Application of Dirac equation with Coulomb-like QCD potential for understanding light and heavy quark meson states mass spectroscopy

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To my beloved Mother who brought me to life and my beloved Father who taught me how to deal with life.

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Contents

Abstract

The Dirac equation has been reviewed with particular reference to its exact solution for the Hydrogen atom, which explains the spin-orbit splitting for Hydrogen-like atoms. Hadron spectroscopy has been reviewed. In particular the hyperfine and spin-orbit splitting in meson spectroscopy are discussed. The hyperfine splitting can be satisfactorily explained by Fermi spin-spin interaction within the framework of one gluon exchange (OGE) potential in QCD. The spin-orbit splitting is experimentally suppressed. This suppression can be qualitatively understood within the framework of Dirac theory where Dirac Hamiltonian shows a dynamical spin symmetry provided that $V_v(r) = V_s(r) + U$, where $V_v(r)$ is the 4th-component of the vector potential and $\mathrm{V}_s(r)$ is the Lorentz scalar. In this case the spin-orbit splitting vanishes.

The spin symmetry is re-derived when the Dirac equation is written in the two-component Pauli-form in a transparent way. Furthermore it is shown that Dirac equation can be solved exactly if both $V_v(r)$ and $V_s(r)$ are Coulomb like. Within this framework the spin-orbit splitting is calculated for $q\bar{Q}$ and $Q\bar{q}$ meson states, where q is light quark and Q is heavy quark.

Chapter 1

Introduction

The quest of man to understand nature and phenomena occurring in the universe is as old as human history. His curiosity led him to modern science to abstract the answer but he is still in search of answers to the unanswered questions in nature. Physics deals with a wide range of phenomena taking place around us, from the smallest subatomic particles to the largest galaxies. Particle physics is the study of the fundamental constituents of matter and the forces governing them. It endeavors to answer the questions: What are the fundamental constituents of matter? How do they interact? What is the nature of these interactions (forces)? How are these constituent particles different from each other? Why are the interactions different on different scales? Can these interactions be unified? Moreover several other questions essential for our understanding of the Universe.

At present level of experimental resolution the smallest particles of matter appear to be quarks and leptons. Quarks are of six flavors namely: up (u), down (d), strange (s), charm (c), bottom (b) and top (t) with different quantum numbers associated with them. Leptons are also of six flavors, to be exact: electron (e), muon (μ), tau (τ), electron neutrino (ν_e), muon neutrino (ν_μ) and tau neutrino (ν_τ), having different physical properties. All quarks are fractionally charged while leptons are integrally charged except neutrinos. All fundamental matter particles have their anti-particles with opposite quantum numbers.

These elementary particles experience the four fundamental forces of nature. These forces are the electromagnetic, strong nuclear, weak nuclear, and gravitational force. Electromagnetic force occurs via exchange of a photon and is experienced by charged particles. Strong nuclear force occurs by the exchange of eight (8) gluons and is responsible for the stability of nuclei. Weak nuclear force is mediated by three particles known as W^+, W^-, and Z^0 and is responsible for the radioactive decays. Gravity is experienced by all particles. All of these force mediators are bosons having integer intrinsic spin. Electromagnetism and gravity have infinite range because their mediators are massless. The strong quark gluon interaction, although mediated by massless gluons, is effectively short range because of confinement of quarks in a hadron. The relative strengths of strong nuclear force, electromagnetic, weak nuclear force and gravity are of the order $1 : 10^{-2} : 10^{-5} : 10^{-40}$ respectively [2]. The universality of these interactions implies that they are gauge forces.

All of the constituent particles of matter and forces governing them are put in a nutshell known as 'Standard Model' and it is the only experimentally tested model so far. The Standard Model classifies all quarks and lepton into three generations. The first generation

$$\begin{pmatrix} u \\ d \end{pmatrix}, \begin{pmatrix} \nu_e \\ e \end{pmatrix}$$

is relavent for the matter in the universe e.g. $p \sim (uud)$, $n \sim (udd)$. The second and third generations

$$\begin{pmatrix} c \\ s \end{pmatrix}, \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}$$

and

$$\begin{pmatrix} t \\ b \end{pmatrix}, \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}$$

do not exist naturally but can be created either in laboratory or in cosmic rays by a collision of particles of first generation.

The Standard Model satisfactorily explains most of the observable phenomena in elementary particle physics. Gravity, being incredibly weak compared to the other forces, is not important while studying microscopic particles and is not described by the standard model. Standard Model incorporates three forces of nature namely strong nuclear force, electromagnetic and weak nuclear force. It is generally believed that the Standard Model will be a part of a final theory which unifies all the forces i.e which combines Standard Model with gravity.

Hadrons are composite particles made up of quarks, which bind together by the strong nuclear force (like atom and molecules are held together by the electromagnetic force). Hadrons are categorized into two families, **Mesons** and **Baryons**.

Mesons are made up of a quark and an anti-quark ($q\bar{q}$) so they have integral spin *i.e.* $J = 0, 1, 2$. While Baryons are made up of three quarks (qqq) so they have half integral spin *i.e.* $J = \frac{1}{2}, \frac{3}{2}$.

One can explain the mass spectrum of hadrons if one assumes that hadrons are made up of quarks and/or anti-quarks. Thus mesons and baryons in the ground state are composite of $(q\bar{q})_{L=0}$ and $(qqq)_{L=o}$ respectively.

Table 1.1: The Quarks

Quark Type (Flavor)	Electric charge	Mass (effective mass in a hadron)
(u, d)	$(\frac{2}{3}, -\frac{1}{3})$	0.33 GeV
(c, s)	$(\frac{2}{3}, -\frac{1}{3})$	1.5 GeV, 0.5 GeV
(t, b)	$(\frac{2}{3}, -\frac{1}{3})$	175 ± 5 GeV, 4.5 GeV

As described earlier, matter has three layers or generations. The first generation (ν_e, e, u, d) makes all the ordinary matter in the universe. For example: proton p (uud), neutron n (udd). The first generation of quarks u and d forms an isospin doublet. The second and third generation of quarks are assigned new quantum numbers as follows: s-quark (strange) with strangeness $S = -1$; c-quark (charm) with charm $C = +1$; b-quark (bottom or beauty) with bottom number $B = -1$, t-quark (top or truth), with top number $T = +1$.

In high energy atomic collisions an atom can be split into its constituents, but a hadron does not split into quarks. The non-splitting of hadron into quarks leads to the idea of quark confinement in hadrons. Besides six flavor of quarks, there is another property of quarks called color. This is required by Pauli Principle. Mesons and Baryons in the ground state are composites of $(q\bar{q})_{L=0}$ and $(qqq)_{L=0}$. As quarks are fermions the total spin of a meson is either 0 or 1, and that of baryon is $\frac{1}{2}$ or $\frac{3}{2}$ i.e $|\Delta^{++}(S_z = \frac{3}{2}) >\sim |u^{\uparrow}u^{\uparrow}u^{\uparrow} >$. This picture violates the Pauli exclusion principle for fermions, that is why another degree of freedom called color is introduced to distinguish between identical quarks. There are three colors namely red, green and blue. The three color charges appear in accordance that there are three quarks in a baryon and if all of them have same quantum numbers they can still be distinguished by assigning different color. So now the picture is modified as $|\Delta^{++}(S_z = \frac{3}{2}) >= \frac{1}{\sqrt{6}}\epsilon_{abc}|u_a^{\uparrow}u_b^{\uparrow}u_c^{\uparrow} >$, where a,b,c are the color indices. To make a hadron we take one quark of each color so that color averages out and all observed particles are color singlets. This also explains why a free quark and an anti-quark are not observed. This is called color confinement.

We have mentioned that quarks carry color and only color singlet particles exist as free particles. Strong color charges are the sources of the strong force between two quarks just as the electric charge is the source of electromagnetic interaction between two electrically charged particles. The three color charges generate the SU(3) group which has 8 generators. The 8 color carrying gluons G_{ab} , $a, b = 1, 2, 3.$ belong to the 8 dimensional adjoint representation of this group. Just as the photon exchange gives electric potential, exchange of gluons gives color potential (see figure 3.1).

The asymptotic freedom in QCD i.e, quark-gluon coupling constant (which is energy dependent due to radiative corrections) decreases with increasing energy or at short distances unlike electromagnetic coupling constant which is increasing. The short range potential of QCD interaction can be treated perturbatively but not the long range potential, which has to be treated non-perturbatively, for which no reliable technique exists as yet. It is the long range part of QCD interaction which would hopefully provide confinement. At present long range part of QCD interaction is treated phenomenologically e.g it is taken to be proportional to 'r' unlike the short range i.e $\frac{1}{r}$. Linear confining potential will prevent quarks to come out of hadrons. Before we consider hadron spectroscopy on the above lines, we summarize the atomic energy levels since the former will be on the same lines.

The energy levels in an atom are associated with the principle quantum number 'n' as described by Bohr's model. The characteristics of an atom are precisely determined by the spectroscopy (study of absorption and emission of light and other radiations). The spectroscopy of hydrogen atom reveals that the energy levels depend only on 'n' and the levels with same 'n' but different 'l' are degenerate. The splitting of the spectral lines into multiple components by small relativistic effects and magnetic coupling between the electron's spin and orbital motion is called as fine structures, it also depends upon the principle quantum number 'n'. The further splitting of spectral lines caused by nucleon-electron spin interaction is called as "hyperfine splitting". A fine structure with fixed angular momentum quantum number J and total angular momentum j is split into hyperfine structure components with minimum degeneracy of $(2j + 1)$.

We will use the relativistic wave equation called Dirac equation for the analysis and the interpretation of the hyperfine splitting in the case of hydrogen atom.

The scheme of the work is as follows: In next chapter after the brief introduction of the Dirac equation it is solved for the Hydrogen atom. In chapter 3, One gluon exchange QCD potential is discussed and hyperfine splitting due to spin-spin Fermi contact term, which naturally occurs in QCD, is elaborated. In chapter 4 we first summarize a dynamical spin symmetry which the Dirac hamiltonian possesses when

$$V_v(r) = V_s(r) + U,$$

where $V_s(r)$ and $V_v(r)$ are respectively the Lorentz scalar and time-component of Lorentz vector [12]. Such symmetry leads to vanishing spin-orbit splitting. This symmetry is re-derived when the Dirac equation is expressed in two component Pauli form. Then an exact solution for the combined scalar and coulomb like vector potentials is derived. Finally the spin-orbit splitting in the spectroscopy of bound states is discussed.

Chapter 2

Dirac equation

In this chapter we will review the Dirac equation and its exact solution for Hydrogen atom. This is needed for our work to be discussed in chapter 4. In the work presented here we have taken $\hbar = c = 1$.

After the discovery of quantum mechanics in 1926-27 the big problem was that the Schrodinger equation

$$i\frac{\partial\Psi}{\partial t} = -\frac{1}{2m}\nabla^2\Psi, \tag{2.1}$$

was not consistent with the special theory of relativity as time and space coordinates are not treated on the same footings. Its natural extension to the Klein-Gordon equation

$$(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2)\Phi = 0, \tag{2.2}$$

had some difficulties of interpretation at that time. Written in covariant form it is

$$(\partial_\mu\partial^\mu + m^2)\Phi = 0. \tag{2.3}$$

The conserved probability current J^μ satisfying

$$\partial_\mu J^\mu = 0, \tag{2.4}$$

or

$$\frac{\partial\rho}{\partial t}+\nabla.\mathbf{J}=0, \tag{2.5}$$

is

$$J^{\mu}=i(\Phi^{*}\partial^{\mu}\Phi-\Phi\partial^{\mu}\Phi^{*}), \tag{2.6}$$

giving

$$\rho=J^{0}=i(\Phi^{*}\frac{\partial\Phi}{\partial t}-\Phi\frac{\partial\Phi^{*}}{\partial t}). \tag{2.7}$$

$$\mathbf{J}=-i(\Phi^{*}\nabla\Phi-\Phi\nabla\Phi^{*}) \tag{2.8}$$

Now a mode with energy E should oscillate as e^{-iEt} i.e

$$\Phi(\mathbf{r},t)\sim e^{-iEt}\varphi(\mathbf{r}), \tag{2.9}$$

$$\rho=2E\varphi^{*}(\mathbf{r})\varphi(\mathbf{r}), \tag{2.10}$$

where

$$E=\pm\sqrt{P^{2}+m^{2}}. \tag{2.11}$$

Thus for $E>0$, ρ is positive definite but for $E<0$, ρ is negative and therefore cannot be interpreted as probability. Thus one encounters two problems

→Energy can be *negative*.

→Probabilty density is negative for negative energies.

These two problems are linked.

Dirac made a break through in 1928. He did it from pure logic by introducing spinor fields in addition to more familiar scalar, vector and tensor fields. By doing so, the two problems are decoupled. ρ is always positive definite but negative energy problem remains. Dirac solved it by proposing Hole Theory, thereby predicting anti-matter. In the Klein-Gordon equation the problem with ρ arises due to time derivative in equation (2.5) which in turn arises because of Klein-Gordon equation being second order in time derivative. Dirac started with a linear equation both with respect to time and space derivatives. Here an analogy with the Maxwell's equations, which are fully relativistic, will be helpful. One can write basic equations of electromagnetism in terms of electromagnetic fields **E** and **B** which have six components and satisfy Maxwell's equations, which are linear in time and space coordinates, or in terms of vector potential A^μ which have four components but satisfy a second order Klein-Gordon-like equation

$$\partial^\mu \partial_\mu A^\nu = 0. \tag{2.12}$$

Each component of **E** and **B** also satisfies such a second order equation.

The above analogy suggests that

(1) We should expect several components of wave functions Ψ_n.

(2) Each component should be related to other by first order linear equations both in space and time derivatives.

(3) Each component should satisfy the Klein-Gordon-equation.

The most general equation one can write satisfying condition (1) and (2) is

$$(i\delta_{\ell n}\frac{\partial}{\partial t} + i(\alpha^i)_{\ell n}\frac{\partial}{\partial x^i} - \beta_{\ell n} m)\Psi_n = 0, \tag{2.13}$$

where $l, n = 1, 2, ...N$ and $i = 1, 2, 3$. One can conveniently write this equation in matrix form

$$(i\frac{\partial}{\partial t} + i\alpha^i\frac{\partial}{\partial x^i} - \beta m)\Psi = 0, \tag{2.14}$$

where Ψ is a column matrix α^i and β are $N \times N$ matrices and repeated indices mean summation.

In order to satisfy condition (3) operate by

$$-i\frac{\partial}{\partial t} + i\alpha^j\frac{\partial}{\partial x^j} - \beta m.$$

One obtains, since $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x_i}$ commute,

$$[\frac{\partial^2}{\partial t^2} - \frac{1}{2}(\alpha^i\alpha^j + \alpha^j\alpha^i)\frac{\partial^2}{\partial x^i \partial x^j} - im(\alpha^j\beta + \beta\alpha^j)\frac{\partial}{\partial x^j} + \beta^2 m^2]\Psi = 0, \tag{2.15}$$

where the term involving $\alpha^i\alpha^j$ has been symmetrized which is permissible since $\frac{\partial}{\partial x^i}$ and $\frac{\partial}{\partial x^j}$ commute.

The above equation reduces to the Klein-Gordon equation provided that

$$\begin{aligned} (\alpha^i\alpha^j + \alpha^j\alpha^i) &= 2\delta^{ij}, \\ \alpha^i\beta + \beta\alpha^j &= 0, \\ \beta^2 &= (\alpha^i)^2 = 1 \quad ; \quad for\ each\ i. \end{aligned} \tag{2.16}$$

Thus $\alpha^1, \alpha^2, \alpha^3$ anti-commute in pair and their square is unity. The equation (2.13) takes the matrix form

$$(i\frac{\partial}{\partial t} + i\alpha.\nabla - \beta m)\Psi = 0, \tag{2.17}$$

or

$$\begin{aligned} i\frac{\partial\Psi}{\partial t} &= H\Psi = (-i\ \alpha.\nabla + \beta m)\Psi, \\ i\frac{\partial\Psi}{\partial t} &= (\alpha.\hat{\mathbf{p}} + \beta m)\Psi, \end{aligned}$$

where $\hat{p} = -i\nabla$. Since the Hamiltonian operator

$$H = \alpha.\hat{p} + \beta m, \tag{2.18}$$

should be hermitian; each of the four matrices α^i and β must be hermitian;

$$\alpha^\dagger = \alpha \quad , \quad \beta^\dagger = \beta. \tag{2.19}$$

It is easy to show that N must be even i.e dimensions of α and β matrices must be even and the lowest dimensional representation which satisfies the relation (2.16) is 4×4 . One of such representations is

$$\begin{aligned} \beta &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad where \;\; I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ \alpha^i &= \begin{pmatrix} 0 & \sigma^i \\ \sigma^i & 0 \end{pmatrix}, \end{aligned}$$

where σ^i are 2×2 Pauli matrices, thus called Pauli representation of $\gamma - matrices$. One can write Dirac equation (2.17) in covariant form by using

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = (\frac{\partial}{\partial t}, \nabla),$$

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0, \tag{2.20}$$

where $\beta = \gamma^0, \gamma^i = \beta\alpha^i = \gamma^0\alpha^i$, so that γ^i are anti-hermitian ; $(\gamma^i)^\dagger = -\gamma^i$ and relations (2.16) can now be written in the compact form;

$$\begin{aligned} \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu &= 2g^{\mu\nu}, \qquad (2.21) \\ \{\gamma^\mu, \gamma^\nu\} &= 2g^{\mu\nu}. \end{aligned}$$

In Pauli representation,

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}. \tag{2.22}$$

Consider the Dirac equation (2.20)

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0.$$

The electromagnetic field defined by the vector potential A^μ is introduced in a gauge invariant way by replacing the ordinary derivative ∂_μ by the covariant derivative

$$\partial_\mu \to D_\mu = \partial_\mu + ieA_\mu,$$

so that equation (2.20) becomes

$$(i\gamma^\mu D_\mu - m)\Psi = 0. \tag{2.23}$$

The Klein-Gordon equation in electromagnetic field is

$$(D_\mu D^\mu + m^2)\Phi = 0. \tag{2.24}$$

The equation (2.23) can be transformed into second order equation similar to Klein-Gordon equation (2.24) by multiplying (2.23) on the left with

$$(-i\gamma^\nu D_\nu + m),$$

so that

$$(\gamma^\mu \gamma^\nu D^\mu D_\nu - m^2)\Psi = 0. \tag{2.25}$$

Using the anti commutation relations (2.21) and introducing

$$\begin{aligned} \sigma^{\mu\nu} &= \frac{i}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu), \\ &= \frac{i}{2}[\gamma^\mu, \gamma^\nu], \end{aligned}$$

which is anti-symmetric in μ, ν; one can reset the above equation as

$$[D^\mu D_\mu + \frac{i}{2}\sigma^{\mu\nu}[D^\mu, D_\nu] + m^2]\Psi = 0. \tag{2.26}$$

It is easy to show that

$$[D_\mu, D_\nu] = ieF_{\mu\nu}, \tag{2.27}$$

where $F_{\mu\nu}$ is electromagnetic field tensor;

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.28}$$

so that our equation becomes

$$(D^\mu D_\mu + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu} + m^2)\Psi = 0. \tag{2.29}$$

This differs from the Klein-Gordon equation (2.24) in the term $\frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}$.

This term can be easily evaluated in terms of electromagnetic fields $\mathbf{E}$ and $\mathbf{B}$ and then equation is remodified as

$$(D^\mu D_\mu + ie\ \sigma.\mathbf{E} - e\ \mathbf{\Sigma}.\mathbf{B} + m^2)\Psi\ = 0, \tag{2.30}$$

where

$$\mathbf{\Sigma} = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma \end{pmatrix} = \gamma^0\gamma. \tag{2.31}$$

Now in Pauli representation α is non-diagonal while every other term is diagonal. It is then preferable to use chiral representation for $\gamma - matrices$.

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{2.32}$$

where other $\gamma - matrices$ are defined to be

$$\begin{aligned} \gamma^0 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}, \\ \alpha^i &= \gamma^0\gamma^i = \begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}. \end{aligned}$$

Then in terms of two components wave functions Ψ_L, Ψ_R;

$$\Psi = \begin{pmatrix} \Psi_L \\ \\ \Psi_R \end{pmatrix}, \tag{2.33}$$

where $\Psi_{L,R} = \frac{1\mp\gamma_5}{2}\Psi$, the equation (2.30) splits into two equations which are completely decoupled.

$$(D^\mu D_\mu I \mp ie\ \sigma.\mathbf{E} - e\ \sigma.\mathbf{B} + m^2 I)\Psi_{L,R} \ = 0, \tag{2.34}$$

where I is a 2×2 unit matrix, which we will not write explicitly in what follows. The above equation differs from Klein-Gordon equation in the term $\sigma.\mathbf{E}$ and $\sigma.\mathbf{B}$.

In order to interpret these terms and see their physical significance we consider two cases.

First we take

$$A^0 = 0, \mathbf{B} = \nabla \times \mathbf{A}, \tag{2.35}$$

and $\mathbf{A}$ is independent of time. For this case the Dirac equation can be solved exactly and we want to derive it here for our future work.

The above equation (2.34) reduces to

$$(\frac{\partial^2}{\partial t^2} - D^2 - e\ \sigma.\mathbf{B} + m^2)\Psi = 0, \tag{2.36}$$

where $D^2 = (\nabla - ie\mathbf{A})^2$ and Ψ is Ψ_L or $\Psi_{R.}$

Now a mode with energy $E = m + \epsilon$ should oscillate in time $\Psi \sim e^{iEt}$. In the non-relativistic limit, kinetic energy $\epsilon << m$, so that for a slowly moving particle:

$$\Psi(\mathbf{r}, t) = \psi(\mathbf{r}, t)e^{-imt}. \tag{2.37}$$

Thus

$$\frac{\partial}{\partial t}(e^{-imt}\psi) = e^{-imt}(-im + \frac{\partial}{\partial t})\psi.$$

Using it again

$$\frac{\partial}{\partial t}[e^{-imt}(-im+\frac{\partial}{\partial t})\psi] = e^{-imt}(-im+\frac{\partial}{\partial t})^2\psi,$$
$$= e^{-imt}(\frac{\partial^2}{\partial t^2} - 2im\frac{\partial}{\partial t} - m^2)\psi.$$

Dropping $\frac{\partial^2}{\partial t^2}$ compared to $2im\frac{\partial}{\partial t}$ equation (2.36) becomes

$$i\frac{\partial\psi}{\partial t} = (-\frac{\nabla^2}{2m} + \frac{1}{m}ie\ \mathbf{A}.\nabla - \frac{e}{2m}\ \sigma.\mathbf{B})\psi, \tag{2.38}$$

where we have used $\nabla.\mathbf{A} = 0$ and have dropped second order term $e^2\mathbf{A}^2$. Taking $\mathbf{B}$ to be uniform

$$\mathbf{A} = \frac{1}{2}\mathbf{B}\times\mathbf{r} \quad satisfies \quad \mathbf{B} = \nabla\times\mathbf{A}.$$

Thus

$$\mathbf{A}.\nabla = \frac{i}{2}\mathbf{B}.\mathbf{L},$$

where

$$\mathbf{L} = -i(\mathbf{r}\times\nabla),$$

is the orbital angular momentum.

Thus finally

$$i\frac{\partial\psi}{\partial t} = [-\frac{\nabla^2}{2m} - \frac{e}{2m}(\mathbf{L}+2\mathbf{S}).\mathbf{B}]\psi, \tag{2.39}$$

where $\mathbf{S} = \frac{1}{2}\sigma$ denotes spin $\frac{1}{2}$ operator. Now $-\frac{e}{2m}\mathbf{L}$ is the magnetic moment due to the orbital motion of the electron of charge $-e$ ($e > 0$). Note the important factor of 2 in front of $\mathbf{S}$, known as magnetic spin anomaly of electron which naturally comes from Dirac equation. The second term gives the intrinsic magnetic moment of a particle of charge $-e$ with spin $\frac{1}{2}$ i.e the magnetic moment of electron is $\mu_e = 1$ or $g = 2$, in terms of Bohr's magnetron $-\frac{e\hbar}{2m}$, in agreement with experiment. This naturally comes out from Dirac equation while earlier it was calculated experimentally. This was regarded as great success in Quantum mechanics.

Now we consider for Hydrogen atom

$$A^0 \neq 0 \quad \mathbf{A} = 0, \tag{2.40}$$

A^0 is the coulomb potential $A^0 = -\frac{Ze}{r}$ so that

$$eA^0 = -\frac{Ze^2}{r} = -\frac{Z\alpha}{r}, \tag{2.41}$$

where $\alpha = e^2 = \frac{1}{137}$ in our units.

$$\mathbf{E} = -\nabla A^0 = -\frac{dA^0}{dr}\hat{r}. \tag{2.42}$$

Then equation (2.34) takes the form

$$[(\frac{\partial}{\partial t} + ieA^0)^2 - \nabla^2 \pm \frac{dA^0}{dr}\sigma.\hat{r} + m^2]\Psi_{L,R} = 0. \tag{2.43}$$

For stationary states

$$\Psi_{L,R} = e^{-iEt}\psi_{L,R}(\mathbf{r}),$$

where

$$\Psi_{L,R}(\mathbf{r}) = Y_{lm}(\theta,\phi)R_{L,R}(r).$$

Thus equation (2.43) becomes

$$[2eEA^0 - e^2A_0^2 - \nabla^2 \pm \frac{dA^0}{dr}\sigma.\hat{r} - (E^2 - m^2)]R_{L,R}(r) = 0, \tag{2.44}$$

where

$$\begin{aligned} \nabla^2 &= \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{r^2}, \\ and \quad L^2 &= l(l+1), \end{aligned}$$

is the square of orbital angular momentum $\mathbf{L}$.

The total angular momentum

$$\mathbf{J} = \mathbf{L} + \mathbf{S},$$

commutes with L^2 and the Hamiltonian. J^2 and J_z have eigenvalues $j(j+1)$ and m respectively, where

$$m = -j, ..., +j \quad ; \; j = \frac{1}{2}, \frac{3}{2}, ...,$$

and can be used to label the states. The integer l takes the values:

$$\begin{aligned} l &= j + \frac{1}{2} = l_+, \\ l &= j - \frac{1}{2} = l_-, \end{aligned}$$

$\sigma.\hat{r}$ is hermitian and has its square $(\sigma.\hat{r})^2 = 1$ so that it has eigenvalues ± 1. In the two dimensional sub-space provided by $l_\pm$, $(\sigma.\hat{r})$ has non-diagonal matrix elements (it is like dipole transition where angular momentum must change by 1 unit), so that,

$$< l_\pm \left| \sigma.\hat{r} \right| l_\pm >= 0. \tag{2.45}$$

Thus in the above subspace the operator

$$\frac{L^2}{r^2} - e^2 A_0^2 \pm \frac{dA_0}{dr} \sigma.\hat{r}, \tag{2.46}$$

is represented by 2×2 matrix

$$\begin{pmatrix} \frac{(j+\frac{1}{2})(j+\frac{3}{2})}{r^2} - e^2 A_0^2 & \pm ie \frac{dA_0}{dr} \\ \pm ie \frac{dA_0}{dr} & \frac{(j-\frac{1}{2})(j+\frac{1}{2})}{r^2} - e^2 A_0^2 \end{pmatrix}, \tag{2.47}$$

which has eigenvalues λ given by

$$[\frac{(j+\frac{1}{2})(j+\frac{3}{2})}{r^2} - e^2 A_0^2 - \lambda][\frac{(j-\frac{1}{2})(j+\frac{1}{2})}{r^2} - e^2 A_0^2 - \lambda] - e^2 (\frac{dA_0}{dr})^2 = 0. \tag{2.48}$$

The solution of the quadratic equation gives

$$\begin{aligned}\lambda &= [\frac{(j+\frac{1}{2})^2}{r^2} - e^2A_0^2] \pm [\frac{(j+\frac{1}{2})^2}{r^4} - e^2(\frac{dA_0}{dr})^2], \\ &= \frac{1}{r^2}[(j+\frac{1}{2})^2 - Z^2\alpha^2]^{\frac{1}{2}}[[(j+\frac{1}{2})^2 - Z^2\alpha^2]^{\frac{1}{2}} \pm 1],\end{aligned} \tag{2.49}$$

where we have used $e^2A_0^2 = \frac{Z^2\alpha^2}{r^2}$ and $e^2(\frac{dA^0}{dr})^2 = \frac{Z^2\alpha^2}{r^4}$. Thus the eigenvalues of the operator

$$\frac{L^2}{r^2} - \frac{Z^2\alpha^2}{r^2} \pm i\frac{Z\alpha}{r^2}\sigma.\hat{r} \quad \text{are} \quad \frac{\bar{\lambda}(\bar{\lambda}+1)}{r^2},$$

where

$$\bar{\lambda} = [(j+\frac{1}{2})^2 - Z^2\alpha^2]^{\frac{1}{2}}; [(j+\frac{1}{2})^2 - Z^2\alpha^2]^{\frac{1}{2}} - 1, \tag{2.50}$$

which may be written as

$$\bar{\lambda} = (j+\frac{1}{2}) - \delta_j,$$

with

$$\delta_j = (j+\frac{1}{2}) - \sqrt{(j+\frac{1}{2})^2 - Z^2\alpha^2}.$$

Thus our equation (2.44) becomes

$$[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + 2\frac{Z\alpha}{r}E + (E^2 - m^2) - \frac{\bar{\lambda}(\bar{\lambda}+1)}{r^2}]R(r) = 0. \tag{2.51}$$

Writing

$$R(r) = \frac{\chi(r)}{r},$$

we have

$$\frac{1}{2m}\frac{d^2\chi}{dr^2} + [\frac{E^2 - m^2}{2m} + \frac{2Z\alpha}{2mr}E - \frac{\bar{\lambda}(\bar{\lambda}+1)}{2mr^2}]\chi(r) = 0. \tag{2.52}$$

This is formally identical with the equation for Hydrogen-like atom in Schrodinger theory i.e

$$\frac{1}{2m}\frac{d^2\chi}{dr^2} + (E + \frac{Z\alpha}{r^2} - \frac{l(l+1)}{2mr^2})\chi = 0, \tag{2.53}$$

with energy eigenvalues,

$$E = -m\frac{Z^2\alpha^2}{2n^2} \ ; \quad n = \acute{n} + l + 1 \ \text{ is a positive integer.}$$

If

$$\begin{aligned}
\alpha &\rightarrow \alpha\frac{E}{m} \quad ,E \rightarrow \frac{E^2 - m^2}{2m}, \\
l &\rightarrow \bar{\lambda} = (j + \frac{1}{2}) - \delta_j, \\
\text{and } n^{'} &= n - (l+1) \rightarrow n - (\bar{\lambda} + 1), \\
&= n - (j + \frac{1}{2}) + \delta_j - 1,
\end{aligned}$$

is an integer. Thus n must be shifted by δ_j.

$$n \rightarrow n = n - \delta_j.$$

Thus

$$\frac{E_{nj}^2 - m^2}{2m} = -\frac{mZ^2\alpha^2}{2(n-\delta_j)^2}, \tag{2.54}$$

or

$$E_{nj} = \pm m\frac{1}{[1 + \frac{Z^2\alpha^2}{(n-\delta_j)^2}]^{\frac{1}{2}}}, \tag{2.55}$$

where $n = 1, 2, ...j$ and $j = \frac{1}{2}, \frac{3}{2}, ..., n - \frac{1}{2}$.

Note that δ_j becomes complex for $Z\alpha > j + \frac{1}{2}$. This catastrophy first occurs for $j = \frac{1}{2}$ i.e for $Z\alpha > 1$ or $Z > 137$. Since α is small therefore for $Z\alpha << 1$, we can expand equation (2.54) in powers of $Z\alpha$. This gives

$$E_{nj} = -m\frac{Z^2\alpha^2}{2n^2}[1 + \frac{Z^2\alpha^2}{n}(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n})] + O(Z^4\alpha^4). \tag{2.56}$$

For Hydrogen atom $Z = 1$,

$$E_n^0 = -\frac{m\alpha^2}{2n^2}. \tag{2.57}$$

Note the important fact that states with the same value of n but different j (e.g $2p_{\frac{1}{2}}$

and $2p_{\frac{3}{2}}$ which were degenerate in Schrodinger theory) now split. It is summarized using (2.55) in the table below:

Table 2.1: Energy levels for several quantum numbers

	n	l	j	E_{nj}
$1S_{\frac{1}{2}}$	1	0	$\frac{1}{2}$	$\mathrm{m}\sqrt{1-Z^2\alpha^2}$
$2S_{\frac{1}{2}}$	2	0	$\frac{1}{2}$	$\mathrm{m}\sqrt{\frac{1+\sqrt{1-Z^2\alpha^2}}{2}}$
$2P_{\frac{1}{2}}$	2	1	$\frac{1}{2}$	$\mathrm{m}\sqrt{\frac{1+\sqrt{1-Z^2\alpha^2}}{2}}$
$2P_{\frac{3}{2}}$	2	1	$\frac{1}{2}$	$\frac{m}{2}\sqrt{4-Z^2\alpha^2}$

We note

→The states of the same principle quantum number n are not degenerate completely anymore.

→The states $2S_{\frac{1}{2}}$ and $2P_{\frac{1}{2}}$ are still degenerate. They have the same quantum number n and j, but a different parity.

→Experiments show that $2S_{\frac{1}{2}}$ has a higher energy than $2P_{\frac{1}{2}}$, known as *Lamb Shift* which can be explained in QED due to quantum relativistic correction.

The eigen energies corresponding to various j and l are presented in the table below when $n = 1, 2$. The mass of the electron and fine structure constant are set according to [9] . Since the spectrum of Hydrogen has been experimentally measured with very

Table 2.2: Energy spectrum for Hydrogen atom

n	n'	j	k	l	*structure*	$E_1(eV)$
1	0	$\frac{1}{2}$	1	0	$1\mathrm{S}_{\frac{1}{2}}$	13.605433412
2	0	$\frac{1}{2}$	2	1	$2\mathrm{P}_{\frac{3}{2}}$	3.400994390
2	1	$\frac{1}{2}$	-1	1	$2\mathrm{P}_{\frac{1}{2}}$	3.401039674
2	1	$\frac{1}{2}$	1	0	$2\mathrm{S}_{\frac{1}{2}}$	3.401039674

high accuracy more significant figures are kept in the calculation result accordingly, as some small distinctions between theory and data only appear at the last few digits of the number.

2.1 Current and charge density of particle in Dirac theory

One can start with the free particle Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\Psi(x) = 0. \tag{2.58}$$

The corresponding equation for the adjoint

$$\bar{\Psi}(x) = \Psi^\dagger(x)\gamma^0,$$

is

$$-i\partial_\mu\bar{\Psi}\gamma_\mu - m\bar{\Psi} = 0. \tag{2.59}$$

Multiply (2.58) on left by $\bar{\Psi}$ and (2.59) on right by Ψ and subtract, one obtains,

$$\begin{aligned} \partial_\mu(\bar{\Psi}i\gamma_\mu\Psi) &= 0, \qquad (2.60) \\ \partial_\mu J^\mu &= 0, \end{aligned}$$

where

$$J_\mu = \bar{\Psi}i\gamma^\mu\Psi,$$

and one can show that it transforms like a four vector under Lorentz transformation. Equation (2.60) takes the form of equation of continuity.

$$\partial^0 J^0 + \nabla.\mathbf{J} = 0,$$

where;

$$J^0 = \bar{\Psi}\gamma^0\Psi = \Psi^\dagger\Psi \geqslant 0 \quad \text{positive definite.}$$

Thus there is no trouble with probability density in Dirac equation.

2.2 Hole theory and existence of Positron (anti-matter)

The difficulty with the negative energy states is solved by Dirac's Hole theory. First we elaborate what is the difficulty?

Consider Hydrogen atom. An electron bound in an atom can emit radiation spontaneously and fall into the continuum of energy of negative energy states. But the ground state of atom is stable. Clearly such a catastrophic transition must be prevented. Dirac solved it by giving a new picture of vacuum regarding it an infinite sea of electrons occupying negative energy states (see figure 2.1).

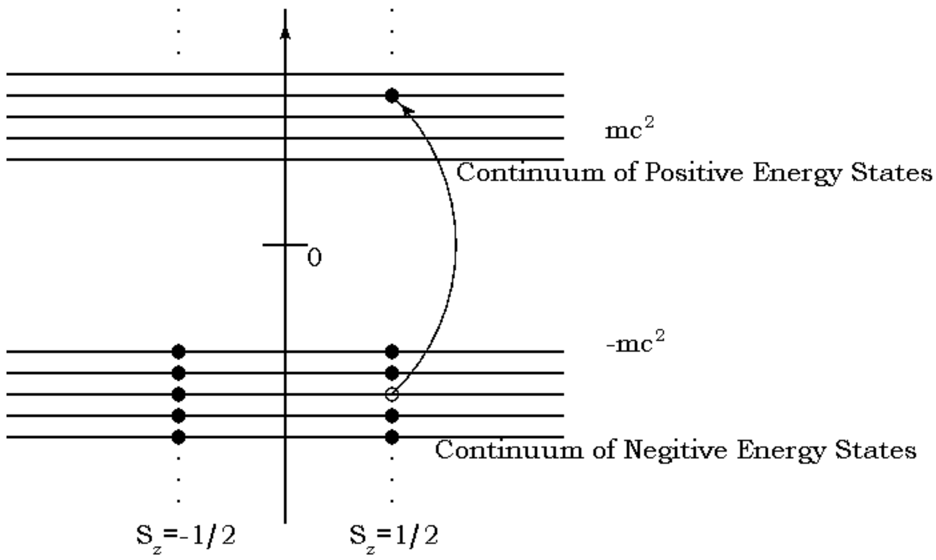

Figure 2.1:

Then Pauli principle would forbid transition of the type mentioned above. The possibility exists that one of the electrons is lifted by a radiation quantum (photon) to a positive energy state, where it becomes observable as an ordinary electron, and the gap or hole in the infinite sea was the Positron (positive charge-opposite to electron due to charge conservation). This is what happens when a radiation quantum of high energy ($\geqslant 2me^2$) disappears with this process, giving rise to e^-e^+ pair.

2.2.1 Feynman diagrams

To describe the above described phenomena, Feynman regarded positron and electron as essentially one particle, differing only in the sense in which they move with respect to time. A positron was an electron moving backward in time (see figure 2.2).

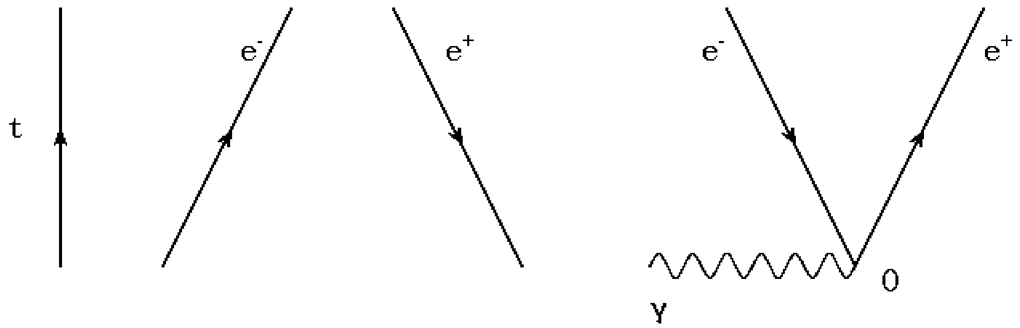

Figure 2.2: Feynman diagram for illustration of electron and positron

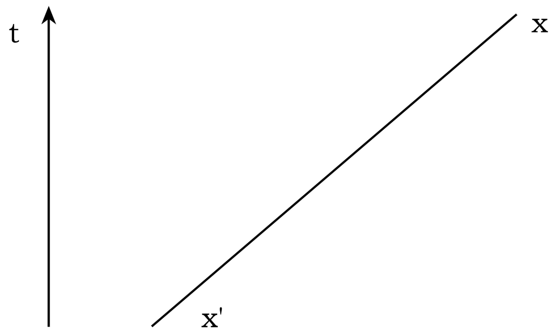

Figure 2.3: Motion of a particle with respect to time

The figure shows that when a radiation quantum (represented by wavy line) interacts with the positron at O it turns the path around and now it moves forward in time hence it appears as an electron.

Feynman introduced the concept of propagators to describe the motion of the particles in QCD and QED. The figure above shows the propagation of a particle from (t', x') where it is created, to a point (t, x) where it is reabsorbed in the vacuum such that $t > t'$. It can also be interpreted as the creation of an anti-particle at (t, x) and reabsorption in vacuum at (t', x').

To conclude, the real justification of Hole theory lies in quantum field theory which takes care of particle creations and annihilations.

For a spin zero particle, the propagator is the Green's function of the Klein-Gordon operator

$$(\partial_\mu \partial^\mu + \mu^2) G(x, x') = -\delta^4(x - x'),$$

which has the solution

$$G(x,x') = i\Delta_F(x-x') = \frac{i}{2\pi^4}\int d^4k \frac{1}{k^2-\mu^2+i\epsilon} e^{-ik.(x-x')}.$$

For spin $\frac{1}{2}$ particle, it is Green's function of Dirac operator (in matrix form)

$$(i\gamma^\mu\partial_\mu + m)G(x,x') = -\delta^4(x-x'),$$

which has the solution

$$\begin{aligned} G(x,x') &= iS_F(x-x'), \\ &= (i\gamma^\mu\partial_\mu + m)i\Delta_F(x-x'), \\ &= \frac{i}{2\pi^4}\int d^4p \frac{\gamma^\mu p_\mu + m}{p^2-m^2+i\epsilon} e^{-ip.(x-x')}. \end{aligned}$$

Thus in momentum space, the spin zero boson and spin -$\frac{1}{2}$ fermion propagators are respectively

$$\frac{i}{k^2-m^2+i\epsilon},$$
$$\frac{i(p+m)}{p^2-m^2+i\epsilon}.$$

Similarly for the spin 1 photon, the propagator is

$$g_{\mu\nu}\frac{i}{q^2},$$

and for a massive spin 1 particle it is

$$g_{\mu\nu}\frac{i}{q^2-m^2},$$

where $g_{\mu\nu}$ comes from polarization sum of spin-1 particle

$$\sum_\alpha \epsilon_\alpha^\mu \epsilon_\alpha^{*\nu} = g_{\mu\nu}.$$

If we attach to the photon (wavy) line, another positron-electron pair at the other end we get figure (2.4), which represents e^+e^- (Bhabba) scattering (annihilation). But there is also another diagram (direct) given along.

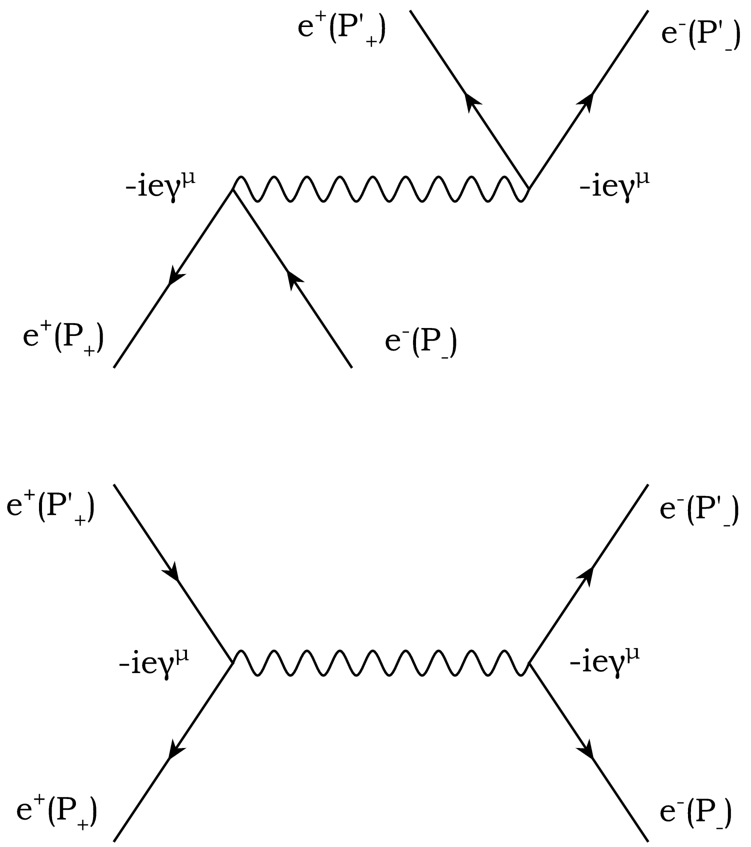

Figure 2.4: Feynman diagram for illustration of electron and positron.

The internal line connecting two vertices represents the propagator and at each vertex we have to associate the factor $-ie\gamma_\mu$ and conserve energy and momentum.

Chapter 3

Quantum chromodynamics, one gluon exchange potential and hadron spectroscopy

As discussed in chapter 1 hadrons are colorless and the binding force between them is provided by the exchange of gluons among the quarks in a hadron. This is similar to the feynman diagram in figure (2.4) where photon is replaced by gluon and charge e by color factor $g_s\frac{\lambda_A}{2}$. The one gluon exchange potential for meson ($q\bar{q}$) system (in the non-relativistic limit) is given by (see figure 3.1):

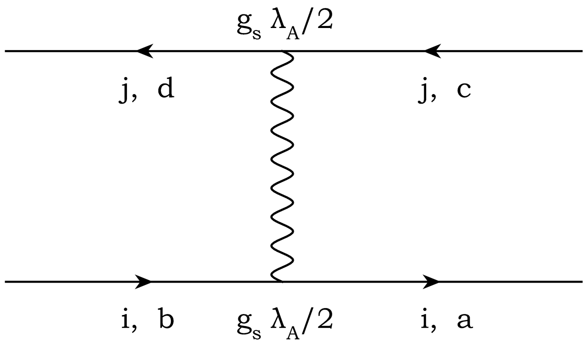

Figure 3.1:

$$V_{ij} = -g_s^2 \frac{1}{4\pi r} \sum_{A=1}^{8} (\frac{\lambda_A}{2})^a_b (\frac{\lambda_A}{2})^d_c \frac{1}{\sqrt{3}} \delta^c_a \frac{1}{\sqrt{3}} \delta^b_d. \tag{3.1}$$

In this equation: i, j are flavor indices; a, b, c, d are color indices; the factors $\frac{1}{\sqrt{3}}\delta_a^c$ and $\frac{1}{\sqrt{3}}\delta_d^b$ occur due to color singlet totally symmetric wave functions for the system. Simplifying it we get

$$V_{ij} = -\frac{4}{3}\frac{\alpha_G}{r} \quad , \quad \alpha_G = \frac{g_s^2}{4\pi} \ . \tag{3.2}$$

One gluon exchange potential for baryon (qqq) is given by (see figure 3.2):

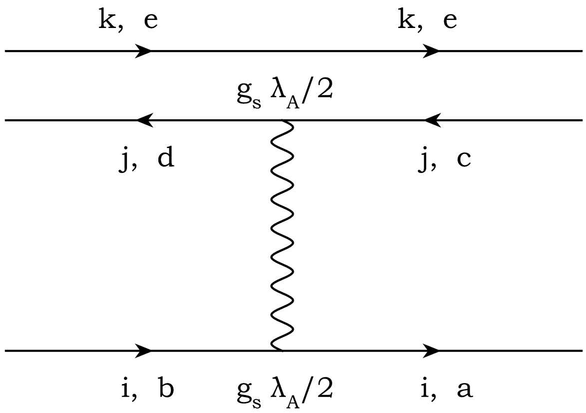

Figure 3.2:

$$V_{ij} = g_s^2 \frac{1}{4\pi r}\frac{\varepsilon_{eac}}{\sqrt{6}}\frac{\varepsilon^{ebd}}{\sqrt{6}}(\frac{\lambda_A}{2})_d^c(\frac{\lambda_A}{2})_b^a. \tag{3.3}$$

The factors $\frac{\varepsilon_{eac}}{\sqrt{6}}$ and $\frac{\varepsilon^{ebd}}{\sqrt{6}}$ occur due to the fact that three quark wave function is totally anti-symmetric in color indices. Simplifying it we get

$$V_{ij} = -\frac{2}{3}\frac{\alpha_G}{r}. \tag{3.4}$$

Hence we conclude that

$$V_{ij}^{q\bar{q}} = 2V_{ij}^{qq}. \tag{3.5}$$

The two body one gluon potential can be written in the compact form as

$$V_{ij} = k_s\frac{\alpha_G}{r} \quad , \quad k_s = \begin{Bmatrix} -\frac{4}{3}q\bar{q} \\ -\frac{2}{3}qq \end{Bmatrix}. \tag{3.6}$$

Note the very important fact that in both cases we have an attractive potential [4]. Without color, V_{ij}^{qq} would have been repulsive. The theory here is called quantum chromodynamics (QCD).

Here due to quantum radiative corrections, α_G is not constant but depends upon the momentum transfer q^2 and decreases logarithmically for increasing q^2. Since q^2 is the momentum conjugate to r, it means that as distance is decreased α_G becomes smaller therefore it is sufficient to use the lowest order one gluon exchange (OGE) as $r \to 0$.

In summary one can use the OGE potential taking into account the running coupling constant $\alpha_G(q^2)$. One may note that OGE is coulomb like and this is not sufficient to confine quarks in a hadron. Since just as one can ionize an atom to knock an electron, similarly quark could be separated from a hadron if sufficient energy is supplied. Thus one takes

$$V_{ij}(r) = V_{ij}^g(r) + V^c(r), \tag{3.7}$$

where $V^c(r)$ is the confining potential (independent of the quark flavor) [4]. $V^c(r)$ is the force of self interaction of color quarks mediated by gluons which increases with distance in such a way that quarks can be confined in hadrons.

In case of large r, QCD perturbation theory fails as the QCD constants become large in the region. It is essentially a non-perturbatively phenomena which still is not satisfactorily handled. Lattice QCD suggests that we have $V^c(r) = \sigma r$ like potential.

3.1 Hyperfine splitting

Now if spin is considered a spin-$\frac{1}{2}$ charged particle having charge eQ_i has magnetic moment

$$e\mu_i = \frac{eQ_i}{2m_i}\sigma_i. \tag{3.8}$$

In quantum mechanics the energy splitting between s-states is given by Fermi-contact term i.e

$$H_{ij}^M = -\frac{8\pi}{3}\alpha\mu_i.\mu_j\delta^3(r_i - r_j). \tag{3.9}$$

In case of QCD we have 8-color magnetic moments

$$g_s\mu_A^i = \frac{g_s}{2m_i}(\frac{\lambda_A}{2})\sigma \quad , A = 1, ..., 8. \tag{3.10}$$

The analogous two particle interaction for QCD is then given by

$$H_{ij} = -\frac{8\pi}{3}\alpha_s\mu_A^{(i)}.\mu_A^{(j)}\delta^3(r_i - r_j), \tag{3.11}$$

so that for color singlets

$$H_{ij} = -\frac{8\pi}{3}\alpha_s k_s \frac{\sigma_i.\sigma_j}{4m_i m_j}\delta^3(r). \tag{3.12}$$

Now for $q\bar{q}$ states

$$\begin{aligned} S &= \frac{1}{2}(\sigma_1 + \sigma_2), \\ s(s+1) &= \frac{3}{2} + \frac{1}{2}\sigma_1.\sigma_2. \end{aligned}$$

Thus for spin singlet $(s = 0), \sigma_1.\sigma_2 = -3$ and for spin triplet $(s = 1)$, $\sigma_1.\sigma_2 = 1$. Hence it immediately follows that

$$m(^3S_1) > m(^1S_0), \tag{3.13}$$

i.e vector meson mass is greater than the corresponding pseudoscalar mass which is in agreement with the experimental results. As seen the hyperfine splitting depends only on quark masses and α_s being energy dependent, may vary slightly with quark masses. As a result such hyperfine splitting get related.

Consider for example $K^*(\rho)$, which are $^3S_1$ states, and $K(\pi)$,which are $^1S_0$ states, unlike the quark content which is the same for K^* and K $(\bar{u}s)$ and that of ρ and π is $(\bar{u}d)$. Then (3.12) predicts

$$\frac{m_K^* - m_K}{m_\rho - m_\pi} \approx \frac{m_u}{m_s} \approx 0.66 \quad (expt\ 0.64), \tag{3.14}$$

where we have used the values of the constituent quark masses $m_u = 336MeV$ and $m_s = 510MeV$ obtained from the magnetic moments of baryons [4]. Further it is clear from (3.12) hyperfine splitting for $\bar{q}Q$ or $\bar{Q}q$ states decreases with the increasing mass of Q and because α_s also decreases as well. This is clearly supported by the experimental data [9] shown below.

$$\begin{aligned}
M(\rho) - M(\pi) &= 644MeV \qquad \text{quark content}: \bar{d}u, \\
M(k^*) - M(k) &= 398MeV \qquad \text{quark content}: \bar{s}u, \\
M_{D^*}(1^-) - M_D(0^-) &= 142MeV \qquad \text{quark content}: \bar{u}c, \\
M_{D_s^*}(1^-) - M_{D_s}(0^-) &= 144MeV \qquad \text{quark content}: \bar{s}c, \\
M_{B^*}(1^-) - M_B(0^-) &= 45.78MeV \qquad \text{quark content}: d\bar{b}, \\
M_{B_s^*}(1^-) - M_{B_S}(0^-) &= 46.5MeV \qquad \text{quark content}: s\bar{b},
\end{aligned}$$

where

$$\begin{aligned}
M_\rho &= 505MeV \qquad M_{K^*} = 892MeV, \\
M_\pi &= 139MeV \qquad M_K = 494MeV, \\
M_D &= 1865MeV \qquad M_{D_S} = 1968MeV, \\
M_B &= 5279.5MeV \qquad M_{B_S} = 5412.8MeV.
\end{aligned}$$

Finally we may remark the spin-spin interaction given in (3.12) arises naturally in OGE, when one considers the next order relativistic corrections to (3.1).

Chapter 4

Symmetries of Dirac equation and spin-orbit splitting

4.1 Spin-orbit splitting

We review the work done earlier which shows that the relativistic symmetry suppresses the quark spin-orbit splitting as seen experimentally (see below). Now as discussed in chapter 2 the solution of Dirac equation with coulomb like potential does give spin-orbit splitting. Since one gluon exchange (OGE) potential is coulomb like, one would expect such a splitting in the spectrum of mesons and baryons. Isgur [10] conjectured that this suppression may be due to accidental cancelation between OGE and Thomas precession effect in semi-relativistic approach. Since Thomas precession comes naturally from Dirac theory, one might look for absence of the spin-orbit coupling due to dynamical symmetries which Dirac Hamiltonian might possess. This indeed is the case as was noted in [11] [12]. Before we discuss the symmetry for the Dirac Hamiltonian, we briefly discuss the experimental spectrum which shows the suppression of the spin-orbit splitting. In $(Q\bar{q})$ or $(q\bar{Q})$ system, where q is light quark and Q is heavy quark, we can combine $\mathbf{j} = \mathbf{l} + \mathbf{s}_q$ with $\mathbf{s}_Q$ to give the total angular momentum $\mathbf{J}$ of the system $\mathbf{J} = \mathbf{j} + \mathbf{s}_Q$. If the heavy quark is infinitely heavy, the angular momentum of light degrees of freedom, $\mathbf{j}$, is separately conserved. Thus for example, when $Q = c$, charm quark, $q = u$ or d, up or down quark, one can have the following multiplets

$$\begin{aligned} l &= 0 \quad [D^*(1^-), D(0^-)]_{j=\frac{1}{2}}, \\ l &= 1 \quad [D_2^*(2^+), D_1(1^+)]_{j=\frac{3}{2}}, \\ & \qquad \quad and \;\; [\mathrm{D}_1^*(1^+), D_0(0^+)]_{j=\frac{1}{2}}, \end{aligned}$$

where (J^P) gives the total angular momentum and parity quantum numbers. The splitting between $j = \frac{3}{2}$ and $j = \frac{1}{2}$ multiplets is due to spin-orbit interaction while the hyperfine splitting between the two members of each multiplet arises from the Fermi term $\mathbf{s}_q.\mathbf{S}_q$ as discussed in chapter 3. The splitting, for D mesons, between $D_1^*(1^+) : 2422.3 \pm 1.3 MeV$ and $D_2^*(2^+) : 2461.1 \pm 1.6 MeV$ is $39 MeV$; for B mesons, between $B_1(1^+) : 5720.7 \pm 2.7$ and $B_2^*(2^+) : 5746.9 \pm 2.9$ is $26 MeV$; for $B_{s_1}(1^+) : 5829.4 \pm 0.7$ and $B_{s_2}(2^+) : 5839.7 \pm 0.6$ is $10 MeV$ [9]. We notice thus the suppression generally increases with the increasing mass of Q.

To conclude we see that the spin orbit splitting is suppressed.

4.2 A dynamical symmetry for the Dirac hamiltonian

If we consider a system of sufficiently heavy anti-quark (quark) and light quark (anti-quark) the dynamics may well be represented by the the motion of the light quark (anti-quark) in the fixed potential provided by the heavy anti-quark (quark). Let us assume that both scalar and vector potentials are present, then the Dirac hamiltonian describing the motion of light quark is

$$H = \alpha.\mathbf{p} + \beta(m + V_s) + V_v + M, \tag{4.1}$$

where $\hbar = c = 1$, and α, β are the usual Dirac matrices, $\mathbf{p}$ is three momentum, m and M are the masses of the light and heavy quarks respectively. If vector and scalar potentials satisfy the relation

$$V_v(r) = V_s(r) + U, \tag{4.2}$$

then Dirac Hamiltonian is invariant under a spin symmetry (called pseudo-spin symmetry)

$$[H, S_i] = 0. \tag{4.3}$$

The generators of this symmetry are given by

$$S_i = \begin{pmatrix} s_i & 0 \\ 0 & \tilde{s}_i \end{pmatrix}, \tag{4.4}$$

where $s_i = \frac{\sigma_i}{2}$ are usual spin generators, σ_i the Pauli matrices, and $\tilde{s}_i = U_p s_i U_p$ with $U_p = \frac{\sigma.\mathbf{p}}{2}$ as helicity operator. Thus Dirac eigenstates can be labeled by the orientation of the spin even though the system may be highly relativistic and the eigenstates with different spin orientations will be degenerate.

For spherically symmetric potentials, the Dirac Hamiltonian has an additional invariant algebra, called orbital angular momentum symmetry,

$$L_i = \begin{pmatrix} l_i & 0 \\ 0 & \tilde{l}_i \end{pmatrix} \tag{4.5}$$

where $\tilde{l}_i = U_p l_i U_p$ and $l_i = (\mathbf{r} \times \mathbf{p})$. It means that the Dirac eigenstates can be labeled with orbital angular momentum as well as spin, and the states with the same orbital angular momentum are degenerate, e.g the states $n_r p_{\frac{1}{2}}$ and $n_r p_{\frac{3}{2}}$ are degenerate where n_r is the radial quantum number.

Thus we have identified a symmetry in the heavy-light quark system which produces spin-orbit degeneracies independent of the details of the potential. If this potential is strong the heavy-light quark system will be very relativistic. The symmetry described here is similar to the symmetry as being responsible for pseudo-spin degeneracies observed in the nuclei [13]. It has been already observed there that the pseudo-spin symmetry improves with increasing energy of the state, for various potentials. A similar behavior may be expected for spin symmetry, consistent with the experimental observations, that the spin-orbit splitting decrease for higher mass states.

4.3 Dirac equation and spin symmetry

We consider the Dirac equation in the presence of a vector potential $V_\nu(r)$ which is the zeroth component of 4-vector V_μ. We also include a scalar potential $V_s(r)$ which is like the mass term. Then the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\Psi = 0,$$

becomes

$$[i\gamma^0(\partial_0 + iV_v(r)) + i\gamma^i\partial_i - m - V_s(r)]\Psi = 0. \tag{4.6}$$

Multiply on the left by

$$[i\gamma^0(\partial_0 + iV_v(r)) + i\gamma^j\partial_j + m + V_s(r)].$$

Since the potentials are independent of time; we have

$$[-\partial_0^2 + 2iV_\nu(r)\partial_0 + V_v^2(r) - \gamma^j\gamma^i(\partial_j\partial_i) + i\gamma^0\gamma^i[\partial_i, V_v] - i\gamma^i[\partial_i, V_s] - (m + V_s(r))^2]\Psi = 0. \tag{4.7}$$

For stationary states, $\frac{\partial}{\partial t} \to iE$, and we use the fact that $\partial_j\partial_i$ is symmetric in j and i, so that

$$\gamma^j\gamma^i(\partial_j\partial_i) = \frac{1}{2}(2g^{ij})\partial_j\partial_i = -\nabla^2.$$

Thus we obtain

$$[\nabla^2 + V_v^2 - V_s^2 - 2EV_v - 2mV_s + i\gamma^0\gamma^i[\partial_i, V_v] - i\gamma^i[\partial_i, V_s] + (E^2 - m^2)]\Psi = 0. \tag{4.8}$$

Now

$$
\begin{aligned}
[\partial_i, V_v] &= \frac{\partial V}{\partial x^i}, \\
[\partial_j, V_s] &= \frac{\partial V}{\partial x^j}.
\end{aligned}
$$

Since potentials are spherically symmetric they depend only on r,

$$\frac{\partial V}{\partial x^i} = \frac{\partial V}{\partial r}(\hat{r})^i.$$

It is convenient to use the chiral representation of $\gamma-$matrices introduced in chapter 2. Then we can write the above equation in two component matrix form

$$[\hat{O} + i \begin{pmatrix} -\sigma.\nabla V_\nu & 0 \\ 0 & \sigma.\nabla V_\nu \end{pmatrix} - i \begin{pmatrix} 0 & \sigma.\nabla V_s \\ -\sigma.\nabla V_s & 0 \end{pmatrix}] \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = 0, \tag{4.9}$$

or

$$[\hat{O} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i\sigma.\hat{r} \begin{pmatrix} -\frac{dV_v}{dr} & \frac{dV_s}{dr} \\ -\frac{dV_s}{dr} & \frac{dV_v}{dr} \end{pmatrix}] \begin{pmatrix} \Psi_L \\ \Psi_R \end{pmatrix} = 0, \tag{4.10}$$

where

$$\hat{O} = \nabla^2 + V_v^2 - V_s^2 - 2EV_v - 2mV_s + (E^2 - m^2). \tag{4.11}$$

The diagonalization of the matrix multiplying $i\sigma.\hat{r}$, since $(\sigma.\hat{r})^2 = 1$, gives the eigenvalues η

$$(\frac{dV_v}{dr} - \eta)(-\frac{dV_v}{dr} - \eta) - (-(\frac{dV_s}{dr})^2) = 0,$$

or

$$\eta = \pm[(\frac{dV_v}{dr})^2 - (\frac{dV_s}{dr})^2]^{\frac{1}{2}}. \tag{4.12}$$

The matrix which diagonalizes this matrix does not effect the first term in the equation (4.9) as it is multiplied by a unit matrix which commutes with every matrix. Denoting the corresponding eigenfunctions by $\Psi_\pm$ which are linear combination of Ψ_L and Ψ_R, we have

$$[\hat{O} \pm i\eta\sigma.\hat{r}]\Psi_{\pm} = 0. \tag{4.13}$$

We note that the eigenvalues η in (4.12) vanish for

$$\frac{dV_v}{dr} = \pm\frac{dV_s}{dr},$$

or

$$V_v(r) = \pm V_s(r) + \mathrm{cons\,tan\,t}. \tag{4.14}$$

Then

$$\hat{O}\Psi_{\pm} = 0, \tag{4.15}$$

where $\hat{O}$ is independent of spin. As a result there is no spin-orbit coupling and the result obtained in section 4.2 is derived in a different and more transparent way. We can solve the equation

$$[\hat{O} \pm i\eta\sigma.\hat{r}]\Psi_{\pm} = 0, \tag{4.16}$$

exactly for the energy eigenvalues as in hydrogen atom discussed in chapter 2 for

$$\begin{aligned} V_v &= -\frac{\alpha_v}{r} + U_v, \\ V_s &= -\frac{\alpha_s}{r} + U_s. \end{aligned}$$

$V_v(r)$ is the OGE potential in which we are interested. The origin of $V_s(r)$ to be coulomb-like is not yet clear. It might arise due to multi-gluon exchanges but nobody so far has shown it to be so and U_v and U_s are constant potentials so that

$$\begin{aligned} \frac{dV_v}{dr} &= \frac{\alpha_v}{r^2}, \\ \frac{dV_s}{dr} &= \frac{\alpha_s}{r^2}. \end{aligned}$$

If $\alpha_s = \alpha_v$, there will be no spin-orbit splitting. For the above case we have

$$\eta = \frac{(\alpha_s^2 - \alpha_v^2)^{\frac{1}{2}}}{r^2}. \quad (4.17)$$

Now in spherical coordinates

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{r^2}.$$

Then the operator $\hat{O}$ becomes

$$\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + 2((E-U_v)\frac{\alpha_v}{r} + (m+U_s)\frac{\alpha_s}{r}) + [(E-U_v)^2 - (m+U_s)^2] + \frac{\alpha_v^2 - \alpha_s^2}{r^2} - \frac{L^2}{r^2}. \quad (4.18)$$

We can now redefine E and m as:

$$\begin{aligned} E' &= E - U_v, \\ m' &= m + U_s. \end{aligned} \quad (4.19)$$

Thus

$$\hat{O} = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{r^2} + \frac{\alpha_v^2 - \alpha_s^2}{r^2} + 2(E'\alpha_v + m'\alpha_s)\frac{1}{r} + (E'^2 - m'^2). \quad (4.20)$$

Accordingly the equation (4.16) becomes

$$[\hat{O} + i\frac{(\alpha_v^2 - \alpha_s^2)^{\frac{1}{2}}}{r^2}\sigma.\mathbf{r}] = 0. \quad (4.21)$$

Note $L^2 = l(l+1)$ and in the subspace provided by

$$\begin{aligned} l &= l_+ = j + \frac{1}{2}, \\ l &= l_- = j - \frac{1}{2}, \end{aligned}$$

the above operator is a 2×2 matrix

$$\frac{1}{r^2}\begin{pmatrix} (j+\frac{1}{2})(j+\frac{3}{2}) & i(\alpha_v^2-\alpha_s^2)^{\frac{1}{2}} \\ -i(\alpha_v^2-\alpha_s^2)^{\frac{1}{2}} & (j-\frac{1}{2})(j+\frac{1}{2}) \end{pmatrix}, \tag{4.22}$$

where we have used the fact that $\sigma.\hat{r}$ has only non-diagonal elements:

$$< l_\pm|\sigma.\hat{r}|l_\pm >= 0, \tag{4.23}$$

and $\sigma.\hat{r}$ has eigenvalues $\pm i$, as $(\sigma.\hat{r})^2 = 1$.

Diagonalization of this matrix gives the eigenvalues (as shown in chapter 2)

$$\frac{1}{r^2}\bar{\lambda}(\bar{\lambda}+1),$$

where

$$\begin{aligned} \bar{\lambda} &= [(j+\frac{1}{2}) - (\alpha_v^2-\alpha_s^2)]^{\frac{1}{2}}, \\ &= [(j+\frac{1}{2}) - (\alpha_v^2-\alpha_s^2)]^{\frac{1}{2}} - 1. \end{aligned}$$

We write

$$\bar{\lambda} = (j \pm \frac{1}{2}) - \delta_j, \tag{4.24}$$

with

$$\delta_j = (j \pm \frac{1}{2}) - [(j \pm \frac{1}{2})^2 - (\alpha_v^2-\alpha_s^2)]^{\frac{1}{2}}. \tag{4.25}$$

Comparing with the radial equation for the hydrogen atom [cf:(2.53) in chapter 2] where in our case

$$\begin{aligned} \alpha &\rightarrow \alpha_v\frac{E'}{m'} + \alpha_s, \\ E &\rightarrow \frac{E'^2 - m'^2}{2m'}, \\ l &\rightarrow \bar{\lambda} = (j+\frac{1}{2}) - \delta_j, \end{aligned} \tag{4.26}$$

$$n' \;=\; n-(l+1) \rightarrow n-(j+\frac{1}{2})+\delta_j-1.$$

Thus n must be shifted by δ_j i.e $n \rightarrow n-\delta_j$ and we obtain energy eigenvalues

$$\frac{E'^2-m'^2}{2m'} = -m' \frac{(\alpha_v \frac{E'}{m'}+\alpha_s)^2}{2(n-\delta_j)^2}.$$

This can be written as

$$E'^2(1+\frac{\alpha_v^2}{(n-\delta_j)^2})+2\frac{\alpha_v\alpha_s}{(n-\delta_j)^2}E'-m'^2(1-\frac{\alpha_s^2}{(n-\delta_j)^2})=0.$$

Solving this quadratic equation we have for E$'$

$$E' = m'[-\frac{\alpha_v\alpha_s}{(n-\delta_j)^2+\alpha_v^2} \pm [\frac{\alpha_v^2\alpha_s^2}{[(n-\delta_j)^2+\alpha_v^2]^2}+\frac{(n-\delta_j)^2-\alpha_s^2}{(n-\delta_j)^2+\alpha_v^2})]^{\frac{1}{2}}]. \tag{4.27}$$

This is an exact result. Re-substituting the values of primed quantities again in (4.19) we have

$$E = U_v + \frac{(m+U_s)}{(n-\delta_j)^2+\alpha_v^2}[-\alpha_v\alpha_s \pm [\alpha_v^2\alpha_s^2+[(n-\delta_j)^2+\alpha_v^2][(n-\delta_j)^2-\alpha_s^2]]^{\frac{1}{2}}]. \tag{4.28}$$

Consider the term

$$\begin{aligned}
& [\alpha_v^2\alpha_s^2+[(n-\delta_j)^2+\alpha_v^2][(n-\delta_j)^2-\alpha_s^2]]^{\frac{1}{2}}, \\
= \; & [\alpha_v^2\alpha_s^2+(n-\delta_j)^4+\alpha_v^2(n-\delta_j)^2-\alpha_s^2(n-\delta_j)^2-\alpha_v^2\alpha_s^2]^{\frac{1}{2}}, \\
= \; & (n-\delta_j)[(n-\delta_j)^2+\alpha_v^2-\alpha_s^2]^{\frac{1}{2}}.
\end{aligned}$$

Substituting it back in the equation

$$E = U_v + \frac{(m+U_s)}{(n-\delta_j)^2+\alpha_v^2}[-\alpha_v\alpha_s \pm (n-\delta_j)[(n-\delta_j)^2+\alpha_v^2-\alpha_s^2]^{\frac{1}{2}}], \tag{4.29}$$

where δ_j has been defined in (4.25).

4.4 Particular cases

(i) If $\alpha_s = 0$,$U_v = U_s = 0$, we recover the formula for hydrogen atom derived in chapter 2 [cf:(2.55)].

(ii) When $V_v = V_s$, we have $\alpha_v = \alpha_s$ and $U_v = U_s$ so that δ_j becomes zero

$$E = U + (m + U)(\frac{-\alpha_v^2 \pm n^2}{\alpha_v^2 + n^2}).$$

Taking only the positive sign in the numerator

$$E = U + (m + U)(1 - \frac{2\alpha_v^2}{\alpha_v^2 + n^2}). \tag{4.30}$$

As the expression for energy is independent of spin term so there is no spin orbit-splitting for this case.

We now discuss spin-orbit splitting based on (4.29) with δ_j [cf: (4.25)] given by

$$\delta_j = (j + \frac{1}{2}) - [(j + \frac{1}{2})^2 - (\alpha_v^2 - \alpha_s^2)]^{\frac{1}{2}}. \tag{4.31}$$

For our case $l = 1, n = 2$ and we have 1^+ and 2^+ mesons corresponding to $j = \frac{1}{2}$ and $j = \frac{3}{2}$. Since experimentally 2^+ mesons are heavier than 1^+ mesons, i.e $E_{2,\frac{3}{2}} > E_{2,\frac{1}{2}}$, one may conclude by looking at (4.31) that $\delta_{\frac{1}{2}} > \delta_{\frac{3}{2}}$. Then (4.31) implies that $(\alpha_v^2 - \alpha_s^2) > 0$ and further since $\delta_{\frac{1}{2}}$ is to be real, $(\alpha_v^2 - \alpha_s^2) < 1$.

For $V_v(r)$ we take OGE potential $V_v(r) = -\frac{\alpha_v}{r}$ i.e put $U_v = 0$ and

$$V_s(r) = U_s(r) - \frac{\alpha_s}{r},$$

Further the mass of $\bar{Q}q$ or $\bar{q}Q$ meson is given by $(n = 2)$

$$M_{2j} = M_Q + E_{2j}, \tag{4.32}$$

where M_Q is the mass of heavy quark. Then the mean mass of $j=\frac{1}{2}, j=\frac{3}{2}$ fermions is

$$\bar{M} = M_Q + \frac{1}{2}(E_{2,\frac{3}{2}} - E_{2,\frac{1}{2}}), \tag{4.33}$$

while the mass splitting is given by

$$\Delta M = (E_{2,\frac{3}{2}} - E_{2,\frac{1}{2}}), \tag{4.34}$$

we rewrite (4.29) (with $U_v = 0$) as

$$E_{2j} = (m + U_s) F_{2j}, \tag{4.35}$$

where

$$F_{2j} = \frac{1}{(2-\delta_j)^2 + \alpha_v^2}[-\alpha_v \alpha_s + (2-\delta_j)[(2-\delta_j)^2 + \alpha_v^2 - \alpha_s^2]^{\frac{1}{2}}]. \tag{4.36}$$

To carry out the numerical work we have to fix m (light quark (u or s) mass), M_Q, α_v, α_s and U_s, where it is known from the mass spectra of $s = 0$ mesons that

$$\begin{aligned} m &= m_u = 330MeV, \\ m_s &= 550MeV, \\ M_c &= 1480MeV, \\ M_b &= 4800MeV, \\ \alpha_v &= \frac{4}{3}\alpha_G, \end{aligned}$$

where α_G is the QCD quark-gluon coupling constant. This coupling is energy dependent, decreasing with increasing energy. Thus one would expect

$$\alpha_v(M_{B_s}) < \alpha_v(M_B) < \alpha_v(M_D),$$

we will take this into consideration.

It is clear from (4.33), (4.35) and (4.36) that once we have calculated $F_{2j}, (m + U_s)$

can be fixed from (4.33). The experimental information on 1^+ and 2^+ is summarized in Table:1.

Table 4.1: Experimental Data

$Mesons$	$Mass$	$Mean\ Mass$	$Mass\ Splitting$
	(MeV)	$M\ MeV$	$\Delta M\ MeV$
$D_1^*(1^+)$	2422	2442	39
$D_2^*(2^+)$	2461		
$B_1(1^+)$	5720	5733	27
$B_2(2^+)$	5747		
$B_{s_1}(1^+)$	5829	5834	11
$B_{s_2}(2^+)$	5840		

Our numerical results are summarized in Tables 2 and 3 for two set of values for α_v and α_s.

Table 4.2: Mass Splitting with $\alpha_s = 0$

$Flavour$	α_v	$F_{\frac{3}{2}} - F_{\frac{1}{2}}$	$\frac{F_{\frac{3}{2}} + F_{\frac{1}{2}}}{2}$	$m + U_s$	$\Delta M(our)$	$\Delta M(expt)$
D	0.85	0.039	0.885	1085	42MeV	39MeV
B	0.8	0.022	0.905	1031	23MeV	26MeV
B_s	0.7	0.011	0.937	1104	12MeV	11MeV

Table 4.3: Mass Splitting with $\alpha_s = 0$

$Flavour$	α_v	$F_{\frac{3}{2}} - F_{\frac{1}{2}}$	$\frac{F_{\frac{3}{2}} + F_{\frac{1}{2}}}{2}$	$m + U_s$	$\Delta M(our)$	$\Delta M(expt)$
D	0.8	0.03	0.835	1152	34MeV	39MeV
B	0.75	0.0215	0.853	1094	23.5MeV	26MeV
B_s	0.65	0.011	0.883	1171	12.5MeV	11MeV

To conclude it is possible to explain the spin-orbit mass splitting, using Dirac equation, with reasonable set of values for α_v, although these values are somewhat larger than those obtained from asymptotic freedom of QCD coupling. However, in potential models α_v and α_s is treated usually as a phenomenological parameter to be fixed by data. We have taken α_s as flavor independent, which is usually taken for confining potential. From tables 2 and 3 , $(m + U_s)$ is almost independent of flavor with a few percent.

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