρ for $b = 0.5$ and $\ell = 0.3$..

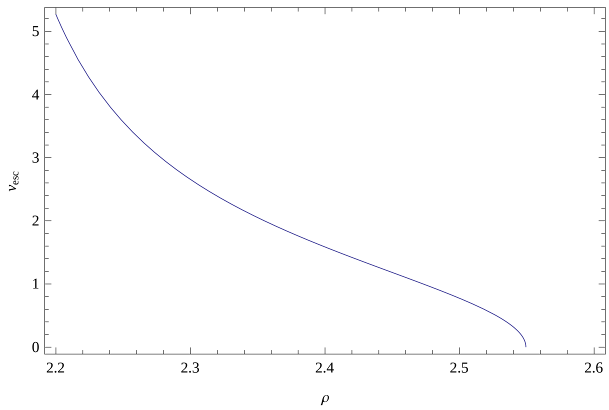

Figure 3.8: Here we plot the escape velocity as a function of ρ for $b = 0.5$ and $\ell = 0.3$..

Chapter 4

Motion of a Charged Particle Around the Slowly Rotating Kerr Black Hole Immersed in a Magnetic Field

4.1 Introduction

In this chapter, we investigate the dynamics of a charged and a neutral particle around a slowly rotating Kerr black hole immersed in a magnetic field. Here we consider a particle moving around black hole in equatorial plane. As it collides with another particle, under what circumstances the particle can escape to infinity is studied. We consider that the magnetic field decreases away from the black hole and gravitational field can be ignorable at large distance from the black hole. Therefore the charged particle moves in a constant magnetic field far away from the black hole. Hence, before reaching infinity where the magnetic field is constant, it will pass through a region where both fields strongly effect its motion. In that region particle's motion is completely unpredictable (chaotic). Its motion is also unpredictable if a particle moving in a non uniform magnetic field in the absence of black hole. [22, 23].

We are extending a previous work (Motion of a charged particle around the magnetized Schwarzschild black hole)[11], by choosing a slowly rotating magnetized Kerr black hole. We are following the same procedure as we have done in previous chapter.First we will discuss the motion of a neutral particle and then we will investigate the dynamics of a charged particle.

4.2 Escape Velocity of a Neutral Particle

We start with the simple case of calculating the escape velocity of the neutral particle in the absence of magnetic field. The Kerr metric is given by equation (2.6.6). For simplicity, we consider the slowly rotating Kerr black hole and neglect the terms involving a^2, then equation (2.6.6) becomes

$$ds^2 = (1-\frac{r_g}{r})dt^2+\frac{2ar_g\sin^2\theta}{r}d\phi dt-\frac{1}{1-\frac{r_g}{r}}dr^2-r^2d\theta^2-r^2\sin^2\theta d\phi^2. \quad (4.2.1)$$

The above metric diverge at $r = 0$ and $r = r_g = 2M$ like Schwarzschild metric. Therefore the event horizon is same for schwarzschild and slowly rotating Kerr black hole. The metric (4.2.1) has the following symmetries

$$\phi = 2\pi - \phi, \quad \theta \to \pi - \theta. \quad (4.2.2)$$

These symmetries imply the following transformations [26].

$$L_z \to -L_z \qquad \text{and} \qquad v_\perp \to -v_\perp. \quad (4.2.3)$$

There are three constants of motion in which two of them are given as [11]

$$\xi_{(t)} = \xi^\mu_{(t)}\partial_\mu = \frac{\partial}{\partial t}, \qquad \xi_{(\phi)} = \xi^\mu_{(\phi)}\partial_\mu = \frac{\partial}{\partial \phi}. \quad (4.2.4)$$

The black hole metric is invariant under time translation and rotation around the symmetry axis. The corresponding conserved quantities are the energy $\mathcal{E}$ per unit mass and the azimuthal angular momentum L_z per unit mass

$$\dot{t} = \frac{1}{r^2 - r_g r}\left[(1-\frac{r_g}{r})L_z + \frac{r_g a}{r}\mathcal{E}\right], \quad (4.2.5)$$

$$\dot{\phi} = \frac{1}{r^2 - r_g r}\big[r^2\mathcal{E} + \frac{r_g a}{r}L_z\big]. \tag{4.2.6}$$

Throughout in this chapter over dot represents differentiation with respect to proper time τ. The third integral of motion is, i.e. [15]

$$L^2 = r^4\dot{\theta}^2 + \frac{L_z^2}{\sin\theta^2} = r^2 v_\perp^2 + \frac{L_z^2}{\sin\theta^2}. \tag{4.2.7}$$

Here we denote $v_\perp \equiv -r\dot{\theta}_o$. Using the normalization condition $u^\mu u_\mu = 1$, we get [25]

$$\dot{r}^2 = \frac{(\mathcal{E}r^2 \mp aL_z)^2}{r^4} - \frac{r^2 - r_g r}{r^4}(r^2 + L_z \mp 2a\mathcal{E}L_z). \tag{4.2.8}$$

In the above equation (4.2.8) the upper sign represents the co-rotation (direction of rotation of particle and black hole is same to each other) and the lower sign represents the contra-rotation (direction of rotation of particle and black hole is opposite to each other)

At the turning points ($\dot{r} = 0$), the equation (4.2.8) becomes quadratic in $\mathcal{E}$ whose solution is

$$\mathcal{E} = \frac{aL_z r_g + \sqrt{r^5(r - r_g) + L_z^2(r^4 - r^3 r_g + a^2 r_g^2)}}{r^3}, \tag{4.2.9}$$

Here $\mathcal{E}$ is the effective potential, $\mathcal{E} = V_{\text{eff}}$.

In case of a Schwarzschild black hole, we discard the negative energy by the condition that the energy of the particle should be positive in the exterior region of black hole. However the negative energy is allowed in Kerr geometry because in case of Kerr black hole if $r^2 << L_z^2$, this condition can always be satisfied by taking the mass of the particle to be small. Inside static limit surface, which is defined by $r = r_{st} = 2M = r_g$, the term $r^4 L^2$ always contributes less as compared to $r_g^2 r^2 L^2$ in equation (4.2.9) (inside static limit surface, we have $r_g > r$). Therefore its square root will always contributes less than $r_g a|L|r$ to $\mathcal{E}$. Thus

$$\mathcal{E} < \frac{(r_g aLr + r_g ar|L|)}{r^4}. \tag{4.2.10}$$

If $L < 0$, then $\mathcal{E} < 0$. At the static limit surface of Kerr black hole, at the equatorial plane $r = r_{st} = 2M = r_g$, there are orbits inside r_{st} in which $L < 0$ then energy is also negative $\mathcal{E} < 0$. Hence we can say that within the static limit surface there are retrograde orbits which have negative energy. It means that energy required to remove a particle from its orbit to infinity is greater than its rest mass.

Consider a particle at the circular orbit $r = r_o$, where r_o is the local minima of the effective potential. The energy and azimuthal angular momentum corresponding to r_o are

$$\begin{aligned} L_{zo} &= \frac{r_o^6 r_g^2}{\sqrt{r_o r_g(2r_o^4 + 18a^2 r_o r_g - 3r_o^3 r_g - 12a^2 r_g^2)}} \\ &\quad +(r_o^6 r_g^2)\Big[\sqrt{6}(a^2 r_o^2 r_g^2)\big(2r_o^5 + 9a^2 r_o^2 r_g - 4r_o^4 r_g \\ &\quad -12a^2 r_g^2 r_o - 2r_o^3 r_g^2 + 4a^2 r_g^3\big)\Big]^{\frac{-1}{2}}, \end{aligned} \tag{4.2.11}$$

$$\mathcal{E}_o = \frac{aL_{zo} r_g + \sqrt{r_o^5(r_o - r_g) + L_{zo}^2(r_o^4 - r_o^3 r_g + a^2 r_g^2)}}{r_o^3}. \tag{4.2.12}$$

Now consider the particle is in a ISCO and collides with another particle. Therefore after the collision it will move within a new plane with respect to the previous one. After collision between particles, three cases are possible for the particles: (i) bounded motion, (ii) captured by black hole and (iii) escape to infinity. The results depends on the collision process. For small change in energy and momentum, orbit of the particle is slightly perturbed. While for large change in energy and momentum, it can go away from initial path and captured by black hole or escape to infinity.

After the collision particle should have new values of energy and momentum $\mathcal{E}$, L_z and the total angular momentum L^2. We simplify the problem by applying the following conditions: (i) the azimuthal angular momentum is fixed and (ii) initial radial velocity remains same after the collision $\dot{r} = 0$. Under these condition only energy can change by which we can determine the motion of the particle. After collision particle acquires an escape velocity $v_\perp$ in orthogonal direction of the equatorial plane [7].

After the collision the momentum and energy of the particle become (at the equatorial plan $\theta = \frac{\pi}{2}$)

$$L^2 = r_o^2 v_\perp^2 + L_z^2, \tag{4.2.13}$$

$$\mathcal{E}_{\text{new}} = \frac{aLr_g + \sqrt{r^5(r - r_g) + L^2(r^4 - r^3 r_g + a^2 r_g^2)}}{r^3}. \tag{4.2.14}$$

These values of momentum and energy are greater then the values of momentum and energy before collision.

We can see from the equation (4.2.14) that as $r \to \infty$, $\mathcal{E}_{\text{new}} \to 1$. So for the unbound motion $\mathcal{E}_{\text{new}} \geq 1$. Physically it means that the energy of the particle exceeds its rest mass energy. Therefore, all the orbits with $\mathcal{E}_{\text{new}} \geq 1$ are unbound in the sense that particle escape to infinity. Conversely for $\mathcal{E}_{\text{new}} < 1$, particle cannot escape to infinity.

Therefore the particle escape to infinity if $\mathcal{E}_{new} \geq 1$, or

$$v_\perp \geq \frac{arr_g + rL_z(r_g - r) + \sqrt{r^2 r_g(r^3 + (a^2 - r^2)r_g)}}{r^2(r - r_g)}. \tag{4.2.15}$$

We get the above expression for velocity v from equation (4.2.14) by putting $\mathcal{E} = 1$ and then solve it for v.

4.3 Charged Particle Around the Slowly Rotating Magnetized Kerr Black Hole

We investigate the motion of a charged particle q (electric charge) in the presence of the magnetic field in the exterior of the black hole. The Killing vector equation is [18]

$$\Box \xi^\mu = 0, \tag{4.3.1}$$

where ξ^μ is a Killing vector in equation (4.3.1) which coincides with the Maxwell equation for 4-potential A^μ in the Lorentz gauge ${A^\mu}_{;\mu} = 0$. The special choice for A^μ is [12]

$$A^\mu = \frac{\mathcal{B}}{2}\xi^\mu_{(\phi)} + a\mathcal{B}\xi^\mu_{(t)}, \tag{4.3.2}$$

Here $\mathcal{B}$ is the magnetic field strength. The 4-potential is invariant under the symmetries which correspond to the Killing vectors, i.e.,

$$L_\xi A_\mu = A_{\mu,\nu}\xi^\nu + A_\nu \xi^\nu_{,\mu} = 0. \tag{4.3.3}$$

A magnetic field vector is defined as [11]

$$\mathcal{B}^\mu = -\frac{1}{2}e^{\mu\nu\lambda\sigma}F_{\lambda\sigma}u_\nu, \tag{4.3.4}$$

where

$$e^{\mu\nu\lambda\sigma} = \frac{\epsilon^{\mu\nu\lambda\sigma}}{\sqrt{-g}}, \quad \epsilon_{0123} = 1, \quad g = det(g_{\mu\nu}). \tag{4.3.5}$$

In equation (4.3.5) $\epsilon^{\mu\nu\lambda\sigma}$ is the Levi Civita symbol. For a local observer at rest (4.2.5)

$$u^\mu_0 = \Big(\frac{1}{\sqrt{(1-\frac{r_g}{r}) + \frac{4am\sqrt{(1-\frac{r_g}{r})}}{r^2\sin\theta}}}\Big)\xi^\mu_{(t)}, \tag{4.3.6}$$

$$u^\mu_3 = \Big(\frac{1}{r\sin\theta\sqrt{(1+\frac{4am}{r^2\sin\theta\sqrt{(1-\frac{r_g}{r})}})}}\Big)\xi^\mu_{(\phi)}. \tag{4.3.7}$$

and the other two components $u^\mu_1 = 0$, at the turning points ($\dot{r} = 0$) and $u^\mu_2 = 0$, at the equatorial plane. From equations (4.3.2) – (4.3.7) we have obtained the components of the magnetic field

$$\begin{aligned}
\mathcal{B}^\mu &= \mathcal{B}\Big[\Big(\cos\theta\Big(\frac{(1-\frac{r_g}{r})}{\sqrt{(1-\frac{r_g}{r}) + \frac{2r_g a\sqrt{(1-\frac{r_g}{r})}}{r^2\sin\theta}}}\Big) \\
&+ \frac{r_g a\sin\theta\cos\theta}{r^5}\Big(\frac{(1-\frac{r_g}{r})}{r\sin\theta\sqrt{(1+\frac{2r_g a}{r^2\sin\theta\sqrt{(1-\frac{r_g}{r})}})}}\Big)\Big)\delta^\mu_r \\
&- \Big(\frac{\sin\theta(1-\frac{r_g}{r})}{r\sqrt{(1-\frac{r_g}{r}) + \frac{2r_g a\sqrt{(1-\frac{r_g}{r})}}{r^2\sin\theta}}}\Big)\delta^\mu_\theta\Big].
\end{aligned} \tag{4.3.8}$$

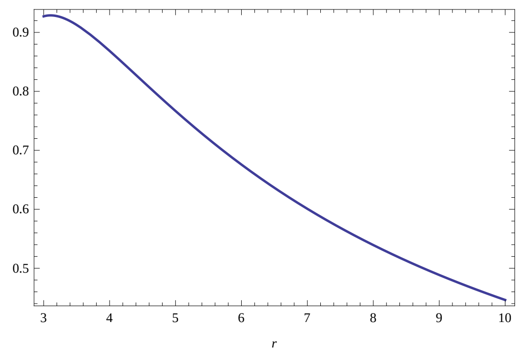

Figure 4.1: Magnetic field B as a function of r for $\theta = \frac{\pi}{2}$. It can be seen from this figure as $r \to \infty$, magnetic field $B \to 0$.

The Lagrangian of the particle of mass m and charge q moving in an external magnetic field of a curved space-time is given by [24]

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu}\dot{x}^{\mu}\dot{x}^{\nu} + \frac{qA_{\mu}}{m}\dot{x}^{\mu}. \tag{4.3.9}$$

Generalized 4-momentum [11] of the particle is defined as $P_\mu = mu_\mu + qA_\mu$. Constants of motion are

$$\dot{t} = \frac{r^3\mathcal{E} + aL_z r_g}{r^2(r - r_g)} - 2aB, \tag{4.3.10}$$

$$\dot{\phi} = \frac{1}{r^2}\left(\frac{a\mathcal{E}}{(r - r_g)} - L_z\right) - B. \tag{4.3.11}$$

Here we denote

$$B \equiv \frac{q\mathcal{B}}{2m}. \tag{4.3.12}$$

We also derive the dynamical equation for r by using Euler Lagrange equation (3.1.3) and constants of motion (4.3.10) and (4.3.11)

$$\begin{aligned}\ddot{r} &= \frac{Ba\mathcal{E}r_g}{r(r - r_g)} \\ &+ \frac{1}{2r^4(r - r_g)}\Big[6B^2r^6 - 2L_z^2(r - r_g)^2 + r^3 r_g(-\mathcal{E}^2 \\ &+ 6B^2 r r_g + \dot{r}^2 - 12B^2r^2)\Big].\end{aligned} \tag{4.3.13}$$

Using normalization condition $u^\mu u_\mu = 1$ and constants of motion given by equation (4.3.10) and (4.3.11) we have calculated the effective potential

$$\begin{aligned}\mathcal{E} &= \frac{1}{r^6(r-r_g)}\Big[2aBr^7 + ar_g r^3\big(2Br^2(r_g-2r)+L_z(r_g-r)\big) \\ &+\Big(a^2r^6(r-r_g)^2\big(r_g(L_z+2Br^2)-2Br^3\big)^2 + \\ &r^9(r-r_g)^3\big(r^2+(L_z+Br^2)^2\big)\Big)^{\frac{1}{2}}\Big]. \end{aligned} \tag{4.3.14}$$

If equation (4.3.14) satisfied initially (at the time of collision), then it is always valid (throughout the motion), provided that $r(\tau)$ is controlled by equation (4.3.13).

Let us discuss the symmetries of equations (4.3.9) − (4.3.14). First, these equations are invariant under the transformations

$$\phi \to -\phi, \quad L_z \to -L_z, \quad B \to -B. \tag{4.3.15}$$

Therefore, without losing the generality, we consider the particle of positive electric charge then the magnetic field $\mathcal{B} > 0$. To consider the particle of negative charge, one should apply the transformations (4.3.15). So, the trajectory of a negatively charge particle is related to positive charge's trajectory by transformation (4.3.15). For $\mathcal{B} > 0$ we will have to study both cases when $L_z > 0$, $L_z < 0$. These cases are physically different: the change of sign of L_z means the change of direction of the Lorentz force on the particle.

The system of equations (4.3.9) − (4.3.14) is invariant with respect to reflection ($\theta \to \pi - \theta$). This transformation retains the initial position of the particle and changes $v_\perp \to -v_\perp$ as $v_\perp \equiv -r\dot{\theta}_o$. Therefore it is sufficient to consider only the positive value of $v_\perp$.

4.4 Dimensionless Form of the Dynamical Equations

To perform the numerical analysis, it is convenient to convert equations into dimensionless form by introducing the following dimensionless quantities: σ, ρ, ℓ, and b [9]:

$$\sigma = \frac{\tau}{r_g}, \ \rho = \frac{r}{r_g}, \ \ell = \frac{L_z}{r_g}, \ b = Br_g. \tag{4.4.1}$$

The equation (4.3.14) become

$$\begin{aligned} \mathcal{E}_o &= \frac{1}{\rho_o^6(\rho_o - 1)}\Big[- a\rho_o^3(\rho_o - 1)(\ell - 2b\rho_o^2(\rho_o - 1)) \\ &+\Big(\rho_o^6(\rho_o - 1)^2(a^2\big(\ell - 2b\rho_o^2(\rho_o - 1)\big)^2) \\ &+\rho_o^3(\rho_o - 1)\big(\rho_o^2 + (\ell + b\rho_o^2)^2\big)\Big)^{\frac{1}{2}}\Big]. \end{aligned} \tag{4.4.2}$$

The energy of the particle moving around the Kerr black hole of radius ρ_o in the equatorial plane is given by equation (4.4.2). For $\rho \to \infty$ and magnetic field is zero at ∞, hence $\mathcal{E} \to 1$ by equation (4.4.2). Dimensionless form of equation (4.3.13) is

$$\begin{aligned} \frac{d^2\rho}{d\sigma^2} &= \frac{1}{2\rho^4(\rho - 1)}\big[\rho^3\big(2a\mathcal{E}b + 6\mathcal{E}^2b^2\rho(\rho - 1)^2\big) \\ &-2\ell(\rho - 1)^2 + \rho^3\frac{d\rho}{d\sigma}\big]. \end{aligned} \tag{4.4.3}$$

We have solved the equation (4.4.3) numerically by using the built in command NDSolve of Mathematica. As ISCO exists at $r = 3r_g$, and using $\rho = \frac{r}{r_g}$ and $\sigma = \frac{\tau}{r_g}$, our initial conditions for solving (4.4.3) become $\rho(1) = 3$ and $\dot{\rho}(1) = 3$. We get the interpolating function $\rho(\sigma)$ as the solution of the equation (4.4.3) which we have plotted in figure 4.2 against σ. In figure 4.3 we have plotted the radial velocity (derivative of the interpolating function) vs σ which shows that the particle will escape to infinity according to the applied initial conditions. As we did before, in the case of a neutral particle, we

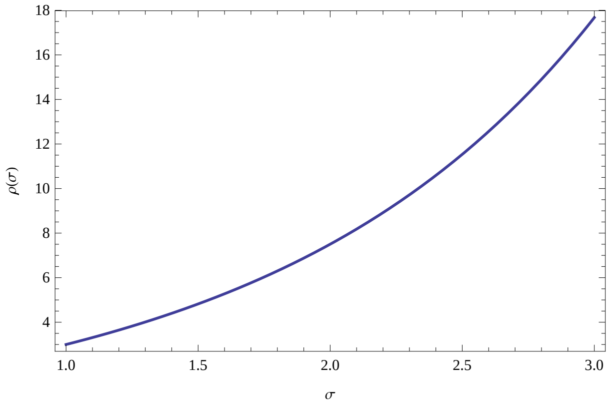

Figure 4.2: Figure shows the graph for $\rho(\sigma)$ vs σ. Here $\mathcal{E} = 1, q = 1, b = 0.5, \ell = 2$, and $a = 0.1$.

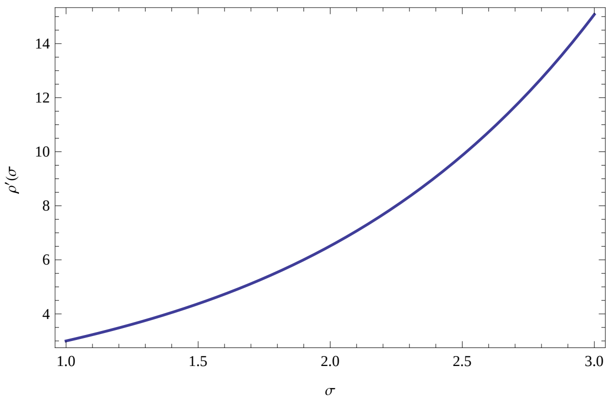

Figure 4.3: Figure shows the graph for $\rho'(\sigma)$ (radial velocity) vs σ. Here $\mathcal{E} = 1, q = 1, b = 0.5, \ell = 2$, and $a = 0.1$.

assume that the collision does not change the azimuthal angular momentum but it will change the transverse velocity. Hence the energy of the particle become $\mathcal{E}_o \to \mathcal{E}$ and it is given by the equation

$$\begin{aligned}\mathcal{E} &= \frac{1}{\rho^6(\rho-1)}\Bigg[-a\rho^3(\rho-1)((\ell+\rho v)-2b\rho^2(\rho-1)) \\ &\quad +\Bigg(\rho^6(\rho-1)^2(a^2\big((\ell+\rho v)-2b\rho^2(\rho-1)\big)^2) \\ &\quad +\rho^3(\rho-1)\big(\rho^2+((\ell+\rho v)+b\rho^2)^2\big)\Bigg)^{\frac{1}{2}}\Bigg] \qquad (4.4.4)\end{aligned}$$

We explained before as $\rho \to \infty$ then the energy $\mathcal{E} \to 1$. For the unbound motion the energy of the particle should be $\mathcal{E} \geq 1$. From (4.4.4) we get four formulae for escape velocity of the particle

$$v^2_{\text{esc}} = \frac{1}{\rho^2(\rho-1)}\Big[\rho(\rho-1)(\ell+b\rho^2) - a\rho + \sqrt{a^2\rho^2 + \rho^4(\rho-1) - 2ab\rho^4(\rho-1)(2\rho-1)}\Big] \quad (4.4.5)$$

$$v^2_{\text{esc}} = \frac{1}{\rho^2(\rho-1)}\Big[\rho(\rho-1)(\ell+b\rho^2) - a\rho - \sqrt{a^2\rho^2 + \rho^4(\rho-1) - 2ab\rho^4(\rho-1)(2\rho-1)}\Big] \quad (4.4.6)$$

The above equations (4.4.5) and (4.4.6) correspond to $\mathcal{E} = 1$. We now discuss the behavior of the particle, when it escapes to asymptotic infinity. For simplicity we consider the particle initially in ISCO. We can express the parameters ℓ and b in term of ρ_o by simultaneously solving $\frac{d\mathcal{E}_o}{d\rho} = 0$ and $\frac{d^2\mathcal{E}_o}{d\rho^2} = 0$ for ℓ and b. But the First and the second derivative of effective potential $\mathcal{E}_l$ are very complicated and we cannot find the explicit expression for ℓ and b in term of ρ.

4.5 Trajectories for Escape Energy

Here we plot only the positive energy $\mathcal{E}_+$ of the particle. We cannot consider the negative energy because it exist only within the static limit surface ($r_{st} = 2M$). There are retrograde orbits which have negative energy in the static limit surface. We have ignored the terms involving a^2 from the metric, hence we have only one horizon ($r = 2M = r_g$). The ISCO are exist only outside the event horizon, therefore we cannot consider the negative energy. For the rotational (angular) variable

$$\frac{d\phi}{d\sigma} = \frac{\ell}{\rho^2} - b + \frac{a\mathcal{E}}{\rho^3(1-\rho)}. \quad (4.5.1)$$

If $\ell > 0$ and $\ell + \mathcal{E} < b$ in equation (4.5.1) then the right hand side is always negative. For $\ell < 0$ the Lorentz force on the charged particle is attractive. This motion is like the oscillation in the radial direction. This motion is very similar to the bound motion of the test particle rotating around the Schwarzschild black hole [4]. Presence of magnetic field might shift the trajectory of the particle away from the black hole. Hence it is easy for a particle to escape when the Lorentz force is attractive. If $\ell > 0$ and $(\ell + \mathcal{E} > b)$ then right hand side of equation (4.5.1) is always positive. For $\ell > 0$ the Lorentz force on the particle is repulsive. The repulsive Lorentz force and the magnetic field might shift the ISCO of the particle very close to the horizon as discussed in [12, 27]. Figures $4.4 - 4.11$ correspond to equation (4.4.2). In figure 4.4 the shaded region corresponds to escape energy (unbound motion) and the unshaded region corresponds to the bound motion. The curved line represents the minimum energy required to escape.

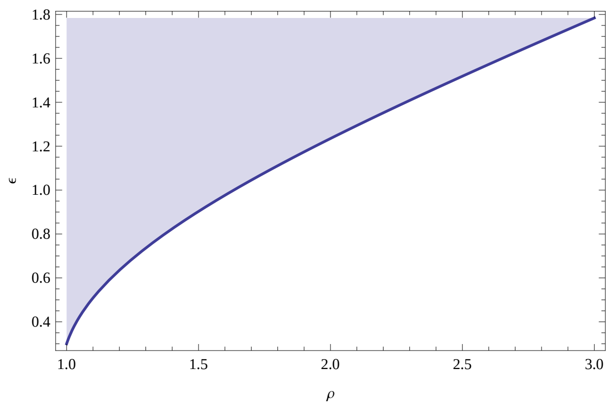

Figure 4.4: Effective potential $\mathcal{E}$ as a function of ρ for $\ell = 0.3$, $b = 0.5$ and $a = 0.1$.

In figure 4.5 we have plotted effective potential (4.4.2) as a function of ρ for $\ell = 10$ and plot is almost like Schwarzschild effective potential as given in [9]. Where $\mathcal{E}_{max}$ corresponds to unstable circular orbit and $\mathcal{E}_{min}$ refers to ISCO. It can be seen from figure 4.7 that as we increase the value of magnetic field, local minima is shifting toward the horizon. The local minima

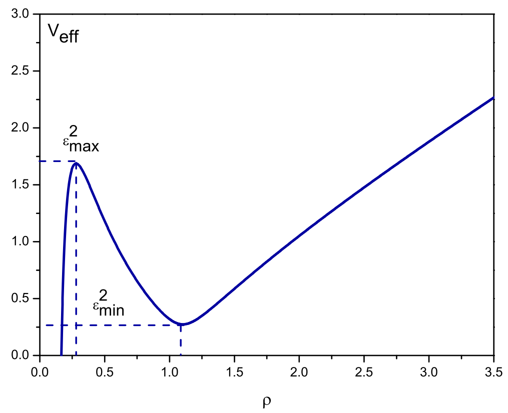

Figure 4.5: Here we plot the effective potential $\mathcal{E}$ against ρ, for $\ell = 10$, $b = 0.5$ and $a = 0.1$. In this figure $\mathcal{E}_{max}$ corresponds to stable circular orbit and $\mathcal{E}_{min}$ corresponds to unstable circular orbit.

is correspond to ISCO. The effective potential $\mathcal{E}$ of a particle moving in a slowly rotating Kerr spacetime is plotted as a function of radial coordinate ρ for different values of ℓ in figure 4.6. Where $\mathcal{E}_{min}$ correspond to ISCO.

Figures 4.8 and 4.9 show the behaviour of energy for both ℓ_+ and ℓ_-. Figure 4.10 show the increase in magnetic field results in the increase in energy. Therefore magnetic field may provide more energy to the particle to escape. Figure 4.11 is for energy vs angular momentum ℓ.

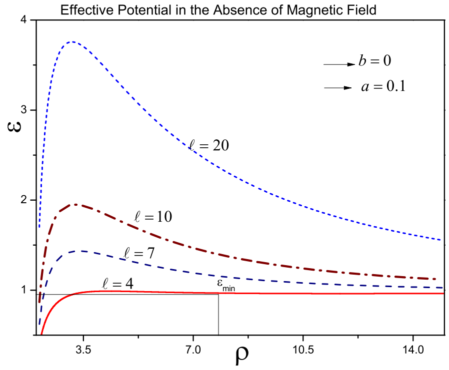

Figure 4.6: The effective potential $\mathcal{E}$ of a particle moving in a slowly rotating Kerr spacetime is plotted as a function of radial coordinate ρ for different values of angular momentum ℓ.

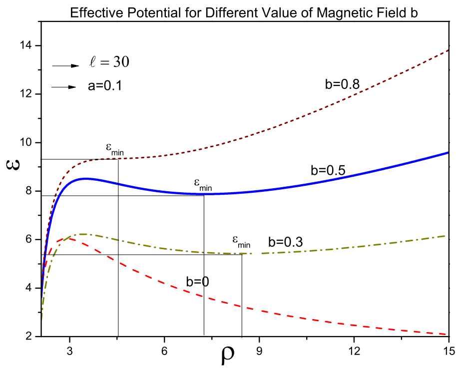

Figure 4.7: The effective potential $\mathcal{E}$ against ρ for different values of magnetic field.

4.6 Trajectories for Escape Velocity

For unbound motion there are four expressions of escape velocity (two positive and two negative). Trajectories for the particle which escape to infinity either $-\infty$ or $+\infty$ are given below. Figures 4.12 and 4.13 corresponds to equations (4.4.5) and (4.4.6). In figure 4.12, the shaded region corresponds to escape velocity of the particle and the solid curved line represents the minimum velocity required to escape to $+\infty$. The unshaded region is for bound motion. In figure 4.13 the shaded region correspond to escape velocity of the particle and the solid curved line represent the minimum velocity

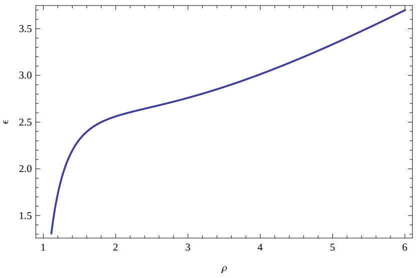

Figure 4.8: Escape energy $\mathcal{E}_+$ against ρ, for $\ell = 5$, $b = 0.5$ and $a = 0.1$.

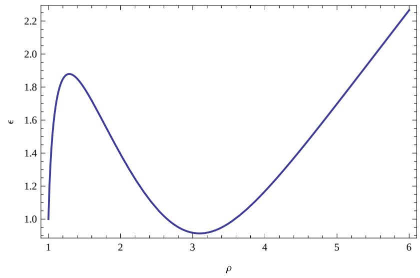

Figure 4.9: Escape energy $\mathcal{E}_+$ vs ρ, for $\ell = -5$, $b = 0.5$ and $a = 0.1$.

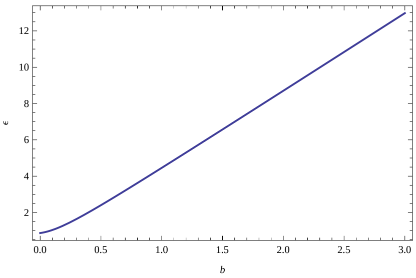

Figure 4.10: Escape energy $\mathcal{E}_+$ against magnetic field b, for $\ell = 0.3$, $\rho = 1.5$ and $a = 0.1$.

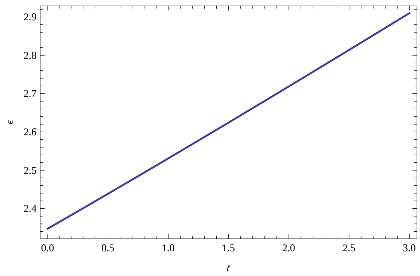

Figure 4.11: Escape energy $\mathcal{E}_+$ against angular momentum ℓ, for $\rho = 1.5$, $b = 0.5$ and $a = 0.1$.

required to escape to $-\infty$. The unshaded region is for bound motion around the black hole. We have plotted the escape velocity v_{esc} for different values of magnetic field b in figure 4.14. Due to the presence of magnetic field in the vicinity of black hole, escape velocity of the particle increases. It become almost same to which we have for the absence of magnetic field $b = 0$. Figure 4.15 show the behaviour of escape velocity v_{esc} for different values of angular momentum ℓ. It is almost same as we get for different values of magnetic field B in figure 4.14.

Figures 4.16 $-$ 4.19 correspond to equation (4.4.6) v_+. Figures 4.16 and 4.17 show the trajectories for the escape velocity have for $\ell = 0.3$ and $\ell = -0.3$. Figure 4.18 is for escape velocity vs magnetic field. It is strong in the accretion disc of the black hole as compare to away from it. Therefore

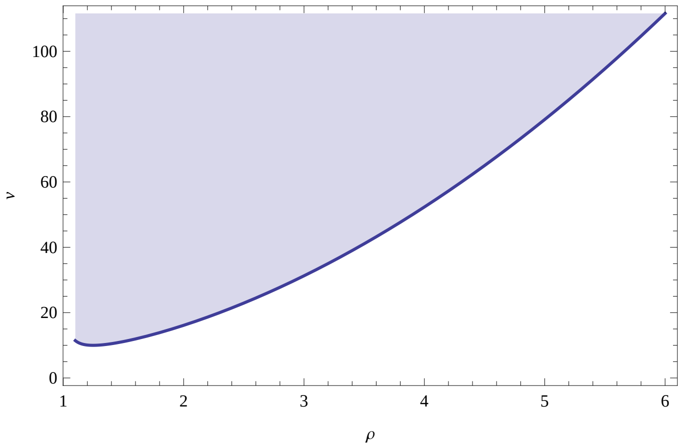

Figure 4.12: Escape velocity as a function of ρ, for $\ell = 0.3$, $b = 0.5$, $\mathcal{E} = 1$ and $a = 0.1$.

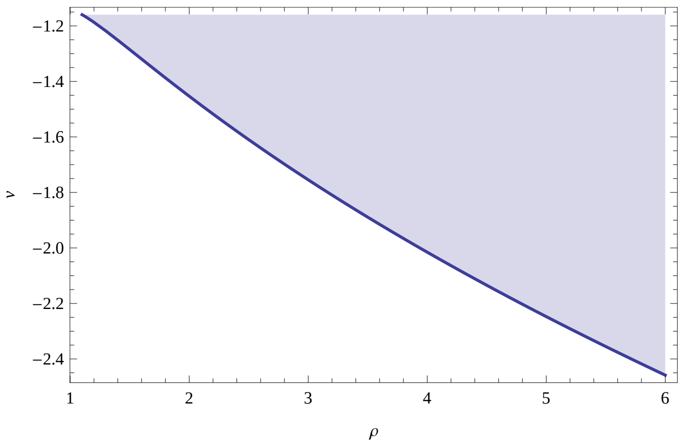

Figure 4.13: Escape velocity as a function of ρ, for $\ell = 0.3$, $b = 0.5$, $\mathcal{E} = 1$ and $a = 0.1$.

magnetic field may provide sufficient energy to the particle to escape. Figure 4.19 we have plotted escape velocity v_+ against angular momentum. Figures 4.20-4.22 correspond to equation 4.4.5 v_+. Figures 4.20 and 4.21 show the trajectories of escape velocity, for $\ell = 0.3$ and $\ell = -0.3$. Figure 4.22 show that the velocity increases linearly as the magnetic field increases.

Figures 4.23-4.26 correspond equation (4.4.6) v_-. Figures 4.23 and 4.24 represent the trajectories for the escape velocity of the particle for $\ell = 0.3$

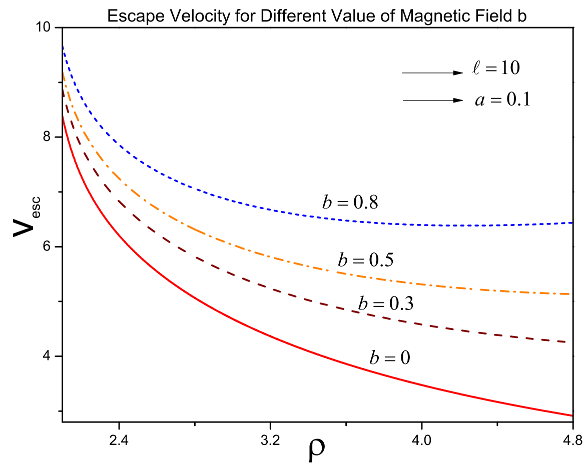

Figure 4.14: Escape velocity v_{esc} for different values of magnetic field b.

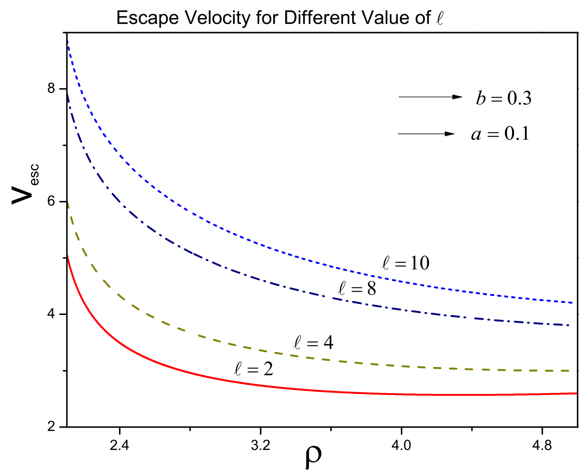

Figure 4.15: Behaviour of escape velocity v_{esc} for different values of angular momentum ℓ.

and $\ell = -0.3$. Figure 4.25 shows that as the magnetic field is increases, the escape velocity also increases. Figure 4.26 shows escape velocity as a function of angular momentum.

Figures 4.27-4.29 correspond to escape velocity as given by equation (4.4.5) v_{-}. Figures 4.27 and 4.28 represent the trajectories of the parti-

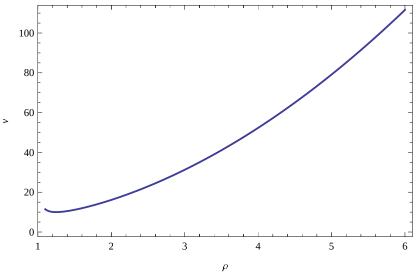

Figure 4.16: Escape velocity v_+ vs rho, for $\ell = 0.3$, $b = 0.5$ and $a = 0.1$.

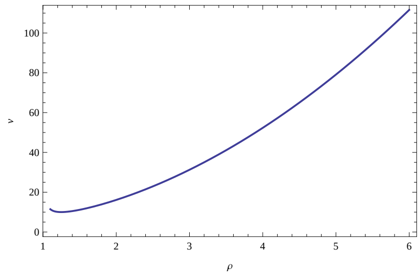

Figure 4.17: Escape velocity v_+ against ρ , for $\ell = -0.3$, $b = 0.5$ and $a = 0.1$.

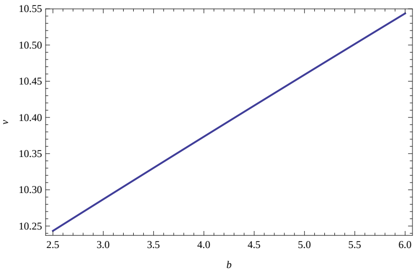

Figure 4.18: Escape velocity v_+ vs magnetic field b, for $\ell = 0.3$, $\rho = 1.3$ and $a = 0.1$.

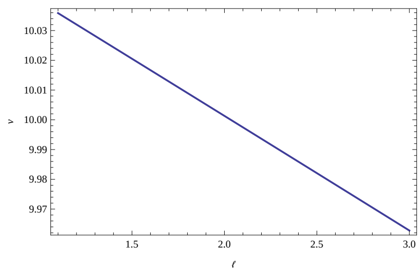

Figure 4.19: Escape velocity v_+ as a function of angular momentum ℓ, for $\rho = 1.3$, $b = 0.5$ and $a = 0.1$.

cle's escape velocity for $\ell = 0.3$ and $\ell = -0.3$. Figure 4.29 show the escape velocity vs magnetic field.

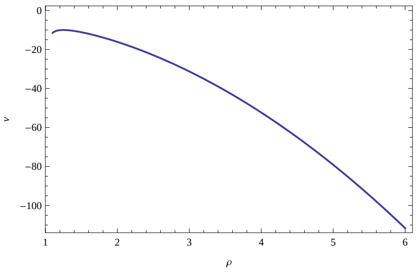

Figure 4.20: Escape velocity v_+ vs ρ, for $\ell = -0.3$, $b = 0.5$ and $a = 0.1$.

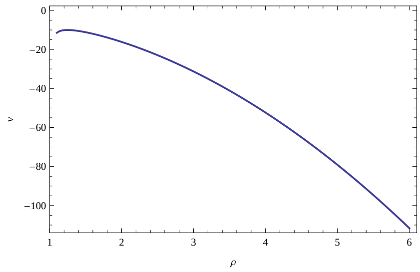

Figure 4.21: Escape velocity (v_+ against ρ, for $\ell = 0.3$, $b = 0.5$ and $a = 0.1$.

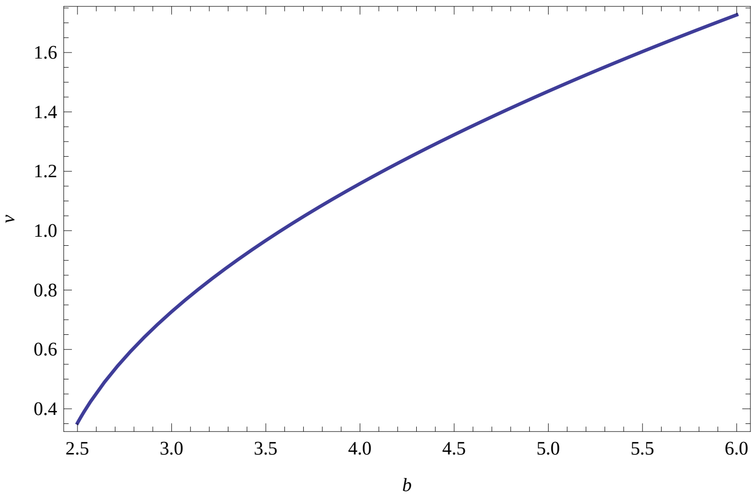

Figure 4.22: Plot of Escape velocity v_+ against magnetic field b, for $\ell = -0.3$, $\rho = 1.3$ and $a = 0.1$.

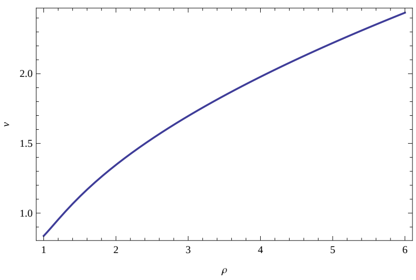

Figure 4.23: Escape velocity v_- against ρ, for $\ell = +0.3$, $b = 0.5$ and $a = 0.1$.

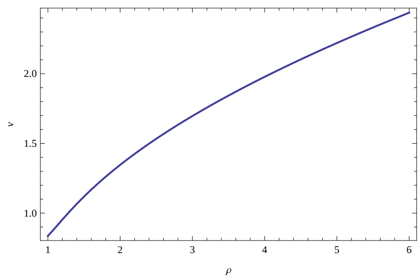

Figure 4.24: Escape velocity v_- vs ρ, for $\ell = -0.3$, $b = 0.5$ and $a = 0.1$.

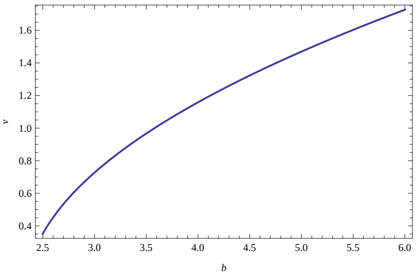

Figure 4.25: Escape velocity v_- as a function of magnetic field b, for $\ell = 0.3$, $\rho = 1.3$ and $a = 0.1$.

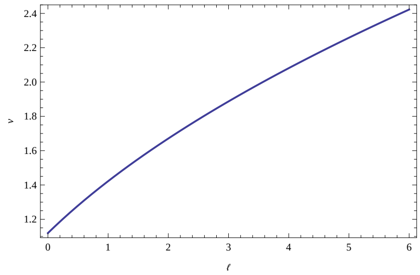

Figure 4.26: Escape velocity v_- vs angular momentum ℓ, for $\rho = 1.3$, $b = 0.5$ and $a = 0.1$.

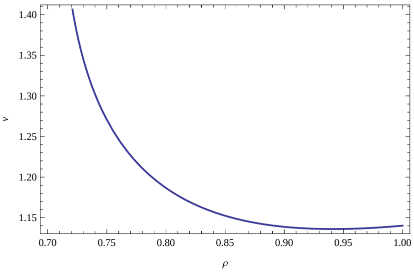

Figure 4.27: Plot of escape velocity v_- as a function of ρ, for $\ell = 0.3$, $b = 0.5$ and $a = 0.1$.

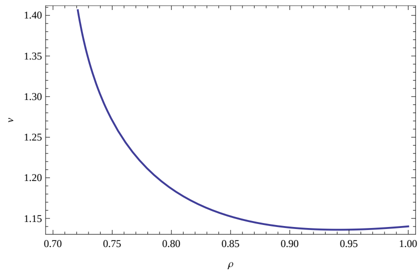

Figure 4.28: Escape velocity v_- against ρ, for $\ell = -0.3$, $b = 0.5$ and $a = 0.1$.

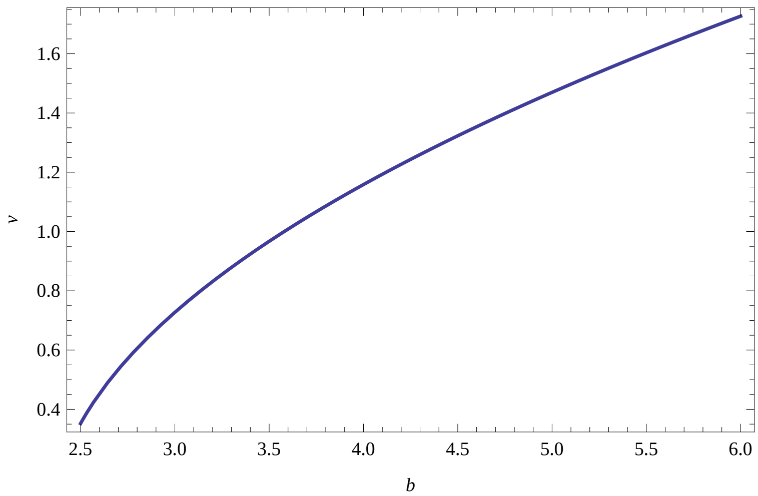

Figure 4.29: Escape velocity v_+ as function of magnetic field b, for $\ell = -0.3$, $\rho = 1.3$ and $a = 0.1$.

Chapter 5

Conclusion

We have studied the dynamics of a neutral and a charged particle around the slowly rotating Kerr black hole which is immersed in a magnetic field. Therefore the particle is under the influence of both gravitational and electromagnetic forces. We have obtained equations of motion by using Lagrangian formalism. We have also calculated the expression for magnetic field present in the vicinity of slowly rotating Kerr black hole. We have calculated the minimum energy for a particle to escape from ISCO. With zero spin, our results reduced to the case of Schwarzschild's black hole [11]. We have also discussed the behavior of energy and escape velocity with angular momentum in the presence of magnetic field. Negative energy is also possible for the Kerr metric inside the static limit. Here we have considered the slowly rotating case, therefore we have only one horizon and did not have ergosphere. It is found that event horizon is the same for both Schwarzschild and slowly rotating Kerr black hole.

It is shown by figures under what conditions particle escape from the black hole vicinity or its motion remain bound. It is presented that for large value of angular momentum, effective potential behaviour is much like Schwarzschild effective potential[9]. It is concluded that magnetic field largely effects the motion of the particle in the vicinity of the black hole and it decreases far away from it. It is found that as we increase the value of magnetic field, local minima of effective potential is shifting toward the horizon.

Particle can escape to $+\infty$ or to $-\infty$ depending on its energy. Escape velocity v_{esc} for different value of magnetic field b is plotted. It is found that due to the presence of magnetic field in the vicinity of black hole escape velocity of the particle increases. It become almost same to which we have for the absence of magnetic field $b = 0$. Behaviour of escape velocity v_{esc} for different value of angular momentum ℓ is almost same as we get for different value of magnetic field b.

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