DEFINING ENERGY IN GRAVITATIONAL WAVES USING THE POST-NEWTONIAN APPROXIMATION AND COMPARISON OF THE GRAVITATIONAL WAVES BEHAVIOR AT DIFFERENT POST-NEWTONIAN ORDERS

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Dedicated to
My Parents

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Abstract

Newton's theory of gravity proposes instantaneous action at a distance and thus violates the law of cause and effect. However, Einstein's General Theory of Relativity predicts the existence of gravitational waves to carry information throughout the fabric of spacetime. Linearizing the Einstein field equations (EFEs), we get a wave equation of gravity. However, this linearization does not help us in defining the energy associated with gravitational waves. In this thesis, the post-Newtonian (PN) approximation method has been used to define the energy associated with gravitational waves. The lowest order post-Newtonian results have been used to reproduce the Hulse-Taylor binary pulsar signal form.

Going to the higher order PN approximations, the formula for the gravitational wave potential has been recalculated correct up to 1.5 PN order. The higher order results (i.e, 2PN, 2.5PN and 3PN) have been taken from literature. The final results for all these PN orders of approximations are compared and the behavior of gravitational waves at different PN orders has been observed.

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1

Introduction

By the end of the 19th century, it became a common view that Physics is on the verge of completion. Some of the great physicists, like William Thomson and Albert Abraham Morley, made direct claims that the only motivation left in Physics is precision and that there would now be no new discoveries. However, the 20^{th} century dawned with a surprise. It was found that for microscopic particles and for particles moving at velocities comparable to the velocity of light, the laws of Physics (known at that time) become inapplicable. Thus, there was a need to develop complete theories for both these regimes. The one formulated for the microscopic World is called "Quantum Physics" while the other, for ultra-high velocities, is called "Special Relativity (SR)".

The Special Theory of Relativity was formulated by Albert Einstein in 1905. It deals with the motion of macroscopic particles moving with uniform velocities. Nearly 10 years later, Einstein was able to formulate a 'General' Theory of Relativity (GR).

In GR, Einstein articulated a picture of gravity which was not as such present in Newton's theory. In it's very nature, GR is a field theory of gravity. It relates geometry to the distribution of matter by a system of partial differential equations known as the Einstein field equations (EFEs),

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \ , \tag{1.1}$$

where $R_{\mu\nu}$, R, $g_{\mu\nu}$, Λ, G, $T_{\mu\nu}$ are the Ricci tensor, Ricci scalar, metric tensor, cosmological constant, gravitational constant and stress-energy tensor respectively. These tensors and scalars are defined in section 1.2 and 1.3. The left hand side of the field equations represents the geometry of spacetime while the right hand side represents the matter distribution within a specific region of spacetime.

A limitation of Newton's theory of gravity is that it proposes instantaneous action at a distance due to gravity. This is because the scalar potential which specifies Newtonian gravity satisfies Poisson's equation and hence the change in potential due to the change in mass density is instantaneous. This violates the principle of cause and effect. On the other hand, GR predicts the existence of gravitational waves to carry information through the fabric of spacetime and hence preserves causality. The idea is that GR pictures gravity as curvature of the fabric of spacetime, and that the accelerated masses can produce disturbance in this fabric. These disturbances can then propagate through the fabric of spacetime in a similar fashion to ripples over the surface of water. These ripples in the fabric of spacetime are called "Gravitational Waves" and they travel at the speed of light.

Another difference between Newton's theory and GR is that the latter is a highly non-linear theory. Newtonian gravity specified by scalar potential leads to linear differential equations. However, the EFEs are a system of coupled non-linear partial differential equations. This highlights the complexity of GR. We can think of a method to linearize the EFEs and following this process we get a wave equation for gravity (section 2.5). In principle, the exact gravitational waves are solutions of the vacuum field equation (where $T_{\mu\nu} = 0$). Here arises a conceptual problem! The source of gravity is mass, which is equivalent to energy (equation 1.23). Therefore, being solutions of the vacuum field equations, gravitational waves (apparently) seem to carry no energy, and this contradicts the very definition of a wave. This problem of energy associated with gravitational waves along with its possible solution(s) is discussed in the 2^{nd} and the 3^{rd} chapter in detail.

Gravitational waves produce a very small effect on a test particle placed in their path. This makes it really hard to detect them. For 100 years after they were first predicted, there was no direct detection of gravitational waves. However, there was strong indirect evidence of the existence of gravitational waves. In 1974, Hulse and Taylor observed a pulsar with an unseen companion named PSR B1913+16. They were able to observe the orbital period decay of the system by emission of gravitational waves [1]. This example is discussed in the 3^{rd} chapter.

In September 2015 a direct observation of a gravitational wave signal was made by LIGO (Laser Interferometer Gravitational wave Observatory) [2]. The observed signal matches with the numerical relativity simulation of binary black hole mergers. This discovery (or observation) is a milestone achieved in the field of Physics. Some people call it "the discovery of the century".

The 1^{st} chapter is devoted to SR and Differential Geometry, which are needed for GR and give an overview of the conservation laws and the Gaussian flux integrals which give us the flux of energy (momentum) radiating out of a system. In the 2^{nd} chapter, a brief review of GR is given and gravitational waves are discussed as a consequence of GR. In the 3^{rd} chapter, I shall discuss the basics of the post-Newtonian approximation and use it to find the approximate solutions of the EFEs. In the 4^{th} chapter, I shall discuss the behavior of gravitational waves emitted by inspiralling compact binaries at higher post-Newtonian orders.

1.1 Special Relativity

Relativity can be regarded as a theory of motion. In Newton's theory of motion, relative velocity of an object is given by simple vector addition of the velocities of the object and that of an observer. Thus if we consider only 1-dimensional motion, then the magnitude of relative velocity of an object can be given by simple addition or subtraction of magnitudes of velocities of the object and the observer. We can write some transformation equations that allow us to develop a relation between a moving frame and a rest frame. In Classical Physics, Galilean transformation equations are used for this purpose. These transformation equations can be written for 1-dimensional motion as,

$$t' = t, \quad x' = x - vt, \quad y' = y, \quad z' = z \ .$$

if the motion is along x-axis. We can get to Newton's velocity addition formula by differentiating x' with respect to t'. This velocity addition method suits well for objects having speeds very small compared to the speed of light. However, it does not hold when we deal with objects moving at speeds comparable to the speed of light.

1.1.1 Pre-relativistic Mechanics

Before Albert Einstein, a lot of work was done by different scientists in the quest of completing the theory of motion. The most important was the work of Hendrik Lorentz and a famous experiment by Michelson and Morley.

In 1909, Lorentz proposed a theory of motion of electrons [3]. He proposed that there must be a transformation of coordinates and a new "local time" parameter must be introduced. The transformation equations thus formed are known as *Lorentz transformation equations.*

Lorentz Transformations

In order to derive the Lorentz transformation equations, let us consider two observers O and O' as shown in Figure (1.1) such that the observer O' is moving rightward with a velocity v relative to the observer O. We consider that at some instance both the observers coincide, i.e, they are displaced a little along the axis perpendicular to the $x-$axis but the are not displaced along the $x-$axis. This instance can be taken as the origin of time, i.e, both start their clocks at this instant. Both these observers send a signal of light in positive and negative $x-$directions as they start moving. Since the speed of light comes out to be the same for all observers, therefore both the signals travel together. Let the observer O measure time and space by the coordinates (t, x, y, z) and the observer O' by (t', x', y', z').

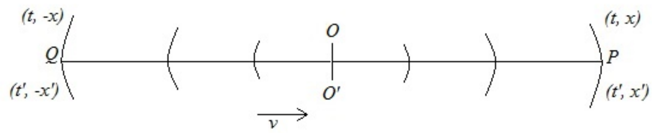

Figure 1.1: Two observers O and O', in different frames which are specified by the coordinates (t, x, y, z) and (t', x', y', z') respectively. Light signals sent by relatively moving observers must travel together. The moving wave-fronts P and Q can be utilized to translate from one frame to another [4].

In order to translate the coordinates from one to the other, we need to find a common point and see how the coordinates of one observer are given in terms of the coordinates of the other observer. We consider that at some instant, the light signal traveling in the positive $x-$direction reach the point P and the other traveling in negative $x-$direction reaches the point Q. The equations for points P and Q according to observer O are given as,

$$ct - x = 0\ , \quad (\text{at } P) \tag{1.2a}$$
$$ct + x = 0\ . \quad (\text{at } Q) \tag{1.2b}$$

Similarly for observer O' are given as,

$$ct' - x' = 0\ , \quad (\text{at } P) \tag{1.3a}$$
$$ct' + x' = 0\ . \quad (\text{at } Q) \tag{1.3b}$$

Since P is given by both the equations (1.2) and (1.3), therefore for both the frames to be physically equivalent we can write,

$$ct' - x' = \lambda(ct - x)\ . \tag{1.4}$$

where λ is constant of proportionality. Similarly for Q we can write,

$$ct' + x' = \mu(ct + x)\ . \tag{1.5}$$

where μ is constant of proportionality. Adding equations (1.4) and (1.5) and dividing by 2 we get,

$$ct' = act - bx\ . \tag{1.6}$$

where a and b are defined as,

$$a = \frac{\lambda + \mu}{2} \ , \tag{1.7a}$$

$$b = \frac{\lambda - \mu}{2} \ . \tag{1.7b}$$

Now subtracting equations (1.4) and (1.5) and dividing by 2 we get,

$$x' = -bct + ax \ . \tag{1.8}$$

In order to find the values of a and b we make use of the fact that the position x of O' according to O can given as $x = vt$, but according to O' it is obviously $x' = 0$. Putting these values in eq. (1.8) we get,

$$0 = -bct + avt \ . \tag{1.9}$$

Since this equation is valid for all values of t, therefore,

$$b = \frac{ax}{c} \ . \tag{1.10}$$

Equations (1.6) and (1.8) then become,

$$ct' = a\left(ct - \frac{v}{c}x\right) \ , \tag{1.11a}$$

$$x' = a\left(x - \frac{v}{c}ct\right) \ . \tag{1.11b}$$

Now, to find out the value of a we make use of the first postulate of special relativity once again. It can be restated as "if we interchange the primed and un-primed variables, it would make no difference." Now let us define x_0 as x at t and x'_0 as x' at t'. Putting $t = 0$ in eq. (1.11b) we get,

$$\frac{x_0}{x'} = \frac{1}{a} \ . \tag{1.12}$$

In order to find out an expression for x'_0 we need to first find out a relationship between x and t when $t' = 0$. The relation thus obtained can then be used to find out an expression for x' at $t' = 0$. Putting $t' = 0$ in eq. (1.11a),

$$ct\big|_{t'=0} = \frac{v}{c}x\Big|_{t'=0} \ . \tag{1.13}$$

Inserting this relation in eq. (1.11b) we get,

$$x'_0 = a\left(x - \frac{v}{c}\frac{v}{c}x\right) \ . \tag{1.14}$$

Thus, we get,

$$\frac{x'_0}{x} = a\left(1 - \frac{v^2}{c^2}\right) \ . \tag{1.15}$$

Since in principle, $x'_0/x = x_0/x'$, therefore,

$$a = \frac{1}{\sqrt{1 - v^2/c^2}} \ . \tag{1.16}$$

Finally we can write the Lorentz transformation equations as,

$$\left.\begin{aligned} t' &= \gamma(t - vx/c^2), \\ x' &= \gamma(x - vt), \\ y' &= y, \\ z' &= z \ , \end{aligned}\right\} \tag{1.17}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$.

The other significant development was Michelson's attempt to measure the velocity of the Earth through the luminiferous aether (using his newly developed interferometer), as it was thought that space is filled with an invisible material (aether) which makes it possible for light to reach the Earth from the Sun. The interferometer consists of a beam splitter, which splits a single beam of light into two beams. These two beams, after reflecting back from mirrors, come back to the same point (in or out of phase). Depending upon the phase difference, these waves interfere constructively or destructively. The constructive interference of two light beams would appear as a bright band, whereas the destructive interference would appear as a dark band on the screen. These bright and dark bands produce interference fringes. Since the relative velocity of Earth is expected to be only in the direction of relative motion, therefore the velocity of light would be different in one direction compared to the other. This would result in a fringe shift. Michelson and Morley expected that a significant fringe shift must be observed because of the relative motion of Earth through the aether, but none was seen. The experiment was repeated again and again, at different locations and at different times of the year, but no fringe shift was observed.

Some explanations were given in the justification of the negative outcome of the Michelson-Morley experiment. One was that the aether may be dragged with the Earth, such that there is no relative velocity of the Earth with respect to the aether. If this was the case then the Earth would lose energy in dragging the aether and would eventually inspiral into the Sun. Another explanation was that the aether does not take any energy during the drag as it is massless. This explanation is even more inappropriate as the aether would then be accelerated infinitely under the action of a small force.

Yet another explanation was given by George Fitzgerald. He proposed that due to some unknown phenomena, there is a contraction in the length of an object in the direction of motion [4]. Thus, for the bodies moving at velocity v the length is not d but d', given as,

$$d' = d\sqrt{1 - v^2/c^2} \ .$$

Following this argument, there would be no path difference for the two light rays and hence no fringe shift would be expected. This idea of contraction of length is called the Lorentz-Fitzgerald contraction.

1.1.2 Einstein's Special Theory of Relativity

In 1905, Albert Einstein came up with an idea that could solve the problems related to the theory of motion. He proposed that in the Classical Kinematics the speed of light ought to be different for different observers. However, Maxwell's equations show that the speed of electromagnetic waves is constant in vacuum. Therefore the idea of observer-dependent speed of light must be dismissed.

He further proposed that if we consider the speed of light to be observer-**independent**, then this consideration directly leads us to the Lorentz transformations. Hence, Einstein put forth his own theory of motion (SR) based on the following postulates:

1. All inertial frames are physically equivalent;

2. Speed of light in vacuum is the same for all observers.

Inertial frames are those in which Newton's 2^{nd} law of motion holds. Thus the laws of physics are the same for a person at rest and a person moving with a uniform velocity. This means that if you are traveling in a car (moving at constant speed in a straight line) then you can flip a coin or pour a drink in a cup just as you could do while sitting on a chair in your room.

The second postulate says that the speed of light in vacuum is observer-independent. If measured by different observers in different reference frames it would come out to be the same. We now look at the basic consequences of the SR.

Time Dilation, Length Contraction and Relativity of Simultaneity

The Lorentz transformations are not directly physically testable. They only refer to the transformation of coordinates. In order to test the Lorentz transformation we need to make predictions in terms of intervals. For example, we can consider what happens to the intervals of time. Let there be an interval of time δt as seen by observer O, i.e, $\delta t = t_1 - t_2$. The position remains the same according to O, therefore $x_1 = x_2$ [4]. Writing down the Lorentz transformations for both the times t_1 and t_2,

$$t_1' = \gamma \left(t_1 - \frac{v}{c^2} x_1 \right) \ , \tag{1.18a}$$

$$t_2' = \gamma \left(t_2 - \frac{v}{c^2} x_2 \right) \ . \tag{1.18b}$$

The time interval $\delta t'$ as seen by the observer O' is $\delta t' = t_1' - t_2'$. Thus,

$$\delta t' = \gamma \left[(t_1 - t_2) - \frac{v}{x^2}(x_1 - x_2) \right] \ . \tag{1.19}$$

Since $x_1 = x_2$, therefore,

$$\delta t' = \frac{\delta t}{\sqrt{1 - v^2/c^2}} \ . \tag{1.20}$$

This is known as the *time dilation* formula. The factor $\sqrt{1 - v^2/c^2}$ is always less than 1. Therefore, the unit of measurement of time in a moving frame would always be larger as compared to the that in a rest frame. This means that a moving clock would run slower as compared to the one at rest.

Einstein proposed that if we synchronize two clocks and then send one of these clocks to a long journey while the other is kept stationary, then when the moved clock is brought back, it would be found lagging behind the one at rest. This does not happen due to some fault in one of the clocks, but this happens because the time has actually passed differently for both the clocks. Einstein put forth this idea in the form of his so-called "thought experiment". Let us replace the clocks with humans beings. To be synchronized, we consider them to be twins. One of the twins becomes an astronaut and goes to a long space journey while the other stays on Earth (say he becomes a politician). After many years, the astronaut comes back to the Earth. When he meets his twin brother, a young astronaut is amazed to see an old politician. This thought experiment says that more time has passed for the politician than that for the astronaut.

We can also think of measuring the spatial intervals. Let the spatial interval for O be defined as $\delta x = x_1 - x_2$. Now the two ends of the spatial interval must be seen simultaneously by the observer O', i.e, $t_1' = t_2'$. Then from eq. (1.17) we can write,

$$t_1 - t_2 = \frac{v}{c^2}(x_1 - x_2) \ . \tag{1.21}$$

Then the spatial interval according to O' is given as,

$$\begin{aligned}
\delta x' &= x'_1 - x'_2 = \gamma\left[(x_1 - x_2) - v(t_1 - t_2)\right] \ , \\
\delta x' &= \gamma\left[(x_1 - x_2) - \frac{v^2}{c^2}(x_1 - x_2)\right] \ , \\
\delta x' &= \gamma \delta x (1 - v^2/c^2) \ , \\
\delta x' &= \delta x \sqrt{1 - v^2/c^2} \ .
\end{aligned} \tag{1.22}$$

This is how we get the Lorentz-Fitzgerald contraction formula.

Relativity of Simultaneity

Let us think of two spatially separated events which occur simultaneously according to an observer O, one occurs at x_1 and the other at x_2. According to another observer O', moving with velocity v relative to O, both the events occur at,

$$t'_1 = \gamma\left(t_1 - \frac{v}{c^2}x_1\right) = \gamma\left(t - \frac{v}{c^2}x\right) \ ,$$

and at,

$$t'_2 = \gamma\left(t_2 - \frac{v}{c^2}x_2\right) = \gamma\left(t - \frac{v}{c^2}x_2\right) \ .$$

The difference in time of both events is not zero for O', but it is,

$$t'_1 - t'_2 = \gamma\frac{v}{c^2}\left(x_2 - x_1\right) \ .$$

This shows that *simultaneity* is relative.

Energy-Mass Equivalence

One of the most astonishing ideas of Einstein was the equivalence of mass and energy. It follows from the fact that as a body moves faster it appears to gain more inertia. Thus the relativistic mass is greater than the rest mass of a body, i.e.

$$m' = m_0/\sqrt{1 - v^2/c^2} \ , \tag{1.23}$$

where m_0 is known to be the rest mass of a body. Squaring both sides and simplifying we get,

$$m'^2c^2 - m'^2v^2 = m_0^2c^2 \ .$$

Taking the differential on both sides we get,

$$2m'c^2dm' - 2m'v^2dm' - 2vm'^2dv = 0 \ ,$$

since the term $m_0^2c^2$ is constant. Further simplification leads us to,

$$c^2dm' = v^2dm' + m'vdv \ . \tag{1.24}$$

The change in kinetic energy 'K.E' is equal to the change in work. i.e.

$$dK = dW = FdS' \ , \tag{1.25}$$

where F is the force defined by Newton's 2^{nd} law of motion as the change in momentum with respect to time and S' is the distance covered by the body. Since the force is defined as the change in momentum and momentum is defined as the product of mass and velocity of a body, therefore we can write,

$$F = \frac{dp}{dt'} = \frac{d(m'v)}{dt'} = m'\frac{dv}{dt'} + v\frac{dm'}{dt'} \ ,$$

Putting this in eq. (1.7) we have,

$$dK = m'\frac{dS'}{dt'}dv + v\frac{dS'}{dt'}dm' \ ,$$

where dS'/dt' is the velocity v of the body. Thus we can write,

$$dK = m'vdv + v^2dm' \ . \tag{1.26}$$

Comparing eq. (1.24) and eq. (1.26) we can write,

$$dK = c^2dm'.$$

Integrating both sides within the limits 0 to K for dK and from m_0 to m for dm' we have,

$$K = mc^2 - m_0c^2 \ .$$

The total energy of a body is the kinetic energy K of the body plus the rest mass energy m_0c^2. Therefore, the total energy E of a moving body is given as,

$$E = mc^2 \ . \tag{1.27}$$

Now this is very beautiful to think about. Recall the time when one tries to memorize the definition of basic sciences from his/her text book in school. There we encounter the phrase, "Physics is the study of matter, energy and their mutual interaction". One concludes from it that Physics deals with the study of two separate things, namely matter and energy. From the energy-mass equivalence we have come to know that both matter and energy are actually the same thing with different appearances.

Einstein's Velocity Addition Formula

Einstein showed that the second postulate of Special theory directly leads to the Lorentz transformations (as we have seen in the previous sections). Therefore, we can differentiate the spatial component x' in Lorentz transformation equations (eq. (1.17)) with respect to the time component t' to get a relativistic velocity addition formula. i.e.

$$u_r = \frac{u \mp v}{1 \mp (uv)/c^2} \ . \tag{1.28}$$

This is also called Einstein's velocity addition formula. The minus and plus sign come from Lorentz and inverse Lorentz transformations respectively (i.e. it depends upon the relative direction of motion of two objects). After putting $u = c$ in this formula we can do simple arithmetic and it comes out that the answer is c again. This shows that there exists a relativistic upper bound in the theory of motion. The speed of light c is the maximum possible speed at which a body can move in vacuum.

This is a mind-blowing fact indeed. If a person is engaged in a debate with a *relativist*, the relativist would say that the speed of light is the maximum possible speed available for anything to travel in a vacuum. The person might ask him, "Think of a particle traveling at some velocity v. It would take a time t for the particle to reach its destination, if it would travel at the speed of light. Suppose the particle has reached its destination earlier than t. This surely means that the particle has traveled faster than c". The relativist would say "No, it didn't." The other person would then ask him to justify the reason for the early arrival of the particle. The relativist would say "Well, the particle didn't reach early because it traveled faster than light. It reached early because moving at relativistic velocity, the time got stretched or the distance got shrunk for the particle. Therefore the particle had more time or had to cover less distance to reach its destination."

1.1.3 The 4-Vector Notation

Soon after the publication of the SR, Hermann Minkowski (who happened to be a teacher of Mathematics to Einstein once) presented a more mathematical form of the theory in the 4-vectors notation. Initially, Einstein seemed to deny the importance of the 4-vector notation and considered it just a waste of time. However, it was of great importance for him in the formulation of his General Theory.

The 4-vector notation is actually a mathematical trick to write different components of a vector under one argument with a certain index. This index can then give different components of a vector. For instance, the position 4-vector can be represented as,

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, x, y, z),$$

where time is taken to be the zeroth dimension (component) of the position 4-vector. This language (or formulation) does not only serves well for the description of the SR but also for the description of the GR. It is not restricted to vectors alone, but is also good for higher rank tensors description.

Invariant Quantities

While discussing "Relativity", it feels a bit weird when the word "invariant" comes in. However, it is a common misconception that SR proposes everything to be relative. There are certain quantities which remain invariant under coordinate transformations.

First of all, scalar quantities are invariant under any kind of coordinate transformation. This means that the magnitude of a vector or a scalar potential remains invariant under the transformation of coordinates.

A vector can be invariant in a sense that during a transformation, the components of the vector transform but not the actual vector. Thus the transformation of the components is recovered by the transformation of the basis vectors, leaving the vector invariant. If we represent a vector '$\mathbf{V}(\mathrm{x})$' in its components form as,

$$\mathbf{V}(\mathrm{x}) = V^x\mathbf{i} + V^y\mathbf{j} + V^z\mathbf{k} = \sum_{i=1}^{3} V^i(x)\mathbf{e}_i = V^i(x)\mathbf{e}_i \ ,$$

then the transformation of a vector is carried out by the transformation law,

$$V'^a(x') = \frac{\partial x'^a}{\partial x^i} V^i(x) \ ,$$

where $\mathbf{e}_i$ represent the basis vectors. Here we have used the Einstein's summation convention, which requires that repeated indices imply summation. A vector satisfying the transformation law discussed above is called a contravariant vector. Similarly a covariant vector transforms by the transformation law,

$$V'_a(x') = \frac{\partial x^i}{\partial x'^a} V_i(x) \ .$$

The difference between a contravariant and a covariant vector is that a contravariant vector transforms as a general vector whereas a covariant vector transforms as the gradient of a scalar field under a coordinate transformation.

Another invariant quantity is a tensor. The invariance of a tensor is same in sense as that of a vector. A 2^{nd} rank tensor can be represented as,

$$\mathbf{K} = K^{ij}(x)\mathbf{e}_i(x)\mathbf{e}_j(x) \ .$$

The transformation laws for a contravariant, covariant and a mixed tensor are given respectively as,

$$K'^{ab}(x') = \frac{\partial x'^a}{\partial x^i}\frac{\partial x'^b}{\partial x^j}K^{ij}(x) \ ,$$
$$K'_{ab}(x') = \frac{\partial x^i}{\partial x'^a}\frac{\partial x^j}{\partial x'^b}K_{ij}(x) \ ,$$
$$K'^a_b(x') = \frac{\partial x'^a}{\partial x^i}\frac{\partial x^j}{\partial x'^b}K^i_j(x) \ .$$

Lorentz Transformations in Four-Vector Notation

We can represent Lorentz transformation equations in 4-vector notation as,

$$x'^\mu = \Lambda^\mu_a x^a \ ,$$

where Λ^μ_a is called the Lorentz transformation matrix. In the matrix form we can write these transformation equations as,

$$\begin{pmatrix} \mathrm{ct}' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \mathrm{ct} \\ x \\ y \\ z \end{pmatrix} \ , \tag{1.29}$$

where $\beta = v/c$.

1.1.4 The Light Cone

Consider an infinitesimal displacement vector, whose magnitude remains invariant under Lorentz transformation, such that,

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = c^2dt^2 - d\mathbf{x}.d\mathbf{x} \ .$$

The magnitude of the vector can be positive, negative, or zero. Corresponding to these conditions, a 4-vector is called time-like, light-like or a space-like vector,

$$\begin{aligned} c^2dt^2 - d\mathbf{x}.d\mathbf{x} > 0 &\quad \text{time-like} \ , \\ c^2dt^2 - d\mathbf{x}.d\mathbf{x} = 0 &\quad \text{light-like} \ , \\ c^2dt^2 - d\mathbf{x}.d\mathbf{x} < 0 &\quad \text{space-like} \ . \end{aligned} \tag{1.30}$$

A time-like vector correspond to the objects which have a velocity less than the velocity of light. This represents the actual path of a physical (massive) object. A light-like vector corresponds to objects having a velocity equal to that of the velocity of light. Only photons follow the light-like path. A space-like vector correspond to objects traveling at velocities greater than the that of the velocity of light. This is not possible for any physical object as the magnitude of vector becomes negative for this condition. This whole idea is represented in the form of a cone structure, which is called the "Light Cone"

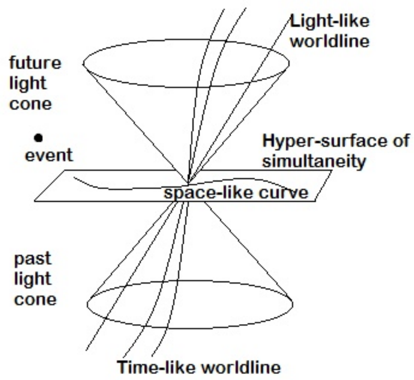

Figure 1.2: The light cone. The lower half of the light cone represents the past whereas the upper half represents the future. Present is represented in the form of hypersurface of simultaneity. An event is a point in the light cone which describes the 'where' and 'when' of an event.

1.2 Tensor Algebra and Tensor Calculus

General Relativity talks in the language of tensors, which makes it essential to study tensors beforehand. Moreover, in GR we have to deal with curved spaces, rather than flat spaces. The operations of flat spaces do not apply to curved spaces. Therefore we need to find out a more abstract definition of a space. These abstract spaces are called *manifolds*.

A manifold of dimension n is a topological space which consists of points that have a neighborhood which locally resembles a Euclidean space ($\mathbb{R}^n$) of dimension n [5]. This means that no matter how complicated is the topology of a space, it looks like $\mathbb{R}^n$ in a local region. The entire manifold is then constructed by smoothly sewing together these local regions. For a space to be regarded as a manifold, it must satisfy the following conditions:

1. It must be a continuum everywhere and must not have many points everywhere;
2. It must be connected everywhere and there must be unique limit points;
3. It must be possible to provide the same number of coordinates for each region of the space.

The examples of a manifold are the Euclidean space ($\mathbb{R}^n$), the n-sphere (S^n), the n-torus (T^n), etc. Two cones connected at their vertices do not form a manifold as there exists a point where it does not locally look like a Euclidean space. A single cone can be thought of a manifold but not a smooth one due to the singularity at its origin. It could be better described by the notion "manifold with boundary".

A tensor '**K**' of rank $(n+m)$ at any point in space can be defined as a set of numbers with indices $n+m$, which transforms over the transformation of coordinates as,

$$K'^{\alpha_1...\alpha_n}_{\beta_1...\beta_m} = \Pi^{\alpha_1}_{\mu_1}...\Pi^{\alpha_n}_{\mu_n}(\Pi^{-1})^{\nu_1}_{\beta_1}...(\Pi^{-1})^{\nu_m}_{\beta_m}K^{\mu_1...\mu_n}_{\nu_1...\nu_m} ,$$

here 'Π' serves as a transformation matrix in general. A tensor of such kind is known as a tensor of contravariant rank n and covariant rank m, and of total rank $n+m$. In fact matrices, vectors and scalars are all specializations of tensors of rank 2,1 and 0 respectively.

A tensor '$K^{\mu\nu}$' is said to be symmetric in the pair of indices (μ,ν), if it remains the same when its indices are interchanged. i.e.

$$K^{\mu\nu} = K^{\nu\mu} .$$

. On the other hand, it is called anti-symmetric if it changes the symbol under the same operation. i.e.

$$K^{\mu\nu} = -K^{\mu\nu} .$$

. Addition (or subtraction) of two tensors can only be carried out if both are of same rank and of same type. Examples of possible additions of tensors are,

$$z^{\mu\nu} = x^{\mu\nu} + y^{\mu\nu} ,$$
$$z^{\alpha}{}_{\beta\gamma} = x^{\alpha}{}_{\beta\gamma} + y^{\alpha}{}_{\beta\gamma} ,$$

whereas following type of combinations for tensors cannot be added,

$$x^{\mu\nu} + y_{\mu\nu} ,$$
$$x^{\mu\nu} + y^{\alpha\beta} ,$$
$$x^{\alpha}{}_{\mu\nu} + y_{\mu\nu} .$$

The addition of two tensors gives a new tensor of the same rank and type. Tensor addition is commutative and associative.

The multiplication of two tensors can only be carried out if one is a contravariant tensor and the other is a covariant tensor in the same index. The product of two tensors is again a tensor whose contravariant and covariant indices comprise all indices of the participants. e.g.

$$K^{\alpha\beta}_{\ \ \gamma} = Q^{\alpha}_{\ \gamma} P^{\beta} \ ,$$
$$K^{\nu} = X_{\mu} P^{\mu\nu} \ .$$

For a tensor of at least 1 contravariant index and 1 covariant index, the internal inner product (or trace) can be defined as,

$$K^{\alpha}_{\ \alpha} \ .$$

This is actually the trace of a matrix $K^{\alpha}_{\ \beta}$ and it is invariant under coordinate transformation. One can also do the contraction of indices for higher rank mixed tensors as,

$$K^{\alpha\beta}_{\ \ \alpha} = P^{\beta} \ .$$

The result is a new tensor 'P^{β}' of rank β. Following this procedure, one can get the order of a tensor reduced from (n, m) to $(n-1, m-1)$. This simplifies the mathematics over the loss of some information.

Now we consider the calculus involving tensors. Tensor Calculus is basically an extension of the Vector Calculus to the tensor fields. The calculus of tensors is not entirely trivial. The presence of indices makes it easy to manipulate things in an easy way but it also makes it too easy to write expressions that have no physical meaning or have bad properties [6].

So far we have been concerned with the properties of tensors at a given point, but in differential geometry the main point of concern are tensor fields where the tensors depend upon the location given by $(\mathbf{x})$. Now we can take the derivative of the tensor fields and the derivatives can also be represented by an index. For example the gradient and the divergence of a tensor field $K^{\mu\nu}$ in the flat space can be given as,

$$K^{\mu\nu}_{\ \ ,\alpha} = \frac{\partial K^{\mu\nu}}{\partial x^{\alpha}} \ ,$$
$$K^{\mu\nu}_{\ \ ,\nu} = \frac{\partial K^{\mu\nu}}{\partial x^{\nu}} \ .$$

For a curved space, simple derivatives do not behave as tensors and they do not have a proper physical meaning. In order to make sense in the curved space, we need to use the covariant derivatives in place of simple derivatives. The covariant derivative gives the total derivative of a vector, i.e. the derivative of the vector components and that of the basis vectors. Let us consider a vector $\mathbf{V} = V^{i}\mathbf{e}_{i}$. If we differentiate $\mathbf{V}$ with respect to a position vector with index k (to differentiate from index i), then we get,

$$\frac{\partial \mathbf{V}}{\partial x^k} = (\frac{\partial V^i}{\partial x^k})\mathbf{e}_i + V^i(\frac{\partial \mathbf{e}_i}{\partial x^k}) \ ,$$

where $\partial \mathbf{e}_i/\partial x^k = \Gamma^j_{ki}\mathbf{e}_j$. Using this in the upper equation and inverting the dummy indices gives us the definition of a covariant derivative. i.e.

$$V^i_{\ ;k} = V^i_{\ ,k} + \Gamma^i_{jk}V^j \ ,$$

where $V^i_{\ ;k}$ represents the covariant derivative of V^i with respect to x^k in a curved space, $V^i_{,k}$ represents the simple derivative of V^i with respect to x^k in flat space and $\Gamma^\alpha_{\mu\nu}$ is called the *Christoffel symbol* which serves as a connection between the manifold and the surface tangent to it. Christoffel symbols are defined in terms of the metric tensor as,

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2}g^{\alpha\beta}(g_{\mu\beta,\nu} + g_{\nu\beta,\mu} - g_{\mu\nu,\beta}) \ . \tag{1.31}$$

The Christoffel symbols contain all the information about the curvature of coordinate system and can therefore be transformed to zero if the coordinates are straightened up. Hence, they are not tensor quantities. Christoffel symbols are also symmetric in the lower pair of indices (i.e. $\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\nu\mu}$). For contravariant and covariant tensors of rank 2, the covariant derivative is written as,

$$\begin{aligned} K^{\mu\nu}_{\ \ ;\alpha} &= K^{\mu\nu}_{\ \ ,\alpha} + \Gamma^\mu_{\alpha\beta}K^{\beta\nu} + \Gamma^\nu_{\alpha\beta}K^{\beta\mu} \ , \\ K_{\mu\nu;\alpha} &= K_{\mu\nu,\alpha} - \Gamma^\beta_{\alpha\mu}K_{\beta\nu} - \Gamma^\beta_{\alpha\nu}K_{\beta\mu} \ . \end{aligned}$$

This can be generalized to the tensors of any rank. The number of times Christoffel symbols appear in the definition of covariant differentiation is equal to the rank of a tensor.

Metric Tensor

Of prime importance in GR is the metric tensor. It comes in the picture from the idea that every point in space is a sort of generalization of the Pythagoras theorem. The distance between two points in Euclidean space is given as the square root of the sum of squares of the distances covered in each direction (or coordinate axis). So, a distance Δs between two points can be written as $\Delta s = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$. This formula encodes the geometry of Euclidean space and it can easily be written in the form of a metric. Now, the idea of Pythagoras can be extended to the geometry of curved spaces. For instance, an infinitesimal displacement **dx** for an arbitrary space can be written as,

$$\mathbf{dx} = \mathbf{e}_\alpha dx^\alpha \ ,$$

Since **dx** is a vector quantity, so its scalar product with itself (i.e. **dx**.**dx**) can be written as,

$$dx^2 = g_{\alpha\beta} dx^\alpha dx^\beta \ , \tag{1.32}$$

where $g_{\alpha\beta}$ is the metric tensor and it is defined as,

$$g_{\alpha\beta} = \mathbf{e}_\alpha . \mathbf{e}_\beta \ .$$

The metric tensor is symmetric and it depends upon the position of coordinates. The inverse metric tensor is the contravariant form of metric tensor,

$$g^{\alpha\beta} = \mathbf{e}^\alpha . \mathbf{e}^\beta \ .$$

The product of the metric tensor with its inverse is the *Kronecker delta function.*

$$g_{\alpha\beta} g^{\beta\gamma} = \delta^\gamma_\alpha \ .$$

The delta function is defined as,

$$\delta^\gamma_\alpha = \begin{cases} 1 & \text{if } \alpha = \beta \ , \\ 0 & \text{if } \alpha \neq \beta \ . \end{cases}$$

Moreover, the metric tensor (in covariant or contravariant form) can be used to raise or lower indices of a tensor. e.g.

$$\begin{aligned} K^\alpha_{\ \nu} &= g^{\alpha\mu} K_{\mu\nu} \ , \\ K_{\mu\nu} &= g_{\alpha\mu} g_{\beta\nu} K^{\alpha\beta} \ . \end{aligned}$$

1.3 Curvature Tensors and Scalars

Before going on to the curvature tensors, let us first go through an overview of the curvature. The curvature of a surface is defined as the amount by which that surface deviates from being a flat plane, or of a curve that deviates from being a straight line. The curvature of a surface can be intrinsic or extrinsic. An intrinsic curvature is one that can be observed from within the surface (i.e. someone existing on the same surface can measure its curvature). However, an extrinsic curvature is one that cannot be observed from within the surface.

The intrinsic curvature of a surface can be measured using the idea of parallel transport. Consider a closed curve of an arbitrary shape on an arbitrary surface. Then draw a vector on the surface such as the tail of the vector lies on some point on the curve. Now transport this vector throughout the curve by keeping the direction of the vector parallel to itself. Observe the vector when it reaches the point where it started. If the vector coincides with its previous image, then the surface has no intrinsic curvature (i.e. the surface is flat) and if the vector does not coincide with its image then the surface is curved.

For two dimensional surfaces, the intrinsic curvature is defined by Gauss' invariant intrinsic curvature (Gaussian curvature). The Gaussian curvature, K, of a surface can be defined as the product of the principle curvatures κ_1 and κ_2 at a given point. i.e.

$$K = \kappa_1.\kappa_2 \ ,$$

To understand the concept of the Gaussian curvature, consider a surface of any shape (surface of a sphere for example). At any point on the surface, we can find a normal vector to the surface. The plane which contains a normal vector is called a normal plane. This normal plane will intersect the surface in a curve. This curve is called a normal section and its curvature is called the normal curvature. The maximum and minimum values of normal curvatures are called principle curvatures. Then finally, the product of two principle curvatures gives the Gaussian curvature of a surface.

The generalization of the Gaussian intrinsic curvature to higher dimensional spaces can be obtained by carrying a basis vector along two different directions in opposite order and then taking the difference of the two results. This can be done by the use covariant derivatives. We have got the knowledge that covariant differentiation is sort of a generalization of partial differentiation. But there is a great difference in both. Covariant differentiation yields a tensor as output whereas partial differentiation does not. Also, the order of differentiation matters while taking covariant derivative of a tensor field. Let us take for example the covariant derivative of an arbitrary tensor K of rank 1,

$$K^a{}_{;c} = K^a{}_{,c} + \Gamma^a_{bc} K^b \ . \tag{1.33}$$

Taking covariant derivative of eq. (1.33) again,

$$K^a{}_{;c\,;d} = (K^a{}_{,c})_{,d} + \Gamma^a_{ed} K^e{}_{;c} + \Gamma^e_{cd} K^a{}_{;e} \ . \tag{1.34}$$

Expanding eq. (1.34) with respect to the derivatives involved we get,

$$K^a{}_{;c\,;d} = (K^a{}_{,c} + \Gamma^a_{bc} K^b)_{,d} + \Gamma^a_{ed}(K^e{}_{,c} + \Gamma^e_{bc} K^b) - \Gamma^e_{cd}(K^a{}_{,e} + \Gamma^a_{be} K^b) \ . \tag{1.35}$$

Now interchanging the order of differentiation in eq. (1.35) we get,

$$K^a{}_{;d\,;c} = (K^a{}_{,d} + \Gamma^a_{bd} K^b)_{,c} + \Gamma^a_{ec}(K^e{}_{,d} + \Gamma^e_{bd} K^b) - \Gamma^e_{dc}(K^a{}_{,e} + \Gamma^a_{be} K^b) \ . \tag{1.36}$$

Subtracting eq. (1.36) from eq. (1.35), balancing the dummy indices and using the symmetry property of Christoffel symbols we get,

$$K^a{}_{;c\,;d} - K^a{}_{;d\,;c} = R^a{}_{bcd}K^b \ , \tag{1.37}$$

where $R^a{}_{bcd}$ is the "Riemann curvature tensor" and is to be,

$$R^a{}_{bcd} = \Gamma^a_{bd,c} - \Gamma^a_{bc,d} + \Gamma^a_{ec}\Gamma^e{}_{bd} - \Gamma^a_{ed}\Gamma^e_{bc} \ . \tag{1.38}$$

It is clear from eq. (1.37) that Riemann curvature tensor is a tensor quantity as the difference of two tensors is again a tensor. Riemann curvature tensor is a tensor of rank 4 and it describes the curvature of a manifold. If $R^a{}_{bcd} = 0$ then the space is flat and if $R^a{}_{bcd} \neq 0$ then the space is curved.

We can transform $R^a{}_{bcd}$ into its covariant form by contraction of indices i.e. $R_{abcd} = g_{af}R^f{}_{bcd}$. It should be noted that $R_{abcd} \neq R^a{}_{bcd}$. The Riemann tensor is anti-symmetric in first two as well as last two indices i.e. $R_{abcd} = -R_{bacd}$ and $R_{abcd} = R_{abdc}$. However, it is symmetric if the first and last pairs are interchanged i.e. $R_{abcd} = R_{cdab}$. Riemann tensor also satisfies the cyclic property,

$$R_{bcd} + R_{acdb} + R_{adbc} = 0.$$

This is known as the *first Bianchi identity*. In a local coordinate system[1] about any arbitrary point P, i.e. $\Gamma^a_{bc}(P) = 0$ but $\Gamma^a_{bc,d}(P) \neq 0$, Riemann curvature tensor can be written as,

$$R_{bacd} = \frac{1}{2}(g_{bc,ad} - g_{ac,bd} + g_{ad,bc} - g_{bd,ac}) \ .$$

The covariant derivative of Riemann tensor can then be written as,

$$R_{abcd;f} = R_{abcd,f} = \frac{1}{2}(g_{bc,ad} - g_{ac,bd} + g_{ad,bc} - g_{bd,ac})_{,f} \ .$$

Permutation over c,d and f cyclically leads us to the result,

$$R_{abcd;f} + R_{abdf;c} + R_{abfc;d} = 0 \ . \tag{1.39}$$

This relation is known as the *second Bianchi Identity*. It holds in all coordinate systems.

Contracting the first and the third index of Riemann tensor we can get a symmetric tensor of rank 2. This 2^{nd} rank symmetric tensor is known as the *Ricci tensor* and it is defined as,

$$R_{ab} = R^d{}_{adb} \ . \tag{1.40}$$

[1] For instance the Riemann normal coordinates, which will be discussed in section 2.4 in detial

From the definition of Riemann tensor in eq. (1.38), the Ricci tensor can be written as,

$$R_{bd} = \Gamma^{a}_{bd,a} - \Gamma^{a}_{ba,d} + \Gamma^{a}_{ea}\Gamma^{e}_{bd} - \Gamma^{a}_{ed}\Gamma^{e}_{ba} \ . \tag{1.41}$$

The Christoffel symbols Γ^{a}_{ba} can be further simplified to the form $(ln\sqrt{|g|})_{,b}$. Hence the Ricci tensor can also be written as,

$$R_{bd} = \Gamma^{a}_{bd,a} - (ln\sqrt{|g|})_{,bd} + (ln\sqrt{|g|})_{,e}\Gamma^{e}_{bd} - \Gamma^{a}_{ed}\Gamma^{e}_{ba} \ . \tag{1.42}$$

The Ricci tensor is the trace of the Riemann tensor. We can further contract the indices of the Ricci tensor using the metric tensor to get the *Ricci Scalar*, which is the trace of the Ricci tensor.

$$R = g^{ab}R_{ab} \ . \tag{1.43}$$

The Riemann curvature tensor is helpful for determining the nature of a singularity. A singularity is a point (or space) where the curvature of a space becomes infinite. The problem with the Riemann tensor is that it is expressed in terms of coordinates. Therefore any problem within the coordinate system would affect the components of the Riemann curvature tensor. Scalars, on the other hand, are invariant under coordinate transformations. Thus, we need to construct scalars from the Riemann tensor. Infinitely many scalars are possible to be constructed from the Riemann tensor. However, symmetry conditions can be used to show that there can be only finitely many possibilities of scalars. These scalars can be written as,

$$\begin{aligned} \mathcal{R}_1 &= g^{ab}R_{ab} \ , \\ \mathcal{R}_2 &= R^{ab}_{\ \ cd}R^{cd}_{\ \ ab} \ , \\ \mathcal{R}_3 &= R^{ab}_{\ \ cd}R^{cd}_{\ \ ef}R^{ef}_{\ \ ab}\cdots \ . \end{aligned} \tag{1.44}$$

If all the independent curvature invariants defined above are finite then the singularity that exists is a coordinate singularity [2]. But if any of the curvature invariants is infinite, then the singularity is essential [3].

1.4 Geodesics

A geodesic is the generalization of the concept of a straight line to manifolds. In order to give a general definition of a straight line we can make use of the idea of

[2]A coordinate singularity is the one that arises due to the poor choice of coordinates. It can be rooted out by an appropriate choice of coordinates.

[3]An essential singularity is the one that is a characteristic of the space. An essential singularity cannot be removed by coordinate transformations

parallel transport once again. Let us take two points on a flat surface and try to connect them by means of arbitrary curves. Then draw a tiny vector tangent to a curve with its tail at one of the two points. Now parallel transport this vector through the curve to reach the other point. If this tangent vector remains tangent to the curve throughout the transport, then the curve is said to be a straight line.

If this process is repeated on a manifold, then we get a generalization of what is called a straight line in a flat space, i.e. a *geodesic*. A tangent vector t^a on a curve λ, parametrized by $x^a(\lambda)$, is defined as,

$$t^a = \frac{\partial x^a}{\partial \lambda} \ . \tag{1.45}$$

In order to transport this tangent vector throughout the curve, we differentiate t^a with respect to λ. If the absolute derivative of t^a vanishes with respect to λ, then the curve is a geodesic. i.e.

$$\frac{Dt^a}{D\lambda} = 0 \ . \tag{1.46}$$

Upper case is used to differentiate it from the ordinary derivative. Let the absolute derivative of a vector V^a on a curve λ be given as,

$$\frac{DV^a}{D\lambda} = \frac{dV^a}{d\lambda} + \Gamma^a_{bc} t^b V^c \ , \tag{1.47}$$

where t^b is tangent vector to the curve. From eq. (1.46) we know that for a curve to be geodesic, the absolute derivative of the tangent vector must vanish. By using t^a from eq. (1.45) into eq. (1.47) to get the absolute derivative of t^a with respect to λ and then using the condition (1.46) we get,

$$\frac{d^2x^a}{d\lambda^2} + \Gamma^a_{bc}\frac{dx^b}{d\lambda}\frac{dx^c}{d\lambda} = 0 \ . \tag{1.48}$$

This is called the geodesics equation and its solutions are called geodesics.

Geodesics as the Shortest Path Between Two Points in a Manifold

We are aware of the fact that the shortest path between two points is a straight line within a flat space. Now we need to find out the shortest path between two points on a manifold. This can be done by using Euler-Lagrange equation. The distance between two points (say A and B) is specified within a manifold by a line element,

$$
\begin{aligned}
S_{AB} &= \int_B^A dS \\
&= \int_B^A g_{ab}(x^c)\dot{x}^a\dot{x}^b dS \\
&= \int_B^A \mathcal{L}[x^a, \dot{x}^a]^b dS \ .
\end{aligned} \tag{1.49}
$$

Then, we can write,

$$
\begin{aligned}
\frac{\partial \mathcal{L}}{\partial x^c} &= g_{ab,c}\dot{x}^a\dot{x}^b \ , \\
\frac{\partial \mathcal{L}}{\partial \dot{x}^c} &= g_{cb}\dot{x}^b + g_{ac}\dot{x}^a \ .
\end{aligned}
$$

The Euler-Lagrange equation for x^c is then,

$$
\begin{aligned}
\frac{d}{dS}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^c}\right) - \frac{\partial \mathcal{L}}{\partial x^c} = \left(\frac{dx^d}{dS}\frac{\partial}{\partial x^d}g_{ac}\right)\dot{x}^a + \left(\frac{dx^d}{dS}\frac{\partial}{\partial x^d}g_{cb}\right)\dot{x}^b \\
+ g_{ac}\frac{\dot{x}^a}{dS} + g_{cb}\frac{\dot{x}^b}{dS} - g_{ab,c}\dot{x}^a\dot{x}^b = 0 \\
= 2g_{cd}\ddot{x}^d + (g_{ac,b} + g_{bc,a} - g_{ab,c})\dot{x}^a\dot{x}^b = 0 \ .
\end{aligned} \tag{1.50}
$$

where the comma in the subscript represents the partial derivative with respect to the position coordinate with the same index i.e. $g_{ac,b} = \partial g_{ac}/\partial x^c$.Inverting the metric in this equation, we get the geodesic equation. This means that geodesics are the shortest paths between two points on a manifold. It must be noted that the shortest path between two points is always a geodesic but a geodesic is not always the shortest path. For example, if a person continue to move on a geodesic on the surface of the sphere, he would eventually reach the point from where he set off his journey. Therefore, it is worthwhile to mention that the geodesics are locally the shortest path between two points on a manifold and not globally.

Lie Derivative

Before we go on to the geodesic deviation, we need to go through the concept of the Lie derivative. The Lie derivative gives the derivative of a tensor field (tensor, vector and scalar fields) in the flow of the tangent vector field. When we differentiate a function in a flat space we can easily write $[f(x+h) - f(x)]/\delta x$. However, this simple derivative does not hold within a manifold as it is not invariant under coordinate transformations. There are two ways of defining an invariant derivative of a tensor along a curve in a manifold. One way is to ignore the effects of coordinatization upon the tensor and perform the derivative on the tensor. This is called *absolute* or *intrinsic derivative* (This was first introduced in eq. 1.46). In actual the derivative is applied on the image of the tensor in the coordinate system. The other way is to pull out the effects of coordinatization and compute the effects of the derivative on the tensor in the manifold. This is called the *Lie derivative* [7]. Acting on a scalar, the Lie derivative is the same as the intrinsic derivative. The advantage of using Lie derivative is that there is no need of computing the Christoffel symbols in order to find out the Lie derivative as we had to do in case of intrinsic derivative.

In order to understand Lie derivatives let us consider a manifold in which a tangent vector field is defined as $t^i(x)$. These tangent vectors satisfy the condition (1.45). The curves of the manifold can be represented as shown in figure (1.3).

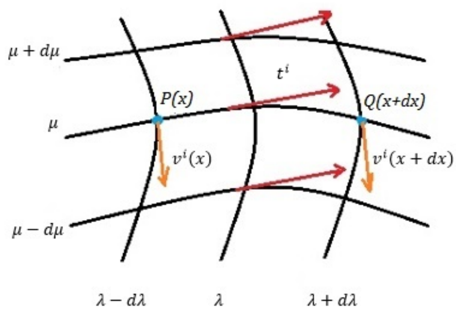

Figure 1.3: A tangent vector field $t^i(x)$ is defined in a manifold that contains different curves. Another vector $v^i(x)$ is also defined on a certain curve in the manifold, which is to be transported along the curve.

Let $P(x)$ and $Q(x+dx)$ be two points on a curve μ that are infinitely separated. Another vector field $v^i(x)$ is defined on the curve μ. We wish to find out the change in v^i as it gets transported from point $P(x)$ to $Q(x+dx)$. Let $x^i+dx^i = x'^i = x^i+t^i d\lambda$, and $v^i(x+dx) = v'^i$, then the transformation law requires,

$$\begin{aligned} v'^i(x') &= x'^i{}_{,j} v^j(x) \\ &= \left(x^i + t^i d\lambda\right)_{,j} v^j(x) \\ &= \left(x^i{}_{,j} + t^i{}_{,j} d\lambda\right) v^j(x) \\ &= (\delta^i_j + t^i{}_{,j} d\lambda) v^j(x) \\ &= v^i(x) + t^i{}_{,j} v^j(x) d\lambda \ . \end{aligned} \tag{1.51}$$

The point $x^i + dx^i$ represents the point Q on the curve, thus we can write the above relation as,

$$v'^i(Q) = v^i(P) + t^i{}_{,j} v^j(P) d\lambda \ . \tag{1.52}$$

We can also find the value of $v^i(x + dx)$ at point $Q(x + dx)$ by using Taylor's expansion. i.e.

$$\begin{aligned} v^i(Q) = v^i(x + dx) &\approx v^i(x) + dx^j v^i{}_{,j}(x) \\ &\approx v^i(x) + t^j v^i{}_{,j}(x) d\lambda \ . \end{aligned} \tag{1.53}$$

Now the comparison of $v'^i(Q)$ with $v^i(Q)$, gives the change of v^i in the flow of t^i. Therefore, the Lie derivative of v^i along μ is given by,

$$\begin{aligned} \pounds_t v^i(P) &= \lim_{d\lambda \to 0} \frac{v^i(Q) - v'^i(Q)}{d\lambda} \\ &= \lim_{d\lambda \to 0} \frac{v^i(P) + t^j v^j{}_{,j}(P) d\lambda - v^i(P - v^j(P) t^j{}_{,j} d\lambda)}{d\lambda} \\ &= t^j v^i{}_{,j}(P) - v^j(P) t^i{}_{,j} \ . \end{aligned} \tag{1.54}$$

The general expression for the Lie derivative of a vector field is given as,

$$\pounds_X Y(p) = \partial_X Y(p) - \partial_Y X(p) \ , \tag{1.55}$$

where ∂_X represents the directional derivative [4] along the direction of X. The general expression for the Lie derivative of a tensor field is given as,

$$\pounds_X T_{ab} = T_{ad} X^d{}_{,b} + T_{db} X^d{}_{,a} + T_{ab,e} X^e \ . \tag{1.56}$$

We can now use the idea of Lie transport in place of parallel transport just as are using Lie derivative in place of intrinsic derivative. The difference between both is that the parallel transport displaces a tensor parallely in the coordinate system whereas the Lie transport displaces the tensor along the curve on the manifold. For a tensor to be Lie transported along a curve, the Lie derivative of it must vanish. i.e.

$$\pounds_X T_{ab} = T_{ad} X^d{}_{,b} + T_{db} X^d{}_{,a} + T_{ab,e} X^e = 0 \ .$$

[4] The directional derivative of a a multi-variable function $f(x^i)$ along a given vector $\mathbf{v}$ is the rate at which the function changes at a given point in the direction of $\mathbf{v}$. It is denoted as,

$$\nabla_{\mathbf{v}} f(x^i) = \mathbf{v}.\nabla f(x^i) \ .$$

If the metric remains invariant when Lie transported along a curve, then the tangent vector to the curve is called *isometry* or a *Killing vector.* i.e.

$$\pounds_X g_{ab} = 0 \ ,$$
$$X^c g_{ab,c} + g_{ac} X^c{}_{,b} + g_{bc} X^c{}_{,a} = 0 \ . \tag{1.57}$$

This is known as the *Killing equation* and its solutions are the Killing vectors.

Geodesic Deviation

If two objects are set to move along two initially parallel trajectories then under the influence of spatially varying gravitational field, the trajectories of both objects would bend towards or away from each producing a relative acceleration between two objects. This phenomenon is known as the *geodesic deviation.*

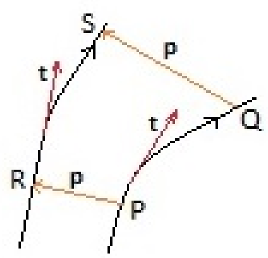

Figure 1.4: Two neighboring geodesics which tend to deviate from each other under the action of a spatially varying gravitational field. The vector tangent to the geodesics is denoted as **t** and the separation between both the geodesics is denoted by a separation vector **p**.

In order to calculate the relative acceleration vector consider two neighboring geodesics with the tangent vector **t** and a separation vector **p** [7]. The relative acceleration vector **A** is defined as,

$$\begin{aligned} A^a = \ddot{p}^a &= \frac{d^2 p^a}{ds^2} \\ &= t^c (t^b p^a{}_{;b})_{;c} \ . \end{aligned} \tag{1.58}$$

The separation vector would be Lie transported along the geodesic. So its Lie derivative must be equal to zero. i.e.,

$$\pounds_t p^a = t^b p^a{}_{;b} - p^b t^a{}_{;b} = 0 \ . \tag{1.59}$$

Using eq. (1.59) in eq. (1.58) we can write,

$$\begin{aligned} A^a &= t^c(p^b t^a{}_{;b})_{;c} \\ &= t^c p^b{}_{;c} t^a{}_{;b} + t^c p^b t^a{}_{;b;c} \\ &= p^c t^b{}_{;c} t^a{}_{;b} + t^c p^b t^a{}_{;b;c} \\ &= p^c(t^b t^a{}_{;b}); c - p^c t^b t^a{}_{;b;c} + t^c p^b t^a{}_{;b;c} \ . \end{aligned} \tag{1.60}$$

The first term on right hand side vanishes from geodesics equation. We can interchange the indices b and c in the third term to get,

$$\begin{aligned} A^a &= -p^c t^b t^a{}_{;b;c} + t^b p^c t^a{}_{;c;b} \\ &= -p^c t^b (t^a{}_{;b;c} - t^a{}_{;c;b}) \ . \end{aligned} \tag{1.61}$$

Now using the eq. (1.37) we can write,

$$A^a = -R^a{}_{bcd} t^b p^c t^d. \tag{1.62}$$

This relation is known as geodesic deviation. It suggests that the relative acceleration between two objects moving along neighboring geodesics depends upon curvature. If $R^a{}_{bcd} = 0$, then the space is flat (i.e. does not have any curvature) and there is no relative acceleration between objects moving along geodesics.

2

Background of General Relativity and Gravitational Waves

General Relativity is a geometric theory of gravity. Of all existing theories of gravity, GR is most consistent with the experimental data and it is accepted as the current theory of gravity. In GR, space and time are treated as a single four-dimensional manifold called *spacetime*. Every point in spacetime specifies an *event* (describing the 'where' and 'when' of an event). The path of a particle in the spacetime is called a *worldline*. The tangent vector to a worldline can be time-like, space-like or light-like (section 1.1.4).

In a four-dimensional spacetime, the components of a diagonalized metric tensor are specified by a signature. The signature of metric tensor used in this dissertation is $(+, -, -, -)$.

2.1 Principles of General Relativity

The basic principles of the General Theory of Relativity are,

1. The Principle of Equivalence;

2. The Principle of General Covariance;

3. The Correspondence Principle.

2.1.1 The Principle of Equivalence

Einstein's Special Theory of Relativity was restricted to the motion of objects moving at constant velocities. This is why SR is also called the restricted theory of relativity. To study dynamics it was necessary to incorporate accelerations into the theory.

The principle of equivalence allows us to add accelerations to the theory of relativity. In order to understand the principle, let us first go through the meanings of inertial and gravitational mass.

Inertial Mass: Inertial mass corresponds to the inertia of a body. i.e. the property of a body of resisting the change in motion. Inertial mass is defined by Newton's second law of motion as,

$$F = m_i a \ .$$

Gravitational Mass: Gravitational mass corresponds to the property of a body by which it imparts/experiences the force of gravity on/by another gravitational mass. Depending upon the force imparted or experienced, gravitational mass can be further subdivided into active and passive gravitational mass.

The *active gravitational mass* corresponds to the property of a body to generate a gravitational field. Therefore, the gravitational potential ϕ due to an active gravitational mass m_a at a distance r from the center of m_a is given by,

$$\phi = \frac{-Gm_a}{r} \ ,$$

where G is Newton's gravitational constant.

The *passive gravitational mass* corresponds to the response of a body to a certain gravitational field. For instance, the response of a massive body placed on the surface of the Earth is defined in terms of the weight of the body,

$$W = m_p g \ ,$$

where g is the gravitational acceleration, as measured on the surface of the Earth.

Repeated experiments from the 17^{th} century showed that the gravitational and inertial the masses are equivalent [8]. Einstein concluded from the equivalence of inertial and gravitational masses that *a frame in a uniform gravitational field is physically equivalent to a frame subjected to a uniform acceleration.* This is known as Einstein's principle of equivalence.

Think of a person in a rocket, and the rocket stands stationary on the surface of the Earth. The rocket has no window to tell the person about the outside environment. The person inside the rocket would feel an acceleration due to gravity of the Earth at a rate equal to $1g = 9.81m/s^2$. Now consider the same kind of rocket going out in free-space with an acceleration of exactly $9.81m/s^2$. There is no way the person in the rocket can tell whether the rocket is standing on the surface of Earth or is accelerating with $1g$ in free-space. This means that a person cannot locally distinguish between a gravitational field and a corresponding acceleration of the frame of reference. Thus, acceleration is physically equivalent to gravity, and in order to incorporate acceleration, we can incorporate gravity in the theory of relativity. In this sense GR is a generalization of both SR and Newton's theory of gravity.

2.1.2 The Principle of General Covariance

The description of the laws of physics in SR was restricted to the inertial frames of reference. This restriction is dropped in GR and all frames of reference are allowed. A generalization of the postulate of SR is that all frames of reference are physically equivalent. This is known as the principle of general covariance.

Another way to put general covariance is that valid physical laws are expressible in tensorial form. A change of frame of reference correspond to a coordinate transformation. Nature does not care about the choice of coordinates, and thus a coordinate transformation must not affect the form of a physical law. Hence the form of a physical law must remain invariant under arbitrary coordinate transformations.

2.1.3 The Correspondence Principle

Newtonian Physics gives a very good description of nature at a scale of macroscopic objects, moving at velocities very small compared to the velocity of light and in the region of weak gravitational field. Therefore, GR must reduce to the Newtonian Physics for the objects moving at velocities very small compared to the velocity of light and in the domain of weak gravitational fields. This principle is known as correspondence principle. The Newtonian limits of GR are $1/c \to 0$ and $G \to 0$.

2.2 The Stress-Energy Tensor and the Einstein Tensor

In GR we deal with the gravitational field. This gravitational field depends upon the distribution of matter in the spacetime. Thus, a mathematical description of matter distribution in spacetime is required. Since matter and energy are equivalent, this mathematical description must incorporate the distribution of energy as well. The energy can be carried by the matter or it can stored in a field. It can be contained in stresses set up on a medium. The *stress-energy tensor* '$T^{\alpha\beta}$' (also called energy-momentum tensor) is a symmetric 2^{nd} rank tensor which serves for this purpose. It acts as a source for the generation of gravitational field and the right hand side of EFEs is built from it [7].

In GR, *dust* and *perfect fluid* are the most commonly used cases of stress-energy tensors. The Dust is a type of matter distribution which consists of non-interacting incoherent matter. It is given by the simplest possible stress-energy tensor defined as,

$$T^{\alpha\beta} = \rho u^\alpha u^\beta \ , \quad \alpha, \beta = 0, 1, 2, 3 \ , \tag{2.1}$$

where ρ is the density of mass-energy and u^α is the 4-velocity defined as $u^\alpha = dx^\alpha / d\tau$, and τ is the proper time.

The perfect fluid is a non-viscous fluid that has zero heat conduction and no force between the particles. It is fully characterized by its pressure p and mass density ρ, defined as,

$$T^{\alpha\beta} = \left(\rho + \frac{p}{c^2}\right) u^\alpha u^\beta - p g^{\alpha\beta} \ . \tag{2.2}$$

If the pressure p of the perfect fluid tends to zero, then the stress-energy of the perfect fluid reduces to the stress-energy of the dust. The conservation of energy-momentum demands that the covariant divergence of stress-energy tensor is zero.

$$T^{\alpha\beta}_{\ \ ;\beta} = 0 \ . \tag{2.3}$$

This expression gives the conservation energy-momentum of matter distribution plus that of the fields. In case of flat spacetime, the above equation reduces to,

$$T^{\alpha\beta}_{\ \ ,\beta} = 0 \ .$$

which is the conventional representation of conservation of energy-momentum as it gives the conservation of energy-momentum of matter alone.

The left hand side of EFEs is built from the Einstein tensor '$G^{\alpha\beta}$', which expresses the curvature of spacetime in the presence of matter. The Einstein tensor must be symmetric and divergence-free such that it must be consistent with the properties of the stress-energy tensor. i.e.

$$\begin{aligned} G^{\alpha\beta} &= G^{\beta\alpha} \ , \\ G^{\alpha\beta}_{\ \ ;\beta} &= 0 \ . \end{aligned}$$

The simplest choice which satisfies the mathematical requirements is $G^{\alpha\beta} = g^{\alpha\beta}$. This is not only divergence-free but is also gradient-free. It gives a constant stress-energy. However stress-energy does not necessarily need to be constant, this is therefore trivial case.

The simplest non-trivial choice would be a linear function of curvature. We start from the Bianchi identity (eq. (1.39)) to reach a divergence-free function,

$$R_{abcd;f} + R_{abdf;c} - R_{abfc;d} = 0 \ .$$

Contracting over a and f we get,

$$R^a_{\ bcd;a} + R_{bd;c} - R_{bc;d} = 0 \ .$$

Again contracting over b and d,

$$R^{a}{}_{c;a} + R_{;c} - R^{d}{}_{c;d} = 0 \ .$$

or

$$\left(R^{a}{}_{c} - \frac{1}{2} R \delta^{a}{}_{c} \right)_{;a} = 0 \ .$$

We can alternatively write as,

$$\left(R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} \right)_{;\beta} = 0 \ . \tag{2.4}$$

Hence the simplest non-trivial, 4-dimensional, divergence free, symmetric function of curvature is,

$$G^{\alpha\beta} = \left(R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} \right) - \Lambda g^{\alpha\beta} \ , \tag{2.5}$$

where Λ comes in as the constant of proportionality. It is called the *cosmological constant* and its value is negligible for non-cosmological systems.

2.3 The Einstein Field Equations

Gravity is not described as a force in GR as it was in Newton's theory. It is rather described as the manifestation of curvature of spacetime. The entire Universe can be considered a 4-dimensional manifold (the spacetime). In a source-free region (where no matter is present), the spacetime possesses no curvature and can be represented as Minkowski spacetime. The shortest path (geodesic) between two points in a flat spacetime is a straight line. The presence of matter produces a curvature in the fabric of spacetime. This curvature then results in the geodesic deviation (section 1.4) which produces a relative acceleration between two neighboring geodesics (eq. (1.61)).Thus the acceleration is related to curvature. We can classically say that "gravitation causes acceleration and it is caused by the presence of matter" [7]. Therefore we need to find a mathematical relationship between matter (energy) distribution and the curvature of spacetime. This relation can be found by exploiting the fact that the Einstein tensor and the stress-energy tensor are both divergence-free,

$$G^{\alpha\beta} = \kappa T^{\alpha\beta} + \Lambda g^{\alpha\beta} \ . \tag{2.6}$$

Using eq. (2.5) we can write,

$$R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta} - \Lambda g^{\alpha\beta} = \kappa T^{\alpha\beta} \ . \tag{2.7}$$

The left hand side of this equation represents the curvature of spacetime and the right hand side represents the distribution of matter in particular region of spacetime. κ is the constant of proportionality and its value can be found by considering the Newtonian limit of the EFEs.

Newtonian Limit of the EFEs

The gravitational potential Φ in the Newton's theory satisfies the Poisson equation. i.e.

$$\nabla^2\Phi = 4\pi G\rho \ . \tag{2.8}$$

Since $1/c \to 0$ in the Newtonian limit, therefore $\dot{x}^0 \to c$, $\dot{x}^i \to u^i$ and $u^i/c \approx 0$. In other words we can say that $\dot{x}^i = 0$. Also there is no time variation so, $\Gamma^0_{00} = \Gamma^0_{ij} = \Gamma^i_{0j} = 0$. Moreover the geodesic equations (1.52) can be broken down into temporal and spatial parts and under the influence of the above mentioned limits, they reduce to the form,

$$\begin{aligned} \ddot{x}^0 &= 0 \ , & (2.9) \\ \ddot{x}^i + \Gamma^i_{00}(\dot{x}^0)^2 &= 0 \ . & (2.10) \end{aligned}$$

The other limit $G \to 0$ implies that the spacetime must reduce to Minkowki spacetime. i.e. $g_{00} \approx -g_{11} \approx -g_{22} \approx -g_{33} \approx 1$ and $g_{\alpha\beta} = 0$ otherwise. The non-zero Christoffel symbols are,

$$\Gamma^0_{0i} = \frac{1}{2}g^{00}g_{00,i}, \quad \Gamma^i_{00} = -\frac{1}{2}g^{ij}g_{00,j} \ . \tag{2.11}$$

Now we can write eq. (2.10) as,

$$\ddot{x}^i = -\frac{1}{2}(\nabla g_{00})c^2 \ , \tag{2.12}$$

where we have taken the limit $g^{ij} = g^{ii} = -1$. Classically we can take,

$$\ddot{x}^i = -\nabla\Phi \ .$$

Eq. (2.12) then reduces to the form,

$$g_{00} = \frac{2\Phi}{c^2} + constant \ .$$

In the limit $r \to 0$, $constant \to 1$. Therefore,

$$g_{00} = 1 + \frac{2\Phi}{c^2} \ . \tag{2.13}$$

We now come to the EFEs. Contracting the indices gives,

$$R^{\alpha}_{\ \alpha} - \frac{1}{\text{ﬄ}2}\delta^{\alpha}_{\alpha}R = R - 2R = -R = \kappa T \ .$$

We can rewrite eq. (2.7) as,

$$R^{\alpha\beta} = \kappa\left(T^{\alpha\beta} - \frac{1}{2}Tg^{\alpha\beta}\right) \ ,$$

and,

$$R^{00} = \kappa\left(T^{00} - \frac{1}{2}Tg^{00}\right) \ . \tag{2.14}$$

Let us consider a gravitational source with no stress or other fields but consists only of matter with density ρ. In the rest frame $T^{00} = \rho c^2$ and $T^{\alpha\beta} = 0$ otherwise. Thus,

$$T = g_{\alpha\beta}T^{\alpha\beta} = g_{00}T^{00} = \rho c^2 \ .$$

Therefore, eq. (2.14) reduces to,

$$R^{00} = \kappa\left(\rho c^2 - \frac{1}{2}\rho c^2\right) = \frac{1}{2}\kappa\rho c^2 \ . \tag{2.15}$$

From the definition of the Ricci tensor (eq. (1.42)) we can write,

$$\begin{aligned} R_{00} = {} & \Gamma^{0}_{00,0} + \Gamma^{i}_{00,i} - \left(\ln\sqrt{|g|}\right)_{,00} + \left(\ln\sqrt{|g|}\right)_{,0}\Gamma^{0}_{00} + \left(\ln\sqrt{|g|}\right)_{,i}\Gamma^{i}_{00} \\ & - (\Gamma^{0}_{00})^2 - 2\Gamma^{0}_{\ 0i}\Gamma^{i}_{00} - \Gamma^{i}_{0j}\Gamma^{j}_{0i} \ . \end{aligned}$$

Putting in the values of non-zero Christoffel symbols we get,

$$\begin{aligned} R_{00} = {} & \left(-\frac{1}{2}g^{ij}g_{00,j}\right)_{,i} + \left(\ln\sqrt{|g|}\right)_{,i}\left(-\frac{1}{2}g^{ij}g_{00,j}\right) , \\ & - 2\left(\frac{1}{2}g^{00}g_{00,i}\right)\left(-\frac{1}{2}g^{ij}g_{00,j}\right) \ , \\ R_{00} \ \approx {} & \ \left(-\frac{1}{2}g^{ij}g_{00,ij}\right) \ . \end{aligned} \tag{2.16}$$

Comparing eq. (2.13), eq. (2.15) and eq. (2.16) we get,

$$\nabla^2\Phi \ \approx \ \frac{1}{2}\kappa\rho c^4 \ . \tag{2.17}$$

The above equation has the same form of Poisson equation as that in Newton's theory of gravity. Therefore we can see that GR does reduce to Newtonian theory in appropriate limits. Comparing eq. (2.8) and eq. (2.17) we can find the value of κ,

$$\kappa = \frac{8\pi G}{c^4} \ . \tag{2.18}$$

2.4 Derivation of the EFEs by the Variational Principle

In GR, the matter distribution serves as the source of spacetime curvature just as the density serves as the source of potential. We can use the variational principle to derive the EFEs. The variational principle states that a physical system must follow the path of evolution for which the action must be minimized. i.e.

$$\delta S = 0 \ . \tag{2.19}$$

The action S, which is a property associated with the dynamics of a system, consists of two parts in GR, a gravitational part S_G and a matter part S_M. The matter part involves the metric tensor as,

$$S_M = \int_V \sqrt{|g|} T^{(matter)} d^4x = \int_V \sqrt{|g|} T^{(matter)}_{\alpha\beta} g^{\alpha\beta} d^4x \ . \tag{2.20}$$

The action for gravity was independently proposed by Einstein and Hilbert [7]. It is purely geometric quantity,

$$S_G = -\gamma \int_V \sqrt{|g|} R d^4x = -\gamma \int_V \sqrt{|g|} R_{\alpha\beta} g^{\alpha\beta} d^4x \ , \tag{2.21}$$

where R is the Ricci scalar, and γ is an arbitrary term that involves the coupling of gravity to matter. Since the total action must remain invariant thus,

$$\delta S = \delta S_G + \delta S_M = 0 \ . \tag{2.22}$$

Let us first consider the variation of gravitational part.

$$\begin{aligned} \delta S_G &= -\gamma \int_V \delta \left(\sqrt{|g|} R_{\alpha\beta} g^{\alpha\beta} \right) \ , \\ &= -\gamma \int_V \left[\left(\delta \sqrt{|g|} \right) R_{\alpha\beta} g^{\alpha\beta} + \sqrt{|g|} \delta R_{\alpha\beta} g^{\alpha\beta} + \sqrt{|g|} R_{\alpha\beta} \delta g^{\alpha\beta} \right] d^4x \ . \end{aligned} \tag{2.23}$$

The variation of $R_{\alpha\beta}$ can be neglected using the *Riemann normal coordinates.* It is a coordinate system where the Christoffel symbols are made negligibly small at a point but its derivative may not be negligible. Consider for instance a function $(x-a)^2$. If we change the coordinate system as $x' = x - a$, then near the origin (i.e. at $x' = 0$) the first derivative, $2x'$, is negligible but the second derivative, 2, is not. Thus in the Riemann normal coordinates $\Gamma^{\alpha}_{\beta\gamma} = 0$ but $\Gamma^{\alpha}_{\beta\gamma,\rho} \neq 0$. Therefore the Ricci tensor can be defined as,

$$R_{\alpha\beta} = \Gamma^{\gamma}_{\alpha\beta,\gamma} - \Gamma^{\gamma}_{\alpha\gamma,\beta} \ , \tag{2.24}$$

and,

$$\delta R_{\alpha\beta} = \delta\Gamma^{\gamma}_{\alpha\beta,\gamma} - \delta\Gamma^{\gamma}_{\alpha\gamma,\beta} \ . \tag{2.25}$$

In our defined coordinate system $g_{\alpha\beta,\gamma} = 0$, therefore,

$$g^{\alpha\beta}\delta R_{\alpha\beta} = (g^{\alpha\beta}\delta\Gamma^{\gamma}_{\alpha\beta} - g^{\alpha\gamma}\delta\Gamma^{\beta}_{\alpha\beta})_{,\gamma} \ . \tag{2.26}$$

The second term in eq. (2.23) then gets the form,

$$\int_V \sqrt{|g|}g^{\alpha\beta}\delta R_{\alpha\beta} = \int_V \sqrt{|g|}(g^{\alpha\beta}\delta\Gamma^{\gamma}_{\alpha\beta} - g^{\alpha\gamma}\delta\Gamma^{\beta}_{\alpha\beta})_{,\gamma}d^4x \ . \tag{2.27}$$

We can use Gauss's divergence theorem to see that only the boundary terms in the above equation contribute to the variation of $R_{\alpha\beta}$. Since the metric and its derivatives vanish at the boundary of region V, therefore,

$$\int_V \sqrt{|g|}g^{\alpha\beta}\delta R_{\alpha\beta} = 0 \ . \tag{2.28}$$

The variation of $\sqrt{|g|}$ comes out to be,

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}g_{\alpha\beta}\delta g^{\alpha\beta} \ . \tag{2.29}$$

Using eq. (2.28) and eq. (2.29) in eq. (2.23) we get variation of gravitational part of the action, i.e.

$$\delta S_G = -\gamma\int_V \left[R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right]\sqrt{|g|}\delta g^{\alpha\beta}d^4x \ . \tag{2.30}$$

whereas the variation of matter part goes in the variation of metric tensor as,

$$\delta S_M = \int_V T_{\alpha\beta}\sqrt{|g|}\delta g^{\alpha\beta}d^4x \ . \tag{2.31}$$

Using eq. (2.30) and eq. (2.31) in eq. (2.22) we get,

$$\delta S = \int_V \left[-\gamma\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right) + T_{\alpha\beta}\right]\sqrt{|g|}\delta g^{\alpha\beta}d^4x = 0 \ . \tag{2.32}$$

This is only true if,

$$-\gamma\left(R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta}\right) + T_{\alpha\beta} = 0 \ .$$

Putting $\gamma = 1/\kappa$ we get the EFEs,

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = \kappa T_{\alpha\beta} \ . \tag{2.33}$$

In order to make this equation consistent with eq. (2.7) we can add a constant to the gravitational Lagrangian and thereby modify the gravitational Lagrangian density to,

$$\mathcal{L}_G = \sqrt{|g|}(R + 2\Lambda) \ . \tag{2.34}$$

The modified form of EFEs after putting the value of κ from eq. (2.18) is,

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} - \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4}T_{\alpha\beta} \ . \tag{2.35}$$

These EFEs are a system of 10 coupled, non-linear, partial differential equations for 10 functions of variables. This highlights the complexities of GR.

2.5 Linearized EFEs

In the previous section, we have seen that the EFEs are non-linear. This makes GR different from the Newton's theory of gravity as the Newton's theory includes linear differential equations. Although any of the consequences of GR come from its non-linearity, yet it is worthwhile to consider the linear approximation of GR. The linearization of the EFEs leads us directly to a wave equation for gravity [7].

In GR, field is the metric tensor. The way it appears in the field equations gives rise to the non-linearity of GR. The appearance of the metric tensor into the field equations cannot be altered. However we can write the curved spacetime metric as the flat Minkowski metric $\eta_{\alpha\beta}$ plus an additional term $h_{\alpha\beta}$. The magnitude of $h^{\alpha\beta}$ must be very small such that the spacetime deviates slightly from being flat. In the linear approximation we require that the higher powers of $h_{\alpha\beta}$ and its derivatives can be neglected. Thus,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \ . \tag{2.36}$$

The inverse metric can be written using the property $g^{\alpha\gamma}g_{\beta\gamma} = \delta^{\alpha}{}_{\beta}$ of metric tensor as,

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} \ . \tag{2.37}$$

There are no derivatives of $\eta_{\alpha\beta}$ as for the moment we can use Cartesian coordinates. Therefore, the Christoffel symbols linearize to,

$$\Gamma^{\gamma}_{\alpha\beta} \approx \frac{1}{2}\eta^{\gamma\rho}(h_{\alpha\rho,\beta} + h_{\beta\rho,\alpha} - h_{\alpha\beta,\rho}) \ . \tag{2.38}$$

Clearly the terms quadratic in Christoffel symbols become quadratic in h and can be neglected. Thus the linearized Ricci tensor is,

$$\begin{aligned} R_{\alpha\beta} &\approx \Gamma^{\gamma}_{\alpha\beta,\gamma} - \Gamma^{\gamma}_{\alpha\gamma,\beta} \\ &\approx \frac{1}{2}[\eta^{\gamma\rho}((h_{\alpha\rho,\beta} + h_{\beta\rho,\alpha} - h_{\alpha\beta,\rho})_{,\gamma}] \\ &\quad - \frac{1}{2}[\eta^{\gamma\rho}((h_{\alpha\rho,\gamma} + h_{\gamma\rho,\alpha} - h_{\alpha\gamma,\rho})_{,\beta}] \\ &\approx \frac{1}{2}\eta^{\gamma\rho}(h_{\alpha\rho,\beta\gamma} + h_{\beta\rho,\alpha\gamma} - h_{\gamma\rho,\alpha\beta} - h_{\alpha\beta,\gamma\rho}) \ . \end{aligned} \tag{2.39}$$

A choice of coordinates can be made to disappear unnecessary terms in the brackets. We can rewrite the Ricci tensor as,

$$R_{\alpha\beta} \approx \frac{1}{2}(h^{\gamma}{}_{\alpha,\beta\gamma} + h^{\gamma}{}_{\beta,\alpha\gamma} - h_{,\alpha\beta}) - \frac{1}{2}\Box h_{\alpha\beta} \ , \tag{2.40}$$

where $h = h^{\alpha}{}_{\alpha}$ and $\Box$ is d'Alembertian operator defined as,

$$\Box = \eta^{\alpha\beta}\partial_\alpha\partial_\beta = \partial^\alpha\partial_\beta = \partial^2/\partial t^2 - \nabla^2 \ .$$

The Ricci scalar can be written by contraction of $R_{\alpha\beta}$ by $g^{\alpha\beta}$. It comes out to be,

$$R = g^{\alpha\beta}R_{\alpha\beta} = (h^{\gamma\rho}{}_{,\gamma\rho} - \Box h) \ . \tag{2.41}$$

We can rewrite the Ricci tensor and scalar as,

$$\begin{aligned} R_{\alpha\beta} &\approx \frac{1}{2}\left(h^{\gamma}{}_{\alpha} - \frac{1}{2}h\delta^{\gamma}{}_{\alpha}\right)_{,\gamma\beta} + \frac{1}{2}\left(h^{\gamma}_{\beta} - \frac{1}{2}h\delta^{\gamma}{}_{\beta}\right)_{,\alpha\gamma} - \frac{1}{2}\Box h_{\alpha\beta} \ , \\ R &= \left(h^{\gamma\rho} - \frac{1}{2}\eta^{\gamma\rho}h\right)_{,\gamma\rho} - \frac{1}{2}\Box h \ , . \end{aligned} \tag{2.42}$$

Putting these values in the field equations we get,

$$\begin{aligned} &\left(h^{\gamma}{}_{\alpha} - \frac{1}{2}h\delta^{\gamma}{}_{\alpha}\right)_{,\gamma\beta} + \left(h^{\gamma}{}_{\beta} - \frac{1}{2}h\delta^{\gamma}{}_{\beta}\right)_{,\alpha\gamma} - \Box h_{\alpha\beta} - \eta_{\alpha\beta}\left(h^{\gamma\rho} - \frac{1}{2}\eta^{\gamma\rho}h\right)_{\gamma\rho} \\ &\qquad - \frac{1}{2}\eta_{\alpha\beta}\Box h = \frac{16\pi G}{c^4}T_{\alpha\beta} \ . \end{aligned} \tag{2.43}$$

To simplify this expression we define a trace-reverse of $h_{\alpha\beta}$ as,

$$\bar{h}^{\gamma}{}_{\alpha} = h^{\gamma}{}_{\alpha} - \frac{1}{2}h\delta^{\gamma}{}_{\alpha} \ . \tag{2.44}$$

Eq. (2.43) now takes the form,

$$\bar{h}^{\gamma}{}_{\alpha,\gamma\beta} + \bar{h}^{\gamma}{}_{\beta,\alpha\gamma} - \Box\bar{h}_{\alpha\beta} - \eta_{\alpha\beta}\bar{h}^{\gamma\rho}{}_{,\gamma\rho} = \frac{16\pi G}{c^4}T_{\alpha\beta}\ . \tag{2.45}$$

These are the basic field equation in linearized form. However this can be further simplified by incorporating another thing which is called the *gauge condition*. We know that in Electrodynamics, we obtained a wave equation for 4-vector potential by incorporating the Lorentz gauge. We can think of a similar kind of gauge condition under which we could get to the wave equation for gravity. There can be a number of choices of gauge condition but we had to make only that choice for which all physical laws remain invariant under the transformation of coordinates. This is why General Covariance is important. Let us see how the linearized field equations look like under a gauge condition.

Gauge Transformation

Let us consider an infinitesimal transformation $x^\alpha \to x'^\alpha = x^\alpha + \xi^\alpha(x^\rho)$ such that we may neglect the terms quadratic in ξ or h. This is the one kind of coordinate transformation which leaves the metric invariant. Therefore,

$$\begin{aligned} ds^2 = g_{\alpha\beta}(x^\rho)dx^\alpha dx^\beta &= g'_{\alpha\beta}(x'^\rho)dx'^\alpha dx'^\beta \\ &= g'_{\alpha\beta}(x'^\rho)(dx^\alpha + \xi^\alpha_{,\mu}dx^\mu)(dx^\beta + \xi^\beta{}_{,\nu}dx^\nu)\ . \end{aligned} \tag{2.46}$$

Using linearization process it is easy to show that,

$$h'_{\alpha\beta} \ \approx\ h_{\alpha\beta} - \xi_{\alpha,\beta} - \xi_{\beta\alpha}\ . \tag{2.47}$$

Thus we have,

$$\bar{h}'^{\gamma}{}_{\alpha} = h^{\gamma}{}_{\alpha} - \frac{1}{2}h\delta^{\gamma}{}_{\alpha} - \eta^{\beta\gamma}\xi_{\alpha,\beta} - \xi^{\gamma}{}_{,\alpha} + \xi^{\gamma}{}_{,\gamma}\delta^{\gamma}{}_{\alpha}\ . \tag{2.48}$$

Differentiating w.r.t x^γ, the last two terms cancel out and we are left with,

$$\bar{h}'^{\gamma}{}_{\alpha,\gamma} = \bar{h}^{\gamma}{}_{\alpha,\gamma} - \eta^{\beta\gamma}\xi_{\alpha,\beta\gamma}\ . \tag{2.49}$$

Choosing ξ_α appropriately we can make,

$$\bar{h}^{\gamma}{}_{\alpha,\gamma} = 0\ . \tag{2.50}$$

This is the analogue of the Lorentz gauge in GR. Using this condition in eq. (2.45) we finally get the d'Alembertian equation for h. i.e.

$$\Box\bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta}\ . \tag{2.51}$$

In the source-free region (where $T_{\alpha\beta} = 0$), this reduces to the classic wave equation.

$$\Box \bar{h}_{\alpha\beta} = 0 \ . \tag{2.52}$$

This does not only verify the existence of gravitational waves but it also give us the information that gravitational waves travel at the speed of light.

2.6 Conservation Laws and Gaussian Flux Integrals

A conservation law states that a certain physical property of a closed system remains conserved as the system evolves in time. Conservation laws refer to the conservation of energy-mass, linear momentum, angular momentum and electric charge. Mathematically, a conservation law is expressed in the form a continuity equation. In Electrodynamics, the conservation of electric charge is expressed as,

$$J^{\alpha}_{\ ,\alpha} = 0 \ , \tag{2.53}$$

where $J^{\alpha} = (c\rho, \vec{j})$ is the electric 4-current. Alternatively, the total charge can be written in integral form [9] and using the Maxwell's equations $F^{\alpha\beta}_{\ \ ,\beta} = 4\pi J^{\mu}$ we can write,

$$Q = \int J^0 d^3x = \frac{1}{4\pi} \int F^{0\beta}_{\ \ ,\beta} d^3x = \frac{1}{4\pi} \int F^{0j}_{\ \ ,j} d^3x = \frac{1}{4\pi} \oint F^{0j} d^2S_j \ . \tag{2.54}$$

In the last part, we have used Gauss' divergence theorem to change volume integral into surface integral. This type of integral is called a Gaussian flux integral as it gives us the flux of a a field F^{oj} out of a closed surface S. The flux integrals work because of the fact that the charge and mass of the source have significant effect on the field that envelop the source.

This idea can also be extended to GR as the external gravitational field is also affected by the energy-momentum of the source. Therefore we can think of developing a similar kind of flux integral for the energy-momentum of a system. The conservation law that gives the conservation of energy-momentum in a flat spacetime is given as,

$$T^{\alpha\beta}_{\ \ ,\beta} = 0 \ . \tag{2.55}$$

where $T^{\alpha\beta}$ is the energy-momentum tensor. Using the knowledge of Electrodynamics as a guide (eqs. 2.53 and 2.54) the total 4-momentum of a system must be of the form,

$$P^{\alpha} = \frac{1}{c} \int T^{\alpha 0} d^3x \ . \tag{2.56}$$

where dx^3 is the volume element of the spacelike hypersurface corresponding to the observer measuring the momentum. In order to write a flux integral for 4-momentum of a system, we need a quantity analogous to $F^{\alpha\beta}$ which have crucial symmetries. This quantity comes out to be,

$$H^{\alpha\mu\beta\nu} = -(\bar{h}^{\alpha\beta}\eta^{\mu\nu} + \eta^{\alpha\beta}\bar{h}^{\mu\nu} - \bar{h}^{\mu\beta}\eta^{\alpha\nu} - \bar{h}^{\alpha\nu}\eta^{\mu\beta}) \ . \tag{2.57}$$

We shall discuss this quantity $H^{\alpha\mu\beta\nu}$ in detail in chapter 3. However we can look upon some important things for now. First of all the quantity $H^{\alpha\mu\beta\nu}$ possesses the same symmetries as the Riemann curvature tensor. i.e

$$\begin{aligned} H^{\alpha\mu\beta\nu} &= H^{\beta\nu\alpha\mu} \ , \\ H^{\alpha\mu\beta\nu} &= -H^{\alpha\mu\nu\beta} = -H^{\mu\alpha\beta\nu} \ , \\ H^{\alpha[\mu\beta\nu]} &= 0 \ . \end{aligned}$$

This quantity $H^{\alpha\mu\beta\nu}$ is related to the stress-energy tensor $T^{\alpha\beta}$ by the equation,

$$H^{\alpha\mu\beta\nu}_{\ \ ,\mu\nu} = \frac{16\pi G}{c^4}(-g)T^{\alpha\beta} \ , \tag{2.58}$$

A complete derivation of eq. (2.58) is given in section 3.1. Comparing equations (2.56) and (2.58) we can write,

$$\begin{aligned} P^\alpha &= \int T^{\alpha 0} d^3x = \frac{c^3}{16\pi G(-g)} \int H^{\alpha\mu 0\nu}{}_{,\mu\nu} d^3x = \frac{c^3}{16\pi G(-g)} \int H^{\alpha\mu 0j}{}_{,\mu j} d^3x \\ &= \frac{c^3}{16\pi G(-g)} \oint_S H^{\alpha\mu 0j}{}_{,\mu} d^2S_j \ . \end{aligned} \tag{2.59}$$

Here the closed 2-surface S completely surrounds the source and lies in 3-surface of constant time x^0. This is a flux integral for the 4-momentum of a system. It can be broken into zeroth and jth components. The zeroth component of 4-momentum gives us the flux for energy whereas the jth component gives us the flux for linear momentum. A similar kind of calculation lead us to a flux integral for angular momentum,

$$J^{\alpha\beta} = \frac{c^3}{16\pi G(-g)} \oint (x^\alpha H^{\beta\mu 0j}_{\ \ ,\mu} - x^\beta H^{\alpha\mu 0j}_{\ \ ,\mu} + H^{\alpha j 0\beta} - H^{\beta j 0\alpha}) \ . \tag{2.60}$$

These flux integrals are in fact a very useful tool. In order to evaluate a flux integrals, we only need to utilize the gravitational field far outside the source. Flux integrals can be used to calculate the 4-momentum and angular momentum for any isolated system when the closed integral is over the surface S in the asymptotically flat region surrounding the source. Although the integrands of flux integrals are not gauge-invariant , the total integrals are and they have physical meaning independent of any coordinate system or gauge. These total integrals are tensors in asymptotically flat region surrounding the source.

2.7 Exact Gravitational Wave Solutions

By linearizing the EFEs, we have reached to a wave equation for gravity. In principle, the solutions of the wave equation could be the exact solutions of the vacuum EFEs. There could be trivial, static solutions which satisfy the Laplace equation. However, they do not represent moving waves. We are interested in the non-static exact solutions to the EFEs. The exact solution for cylindrical gravitational waves was given in 1937 [10] whereas the plane wave solution was given in 1957 [11].

2.7.1 Exact Solution of the Cylindrical Gravitational Waves

Einstein and Rosen came up with the cylindrical gravitational wave solution in 1937 [10]. A cylinderically symmetric metric can be written as,

$$ds^2 = e^{2(\gamma-\psi)}(c^2dt^2 - d\rho^2) - e^{-2\psi}\rho^2 d\phi^2 - e^{2\psi}dz^2 \ . \tag{2.61}$$

where γ and ψ are two arbitrary functions of time t and ρ is the cylindrical radial coordinate. The non-zero Christoffel symbols for this metric are:

$$\left.\begin{aligned}
&\Gamma^0_{00} = \Gamma^0_{11} = \Gamma^1_{10} = \Gamma^1_{01} = \dot{\gamma} - \dot{\psi} \ , \\
&\Gamma^0_{01} = \Gamma^0_{10} = \Gamma^1_{00} = \Gamma^1_{11} = \gamma' - \psi' \ , \\
&\Gamma^0_{22} = -\rho^2\dot{\psi}e^{-2\gamma} \ , \\
&\Gamma^0_{33} = \dot{\psi}e^{2(2\psi-\gamma)} \ , \\
&\Gamma^1_{22} = \rho(\rho\psi' - 1)e^{-2\gamma} \ , \\
&\Gamma^1_{33} = -\psi' e^{2(2\psi-\gamma)} \ , \\
&\Gamma^2_{02} = \Gamma^2_{20} = -\dot{\psi} \ , \\
&\Gamma^2_{12} = \Gamma^1_{21} = -(\psi' - 1/\rho) \ , \\
&\Gamma^3_{03} = \Gamma^3_{30} = \dot{\psi} \ , \\
&\Gamma^3_{13} = \Gamma^3_{31} = \psi' \ , \\
&\Gamma^\mu_{\mu 0} = \Gamma^\mu_{0\mu} = \left(\ln\sqrt{|g|}\right)_{,0} = 2(\dot{\gamma} - \dot{\psi}) \ , \\
&\Gamma^\mu_{\mu 1} = \Gamma^\mu_{1\mu} = \left(\ln\sqrt{|g|}\right)_{,1} = 2(\gamma' - \psi') + 1/\rho \ ,
\end{aligned}\right\} \tag{2.62}$$

where the prime refers to the derivative with respect to ρ and the dot represents the derivative with respect to ct. The components of Ricci tensor can be obtained using eq. (1.42). The non-zero components of Ricci tensor are:

$$\left.\begin{aligned} R_{00} &= -(\ddot{\gamma} - \ddot{\psi}) + (\gamma'' - \psi'') + \frac{1}{\rho}(\gamma' - \psi') - 2\dot{\psi}^2 \ , \\ R_{11} &= (\ddot{\gamma} - \ddot{\psi}) - (\gamma'' - \psi'') + \frac{1}{\rho}(\gamma' + \psi') - 2\psi'^2 \ , \\ R_{22} &= \rho^2 e^{-2\gamma}(-\ddot{\psi} + \psi'' + \frac{1}{\rho}\psi') \ , \\ R_{01} &= \frac{1}{\rho}\dot{\gamma} - 2\dot{\psi}\psi' \ . \end{aligned}\right\} \tag{2.63}$$

We are interested in finding the exact solution of the EFEs. Since the exact solutions of the EFEs satisfy,

$$R_{\mu\nu} = 0 \ ,$$

therefore we have,

$$-(\ddot{\gamma} - \ddot{\psi}) + (\gamma'' - \psi'') + \frac{1}{\rho}(\gamma' - \psi') - 2\dot{\psi}^2 = 0 \ , \tag{2.64}$$

$$(\ddot{\gamma} - \ddot{\psi}) - (\gamma'' - \psi'') + \frac{1}{\rho}(\gamma' + \psi') - 2\psi'^2 = 0 \ , \tag{2.65}$$

$$\rho^2 e^{-2\gamma}(-\ddot{\psi} + \psi'' + \frac{1}{\rho}\psi') = 0 \ , \tag{2.66}$$

$$\frac{1}{\rho}\dot{\gamma} - 2\dot{\psi}\psi' = 0 \ . \tag{2.67}$$

Eq. (2.66) is a second order linear differential equation representing the conventional form of the cylindrical wave equation. The solution of this equations includes two arbitrary constants. One corresponds to the ingoing cylindrical wave and the other to the outgoing cylindrical wave. We Retain only to the outgoing waves which have the amplitude A and frequency ω, and thus we get the solution,

$$\psi(t, \rho) = A[J_0(x)\cos\omega t + N_0(x)\sin\omega t] \ , \tag{2.68}$$

where $x = \omega\rho/c$ and J_0 and N_0 are the zeroth order Bessel and Neumann functions respectively. To solve the other equations we can add eq. (2.64) and eq. (2.65). Thus we get,

$$\gamma' = \rho(\dot{\psi}^2 + \psi'^2) \ . \tag{2.69}$$

Equations (2.69) and (2.67) give the space and the time derivatives of $\gamma(t, \rho)$ in terms of functions that are known through eq. (2.60). To get the required solution we need to integrate with respect to space and time using the standard formulas for the integrals of the Bessel's and the Neumann's functions. The final solution for γ then comes out to be [7],

$$\begin{aligned}\gamma(\rho, t) =&\frac{1}{2}A^2 x\{J_0(x)J_0'(x) + N_0(x)N_0'(x)\\ &+ x[J_0(x)^2 + N_0(x)^2 + J_0'(x)^2 + N_0'(x)^2]\\ &+ [J_0(x)J_0'(x) - N_0(x)N_0'(x)]\cos 2\omega t\\ &+ [J_0(x)J_0'(x) - N_0(x)N_0'(x)]\sin 2\omega t\}\\ &- \frac{2}{\pi}A^2\omega t \ .\end{aligned} \tag{2.70}$$

The prime now refers to the derivative with respect to x, not ρ.

2.7.2 Exact Solution of Plane Gravitational Waves

In 1957 Bondi and Robinson obtained the plane wave solution for gravitational waves [11]. In order to reach the plane wave solution, we consider a line element which possesses the symmetries of a plane and represents a wave going in the x-direction. All the coefficients are the functions of $(ct - x)$, which is represented by u,

$$ds^2 = e^{2\alpha(u)}(c^2dt^2 - dx^2) - u^2(e^{2\beta(u)}dy^2 + e^{-2\beta(u)}dz^2) \ . \tag{2.71}$$

Here x, y and z are not the usual Cartesian coordinates but are rectangular coordinates in a curved spacetime. The non-zero Christoffel symbols are:

$$\left.\begin{aligned}&\Gamma^0_{00} = \Gamma^0_{11} = \Gamma^1_{01} = \Gamma^1_{10} = \alpha'(u) \ ,\\ &\Gamma^0_{01} = \Gamma^0_{10} = \Gamma^1_{00} = \Gamma^1_{11} = -\alpha'(u) \ ,\\ &\Gamma^0_{22} = \Gamma^1_{22} = u(u\beta' + 1)e^{2(\beta-\alpha)} \ ,\\ &\Gamma^0_{33} = \Gamma^1_{33} = u(-u\beta' + 1)e^{-2(\beta+\alpha)} \ ,\\ &\Gamma^2_{02} = \Gamma^2_{20} = -\Gamma^2_{12} = \Gamma^2_{21} = (\beta' - 1/u) \ ,\\ &\Gamma^3_{03} = \Gamma^3_{30} = \Gamma^3_{13} = \Gamma^3_{31} = (\beta' + 1/u) \ .\end{aligned}\right\} \tag{2.72}$$

where prime indicates the derivative with respect to u. We can make use of eq. (1.42) again to obtain the components of the Ricci tensor. The non-zero components of the Ricci tensor are:

$$\left.\begin{aligned}R_{00} &= 4\frac{\alpha'(u)}{u} - 2\beta'(u)^2 \ ,\\ R_{11} &= 4\frac{\alpha'(u)}{u} - 2\beta'(u)^2 \ ,\\ R_{01} &= -4\frac{\alpha'(u)}{u} + 2\beta'(u)^2 \ .\end{aligned}\right\} \tag{2.73}$$

We see that $R_{00} = R_{11} = -R_{01}$. Therefore we have only one Einstein equation,

$$4\frac{\alpha'(u)}{u} - 2\beta'(u)^2 = 0 \ . \tag{2.74}$$

From this we get the exact plane gravitational wave solution,

$$\alpha'(u) = \frac{1}{2}u\beta'(u)^2 \ . \tag{2.75}$$

Thus any $\alpha(u)$ and $\beta(u)$ are the exact plane gravitational wave solutions if they satisfy eq. (2.67). Moreover, eq. (2.63) describes a linearly polarized plane gravitational wave. The more general, circularly polarized plane gravitational wave is given by the line element,

$$\begin{aligned} ds^2 = e^{2\alpha}(dt^2 - dx^2) - u^2[(dy^2 + dz^2)\cosh 2\beta \\ + (dy^2 - dz^2)\sinh 2\beta\cos 2\theta - 2\sinh 2\beta\sin 2\theta dydz] \ , \end{aligned} \tag{2.76}$$

where α, β and θ are arbitrary functions of u. The condition for the exact solution in vacuum in that case comes out to be,

$$2\alpha'(u) = u[\beta'(u)^2 + \theta'(u)^2\sinh^2 2\beta(u)] \ . \tag{2.77}$$

It can be easily verified that for $\theta = 0$, eq. (2.69) reduces to eq. (2.67) and eq. (2.68) reduces to eq. (2.63).

3

Post-Newtonian Approximation

In section 2.5 we have seen that the linearization of the EFEs leads us to a wave equation (2.52) for gravity. The gravitational waves are exact solutions of the vacuum EFEs (section 2.7). However, there is a conceptual problem with eq. (2.52). The source of gravitational waves is matter (which is equivalent to energy). Therefore, being solutions of the vacuum EFEs, gravitational waves appear to carry no energy. However, it has been shown that gravitational waves impart momentum on a test particle placed in their path. Weber and Wheeler [12] showed that the cylinderical gravitational waves impart momentum on a test particle placed in their path. Later Ehler and Kundt [13] extended the demonstration for a test particle placed in the path of plane gravitational waves. The general formula for momentum imparted on a test particle in an arbitrary spacetime was given by Qadir and Sharif [14].

Though it is obvious that gravitational waves do carry energy in order to impart momentum to a test particle, no clear measure was available for the energy they carried. The problem is that energy is not generally conserved in GR, it is only conserved in a spacetime that possesses a time-like isometry. Another problem is that the equation (2.52) comes out as a result of the linearization of the EFEs. Thus it is correct only up to the terms linear in $\bar{h}$. Therefore, in order to define energy associated with the gravitational waves, we can think of re-inserting the non-linear terms back into the eq. (2.52). The method used for this purpose is called the *post-Newtonian approximation*.

The post-Newtonian (PN) approximation is a method that gives approximate solutions of the EFEs. The approximations are given in the form of expansions in small parameters which give orders of deviation from Newton's theory of gravity (thus the name "post-Newtonian").

3.1 Landau-Lifshitz Formalism

Lev Landau and Evgeny Lifshitz [15] provided a method by which we can reintroduce the non-linear terms into the EFEs. They started not with the idea of linearizing the EFEs but with the idea of conservation of energy and momentum.

The conservation of a physical quantity is given by the equation of continuity in a flat spacetime. We have encountered the example for the conservation of electric charge many times in electrodynamics. In the 4-vector notation, the conservation of electric charge can simply be written as the four divergence of current density (i.e. $J^{\alpha}{}_{,\alpha} = 0$). In GR the spacetime is not always flat. Therefore the four divergence must be replaced by the covariant divergence in order to give pure physical meaning in a curved spacetime. Thus the conservation of energy-momentum in a curved spacetime is given by,

$$T^{\alpha\beta}_{\ \ ;\beta} = 0 \ . \tag{3.1}$$

However, this does not generally express the conservation of energy-momentum of the matter distribution. It rather expresses the conservation of energy-momentum of the matter distribution plus that of the gravitational field. In order to determine the conserved 4-momentum of the matter plus that of the gravitational field we make use of Riemann normal coordinates once again which we have previously used in the section 2.4. We have seen that using the Riemann normal coordinates, all the Christoffel symbols vanish in a given locality but their 1^{st} derivatives survive, therefore eq. (3.1) reduce to,

$$T^{\alpha\beta}_{\ \ ,\beta} = 0 \ . \tag{3.2}$$

Since the 4-divergence of the stress-energy tensor is zero, therefore we can write it in terms of an antisymmetric quantity $A^{\alpha\beta\nu}$ as,

$$T^{\alpha\beta} = A^{\alpha\beta\nu}_{\ \ \ \ ,\nu} \ , \tag{3.3}$$

the quantity $A^{\alpha\beta\nu}$ is antisymmetric in the last pair of indices, i.e. in β and ν. This condition is analogous to the idea of the introduction of the magnetic vector potential in Electrodynamics[1]. Since the curl of a vector potential yields magnetic field (which is a flux density) therefore we can use this analogy in eq. (3.3), where

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[1] From Maxwell's equation we know that the divergence of magnetic field is zero, i.e.

$$\nabla.\mathbf{B} = B^{i}{}_{,i} = 0 \ .$$

Therefore we can write the magnetic field as the curl of a vector potential (i.e. the magnetic vector potential),

$$\mathbf{B} = \nabla \times \mathbf{A} = \epsilon_{ijk} A^{j}{}_{,k} \hat{i} \ ,$$

such that the divergence of the curl of the magnetic vector potential vanishes, satisfying the Maxwell's equation. This idea can be generalized to quantities with n indices.

the 4-divergence of an antisynnmetric potential term yields the stress-energy tensor (which contains the density and flux of the energy and momentum in spacetime). We can reach to this form of equation by applying the Riemann normal coordinate condition to the EFEs. We make use of the EFEs of the form,

$$T^{\alpha\beta} = \frac{c^4}{8\pi G}\left[R^{\alpha\beta} - \frac{1}{2}g^{\alpha\beta}R\right] \ . \tag{3.4}$$

Under the choice of Riemann normal coordinates, only the first two terms, in the definition of Ricci tensor (eq. (1.41)), will survive. Thus we have,

$$R^{\alpha\beta} = \frac{1}{2}g^{\alpha\mu}g^{\beta\nu}g^{\lambda\rho}\left\{g_{\lambda\rho,\mu\nu} + g_{\mu\nu,\lambda\rho} - g_{\lambda\nu,\mu\rho} - g_{\mu\rho,\lambda\nu}\right\} \ . \tag{3.5}$$

Reducing this to get Ricci scalar and putting both of these in eq. (3.4) we get,

$$T^{\alpha\beta} = \left\{\frac{c^4}{16\pi G}\frac{1}{(-g)}\left[(-g)(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\beta\mu})\right]_{,\mu}\right\}_{,\nu} \ . \tag{3.6}$$

The quantity $\left[(-g)(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\beta\mu})\right]_{,\mu}$ is the antisymmetric quantity which comes in place for $A^{\alpha\beta\nu}$. The terms within the square brackets can be represented by an antisymmetric quantity having four indices (let $H^{\alpha\mu\beta\nu}$). Since all the first derivatives of $g^{\alpha\beta}$ vanish at the point under consideration, the factor $1/(-g)$ can be taken out of the derivative w.r.t ν. Therefore,

$$H^{\alpha\mu\beta\nu}{}_{,\mu\nu} = \frac{16\pi G}{c^4}(-g)T^{\alpha\beta} \ , \tag{3.7}$$

where,

$$H^{\alpha\mu\beta\nu} = (-g)(g^{\alpha\beta}g^{\mu\nu} - g^{\alpha\nu}g^{\beta\mu}) \ . \tag{3.8}$$

We can also write it in the form,

$$H^{\alpha\mu\beta\nu} = \mathfrak{g}^{\alpha\beta}\mathfrak{g}^{\mu\nu} - \mathfrak{g}^{\alpha\nu}\mathfrak{g}^{\beta\mu} \ , \tag{3.9}$$

where,

$$\mathfrak{g}^{\alpha\beta} = \sqrt{-g}g^{\alpha\beta} \ . \tag{3.10}$$

We denote $\mathfrak{g}$ by gothic font style in order to differentiate it from the regular inverse metric tensor and we call it the *gothic inverse*. It is Eq. (3.7), derived under the local coordinate condition ($g^{\alpha\beta}_{,\gamma}$), is not valid for any arbitrary choice of coordinates. In general the equality does not hold and the difference between the quantities on both the right and the left hand sides is different from zero. Thus we can modify the eq. (3.7) for any arbitrary choice of coordinates as,

$$H^{\alpha\mu\beta\nu}{}_{,\mu\nu} = \frac{16\pi G}{c^4}(-g)\left[T^{\alpha\beta} + t^{\alpha\beta}_{LL}\right] \ . \tag{3.11}$$

This is an alternative form of the EFEs. On the right hand side of the equation we have a quantity $t^{\alpha\beta}_{LL}$ adding up to the energy-momentum $T^{\alpha\beta}$ of the matter distribution as a source term. This quantity represents the energy-momentum of the gravitational field and it is called the *gravitational energy-momentum pseudotensor* or the *Landau-Lifshitz pseudotensor*. Thus the advantage of using this representation is that now we can see the effects due to gravity coming in our equation. The exact expression for $t^{\alpha\beta}_{LL}$ can be obtained by using the regular full form of the EFEs for the value of $T^{\alpha\beta}$ and eq. (3.9) for the value of $H^{\alpha\mu\beta\nu}$.

$$\begin{aligned}\frac{16\pi G}{c^4}(-g)t^{\alpha\beta}_{LL} =& \mathfrak{g}^{\alpha\beta}_{\ ,\lambda}\mathfrak{g}^{\lambda\mu}_{\ ,\mu} - \mathfrak{g}^{\alpha\lambda}_{\ ,\lambda}\mathfrak{g}^{\beta\mu}_{\ ,\mu} + \frac{1}{2}g^{\alpha\beta}g_{\lambda\mu}\mathfrak{g}^{\lambda\nu}_{\ ,\rho}\mathfrak{g}^{\rho\mu}_{\ ,\nu} \\ & - (g^{\alpha\lambda}g_{\mu\nu}\mathfrak{g}^{\beta\nu}_{\ ,\rho}\mathfrak{g}^{\mu\rho}_{\ ,\lambda} + g^{\beta\lambda}g_{\mu\nu}\mathfrak{g}^{\alpha\nu}_{\ ,\rho}\mathfrak{g}^{\mu\rho}_{\ ,\lambda}) + g_{\lambda\mu}g^{\nu\rho}\mathfrak{g}^{\alpha\lambda}_{\ ,\nu}\mathfrak{g}^{\beta\mu}_{\ ,\rho} \\ & + \frac{1}{8}(2g^{\alpha\lambda}g^{\beta\mu} - g^{\alpha\beta}g^{\lambda\mu})(2g_{\nu\rho}g_{\sigma\tau} - g_{\rho\sigma}g_{\nu\tau})\mathfrak{g}^{\nu\tau}_{\ ,\lambda}\mathfrak{g}^{\rho\sigma}_{\ ,\mu} \ . \end{aligned} \tag{3.12}$$

Since $H^{\alpha\mu\beta\nu}$ is antisymmetric in β and ν, therefore the equation

$$H^{\alpha\mu\beta\nu}_{\quad\ ,\mu\nu\beta} = 0 \ , \tag{3.13}$$

holds an identity [16]. Comparing this with eq. (3.11) we can write,

$$\Big[(-g)(T^{\alpha\beta} + t^{\alpha\beta}_{LL})\Big]_{,\beta} = 0 \ . \tag{3.14}$$

This expression is equivalent to the conservation law represented in eq. (3.1). We can use the idea of Gaussian flux integrals, introduced in section 2.6, to find out the total 4-momentum of matter plus that of the fields of a system and can eventually obtain a formula for the energy radiated out of a system via emission of gravitational waves.

3.1.1 Integral Conservation Identities

Since it includes a partial derivative, eq. (3.14) can be converted into an integral identity. We consider a 3-dimensional region V with a fixed (time-independent) domain of the spatial component x^j, bounded by a 2-dimensional surface S. We assume that V contains atleast some of the matter (so that $T^{\alpha\beta}$ is non-zero somewhere within V). However the surface S does not intersect any of the matter (so that $T^{\alpha\beta}$ is zero everywhere on S). From eq. (3.14) in comparison with eq. (2.56) we can write the total 4-momentum $P^\alpha[V]$ of the system within the volume V as,

$$P^\alpha[V] = \frac{1}{c}\int_V (-g)(T^{\alpha 0} + t^{\alpha 0}_{LL})d^3x \ . \tag{3.15}$$

This total 4-momentum can be decomposed into its temporal and spatial components. The zeroth component gives us the energy as $E[V] = cP^0[V]$. Therefore we can write the expression for energy as,

$$E[V] = \int_V (-g)(T^{00} + t^{00}_{LL}) d^3x \ . \tag{3.16}$$

The linear momentum is given by,

$$P^j[V] = \frac{1}{c}\int_V (-g)(T^{j0} + t^{j0}_{LL}) d^3x \ . \tag{3.17}$$

Similarly we can define the total angular momentum $J^{\alpha\beta}[V]$ associated with the region V as,

$$J^{\alpha\beta}[V] = \frac{1}{c}\int_V \left[x^\alpha(-g)(T^{\beta 0} + t^{\beta 0}_{LL}) - x^\beta(-g)(T^{\alpha 0} + t^{\alpha 0}_{LL})\right] d^3x \ . \tag{3.18}$$

In the flat spacetime, using the Lorentzian coordinates, P^α defined by eq. (3.15) would have a firm interpretation as the total momentum vector associated with the energy-momentum tensor $T^{\alpha\beta}$. However in the curved spacetime and in the non-Lorentzian coordinate system it does not have any direct physical meaning, since the pseudotensor cannot be defined at every point in V. We can think of a method to convert the volume integral into surface integral in order to extract some physical meaning out of it. Taking the time derivative on both sides of eq. (3.15) we have,

$$\dot{P}^\alpha[V] = \frac{1}{c}\int_V \left[(-g)(T^{\alpha 0} + t^{\alpha 0}_{LL})\right]_{,0} d^3x \ . \tag{3.19}$$

From eq. (3.14) we can write,

$$\frac{1}{c}\left[(-g)(T^{\alpha 0} + t^{\alpha 0}_{LL})\right]_{,0} = -\left[(-g)(T^{\alpha j} + t^{\alpha j}_{LL})\right]_{,j} \ . \tag{3.20}$$

Thus we can write,

$$\dot{P}^\alpha[V] = -\int_V [(-g)(T^{\alpha j} + t^{\alpha j}_{LL})]_{,j} d^3x \ . \tag{3.21}$$

Using Gauss's divergence theorem this reduces to,

$$\dot{P}^\alpha[V] = -\oint_S (-g)(T^{\alpha j} + t^{\alpha j}_{LL}) dS_j \ . \tag{3.22}$$

Since S does not intersect any of the matter, therefore the integration of $T^{\alpha j}$ over dS_j would be zero. Therefore,

$$\dot{P}^\alpha[V] = -\oint_S (-g) t^{\alpha j}_{LL} dS_j \ . \tag{3.23}$$

This is a Gaussian flux integral which gives the flux of the total 4-momentum of a system out of the surface S. We can separate out the zeroth component of $\dot{P}^\alpha[V]$ again and that gives us the flux of the energy radiating out of a system.

$$\dot{E}[V] = -c \oint_S (-g) t_{LL}^{0j} dS_j \ . \tag{3.24}$$

Discussion About the Pseudotensor

A significant property of $t_{LL}^{\alpha\beta}$ is that it does not represent a tensor as it depends, for its definition, on the choice of coordinates. It can vanish in one coordinate system and survive in another (In fact it *does* vanish in Riemann normal coordinates). It is rather called a "pseudotensor".

The energy of the gravitational field cannot be defined locally. There is no local gravitational energy-momentum. The energy-momentum, in principle, has weight. It curves spacetime. It comes as a source term on the right hand side of the EFEs. It produces a relative geodesic deviation between two neighboring geodesics in a spacetime. None of this property is satisfied by the local gravitational energy-momentum. The reason is that in a given locality one can always choose a frame of reference in which all the local gravitational fields disappear. No gravitational field means no gravitational energy-momentum. This highlights the strength of Einstein's principle of equivalence.

However the contribution of gravitational forces to a system cannot be denied. There always is an influence on massive bodies in a concentration of gravitational waves. We can deduce from all this that there is no issue over the existence gravitational energy but over its localization. One can therefore look for the global effects due to $t_{LL}^{\alpha\beta}$ and not for the local effects. This is the lesson for the non-uniqueness of $t_{LL}^{\alpha\beta}$ [9].

It is worth mentioning that the Landau-Lifshitz pseudotensor is not a generalization of the concept of a pseudo-vector[2] to the tensor fields. It is named *pseudo*-tensor just to **emphasize** the fact that it does not represent a physical tensor.

3.2 Relaxed Field Equations

We have seen that the antisymmetric quantity $H^{\alpha\mu\beta\nu}$ which builds the left hand side of the field equations (eq. (3.11)) is defined in terms of the gothic inverse $\mathfrak{g}^{\alpha\beta}$.

[2]A pseudo-vector is a quantity that transforms like a vector under a proper rotation, but in 3 dimensions it gains an additional sign flip under improper rotations like reflection. Examples of pseudo-vectors are magnetic field, torque, angular velocity and angular momentum.

We now take this gothic inverse to serve as our main variable on part of the metric tensor $g_{\alpha\beta}$. We can define a potential as,

$$h^{\alpha\beta} = \eta^{\alpha\beta} - \mathfrak{g}^{\alpha\beta} \ . \tag{3.25}$$

Moreover we can impose the harmonic coordinate conditions,

$$\mathfrak{g}^{\alpha\beta}_{\ \ ,\beta} = 0 \ , \tag{3.26}$$

on the gothic inverse and on the potential $h^{\alpha\beta}$ as,

$$h^{\alpha\beta}_{\ \ ,\beta} = 0 \ . \tag{3.27}$$

Using these coordinate conditions we can simplify the field equations. Putting $\mathfrak{g}^{\alpha\beta}$ from eq. (3.25) into the definition of $H^{\alpha\mu\beta\nu}$ and taking the derivative one-by-one with respect to x^μ and x^ν and using the harmonic coordinate conditions we get,

$$H^{\alpha\mu\beta\nu}_{\qquad ,\mu\nu} = -\Box h^{\alpha\beta} + h^{\mu\nu} h^{\alpha\beta}_{\ \ ,\mu\nu} - h^{\alpha\nu}_{\ \ ,\mu} h^{\beta\mu}_{\ \ ,\nu} \ , \tag{3.28}$$

where $\Box = \eta^{\mu\nu}\partial_{\mu\nu}$. The right hand side of the of eq. (3.11) remains almost the same. However, the use of harmonic coordinate conditions make a slight difference to the definition of $t^{\alpha\beta}_{LL}$, since the first two terms on the right hand side of the eq. (3.12) vanish. Using eq. (3.28) with eq. (3.11) and isolating the d'Alembertian operator on the left hand side we finally get to the expression,

$$\Box h^{\alpha\beta} = -\frac{16\pi G}{c^4} \tau^{\alpha\beta} \ , \tag{3.29}$$

where,

$$\tau^{\alpha\beta} = (-g)(T^{\alpha\beta} + t^{\alpha\beta}_{LL} + t^{\alpha\beta}_{H}) \ , \tag{3.30}$$

in which,

$$\frac{16\pi G}{c^4}(-g) t^{\alpha\beta}_{H} = h^{\alpha\nu}_{\ \ ,\mu} h^{\beta\mu}_{\ \ ,\nu} - h^{\mu\nu} h^{\alpha\beta}_{\ \ ,\mu\nu} \ . \tag{3.31}$$

Equation (3.29) is a wave equation for gravity similar to eq. (2.51) which we got through linearization of the EFEs. The difference is that now it includes the non-linear terms in $t^{\alpha\beta}_{LL}$ and $t^{\alpha\beta}_{H}$ and thus includes the effects due to gravity. Since the wave eq. (3.29) is derived under the use harmonic coordinate conditions (3.27), therefore both these equations together give the complete description of the EFEs. The solutions of the wave eq. (3.29) must satisfy the harmonic coordinate conditions (3.27) in order to be called the solutions of the EFEs. Equation (3.29) taken independently does not give the full description of the EFEs, it is therefore called the *relaxed Einstein field equations*.

3.3 Formal Solution of the Relaxed Field Equations

The formal solution of the wave eq. (3.29) is of the form,

$$h^{\alpha\beta}(x) = \frac{4G}{c^4}\int \mathcal{G}(x,x')\tau^{\alpha\beta}(x')d^4x' \ , \tag{3.32}$$

where $x = (ct, \mathbf{x})$ is the position 4-vector of the field point of the observer and $x' = (ct', \mathbf{x}')$ is the position 4-vector of the source point. $\mathcal{G}(x, x')$ is the retarded Green's function of the wave operator which satisfies the relation,

$$\Box\mathcal{G}(x,x') = -4\pi\delta(x-x') = -4\pi\delta(ct-ct')(\mathbf{x}-\mathbf{x}') \ . \tag{3.33}$$

The retarded Green's function has a property that it vanishes when x is in the past of x'. Since an event cannot be observed before it has happened, therefore the retarded Green's function preserves causality. Green's function is given explicitly by,

$$\mathcal{G}(x,x') = \frac{\delta(ct-ct'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|} \ , \tag{3.34}$$

where $|\mathbf{x}-\mathbf{x}'| = \sqrt{(\mathbf{x}-\mathbf{x}')^2 + (\mathbf{y}-\mathbf{y}')^2 + (\mathbf{z}-\mathbf{z}')^2}$ is the Euclidean distance between the field and the source point. The solution of the wave equation now takes the form,

$$h^{\alpha\beta}(x) = \frac{4G}{c^4}\int \frac{\delta(ct-ct'-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}\tau^{\alpha\beta}(x')d^4x' \ . \tag{3.35}$$

The integration over d^4x' can also be written as $cdt'd^3\mathbf{x}'$. Performing the integration over cdt' and using the property of the delta function we have,

$$h^{\alpha\beta}(t,\mathbf{x}) = \frac{4G}{c^4}\int \frac{\tau^{\alpha\beta}(t-|\mathbf{x}-\mathbf{x}'|/c,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}' \ . \tag{3.36}$$

This is the retarded solution of the wave equation and the integration extends over the past light cone $\mathcal{C}(\mathbf{x})$ of the field point $x = (ct, \mathbf{x})$.

3.3.1 Iteration of the Relaxed Field Equations

The exact solution of the relaxed field equations is very difficult to obtain. Therefore, we try to obtain its solution by successive approximations. We have a formal solution in the form of eq. (3.36) which needs to be integrated to obtain a complete solution. Moreover, the variable $\tau^{\alpha\beta}$ on the right hand side contains $h^{\alpha\beta}$ itself in the definitions of $t^{\alpha\beta}_{LL}$ and $t^{\alpha\beta}_{H}$, therefore the solution must be an iterative one. We base our approximation on the very same idea that we used during the linearization of the field equations. We treat flat spacetime as the background metric (initial condition for the zeroth iteration) that varies slightly from being flat. The perturbed metric is then treated as the background metric for the next iteration and the deviation of spacetime from being flat increases with the increase in the strength of the field. To construct the metric we recall eq. (2.36), i.e,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \ . \tag{3.37}$$

The inverse of this perturbed metric tensor can be written as an inverse binomial expansion,

$$g^{\alpha\beta} = \eta^{\alpha\beta} - h^{\alpha\beta} + h^{\alpha}_{\gamma} h^{\beta\gamma} + \ldots \ . \tag{3.38}$$

One can also consider the inverse metric tensor to be the inverse Minkowski metric plus some inverse perturbative term $f^{\alpha\beta}$, i.e,

$$g^{\alpha\beta} = \eta^{\alpha\beta} + f^{\alpha\beta} \ . \tag{3.39}$$

Comparing equations (3.38) and (3.39), the inverse perturbative term $f^{\alpha\beta}$ can be written in the form,

$$f^{\alpha\beta} = h^{\alpha\beta} - h^{\alpha}_{\gamma} h^{\beta\gamma} + h^{\alpha\gamma} h_{\gamma\delta} h^{\beta\delta} + \ldots \ . \tag{3.40}$$

Therefore, the gravitational wave potential $h^{\alpha\beta}$ can also be written in the form of a formal expansion,

$$h^{\alpha\beta} = G k_1^{\alpha\beta} + G^2 k_2^{\alpha\beta} + G^3 k_3^{\alpha\beta} \ldots \ . \tag{3.41}$$

of the fields. This type of expansion in powers of G is known as post-Minkowskian expansion as it gives the orders of deviation of the spacetime from being Minkowskian spacetime. Since the approximation needs to be an iterative one therefore $h^{\alpha\beta}$ meets the following solutions,

$$\begin{aligned} h_0^{\alpha\beta} &= 0 \ , \\ h_1^{\alpha\beta} &= G k_1^{\alpha\beta} \ , \\ h_2^{\alpha\beta} &= G k_1^{\alpha\beta} + G^2 k_2^{\alpha\beta} \ , \end{aligned}$$

and so on. The post-Minkowskian approximation works parallel with the PN approximation. In principle we can solve by putting eq. (3.41) on the right hand side of the eq. (3.36) and then plucking out the same powers. However, it is easier to solve it through iterations. In the zeroth iteration we take $h_0^{\alpha\beta} = 0$ and doing so we get $g^{\alpha\beta} = \eta^{\alpha\beta}$ which is the Minkowskian spacetime metric. Also we get $t_{LL}^{\alpha\beta} = t_H^{\alpha\beta} = 0$ and thus $\tau_0^{\alpha\beta} = T^{\alpha\beta}$. The solution of eq. (3.36) then gives $h_1^{\alpha\beta}$, which can then be used in the definitions of $t_{LL}^{\alpha\beta}$ and $t_H^{\alpha\beta}$ to obtain the value of $\tau_1^{\alpha\beta}$. This $\tau_1^{\alpha\beta}$ is used on the right hand side of the eq. (3.36), the solution of which gives us $h_2^{\alpha\beta}$. In the second iteration $h_2^{\alpha\beta}$ is used on the right hand side and the process continues to the n^{th} iteration. To simplify our calculations we can impose the condition $(h_n^{\alpha\beta})_{,\beta} = 0$ on the iterated solutions of the relaxed field equations.

3.4 Integration Domains

In the previous section we saw that the formal solution of the relaxed field equations leads us to eq. (3.36), in which integration is to be carried out over the past light cone $\mathcal{C}(x)$. Other than integrating over the entire past light cone, we can partition the domain $\mathcal{C}(\mathbf{x})$ into the near-zone domain $\mathcal{N}(x)$ and the wave-zone domain $\mathcal{W}(x)$. In order to differentiate between both these domains we first consider the following scaling quantities:

$$r'_c = \text{characteristic length scale of the source} , \quad (3.42a)$$

$$t'_c = \text{characteristic timescale of the source} , \quad (3.42b)$$

$$\omega'_c = \text{characteristic frequency of the source} , \quad (3.42c)$$

$$\lambda' c = \text{characteristic wavelength of the radiation} . \quad (3.42d)$$

The near and the wave zones are defined as,

$$\text{near-zone} = r'_c < \lambda'_c = \frac{2\pi c}{\omega'_c} = ct'_c , \quad (3.43a)$$

$$\text{wave-zone} = r'_c > \lambda'_c = \frac{2\pi c}{\omega'_c} = ct'_c . \quad (3.43b)$$

Thus, the near-zone is the region of space in which $r'_c = |\mathbf{x}'|$ is smaller than the characteristic wavelength of the radiations emitted by the source whereas the wave-zone is the region in which r'_c is large compared to the wavelength of the radiation. In order to differentiate both these regions we consider the past light cone $\mathcal{C}(x)$ as shown in Figure (3.1). Let a 3-dimensional ball of radius R sweep a world tube $\mathcal{D}$ through the light cone such that the magnitude of R is of the order of the magnitude of λ'_c. Any point within this world tube is considered to be in the near-zone and the rest of the lightcone is the wave-zone. The near-zone and the wave-zone join together to complete the light cone of the field point x, i.e. $\mathcal{N}(x) + \mathcal{W}(x) = \mathcal{C}(x)$. The complete solution of eq. (3.36) is then given as,

$$h^{\alpha\beta}(t, \mathbf{x}) = h^{\alpha\beta}_{\mathcal{N}}(t, \mathbf{x}) + h^{\alpha\beta}_{\mathcal{W}}(t, \mathbf{x}) . \quad (3.44)$$

A slow-motion condition also comes in handy when the characteristic velocity of the source is very small in comparison to the speed of light. i.e,

$$v'_c << c . \quad (3.45)$$

The characteristic velocity can be defined as $v'_c = r'_c/t'_c$. Putting this in the above equation and comparing it with eq. (3.43) we get the slow-motion condition. i.e,

$$r'_c << \lambda'_c . \quad (3.46)$$

This equation states that the source must be situated deep within the near-zone when the slow-motion condition is in effect. We can therefore carry out the integration over the near-zone alone.

3.4.1 Integration Over the Near-Zone

The slow-motion condition allows us to calculate the effect due to near-zone contribution of the source. Since the integration is to be carried over the near-zone domain, therefore we can write eq. (3.36) as,

$$h^{\alpha\beta}_{\mathcal{N}}(t, \mathbf{x}) = \frac{4G}{c^4} \int_{\mathcal{N}} \frac{\tau^{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' \ . \tag{3.47}$$

The near-zone domain can be further subdivided into two parts depending upon the location of the field point.

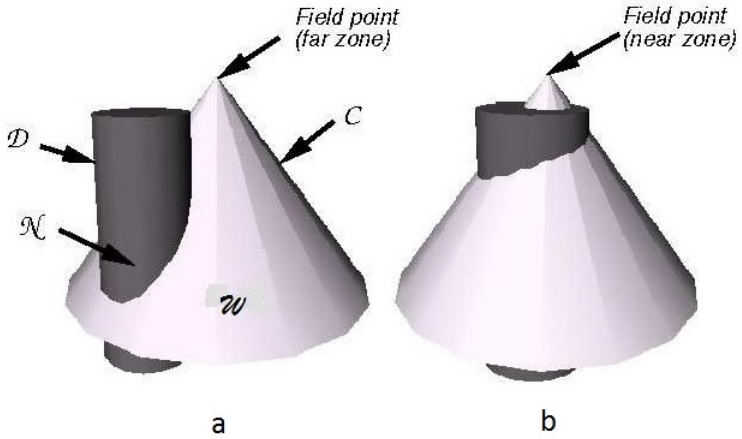

Figure 3.1: Near-zone integration domain for the retarded solution of the relaxed field equations. $\mathcal{C}$ is the past light cone of the field point x, $\mathcal{D}$ is the world tube swept out by the ball of radius R. $\mathcal{N}$ is the near-zone region of spacetime and $\mathcal{W}$ is the wave-zone region. Figure (a) shows the integration domain when the field point lies in the wave-zone whereas figure (b) shows the domain when field point lies in the near-zone along with the source point. [17].

Wave-Zone Field Point

In order to evaluate eq. (3.47), when the field point x is situated in the wave-zone such that $r = |\mathbf{x}| > R$, we introduce a modified integrand,

$$\begin{aligned} \frac{\tau^{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} &= \int \frac{\tau^{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{y})}{|\mathbf{x} - \mathbf{x}'|} \delta(\mathbf{y} - \mathbf{x}') d^3\mathbf{y} \\ &= \int g(\mathbf{x}, \mathbf{x}', \mathbf{y}) \delta(\mathbf{y} - \mathbf{x}') d^3\mathbf{y} \ , \end{aligned} \tag{3.48}$$

in which we treat $\mathbf{x}'$ and $\mathbf{y}$ as independent variables. Since the source point $\mathbf{x}'$ lies within the near-zone, it is considered to be a very small vector. We can therefore express g as a Taylor expansion about $\mathbf{x}' = 0$.

$$g(\mathbf{x}, \mathbf{x}', \mathbf{y}) = g(\mathbf{x}, 0, \mathbf{y}) + \frac{\partial g}{\partial x'^j} x'^j + \frac{1}{2} \frac{\partial^2 g}{\partial x'^j x'^k} x'^j x'^k + ... \, . \tag{3.49}$$

All the derivatives are evaluated over $\mathbf{x}' = 0$. The dependence of g on $\mathbf{x}'$ is only through $|\mathbf{x} - \mathbf{x}'|$, thus $\partial g/\partial x'^j = -\partial g/\partial x^j$. Our Taylor expansion then takes the form,

$$g(\mathbf{x}, \mathbf{x}', \mathbf{y}) = g(\mathbf{x}, 0, \mathbf{y}) - \frac{\partial g}{\partial x^j} x'^j + \frac{1}{2} \frac{\partial^2 g}{\partial x^j x^k} x'^j x'^k + ... \, . \tag{3.50}$$

The derivatives are still carried out over $\mathbf{x}' = 0$ but because the differentiation is with respect to $\mathbf{x}$, we can set $\mathbf{x}' = 0$ in g before taking the derivatives. Then g becomes the function of $|\mathbf{x} - \mathbf{x}'| \to |\mathbf{x}| = r$, then we have

$$g(\mathbf{x}, \mathbf{x}', \mathbf{y}) = g(r, 0, \mathbf{y}) - \frac{\partial g(r, 0, \mathbf{y})}{\partial x^j} x'^j + \frac{1}{2} \frac{\partial^2 g(r, 0, \mathbf{y})}{\partial x^j x^k} x'^j x'^k + ... \, . \tag{3.51}$$

Keeping all the terms of the Taylor expansion we can write,

$$y = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} x'^L \partial_L g(r, 0, \mathbf{y}) \ , \tag{3.52}$$

where $x'^L = x'^j x'^k x'^l ...$ and $\partial_L = \partial_j \partial_k \partial_l ...$. Putting the value of $g(\mathbf{x}, \mathbf{x}', \mathbf{y})$ back in eq. (3.48) and using the property of delta function we have,

$$\frac{\tau^{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|/c, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} x'^L \partial_L \frac{\tau^{\alpha\beta}(t - r/c, \mathbf{x}')}{r} \ . \tag{3.53}$$

Putting back into eq. (3.47) we arrive to the result,

$$h^{\alpha\beta}_{\mathcal{N}}(t, \mathbf{x}) = \frac{4G}{c^4} \left\{ \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \partial_L \left[\frac{1}{r} \int_{\mathcal{M}} \tau^{\alpha\beta}(t_r, \mathbf{x}') \mathbf{x}'^L d^3\mathbf{x}' \right] \right\} \ , \tag{3.54}$$

where $t_r = t - r/c$ is the retarded time variable. Since the temporal dependence of the source function no longer involves $\mathbf{x}'$ (the variable of integration), the domain of integration has therefore become the surface of constant time bounded by the sphere $r' = R$. We denote this domain by $\mathcal{M}$. This relation is valid everywhere within the wave-zone. However it simplifies when $r \to 0$, that is when $h^{\alpha\beta}_{\mathcal{N}}$ is evaluated in the far-away wave-zone. In that case we keep only the dominant r^{-1} terms.

$$h^{\alpha\beta}_{\mathcal{N}}(t, \mathbf{x}') = \frac{4G}{c^4 r} \sum_{l=0}^{\infty} \frac{(-1)^l}{l!} \int_{\mathcal{M}} \partial_L \tau^{\alpha\beta}(t_r, \mathbf{x}') \mathbf{x}'^L d^3\mathbf{x}' + \mathcal{O}(r^{-2}) \ . \tag{3.55}$$

The dependence of $\tau^{\alpha\beta}$ on x^j is contained in t_r, so that $\partial_j \tau^{\alpha\beta}(t_r, \mathbf{x}') = -c^{-1}\tau^{\alpha\beta(1)}\partial_j r = -c^{-1}\tau^{\alpha\beta(1)} n_j$, where $\tau^{\alpha\beta(1)}$ denotes the first derivative of $\tau^{\alpha\beta}$ with respect to t_r and $n_j = \partial_j r$. Taking the multiple derivatives of $\tau^{\alpha\beta}$ we get a relation,

$$\partial_L \tau^{\alpha\beta}(t_r, \mathbf{x}') = (-1)^l c^{-l} \tau^{\alpha\beta(l)} n_L + \mathcal{O}(r^{-1}) \ .$$

Inserting into the previous equation we finally get,

$$h^{\alpha\beta}_{\mathcal{N}}(t, \mathbf{x}) = \frac{4G}{c^4 r} \sum_{l=0}^{\infty} \frac{1}{l! c^l} n_L \left(\frac{d}{dt_r} \right)^l \int_{\mathcal{M}} \tau^{\alpha\beta}(t_r, \mathbf{x}') \mathbf{x}'^L d^3\mathbf{x}' + \mathcal{O}(r^{-2}) \ . \tag{3.56}$$

We can see in this expression that each successive term comes with an additional factor v_c/c, this signifies that each term is smaller than its preceeding term by a factor $v_c/c << 1$. This is where the PN expansion appears in our calculations for the first time. The PN expansion parameter is the square of ratio of the magnitude of the velocity of the source to the speed of light (which in this case can also be called the "speed of gravity"), i.e, $(v/c)^2$. For the magnitude of the velocity of the source to be very small compared to the speed of gravity it can be written as $(1/|c|)^2$. The PN numbers are assigned accordingly. The lowest term happens to be the Newtonian term, which is a given a 0^{th} order. The next term, which is $1/|c|$ times smaller than the Newtonian term, is of the order of 0.5 PN. The further next term, which is $1/|c|^2$ times smaller than the Newtonian term, is of the order of 1 PN and the numbering goes on. Moreover the integrals involved denote the multipole moments, defined as,

$$\mathcal{I}^L(t_r) = \int_{\mathcal{M}} c^{-2} \tau^{00} x^L d^3x, \tag{3.57}$$

where L is a multi-index which contains a series of individual indices j, i.e, $x^L = x^{j_1} x^{j_2} ... x^{j_l}$. We can thus say that the PN expansion is a kind of multipole expansion.

Near-Zone Field Point

Now we consider the case when both the field and the source points are situated within the near-zone. In that case $|\mathbf{x} - \mathbf{x}'|$ can be treated as a small quantity and we can Taylor expand the time dependence of the source function,

$$\tau^{\alpha\beta}(t - |\mathbf{x} - \mathbf{x}'|/c) = \tau^{\alpha\beta}(t) - \frac{1}{c}\frac{\partial \tau^{\alpha\beta}}{\partial t}|\mathbf{x} - \mathbf{x}'| + \frac{1}{2}\frac{1}{c^2}\frac{\partial^2 \tau^{\alpha\beta}}{\partial t^2}|\mathbf{x} - \mathbf{x}'|^2 + ... \ , \tag{3.58}$$

where all the derivatives are carried out at time t. Substituting in eq. (3.47) we have,

$$h^{\alpha\beta}_{\mathcal{N}}(t, \mathbf{x}') = \frac{4G}{c^4} \left\{ \sum_{l=0}^{\infty} \frac{(-1)^l}{l! c^l} \left(\frac{d}{dt} \right)^l \int_{\mathcal{M}} \tau^{\alpha\beta}(t, \mathbf{x}') |\mathbf{x} - \mathbf{x}'|^{l-1} d^3\mathbf{x}' \right\} \ . \tag{3.59}$$

The domain of integration is again $\mathcal{M}$, a surface of constant time bounded externally by sphere $r'_c = R$. However, the differentiation is now carried out at time t rather than retarded time t_r.

3.4.2 Integration Over the Wave-Zone

In the previous section we developed a method to evaluate the near-zone contribution to the solution of eq. (3.36). Now we wish to develop a method to evaluate the wave-zone contribution of the source to the solution.

The integration over $\mathcal{W}$ must reflect the nature of the integrand in the wave-zone. Since the wave-zone correspond to the entire region of the past light cone except $\mathcal{D}$, therefore it must be taken into consideration that integration must be carried out over a null-cone rather than a surface of constant time. For the slow motion condition, the source term must lie deep within the near-zone and therefore it has no contribution in the wave-zone. However the wave-zone contribution comes from the potentials that are themselves the solutions of the wave equation, i.e, the waves from the source serve as a source in the wave-zone to give wave-zone contribution. Any point on $\mathcal{W}$ is therefore a function of the retarded time variable and hence the integration carried out over retarded time as variable of integration. The strategy is to express the integral of eq. (3.36) in terms of spherical coordinates (r', θ', ϕ') and then switch the variables from r' to $u' = ct' - r'$ in order to perform the integration over retarded time variable.

This strategy leads us to a nice geometrical representation. A surface $u' =$ constant is a future directed null cone $\mathcal{F}$ that emerges from $r' = 0$. It intersects $\mathcal{C}(x)$ on a two dimensional surface $\mathcal{S}(u')$ parameterized by the angular variables θ' and ϕ'. Integration on $\mathcal{C}(x)$ can therefore be achieved by integrating over $\mathcal{S}(u')$ and then adding the contributions from each relevant $\mathcal{F}$. The limits of integration range from $u' = -\infty$ to $u' = u = ct - r$, whereas θ' and ϕ' are integrated over all allowed range.

Since $ct' = ct - |\mathbf{x} - \mathbf{x}'|$ on $\mathcal{C}(x)$ and $ct' = u' + r'$ on $\mathcal{F}$, therefore,

$$u' = ct - r' - |\mathbf{x} - \mathbf{x}'| \ , \tag{3.60}$$

where u' and t are constant. This equation can be solved for r' and expressing it as a function of θ' and ϕ' we can write,

$$r'(u', \theta', \phi') = \frac{(ct - u')^2 - r^2}{2(ct - u' - \mathbf{n}'.\mathbf{x})} \ , \tag{3.61}$$

where $\mathbf{n}' = \mathbf{x}'/r'$. Next, we change the variables in eq. (3.36) from r' to u' using,

$$\frac{\partial u'}{\partial r'} = \mathbf{n}'.\nabla' u' = \frac{u' - ct + \mathbf{n}'.\mathbf{x}}{|\mathbf{x} - \mathbf{x}'|} \ . \tag{3.62}$$

Now the wave-zone contribution to the integral of eq.(3.36) takes the form,

$$h^{\alpha\beta}_{\mathcal{W}}(t,\mathbf{x}) = \frac{4G}{c^4}\int_{-\infty}^{u} du' \oint_{\mathcal{S}} \frac{\tau^{\alpha\beta}((u'+r')/c,\mathbf{x}')}{ct-u'-\mathbf{n}'.\mathbf{x}} r'(u',\theta',\phi')^2 d\Omega' \ . \tag{3.63}$$

To proceed further, we restrict our attention to the source function of the form,

$$\tau^{\alpha\beta}(\mathbf{x}') = \frac{1}{4\pi}\frac{f^{\alpha\beta}(t'_r)}{r'^n} n'^{\langle L\rangle} \ , \tag{3.64}$$

where $f^{\alpha\beta}$ is an arbitrary function of retarded time $t'_r = t' - r'/c$, n is an arbitrary integer and $n'^{\langle L\rangle}$ is the symmetric trace free product of l radial vectors $n'^j = x'^j/r$. These are related closely to the spherical-harmonic functions as $Y_{lm}(\theta',\phi')$ as,

$$n'^{\langle L\rangle} = N_l \sum_{m=-l}^{l} \mathcal{Y}^{\langle L\rangle}_{lm} Y_{lm}(\theta',\phi') \ , \tag{3.65}$$

where $\mathcal{Y}^{\langle L\rangle}_{lm}$ is a constant STF tensor which satisfies the identity $\mathcal{Y}^{\langle L\rangle}_{l,-m} = (-1)^m \mathcal{Y}^{*\langle L\rangle}_{lm}$ and N_l is a normalization constant defined as [16],

$$N_l = \frac{4\pi l!}{(2l+1)!!} \ .$$

Substituting eq. (3.64) in eq. (3.63) we get,

$$h^{\alpha\beta}_{\mathcal{W}}(t,\mathbf{x}) = \frac{G}{c^4\pi}\int_{-\infty}^{u} du' f^{\alpha\beta}(u'/c) \oint_{\mathcal{S}} \frac{n'^{\langle L\rangle}}{r'(u',\theta',\phi')^{n-2}} \frac{d\Omega'}{ct-u'-\mathbf{n}'.\mathbf{x}} \ . \tag{3.66}$$

The angular integration can be simplified by orienting the coordinate axes such as the selected field point is aligned with the z-direction, so that $\mathbf{n} = \hat{e}_z$. Using eq. (3.65) in eq. (3.66) and integrating over ϕ', we observe that the rest of the integrand is independent of ϕ' and the only surviving term in the sum is $m = 0$. Inserting $Y_{l0} = [(2l+1)/4\pi]^{1/2} P_l(\cos\theta')$ and $\mathcal{Y}^{\langle L\rangle}_{l0} = [4\pi/(2l+1)]^{1/2} N_l^{-1} e_z^{\langle L\rangle}$ within the integral we obtain,

$$h^{\alpha\beta}_{\mathcal{W}}(t,\mathbf{x}) = \frac{2G}{c^4} n^{\langle L\rangle} \int_{-\infty}^{u} du' f^{\alpha\beta}(u'/c) \int_{\mathcal{S}} \frac{P_l(\xi)}{r'(u',\xi)^{n-2}(ct-u'-r\xi)} d\xi \ . \tag{3.67}$$

where $\xi = \cos\theta'$ and,

$$r'(u',\xi) = r(u',\theta',\phi') = \frac{(ct-u')^2 - r^2}{2(ct-u'-r\xi)} \ . \tag{3.68}$$

Switching integration variables back to r' from ξ, we get,

$$h^{\alpha\beta}_{\mathcal{W}}(t,\mathbf{x}) = \frac{2G}{c^4 r} n^{\langle L\rangle} \int_{-\infty}^{u} du' f^{\alpha\beta}(u'/c) \int_{\mathcal{S}} \frac{P_l(\xi)}{r'^{(n-1)}} dr' \ . \tag{3.69}$$

In this equation ξ is function of r'. Now we can see that when we remove the restriction of $\mathbf{n}$ to aligned with the z-direction, the angular dependence of $h^{\alpha\beta}_{\mathcal{W}}$ is contained in $n^{\langle L\rangle}$. But since the remaining integral is now independent of all angles, therefore the orientation of coordinate axis has become irrelevant. We may now take $\mathbf{n}$ to a point in any arbitrary direction specified by the polar angles θ and ϕ. The potential $h^{\alpha\beta}_{\mathcal{W}}$ will then become a function of (t, r, θ, ϕ) in which the dependence of t is contained in $u = ct - r$.

To complete the wave-zone integration, we must give the explicit description of the surface $\mathcal{S}(u')$ and specify the limits of integration over r'. The specific limits depend upon whether the field point is in near-zone or in wave-zone.

Wave-Zone Field Point

We first consider the case when the field point lies within the wave-zone, so that $r > R$. From figure (3.2a) we see that when $u' < u - 2R$, $\mathcal{S}(u')$ does not encounter the boundary of the near-zone and in this case ξ ranges from $\xi = -1$, at which $r' = \frac{1}{2}(ct - u' - r) = \frac{1}{2}(u - u')$, to $\xi = 1$, at which $r' = \frac{1}{2}(ct - u' + r) = \frac{1}{2}(u - u') + r$. When $u' > u - 2R$, $\mathcal{S}(u')$ runs into the boundary of the near-zone and in this case the lower bound on r' must be $r' = R$ and ξ must be greater than -1. The upper bound on r' is still $r' = \frac{1}{2}(u - u') + r$. In part (c), the integration terminates at $u = u'$. Let us define $s = \frac{1}{2}(u - u')$ and these functions,

$$\mathrm{A}(s,r) = \int_{R}^{r+s} \frac{P_l(\xi)}{r'^{(n-1)}} dr' \ , \tag{3.70a}$$

$$\mathrm{B}(s,r) = \int_{s}^{r+s} \frac{P_l(\xi)}{r'^{(n-1)}} dr' \ . \tag{3.70b}$$

We finally get to the expression,

$$h^{\alpha\beta}_{\mathcal{W}}(t,r,\theta,\phi) = \frac{n^{\langle L\rangle}}{r} \left\{ \int_0^R ds f^{\alpha\beta}(t_r - 2s/c) A(s,r) + \int_R^{\infty} ds f^{\alpha\beta}(t_r - 2s/c) B(s,r) \right\} \ , \tag{3.71}$$

for the wave-zone contribution to the potential $h^{\alpha\beta}$ when the field point lies within the wave-zone. The quantity ξ can be determined by using eq. (3.68) and inserting $u = ct - r$ and $s = \frac{1}{2}(u - u')$. It comes out to be,

$$\xi = \frac{r + 2s}{r} - \frac{2s(r+s)}{rr'}. \tag{3.72}$$

Figure 3.2: Wave-zone integration domain for the retarded solution of the EFEs. Part (a) corresponds to $u' < u - 2R$; the integration runs from $\xi = -1$ to $\xi = 1$. Part (b) corresponds to $u' > u - 2R$; the intersection $\mathcal{S}(u')$ terminates at A', the boundary of the near-zone. Part (c) corresponds to $u' = u$; the cones are tangent and $\mathcal{S}(c)$ runs from the edge of the near-zone to x.

This yeilds $\xi = -1$ when $r' = r + s$ and $\xi = 1$ when $r' = s$.

Near-Zone Field Point

Now we consider the case when the field point lies in the near-zone, so that $r < R$. This is the case shown in figure (3.2b) where $u' > u - 2R$. Now the integration terminates at the point where the surface $\mathcal{S}(u')$ runs into the boundary of the near-zone. In this case the minimum value of $s = \frac{1}{2}(u - u')$ is $R - r$. Thus we obtain the final expression for the wave-zone contribution to the potential $h^{\alpha\beta}$ when the field point lies in the near-zone as,

$$h^{\alpha\beta}_{\mathcal{W}}(t, r, \theta, \phi) = \frac{n^{\langle L\rangle}}{r}\left\{\int_{R-r}^{R} dsf^{\alpha\beta}(t_r - 2s/c)A(s, r) + \int_{R}^{\infty} dsf^{\alpha\beta}(t_r - 2s/c)B(s, r)\right\}\ , \tag{3.73}$$

3.5 Einstein's Quadrupole Formula

In the previous section we have obtained a general form for the approximate solution of the relaxed field equations in different regions of the past light cone. Moreover, eq. (3.56) shows that the PN expansion is a kind of multipole expansion. We can proceed further on the same footings and do the iteration of the relaxed field equations for all the cases discussed in the previous section but we restrict our attention to the case when the field point lies in the wave-zone. The reason is that our primary motive is to calculate the energy radiated out of a system via emission of gravitational waves, and we wish to use Gaussian flux integrals for that purpose. We have already discussed in the section 2.6 that Gaussian flux integrals give physical meaning in an asymptotically flat region surrounding the source. Therefore, we consider only the case when the source is hidden deep within the near-zone and the field point lies in the far-away wave-zone. Moreover, we shall see in this section that the dominant term for gravitational wave potential in the multipole expansion is the quadrupole term.

3.5.1 General Structure of Potentials

Before going towards the iteration of the field equations we look at the general structure of the potentials developed in the previous section. From eq. (3.54) we can see that the near-zone contribution to the gravitational potential, when the field point lies in the wave-zone is given by,

$$h^{\alpha\beta}_{\mathcal{N}}(t,\mathbf{x}) = \frac{4G}{c^4}\left\{\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{\alpha\beta}(t_r,\mathbf{x}')x'^L d^3\mathbf{x}'\right]\right\} ,$$

Expanding few terms and after some manipulations we can simplify the components of the gravitational potentials as,

$$h^{00}_{\mathcal{N}} = \frac{4GM_0}{c^2 r} + \frac{4G}{c^2}\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{\mathcal{I}^L(t_r)}{r}\right] , \tag{3.74a}$$

$$h^{0j}_{\mathcal{N}} = -\frac{2G}{c^3}\frac{(n\times J_0)^j}{r^2} - \frac{2G}{c^3}\partial_k\left[\frac{\dot{\mathcal{I}}^{jk}(t_r)}{r}\right]$$

$$\frac{4G}{c^4}\sum_{l=2}^{\infty}\frac{(-1)^l}{l!}\partial_l\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{0j}(t_r,\mathbf{x}')x'^L d^3\mathbf{x}'\right] , \tag{3.74b}$$

$$h^{jk}_{\mathcal{N}} = \frac{2G}{c^4}\frac{\ddot{\mathcal{I}}^{jk}(t_r)}{r} + \frac{4G}{c^4}\sum_{l=1}^{\infty}\frac{(-1)^l}{l!}\partial_L\left[\frac{1}{r}\int_{\mathcal{M}}\tau^{jk}(t_r,\mathbf{x}')x'^L d^3\mathbf{x}'\right] , \tag{3.74c}$$

in which the over dots indicate differetiation with respect to $t_r = t - r/c$. The wave-zone contribution to the gravitational potential is given by eq. 3.71,

$$h^{\alpha\beta}_{\mathcal{W}}(t,r,\theta,\phi) = \frac{n^{\langle L\rangle}}{r}\left\{\int_0^R ds f^{\alpha\beta}(t_r - 2s/c)A(s,r) + \int_R^{\infty} ds f^{\alpha\beta}(t_r - 2s/c)B(s,r)\right\} .$$

These contributions shall be evaluated in later up to the requirement.

3.5.2 First Iteration

As discussed earlier, the iteration of the field equations is carried out using post-Minkowskian approximation. Also the post-Minkowskian approximation works parallel with the PN approximation. Therefore, we take the fact that the 0^{th} iteration gives us the solution of the relaxed field equations correct up to Newtonian order. The 1^{st} iteration gives solution correct up to 0.5 PN and the 2^{nd} iteration gives the solution correct up to 1 PN.

To perform the first iteration of the relaxed field equations, we put $h_0^{\alpha\beta} = 0$ on the right hand side of the eq. (3.54). In that case we have $\tau^{\alpha\beta} = T^{\alpha\beta}$. We can make this substitution in eq. (3.54) and evaluate the multipole moments explicitly, but we don't need to do that right now. We just keep the multipole moments as it is and go for the second iteration. We will see later that only $h_{\mathcal{N}}^{00}$ component of the near-zone contribution to the potential is required for the preparation of the second iteration. Therefore, after the first iteration, we can write the near-zone contribution to the gravitational potential when the field point lies in the wave-zone as,

$$h_{\mathcal{N}}^{00} = \frac{4G}{c^2}\left\{\frac{\mathcal{I}(t_r)}{r} - \left[\frac{\mathcal{I}^j(t_r)}{r}\right]_{,j} + \frac{1}{2}\left[\frac{\mathcal{I}^{jk}(t_r)}{r}\right]_{,jk} ...\right\} . \tag{3.75}$$

The PN numbering is very subtle in this case. The first term on the right hand side of the eq. (3.75) is the monopole term and we naturally assign it 0 PN order to this term. The second (dipole) term is of 0.5 PN order and the third (quadrupole) term is of 1 PN order.

Since $\tau^{\alpha\beta} = T^{\alpha\beta}$ for $h^{\alpha\beta} = 0$, it does not extend beyond the near-zone. The wave-zone contribution $h_{\mathcal{W}}^{\alpha\beta}$ to the gravitational potential vanishes in the first iteration.

3.5.3 Second Iteration

Knowing the fact that the multipole moments bring an additional factor of v_c/c to the PN ordering we can write the different components of the near-zone contribution to the gravitational potential,

$$h_{\mathcal{N}}^{00} = \frac{4GM}{c^2 r} + \frac{2G}{c^2}\left[\frac{\mathcal{I}^{jk}(t_r)}{r}\right]_{,jk} - \frac{2G}{3c^2}\left[\frac{\mathcal{I}^{jkn}(t_r)}{r}\right]_{,jkn} + ... \ , \tag{3.76a}$$

$$\begin{aligned} h_{\mathcal{N}}^{0j} = &- \frac{2G}{c^3}\frac{(n\times J)^j}{r^2} - \frac{2G}{c^3}\left[\frac{\dot{\mathcal{I}}^{jk}(t_r)}{r}\right]_{,k} \\ &- \frac{G}{3c^3}\left[\frac{\dot{\mathcal{I}}^{jkn}(t_r) - 2\epsilon^{mjk}\mathcal{J}^{mn}(t_r)}{r}\right]_{,kn} + ... \ , \end{aligned} \tag{3.76b}$$

$$h^{jk}_{\mathcal{N}} = \frac{2G}{c^4}\frac{\ddot{\mathcal{I}}^{jk}(t_r)}{r} + \frac{2G}{3c^4}\left[\frac{\dddot{\mathcal{I}}^{jkn}(t_r) + 4\epsilon^{mn(j}\ddot{\mathcal{J}}^{m|k)}(t_r)}{r}\right]_{,n} + \ldots \ , \tag{3.76c}$$

The 1^{st} term in the expression of $h^{00}_{\mathcal{N}}$ is of 0 PN, the 2^{nd} term is of 1 PN and the 3^{rd} term is of 1.5 PN order. In the expression of $h^{0j}_{\mathcal{N}}$, the 1^{st} and the 2^{nd} terms are of 1 PN and the 3^{rd} term is of 1.5 PN order. In $h^{jk}_{\mathcal{N}}$, the 1^{st} term is of 1 PN and the 2^{nd} term is of 1.5 PN order. The multipole moments involved in here are functions of retarded time. Formally, the expression should have been evaluated by putting $\tau_1^{\alpha\beta}$ (obtained after the first iteration) in the relaxed field equations and performing the second iteration. However, the multipole moments appear at 1 and 1.5 PN orders. Therefore, any correction to the multipole moments will appear in the higher order PN approximation. We can truncate $\tau^{\alpha\beta}$ to its leading order only. Then the multipole moments can be defined as,

$$\mathcal{I}^L(t_r) = \int_{\mathcal{M}} c^{-2}\tau^{00}(t_r, \mathbf{x})x^L d^3\mathbf{x} + \mathcal{O}(c^{-4}) \ , \tag{3.77a}$$

$$\mathcal{J}^{jL}(t_r) = \epsilon^{jab}\int_{\mathcal{M}} c^{-1}\tau^{0b}(t_r, \mathbf{x})x^{aL} d^3\mathbf{x} + \mathcal{O}(c^{-3}) \ . \tag{3.77b}$$

Now we move towards the computation of wave-zone contribution to the gravitational potential. In the first iteration we saw that the wave-zone contribution vanishes. To perform the second iteration we use the solution of the first iteration in the definition of $\tau_1^{\alpha\beta}$ and solve the relaxed field equations. To the required degree of accuracy, the components of the Landau-Lifshitz pseudotensor come out to be,

$$\frac{16\pi G}{c^4}(-g)t^{00}_{LL} = -\frac{7}{8}\partial_j h^{00}\partial^j h^{00} + \mathcal{O}(c^{-6}) \ , \tag{3.78a}$$

$$\frac{16\pi G}{c^4}(-g)t^{0j}_{LL} = \frac{3}{4}\partial^j h^{00}\partial_0 h^{00} + (\partial^j h^{ok} - \partial^k h^{0j})\partial_k h^{00} + \mathcal{O}(c^{-7}) \ , \tag{3.78b}$$

$$\frac{16\pi G}{c^4}(-g)t^{jk}_{LL} = \frac{1}{4}\partial^j h^{00}\partial^k h^{00} - \frac{1}{8}\delta^{jk}\partial_n h^{00}\partial^n h^{00} + \mathcal{O}(c^{-6}) \ , \tag{3.78c}$$

whereas the components of $t_H^{\alpha\beta}$ are,

$$\frac{16\pi G}{c^4}(-g)t^{00}_{H} = \mathcal{O}(c^{-6}) \ , \tag{3.79a}$$

$$\frac{16\pi G}{c^4}(-g)t^{0j}_{H} = \mathcal{O}(c^{-7}) \ , \tag{3.79b}$$

$$\frac{16\pi G}{c^4}(-g)t^{jk}_{H} = \mathcal{O}(c^{-6}) \ . \tag{3.79c}$$

By virtue of the requirement of 1.5 PN accuracy, the only relevant piece of first iterated potential is the Newtonian term in $h^{00}_{\mathcal{W}}$, i.e,

$$h^{00}_{\mathcal{W}} = \frac{4GM}{c^2 r} + \mathcal{O}(c^{-4}) \ . \tag{3.80}$$

Putting this in eq. (3.78) we get the components of $\tau_1^{\alpha\beta}$,

$$\tau_1^{00} = -\frac{7GM^2}{8\pi r^4} + \mathcal{O}(c^{-2}) \;, \tag{3.81a}$$

$$\tau_1^{0j} = \mathcal{O}(c^{-3}) \;, \tag{3.81b}$$

$$\tau_1^{00} = \frac{GM^2}{4\pi r^4}\left(n^j n^k - \frac{1}{2}\delta^{jk}\right) \;, \tag{3.81c}$$

where $n^j = x^j/r$.

In order to calculate $h_{\mathcal{W}}^{\alpha\beta}$ we first decompose the effective stress tensor of eq. (3.81 c) in terms of STF angular tensors. We invoke the identity $n^j n^k = n^{\langle jk\rangle} + \frac{1}{3}\delta^{jk}$. Then,

$$\tau_1^{jk} = \frac{GM^2}{4\pi r^4}\left(n^{\langle jk\rangle} - \frac{1}{6}\delta^{jk}\right) \;. \tag{3.82}$$

Now this is of the form of eq. (3.64), and we identify $f_{l=0}^{00}$ with $-\frac{7}{2}GM^2$, $f_{l=2}^{jk}$ with GM^2 and $f_{l=0}^{jk}$ with $-\frac{1}{6}GM^2\delta^{jk}$. In each case we have $n = 4$. Using eq. (3.70) we can write the components of $h_{\mathcal{W}}^{\alpha\beta}$ for $l = 0$,

$$h_{\mathcal{W}}^{00} = 7\left(\frac{GM}{c^2 r}\right)^2\left(1 - 2\frac{r}{R}\right) \;, \tag{3.83a}$$

$$h_{\mathcal{W}}^{jk} = \frac{1}{3}\left(\frac{GM}{c^2 r}\right)^2 \delta^{jk}\left(1 - 2\frac{r}{R}\right) \;, \tag{3.83b}$$

and for $l = 2$,

$$h_{\mathcal{W}}^{00} = \left(\frac{GM}{c^2 r}\right)^2 n^{\langle jk\rangle}\left(1 - \frac{4R}{5r}\right) \;. \tag{3.84}$$

Discarding all terms including R we can write the final expression for the components of wave-zone contribution to the gravitational potential when the field point lies in the wave-zone as,

$$h_{\mathcal{W}}^{00} = 7\left(\frac{GM}{c^2 r}\right)^2 \;, \tag{3.85a}$$

$$h_{\mathcal{W}}^{jk} = \frac{1}{3}\left(\frac{GM}{c^2 r}\right)^2 n^j n^k \;. \tag{3.85b}$$

Combining the results of near-zone contribution in eq. (3.76) and wave-zone contribution in eq. (3.85) using eq. (3.44) we get the final expression for the components gravitational potential,

$$h^{00} = \frac{4G}{c^2}\left[\frac{M}{r} + \frac{1}{2}\left(\frac{\mathcal{I}^{jk}}{r}\right)_{,jk} - \frac{1}{6}\left(\frac{\mathcal{I}^{jkn}}{r}\right)_{,jkn} + \frac{7}{4}\frac{GM^2}{c^2r^2} + ...\right] , \tag{3.86a}$$

$$h^{0j} = \frac{4G}{c^3}\left[-\frac{1}{2}\frac{(n \times J)^j}{r^2} - \frac{1}{2}\left(\frac{\dot{\mathcal{I}}^{jk}}{r}\right)_{,k} - \frac{1}{12}\left(\frac{\dot{\mathcal{I}}^{jkn} - 2\epsilon^{mjk}\mathcal{J}^{mn}}{r}\right)_{,kn} + ...\right] , \tag{3.86b}$$

$$h^{jk} = \frac{4G}{c^4}\left[\frac{1}{2}\frac{\ddot{\mathcal{I}}^{jk}}{r} - \frac{1}{6}\left(\frac{\ddot{\mathcal{I}}^{jkn} + 2\epsilon^{mnj}\dot{\mathcal{J}}^{mk} + 2\epsilon^{mnk}\dot{\mathcal{J}}^{mj}}{r}\right)_{,n} + \frac{GM^2}{4r^2}n^jn^k + ...\right] . \tag{3.86c}$$

So far we have obtained the structure of gravitational potential, correct upto the order of 1.5 PN, when the field point is situated in the wave-zone. Before going further we make use of a specific gauge in order to remove ambiguities with the degrees of freedom of gravitational waves.

3.5.4 Transverse-Traceless Gauge

The simplest solution of the gravitational wave-equation (2.52) is the monochromatic plane-wave solution [9],

$$\bar{h}_{\alpha\beta} = \Re[A_{\alpha\beta}\exp(\imath k_\mu x^\mu)] , \tag{3.87}$$

where $A_{\alpha\beta}$ represents the amplitude along with the wave polarization and k_μ is the wave-vector, whereas $\Re$ stands for the real part. Both $A_{\alpha\beta}$ and k_μ satisfy the conditions,

$$k_\mu k^\mu = 0, \tag{3.88a}$$

$$A_{\alpha\mu}k^\mu = 0. \tag{3.88b}$$

The amplitude of this plane-wave appears to have six independent components (ten minus four orthogonality constraints). But the gravitational field has two degrees of freedom, not six. Therefore, we need to reduce the degrees of freedom by introducing a specific gauge. The gauge chosen for this purpose is the *transverse-traceless gauge* or TT-gauge.

In order to understand the imposition of this gauge, let us select a 4-velocity **u** defined throughout the spacetime under consideration. By a specific gauge transformation, we can impose the condition,

$$A_{\alpha\beta}u^\beta = 0 . \tag{3.89}$$

This adds three constraints to $A_{\alpha\beta}$ as the fourth constraint is already contained in the orthogonality condition. For the last constraint we have,

$$A^{\alpha}_{\alpha} = 0 \ . \tag{3.90}$$

Now we have eight constraints in total, i.e. $A_{\alpha\beta}u^{\beta} = A_{\alpha\beta}k^{\beta} = A^{\alpha}_{\alpha} = 0$. Thus the two remaining components of $A_{\alpha\beta}$ give the two degrees of freedom of gravitational waves. We can restate these contraints in the Lorentz frame (where $u^0 = 1$ and $u^j = 0$) and in a form where k^{μ} does not appear explicitly as,

$$h_{\alpha 0} = 0 \ , \quad \text{i.e. only spatial components survive;} \tag{3.91a}$$

$$h_{kj,j} = 0 \ , \quad \text{i.e. spatial components are divergence-free;} \tag{3.91b}$$

$$h^{k}{}_{k} = 0 \ , \quad \text{i.e. spatial components are trace-free.} \tag{3.91c}$$

It must be noted that under the TT-gauge, there is no difference in $h_{\alpha\beta}$ and $\bar{h}_{\alpha\beta}$ since the trace of $h_{\alpha\beta}$ is zero. Applying these contraints to eq. (3.86) and the fact that we are interested in the result correct upto the order of 1 PN, we can write the final expression for gravitational potential as,

$$h^{jk} = \frac{2G}{c^4 r}\ddot{\mathcal{I}}^{jk} \ . \tag{3.92}$$

This is known as the *Einstein's quadrupole formula.* It is the solution of the EFEs correct upto the order of 1 PN.

3.6 Hulse-Taylor Binary and the Indirect Detection of Gravitational Waves

In 1974, Russel Alan Hulse and Joseph Hooton Taylor observed pulsed radio wave emissions from a source using the Arecibo 305m antenna [18]. The source was identified to be a pulsar [3]. After timing the pulses from the pulsar, Hulse and Taylor observed that there was a systematic variation in the arrival time of pulses. From the timing observation of the pulses it was realized that the pulsar was actually in a binary orbit along with an other companion, this other companion was later confirmed to be a neutron star [1]. Observing the systematic variation in the pulse timing it was also realized that the orbit of the binary system shrunk gradually. The only thing responsible for shrinking the orbit of a binary system

[3] A pulsar in actual is a neutron star, which is highly magnetized and is rotating about its axis at a very high rate. It emits electromagnetic radiations from its poles, which can be observed in the form of pulses every time the pole of the rotating neutron star comes in the line of sight of the observer. Therefore, it is called a *pulsar.*

was the emission of gravitational waves, which were predicted way back by Albert Einstein [19] [20]. Therefore, this binary system served as a fantastic example to detect the emission of gravitational waves out of a system by observing the orbital decay of the system. We are going to develop a formula for orbital decay of a binary system using the approximate solution of the EFEs upto 1 PN order (i.e. using the Einstein quadrupole formula).

3.6.1 Energy Radiated Out of a System Via Emission of Gravitational Waves

In the section 3.1.1 we have seen that we can define the energy radiated out of a system, via emmision of gravitational waves in an asymptotically flat region surrounding the source, in the form of Gaussian flux integrals. Eq. (3.24) shows that the flux of energy is defined in terms of the Landau-Lifshitz pseudotensor, whereas the pseudotensor itself depends on $h_{\alpha\beta}$ for its definition. Therefore, we can use the Einstein's quadrupole formula (eq. (3.92)) to find the values of different components of $t_{LL}^{\alpha\beta}$ and that can be used in eq. (3.24) to obtain the amount of energy radiated out of a system.

Stress-Energy Pseudotensor for a Wave Propagating in Positive z-direction

Let us consider a plane gravitational wave propagating in positive z-direction. Then according to the constraints of TT-gauge (eq. (3.91)), the only non-zero components of $h_{\alpha\beta}$ will be,

$$\left.\begin{aligned} h_{xx} &= -h_{yy} \ , \\ h_{xy} &= h_{yx} \ . \end{aligned}\right\} \tag{3.93}$$

Therefore we can write the components of $h_{\alpha\beta}$ in the form,

$$h_{\alpha\beta} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & h_{xx} & h_{xy} & 0 \\ 0 & h_{xy} & -h_{xx} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \ . \tag{3.94}$$

We know that $h_{\alpha\beta} = \bar{h}_{\alpha\beta}$ under the TT-gauge, therefore the metric tensor is defined as,

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta} \ . \tag{3.95}$$

In the matrix form, we can write the metric tensor components as,

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 + h_{xx} & h_{xy} & 0 \\ 0 & h_{xy} & -1 - h_{xx} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \ . \tag{3.96}$$

We can see that the only non-zero components of the metric tensor are,

$$\left.\begin{aligned} g_{00} &= 1 \ , \\ g_{11} &= -1 + h_{xx} \ , \\ g_{12} &= h_{xy} \ , \\ g_{21} &= h_{xy} \ , \\ g_{22} &= -1 - h_{xx}, \\ g_{33} &= -1 \ . \end{aligned}\right\} \tag{3.97}$$

Another thing that must be taken into account is that $h_{\alpha\beta}$ is function of t and z alone as from eq. (3.92) we have,

$$h^{jk}(t,r) = \frac{2G}{c^4 r}\ddot{\mathcal{I}}^{jk}(t_r) \ , \tag{3.98}$$

where $t_r = t - r/c$ is the retarded time. For the wave travelling in positive z-direction we have $t_r = t - z/c$. Thus, the only non-zero derivatives of the components of metric tensor are,

$$\left.\begin{aligned} g_{11,0} &= \frac{1}{c}h_{xx,t} \ , \\ g_{12,0} &= \frac{1}{c}h_{xy,t} \ , \\ g_{21,0} &= \frac{1}{c}h_{xy,t} \ , \\ g_{22,0} &= -\frac{1}{c}h_{xx,t} \ , \\ g_{11,3} &= h_{xx,z} \ , \\ g_{12,3} &= h_{xy,z} \ , \\ g_{21,3} &= h_{xy,z} \ , \\ g_{22,3} &= -h_{xx,z} \ . \end{aligned}\right\} \tag{3.99}$$

We can use the identity $g_{\alpha\beta}g^{\alpha\beta} = \delta^{\alpha}_{\ \beta}$ to find the components of inverse metric tensor. The matrix form of the inverse metric tensor is,

$$g^{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{-1-h_{xx}}{1-h_{xx}^2-h_{xy}^2} & \frac{-h_{xy}}{1-h_{xx}^2-h_{xy}^2} & 0 \\ 0 & \frac{-h_{xy}}{1-h_{xx}^2-h_{xy}^2} & \frac{-1+h_{xx}}{1-h_{xx}^2-h_{xy}^2} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \tag{3.100}$$

Non-zero components of inverse metric tensor, correct upto second order of $h_{\alpha\beta}$, are,

$$\left.\begin{aligned} g^{00} &= 1, \\ g^{11} &= -1 - h_{xx} + h_{xx}^2 + h_{xy}^2 \ , \\ g^{12} &= -h_{xy} \ , \\ g^{21} &= -h_{xy} \ , \\ g^{22} &= -1 + h_{xx} + h_{xx}^2 + h_{xy}^2 \ , \\ g^{33} &= -1 \ . \end{aligned}\right\} \tag{3.101}$$

Now the non-zero Christoffel symbols are,

$$\left.\begin{aligned} \Gamma^0_{11} &= -\frac{1}{2c} h_{xx,t} \ , \\ \Gamma^0_{12} &= -\frac{1}{2c} h_{xy,t} \ , \\ \Gamma^0_{22} &= \frac{1}{2c} h_{xx,t} \ , \\ \Gamma^3_{11} &= \frac{1}{2} h_{xx,z} \ , \\ \Gamma^3_{12} &= \frac{1}{2} h_{xy,z} \ , \\ \Gamma^3_{22} &= -\frac{1}{2} h_{xx,z} \ , \\ \Gamma^1_{10} &= -\frac{1}{2c} h_{xx,t} - \frac{1}{2c} h_{xx} h_{xx,t} - \frac{1}{2c} h_{xy} h_{xy,t} \ , \\ \Gamma^1_{20} &= -\frac{1}{2c} h_{xy,t} - \frac{1}{2c} h_{xx} h_{xy,t} + \frac{1}{2c} h_{xy} h_{xx,t} \ , \\ \Gamma^2_{10} &= -\frac{1}{2c} h_{xy,t} + \frac{1}{2c} h_{xx} h_{xy,t} - \frac{1}{2c} h_{xy} h_{xx,t} \ , \\ \Gamma^2_{20} &= \frac{1}{2c} h_{xx,t} - \frac{1}{2c} h_{xx} h_{xx,t} - \frac{1}{2c} h_{xy} h_{xy,t} \ , \\ \Gamma^1_{13} &= -\frac{1}{2} h_{xx,z} - \frac{1}{2} h_{xx} h_{xx,z} - \frac{1}{2} h_{xy} h_{xy,z} \ , \\ \Gamma^1_{23} &= -\frac{1}{2} h_{xy,z} - \frac{1}{2} h_{xx} h_{xy,z} + \frac{1}{2} h_{xy} h_{xx,z} \ , \\ \Gamma^2_{13} &= -\frac{1}{2} h_{xy,z} + \frac{1}{2} h_{xx} h_{xy,z} - \frac{1}{2} h_{xy} h_{xx,z} \ , \\ \Gamma^2_{23} &= \frac{1}{2} h_{xx,z} - \frac{1}{2} h_{xx} h_{xx,z} - \frac{1}{2} h_{xy} h_{xx,z} \ . \end{aligned}\right\} \tag{3.102}$$

The rest of the Christoffel symbols are zero. We are considering a wave propagating in positive z-direction, therefore t^{03} component of the Landau-Lifshitz pseudotensor is our interest. Thus, we try to calculate R_{03} component of the Ricci tensor. From the definition of Ricci tensor we can write,

$$R_{03} = \Gamma^{\alpha}_{03,\alpha} - \Gamma^{\alpha}_{0\alpha,3} + \Gamma^{\alpha}_{\beta\alpha}\Gamma^{\beta}_{03} - \Gamma^{\alpha}_{\beta 3}\Gamma^{\beta}_{\alpha 0} \ . \tag{3.103}$$

From eq. (3.103) we know that,

$$\Gamma^{\alpha}_{03} = 0 \implies \Gamma^{\alpha}_{03,\alpha} = 0 \ . \tag{3.104}$$

Then,

$$R_{03} = -\Gamma^{\alpha}_{0\alpha,3} - \Gamma^{\alpha}_{\beta 3}\Gamma^{\beta}_{\alpha 0} \ . \tag{3.105}$$

Expanding over all possible values of the indices α and β we can write R_{03} in terms of non-zero Christoffel symbols as,

$$R_{03} = -\Gamma^{1}_{01,3} - \Gamma^{2}_{02,3} - \Gamma^{1}_{13}\Gamma^{1}_{10} - \Gamma^{1}_{23}\Gamma^{2}_{10} - \Gamma^{2}_{13}\Gamma^{1}_{20} - \Gamma^{2}_{23}\Gamma^{2}_{20} \ . \tag{3.106}$$

Putting values of the Christoffel symbols from eq. (3.102) in eq. (3.106) and after simplification we get the value of R_{03} correct up to the 2^{nd} order in $h_{\alpha\beta}$,

$$R_{03} = \frac{1}{c}\left[h_{xx}h_{xx,tz} + h_{xy}h_{xy,tz} + \frac{1}{2}h_{xx,t}h_{xx,z} + \frac{1}{2}h_{xy,t}h_{xy,z}\right] \ . \tag{3.107}$$

Knowing the fact that $h_{\alpha\beta}$ depends upton 'z' only through $t - z/c$ we can write $h_{\alpha\beta,z} = -\frac{1}{c}h_{\alpha\beta,t}$. Hence,

$$R_{03} = \frac{1}{c^2}\left[-h_{xx}\ddot{h}_{xx} - h_{xy}\ddot{h}_{xy} - \frac{1}{2}\dot{h}^2_{xx} - \frac{1}{2}\dot{h}^2_{xy}\right] \ . \tag{3.108}$$

Similarly we can find the other components of Ricci tensor correct upto the 2^{nd} order in $h_{\alpha\beta}$. Other components of Ricci tensor come out to be,

$$R_{00} = R_{33} = -R_{03} \ , \tag{3.109a}$$
$$R_{01} = R_{02} = R_{31} = R_{32} = R_{11} = R_{12} = R_{22} = 0 \ . \tag{3.109b}$$

Then the Ricci scalar becomes,

$$R = g^{\alpha\beta}R_{\alpha\beta} = 0 \ . \tag{3.110}$$

Therefore the Einstein tensor can be written equal to Ricci tensor i.e. $R_{\alpha\beta} = G_{\alpha\beta}$. Raising the indices we can write the components of Einstein tensor as [21],

$$G^{00} = G^{33} = -G^{03} = \frac{1}{c^2}\left[h_{xx}\ddot{h}_{xx} + h_{xy}\ddot{h}_{xy} + \frac{1}{2}\dot{h}^2_{xx} + \frac{1}{2}\dot{h}^2_{xy}\right] \ . \tag{3.111}$$

Now we are in a position to find out the components of the pseudotensor $t^{\alpha\beta}_{LL}$. From eq. (3.11) and eq. (2.6) we can write,

$$H^{\alpha\mu\beta\nu}{}_{,\mu\nu} = 2(-g)G^{\alpha\beta} + \frac{16\pi G}{c^4}(-g)t^{\alpha\beta}_{LL} \ . \tag{3.112}$$

Talking of plane gravitational waves, we have $H^{\alpha\mu\beta\nu}{}_{,\mu\nu} = 0$. Then,

$$t^{\alpha\beta}_{LL} = \frac{-c^4}{8\pi G}G^{\alpha\beta} \ , \tag{3.113}$$

where $G^{\alpha\beta}$ is the Einstein tensor and G is the gravitational constant. From eq. (3.111) we can write,

$$t^{00} = t^{33} = -t^{03} = \frac{-c^2}{8\pi G}\left[h_{xx}\ddot{h}_{xx} + h_{xy}\ddot{h}_{xy} + \frac{1}{2}\dot{h}^2_{xx} + \frac{1}{2}\dot{h}^2_{xy}\right] \ . \tag{3.114}$$

We can average out the values of $t^{\alpha\beta}_{LL}$ over a wave-period. It gives no meaning if averaged over a fraction of wave-period, since the result is then gauge dependent. Moreover, we cannot perform operations to measure mass (or correspondingly energy) in far field wave-zone for less than a wave-period. This is because locally spacetime does not posses energy. Averaging out over a wave-period, the values of $t^{\alpha\beta}_{LL}$ come out to be,

$$\begin{aligned}\left\langle t^{00}\right\rangle = \left\langle t^{33}\right\rangle = -\left\langle t^{03}\right\rangle =& \frac{-c^2}{8\pi G}\left\langle -\dot{h}^2_{xx} - \dot{h}^2_{xy} + \frac{1}{2}\dot{h}^2_{xx} + \frac{1}{2}\dot{h}^2_{xy}\right\rangle \\ =& \frac{c^2}{16\pi G}\left\langle \dot{h}^2_{xx} + \dot{h}^2_{xy}\right\rangle \ ,\end{aligned} \tag{3.115}$$

or,

$$c(-g)\left\langle t^{03}\right\rangle = \frac{c^3}{16\pi G}(-g)\left\langle \dot{h}^2_{xx} + \dot{h}^2_{xy}\right\rangle \ , \tag{3.116}$$

where g is the metric determinant and the value of $(-g) = 1 - \dot{h}^2_{xx} - \dot{h}^2_{xy}$. Multiplying these terms inside the averaged value and keeping only the quadratic terms we get,

$$c(-g)\left\langle t^{03}\right\rangle = \frac{c^3}{16\pi G}\left\langle \dot{h}^2_{xx} + \dot{h}^2_{xy}\right\rangle \ . \tag{3.117}$$

Now using the Einstein quadrupole formula (eq. (3.92)) we can write,

$$c(-g)\left\langle t^{03}\right\rangle = \frac{G}{4\pi c^5 r^2}\left\langle \dddot{\mathcal{I}}^2_{xx} + \dddot{\mathcal{I}}^2_{xy}\right\rangle \ . \tag{3.118}$$

The flux of energy is then given by eq. (3.24) as,

$$-\left\langle \dot{E}\right\rangle = -\left\langle P\right\rangle = c\oint(-g)\left\langle t^{03}\right\rangle dS_3 \ . \tag{3.119}$$

3.6.2 Power Radiated Out in a Solid Angle

Using eq. (3.119) and eq. (3.118) we can write the power radiated out of a system in a solid angle Ω. i.e,

$$\frac{dP}{d\Omega} = -\frac{G}{4\pi c^5}\left\langle \dddot{\mathcal{I}}^2_{xx} + \dddot{\mathcal{I}}^2_{xy} \right\rangle \ , \tag{3.120}$$

where $\mathcal{I}_{jk}$ is the mass quadrupole moment defined as,

$$\mathcal{I}_{jk} = c^{-2}\int T^{00} x_j x_k d^3x \ ,$$

for continuous distribution of matter. If we consider discrete matter distribution than we can replace $\mathcal{I}_{jk}$ by $\mathcal{Q}_{jk}$, which is defined as,

$$\mathcal{Q}_{jk} = \sum_\alpha m_\alpha x_{\alpha j} x_{\alpha k} \ . \tag{3.121}$$

Then we can write the formula for power radiated out in a solid angle Ω as,

$$\frac{dP}{d\Omega} = -\frac{G}{8\pi c^5}\left(\frac{d^3\mathcal{Q}_{ij}}{d^3t} e_{ij}\right) \ , \tag{3.122}$$

where e_{ij} is the unit polarization tensor which obeys the following conditions,

$$e_{ij} = e_{ji} \ , \quad e_{ii} = 0 \ , \quad k_i e_{ij} = 0 \ , \quad e_{ij} e_{ij} = 1 \ . \tag{3.123}$$

The sum is running over masses m_α in our system. It must be noted that the result is independent of all kinds of stresses present. Summing eq. (3.122) over two allowed polarizations we get,

$$\sum_{pol}\frac{dP}{d\Omega} = -\frac{G}{8\pi c^5}\left[\frac{d^3\mathcal{Q}_{ij}}{d^3t}\frac{d^3\mathcal{Q}_{ij}}{d^3t} - 2n_i\frac{d^3\mathcal{Q}_{ij}}{d^3t}n_k\frac{d^3\mathcal{Q}_{kj}}{d^3t} - \frac{1}{2}\left(\frac{d^3\mathcal{Q}_{ii}}{d^3t}\right)^2 \right.$$
$$\left. + \frac{1}{2}\left(n_i n_j\frac{d^3\mathcal{Q}_{ij}}{d^3t}\right)^2 + \frac{d^3\mathcal{Q}_{ii}}{d^3t}n_j n_k\frac{d^3\mathcal{Q}_{jk}}{d^3t}\right] \ , \tag{3.124}$$

or,

$$\sum_{pol}\frac{dP}{d\Omega} = -\frac{G}{8\pi c^5}\left[\frac{d^3\mathcal{Q}_{ij}}{d^3t}\frac{d^3\mathcal{Q}_{ij}}{d^3t} - 2n_i\frac{d^3\mathcal{Q}_{ij}}{d^3t}n_k\frac{d^3\mathcal{Q}_{kj}}{d^3t} - \frac{1}{2}\frac{d^3\mathcal{Q}_{ii}}{d^3t}\frac{d^3\mathcal{Q}_{jj}}{d^3t} \right.$$
$$\left. + \frac{1}{2}n_i n_j n_k n_l\frac{d^3\mathcal{Q}_{ij}}{d^3t}\frac{d^3\mathcal{Q}_{lk}}{d^3t} + \frac{d^3\mathcal{Q}_{ii}}{d^3t}n_j n_k\frac{d^3\mathcal{Q}_{jk}}{d^3t}\right] \ , \tag{3.125}$$

where $\hat{n}$ is the unit vector in the direction of radiation. The total rate of radiation is obtained after integrating over all directions of emission. To do so we integrate over d^2n using the following identities,

$$\int_{S^2} d^2n = 4\pi \ , \tag{3.126a}$$

$$\int_{S^2} n_j n_k d^2n = \frac{4\pi}{3}\delta_{jk} \ , \tag{3.126b}$$

$$\int_{S^2} n_i n_j n_l n_k d^2n = \frac{4\pi}{15}\left(\delta_{ij}\delta_{kl} + \delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl}\right) \ . \tag{3.126c}$$

Finally we get the formula for the total amount of power radiated out by a system Via Emission of gravitational waves, i.e,

$$P = -\frac{G}{5c^5}\left[\frac{d^3\mathcal{Q}_{ij}}{d^3t}\frac{d^3\mathcal{Q}_{ij}}{d^3t} - \frac{1}{3}\frac{d^3\mathcal{Q}_{ii}}{d^3t}\frac{d^3\mathcal{Q}_{jj}}{d^3t}\right] \ . \tag{3.127}$$

3.6.3 Gravitational Radiation From Two Point Masses in a Keplerian Orbit

Let us consider two masses m_1 and m_2 having coordinates $(d_1\cos\psi, d_1\sin\psi)$ and $(-d_2\cos\psi, -d_2\sin\psi)$ respectively in the xy-plane as shown in figure (3.3). The

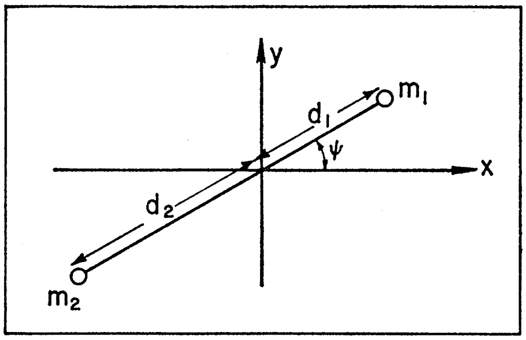

Figure 3.3: Coordinate system for two point masses in a Keplerian orbit [22].

origin is taken as the center of mass. Therefore,

$$d_1 = \left(\frac{m_2}{m_1+m_2}\right)d \ , \quad d_2 = \left(\frac{m_1}{m_1+m_2}\right)d \ . \tag{3.128}$$

The non-zero components of $\mathcal{Q}_{ij}$ are,

$$\mathcal{Q}_{xx} = \mu d^2 \cos^2\psi \ , \tag{3.129a}$$

$$\mathcal{Q}_{yy} = \mu d^2 \sin^2\psi \ , \tag{3.129b}$$

$$\mathcal{Q}_{xy} = \mathcal{Q}_{yx} = \mu d^2 \sin\psi\cos\psi \ , \tag{3.129c}$$

where μ is the reduced mass $(m_1 m_2)/(m_1+m_2)$. For Kepler motion, the equation of orbit is,

$$d = \frac{a(1-e^2)}{1+e\cos\psi} \ , \tag{3.130}$$

where a is the semi-major axis and e is the eccentricity of the orbit. Also the angular velocity for Kepler motion is given by,

$$\dot{\psi} = \frac{[G(m_1+m_2)a(1-e^2)]^{1/2}}{d^2} \ . \tag{3.131}$$

Using these equations we can find out the values of 3^{rd} order derivatives,

$$\frac{d^3\mathcal{Q}_{xx}}{dt^3} = \beta(1+e\cos\psi)^2(2\sin 2\psi + 3e\sin\psi\cos^2\psi) \ , \tag{3.132a}$$

$$\frac{d^3\mathcal{Q}_{yy}}{dt^3} = -\beta(1+e\cos\psi)^2\left[2\sin 2\psi + e\sin\psi(1+3\cos^2\psi)\right] \ , \tag{3.132b}$$

$$\frac{d^3\mathcal{Q}_{xy}}{dt^3} = \frac{d^3\mathcal{Q}_{yx}}{dt^3} = -\beta(1+e\cos\psi)^2\left[2\sin 2\psi - e\sin\psi(1-3\cos^2\psi)\right] \ , \tag{3.132c}$$

where β is defined as,

$$\beta^2 = \frac{4G^3 m_1^2 m_2^2(m_1+m_2)}{a^5(1-e^2)^5} \ . \tag{3.133}$$

The total power radiated out by this system in form of gravitational waves is,

$$P = -\frac{8}{15}\frac{G^4}{c^5}\frac{m_1^2 m_2^2(m_1+m_2)}{a^5(1-e^2)^5}(1+e\cos\psi)^4\left[12(1+e\cos\psi)^2 + e^2\sin^2\psi\right] \ . \tag{3.134}$$

Averaging this equation over one period of elliptical motion we get the average rate at which the system radiates energy. i.e,

$$\langle P\rangle = -\frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2(m_1+m_2)}{a^5(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \ . \tag{3.135}$$

If we consider the motion in a circular orbit, then the equation reduces to,

$$\langle P\rangle = -\frac{32}{5}\frac{G^4}{c^5}\frac{m_1^2 m_2^2(m_1+m_2)}{a^5} \ , \tag{3.136}$$

here a represents the radius of the circular orbit rather than the semi-major axis of an ellipse. Thus the average power radiated out two masses orbiting a common center of mass in an elliptical orbit is equal to the power radiated from two masses in a circular orbit (which has the same radius as the semi-major axis of the elliptical orbit) times an enhanced factor,

$$f(e) = \frac{1 + (73/24)e^2 + (37/96)e^4}{(1-e^2)^{7/2}} \ . \tag{3.137}$$

3.6.4 Orbital Decay for Two Point Masses in Keplerian Orbit Via Emission of Gravitational Waves

In the previous section we have seen that the emission of gravitational waves from a system takes energy out of the system at a rate given by eq. (3.135). Therefore the emission of gravitational waves results in the reduction of orbital period for two point masses in Keplerian orbit. To find the formula for the rate at which the orbital period reduces we make use of Kepler's 3^{rd} law of planetary motion, i.e,

$$P_b^2 = \frac{4\pi^2}{G(m_1+m_2)} a^3 \ , \tag{3.138}$$

where P_b is the orbital period. We can see that the orbital time period depends upon the semi-major axis a, so we can find out the formula for reduction of semi-major axis first. We know from classical two-body problem that the semi-major axis is related to energy by the formula [23],

$$a = -\frac{Gm_1m_2}{2E} \ . \tag{3.139}$$

Using eq. (3.119), eq. (3.135) and eq. (3.139) we can write the formula for reduction of semi-major axis [24],

$$\left\langle \frac{da}{dt} \right\rangle = -\frac{64}{5}\frac{G^3 m_1 m_2 (m_1+m_2)}{c^5 a^3 (1-e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) \ . \tag{3.140}$$

Now using Kepler's 3^{rd} law, we can write the formula for orbital period decay rate,

$$\left\langle \frac{dP_b}{dt} \right\rangle = -\frac{192}{5}\frac{G^{5/3} m_1 m_2 (m_1+m_2)^{-1/3}}{5c^2 (1-e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right) \left(\frac{P_b}{2\pi}\right)^{-5/3} \ . \tag{3.141}$$

3.6.5 Orbital Decay of Hulse Taylor Binary PSR B1913+16

Having developed a general formula for the orbital decay of a binary system via emission of gravitational waves, we are now in a position to study the example of the Hulse-Taylor binary PSR B1913+16. The orbitals parameters involved in eq. (3.141) are given by [1],

$$\begin{aligned} m_1 =& 2.86311 \times 10^{30} kg \ , \\ m_2 =& 2.7613 \times 10^{30} kg \ , \\ e =& 0.6171334 \ , \\ P_b =& 27906.979585910402 s \ . \end{aligned}$$

Knowing these values and the values of the constants G and c, we can plot the solution of eq. (3.141), Figure (3.4).

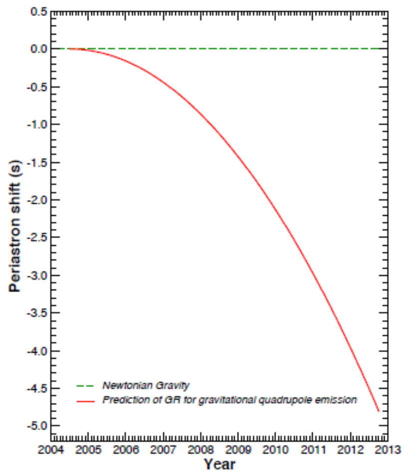

Figure 3.4: The commulative period shift of the Hulse-Taylor binary system, which shows the orbital decay as a result of emission of gravitational waves [25].

4

Gravitational Waves from Inspiralling Compact Binaries

Inspiralling compact binaries, containing neutron stars and/or black holes, are likely to become the bread-and-butter sources of gravitational waves for the detectors LIGO*,* VIRGO*,* GEO *and* KARGA *on ground, and also e*LISA *in space* [26].

Luc Blanchet

In the previous chapter we have developed a general formalism for studying gravitational waves (i.e. the PN approximation) and used the lowest-order PN results to reproduce the Hulse-Taylor binary (section 3.6). Now we wish to study gravitational waves at successive PN orders to higher degree of accuracy. Since inspiralling compact binaries are considered to be an ideal source for the generation of detectable gravitational waves, therefore we wish to use the PN results at each successive order to study gravitational waves produced by inspiralling compact binaries. In this chapter I shall discuss the behavior of gravitational waves at successive PN orders starting from 0PN upto 3PN order. I have recalculated explicitly the relation for the gravitational wave potential correct upto 1.5PN order, whereas the higher order relations (i.e. 2PN, 2.5PN and 3PN) have been imported directly from literature.

4.1 Gravitational Waves at 0PN Order

In the previous chapter we have discussed the convention of PN counting according to which the quadrupole term in gravitational wave potential is the 1PN term. The reason for this is that the general expression for the gravitational wave potential $h^{\alpha\beta}$ (eq. (3.86)) contains a mass term at the lowest order, which is taken to be Newtonian term (or 0PN). The quadrupole term is greater than this term by a factor $(v/c)^2$, and hence it is assigned the rank of 1PN. However, in the upcoming sections the PN counting differs from the convention adopted previously. Now we are interested only in the spatial components of the gravitational wave potential, since only the spatial components survive under the TT-gauge. The leading term in the spatial components of the gravitational wave potential is the quadrupole term, hence it is convenient to reset the PN counting to assign 0PN rank to the quadrupole term. The higher terms will then be assigned a rank of 0.5PN, 1PN, 1.5PN and so on, correspondingly. Under this new convention of PN ordering, the lowest order gravitational wave potential is given by,

$$h^{jk} = \frac{2G}{c^4 r}\ddot{I}^{jk} \ . \tag{4.1}$$

When the internal structure of each body in the system can be ignored, we can adopt the point-mass description (first used in section 3.6), in which,

$$\rho = \sum_A M_A \delta(\mathbf{x} - \mathbf{r}_A) \ , \tag{4.2}$$

where M_A is the total mass-energy of the body identified by the label A and r_A is the position vector of the body A evaluated at retarded time t_r. In this case the quadrupole moment becomes,

$$I^{jk} = \sum_A M_A r_A^j r_A^k \ . \tag{4.3}$$

The dynamics of the system are governed by Newton's equations of motion. i.e,

$$\mathbf{a}_A = -\sum_{B\neq A} \frac{GM_B}{r_{AB}^2}\mathbf{n}_{AB} \ , \tag{4.4}$$

where $\mathbf{r}_{AB} = \mathbf{r}_A - \mathbf{r}_B$, $r_{AB} = |\mathbf{r}_A - \mathbf{r}_B|$ and $\mathbf{n}_{AB} = \mathbf{r}_{AB}/r_{AB}$. Differentiating eq. (4.3) twice with respect to retarded time t_r and using the equation of motion (eq. (4.4)) we get,

$$\frac{1}{2}\ddot{I}^{jk} = -\frac{1}{2}\sum_A \sum_{B\neq A} \frac{GM_A M_B}{r_{AB}} n_{AB}^j n_{AB}^k + \sum_A M_A v_A^j v_A^k \ , \tag{4.5}$$

where $v_A^j = dr_A^j/dt_r$ is the velocity vector of the body A.

4.1.1 Binary System

We consider the gravitational waves emitted by a binary system of orbiting bodies using the lowest order PN approximation. The position of the first body of mass m_1 is $\mathbf{r}_1$ with respect to system's center of mass, and its velocity is $\mathbf{v}_1$. Similarly, the position of the second body with mass m_2 is given by the position vector $\mathbf{r}_2$ and its velocity is given as $\mathbf{v}_2$. We can write the position vectors in terms of the separation vector $\mathbf{r} = \mathbf{r}_{12} = \mathbf{r}_1 - \mathbf{r}_2$ as,

$$\mathbf{r}_1 = \frac{m_2}{m}\mathbf{r} \ , \quad \mathbf{r}_2 = \frac{m_1}{m}\mathbf{r} \ , \tag{4.6}$$

where $m = m_1 + m_2$ is the total mass of the system. Similarly we can write the velocities as,

$$\mathbf{v}_1 = \frac{m_2}{m}\mathbf{v} \ , \quad \mathbf{v}_2 = \frac{m_1}{m}\mathbf{v} \ , \tag{4.7}$$

where $\mathbf{v} = \mathbf{v}_1 - \mathbf{v}_2$ is the relative velocity of m_1 with respect to m_2. We introduce another quantity, for later use, i.e,

$$\eta = \frac{m_1 m_2}{(m_1 + m_2)^2} \ . \tag{4.8}$$

This is called the symmetric mass ratio of the system. Making all these substitutions in eq. (4.5) we get,

$$\frac{1}{2}\ddot{I}^{jk} = \eta m[v^j v^k - (Gm/r)n^j n^k] \ , \tag{4.9}$$

and using this in the quadrupole formula (eq. (4.1)) we obtain,

$$h^{jk} = \frac{4G\eta m}{c^4 r}\left(v^j v^k - \frac{Gm}{r}n^j n^k\right) \ . \tag{4.10}$$

Now we need expressions for v^j and r^j. To describe the orbital parameter, we introduce an orbit-adapted coordinates system (x, y, z) which possesses the following properties. Firstly, the origin of the coordinate system coincides with the system's center of mass. Secondly, the orbital plane coincides with the xy-plane and the angular momentum points in the z-direction. Thirdly, the major axis of the system coincides with the x-axis and the minor axis of the system coincides with the y-axis. The relative orbit is described by Kepler's equations,

$$r = \frac{p}{1 + e\cos\phi} \ , \quad \dot{\phi} = \sqrt{\frac{Gm}{p^3}}(1 + e\cos\phi)^2 \ , \tag{4.11}$$

where ϕ is the angle from the x-axis (also known as the true-anomaly), p is the system's semi-latus rectum and e is eccentricity of the orbit. In the orbit-adapted frame, we have the unit vector $\mathbf{n}$ and $\boldsymbol{\lambda}$ defined as,

$$\mathbf{n} = [\cos\phi, \sin\phi, 0] \ , \quad \boldsymbol{\lambda} = [-\sin\phi, \cos\phi, 0] \ , \tag{4.12}$$

which form the basis of the frame. The position and the velocity vectors are defined in terms of these basis vectors as,

$$\mathbf{r} = r\mathbf{n} \ , \quad \mathbf{v} = \dot{r}\mathbf{n} + r\dot{\phi}\boldsymbol{\lambda} \ . \tag{4.13}$$

Using eq. (4.11) and eq. (4.13) in eq. (4.10) we finally get the expression of gravitational wave potential for a binary system,

$$h^{jk} = \frac{4\eta(Gm)^2}{c^4 rp}\big[-(1+e\cos\phi - e^2 sin^2\phi)n^j n^k \\ +e\sin\phi(1+e\cos\phi)(n^j\lambda^k + \lambda^j n^k) + (1+e\cos\phi)^2\lambda^j\lambda^k\big] \ . \tag{4.14}$$

4.1.2 Polarizations

The geodesic deviation in the transverse direction provides a way to study the polarization of gravitational waves. Since gravitational waves are tensorial in nature and they have two degrees of freedom, therefore gravitational waves can have two independent modes of polarization, represented as "+" or "×" polarizations. Let us consider the case of a plane gravitational wave first. From eq. (3.87) we can write the tranverse-traceless components of the $h_{\mu\nu}$ for a monochromatic plane gravitational wave propagating in positive z-direction as,

$$h_{xx} = -\, h_{yy} = \Re[A_+ \exp(-\imath\omega(t-z))] \ , \tag{4.15a}$$

$$h_{xy} = h_{yx} = \Re[A_\times \exp(-\imath\omega(t-z))] \ . \tag{4.15b}$$

The amplitudes A_+ and $A_\times$ are two independent modes of gravitational wave polarizations. Just as electromagnetic waves, gravitational waves can also be resolved into linearly or circularly polarized components.

For *linearly polarized waves*, the unit polarization vectors of electromagnetic theory are $\mathbf{e}_x$ and $\mathbf{e}_y$. A test charge hit by a plane wave with x-polarization oscillates in the x-direction and the one hit by a plane wave with y-polarization oscillates in the y-direction. Using the analogy of electromagnetic waves, the unit polarization tensors for gravitational waves are,

$$\mathbf{e}_+ = \mathbf{e}_x \otimes \mathbf{e}_x - \mathbf{e}_y \otimes \mathbf{e}_y \ , \tag{4.16a}$$

$$\mathbf{e}_\times = \mathbf{e}_x \otimes \mathbf{e}_y + \mathbf{e}_y \otimes \mathbf{e}_x \ , \tag{4.16b}$$

The plane gravitational wave having $A_\times = 0$ has $\mathbf{e}_+$ polarization and it can be written as,

$$h_{jk} = \Re[A_+ \exp(-\imath\omega(t-z)\mathrm{e}_{+jk})] \ . \tag{4.17}$$

The effect of geodesic deviation produced by a linearly '+' polarized plane gravitational wave between two test particles depends upon the direction of their separation. In order to see the effect in all directions at one we can consider a circular ring of test particles in the transverse $xy-$plane surrounding a center particle. As the plane wave, with a polarization $\mathbf{e}_x$, passes through this geometry in the positive z-direction, it deforms the ring into an ellipse with axes in the x and y directions that pulsate in and out. On the other hand when a wave with $\mathbf{e}_\times$ passes, it deforms the ring at an angle of 45 degrees from the x and y axes.

For *circularly polarized waves*, the unit polarization vectors of electromagnetic theory are,

$$\mathbf{e}_R = \frac{1}{\sqrt{2}}(\mathbf{e}_x + \iota \mathbf{e}_y) \ , \tag{4.18a}$$

$$\mathbf{e}_L = \frac{1}{\sqrt{2}}(\mathbf{e}_x - \iota \mathbf{e}_y) \ . \tag{4.18b}$$

Using this analogy, the unit polarization tensors for circularly polarized plane gravitational waves can be written as,

$$\mathbf{e}_R = \frac{1}{\sqrt{2}}(\mathbf{e}_+ + \iota \mathbf{e}_\times) \ , \tag{4.19a}$$

$$\mathbf{e}_L = \frac{1}{\sqrt{2}}(\mathbf{e}_+ - \iota \mathbf{e}_\times) \ . \tag{4.19b}$$

The geodesic deviation produced in a ring of test particles surrounding a center particle by linearly and circularly polarized plane gravitational waves can be visualized in figure (4.1),

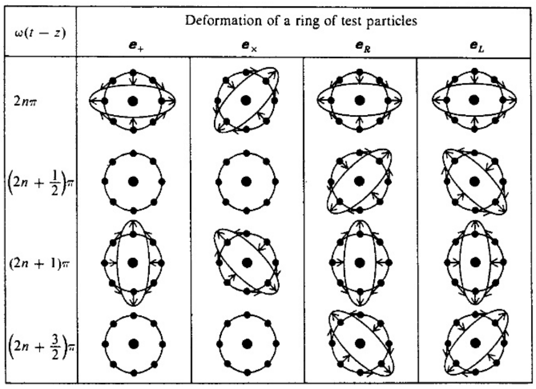

Figure 4.1: The geodesic deviation visualized as the deformation in the shape of a ring of test particles placed in the path of plane gravitational waves for different polarizations [9].

Now we wish to consider the polarizations of gravitational waves produced by compact binaries at 0PN order. In order to do so we define a "detector-adapted" frame (X, Y, Z) in addition to the "orbit-adapted frame" (x, y, z). This detector-adapted frame possesses the following properties. First is that the origin of these coordinates system co-incide with the origin of the frame (x, y, z). Second is that the $Z-$axis points in the direction of the the gravitational-wave detector, at which these polarizations are to be measured. Third is that the $XY-$plane is orthogonal to the $Z-$axis and coincides with the plane of the sky from the detector's point of view, and the $X-$axis is aligned with the line of nodes (the line at which the orbital plane cuts the reference plane). We use the convention that the $X-$axis points towards the ascending node (the point at which the orbit cuts the plane from below). The entire construction is demonstrated in the figure (4.2). The new

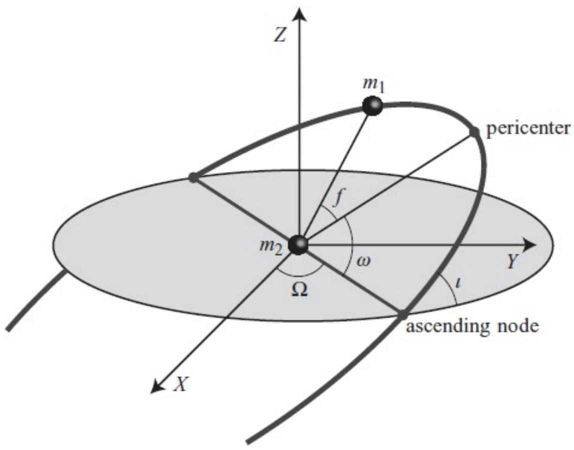

Figure 4.2: Orbit-adapted frame as viewed in the detector-adapted frame. The angle Ω between the line of nodes and the $X-$direction is the *logitude of ascending node*. The angle ω is the angle between the direction of pericenter and the line of nodes, as measured in the orbital plane. The angle ι is the inclination of $xy-$plane from the $XY-$plane. Finally, angle f is the angle between the direction of the system's center of mass and the separation vector $\mathbf{r}$ [16].

coordinate directions are described by,

$$\mathbf{e}_X = [\cos\omega, \sin\omega, 0] \ , \tag{4.20a}$$

$$\mathbf{e}_Y = [\cos\iota\sin\omega, \cos\iota\cos\omega, -\sin\iota] \ , \tag{4.20b}$$

$$\mathbf{e}_Z = [\sin\iota\sin\omega, \sin\iota\cos\omega, \cos\iota] \ , \tag{4.20c}$$

We can take the condition that the $X-$axis of the detector-adapted frame be aligned with the ascending node (figure 4.2), so that we can take the angle Ω to be equal to 0. The unit vectors $\mathbf{n}$ and $\boldsymbol{\lambda}$, when observed from the detector-adapted coordinates (X, Y, Z), are given by,

$$\mathbf{n} =[\cos(\omega+\phi), \cos\iota\sin(\omega+\phi), \sin\iota\sin(\omega+\phi)] \ , \tag{4.21a}$$
$$\boldsymbol{\lambda} =[-\sin(\omega+\phi), \cos\iota\cos(\omega+\phi), \sin\iota\cos(\omega+\phi)] \ . \tag{4.21b}$$

Considering that the gravitational waves travel from binary system to the detector along $Z-$axis, $\mathbf{e}_X$ and $\mathbf{e}_Y$ can be taken as the vectorial basis in transverse subspace. Under these considerations the h_+ and $h_\times$ polarizations can be computed as,

$$h_+ =\frac{1}{2}(\mathrm{e}_{jX}\mathrm{e}_{kX} - \mathrm{e}_{jY}\mathrm{e}_{kY})h^{jk} \ , \tag{4.22a}$$
$$h_\times =\frac{1}{2}(\mathrm{e}_{jX}\mathrm{e}_{kY} - \mathrm{e}_{jY}\mathrm{e}_{kX})h^{jk} \ , \tag{4.22b}$$

Combining equations (4.14), (4.20), (4.21) and (4.22) we can finally write the gravitational wave polarizations at 0PN order as,

$$h_+ = -h_0(1+\cos^2\iota)\left[\cos(2\phi+2\omega) + \frac{5}{4}e\cos(\phi+2\omega) + \frac{1}{4}e\cos(3\phi+2\omega) + \frac{1}{2}e^2\cos 2\omega\right] + \frac{1}{2}e\sin^2\iota(\cos\phi + e) \ , \tag{4.23a}$$
$$h_\times = -2h_0\cos\iota\left[\sin(2\phi+2\omega) + \frac{5}{4}e\sin(\phi+2\omega) + \frac{1}{4}e\sin(3\phi+2\omega) + \frac{1}{2}e^2\sin 2\omega\right] \ , \tag{4.23b}$$

where e is the eccentricity of the orbit and,

$$h_0 = \frac{2\eta(Gm)^2}{c^4 rp} \ , \tag{4.24}$$

is the gravitational wave amplitude. We can further consider the case when the orbit of binary system is circular, i.e, $e = 0$ and ϕ increases linearly with time. Then the gravitational wave polarizations reduce to,

$$h_+ =- h_0(1+\cos^2\iota)\cos 2(\phi t + \omega) \ , \tag{4.25a}$$
$$h_\times =- 2h_0\cos\iota\sin 2(\phi t + \omega) \ . \tag{4.25b}$$

4.1.3 Gravitational Waves Detection by Interferometer Detectors

As discussed earlier, the first ever direct detection of a gravitational wave signal was made my LIGO in September 2015 [2]. The hype got paid when Rainer Weiss, Barry C. Barish and Kip S. Thorne were awarded the Nobel prize in Physics (2017) for "*for decisive contributions to the LIGO detector and the observation of gravitational waves*" [27]. A point of interest in here is that LIGO, in actual, has a geometry similar to Michelson's interferometer. It is therefore advantageous to discuss the method by which an interferometer detector detects a gravitational wave signal.

A laser interferometer consists of a laser source, a beam splitter, two highly reflective end mirrors mounted on test masses imagined to be freely moving in spacetime and a light detector 4.3. A light ray coming from the source is splitted into two beams after passing through the beam splitter. These splitted light beams travel a sufficient distance of 4km (in the case of LIGO) going forward in their specified direction, and after reflecting from the end mirror cover the same amount of distance backwards. The reflected beams then combine together and interfere constructively or destructively, depending upon the difference of phase between them. Initially, the interferometer is set in a configuration that the reflected beams interfere destructively (phase difference is zero), and there is no signal received on the light detector. When a gravitational wave passes, it stretches and/or squeezes the arms lengths of the interferometer, thus altering the amount of distance traveled by the light beams. In that case, the phase difference does not remains zero when the interference occurs, and a signal will be received at the light detector.

Let us assume the length of the arms of the interferometer be L_1 and L_1. The

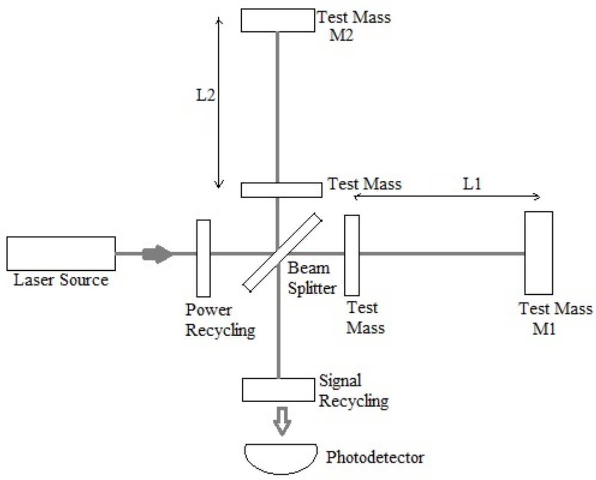

Figure 4.3: Schematic representation of laser interferometer gravitational wave detector.

light ray traveling along L_1 covers a total distance of $2L_1$ and similarly the light ray traveling along L_2 covers a total distance of $2L_2$. The time taken by both light rays to come back to the beam splitter after getting reflected back from the end mirrors is given respectively as, $2L_x/c$ and $2L_y/c$. Then the phase difference is given by,

$$\Delta\Phi = 2\pi\nu(2L_1/c - 2L_2/c) \ , \tag{4.26}$$

where ν is the frequency of laser light. When no gravitational wave is passing, the length of both the arms is constant $L_x = L_y = L_0$ and hence the phase difference is zero. However, when a gravitational wave passes through the detector, it changes the lengths of arms of the detector. In order to find out the change in length, we assume that the position of the end mirror $M1$ from the test mass is specified by a coordinate $\vec{a}_x$, whereas that of $M2$ is specified by $\vec{a}_y$. In the absence of any gravitational wave we can write the coordinates as $\vec{a}_x = L_0\mathbf{e}_1$ and $\vec{a}_y = L_0\mathbf{e}_2$, $\mathbf{e}_1$ and $\mathbf{e}_2$ are the unit vectors in the directions of L_1 and L_2 respectively. We say that when a gravitational wave passes it perturbs the length of the arms. Then the new coordinates can be written as,

$$a_1^j = L_0\left(\mathrm{e}_1^j + \frac{1}{2}h_{TT}^{jk}\mathrm{e}_{k1}\right) \ , \tag{4.27a}$$

$$a_2^j = L_0\left(\mathrm{e}_2^j + \frac{1}{2}h_{TT}^{jk}\mathrm{e}_{k2}\right) \ . \tag{4.27b}$$

Then perturbed length of each can be written as,

$$L_1 = L_0\left(1 + \frac{1}{2}h_{TT}^{jk}\mathrm{e}_{j1}\mathrm{e}_{k1}\right) \ , \tag{4.28a}$$

$$L_2 = L_0\left(1 + \frac{1}{2}h_{TT}^{jk}\mathrm{e}_{j2}\mathrm{e}_{k2}\right) \ . \tag{4.28b}$$

The phase difference can be given by putting these in eq. (4.26) as,

$$\Delta\Phi = \frac{2\pi\nu L_0}{c}(\mathrm{e}_{j1}\mathrm{e}_{k1} - \mathrm{e}_{j2}\mathrm{e}_{k2})h_{TT}^{jk} \ . \tag{4.29}$$

The output of the interferometer detector is given in the form of gravitational wave-strain as a difference in lengths of its original arms [2], i.e, $\Delta L = L_1 - L_y 2h(t)L_0$. We can write down the formula for strain by the use of eq. (4.28) as,

$$h(t) = \frac{1}{2}(\mathrm{e}_{j1}\mathrm{e}_{k1} - \mathrm{e}_{j2}\mathrm{e}_{k2})h_{TT}^{jk} \ . \tag{4.30}$$

In order to calculate $h(t)$, we decompose h_{TT}^{jk} in transverse basis of the detector-adapted frame $(\mathbf{e}_X, \mathbf{e}_Y, \mathbf{e}_Z)$. It can be easily proved from eq. (4.22) that,

$$h_{TT}^{jk} = (\mathrm{e}_X^j\mathrm{e}_X^k - \mathrm{e}_Y^j\mathrm{e}_Y^k)h_+ + (\mathrm{e}_X^j\mathrm{e}_Y^k + \mathrm{e}_Y^j\mathrm{e}_X^k)h_\times \ . \tag{4.31}$$

Putting in eq. (4.30) we can write,

$$h(t) = F_+ A_+(t) + F_\times A_\times(t) \ , \tag{4.32}$$

where,

$$F_+ = \frac{1}{2}(\mathrm{e}_{j1}\mathrm{e}_{k1} - \mathrm{e}_{j2}\mathrm{e}_{k2})(\mathrm{e}^j_X \mathrm{e}^k_X - \mathrm{e}^j_Y \mathrm{e}^k_Y) \ , \tag{4.33a}$$

$$\text{and} \quad F_\times = \frac{1}{2}(\mathrm{e}_{j1}\mathrm{e}_{k1} - \mathrm{e}_{j2}\mathrm{e}_{k2})(\mathrm{e}^j_X \mathrm{e}^k_Y + \mathrm{e}^j_Y \mathrm{e}^k_X) \ . \tag{4.33b}$$

In order to find out the exact expressions for F_+ and $F_\times$ we need to correlate the detector basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ with the transverse basis $(\mathbf{e}_X, \mathbf{e}_Y, \mathbf{e}_Z)$. We start by assuming that the two sets of basis are not aligned in any dimension, and the direction of the source of incoming gravitational waves is in the direction $-\mathbf{N} = \mathbf{e}_X$. In order to align the detector basis with the we perform 3 euler rotations one-by-one. The First rotation by an angle α around the $\mathbf{e}_3$ axis to align the $\mathbf{e}_1$ axis with in the direction of projection of $-\mathbf{N}$ in $\mathbf{e}_1 - \mathbf{e}_2$ plane. This angle α is known as the right *ascension*. The Second rotation by an angle δ around the new $\mathbf{e}_2$ axis to align $\mathbf{e}_3$ in the direction of $-\mathbf{N}$. This angle δ is known as the *declination*. The third rotation is then performed by an angle χ around the axis of $\mathbf{N}$ to align new $\mathbf{e}_1$ with $\mathbf{e}_X$. This rotation is specified by the incoming gravitational wave polarization. Performing all these transformations, the relation between these basis is given as,

$$\begin{aligned}\mathbf{e}_1 =& (\cos\alpha\cos\delta\cos\chi - \sin\alpha\sin\chi)\mathbf{e}_X + (\cos\alpha\cos\delta\sin\chi \\ & + \sin\delta\cos\chi)\mathbf{e}_Y - (\sin\alpha\cos\delta)\mathbf{e}_Z \ ,\end{aligned} \tag{4.34a}$$

$$\begin{aligned}\mathbf{e}_1 =& (\cos\alpha\sin\delta\cos\chi + \cos\delta\sin\chi)\mathbf{e}_X + (\cos\alpha\sin\delta\sin\chi \\ & - \cos\delta\cos\chi)\mathbf{e}_Y - (\sin\alpha\sin\delta)\mathbf{e}_Z \ ,\end{aligned} \tag{4.34b}$$

$$\mathbf{e}_3 = (-\sin\alpha\cos\chi)\mathbf{e}_X - (\sin\alpha\sin\chi)\mathbf{e}_Y - (\cos\alpha)\mathbf{e}_Z \ . \tag{4.34c}$$

Putting back in eq. (4.33) we get the final expression for $F_{+,\times}$,

$$F_+ = \frac{1}{2}(1 + \cos^2\alpha)\cos 2\delta\cos 2\chi - \cos\alpha\sin 2\delta\sin 2\chi \ , \tag{4.35a}$$

$$F_\times = \frac{1}{2}(1 + \cos^2\alpha)\cos 2\delta\sin 2\chi + \cos\alpha\sin 2\delta\cos 2\chi \ . \tag{4.35b}$$

Using eq. (4.25) and eq. (4.35) in eq. (4.32), we can find out a relation for the strain produced in the length of the arms of the detector when a gravitational wave passes through it. Since the strain $h(t)$ is dependent upon time, therefore we are interested in plotting a strain-time graph.

Strain Plot for GW150914 Signal at 0PN

We use the data provided by LIGO for the first gravitational wave discovery, the source of which happens to be a binary black hole merger. This detection is named "GW150914", in which GW is an acronym of gravitational waves and the numbers specify the date of detection (i.e, September 14, 2015). According to published data [28], we can write the masses of the black holes as $m_1 = 7.1604\times10^{31}$kg and $m_2 = 5.7681\times10^{31}$ kg. The luminosity distance as $r = 1.35784\times10^{25}$ m. The orbital plane of the binary is found to strongly misaligned to the line of sight, the maximum probability of inclination angle is at $\iota = 155$ deg. The angles α and δ can be found by viewing the location of the event in the sky. The sky map of GW150914 corresponds to a projected region of two-dimensional credible region of 150deg^2 (with a 50 % probability) and 610deg^2 (with a probability of 90 %). Viewing the sky map, we have a lot of possible values for α and δ. However, we choose values from the most colored regions, then $\alpha = -70$ deg and $\delta = 0.15$ deg. Assuming the linear polarization of the incoming gravitational waves we can write $\chi = 90$ deg. Using all these values in eq. (4.32), we can plot a strain-time graph (Fig4.4) as,

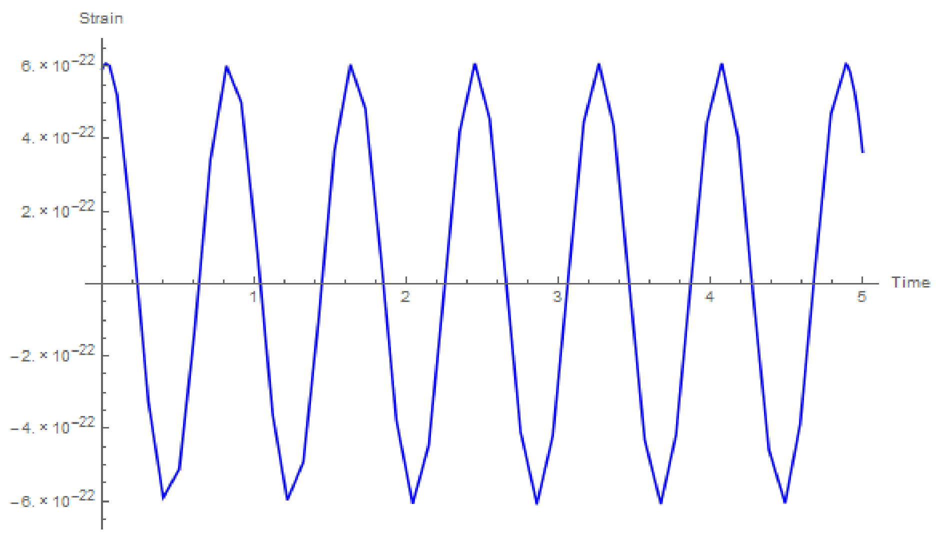

Figure 4.4: The strain versus time plot for the gravitational wave signal GW150914 correct up to the lowest (0PN) order.

4.2 Gravitational Waves at 1.5PN Order

We have seen in the previous chapter that the Einstein quadrupole formula serves well to describe the motion of compact binaries at the early stage of the inspiral phase (section 3.6). Now we extend our approach from the quadrupole formula and study the behavior of gravitational waves at 1.5PN order. The approach is the same that we need to integrate the relaxed field equations (3.29) within the near-zone and the wave-zone domains.

4.2.1 Near-Zone Contribution

From eq. (3.56) we can see that the general form of spatial components of the near-zone wave potential can be written as,

$$h^{jk}_{\mathcal{N}}(t,\mathbf{x}) = \frac{4G}{c^4 r}\sum_{l=0}^{\infty}\frac{(-1)^l}{l!c^l}n_L\left(\frac{d}{dt_r}\right)^l\int_{\mathcal{M}}\tau^{jk}(t_r,\mathbf{x}')\mathbf{x}'^L d^3\mathbf{x}' + \mathcal{O}(r^{-2}) \ . \tag{4.36}$$

Expanding the first few terms we can write,

$$\begin{aligned} h^{jk}_{\mathcal{N}} = \frac{4G}{c^4 r}\Bigg[&\int_{\mathcal{M}}\tau^{jk}d^3\mathbf{x}' - \frac{1}{c}N_a\frac{d}{dt_r}\int_{\mathcal{M}}\tau^{jk}x'^a d^3\mathbf{x}' + \frac{1}{2c^2}N_aN_b\frac{d^2}{dt_r^2}\int_{\mathcal{M}}\tau^{jk}x'^a x'^b d^3\mathbf{x}' \\ &-\frac{1}{6c^3}N_aN_bN_c\frac{d^3}{dt_r^3}\int_{\mathcal{M}}\tau^{jk}x'^a x'^b x'^c d^3\mathbf{x}'...\Bigg] \ . \end{aligned} \tag{4.37}$$

In order to simplify the integrals involved we can make use of the following identities which come as a consequence of the harmonic gauge conditions,

$$\tau^{jk} = \frac{1}{2c^2}\frac{\partial^2}{\partial t_r^2}(\tau^{00}x^jx^k) + \frac{1}{2}[\tau^{jp}x^k + \tau^{kp}x^j - (\tau^{pq}x^jx^k)_{,q}]_{,p} \ , \tag{4.38a}$$

$$\begin{aligned} \tau^{jk}x^a =& \frac{1}{2c}\frac{\partial}{\partial t_r}(\tau^{0j}x^kx^a + \tau^{0k}x^jx^a - \tau^{0a}x^jx^k) \\ & \frac{1}{2}(\tau^{jp}x^kx^a + \tau^{kp}x^jx^a - \tau^{ap}x^jx^k)_{,p} \ . \end{aligned} \tag{4.38b}$$

Making these substitutions in eq. (4.37) and introducing some new notations we get,

$$\begin{aligned} h^{jk}_{\mathcal{N}} = &\frac{2G}{c^4 R}\frac{\partial^2}{\partial t_r^2}\left[\mathcal{Q}^{jk} + \mathcal{Q}^{jka}N_a + \mathcal{Q}^{jkab}N_aN_b + \frac{1}{3}\mathcal{Q}^{jkabc}N_aN_bN_c + ...\right] \\ &\frac{2G}{c^4 r}[P^{jk} + P^{jka}N_a] \ . \end{aligned} \tag{4.39}$$

where the relative multipole moments are defined as,

$$\begin{aligned}
\mathcal{Q}^{jk} =& \frac{1}{c^2}\int_{\mathcal{M}} \tau^{00} x'^j x'^k d^3\mathbf{x}' \ , && (4.40a)\\
\mathcal{Q}^{jka} =& \frac{1}{c}\int_{\mathcal{M}} (c^{-1}\tau^{0j} x'^k x'^a + c^{-1}\tau^{0k} x'^j x'^a - c^{-1}\tau^{0a} x'^j x'^k) d^3\mathbf{x}' \ , && (4.40b)\\
\mathcal{Q}^{jkab} =& \frac{1}{c^2}\int_{\mathcal{M}} \tau^{jk} x'^a x'^b d^3\mathbf{x}' \ , && (4.40c)\\
\mathcal{Q}^{jkabc} =& \frac{1}{c^3}\frac{d}{dt_r}\int_{\mathcal{M}} \tau^{jk} x'^a x'^b x'^c d^3\mathbf{x}' \ , && (4.40d)\\
P^{jk} =& \oint_{\partial\mathcal{M}} [\tau^{jp} x'^k + \tau^{kp} x'^j - (\tau^{pq} x'^j x'^k)_{,q'}] dS_p \ , && (4.40e)\\
P^{jka} =& \frac{1}{c}\frac{d}{dt_r}\oint_{\partial\mathcal{M}} (\tau^{jp} x'^k x'^a + \tau^{kp} x'^j x'^a - \tau^{ap} x'^j x'^k) dS_p \ . && (4.40f)
\end{aligned}$$

In order to find out the values of these multipole moments, we make use of the idea of matter distribution that consists of a perfect fluid. The components of the stress-energy tensor for a perfect fluid correct upto 1.5PN order are given as [16],

$$\begin{aligned}
c^{-2}(-g)T_1^{00} =& \rho[1 + \frac{1}{c^2}(\frac{1}{2}v^2 + 3U + \Pi) + \mathcal{O}(c^{-4})] \ , && (4.41a)\\
c^{-1}(-g)T_1^{0j} =& \rho v^j[1 + \frac{1}{c^2}(\frac{1}{2}v^2 + 3U + \Pi + \frac{P}{\rho}) + \mathcal{O}(c^{-4})] \ , && (4.41b)\\
(-g)T_1^{jk} =& \rho v^j v^k + P\delta^{jk} + \mathcal{O}(c^{-4}) \ . && (4.41c)
\end{aligned}$$

where the number 1 in the subscript denotes that these values are obtained after the first iteration of the Einstein field equations, Π is the internal energy per unit mass of the fluid and U is the Newtonian gravitational potential defined as,

$$U(t,\mathbf{x}) = G\int \frac{\rho(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3\mathbf{x}' \ . \tag{4.42}$$

The components of the Landau-Lifshitz pseudotensor are given as,

$$\begin{aligned}
c^{-2}(-g)t_{LL}^{00} =& \frac{-1}{4\pi G c^2}\left(\frac{7}{2}\partial_j U \partial^j U\right) + \mathcal{O}(c^{-4}) \ , && (4.43a)\\
c^{-1}(-g)t_{LL}^{0j} =& \frac{1}{4\pi G c^2}\left[3\partial_t U \partial^j U + 4(\partial^j U^k - \partial^k U^j)\partial_k U\right] + \mathcal{O}(c^{-4}) \ , && (4.43b)\\
(-g)t_{LL}^{jk} =& \frac{1}{4\pi G}(\partial^j U \partial^k U - \frac{1}{2}\delta^{jk}\partial_n U \partial^n U) + \mathcal{O}(c^{-2}) \ , && (4.43c)
\end{aligned}$$

where,

$$U^j(t,\mathbf{x}) = G\int \frac{\rho v^j(t,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|} d^3\mathbf{x}' \ . \tag{4.44}$$

The harmonic gauge conditions, however, do not make significant contributions upto 1.5PN order. The components of $t_H^{\alpha\beta}$ can therefore be written as,

$$\frac{16\pi G}{c^4}(-g)t_H^{00} = \mathcal{O}(c^{-6}) \ , \tag{4.45a}$$

$$\frac{16\pi G}{c^4}(-g)t_H^{0j} = \mathcal{O}(c^{-7}) \ , \tag{4.45b}$$

$$\frac{16\pi G}{c^4}(-g)t_H^{jk} = \mathcal{O}(c^{-6}) \ , \tag{4.45c}$$

Finally we can write the components of $\tau^{\alpha\beta}$ using equations (4.41), (4.43) and (4.45) and using them in eq. (3.30),

$$c^{-2}\tau_1^{00} = \rho[1 + \frac{1}{c^2}(\frac{1}{2}v^2 + 3U + \Pi)] - \frac{1}{4\pi Gc^2}\left(\frac{7}{2}\partial_j U \partial^j U\right) + \mathcal{O}(c^{-4}) \ , \tag{4.46a}$$

$$c^{-1}\tau_1^{0j} = \rho v^j[1 + \frac{1}{c^2}(\frac{1}{2}v^2 + 3U + \Pi + \frac{P}{\rho})] + \frac{1}{4\pi Gc^2}\left[3\partial_t U\partial^j U + 4(\partial^j U^k - \partial^k U^j)\partial_k U\right] + \mathcal{O}(c^{-4}) \ , \tag{4.46b}$$

$$\tau_1^{jk} = \rho v^j v^k + P\delta^{jk} + \frac{1}{4\pi G}(\partial^j U \partial^k U - \frac{1}{2}\delta^{jk}\partial_n U \partial^n U) + \mathcal{O}(c^{-2}) \ . \tag{4.46c}$$

Radiative Quadrupole Moment

Now we turn our attention to the radiative multipole moments defined in eq. (4.40). The first one being the radiative quadrupole moment. In order to find out the value of $\mathcal{Q}^{jk}$ we can put the value of τ^{00} from eq. (4.46) into eq. (4.40). Also to find out the value of τ^{00}, we need to know the potentials involved in its definition. For that reason we consider a system of point particles, such that,

$$\rho = \sum_A M_A \delta(\mathbf{x} - r_A) \ , \tag{4.47a}$$

$$\Pi = 0 \ , \tag{4.47b}$$

$$U = \sum_B GM_B|\mathbf{x} - r_B|^{-1} \ . \tag{4.47c}$$

Then we can write,

$$c^{-2}\tau^{00} = \sum_A M_A\left[1 + \frac{v_A^2}{2c^2} + \frac{3U_A}{c^2}\right]\delta(\mathbf{x} - r_A) - \frac{7}{8\pi Gc^2}\partial_p U \partial^p U + \mathcal{O}(c^{-4}) \ . \tag{4.48}$$

We can divide the radiative quadrupole moment into a matter part and a field contribution part as,

$$\mathcal{Q}^{jk} = \mathcal{Q}^{jk}[M] + \mathcal{Q}^{jk}[F] + \mathcal{O}(c^{-4}) \ . \tag{4.49}$$

Both the matter part and the field contribution part after integration come out to be,

$$\mathcal{Q}^{jk}[M] = \sum_A M_A \left(1 + \frac{v_A^2}{2c^2} + \frac{3U_A}{c^2}\right) r_A^j r_A^k \ , \tag{4.50a}$$

$$\mathcal{Q}^{jk}[F] = \frac{-7}{2c^2} \sum_A M_A U_A r_A^j r_A^k \ . \tag{4.50b}$$

Using equations (4.50) in eq. (4.49) we get the final expression for the radiative quadrupole moment,

$$\mathcal{Q}^{jk} = \sum_A M_A \left[1 + \frac{v_A^2}{2c^2} - \frac{U_A}{2c^2}\right] r_A^j r_A^k \ . \tag{4.51}$$

We can proceed further on the same footings to find out the values of higher radiative multipole moments.

Radiative Octopole Moment

The value of radiative octopole moment is given by the expression,

$$\mathcal{Q}^{jka} = A^{jka} + A^{kja} - A^{ajk} \ , \tag{4.52}$$

where,

$$\begin{aligned} A^{jka} =& \frac{1}{c} \sum_A M_A \left[1 + \frac{v_A^2}{2c^2}\right] v_A^j r_A^k r_A^a - \frac{1}{2c^3} \sum_A \sum_{B \neq A} \frac{G M_A M_B}{r_{AB}} [(\mathbf{n}_{AB}.\mathbf{v}_A) \\ & \times n_{AB}^j r_A^k r_A^a + v_A^j r_A^k r_A^a] + \frac{1}{2c^3} \sum_A \sum_{B \neq A} G M_A M_B [(\mathbf{n}_{AB}.\mathbf{v}_B) n_{AB}^j n_{AB}^{(k} r_A^{a)} \\ & - 7 n_{AB}^j v_A^{(k} r_A^{a)} + 7 v_A^j n_{AB}^{(k} r_A^{a)}] - \frac{1}{6c^3} \sum_A \sum_{B \neq A} G M_A M_B r_{AB} [(\mathbf{n}_{AB}.\mathbf{v}_A) \\ & \times n_{AB}^j n_{AB}^k n_{AB}^a - 11 n_{AB}^j n_{AB}^{(k} v_A^{a)} + 11 v_A^j n_{AB}^k n_{AB}^a] + \mathcal{O}(c^{-5}) \ . \end{aligned} \tag{4.53}$$

Radiative 4-pole and 5-pole Moments

The values of radiative 4-pole and 5-pole moments are given as,

$$\begin{aligned} \mathcal{Q}^{jkab} =& \frac{1}{c^2} \sum_A M_A v_A^j v_A^k r_A^a r_A^b - \frac{1}{2c^2} \sum_A \sum_{B \neq A} \frac{G M_A M_B}{r_{AB}} n_{AB}^j n_{AB}^k r_A^a r_A^b \\ & \frac{1}{12c^2} \sum_A \sum_{B \neq A} G M_A M_B r_{AB} n_{AB}^j n_{AB}^k (n_{AB}^a n_{AB}^b - \delta^{ab}) + \mathcal{O}(c^{-4}) \end{aligned} \tag{4.54}$$

$$\begin{aligned}\mathcal{Q}^{jkabc} =&\frac{1}{c^3}\frac{\partial}{\partial t_r}[\sum_A M_A v_A^j v_A^k r_A^a r_A^b r_A^c - \frac{1}{2}\sum_a\sum_{B\neq A}\frac{GM_A M_B}{r_{AB}} n_{AB}^j n_{AB}^k r_A^a r_A^b r_A^c \\ &+ \frac{1}{4}\sum_A\sum_{B\neq A} GM_A M_B r_{AB} n_{AB}^j n_{AB}^k r_A^{(a}\{n_{AB}^b n_{AB}^{c)} - \delta^{bc)}\}] + \mathcal{O}(c^{-5}) \ . \quad (4.55)\end{aligned}$$

All the multipole moments are expressed in terms of mass-energy M_A of each body in a system of bodies, its position $\mathbf{r}_A$ and velocity $\mathbf{v}_A$ evaluated over the retarded time t_r. Therefore, the radiative multipole moments are functions of retarded time t_r, $r_{AB} = |\mathbf{r}_A - \mathbf{r}_B|$ is the distance between bodies A and B and $\mathbf{n}_{AB} = (\mathbf{r}_A - \mathbf{r}_B)/r_{AB}$ is the unit vector that points from body B to body A. However, the contributions due to P^{jk} and P^{jka} vanish upto 1.5PN order.

4.2.2 Wave-Zone Contribution

We have seen in the previous chapter that the wave-zone contribution to h^{jk} can be obtained by considering τ^{jk} as the sum of the terms of the form,

$$\tau^{jk}[l,m] = \frac{1}{4\pi}\frac{f^{jk}(t_r)}{r^m} n^{\langle L\rangle} \ . \quad (4.56)$$

where f is an arbitrary function of retarted time t_r, m is an arbitrary integer and $n^{\langle L\rangle}$ is an angular STF tensor of degree l, which satisfies the following identities,

$$n^{\langle jk\rangle} = n^j n^k - \frac{1}{3}\delta^{jk} \ , \quad (4.57a)$$

$$n^{\langle jka\rangle} = n^j n^k n^a - \frac{1}{5}(\delta^{jk} n^a + \delta^{ja} n^k \delta^{ka} n^j) \ , \quad (4.57b)$$

$$\begin{aligned}n^{\langle jkab\rangle} =& n^j n^k n^a n^b - \frac{1}{7}(\delta^{jk} n^a n^b + \delta^{ja} n^k n^b + \delta^{jb} n^k n^a + \delta^{ka} n^j n^b + \delta^{kb} n^j n^a + \delta^{ab} n^j n^k) \\ &\frac{1}{35}(\delta^{jk}\delta^{ab} + \delta^{ja}\delta^{kb} + \delta^{jb}\delta^{ka}) \ . \quad (4.57c)\end{aligned}$$

A general form for wave-zone contribution $h_{\mathcal{W}^{jk}}$ is given by eq. (3.70) as,

$$h^{jk}_{\mathcal{W}}[l,m] = \frac{4G}{c^4 r} n^{\langle L\rangle}\left[\int_0^r ds f^{jk}(t_r - 2s/c) A(s,r) + \int_r^\infty dS f^{jk}(t_r - 2s/c) B(s,r)\right] \ . \quad (4.58)$$

where $A(s,r)$ and $B(s,r)$ are both defined in terms of Legendre polynomials given in eq. (3.70) as,

$$A(s,r) = \int_R^{r+s}\frac{P_l(\xi)}{p^{m-1}}dp \ ; \quad B(s,r) = \int_S^{r+s}\frac{P_l(\xi)}{p^{m-1}}dp \ . \quad (4.59)$$

where the argument of Legendre polynomial ξ is defined as,

$$\xi = \frac{r+2s}{r} - \frac{2s(r+s)}{rp} \ . \tag{4.60}$$

Now we can attempt to reach such forms of equations using the source terms.

4.2.3 Construction of the Source Terms

Since the wave-zone doesn't contain any source interaction, therefore we expect to get only the contributions due to the fields from the wave-zone field. Therefore, τ^{jk} can be defined only in terms of t^{jk}_{LL} and t^{jk}_{H} with no trace of T^{jk}. The Landau-Lifshitz pseudotensor is given as,

$$\begin{aligned}(-g)t^{jk}_{LL} = \frac{c^4}{16\pi G}\Bigg[&\frac{1}{4}\partial^j h^{00}\partial^k h^{00} + \partial^j h^{00}\partial_0 h^{0k} + \partial^k h^{00}\partial_0 h^{0j} + \frac{1}{4}\partial^j h^{00}\partial^k h^p_p \\ &\frac{1}{4}\partial^k h^{00}\partial^j h^p_p + ...\Bigg] \ ,\end{aligned} \tag{4.61}$$

and the contribution due to the harmonic gauge conditions is,

$$(-g)t^{jk}_{H} = \frac{c^4}{16\pi G}[-h^{00}\partial_{00}h^{jk} + ...] \ . \tag{4.62}$$

We can use the identity $\partial_{00} = c^{-2}\partial_{tt}$, in order to simplify our expressions. Using equations (4.61) and (4.62) we can write the value of τ^{jk}.

$$\begin{aligned}\tau^{jk} = \frac{c^4}{16\pi G}\Bigg[&\frac{1}{4}\partial^j h^{00}\partial^k h^{00} + \partial^j h^{00}\partial_0 h^{0k} + \partial^k h^{00}\partial_0 h^{0j} + \frac{1}{4}\partial^j h^{00}\partial^k h^p_p \\ &\frac{1}{4}\partial^k h^{00}\partial^j h^p_p - \frac{1}{c^2}h^{00}\partial_{tt}h^{jk}...\Bigg] \ .\end{aligned} \tag{4.63}$$

In order to find the derivatives involved, we have previously calculated (eq. 3.86) that the components of h^{jk} in the wave-zone, correct upto 1PN order come out to be

$$h^{00} = \frac{4G}{c^2}\left[\frac{M}{r} + \frac{1}{2}\left(\frac{\mathcal{I}^{jk}}{r}\right)_{jk} + ...\right] \ , \tag{4.64a}$$

$$h^{0j} = \frac{4G}{c^3}\left[\frac{-1}{2}\mathcal{J}^{jk}\frac{n^k}{r} - \frac{1}{2}\left(\frac{\dot{\mathcal{I}}^{jk}}{r}\right) + ...\right] \ , \tag{4.64b}$$

$$h^{jk} = \frac{4G}{c^4}\left[\frac{1}{2}\frac{\dddot{\mathcal{I}}^{jk}}{r} + ...\right] \ . \tag{4.64c}$$

The respective derivatives can then be written as,

$$\partial^j h^{00} = \frac{4G}{c^2}\left[\frac{-M}{r^2}n^j + \frac{1}{2}\partial^j_{pq}\left(\frac{\mathcal{I}^{pq}}{r}\right) + ...\right] , \tag{4.65a}$$

$$\partial_t h^{0j} = \frac{4G}{c^3}\left[\frac{-1}{2}\partial_p\left(\frac{\ddot{\mathcal{I}}^{jp}}{r} + ...\right)\right] , \tag{4.65b}$$

$$\partial^j h^p_p = \frac{4G}{c^4}\left[\frac{-1}{2}\frac{\ddot{\mathcal{I}}}{r^2}n^j + ...\right] , \tag{4.65c}$$

$$\partial_{tt} h^{jk} = \frac{4G}{c^4}\left[\frac{1}{2}\mathcal{I}^{(4)jk} + ...\right] . \tag{4.65d}$$

where $\mathcal{I}^{(4)jk}$ represents the 4^{th} time derivative of $\mathcal{I}^{jk}$ and $\ddot{\mathcal{I}} = \ddot{\mathcal{I}}^{pp}$. Using these values in eq. (4.63) we get,

$$\tau^{jk} = \frac{GM}{4\pi r^2}\left[\frac{M}{r^2}n^j n^k - n^{(j}\partial^{k)}_{pq}\left(\frac{\mathcal{I}^{pq}}{r}\right) + \frac{4}{c^2}n^{(j}\partial_p\left(\frac{\ddot{\mathcal{I}}^{k)p}}{r}\right)\right.$$
$$\left.\frac{1}{c^2}\left(\frac{\ddot{\mathcal{I}}}{r^2} + \frac{1}{c}\frac{\dddot{\mathcal{I}}}{r}\right)n^j n^k - \frac{2}{c^4}\mathcal{I}^{(4)jk} + ...\right] \tag{4.66}$$

We still got some derivatives to deal with. In order to calculate them, we recall that $\partial_j r = n_j$ and $\partial_j n_k = r^{-1}(\delta_{jk} - n_j n_k)$. We know that $\mathcal{I}^{jk}$ depends upon the spatial components through $t_r = t - r/c$, so that $\partial_p \mathcal{I}^{jk} = -c^{-1}\dot{\mathcal{I}}^{jk} n_p$. Using these rules we can find out the derivatives,

$$\partial_p\left(\frac{\ddot{\mathcal{I}}^{jk}}{r}\right) = -\left(\frac{\ddot{\mathcal{I}}^{jk}}{r^2} + \frac{1}{c}\frac{\dddot{\mathcal{I}}^{jk}}{r}\right)n_p , \tag{4.67a}$$

$$\partial^j_{pq}\left(\frac{\mathcal{I}^{pq}}{r}\right) = -\left(15\frac{\mathcal{I}^{pq}}{r^4} + \frac{15}{c}\frac{\dot{\mathcal{I}}^{pq}}{r^3} + \frac{6}{c^2}\frac{\ddot{\mathcal{I}}^{pq}}{r^2} + \frac{1}{c^3}\frac{\dddot{\mathcal{I}}^{pq}}{r}\right)n^j n_p n_q$$
$$+\left(3\frac{\mathcal{I}^{pq}}{r^4} + \frac{3}{c}\frac{\dot{\mathcal{I}}^{pq}}{r^3} + \frac{1}{c^2}\frac{\ddot{\mathcal{I}}^{pq}}{r^4}\right)(n^j\delta_{pq} + \delta^j_p n_q + \delta^j_q n_p) . \tag{4.67b}$$

Using these results eq. (4.66) becomes,

$$\tau^{jk} = \frac{GM^2}{4\pi r^4}n^j n^k + \frac{GM}{4\pi r^2}\left[\left(15\frac{\mathcal{I}^{pq}}{r^4} + \frac{15}{c}\frac{\dot{\mathcal{I}}^{pq}}{r^3} + \frac{6}{c^2}\frac{\ddot{\mathcal{I}}^{pq}}{r^2} + \frac{1}{c^3}\frac{\dddot{\mathcal{I}}^{pq}}{r}\right)n^j n^k n_p n_q\right.$$
$$-\left(3\frac{\mathcal{I}}{r^4} + \frac{3}{c}\frac{\dot{\mathcal{I}}}{r^3} + \frac{1}{c^3}\frac{\dddot{\mathcal{I}}}{r}\right)n^j n^k - \left(3\frac{\mathcal{I}^{jp}}{r^4} + \frac{3}{c}\frac{\dot{\mathcal{I}}^{jp}}{r^3} + \frac{3}{c^2}\frac{\ddot{\mathcal{I}}^{jp}}{r^2} + \frac{2}{c^3}\frac{\dddot{\mathcal{I}}^{jp}}{r}\right)n^k n_p$$

$$
- \left(3\frac{\mathcal{I}^{kp}}{r^4} + \frac{3}{c}\frac{\dot{\mathcal{I}}^{kp}}{r^3} + \frac{3}{c^2}\frac{\ddot{\mathcal{I}}^{kp}}{r^2} + \frac{2}{c^3}\frac{\dddot{\mathcal{I}}^{kp}}{r} \right) n^j n_p - \frac{2}{c^4}\mathcal{I}^{(4)jk} + ... \Bigg] \ . \tag{4.68}
$$

The final step is to express the angular dependence of τ^{jk} in terms of the STF tensor $n^{\langle L\rangle}$. We make use of the identities introduced in eq. (4.57) and discard the terms proportional to δ^{jk}, thus arriving to the expression,

$$
\begin{aligned}
\tau^{jk} =& \frac{GM^2}{4\pi r^4} n^{\langle jk\rangle} + \frac{GM}{4\pi r^2} \Bigg[\left(15\frac{\mathcal{I}_{pq}}{r^4} + \frac{15}{c}\frac{\dot{\mathcal{I}}_{pq}}{r^3} + \frac{6}{c^2}\frac{\ddot{\mathcal{I}}_{pq}}{r^2} + \frac{1}{c^3}\frac{\dddot{\mathcal{I}}_{pq}}{r} \right) n^{\langle jkpq\rangle} \\
& - \left(\frac{6}{7}\frac{\mathcal{I}}{r^4} + \frac{6}{7c}\frac{\dot{\mathcal{I}}}{r^3} - \frac{6}{7c^2}\frac{\ddot{\mathcal{I}}}{r^2} - \frac{8}{7c^3}\frac{\dddot{\mathcal{I}}}{r} \right) n^{\langle jk\rangle} + \left(\frac{9}{7}\frac{\mathcal{I}_p^{(j}}{r^4} + \frac{9}{7c}\frac{\dot{\mathcal{I}}_p^{(j}}{r^3} - \frac{9}{7c^2}\frac{\ddot{\mathcal{I}}_p^{(j}}{r^2} \right. \\
& \left. - \frac{12}{7c^3}\frac{\dddot{\mathcal{I}}_p^{(j}}{r} \right) n^{\langle k)p\rangle} - \frac{6}{5c^2}\frac{\ddot{\mathcal{I}}^{\langle jk\rangle}}{r^2} - \frac{6}{5c^3}\frac{\dddot{\mathcal{I}}^{\langle jk\rangle}}{r} - \frac{2}{c^4}\mathcal{I}^{(4)jk} + ... \Bigg] \ .
\end{aligned} \tag{4.69}
$$

This expression is the sum of terms that have a structure similar to eq. (4.56). Therefore, from this expression we can find out the appropriate function 'f' for each value of l and m.

Each term of τ^{jk} makes contribution to the gravitational wave field h^{jk} through eq. (4.58). To see how the integrals are evaluated we look at the case where $l = 0$ and $m = 3$. We begin by extracting the relevant piece of τ^{jk}, i.e,

$$
\frac{GM}{4\pi r^2} \left(\frac{-6}{5c^3}\frac{\dddot{\mathcal{I}}^{\langle jk\rangle}}{r} \right) \ .
$$

In this case the function f can be written as,

$$
f(t_r) = \frac{-6GM}{5c^3}\dddot{\mathcal{I}}^{\langle jk\rangle} \ . \tag{4.70}
$$

Next, we get the functions $A(s,r)$ and $B(s,r)$ for $l = 0$ and $m = 3$,

$$
A(s,r) = \frac{1}{\mathcal{R}} - \frac{1}{r+s} \ , \quad B(s,r) = \frac{1}{s} - \frac{1}{r+s} \ . \tag{4.71}
$$

We now calculate the integrals of these functions as,

$$
\begin{aligned}
F_A =& \frac{1}{r}\int_0^{\mathcal{R}} f(t_r - 2s/c)ds - \int_0^{\mathcal{R}} f(t_r - 2s/c)d\ln(r+s) \ , \\
=& - f(t_r - 2\mathcal{R}/c)\ln(r+\mathcal{R}) + f(t_r)\ln r + \frac{1}{\mathcal{R}}\int_0^{\mathcal{R}} f(t_r - 2s/c)ds \\
& - \frac{2}{c}\int_0^{\mathcal{R}} \dot{f}(t_r - 2s/c)\ln(\frac{r+s}{s})ds - \frac{2}{c}\int_0^{\mathcal{R}} \dot{f}(t_r - 2s/c)\ln ds \ ,
\end{aligned} \tag{4.72}
$$

and,

$$\begin{aligned} F_B =& -\int_{\mathcal{R}}^{\infty} f(t_r - 2s/c) d \ln \frac{r+s}{s} \ , \\ =& f(t_r - 2\mathcal{R}/c) \ln \frac{r+\mathcal{R}}{\mathcal{R}} - \frac{2}{c}\int_{\mathcal{R}}^{\infty} \dot{f}(t_r - 2s/c) \ln \frac{r+s}{s} ds \ , \end{aligned} \tag{4.73}$$

assuming that $f(t_r - 2s/c)$ goes to zero fast enough as $s \to \infty$ such that it ensures that there is no boundary term as $s = \infty$. The sum of F_A and F_B then comes out to be,

$$\begin{aligned} F =& - f(t_r - 2\mathcal{R}/c) \ln \mathcal{R} + f(t_r) \ln r + \frac{1}{\mathcal{R}} \int_0^{\mathcal{R}} f(t_r - 2s/c) ds \\ & - \frac{2}{c}\int_0^{\mathcal{R}} \dot{f}(t_r - 2s/c) \ln s ds - \frac{2}{c}\int_0^{\infty} \dot{f}(t_r - 2s/c) \ln \frac{r+s}{s} ds \ . \end{aligned} \tag{4.74}$$

In order to simplify this result, we can exploit the fact that we may remove all the $\mathcal{R}-$dependent terms and its derivatives as an infinite Taylor series in the power of s and evaluate the two integrals from $s = 0$ to $s = \mathcal{R}$. They combine to give $f(t_r)$, plus the terms that can be neglected because they come with explicit factors of $\mathcal{R}$. After expanding $f(t_r - 2\mathcal{R}/c)$ in powers of $\mathcal{R}$, we also find out that,

$$F = f(t_r)[1 + \ln(r/\mathcal{R})] - \frac{2}{c}\int_0^{\infty} \dot{f}(t_r - 2s/c) \ln \frac{r+s}{s} ds \ . \tag{4.75}$$

The final expression for $h^{jk}_{\mathcal{W}}[0,3]$ is,

$$h^{jk}_{\mathcal{W}}[0,3] = \frac{4GM}{c^4 r}\left[-\frac{6G}{5c^3}\left\{1 + \ln(r/\mathcal{R})\right\} \dddot{\mathcal{I}}^{\langle jk \rangle} + \frac{12}{5} K^{jk} \right] \ , \tag{4.76}$$

where,

$$K^{jk}(t_r, r) = \frac{G}{c^4}\int_o^{\infty} \mathcal{I}^{(4)jk}(t_r - 2s/c) \ln \frac{r+s}{s} ds \ , \tag{4.77}$$

is the *tail integral* which involves the entire past history of the system from infinite past (at $s = \infty$) to the current retarded time (at $s = 0$). The same technique can be used to find the other contributions to $h^{jk}_{\mathcal{W}}$. These contributions come out to be,

$$h^{jk}_{\mathcal{W}}[0,2] = \frac{4GM}{c^4 r}\left[-2K^{jk}\right] \ , \tag{4.78a}$$

$$h^{jk}_{\mathcal{W}}[0,4] = \frac{4GM}{c^4 r}\left[\frac{6G}{5c^3}\left\{\frac{3}{2} + \ln(r/\mathcal{R})\right\}\dddot{\mathcal{I}}^{\langle jk\rangle} - \frac{12}{5}K^{jk}\right] \ , \tag{4.78b}$$

$$h^{jk}_{\mathcal{W}}[2,3] = \frac{4GM}{c^4 r}\left[\left\{-\frac{2G}{7c^3}\dddot{\mathcal{I}}^{j}_{p}\right\} n^{\langle pk\rangle} + \left\{-\frac{2G}{7c^3}\dddot{\mathcal{I}}^{k}_{p}\right\} n^{\langle pj\rangle}\right] \ , \tag{4.78c}$$

$$h^{jk}_{\mathcal{W}}[2,4] = \frac{4GM}{c^4 r}\left[\left\{-\frac{3G}{28c^3}\dddot{\mathcal{I}}^{j}_{p}\right\} n^{\langle pk\rangle} + \left\{-\frac{3G}{28c^3}\dddot{\mathcal{I}}^{k}_{p}\right\} n^{\langle pj\rangle}\right] \ , \tag{4.78d}$$

$$\begin{aligned} h^{jk}_{\mathcal{W}}[2,5] =& \frac{4GM}{c^4 r}\left[\frac{G}{c^3}\left\{\frac{47}{700} + \frac{3}{35}\ln(r/\mathcal{R})\right\}\dddot{\mathcal{I}}^{j}_{p} - \frac{6}{35}K^{j}_{p}\right] n^{\langle pk\rangle} \\ & \frac{4GM}{c^4 r}\left[\frac{G}{c^3}\left\{\frac{47}{700} + \frac{3}{35}\ln(r/\mathcal{R})\right\}\dddot{\mathcal{I}}^{k}_{p} - \frac{6}{35}K^{k}_{p}\right] n^{\langle pj\rangle} \ , \end{aligned} \tag{4.78e}$$

$$\begin{aligned} h^{jk}_{\mathcal{W}}[2,6] =& \frac{4GM}{c^4 r}\left[\frac{G}{c^3}\left\{\frac{-97}{700} - \frac{3}{35}\ln(r/\mathcal{R})\right\}\dddot{\mathcal{I}}^{j}_{p} + \frac{6}{35}K^{j}_{p}\right] n^{\langle pk\rangle} \\ & \frac{4GM}{c^4 r}\left[\frac{G}{c^3}\left\{\frac{-97}{700} - \frac{3}{35}\ln(r/\mathcal{R})\right\}\dddot{\mathcal{I}}^{k}_{p} + \frac{6}{35}K^{k}_{p}\right] n^{\langle pj\rangle} \ , \end{aligned} \tag{4.78f}$$

$$h^{jk}_{\mathcal{W}}[4,3] = \frac{4GM}{c^4 r}\left[\frac{G}{20c^3}\dddot{\mathcal{I}}_{pq}\right] n^{\langle jkpq\rangle} \ , \tag{4.78g}$$

$$h^{jk}_{\mathcal{W}}[4,4] = \frac{4GM}{c^4 r}\left[\frac{G}{30c^3}\dddot{\mathcal{I}}_{pq}\right] n^{\langle jkpq\rangle} \ , \tag{4.78h}$$

$$h^{jk}_{\mathcal{W}}[4,5] = \frac{4GM}{c^4 r}\left[\frac{G}{42c^3}\dddot{\mathcal{I}}_{pq}\right] n^{\langle jkpq\rangle} \ , \tag{4.78i}$$

$$h^{jk}_{\mathcal{W}}[4,6] = \frac{4GM}{c^4 r}\left[\frac{G}{56c^3}\dddot{\mathcal{I}}_{pq}\right] n^{\langle jkpq\rangle} \ . \tag{4.78j}$$

Adding all these contributions we get the final result as,

$$\begin{aligned} h^{jk}_{\mathcal{W}} = \frac{4GM}{c^4 r} & \left[\frac{3G}{5c^3}\dddot{\mathcal{I}}^{\langle jk\rangle} - 2K^{jk} - \frac{13G}{28c^3}\left(\dddot{\mathcal{I}}^{j}_{p} n^{\langle pk\rangle} + \dddot{\mathcal{I}}^{k}_{p} n^{\langle pj\rangle}\right)\right. \\ & \left. \frac{G}{8c^3} n^{\langle jkpq\rangle}\right] \ . \end{aligned} \tag{4.79}$$

We can further remove the terms which won't survive the TT-projection. For example, we can write,

$$\dddot{\mathcal{I}}_{p} n^{\langle pk\rangle} = \dddot{\mathcal{I}}^{j}_{p}(n^p n^k - \frac{1}{3}\delta^{pk}) \overset{TT}{=} -\frac{1}{3}\dddot{\mathcal{I}}^{\langle jk\rangle} \ , \tag{4.80a}$$

$$\dddot{\mathcal{I}}_{pq} n^{\langle jkpq\rangle} \overset{TT}{=} \frac{2}{35}\dddot{\mathcal{I}}^{\langle jk\rangle} \ . \tag{4.80b}$$

Using this condition we get the final expression for the wave-zone contribution to the gravitational wave field as,

$$h^{jk}_{\mathcal{W}} = \frac{4GM}{c^4 r}\left[\frac{11}{12}\frac{G}{c^3}\dddot{\mathcal{I}}^{\langle jk\rangle} - 2K^{jk}\right] . \tag{4.81}$$

Gravitational Wave-Tails

In eq. (4.77), we encountered the tail-integral for the first time. This gives us the insight that *gravitational wave-tail* effects find their appearance in the total contribution of the gravitational wave potential at 1.5PN order.

The gravitational wave-tails are basically the memory of the waves that have already passed through the wave-zone and have distorted the spacetime (i.e. the former post-Minkowskian contributions), such that this distorted spacetime serves as the background curvature for the latter wave. These wave-tails are produced as a result of back-scattering[1] of the outgoing gravitational radiations off the curved spacetime associated with the total mass of the system [29].

4.2.4 Binary System

Now we wish to study the case of the binary system again, this time the degree of accuracy is 1.5PN rather 0PN in the previous section 4.1. The description of the binary system is the same as before, i.e, both the masses m_1 and m_2 have position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ and velocities $\mathbf{v}_1$ and $\mathbf{v}_2$ respectively. We make use of the center of mass frame as well.

To the required degree of accuracy, the position vectors are given as,

$$\mathbf{r}_1 = \frac{m_2}{m}\mathbf{r} + \frac{\eta\Delta}{2c^2}\left(v^2 - \frac{Gm}{r}\right)\mathbf{r} + \mathcal{O}(c^{-4}) , \tag{4.82a}$$

$$\mathbf{r}_2 = -\frac{m_1}{m}\mathbf{r} + \frac{\eta\Delta}{2c^2}\left(v^2 - \frac{Gm}{r}\right)\mathbf{r} + \mathcal{O}(c^{-4}) . \tag{4.82b}$$

whereas the velocities of both the objects are given as,

$$\mathbf{v}_1 = \frac{m_2}{m}\mathbf{v} + \frac{\eta\Delta}{2c^2}\left[\left(v^2 - \frac{Gm}{r}\right)\mathbf{r} - \frac{Gm}{r}\dot{r}\mathbf{n}\right] + \mathcal{O}(c^{-4}) , \tag{4.83a}$$

$$\mathbf{v}_2 = -\frac{m_1}{m}\mathbf{v} + \frac{\eta\Delta}{2c^2}\left[\left(v^2 - \frac{Gm}{r}\right)\mathbf{r} - \frac{Gm}{r}\dot{r}\mathbf{n}\right] + \mathcal{O}(c^{-4}) , \tag{4.83b}$$

[1]Back-scattering of a wave is similar to the reflection of a wave, i.e, the wave get reflected back into the same medium from where it comes. However, the difference is that back-scattering is a kind of diffused reflection due to scattering, and not a pure reflection.

where $\dot{r} = \mathbf{n.v}$ is the radial component of the velocity vector and $\mathbf{n} = \mathbf{r}/r$ is the unit radial vector that points from body 2 to 1. The relative acceleration vector $\mathbf{a} = \mathbf{a}_1 - \mathbf{a}_2$ is given as [16],

$$\mathbf{a} = \frac{-Gm}{r^2}\mathbf{n} - \frac{Gm}{c^2r^2}\{\left[(1+3\eta)v^2 - \frac{3}{2}\eta\dot{r}^2 - 2(2+\eta)\frac{Gm}{r}\right]\mathbf{n} \\ - 2(2-\eta)\dot{r}\mathbf{v}\} + \mathcal{O}(c^{-4}) \ . \tag{4.84}$$

We can express radiative multipole moments defined in equations (4.51), (4.52), (4.54) and (4.55) for binary system as,

$$\mathcal{Q}^{jk} = \eta m\left[1 + \frac{1}{2}(1-3\eta)\frac{v^2}{c^2} - \frac{1}{2}(1-2\eta)\frac{Gm}{c^2r} + \mathcal{O}(c^{-4})\right]r^jr^k \ , \tag{4.85a}$$

$$\begin{aligned}\mathcal{Q}^{jka} = \frac{\eta m\Delta}{c}&\left[r^jr^kv^a - (v^jr^k + r^jv^k)r^a - \left\{\frac{1}{2}(1-5\eta)\frac{v^2}{c^2}\right.\right.\\ &+\left.\frac{1}{6}(7+12\eta)\frac{Gm}{c^2r}\right\}(v^jr^k + r^jv^k)r^a + \left\{\frac{1}{2}(1-5\eta)\frac{v^2}{c^2}\right.\\ &+\left.\frac{1}{6}(17+12\eta)\frac{Gm}{c^2r}\right\}r^jr^kv^a + \frac{1}{6}(1-6\eta)\frac{Gm}{c^2r}\dot{r}n^jr^kr^a\\ &\left.+\mathcal{O}(c^{-4})\right] \ ,\end{aligned} \tag{4.85b}$$

$$\begin{aligned}\mathcal{Q}^{jkab} = \frac{\eta m}{c^2}&\left[(1-3\eta)v^jv^kr^ar^b - \frac{1}{3}(1-3\eta)\frac{Gm}{r}n^jn^kr^ar^b\right.\\ &\left.-\frac{1}{6}\frac{Gm}{r}r^jr^k\delta^{ab} + \mathcal{O}(c^{-2})\right] \ .\end{aligned} \tag{4.85c}$$

$$\begin{aligned}\mathcal{Q}^{jkabc} = \frac{\eta m\Delta}{c^3}\frac{\partial}{\partial t_r}&\left[-(1-2\eta)v^jv^kr^ar^br^c + \frac{1}{4}(1-2\eta)\frac{Gm}{r}n^jn^kr^ar^br^c\right.\\ &\left.\frac{1}{4}\frac{Gm}{r}r^jr^kr^{(a}\delta^{bc)} + \mathcal{O}(c^{-2})\right] \ .\end{aligned} \tag{4.85d}$$

From eq. (4.39) we can see that in order to find out the near-zone contribution to the gravitational wave potential we have to calculate the 2^{nd} time derivatives of the radiative multipole moments. These derivatives when by the identities,

$$v\dot{v} = -\frac{Gm}{r^2}\dot{r} + \mathcal{O}(c^{-2}) \ , \quad r\ddot{r} = v^2 - \dot{r}^2 - \frac{Gm}{r} + \mathcal{O}(c^{-2}) \ , \tag{4.86}$$

come out to be,

$$\ddot{\mathcal{Q}}^{jk} = 2\eta m(v^jv^k - \frac{Gm}{r}n^jn^k) + \frac{\eta m}{c^2}\left[\left\{-\frac{1}{2}(7+2\eta)v^2 + \frac{3}{2}(1-2\eta)\dot{r}^2\right.\right.$$

$$+ \frac{19}{2}\frac{Gm}{r}\bigg\} \frac{Gm}{r} n^j n^k + \left\{ (1-3\eta)v^2 - (1-2\eta)\frac{Gm}{r} \right\} v^j v^k$$

$$+ (3+2\eta)\frac{Gm}{r}\dot{r}(v^j n^k + n^j v^k) \bigg] + \mathcal{O}(c^-4) \ , \tag{4.87a}$$

$$\ddot{\mathcal{Q}}^{jka} = \frac{\eta m \Delta}{c}\bigg[-3\frac{Gm}{r}\dot{r} n^j n^k n^a + \frac{3Gm}{r}(v^j n^k + n^j v^k) n^a + \frac{Gm}{r} n^j n^k v^a$$

$$-2v^j v^k v^a \bigg] + \frac{\eta m \Delta}{c^3}\bigg[\bigg\{ \frac{3}{2}(2-\eta)v^2 + \frac{9}{2}(1+\eta)\dot{r}^2 - \frac{1}{3}(31-9\eta)$$

$$\times \frac{Gm}{r}\bigg\} \frac{Gm}{r}(v^j n^k + n^j v^k) n^a - (15+2\eta)\frac{Gm}{r}\dot{r} v^j v^k n^a$$

$$+ \left\{ -\frac{3}{2}(4-3\eta)v^2 + \frac{5}{2}(1-3\eta)\dot{r}^2 \frac{2}{3}(29-3\eta)\frac{Gm}{r} \right\} \frac{Gm}{r}\dot{r} n^j n^k n^a$$

$$+ \left\{ \frac{1}{2}(4-\eta)v^2 - \frac{3}{2}(1-\eta)\dot{r}^2 - \frac{1}{3}(25-3\eta)\frac{Gm}{r} \right\} \frac{Gm}{r} n^j n^k v^a$$

$$-(3+2\eta)\frac{Gm}{r}\dot{r}(v^j n^k + n^j v^k) v^a \left\{ -(1-5\eta)v^2 + (1-4\eta)\frac{Gm}{r} \right\}$$

$$v^j v^k v^a \bigg] + \mathcal{O}(c^{-5}) \ , \tag{4.87b}$$

$$\ddot{\mathcal{Q}}^{jkab} = \frac{\eta m}{c^2}\bigg[5(1-3\eta)\frac{Gm}{r}\dot{r}(v^j n^k + n^j v^k) n^a n^b + (1-3\eta)\bigg(v^2 - 5\dot{r}^2$$

$$+ \frac{7}{3}\frac{Gm}{r}\bigg) \frac{Gm}{r} n^j n^k n^a n^b - \frac{14}{3}(1-3\eta)\frac{Gm}{r} v^j v^k n^a n^b + 2(1-3\eta)$$

$$\times \frac{Gm}{r}(v^j n^k + n^j v^k)(v^a n^b + n^a v^b) + 2(1-3\eta) v^j v^k v^a v^b + \frac{1}{6}$$

$$\times \frac{Gm}{r}\left(v^2 - 3\dot{r}^2 + \frac{Gm}{r} \right) n^j n^k \delta^{ab} + \frac{1}{3}\frac{Gm}{r}\dot{r}(v^j n^k + n^j v^k)\delta^{ab}$$

$$-\frac{1}{3}\frac{Gm}{r} v^j v^k \delta^{ab} \bigg] + \mathcal{O}(c^{-4}) \ , \tag{4.87c}$$

$$\ddot{\mathcal{Q}}^{jkabc} = \frac{\eta m \Delta}{c^3}\bigg[-\frac{1}{4}(1-2\eta)\left(21v^2 - 105\dot{r}^2 + 44\frac{Gm}{r} \right) \frac{Gm}{r}(v^j n^k + n^j v^k)$$

$$\times n^a n^b n^c + \frac{1}{4}(1-2\eta)\left(45v^2 - 105\dot{r}^2 + 90\frac{Gm}{r} \right) \frac{Gm}{r}\dot{r} n^j n^k n^a n^b n^c$$

$$- \frac{51}{2}(1-2\eta)\frac{Gm}{r}\dot{r} v^j v^k n^a n^b n^c - \frac{27}{2}(1-2\eta)\frac{Gm}{r}\dot{r}(v^j n^k + n^j v^k)$$

$$\times (v^a n^b n^c + n^a v^b n^c + n^a n^b v^c) - \frac{1}{4}(1-2\eta)\left(9v^2 - 45\dot{r}^2 + 28\frac{Gm}{r} \right)$$

$$\times \frac{Gm}{r} n^j n^k (v^a n^b n^c + n^a v^b n^c + n^a n^b v^c) + \frac{29}{2}(1-2\eta)\frac{Gm}{r} v^j v^k$$

$$
\begin{aligned}
&(v^a n^b n^c + n^a v^b n^c + n^a n^b v^c) - \frac{15}{2}(1-2\eta)\frac{Gm}{r}(v^a n^b n^c + n^a v^b n^c + n^a n^b v^c) \\
&- 6(1-2\eta)v^j v^k v^a v^b v^c - \frac{9}{2}(1-2\eta)\frac{Gm}{r}\dot{r} n^j n^k (v^a v^b n^c + v^a n^b v^c \\
&+ n^a v^b v^c) + \frac{3}{2}(1-2\eta)\frac{Gm}{r} n^j n^k v^a v^b v^c + \frac{1}{4}\left(9v^2 - 15\dot{r}^2 + 10\frac{Gm}{r}\right) \\
&\times \left(\frac{Gm}{r}\dot{r} n^j n^k n^{(a}\delta^{bc)}\right) - \frac{1}{4}\left(3v^2 - 9\dot{r}^2 + 4\frac{Gm}{r}\right)\frac{Gm}{r}(v^j n^k + n^j v^k) n^{(a}\delta^{bc)} \\
&- \frac{1}{4}\left(3v^2 - 9\dot{r}^2 + 4\frac{Gm}{r}\right)\frac{Gm}{r} n^j n^k v^{(a}\delta^{bc)} - \frac{3}{2}\frac{Gm}{r}\dot{r} v^j v^k n^{(a}\delta^{bc)} \\
&- \frac{3}{2}\frac{Gm}{r}\dot{r}(v^j n^k + n^j v^k) v^{(a}\delta^{bc)} + \frac{3}{2}\frac{Gm}{r}\dot{r} v^j v^k v^{(a}\delta^{bc)}\Bigg] + \mathcal{O}(c^{-5}) \ . \qquad (4.87\text{d})
\end{aligned}
$$

In addition we have that,

$$
\begin{aligned}
\mathcal{I}^{(4)jk} =& 2\eta m \frac{Gm}{r^3}\left[\left(3v^2 - 15\dot{r}^2 + \frac{Gm}{r}\right) n^j n^k + 9\dot{r}(v^j n^k + n^j v^k) \right. \\
& \left. -4 v^j v^k\right] + \mathcal{O}(c^{-2}) \ . \qquad (4.88)
\end{aligned}
$$

Now using all these derivatives back in eq. (4.39) and eq. (4.81) and then adding the results for near-zone and the wave-zone contributions we finally get the value of gravitational wave potential h^{jk} correct upto 1.5PN order,

$$
\begin{aligned}
h^{jk}(t,\mathbf{x}) = & \frac{2\eta G m}{c^4 r}[A^{jk}[0PN] + A^{jk}[0.5PN] + A^{jk}[1PN] + A^{jk}[1.5PN] \\
& + A^{jk}[tail] + \mathcal{O}(c^{-4})] \ , \qquad (4.89)
\end{aligned}
$$

where we have separated out all the terms corresponding to their PN order under the same argument. These separated terms are,

$$
A^{jk}[0PN] = 2\left[v^j v^k - \frac{Gm}{r} n^j n^k\right] \ , \qquad (4.90\text{a})
$$

$$
\begin{aligned}
A^{jk}[0.5PN] = & \frac{\Delta}{c}\left[\frac{3Gm}{r}(\mathbf{n}.\mathbf{N})(v^j n^k + n^j v^k - \dot{r} n^j n^k) + (\mathbf{v}.\mathbf{N})(-2v^j v^k \right. \\
& \left. + \frac{Gm}{r} n^j n^k)\right] \ , \qquad (4.90\text{b})
\end{aligned}
$$

$$
\begin{aligned}
A^{jk}[1PN] = & \frac{1}{c^2}\left[\frac{1}{3}\left\{3(1-3\eta)v^2 - 2(2-3\eta)\frac{Gm}{r}\right\} v^j v^k + \frac{2}{3}(5+3\eta) \right. \\
& \times \frac{Gm}{r}\dot{r}(v^j n^k + n^j v^k) + \frac{1}{3}\frac{Gm}{r}\left\{-(10+3\eta)v^2 + 3(1-3\eta)\dot{r}^2 \right. \\
& \left. + 29\frac{Gm}{r}\right\} n^j n^k + \frac{2}{3}(1-3\eta)(\mathbf{v}.\mathbf{N})^2\left(3v^j v^k - \frac{Gm}{r} n^j n^k\right)
\end{aligned}
$$

$$+\frac{4}{3}(1-3\eta)(\mathbf{v}.\mathbf{N})(\mathbf{n}.\mathbf{N})\frac{Gm}{r}\{-4(v^jn^k+n^jv^k)+3\dot{r}n^jn^k\}$$
$$+\frac{1}{3}(1-3\eta)(\mathbf{n}.\mathbf{N})^2\frac{Gm}{r}\{-14v^jv^k+15\dot{r}(v^jn^k+n^jv^k)$$
$$+\left(3v^2-15\dot{r}^2+7\frac{Gm}{r}\right)n^jn^k\}\Bigg] \ , \qquad (4.90c)$$

$$A^{jk}[1.5PN]=\frac{\Delta}{c^3}\left[\frac{1}{12}(\mathbf{v}.\mathbf{N})\left\{-6[2(1-5\eta)v^2-(3-8\eta)\frac{Gm}{r}]v^jv^k\right.\right.$$
$$-6(7+4\eta)\frac{Gm}{r}\dot{r}(v^jn^k+n^jv^k)+\frac{Gm}{r}[3(7-2\eta)v^2$$
$$\left.-9(1-2\eta)\dot{r}^2-4(26-3\eta)\frac{Gm}{r}]n^jn^k\right\}$$
$$+\frac{1}{2}(1-2\eta)(\mathbf{v}.\mathbf{N})^3\left\{-4v^jv^k+\frac{Gm}{r}n^jn^k\right\}$$
$$+\frac{3}{2}(1-2\eta)(\mathbf{v}.\mathbf{N})^2(\mathbf{n}.\mathbf{N})\frac{Gm}{r}\left\{5(v^jn^k+n^jv^k-3\dot{r}n^jn^k)\right\}$$
$$+\frac{1}{4}(1-2\eta)(\mathbf{v}.\mathbf{N})(\mathbf{n}.\mathbf{N})^2\frac{Gm}{r}\{58v^jv^k-54\dot{r}(v^jn^k+n^jv^k)$$
$$-[9v^2-45\dot{r}^2+28\frac{Gm}{r}]n^jn^k\}+\frac{1}{12}(1-2\eta)(\mathbf{n}.\mathbf{N})^3$$
$$\times\frac{Gm}{r}\{-102\dot{r}v^jv^k-[21v^2-105\dot{r}^2+44\frac{Gm}{r}](v^jn^k+n^jv^k)$$
$$\left.+15\dot{r}[3v^2-7\dot{r}^2+6\frac{Gm}{r}]\}\right] \ , \qquad (4.90d)$$

$$A^{jk}[tail]=\frac{4Gm}{c^3}\int_0^\infty\left[\frac{Gm}{r}\left\{\left(3v^2-15\dot{r}^2+\frac{Gm}{r}\right)n^jn^k+9\dot{r}(v^jn^kn^jv^k)\right.\right.$$
$$\left.\left.-4v^jv^k\right\}\right]_{(t_r-2s/c)}\left[\ln(\frac{r+s}{s})+\frac{11}{2}\right]ds \ . \qquad (4.90e)$$

4.2.5 Polarizations

In order to find out the '+' and '×' polarizations of gravitational wave potential h^{jk} correct upto 1.5PN order, we make use of the detector-adapted frame once again. The polarizations of gravitational wave potential can then be given by,

$$h_{+,\times}=\frac{2\eta Gm}{c^4r}[A_{+,\times}[0PN]+A_{+,\times}[0.5PN]+A_{+,\times}[1PN]$$
$$+A_{+,\times}[1.5PN]+A_{+,\times}[tail]+\mathcal{O}(c^{-4})] \ . \qquad (4.91)$$

The values of $[0PN]$ and $[tail]$ contributions are given as,

$$
\begin{aligned}
A_+[0PN] = & \frac{1}{2}\left[\dot{r}^2 + (r\dot{\phi})^2 - \frac{Gm}{r}\right]\sin^2\iota + \frac{1}{2}\left[\dot{r}^2 - (r\dot{\phi})^2 - \frac{Gm}{r}\right] \\
& \times(1+\cos^2\iota)\cos 2\psi - \dot{r}^2(r\phi)(1+\cos^2\iota)\sin 2\psi \ , \qquad (4.92a) \\
A_\times[0PN] = & \left[\dot{r}^2 - (r\dot{\phi})^2 - \frac{Gm}{r}\right]\cos\iota\sin 2\psi + 2\dot{r}(r\dot{\phi})\cos\iota\cos 2\psi \ , \qquad (4.92b)
\end{aligned}
$$

and,

$$
\begin{aligned}
A_+[tail] = & \frac{Gm}{c^3}\sin^2\int_0^\infty \left[\frac{Gm}{r^3}\left\{2\dot{r}^2 - (r\dot{\phi})^2 - \frac{Gm}{r}\right\}\right]_{t_r-2s/c}\chi ds \\
& \frac{Gm}{c^3}(1+\cos^2\iota)\int_0^\infty \left[\frac{Gm}{r^3}\left\{2\dot{r}^2 + 7(r\dot{\phi})^2 - \frac{Gm}{r}\right\}\cos 2\psi\right]_{t_r-2s/c}\chi ds \\
& - 10\frac{Gm}{c^3}(1+\cos^2\iota)\int_0^\infty \left[\frac{Gm}{r^3}\dot{r}(r\dot{\phi})\sin 2\psi\right]_{t_r-2s/c}\chi ds \ , \qquad (4.93a) \\
A_\times[tail] = & \frac{2Gm}{c^3}\cos\iota\int_0^\infty \left[\frac{Gm}{r^3}\left\{2\dot{r}^2 + 7(r\dot{\phi})^2 + \frac{Gm}{r}\right\}\sin 2\psi\right]_{t_r-2s/c}\chi ds \\
& + 20\frac{Gm}{c^3}\cos\iota\int_0^\infty \left[\frac{Gm}{r^3}\dot{r}(r\dot{\phi})\cos 2\psi\right]_{t_r-2s/c}\chi ds \ . \qquad (4.93b)
\end{aligned}
$$

where $\psi = \phi(t_r) + \omega$ and $\chi = \ln\left(\frac{r+s}{s} + \frac{11}{12}\right)$. The other contributions are too lengthy to be presented here. In order to reduce them, we directly go to the circular orbits approximation, where $\dot{r} = 0$.

The circular orbit approximation is sufficiently valid for inspiralling compact binaries. The reason is that as the binary system evolves, it loses energy as a result of emission of gravitational waves. The orbit separation decreases slowly and the orbit tends to get circular. The angular velocity of an orbit of constant radius is given by,

$$
\Omega^2 = \phi^2 = \frac{Gm}{r^3}\left[1 - (3-\eta)\frac{Gm}{c^2 r} + \mathcal{O}(c^{-4})\right] \ . \qquad (4.94)
$$

This is just a PN generalization of the usual Keplerian relation $\Omega^2 = Gm/r^3$. The orbital velocity $v = r\Omega$ is given as,

$$
v^2 = \frac{Gm}{r}\left[1 - (3-\eta)\frac{Gm}{c^2 r} + \mathcal{O}(c^{-4})\right] \ . \qquad (4.95)
$$

Moreover we can think of an alternative PN expansion parameter other than $(v/c)^2$. We choose this parameter to be,

$$
\beta = \left(\frac{Gm\Omega}{c^3}\right)^{1/3} \ .
$$

It has several advantages, as out calculations are dependent on Ω. In terms of expansion parameter β, the polarizations of gravitational wave potential are given as,

$$h_{+,\times} = \frac{2\eta Gm}{c^2 r}\left(\frac{Gm\Omega}{c^3}\right)[H_{+,\times}[0PN] + \beta H_{+,\times}[0.5PN] + \beta^2 H_{+,\times}[1PN] + \beta^3 H_{+,\times}[1.5PN] + \beta^3 H_{+,\times}[tail] + \mathcal{O}(\beta^4)] \ . \tag{4.96}$$

where $H_{+,\times}$ represent the scale free polarizations of the gravitational wave potential. These polarizations are given as,

$$H_+[0PN] = -(1+\cos^2\iota)\cos 2\psi \ , \tag{4.97a}$$

$$H_+[0.5PN] = -\Delta\left[\frac{1}{8}\sin\iota(5+\cos^2\iota)\cos\psi + \frac{9}{8}\sin\iota(1+\cos^2\iota)\cos 3\psi\right] \ , \tag{4.97b}$$

$$H_+[1PN] = \frac{1}{6}\left[(19+9\cos^2\iota - 2\cos^4\iota) - (19 - 11\cos^2\iota - 6\cos^4\iota)\eta\right] \times \cos 2\psi - \frac{4}{3}(1-3\eta)\sin^2\iota(1+\cos^2\iota)\cos 4\psi \ , \tag{4.97c}$$

$$H_+[1.5PN] = \Delta\left[\frac{1}{192}\sin\iota\left[(57+60\cos^2\iota - \cos^4\iota) - 2(49 - 12\cos^2\iota - \cos^4\iota)\eta\right]\cos\psi - \frac{9}{128}\sin\iota\left[(73+40\cos^2\iota - 9\cos^4\iota) - 2(25 - 8\cos^2\iota - 9\cos^4\iota)\eta\right]\cos 3\psi + \frac{625}{384}(1-2\eta)\sin^3\iota (1+\cos^2\iota)\cos 5\psi\right] \ , \tag{4.97d}$$

$$H_+[tail] = -4(1+\cos^2\iota)\left[\frac{\pi}{2}\cos 2\psi\left\{\gamma + \ln\left(\frac{4\Omega r}{c}\right)\right\}\sin 2\psi\right] \ . \tag{4.97e}$$

and,

$$H_\times[0PN] = -2\cos\iota\sin 2\psi \ , \tag{4.98a}$$

$$H_\times[0.5PN] = -\Delta\left[\frac{3}{4}\sin\iota\cos\iota\sin\psi + \frac{9}{4}\sin\iota\cos\iota\sin 3\psi\right] \ , \tag{4.98b}$$

$$H_\times[1PN] = \frac{1}{3}\cos\iota\left[(17-4\cos^2\iota) - (13-12\cos^2\iota)\eta\right]\sin 2\psi - \frac{8}{3}(1-3\eta)\sin^2\iota\cos\iota\sin 4\psi \ , \tag{4.98c}$$

$$H_\times[1.5PN] = \Delta\left[\frac{1}{96}\sin\iota\cos\iota\left[(63-5\cos^2\iota) - 2(23-5\cos^2\iota)\eta\right]\sin\psi - \frac{9}{64}\sin\iota\cos\iota\left[(67-15\cos^2\iota) - 2(19-15\cos^2\iota)\eta\right]\right.$$

$$\times \sin 3\psi + \frac{625}{192}(1-2\eta)\sin^3\iota\cos\iota\sin 5\psi \Bigg] \ , \tag{4.98d}$$

$$H_\times[tail] = -\,8c\left[\frac{\pi}{2}\sin 2\psi - \left\{\gamma + \ln\left(\frac{4\Omega r}{c}\right)\right\}\cos 2\psi\right] \ . \tag{4.98e}$$

4.2.6 Strain Plot for GW150914 Signal at 1.5PN

Using the above results into eq. (4.32) and using the data of GW150914 again, we can plot a strain-time graph at 1.5PN order,

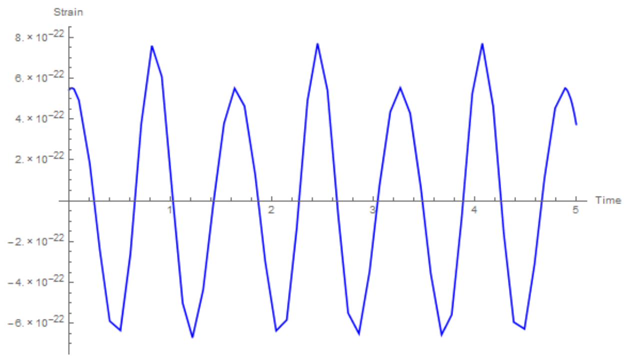

Figure 4.5: The strain versus time plot for the gravitational wave signal GW150914 where the PN degree of accuracy is 1.5PN.

4.3 Beyond 1.5PN Order

In order to go at higher degrees of PN accuracy, we extend our results beyond 1.5PN order. Further calculations are extremely lengthy and complex to be solved here therefore I would present only the end results at each PN order above 1.5PN.

4.3.1 Gravitational Waves at 2PN Order

The scale free '+' and '×' polarizations of the gravitational wave potential correct upto 2PN order can be written in the same way as eq. (4.97) and eq. (4.98). The lower order contributions are the same as before, only the 1.5PN contribution can be written alternatively absorbing within the wave-tail contribution. Therefore I take the liberty to write only the 1.5PN and 2PN contributions here [30], i.e,

$$\begin{aligned}
H_+[1.5PN] =&\Delta \left[\frac{1}{192}\sin\iota\left[(57+60\cos^2\iota-\cos^4\iota)-2(49-12\cos^2\iota\right.\right.\\
&\left.-\cos^4\iota)\eta\right]\cos\psi-\frac{9}{128}\sin\iota\left[(73+40\cos^2\iota-9\cos^4\iota)\right.\\
&\left.-2(25-8\cos^2\iota-9\cos^4\iota)\eta\right]\cos 3\psi+\frac{625}{384}(1-2\eta)\sin^3\iota\\
&\left.\times(1+\cos^2\iota)\cos 5\psi\right]-2\pi(1+\cos^2\iota)\cos 2\psi\ , \qquad (4.99a)\\
H_+[2PN] =&\frac{1}{120}[(22+396\cos^2\iota+145\cos^4\iota-5\cos^6\iota)+\frac{5}{3}(706\\
&-216\cos^2\iota-251\cos^4\iota+15\cos^6\iota)\eta-5(98-108\cos^2\iota\\
&\left.+7\cos^4\iota+5\cos^2\iota)\eta^2\right]\cos 2\psi+\frac{2}{15}\sin^2\iota\left[(59+35\cos^2\iota\right.\\
&-8\cos^4\iota)-\frac{5}{3}(131+59\cos^2\iota-24\cos^4\iota)\eta+5(21-3\cos^2\iota\\
&\left.-8\cos^4\iota)\eta^2\right]\cos 4\psi-\frac{81}{40}(1-5\eta+5\eta^2)\sin^4\iota(1+\cos^2\iota)\\
&+\times\cos 6\psi\frac{\Delta}{40}\sin\iota\left[\{11+7\cos^2\iota+10(5+\cos^2\iota)\ln 2\}\sin\psi\right.\\
&-5\pi(5+\cos^2\iota)\cos\psi-27\{7-10\ln(3/2)\}(1+\cos^2\iota)\\
&\times\sin 3\psi+135\pi(1+\cos^2\iota)\cos 3\psi]\ , \qquad (4.99b)
\end{aligned}$$

and,

$$\begin{aligned}
H_\times[1.5PN] =&\Delta\left[\frac{1}{96}\sin\iota\cos\iota\left[(63-5\cos^2\iota)-2(23-5\cos^2\iota)\eta\right]\sin\psi\right.\\
&-\frac{9}{64}\sin\iota\cos\iota\left[(67-15\cos^2\iota)-2(19-15\cos^2\iota)\eta\right]\sin 3\psi\\
&\left.+\frac{625}{192}(1-2\eta)\sin^3\iota\cos\iota\sin 5\psi\right]-4\pi\cos\iota\sin 2\psi\ , \qquad (4.100a)\\
H_\times[2PN] =&\frac{1}{60}\cos\iota[(68+226\cos^2\iota-15\cos^4\iota)+\frac{5}{3}(572-490\cos^2\iota\\
&+45\cos^4\iota)\eta-5(56-70\cos^2\iota+15\cos^4\iota)]\sin 2\psi\\
&+\frac{4}{15}\cos\iota\sin^2\iota[(55-12\cos^2\iota)-\frac{5}{3}(119-36\cos^2\iota)\eta\\
&+5(17-12\cos^2\iota)\eta^2]\sin 4\psi-\frac{81}{20}(1-5\eta+5\eta^2)\cos\iota\sin^4\iota\\
&\times\sin 6\psi-\frac{3}{20}\sin\iota\cos\iota\Delta[\{3+10\ln 2\}\cos\psi+5\pi\sin\psi\\
&-9\{7-10\ln(3/2)\}\cos 3\psi-45\pi\sin 3\psi]\ . \qquad (4.100b)
\end{aligned}$$

The gravitational wave polarizations can then be given as,

$$h_{+,\times} = \frac{2\eta Gm}{c^r}\left(\frac{Gm\Omega}{c^3}\right)^{2/3}\left[H_{+,\times}[0PN] + \beta H_{+,\times}[0.5PN] + \beta^2 H_{+,\times}[1PN] + \beta^3 H_{+,\times}[1.5PN] + \beta^4 H_{+,\times}[2PN] + \mathcal{O}(\beta^5)\right] . \tag{4.101}$$

4.3.2 Strain Plot for GW150914 Signal at 2PN

The plot of strain produced per unit length in the arms of the detector correct upto 2PN order for gravitational wave signal GW150914 is given in Figure (4.6).

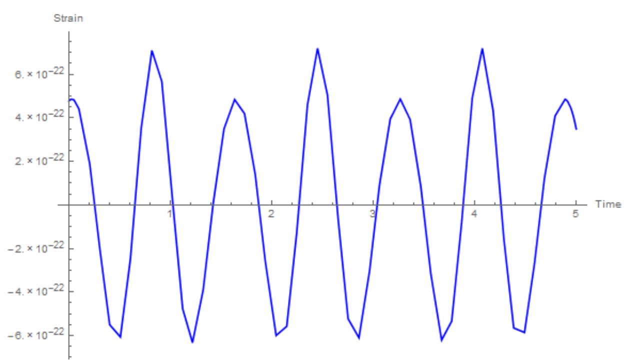

Figure 4.6: The strain versus time plot for the gravitational wave signal GW150914, where the PN degree of accuracy is 2PN.

4.3.3 Gravitational Waves at 2.5PN Order

The 2.5PN contribution of the scale free '+' and '×' polarizations of the gravitational wave potential are given as [31],

$$
\begin{aligned}
H_{+}[2.5PN] =& \Delta \sin\iota \cos\psi \left[\frac{1771}{5120} - \frac{1671}{5120}\cos^2\iota + \frac{217}{9216}\cos^4\iota - \frac{1}{9216}\cos^6\iota \right. \\
& + \left(\frac{681}{256} + \frac{13}{768}\cos^2\iota - \frac{35}{768}\cos^4\iota + \frac{1}{2304}\cos^6 \right)\eta \\
& \left. + \left(-\frac{3451}{9216} + \frac{673}{3072}\cos^2\iota - \frac{5}{9216}\cos^4\iota - \frac{1}{3072}\cos^6 \right)\eta^2 \right] \\
& + \pi\cos 2\psi \left[\frac{19}{3} + 3\cos^2\iota - \frac{2}{3}\cos^4\iota + \left(-\frac{19}{3} + \frac{11}{3}\cos^2\iota \right.\right. \\
& \left.\left. + 2\cos^4\iota \right) \right] + \Delta\sin\iota\cos 3\psi \left[\frac{3537}{1024} - \frac{22977}{5120}\cos^2\iota \right. \\
& - \frac{15309}{5120}\cos^4\iota + \frac{729}{5120}\cos^6\iota + \left(-\frac{23829}{1280} + \frac{5529}{1280}\cos^2\iota \right. \\
& \left. + \frac{7749}{1280}\cos^4\iota - \frac{729}{1280}\cos^6\iota \right)\eta + \left(\frac{29127}{5120} - \frac{27267}{5120}\cos^2\iota \right. \\
& \left. - \frac{1647}{5120}\cos^4\iota + \frac{2187}{5120}\cos^6\iota \right)\eta^2 + \cos 4\psi \left[-\frac{16\pi}{3}(1+\cos^2\iota) \right. \\
& \left. \sin^2\iota(1-3\eta) \right] + \Delta\sin\iota\cos 5\psi \left[-\frac{108125}{9216} + \frac{40625}{9216}\cos^2\iota \right. \\
& + \frac{83125}{9216}\cos^4\iota - \frac{15625}{9216}\cos^6\iota + \left(\frac{8125}{256} - \frac{40625}{2304}\cos^2\iota \right. \\
& \left. + -\frac{48125}{2304}\cos^4\iota + \frac{15625}{2304}\cos^6\iota \right)\eta \left(-\frac{119375}{9216} + \frac{40625}{3072}\cos^2\iota \right. \\
& \left.\left. + \frac{44375}{9216}\cos^4\iota - \frac{15625}{3072}\cos^6\iota \right)\eta^2 \right] + \Delta\cos 7\psi \left[\frac{117649}{46080}\sin^5\iota \right. \\
& \left. \times(1+\cos^2\iota)(1-4\eta+3\eta^2) \right] + \sin 2\psi \left[-\frac{9}{5} + \frac{14}{5}\cos^2\iota \right. \\
& \left. + \frac{7}{5}\cos^4\iota + \left(\frac{96}{5} - \frac{8}{5}\cos^2\iota - \frac{28}{5}\cos^4\iota \right) \right] + \sin^2\iota(1+\cos^2\iota) \\
& \times\sin 4\psi \left[\frac{56}{5} - \frac{32\ln 2}{3} - \left(\frac{1193}{30} - 32\ln 2 \right)\eta \right], \qquad (4.102a)
\end{aligned}
$$

$$
H_{\times}[2.5PN] = \frac{6}{5}\cos\iota\sin^2\iota\eta + \cos\iota\cos 2\psi \left[2 - \frac{22}{5}\cos^2\iota + \left(-\frac{154}{5} + \frac{94}{5} \right)\eta \right]
$$

$$
\begin{aligned}
&+ \cos\iota \sin^2\iota \cos 4\psi \left[-\frac{112}{5} + \frac{64}{3}\ln 2 + \left(\frac{1193}{15} - 64\ln 2\right)\eta\right] \\
&+ \Delta \sin\iota \cos\iota \sin\psi \left[-\frac{913}{7680} + \frac{1891}{11520}\cos^2\iota - \frac{7}{4608}\cos^4\iota \right.\\
&+ \left(\frac{1165}{384} - \frac{235}{576}\cos^2\iota + \frac{7}{1152}\cos^4\iota\right)\eta + \left(-\frac{1301}{4608} + \frac{301}{2304}\cos^2\iota \right.\\
&\left.\left. -\frac{7}{1536}\cos^4\iota\right)\right] + \pi\cos\iota\sin 2\psi \left[\frac{34}{3} - \frac{8}{3}\cos^2\iota - \left(\frac{26}{3} - 8\cos^2\iota\right)\eta\right] \\
&+ \Delta \sin\iota \cos\iota \sin 3\psi \left[\frac{12501}{2560} - \frac{12069}{1280}\cos^2\iota + \frac{1701}{2560}\cos^4\iota \right.\\
&+ \left(-\frac{19581}{640} + \frac{7821}{320}\cos^2\iota - \frac{1701}{640}\cos^4\iota\right)\eta + \left(\frac{18903}{2560} - \frac{11403}{1280}\cos^2\iota \right.\\
&\left.\left. + \frac{5103}{2560}\cos^4\iota\right)\right] + \cos\iota\sin^2\iota\sin 4\psi \left[-\frac{32\pi}{3}(1-3\eta)\right] \\
&+ \Delta \sin\iota \cos\iota \sin 5\psi \left[-\frac{101875}{4608} + \frac{6875}{256}\cos^2\iota - \frac{21875}{4608}\cos^4\iota \right.\\
&+ \left(\frac{66875}{1152} - \frac{44375}{576}\cos^2\iota + \frac{21875}{1152}\cos^4\iota\right)\eta + \left(-\frac{100625}{4608} \right.\\
&\left.\left. +\frac{83125}{2304}\cos^2\iota - \frac{21875}{1536}\cos^4\iota\right)\right] + \Delta \sin^5\iota \cos\iota \sin 7\psi \\
&\times \left[\frac{117649}{23040}(1 - 4\eta + 3\eta^2)\right] . \qquad (4.102b)
\end{aligned}
$$

The gravitational waves polarizations correct upto 2.5PN order can then be given as,

$$
\begin{aligned}
h_{+,\times} =& \frac{2\eta G m}{c^r}\left(\frac{Gm\Omega}{c^3}\right)^{2/3} [H_{+,\times}[0PN] + \beta H_{+,\times}[0.5PN] + \beta^2 H_{+,\times}[1PN] \\
&+ \beta^3 H_{+,\times}[1.5PN] + \beta^4 H_{+,\times}[2PN] + \beta^5 H_{+,\times}[2.5PN] + \mathcal{O}(\beta^6)] . \qquad (4.103)
\end{aligned}
$$

4.3.4 Strain Plot for GW150914 Signal at 2.5PN

The strain per unit length plot correct upto 2.5PN order is shown in Figure (4.7),

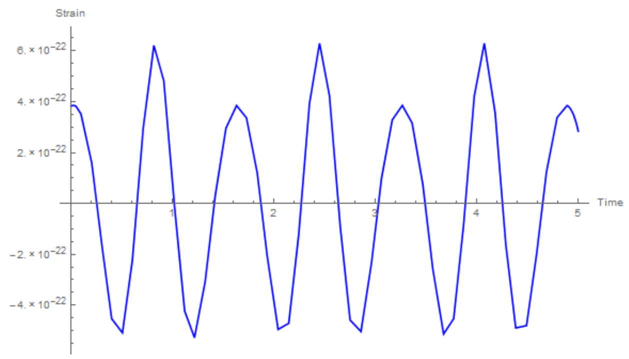

Figure 4.7: The strain versus time plot for the gravitational wave signal GW150914, where the PN degree of accuracy is 2.5PN.

4.3.5 Gravitational Waves at 3PN Order

The 3PN contribution to the scale free polarizations of gravitational wave potential are given as [32] ,

$$\begin{aligned}
H_+[3PN] =& \Delta\pi \sin\iota\cos\psi\left[\frac{19}{64}+\frac{5}{16}\cos^2\iota-\frac{1}{192}\cos^4\iota+\left(-\frac{19}{96}+\frac{3}{16}\cos^2\iota\right.\right.\\
&\left.\left.+\frac{1}{96}\cos^4\iota\right)\right]+\cos 2\psi\left[-\frac{465497}{11025}+\left(\frac{865}{105}-\frac{2\pi^2}{3}\right.\right.\\
&\left.+\frac{428}{105}\ln(16\beta^2)\right)(1+\cos^2\iota)-\frac{3561541}{88200}\cos^2\iota-\frac{943}{720}\cos^4\iota\\
&+\frac{169}{720}\cos^6\iota-\frac{1}{360}\cos^8\iota+\left(\frac{2209}{360}-\frac{41\pi^2}{96}(1+\cos^2\iota)\,\eta\right.\\
&\left.+\frac{2039}{180}\cos^2\iota+\frac{3311}{720}\cos^4\iota-\frac{853}{720}\cos^6\iota+\frac{7}{360}\cos^8\iota\right)\\
&+\left(\frac{12871}{540}-\frac{1583}{60}\cos^2\iota-\frac{145}{108}\cos^4\iota+\frac{56}{45}\cos^6\iota-\frac{7}{180}\cos^8\iota\right)\eta^2\\
&+\left(-\frac{3277}{810}+\frac{19661}{3240}\cos^2\iota-\frac{281}{144}\cos^4\iota-\frac{73}{720}\cos^6\iota+\frac{7}{360}\cos^8\iota\right)\\
&\left.\times\eta^3\right]+\Delta\pi\sin\iota\cos 3\psi\left[-\frac{1971}{128}-\frac{135}{16}\cos^2\iota+\frac{243}{128}\cos^4\iota+\left(\frac{567}{65}\right.\right.\\
&\left.\left.-\frac{81}{16}\cos^2\iota-\frac{243}{64}\cos^4\iota\right)\eta\right]+\sin^2\iota\cos 4\psi\left[-\frac{2189}{210}+\frac{1123}{210}\cos^2\iota\right.\\
&+\frac{56}{9}\cos^4\iota-\frac{16}{45}\cos^6\iota+\left(\frac{6271}{90}-\frac{1969}{90}\cos^2\iota-\frac{1432}{45}\cos^4\iota\right.
\end{aligned}$$

$$+\frac{112}{45}\cos^6\iota\Big)\eta+\Big(-\frac{3007}{27}+\frac{3493}{135}\cos^2\iota+\frac{1568}{45}\cos^4\iota-\frac{224}{45}$$
$$\times\cos^6\iota\Big)\eta^2+\left(\frac{161}{6}-\frac{1921}{90}\cos^2\iota-\frac{184}{45}\cos^4\iota+\frac{112}{45}\cos^6\iota\right)\eta^3\Big]$$
$$+\Delta\cos 5\psi\left[\frac{3125\pi}{384}\sin^3\iota(1+\cos^2\iota)(1-2\eta)\right]+\sin^4\iota\cos 6\psi$$
$$\times\Big[\frac{1377}{80}+\frac{891}{80}\cos^2\iota-\frac{729}{280}\cos^4\iota+\Big(-\frac{7857}{80}-\frac{891}{16}\cos^2\iota$$
$$+\frac{729}{40}\cos^4\iota\Big)\eta+\left(\frac{567}{4}+\frac{567}{10}\cos^2\iota-\frac{729}{20}\cos^4\iota\right)\eta^2$$
$$+\left(-\frac{729}{16}-\frac{243}{80}\cos^2\iota+\frac{729}{40}\cos^4\iota\right)\eta^3\Big]$$
$$+\cos 8\psi\left[-\frac{1024}{315}\sin^6\iota(1+\cos^2\iota)(1-7\eta+14\eta^2-7\eta^3)\right]$$
$$+\Delta\sin\iota\sin\psi\Big[-\frac{2159}{40320}-\frac{19\ln 2}{32}+\left(-\frac{95}{244}-\frac{5\ln 2}{8}\right)\cos^2\iota$$
$$+\left(\frac{181}{13440}+\frac{\ln 2}{96}\right)\cos^4\iota+\Big\{\frac{81127}{10080}+\frac{19\ln 2}{48}+\left(-\frac{41}{48}-\frac{3\ln 2}{8}\right)$$
$$\times\cos^2\iota+\left(-\frac{313}{480}-\frac{\ln 2}{48}\right)\cos^4\iota\Big\}\Big]+\sin 2\psi\left[-\frac{428\pi}{105}(1+\cos^2\iota)\right]$$
$$+\Delta\sin\iota\sin 3\psi\Big[\frac{205119}{8960}-\frac{1971}{64}\ln(3/2)+\left(\frac{1917}{224}-\frac{135}{8}\ln(3/2)\right)$$
$$\times\cos^2\iota+\left(-\frac{43983}{8960}+\frac{243}{64}\ln(3/2)\right)\cos^4\iota+\Big\{-\frac{54869}{960}+\frac{567}{32}\ln(3/2)$$
$$+\left(-\frac{923}{80}-\frac{81}{8}\ln(3/2)\right)\cos^2\iota+\left(\frac{41851}{2880}-\frac{243}{32}\ln(3/2)\right)\cos^4\iota\Big\}\Big]$$
$$+\Delta\sin^3\iota(1+\cos^2\iota)\sin 5\psi\Big[\frac{113125}{5376}-\frac{3125}{192}\ln(5/2)$$
$$+\left(\frac{17639}{320}-\frac{3125}{96}\ln(5/2)\right)\eta\Big]\ , \qquad (4.104a)$$

$$H_\times[3PN]=\Delta\sin\iota\cos\iota\cos\psi\Big[\frac{11617}{20160}+\frac{21}{16}\ln 2+\left(-\frac{251}{2240}-\frac{5}{48}\ln 2\right)\cos^2\iota$$
$$+\left(-\frac{48239}{5040}-\frac{5}{24}\ln 2+\left(\frac{727}{240}+\frac{5}{24}\ln 2\right)\cos^2\iota\right)\Big]$$
$$+\cos\iota\cos 2\psi\left[\frac{856\pi}{105}\right]+\Delta\sin\iota\cos\iota\cos 3\psi\Big[\frac{36801}{896}+\frac{1809}{32}\ln(3/2)$$

$$
\begin{aligned}
&+\left(-\frac{65097}{4480}-\frac{405}{32}\ln(3/2)\right)\cos^2\iota+\left(\frac{28445}{892886}-\frac{405}{16}\ln(3/2)\right.\\
&\left.\left.+\left(-\frac{7137}{160}+\frac{3125}{48}\ln(3/2)\right)\cos^2\iota\eta\right)\right]+\Delta\sin^2\iota\cos\iota\cos 5\psi\\
&\times\left[\frac{113125}{2688}-\frac{3125}{96}\ln(5/2)+\left(-\frac{17639}{160}+\frac{3125}{48}\ln(5/2)\right)\cos^2\iota\right]\\
&+\Delta\pi\sin\iota\cos\iota\sin\psi\left[\frac{21}{32}+\frac{5}{96}\cos^2\iota+\left(-\frac{5}{48}-\frac{5}{48}\cos^2\iota\right)\eta\right]\\
&+\cos\iota\sin 2\psi\left[-\frac{3620761}{44100}+\frac{1712C}{105}-\frac{41\pi^2}{48}\right.\\
&+\frac{856}{105}\ln(16\beta^2)-\frac{3413}{1260}\cos^2\iota+\frac{2909}{2520}\cos^4\iota-\frac{1}{45}\cos^6\iota\\
&+\left(\frac{743}{90}-\frac{41\pi^2}{48}+\frac{3391}{180}\cos^2\iota-\frac{2287}{360}\cos^4\iota+\frac{7}{45}\cos^6\iota\right)\eta\\
&+\left(+\frac{7919}{270}-\frac{5426}{135}\cos^2\iota+\frac{382}{45}\cos^4\iota-\frac{14}{45}\cos^6\iota\right)\eta^2\\
&\left.+\left(-\frac{6457}{1620}+\frac{1109}{180}\cos^2\iota-\frac{281}{120}\cos^4\iota+\frac{7}{45}\cos^6\iota\right)\eta^3\right]\\
&+\Delta\sin\iota\cos\iota\sin 3\psi\left[-\frac{1809}{64}+\frac{405}{64}\cos^2\iota+\left(\frac{405}{32}\right.\right.\\
&\left.\left.-\frac{405}{32}\cos^2\iota\right)\eta\right]+\sin^2\iota\cos\iota\sin 4\psi\left[-\frac{1781}{105}+\frac{1208}{63}\cos^2\iota\right.\\
&-\frac{64}{45}\cos^4\iota+\left(\frac{5207}{45}-\frac{536}{5}\cos^2\iota+\frac{448}{45}\cos^4\iota\right)\eta\\
&+\left(-\frac{24838}{135}+\frac{2224}{15}\cos^2\iota-\frac{64}{45}\cos^4\iota\right)\eta^2+\left(\frac{1703}{45}-\frac{1976}{45}\cos^2\iota\right.\\
&\left.\left.+\frac{448}{45}\cos^4\iota\right)\eta^3\right]+\Delta\sin 5\psi\left[\frac{3125\pi}{192}\sin^3\iota\cos\iota(1-2\eta)\right]\\
&+\sin^4\iota\cos\iota\sin 6\psi\left[\frac{9153}{280}-\frac{243}{35}\cos^2\iota+\left(-\frac{7371}{40}+\frac{243}{5}\cos^2\iota\right)\eta\right.\\
&\left.+\left(\frac{1296}{5}-\frac{486}{5}\cos^2\iota\right)\eta^2+\left(-\frac{3159}{40}+\frac{243}{5}\cos^2\iota\right)\eta^3\right]\\
&+\sin 8\psi\left[-\frac{2048}{315}\sin^6\iota\cos\iota(1-7\eta+14\eta^2-7\eta^3)\right]\ .
\end{aligned}
\tag{4.104b}
$$

Strain Plot for GW150914 at 3PN

The plot of strain produced per unit length in the arms of a detector correct up to 3PN order is given in Figure (4.8). We have considered the behavior of

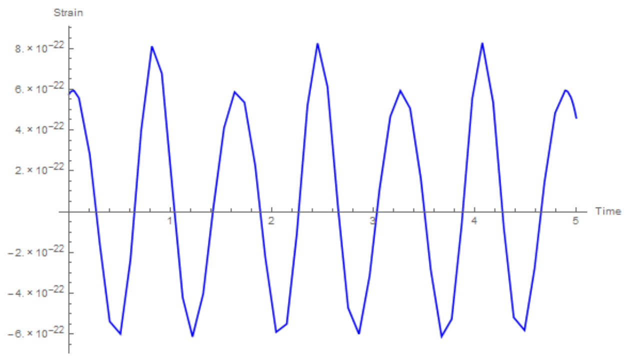

Figure 4.8: The strain versus time plot for gravitational wave signal GW150914, where the PN degree of accuracy is 2.5PN.

gravitational waves emitted by a system of compact binaries, in a detector-adapted frame of reference correct upto 3PN orders. Moreover, we have seen the strain plots for each successive PN order. Now we can compare these different results with each other and study the trend which the PN orders follow as they go to higher and higher degree of accuracy. This would be the part of the conclusion of this dissertation.

5

Conclusion

In chapter 4, we have seen the plots of gravitational wave potential at different PN orders, ranging from 0PN up to 3PN. We can now plot all these contributions in a single plot and see how the gravitational waves potential at one PN order differ from the others (Figure 5.1).

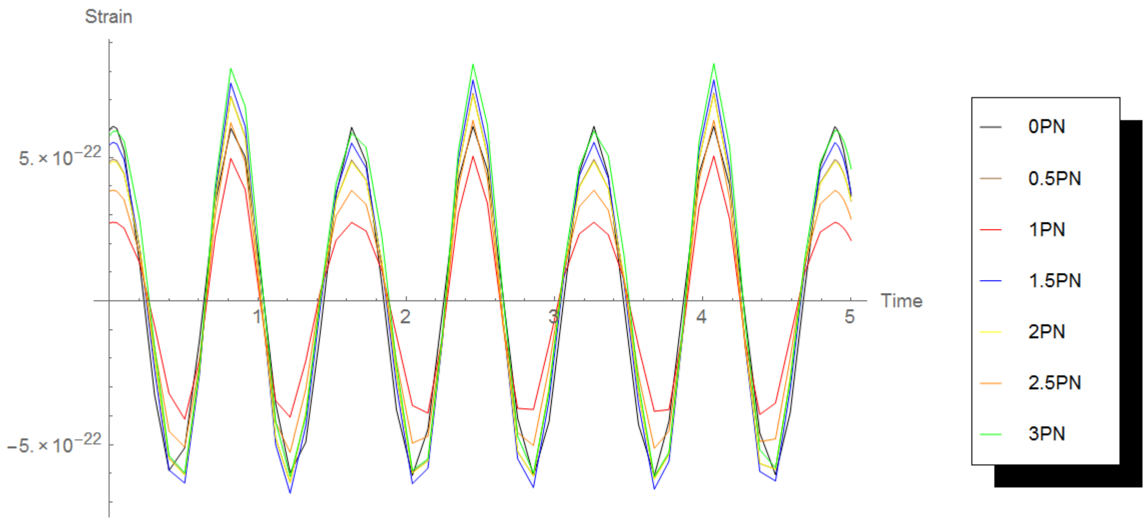

Figure 5.1: The strain versus time plot for GW150914 signal. The PN degree of accuracy ranging from 0PN up to 3PN.

A crest and a trough of the complete waveform in (Figure 5.1) have been maximized in order to have a closer look. These maximized portions of the actual plot are shown in (Figure 5.2) and (Figure 5.3),

Looking closely at the start of the crest (Figure 5.2) we can see that the gravitational wave potential amplitude is high at 0PN order, then it drops at

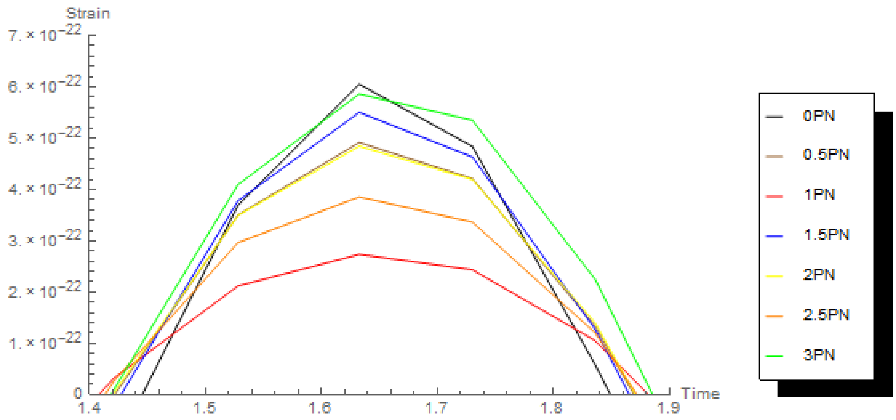

Figure 5.2: Zoomed in image of the first crest from Figure 5.1.

0.5PN and has the minimum value at 1PN. Again at 1.5PN order, the amplitude goes higher and comes down for 2PN and 2.5PN. Finally at 3PN, the amplitude experiences a rise again.

Similarly, when we look at the trough (Figure 5.3), we can see that 0PN has a maximum amplitude. It drops at 0.5PN and goes to a minimum value at 1PN. At 1.5PN, the amplitude rises again and falls when going to 2PN and 2.5PN. At 3PN, the amplitude rises again.

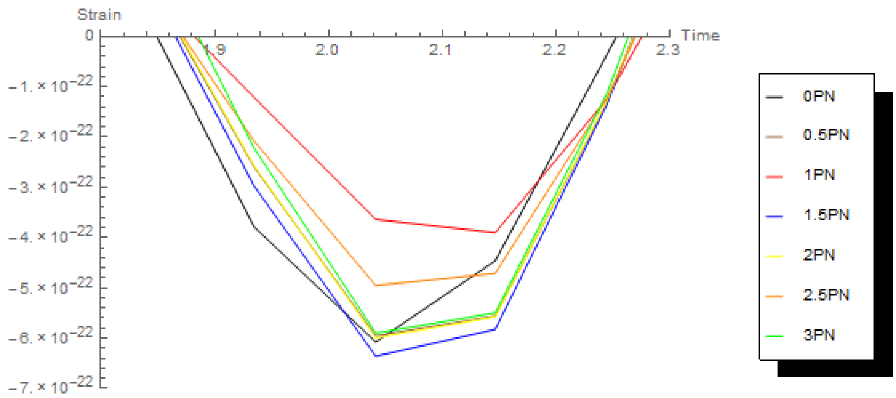

Figure 5.3: Zoomed in image of the second trough from Fig 5.1.

5.1 Visible Trend

Looking closer at the combined plot of gravitational wave potential at different PN orders (Figure 5.1) we can look for a trend, according to which the amplitude increases or decreases. We see that the amplitude of the gravitational wave potential has a high value at the lowest (0) PN order. It decreases as we go up to 0.5PN and 1PN orders. However as we reach 1.5PN PN order, there is an increase in the amplitude. We have seen in section 4.2 that 1.5PN is the order at which the gravitational wave-tails effects appear. Therefore, we can say that the increase in amplitude is due to the effects of gravitational wave-tails. This makes sense as L. Blanchet and G. Schafer have shown that the effect of wave-tails is that they increase the amplitude of the Fourier components the waves [29].

Going beyond 1.5PN order, the amplitude decreases again as we go to 2PN and further up to 2.5PN order. At 3PN order, there is an increase in the amplitude once again. This is the point at which the *tails-of-tails* come in place to contribute to the gravitational wave potential. The so-called tails-of-tails are the contributions to the gravitational wave potential which act as memory of gravitational wave-tails (thus the name "tails-of-tails"). We can thus say that the increase in amplitude once again is due to the effects of tails-of-tails.

Following this trend we can say that if we go to the higher PN orders, further increase in amplitude might occur at 4.5PN, 6PN, 7.5PN and so on (up to the point where the iteration terminates (if it does)). These increase in amplitudes might arise due to effects of something like tails of the tails-of-tails (i.e, the memory of tails-of-tails) and so on.

Appendices

Appendix A

Christoffel Symbols

Mathematica codes for finding the Christoffel symbols.

```
ClearAll[coord, metric, inversemetric, listaffine, affine, t, r,
\[Theta], \[Phi], z, i, j, k, l, s, n, a, b, c, u, v, w, x, ha, hb];
n = 4;
coord = {t, x, y, z};
ha = hx[t, x, y, z];
hb = hy[t, x, y, z];
```

Displaying metric tensor:

```
metric = {{1, 0, 0, 0}, {0, -1 + ha, hb, 0}, {0, hb, -1 - ha, 0}, {0,
    0, 0, -1}};
MatrixForm[metric]
```

Displaying inverse metric tensor:

```
inversemetric = {{1, 0, 0, 0}, {0, -1 - ha + ha^2 + hb^2, -hb,
    0}, {0, -hb, -1 + ha + ha^2 + hb^2, 0}, {0, 0, 0, -1}};
MatrixForm[inversemetric]
```

Calculating and displaying Christoffel symbols:

```
affine := affine = With[{n = 4}, Simplify[Table[1/2 \!\(
\*UnderoverscriptBox[\(\[Sum]\), \(s = 1\), \(n\)]\(inversemetric[[i,
          s]]\ \((\(-
\*SubscriptBox[\(\[PartialD]\), \({coord[[s]]}\)]metric[[j, k]]\) +
\*SubscriptBox[\(\[PartialD]\), \({coord[[k]]}\)]metric[[s, j]] +
\*SubscriptBox[\(\[PartialD]\), \({coord[[j]]}\)]metric[[s,
             k]])\)\)\), {i, 1, n}, {j, 1, n}, {k, 1, n}]]];
```

```
listaffine :=
 With[{n = 4},
  Table[If[affine[[i, j, k]] =!=
     0, {ToString[\[CapitalGamma][
       coord[[i]] \[LowerRightArrow] coord[[j]], coord[[k]]]],
     affine[[i, j, k]]}], {i, 1, n}, {j, 1, n}, {k, 1, n}]]
TableForm[Partition[DeleteCases[Flatten[listaffine], Null], 2],
 TableSpacing -> {2, 2}]
```

Appendix B

Radiative Multipole Moments

```
ClearAll[\[Rho], Pjj, Ptj, Pjkk, Pkjk, Pnn, Pjk, U, MA, MB, x,
G, c, Uj, Uk, Qjk, Qjka, Qjkab, Qjkabc, rA, rB, Ajka, Akja,
Aajk, T00, T0j, Tjk, nAB, rAB, vAB];
\[Rho] = MA*DiracDelta[x - rA];
Pjj = 4*\[Pi]*G*MA*U*DiracDelta[x - rA];
T00 = \[Rho]*(1 + v^2/(2*c^2) + 3*U/c^2) - 1/(4*\[Pi]*G*c^2)
*(7/2*Pjj);
```

Radiative Quadrupole Moment:

```
Qjk = Integrate[T00*Subscript[x, j]*Subscript[x, k], {x, -Infinity,
 Infinity}]
```

```
Ptj = 2/3*\[Pi]*G*MA*
   MB*((nAB.vAB)*Subscript[nAB, j]*(Subscript[nAB, k]*Subscript[rA,
    a] + Subscript[nAB, a]*Subscript[rA, k]) -  7*Subscript[nAB, j]
        *(Subscript[vAB, k]*Subscript[rA, a] +Subscript[vAB, a]
        *Subscript[rA, k]) +7*Subscript[vA, j]*(Subscript[nAB, k]
        Subscript[rA, a] +  Subscript[nAB, a] Subscript[rA, k]));
Pjkk = 2/3*\[Pi]*G*MA*
   MB*((nAB.vAB)*Subscript[nAB, j]*Subscript[nAB, k]*Subscript[nAB,
      a] - 11*Subscript[nAB, j]*(Subscript[nAB, k]*Subscript[vA, a]
       +  nABa*Subscript[vA, k]) + 11*Subscript[vA, j]*Subscript
       [nAB, k]*Subscript[nAB, a]);
T0j = \[Rho]*Subscript[v, j]*(1 + v^2/(2*c^2) + 3*U/c^2) + 1/(4*
\[Pi]*G*c^2) *(3*Ptj + 4*Pjkk);
```

Radiative Octopole Moment:

```
Ajka = Integrate[T0j*Subscript[x, k]*Subscript[x, a], {x, -Infinity,
 Infinity}]

Pjk = 2*\[Pi]*G^2*MA*MB/rAB*Subscript[nAB, j]*Subscript[nAB, k]*
   Subscript[rA, a]*Subscript[rA, b];
Pnn = 1/3*\[Pi]*G^2*MA*MB*rAB*Subscript[nAB, j]*Subscript[nAB,
   k] (Subscript[nAB, a] Subscript[nAB, b] - KroneckerDelta[a, b]);
Tjk = \[Rho]*Subscript[v, j]*Subscript[v, k] + 1/(4*\[Pi]*G)*(Pjk
 - 1/2*Pnn);
```

Radiative 4-pole and 5-pole Moments:

```
Qjkab = Integrate[
  Tjk*Subscript[x, a]*Subscript[x, b], {x, -Infinity, Infinity}]
Qjkabc = Integrate[
  Tjk*Subscript[x, a]*Subscript[x, b]*Subscript[x, c], {x, -Infinity,
   Infinity}]
```

Appendix C

Wave Zone Terms

Mathematica codes to find out the terms $A(s, r)$ and $B(s, r)$ as used in the definition of $h^{jk}_{\mathcal{W}}$.

```
ClearAll[A, B, Fa, Fb, fx, ft, fs, l, \[Xi], t, p, s, R];
A= Integrate[LegendreP[l, \[Xi]]/p^(n - 1), {p, r, R + s}];
B = Integrate[LegendreP[l, \[Xi]]/p^(n - 1), {p, s, R + s}];
Do[Print[A], {l, 0, 0}, {n, {3, 4}}]
Do[Print[A], {l, 2, 2}, {n, 3, 6}]
Do[Print[A], {l, 4, 4}, {n, 3, 6}]
Do[Print[B], {l, 0, 0}, {n, {3, 4}}]
Do[Print[B], {l, 2, 2}, {n, 3, 6}]
Do[Print[B], {l, 4, 4}, {n, 3, 6}]
```

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