Dynamics of Coherent State Wave Packets in Gravitational Cavity

by

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Dedicated to my beloved family

and all whose prayers, encouragement and love made all this possible.

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Acknowledgements

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Abstract

The notion of coherent states was originated in 1926 for harmonic oscillator in the context of classical-quantum correspondence of dynamical systems. They played a vital role in numerous areas of physics, especially, in mathematical physics, quantum mechanics, quantum optics and quantum information. Despite being purely quantum mechanical by construction, the coherent states of harmonic oscillator exhibit a classical-like dynamical behavior. However, the coherent states for the systems other than harmonic oscillator, known as generalized coherent states, may exhibit several non-classical dynamical features. In this dissertation the generalized coherent states for the gravitational cavity are constructed and their dynamical characteristics are investigated.

In order to investigate the dynamical behavior, we first construct the time evolved coherent state and calculate the autocorrelation function. In the short time evolution up to few classical periods, the coherent states exhibit classical-like behavior. Afterwards, the dephasing between various constituting eigenstates, due to the nonlinearity of the energy spectrum, dominates and the coherent state observes a collapse. Later on, the coherent state reconstructs itself under the condition of phase matching and the phenomenon of quantum revivals takes place. Secondly, the time evolution of the position space probability density is studied that manifests the formation of interference structures, known as quantum carpets. Moreover, the phase space properties of the constructed coherent states are studied by means of Wigner function. The negativity of the Wigner function reflects the nonclassical behavior of these states.

Chapter 1

INTRODUCTION

The mysterious nature of the microphysical world exposed the limitations of celebrated classical physics, at the end of the nineteenth century, in explaining various phenomena at atomic or subatomic level. Therefore, new ideas were essentially required to understand the phenomena at the microscopic level. This essence gave birth to new theory known as quantum physics. However, the classical-quantum correspondence of physical systems has been a much debated topic ever since the emergence of quantum mechanics. In what follows, we review the historical development of quantum mechanical states, namely coherent states, in the perspective of classical limit of a quantum mechanical system.

1.1 The Coherent States

Based on the ideas of Max-Plank, Albert Einstein and Louis de Broglie, Erwin Schrödinger developed the *wave mechanics* wherein the state of a microscopic particle is represented by a *wave packet* which is a superposition of complex valued functions namely wave functions. The dynamical state of a particle, in classical mechanics, is represented by a point in phase-space, whereas a wave packet, being delocalized, cannot have a similar phase-space representation. Therefore, to establish the connection between the dynamics of a point particle and that of a wave packet was a serious challenge. In 1926, Schrödinger [1] succeeded to construct the quantum wave packets for the harmonic oscillator whose centroids follow the classical trajectories. Moreover, these quantum mechanical states minimize the Heisenberg uncertainty relation and therefore known as minimum uncertainty states.

The minimum uncertainty states, introduced by Schrödinger, remained dormant for more than three decades till 1963 when Roy Glauber used these states in the description of coherent electromagnetic field, hence named as the coherent states. In fact, he refined these states in a different way by making use of the underlying algebra of harmonic oscillator, namely Heisenberg-Wyle algebra. In its classical description, the energy of a single-mode electromagnetic field is expressed by the Hamiltonian of a classical harmonic oscillator, also known as radiation oscillator. The canonical quantization of the radiation oscillators give rise to the quantized radiation energy as a multiple of $\hbar\omega$, where ω is the frequency of the radiation. An eigenstate of the quantized oscillator Hamiltonian corresponds to a state of the field in which there are definite number of photons. However, the photon number states do not express the actual state of the radiation field because the expectation values of the electric field and the magnetic field vanish with respect to these number states. Glauber's coherent states are perfectly suited to represent the quantum mechanical state of the radiation field which closely resembles to its classical description. He defined the coherent states in three different ways [2] as; i) an eigenstate of annihilation operator, ii) the displaced vacuum state and, iii) the state minimizing the uncertainty relation. However, all these definitions of coherent states are equivalent.

Due to its equally spaced energy spectrum, the harmonic oscillator has a very special place in the foundation of many quantum mechanical theories, so is the construction of its coherent states. The coherent states of the harmonic oscillator exhibit a set of very important properties [3]. One of the most striking features of the coherent states is their temporal stability, which means that a coherent state remains coherent under time evolution. These states are nonorthogonal but yet hold the completeness relation. Another very unusual property of the coherent is the over-completeness, which means that any coherent state can be represented in terms of other ones. Hence there are more than enough states available to represent one coherent state in terms of the other. Although a complete orthonormal system of basis vector in Hilbert space is one of the main concepts in functional analysis and mathematical physics [4].

However, in many quantum physics problems the non-orthogonal and over-complete systems of state vectors are very helpful. Besides their applications in quantum optics, the coherent states have been successfully deployed in many other areas of physics [5], such as, radio-physics, superfluidity, non-ideal Bose gases, etc. In the Heisenberg model of ferromagnetism, they are used to describe spin waves, in quantum electrodynamics, they describe soft photon clouds around charged particles. In the nonlinear field theories, they give an approximate quantum description of localized field states.

1.2 Generalizations of the Coherent States

Due to abundant applications of the coherent states in various areas of physics and mathematics, there has been a strong motivation to generalize the notion of coherent states for the systems other than harmonic oscillator. Every generalization scheme has extended one of the above mentioned definitions of coherent states for general systems. However, most of the early generalizations were made by making use of the definitions based on the underlying algebra of the system. Definition of coherent states as an eigenstate of annihilation operator was generalized by Barut and Girardello [6] for non-compact groups and satisfies the definition of coherent states as annihilation operators eigenstates and are known as Barut-Girardello coherent states. The displacement operator definition of coherent states was the takeoff point of Gilmore and Perelomov [7] due to its clear group-theoretical flavor and was used for constructing the coherent states employing Lie groups. However, this approach is not applicable to all Lie groups; in particular, it is invalid for compact groups. In contrast, Nieto [8] and Simmons [9] generalized the idea, based on definition as the minimum uncertainty states, for more general coherent states adapted to a local potential with at least one confined region.

The initial developments in constructing the generalized coherent states explicitly dependent on the underlying algebra of the system. Therefore, a strong urge was felt to develop some sort of technique that would be suitable for the systems that do not have a well defined algebra. In 1996, Klauder presented a new concept for constructing the generalized coherent state of the system executing discrete, degenerate energy spectra [10]. Later on in 1999, Gazeau and Klauder extended this idea for the systems with bounded below discrete, non-degenerate and continuous energy spectra. These states hold a set of properties and were named as Gazeau-Klauder coherent states. This methodology gets more praise due to its algebraic independence. The coherent states for a vast range of Hamiltonian systems were constructed using this method such as, the infinite square well [11], the pseudoharmonic oscillator [12], the Pöschl-Teller potential [13], the power law potentials [14], the triangular well potential [15], the Morse potential [16, 17], and single mode periodic potential systems. Furthermore, another generalization technique for constructing coherent states was introduced by R. F. Fox [18], in which he used a Gaussian function to approximate the behavior of the coherent states. This formalism was named as Gaussian Klauder coherent states.

1.3 Dynamical Properties

Over the last one and a half decades the theoretical analysis, numerical prediction and experimental verification of the occurrence of wave packet revivals in quantum systems has flourished a lot. An important tool used to measure the phenomena of wave packet revivals [20] in coherent states is the autocorrelation function $A(t)$ which measure the overlap of time dependent coherent states with its initial one [11, 15, 19]. The maximum value of autocorrelation function is unity which means that the initial coherent state is its self properly normalized, but in general it has the value less than unity that is at the later time as the coherent state develops in time different energy or momentum components will contribute differently. We can then describe quantum revivals as the periodic recurrence of the quantum wave function from its initially localized form to that evolved with time either in the terms of fractional revivals or regaining of its initial state as in the beginning so named as full revival. In case of a wave function periodic in time the full revival occurs for every period. For example, in case of harmonic oscillator, infinite square well, while fractional revivals occur for triangular well potential, the hydrogen atom and many other quantum systems [20]. Besides being useful to measure the time evolution of a quantum state it can also be used to measure the similarity between the two states that is we can measure its magnitude and phase using it which plays the complementary role to this issue as addressed by the position and momentum space approaches.

In case of classical mechanics the position and momentum of a particle have a definite value so can be represented by a point in phase space. Similarly for an ensemble of particles, the probability of finding a particle at a certain position in phase space is specified by a probability distribution. This interpretation, however, fails in the case of quantum mechanics due to the limitations of the uncertainty principle. In order to solve this problem E. P. Wigner [21] in 1932 introduce another tool to measure the joint probability distribution in the position and momentum space known as Wigner function. Though in the classical limit it approaches the classical distribution function so it cannot be explicitly interpreted as probability density in quantum mechanical context. Unlike usual probability it takes the negative value which clearly shows its non-classical behavior so named as qusai-probability distribution [22]. The non-classicality of coherent states plays an important role in quantum physics and have many applications in quantum information and quantum communication such as quantum teleportation [23], quantum computation [24], quantum cryptography [25] and interferometric measurements [26].

1.4 The Outline

Our task ahead is to construct the generalized coherent state for gravitational cavity and to study its different dynamical properties. For that we had tracked the following routes that are in Chapter 2, we will discuss the quantum gravitational cavity and its physical interpretation in the form of quantum bouncer. The governing potential inside the cavity can be approximated as triangular well potential which is a linear potential and the solution of Schrödinger equation for it, leads to the Airy function. We then calculate the matrix element for the two canonical variables, that is position and momentum, which will be helpful in calculating the uncertainty relationship ahead. In Chapter 3, we will give a detailed description of coherent states for harmonic oscillator that is its different definitions and the properties it holds. Moreover, we will discuss the different generalization techniques, keeping in view our system that does not exhibit a well defined algebra. Chapter 4, is dedicated to the main idea of thesis that is constructing the generalized coherent state for gravitational cavity using Gazeau-Klauder formalism and study its different statistical and dynamical properties. In the end, we will conclude the whole thesis.

Chapter 2

The Quantum Dynamics in the Gravitational Cavity

2.1 Introduction

Gravitational cavity is a very important quantum mechanical system which can be described as a reflecting surface on which particle bounce elastically under the influence of gravitational field. This chapter comprises of three chapters. In Sec. 1, we will discuss the quantum gravitational cavity and explain it with the help of its physical interpretation that is the quantum bouncer. We then approximate the potential inside the quantum gravitational cavity as the triangular well potential which can be described as a potential with a constant slope having an infinite barrier on one side. We resolve the potential in Sec. 2 using Schrödinger which leads to the equation as an Airy differential equation. The solution of which is a special function known as Airy function. Sec.3, comprises of matrix elements of the system for position and momentum which will be then used in finding the uncertainty relation ahead.

2.2 The Quantum Gravitational Cavity

In the past few years, many experiments have been carried out using atoms which show that they can be held almost stationary [27], made to exhibit quantum interference [28] and can also trap in quantum wells [29]. The basic aim behind these experiments was to show that though the internal structure of an atom is quite complicated, yet they produce observable quantum interference and to highest precision facilitate the experimental techniques. With this the periodically driven quantum systems have received much attention due to the existence of Anderson-like localization [30, 31]. Now here we will consider a very important physical system that is the quantum gravitational cavity in which the atoms are periodically driven under the influence of the gravitational field. The concept of it can be very well understood with the help of its physical interpretation given below.

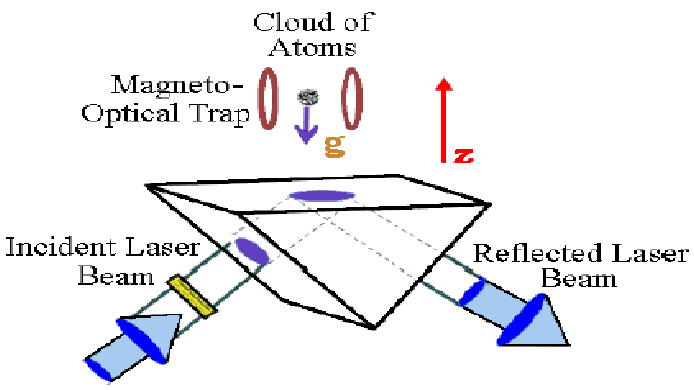

Figure 2.1: Experimental arrangement of quantum bouncer

Here is an experimental demonstration of "quantum bouncer" which is a renowned physical interpretation of the phenomena undergoing inside the gravitational cavity. The experimental realization of such a system is made possible due to the development of techniques of cooling the atoms. In this experiment we cooled and trapped a cloud of atoms in magneto-optical trap. This was then positioned at a certain altitude before the start of the experiment which on switching off starts moving towards the reflecting surface with constant gravitational acceleration. The reflecting surface is provided by an evanescent filed made by a glass prism undergoing total internal reflection of the laser beam on the other side. The evanescent wave field is modulated with the help of acousto-optic modulator which results in the intensity modulation of the incident laser light [32]. Hence, as the atoms strikes the surface, they bounce off and moves along the gravitational field as demonstrated in the figure above. Let us consider an atom in a gravitational cavity such that the atom moves in the direction of gravity in the positive z-direction and bounces back as it hits the reflecting surface. The reflecting surface is due to the evanescent field provided by an electromagnetic field of electric strength $E(z,t) = e_x E_0 e^{-\omega_L z/c} e^{-i\omega_L t} + c.c.$ The evanescent field is modulated by $E_o = \zeta_o exp\ (\epsilon\ sin(\omega t))$ where ω is the driving frequency. For the one dimensional case, the Hamiltonian is given by [32]

$$\hat{H} = \frac{\hat{p}^2}{2m} + mgz + \frac{\hbar\Omega_R}{4} e^{-2\omega_L z/c + \epsilon sin(\omega t)}, \tag{2.1}$$

where $\hat{p}$ is momentum, g is the gravitational acceleration and m is the mass. The strength of influence of the applied electric field is characterized by Rabi-frequency Ω_R [33]. The potential generated inside the quantum gravitational field can be approximated as triangular well potential with an infinite potential at $z = \frac{c\epsilon}{2\omega_L}$ of the reflecting surface. By virtue of this approximation our result becomes independent of the Rabi-frequency Ω_R and laser frequency ω_L So the potential inside the gravitational cavity can be approximated as triangular well potential which is a linear potential, mathematically defined as

$$V(z) = \begin{cases} mgz & \text{if} \quad z > 0, \\ \infty & \text{if} \quad z \leq 0. \end{cases} \tag{2.2}$$

The Hamiltonian for the system is defined as

$$H = \frac{P^2}{2m} + V(z), \tag{2.3}$$

where m defines the mass of the particle.

2.3 Eigenenergies and Eigenfunctions

The time-independent Schrödinger equation that corresponds to the Hamiltonian given by Eq.(2.3) is

$$\frac{-\hbar^2}{2m}\frac{d^2\Psi_n}{dz^2} + mgz\Psi_n = E_n\Psi_n, \tag{2.4}$$

with the governing boundary condition as

$$\Psi_n(0) = 0. \tag{2.5}$$

To find out the solution of Eq.(2.4) it is better to rescale the energy and position variable as follows. Let the "characteristic gravitational length" be defined as

$$l_g = \left(\frac{\hbar^2}{2m^2g}\right)^{1/3}. \tag{2.6}$$

Let $\xi = z/l_g$ and $\xi_n = E_n/(mgl_g)$ are the rescaled energy and position variables, respectively. Then Eq.(2.4) becomes

$$\frac{d^2\Psi_n}{d\xi^2} - (\xi - \xi_n)\Psi_n = 0. \tag{2.7}$$

The above equation is called "Airy Differential equation". Firstly, we will now see how to solve this differential equation. Various techniques have been developed to solve this equation. One very easy approach is via Fourier transform method. This method is advantageous because it will convert a differential equation of second order into first which will then become a lot easier to solve. For convenience let $u = \xi - \xi_n$ then the above equation becomes

$$\frac{d^2y(u)}{du^2} = uy(u). \tag{2.8}$$

By applying on both sides the Fourier transform, we get

$$F(y''(u)) = F(uy(u)), \tag{2.9}$$

remembering the definition that the Fourier transform of position is momentum we have

$$F(y(u)) = \widehat{y}(p) = \int_{-\infty}^{+\infty} e^{-ipu}y(u)du, \tag{2.10}$$

and the Fourier inverse of it is

$$y(u) = F^{-1}(\widehat{y}(p)) = \int_{-\infty}^{+\infty} e^{ipu}\widehat{y}(p)du. \tag{2.11}$$

In order to resolve the left hand side of Eq.(2.9) we will take the derivative of Eq.(2.11), that is

$$\frac{dy(u)}{du} = \frac{d}{du}\left(\int_{-\infty}^{+\infty} e^{ipu}\widehat{y}(p)du\right), \tag{2.12}$$

$$= \int_{-\infty}^{+\infty} (ip)e^{ipu}\widehat{y}(p)du, \tag{2.13}$$

again, taking the derivative we get

$$\frac{dy^2(u)}{du^2} = \int_{-\infty}^{+\infty} (ip)^2 e^{ipu}\widehat{y}(p)du, \tag{2.14}$$

$$= F^{-1}((ip)^2\widehat{y}(p)), \tag{2.15}$$

thus,

$$F\left(\frac{dy^2(u)}{du^2}\right) = -(p)^2\widehat{y}(p). \tag{2.16}$$

Now evaluating the right hand side of Eq.(2.9)

$$\frac{d\widehat{y}(p)}{dp} = \frac{d}{dp}\left(\int_{-\infty}^{+\infty} y(u)e^{-ipu}du\right), \tag{2.17}$$

$$= \int_{-\infty}^{+\infty} (-iu)y(u)e^{-ipu}du, \tag{2.18}$$

$$\frac{d\widehat{y}(p)}{dp} = -iF(uy(u)), \tag{2.19}$$

multiplying i on both sides, we get

$$i\frac{d\widehat{y}(p)}{dp} = F(uy(u)). \tag{2.20}$$

Equating Eq.(2.16) and Eq.(2.20) we get the Airy Equation that is Eq.(2.9) as,

$$-(p)^2\widehat{y}(p) = i\frac{d\widehat{y}(p)}{dp}. \tag{2.21}$$

This is a first order homogeneous differential equation and here we can see the advantage behind using Fourier transform. Integrating on both sides,

$$i\int \frac{d\widehat{y}(p)}{dp} = -\int (p)^2\widehat{y}(p), \tag{2.22}$$

$$i\ln\widehat{y} = -\frac{p^3}{3}. \tag{2.23}$$

Now taking anti-log, we get

$$\widehat{y}(p) = e^{ip^3/3}, \tag{2.24}$$

and converting it again to the position basis by the application of the inverse Fourier transform

$$y(u) = F^{-1}(e^{ip^3/3}), \tag{2.25}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{ip^3/3}e^{ipu}dp \equiv Ai(u). \tag{2.26}$$

As the Fourier transform changes a second order differential equation to the first one, so half of the solution is vanished by the dimension count at first step and we are left with only single function instead of double. The solutions which are not proportional to Ai (i.e., $\beta Ai + \gamma Bi$ with $\gamma \neq 0$) grows, so rapidly that their Fourier transform are not well defined at infinity as shown graphically in Fig(2.3).

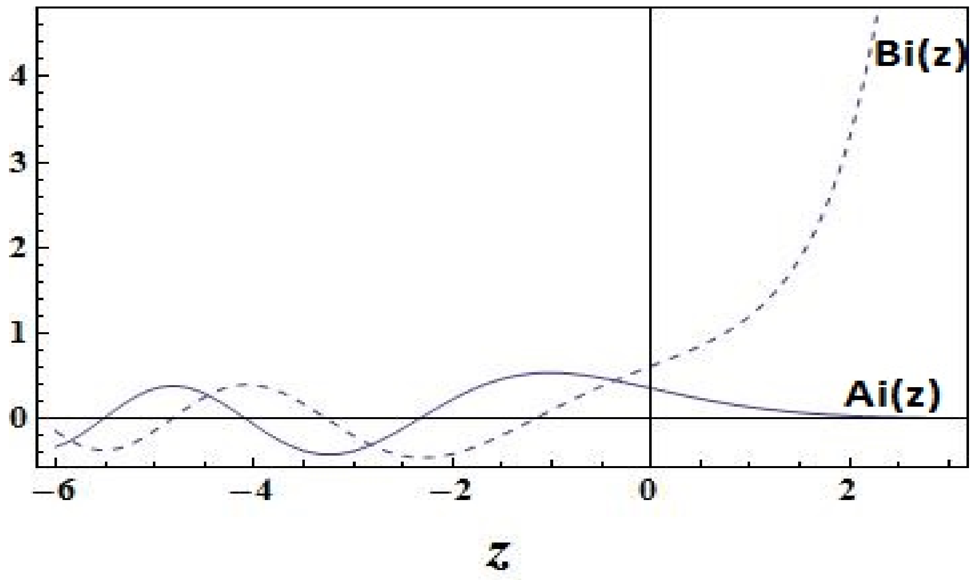

Figure 2.2: Graph of Airy function Ai(solid curve) and Bi(dashed curve).

The solution to Eq. (2.7) are Airy function, Ai or Bi, of the variable $(\xi - \xi_n)$. Since in our case the system is not bounded above, so the Bi function which goes to infinity for the positive argument is discarded in our case. The boundary condition Eq. (2.6) puts a condition on ξ_n which must be chosen so as $Ai(-\xi_n) = 0$. Let the zeros of Airy function are

$-\xi_n$ where $n = 0, 1, 2, 3....$ and $\xi_n > 0$ one then finds the solution as

$$\Psi_n(\xi) = N_n Ai(\xi - \xi_n), \tag{2.27}$$

where N_n is the normalization constant.

A precise analytic expression for the zeros of Airy function ξ_n are not available but one can approximate it for large n [34] as

$$\xi_n = \left[\frac{3\pi}{2}\left(n - \frac{1}{4}\right)\right]^{2/3}. \tag{2.28}$$

The normalization constant N_n can then be evaluated as

$$N_n = \left[\int_0^{+\infty} Ai^2(\xi - \xi_n) d\xi\right]^{-1/2}, \tag{2.29}$$

squaring on both side, we get

$$\frac{1}{N_n^2} = \int_0^{+\infty} Ai^2(\xi - \xi_n) d\xi. \tag{2.30}$$

Let us suppose $v = \xi - \xi_n$ than $dv = d\xi$ than the limits of integration would then be $v \to \infty$ as $\xi \to \infty$ and $v \to -\xi_n$ as $\xi \to 0$ making these substitutions we will have

$$\frac{1}{N_n^2} = \int_{-\xi_n}^{+\infty} Ai^2(v) dv, \tag{2.31}$$

using the identity,

$$\int Ai^2(v) du = vAi^2(v) - Ai'^2(v), \tag{2.32}$$

the above equation simplifies to

$$\frac{1}{N_n^2} = z_n Ai^2(-\xi_n) - Ai'^2(-\xi_n). \tag{2.33}$$

Since, $Ai^2(v)$ and $Ai'^2(v) \to 0$. Here, $v \to \infty$ and $Ai(-\xi_n) = 0$ and the zeros of airy function holds the following property that is $Ai'^2(-\xi_n) \to [\frac{3}{8\pi^2}(4n-1)]^{-1/3}$ as $n \to \infty$ so, we evaluated the normalization constant as

$$N_n = \left[\frac{8\pi^2}{3(4n-1)}\right]^{1/6}. \tag{2.34}$$

The energy eigenvalues of Eq(2.6) are

$$E_n = \xi_n F l_g. \tag{2.35}$$

Substituting the value from Eq.(2.28) and Eq.(2.6) we get

$$E_n = F \left(\frac{\hbar^2}{2mF} \right)^{1/3} \left[\frac{3\pi}{2} \left(n - \frac{1}{4} \right) \right]^{2/3}. \tag{2.36}$$

Hence,

$$E_n = \omega[(n - 1/4)]^{2/3} \simeq \omega(n)^{2/3}. \tag{2.37}$$

where $\omega = (3\pi m g \hbar / 2\sqrt{2m})^{2/3}$ owns the dimensions of energy. Now plotting the wave function as a function of rescaled position for different values of n as shown below.

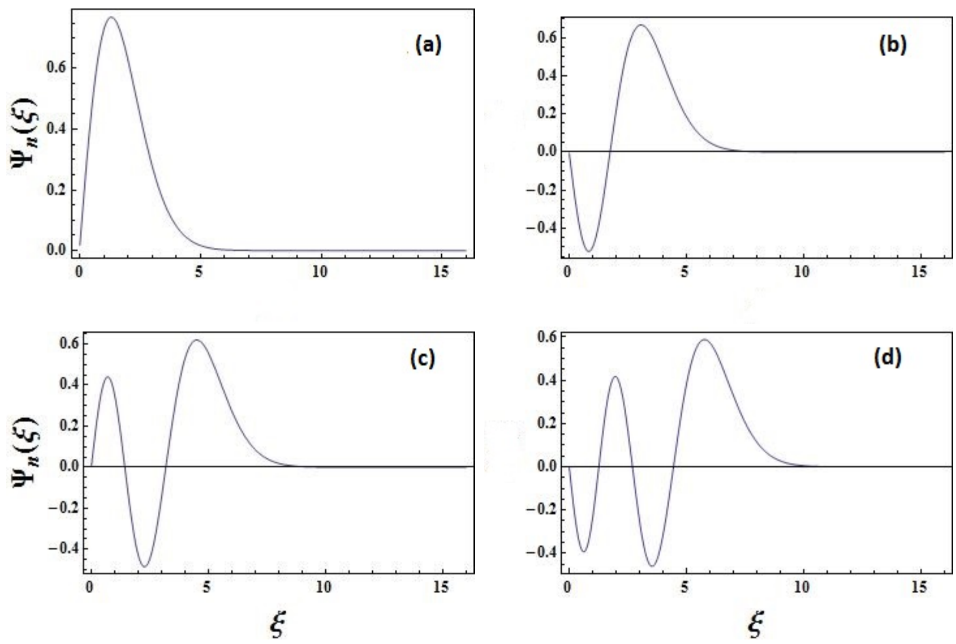

Figure 2.3: Wave function as a function of rescaled position for $n = 1, 2, 3, 4$ from (a) to (d) respectively.

2.4 Matrix Elements

A matrix element plays a very significant role in quantum mechanics. They can be utilized in evaluating the different statistical and phase space properties of a system. Before going to the matrix elements we will calculate certain expression which would be helpful in the evaluation of matrix elements ahead [35].

$$\int_0^\infty d\xi Ai^2(\xi-\xi_n) = \xi[Ai(\xi-\xi_n)]^2|_0^\infty - 2\int_0^\infty d\xi\xi Ai(\xi-\xi_n)Ai^{'}(\xi-\xi_n), \tag{2.38}$$

on applying the limit the first term goes to zero and in the second term we can substitute $Ai^{''}(\xi-\xi_n) = \xi Ai(\xi-\xi_n)$ consequently, the above equation reduces to

$$= -2\int_0^\infty d\xi Ai^{''}(\xi-\xi_n)Ai^{'}(\xi-\xi_n), \tag{2.39}$$

$$= -[Ai^{'}(\xi-\xi_n)]^2|_0^\infty, \tag{2.40}$$

finally, we get the relation

$$\int_0^\infty d\xi Ai^2(\xi-\xi_n) = [Ai^{'}(\xi_n)]^2. \tag{2.41}$$

The above expression provides a relation for the normalization constant. Putting the value from above equation in Eq.(2.29) we get,

$$N_n = |Ai^{'}(\xi_n)|^{-1}. \tag{2.42}$$

Here, the above expression illustrates that the normalization constant for the nth quantum state is linked to the derivative of nth zero of the Airy function. As the Airy function is oscillatory in the negative half plane subsequently the sign of its derivative at $-\xi_n$ changes

$$Ai^{'}(\xi_n) = (-1)^{n+1}|Ai^{'}(\xi_n)|. \tag{2.43}$$

This result leads to an important fact that all normalized quantum states for gravitational cavity have derivative ± 1 at the origin that is

$$\psi_n^{'}(0) = (-)^{n+1}. \tag{2.44}$$

Now for calculating the matrix element of ξ and P we will proceed as:

2.4.1 Matrix Element of Position

We shall first calculate the cross matrix element of position (ξ) and then the diagonal one's.

2.4.1.1 For Cross Matrix Element ξ and ξ^2

To calculate the cross matrix element for ξ we need to evaluate the following expression that is,

$$\langle n|\xi|m\rangle = N_n N_m \int_0^\infty \xi Ai(\xi - \xi_m) Ai(\xi - \xi_n) d\xi. \tag{2.45}$$

Now using the formula [36]

$$\begin{aligned}
\int & xA[\beta(x+\gamma_1)]B[\beta(x+\gamma_2)]dz \\
&= -\frac{\gamma_1+\gamma_2+2x}{\beta^3(\gamma_1-\gamma_2)^2}A[\beta(x+\gamma_1)]B[\beta(x+\gamma_2)] \\
&\quad + \left[\frac{x}{\beta^2(\gamma_1-\gamma_2)} + \frac{2}{\beta^5(\gamma_1-\gamma_2)^3}\right] \\
&\quad \{A^{'}[\beta(x+\gamma_1)]B[\beta(x+\gamma_2)] \\
&\quad -A[\beta(x+\gamma_1)]B^{'}[\beta(x+\gamma_2)]\} \\
&\quad + \frac{2}{\beta^4(\gamma_1-\gamma_2)^2}A^{'}[\beta(x+\gamma_1)]B^{'}[\beta(x+\gamma_2)].
\end{aligned} \tag{2.46}$$

Here

$$A[\beta(x+\gamma_1)] = Ai(\xi - \xi_m),$$

$$B[\beta(x+\gamma_2)] = Ai(\xi - \xi_n),$$

so the above equation becomes,

$$\langle n|\xi|m\rangle = N_nN_m\Big| -\frac{-\xi_m-\xi_n+2\xi}{(\xi_n-\xi_m)^2}Ai[(\xi-\xi_m)]Ai[(\xi-\xi_n)]$$
$$+\left[\frac{\xi}{(-\xi_n+\xi_m)}+\frac{2}{(-\xi_n+\xi_m)^3}\right]$$
$$\{Ai^{'}[(\xi-\xi_n)]Ai[(\xi-\xi_m)]$$
$$-Ai[(\xi-\xi_n)]Ai^{'}[(\xi-\xi_m)]\}$$
$$+\frac{2}{(\xi_n-\xi_m)^2}Ai^{'}[(\xi-\xi_n)]Ai^{'}[(\xi-\xi_m)]\Big|_0^\infty. \quad (2.47)$$

As limit tends to infinity, $Ai(\xi-\xi_m) = Ai(\xi-\xi_n) = Ai^{'}(\xi-\xi_m) = Ai^{'}(\xi-\xi_n) \to 0$ and for $\xi = 0$ the Zeros of Airy function holds the following property that is $Ai(-\xi_n) = Ai(-\xi_m) = 0$. Therefore simplifying the above equation we get

$$= N_nN_m\frac{2(-1)^{n+m+1}}{(\xi_n-\xi_m)^2}Ai^{'}(-\xi_m)Ai^{'}(-\xi_n). \quad (2.48)$$

We know that

$$N_n = \frac{1}{Ai^{'}(-\xi_n)} \quad , \quad N_m = \frac{1}{Ai^{'}(-\xi_m)}$$

putting these values we get

$$\langle n|\xi|m\rangle = \frac{2(-1)^{n+m+1}}{(\xi_n-\xi_m)^2}. \quad (2.49)$$

To calculate the cross matrix elements for ξ^2 we need to evaluate the following expression that is,

$$\langle n|\xi^2|m\rangle = N_nN_m\int_0^\infty \xi^2 Ai(\xi-\xi_m)Ai(\xi-\xi_n)d\xi. \quad (2.50)$$

Now using the formula [36]

$$\int x^2 A[\beta(x+\gamma_1)]B[\beta(x+\gamma_2)]dx = \frac{4}{(\gamma_2-\gamma_1)^2}$$
$$\Big\{-\frac{1}{\beta^3}\left[x^2+cx+\frac{3(\gamma_1+\gamma_2)}{\beta^3(\gamma_2+\gamma_1)^2}\right]A[\beta(x+\gamma_1)]B[\beta(x+\gamma_2)]$$
$$-\left[\frac{(\gamma_2-\gamma_1)x^2}{4\beta^2}+\frac{3x+\gamma_2+c}{\beta^5(\gamma_2-\gamma_1)}\right]A'[\beta(x+\gamma_1)]B[\beta(x+\gamma_2)]$$
$$+\left[\frac{(\gamma_2-\gamma_1)x^2}{4\beta^2}+\frac{3x+\gamma_2+c}{\beta^5(\gamma_2-\gamma_1)}\right]A[\beta(x+\gamma_1)]B'[\beta(x+\gamma_2)]$$
$$+\frac{1}{\beta^4}\left[x+\frac{6}{\beta^3(\gamma_2-\gamma_1)^2}\right]A'[\beta(x+\gamma_1)]B'[\beta(x+\gamma_2)]\Big\}. \quad (2.51)$$

with

$$c = \frac{(\gamma_2-\gamma_1)^2(\gamma_2-\gamma_1)+12/\beta^3}{2(\gamma_2-\gamma_1)^2}.$$

Now putting values in the above formula we have

$$= \Big|N_nN_m\frac{4}{(\xi_n-\xi_m)^2} * \Big\{[\xi^2+c\xi+\frac{3(-\xi_n-\xi_m)}{(\xi_n+\xi_m)^2}]Ai'(\xi-\xi_m)Ai(\xi-\xi_m)$$
$$-\left[\frac{(-\xi_n+\xi_m)\xi^2}{4}+\frac{3\xi-\xi_n+c}{(-\xi_n+\xi_m)}\right]Ai(\xi-\xi_m)Ai'(\xi-\xi_m)$$
$$+\left[\xi+\frac{6}{(\xi_n-\xi_m)^2}\right]Ai'(\xi-\xi_m)Ai'(\xi-\xi_m)\Big\}\Big|_0^{\infty}, \quad (2.52)$$

on applying the limit we get

$$= N_nN_m(-1)^{n+m+1}\left[\frac{4}{(\xi_n-\xi_m)^2} * \frac{6}{(\xi_n-\xi_m)^2}\right]Ai'(-\xi_n)Ai'(-\xi_m).$$

Since

$$N_n = \frac{1}{Ai'(-\xi_n)} \quad , \quad N_m = \frac{1}{Ai'(-\xi_m)}$$

Therefore,

$$\langle n|\xi^2|m\rangle = \frac{24(-1)^{n+m+1}}{(\xi_n-\xi_m)^4}. \quad (2.53)$$

2.4.1.2 For Diagonal Matrix Element ξ and ξ^2

To calculate the diagonal matrix element for ξ we need to evaluate the following expression that is,

$$\langle n|\xi|n\rangle = N_nN_n\int_0^{\infty}\xi Ai(\xi-\xi_n)Ai(\xi-\xi_n)d\xi. \quad (2.54)$$

Now using the formula [36]

$$\int xA[\beta(x+\gamma)]B[\beta(x+\gamma)]dx$$
$$= \frac{1}{3}(x^2 - x\gamma - 2\gamma^2)A[\beta(x+\gamma)]B[\beta(x+\gamma)]$$
$$+ \frac{1}{6\beta^2}\{A^{'}[\beta(x+\gamma)]B[\beta(x+\gamma)]$$
$$+ A[\beta(x+\gamma)]B^{'}[\beta(x+\gamma)]\}$$
$$+ \frac{2\gamma - x}{3\beta}A^{'}[\beta(x+\gamma)]B^{'}[\beta(x+\gamma)]. \quad (2.55)$$

Here $A(x)$ and $B(x)$ are any two linear combinations of Airy functions. So the above equation will become,

$$\langle n|\xi|n\rangle = N_n^2 \Big| \frac{1}{3}(\xi^2 + \xi\xi_n - 2\xi_n^2)Ai[(\xi-\xi_n)]Ai[(\xi-\xi_n)]$$
$$+ \frac{1}{6}\{Ai^{'}[(\xi-\xi_n)]Ai[(\xi-\xi_n)] + Ai[(\xi-\xi_n)]Ai^{'}[(\xi-\xi_n)]\}$$
$$+ \frac{-2\xi_n - \xi}{3}Ai^{'}[(\xi-\xi_n)]Ai^{'}[(\xi-\xi_n)]\Big|_0^{\infty}, \quad (2.56)$$

on applying the limit it simplifies to

$$= N_n^2 \left(\frac{2\xi_n}{3}\right)(Ai^{'}(-\xi_n))^2. \quad (2.57)$$

we know $N_n = \frac{1}{Ai^{'}(-\xi_n)}$. Hence,

$$\langle n|\xi|n\rangle = \frac{2}{3}\xi_n. \quad (2.58)$$

The direct physical interpretation of above expression can be given by the virial theorem of classical mechanics for linear potential as $\langle T\rangle = 1/2\langle V\rangle$, which is equivalent to $E = \frac{3}{2}\langle V\rangle$. The potential energy and total energy in normalized units are $V = \xi^{'}$, $E = \xi_n$. Hence the above expression is consistent with classical virial theorem.

Now calculating the expression for the diagonal elements of ξ^2 we need to evaluate the following expression.

$$\langle n|\xi^2|n\rangle = N_n N_n \int_0^{\infty} \xi^2 Ai(\xi-\xi_n)Ai(\xi-\xi_n)d\xi. \quad (2.59)$$

Now using the formula [36]

$$\int xA[\beta(x+\gamma)]B[\beta(x+\gamma)]dx$$
$$= \frac{1}{15}(3x^3 - x^2\gamma + 4x\gamma^2 + 8\gamma^3 - 3\beta^{-3}) * A[\beta(x+\gamma)]B[\beta(x+\gamma)]$$
$$+ \frac{3x-2\gamma}{15\beta^2}A^{'}[\beta(x+\gamma)]B[\beta(x+\gamma)] + A[\beta(x+\gamma)]B^{'}[\beta(x+\gamma)]$$
$$- \frac{3x^2 - 4x\gamma + 8\gamma^2}{15\beta}A^{'}[\beta(x+\gamma)]B^{'}[\beta(x+\gamma)]. \quad (2.60)$$

Putting the values in above formula we have,

$$= N_n N_n | \frac{1}{15}(3\xi^3 + \xi^2\xi_n + 4\xi\xi_n^2 - 8\xi_n^3 - 3) * Ai[(\xi - \xi_n)]Ai[(\xi - \xi_n)]$$
$$+ \frac{3\xi + 2\xi_n}{15}\{Ai^{'}[(\xi - \xi_n)]Ai[(\xi - \xi_n)] + Ai[(\xi - \xi_n)]Ai^{'}[(\xi - \xi_n)]\}$$
$$- \frac{3\xi^2 + 4\xi\xi_n + 8\xi_n^2}{15}Ai^{'}[(\xi - \xi_n)]Ai^{'}[(\xi - \xi_n)]|_0^\infty, \quad (2.61)$$

Applying the limit it reduces to,

$$= N_n^2 \left(\frac{8}{15}\xi_n^2 Ai^{'}(-\xi_n)Ai^{'}(-\xi_n) \right), \quad (2.62)$$

as $N_n = \frac{1}{Ai^{'}(-\xi_n)}$. Therefore,

$$\langle n|\xi^2|n\rangle = \frac{8}{15}\xi_n^2. \quad (2.63)$$

2.4.2 Matrix Element of Momentum

Now we shall first calculate the cross matrix element of momentum (P) and then the diagonal ones.

2.4.2.1 For Cross Matrix Element P and P^2

To calculate the cross matrix element for P we need to calculate the following expression,

$$\langle n|\frac{d}{d\xi}|m\rangle = N_n N_m \int_0^\infty Ai(\xi - \xi_n)\frac{d}{d\xi}Ai(\xi - \xi_m)d\xi. \quad (2.64)$$

$$\langle n|\frac{d}{d\xi}|m\rangle = \frac{2(-1)^{n+m}}{\xi_n - \xi_m}. \quad (2.65)$$

In order to calculate the matrix element of P^2 we first evaluate

$$\langle n|\frac{d^2}{d\xi^2}|m\rangle = N_n N_m \int_0^\infty \xi^2 Ai(\xi-\xi_n)\frac{d^2}{d\xi^2}Ai(\xi-\xi_m)d\xi. \tag{2.66}$$

Since we know that

$$Ai^{''}(\xi) = \xi Ai(\xi),$$

so the above equation becomes,

$$\langle n|\frac{d^2}{d\xi^2}|m\rangle = N_n N_m \int_0^\infty \xi Ai(\xi-\xi_m)Ai(\xi-\xi_n)d\xi, \tag{2.67}$$

Now using the formula

$$\begin{aligned}
&\int xA[\beta(x+\gamma_1)]B[\beta(x+\gamma_2)]dz \\
&\qquad = -\frac{\gamma_1+\gamma_2+2x}{\beta^3(\gamma_1-\gamma_2)^2}A[\beta(x+\gamma_1)]B[\beta(x+\gamma_2)] \\
&\qquad\quad + \left[\frac{x}{\beta^2(\gamma_1-\gamma_2)} + \frac{2}{\beta^5(\gamma_1-\gamma_2)^3}\right] \\
&\qquad\quad \{A^{'}[\beta(x+\gamma_1)]B[\beta(x+\gamma_2)] \\
&\qquad\quad -A[\beta(x+\gamma_1)]B^{'}[\beta(x+\gamma_2)]\} \\
&\qquad\quad + \frac{2}{\beta^4(\gamma_1-\gamma_2)^2}A^{'}[\beta(x+\gamma_1)]B^{'}[\beta(x+\gamma_2)].
\end{aligned} \tag{2.68}$$

where,

$$A[\beta(x+\gamma_1)] = Ai(\xi-\xi_m),$$

$$B[\beta(x+\gamma_2)] = Ai(\xi-\xi_n).$$

Above equation then becomes,

$$\begin{aligned}
\langle n|\frac{d^2}{d\xi^2}|m\rangle = \Big| &-\frac{-\xi_m-\xi_n+2\xi}{(\xi_n-\xi_m)^2}Ai[(\xi-\xi_m)]Ai[(\xi-\xi_n)] \\
&+ \left[\frac{-\xi}{(-\xi_n+\xi_m)} + \frac{2}{(-\xi_n+\xi_m)^3}\right] \\
&\{Ai^{'}[(\xi-\xi_n)]Ai[(\xi-\xi_m)] \\
&-Ai[(\xi-\xi_n)]Ai^{'}[(\xi-\xi_m)]\} \\
&+ \frac{2}{(\xi_n-\xi_m)^2}Ai^{'}[(\xi-\xi_n)]Ai^{'}[(\xi-\xi_m)]\Big|_0^\infty.
\end{aligned} \tag{2.69}$$

As the limit approaches infinity $Ai(\xi-\xi_m) = Ai(\xi-\xi_n) = Ai^{'}(\xi-\xi_m) = Ai^{'}(\xi-\xi_n) \to 0$ and for $\xi = 0$ from the properties of Zeros of Airy Function we have $Ai(-\xi_n) = Ai(-\xi_m) = 0$. Therefore simplifying the above equation we get

$$\langle n|\frac{d^2}{d\xi^2}|m\rangle = N_n N_m \frac{2(-1)^{n+m+1}}{(\xi_n-\xi_m)^2} Ai^{'}(-\xi_m) Ai^{'}(-\xi_n). \tag{2.70}$$

We know that

$$N_n = \frac{1}{Ai^{'}(-\xi_n)} \quad , \quad N_m = \frac{1}{Ai^{'}(-\xi_m).}$$

Putting these values we get

$$\langle n|\frac{d^2}{d\xi^2}|m\rangle = \frac{2(-1)^{n+m+1}}{(\xi_n-\xi_m)^2}. \tag{2.71}$$

So matrix element for P^2 would be

$$\langle n|P^2|m\rangle = (\hbar)^2 \frac{2(-1)^{n+m}}{(\xi_n-\xi_m)^2}. \tag{2.72}$$

2.4.2.2 For Diagonal Matrix Element P and P^2

To calculate the diagonal element for P we need to evaluate the following expression,

$$\langle n|\frac{d}{d\xi}|n\rangle = N_n N_n \int_0^\infty Ai(\xi-\xi_n)\frac{d}{d\xi}Ai(\xi-\xi_n)d\xi. \tag{2.73}$$

$$= N_n^2 \int_0^\infty Ai(\xi-\xi_n)Ai^{'}(\xi-\xi_n)d\xi, \tag{2.74}$$

$$= \frac{1}{2}N_n^2 |Ai^2(\xi-\xi_n)|_0^\infty, \tag{2.75}$$

on applying the limit we get

$$\langle n|\frac{d}{d\xi}|n\rangle = 0. \tag{2.76}$$

Therefore,

$$\langle n|P|n\rangle = 0. \tag{2.77}$$

Now to find matrix element P^2 we first need to evaluate the following expression that is,

$$\langle n|\frac{d^2}{d\xi^2}|n\rangle = N_n N_n \int_0^\infty Ai(\xi-\xi_n)\frac{d^2}{d\xi^2}Ai(\xi-\xi_n)d\xi. \tag{2.78}$$

using integration by parts

$$= N_n^2 (Ai(\xi-\xi_n)Ai^{'}(\xi-\xi_n)|_0^\infty - \int_0^\infty [Ai^{'}(\xi-\xi_n)]^2 d\xi), \tag{2.79}$$

on applying the limit the first term get zero and on second term again using by parts we have

$$= N_n^2(-\xi[Ai^{'}(\xi-\xi_n)]^2|_0^\infty + 2\int_0^\infty Ai^{''}(\xi-\xi_n)Ai^{'}(\xi-\xi_n)\xi d\xi), \tag{2.80}$$

we know that $Ai^{''}(\xi-\xi_n) = \xi Ai(\xi-\xi_n)$. Again taking the derivative we get,

$$Ai^{'''}(\xi-\xi_n) = Ai(\xi-\xi_n) + \xi Ai^{'}(\xi-\xi_n).$$

Now substituting this value above

$$= N_n^2(\xi_n[Ai^{'}(-\xi_n)]^2 + 2\int_0^\infty Ai^{''}(\xi-\xi_n)Ai^{'''}(\xi-\xi_n)d\xi - 2\int_0^\infty Ai(\xi-\xi_n)Ai^{''}(\xi-\xi_n)d\xi), \tag{2.81}$$

the third term is the same as we are evaluating so, taking it on the left hand side we get

$$\int_0^\infty Ai(\xi-\xi_n)Ai^{''}(\xi-\xi_n)d\xi = \frac{1}{3}N_n^2\Big(\xi_n[Ai^{'}(-\xi_n)]^2 + 2\frac{[Ai^{''}(\xi-\xi_n)]^2}{2}\Big|_0^\infty\Big). \tag{2.82}$$

Now using

$$Ai^{''}(\xi-\xi_n) = \xi Ai(\xi-\xi_n)$$

than

$$\langle n|\frac{d^2}{d\xi^2}|n\rangle = \frac{1}{3}N_n^2\Big(\xi_n[Ai^{'}(-\xi_n)]^2 + [\xi Ai(\xi-\xi_n)]^2\Big|_0^\infty\Big), \tag{2.83}$$

on applying the limit it simplifies to,

$$\langle n|\frac{d^2}{d\xi^2}|n\rangle = \frac{\xi_n}{3}N_n^2[Ai^{'}(-\xi_n)]^2, \tag{2.84}$$

we know that $N_n = [Ai^{'}(-\xi_n)]^{-1}$ substituting back in above equation

$$\langle n|\frac{d^2}{d\xi^2}|n\rangle = \frac{1}{3}\xi_n. \tag{2.85}$$

Hence,

$$\langle n|P^2|n\rangle = \frac{-1}{3}(\hbar)^2\xi_n. \tag{2.86}$$

Now doing all this cumbersome effort we have calculated the matrix element of position and momentum for the diagonal and off diagonal terms up to the square power. Clearly this effort can be continued indefinitely, or as long as the time and requirement permits.

Chapter 3

THE COHERENT STATES AND THEIR GENERALIZATIONS

3.1 Introduction

Coherent states play a central role in quantum mechanics and in general to quantum optics and in particular in quantum information [37]. The theory of coherent states was first introduced by Erwin Schrödinger [1] in 1926 as part of his description of wave mechanics. He described the non-stationary states of the harmonic oscillator that retain the shape of wave packet and owns a classical motion. So they can be defined as the states which mimic in the best possible way the classical trajectory of a particle for the harmonic oscillator. After Schrödinger this idea didn't receive much attention until R. J. Glauber in 1963 defined them as quantum-mechanical equivalent of a classical monochromatic electromagnetic wave [2]. Realizing their importance in quantum optics he was awarded the Nobel prize for physics in 2005. During the past two and a half decades, there have been breath taking development in the field of coherent states and its application. Since then, throughout quantum physics coherent states and its generalizations have been disseminated.

In Sec. 2 of this chapter, we will review the coherent states for harmonic oscillator which includes its different ways of construction and the properties it holds. Since we know that the electromagnetic fields can be treated as a collection of the uncoupled harmonic oscillator. Here, for simplicity, we will discuss a single mode field the results of which can then be straightforwardly extended to n-modes.In Sec. 3, we will discuss the different method of generalization keeping in view our system which do not exhibit a well defined algebra.

3.2 Coherent State for Harmonic Oscillator

The idea of coherent state for harmonic oscillator came from the fact that since for the classical oscillator the position and momentum behave as a sinusoidal function, but when we come to the quantum oscillator calculating their expectation value vanishes for energy eigenstates. Moreover, contrary to the ground state which is minimum uncertain, the uncertainty in higher states increases as the value of n increases, which is against the correspondence principle which expect it to behave more classically there. In order to cater these issues, coherent states came into being. But before going to them, we will firstly give an overview of the governing algebra of a harmonic oscillator. The Hamiltonian of a harmonic oscillator having quadratic potential is

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{q}^2. \tag{3.1}$$

Where m is the mass of the particle and ω is the corresponding frequency. The quantization of this Hamiltonian can be done by using the two operators that is

$$\hat{a} = \frac{1}{\sqrt{2\hbar\omega m}}(i\hat{p} + m\omega\hat{q}), \tag{3.2}$$

$$\hat{a}^\dagger = \frac{1}{\sqrt{2\hbar\omega m}}(-i\hat{p} + m\omega\hat{q}). \tag{3.3}$$

Which are the annihilation and creation operator respectively. There commutation relation can be calculated straight forwardly using the above definition

$$[\hat{a}, \hat{a}^\dagger] = \frac{-i}{\hbar}[\hat{q}, \hat{p}] = 1. \tag{3.4}$$

Now we can represent the Hamiltonian in Eq. (3.1) in term of these operators as

$$\hat{H} = \hbar\omega\Big(\hat{a}^\dagger\hat{a} + \frac{1}{2}\Big), \tag{3.5}$$

satisfying the following commutation relations

$$[\hat{H}, \hat{a}] = -\hbar\omega\hat{a} \qquad , \qquad [\hat{H}, \hat{a}^\dagger] = -\hbar\omega\hat{a}^\dagger. \tag{3.6}$$

Since we know that $\hat{H}|n\rangle = E_n|n\rangle$ where E_n denotes the eigenenergy of the system. Now multiplying $\hat{a}$ on both side, we will see that it will always annihilate the energy of the state thus named as annihilation operator.

$$\hat{H}\hat{a}|n\rangle = (\hat{a}\hat{H} - \hbar\omega\hat{a})|n\rangle = (E_n - \hbar\omega)\hat{a}|n\rangle. \tag{3.7}$$

Similarly, creation operator will create the same amount of energy as

$$\hat{H}\hat{a}^\dagger|n\rangle = (\hat{a}^\dagger\hat{H} + \hbar\omega\hat{a}^\dagger)|n\rangle = (E_n + \hbar\omega)\hat{a}^\dagger|n\rangle. \tag{3.8}$$

So named as creation operator. The number operator is defined as $N = \hat{a^\dagger}\hat{a}$. Applying it to the number states we get

$$\hat{N}|n\rangle = n|n\rangle, \tag{3.9}$$

and holds the following commutation relation.

$$[\hat{N}, \hat{a}] = -\hat{a} \qquad , \qquad [\hat{N}, \hat{a}^\dagger] = \hat{a}^\dagger. \tag{3.10}$$

Using them, we can find the two important relations,

$$\hat{a}|n| = \sqrt{n}|n-1\rangle \tag{3.11}$$

$$\hat{a}^\dagger|n| = \sqrt{n+1}|n+1\rangle \tag{3.12}$$

Now, having all this algebra in hand, we are now in a position to construct coherent states for harmonic oscillator as described below.

3.2.1 Eigenstates of Annihilation Operator

In order to get non-zero value of position and momentum expectation value or equivalently of annihilation and creation operator, we need to have such a superposition number state so that the expectation value is a non zero. A unique way to achieve this requirement is to have such a state which is an eigenstate of annihilation operator. Let that state be $|\alpha\rangle$ which is a coherent state such that

$$\hat{a}|\alpha\rangle = \alpha|\alpha\rangle, \tag{3.13}$$

Where α is a complex number. Since $\hat{a}$ has been non-hermitian operator, its eigenvalues can be, and are complex and in analogue we can say

$$\langle\alpha|\hat{a}^\dagger = (\alpha)^*\langle\alpha|. \tag{3.14}$$

But, coherent state is not an eigenstate of creation operator. Since number state form a complete set of orthonormal basis so we can represent a coherent state in terms of them by using the following approach

$$|\alpha\rangle = \sum_{0}^{\infty} |n\rangle\langle n|\alpha\rangle, \tag{3.15}$$

where $\langle n|\alpha\rangle$ denotes the probability amplitude. As $\langle n|\hat{a} = \sqrt{n+1}\langle n+1|$ so from Eq.(3.13) we observe

$$\langle n|\hat{a}|\alpha\rangle = \sqrt{n+1}\langle n+1|\alpha\rangle = \alpha\langle n|\alpha\rangle, \quad \Rightarrow \langle n|\alpha\rangle = \frac{\alpha}{\sqrt{n}}\langle n-1|\alpha\rangle, \tag{3.16}$$

so on iterating we will have,

$$\langle n|\alpha\rangle = \frac{(\alpha)^n}{\sqrt{n!}}\langle 0|\alpha\rangle, \tag{3.17}$$

Substituting this value above, we get

$$|\alpha\rangle = \langle 0|\alpha\rangle \sum_{0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}}|n\rangle. \tag{3.18}$$

With normalizing it, we will have $\langle 0|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}$. Hence the coherent states can be represented in terms of the number state as

$$|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{3.19}$$

3.2.2 Displaced Vacuum States

The definition of displacement-operator coherent states requires that the wave packet keeps its shape and oscillates with some classical motion. The name refers to as a method of generating the states, whereby the displacement operator acts upon the ground state of the unperturbed system. This has the effect of "displacing" the system from its equilibrium. In case of the quantum harmonic oscillator, this causes the wave packet to oscillate within the potential. The unitary displacement operator for a complex number α is defined as

$$\hat{D}(\alpha) = e^{(\alpha\hat{a}^\dagger - \alpha^*\hat{a})}, \tag{3.20}$$

which holds the following relation

$$\hat{D}(-\alpha) = \hat{D}(\alpha)^\dagger = \hat{D}(\alpha)^{-1}.$$

The coherent states can then be defined as

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle. \tag{3.21}$$

In order to derive the above relation we will use the $Baker - Haussdorf\ formula$ given for any two operators as

$$e^{\hat{X}+\hat{Y}} = e^{\hat{X}}e^{\hat{Y}}e^{-\frac{1}{2}[\hat{X},\hat{Y}]}. \tag{3.22}$$

provided $[\hat{X},[\hat{X},\hat{Y}]] = [\hat{Y},[\hat{X},\hat{Y}]] = 0$ when $[\hat{X},\hat{Y}] \neq 0$. Here we have $\hat{X} = \alpha\hat{a}^\dagger$ and $\hat{Y} = \alpha^*\hat{a}$ and $[\hat{X},\hat{Y}] = |\alpha|^2$ using the above identity we can write

$$\hat{D}(\alpha) = e^{-\frac{1}{2}|\alpha|^2}e^{(-\alpha^*\hat{a})}e^{(\alpha\hat{a}^\dagger)}. \tag{3.23}$$

Now applying the above form of displacement operator upon the vacuum state of a harmonic oscillator and using the series expansion of the exponentials we see that

$$e^{(\alpha\hat{a}^\dagger)}|0\rangle = \sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle, \tag{3.24}$$

and

$$e^{(-\alpha^*\hat{a})}|0\rangle = 0, \tag{3.25}$$

Hence,

$$|\alpha\rangle = \hat{D}|0\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle. \tag{3.26}$$

Now the equation is exactly the same as that obtained using the annihilation operator definition which shows the equivalence of the two definitions.

3.2.3 Minimum-Uncertainty States

As we know that coherent states for the ground state of harmonic oscillator are minimum uncertain. Except, of it when we calculate the uncertainty relation for higher order states we see that the uncertainty is larger for higher order terms which is against the "correspondence principle" which expect them to behave more classically. Coherent states are minimum uncertain to prove that we will take the dimensionless position $\hat{X}_1$ and momentum $\hat{X}_2$ operators of harmonic oscillator that is

$$\hat{X}_1 = \frac{1}{2}(\hat{a} + \hat{a}^\dagger), \qquad \hat{X}_2 = \frac{1}{2}(\hat{a} - \hat{a}^\dagger). \tag{3.27}$$

Now calculating the expectation value of position with respect to the coherent states as

$$\langle \hat{X}_1 \rangle = \frac{1}{2}\langle \alpha | \hat{a} + \hat{a}^\dagger | \alpha \rangle = \frac{1}{2}(\alpha + \alpha^*), \tag{3.28}$$

and the expectation value of $\hat{X}^2$ would then be

$$\langle \hat{X}_1^2 \rangle = \frac{1}{2}\langle \alpha | (\hat{a} + \hat{a}^\dagger)^2 | \alpha \rangle = \frac{1}{2}(1 + \alpha^2 + \alpha^{*2} + 2\alpha\alpha^*). \tag{3.29}$$

Then the uncertainty in $\hat{X}_1$ would be

$$\Delta \hat{X}_1 = \sqrt{\langle \alpha | \hat{X}_1^2 | \alpha \rangle - (\langle \alpha | \hat{X}_1 | \alpha \rangle)^2} = \sqrt{\frac{1}{2}} \tag{3.30}$$

On similar footings we can find the uncertainty in momentum which will give us

$$\Delta \hat{X}_2 = \sqrt{\langle \alpha | \hat{X}_2^2 | \alpha \rangle - (\langle \alpha | \hat{X}_2 | \alpha \rangle)^2} = \sqrt{\frac{1}{2}} \tag{3.31}$$

The uncertainty relation would then be

$$\Delta \hat{X}_1 \Delta \hat{X}_2 = \frac{1}{2}. \tag{3.32}$$

Hence, in case of coherent states of the harmonic oscillator, the uncertainty in the $\hat{X}_1$-operator and the $\hat{X}_2$-operator is minimum. If this is not the case, the states are said to be a squeezed coherent state.

3.3 Basic Properties of Coherent States

3.3.1 Non-Orthogonality of Coherent States

As we have earlier defined, coherent states as annihilation operator eigenstates owning a complex value. This clearly shows that the operator $\hat{a}$ is not hermitian and we cannot automatically say whether the coherent states are orthogonal or not. To see the result all we need is to calculate the overlap of any two coherent states let it be $\langle\gamma|\beta\rangle$ such that

$$\langle\gamma|\beta\rangle = \langle 0|\hat{D}^{\dagger}(\gamma)\hat{D}(\beta)|0\rangle, \tag{3.33}$$

where

$$\hat{D}(\beta) = e^{-1/2|\beta|^2} e^{\beta\hat{a}^{\dagger}} e^{-\beta^*\hat{a}}.$$

and

$$\hat{D}^{\dagger}(\gamma) = e^{-1/2|\gamma|^2} e^{-\gamma\hat{a}^{\dagger}} e^{\gamma^*\hat{a}}.$$

Now substituting the above two values in Eq.(3.33) we get

$$\langle\gamma|\beta\rangle = \langle 0|e^{-\gamma\hat{a}^{\dagger}} e^{\gamma^*\hat{a}} e^{\beta\hat{a}^{\dagger}} e^{-\beta^*\hat{a}}|0\rangle e^{-1/2(|\gamma|^2+|\beta|^2)}. \tag{3.34}$$

Using Taylor series for the expansions of exponential

$$e^{-\gamma\hat{a}^{\dagger}} = (1 - (\gamma\hat{a}^{\dagger}) + ...),$$

$$e^{-\beta*\hat{a}} = (1 - (\beta^*\hat{a}) + ...),$$

and operating them on left and right sides respectively, we get $\langle 0|\hat{a}^{\dagger} = 0$ and $\beta^*\hat{a}|0\rangle = 0$. All other higher terms would also be zero. All we are left with is a single term contribution from both sides that is

$$\langle\gamma|\beta\rangle = \langle 0|e^{\gamma^*\hat{a}} e^{\beta\hat{a}^{\dagger}}|0\rangle e^{-1/2(|\beta|^2+|\gamma|^2)}. \tag{3.35}$$

Again, using the Taylor expansion for the exponentials

$$\langle\gamma|\beta\rangle = \langle 0|(1 + (\gamma^*\hat{a}) + \frac{1}{2!}(\gamma^*\hat{a})^2 + ...)(1 + (\beta\hat{a}^{\dagger}) + \frac{1}{2!}(\beta\hat{a}^{\dagger})^2 + ...)|0\rangle e^{-1/2(|\gamma|^2+|\beta|^2)},$$

on simplifying, we get,

$$= \Big[1 + \frac{\beta\gamma^*}{1!}\sqrt{1!} + \frac{(\beta\gamma^*)^2}{2!}\sqrt{2!} + ...\Big]e^{-1/2(|\gamma|^2+|\beta|^2)},$$

closing the above series, we get the exponential as,

$$\langle\gamma|\beta\rangle = e^{\beta\gamma^* - 1/2(|\gamma|^2+|\beta|^2)}, \tag{3.36}$$

on taking the modulus square we have,

$$|\langle\gamma|\beta\rangle|^2 = (\langle\gamma|\beta\rangle) * (\langle\gamma|\beta\rangle^*), \tag{3.37}$$

$$= e^{(\beta\gamma^* + \beta^*\gamma - (|\gamma|^2+|\beta|^2))},$$

$$|\langle\gamma|\beta\rangle|^2 = e^{-|\gamma-\beta|^2}. \tag{3.38}$$

So, the coherent states are not orthogonal. They are orthogonal only if $|\gamma - \beta|^2$ is infinity or distance between them is very large.

3.3.2 Completeness

Coherent states bear a completeness relation except for the fact that they are not orthogonal. The completeness relation can be defined over a complex plane β as

$$\frac{1}{\pi}\int d^2\beta|\beta\rangle\langle\beta| = 1. \tag{3.39}$$

It can be proved using the number state representation of coherent state that is taking the right hand side and substituting the values we get

$$\frac{1}{\pi}\int d^2\beta|\beta\rangle\langle\beta| = \frac{1}{\pi}\frac{1}{\sqrt{n!m!}}\int d^2\beta\beta^n(\beta^*)^m e^{-|\beta|^2}|n\rangle\langle m|. \tag{3.40}$$

The above integral can be solved by making the substitution $\beta = re^{i\theta}$ than $d^2\beta = rdrd\theta$ and $|\beta|^2 = r^2$ evaluating the above integral that is

$$\int d^2\beta\beta^n(\beta^*)^m e^{-|\beta|^2} = \int_0^\infty rdre^{-r^2}r^n r^m \int_0^{2\pi} d\theta e^{(m-n)\theta}, \tag{3.41}$$

where $\int_0^{2\pi} d\Theta e^{(m-n)\theta} = 2\pi\delta_{nm}$, and for $n = m$

$$= 2\pi\int_0^\infty dr\ re^{-r^2}r^{2n}.$$

Putting $r^2 = t$ and $2rdr = dt$

$$= \pi \int_0^\infty t^n e^t dt, \tag{3.42}$$

where the above integral is equal to the Gamma function. So the above equation becomes

$$= \pi n!. \tag{3.43}$$

Substituting this value in Eq. (3.40) under the condition that $n = m$

$$\frac{1}{\pi} \int d^2\beta |\beta\rangle\langle\beta| = \frac{1}{\pi n!} \sum_{n=0}^{\infty} |n\rangle\langle n|(\pi n!), \tag{3.44}$$

so

$$\frac{1}{\pi} \int d^2\beta |\beta\rangle\langle\beta| = 1. \tag{3.45}$$

Hence coherent states are said to form a complete set. With the help of this property we can write any quantum mechanical state in terms of coherent states.

3.3.3 Over Completeness

The unusual property of coherent is its over-completeness by that we mean that together with the property of non orthogonality and over completeness, we can not only represent any quantum mechanical state in terms of coherent states but we can also write one coherent state in terms of another coherent state. Suppose a system in a quantum state $|\beta\rangle$, then by non-orthogonality of coherent states there is a nonzero chance that the system is in quantum state $|\gamma\rangle$ because $|\langle\gamma|\beta\rangle|^2 \neq 0$ when $\beta \neq \gamma$. Consequently, the number of coherent states is greater than that needed for basis, therefore coherent states are said to be over-complete. Let's consider a coherent state $|\gamma\rangle$ then

$$|\gamma\rangle = 1|\gamma\rangle = \frac{1}{\pi} \int d^2\beta |\beta\rangle\langle\beta|\gamma\rangle, \tag{3.46}$$

and we know that

$$\langle\beta|\gamma\rangle = e^{-1/2(|\beta|^2+|\gamma|^2)+\beta\gamma^*}, \tag{3.47}$$

so,

$$|\gamma\rangle = \frac{1}{\pi} \int d^2\beta e^{-1/2(|\beta|^2+|\gamma|^2)+\beta\gamma^*} |\beta\rangle. \tag{3.48}$$

The above equation shows that one coherent state can be represented in terms of another this is referred to as over-completeness of coherent states.

3.3.4 Temporal Stability

Coherent states are temporally stable by that we mean that under time evolution a coherent state will remain coherent. Let us consider a coherent state $|\beta(0)\rangle$ at time $t = 0$. By applying the time evolution operator, we get the time evolved coherent state represented as $|\beta(t)\rangle$ such that

$$|\beta(t)\rangle = U(t)|\beta(0)\rangle, \tag{3.49}$$

where for the harmonic oscillator $U(t) = e^{-iHt/\hbar}$ with $H = \hbar\omega(\hat{a}^\dagger a + 1/2)$

$$|\beta(t)\rangle = e^{-i(\hat{a}^\dagger a+1/2)\omega t}|\beta(0)\rangle, \tag{3.50}$$

we know that

$$|\beta(0)\rangle = e^{-1/2|\beta|^2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}|n\rangle, \tag{3.51}$$

putting this value in the above equation

$$|\beta(t)\rangle = e^{-1/2|\beta|^2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}e^{-i(\hat{a}^\dagger\hat{a}+1/2)\omega t}|n\rangle, \tag{3.52}$$

$$= e^{-1/2|\beta^2|}e^{-i\omega t/2}\sum_{n=0}^{\infty}\frac{\beta^n}{\sqrt{n!}}e^{-i\omega tn}|n\rangle,$$

$$= e^{-1/2|\beta|^2}e^{-i\omega t/2}\sum_{n=0}^{\infty}\frac{(\beta^n e^{-i\omega t})n}{\sqrt{n!}}|n\rangle,$$

$$|\beta(t)\rangle = e^{-i\omega t/2}[e^{-1/2|\beta|^2}\sum_{n=0}^{\infty}\frac{\beta'^n}{\sqrt{n!}}|n\rangle]. \tag{3.53}$$

where $\beta' = \beta e^{-i\omega tn}$ and $|\beta'| = |\beta|$. Hence, only a phase shift came so coherent states are temporally stable.

3.4 The Generalized Coherent States

The coherent states of the electromagnetic field have been a strong source of stimulation in the development of coherent states. Many attempts have been made to generalize the field coherent states [38], by the generalization of three definitions, as explained in the previous section. In doing so one discovers that different generalization will lead to different outcomes and in fact some of the definitions cannot be generalized [8, 6, 7]. But since these generalizations have the algebraic dependence so attempts have been made to construct the methods which are algebraically independent and can be used to a construct coherent state for any system. Recently two such techniques have been introduced such as Gazeau-Klauder coherent states [39] and Gaussian-Klauder coherent state [40]. In the preceding section we have presented harmonic oscillator as an explicit example of field coherent state. Now in this section keeping in view the eigen value and eigen-functions of our system we will see the method of generalization for the construction of coherent state for any dynamical system without explicit dependence of group algebra.

3.4.1 Gazeau-Klauder Coherent States

In 1999, an important class of generalized coherent state known as *Gazeau−Klauder coherent states* were introduced by Gazeau and Klauder which for any arbitrary quantum mechanical system can be denoted by $|J, \varphi\rangle$ [39]. Due to its algebraic independence, this formalism receives much attention. For any Hamiltonian with discrete (non-degenerate) eigenvalues $e_n \geq 0$ these states are defined as,

$$|J, \varphi\rangle = (\mathcal{N}(J))^{-1} \sum_{n=1}^{\infty} \frac{J^{n/2} e^{-ie_n\varphi}}{\sqrt{\rho_n}} |n\rangle, \quad J \geq 0, \quad -\infty < \varphi < \infty \tag{3.54}$$

where $\mathcal{N}(J)$ denotes the normalization constant. Note that the above relation is very similar to the coherent state for harmonic oscillator except that we have ρ_n in place of $n!$. The orthonormal vector $\{|n\rangle\}_0^\infty$ satisfy the eigenvalue equation:

$$\hat{H}|n\rangle = E_n|n\rangle \equiv \hbar\omega e_n|n\rangle = e_n|n\rangle. \tag{3.55}$$

where $\omega > 0, \hbar \equiv 1$. The eigenvalue for the Hamiltonian $\hat{H}$ are dimensionless energies $0 = e_1 < e_2....$ Following requirements should be satisfied by these states: a)*continuity of labeling*, b)*resolution of the identity*, c)*temporal stability* and d)*action identity*. The last two conditions require $\rho(n) = [e_n]!$. Now finding the value of the normalization constant

$$\langle J, \varphi|J, \varphi\rangle = (\mathcal{N}(J))^{-2} \sum_{n,m=0}^{\infty} \Big(\frac{J^{\frac{n+m}{2}} e^{-i(e_n - e_m)\varphi}}{\rho_n \rho_m}\Big)\langle n|m\rangle, \tag{3.56}$$

where $\delta_{n,m} = \langle m|n\rangle$ and for $n = m$ we have

$$(\mathcal{N}(J))^2 = \Big(\sum_{n=0}^{\infty} \frac{J^{2n}}{\rho_n}\Big). \tag{3.57}$$

which properly normalizes the above state. Thus $\rho_n,\ n = 0, 1, 2....$ must satisfy the term $\sum_{n=0}^{\infty}(J^{2n}/\rho_n)$ is convergent. The range of variable J is fixed as $0 \leq J \leq R$ where R shows the radius of convergence of this series. Thus the radius of convergence is defined as $R = \lim_{n\to\infty}(\rho_n)^{1/n}$ and can also be given by the ratio test for convergence as

$$R = \lim_{n\to\infty}\left(\frac{\rho_{n+1}}{\rho_n}\right). \tag{3.58}$$

Evidently, this R maybe finite or infinite and depends upon the large n behavior of ρ_n. For the property of resolution of unity we need to see that

$$I = \int |J,\varphi\rangle\langle J,\varphi| d\mu(J,\varphi) = \sum_{n=0}^{\infty} |n\rangle\langle n|. \tag{3.59}$$

Let us assume that the no. of $\rho_n,\ n = 0, 1, 2....$ arises from a probability distribution function $\rho(J)$ by

$$\rho_n = \int_0^R \rho(J)J^n dJ, \tag{3.60}$$

where $\rho(J) \geq 0$ is called a probability density function. In other words, ρ_n is the n^{th} moment of this density function.we assume that the moment exists and $\rho_0 = 1,\ \rho_n < \infty$.

Now let us consider the example of Harmonic oscillator where $\rho_n = n!$. The radius of convergence for it can be calculated as

$$\lim_{n\to\infty}\frac{\rho_{n+1}}{\rho_n} = \lim_{n\to\infty}\frac{(n+1)!}{n!}, \tag{3.61}$$

$$= \lim_{n\to\infty}(n+1) = \infty, \tag{3.62}$$

Thus, we are required to find $\rho(J)$ such that

$$\rho_n = n! = \int_0^{\infty} \rho(J)J^n dJ, \tag{3.63}$$

Hence, $\rho(J) = e^{-J}$ where $\rho(J) > 0 for\ all J$ and

$$\int_0^{\infty} e^{-J}J^n dJ = \Gamma(n+1) = n!, \tag{3.64}$$

we need to obtain $d\mu(J,\varphi)$ to satisfy the resolution of unity. First, we define

$$\int ...d\nu(\varphi) = \lim_{\Gamma\to\infty}\frac{1}{2\Gamma}\int_{-\Gamma}^{\Gamma}....d\varphi, \tag{3.65}$$

and we calculate

$$\int |J,\varphi\rangle\langle J,\varphi|d\nu(\varphi) = \lim_{\Gamma\to\infty}\frac{1}{2\Gamma}\int_{-\Gamma}^{\Gamma}\sum_{n,m=0}^{\infty} J^{(n+m)/2}\frac{e^{-i\varphi(\rho_m-\rho_n)}|m\rangle\langle n|}{\mathcal{N}^2(J)\sqrt{\rho_n\rho_m}}. \tag{3.66}$$

Considering

$$\int_{-\Gamma}^{\Gamma} e^{-i\varphi(e_m-e_n)}d\varphi = 0, \quad if \quad n\neq m$$

and

$$\lim_{\Gamma\to\infty}\frac{1}{2\Gamma}\int_{-\Gamma}^{\Gamma} e^{-i\varphi(e_m-e_n)}d\varphi = 0, \quad if \quad n = m$$

Then this integral covers the whole range of angle variable (φ) that varies from $-\infty<\varphi<\infty$.
We arrive at

$$|J,\varphi\rangle\langle J,\varphi|d\nu(\varphi) = \frac{1}{(\mathcal{N}(J))^2}\sum_{n=0}^{\infty}\frac{J^n}{\rho^n}|n\rangle\langle n|. \tag{3.67}$$

To complete the resolution of unity we define

$$K(J) = [\mathcal{N}(J)]^2\rho(J), \quad for \quad 0\leq J<R$$

and

$$K(J) = \rho(J) = 0, \quad for \quad J>R$$

where $K(J)$ is expressed in terms of the function $\rho(J)$ which results in the moment ρ_n and $\mathcal{N}(J)$ is the normalization constant. Then

$$\frac{1}{\rho_n}\int_0^R K(J)(\mathcal{N}(J))^{-2}J^n dJ,$$

$$= \frac{1}{\rho_n}\int_0^R (\mathcal{N}(J))^2\rho(J)(\mathcal{N}(J))^{-2}J^n dJ,$$

$$= \frac{1}{\rho_n}\int_0^R \rho(J)J^n dJ = \frac{1}{\rho_n}\rho(n) = 1.$$

Hence, the resolution of unity is satisfied.

The information we set for the model, then leads to action angle identity that is

$$H(J,\varphi) = \langle J,\varphi|H|J,\varphi\rangle = \omega J. \tag{3.68}$$

Now evaluating it out

$$H|J,\varphi\rangle = \frac{H}{\mathcal{N}(J)}\left[\sum_{n=0}^{\infty}\frac{J^{n/2}e^{-ie_n\varphi}}{\sqrt{\rho_n}}|n\rangle\right], \tag{3.69}$$

$$= \frac{1}{\mathcal{N}(J)}\left[\sum_{n=0}^{\infty}\frac{J^{n/2}e^{-ie_n\varphi}\omega e_n}{\sqrt{\rho_n}}|n\rangle\right], \tag{3.70}$$

then

$$\langle J,\varphi|H|J,\varphi\rangle = \frac{1}{(\mathcal{N}(J))^2}\left[\sum_{n,m=0}^{\infty}\frac{J^{n+m/2}e^{-i(e_n-e_m)\varphi}\omega e_n}{\sqrt{\rho_n\rho_m}}\langle m|n\rangle\right], \tag{3.71}$$

Since

$$\frac{\omega}{(\mathcal{N}(J))^2} = \sum_{n=0}^{\infty}\left(\frac{J^n e_n}{\rho_n}\right).$$

So the above equation becomes

$$= \omega\left(\left(\sum_{n=0}^{\infty}\frac{J^n e_n}{\rho_n}\right)\Big/\left(\sum_{n=0}^{\infty}\frac{J^n}{\rho_n}\right)\right). \tag{3.72}$$

To satisfy the above condition, we demand that

$$\sum_{n=0}^{\infty}\frac{J^n e_n}{\rho_n} = J\sum_{n=0}^{\infty}\frac{J^n}{\rho_n} = \sum_{n=0}^{\infty}\frac{J^{n+1}}{\rho_n} = \sum_{n=1}^{\infty}\frac{J^n}{\rho_{n-1}}. \tag{3.73}$$

On comparing both sides, we get

$$\frac{e_n}{\rho_n} = \frac{1}{\rho_{n-1}}.$$

Recalling $e_0 = 0$ we get $e_n = (\rho_n)/(\rho_{n-1})$. Therefore, $\rho_0 = 1$ which leads to the relation

$$\rho_n = e_1 e_2 .....e_n. \tag{3.74}$$

Hence the above equation gives ρ_n in terms of $e_n's$. Then R is determined as the radius of convergence for the series $\sum_{n=0}^{\infty} J^n/\rho n$ and $dJ\ d\varphi_\nu = d(J,\varphi)$ is the integration parameter. $K(J)$ is determined in term of $\rho(J)$ when

$$\int_0^R \rho(J)J^n dJ = \rho_n,$$

The only unknown is the moment generating function to be determined from the numbers ρ_n. We need to emphasize that $\rho(J)$ so determined may not be positive throughout, then the coherent state construction would not be possible. Now finding the connection of $\rho(J)$ with $\{\rho_n\}$. Indeed

$$\int_0^\infty \frac{J^n K(J)dJ}{\mathcal{N}^2(J)} = \rho_n, \tag{3.75}$$

The *Mellin transform* for a complex s of a function $f(x)$ is defined as

$$f^*(s) = \int_0^\infty x^{s-1} f(x)dx \equiv \mathfrak{M}(f(x); s). \tag{3.76}$$

and its inverse is then

$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} f^*(s)ds \equiv \mathfrak{M}^{-1}(f^*(s); x). \tag{3.77}$$

here c is the integration over the imaginary axis. Then

$$\frac{K(J)}{\mathcal{N}^2(J)} = \mathfrak{M}^{-1}(\rho(s-1); J), \tag{3.78}$$

we have generalized ρ_n to a function $\rho(s-1)$ of a complex variable s and then we calculate its inverse Mellin transform in the $K(J)$ we needed. The problem of determining the moment generating function for the moment problem for coherent state on a plane $R = \infty$ (Stieltjes moment problem) or coherent state on a disk $R < \infty$ (Hausdorff moment problem) these are related by inverse Mellin transformation [41].

3.4.2 Gaussian-Klauder Coherent States

In 1999 R. F. Fox introduced the generalized Gaussian-Klauder coherent states [40]. These states were introduced in order to criticize the notion that a Gaussian function cannot be used to approximate a coherent state. The basic idea was that a Gaussian function is appropriate for the construction of generalized Gaussian-Klauder coherent states and the key advantage of these states was that they satisfy the resolution of unity. Initially, these states were constructed for the harmonic oscillator and Rydberg atom. The generalized form of such states [18] is

$$|J,\varphi\rangle = (\mathcal{N}(J))^{-1/2}\sum_0^\infty e^{-\frac{(n-J)^2}{4\sigma^2}} e^{ie_n\varphi}|n\rangle, \quad J \geq 0, \quad -\infty < \varphi < \infty \tag{3.79}$$

where, ρ_n are the moments of positive weight function of the system and

$$\mathcal{N}(J) = e^{-\frac{(n-J)^2}{4\sigma^2}}. \tag{3.80}$$

where σ is the width of the Gaussian.

Chapter 4

Generalized Coherent States for the Gravitational Cavity

As discussed earlier in the previous chapter the potential inside gravitational cavity do not have explicitly defined algebraic structure. Therefore, Gazeau-Klauder(GK) formulism is well suited to construct generalized coherent states for such systems. After obtaining the eigenvalues and eigenfunction of gravitational cavity and discussing the GK coherent states, we are now in a position to construct GK coherent states for gravitational cavity.

This chapter is organized as follow: In Sec. 1, Gazeau-Klauder coherent states for gravitational cavity are constructed. we will then discuss in Sec. 2, the statistical properties of our coherent states. Furthermore Sec. 3, comprises of spatiotemporal properties of the GK coherent states which will include the autocorrelation function, the position space probability density and its time evolution. Lastly, we will study the Phase space picture using the Wigner function.

4.1 GK Coherent States for Gravitational Cavity

We have discussed the potential inside the gravitational cavity in great detail in chapter 2. Now following the same procedure as discussed in the previous chapter, we will construct the GK coherent states for our system [15]. Since we know that the Hamiltonian for our system is bounded from below and posses a discrete spectrum with non-degenerate eigenstates. Keeping in view the energy spectrum of our system as computed in Eq.(2.37) where ω has the dimensions of energy. We can express H as

$$\hat{H} = \omega\hat{\kappa}, \tag{4.1}$$

where $\hat{\kappa}$ is the dimensionless Hamiltonian. The eigenstates $|n\rangle$ are orthonormal basis and fulfills the equation,

$$\hat{\kappa}|n\rangle = e_n|n\rangle. \tag{4.2}$$

Here e_n are the dimensionless eigenenergies, such that $e_0 = 0$ and $e_n < e_{n+1}$. By comparing Eq.(2.37) and Eq.(4.1) we define e_n for $n = 0, 1, 2....$ as

$$e_n = n^{2/3}, \tag{4.3}$$

The general form of *Gazeau − Klauder coherent states* with φ and J as real parameters, such that $-\infty < \varphi < \infty$ and $J \geq 0$ is defined as;

$$|J,\varphi\rangle = C(J)\sum_{n=0}^{\infty}\frac{J^{n/2}e^{-ie_n\varphi}}{\sqrt{\rho(n)}}|n\rangle, \tag{4.4}$$

we know that $\rho(n)$ is defined as the product of all the eigen energies e_n as

$$\rho(n) = e_n.e_{n-1}.e_{n-2}......e_1 \tag{4.5}$$

putting the value of e_n we get

$$\rho(n) = (n!)^{2/3} = n!f(n!), \tag{4.6}$$

where $f(n!) = 1/(n!)^{1/3}$. Hence,the coherent state for gravitational cavity can be expressed as

$$|J,\varphi\rangle = C(J)\sum_{n=0}^{\infty}\frac{J^{n/2}e^{-ie_n\varphi}}{\sqrt{n!f(n!)}}|n\rangle. \tag{4.7}$$

Now evaluating the normalization constant first that is

$$\langle J,\varphi|J,\varphi\rangle = (C(J))^2\sum_{n,m=0}^{\infty}\frac{J^{n+m/2}e^{-i(e_n-e_m)\varphi}}{\sqrt{n!f(n!)}}\langle m|n\rangle, \tag{4.8}$$

where, $\langle m|n\rangle = \delta_{n,m}$ for $n = m$ we evaluate the normalization constant which properly normalizes our coherent states.

$$C(J) = \Big(\sum_{n=0}^{\infty} \frac{J^n}{n!f(n!)}\Big)^{-1/2}. \tag{4.9}$$

These states satisfies the set of properties necessary to be called GK coherent states [15].

4.2 Statistical Properties

In order to gain insight of the statistical properties of a coherent state, we need to find the variance first and for that we need to calculate the expectation value of $\hat{N}$ and $\hat{N}^2$ that is

$$\langle n\rangle = \langle J,\varphi|\hat{N}|J,\varphi\rangle = |C(J)|^2 \sum_{n=0}^{\infty} \frac{J^2}{f(n!)n!} n, \tag{4.10}$$

and

$$\langle n^2\rangle = \langle J,\varphi|\hat{N}|J,\varphi\rangle = |C(J)|^2 \sum_{n=0}^{\infty} \frac{J^2}{f(n!)n!} n^2, \tag{4.11}$$

respectively. The variance will be then

$$\sigma^2 = \langle n^2\rangle - \langle n\rangle^2, \tag{4.12}$$

$$\sigma^2 = |C(J)|^2 \sum_{0}^{\infty} \frac{J^2}{f(n!)n!} n^2 - |C(J)|^4 \sum_{n=0}^{\infty} \frac{J^4}{(f(n!)n!)^2} n^2. \tag{4.13}$$

In the case of coherent state of harmonic oscillator [3] the mean and variance are equivalent that is

$$\langle n\rangle_{cs} = \sigma^2_{cs} = \frac{|J|^{2n}}{n!} e^{-|J|^2}, \tag{4.14}$$

which is clearly the Poisson distribution. However, for the case of gravitational cavity the variance is always greater than the mean as plotted in the Fig(4.2).

4.2.1 Weighting Distribution

The weighting distribution of the coherent states wave packets in gravitational cavity is calculated as

$$P(n) = |\langle n|J,\varphi\rangle|^2, \tag{4.15}$$

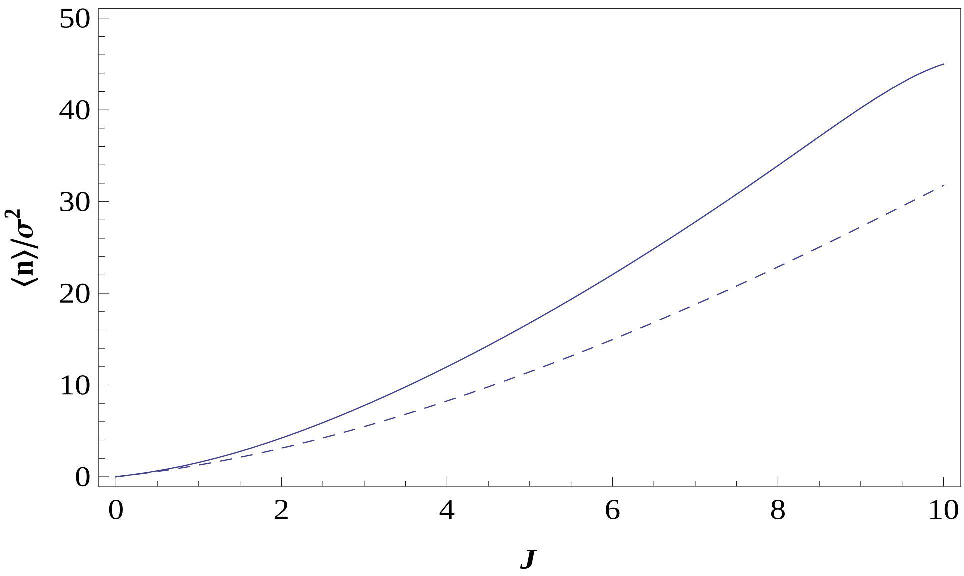

Figure 4.1: The variance σ^2 (solid curve) and mean $\langle n \rangle$ (dashed curve) as a function of coherent state parameter J

$$\langle n|J,\varphi\rangle = C(J)\frac{J^{n/2}e^{-ie_n\varphi}}{\sqrt{n!f(n!)}}, \tag{4.16}$$

on taking the modulus square we get

$$P(n) = |C(J)|^2\frac{J^n}{n!f(n!)}. \tag{4.17}$$

Now in order to do a comparison we have plotted together the weighting distribution of harmonic oscillator that is the Poisson distribution, and that in the present case as a function of n as can be viewed in the Fig(4.2.1). It is clear from the graphs that the weighting distribution depends only on the parameter J and its mean $\langle n \rangle$ which increases correspondingly with J. The distribution for our coherent states is broader as compare to the Poisson distribution of a harmonic oscillator for a particular value of $\langle n \rangle$.

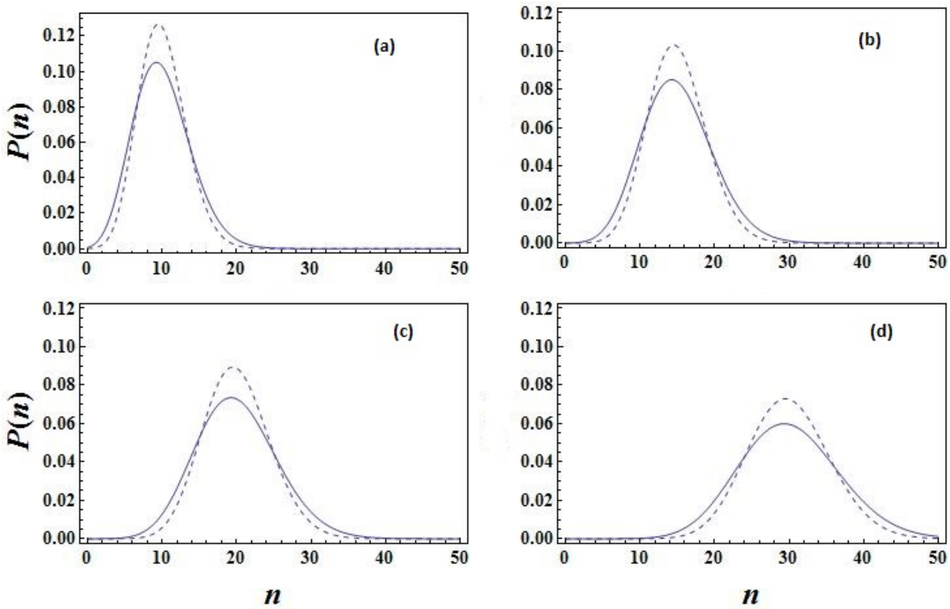

Figure 4.2: The weighting distribution $P(n)$ for gravitational cavity (bar chart) and harmonic oscillator(solid curve) as a function of n for: (a) $J = 4.56$, (b) $J = 6.02$,(c) $J = 7.31$, and (d) $J = 9.62$.

4.2.2 Mandel Q Parameter

A simple way to measure the nature of the photon statistics of any state is done by calculating the so-called Q-parameter [37, 42] as

$$Q = \frac{\sigma^2}{\langle n \rangle} - 1 \tag{4.18}$$

For a state with $Q > 0$ the statistics are super-Poissionian, and in the range $-1 < Q < 0$ it is sub-Poissonian. Obviously, $Q = 0$ for the coherent states of harmonic oscillator. A convenient way to characterize the non-classicality of a state is via the Q-parameter which is negative whenever, the statistics are sub-Poissonian. For a Fock state the Q-parameter takes on its greatest possible negative value of -1. Now calculating the Mandel's parameter for the coherent states wave packet in gravitational cavity and plotting it out as a function of J we see that it exhibits the super-Poissionian distribution as shown in the figure below.

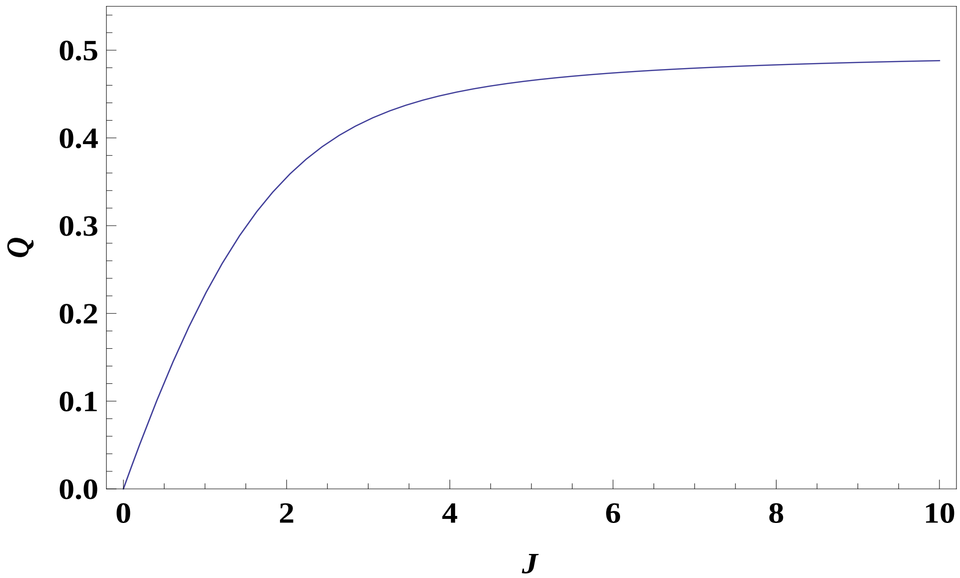

Figure 4.3: Mandel parameter Q as a function of J.

4.2.3 Second-order Correlation Function

The second-order correlation function $g^2(\tau)$ is basically a mathematical entity for characterizing the joint probability of detecting one photon followed by another at some fixed position within the delay time τ [43]. For a single-mode field, $g^2(\tau) = g^2(0) =$ constant and thus there can be no photon bunching (or anti-bunching for that matter). Anti-bunching and bunching occur only for multi-mode fields. For harmonic oscillator coherent states, $g^2(0) = 1$. For $g^2(0) > 1$, the probability of detecting a second photon decreases and this indicates a bunching of photons. For a thermal field, $g^2(0) = 2$ contrary to it, for $g^2(0) < 1$, the probability of detecting a second photon increases and this is called photon anti-bunching

For a classical field state one must have $g^2(0) \geq 1$ but for a nonclassical field state it is possible to have $g^2(0) < 1$, which, may be interpreted as a quantum mechanical violation of the Cauchy inequality. One may notice that the condition $g^2(0) < 1$ is the condition in sub-Poissonian statistics of the Q-parameter to be negative. Indeed Q and $g^2(0)$ [44], for a single-mode field are simply related:

$$g^2(0) = \frac{\langle n^2 \rangle - \langle n \rangle}{\langle n \rangle^2} = 1 + \frac{Q}{\langle n \rangle} \tag{4.19}$$

The fact that $Q < 0$ when $g^2(0) < 1$ has led to some confusion regarding the relationship of sub-Poissonian statistics and photon anti-bunching. Note that $g^2(0)$ will be less than unity whenever $\langle(\Delta n)^2\rangle < \langle n\rangle$. (For a number state $\langle(\Delta n)^2\rangle = 0$.) States for which this condition holds are sub-Poissonian. (States that possess sub-Poissonian statistics are also nonclassical) Since $g^2(0)$ is constant for the single-mode field, photon anti-bunching does not occur, the requirement for it to occur being $g^2(0) < 1$. The point is that photon anti-bunching and sub-Poissonian statistics are different effects, although they have often been confused as being essentially the same thing but they are not. Now using the above formula and calculating the second order coherence for coherent states wave packets in gravitational cavity and plotting it out as a function of α we see that it exhibits the bunching phenomena.

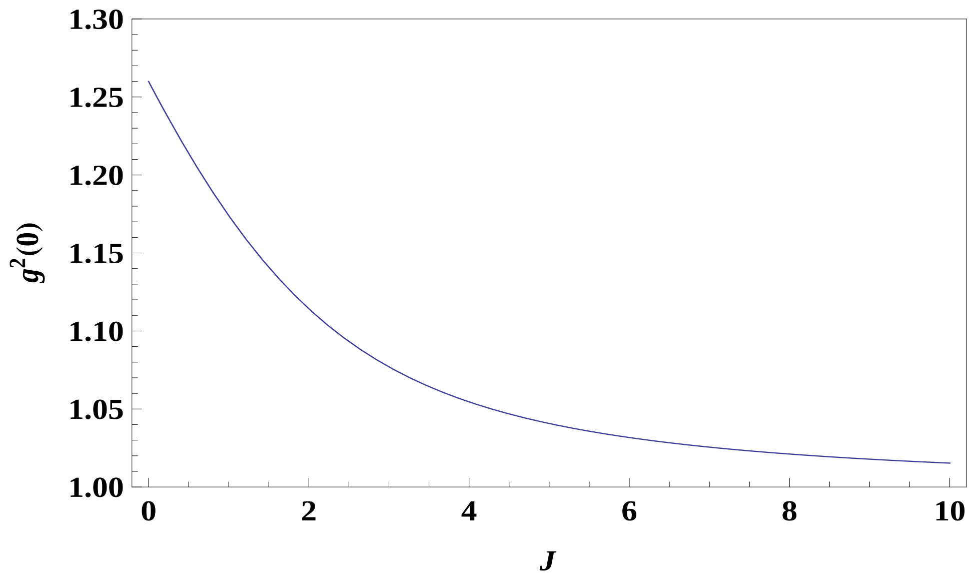

Figure 4.4: Second-order correlation function $g^2(0)$ verses J.

4.2.4 Uncertainty Relation

In order to calculate the time independent uncertainty product we first calculate the expectation value of position with respect to coherent states

$$\langle \xi \rangle = \langle J, \varphi | \xi | J, \varphi \rangle = |C(J)|^2 \sum_{n,m=0}^{\infty} \frac{J^{(n+m)/2} e^{-i(e_n - e_m)\varphi}}{\sqrt{\rho(n)\rho(m)}} \langle n | \xi | m \rangle, \tag{4.20}$$

As we have already evaluated the matrix elements in Chapter 2, putting the value

$$\langle n | \xi | m \rangle = \frac{2(-1)^{n+m+1}}{(\xi_n - \xi_m)^2},$$

substituting this value above, we get,

$$\langle \xi \rangle = |C(J)|^2 \sum_{n,m=0}^{\infty} \frac{J^{(n+m)/2} e^{-i(e_n - e_m)\varphi}}{\sqrt{\rho(n)\rho(m)}} * \frac{2(-1)^{n+m+1}}{(\xi_n - \xi_m)^2}. \tag{4.21}$$

Similarly, the expectation value of ξ^2 would be,

$$\langle \xi^2 \rangle = \langle J, \varphi | \xi | J, \varphi \rangle = |C(J)|^2 \sum_{n,m=0}^{\infty} \frac{J^{(n+m)/2} e^{-i(e_n - e_m)\varphi}}{\sqrt{\rho(n)\rho(m)}} \langle n | \xi^2 | m \rangle, \tag{4.22}$$

where,

$$\langle n | \xi^2 | m \rangle = \frac{24(-1)^{n+m+1}}{(\xi_n - \xi_m)^4}, \tag{4.23}$$

so,

$$\langle \xi^2 \rangle = |C(J)|^2 \sum_{n,m=0}^{\infty} \frac{J^{(n+m)/2} e^{-i(e_n - e_m)\varphi}}{\sqrt{\rho(n)\rho(m)}} * \frac{24(-1)^{n+m+1}}{(\xi_n - \xi_m)^4}. \tag{4.24}$$

On similar footing we calculate expectation value of momentum we get,

$$\langle P \rangle = (-i)|C(J)|^2 \sum_{n,m=0}^{\infty} \frac{J^{(n+m)/2} e^{-i(e_n - e_m)\varphi}}{\sqrt{\rho(n)\rho(m)}} * \frac{(-1)^{n+m}}{\xi_n - \xi_m}. \tag{4.25}$$

and

$$\langle P^2 \rangle = (-i^2)|C(J)|^2 \sum_{n,m=0}^{\infty} \frac{J^{(n+m)/2} e^{-i(e_n - e_m)\varphi}}{\sqrt{\rho(n)\rho(m)}} * \frac{2(-1)^{n+m}}{(\xi_n - \xi_m)^2}. \tag{4.26}$$

Now using the above relation we can find the variance that is

$$\Delta \xi = (\langle \xi^2 \rangle - \langle \xi \rangle^2)^{1/2}. \tag{4.27}$$

$$\Delta P = (\langle P^2 \rangle - \langle P \rangle^2)^{1/2}. \tag{4.28}$$

On multiplying it, we will get the time independent uncertainty relation.

4.3 Spatiotemporal characteristics

With the advent of modern computational technology, robust numerical calculations of different time-dependent phenomena in quantum mechanics are now common, as is the visualization of the resulting effects. With this, we not only illustrate the wave packets, spread with time, but also extending it to more novel phenomena such as wave packet revivals [20]. The concept of wave packet revivals can be understood by considering a quantum wave packet solutions which are initially highly localized and exhibit quasi-classical behavior, then disperse in time to a so-called collapsed phase, and latter reform to something very much similar to its initial state. Which can then be extended to time-development of such solutions in terms of their position and momentum uncertainties and expectation values. This type of expectation value analysis, coupled with existing numerical, analytical, and visualization studies can then help to form a more complete picture of the model. In this section we will discuss the spatiotemporal properties of our system.

4.3.1 Auto-correlation Function

A standard tool used to probe the approximate return of the wave packet to its initial state is the auto-correlation function [19] defined as

$$A(t) = \langle J, \varphi | J, \varphi, t \rangle \tag{4.29}$$

Numerically, the value of $|A(t)|^2$ varies between 1 and 0. The maximum value $|A(t)|^2 = 1$ is reached when time evolved state exactly matches the initial state, and the minimum value 0 corresponds to non overlapping that is $|J, \varphi, t\rangle$ is far away from the initial state. For the oscillator, one can argue that because the wave packets never collapse and are exactly periodic, there are no revivals. Now calculating the autocorrelation for our coherent state. The time evolved coherent state for our system can be evaluated by applying the time evolution operator that is $U(t) = e^{-iHt} = e^{-i\omega\hat{\kappa}t}$ we then have,

$$|J, \varphi, t\rangle = C(J) \sum_{n=0}^{\infty} \frac{J^{n/2} e^{-ie_n(\varphi+\omega t)}}{\sqrt{\rho_n}} |n\rangle. \tag{4.30}$$

Then the auto-correlation function for the coherent states wave packet inside the gravitational cavity would be

$$A(t) = \langle J, \varphi | J, \varphi, t \rangle = \sum_{n=0}^{\infty} P(n) e^{-i\varepsilon_n t} \tag{4.31}$$

here $\varepsilon_n = \omega e_n$, and $P(n)$ is the weighting distribution which shows the initial localization of the coherent states with the help of mean and variance. By the revival of wave function we mean that when the time evolved wave function closely resembles its initial form. A fractional revival occurs when the time evolved wave function can be describe out as a collection of spatially distributed sub-wave functions, each of which closely resembles the shape of the initial shape. Let the coherent state be initially localized at $\langle n \rangle$ such that $\Delta n \ll \langle n \rangle$ than taking the Taylor expansion of ε_n around n that is

$$\varepsilon_n = \varepsilon_{\langle n \rangle} + \Big| \frac{\partial \varepsilon_n}{\partial n} \Big|_{n=\langle n \rangle} \frac{(n - \langle n \rangle)}{1!} + \Big| \frac{\partial^2 \varepsilon_n}{\partial n^2} \Big|_{n=\langle n \rangle} \frac{(n - \langle n \rangle)^2}{2!} + ... \tag{4.32}$$

Which can then be reduces to the formula given below which shows the revival of wave packet at various time [11] that is

$$T_r = 2\pi \Big(\frac{1}{r!} \Big| \frac{\partial^r \varepsilon_n}{\partial n^r} \Big|_{n=\langle n \rangle} \Big)^{-1} \tag{4.33}$$

These derivatives defines different time scales for $r = 1, 2, 3....$ namely, *classical period* $T_{cl} = 2\pi/\varepsilon_n^{'}$; the *revival time* $T_{rev} = 2\pi/\frac{1}{2}\varepsilon_n^{''}$; the *superrevival time* $T_{sup} = 2\pi/\frac{1}{6}\varepsilon_n^{'''}$; and so on. Since the energy spectrum in the case of gravitational cavity is non-linear, so an exact revival of the initial state does not occur. The revival times depend on $\langle n \rangle$ and therefore strongly depends upon α with increasing r.

On plotting it out we see that the coherent states wave packets in the gravitational cavity follow the classical trajectories for few classical periods and then undergo a collapse. But, in general case, fractional "super-revivals" and fractional revivals appear as periodic peaks in $|A(t)|^2$ and the periods in that case are rational fractions of the classical period round-trip T_{cl} and T_{rev} the revival time.

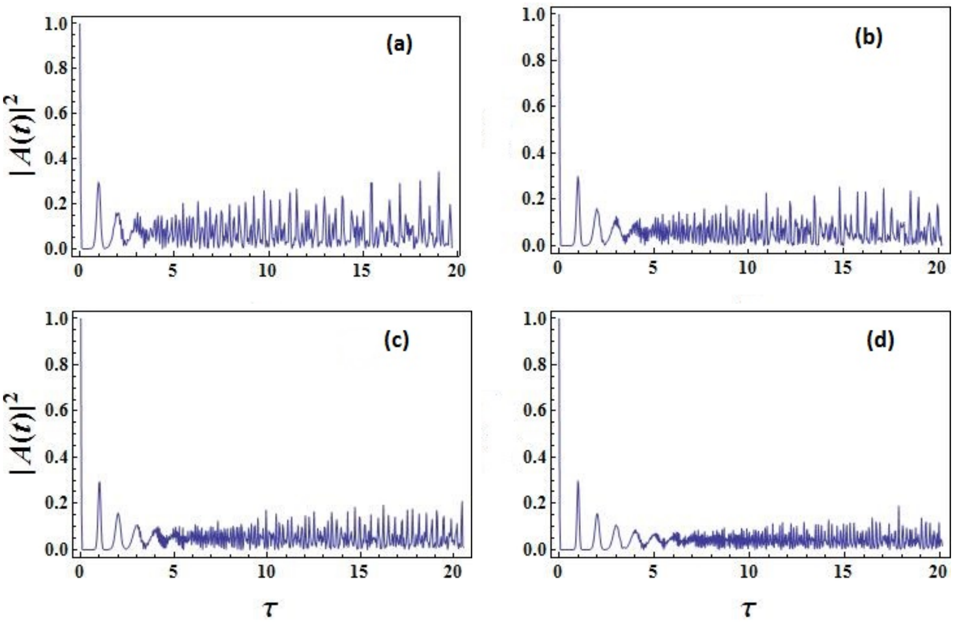

Figure 4.5: The Modulus squared of the autocorrelation function, $|A(t)|^2$, as a function of scaled time, $\tau = t/T_{cl}$, for: (a) $\alpha = 4.56$, (b) $\alpha = 6.02$,(c) $\alpha = 7.31$, and (d) $\alpha = 9.62$.

4.3.2 Probability Density

4.3.2.1 Position Space Probability Density

A very important property to investigate the spatiotemporal evolution of coherent state is the probability density function [11] which is mathematically described as,

$$P(\xi, t) = |\langle \xi | J, \varphi, t \rangle|^2. \tag{4.34}$$

Evaluating it for the coherent states of gravitation cavity where

$$\langle \xi | J, \varphi, t \rangle = C(J) \sum_{n=0}^{\infty} \frac{J^{n/2} e^{-ie_n(\varphi+\omega t)}}{\sqrt{\rho_n}} \langle \xi | n \rangle, \tag{4.35}$$

then the probability density is given as

$$P(\xi, t) = |C(J) \sum_{n=0}^{\infty} \frac{J^{n/2} e^{-ie_n(\varphi+\omega t)}}{\sqrt{\rho_n}} \psi_n(\xi)|^2. \tag{4.36}$$

where $\psi_n(\xi)$ is the eigenfunction of gravitational cavity. On plotting it out as a function of rescaled position ξ at time $t = 0$ for different values of J we see the position space probability density as shown below.

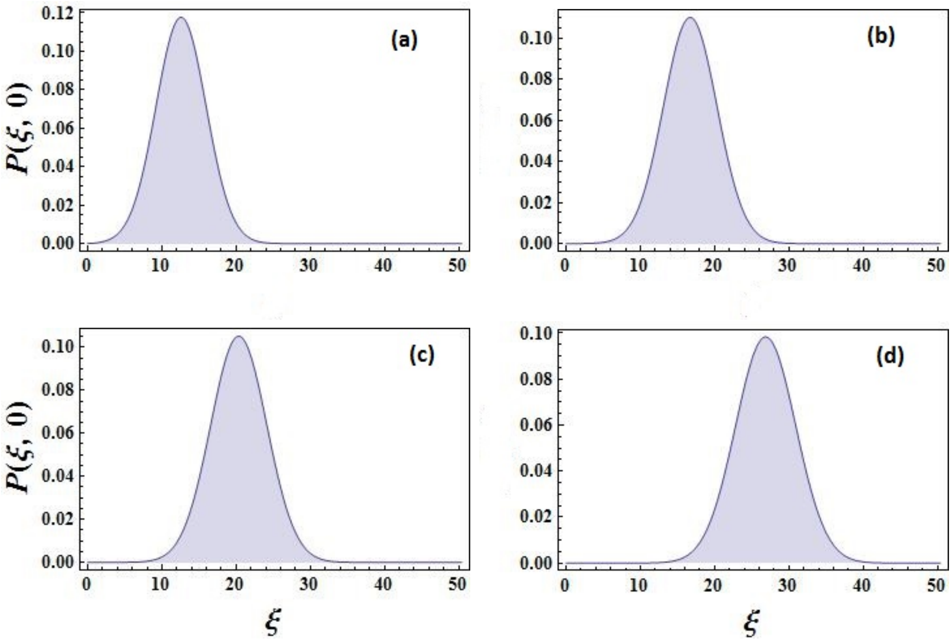

Figure 4.6: The probability density $P(\xi)$ as a function of rescaled position for (a) $J = 4.56$, (b) $J = 6.02$,(c) $J = 7.31$, and (d) $J = 9.62$.

4.3.2.2 Time Evolved Probability Density

The space time evolution of the probability density as calculated in Eq. (4.36) depends both on the structure of energy spectrum e_n and the nature of eigenstates $\psi_n(\xi)$ of the physical system, therefore it exhibits an interference structure. This interference is in fact, due to the time dependent part of the probability density. Now taking the snap shots of the probability density at different times. We see that the probability density splits into multiple peaks which exhibits a single peak at $t = 0$ as shown in Fig. (4.3.2.2). We can also see the time evolution of probability density which demonstrates the multiple splitting and interference pattern resulting in quantum carpets [45, 32]. As shown in Fig. (4.3.2.2) for 2 and 20 classical periods respectively where dark region execute minimum probability and light region execute maximum probability.

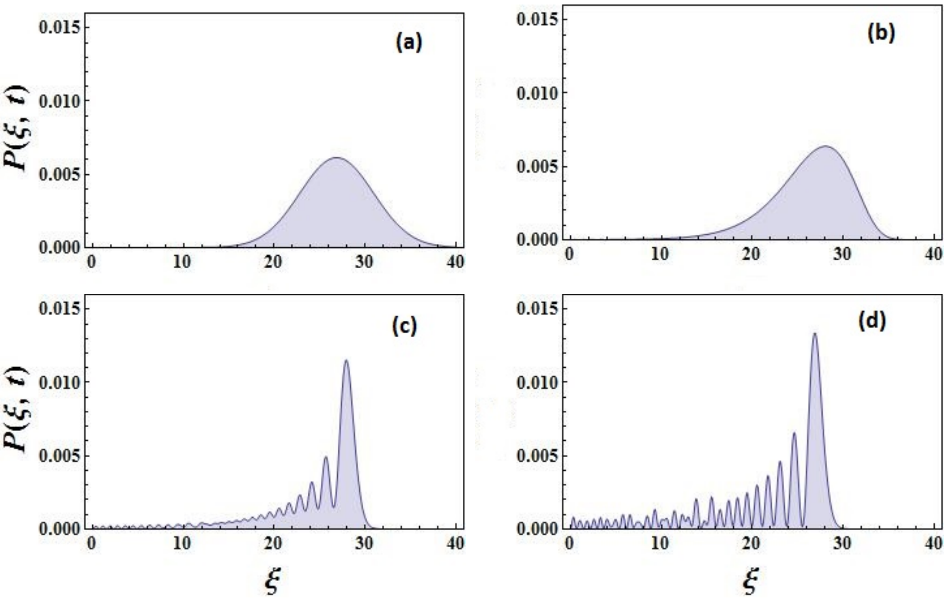

Figure 4.7: Snap shot of probability density vs rescaled position at $\tau = 0, 1, 2, 3$ from (a) to (d) respectively.

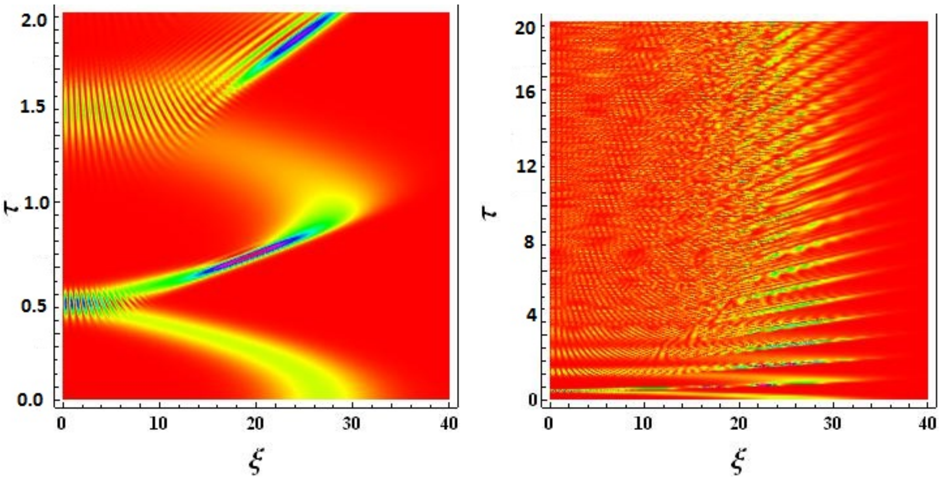

Figure 4.8: Quantum carpet for 2 and 20 classical periods respectively executing large probability in light region and low probability in the dark one.

4.3.3 Phase Space Picture Wigner Function

The Wigner function $W(x,p)$ was introduced in 1932 by E. P. Wigner [21]. With the help of this function we can represent a quantum state $|\psi\rangle$ which can be interpreted as the quantum equivalent of a "density" of traditional phase space. For any arbitrary density operator $\hat{\rho}$ it can be defined [3] as

$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\Big\langle x+\frac{1}{2}y|\hat{\rho}|x-\frac{1}{2}y\Big\rangle e^{\frac{ipy}{\hbar}}dy \tag{4.37}$$

where $|q \pm \frac{1}{2}y\rangle$ are eigenkets of the position operator and $\hat{\rho} = |\psi\rangle\langle\psi|$ then

$$W(x,p) = \frac{1}{2\pi\hbar}\int_{-\infty}^{\infty}\Psi^*\Big(x-\frac{1}{2}y\Big)\Psi\Big(x+\frac{1}{2}y\Big)e^{\frac{ipy}{\hbar}}dy \tag{4.38}$$

where $\langle x + \frac{1}{2}y|\psi\rangle = \Psi(q + \frac{1}{2}y)$ and vice versa. Without going into much detail, $W(x,p)$ is a real function, but is not always positive. Since the Wigner function depends upon both the position $\hat{x}$ and momentum $\hat{p}$ which holds an uncertainty relation that is $[\hat{x},\hat{p}] = i\hbar$. Due to the incompatibility of these observable a joint probability of them cannot be measure which is in accordance with famous Heisenberg uncertainty relation. The appearance of negative value refers to as the non-classicality of the Wigner function. Hence the Wigner distribution is named as quasi probability distribution [22]. Consequently, $W(x,p)$ cannot be considered as a true density in phase space. Though it owns the usual properties of a probability distribution. Integrating over position we get probability density in momentum space

$$\int_{-\infty}^{\infty} W(x,p)dq = |\langle x|\psi\rangle|^2 \tag{4.39}$$

Similarly, integrating over p we get the probability density of position that is

$$\int_{-\infty}^{\infty} W(x,p)dp = |\langle p|\psi\rangle|^2 \tag{4.40}$$

Now calculating the Wigner function for our system using Eq. (4.38). On plotting it out we see the negative values which shows the non-classical behavior of the Wigner function. For smaller values of J the non classicality is minimum but, as the value J increases, it becomes more and more prominent.

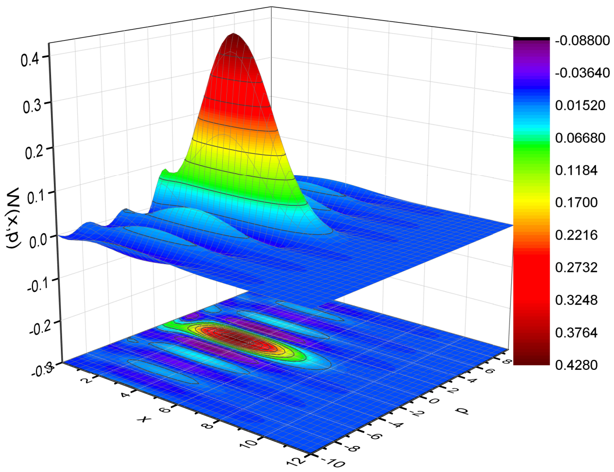

Figure 4.9: Wigner function for $J = 2$.

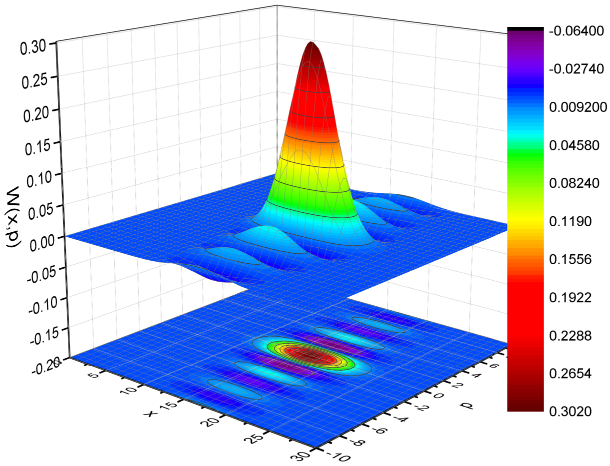

Figure 4.10: Wigner function for $J = 9.62$.

Chapter 5

SUMMARY AND CONCLUSIONS

The thesis focuses on the construction of generalized coherent states for gravitational cavity and their dynamical properties. Firstly, the gravitation cavity is described schematically as well as mathematically. The Schrödinger equation corresponding to the gravitational cavity is mapped onto the Airy differential equation whose solutions give the eigenstates in the form of Airy functions and the eigenenergies as negative zeros of Airy functions. Secondly, using these eigenenergies and eigenstates, the generalized coherent states are constructed following Gazeau-Klauder formalism and their various properties are analyzed. Before proceeding to the dynamical features of the constructed coherent states, we study various statistical characteristics such as, weighting distribution, mean occupation number, second order correlation function and Mandel Q-parameter. It is found that the coherent states of the gravitational cavity exhibit the super-Poissionian distribution which is broader than the Poissonian one. The analysis of the second order correlation function reveals that the constructed coherent states show the bunching effect.

Finally, we analyze the dynamical characteristics of these coherent states with a special focus on classical-quantum transition of various phenomena. In order to investigate the dynamical behavior, the time evolved coherent state are constructed to calculate the auto-correlation function. In their short time evolution up to few classical periods, the coherent states exhibit classical-like behavior. Afterwards, the dephasing between various constituting eigenstates, due to the nonlinearity of the energy spectrum, dominates and the coherent state observes a collapse. Later on, the coherent state reconstructs itself under the condition of phase matching and the phenomenon of quantum revivals takes place. In addition, the position space probability density is studied as a function of time. The time evolution of the probability density undergoes to the constructive and destructive interferences that manifests the interesting interference structures, known as quantum carpets. Moreover, the phase space properties of the constructed coherent states are studied by means of Wigner function. The negativity of the Wigner function reflects the transition from classical to the nonclassical behavior of these states.

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