Phase Transition Model for Generalized Bose-Einstein Condensation

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This Dissertation is dedicated to my parents

for their endless love, support and encouragement.

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Abstract

In this thesis, we have introduced generalized statistics in which different states have different maximum occupation numbers. Bose-Einstein, Fermi-Dirac and Gentile statistics are its special cases. With the help of this statistics and the use of homogeneous form of Riemann-Hilbert problem we calculate an explicit expression for fugacity. Its calculation is a key point in the solution of our model. It is because finding discontinuity in its derivative will tell us whether there is a phase transition or not. Bose-Einstein condensation is the first purely statistically derived example of a phase transition. In this thesis, the transition is generalized Bose-Einstein condensation.

In the first chapter, a brief review of some basic concepts of statistical physics are explained. These include entropy of a system, thermodynamic potentials, Bose-Einstein and Fermi-Dirac statistics. In addition, some light is also shed on the concepts of Bose-Einstein condensation and phase transitions, especially to first order phase transition.

The second chapter is devoted to the detailed study of Gentile statistics, first introduced by Gentile in 1924. In this chapter, partition function is derived with the help of multinomial theorem. After calculation of grand partition function all thermodynamic properties are easily calculated. The properties at low temperature and high densities are also discussed with the contribution of ground state.

In the last chapter, the work is concluded with some other examples of statistics used besides that of Fermi-Dirac, Bose-Einstein and Gentile statistics.

Chapter 1

Introduction

In this dissertation, the phase transition model is solved by calculating an explicit expression for the fugacity for an idealized gas obeying generalized statistics. It is done with the help of solution of mathematical problem known as Riemann-Hilbert problem. Moreover, the thermodynamic properties of Gentile statistics are also calculated. Since, thermodynamics play an important role in the study of statistical mechanics, therefore, the basic concepts of thermodynamics are reviewed first in this chapter.

1.1 Statistical Mechanics

A condensed material includes a large number of particles composed of electrons and ions of the order of 10^{23}. On this very large scale, it is impossible to characterize all information of each individual particle. We know that in order to describe some physical quantity of a system, we need to know about quantities making the system. For example, we know that specific heat of different materials is different. It is different because microscopically, the materials are different. The macroscopic quantities in materials result from microscopic properties. As another example, pressure in a gas results from the collisions of molecules with the walls of the container. Similarly, temperature of a material depends upon the mean kinetic energy of particles.

The basic aim of statistical mechanics is to derive all the thermodynamic properties of a given system needed to describe it, by relating the microscopic properties of individual particles to macroscopic material. From dynamical point of view, a microstate can be defined completely by specifying all the dynamical variables of the system such as position and momentum. However, some physical quantities like temperature, pressure, energy, etc, are needed to describe the macrostate of the system because their properties do not depend on every little detail of each atom. Therefore, we can say that in statistical mechanics, we try to understand the properties of complex systems even if we do not know all the information about the system.

Statistical mechanics is often divided into classical and quantum statistical mechanics. Our work in this dissertation is based on quantum statistical mechanics, therefore, we will have only a brief introduction of classical statistical mechanics.

1.2 Classical Statistical Mechanics

Matter is composed of atoms, and atoms as we know must obey quantum statistics but still classical statistics can be reasonable and instructive in many cases. Therefore, we start with classical phase space. A state of a system is completely and uniquely defined by $3N$ canonical coordinates, $q_1, q_2, q_3, ...q_{3N}$ and $3N$ canonical momenta, $p_1, p_2, p_3, ...p_{3N}$. A total of $6N$ variables are collectively denoted by the abbreviation (p, q) and constitute phase space of the system also known as $6N$ dimensional $\Gamma-$space. Each phase point represents a microstate of the system in classical statistical mechanics and these states form a continuous set of points in phase space. The motion of a phase point in $\Gamma-$space is governed by following set of equations called canonical equations of motion or Hamilton's equations of motion:

$$\frac{\partial H(p,q)}{\partial p_i} = \dot{q_i} \qquad , \qquad \frac{\partial H(p,q)}{\partial q_i} = -\dot{p_i}, \tag{1.1}$$

where, $H\left(p(t), q(t)\right)$ is defined as the total energy of the system and is a function of phase space points (p, q) and time [1].

When the Hamiltonian does not depend explicitly on time, the trajectory which the phase point follows is called a phase orbit. The surface in $\Gamma-$space which satisfies the condition $H(p, q) = E$, defines an orbit with constant energy given by the law of energy conservation. Thus, the phase space trajectory always moves on a constant energy curve. The classical statistical mechanics is governed by Maxwell-Boltzmann statistics.

1.3 Entropy

Entropy (S) was first defined by Clausius in the form of equation $S = Q/T$. At that time, the idea that gas is made up of small molecules and temperature results from average kinetic energy had not yet appeared. In the equation, ($S = Q/T$) a system can have temperature only if it is in a thermodynamic equilibrium. Thus, we can define entropy only for those systems, which are in a thermodynamic equilibrium from classical thermodynamic point of view. Therefore, if temperature is constant, we can differentiate above equation to get another equation of entropy as

$$\Delta S = \frac{\Delta Q}{T}; \qquad \Delta S = S_2 - S_1 \ , \ \Delta Q = Q_2 - Q_1 \ , \tag{1.2}$$

where ΔS represents a small change in the entropy of a system 1 and 2. It is a well known fact that when heat enters a system, ΔQ is negative; which means entropy also decreases. On the other hand if ΔQ is positive, heat leaves the system and entropy increases. This is how S is defined in classical thermodynamic.

Entropy in statistical mechanics was defined by Ludwig Boltzmann and J. Willard Gibbs in the 1870's by analyzing the statistical behavior of microscopic components of the system, while in classical thermodynamics, single systems are dealt. Below is given the general form of entropy

$$S = -k \sum_i p_i \ln p_i \ , \tag{1.3}$$

where p_i is the probability of a microstate in which given particle is in state i, while all of p_i evaluated for the same microstate of the system. The constant k is called Boltzmann constant with value $1.380 \times 10^{-23} J/K$. Because p_i is always between 0 and 1 and log of numbers less than one is negative, therefore, both signs cancel with each other to make entropy positive.

Next we want to relate the definition of entropy with the information theory. This work was done by Claude Shannon in 1948. If a system has uniform probability distribution, it means that we have no information about the system, i.e., in which microstate our system is. Thus, we have least information about the system.

If W is the number of microstates of a system, then the probability of finding the system in a certain state is

$$p_i = p = \frac{1}{W} \tag{1.4}$$

Thus, the entropy from above is

$$S = -k \sum_{i=1}^{W} \frac{1}{W} \ln(\frac{1}{W}), \tag{1.5}$$

$$= -kW \left[\frac{1}{W} \ln(\frac{1}{W}) \right], \tag{1.6}$$

$$S = -k \ln(\frac{1}{W}) = k \ln W \ . \tag{1.7}$$

This equation states that the larger the number of states, the larger the entropy. Also, a) S is maximum when W is maximum b) S is minimum when W is minimum. Specifically, when $W = 1$, $S = 0$, i.e., no randomness at all [2].

This definition of entropy looks similar to the previous one, but this definition is for a system which obeys principle of equal apriori probability (it will be discussed in detail in the section No. 1.5). Eq. (1.3), is however, more general and applies to equilibrium and non equilibrium systems equally.

Entropy is an extensive quantity, because it depends on the amount of material in a system. For example, let us suppose that two systems, like a gas in a large container, is separated from the middle. Let their entropies be represented by S_1 and S_2 respectively. Then we remove that separation and allow the gas to mix. The combined system will have more entropy than the single one ($S_1 + S_2 = S$).

1.4 Thermodynamic Potentials

Thermodynamic potentials are very important in Thermodynamic systems. These include internal energy, Helmholtz free energy, enthalpy, and Gibbs free energy [2]. Only internal energy and Helmholtz free energy are discussed below.

1.4.1 Internal Energy

In a closed system for an infinitesimal process, first law of thermodynamics reads as

$$dU = \delta Q - \delta W, \tag{1.8}$$

here, U is called internal energy and is a function of S and V. δQ is a small amount of heat added in a system and δW is small amount of work done by the system. From second law of thermodynamics we have, TdS= δQ and $\delta W = pdV$, if pV is the work done. Then above equation reads as

$$\left(\frac{\partial U}{\partial S}\right)_V dS + \left(\frac{\partial U}{\partial V}\right)_S dV = TdS - pdV. \tag{1.9}$$

or

$$\left(\frac{\partial U}{\partial S}\right)_V = T, \tag{1.10}$$

and

$$\left(\frac{\partial U}{\partial V}\right)_S = -p. \tag{1.11}$$

1.4.2 Helmholtz Free Energy

It can be defined as the useful work that can be obtained from a closed thermodynamic system at constant temperature and volume. It is important in determining the equilibrium state of a system. The physical meaning of the free energy F is that in an isothermal transformation, the change of the free energy is the negative of the maximum possible work done by the system. Mathematically, it is written as

$$F = U - TS \ , \tag{1.12}$$

where F is a function of T and V. Then

$$dF = dU - TdS - SdT \ , \tag{1.13}$$

or

$$\left(\frac{\partial F}{\partial T}\right)_V dT + \left(\frac{\partial F}{\partial V}\right)_T dV = TdS - pdV - TdS - SdT \ . \tag{1.14}$$

By comparing both sides of Eq. (1.14), we get

$$\left(\frac{\partial F}{\partial T}\right)_V = -S \quad , \quad \left(\frac{\partial F}{\partial V}\right)_T = -p \; . \tag{1.15}$$

These two equations give another thermodynamic definition of entropy and pressure.

1.5 Ensembles

As already mentioned, for a gas of N particles, a microstate of a system is represented by a point in $6N$ dimensional phase space. Thus, if we have to consider all possible states which a gas can have, then we would have a large collection of points in the phase space. In this way, we have a large number of identical copies of a system but each copy may be in different possible microstate. These identical copies may be characterized by the same set of quantities such as pressure, volume, etc., and is called an ensemble of a system. A gas goes from one point to the next in phase space when it evolves in time, in a specific sequence. Any macroscopic quantity of the gas is not measured instantaneously. In fact, it is to be measured over a finite time which is usually long as compared to time scale of motion of particles of the gas. In this way, the quantity which we measure is a time-averaged quantity. When we are measuring time average of a quantity, what we are doing is looking at different values which that quantity takes during its time evolution from one microstate to the other. After that, we take an average of all the values of that quantity. For doing so, we are not concerned with the sequence of going from one state to the other, rather we can just consider the phase points of the ensemble over which the system goes and then take the average. We can then say that the time average can be replaced by the whole ensemble average [2].

By doing so, there are some conditions to be satisfied. First of all, replacing time average by the ensemble average, simply assumes that the system is visiting all the phase points during its time evolution. This is simply the ergodic hypothesis which states that: “over long periods of time, the time which a system spent in some particular region of phase space of microstates with the same energy is proportional to the phase volume of this region.” In other words, all states appear with equal probability. This hypothesis is considered as a pillar of statistical mechanics.

Not all surfaces are ergodic. When a system is ergodic, statistical physics applies to its ensemble. In case of non ergodic system, statistical physics also applies to the time average of the system. Three types of ensembles are used in statistical mechanics: micro-canonical, canonical and grand canonical ensemble.

1.5.1 Micro-canonical Ensemble

It is an ensemble formed by isolated systems which have the same energy. "Same energy" means that energy of all systems lie between certain region E and E+ΔE. The other restrictions are: the volume V and number of particles N are fixed. The probability for each accessible microstate in this ensemble is the same. Thus, if Ω is the number of microstates, then at any time the probability of finding the system at random from the ensemble would be $1/\Omega$. The total number of states available to the system is actually the measure of the entropy of the system

$$S = k \ln \Omega. \tag{1.16}$$

which is maximum under this condition. This formula of entropy is the same as given by Boltzmann, except that here Ω is used instead of W for the number of available states. Here $\Omega(N, V, E)$, is actually the partition function of this ensemble.

The micro-canonical ensemble (MCE) has many drawbacks in its use. At first, we cannot have a system where all microstates have the same energy because it is not always easy. Infact it would be easier if average values are calculated with a distribution function that include a range of energies. The MCE corresponds to a system which has its energy completely isolated from the rest of universe. This is usually not the way a thermodynamic system is prepared. Infact, we usually know about the temperature of a system, that is, we have information about the average energy of the system. In other words, we do not usually deal with completely isolated systems, but we often deal with systems kept in contact with a heat reservoir at a given temperature, so that its energy varies somewhat from instant to instant but its time average is known [2].

As we noted, MCE is a collection of similar systems, all with exactly the same energy. So, it is an ensemble which would be impossible to construct in general. Following is a more general ensemble which would be much easier to form.

1.5.2 Canonical Ensemble

Let us prepare an ensemble in which each system has fixed number of particles N, enclosed in a volume V [2]. Each system is placed in a heat reservoir (heat bath) with which it would be in equilibrium. Such a system has a distribution function which should satisfy the following conditions

$$S = -k_\beta \sum_i p_i \ln p_i, \tag{1.17}$$

subject to conditions: $\sum_i p_i = 1$ and $\sum_i p_i E_i = 1$. Here 'i' represents all possible states of the system. We try to solve p_i by use of Lagrange multipliers. We require

$$S + \alpha_a \sum_i p_i + \alpha_b \sum_i p_i E_i \ , \tag{1.18}$$

to be maximum and α_a and α_b are adjusted so that

$$\sum_i p_i = 1 \quad , \quad and \qquad \sum_i p_i E_i = 1 \ . \tag{1.19}$$

Taking the partial derivatives of Eq. (1.19) equation with respect to $p_i's$ and then setting them equal to zero, we get

$$-k_B \ln p_i - k_B + \alpha_a + \alpha_a E_i = 0 \ , \tag{1.20}$$

or

$$p_i = exp\left[\frac{\alpha_a - k_B + \alpha_b E_i}{k_B}\right] \ . \tag{1.21}$$

Taking the summation on both sides gives

$$\exp(\frac{\alpha_a}{k_B} - 1) \sum_i \exp(\frac{\alpha_b E_i}{k_B}) = 1 \ , \tag{1.22}$$

Using the notation

$$\exp(\frac{\alpha_b E_i}{k_B}) = \frac{1}{Z} \ , \tag{1.23}$$

we can write Eq.(1.22) as

$$Z = \exp(1 - \frac{\alpha_a}{k_B}) \ . \tag{1.24}$$

Next, we have to adjust Lagrange multiplier α_b such that U and S are related to T like in the formula $F = U - TS$. Re-arranging it, gives

$$S - \frac{U}{T} = -(\frac{F}{T}) \tag{1.25}$$

It can be done if we take average value of Eq.(1.20), then multiply it by p_i and sum over i. Thus we obtain

$$-k_B \sum_i p_i \ln p_i - (k_B - \alpha_a) \sum_i p_i + \alpha_b \sum_i p_i E_i = 0 \ , \tag{1.26}$$

or

$$S + \alpha_b U = k_B - \alpha_a = k_B \ln Z \ . \tag{1.27}$$

By comparing with Eq.(1.25), we find

$$\alpha_b = -(\frac{1}{T}) \qquad , \qquad k_B \ln Z = -(\frac{F}{T}) \ . \tag{1.28}$$

Thus, p_i becomes

$$p_i = (\frac{1}{Z}) \exp(\frac{-E_i}{k_B T}) \ , \tag{1.29}$$

or

$$Z = \sum_i \exp(\frac{-E_i}{k_B T}), \tag{1.30}$$

and is called the partition function of the canonical ensemble. Once partition function is found, it is easy to calculate all thermodynamics properties of a system, like Helmholtz free energy as $F = -k_B T \ln Z$. With the help of F, entropy and Gibbs potential can be calculated by the use of Maxwell's relations. Here Z is a function of T, V and N. The quantity N specifies the list of microstates used in the sum.

So, canonical ensemble is an ensemble formed when a system has fixed V and N and it has a well defined temperature (connected to an infinite heat bath). In most of the cases, the system consists of N sub-systems which are separate and independent having α_j degrees of freedom such that

$$\Phi = \sum_{j=1}^{N} \alpha_j \ . \tag{1.31}$$

Now, consider a system of distinguishable particles. In this case canonical partition function becomes

$$Z(N,V,T) = \sum_i e^{-E_i/k_BT} = \sum_{a,b,c=i} e^{-(\epsilon_a^\alpha+\epsilon_b^\beta+\epsilon_c^\gamma+\ldots)/k_BT}, \tag{1.32}$$

$$\sum_a e^{-\epsilon_a^\alpha}/k_BT \sum_b e^{-\epsilon_b^\beta}/k_BT \sum_c e^{-\epsilon_c^\gamma}/k_BT = z_\alpha z_\beta z_\gamma, \tag{1.33}$$

where $z(V,T) = \sum_a e^{-\epsilon_a^\alpha}/k_BT$ is the partition function of single particle and a is the number of particles in a state. If energy states of all particles are the same, then

$$Z(T,V,N) = [z(V,T)]^N \ . \tag{1.34}$$

This is the distribution function of distinguishable particles. For the case of indistinguishable particles we have

$$Z(T,V,N) = \frac{z^N}{N!} \ . \tag{1.35}$$

But it is only possible if quantum states at room temperature available to a particle are much greater than the number of particles in the system. In this case the chance of any two particles in the same state would be negligible.

1.5.3 Grand Canonical Ensemble

Dealing with the systems with a given temperature, a parameter that we can control in experiments, is easier than the systems where energy has to be kept constant, which is physically impossible. Therefore, we prefer canonical ensemble to micro-canonical ensemble. Yet it is a great problem as well to keep the number of particles fixed and it makes the situation difficult to handle. It is because the number of particles in a macroscopic system is never precisely known. A huge amount of particles combine to make a macroscopic system, then all we know is the average number of particles from experiments. This was the basic idea which motivated physicist to introduce grand canonical ensemble in which the system can have any number of particles with average number determined by the conditions external to the system. This may be considered analogous to the situation in CE, where the average energy of the system is determined by the temperature of heat reservoir with which it is in contact. A microstate of GCE is specified by the number of particles N that the system has.

Let us consider an open, isothermal system whose thermodynamic state is specified by V, T and μ as given in Ref.[1]. The distribution function that such a system must satisfy containing N molecules in the energy state $E_{N,i}$ is

$$p_{N,i}(V,T,\mu) = \frac{e^{-E_{N,i}/kT} e^{N\mu/kT}}{\Xi(V,T,\mu)} \ , \tag{1.36}$$

where $\Xi(V,T,\mu)$ is the grand partition function defined as

$$\Xi(V,T,\mu) = \sum_{N,i} e^{-E_{N,i/kT}} e^{N\mu}/kT \ . \tag{1.37}$$

An alternative form of Ξ is

$$\begin{aligned} \Xi(V,T,\mu) &= \sum_N \left[e^{N\mu/kT} \sum_i e^{-E_{N,i/kT}} \right] \ , &(1.38)\\ &= \sum_N Z(N,V,T) e^{N\mu/kT} \ . &(1.39) \end{aligned}$$

Here Z is the canonical partition function and $e^{\mu/kT}$ is called the fugacity. By using this partition function we can calculate all other thermodynamic properties of the system.

1.6 Quantum Statistics

In this Section, quantum statistics of one component macroscopic system of identical and indistinguishable particles without inter-particle interactions is discussed. It means that they do not exert forces on each other , in a volume V at temperature T. Let us denote energy eigenfunctions as $\psi_1, \psi_2, \psi_3, ...$ and let the energy eigenvalues of single particle are $, \epsilon_1, \epsilon_2, \epsilon_3.....$ In a system with two identical particles, it is impossible to detect an exchange. Let ψ be the wave function representing the state of a system of N identical and indistinguishable particles. Then we can easily deduce from above mentioned experimental fact that ψ may behave in one of two ways: when the coordinates of two particles are exchanged in the function ψ , it may remain unchanged or it may change sign. We say that ψ is symmetrical in the coordinates of identical particles in the former case and anti-symmetrical in the later case. One of the fundamental postulates of quantum mechanics is that (a) the only states available to real systems of indistinguishable particles are those represented by wave functions which are either symmetrical or antisymmetrical, and (b) particles with half integral spin (e.g., electrons, protons, etc) have antisymmetric wave functions, while those with integer spins (e.g., photons) have symmetrical wave functions. Nuclei, atoms, ions and molecules made up of an odd number of electrons, protons and neutrons (e.g., He^3) are in the antisymmetrical class, and those with an even numbers (e.g., H_2, He^4) are in the symmetrical class. The above restrictions on accessibility are applicable whether or not there are inter particle forces. In either way canonical partition function is the same, i.e.,

$$Z = \sum_j e^{-E_j/kT} \ , \tag{1.40}$$

where E_j are the eigenvalues of the operator H of the system. But because of the above restriction on the wave function, we will include only those energy eigenstates associated with energy eigenfunctions with the correct symmetry properties. In other words, the sum is over all accessible energy states; those states which are inaccessible because of symmetry restrictions are omitted from the sum [1].

One way to define Bose-Einstein statistics and Fermi-Dirac statistics is that wave function is symmetrical in BE case and antisymmetrical in FD case. As, an example let us consider a system with three particles, $N = 3$. Let us use canonical partition function

$$Z = \sum_i \Omega_i e^{-E_i/kT} \ . \tag{1.41}$$

In the above sum is over all energy levels instead of states. Now consider the system energy level is $E = 9\delta$ (δ is a constant). It means that 9 units of energy are to be divided among three particles. In this way the total number of ways that the particles can be divided are 28, if we ignore symmetry. That is, the degeneracy of system energy level $E = 9\delta$ is $\Omega_D = 28$ (D is for distinguishable). Thus, one term in above equation is

$$Z = 28e^{-9\delta/kT} \,. \tag{1.42}$$

For FD case, only three linear combinations out of seven permutations are allowed because of anti symmetric property of wave function. Therefore, in this example, there are only three antisymmetrical system energy states belonging to the eigenvalue E= 9δ; that is, $\Omega_{FD} = 3$. Thus one term will be

$$Z = 3e^{-9\delta/kT} \,. \tag{1.43}$$

From here one can make a general statement which is called Pauli exclusion principle (which forms the basis of periodic table). It states that each particle quantum state can be occupied by either one or zero particle at a time in Fermi Dirac statistics; *number can be* $0, 1$ *only.*

For Bose-Einstein (BE) case, one linear combination out of each seven different ways of arranging particles in the system are allowed because of symmetric property of wave function. Therefore, $\Omega_{BE} = 7$ and another term will be

$$Z = 7e^{-9\delta/kT} \,. \tag{1.44}$$

Here, Ω_{BE} is the total number of ways of dividing E among N particles, treating the particles as indistinguishable and without restricting the number of particle in any state.

From above example one can say that in BE case there is no restriction on the number of particles in a particle quantum state; the number can be $0, 1, 2, 3...N$. It is also obvious that

$$\Omega_{BE} \geq \frac{\Omega_D}{N!} \geq \Omega_{FD} \,. \tag{1.45}$$

If energy is very large (high temperature), the number of available states are large compared to the total number of particles in the system. Thus, the vast majority of sets of particle quantum numbers will all have particle quantum numbers different. So, for large values of E we have $\Omega_{BE} \longrightarrow \frac{\Omega_D}{N!} \longleftarrow \Omega_{FD}$. This limiting statistics is referred as Boltzman or classical statistics (particles are treated as distinguishable).

Next, thermodynamic properties of both statistics are calculated with the help of grand partition function because the number of particles in these systems is very large and cannot be easily fixed. For an N-particle state, the sum of all the n_p obeys

$$\sum_p n_p = N \ . \tag{1.46}$$

and the energy eigenvalue of this N-particle state is

$$E\{n_p\} = \sum_p n_p \epsilon_p \ . \tag{1.47}$$

Here, p denotes the momentum eigenvalue of the single particle. The grand partition function can now readily be calculated

$$\begin{aligned} \Xi(z, V, T) &= \sum_{N=0}^{\infty} z^N Q_N(V, T) = \sum_{N=0}^{\infty} \sum_{\substack{n_p \\ \Sigma n_p = N}} z^N e^{-\beta E_p} \ , & (1.48) \\ \Xi &= \sum_{N=0}^{\infty} \sum_{\substack{n_p \\ \Sigma n_p = N}} z^{\sum_p n_p} e^{-\beta \sum_p n_p \epsilon_p} \ , & (1.49) \\ \Xi &= \sum_{N=0}^{\infty} \sum_{\substack{n_p \\ \Sigma n_p = N}} \prod_p \left(z e^{-\beta \epsilon_p} \right)^{n_p} \ . & (1.50) \end{aligned}$$

Here, double summation is equivalent to summing each n_p independently. Every term in one case appears once and only once in the other, and vice versa. Therefore,[3]

$$\begin{aligned} \Xi &= \sum_{n_0} \sum_{n_1} ...[(ze^{-\beta\epsilon_0})^{n_0} (ze^{-\beta\epsilon_1})^{n_1} ...] \ , & (1.51) \\ &= \left[\sum_{n_0} (ze^{-\beta\epsilon_0})^{n_0} \right] \left[\sum_{n_1} (ze^{-\beta\epsilon_1})^{n_1} \right] \ , & (1.52) \\ &= \prod_p \left[\sum_n (ze^{-\beta\epsilon_p})^{n_p} \right] \ , & (1.53) \end{aligned}$$

where the Σ_n extends over the values $n = 0, 1, 2, 3, ...$ for bosons and $n = 0, 1$ for fermions. Expanding for fermions Eq. (1.53) becomes

$$\Xi = \prod_p (1 + ze^{-\beta\epsilon_p}) \ , \tag{1.54}$$

where first term is for $n = 0$ and second term for $n = 1$. The equation of state can be found by taking natural log of the partition function as

$$\frac{PV}{kT} = \ln \Xi(z, V, T) = \sum_p \ln(1 + ze^{-\beta\epsilon_p}) \ . \tag{1.55}$$

For average number of particles

$$N = z \ln \frac{\partial}{\partial z} \ln \Xi(z, V, T) \ , \tag{1.56}$$

or

$$N = \sum_p \frac{ze^{-\beta\epsilon_p}}{1 + ze^{-\beta\epsilon_p}} = \sum_p n_p \ , \tag{1.57}$$

where

$$n_p = \frac{ze^{-\beta\epsilon_p}}{1 + ze^{-\beta\epsilon_p}} \ . \tag{1.58}$$

When $\exp(\frac{\mu}{kT}) = z$ and $\beta = \frac{1}{kT}$, Eq. (1.58) can also be written as

$$n_p = \frac{1}{e^{\frac{(\epsilon_p - \mu)}{kT}} + 1} \ . \tag{1.59}$$

This is Fermi-Dirac distribution law.

For bosons Eq. (1.53) gives

$$\Xi = \frac{1}{1 - ze^{-\beta\epsilon_p}} \ , \tag{1.60}$$

and the equation of state will be

$$\frac{PV}{kT} = -\sum_p \ln(1 - ze^{-\beta\epsilon_p}) \ . \tag{1.61}$$

For mean number of particles

$$N = \sum_p \frac{ze^{-\beta\epsilon_p}}{1 - ze^{-\beta\epsilon_p}} = \sum_p n_p \ , \tag{1.62}$$

where

$$n_p = \frac{ze^{-\beta\epsilon_p}}{1 - ze^{-\beta\epsilon_p}} \ . \tag{1.63}$$

The fugacity z is non negative for both ideal Fermi gas and ideal Bose gas because, if it was negative then Eqs. (1.57) and (1.62) cannot be satisfied for positive N. Eq. (1.63) can also be written as

$$n_p = \frac{1}{e^{\frac{(\epsilon_p - \mu)}{kT}} - 1}. \tag{1.64}$$

This is Bose-Einstein distribution law [3]. The condition for convergence of this series requires $\mu < \epsilon_p$ for all p. Now we may associate with each of the discrete p values a volume element of size $V = 2\Pi\hbar/L^3$ because momentum quantum numbers can take on the following values

$$p = \frac{2\Pi\hbar}{L}(v_1, v_2, v_3), \qquad v_\alpha = 0, \ \pm 1, ... \tag{1.65}$$

Hence, in the limit of $V \longrightarrow \infty$ sums over p can be replaced by integral.

$$\sum_p = g\frac{V}{(2\Pi\hbar)^3}\int d^3p \tag{1.66}$$

where g is the degeneracy factor given by

$$g = 2s + 1, \tag{1.67}$$

as a result of spin independence of the single particle energy ϵ_p.

1.6.1 Ideal Fermi Gas

From Eq. (1.57), average number of particles [4] become

$$N = g\frac{V}{(2\Pi\hbar)^3}\int d^3pn(\epsilon_p) \ = \ g\frac{V}{(2\Pi\hbar)^3}\int_0^\infty d^3pn(\epsilon_p) \ , \tag{1.68}$$

$$= \ \frac{gVm^{\frac{3}{2}}}{\sqrt{2}\Pi^2\hbar^3}\int_0^\infty \frac{\sqrt{\epsilon}}{e^{\beta(\epsilon-\mu)-1}}d\epsilon \ , \tag{1.69}$$

where we have put $\epsilon = \frac{p^2}{2m}$ and $v = V/N$ (specific volume). Now substituting $x = \beta\epsilon$. Thus, above equation reduces to

$$\begin{aligned} \frac{1}{v} &= \frac{1}{\lambda^3}\frac{2g}{\sqrt{\Pi}}\int_0^\infty \frac{\sqrt{x}}{e^x z^{-1} - 1}dx \ , && (1.70) \\ \frac{1}{v} &= \frac{g}{\lambda^3} f_{\frac{3}{2}}(z) \ . && (1.71) \end{aligned}$$

where λ is thermal wavelength. Here we have made use of Fermi Dirac integral which is

$$f_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}}{e^x z^{-1} - 1}dx \ . \quad (1.72)$$

Equation of state comes out to be

$$\frac{P}{kT} = \frac{g}{\lambda^3} f_{\frac{5}{2}}(z) \ . \quad (1.73)$$

1.6.2 Ideal Bose Gas

When z approaches 1, we must be more careful in taking the limit of $\sum_p \longrightarrow \int dp^3$ for the ideal Bose gas as the summand in Eqs. (1.61) and (1.62) diverges as $z \to 1$. Therefore, we split off the term $p = 0$ in these equations and replace the remaining term by integral. Also, using Bose-Einstein integral, [4] which is

$$g_\nu(z) = \frac{1}{\Gamma(\nu)}\int_0^\infty \frac{x^{\nu-1}}{e^x z^{-1} + 1}dx \ . \quad (1.74)$$

Eqs. (1.61) and (1.62) can also be written as

$$\frac{P}{kT} = \frac{g}{\lambda^3} g_{\frac{5}{2}}(z) - \frac{1}{V}\ln(1 - z) \ , \quad (1.75)$$

and

$$\frac{1}{v} = \frac{g}{\lambda^3} g_{\frac{3}{2}}(z) + \frac{1}{V}\frac{z}{1 - z}, \quad (1.76)$$

where the last term corresponds to $p = 0$. The $p = 0$ term in fermions does not need any special treatment, since the average occupation number can have the value 1 at most.

1.7 Bose-Einstein Condensation

Eqs. (1.75) and (1.76) give the equation of state and specific volume of an ideal Bose gas containing N particles each having mass m contained in a volume V. In order to study the properties of ideal Bose gas in more detail we must find fugacity z as a function of temperature and specific volume by solving Eq. (1.76). For this purpose we focus our attention on the properties of non relativistic ideal Bose gas at low temperature with spin 0, ($g = 1$). The dispersion relation used will be $\epsilon_p = \frac{p^2}{2m}$. In doing so we will have to study the properties of Bose function $g_{\frac{3}{2}}(z)$ in detail, [3] which is a special case of more general functions

$$g_n(z) = \sum_{l=1}^{\infty} \frac{z^l}{l^n} \ . \tag{1.77}$$

It is obvious that the fugacity of bosons is limited to $z \leq 1$, that is, the radius of convergence of above equation is given by

$$0 \leq z \leq 1 \ . \tag{1.78}$$

For fermions fugacity is limited to $0 \leq z \leq \infty$. The value of $g_{3/2}(z)$ at $z = 1$ is calculated as follows

$$g_{\frac{3}{2}}(z) = z + \frac{z^2}{2^{\frac{3}{2}}} + \frac{z^3}{3^{\frac{3}{2}}} + \ldots \ , \tag{1.79}$$

$$g_{\frac{3}{2}}(1) = \sum_{l=1}^{\infty} \frac{1}{l^{\frac{3}{2}}} = \zeta(\frac{3}{2}) = 2.612, \tag{1.80}$$

where $\zeta(x)$ is Riemann zeta function.Thus, for all z between 0 and 1,

$$g_{\frac{3}{2}}(z) \leq 2.612 \ , \tag{1.81}$$

or

$$\frac{\lambda^3}{v} > 2.612 \ . \tag{1.82}$$

We have to find z as a function of T, or more expediently, of $\frac{v}{\lambda^3}$. When temperature decreases $\frac{v}{\lambda^3}$ decreases and thus z increases, until at $\frac{v}{\lambda^3} = \frac{1}{2.612}$, it reaches its maximum value $z = 1$. For a given specific volume, this defines a characteristic temperature given by

$$\lambda_c^3 = v g_{\frac{3}{2}}(1) \ , \tag{1.83}$$

$$kT_c = \frac{\frac{2\Pi\hbar^2}{m}}{(2.612v)^{\frac{2}{3}}} \ . \tag{1.84}$$

At critical temperature the thermal wavelength is of the same order of magnitude as of inter-particle separation [5]. As stated earlier, the occupation number n_p satisfies the condition $\sum_p n_p = N$, which now in the case of bosons become

$$N = \frac{1}{z^{-1}-1} + \sum_{p\neq 0} n(\epsilon_p) = \frac{1}{z^{-1}-1} + \frac{V}{(2\Pi\hbar)^3}\int dp^3 n(\epsilon_p) \ . \tag{1.85}$$

Thus, for bosons we get

$$N = \frac{1}{z^{-1}-1} + N\frac{v}{\lambda^3} g_{\frac{3}{2}}(z) \ , \tag{1.86}$$

or using Eq. (1.84)

$$N = \frac{1}{z^{-1}-1} + N\left(\frac{T}{T_c}\right)^{\frac{3}{2}} \frac{g_{\frac{3}{2}}(z)}{g_{\frac{3}{2}}(1)} \ . \tag{1.87}$$

The total number of particles N is then the sum of two terms: number of particles in the ground state

$$N_0 = \frac{1}{z^{-1}-1} \ , \tag{1.88}$$

and the number of particles in the excited state

$$N_1 = N\left(\frac{T}{T_c}\right)^{\frac{3}{2}} \frac{g_{\frac{3}{2}}(z)}{g_{\frac{3}{2}}(1)} \ . \tag{1.89}$$

In the limit when $T >> T_c$, Eq. (1.87) gives value for $z < 1$. The first term on the right hand side of Eq. (1.87) becomes finite and therefore can be neglected as compared to N (we can say that it can always be neglected for $p \neq 0$). The particles spread thinly over all the levels. Then z follows from

$$g_{\frac{3}{2}}(z) = 2.612\left(\frac{T}{T_c}\right)^{\frac{3}{2}} \ . \tag{1.90}$$

In the limit when $T << T_c$ all of the particles which are no longer in the excited state can occupy the single level with $p = 0$ or equivalently ground state. When z is so close to one, we can put it equal to one in the second term of Eq. (1.87) and find value of N_0, which comes out to be

$$N_0 = N\left(1 - \left(\frac{T}{T_c}\right)^{\frac{3}{2}}\right) \ . \tag{1.91}$$

Let us define v_0 as condensate fraction in the thermodynamic limit as

$$v_0 = \lim_{N\to\infty}\left(\frac{N_0}{N}\right) \ . \tag{1.92}$$

Summarizing the above results we get

$$v_0 = \begin{cases} 0 \ , & T >> T_c \\ 1 - \left(\frac{T}{T_c}\right)^{\frac{3}{2}} , & T << T_c \end{cases}$$

Below T_c the ground state is macroscopically occupied [5]. Between the high temperature phase, where the bosons are distributed over the whole spectrum of momentum values, corresponding to Bose distribution function, and the phase in which $p = 0$ term is macroscopically occupied, a phase transition takes place. This is called Bose Einstein condensation of an ideal Bose gas. It was first generally predicted by Satyendra Nath Bose and Albert Einstein in 1925. In 1938 Fritz London described BEC as a mechanism for superfluidity in He^4 and superconductivity. But gaseous condensates show very different behavior from that in liquid helium. For example, more than 99 percent of alkali atoms form condensate at $T = 0$, while for liquid helium the fraction is only 10 percent.

The first gaseous condensate was produced by Eric Cornell and Carl Wiemann, by using gas of rubidium atoms cooled to 170nk. For this achievements they were awarded nobel prize in Physics in 2001.

1.8 Phase Transitions

1.8.1 Definition

The different chemical substances within a system are called its components. In the case of a single chemical substance, in contrast, one refers to a single component system or a pure system. These components of a system can occur in different physical forms or structures, which are termed as phases. Within a thermodynamic system and states of matter, a phase has same physical properties.

When a substance changes phase from one to another, phase transition takes place. This happens when there is a change in external parameters such as temperature, pressure, volume, entropy, etc, of a system. The value at which phase transition takes place is called a transition point. This process is very important in various natural processes and in industry as well. Examples include, liquid evaporation, formation of ice on sheets. In metallurgy various materials are constructed using this phenomenon.

1.8.2 Classification of Phase Transitions

The thermodynamic potentials which are used to describe phase transitions are Helmholtz free energy $H(T,V,N)$, Gibbs free energy $G(T,P,N)$, and chemical potential $\mu(P,T,V)$ defined as [2]

$$\begin{aligned} F &= U - TS, && (1.93)\\ G &= H - TS, && (1.94)\\ \mu &= E - TS + PV. && (1.95) \end{aligned}$$

Phase transitions occur when these potentials show singularities/non-analytic behavior. The type of that singularity determines, which order of transition would take place. Two types of transitions are encountered in nature: one is called first order phase transition and the other is higher order phase transition usually second order phase transitions.

When first order derivative of free energy is discontinuous with respect to some thermodynamic variable, then it is classified as “first order phase transition.”

When first order derivative of free energy is continuous with respect to some thermodynamic variable, but second order derivative is discontinuous, then it will be a “second order phase transition.”

The thermodynamic potential which is most commonly used to describe phase transitions among three potentials is Gibbs potential because it depends on two intensive variables T and P. These two variables are the same for the two phases that co-exist at a first order phase transition line. It is also defined as, $g = \mu N$. It is important because another condition for equilibrium besides T and P are constant is, that μ in the first and second phase must be equal.

1.8.3 Order Parameter

In modern classification scheme, transitions are described by the behavior of some central quantity called “order parameter”. This is a quantity which has value 0 in one phase by definition and assumes finite values in the other. In most cases we have natural choice for the order parameter. For example, in gas liquid transition a natural choice may be the difference of densities of gas and liquid, while it may be magnetization in paramagnet- ferromagnetic transition. Here is then another way of defining first and second order transitions. If the change in the order parameter is finite in phase transition, then we are dealing with former one and if it tends to zero value or departs from it in a continuous fashion then we are dealing with later ones. So this is accepted that there are "discontinuous" first order transitions with jumps in their physical properties as well as continuous second order transitions without jumps.

1.8.4 Clausius Clapeyron Equation

Suppose that in a one component system two phases co-exist with the given T and $P(T)$, for example gas-liquid mixture. This pressure $P(T)$ is called vapor pressure at the temperature T. Then The Gibbs potential of this state must be at minimum. Thus, if any of the parameters other than T and P are changed, we must have $\delta G = 0$. Let μ_1 be the chemical potential of the liquid in state 1, and μ_2 be the chemical potential of the gas in state 2. The total Gibbs potential of this gas-liquid mixture [3] can then be written as

$$G = N_1\mu_1 + N_2\mu_2 \ , \tag{1.96}$$

where N_i is the number of particles in phase i, where $(i = 1, 2)$, and $N = N_1 + N_2$ is fixed. Thus, condition for equilibrium is

$$\mu_1 = \mu_2. \tag{1.97}$$

Now, chemical potentials $\mu_1(T,P)$ and $\mu_2(T,P)$ are are two state functions of liquid and gas respectively. In each phase we have

$$\left(\frac{\partial G}{\partial T}\right)_p = -S \ , \qquad \text{(entropy per unit mass)}$$

$$\left(\frac{\partial G}{\partial p}\right)_T = V \ . \qquad \text{(volume per unit mass)}$$

From these equations it is clear that first order derivatives of μ are different, which is necessary condition of first order phase transition.

$$-\left[\frac{\partial(\mu_2-\mu_1)}{\partial T}\right]_p = S_2 - S_1 \ , \tag{1.98}$$

$$\left[\frac{\partial(\mu_2-\mu_1)}{\partial p}\right]_T = V_2 - V_1 \ . \tag{1.99}$$

Both entropy and volume are discontinuous here. Here ΔS gives the amount of heat L that is transferred from the system to the environment as $L = \delta Q = T\Delta s$ and is called the latent heat. From this we arrive at Clausius-Clapeyron equation

$$\left(\frac{dP}{dT}\right) = \frac{S_2 - S_1}{V_2 - V_1} = \frac{\Delta S}{\Delta V} \ , \tag{1.100}$$

or

$$\left(\frac{dP}{dT}\right) = \frac{L}{T\Delta V} \ . \tag{1.101}$$

It is also possible that $\Delta S = 0$ and $\Delta V = 0$ in some cases. If this is the case then the first order derivatives of μ are continuous, while second order are not, in which case second order transitions take place. Ehrenfest defines a phase transition to be of nth order [3] if at the transition point,

$$\frac{\partial^n \mu_1}{\partial T^n} \neq \frac{\partial^n \mu_2}{\partial T^n} \qquad \text{and} \qquad \frac{\partial^n \mu_1}{\partial p^n} \neq \frac{\partial^n \mu_2}{\partial p^n} \ .$$

whereas all lower derivatives are equal. But many examples of transitions cannot be explained by the same procedure. For example, in many cases specific heat diverges at transition point, like in λ transition in ferromagnets. Since specific heat is is related to second order of μ, these examples cannot be characterized by the behavior of μ, because they do not exist.

Our task in the next chapters is to solve phase transition model. But before this a detailed discussion is done on Gentile statistics in the next chapter. First of all, grand partition function is calculated and then the thermodynamic behavior of an ideal gas obeying Gentile statistics is discussed with the contribution of the ground state.

Chapter 2

Gentile Statistics with Large Maximum Occupation Number

2.1 Introduction

We know that in Bose-Einstein statistics, maximum occupation of particles in a state is infinite and in Fermi-Dirac statistics, a state can accommodate either 0 or 1 particle. There is another kind of statistics called Gentile statistics in which maximum occupation number of particles can take on any values between $1 < n < \infty$, i.e., neither 1 nor ∞ but a finite number n and it can reduce to BE or FD statistics depending on the value of n [8].

If N is the total number of particles in a system, then it is usually believed that when maximum occupation number approaches the total number of particles then Gentile statistics reduces to Bose-Einstein statistics. This is indeed true only when $z < 1$. However when $z > 1$ this statement is no more valid. Here z is called fugacity. In this dissertation we have discussed the case when $n \to N$ and $z > 1$, we will see that the Bose Einstein case cannot be recovered from Gentile statistics.

In literature, when thermodynamic quantities of Gentile statistics are discussed, the contribution from ground state is usually ignored. We know that we can ignore such contribution in a fermionic system but in bosons we cannot do so, *especially at low temperatures and high densities.* In this dissertation, we will calculate thermodynamic properties with dispersion relation $E = \frac{p^s}{2m}$ in a ν dimensional space. Here ν and s are arbitrary integers. Our result shows that for the case $\nu > s$ the contribution of the ground state depends sensitively on maximum occupation number n.

In section 2, we first calculate an expression for partition function in Gentile statistics. Thermodynamic properties such as internal energy, entropy, specific heat etc will be discussed with contribution from ground state and at low temperatures and high densities in sections 3 and 4 of this dissertation respectively.

2.2 The Partition Function and the Case of $n \longrightarrow N$ with $z > 1$ in Gentile Statistics

We first calculate partition function for Gentile statistics by the use of multinomial theorem. Starting with counting the number of states we express the grand partition function as follows:

$$\Xi = \prod_l \Xi_l = \prod_l \sum_{a_l} \Omega_l e^{-(\alpha+\beta)a_l} \ , \tag{2.1}$$

where

$$\Xi_l = \sum_{a_l} \Omega_l e^{-(\alpha+\beta)a_l} \ , \tag{2.2}$$

and Ω_l is the number of quantum states given by

$$\Omega_l = \sum_{\alpha_0} .... \sum_{\alpha_n} \frac{\omega_l!}{\prod_{i=0}^{n} \alpha_i!} \ . \tag{2.3}$$

Here $i = 0, 1, 2....n$ and α_i denotes the number of states which contain i particles. The set of α_i must satisfy the following condition

$$\sum_i \alpha_i = \omega_l \quad , \qquad \sum_i \alpha_i i = a_l \ . \tag{2.4}$$

Here we use the concept of Multinomial theorem, which states that, "for any positive integer x and non-negative y, the multinomial formula tells us how a sum with x terms expands when raised to an arbitrary power y

$$(m_1 + m_2 + m_3.....m_x)^y = \sum_{k_1+k_2+k_3..k_x=y} \binom{y}{k_1, k_2, k_3...k_x} \prod_{1\le t\le x} m_t^{kt} \ , \tag{2.5}$$

where

$$\binom{y}{k_1, k_2, k_3...k_y} = \frac{y!}{k_1!k_2!k_3!...k_x!}, \tag{2.6}$$

is a multinomial coefficient".

In our case we can say that the coefficient Ω of $m^a(0 \leq a \leq y\omega)$ in the expansion of $(1+x+x^2+x^3....+x^n)^\omega$ is

$$\Omega = \sum_{\alpha_0} .... \sum_{\alpha_n} \frac{\omega!}{\prod_{i=0}^{n} \alpha_i!}. \tag{2.7}$$

where, the set of α_i must satisfy

$$\sum_{i=0}^{n} \alpha_i = \omega \ , \quad \sum_{i=0}^{n} \alpha_i i = a. \tag{2.8}$$

and the summation runs over all possible states. By using the above definitions we can say that Ω_l is the coefficient of the term $e^{-(\alpha+\beta\varepsilon_l)a_l}$ such that

$$e^{-(\alpha+\beta\varepsilon_l)a_l} = \left[1 + e^{-(\alpha+\beta\varepsilon_l)a_l} + e^{-2(\alpha+\beta\varepsilon_l)a_l} + ... + e^{-n(\alpha+\beta\varepsilon_l)a_l}\right]^{\omega_l}. \tag{2.9}$$

It is a finite geometric series with $a = 1$ and $r = e^{-(\alpha+\beta\varepsilon_l)a_l}$. So by making use of formula

$$\sum_{k=0}^{n} ar^k = \frac{a(r^{n+1}-1)}{r-1}, \tag{2.10}$$

$$\Xi_l = \left[\frac{e^{-(n+1)(\alpha+\beta\varepsilon_l)}}{e^{-(\alpha+\beta\varepsilon_l)a_l} - 1}\right]^{\omega_l}. \tag{2.11}$$

Taking $e^{-\alpha} = z$, where z is the fugacity, the above equation becomes

$$\Xi_l = \left[\frac{z^{n+1}e^{-(n+1)\beta\varepsilon_l} - 1}{ze^{-\beta\varepsilon_l} - 1}\right]^{\omega_l}. \tag{2.12}$$

Putting value of Ξ_l in Eq.(2.1), we get grand partition function of Gentile statistics as

$$\Xi = \prod_l \left[\frac{z^{n+1}e^{-(n+1)\beta\varepsilon_l} - 1}{ze^{-\beta\varepsilon_l} - 1}\right]^{\omega_l}. \tag{2.13}$$

Here it is obvious that when $z < 1$, Ξ_l diverges at $n = \infty$ and only in this case Gentile statistics is reduced to Bose case. This equation also holds for $z > 1$ and $n >> 1$ and in section 4, we will see that the behavior of this statistics is very different from that of BE statistics.

2.3 Thermodynamics with the Contribution of Ground State

Taking log on both sides of Eq. (2.13)

$$\ln \Xi = \sum_l \ln \Xi_l = \sum_l \omega_l \left[\ln(z^{n+1} e^{-(n+1)\beta\varepsilon_l} - 1) - \ln(z e^{-\beta\varepsilon_l} - 1) \right]. \tag{2.14}$$

We will evaluate the above by replacing the summation by corresponding integral

$$\sum_l \rightarrow \frac{2gV\Pi^{\frac{\nu}{2}}}{h^\nu \Gamma(\frac{\nu}{2}) s} \int (2m)^{(\frac{\nu}{s})} \epsilon^{(\frac{\nu}{s}-1)} d\epsilon, \tag{2.15}$$

where we have used the dispersion relation $E = \frac{p^s}{2m}$ in ν dimensional space, ν and s are arbitrary integers. We know that thermal wavelength is defined as $\lambda = (\frac{h^s}{2\Pi^{\frac{s}{2}} mkT})^{\frac{1}{s}}$. Simplifying λ gives

$$\begin{aligned} \frac{1}{h} &= \frac{1}{(2mkT)^{\frac{1}{s}} \Pi^{\frac{1}{2}} \lambda}, && (2.16) \\ \frac{2m^{\frac{\nu}{s}} \Pi^{\frac{\nu}{2}}}{h^\nu} &= \frac{1}{\lambda^\nu (kT)^{\frac{\nu}{s}}}. && (2.17) \end{aligned}$$

So the above summation becomes

$$\sum_l \longrightarrow \frac{2gV}{\Gamma(\frac{\nu}{2})} \frac{1}{\lambda^\nu (kT)^{\frac{\nu}{s}} s} \int \epsilon^{(\frac{\nu}{s}-1)} (d\epsilon). \tag{2.18}$$

Further taking $\epsilon = kT\xi$ so that $d\epsilon = kTd\xi$, Eq. (2.18) becomes

$$\begin{aligned} \sum_l &= \frac{2gV}{\Gamma(\frac{\nu}{2})} \frac{1}{\lambda^\nu s} \int \frac{(kT\xi)^{\frac{\nu}{s}-1} (kT) d\xi}{(kT)^{\frac{\nu}{s}}}, && (2.19) \\ &= \frac{2gV}{\Gamma(\frac{\nu}{2})} \frac{1}{\lambda^\nu s} \int \xi^{(\frac{\nu}{s}-1)} d\xi. && (2.20) \end{aligned}$$

Rearranging and using the above in Eq. (2.14), we find

$$\sum_l \ln \Xi_l = \frac{2gV}{\Gamma(\frac{\nu}{2}) \lambda^\nu s} \int_0^\infty \left[\ln(z^{n+1} e^{-(n+1)\xi} - 1) - \ln(z e^{-\xi} - 1) \right] \xi^{\frac{\nu}{s}-1} d\xi. \tag{2.21}$$

integrating first term by parts gives

$$\begin{aligned}\int_0^\infty \left[\ln(z^{n+1}e^{-(n+1)\xi}-1)\right]\xi^{\frac{\nu}{s}-1}d\xi &= \left[\frac{\xi^{\frac{\nu}{s}}}{\frac{\nu}{s}}\ln(z^{n+1}e^{-(n+1)\xi}-1)\right]_0^\infty \\ &- \frac{s}{\nu}\int_0^\infty \xi^{\frac{\nu}{s}}\frac{-(n+1)z^{n+1}e^{-(n+1)\xi}}{z^{n+1}.e^{-(n+1)\xi}-1}d\xi \ . \end{aligned} \tag{2.22}$$

By putting limits, first term will give zero. So we are left with

$$\int_0^\infty \left[\ln(z^{n+1}e^{-(n+1)\xi}-1)\right]\xi^{\frac{\nu}{s}-1}d\xi = \frac{(n+1)s}{\nu}\int_0^\infty \frac{\xi^{\frac{\nu}{s}}d\xi}{1-z^{-(n+1)}e^{(n+1)\xi}} \ . \tag{2.23}$$

Similarly considering the second integral

$$\int_0^\infty \left[\ln(ze^{-\xi}-1)\right]\xi^{\frac{\nu}{s}-1}d\xi = \int_0^\infty \ln(ze^{-\xi}-1)\xi^{\frac{\nu}{s}-1}d\xi \ , \tag{2.24}$$

$$= \left[\frac{\xi^{\frac{\nu}{s}}}{\frac{\nu}{s}}\ln(ze^{-\xi}-1)\right]_0^\infty - \frac{s}{\nu}\int_0^\infty \xi^{\frac{\nu}{s}}.\frac{-ze^{-\xi}d\xi}{ze^{-\xi}-1} \ , \tag{2.25}$$

$$= \frac{s}{\nu}\int_0^\infty \xi^{\frac{\nu}{s}}.\frac{1}{1-z^{-1}e^{\xi}}d\xi \ . \tag{2.26}$$

Using these both results in Eq. (2.21) we get

$$\begin{aligned}\sum_l \ln \Xi_l &= \frac{2gV}{\Gamma(\frac{\nu}{2})\lambda^\nu s}\Big[\{\frac{(n+1)s}{\nu}\int_0^\infty \frac{\xi^{\frac{\nu}{s}}d\xi}{1-z^{-(n+1)}e^{(n+1)\xi}}\} \\ &- \{\frac{s}{\nu}\int_0^\infty \xi^{\frac{\nu}{s}}.\frac{1}{1-z^{-1}e^{\xi}}d\xi\}\Big], \end{aligned} \tag{2.27}$$

$$= \frac{2gV}{\Gamma(\frac{\nu}{2})\lambda^\nu \nu}\int_0^\infty \left[\frac{1}{z^{-1}e^{\xi}-1} - \frac{n+1}{z^{-(n+1)}e^{(n+1)\xi}-1}\right]\xi^{\frac{\nu}{s}}d\xi. \tag{2.28}$$

For contribution of ground state of the system put $\epsilon = 0$ in Eq. (2.13)

$$\sum_l \ln \Xi_l = \ln\left(\frac{z^{n+1}-1}{z-1}\right) = \ln\frac{1-z^{(n+1)}}{1-z}. \tag{2.29}$$

Finally we get

$$\sum_l \ln \Xi_l = \frac{2gV}{\Gamma(\frac{\nu}{2})\lambda^\nu \nu} \int_0^\infty \left[\frac{1}{z^{-1}e^{\xi} - 1} - \frac{n+1}{z^{-(n+1)}e^{(n+1)\xi} - 1} \right] \xi^{\frac{\nu}{s}} d\xi + \ln \frac{1 - z^{(n+1)}}{1 - z}, \tag{2.30}$$

where g is a weight factor that arises from the internal structure of the particles. Here the last term represents the contribution of the ground state of the system. The number of particles which are accommodated in a state decide the contribution of that state to the system. When temperature limit of particles become low, they tend to occupy lowest possible state. Therefore, at $T \longrightarrow 0$, particle number of ground state nearly equals the maximum occupation number of the state n. For fermions, contribution from ground state can be ignored because $n = 1$, but for bosons we cannot ignore it because $n = \infty$ in the low temperature limit. When $\nu > s$, the term $\epsilon = 0$ term diverges in Bose case and causes phase transition. However in gentile statistics n is not infinite so $\epsilon = 0$ term does not diverge and no such phase transition occurs. When n is very large, the influence of such a term becomes very important. In the following we will show this influence on the thermodynamic quantities.

We come across two type of integrals in the literature. They are Bose-Einstein integrals and Fermi-Dirac integrals and are represented as

$$g_\sigma(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \frac{x^{\sigma-1}}{z^{-1}e^x - 1} dx, \tag{2.31}$$

and

$$f_\sigma(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \frac{x^{\sigma-1}}{z^{-1}e^x + 1}. \tag{2.32}$$

Similarly, we can introduce such an integral for Gentile statistics as

$$h_\sigma(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty \left[\frac{1}{z^{-1}e^x - 1} - \frac{n+1}{z^{-(n+1)}e^{(n+1)x} - 1} \right] x^{\sigma-1} dx. \tag{2.33}$$

When $n = 1$ or ∞ this integral will reduce to Bose-Einstein or Fermi-Dirac integral respectively. Here we use a basic relationship between $h_\sigma(z)$ and $h_{\sigma-1}(z)$, which is

$$h_{\sigma-1}(z) = z\frac{\partial}{\partial z} h_\sigma(z). \tag{2.34}$$

Its proof is as follows:

Integrating Eq. (2.33) by parts w.r.t x, gives

$$h_{\sigma}(z)=\frac{1}{\Gamma(\sigma+1)}\int_0^{\infty}x^{\sigma}\left[\frac{z^{-1}e^{x}}{(z^{-1}e^{x}-1)^2}+\frac{(n+1)^2z^{-(n+1)}e^{(n+1)x}}{(z^{-(n+1)}e^{(n+1)x}-1)^2}\right]dx. \tag{2.35}$$

On replacing σ by $\sigma-1$, we have

$$h_{\sigma-1}(z)=\frac{1}{\Gamma(\sigma)}\int_0^{\infty}x^{\sigma-1}\left[\frac{z^{-1}e^{x}}{(z^{-1}e^{x}-1)^2}+\frac{(n+1)^2z^{-(n+1)}e^{(n+1)x}}{(z^{-(n+1)}e^{(n+1)x}-1)^2}\right]dx. \tag{2.36}$$

Again calculating $\frac{\partial}{\partial z}h_{\sigma}(z)$, we get

$$\frac{\partial}{\partial z}h_{\sigma}(z)=\frac{1}{\Gamma(\sigma)}\int_0^{\infty}x^{\sigma-1}\left[\frac{z^{-2}e^{x}}{(z^{-1}e^{x}-1)^2}+\frac{(n+1)^2z^{-(n+1)}e^{(n+1)x}}{(z^{-n-2}e^{(n+1)x}-1)^2}\right]dx, \tag{2.37}$$

$$z\frac{\partial}{\partial z}h_{\sigma}(z)=\frac{1}{\Gamma(\sigma)}\int_0^{\infty}x^{\sigma-1}\left[\frac{z^{-1}e^{x}}{(z^{-1}e^{x}-1)^2}+\frac{(n+1)^2z^{-(n+1)}e^{(n+1)x}}{(z^{-(n+1)}e^{(n+1)x}-1)^2}\right]dx. \tag{2.38}$$

Hence Eq. (2.34) is proved and we will use it later.

Thus the grand partition function in Eq. (2.30) can be expressed in terms of $h_{\frac{\nu}{s}}(z)$. Infact

$$\begin{aligned}
h_{(\frac{\nu}{s}+1)}(z) &= \frac{1}{\Gamma(\frac{\nu}{s}+1)}\int_0^{\infty}\left[\frac{1}{z^{-1}e^{x}-1}-\frac{n+1}{z^{-(n+1)}e^{-(n+1)x}}\right]x^{\frac{\nu}{s}}dx, && (2.39)\\
h_{(\frac{\nu}{s}+1)}(z)\Gamma(\frac{\nu}{s}+1) &= \int_0^{\infty}\left[\frac{1}{z^{-1}e^{x}-1}-\frac{n+1}{z^{-(n+1)}e^{-(n+1)x}}\right]x^{\frac{\nu}{s}}dx, && (2.40)\\
\ln\Xi=\sum_l \ln\Xi_l &= \frac{2gV}{\Gamma(\frac{\nu}{2})\lambda^{\nu}\nu}h_{(\frac{\nu}{s}+1)}(z)\Gamma(\frac{\nu}{s}+1)+\ln\frac{z^{(n+1)}-1}{z-1}. && (2.41)
\end{aligned}$$

When $n \longrightarrow \infty$ and $z < 1$, the last term of above equation which describes contribution from ground state , returns to Bose case i.e, $-\ln(1-z)$; for fermions it can be ignored as $n = 1$. In Gentile statistics $1 < n < \infty$, contribution of ground state depends on value of n. When n is close to 1, contribution from ground state can be ignored like in Fermi case and when n approaches total number of particles it becomes very important. Using this partition function we can find all thermodynamic properties like internal energy, entropy, heat capacity etc. First of all we calculate equation of state,

$$\begin{aligned} \frac{P}{kT} &= \frac{1}{V}\ln\Xi, && (2.42)\\ &= \frac{2g}{\lambda^{\nu}}\left(\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}h_{\frac{\nu}{s}+1}(z)\right)+\frac{1}{V}\left(\ln\frac{z^{n+1}-1}{z-1}\right). && (2.43)\end{aligned}$$

For specific volume, we require

$$\frac{1}{v}=\frac{N}{V}=\frac{1}{V}z\frac{\partial}{\partial z}\ln\Xi. \tag{2.44}$$

Now

$$\begin{aligned} z\frac{\partial}{\partial z}\ln\frac{z^{(n+1)-1}}{z-1} &= z\left[\frac{(n+1)z^{n}}{z^{n+1}-1}-\frac{1}{z-1}\right], && (2.45)\\ &= \frac{(n+1)z^{n+1}}{z^{n+1}-1}-\frac{z}{z-1}, && (2.46)\\ &= \frac{(n+1)(z^{n+1}-1)+(n+1)}{z^{n+1}-1}-\frac{z+1-1}{z-1}, && (2.47)\\ n-\left(\frac{1}{z-1}-\frac{n+1}{z^{n+1}-1}\right) &= N_0. && (2.48)\end{aligned}$$

N_0 is the average occupation number of the ground state. When $n=1$ it reduces to $N_0=\frac{z}{1+z}$, for the case of fermions and for $n=\infty$ we have $N_0=\frac{z}{1-z}$, exactly as for the case of bosons.

$$\frac{N}{V}=\frac{2g}{\lambda^{\nu}}\left(\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}h_{\frac{\nu}{s}}(z)\right)+\frac{N_0}{V} \tag{2.49}$$

The internal energy of ideal gas is calculated as,

$$\frac{U}{N}=-\frac{\partial}{\partial\beta}\ln\Xi, \tag{2.50}$$

$$\begin{aligned} \frac{U}{N} &= -\frac{\frac{(2m\Pi)^{\frac{\nu}{s}}}{h^{\nu}}\cdot\frac{\nu}{s}(kT)^{\frac{\nu}{s}-1}\Gamma(\frac{\nu}{s}+1)h_{\frac{\nu}{s}+1}(z)\frac{2gV}{\nu\Gamma\frac{\nu}{2}}}{\frac{2gh^{\nu}}{(2m\Pi)^{\frac{\nu}{s}}(kT)^{\frac{\nu}{s}}\nu\Gamma\frac{\nu}{2}}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}h_{\frac{\nu}{s}}(z)+N_o}, \\ &= \frac{\frac{\nu}{s}(kT)\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)}}{1+\frac{N_0}{N-N_0}}, \\ &= \frac{\nu}{s}(kT)\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)}\left(\frac{N-N_0}{N}\right), \\ \frac{U}{N} &= \left(1-\frac{N_0}{N}\right)\frac{\nu}{s}(kT)\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)}, && (2.51)\end{aligned}$$

where we have used value of λ because it also contains temperature. For Helmholtz free energy of the gas, we get,

$$\begin{aligned}
F &= N\mu - PV, && (2.52)\\
&= NkT\ln z - kT\frac{2g}{\lambda^{\nu}}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma\frac{\nu}{2}}h_{\frac{\nu}{s}+1}(z) + \ln\frac{z^{n+1}-1}{z-1}, &&\\
&= NkT\left[\ln z - \frac{2g}{\lambda^{\nu}}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma\frac{\nu}{2}}\frac{h_{\frac{\nu}{s}+1}(z)}{\frac{2g}{\lambda^{\nu}}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}h_{\frac{\nu}{s}}(z)\left(1+\frac{N_0}{N-N0}\right)}\right] &&\\
&+ \frac{1}{N}\ln\frac{z^{n+1}-1}{z-1}, &&\\
F &= NkT\left[\ln z - \left(1-\frac{N_0}{N}\right)\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)} + \frac{1}{N}\ln\frac{z^{n+1}-1}{z-1}\right]. && (2.53)
\end{aligned}$$

The entropy of the gas is given by

$$\begin{aligned}
S &= \frac{U-F}{T}, &&\\
&= \frac{NkT}{T}\Big[\left(1-\frac{N_0}{N}\right)\frac{\nu}{s}\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)} - \ln z + \frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)}\left(1-\frac{N_0}{N}\right) &&\\
&+ \frac{1}{N}\ln\frac{z^{n+1}-1}{z-1}\Big], &&\\
S &= Nk\left[\left(1-\frac{N_0}{N}\right)\left(\frac{\nu}{s}+1\right)\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)} - \ln z + \frac{1}{N}\ln\frac{z^{n+1}-1}{z-1}\right]. && (2.54)
\end{aligned}$$

For specific heat capacity of the gas we start with Eq. (2.51), which is

$$U = (N-N_0)\frac{\nu}{s}(kT)\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)}. \tag{2.55}$$

By putting value of $(N-N_0)$ from Eq. (2.49) it gives

$$U = \frac{2g}{\lambda^{\nu}}\frac{kT}{s}\frac{\Gamma(\frac{\nu}{s}+1)}{\Gamma(\frac{\nu}{2})}h_{\frac{\nu}{s}+1}(z). \tag{2.56}$$

Now T dependence is in T/λ^{ν} and in $h_{\frac{\nu}{s}+1}(z)$.

$$\begin{aligned}\frac{T}{\lambda^\nu} &= T\left(\frac{2\Pi^{\frac{\nu}{2}}mkT}{h^s}\right)^{\frac{\nu}{s}} = const.\left(T^{\frac{\nu}{s}+1}\right), && (2.57)\\ \frac{d}{dT}\left(\frac{T}{\lambda^\nu}\right) &= \frac{\frac{\nu}{s}+1}{\lambda^\nu}. && (2.58)\end{aligned}$$

Then from Eq. (2.55)

$$\begin{aligned}\frac{\partial U}{\partial T} &= \frac{\nu}{s}Nk\left(1-\frac{N_0}{N}\right)\left(\frac{\nu}{s}+1\right)\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)},\\ \frac{\partial}{\partial T}h_{\frac{\nu}{s}+1}(z) &= \frac{\partial}{\partial z}\left(h_{\frac{\nu}{s}+1}(z)\right)\frac{\partial z}{\partial T},\\ &= \frac{1}{z}\left[z\frac{\partial}{\partial z}\left(h_{\frac{\nu}{s}+1}(z)\right)\frac{\partial z}{\partial T}\right],\\ &= \frac{1}{z}\left(h_{\frac{\nu}{s}}(z)\right)\frac{\partial z}{\partial T}. && (2.59)\end{aligned}$$

Putting in R.H.S of Eq. (2.55) we find

$$U = \frac{\nu}{s}Nk\left(1-\frac{N_0}{N}\right)\frac{T}{z}\frac{\partial z}{\partial T}\ . \qquad (2.60)$$

Now N does not depend upon $'T'$. So

$$\begin{aligned}\frac{dN}{dT} &= \frac{\partial N}{\partial z}.\frac{\partial z}{\partial T}+\frac{\partial N}{\partial T} = 0, && (2.61)\\ \frac{\partial N}{\partial z}.\frac{\partial z}{\partial T} &= \left[\frac{2g}{\lambda^\nu}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}.\frac{1}{z}h_{\frac{\nu}{s}-1}(z)+\frac{1}{(z-1)^2}-\frac{(n+1)^2z^n}{(z^{n+1}-1)^2}\right]\frac{\partial z}{\partial T}, && (2.62)\\ \frac{\partial N}{\partial T} &= \frac{2g}{\lambda^\nu}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}.\frac{1}{z}h_{\frac{\nu}{s}}(z),\left(\frac{\nu}{s}\right).\frac{1}{T}, && (2.63)\\ &\frac{2g}{\lambda^\nu}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}.\frac{1}{z}h_{\frac{\nu}{s}}(z) = N-N_0. && (2.64)\end{aligned}$$

So Eq. (2.61) becomes

$$\begin{aligned}
[(N-N_0)\frac{1}{z}\frac{h_{\frac{\nu}{s}-1}(z)}{h_{\frac{\nu}{s}}(z)} &+ \frac{1}{(z-1)^2}-\frac{(n+1)^2 z^n}{(z^{n+1}-1)^2}]\frac{\partial z}{\partial T}+\frac{1}{T}(N-N_0)\frac{\nu}{s}=0\ , && (2.65)\\
\frac{\partial z}{\partial T} &= \frac{-\frac{1}{T}(N-N_0)\frac{\nu}{s}}{(N-N_0)\frac{1}{z}\frac{h_{\frac{\nu}{s}-1}(z)}{h_{\frac{\nu}{s}}(z)}+\frac{1}{(z-1)^2}-\frac{(n+1)^2 z^n}{(z^{n+1}-1)^2}}\ , && (2.66)\\
&= \frac{\frac{-1}{T}\frac{\nu}{s}}{\frac{h_{\frac{\nu}{s}-1}(z)}{h_{\frac{\nu}{s}(z)}z}\left[1+\frac{z}{N-N_0}\frac{h_{\frac{\nu}{s}}(z)}{h_{\frac{\nu}{s}-1}(z)}\left(\frac{1}{(z-1)^2}-\frac{(n+1)^2 z^n}{(z^{n+1}-1)^2}\right)\right]}\ , && (2.67)\\
&= \frac{z}{T}\frac{-\nu}{s}\frac{h_{\frac{\nu}{s}}(z)}{h_{\frac{\nu}{s}-1}}(z)\\
&\quad \times\left[1+\frac{1}{N-N_0}\frac{h_{\frac{\nu}{s}}(z)}{h_{\frac{\nu}{s}-1}(z)}\left(\frac{z}{(z-1)^2}-\frac{(n+1)^2 z^{n+1}}{(z^{n+1}-1)^2}\right)\right]^{-1} && (2.68)
\end{aligned}$$

Combining Eq. (2.59) and Eq. (2.68) gives the value of specific heat of the gas. In all of these equations, the factor $\left(1-\frac{N_0}{N}\right)$ appears due to the influence of ground state. Rearranging Eq. (2.49) we have

$$\begin{aligned}
\frac{\lambda^\nu}{v} &= \frac{2g}{\lambda^\nu}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}h_{\frac{\nu}{s}}(z)\left[1+\frac{N_0}{N-N_0}\right]\ , && (2.69)\\
&= \frac{2g}{\lambda^\nu}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}h_{\frac{\nu}{s}}(z)\left[\frac{1}{1-\frac{N_0}{N}}\right]\ . && (2.70)
\end{aligned}$$

Here we are dealing with $h_{\frac{\nu}{s}}(z)$. So we have to determine the range of the fugacity. For this purpose we rewrite the integrand of Eq. (2.39) as

$$I=\frac{1}{\xi-1}-\frac{n+1}{\xi^{n+1}-1}, \qquad (2.71)$$

where $\xi=z^{-1}e^x$. Now writing $\xi^{n+1}=(\xi-1)(\xi^n+\xi^{n-1}+.......+1)$ in above equation, gives

$$\begin{aligned}
I &= \frac{(\xi^n+\xi^{n-1}+.......+1)-(n+1)}{(\xi-1)(\xi^n+\xi^{n-1}+.......+1)}\ , && (2.72)\\
&= \frac{\xi^n+\xi^{n-1}+.......-n}{(\xi-1)(\xi^n+\xi^{n-1}+.......+1)}\ , && (2.73)\\
&= \frac{\xi^{n-1}+2\xi^{n-2}+3\xi^{n-3}.......+(n-1)\xi+n}{\xi^n+\xi^{n-1}+.......+1}\ . && (2.74)
\end{aligned}$$

From last equation, it can be seen that for any real values of z, $h_{\frac{\nu}{s}}(z)$ is a bounded, positive function of z. It is also obvious that if we expect $h_{\frac{\nu}{s}}(z)$ to return to the BE integral, the necessary condition $z < 1$ is needed so that $h_{\frac{\nu}{s}}(z)$ is bounded.

2.4 Low Temperatures and High Densities

In this section, we will discuss the behavior of the statistics for the case $z >> 1$. In this case the results will not return to Bose case even when $n \longrightarrow N$. To obtain Bose-Einstein statistics from Gentile statistics, one needs not only to perform the limit $n \longrightarrow N$ but also restrict the fugacity $z \leq 1$ To study the behavior of $h_\sigma(z)$ in Eq. (2.33) for large z, we introduce a variable $t = \ln z$. Evaluating first integral of $h_\sigma(z)$ in the powers of t we get

$$\begin{aligned} I(t) &= \int_0^\infty \frac{\xi_1^{\frac{\nu}{s}}}{z^{-1}e^{\xi_1} - 1} d\xi_1, & (2.75) \\ I(t) &= \int_0^\infty \frac{\xi_1^{\frac{\nu}{s}}}{e^{\xi_1 - t} - 1} d\xi_1, & (2.76) \\ &= \left[\frac{\xi_1^{\frac{\nu}{s}+1}}{(e^{\xi_1 - t} - 1)(\frac{\nu}{s}+1)} \right]_0^\infty - \int_0^\infty \frac{\xi_1^{\frac{\nu}{s}+1}}{(\frac{\nu}{s}+1)} \frac{e^{\xi_1 - t}}{(e^{\xi_1 - t} - 1)^2} d\xi. & (2.77) \end{aligned}$$

By putting limits first term gives zero

$$I(t) = -\int_0^\infty \frac{\xi_1^{\frac{\nu}{s}+1}}{(\frac{\nu}{s}+1)} \frac{e^{\xi_1 - t}}{(e^{\xi_1 - t} - 1)^2} d\xi_1 \ . \tag{2.78}$$

Using another variable $\xi_1 - t = \xi$

$$I(t) = -\int_{-t}^\infty \frac{(\xi + t)^{\frac{\nu}{s}+1}}{(\frac{\nu}{s}+1)} \frac{e^{\xi}}{(e^{\xi} - 1)^2} d\xi \ . \tag{2.79}$$

We expand $(\xi + t)^{\frac{\nu}{s}+1}$ in power equation. So it will give

$$I(t) = \frac{-1}{\frac{\nu}{s}+1} \sum_{j=0,1,2,3,\ldots} \binom{\frac{\nu}{s}+1}{j} t^{\frac{\nu}{s}+1-j} \int_{-t}^\infty \frac{\xi^j e^{\xi}}{(e^{\xi} - 1)^2} d\xi \ . \tag{2.80}$$

Let us first consider $j = 0$ term in I(t), which is

$$
\begin{aligned}
I(t) &\simeq \frac{-1}{\frac{\nu}{s}+1} t^{\frac{\nu}{s}+1} \int_{-\infty}^{\infty} \frac{e^{\xi}}{(e^{\xi}-1)^2} d\xi \ , && (2.81)\\
&\simeq \frac{-t^{\frac{\nu}{s}+1}}{\frac{\nu}{s}+1} \left[\frac{-1}{e^{\xi}-1}\right]_{-t}^{\infty}, && (2.82)\\
I(t) &\simeq \frac{t^{\frac{\nu}{s}+1}}{\frac{\nu}{s}+1}. && (2.83)
\end{aligned}
$$

Then

$$
I(t) = \frac{-t^{\frac{\nu}{s}+1}}{(\frac{\nu}{s}+1)^2} \sum_{j=0,1,2,3,\ldots} \binom{\frac{\nu}{s}+1}{j} t^{\frac{\nu}{s}+1-j} \int_{-t}^{\infty} \frac{\xi^j e^{\xi}}{(e^{\xi}-1)^2} d\xi \ . \qquad (2.84)
$$

We take $t \to \infty$ as an approximation. Further since $\frac{e^{\xi}}{(e^{\xi}-1)^2}$ is symmetric under $\xi \to -\xi$. So we need here only $j = 2, 4, 6, \ldots..$ Therefore above equation becomes

$$
I(t) = \frac{-t^{\frac{\nu}{s}+1}}{(\frac{\nu}{s}+1)^2} + 2 \sum_{j=2,4,6,\ldots} \binom{\frac{\nu}{s}+1}{j} t^{\frac{\nu}{s}+1-j} \int_{0}^{\infty} \frac{\xi^j e^{\xi}}{(e^{\xi}-1)^2} d\xi \ . \qquad (2.85)
$$

where

$$
\frac{e^{\xi}}{(e^{\xi}-1)^2} = \frac{1}{e^{\xi}-1} + \frac{1}{(e^{\xi}-1)^2}
$$

. By using formulae

$$
\begin{aligned}
\int_0^{\infty} \frac{x^{\nu-1}}{e^{\mu x}+1} dx &= \frac{1}{\mu^{\nu}} (1-2^{1-\mu}) \Gamma(\nu) \zeta(\nu) \ , \qquad [Re\ \mu > 0\ , Re\ \nu > 0] && (2.86)\\
\int_0^{\infty} \frac{x^{\nu-1}}{(e^{x}-1)^2} dx &= \Gamma(\nu) \left[\zeta(\nu-1) - \zeta(\nu)\right] \ , \qquad [Re\ \nu > 2] && (2.87)
\end{aligned}
$$

we get

$$
\begin{aligned}
I(t) &= \frac{-t^{\frac{\nu}{s}+1}}{(\frac{\nu}{s}+1)^2} + 2 \sum_{j=2,4,6,\ldots} \binom{\frac{\nu}{s}+1}{j} t^{\frac{\nu}{s}+1-j} \Big[\Gamma(j+1)\zeta(j+1) \\
&+ \zeta(j) - \zeta(j+1)\Big], && (2.88)\\
I(t) &= \frac{-t^{\frac{\nu}{s}+1}}{(\frac{\nu}{s}+1)^2} + 2 \sum_{j=2,4,6,\ldots} \binom{\frac{\nu}{s}+1}{j} t^{\frac{\nu}{s}+1-j} \Big[\Gamma(j+1)\zeta(j)\Big]. && (2.89)
\end{aligned}
$$

Replacing j by $j+1$

$$I(t) = \frac{-t^{\frac{\nu}{s}+1}}{(\frac{\nu}{s}+1)^2} + 2\sum_{j=1,3,5,\ldots} \binom{\frac{\nu}{s}+1}{j+1} t^{\frac{\nu}{s}-j} \Big[\Gamma(j+2)\zeta(j+1)\Big] \ . \tag{2.90}$$

Now $\Gamma(j+2) = (j+1)\Gamma(j+1)$. Thus

$$I(t) = \frac{-t^{\frac{\nu}{s}+1}}{(\frac{\nu}{s}+1)^2} + 2\sum_{j=1,3,5,\ldots} \left(\frac{(\frac{\nu}{s}+1)!}{(j+1)!(\frac{\nu}{s}-j)}\right) t^{\frac{\nu}{s}-j} \Big[(j+1)\Gamma(j+1)\zeta(j+1)\Big] \ , \tag{2.91}$$

$$I(t) = \frac{-t^{\frac{\nu}{s}+1}}{\frac{\nu}{s}+1} + 2\sum_{j=1,3,5,\ldots} \left(\frac{(\frac{\nu}{s})!}{j!(\frac{\nu}{s}-j)}\right) t^{\frac{\nu}{s}-j} \Big[\Gamma(j+1)\zeta(j+1)\Big] \ . \tag{2.92}$$

Finally we get

$$I(t) = \frac{-t^{\frac{\nu}{s}+1}}{\frac{\nu}{s}+1} + 2\sum_{j=1,3,5,\ldots} \binom{\frac{\nu}{s}}{j} t^{\frac{\nu}{s}-j} \Big[\Gamma(j+1)\zeta(j+1)\Big] \ . \tag{2.93}$$

In all of the above, $\zeta(j+1)$ is the Riemann Zeta function. Following similar steps, the second integral of Eq. (2.33) comes out to be

$$I^{'} = \int_0^\infty \frac{(n+1)\xi_1^{\frac{\nu}{s}}}{z^{-(n+1)}e^{(n+1)\xi_1} - 1} d\xi_1. \tag{2.94}$$

By putting $t = \ln z$, it becomes

$$I^{'} = \int_0^\infty \frac{(n+1)\xi_1^{\frac{\nu}{s}}}{e^{(n+1)(\xi_1 - t)} - 1} d\xi_1, \tag{2.95}$$

Thus, its solution gives following result

$$I^{'} = -(n+1)\frac{t^{\frac{\nu}{s}+1}}{\frac{\nu}{s}+1} + 2\sum_{j=1,3,5,\ldots} \binom{\frac{\nu}{s}}{j} t^{\frac{\nu}{s}-j} \frac{1}{(n+1)^j} \Big[\Gamma(j+1)\zeta(j+1)\Big] \ . \tag{2.96}$$

So integral of Eq. (2.33) becomes

$$
\begin{aligned}
h_\sigma(z) \quad &= \quad \frac{1}{\Gamma(\sigma)}\Big[\frac{-t^{\frac{\nu}{s}+1}}{\frac{\nu}{s}+1} + 2\sum_{j=1,3,5,\dots}\binom{\frac{\nu}{s}}{j} t^{\frac{\nu}{s}-j}\Gamma(j+1)\zeta(j+1) + \frac{t^{\frac{\nu}{s}+1}}{\frac{\nu}{s}+1} + \frac{nt^{\frac{\nu}{s}+1}}{\frac{\nu}{s}+1} \\
&- \quad 2\sum_{j=1,3,5,\dots}(\genfrac{}{}{0pt}{}{\frac{\nu}{s}}{j})t^{\frac{\nu}{s}-j}\frac{1}{(n+1)^j}\Gamma(j+1)\zeta(j+1)\Big] \ . \qquad (2.97)
\end{aligned}
$$

For $j = 1$

$$
h_\sigma(z) = \frac{1}{\sigma\Gamma(\sigma)}nt^\sigma\left[1 + \frac{\sigma}{nt^\sigma}2(\frac{\nu}{s})t^{\frac{\nu}{s}-1}\Gamma(2)\zeta(2) - 2\frac{\sigma}{nt^\sigma(n+1)}(\frac{\nu}{s})t^{\frac{\nu}{s}-1}\Gamma(2)\zeta(2)\right] \ , \qquad (2.98)
$$

$$
= \frac{1}{\Gamma(\sigma+1)}nt^\sigma\left[1 + 2\sigma(\sigma-1)t^{-2}\Gamma(2)\zeta(2)\frac{1}{n}\left(1 - \frac{1}{n+1}\right)\right] \ . \qquad (2.99)
$$

where $\Gamma(2) = 1$. We know that

$$
\zeta(p) = \sum_{k=1}^{\infty}\frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots\dots \qquad Re(p) > 1 \ , \qquad (2.100)
$$

where

$$
\sum_{k=1}^{\infty}\frac{1}{k^{2n}} = \frac{2^{2n-1}.\Pi^{2n}|B_{2n}|}{2n!} \ . \qquad (2.101)
$$

When $n = 1$ we have

$$
\zeta(2) = \sum_{k=1}^{\infty}\frac{1}{k^{2n}} = \frac{2\Pi^2}{2!}B_2, \qquad (2.102)
$$

where B is nth Bernoulli's number and $B_2 = \frac{1}{6}$ and

$$
\zeta(2) = \frac{\Pi^2}{6}. \qquad (2.103)
$$

So $h_\sigma(z)$ becomes

$$
h_\sigma(z) = \frac{1}{\Gamma(\sigma+1)}nt^\sigma\left[1 + \sigma(\sigma-1)t^{-2}2\left(\frac{\Pi^2}{6}\right).\frac{1}{n}\left(\frac{n}{n+1}\right)\right] \ , \qquad (2.104)
$$

$$
= \frac{1}{\Gamma(\sigma+1)}nt^\sigma\left[1 + \frac{\Pi^2}{3}\sigma(\sigma-1)t^{-2}.\frac{1}{n+1}\right]. \qquad (2.105)
$$

For $j = 3$,

$$h_\sigma(z)=\frac{1}{\Gamma(\sigma+1)}nt^\sigma\left[\frac{\sigma}{nt^\sigma}2\binom{\frac{\nu}{s}}{3}t^{\frac{\nu}{s}-3}\Gamma(4)\zeta(4)-\frac{1}{(n+1)^3}\frac{\sigma}{nt^\sigma}2\binom{\frac{\nu}{s}}{3}t^{\frac{\nu}{s}-3}\Gamma(4)\zeta(4)\right]. \tag{2.106}$$

$$\begin{aligned}\Gamma(4) &= 6\ ,\\ \zeta(4) &= \sum_{k=1}^{\infty}\frac{1}{k^4}=\frac{2^3\Pi^4}{4!}|B_4|.\end{aligned}$$

Here $|B_4|=-\frac{1}{30}$. So we get

$$\zeta(4)=-\frac{\Pi^4}{90}. \tag{2.107}$$

and

$$\binom{\frac{\nu}{s}}{3}=(\frac{\nu}{s})(\frac{\nu}{s}-1)(\frac{\nu}{s}-2)/3!=\frac{(\sigma-1)(\sigma-2)(\sigma-3)}{3!}, \tag{2.108}$$

$$h_\sigma(z)=\frac{1}{\Gamma(\sigma+1)}nt^\sigma\sigma(\sigma-1)(\sigma-2)(\sigma-3)t^{-4}\frac{6\Pi^4}{90n}\left[1-\frac{1}{(n+1)^3}\right]. \tag{2.109}$$

Combining Eqs. (2.105) and (2.109) we get

$$\begin{aligned}h_\sigma(z) &= \frac{1}{\Gamma(\sigma+1)}nt^\sigma\Big[1+\frac{\Pi^2}{3}\sigma(\sigma-1)t^{-2}.\frac{1}{n+1}+\sigma(\sigma-1)(\sigma-2)\\ &\quad(\sigma-3)t^{-4}\frac{6\Pi^4}{45n}\{1-\frac{1}{(n+1)^3}\}\Big].\end{aligned} \tag{2.110}$$

By putting in Eq. (2.49), it gives

$$\begin{aligned}\frac{N}{V} &= \frac{2g}{\lambda^\nu}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}\frac{1}{\Gamma(\frac{\nu}{s}+1)}t^{\frac{\nu}{s}}n\Big[1+\frac{\Pi^2}{3}\frac{1}{(n+1)}\frac{\nu}{s}(\frac{\nu}{s}-1)t^{-2}\\ &+ \frac{\Pi^4}{45n}\left(1-\frac{1}{(n+1)^3}\right)\left(\frac{\nu}{s}\right)\left(\frac{\nu}{s}-1\right)\left(\frac{\nu}{s}-2\right)\left(\frac{\nu}{s}-3\right)t^{-4}+......\Big]\ ,\end{aligned} \tag{2.111}$$

$$\begin{aligned} \frac{N}{V} &= \frac{2g}{\lambda^{\nu}}\frac{1}{\nu\Gamma(\frac{\nu}{2})}t^{\frac{\nu}{s}}n\Big[1+\frac{\Pi^2}{3}\frac{1}{(n+1)}\frac{\nu}{s}(\frac{\nu}{s}-1)t^{-2} \\ &+ \frac{\Pi^4}{45n}\left(1-\frac{1}{(n+1)^3}\right)\left(\frac{\nu}{s}\right)\left(\frac{\nu}{s}-1\right)\left(\frac{\nu}{s}-2\right)\left(\frac{\nu}{s}-3\right)t^{-4}+......\Big]. \end{aligned} \tag{2.112}$$

For other thermodynamic properties at low temperatures and high densities we calculate expansion of chemical potential. When z is very is large, then we have, $N_0 = n$. Thus, Eq. (2.49) becomes

$$\frac{N-N_0}{V} = \frac{N-n}{V} = \frac{2g}{\lambda^{\nu}}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}h_{\frac{\nu}{s}}(z). \tag{2.113}$$

From Eq. (2.110), taking only first term

$$\Gamma(\frac{\nu}{s}+1)h_{\frac{\nu}{s}}(z) = t^{\frac{\nu}{s}}.n\left[1+\frac{\Pi^2}{3}\frac{1}{(n+1)}\frac{\nu}{s}(\frac{\nu}{s}-1)t^{-2}\right] , \tag{2.114}$$

$$\frac{N-n}{V} = t^{\frac{\nu}{s}}.n\frac{2g}{\lambda^{\nu}}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}\left[1+\frac{\Pi^2}{3}\frac{1}{(n+1)}\frac{\nu}{s}(\frac{\nu}{s}-1)t^{-2}\right]. \tag{2.115}$$

For simplicity first of all consider only

$$\frac{N-n}{V} = \frac{t^{\frac{\nu}{s}}}{\lambda^{\nu}}, \tag{2.116}$$

where $t = \ln z$ and $\mu = kT\ln z$ or $t = \frac{\mu}{kT}$. By putting these values in Eq. (2.116) we get

$$\frac{N-n}{V} = \frac{2g}{h^{\nu}}\frac{\Pi^{\frac{\nu}{2}}(2mkT)^{\frac{\nu}{s}}}{\Gamma(\frac{\nu}{2}+1)}\left(\frac{\mu}{kT}\right)^{\frac{\nu}{s}}.n , \tag{2.117}$$

$$\mu = \left[\left(\frac{N-n}{nV}\right)\frac{h^{\nu}}{g}\frac{\Gamma(\frac{\nu}{2}+1)}{\Pi^{\frac{\nu}{2}}(2m)^{\frac{\nu}{s}}}\right]^{\frac{s}{\nu}} , \tag{2.118}$$

$$= \left(\frac{N-n}{nN}\right)^{\frac{s}{\nu}}\frac{h^s}{2m}\left[\left(\frac{N}{V}\right)\frac{\Gamma(\frac{\nu}{2}+1)}{g\Pi^{\frac{\nu}{2}}}\right]^{\frac{s}{\nu}} , \tag{2.119}$$

$$= \left(\frac{N-n}{nN}\right)^{\frac{s}{\nu}}\frac{\hbar^s}{2m}\frac{2^s\Pi^s}{\Pi^{\frac{s}{2}}}\left[\left(\frac{N}{V}\right)\frac{\Gamma(\frac{\nu}{2}+1)}{g}\right]^{\frac{s}{\nu}} , \tag{2.120}$$

$$= \left(\frac{N-n}{nN}\right)^{\frac{s}{\nu}}\frac{\hbar^s}{2m}\left[\left(\frac{N}{V}\right)\frac{2^{\nu}\Pi^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2}+1)}{g}\right]^{\frac{s}{\nu}} . \tag{2.121}$$

We make an approximation $\mu = \epsilon_f$, where ϵ_f is the chemical potential at absolute zero and is called Fermi energy. So

$$\epsilon_f = \epsilon_f^{fermion} \left(\frac{N-n}{Nn} \right)^{\frac{s}{\nu}} \quad , \tag{2.122}$$

where

$$\epsilon_f^{fermion} = \frac{h^s}{2m} \left[\left(\frac{N}{V} \right) \frac{2^\nu \Pi^{\frac{\nu}{2}} \Gamma(\frac{\nu}{2}+1)}{g} \right]^{\frac{s}{\nu}} \quad . \tag{2.123}$$

In other words, ϵ_f can be regarded as an analogue of the Fermi energy in Gentile statistics and it will return to $\epsilon_f^{fermion}$ when $n = 1$. Putting $\epsilon_f^{fermion}$, along with other terms in Eq. (2.115) gives the value of chemical potential as

$$\left(\frac{N-n}{V} \right) = \frac{g\Pi^{\frac{\nu}{2}}}{h^\nu} \frac{n(2m)^{\frac{\nu}{s}}}{\Gamma(\frac{\nu}{2}+1)} \mu^{\frac{\nu}{s}} \left[1 + \frac{\Pi^2}{3} \frac{1}{(n+1)} \frac{\nu}{s} \left(\frac{\nu}{s} - 1 \right) \left(\frac{kT}{\mu} \right)^2 .... \right] \quad , \tag{2.124}$$

$$\epsilon_f^{\frac{\nu}{s}} = \mu^{\frac{\nu}{s}} \left[1 + \frac{\Pi^2}{3} \frac{1}{(n+1)} \frac{\nu}{s} \left(\frac{\nu}{s} - 1 \right) \left(\frac{kT}{\mu} \right)^2 .... \right] \quad , \tag{2.125}$$

$$\mu^{\frac{\nu}{s}} = \epsilon_f^{\frac{\nu}{s}} - \frac{\Pi^2}{3} \frac{1}{(n+1)} \frac{\nu}{s} \left(\frac{\nu}{s} - 1 \right) (kT)^2 (\epsilon_f)^{\frac{\nu}{s}-2} .. \,, \tag{2.126}$$

or

$$\mu^{\frac{\nu}{s}} = \epsilon_f^{\frac{\nu}{s}} \left[1 - \frac{\Pi^2}{3} \frac{1}{(n+1)} \frac{\nu}{s} \left(\frac{\nu}{s} - 1 \right) \left(\frac{kT}{\epsilon_f} \right)^2 + ... \right] \quad , \tag{2.127}$$

$$\mu = \epsilon_f \left[1 - \frac{\Pi^2}{3} \frac{1}{(n+1)} \left(\frac{\nu}{s} - 1 \right) \left(\frac{kT}{\epsilon_f} \right)^2 + ... \right] \quad . \tag{2.128}$$

Other thermodynamic properties can be determined by using Eq. (2.128). First we calculate internal energy of ideal gas at low temperatures and high densities. Starting from Eq. (2.51)

$$\frac{U}{N} = \frac{\nu}{s}(kT) \frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)} \left(\frac{N-n}{N} \right) \quad , \tag{2.129}$$

$$= \frac{\nu}{s}(kT) \left[\frac{\frac{1}{\Gamma(\frac{\nu}{s}+2)} t^{(\frac{\nu}{s}+1)} .n\{1 + \frac{\Pi^2}{3} \frac{1}{(n+1)} \frac{\nu}{s} (\frac{\nu}{s}+1) t^{-2} + ...\}}{\frac{1}{\Gamma(\frac{\nu}{s}+1)} t^{\frac{\nu}{s}} .n\{1 + \frac{\Pi^2}{3} \frac{1}{(n+1)} \frac{\nu}{s} (\frac{\nu}{s}-1) t^{-2}\} + ...} \right] \left(\frac{N-n}{N} \right), \tag{2.130}$$

$$\frac{U}{N} = \left(\frac{N-n}{N} \right) \frac{\nu}{s}(kT) \frac{1}{(\frac{\nu}{s}+1)} .t \left[1 + \frac{\Pi^2}{3} \frac{1}{(n+1)} \frac{2v}{s} t^{-2} + ... \right] \quad . \tag{2.131}$$

Putting value from Eq. (2.128) because $kTt = \mu$

$$\begin{aligned}
\frac{U}{N} &= \left(\frac{N-n}{N}\right)\frac{1}{\left(\frac{\nu}{s}+1\right)}\frac{\nu}{s}\left[\epsilon_f\{1-\frac{\Pi^2}{3}\frac{1}{(n+1)}(\frac{\nu}{s}-1)\left(\frac{kT}{\epsilon_f}\right)^2\}....\right] \\
&\times \left[1+\frac{\Pi^2}{3}\frac{1}{(n+1)}\frac{2v}{s}\left(\frac{kT}{\epsilon_f}\right)^2...\right], \qquad (2.132) \\
&= \left(\frac{N-n}{N}\right)\frac{1}{(\frac{s}{\nu}+1)}\epsilon_f\left[1+\frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{kT}{\epsilon_f}\right)^2(\frac{2\nu}{s}-\frac{\nu}{s}+1)\right], \qquad (2.133)
\end{aligned}$$

$$= \left(\frac{N-n}{N}\right)\frac{1}{(\frac{s}{\nu}+1)}\epsilon_f\left[1+\frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{kT}{\epsilon_f}\right)^2(\frac{2\nu}{s}-\frac{\nu}{s}+1)\right], \qquad (2.134)$$

So,

$$\frac{U}{N} = \left(\frac{N-n}{N}\right)\frac{1}{(\frac{s}{\nu}+1)}\epsilon_f\left[1+\frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{kT}{\epsilon_f}\right)^2(\frac{\nu}{s}+1)...\right]. \qquad (2.135)$$

When z is very large Eq.(2.43) becomes,

$$\frac{P}{kT} = \frac{2g}{\lambda^\nu}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}h_{\frac{\nu}{s}+1}(z)+\frac{n}{V}\ln z\ . \qquad (2.136)$$

But from Eq. (2.49), we have

$$\frac{2g}{\lambda^\nu}\frac{\Gamma(\frac{\nu}{s}+1)}{\nu\Gamma(\frac{\nu}{2})}h_{\frac{\nu}{s}(z)} = \frac{N-n}{V}. \qquad (2.137)$$

Putting in Eq. (2.49), it becomes

$$\begin{aligned}
\frac{P}{kT} &= \frac{N-n}{V}\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)}+\frac{n}{V}\ln z, \qquad (2.138) \\
P &= \frac{N-n}{V}\left[\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)}kT\right]+\frac{n}{V}\mu, \qquad (2.139) \\
P &= \frac{N-n}{V}\left[\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)}kT\right]+\frac{n}{V}\left[\epsilon_f\{1-\frac{\Pi^2}{3}\frac{1}{(n+1)}(\frac{\nu}{s}-1)(\frac{kT}{\epsilon_f})^2....\}\right]. \qquad (2.140)
\end{aligned}$$

From Eq. (2.129) we know that,

$$(kT)\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)}\left(\frac{N-n}{N}\right)=\frac{U}{N}\frac{s}{\nu}\ . \tag{2.141}$$

Putting value of $\frac{U}{N}$ in above equation from Eq. (2.135)

$$\begin{aligned}(kT)\frac{h_{\frac{\nu}{s}+1}(z)}{h_{\frac{\nu}{s}}(z)}\left(\frac{N-n}{N}\right) &= \frac{s}{\nu}\left(\frac{N-n}{N}\right)\frac{1}{(\frac{s}{\nu}+1)}\epsilon_f\Big[1+\frac{\Pi^2}{3} \\ &\times \frac{1}{(n+1)}\left(\frac{kT}{\epsilon_f}\right)^2\left(\frac{\nu}{s}+1\right)...\Big]\ .\end{aligned} \tag{2.142}$$

Putting Eqs. (2.141) and (2.142) in Eq. (2.140) gives value of P.

$$\begin{aligned}P &= \left(\frac{N-n}{V}\right)\frac{1}{(\frac{\nu}{s}+1)}\epsilon_f\left[1+\frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{kT}{\epsilon_f}\right)^2\left(\frac{\nu}{s}+1\right)...\right] \\ &+ \frac{n}{V}\left[\epsilon_f\{1-\frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{\nu}{s}-1\right)\left(\frac{kT}{\epsilon_f}\right)^2....\}\right]\end{aligned} \tag{2.143}$$

The free energy of the system can be calculated from Eq. (2.135)

$$\frac{F}{N}=\mu-\frac{PV}{N}\ ,$$

$$\begin{aligned}\frac{F}{N} &= \epsilon_f\left[1-\frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{\nu}{s}-1\right)(\frac{kT}{\epsilon_f})^2....\right]-\left(\frac{N-n}{N}\right)\frac{1}{(\frac{\nu}{s}+1)}\epsilon_f\left[1+\frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{kT}{\epsilon_f}\right)^2\left(\frac{\nu}{s}+1\right)...\right] \\ &- \frac{n}{N}\left[\epsilon_f\{1-\frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{\nu}{s}-1\right)\left(\frac{kT}{\epsilon_f}\right)^2....\}\right]\ ,\end{aligned}$$

$$\begin{aligned}&= \epsilon_f\left[1-\frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{\nu}{s}-1\right)\left(\frac{kT}{\epsilon_f}\right)^2...\right]\left(1-\frac{n}{N}\right)-\left(\frac{N-n}{N}\right)\frac{1}{(\frac{\nu}{s}+1)} \\ &\times\epsilon_f\left[1+\frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{kT}{\epsilon_f}\right)^2(\frac{\nu}{s}+1)...\right]\ ,\end{aligned} \tag{2.144}$$

$$\begin{aligned}\frac{F}{N} &= \epsilon_f\left(\frac{N-n}{N}\right)\left[\left(1-\frac{1}{(\frac{\nu}{s}+1)}\right)-\frac{\Pi^2}{3}\frac{1}{(n+1)}\{\left(\frac{\nu}{s}-1+1\right)\left(\frac{kT}{\epsilon_f}\right)^2\}\right]\ , \\ \frac{F}{N} &= \epsilon_f\left(\frac{N-n}{N}\right)\left[\frac{1}{(\frac{\nu}{s}+1)}-\frac{\Pi^2}{3}\frac{1}{(n+1)}\frac{\nu}{s}\left(\frac{kT}{\epsilon_f}\right)^2...\right]\ .\end{aligned}$$

Put $\frac{\nu}{s} = \frac{(\frac{\nu}{s}+1)}{(\frac{s}{\nu}+1)}$ in above equation

$$\frac{F}{N} = \epsilon_f \left(\frac{N-n}{N}\right) \frac{1}{(\frac{s}{\nu}+1)} \left[1 - \frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{\nu}{s}+1\right)\left(\frac{kT}{\epsilon_f}\right)^2 ...\right] . \tag{2.145}$$

Entropy of the system is given by

$$S = \frac{U-F}{T} .$$

$$\begin{aligned}
S &= \frac{N}{T}\left[\left(\frac{N-n}{N}\right)\frac{1}{(\frac{s}{\nu}+1)}\epsilon_f\{1 + \frac{\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{kT}{\epsilon_f}\right)^2\left(\frac{\nu}{s}+1\right)\right. \\
&- \left. 1 + \frac{\Pi^2}{3}\frac{1}{(n+1)}(\frac{\nu}{s}+1)\left(\frac{kT}{\epsilon_f}\right)^2\}\right] , \qquad (2.146) \\
&= \frac{N}{T}\left[\left(\frac{N-n}{N}\right)\left(\frac{\nu}{s+\nu}\right)\epsilon_f\frac{2\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{s}{\nu}+1\right)\left(\frac{kT}{\epsilon_f}\right)\left(\frac{kT}{\epsilon_f}\right)\right] , \qquad (2.147) \\
\frac{S}{Nk} &= \left(\frac{N-n}{N}\right)\frac{2\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{\nu}{s}\right)\left(\frac{kT}{\epsilon_f}\right) . \qquad (2.148)
\end{aligned}$$

From Eq. (2.135) specific heat can be obtained.

$$C_v = \frac{\partial U}{\partial T} , \tag{2.149}$$

$$\begin{aligned}
\frac{1}{N}\frac{\partial U}{\partial T} &= \frac{\Pi^2}{3}\frac{1}{(n+1)}(2T)\left(\frac{N-n}{N}\right)\left(\frac{\nu}{s+\nu}\right)\left(\frac{s+\nu}{\nu}\right)\left(\frac{k^2}{\epsilon_f}\right) , \qquad (2.150) \\
\frac{C_v}{Nk} &= \frac{2\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{N-n}{N}\right)\left(\frac{\nu}{s}\right)\left(\frac{kT}{\epsilon_f}\right) . \qquad (2.151)
\end{aligned}$$

Next, let us compare it with the value of specific heat of fermions. First putting value of μ from Eq. (2.122) into Eq. (2.152) we obtain

$$\begin{aligned}
\frac{C_v}{Nk} &= \frac{2\Pi^2}{3}\frac{1}{(n+1)}\left(\frac{N-n}{N}\right)\left(\frac{\nu}{s}\right)\left(\frac{kT}{\epsilon_f^{fermion}(\frac{N-n}{Nn})^{\frac{s}{\nu}}}\right), \qquad (2.152) \\
&= \frac{1}{n+1}\frac{2\Pi^2}{3}n^{\frac{s}{\nu}}\left(\frac{N-n}{N}\right)^{1-\frac{s}{\nu}}\left(\frac{\nu}{s}\right)\frac{kT}{\epsilon_f^{fermion}} . \qquad (2.153)
\end{aligned}$$

In order to calculate $(C_v)_{fermion}$ in thermodynamic limit, put $n = 1$ in above equation

$$\frac{(C_v)_{fermion}}{Nk} = \frac{2\Pi^2}{3}\frac{\nu}{s}\frac{kT}{\epsilon_f^{fermion}}\frac{1}{2} \ , \tag{2.154}$$

Thus,

$$C_v = (C_v)_{fermion}\left(\frac{N-n}{N}\right)^{1-\frac{s}{\nu}} n^{\frac{s}{\nu}}\frac{2}{n+1} \ . \tag{2.155}$$

Or putting

$$\left(\frac{N-n}{N}\right)^{1-\frac{s}{\nu}} n^{\frac{s}{\nu}}\frac{2}{n+1} = \eta_1, \tag{2.156}$$

we get

$$C_v = \eta_1(C_v)_{fermion}. \tag{2.157}$$

It means that specific heat of Gentile statistics differs from specific heat of FD statistics by a factor of η_1. Also, it is obvious that a factor of $(N - n)$ is appearing in each of the above expressions. This factor represents the contribution of ground state. When n is of the order of N, the thermodynamic properties are strongly suppressed by this factor. The results also show that to obtain Bose-Einstein statistics from Gentile statistics, one needs not only to perform the limit $n \to N$, but also restrict the fugacity $z \leq 1$.

In the next chapter, first generalized statistics is introduced. Then after discussing homogeneous form of Riemann-Hilbert problem, an exact expression for the fugacity is solved and phase transition point and phase transition temperature are calculated. We will show that, beyond the Bose case, there are still other systems that can display BEC type phase transitions.

Chapter 3

Generalized Statistics and Generalized Bose-Einstein Condensation

3.1 Introduction

When temperature is decreased and density becomes very high, all atoms tend to go to lowest energy quantum state. Maximum occupation number of this quantum state is infinite. As a result of this accumulation, large number of atoms form a condensate, known as the Bose-Einstein condensate. This is an example of first order phase transition already discussed in first chapter.

We present in this dissertation generalized Bose-Einstein condensation type phase transition model. This model is derived purely statistically. We first derive an expression for fugacity and then by analyzing discontinuity in its derivative, determine whether there is a phase transition or not [9].

When dealing with the gas phase i.e., when temperature is very high and density is low, all quantum states are well defined and we are in the classical regime. On the other hand, when this gas condenses by lowering the temperature, velocity of particles become low and they try to occupy some lower energy state usually the ground state. We can say that a large number of particles occupy a single quantum state in condensed phase. This occupation number can be of the order of the total number of particles in the system. In this way, this single quantum state can be used to describe all thermodynamic properties of a system in condensed phase. This is exactly the case of Bose-Einstein condensation. In this dissertation, we will show that BEC type phase transition can occur in an ideal gas of any dimension obeying generalized statistics, while BEC can occur in only three dimensional ideal Bose gases.

A generalized statistics is one in which different quantum states have different maximum occupation number. Bose-Einstein, Fermi-Dirac and Gentile statistics are all its special cases. We will show that BEC type phase transition can occur in such systems, in which maximum occupation number of only one state is infinite like that in BE case and all other states is 1, like in FD case. The model is based on the calculation of an explicit expression of fugacity with the help of generalized statistics. The key idea of solving an expression of fugacity is based on homogeneous Riemann-Hilbert (RH) problem, which is defined as a boundary value problem of finding an analytic function which satisfies a given jump condition along an oriented contour on which analyticity is lost [7].

Thus, we have to find solution of an analytic function say, for example, $\Psi(\zeta)$ with boundary condition

$$\Psi^{+}(\zeta) = G(\zeta)\Psi^{-}(\zeta) + g(\zeta) \quad \zeta \in L \tag{3.1}$$

where L is a union of finite number of simple smooth arcs. $\Psi^{+}(\zeta)$ denotes the limit of Ψ as ζ approaches L from right and $\Psi^{-}(\zeta)$ denotes the limit of Ψ as ζ approaches L from left. $G(\zeta) \neq 0$ on L everywhere. Further $G(\zeta)$ and $g(\zeta)$ are functions which satisfy following condition called Holder condition [6].

$$|f(\zeta_2) - f(\zeta_1)| = A|\zeta_2 - \zeta_1|^{\mu}. \tag{3.2}$$

Here A and μ are greater than zero. $\Psi(\zeta)$ is a sectionally analytic function which means that it is analytic in each region which does not contain boundary points of L and is continuous on L from left and from right, except at some possible ends of L. $\Psi(\zeta)$ should satisfy following condition

$$|\Psi(\zeta)| < \frac{const.}{|\zeta - c_m|^{\beta_m}} \qquad \beta_m < 1, \tag{3.3}$$

near such ends. Here, β_m is a constant corresponding to mth end c_m.

In this problem we will use homogeneous form of Riemann-Hilbert problem, in which $g(\zeta) = 0$, i.e., we have to find a function $\Psi(\zeta)$ from the jump on two sides of boundary L, i.e.,

$$G(\zeta) = \frac{\Psi^{+}(\zeta)}{\Psi^{-}(\zeta)}. \tag{3.4}$$

Further results also show that if total number of particles and volume are infinite, only then phase transition can occur, i.e., in thermodynamic limit. First of all we start with the discussion of generalized statistics in which different states have different maximum occupation number of particles in a quantum state.

3.2 The Generalized Statistics

If we define n_i as the maximum occupation number of ith quantum state. This number will have any integer value or infinity. The grand partition function of generalized statistics in terms of the chemical potential is

$$\Xi(T, V, \mu) = \prod_{i=0}^{\infty} \frac{1 - e^{-\beta(n_i+1)(\varepsilon_i - \mu)}}{1 - e^{-\beta(\varepsilon_i - \mu)}}, \tag{3.5}$$

wherein in the above expression T, V and μ represent the temperature, volume, and the chemical potential of the system. Further $\beta = \frac{1}{k_B T}$ and ε_i denotes the energy of the ith state. Put $e^{\beta\mu} = z$ in the above equation. For obtaining equation of state we have

$$k_B T \ln \Xi = PV. \tag{3.6}$$

So, from Eq. (3.5) we have

$$\frac{PV}{k_B T} = \sum_{i=0}^{\infty} \ln \frac{1 - e^{-\beta(n_i+1)(\varepsilon_i)} z^{n_i+1}}{1 - z e^{-\beta\varepsilon_i}} \;. \tag{3.7}$$

When $n_i = 1$,

$$\frac{PV}{k_B T} = \sum_{i=0}^{\infty} \ln \frac{1 - e^{-2\beta\varepsilon_i} z^2}{1 - z e^{-\beta\varepsilon_i}}$$

,

$$= \sum_{i=0}^{\infty} \ln(1 + z e^{-\beta\varepsilon_i}), \qquad \text{(Fermi case)} \;.$$

When $n_i = \infty$

$$\frac{PV}{k_BT} = \sum_{i=0}^{\infty} \ln \frac{1}{(1 - ze^{-\beta\varepsilon_i})}, \qquad \text{(Bose case)}.$$

When $n_i = n$, the equation of state will reduce to that of the Gentile statistics case. Now for the total number of particles N, we have

$$N = z\frac{\partial}{\partial z} \ln \Xi(T, V, z) \ , \tag{3.8}$$

$$\begin{aligned} z\frac{\partial}{\partial z}\left[\sum_{i=0}^{\infty} \ln \frac{1 - e^{-\beta(n_i+1)(\varepsilon_i)} z^{n_i+1}}{1 - ze^{-\beta\varepsilon_i}}\right] &= z\frac{\partial}{\partial z}\Big[\sum_{i=0}^{\infty} \ln(1 - e^{-\beta(n_i+1)(\varepsilon_i)} z^{n_i+1}) \\ &- \ln(1 - ze^{-\beta\varepsilon_i})\Big] \ , \\ &= z\sum_{i=0}^{\infty}\left[\frac{-(n_i+1)z^{n_i}e^{-\beta(n_i+1)\varepsilon_i}}{1 - e^{-\beta(n_i+1)\varepsilon_i} z^{n_i+1}} + \frac{e^{-\beta_i}}{1 - ze^{-\beta\varepsilon_i}}\right] \ , \\ &= \sum_{i=0}^{\infty}\left[\frac{1}{z^{-1}e^{\beta\varepsilon_i} - 1} - \frac{n_i+1}{e^{-\beta\varepsilon_i(n_i+1)} z^{n_i+1} - 1}\right] \ . \end{aligned}$$

Again by putting $n_i = 1, \infty$ and n we recover Fermi, Bose and Gentile cases respectively. We replace summation by the integral on right side of Eq. (3.7)

$$\sum_l \rightarrow \frac{2gV\Pi^{\frac{\nu}{2}}}{h^{\nu}\Gamma(\frac{\nu}{2})s} \int (2m)^{(\frac{\nu}{s})} \epsilon^{(\frac{\nu}{s}-1)} d\epsilon \ . \tag{3.9}$$

Performing almost the same calculations like we did for Gentile statistics in the previous chapter, we find

$$\sum_{i=0}^{\infty} \ln \Xi = \frac{2V}{\Gamma(\frac{\nu}{2})\lambda^{\nu}\nu} \int_0^{\infty} \left[\frac{1}{z^{-1}e^{\xi} - 1} - \frac{n_i+1}{z^{-(n_i+1)}e^{(n_i+1)\xi} - 1}\right] \xi^{\frac{\nu}{s}} d\xi \ . \tag{3.10}$$

Like Bose-Einstein and Fermi-Dirac integrals $g_\sigma(z)$ and $f_\sigma(z)$ we introduce the function

$$h_\sigma(z) = \frac{1}{\Gamma(\sigma)} \int_0^{\infty} \left[\frac{1}{z^{-1}e^{x} - 1} - \frac{n+1}{z^{-(n+1)}e^{(n+1)x} - 1}\right] x^{\sigma-1} dx \ . \tag{3.11}$$

So, Eq. (3.10) can be written as

$$\sum_{0}^{\infty} \ln \Xi = \frac{2V}{\Gamma(\frac{\nu}{2})\lambda^{\nu}\nu} \Gamma(\sigma) h_\sigma(z) \ . \tag{3.12}$$

Define $N_\lambda = \frac{2\Gamma(\frac{\nu}{s})V}{s\Gamma(\frac{\nu}{2})\Lambda^\nu}$, and $\lambda = (\frac{h^s}{2\Pi^{\frac{s}{2}} mkT})^{\frac{1}{s}}$, equation of state for an ideal gas in our case with dispersion relation $\varepsilon = \frac{p^s}{2m}$ will become

$$\frac{PV}{k_B T} = N_\lambda h_\sigma(z) - \ln\left[(1 - ze^{-\beta\varepsilon_k})^{n+1}\right], \tag{3.13}$$

$$N = N_\lambda h_{\frac{\nu}{s}}(z) - \frac{n+1}{(ze^{-\beta\varepsilon_k})^{n+1} - 1}. \tag{3.14}$$

As we know that

$$g_\sigma(z) = \frac{1}{\Gamma(\sigma)} \int_0^\infty (\frac{1}{z^{-1}e^x - 1}) dx. \tag{3.15}$$

Thus, we can write $h_\sigma(z)$ as

$$h_\sigma(z) = g_\sigma(z) - (n_i + 1)^{-(\sigma-1)} g_\sigma(z^{n_i+1}). \tag{3.16}$$

In the next section, we will calculate an equation for the fugacity from equation of state to determine whether we can obtain a BEC type phase transition for our model or not by determining phase transition temperature and discontinuity in the derivative of fugacity. We will consider an ideal gas obeying generalized statistics in which only one state has maximum occupation number as infinity and for all other states it is n (any integer). In other words $n_i = n$ and $n_k = \infty$ where $i \neq k$. More generally, the case which will be considered, when $k = 0$, or $n_0 = \infty$, and $n_i = n$ where $(i \neq 0)$. Also results will be discussed for $n = 1$.

3.3 An Explicit Expression For Fugacity

We can calculate fugacity z from Eq. (3.14)using the homogeneous form of RH problem. Considering the case of $n_k = \infty$, and $n_i = n$ $(i \neq k)$, we introduce a real function

$$\begin{aligned} \Psi(z) &= \frac{N_\lambda}{N} h_{\frac{\nu}{s}}(z) - \frac{1}{N} \frac{n+1}{(z^{-1}e^{-\beta\varepsilon_k})^{n+1} - 1} - 1, & (3.17) \\ &= f(z) - 1, & (3.18) \end{aligned}$$

where

$$f(z) = \frac{N_\lambda}{N} h_{\frac{\nu}{s}}(z) - \frac{1}{N} \frac{n+1}{(z^{-1}e^{-\beta\varepsilon_k})^{n+1} - 1}. \tag{3.19}$$

Clearly Eq. (3.14) corresponds to $\Psi(z) = 0$, or we can say that fugacity z is the zero of the function $\Psi(z)$ and problem of solving z is converted into the problem of finding the zero of $\Psi(z)$. In order to find the zero of $\Psi(z)$, we make analytic continuation of the function $\Psi(z)$ to the overall complex plane (Riemann zeta plane). The analytically continued function [7] is then

$$\Psi(\zeta) = f(\zeta) - 1, \tag{3.20}$$

where

$$f(\zeta) = \frac{N_\lambda}{N} h_{\frac{\nu}{s}}(\zeta) - \frac{1}{N} \frac{n+1}{(\zeta^{-1} e^{-\beta \varepsilon_k})^{n+1} - 1}. \tag{3.21}$$

So, if z is the zero of real function $\Psi(z)$, so is the zero of complex function $\Psi(\zeta)$ on the real axis. The function $\Psi(\zeta)$ may not have only one zero corresponding to fugacity. There can be other zeros as well in the complex ζ-plane. We denote the zeros as ω_i, where $i = 1, 2, ...n_{0-1}$. Here n_0 is the total number of zeros. Besides this, $\Psi(\zeta)$ can also have singularities in the complex zeta plane. A singularity is a point at which a function is not defined. In our case , however, we will see that $\Psi(\zeta)$ will have only non isolated singularities also called poles. They form some rays in the complex ζ-plane. These rays form the boundary of analytic region of $\Psi(\zeta)$ with origin of these rays on a unit circle.

Now we are in a position to define $\Psi(\zeta)$ as

$$\Psi(\zeta) = \eta \frac{(\zeta - z) \prod_{i=1}^{n_{zero-1}} (\zeta - \omega_i)}{\prod_{j=1}^{n_p} (\zeta - \rho_j) \prod_{m=1}^{n_b} (\zeta - c_m)^{k_m}} \phi(\zeta). \tag{3.22}$$

Those regions in which there are no points of arcs and has no zeros and singularities, $\Psi(\zeta)$ is analytic. In this expression η is a constant, z and ω_i are zeros of $\Psi(\zeta)$, n_{zero} is the total number of zeros, n_p is the number of poles and ρ_j is a pole of $\Psi(\zeta)$, c_m is the end point that is different from the boundary of analytic region of $\Psi(\zeta)$, n_b is the total number of points that are different from the boundary and the constant k_m is used to balance the degree of divergence on either side of this equation at the mth end point c_m. $\phi(\zeta)$ is a function in the complex $\zeta-$plane which is analytic in each region not containing the points of arcs and zeros.

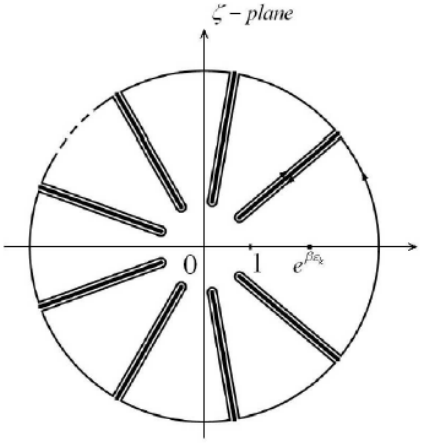

fig. 1(a): Analytic region of Ψ(ζ), when n is even.

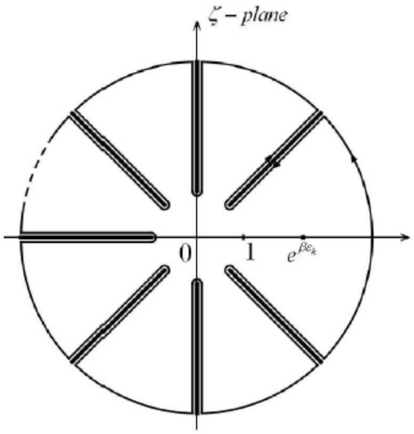

fig. 1(b): Analytic region of Ψ(ζ), when n is odd.

Infact, if we know the end points c_m, poles ρ_j, zeros ω_i and the values of constants k_m, η, etc, we can find an expressions for fugacity from above equation as

$$\eta(\zeta - z)\prod_{i=1}^{n_{zero-1}}(\zeta-\omega_i) = \prod_{j=1}^{n_p}(\zeta-\rho_j)\prod_{m=1}^{n_b}(\zeta-c_m)^{k_m}\frac{\Psi(\zeta)}{\phi(\zeta)}. \tag{3.23}$$

Then the problem of solving for z is converted into the problem of solving for zeros, poles, $\phi(\zeta)$, etc. To determine $\phi(\zeta)$ we first analyze the behavior of singularities of $\Psi(\zeta)$ in ζ-plane (see fig 1). From Eq. (3.21) we can see that the singularities of $\Psi(\zeta)$ is determined by singularities of $h_\sigma(\zeta)$ which is analytic continuation of $h_\sigma(z)$. Further we know that

$$h_\sigma(\zeta) = g_\sigma(\zeta) - \frac{1}{(n_i+1)^{(\sigma-1)}}g_\sigma(\zeta^{n_i+1}) \ . \tag{3.24}$$

It means that $h_\sigma(\zeta)$ depends on BE integral $g_\sigma(\zeta)$ which is just Jonquiére function also known as the poly-algorithm function $Li_\sigma(\zeta)$. Now we know that $Li_\sigma(\zeta)$ is analytic in the region with the boundary along positive real axis from 1 to ∞. In other words on the positive real axis $h_\sigma(\zeta)$ has no singularities or we can say that in this region it is analytic, i.e., its boundary is from $[1, \infty)$. Consequently, the boundary of $h_\sigma(\zeta)$ and $\Psi(\zeta)$ consist of n rays with origins on a unit circle, denoted as L_m, $m = 1, 2, 3....n$, i.e., the mth ray L_m is $[e^{\frac{2\Pi im}{n+1}}, \infty e^{\frac{2\Pi im}{n+1}})$ and

n is the total number of arcs. Then the boundary has only one point different from infinity given below

$$c_m = e^{\frac{2\Pi im}{n+1}} , \qquad\qquad m = 1, 2, ..., n \; . \tag{3.25}$$

Therefore we have $n_b = n$.

The isolated singularity. From Eq. (3.21) we can see that $\Psi(\zeta)$ has only one isolated singularity, which is

$$\rho = e^{\beta\varepsilon_k} \; . \tag{3.26}$$

Calculating $\phi(\zeta)$ needs the result of homogeneous RH problem which is

$$\phi(\zeta) = e^{\gamma(\zeta)} \prod_{m=1}^{n} (\zeta - c_m)^{\lambda_m} \; , \tag{3.27}$$

where

$$\gamma(\zeta) = \frac{1}{2\Pi\imath} \int_{L_m}^{\infty} \frac{\ln G(x)}{x - \zeta} dx \; , \tag{3.28}$$

$$= \frac{1}{2\Pi i} \sum_{m=1}^{n} e^{\frac{2\Pi im}{n+1}} \int_{1}^{\infty} \frac{\ln G(xe^{\frac{2\Pi im}{n+1}})}{xe^{\frac{2\Pi im}{n+1}} - \zeta} dx \; . \tag{3.29}$$

x is a variable which goes from $1 \longrightarrow \infty$. For calculating $\phi(\zeta)$, we first need to calculate the jump at the two sides of the boundary

$$G(\zeta) = \frac{\phi^{+}(\zeta)}{\phi^{-}(\zeta)} \; . \tag{3.30}$$

From Eq. (3.22) we can see that only non isolated singularities of $\Psi(\zeta)$ form the boundary of analytic region of $\phi(\zeta)$. Thus, the jump of $\phi(\zeta)$ and $\Psi(\zeta)$ is same. So we can write

$$G(\zeta) = \frac{\Psi^{+}(\zeta)}{\Psi^{-}(\zeta)} \; . \tag{3.31}$$

Now the jump of $\Psi(\zeta)$ on the boundary according to Eq.(3.21) is determined by the jump of $h_\sigma(\zeta)$ and $g_\sigma(\zeta)$ (which is simply $Li_\sigma(\zeta)$). The imaginary part of $Li_\sigma(\zeta)$ has a discontinuity on the boundary [6] is given by

$$\begin{aligned} ImLi_\sigma(x+\imath\delta) &= \frac{\Pi}{\Gamma(\sigma)}(\ln x)^{\sigma-1}\ , && (3.32)\\ ImLi_\sigma(x-\imath\delta) &= \frac{-\Pi}{\Gamma(\sigma)}(\ln x)^{\sigma-1}, \qquad\qquad x\in[1,\infty)\ . && (3.33)\end{aligned}$$

Here δ represents a small positive quantity. So, the values of $h_\sigma(\zeta)$ on the two sides of L_m are

$$h_\sigma(xe^{\frac{2\Pi im}{n+1}}) = g_\sigma(xe^{\frac{2\Pi im}{n+1}}) - \frac{1}{(n+1)^{(\sigma-1)}}g_\sigma(xe^{\frac{2\Pi im}{n+1}})^{n+1} - \frac{\imath\Pi}{\Gamma(\sigma)}(\ln x)^{\sigma-1}, \tag{3.34}$$

where

$$g_\sigma(x) = \frac{1}{\Gamma(\sigma)}\int_0^\infty \frac{t^{\sigma-1}}{x^{-1}e^t-1}dt. \tag{3.35}$$

Putting Eq. (3.34) into Eq. (3.21), we get

$$\begin{aligned} \Psi^{\pm}(xe^{\frac{2\Pi im}{n+1}}) &= \frac{N_\lambda}{N}\left[g_{\frac{\nu}{s}}(xe^{\frac{2\Pi im}{n+1}}) - \frac{1}{(n+1)^{(\frac{\nu}{s}-1)}}g_{\frac{\nu}{s}}(x)^{n+1}\right]\\ &+ \frac{1}{N}\frac{n+1}{(x^{-1}e^{\beta\varepsilon_k})^{n+1}} - 1 - \frac{\imath\Pi}{\Gamma(\frac{\nu}{s})}(\ln x)^{\frac{\nu}{s}-1}. && (3.36)\end{aligned}$$

For the case $n_k=\infty$ and $n_i=1$ this equation reduces to

$$\Psi^{\pm}(-x) = \frac{N_\lambda}{N}\left[g_{\frac{\nu}{s}}(-x) - \frac{1}{2^{(\frac{\nu}{s}-1)}}g_{\frac{\nu}{s}}(x)^2\right] + \frac{1}{N}\frac{2}{(x^{-1}e^{\beta\varepsilon_k})^2} - 1 - \frac{\imath\Pi}{\Gamma(\frac{\nu}{s})}(\ln x)^{\frac{\nu}{s}-1}\ . \tag{3.37}$$

Now for the first two terms on R.H.S above, we have

$$g_{\frac{\nu}{s}}(-x) - \frac{1}{2^{(\frac{\nu}{s}-1)}}g_{\frac{\nu}{s}}(x)^2 = \frac{1}{\Gamma(\frac{\nu}{s})}\int_0^\infty \frac{t^{\frac{\nu}{s}-1}}{-\frac{1}{x}e^t-1}dt - \frac{1}{2^{(\frac{\nu}{s}-1)}}\frac{1}{\Gamma(\frac{\nu}{s})}\int_0^\infty \frac{t^{\frac{\nu}{s}-1}}{-\frac{1}{x^2}e^t-1}\ , \tag{3.38}$$

or

$$\begin{aligned} \frac{1}{-\frac{1}{x^2}e^t-1} &= \frac{1}{(-\frac{1}{x}e^{\frac{t}{2}}-1)(-\frac{1}{x}e^{\frac{t}{2}}+1)}, && (3.39)\\ &= \frac{1}{2}\left[\frac{1}{(-\frac{1}{x}e^{\frac{t}{2}}-1)} - \frac{1}{(-\frac{1}{x}e^{\frac{t}{2}}+1)}\right]. && (3.40)\end{aligned}$$

So

$$=\frac{1}{\Gamma(\frac{\nu}{s})}\left[\int_0^\infty \frac{t^{\frac{\nu}{s}-1}}{-\frac{1}{x}e^{t}-1}dt-\frac{1}{2^{(\frac{\nu}{s}-1)}}\left(-\frac{1}{2}\int_0^\infty \frac{t^{\frac{\nu}{s}-1}}{+\frac{1}{x}e^{\frac{t}{2}}+1}dt-\frac{1}{2}\int_0^\infty \frac{t^{\frac{\nu}{s}-1}}{-\frac{1}{x}e^{\frac{t}{2}}+1}dt\right)\right] . \tag{3.41}$$

Let us take $t = y$ in the first integral and $\frac{t}{2} = y$ in second integral, $dt = 2dy$

$$=\frac{1}{\Gamma(\frac{\nu}{s})}\left[\int_0^\infty \frac{y^{\frac{\nu}{s}-1}}{-\frac{1}{x}e^{y}-1}dy+\int_0^\infty \frac{y^{\frac{\nu}{s}-1}}{+\frac{1}{x}e^{y}+1}dy+\int_0^\infty \frac{y^{\frac{\nu}{s}-1}}{-\frac{1}{x}e^{y}+1}dy\right] , \tag{3.42}$$

Then

$$\int_0^\infty \frac{y^{\frac{\nu}{s}-1}}{-\frac{1}{x}e^{y}+1}dy = f_{\frac{\nu}{s}}(-x) . \tag{3.43}$$

Where $f_{\frac{\nu}{s}}(-x)$ is analytically continued FD integral. So, Eq. (3.37) becomes as

$$\Psi^{\mp}(-x) = \frac{N_\lambda}{N}f_{\frac{\nu}{s}}(-x)+\frac{1}{N}\frac{2}{(x^{-1}e^{\beta\varepsilon_k})^2}-1-\frac{\imath\Pi}{\Gamma(\frac{\nu}{s})}(\ln x)^{\frac{\nu}{s}-1} . \tag{3.44}$$

Next we calculate the value of λ_m when λ_m is an integer determined by the condition [6]

$$\begin{array}{lll} \mp Re\dfrac{\ln G(c_m)}{2\Pi\imath}+\lambda_m = o & , & if \mp Re\dfrac{\ln G(c_m)}{2\Pi\imath} \text{ is an integer} \\ -1 < \mp Re\dfrac{\ln G(c_m)}{2\Pi\imath} < 1 & & \text{otherwise.} \end{array}$$

Above condition gives

$$-1 < \mp Re\frac{\ln G(c_m)}{2\Pi\imath}+\lambda_m < 0 . \tag{3.45}$$

where “ $-$ ” and “ $+$ ” signs are taken for the starting and end point of certain ray L_m respectively. At end points $\phi^{+}(\zeta)$ and $\phi^{-}(\zeta)$ are equal, so their ratio is 1. Therefore, at end points we have

$$G(e^{\frac{2\Pi im}{n+1}}) = G(\infty) = 1 . \tag{3.46}$$

Taking $\ln G(\infty) = 0$, we are left with

$$\ln G(e^{\frac{2\Pi im}{n+1}}) = \ln|e^{\frac{2\Pi im}{n+1}}| + iargG(e^{\frac{2\Pi im}{n+1}}) , \tag{3.47}$$

$$= -2\Pi i = -1 = \lambda_m \ . \tag{3.48}$$

Therefore,

$$\phi(\zeta) = e^{\gamma(\zeta)} \prod_{m=1}^{n} \frac{1}{\zeta - e^{\frac{2\Pi i m}{n+1}}} \ , \tag{3.49}$$

$$= e^{\gamma(\zeta)} \frac{\zeta - 1}{\zeta^{n+1} - 1} \ . \tag{3.50}$$

The value of k_m: We choose the parameter k_m such that it ensures the degree of divergence on the two sides of Eq.(3.22) is same at the end point c_m. Considering the origin of L_m, we see that $c_m = e^{2i\Pi m}/(n+1)$. $g_{\nu/s}(x^{n+1})$ and $(\ln x)^{\nu/s-1}$ are convergent for $\nu/s > 1$,, and when $\nu/s \leq 1$, the degree of divergence of $g_{\nu/s}(x^{n+1})$ and $(\ln x)^{\nu/s-1}$ are less than 1. Thus we have $k_m = 0$. Finally put all results in Eq. (3.23)

$$\eta(\zeta - z) \prod_{i=1}^{n_{zero-1}} (\zeta - \omega_i) = e^{-\gamma(\zeta)} \frac{\zeta^{n+1} - 1}{\zeta - 1} (\zeta - e^{\beta \varepsilon_k}) \Psi(\zeta) \ . \tag{3.51}$$

If now we know the value of η and ω_i, we can easily find z. The problem of solving for ω_i is as difficult as for solving z itself. Therefore, we have to find an alternative way. For this purpose we form certain set of equations and then solving them together we can eliminate η and ω_i and can find an explicit expression for fugacity z. For our case $n_k = \infty$ and $n_i = 1$ this equation reduces to

$$\eta(\zeta - z)(\zeta - \omega) = e^{-\gamma(\zeta)} (\zeta + 1)(\zeta - e^{\beta \varepsilon_k}) \Psi(\zeta) \ . \tag{3.52}$$

For this purpose, we need three equations to solve for z. Substituting $\zeta = 0$, in above equation. This requires the value of $\Psi(0)$ from Eq. (3.21).

$$\Psi(0) = \frac{N_\lambda}{N} h_{\frac{\nu}{s}}(0) + 0 - 1 \ , \tag{3.53}$$

$$h_{\frac{\nu}{s}}(0) = g_{\frac{\nu}{s}}(0) - \frac{1}{(n+1)^{\sigma-1}} g_{\frac{\nu}{s}}(0) \ , \tag{3.54}$$

$$g_{\frac{\nu}{s}}(0) = Li_{\frac{\nu}{s}}(0) \ , \tag{3.55}$$

$$Li_s(z) \sum_{k=1}^{\infty} \frac{z^k}{k^s} = z + \frac{z^2}{2^s} + \frac{z^3}{3^s} + \ldots.. \ , \tag{3.56}$$

$$Li_{\frac{\nu}{s}}(0) = 0 \ . \tag{3.57}$$

Thus, $h_{\frac{\nu}{s}}(0) = 0$ and therefore we get

$$\Psi(0) = -1 \ . \tag{3.58}$$

Now putting the values of $\zeta = 0$ and $\Psi(0)$ in Eq. (3.52), it becomes

$$\eta z \omega = e^{\beta\varepsilon_k - \gamma(0)} \ . \tag{3.59}$$

Taking derivative of Eq. (3.52) w.r.t ζ gives

$$\begin{aligned} \eta(-z)(-\omega) &= -\gamma(\zeta) e^{-\gamma(\zeta)}(\zeta+1)(\zeta - e^{\beta\varepsilon_k})\Psi(\zeta) + e^{-\gamma(\zeta)}(\zeta - e^{\beta\varepsilon_k})\Psi(\zeta) \\ &+ e^{-\gamma(\zeta)}(\zeta+1)\Psi(\zeta) + e^{-\gamma(\zeta)}(\zeta+1)(\zeta - e^{\beta\varepsilon_k})\Psi'(\zeta) \ . \end{aligned} \tag{3.60}$$

We take $\zeta = 0$ in above equation and find that

$$\begin{aligned} \eta(-z)(-\omega) &= -\gamma'(0) e^{-\gamma(0)} e^{\beta\varepsilon_k} + e^{-\gamma(0)} e^{\beta\varepsilon_k} \\ &- e^{-\gamma(0)} - e^{-\gamma(0)}(e^{\beta\varepsilon_k})\left(\frac{N_\lambda}{N}\right) \ , \end{aligned} \tag{3.61}$$

$$-\frac{1}{\omega} - \frac{1}{z} = \frac{e^{-\gamma(0)} e^{\beta\varepsilon_k}}{\eta z \omega}\left[-\gamma'(0) + 1 - e^{-\beta\varepsilon_k} - \frac{N_\lambda}{N}\right] \ , \tag{3.62}$$

$$-\frac{1}{\omega} - \frac{1}{z} + \gamma'(0) - 1 + e^{-\beta\varepsilon_k} = -\frac{N_\lambda}{N} \ . \tag{3.63}$$

Taking

$$\gamma'(0) - 1 + \frac{N_\lambda}{N} = \eta_\lambda \ , \tag{3.64}$$

Eq. (3.63) reads as

$$\frac{1}{\omega} = \eta_\lambda + e^{-\beta\varepsilon_k} - \frac{1}{z} \ . \tag{3.65}$$

Now putting $\zeta = e^{\beta\varepsilon_k}$ in Eq. (3.52), gives

$$\eta(e^{\beta\varepsilon_k} - z)(e^{\beta\varepsilon_k} - \omega) = -e^{-\gamma(e^{\beta\varepsilon_k})}(e^{\beta\varepsilon_k} + 1)\frac{e^{\beta\varepsilon_k}}{N} \ . \tag{3.66}$$

Divide Eq. (3.66) by Eq. (3.59)

$$\frac{(e^{\beta\varepsilon_k}-z)(e^{\beta\varepsilon_k}-\omega)}{z\omega} = -\frac{(e^{\beta\varepsilon_k}+1)e^{\beta\varepsilon_k-\gamma(0)}}{N} \ , \tag{3.67}$$

$$\begin{aligned}
(e^{\beta\varepsilon_k}-z)(\frac{e^{\beta\varepsilon_k}}{\omega}-1) &= -\frac{z(e^{\beta\varepsilon_k}+1)}{Ne^{\gamma(e^{\beta\varepsilon_k})-\gamma_0}} \ , & (3.68)\\
(e^{\beta\varepsilon_k}-z)\left[e^{\beta\varepsilon_k}(\eta_\lambda+e^{-\beta\varepsilon_k}-\frac{1}{z})-1\right] &= -\frac{z(e^{\beta\varepsilon_k}+1)}{Ne^{\gamma(e^{\beta\varepsilon_k})-\gamma_0}} \ , & (3.69)\\
(e^{\beta\varepsilon_k}-z)\left[e^{\beta\varepsilon_k}(\eta_\lambda-\frac{1}{z})\right] &= -\frac{z(e^{\beta\varepsilon_k}+1)}{Ne^{\gamma(e^{\beta\varepsilon_k})-\gamma_0}} \ , & (3.70)\\
z^2\left[-\eta_\lambda+\frac{(1+e^{-\beta\varepsilon_k})}{Ne^{\gamma(e^{\beta\varepsilon_k})-\gamma_0}}\right]+z(\eta_\lambda e^{\beta\varepsilon_k}+1)-e^{\beta\varepsilon_k} &= 0 \ , & (3.71)\\
z^2\left[-\eta_\lambda e^{-\beta\varepsilon_k}+\frac{(1+e^{-\beta\varepsilon_k})e^{-\beta\varepsilon_k}}{Ne^{\gamma(e^{\beta\varepsilon_k})-\gamma_0}}\right]+z(\eta_\lambda+e^{-\beta\varepsilon_k})-1 &= 0 \ . & (3.72)
\end{aligned}$$

Thus above expression is equal to the quadratic formula, $az^2+bz+c=0$ when

$$\begin{aligned}
a &= \left[-\eta_\lambda e^{-\beta\varepsilon_k}+\frac{(1+e^{-\beta\varepsilon_k})e^{-\beta\varepsilon_k}}{Ne^{\gamma(e^{\beta\varepsilon_k})-\gamma_0}}\right] \ , & (3.73)\\
b &= (\eta_\lambda+e^{-\beta\varepsilon_k}) \ , & (3.74)\\
c &= -1 \ . & (3.75)
\end{aligned}$$

Solving it we get

$$z=\frac{-b\pm\sqrt{b^2-4ac}}{2a} \ . \tag{3.76}$$

By rationalizing

$$\begin{aligned}
z &= \frac{-4ac}{2a(b\pm\sqrt{b^2-4ac})} \ , & (3.77)\\
&= \frac{2}{b\pm\sqrt{b^2-4ac}} \ , & (3.78)\\
&= 2(b\pm\sqrt{b^2-4ac})^{-1} \ , & (3.79)\\
&= 2\left[(\eta_\lambda+e^{-\beta\varepsilon_k})\pm\sqrt{(\eta_\lambda+e^{-\beta\varepsilon_k})^2+4\left(-\eta_\lambda e^{-\beta\varepsilon_k}+\frac{(1+e^{-\beta\varepsilon_k})e^{-\beta\varepsilon_k}}{Ne^{\gamma(e^{\beta\varepsilon_k})-\gamma_0}}\right)}\right]^{-1} & (3.80)
\end{aligned}$$

Finally

$$z \;=\; 2\left[(\eta_\lambda + e^{-\beta\varepsilon_k}) + \sqrt{(\eta_\lambda - e^{-\beta\varepsilon_k})^2 + \frac{4(1+e^{-\beta\varepsilon_k})e^{-\beta\varepsilon_k}}{Ne^{\gamma(e^{\beta\varepsilon_k})-\gamma_0}}}\right]^{-1}. \tag{3.81}$$

Fugacity is always positive as it is an exponential function. Therefore, we have ignored minus sign.

3.4 Phase Transition in Thermodynamic Limit

We know that the phase transitions take place whenever there is a discontinuity in the derivative of a certain parameter describing a system. Here it is easily seen that in the thermodynamic limit, i.e., N$\longrightarrow\infty$, there is discontinuity in the first order derivative of fugacity calculated in Eq. (3.81). Thus, it means that there is a phase transition but thermodynamic limit is a necessary condition because fugacity given in Eq. (3.81) is a smooth function and there is no singularity and there is no phase transition regardless of how low the temperature is. For N$\longrightarrow\infty$, Eq. (3.81) becomes

$$z = 2\frac{1}{\eta_\lambda + e^{-\beta\varepsilon_k} + |\eta_\lambda + e^{-\beta\varepsilon_k}\,|}, \tag{3.82}$$

$$v_0 = \begin{cases} e^{\beta\varepsilon_k}, & \text{when} \;\; \eta_\lambda < e^{-\beta\varepsilon_k}, \\ \frac{1}{\eta_\lambda}\;, & \text{when} \;\; \eta_\lambda > e^{-\beta\varepsilon_k}. \end{cases}$$

This equation tells us that there is a discontinuity in the derivative of fugacity and the discontinuous point appears at

$$\eta_\lambda = e^{-\beta\varepsilon_k}, \tag{3.83}$$

which is just the phase transition point.

3.5 The Case of $n_0 = \infty$ and $n_i = 1(i \neq 0)$: The Phase Transition Temperature

We now consider the case in which the state whose maximum occupation number is infinite is ground state, $\varepsilon_k = \varepsilon_0 = 0$. In any dimension, there must exist a phase transition. It can be seen directly by observing discontinuity in the derivative of fugacity z from equation Eq. (3.83)

$$\eta_\lambda = 1. \tag{3.84}$$

In order to calculate the transition temperature, from Eq. (3.14) we see that

$$\begin{aligned}
N &= N_\lambda h_{\frac{\nu}{s}}(z), && (3.85)\\
&= \left(\frac{2\Gamma(\frac{\nu}{s})V}{s\Gamma(\frac{\nu}{2})\lambda^\nu}\right) h_{\frac{\nu}{s}}(z), && (3.86)\\
&= \frac{2\Gamma(\frac{\nu}{s})V}{s\Gamma(\frac{\nu}{2})}\frac{(2\Pi^{\frac{s}{2}}mkT)^{\frac{\nu}{s}}}{h^\nu} h_{\frac{\nu}{s}}(z), && (3.87)
\end{aligned}$$

where we have first put value of N_λ and then λ. Now solving for T gives

$$T_c = \frac{h^s}{2\Pi^{\frac{s}{2}}mk}\left[\frac{N}{V}\frac{s\Gamma(\frac{\nu}{2})}{2\Gamma(\frac{\nu}{s})h_{\frac{\nu}{s}}(z)}\right]^{\frac{s}{\nu}}. \tag{3.88}$$

Now we will calculate $h_{\frac{\nu}{s}}(z)$ separately. As we know that $h_{\frac{\nu}{s}}(z)$ depends upon $g_{\frac{\nu}{s}}(z)$. Transition occurs at $\eta_\lambda = 1$ and $z = 1$. This makes $g_{\frac{\nu}{s}}(z)=\zeta(\frac{\nu}{s})$, where ζ is Riemann-zeta function.

$$\zeta\left(\frac{\nu}{s}\right) = \sum_{k=1}^{\infty}\frac{1}{k^{\frac{\nu}{s}}}. \tag{3.89}$$

Thus, at $n = 1$, $h_{\frac{\nu}{s}}(z)$ from Eq. (2.33) reads as

$$h_{\frac{\nu}{s}}(z) = \frac{1}{\Gamma(\sigma)}\int_0^\infty \left[\frac{1}{z^{-1}e^x - 1} - \frac{2}{z^{-2}e^{2x} - 1}\right] x^{\sigma-1}dx. \tag{3.90}$$

Replacing $2x$ by x' in second integral

$$h_\sigma(z) = \frac{1}{\Gamma(\sigma)}\int_0^\infty \left[\frac{1}{z^{-1}e^x - 1}x^{\sigma-1}dx - \int_0^\infty \frac{2^{1-\sigma}}{z^{-2}e^{x'} - 1}x'^{\sigma-1}dx'\right], \tag{3.91}$$

$$g_\sigma(1) = \zeta(\sigma). \tag{3.92}$$

Then,

$$h_{\sigma}(z)\mid_{z\to 1}=\zeta(\sigma)-2^{1-\sigma}\zeta(\sigma), \tag{3.93}$$

or

$$h_{\frac{\nu}{s}}(z)\mid_{z\to 1}=\zeta(\frac{\nu}{s})(1-2^{1-\sigma}). \tag{3.94}$$

Thus phase transition temperature reads as

$$T_c=\frac{h^s}{2\Pi^{\frac{s}{2}}mk}\left[\frac{N}{V}\frac{s\Gamma(\frac{\nu}{2})}{2\Gamma(\frac{\nu}{s})\zeta(\frac{\nu}{s})(1-2^{1-\sigma})}\right]^{\frac{s}{\nu}}. \tag{3.95}$$

Now the total number of particles from (3.14) in the excited state is

$$N_e=N_\lambda f_{\frac{\nu}{s}(z)}. \tag{3.96}$$

Comparing with T_c gives

$$N_e=N. \tag{3.97}$$

This is the condition that determines the PT temperature in an ideal Bose gas for BEC. In our model, this results comes from mathematical rigorous calculation rather than being put in by hand.

We can conclude that when PT occurs, the macroscopic properties of a system will begin to be controlled by a unique quantum state (here it is the ground state). In condensed phase the quantum state with infinite maximum occupation number dominates. This is a BEC type phase transition.

Chapter 4

Conclusion and Summary

In this thesis we have constructed an exactly solvable phase transition model with the help of generalized statistics. In the first chapter, detailed discussion is done on the basics of statistical thermodynamics. An introduction to Bose-Einstein and Fermi-Dirac statistics is also given. In addition to this phase transitions are also discussed in detail which is essentially required for the understanding of this thesis.

The second chapter is totally based on an intermediate statistics, called Gentile statistics, in which the maximum occupation number can take on any random values. All of its thermodynamic properties are obtained with the contribution of ground state. At the same time stress is also laid on the properties of the system at low temperature and high densities.

The third chapter is focussed on the main idea of the thesis. It consists of the construction of a phase transition model which displays Bose-Einstein condensation type phase transition. First of all generalized statistics is introduced in which the maximum occupation number can take on different values. When this maximum occupation number takes values $1, \infty$ or any integer between 1 and ∞, the generalized statistics is reduced to that of Fermi-Dirac, Bose-Einstein and Gentile statistics respectively. The main idea in the solution of this model is the calculation of an exact expression for fugacity with the help of homogeneous form of RH problem. We observe that the discontinuity occurs in the derivative fugacity which results in the determination of phase transition and the phase transition point. With the help of this model we also see that the thermodynamic limit is a necessary condition for phase transition to occur in ideal systems.

Some systems are known to obey statistics other than Fermi-Dirac and Bose-Einstein statistics. The concept of bosons and fermions have appeared together with quantum mechanics itself and till quite recently these have been considered as the only logical possibilities. Now,we know that quantum statistics is not restricted to that of Fermi-Dirac and Bose-Einstein statistics. Indeed, a continuous interpolation between these two is possible. For example, quite recently it has been observed that anyons obey fractional statistics [10, 11, 12].

Moreover, in the model presented in the present thesis, we have both fermionic and bosonic states in our systems. In a Bosonic system if each boson consists of two fermions , then in the system there must simultaneously exist both fermions and bosons due to the fact that there exists an 'ionization' energy. As long as the temperature of the system is not absolute zero, there is always a certain proportion of particles having energies larger than 'ionization' energy and behaving as fermions. In this case the particles in the low lying state behave as bosons and particles in the high lying state behave as fermions. This leads us to the conclusion that a composite system may not accurately be expandable in terms of Bose-Einstein or Fermi-Dirac statistics. In such a composite system, a generalized statistics, like the one presented in this thesis may be useful.

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