Primordial Black Holes in an Accelerating Universe - Revisited

by

Sumaira Naz

Centre for Advanced Mathematics and Physics (CAMP)
National University of Sciences and Technology (NUST)
H-12, Islamabad, Pakistan

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Prof. Dr. Asghar Qadir

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Dedicated

To my dear brothers
JAWAD and SHAZAD,
our little angel
SAMAN
and my best friend
SARA

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Sumaira Naz

Abstract

Hawking had shown that black holes radiate with a temperature inversely proportional to their mass, thereby losing energy and hence mass. For sufficiently small masses (less than 10^{15}g) the black hole would evaporate today and hence has a "life" equal to the present age of the universe. The study of primordial black holes (thought to be formed within 1s after big bang) is of cosmological interest because of their wide mass range (10^{-5}g to 10^{38}g). Evaporation rate becomes significant due to their small mass. In particular, primordial black holes of mass $\sim 10^{15}$g should be evaporating today. Since this is not observed, it could be concluded that primordial black holes of this mass did not form in adequate number or they might have accreted matter and radiation and grown in size instead of vanishing. Dark energy is considered responsible for the accelerated expansion of the universe. Observational data support the existence of an exotic form of dark energy called phantom energy. It decreases the mass of a black hole by accretion when it is taken to be a perfect fluid. In 2010 Jamil and Qadir showed that this energy enhances the rate of evaporation. Thus, to have a primordial black hole evaporating today, its initial mass should be larger than 10^{15}g or the primordial black holes of mass $\sim 10^{15}$g should evaporate earlier. Infact, it was claimed that a primordial black hole would be ten orders of magnitude larger to have it evaporating today. This effect is revisited and the correction term in the lifetime for primordial black holes is computed. It is found that the effect of phantom energy on the lifetime of a 10^{15}g primordial black hole is negligible but the evaporation rate due to phantom energy accretion is considerable for higher mass black holes ($\sim 10^{40}$g). The mass at which the effect of phantom energy equals the effect of Hawking radiation (called the transition mass) has also been discussed.

Contents

Chapter 1

Introduction

A black hole (BH) is the highest possible density object with a gravitational field that is so intense that once captured not even light can escape from its influence. Thus the light falling in cannot be reflected back to make the BH visible. Therefore BHs can grow in mass by accreting matter and radiation. The boundary of the region around a BH in which it can "engulf" the particles and the objects, is called event horizon. BHs are formed due to the gravitational collapse of any stellar object like a massive star and are considered to be residing at the centre of each galaxy and in galactic haloes. Their mass ranges from 10^{39}g to 10^{43}g. BHs are also considered to be formed within the first second of the universe from primordial density fluctuations without the formation of any other stellar object initially. These are called primordial black holes (PBHs) and the range of their initial mass (mass at the time of formation) is $\sim 10^{-5}$g to $\sim 10^{33}$g. Once a BH achieves a stable condition after formation, its mass, charge and angular momentum are the three physical macroscopic parameters which describe it completely. BHs are considered to be the most perfectly thermal objects but the description of their thermal properties needs a few microscopic parameters that are not known [1].

The formulation of general relativity (GR) and quantum mechanics (QM) in the 20^{th} century were turning points in the history of physics. However, the two "revolutions" do not support each other. Although they seem compatible to some extent but, while relativity provides an excellent description of mechanics in the presence of gravity, there is no quantum field theory of gravity constructed yet. Strictly speaking, no attempts to unify GR and QM have met with any significant success [2].

In 1974, Hawking argued that BHs can emit radiation due to quantum effects resulting in decrease of mass. Infact, isolated BHs evaporate completely in a finite time [3]. The radiation that is emitted in this process is called Hawking radiation (HR). Hence QM, GR and thermodynamics are discussed simultaneously for the first time in the context of HR.

The only type of matter known till 1933 was luminous matter. But the problem of "missing matter" (to be discussed later in detail) and the phenomenon of gravitational lensing (bending of light by invisible objects) led to the concept of dark matter. Luminous matter is 4 percent and dark matter is 23 percent of the total mass-energy of the universe.

Dark energy is a hypothetical form of energy which accounts for 73 percent of the total mass-energy of the universe. The equation of state (EoS) parameter for dark energy is defined as: $\omega \equiv p/\rho c^2$. Its values $\frac{1}{3}$, 0 and -1 correspond to the "radiation", "dust" and "cosmological constant" respectively (explanation is given in Chapter 2). There are indications that when this dark energy is taken to be a perfect fluid, any value of ω less than $-\frac{1}{3}$ (to be discussed in detail later) explains the acceleration in the expansion rate of the universe. On the basis of certain astronomical data, Caldwell and his co-workers suggested $\omega < -1$, which violates the dominant energy condition $\rho c^2 \geq |p|$. This type of dark energy is called phantom energy [4]. The accretion of phantom energy onto any gravitationally bound object may change the physics of several relevant phenomena. It has been shown that phantom energy can even cause the universe to rip apart, causing a "cosmic dooms day" [5].

In this dissertation, I study the effect of accretion of phantom energy on PBHs in addition to HR (ignoring the matter and radiation accretion). For simplicity, I have taken the case of uncharged and non-rotating PBHs. Although there is no evidence of the formation of PBHs, their study is of great interest because their formation could provide a unique probe of the early Universe (if $M_{PBH} < 10^{15}$ g), gravitational collapse (if $M_{PBH} > 10^{15}$ g) and high energy physics (if $M_{PBH} \sim 10^{15}$ g).To have an isolated PBH evaporating today through HR, its initial mass would have been less than 10^{15}g and it must have formed within the first $\sim 10^{-23}$s [3]. Any PBH of initial mass higher than 10^{15}g will evaporate later. It was shown that the accretion of phantom energy on a BH, in contrast to the accretion of matter and radiation, effectively decreases its mass [7, 8, 9, 10]. Due to the smaller mass of PBHs evaporating today (i.e., PBHs of initial mass 10^{15}g), they can exhibit significant behavior under the accretion of phantom energy [3]. In particular, Jamil and Qadir [10] have discussed the decrease in mass of a PBH due to the combined effect of accretion of phantom energy and Hawking evaporation. A relation between the time and mass of a PBH (which evaporates at $t = t_o$) was derived and it was said that the initial mass of a PBH is required to be increased by 10 orders of magnitude in order to have it evaporating today when the effect of phantom energy is also taken in to account in addition to HR. Here I revisit the effect of the accretion of phantom energy on the value of initial mass of PBHs evaporating today. This work is related to the qualitative study of GR, HR, BHs (PBHs in particular) and phantom energy. Thus I shall review all these topics in more detail to provide necessary background.

The plan of the dissertation is as follows. In the remaining part of Chapter 1, I shall briefly describe some concepts of Special Relativity (SR) needed for the later discussion of GR. I shall discuss Cosmology in Chapter 2 and HR and PBHs in Chapter 3. The effect of phantom energy accretion on a BH is reviewed in Chapter 4 referring to 3 papers [8, 9, 10]. I have reviewed [10] thoroughly and pointed a few mistakes of the paper. The corrections of these mistakes lead to considerable changes in the results of [10] and are given in Chapter 5. Then, in the same chapter, I shall give the calculations of the *transition mass* and the analysis done to find the value of initial mass of a PBH evaporating today under the combined effect of HR and phantom energy accretion from [11]. Also, I shall discuss some recent work done to find the limit of the mass of a BH to exhibit observable effect of phantom energy by its evaporation [1]. The last chapter is a summary of my research work. Also, roman indices run over 1, 2, 3 and Greek indices run over 0, 1, 2, 3 in this dissertation.

1.1 Brief Review of Special Relativity

Special relativity is the theory about relation of observations of physics of all the phenomena in different inertial frames. It connects space and time, matter and energy, electricity and magnetism and provides links that are crucial to our understanding of the physical universe [13]. SR has made a number of predictions which have been proved to be true experimentally like time dilation ("faster clock ticks slower": established in an experiment using atomic clocks) etc. Here I shall

[1] I am working on this project with Maqbool Ahmed. A part of this work has been published in [12]. Here I shall give the results obtained so far.

describe the concepts which are introduced in SR and are also used in GR. But first I shall state the two basic principles of SR namely principle of relativity and constancy of the speed of light.

1.1.1 Principle of Relativity

This principle states that the laws of physics are the same in all inertial frames of reference.

This principle discards the idea of existence of any universal frame of reference. This can be understood by the argument that if the laws of physics were different for any two observers in relative motion, the observers could find from these differences which of them was “stationary” in space and which was “moving”.

1.1.2 Constancy of Speed of Light

This principle states that the speed c of light in free space has the same value in all inertial frames of reference.

As a consequence, c is the upper bound for the speed of all objects.

1.1.3 Relativity of Simultaneity

One of the very important concepts introduced by SR is the relativity of simultaneity, i.e., the events which are simultaneous for one inertial observer are not simultaneous for other inertial observers [14].

Consider two observers A and B. Consider the rest frame of A and let B be moving with uniform velocity with respect to A as shown in Fig. 1.1. Fig. 1.2 shows two of the possible world lines for light signals in this frame. I have taken one unit equal to one light second and one second on horizontal and vertical axis respectively. Therefore all the possible world lines for light are parallel lines at angle 45^o or 135^o with the positive X-axis.

Now consider two events P and Q, which occur simultaneously with respect to A as shown in Fig. 1.3. T_1P and T_1Q are the light signals sent by A at time t_1 to illuminate P and Q respectively. Clearly, A receives the two signals PT_2 and QT_2 simultaneously (say at time t_2). But for B, the matter is not the same. Fig. 1.4 shows that B can illuminate P and Q by sending light signals T_3P and T_4Q at different times, say t_3 and t_4 respectively. B receives these signals QT_5 and

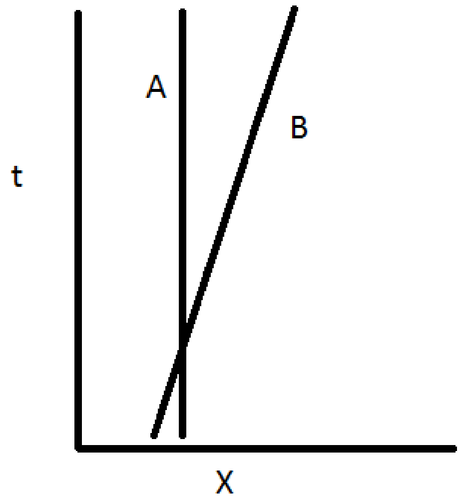

Figure 1.1: Rest frame of the observer A; observer B moves with velocity $\underline{v}$ in the X-direction in this frame .

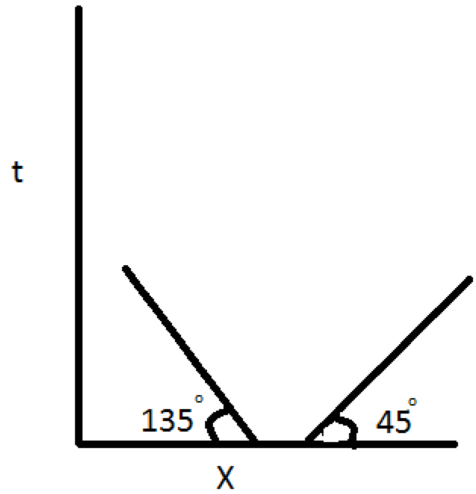

Figure 1.2: World lines of all possible light signals (with respect to angle) that can be sent on either side of direction of motion .

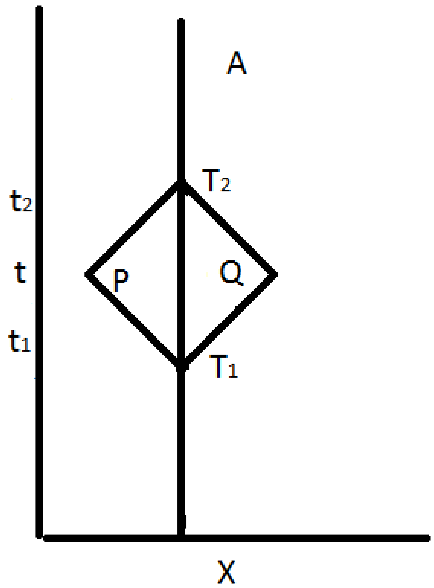

Figure 1.3: Observation made by A; both events (P and Q) are simultaneous since the light signals are received at the same time t_2 .

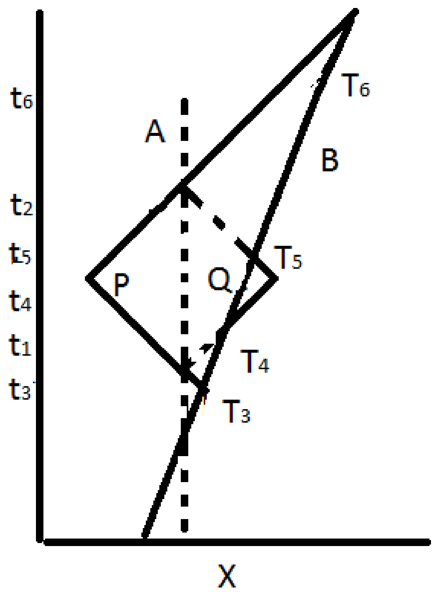

Figure 1.4: Observation made by B; both events (Q and P) are non-simultaneous since the light signals are received at different time t_5 and t_6 respectively .

PT_6 at different times t_5 and t_6 respectively. Therefore for B, Q occurred before P. Hence it is concluded that simultaneity is relative.

This relativity of simultaneity led Einstein to abolish the concept of the coordinates being spatial only. Thus, time cannot be taken absolute as was assumed since Newton. It is rather taken as a coordinate changing the 3−dimensional Newtonian space to a 4−dimensional Minkowski spacetime $(ct,\ x,\ y,\ z) \equiv (x^0,\ x^1,\ x^2,\ x^3)$ where $x^0 \ = \ ct$, c being speed of light.

1.1.4 Interval

In Galilean space, the square of the distance between two points (interval) $(x_1,\ y_1,\ z_1)$ and $(x_2,\ y_2,\ z_2)$ is given by

$$I = s^2 \ = \ (x_1 \ - \ x_2)^2 \ + \ (y_1 \ - \ y_2)^2 \ + \ (z_1 \ - \ z_2)^2 \ . \tag{1.1.1}$$

When time is also included as a coordinate (Minkowski spacetime) then the square of the distance between two events $(t_1,\ x_1,\ y_1,\ z_1)$ and $(t_2,\ x_2,\ y_2,\ z_2)$ is given by

$$s^2 \ = \ c^2(t_1 \ - \ t_2)^2 \ - \ (x_1 \ - \ x_2)^2 \ - \ (y_1 \ - \ y_2)^2 \ - \ (z_1 \ - \ z_2)^2 \ . \tag{1.1.2}$$

When the events are separated infinitesimally, the interval can be written as

$$ds^2 \ = \ c^2dt^2 \ - \ dx^2 \ - \ dy^2 \ - \ dz^2 \ . \tag{1.1.3}$$

This interval provides geometry for SR.

1.1.5 Lorentz Transformation

Consider two observers A and B with their rest frames S with coordinates $(t,\ x,\ y,\ z)$ and S' with coordinates $(t',\ x',\ y',\ z')$ respectively such that S' is moving with velocity $\mathbf{v}$ with respect to S along the X-axis. Let A fire a shot in the direction of the negative X-axis with velocity $-\mathbf{u}$ with respect to his rest frame S. Then by the Galilean transformation law of velocities (derived from the Galilean transformation of coordinates; $x' \ = \ x \ - \ x_0$, x_0 being the distance of S' from S), velocity of the bullet accordingto B (say $\mathbf{u}'$) is

$$\mathbf{u}' \ = \ - \ \mathbf{u} \ - \ \mathbf{v} \ . \tag{1.1.4}$$

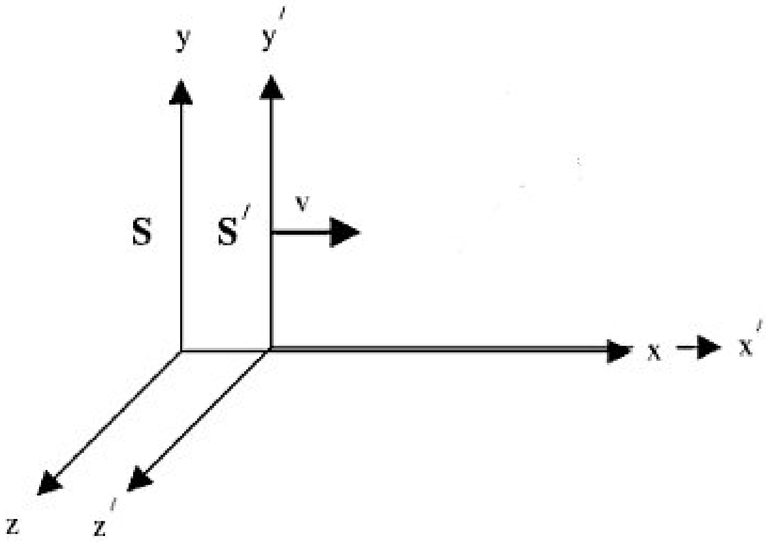

Figure 1.5: Rest frame S of the observer A in which rest frame S' of the observer B moves with velocity $\underline{v}$ in the X-direction .

Now suppose that A sends a light signal in the direction of the negative X-axis, then by the above equation we would conclude that B should find the speed of the light signal to be greater than c. But this is inconsistent with the basic postulate of SR about the speed of light being the same for all inertial observers. Moreover, the Galilean transformation takes time to be absolute. Therefore we cannot use the Galilean transformation in the formulation of SR. Thus we derive a new transformation law for coordinates to use in place of the Galilean transformation laws. These transformation laws were derived by Lorentz and therefore are called Lorentz transformations.

By the principle of relativity, if A sees a free particle traveling in a straight line with constant speed, then so will B. Since straight lines are mapped onto straight lines, it suggests that the transformation between the two frames is linear. Thus, the transformation from frame S to S' can be written as [14]

$$x'^{\mu} = L^{\mu}_{\nu} x^{\nu} , \tag{1.1.5}$$

where L is a 4×4 matrix whose components depend only on v and c. As v has been taken in the direction of the X-axis

$$y' = y , \ z' = z . \tag{1.1.6}$$

Let both the observers start their clocks ($t' = t = 0$) and emit a light signal when origins of S and S' coincide. 'Coincide' means that there is no displacement along the X-axis but a negligible displacement along the axis which is perpendicular to the X-axis [15]. According to S, the light signal moves radially outward from the origin with speed c. The wavefront of light will constitute a sphere satisfying the following condition for the events comprising this sphere

$$s^2 = c^2t^2 - x^2 - y^2 - z^2 = 0 . \tag{1.1.7}$$

S' also must see the spherical wavefront moving out with speed c (according to the 2^{nd} postulate of SR) and satisfying following condition

$$s'^2 = c^2t'^2 - x'^2 - y'^2 - z'^2 = 0 . \tag{1.1.8}$$

Thus, under a transformation connecting the two frames

$$s = 0 \Leftrightarrow s' = 0 , \tag{1.1.9}$$

Since the transformation is linear

$$s = ns' . \tag{1.1.10}$$

Now assuming that the space is isotropic (i.e. the same in all directions) which allows me to reverse the direction of the axes. Moreover, considering the motion from B's point of view the principle of relativity becomes

$$s' = ns . \tag{1.1.11}$$

Eqs. (1.1.10) and (1.1.11) imply that $n^2 = 1$ or $n = \pm 1$. In the limit as $v \longrightarrow 0$, the two frames coincide and $s' \rightarrow s$. Thus, it can be concluded that $n = 1$ or $s = s'$. Using Eqs. (1.1.6) gives

$$c^2t^2 - x^2 = c^2t'^2 - x'^2 . \tag{1.1.12}$$

Now introducing imaginary time coordinates T and T' defined by

$$\begin{aligned} T &= ict , \\ T' &= ict' . \end{aligned} \tag{1.1.13}$$

Using Eqs. (1.1.13) in Eq. (1.1.12) gives

$$x^2 \ + \ T^2 \ = \ x'^2 \ + \ T'^2 \ . \tag{1.1.14}$$

In the 2-dimensional $(x, \ T)$ space, the quantity $x^2 \ + \ T^2$ represents the distance of a point from the origin and it will remain invariant only under a rotation in the $(x, \ T)$ space. Suppose the rotation is in anti-clockwise direction and by an angle θ. The transformed coordinates are

$$\begin{aligned} x' &= xcos\theta \ + \ Tsin\theta \ , \\ T' &= - \ xsin\theta \ + \ Tcos\theta \ . \end{aligned} \tag{1.1.15}$$

The origin of S' satisfies

$$x' \ = \ 0 \tag{1.1.16}$$

and

$$x \ = \ vt \ . \tag{1.1.17}$$

Using the first of equation (1.1.13) gives

$$x \ = \ - \frac{ivT}{c} \ . \tag{1.1.18}$$

Using Eqs. (1.1.16) and (1.1.18) in first equation of (1.1.15) we find

$$\tan\theta \ = \ \frac{iv}{c} \ . \tag{1.1.19}$$

Now

$$\begin{aligned} \gamma \quad &\equiv \ \cos\theta \ = \ \frac{1}{\sec\theta} \\ \cos\theta \quad &= \ \frac{1}{\sqrt{1 \ + \ \tan^2\theta}} \\ &= \ \frac{1}{\sqrt{1 \ - \ \frac{v^2}{c^2}}} \ . \end{aligned} \tag{1.1.20}$$

Eq. (1.1.15) can be written as

$$\begin{aligned} x' &= \ \cos\theta(x \ + \ T\tan\theta) \ , \\ T' &= \ \cos\theta(- \ x\tan\theta \ + \ T) \ . \end{aligned} \tag{1.1.21}$$

Using Eqs. (1.1.19) and $\gamma \equiv \cos\theta$ in above equation

$$x' = \gamma[x + ict(iv/c)]$$
$$= \gamma(x - vt) , \tag{1.1.22}$$
$$T' = \gamma[- x(iv/c) + ict] . \tag{1.1.23}$$

Since $T' = ict'$, therefore

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) . \tag{1.1.24}$$

Eqs. (1.1.6), (1.1.22) and (1.1.24) with the value of γ given by Eq. (1.1.20) are the Lorentz transformation laws for coordinates.

1.1.6 Law of Velocity Transformation

Law of velocity addition derived from Lorentz transformation must be consistent with the postulates of SR. In differential form, Lorentz transformations can be written as [15]

$$\begin{aligned} dt' &= \gamma(dt - vdx/c^2) , \\ dx' &= \gamma(dx - vdt) , \\ dy' &= dy , \\ dz' &= dz . \end{aligned} \tag{1.1.25}$$

Now the components of velocity of any object in frame S are

$$u_x = \frac{dx}{dt} , \ u_y = \frac{dy}{dt} , \ u_z = \frac{dz}{dt} . \tag{1.1.26}$$

In frame S', the components of the velocity are

$$u'_x = \frac{dx'}{dt'} , \ u'_y = \frac{dy'}{dt'} , \ u'_z = \frac{dz'}{dt'} . \tag{1.1.27}$$

Using Eqs. (1.1.25), we get following equations for transformation of components of velocity of any object from frame S to S':

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - u_x v/c^2} , \\ u'_y &= \frac{u_y\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} , \\ u'_z &= \frac{u_z\sqrt{1 - v^2/c^2}}{1 - u_x v/c^2} . \end{aligned} \tag{1.1.28}$$

Now taking $u_x = -c\hat{x}$ (where $\hat{x}$ denotes the direction of the X-axis) in Eq. (1.1.29) gives $u'_x = -c\hat{x}$ which is consistent with the second postulate of SR. Thus, contrary to the Galilean transformation, Lorentz transformations give results that are consistent with the basic postulates of SR. Rearranging the first equation gives the transformation law of the x-component of velocity from frame S' to S

$$u_x = \frac{u'_x + v}{1 + u'_x v/c^2} . \tag{1.1.29}$$

which is needed for later use.

1.1.7 Relativistic Mass

In classical mechanics, mass of an object is also constant like absolute time and is defined as the quantity of matter. In relativistic mechanics, this concept of mass is also modified to "relativistic mass" by considering the mass to be an observer dependent quantity. To prove this, I am following Einstein's original derivation in more modern notation.

Consider two identical particles which collide inelastically and stick together after collision. Consider two reference frames K and K' such that in the former frame one of the particles is at rest and other one is moving with velocity u and after collision the combined particle moves with velocity U. Let K' be the center-of-mass frame. Thus, in K' both particles move with equal and opposite velocities U before collision and the combined mass is at rest after collision. Obviously then K' is moving with velocity U with respect to K. Let m_0, m_u and M_U be the masses of particles with velocities 0, u and U respectively in frame K and M_0 be the rest mass of combined object in frame K'. Assuming the conservation of mass and linear momentum in frame K, we can write

$$m_0 \; + \; m_u \quad = \; M_U \; , \tag{1.1.30}$$

$$0 \; + \; m_u u \quad = \; M_U U \; . \tag{1.1.31}$$

Eliminating M_U from both equations

$$m_u \; = \; m_0 \left(\frac{U}{u \; - \; U} \right) \; . \tag{1.1.32}$$

Consider one of the particles that is moving with velocities u and U in frames K and K' respectively. Now K' is itself moving with velocity U with respect to K. Thus law of addition of velocities, stated in Eq. (1.1.29), gives

$$u \; = \; \frac{2U}{1 \; + \; \frac{U^2}{c^2}} \; . \tag{1.1.33}$$

Solving Eq. (1.1.33) for U gives quadratic equation

$$U^2 \; - \; \left(\frac{2c^2}{u} \right) U \; + \; c^2 \; = \; 0 \; , \tag{1.1.34}$$

which has roots

$$\begin{aligned} U \quad &= \; \frac{c^2}{u} \; \pm \; \left[\left(\frac{c^2}{u} \right)^2 \; - \; c^2 \right]^{\frac{1}{2}} \\ &= \; \frac{c^2}{u} \left[1 \; \pm \; \left(1 \; - \; \frac{u^2}{c^2} \right)^{\frac{1}{2}} \right] \; . \end{aligned} \tag{1.1.35}$$

In the limit $u \rightarrow 0$, this must produce finite result. This suggests to take negative sign. Substituting the value in Eq.(1.1.32)

$$m_u \; = \; \frac{m_0}{\sqrt{1 \; - \; \frac{u^2}{c^2}}} \; . \tag{1.1.36}$$

Thus, it can be inferred from Eq. (1.1.36) that the mass of a relativistic particle increases indefinitely with increase in its speed. As $u \rightarrow c$, inertia (resistance to acceleration) tends to infinity. This is naturally expected if c is to be the maximum speed. Thus, the concept of relativistic mass is consistent with the constancy of speed of light.

1.1.8 Relativistic Linear Momentum

Relativistic linear momentum p is defined as the product of relativistic mass and velocity (m_u and u respectively with reference to the previous section).i.e.

$$\begin{aligned}\mathbf{p}(\mathbf{u}) &= m_u \mathbf{u} \\ &= \gamma m_0 \mathbf{u} \qquad (1.1.37) \\ &= \frac{m_0 \mathbf{u}}{\sqrt{1 - \frac{\mathbf{u}^2}{c^2}}} , \qquad (1.1.38)\end{aligned}$$

where γ is defined by Eq. (1.1.20). It is quite obvious that this expression also reduces to the classical Newtonian momentum when $u << c$.

1.1.9 Relativistic Second Law of Motion

In relativity, second law of motion is defined as

$$\begin{aligned}\mathbf{F} &= \frac{d\mathbf{p}}{dt} \\ &= \frac{d}{dt}(\gamma m_0 \mathbf{u}) , \qquad (1.1.39)\end{aligned}$$

where Eq. (1.1.37) has been used. This is more complicated than the classical formula $F = ma$ because γ is a function of velocity. Eq. (1.1.39) also reduces to the classical formula in the limit $v << c$.

1.1.10 Mass-Energy Equivalence

Having developed the concepts of relativistic mass, relativistic momentum and relativistic second law of motion, I can now move to the famous Einstein mass-energy equivalence relation). Consider a particle with rest mass m_0, relativistic mass $m_u = \gamma m_0$ which is moving along the X-axis under the effect of a variable force F acting in the same direction. Assuming that there is no other force acting on the particle, the work done on the particle can be written as

$$W = Fs . \qquad (1.1.40)$$

Since the work done by the force is being converted in the kinetic energy of the particle, thus

$$K_r = \int_0^s F \, ds , \qquad (1.1.41)$$

where K_r is the relativistic kinetic energy. The non-relativistic mechanics kinetic energy (K) of such a particle is

$$K = \frac{1}{2} m_0 u^2 \ , \tag{1.1.42}$$

Substituting the relativistic expression of force given in Eq. (1.1.39) in Eq. (1.1.41) gives relativistic formula for kinetic energy to be

$$\begin{aligned} K_r &= \int_0^s \frac{d(\gamma m_0 u)}{dt} \, ds \\ &= \int_0^{mu} u d(\gamma m_0 u) \\ &= m_0 \int_0^u u \, d(\gamma u) \ , \end{aligned} \tag{1.1.43}$$

where $ds/dt = u$ is used. Now integrating Eq. (1.1.43) by parts (i.e. using $\int x \, dy = xy - \int y \, dx$) gives

$$\begin{aligned} K_r &= \gamma m_0 u^2 - m_0 \int_0^u 1/\sqrt{1 - \frac{u^2}{c^2}} \, u \, du \\ &= \frac{m_0 u^2}{\sqrt{1 - \frac{u^2}{c^2}}} + \left[m_0 c^2 \sqrt{1 - \frac{u^2}{c^2}} \, \right]_0^u \\ &= \frac{m_0 c^2}{\sqrt{1 - \frac{u^2}{c^2}}} - m_0 c^2 \\ &= \gamma m_0 c^2 - m_0 c^2 \\ &= mc^2 - m_0 c^2 \ . \end{aligned} \tag{1.1.44}$$

The above equation shows that the kinetic energy of a relativistic particle is equal to the difference between $\gamma m_0 c^2$ and $m_0 c^2$. Thus, mc^2 can be interpreted as the total energy of the particle. Rearranging Eq. (1.1.44) gives

$$E \equiv mc^2 = m_0 c^2 + K_r \ . \tag{1.1.45}$$

Thus, if the particle is at rest and $K_r = 0$, the particle has non-zero energy $E = m_0 c^2$. This energy $m_0 c^2$ is called the rest mass energy of the particle which shows that mass and energy are equivalent concepts and there corresponds a huge amount of energy to a small amount of mass at rest. Hence Eq. (1.1.45) is called the relativistic mass-energy equivalence formula.

1.2 General Relativity as a Field Theory of Gravity

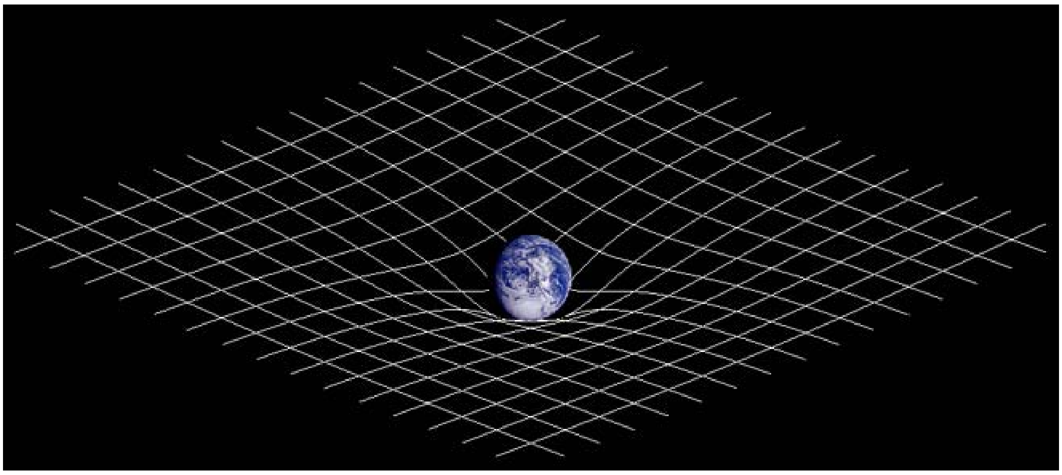

Figure 1.6: The spacetime curvature caused by the presence of matter [16] .

Special relativity deals with inertial frames and constant speeds (i.e. no acceleration) only. But objects moving under the effect of gravity are accelerated. Therefore Einstein did not stop at the formulation of SR but went on working for generalizing his special or "restricted" theory to such a theory which could incorporate acceleration as well. In 1915, he finally succeeded in the formulation of such a (geometrical) theory called the General Theory of Relativity.

GR describes gravity as the curvature of spacetime caused by matter and energy, due to which objects follow a curved path. It is described by 10 coupled partial differential equations called the Einstein Field Equations (EFEqs). But before going to these equations, I shall describe basic principles of GR which can be regarded as the generalization of the basic principles of SR.

1.2.1 Principles of GR

General relativity is based on two principles: principle of equivalence and principle of general covariance.

Principle of Equivalence

Inertial mass of a body is its resistance to any change in its motion caused by an external force. In Newton's 2nd law of motion this mass (m_i) is related to the applied force (F) and resultant acceleration ($\mathbf{a}$) as

$$\mathbf{F} = m_i \mathbf{a} . \tag{1.2.1}$$

Gravitational mass, m_g of an object is the measure of the force with which earth attracts object and is given as

$$\mathbf{F} = -\frac{GMm_g}{r^3}\mathbf{r} , \tag{1.2.2}$$

where M is the mass of the earth, G is gravitational constant and its value is $6.67 \times 10^{-8}\mathrm{cm}^3\mathrm{g}^{-1}\mathrm{s}^{-2}$, $\mathbf{r}$ is the vector from the gravitational source of mass M to the place where a test particle of mass m_g is situated and r is the magnitude of $\mathbf{r}$. The principle of equivalence states that the gravitational mass and inertial mass are equivalent.

Principle of General Covariance

This principle is the generalization of the principle of relativity stated in section 1.1.1 and is stated as: all frames of reference are physically equivalent. This stresses that all valid physical laws should be written in tensorial form. Change of a frame of reference generally corresponds to a non-linear coordinate transformation but the converse is not always true since coordinate systems can be transformed without changing the frame of reference, e.g., transformation from cartesian coordinates to the spherical coordinates does not change the frame of reference.

1.2.2 Metric Tensor and Stress-Energy Tensor

General relativity is a geometrical theory formulated in terms of tensors. Therefore before going to EFEqs, I shall define the following two tensors: the metric tensor ($g^{\alpha\beta}$) and the stress-energy tensor ($T^{\alpha\beta}$). The former gives the coordinates of a spacetime in terms of mass and gravity while the latter is the generalization of pressure (or force), as will be shown in the next few sections of this chapter. The concept of both these tensors is very important for relating spacetime curvature to gravity.

Metric Tensor

The interval between two events in a 4-dimensional spacetime with Cartesian coordinates $(x^0,\ x^1,\ x^2,\ x^3)$ and $(x^0 + dx^0,\ x^1 + dx^1,\ x^2 + dx^2,\ x^3 + dx^3)$ is defined as

$$ds^2 \ = \ (dx^0)^2 \ - \ (dx^1)^2 \ - \ (dx^2)^2 \ - \ (dx^3)^2 \ , \qquad (1.2.3)$$

Eq. (1.2.3) retains the same form under Lorentz transformation which transforms one inertial reference frame to the other inertial frame. However when we transform from an inertial frame to a non-inertial frame, the interval ds can not be written in a simplified form like Eq. (1.2.3) (i.e. the sum of the squares of the coordinates), rather it appears in a quadratic form which can in general be written as [17]

$$ds^2 \ = \ g_{\mu\nu} dx^\mu dx^\nu \ , \qquad (1.2.4)$$

where g_{ik} are the coefficients which are functions of the spacetime coordinates $(x^0,\ x^1,\ x^2,\ x^3)$ and are defined as the components of a tensor, called metric tensor. g_{ik} is clearly symmetric in change of indices as dx^i commute. Thus there are 10 coordinates of g_{ik}. Comparing Eqs. (1.2.3) and (1.2.4), we can write for inertial frames $g_{00} = 1,\ g_{11} = g_{22} = g_{33} = -1,\ g_{ik} = 0 \ \forall \ i \neq k$.

Geometrical properties of the spacetime are not fixed properties of the spacetime, rather they are determined by the gravitational field of the objects (or particles) present. In a gravitational field, values of metric tensor components cannot be reduced to their values in Galilean space globally by any transformation. This type of spacetime is called curved. Following are a few quantities defined in terms of the metric tensor.

In flat spacetime, we know the transformation laws from one point to the other. Now we need to specify how basis vectors change from the point $(x^0,\ x^1,\ x^2,\ x^3)$ to the neighboring point $(x^0 + dx^0,\ x^1 + dx^1,\ x^2 + dx^2,\ x^3 + dx^3)$ in curved spacetime. Let $\mathbf{e}_\alpha$ are the basis vectors. The transformation law is [18]

$$d\mathbf{e}_\mu \ = \ \Gamma^\lambda_{\mu\nu} \mathbf{e}_\lambda dx^\nu \ , \qquad (1.2.5)$$

where the coefficients Γ^k_{ij} are called *Christoffel symbols*. Physically, Christoffel symbols describe parallel transport in manifolds. Alternatively, Christoffel symbols are also defined as follows. If the derivative of a basis vector with respect to any coordinate axis is written as linear combination of basis vectors, Christoffel symbol (of $2nd$ kind) are the coefficients of basis vectors [19]

$$\frac{\partial \mathbf{e}_\alpha}{\partial x_\beta} = \Gamma^\mu_{\alpha\beta}\mathbf{e}_\mu \ . \tag{1.2.6}$$

Mathematically this is the same as is Eq. (1.2.5). Since the components of the metric tensor are the coefficients of the coordinate axis, Christoffel symbols can be defined in terms of the metric tensor as

$$\Gamma^\mu_{\nu\gamma} = \frac{1}{2} g^{\mu\rho}[g_{\rho\nu,\gamma} + g_{\rho\gamma,\nu} - g_{\nu\gamma,\rho}] \ , \tag{1.2.7}$$

where $_{,k}$ represents partial derivative with respect to x^k.

Riemann curvature tensor describes the curvature of spacetime. It is defined in terms of Christoffel symbol as

$$R^\alpha_{\beta\gamma\delta} = \Gamma^\alpha_{\beta\delta,\gamma} - \Gamma^\alpha_{\beta\gamma,\delta} + \Gamma^\mu_{\beta\delta}\Gamma^\alpha_{\mu\gamma} - \Gamma^\mu_{\beta\gamma}\Gamma^\alpha_{\mu\delta} \ . \tag{1.2.8}$$

Ricci curvature tensor is the resultant of contraction of indices of the Reimann curvature tensor, i.e., $R_{\alpha\beta} = R^\rho_{\alpha\rho\beta}$. Thus

$$\begin{aligned} R_{\alpha\beta} &= \partial_\rho\Gamma^\rho_{\beta\alpha} - \partial_\beta\Gamma^\rho_{\rho\alpha} + \Gamma^\rho_{\rho\chi}\Gamma^\chi_{\beta\alpha} - \Gamma^\rho_{\beta\chi}\Gamma^\chi_{\rho\alpha} \\ &= 2\Gamma^\rho_{[\beta,\rho]} + 2\Gamma^\rho_{\chi[\rho}\Gamma^\chi_{\beta]\alpha} \ , \end{aligned} \tag{1.2.9}$$

where $[\mu,\nu] \equiv \frac{1}{2}(\mu\nu - \nu\mu)$.

Ricci scalar is the resultant of the contraction of Ricci curvature tensor and metric tensor, i.e.,

$$R = g^{\mu\nu}R_{\mu\nu} \ . \tag{1.2.10}$$

Stress-Energy Tensor

In the rest frame in Minkowski spacetime, the 4-dimensional tensor is

$$T^{\mu\nu} = \rho c^2 \delta^\mu_0 \delta^\nu_0 + \sigma^{ij}\delta^\mu_i \delta^\nu_j \ . \tag{1.2.11}$$

Classically, in 3-dimensions, stress is a generalization of the concept of pressure (force per unit area). In a reference frame other than the center of mass reference frame, both energy and volume change, therefore energy density cannot be represented by some component of a vector. For curved spacetime, we can generalize Eq. (1.2.11) for an arbitrary manifold and frame as [2]

$$T^{\mu\nu} \;=\; \rho u^{\mu} u^{\nu} \;+\; \sigma^{ij} \delta_i^{\mu} \delta_j^{\nu} \;, \tag{1.2.12}$$

where u^{μ} is the 4-velocity. Energy-momentum tensor has energy, a constant-time surface (across which flux is defined as energy density) and momentum flux as components. Thus,

$$T^{\mu\nu} \;=\; flux\ of\ \mu\ momentum\ across\ a\ surface\ of\ constant\ x^{\nu} \;, \tag{1.2.13}$$

where μ-momentum is the μ^{th} component of 4-momentum. Stress energy tensor has units of energy density and is represented as a 4×4 matrix,

$$T^{\mu\nu} = \begin{bmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{bmatrix} \tag{1.2.14}$$

$T^{00} \;=\; \rho \;=$ energy density divided by the speed of light squared, i.e., density of relativistic mass,
$T^{ij} \;=\;$ flux of i-momentum across j-surface,
for $i = j$, T^{ij} is the normal stress (normal stress is pressure in an isotropic medium) and for $i \neq j$, this is shear stress.
$T^{0i} \;=\; T^{i0}$ means that the flux of the relativistic mass across the x^i surface is equivalent to the density of the ith component of linear momentum. In a stress-free arbitrary frame, $T^{\mu\nu}$ gives total energy and momentum of any portion of fluid.

Conservation of Mass-Energy and Momentum

It is known from classical mechanics that mass is conserved, i.e., it can neither be created nor destroyed. Hence the total time derivative of matter density (ρ) must be zero

$$\frac{D\rho}{dt} \;=\; 0 \;. \tag{1.2.15}$$

It can be written in terms of partial derivatives with respect to time and position as

$$\frac{\partial \rho}{\partial t} \;+\; \nabla.(\rho \mathbf{u}) \;=\; 0 \;. \tag{1.2.16}$$

Now consider the conservation of momentum which says that the total force on a particle in equilibrium is zero. The body force ($\rho a \;\equiv\; \rho D\mathbf{u}/Dt$) on a matter element must be balanced by the release of tension (i.e, $\sigma^{ij}_{,j}$). Thus

$$\rho\frac{Du^i}{Dt} \; + \; \sigma^{ij}_{,j} \; = \; 0 \; . \tag{1.2.17}$$

In the discussion above, if the relativistic mass energy or energy density is considered in place of the rest-mass density then Eqs. (1.2.16 - 1.2.17) can be written as

$$\begin{aligned} (\rho u^\nu)_{,\nu} \quad &= \; 0 \; , & (1.2.18) \\ \rho u^\nu u^k_{,\nu} \; + \; \sigma^{kj}_{,j} \quad &= \; 0 \; . & (1.2.19) \end{aligned}$$

Now consider the divergence of the stress-energy tensor in an approximately Minkowski space in Cartesian coordinates. Since the Christoffel symbols are zero, the divergence is replaced by the partial derivative. Taking the divergence (partial derivative) of Eq. (1.2.12)

$$\begin{aligned} T^{0\nu}_{,\nu} \quad &= \; (\rho c u^\nu)_{,\nu} \; + \; \sigma^{ij}\delta^0_i\delta^\nu_i)_{,\nu} \\ &= \; c(\rho u^\nu)_{,\nu} \; , \end{aligned} \tag{1.2.20}$$

where $u^\mu \;\; = \;\; c$ has been used. Considering Eq. (1.2.18), the above expression equates to zero. Similarly consider the divergence of $T^{k\nu}$ in Eq. (1.2.12)

$$\begin{aligned} T^{k\nu}_{,\nu} \quad &= \; (\rho u^k u^\nu)_{,\nu} \; + \; (\sigma^{ij}\delta^k_i\delta^\nu_j)_{,\nu} \\ &= \; u^k(\rho u^\nu)_{,\nu} \; + \; [\rho u^\nu u^k_{,\nu} \; + \; \sigma^{kj}_{,j}] \; . \end{aligned} \tag{1.2.21}$$

Both terms on the right hand side of the above equation are zero by Eqs. (1.2.18 - 1.2.19). Therefore $T^{k\nu}_{,\nu} \;\; = \;\; 0$. Hence the laws of conservation of mass-energy and momentum could be stated as the requirement that the stress-energy tensor be divergence free. For an arbitrary spacetime in arbitrary coordinates the natural generalization of this requirement is that the stress-energy tensor is divergence-free, i.e.

$$T^{\mu\nu}_{;\nu} \; = \; 0 \; . \tag{1.2.22}$$

Stress-Energy Tensor for Fields

The stress-energy tensor for fields must be a quantity which has units of energy density and is divergence free (i.e. conserved) [2]. Also, such a tensor should reduce to usual stress-energy tensor for a field describing a fluid.

Let $L[\phi, \phi_{,\rho}]$ be the Lagrangian density of a scalar field. Then

$$\frac{\partial L}{\partial x^\mu} = \frac{\partial \phi}{\partial x^\mu}\frac{\delta L}{\delta \phi} + \frac{\partial \phi_{,\nu}}{\partial x^\mu}\frac{\delta L}{\delta \phi_{,\nu}} . \tag{1.2.23}$$

Using $\delta L/\delta\phi = (\delta L/\delta\phi_{,\nu})_{;\nu}$ and $\partial\phi_{,\nu}/\partial x_\mu = (\partial\phi/\partial x_\mu)_{;\nu}$ in Eq. (1.2.23), we get

$$\frac{\partial L}{\partial x^\mu} = \frac{\partial \phi}{\partial x^\mu}\left(\frac{\delta L}{\delta \phi_{,\nu}}\right)_{;\nu} + \left(\frac{\partial \phi}{\partial x^\mu}\right)_{;\nu}\frac{\delta L}{\delta \phi_{,\nu}} , \tag{1.2.24}$$

$$\frac{\partial L}{\partial x^\mu} = \left(\frac{\partial \phi}{\partial x^\mu}\frac{\delta L}{\delta \phi_{,\nu}}\right)_{;\nu} = (\delta^\nu_\mu L)_{;\nu} , \tag{1.2.25}$$

$$\left(\frac{\partial \phi}{\partial x^\mu}\frac{\delta L}{\delta \phi_{,\nu}} - \delta^\nu_\mu L\right)_{;\nu} = 0 . \tag{1.2.26}$$

The divergence free quantity in above equation has the units of energy density. Hence we can define it as stress energy tensor

$$T^\nu_\mu = \frac{\partial \phi}{\partial x^\mu}\frac{\delta L}{\delta \phi_{,\nu}} - \delta^\nu_\mu L . \tag{1.2.27}$$

Stress-Energy Tensor for the Electromagnetic Field

This is known from electromagnetism that the generalized curl of the vector field is given by

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} , \tag{1.2.28}$$

where A_μ is the 4-vector potential. The corresponding Lagrangian density is [2]

$$L = \frac{1}{16\pi}[F_{\mu\nu}F^{\mu\nu} + j_\mu A^\mu] , \tag{1.2.29}$$

where j_μ is the four vector current density which is zero for a source-free region. Thus Eq. (1.2.27 - 1.2.29) yield

$$T^{\mu\nu} = -\frac{1}{4\pi}[F^{\mu\rho}F^\nu_\rho - \frac{1}{4}g^{\mu\nu}F^{\rho\pi}F_{\rho\pi}] . \tag{1.2.30}$$

1.2.3 The Einstein Field Equations

The set of Einstein field equations is analogous to classical Poisson equation [19]

$$\nabla^2\phi = 4\pi G\rho , \tag{1.2.31}$$

where ρ is the density of mass. In section 1.1.10, it has been shown that the relativistic generalization of mass is the total energy (including rest mass). EFEqs can be derived from action principle. The action for gravitational field is

$$S = S_m + S_g , \tag{1.2.32}$$

where S_m and S_g are the parts of the action due to matter and gravity (spacetime) respectively. Since the curvature of spacetime is described by Ricci tensor and metric tensor, S_g is defined as

$$\begin{aligned} S_g &= -\frac{1}{\kappa}\int L_g d\Omega \\ &= -\frac{1}{\kappa}\int R\sqrt{|g|}d\Omega \\ &= -\frac{1}{\kappa}\int g^{\mu\nu}R_{\mu\nu}\sqrt{|g|}d\Omega, \end{aligned} \tag{1.2.33}$$

where $1/\kappa$ is the coupling constant of matter and gravity which has the value $8\pi G/c^4$ and Ω is the 4-volume.

Also, the stress energy tensor (due to matter) is defined in terms of Lagrangian. Therefore S_m is defined as

$$S_m = \int L\sqrt{|g|}d\Omega . \tag{1.2.34}$$

Taking δS_g in Eq. (1.2.33) gives

$$\begin{aligned} \delta S_g &= -\frac{1}{\kappa}\delta\int g^{\mu\nu}R_{\mu\nu}\sqrt{|g|}d\Omega , \\ &= \int\Big(\delta g^{\mu\nu}R_{\mu\nu}\sqrt{|g|} + g^{\mu\nu}\delta R_{\mu\nu}\sqrt{|g|} \\ &\quad + g^{\mu\nu}R_{\mu\nu}\delta\sqrt{|g|}\Big)\, d\Omega . \end{aligned} \tag{1.2.35}$$

Using Eq. (1.2.10) and $\delta\sqrt{\mid g\mid} = -\frac{1}{2}\sqrt{\mid g\mid}g_{\mu\nu}\delta g^{\mu\nu}$ in third term on right hand side of above equation gives

$$\begin{aligned} \delta S_g &= \int\Big[\delta g^{\mu\nu}R_{\mu\nu}\sqrt{|g|} + g^{\mu\nu}\delta R_{\mu\nu}\sqrt{|g|} \\ &\quad + R\Big(-\frac{1}{2}\sqrt{\mid g\mid}g_{\mu\nu}\delta g^{\mu\nu}\Big)\Big]\, d\Omega , \\ &= \int\delta g^{\mu\nu}R_{\mu\nu}\sqrt{|g|}d\Omega + \int\Big(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\Big)\sqrt{\mid g\mid}\delta g^{\mu\nu}d\Omega \end{aligned} \tag{1.2.36}$$

The first term on the right hand side of above equation vanishes which can be shown as follows. By Eq. (1.2.9)

$$\begin{aligned} \delta R_{\alpha\beta} &= \delta(\Gamma^{\lambda}_{\alpha\beta,\lambda} - \Gamma^{\lambda}_{\alpha\lambda,\beta}) \\ &= (\delta\Gamma^{\lambda}_{\alpha\beta})_{,\lambda} - (\delta\Gamma^{\lambda}_{\alpha\lambda})_{,\beta} \ . \end{aligned} \tag{1.2.37}$$

Therefore

$$g^{\alpha\beta}\delta R_{\alpha\beta} = (g^{\alpha\beta}\delta\Gamma^{\lambda}_{\alpha\beta} - g^{\alpha\lambda}\delta\Gamma^{\beta}_{\alpha\beta})_{,\lambda} \ , \tag{1.2.38}$$

where β and λ has been interchanged in the 2^{nd} term on right hand side of the above equation. Integrating over the volume element yields

$$\int g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{|g|}d\Omega = \int (g^{\alpha\beta}\delta\Gamma^{\lambda}_{\alpha\beta} - g^{\alpha\lambda}\delta\Gamma^{b}_{\alpha\beta})_{,\lambda}\sqrt{|g|}d\Omega \ . \tag{1.2.39}$$

Considering such a frame where metric and its derivatives vanish at the boundary, right hand side of Eq. (1.2.39) becomes zero

$$\int (g^{\alpha\beta}\sqrt{|g|}\delta R_{\alpha\beta})d\Omega = 0 \ . \tag{1.2.40}$$

Thus, Eq. (1.2.36) becomes

$$\delta S_g = \int \left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R\right)\sqrt{|\,g\,|}\delta g^{\mu\nu}d\Omega \ . \tag{1.2.41}$$

Now Eq. (1.2.34) gives

$$\begin{aligned} \delta S_m &= \delta\int L\sqrt{|g|}d\Omega \ , \\ &= \int \delta\left(L\sqrt{|g|}\right)d\Omega \ , \\ &= \int \left(\frac{\partial L}{\partial g^{\mu\nu}}\sqrt{|g|}\delta g^{\mu\nu} + \frac{\partial L}{\partial g^{\mu\nu}_{,\rho}}\sqrt{|g|}\delta g^{\mu\nu}_{,\rho}\right)d\Omega \ . \end{aligned} \tag{1.2.42}$$

Consider

$$\begin{aligned} \frac{\partial L}{\partial g^{\mu\nu}_{,\rho}}\sqrt{|g|}\delta g^{\mu\nu}_{,\rho} &= \frac{\partial L}{\partial g^{\mu\nu}_{,\rho}}\frac{\partial}{\partial x^{\rho}}\sqrt{|g|}\delta g^{\mu\nu} \\ &= \frac{\partial}{\partial x^{\rho}}\frac{\partial L}{\partial g^{\mu\nu}_{,\rho}}\sqrt{|g|}\delta g^{\mu\nu} - \frac{\partial}{\partial x^{\rho}}\frac{\partial L}{\partial g^{\mu\nu}_{,\rho}}\sqrt{|g|}\delta g^{\mu\nu} \ . \end{aligned} \tag{1.2.43}$$

Integrating

$$\int \frac{\partial L}{\partial g^{\mu\nu}_{,\rho}} \sqrt{|g|}\delta g^{\mu\nu}_{,\rho} d\Omega \;=\; \int \frac{\partial}{\partial x^\rho}\left(\frac{\partial L}{\partial g^{\mu\nu}_{,\rho}} \sqrt{|g|}\delta g^{\mu\nu}\right) d\Omega \;-\; \int \frac{\partial}{\partial x^\rho}\frac{\partial L}{\partial g^{\mu\nu}_{,\rho}}\delta g^{\mu\nu} d\Omega$$
$$= \int \omega^{\rho}_{;\rho}\sqrt{|g|}d\Omega$$
$$- \int \frac{\partial}{\partial x^\rho}\left(\frac{\partial L}{\partial g^{\mu\nu}_{,\rho}}\right)\sqrt{|g|}\delta g^{\mu\nu} d\Omega \;, \tag{1.2.44}$$

where $\omega^\rho \equiv \partial L \,/\, \partial g^{\mu\nu}_{,\rho}\delta g^{\mu\nu}$. By Gauss divergence theorem,

$$\int \omega^{\rho}_{;\rho}\sqrt{|g|}d\Omega \;=\; \oint \omega^\rho \sqrt{|g|}ds_\rho \;, \tag{1.2.45}$$

which vanishes. Thus. Eq. (1.2.44) reduces to

$$\int \frac{\partial L}{\partial g^{\mu\nu}_{,\rho}} \sqrt{|g|}\delta g^{\mu\nu}_{,\rho} d\Omega \;=\; -\; \int \frac{\partial}{\partial x^\rho}\frac{\partial L}{\partial g^{\mu\nu}_{,\rho}}\sqrt{|g|}\delta g^{\mu\nu} d\Omega \;. \tag{1.2.46}$$

Thus, Eq. (1.2.42) becomes

$$\delta S_m \;=\; \delta \int L\sqrt{|g|}d\Omega \;,$$
$$= \int \frac{\partial L}{\partial g^{\mu\nu}}\sqrt{|g|}\delta g^{\mu\nu} \;-\; \int \frac{\partial}{\partial x^\rho}\frac{\partial L}{\partial g^{\mu\nu}_{,\rho}}\sqrt{|g|}\delta g^{\mu\nu} d\Omega \;,$$
$$= \int \left(\frac{\partial L}{\partial g^{\mu\nu}} \;-\; \frac{\partial}{\partial x^\rho}\frac{\partial L}{\partial g^{\mu\nu}_{,\rho}}\right)\sqrt{|g|}\delta g^{\mu\nu} d\Omega \;. \tag{1.2.47}$$

But

$$\frac{\partial L}{\partial g^{\mu\nu}}\delta g^{\mu\nu} \;-\; \frac{\partial}{\partial x^\rho}\frac{\partial L}{\partial g^{\mu\nu}_{,\rho}}\delta g^{\mu\nu} \;\equiv\; T_{\mu\nu} \;. \tag{1.2.48}$$

Therefore

$$\delta S_m \;=\; \int_\Omega T_{\mu\nu}\delta g^{\mu\nu} d\Omega \;. \tag{1.2.49}$$

Combining Eqs. (1.2.32), (1.2.34) and (1.2.49) and taking $\delta S \;=\; 0$ gives

$$\delta S \;=\; -\frac{1}{\kappa}\int_\Omega \left(R_{\mu\nu} \;-\; \frac{1}{2}g_{\mu\nu}R\right)\sqrt{|\,g\,|}\delta g^{\mu\nu}d\Omega \;+\; \int_\Omega T_{\mu\nu}\sqrt{|\,g\,|}\delta g^{\mu\nu}d\Omega \;,$$
$$= \int_\Omega \left[-\;\frac{1}{\kappa}\left(R_{\mu\nu} \;-\; \frac{1}{2}g_{\mu\nu}R\right) \;+\; T_{\mu\nu}\right]\delta g^{\mu\nu}\sqrt{|\,g\,|}d\Omega \;,$$
$$= 0 \;. \tag{1.2.50}$$

Since the above equation is true for arbitrary $\delta g^{\mu\nu}$ and volume element $d\Omega$, the integrant must be zero in general, i. e.,

$$\varepsilon_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + N_{\mu\nu} = \kappa T_{\mu\nu} , \tag{1.2.51}$$

where $\varepsilon_{\mu\nu}$ is called the Einstein tensor and $N_{\mu\nu}$ is the constant of integration.

Equivalence of mass and energy from SR suggests that each form of energy must act as a source of gravitational field. ε tells spacetime how to curve and $T_{\mu\nu}$ tells matter how to move. $N_{\mu\nu}$ should have a universal value. Also, for the conservation of stress energy tensor it is required that the covariant derivative of $N_{\mu\nu}$ also vanishes. Therefore $N_{\mu\nu} \equiv \Lambda g_{\mu\nu}$ where Λ is called cosmological constant. $g_{\mu\nu}$ ensures the conservation of $T_{\mu\nu}$ defined by Eq. (1.2.22). Now the EFEqs can be written as

$$\varepsilon_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \kappa T_{\mu\nu} . \tag{1.2.52}$$

To obtain static solutions, there must be some repulsive force which balances the gravitational attraction. To obtain such a result, Einstein suggested the negative value of Λ. Later the discovery of dynamical universe made Einstein to withdraw this cosmological constant calling it the greatest blunder of his life. (But this was not as big a mistake as he thought, as we shall see in Chapter 2.)

Alternatively, the factor Λ can also be introduced in the gravitational Lagrangian density to get the same EFEqs given by (1.2.52), i.e., one can take

$$L_g = \sqrt{|g|}(R + 2\Lambda) . \tag{1.2.53}$$

1.3 Solutions of Einstein Field Equations

The solutions of EFEqs are the metrics of spacetime. These metrics describe the inertial motion of objects in spacetime. The exact solutions of EFEqs are not always known without approximations. The solution of EFEqs for uncharged non-rotating point perticle is called the Schwarzschild metric and that for case of a charged particle is called the Reissner-Nordstrom Metric.

1.3.1 Schwarzschild's Solution of Einstein Field Equation

Consider a neutral point particle of mass m which is at rest at origin. The solution of EFEqs for this particle (Schwarzchild solution) is obtained as follows.

Exterior Solution

Schwarzschild exterior solution is obtained by taking the following assumptions (in addition to the conditions of zero charge and zero angular momentum).

(i) The metric is spherically symmetric.

(ii) There is no matter in the region under consideration, i.e., $T_{\mu\nu} = 0$.

The most general form of such a metric (in spherical-polar coordinates) is [2]

$$ds^2 = e^{\nu(r)}(cdt)^2 - e^{\chi(r)}(dr)^2 - r^2 d\Omega^2 , \tag{1.3.1}$$

where $d\Omega$ is the solid angle subtended at the origin and is equal to $d\theta^2 + \sin^2\theta d\phi^2$, ν and χ are arbitrary functions of radial coordinate r only (since for point particle field does not change with time). In matrix form this metric is

$$g_{\mu\nu} = \begin{bmatrix} e^{\nu(r)} & 0 & 0 & 0 \\ 0 & -e^{\chi(r)} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} . \tag{1.3.2}$$

Clearly $|g| = -e^{\nu+\chi}r^4\sin^2\theta$. Therefore, $ln\sqrt{|g|} = \frac{1}{2}(\nu + \chi) + 2lnr + ln\sin\theta$. Thus the derivatives of the metric coefficients are

$$\begin{aligned} g_{00,1} &= \nu' e^{\nu},\ g_{11,1} = -\chi' e^{\chi},\ g_{22,1} = -2r,\ g_{33,1} = -2r\sin^2\theta, \\ g_{33,2} &= -2r^2\sin\theta\cos\theta,\ g_{\chi\nu,\rho} = 0 \quad otherwise. \end{aligned} \tag{1.3.3}$$

The derivatives of the logarithm of the determinant of the metric tensor are

$$\begin{aligned} (ln\sqrt{|g|})_{,1} &= \frac{1}{2}(\nu' + \chi') + \frac{2}{r} , \\ (ln\sqrt{|g|})_{,11} &= \frac{1}{2}(\nu'' + \chi'') - \frac{2}{r^2} , \\ (ln\sqrt{|g|})_{,2} &= \cot\theta, (ln\sqrt{|g|})_{,22} = \csc^2\theta, \\ (ln\sqrt{|g|})_{,\mu} &= (ln\sqrt{|g|})_{,\chi\nu} = 0 \quad otherwise. \end{aligned} \tag{1.3.4}$$

Now evaluating the Christoffel symbols, following are found to be zero

$$\begin{aligned} \Gamma^0_{00} &= \Gamma^0_{ij} = \Gamma^i_{0j} = \Gamma^3_{us} = \\ \Gamma^u_{s2} &= \Gamma^u_{s3} = \Gamma^3_{33} = 0 , \end{aligned} \tag{1.3.5}$$

where $i,\ j\ =\ 1,\ 2,\ 3;\ u,\ s\ =\ 0,\ 1,\ 2$. The non-zero Christoffel symbols are

$$\begin{aligned}
\Gamma^0_{01} &= \frac{1}{2}\nu',\ \Gamma^1_{00} = \frac{1}{2}\nu' e^{(\nu\ -\ \chi)},\ \Gamma^1_{11} = \frac{1}{2}\chi',\ \Gamma^1_{22} = -\ re^{-\chi}\ , \\
\Gamma^1_{33} &= -\ r\sin^2\theta e^{-\chi},\ \Gamma^2_{12} = \frac{1}{r} = \Gamma^3_{13}\ , \\
\Gamma^2_{33} &= -\ r\sin\theta\cos\theta,\ \Gamma^3_{23} = \cot\theta\ . \qquad (1.3.6)
\end{aligned}$$

For vacuum, Eq. (1.2.52) reduces to $R_{\mu\nu} = 0$.

$$\begin{aligned}
R_{\mu\nu} &= \Gamma^\rho_{\mu\nu,\rho} - (ln\sqrt{|\ g\ |})_{,\mu\nu} + (ln\sqrt{|\ g\ |})_{,\rho}\Gamma^\rho_{\mu\nu} \\
&- \Gamma^\rho_{\pi\mu}\Gamma^\pi_{\rho\nu} = 0\ . \qquad (1.3.7)
\end{aligned}$$

Now computing the components of Ricci tensor

$$\begin{aligned}
R_{00} &= \Gamma^\rho_{00,\rho} - (ln\sqrt{|g|})_{,00} + (ln\sqrt{|g|})_{,\rho}\Gamma^\rho_{00} - \Gamma^\rho_{\pi 0}\Gamma^\pi_{\rho 0} \\
&= \left(\frac{1}{2}\nu' e^{\nu-\chi}\right)' + \left[\frac{1}{2}(\nu' + \chi') + \frac{2}{r}\right]\left(\frac{1}{2}\nu' e^{\nu-\chi}\right) - 2\left(\frac{1}{2}\nu'\right)\left(\frac{1}{2}\nu' e^{\nu-\chi}\right) \\
&= \frac{1}{2}\left[\nu'' + \frac{1}{2}\nu'\left(\nu' - \chi'\right) + \frac{2}{r}\nu'\right]e^{\nu-\chi} = 0\ . \qquad (1.3.8)
\end{aligned}$$

$$\begin{aligned}
R_{11} &= \Gamma^\rho_{11,\rho} - \left(ln\sqrt{|g|}\right)_{,11} + \left(ln\sqrt{|g|}\right)_{,\rho}\Gamma^\rho_{11} - \Gamma^\rho_{\pi 1}\Gamma^\pi_{\rho 1} \\
&= \frac{1}{2}\left[\nu'' + \frac{1}{2}\nu'(\nu' - \chi') - \frac{2}{r}\chi'\right] = 0\ . \qquad (1.3.9)
\end{aligned}$$

$$\begin{aligned}
R_{22} &= \Gamma^\rho_{22,\rho} - (ln\sqrt{|g|})_{,22} + (ln\sqrt{|g|})_{,\rho}\Gamma^\rho_{22} - \Gamma^\rho_{\pi 2}\Gamma^\pi_{\rho 2} \\
&= (-\ re^{-\chi})' + 1 = 0\ . \qquad (1.3.10)
\end{aligned}$$

$$\begin{aligned}
R_{33} &= \Gamma^\rho_{33,\rho} - \left(ln\sqrt{|g|}\right)_{,33} + \left(ln\sqrt{|g|}\right)_{,\rho}\Gamma^\rho_{33} - \Gamma^\rho_{\pi 3}\Gamma^\pi_{\rho 3} \\
&= [(-\ re^{-\chi})' + 1]\sin^2\theta = 0\ . \qquad (1.3.11)
\end{aligned}$$

Solving Eqs. (1.3.8 - 1.3.11) gives

$$e^\nu = 1 - \frac{2Gm}{c^2 r}\ , \qquad (1.3.12)$$

$$e^\chi = e^{-\nu} = \left(1 - \frac{2Gm}{c^2 r}\right)^{-1}\ . \qquad (1.3.13)$$

Thus the components of the metric tensor are

$$g_{\mu\nu} = \begin{bmatrix} 1-\frac{2Gm}{c^2r} & 0 & 0 & 0 \\ 0 & -(1-\frac{2Gm}{c^2r})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} , \qquad (1.3.14)$$

or

$$ds^2 = \left(1 - \frac{2Gm}{c^2r}\right)(cdt)^2 - \left(1 - \frac{2Gm}{c^2r}\right)^{-1}(dr)^2 - r^2d\Omega^2 . \qquad (1.3.15)$$

In the limit of large r, above metric redcuces to that for weak Newtonian fields. Thus Eq. (1.3.15) is a relativistic generalization of Newtonian gravity. Also, this metric becomes singular in radial coordinate at

$$r = \frac{2Gm}{c^2} \equiv r_s, \qquad (1.3.16)$$

where r_s is called Schwarzchild radius. Objects having $r = r_s$ are the BHs (Schwarzchild BHs). Hence GR predicts the existence of BHs.

Interior Solution

Consider an isotropic fluid in the shape of a sphere of radius a centered at the origin. Again, due to the spherical symmetry I shall use the spherical coordinates. Let ρ be the density of the fluid and is function of radial coordinate r. The stress-energy tensor for the matter inside the sphere is given by

$$T^{\nu}_{\mu} = [\rho(r) + p(r)/c^2]u^{\nu}u_{\mu} - \delta^{\nu}_{\mu}p(r)/c^2 , \qquad (1.3.17)$$

where p is the pressure of the fluid. Also $u^{\nu}u_{\mu} = \delta^{\nu}_0\delta^0_{\mu}$. Using Eq. (1.2.22)

$$[\rho(r) + p(r)/c^2]\delta^0_{\mu} + [\rho(r) + p(r)/c^2]\left(ln\sqrt{\mid g \mid}\right)\delta^0_{\mu} - [p(r)/c^2]_{,\nu} - [\rho(r) + p(r)/c^2]\Gamma^0_{0\mu} = 0 . \qquad (1.3.18)$$

Since no time dependent term is involved in the first two terms on the left hand side of above equation, these are zero. The next two terms are also identically zero for $\mu \neq 1$. Thus, the only new equation is for $\mu = 1$

$$\frac{1}{2}[\rho(r)c^2 + p(r)]\nu'(r) + p'(r) = 0 . \qquad (1.3.19)$$

Evaluating Ricci scalar and Ricci tensors for $\mu = 0 = \nu$ and $\mu = 1 = \nu$ and using Eq. (1.2.52) results in

$$\left[\frac{\xi'(r)}{r} - \frac{1}{r^2}\right] e^{-\xi(r)} + \frac{1}{r^2} = \kappa c^2 \rho(r) \tag{1.3.20}$$

and

$$\left[\frac{\nu'(r)}{r} + \frac{1}{r^2}\right] e^{-\xi(r)} - \frac{1}{r^2} = -\kappa p(r)\ . \tag{1.3.21}$$

Eqs. (1.3.6 - 1.3.8) are in terms of four unknown functions and their derivatives. Therefore we need another equation in terms of these functions. Generally, the equation of state (as mentioned earlier that it will be discussed in detail in chapter 2) serves the purpose of eliminating one of the two variables: ρ and p, in terms of the other variable. For phantom energy, one is a multiple of the other which will be discussed in Chapter 2.

1.3.2 The Reissner-Nordstr$\ddot{o}$m Metric

The Reissner-Nordström metric is the solution for the gravitational field produced by a charged massive particle. Consider a point mass m carrying charge Q be at rest at origin. Obviously, metric is spherically symmetric and static. We need to solve the coupled Einstein-Maxwell field equations. The gravitational field enters into the Maxwell equations as we require the covariant divergence of the field tensor to be zero. The electromagnetic stress-energy tensor acts as a source term for the gravitational field. Physically, the energy distribution due to the electromagnetic field has an effective mass which causes a gravitational field. Electromagnetic 4$-$vector potential is taken to be [2]

$$A_\mu = \left(\frac{Q}{cr}, \underline{0}\right)\ . \tag{1.3.22}$$

Thus

$$F_{\mu\nu} = -F_{\nu\mu} = 2\delta^0_{[\mu}\delta^1_{\nu]}\frac{Q}{cr^2}\ . \tag{1.3.23}$$

The components of mixed form of metric tensor are

$$\begin{aligned} T^0_0 &= T^1_1 = -T^2_2 = -T^3_3 \\ &= Q^2 e^{-(\nu+\chi)/8\pi c^2 r^4} \end{aligned} \tag{1.3.24}$$

and

$$T^{\mu}_{\nu} \;=\; 0 \qquad\qquad \forall \quad \mu \neq \nu \;. \tag{1.3.25}$$

The components of Ricci tensor $R_{\mu\nu}$ are the same as given in Eqs. (1.3.8 - 1.3.11). Solving EFEqs using $T^0_0 \;=\; T^1_1$ and Eqs. (1.3.8-1.3.9) gives

$$\nu'(r) \;+\; \xi'(r) \;=\; 0 \;. \tag{1.3.26}$$

Thus

$$\nu(r) \;+\; \xi(r) \;=\; 0 \;. \tag{1.3.27}$$

Now the EFEq for $2-2$ component is $R^2_2 \;=\; \kappa T^2_2$. Using Eq. (1.3.10) gives

$$-\;\frac{1}{r^2}[(-re^{-\xi})' \;+\; 1] \;=\; -\;\frac{8\pi G}{c^2}\frac{Q^2}{8\pi c^2 r^4} \;. \tag{1.3.28}$$

Solving the above equation yields

$$e^{\nu(r)} \;=\; e^{-\xi(r)} \;=\; 1 \;-\; \frac{2Gm}{c^2 r} \;+\; \frac{GQ^2}{c^4 r^2} \;. \tag{1.3.29}$$

Thus the components of the metric tensor are

$$g_{\mu\nu} = \begin{bmatrix} 1-\frac{2Gm}{c^2r}+\frac{GQ^2}{c^4r^2} & 0 & 0 & 0 \\ 0 & -(1-\frac{2Gm}{c^2r}+\frac{GQ^2}{c^4r^2})^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2\sin^2\theta \end{bmatrix} , \tag{1.3.30}$$

or

$$ds^2 \;=\; \left(1 \;-\; \frac{2Gm}{c^2 r} \;+\; \frac{GQ^2}{c^4 r^2}\right) c^2 dt^2 - \left(1 \;-\; \frac{2Gm}{c^2 r} \;-\; \frac{GQ^2}{c^4 r^2}\right)^{-1} dr^2 - r^2 d\Omega^2 \;. \tag{1.3.31}$$

In the case where $Q \;=\; 0$, above solution reduces to Schwarzchild solution which is required. Also, the objects satisfying $1 \;-\; \frac{2Gm}{c^2r} \;-\; \frac{GQ^2}{c^4r^2} = 0$ condition are the charged BHs.

Chapter 2

Cosmology

The name cosmology comes from two Greek words "cosmos" ($\kappa\omega\sigma\mu\omega\sigma$) and "logos" ($\lambda\omega\gamma\omega\sigma$) [2]. The former means "the Universe" and the latter means "for the study of". Thus cosmology is the study of the universe. It was originally a field of theology and non-scientific philosophy that was formulated in order to answer the questions about the purpose of life and the universe. But the revolution brought by relativity (GR in particular) changed its basis from religious philosophy to science as Einstein chose cosmology for the application of GR.

2.1 Standard Model of Cosmology

The standard model of cosmology is based on GR and the *Cosmological Principle* which states that the universe is homogeneous and isotropic. Homogeneity means that it is independent of the position of the observer and isotropy implies that the matter is distributed uniformly in all directions. More precisely, homogeneity means that all points in space have the same physical conditions (like density and temperature). The former can be regarded as translational invariance and latter as rotational invariance [20]. The difference between the two terminologies is clear from Fig. 2.1 − 2.4.

The standard model is the simplest model and is compatible with the observational data. No other model gives better statistical fit.

2.2 Brief History of the Universe

The standard model of cosmology states that the universe started with a "big bang" 13.75 billion years ago and is expanding and cooling since then. The

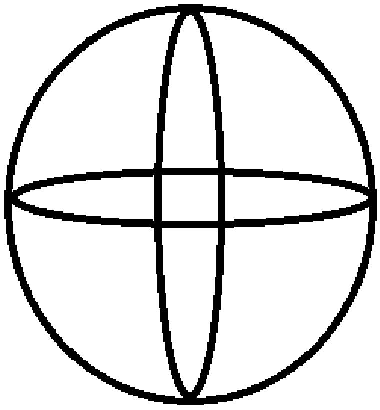

Figure 2.1: Homogeneous and isotropic: matter is distributed uniformly and there is no preferred direction as well .

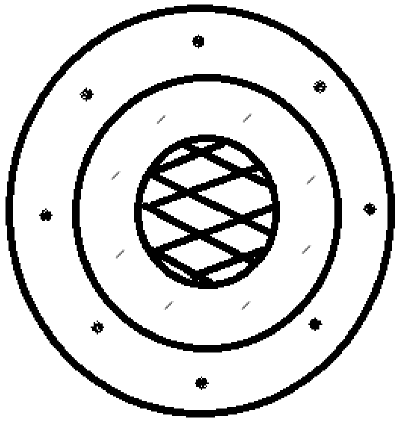

Figure 2.2: Inhomogeneous but isotropic: matter distribution is non-uniform in the sphere but is similar in all the directions .

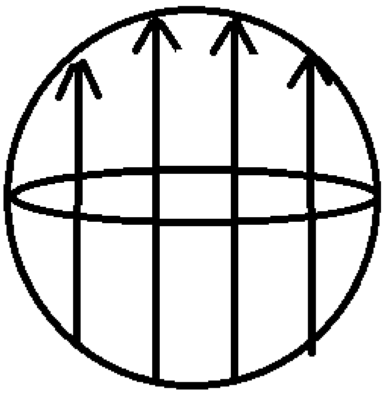

Figure 2.3: Homogeneous but anisotropic: the vertical direction is the preferred direction here .

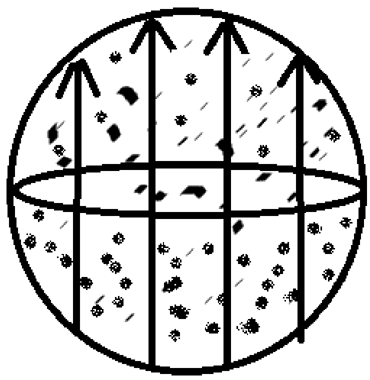

Figure 2.4: Inhomogeneous and anisotropic: matter distribution is non-uniform and vertical direction is the preferred direction .

expansion of the universe was decelerating at first but then there was a transition from this deceleration phase to the present accelerating phase. At 1s after the big bang, the temperature and density of the universe were 10 billion Kelvin and 1.8817×10^8g/cm^3 respectively which dropped to 3000K and 1.5242×10^{-18}kg/cm^3 after $380,000$ years. At the latter temperature, the free electrons joined atoms breaking the thermal contact between matter and radiation. This stage is called the surface of last scattering. Whatever radiation existed at that time has since been enormously red shifted but it still fills the space [21]. This is called cosmic microwave background (CMB).

Light nuclei like hydrogen (H) and helium (He) were formed 3 minutes after the big bang. The formation of these nuclei is called big bang nucleosynthesis. Nucleosynthesis must have been blocked by free radiation. The temperature of the radiation is such that it blocks the formation of heavier elements but allows it for the light elements in the observed ratio. This provides the total amount of baryons and the temperature of the universe at that time. This also relates the physics of the early universe cosmology to particle physics and thermodynamics. The earlier period of the universe is called radiation-dominated era when energy density of radiation was more than that of matter. After the surface of last scattering, matter-dominated era started. The present era is called dark-energy dominated era and will be explained in the last section of this chapter.

2.3 Evidence of the Isotropy and Homogeneity of the Universe

Evidence of the homogeneity and isotropy of the universe comes from the discovery of the CMB by two engineers Penzias and Wilson in 1965. They were trying to develop an extremely low noise super-cooled microwave antenna for Bell Laboratories. They found a background noise which they failed to remove by doing all sort of technical precautionary measures. Later they found that this noise was independent of the direction of the antenna: it existed even when the antenna was mounted pointing towards the empty sky. The amount of noise corresponded to what would be expected of radiation from a black body "heated" to 3.5K. Dicke, an experimental relativist, identified it as the CMB predicted by the standard model. Data collected by different projects have given the temperature of this radiation to be 2.7306K in all directions [2].

The same temperature of CMB in all directions is also evidence of isotropy. This isotropy, in turn, also implies that the universe must have been homogeneous in the past (although it is inhomogeneous at present) otherwise such a high degree of isotropy would not have been observed. Here I would also mention about the anisotropy in CMB temperature, δT. The ratio of this anisotropy to the temperature of CMB comes out to be 2×10^{-5}. Therefore, anisotropy is neglected and the universe is assumed to be homogeneous and isotropic. This anisotropy actually indicates the clumping of matter for structure formation which was resisted by radiation pressure.

2.4 Hubble Law and Expansion of the Universe

In 1929, Edwin Hubble found observationally that the galaxies are moving away from us and the speed of recession increases with the increasing distance of a galaxy from us, i.e., the speed of recession is directly proportional to the distance between the observer and the galaxy. In an expanding isotropic universe, the relative velocities of observers obey the Hubble law: the velocity of an observer B with respect to another observer A is

$$\mathbf{v}_{B(A)} = H_{(t)}\mathbf{r}_{BA} , \tag{2.4.1}$$

where $\mathbf{r}_{BA}$ is the vector pointing from A to B and the Hubble parameter $H(t)$ depends only on time. Its present value is 73.8 ± 2.4km/s/Mpc and is denoted by H_0. Since in a homogeneous and isotropic universe there are no *privileged* points, therefore the expansion appears the same to all observers wherever they are located. The Hubble law is in complete agreement with it. It can be proved as follows.

Consider another observer C. By Hubble law, the velocity of C with respect to A is

$$\mathbf{v}_{C(A)} = H_{(t)}\mathbf{r}_{CA} . \tag{2.4.2}$$

From Fig. 2.5 and Eqs. (2.4.1 - 2.4.2), the velocity of C with respect to B can be written as

$$\begin{aligned} \mathbf{v}_{C(B)} &= \mathbf{v}_{C(A)} - \mathbf{v}_{B(A)} \\ &= H_{(t)}(\mathbf{r}_{CA} - \mathbf{r}_{BA}) \\ &= H(t)\mathbf{r}_{CB} . \end{aligned} \tag{2.4.3}$$

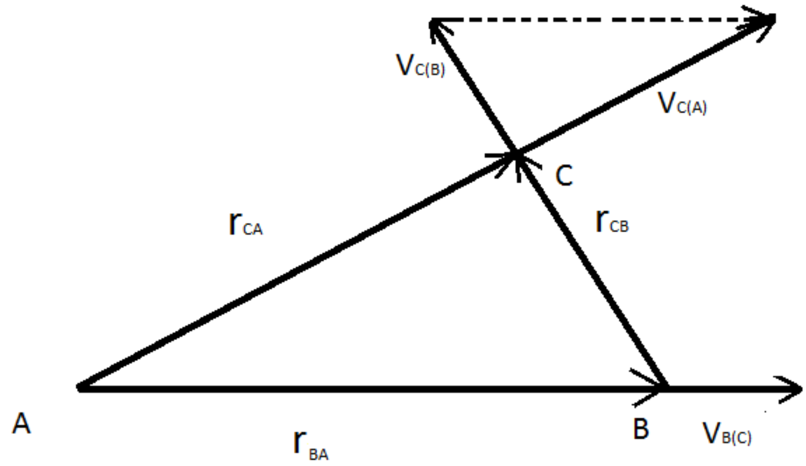

Figure 2.5: Position vectors and velocities of observer: C with respect to A and B, B with respect to A [23] .

Thus, B sees precisely the same expansion law as observer A. Now consider a 2-dimensional sphere as shown in Fig. 2.6. Let the angle between points A and B be θ_{AB} and $a(t)$ be the radius of the sphere. It is obvious that θ_{AB} does not change as $a(t)$ is increased but the distance (length of the arc AB) between the two points definitely changes. Therefore

$$r_{AB}(t) = a(t)\theta_{AB} , \tag{2.4.4}$$

implying a relative speed

$$v_{AB} = \dot{r}_{AB} , \tag{2.4.5}$$

where "dot" represents derivative with respect to time. Since θ_{AB} is constant, from Eq. (2.4.4)

$$\dot{r}_{AB}(t) = \dot{a}(t)\theta_{AB} . \tag{2.4.6}$$

From Eqs. (2.4.5 - 2.4.6)

$$\begin{aligned} v_{AB} &= \dot{a}(t)\theta_{AB} \\ &= \frac{\dot{a}}{a} r_{AB} , \end{aligned} \tag{2.4.7}$$

where Eq. (2.4.4) has been used in the last step of (2.4.7). Comparing Eq. (2.4.7)

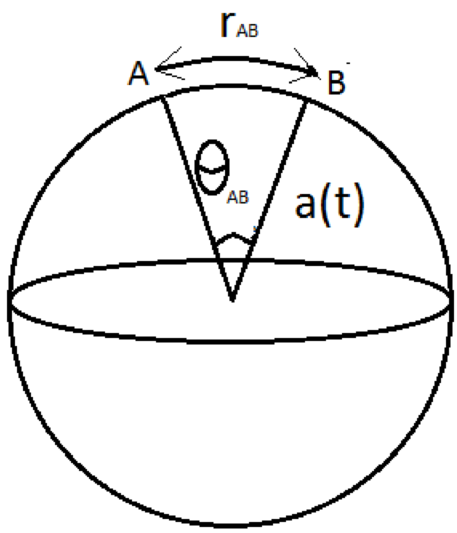

Figure 2.6: A hyper-sphere: analogy of Hubble law. Increasing the radius (scale factor) causes an "expansion" of the sphere resulting in an increase of distance between points A and B [23] .

with Hubble law, it can be stated that

$$H(t) = \frac{\dot{a}}{a} . \tag{2.4.8}$$

Thus in a homogeneous and isotropic universe the distance between observers A and B can be written as

$$\dot{\mathbf{r}}_{BA} = H(t)\mathbf{r}_{BA} . \tag{2.4.9}$$

Integrating the above equation

$$\mathbf{r}_{BA}(t) = a(t)c_{BA} , \tag{2.4.10}$$

where

$$a(t) = e^{(\int H(t)dt)} , \tag{2.4.11}$$

is called the scale factor and is analogous to the radius of a 2-sphere [23] and c_{BA} is a constant of integration. It is an analogue of θ_{BA} and can be interpreted as the distance between A and B at some particular time. It is called the co-moving coordinate of B, assuming a coordinate system centered at A. Here perfect homogeneity and isotropy has been assumed. Thus, the Hubble law holds at sufficiently large scales in the real universe.

2.5 Evidence of the Accelerated Expansion of the Universe

Evidence of the accelerated expansion of the universe comes from the observation of supernovae of type Ia. This type of supernovae result from a white dwarf star in a binary system. Matter transfers from the normal star to the white dwarf until the white dwarf attains a critical mass (the Chandrasekhar limit: 1.5 solar mass) and undergoes a thermonuclear explosion. Because all white dwarfs achieve the same mass before exploding, they all achieve the same luminosity (since the luminosity is directly proportional to the radius and hence to the mass as well) and can be used as "standard candles". Thus by observing their apparent brightness, their distance can be determined using the $1/r^2$ law.

The light from the supernova was expected to be red-shifted because of the expansion of the universe but it was found that it was 25 percent more fainter than expected. This means that it was more distant than expected. Hence it is concluded that the universe is accelerating in its expansion. By measuring this redshift from the spectrum of the supernova, it can be determined how much the Universe has expanded since the explosion. By studying many supernovae at different distances, the history of the expansion of the universe can be known.

2.6 Perfect Fluid

A perfect fluid is defined as having at each point a velocity $\mathbf{v}$ such that an observer moving with this velocity sees the fluid around him as isotropic. This will be the case if the mean free path between collisions of the particles of the system under consideration is small compared with the length scales used by the observer [21]. For example, an oscillation will propagate in air if its wavelength is large compared with the mean free path of the particles of air, but at very short wavelengths viscosity becomes important and air stops acting like a perfect fluid. The stress-energy tensor for a perfect fluid is

$$T^{\mu}_{\nu} = \left(\frac{p}{c^2} + \rho\right) u^{\mu} u_{\nu} - p g^{\mu}_{\nu} , \tag{2.6.1}$$

where p is the pressure, ρc^2 is the energy density and u^{μ} is the four-velocity of the fluid. Many macroscopic physical systems can be regarded as perfect fluids in the above mentioned limit provided that the pressure is isotropic.

2.7 Equation of State

As mentioned in section 1.3.1, an EoS that gives a relationship between pressure (p) and energy density (ρc^2) of the stuff that fills up the space around the particle is needed. For substances of cosmological interest this space is the observable universe. For an ideal gas it is

$$\omega \equiv \frac{p}{\rho c^2} , \tag{2.7.1}$$

where ω is a dimensionless constant called the EoS parameter, with $0 \leq \omega \leq 1/3$. The lower limit is for the dust while the upper limit is for the electromagnetic radiation (as will be shown in this section).

Consider a low density gas of massive particles. This gas obeys the perfect gas law given by

$$p = \frac{\rho}{\mu} k_b T , \tag{2.7.2}$$

where μ is the mean mass of the gas particles, T is the temperature of the gas and k_b is Boltzmann constant. Substituting $\rho = p/\omega c^2$ in above equation

$$\omega = \frac{k_b T}{\mu c^2} . \tag{2.7.3}$$

Since for an ideal gas, T and root mean square thermal velocity, $< v^2 >$, are related as

$$3k_b T = \mu < v^2 > . \tag{2.7.4}$$

Thus, Eq. (2.7.3) becomes

$$\omega = \frac{< v^2 >}{3c^2} . \tag{2.7.5}$$

2.7.1 EoS for Dust Particles

We can apply the limit $v << c$ in Eq. (2.7.5) for ordinary gases (e.g. nitrogen atoms have $< v^2 >\sim 500$m/s yielding $\omega \sim 10^{-12}$) as well as for the gases of astronomical interests (e.g. electrons in ionized hydrogen provided that $T << 6 \times 10^9$K and protons provided that $T << 10^{13}$K). Thus, it can be approximated that $\omega = 0$ for non-relativistic massive particles. This corresponds to $p = 0$, i.e., a pressureless gas.

2.7.2 EoS for Photons

A gas of photons is relativistic by definition. Although photons have no mass but they have momentum and hence exert pressure. Taking $v = c$ in Eq. (2.7.5), the EoS of photons is obtained to be

$$p = \frac{1}{3}\rho c^2 . \tag{2.7.6}$$

Alternatively, consider $u^\mu = c$ for photons in Eq. (2.6.1), the trace of this tensor is

$$\begin{aligned} 0 &= p + \rho c^2 - 4p \\ &= \rho c^2 - 3p . \end{aligned} \tag{2.7.7}$$

Hence $\omega = 1/3$ for photons. All gases of highly relativistic particles ($< v^2 >\sim c^2$) also has the same value of EoS parameter. A mildly relativistic gas has $0 < \omega < 1/3$.

2.7.3 Dominant Energy Condition

As c is the upper limit of speed which corresponds to $\omega = 1/3$, therefore it can be stated generally that $\omega < 1$. Considering the negative pressure (which acts as repulsive force), it can be re-stated as $|\omega| < 1$ or $\rho c^2 \geq |p|$. This is called the dominant energy condition.

Here, I should emphasise that negative pressure is not "gravitational repulsion": gravity is an attractive force in any case. Negative pressure refers to the repulsion caused by any other thing which is totally different from normal matter and energy and strictly speaking unknown as yet.

2.8 Robertson-Walker Metric and Einstein-Friedmann Model of the Universe

The interval defined in Eq. (1.1.3) applies only within the context of SR since it deals with the special case when the spacetime is not curved because of the presence of matter and energy. When gravity is added, the spacetime metric changes as derived in sections 1.3.1 and 1.3.2 for point mass (uncharged as well as charged). I have also discussed these metrics as BH solutions. But what would be the form of metric tensor for such a universe that is spatially homogeneous and isotropic at all time and distances are allowed to expand (or contract) as a function of time. The answer to this question was provided by Robertson and Walker independently of each other and the metric derived is named as Robertson-Walker (RW) metric and is given by [2]

$$ds^2 \; = \; c^2 dt^2 \; - \; a^2(t) d\sigma^2 \; , \tag{2.8.1}$$

where $d\sigma^2$ is given by

$$d\sigma^2 \; = \; d\chi^2 \; + \; f_k^2(\chi) d\Omega^2 \; . \tag{2.8.2}$$

$f_k(\chi) = \sin\chi$ for $k = +1$, χ for $k = 0$ and $\sinh\chi$ for $k = -1$. $k = 0$ corresponds to a "flat" universe, $k = 1$ corresponds to a "closed" universe and $k = -1$ represents an "open" universe.

Einstein-Friedmann Model of the Universe was independently formulated first by Einstein and later by Friedmann as Einstein had discarded it himself. Taking the universe to be homogeneous, isotropic and filled with dust. Consider the metric given by Eqs. (2.8.1) and (2.8.2)

$$\begin{aligned} g_{00} &= 1 \; , \\ g_{11} &= -\, a^2(t) \; , \\ g_{22} &= -\, a^2(t) f_k^2(\chi) \; , \\ g_{33} &= -\, a^2(t) f_k^2(\chi) \sin^2\theta \; , \\ g_{\mu\mu} &= (g_{\mu\mu})^{-1} \; . \end{aligned} \tag{2.8.3}$$

Non-zero Christoffel symbols are

$$\begin{aligned} \Gamma^0_{11} &= a\dot{a} \; , \; \Gamma^0_{22} = a\dot{a} f^2 \; , \; \Gamma^0_{22} = a\dot{a} f^2 \sin^2\theta \; , \\ \Gamma^1_{22} &= -ff' \; , \; \Gamma^1_{33} = -ff' \sin^2\theta \; , \\ \Gamma^1_{01} &= \Gamma^2_{02} = \Gamma^3_{03} = \dot{a}/a \; , \\ \Gamma^2_{12} &= \Gamma^3_{13} = f'/f \; , \; \Gamma^3_{23} = \cot\theta \; , \; \Gamma^2_{33} = -\sin\theta\cos\theta \; . \end{aligned} \tag{2.8.4}$$

Non-zero components of the Einstein tensor are only the diagonal components and are [2]

$$R_{00} = -3\ddot{a}/a \ , \tag{2.8.5}$$
$$R_{11} = a\ddot{a} + 2\dot{a}^2 - 2f''/f \ , \tag{2.8.6}$$
$$R_{22} = (a\ddot{a} + 2\dot{a}^2)f^2 - (ff'' + f'^2) + 1 \ , \tag{2.8.7}$$
$$R_{33} = R_{22}\sin^2\theta \ . \tag{2.8.8}$$

Taking $k = 1$: $f = \sin\chi$, Eq. (2.8.7) gives $R_{22} = R_{11}\sin^2\chi$. Similarly for $k = -1$: $f = \sinh\chi$, Eq. (2.8.7) gives $R_{22} = R_{11}\sinh^2\chi$. Thus, it can be generalized as

$$R_{22} = R_{11} f_k^2(\chi) \ . \tag{2.8.9}$$

For both choices of k

$$\begin{aligned} \frac{-f_k''(\chi)}{f_k(\chi)} &= +k \\ &= \frac{1 - f_k'^2(\chi)}{f_k^2(\chi)} \ . \end{aligned} \tag{2.8.10}$$

Ricci scalar comes out to be [2]

$$R = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right) \ . \tag{2.8.11}$$

Thus EFEqs are [2]

$$3\left[\frac{\dot{a}^2}{a^2} + \frac{kc^2}{a^2}\right] = 8\pi G\rho(t) \ , \tag{2.8.12}$$

and

$$2a\ddot{a} + \dot{a}^2 + kc^2 = -\frac{8\pi G p(t)}{c^2} \ . \tag{2.8.13}$$

Taking $k = 0$ in Eq. (2.8.12) gives energy density (ρ) to be $3H^2/8\pi G$. This value of the energy density is called critical density and is required for a flat universe. Its value is 10^{-29}g/cm^3. The ratio of the density of the universe to the critical density is defined as *density parameter* Ω. The value of $\Omega > 1$ corresponds to a closed universe and the value of $\Omega < 1$ corresponds to an open universe. The value of the energy density of the universe is very close to its critical value today, i.e., $\Omega \approx 1$.

2.9 Inflation

In section 2.3, the argument for the temperature in all parts of the space to be the same was provided. This tells clearly that all the parts of the universe must have been in thermal contact in the past. But the co-moving distance (i.e., the distance which does not change with the expansion of the universe) over which the cosmic interactions can occur before the release of CMB is less than the co-moving distance at which the radiation travels after the decoupling [22]. This means that these microwaves were never in thermal contact in the past. This contradiction is called the horizon problem which the standard model or the big bang cosmology failed to solve.

There is another problem called the flatness problem which is left unsolved by the standard model. As mentioned in the last section, the value of the density parameter is ~ 1 today. At the time of nucleosynthesis, its value must have been more close to unity. This suggests extremely fine tuned initial conditions which is extremely unlikely [22].

To solve these problems, Guth proposed an inflationary model in 1981 which suggests the exponential expansion of the universe in first 10^{-34}s. Inflation does not change or affect the standard model, it is just an add-on to this model. Inflation answers the questions that arise in both the above mentioned problems as follows. During inflation, the different parts of the universe must have been in a tiny region much smaller than the horizon. The critical density was higher during inflation and it was driven towards 1 by the end of inflation rather than away from it [22].

2.10 Composition of the Universe

So far I have mentioned the presence of a large amount of H, some He and traces of heavier elements in the universe. Also, CMB is evidence of the existence of photons. Now for each particle there exists an *anti-particle* having some properties (like mass, spin, etc.) identicle and others (like charge, helicity, etc.) opposite. For example, the anti-particle of the electron is the positron, that of the neutrino is the anti-neutrino etc. Anti-particles can combine to form anti-atoms, e.g., a positron revolving about an anti-proton is an anti-hydrogen atom. This group of all anti-atoms is called *anti-matter*. Very small amounts of anti-matter have been made in the laboratory.

In general, when anti-particles meet their particle counterparts they annihilate each other and give two photons. The question arises whether there could be a significant quantity of anti-matter in the universe. Since the matter and anti-matter would annihilate there can be no significant amount of anti-matter locally. It might be argued that a neighbouring star may be composed of anti-matter. However, this is ruled out because stellar wind from it would annihilate our solar wind and produce a sheet of very high energy photons that should then be seen as γ-rays on earth. This is not seen. One could propose that another galaxy may be made of anti-matter but that is ruled out by a similar argument, which further extends to supercluster of galaxies and thence to filaments. As such it is taken that there is no significant contribution by anti-matter. Infact, there are stringent limits on the quantity (ratio of anti-matter to the matter is $< 1.7 \times 10^{-6}$) [24].

In 1933, Fritz Zwicky concluded that the spiral arms of the galaxies should not be stable against orbital rotation due to the centrifugal force. The centrifugal force (repulsive: acting outwards) on a particle of mass m distant r from the central axis and with angular velocity ω is $F_{re} = mr\omega^2$. The gravitational force (attractive: acting inwards) is $F_{att} = GMm/r^2$. For observed stability, $F_{att} \geq F_{re}$. This implies

$$M \;\geq\; r^3\omega^2/G \; . \tag{2.10.1}$$

Now, in the spiral arms $v = r\omega \approx 250$km/s and distances go up to 10^{18}km [2]. This gives $M \sim 10^{45}$g. The total luminous matter in the galaxies is much less than this. Therefore most of the mass in the galaxies must be "dark". Inter-galactic dust, comets, planetary size freely floating bodies (that are called "Jupiters" or as John Wheeler called them "brownies") etc are viable candidates for it. This type of dark matter is called baryonic dark matter in accordance with its composition of baryons (neutron, proton, hydrogen and other light nuclei).

Evidence for dark matter also comes from the difference between the expected and the observed rotation curves. A rotation curve of a galaxy is the graph obtained by plotting the distance of stars from the center of the galaxy and their orbital speed. It was expected that such a curve should slope down but the observational data gave a rotation curve that is flat. This is called galaxy rotation problem and was pointed out for the first time be Vera Rubin [25]. Flat rotation curves indicated the presence of matter that was non-luminous.

The bending of light by invisible objects (called gravitational lensing) indicated the presence of dense massive objects at the center of a galaxy. These missing invisible objects are called "MACHOs" (massive compact halo objects). These are also included in baryonic type of dark matter. Another suggestion is of PBHs. But there are limitations on maximum amount of baryonic dark matter from big bang nucleosynthesis. Therefore a large amount of missing dark matter must be of non-baryonic type. Particle physics suggests some non-baryonic type of particles as candidates for dark matter, e.g., neutrinos (named as *left-handed* because their spin points in the opposite direction from their angular momentum). Neutrinos were assumed to be massless particles but now there are evidences showing that they have very small mass [26]. Also, neutrinoes have very high velocity and are called *hot dark matter*. But the structure formation suggests that the dark matter must be *cold dark mtter*, i.e., it must consist of non-relativistic particles. Thus, neutrinoes are no longer candidates for dark matter.

2.11 Dark Energy: an Explanation of the Observed Accelerated Expansion

In the early 1990's, there were two ideas about the energy density present in the universe: the universe might have enough energy density to stop its expansion and recollapse or it might have so little energy density that it would never stop expanding. But gravity was certain to slow the expansion as time went on. The universe is full of matter and the attractive force of gravity pulls all matter together. Although the slowing had not been observed, theoretically the universe had to slow. It has already been mentioned in section 2.5 that the observations of very distant type Ia supernovae showed that, a long time ago, the Universe was actually expanding more slowly than it is today.

Considering the RW metric, a strong constant negative pressure in all the universe causes an acceleration in universe expansion if the universe is already expanding, or a deceleration in universe contraction if the universe is already contracting. More exactly, the second derivative of the universe scale factor is positive if the equation of state of the universe is such that it has negative pressure.

The unsuccessful attempts of quantum gravity predicted a cosmological constant of Plank's scale while the observational value is about 126 or 127 order of magnitude different from it. This was called the worst prediction by Hawking. Thus the above mentioned repulsion was hard to explain. Then it was suggested that there was some strange kind of energy-fluid called dark energy that filled up the space. Dark energy is assumed to be a perfect fluid and it does not couple with ordinary matter and energy. In-fact, this energy behaves opposite to the normal matter and energy and over comes the gravitational attraction resulting in an accelerated expansion of the universe.

A 2011 survey of more than 200, 000 galaxies appears to confirm the existence of dark energy, although the exact physics behind it remains unknown. The thing that is needed to decide between dark energy possibilities (a property of space or a new dynamic fluid) is better experimental data. Measuring the EoS for dark energy is one of the biggest efforts in observational cosmology today which can decide between all the candidates responsible for the observed acceleration of the universe.

2.11.1 Cosmological Constant

The cosmological constant is a constant energy density filling up the space homogeneously. It is physically equivalent to vacuum energy. The EoS for this is $\omega = -1$. Adding the cosmological constant to cosmology's standard RW metric leads to the Lambda-CDM model which has been actually called the standard model of cosmology. EFEqs including cosmological constant are already given in section 1.2.6. Friedmann equations including the cosmological constant term are

$$3\left[\frac{\dot{a}^2}{a^2} + \frac{k}{a^2}\right] - \Lambda c^2 = 8\pi G\rho(t) \, , \tag{2.11.1}$$

which is one of the Friedmann equation. 2^{nd} one is

$$-2a\ddot{a} - \dot{a}^2 - kc^2 + 2\Lambda c^2 a^2 = \frac{8\pi G}{c^4} p a^2 \, , \tag{2.11.2}$$

2.11.2 Scalar Fields

Another proposed form of dark energy is such scalar fields whose energy density can vary in space and time. Lagrangian for such a scalar field is given by

$$L_\Phi = \frac{n}{2}\partial_\mu \Phi \partial^\mu \Phi - V(\Phi) \, , \tag{2.11.3}$$

where the scalar field Φ is time dependent and spatially homogeneous and n takes the value 1 or -1 as will be specified later. Contributions from scalar fields that are constant in space are usually also included in the cosmological constant.

Quintessence

Quintessence is such a scalar field whose EoS includes kinetic energy ($\frac{1}{2}\dot{\Phi}^2$) and potential energy ($V(\Phi)$) causing it to be dynamical. In case of the scalar field being quintessence, $n = 1$ in Eq. (2.11.3) and EoS is

$$\omega \;=\; \frac{\frac{1}{2}\dot{\Phi}^2 \;-\; V(\Phi)}{\frac{1}{2}\dot{\Phi}^2 \;+\; V(\Phi)} \tag{2.11.4}$$

and its range is $-1 < \omega < 0$. For constant fields, $\omega = -1$ which is the cosmological constant. Scalar fields which do change in space can be difficult to distinguish from a cosmological constant because the change may be extremely slow.

Chaplygin Gas

This is such a type of dark energy which changes its behavior with time depending on the scale factor of the expansion of the universe. At sufficiently early times, it behaves like a pressureless gas (dust) and for large $a(t)$, it behaves like a cosmological constant. Its EoS is defined to be

$$p \;=\; -\frac{X}{\rho c^2}\;, \tag{2.11.5}$$

where X is a non-zero constant parameter whose value can be either positive or negative.

Phantom Energy

So far I have discussed the cases with the EoS parameter in the range $0 \leq \omega \leq 1/3$ (dust and radiation), $\omega = -1$ (cosmological constant) and $-1 < \omega < 0$ (quintessence). Phantom energy is a type of dark energy suggested by Caldwell and his co-workers [4] with EoS $\omega < -1$ which is supported by the experimental data. For phantom energy $n = -1$ in Eq. (2.11.3). Thus EoS is

$$\omega \;=\; \frac{-\frac{1}{2}\dot{q}^2 \;-\; V(q)}{-\frac{1}{2}\dot{q}^2 \;+\; V(q)} \;=\; -\;\frac{V(q) \;+\; \frac{1}{2}\dot{q}^2}{V(q) \;-\; \frac{1}{2}\dot{q}^2} \;<\; -1\;. \tag{2.11.6}$$

The most striking property of phantom energy is the variation of its energy density with the scale factor:

$$\rho c^2 \;\propto\; a^{3|1+\omega|}\;. \tag{2.11.7}$$

This is clearly in contrast with the behavior of ordinary matter whose energy density decreases with the expansion of the universe. Thus, the energy density of phantom energy will become infinite in finite time causing the universe and all its contents (galaxies, solor system and then the atoms) to rip apart in their constituent particles. This singularity is commonly called the *big rip* [5].

In my present research work, I shall consider phantom energy as a cause of accelerated expansion of the universe and study the effect of phantom energy on the lifetime of a Schwarzchild BH (PBH in particular) evaporating due to Hawking evaporation. Considering the neutrino mass to be the one which is claimed in the result of a double beta decay experiment, analysis of astronomical data gives that $-1.67 < \omega < -1.05$ [26]. For our calculations, we shall use these values of ω.

Chapter 3

Hawking Radiation and Primordial Black Holes

In this chapter, first I shall discuss basic BH thermodynamics (for the Schwarzchild BH in particular). A background knowledge of classical thermodynamics and statistical mechanics is assumed on the part of the reader and I shall mention the laws of thermodynamics only for revision. Then I shall discuss the analogy of these laws for BHs; HR will come as a part of BH thermodynamics. I end this chapter with a discussion of PBHs.

3.1 The Laws of Thermodynamics

In thermodynamics, temperature is defined in the form of a law which states that if any system is in thermal equilibrium with two isolated systems, the two systems must be in thermal equilibrium with each other. It is obvious that this phenomenon is a consequence of the three systems being at the same temperature. This law is called the zeroth law of thermodynamics. The reason for such a "numbering" is that this law was stated after the first and second laws of thermodynamics but being at the base of the others needed to be placed before the first law. This law can also be stated as: if a system is in thermal equilibrium, its temperature is the same in all parts of the system; as otherwise heat must have flowed from the region of higher temperature to the region of lower temperature.

The sum of all forms of kinetic and potential energies of a system is called its internal energy. Also, energy is required by an object or a system to do some work. Energy is supplied to a thermodynamic system in the form of heat. The first law of thermodynamics states that the amount of heat supplied to a system is equal to the amount of work done by the system on its surroundings and the change in internal energy of the system. Mathematically,

$$đE = dU + đW , \tag{3.1.1}$$

where $đ$ is inexact differential, $đE$ is the energy added to the system, dU is the change in internal energy of the system and $đW$ is the amount of work done by the system.

Whenever energy is extracted from a (high temperature) heat reservoir to perform some work, not all of it can be converted to work; a part of this heat energy must be delivered to another (low temperature) heat reservoir, called a sink. This is called the second law of thermodynamics and is paraphrased as "there is no such thing as free lunch, one has to pay the bill" [27] (in the form of the energy disposed off to the sink).

The energy delivered to the low temperature reservoir becomes unavailable for performing work. Hence there is a net "disorder" produced in the system. This disorder is called the *entropy* of the system. The net change in entropy in a thermodynamical process is always non-negative. Entropy is an *extensive* variable (sensitive to the change in the extent or size of the system) and temperature of the system is the corresponding *intensive* variable. Mathematically

$$dS = \frac{đE}{T} , \tag{3.1.2}$$

where dS is the change in entropy of the system, $đE$ is the heat supplied to the system and T is the absolute temperature of the system.

The third law of thermodynamics states that absolute zero temperature can not be achieved in a finite number of isothermal (constant temperature) and adiabatic (constant heat) steps of cooling.

3.2 Black Hole Thermodynamics

To arrive at the analogy of the thermodynamic laws for BHs, I shall start with a paradox propounded by Penrose. Suppose there is a civilization living far away from the event horizon of a BH. It produces energy for its use by some mechanism and wastes energy, which is required to be there by second law of thermodynamics, is in form of radiation and is stored in a box that can be opened on top and bottom as per requirement. Now suppose that a spring is attached to the top of

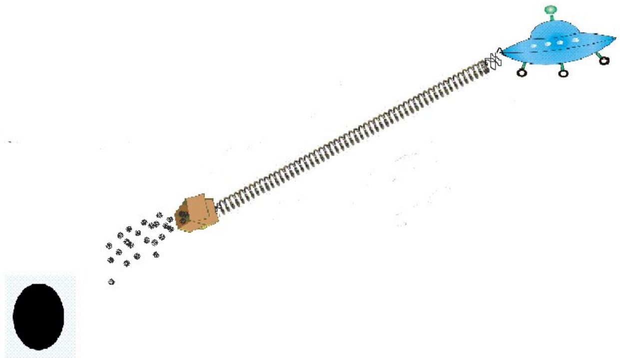

Figure 3.1: Paradox violating second law of thermodynamics .

this box. The box is lowered to the BH and its lower end is opened. The situation is shown in Fig. 3.1. The radiation will fall into the BH. Now the empty box can be lifted back. In this way usable energy has been "extracted" from the BH since the box is lighter when it is pulled back. This can be calculated as follows. Consider the Plank spectrum with function $f(\nu)$ where ν is the frequency of a particular radiation. Now from the Einstein energy-mass equivalence relationship and $E_\nu = h\nu$, it can be written

$$m_\nu = \frac{h\nu}{c^2} . \tag{3.2.1}$$

Let V be the volume of the box, then the energy obtained in each step is [28]

$$E = V \int_0^\infty \frac{h\nu}{c^2} f(\nu) dV . \tag{3.2.2}$$

This is contradictory with the second law; the civilization is being paid for having a lunch [27] as it gets extra usable energy in addition to getting rid of the waste energy !

3.2.1 Entropy of a Black Hole

It is clear that to save second law of thermodynamics in the vicinity of a BH, it is needed to introduce the concept of entropy for a BH. This made Bekenstein to consider that in the above mentioned paradox, mass of the BH is increased by the addition of radiation to it. This in turn increases the radius of the BH as is given by Eq. (1.3.16). Ultimately, the surface area of the BH, being proportional to the square of the radius, increases. Thus, entropy of a BH can be described in terms of its area. Now it was required to define an intensive variable analogous to the absolute temperature. It was proposed that surface gravity could serve this purpose since it is non-negative. Surface gravity of a BH is given by

$$\kappa = \frac{c^4}{4GM} \ . \qquad (3.2.3)$$

Thus the above stated paradox can be resolved by stating that although the entropy of the civilization decreases but the net entropy of the civilization and its surroundings (including BH) is non-decreasing. Entropy associated with the BH is

$$S = \frac{k_B A_H}{4 l_P^2} \ , \qquad (3.2.4)$$

where k_B is Boltzman constant, $A_H \equiv 16\pi GM^2/c^4$ is the area of event horizon and $l_P \equiv \sqrt{G\hbar/c^3} \sim 10^{-33}$cm is Plank length. Therefore $S \sim M^2$ for BHs.

3.2.2 Hawking Radiation

In classical thermodynamics, entropy and the associated temperature are related as

$$\frac{1}{T} = \frac{\partial S}{\partial E} \ . \qquad (3.2.5)$$

Since for BHs, $S \sim M^2$, therefore

$$\frac{1}{T} = \frac{\partial S}{\partial E} \sim \frac{\partial M^2}{\partial M} \sim M \ . \qquad (3.2.6)$$

If the BH has a certain temperature, then it must radiate like a hot body. This is impossible classically. Quantum effects are usually ignored in calculations of the formation and evolution of BHs because the radius of curvature of spacetime outside the event horizon is very large compared to the Planck length.

In 1973, Hawking visited Moscow where he was showed by Soviet scientists Yakov Zeldovich and Alexander Starobinsky that the rotating BH should create and emit particles by uncertainity principle. On his return, Hawking started studying quantum effects for BHs. Although the quantum mechanical effects are small for BHs but they become considerable when taken over a long period of time such as the age of the universe ($\sim 10^{17}$)s. In an attempt to construct a theory of quantum gravity, as a "half-way-house", Hawking tried quantizing scalar fields in a classical curved spacetime background of a Schwarzschild BH. When temperature of a BH exceeds the rest mass energy of a particle of certain type, it emits particles and antiparticles of that particular type in addition to photons. Quantum mechanically, it can be thought like the tunneling of particles through a potential well.

Primarily, the idea of emission of particles from a BH was originally about the rotating BH, yet it was found that the BHs continue to radiate even when they become effectively static. It was found by Hawking that a BH will create and emit particles at such a rate as if it were a black body with temperature inversely proportional to mass which is given by [6]

$$T = \frac{\hbar c^3}{8\pi M k_B} \approx 10^{-7}\left(\frac{M}{M_s}\right)^{-1} K\ , \tag{3.2.7}$$

where M_s and M are the masses of the sun and BH respectively. A 30 solar mass BH has temperature 2×10^{-9}K. To radiate, temperature of a BH should be greater than the background radiation (CMB) which corresponds to a specific value of mass. Currently, a 10^5g BH would be in thermal equilibrium with its surroundings. Obviously a BH of mass greater than this will not radiate. But as the universe will cool down more with its expansion, it will start radiating after some time from today.

3.2.3 Evaporation Rate of Black Holes

The mass of a BH decreases by emission of radiation, thus the process is called BH evaporation. This decrease in mass results in the increase in the surface gravity of the BH which consequently increases the rate of emission. This increase in emission rate is clear from the following relation for the rate of emission of radiation from a BH [3]

$$\left.\frac{dM}{dt}\right|_{hr} = -\frac{\hbar c^4}{G^2}\frac{\alpha}{M^2}\ , \tag{3.2.8}$$

where the subscript hr stands for HR and α is the spin parameter of emitting particles. Eq. (3.2.8) shows that the emission becomes drastic near the end of mass of BHs. On integrating Eq. (3.2.8), following formula is obtained

$$T_{hr} = \frac{G^2}{3\hbar\alpha c^4}M_i^3\ . \tag{3.2.9}$$

Hawking had calculated that the energy emitted in the last 0.1s before the vanishing of a BH would be about 10^{30} erg. Thus BHs end with an explosion.

When a BH evaporates, its surface area and consequently entropy decreases. But entropy of surrounding increases keeping a net change of entropy to be non-negative and hence satisfying the laws of BH thermodynamics.

Hawking Radiations from Smaller Mass Black Holes

The effect of the HR is too small for larger mass BHs (like solar mass BHs) as compared to the case of smaller mass BHs since the evaporation time is proportional to M^3 as given in Eq. (3.2.9) The evaporation time for a 30 solar mass BH is 10^{61} times the age of the Universe. But smaller micro BHs would dissipate faster.

Luminosity of a Black Hole

Luminosity of a black hole is given by the Stephen-Boltzman law for black body radiation

$$L = A\sigma T^4 , \tag{3.2.10}$$

where A is the surface area, σ is Stefan-Boltzman constant given by $\sigma = \pi^2 k^4/60c^2\hbar^3$. The BH luminosity is given by

$$L = A_H(n_{eff}/2)\sigma T^4 , \tag{3.2.11}$$

where n_{eff} is the effective number of the types of the emitted particles. In case of no other particle emission except the photon, $n_{eff} = 2$ because of the polarization of photons.

Since $r_s \propto M^2$ and $T \propto 1/M$, Eq. (3.2.11) implies $L \propto 1/M^2$. Thus larger mass BHs are dimmer. An evaporating BH can be detected from earth only if it is at a distance equal to that of the nearest star.

3.3 Gravitational Collapse

Gravitational collapse is the inward fall of an object (or a clump of matter) due to higher (attractive) gravitational force than the outward (repulsive) pressure. This process of collapse continues until there is again an equilibrium established between gravity and the pressure. The process of gravitational collapse is the cause of the formation of stellar objects like white dwarfs, neutron stars and BHs.

The end product of gravitational collapse is determined by the mass of the star which starts to collapse. Huge clouds of dust (about 21 light years across), called nebula, result in the formation of stars by the process of gravitational collapse. Inside star, heavier elements are produced by the process of thermonuclear fusion. This process also provides repulsive pressure to the gravity. When a star has burnt all its fuel (hydrogen), the repulsive force is switched off and it starts to collapse under gravity until it becomes white dwarf. If the initial mass of the star was less than $1.5M_s$, the equilibrium between the pressure and gravity will be attained. But if the mass is greater than this limit, it will collapse further to become a neutron star where there is again an equilibrium is established between the two forces. Now if the initial mass of the star was more than $3.2M_s$, it will collapse into a BH. These mass limits were found for the first time by Chandrasekhar and named after him as Chandrasekhar limit [29].

3.4 Accretion onto a Black Hole

Once a BH is formed because of gravitational collapse (of stellar objects or dense regions), it can still accumulate more mass from its surroundings and hence can grow in mass and size. This process of accumulating mass is called accretion.

Due to its highest gravitational force, any particle or object passing by is influenced by it and starts to spiral down into the BH. Sometime it is ejected back even then a fraction of mass is retained within the BH. This phenomenon was predicted theoretically but has been luckily captured recently. When a BH accretes mass comparable to its initial mass, its spin changes.

When the objects or particles start spiralling about a BH, they spiral in to the BH. These spirals of different particles lying in the same plane form of a disk called *accretion disk*. The particles in this disk emit radiation making the disk visible (in infra-red region) at the cost of their gravitational binding energy. Binary star system is also a phenomenon of accretion and has already been discussed in Chapter 2.

3.5 Primordial Black Holes

Primordial black holes are thought to be formed in the beginning of the universe mainly because of gravitational collapse of high density regions. It should be noted that only the high density of the universe is not the reason of PBH formation; rather there are density and pressure fluctuations (primordial inhomogeneities) which are spherically symmetric and undergo gravitational collapse whenever gravitational force over comes the pressure. When the pressure is less than the gravitational force, EoS is said to be *soft*. This can happen only when the particles are massive and non-relativistic [6]. Thus, the role of pressure to resist gravitational collapse is quite ignorable and all the fraction of the matter in the universe which is in form of spherical dense regions undergoes gravitational collapse in that period of soft EoS. Infact, this was found by Sir James Jeans that under this condition a dense region can accrete more matter from surroundings and become more dense resulting in the formation of a gravitationally bound object by collapsing. The instability is called Jeans gravitational instability. Acoustic perturbations across a Jeans length cause a Jeans mass to collapse over a characteristic time.

Here I would also clear that this gravitational collapse resulting in the formation of PBHs is different from gravitational collapse of stellar objects like neutron stars, white dwarfs etc. This is because at the time of PBH formation, atoms even nucleons are not formed. Matter density consists of more fundamental particles (quark-gluon plasma which is not to be discussed here). There are other mechanisms too which are thought to lead to PBH formation like *bubble collision*, collapse of *cosmic loops* and collapse of *domain walls* but I shall not discuss them here.

3.5.1 Mass Range of PBHs and its Significance

Density of a BH is obviously $M/r_s^3 \sim M^{-2}$. By comparing this density with the density of the universe at a time t after the big bang, initial mass of PBHs (M_{PBH}) is found to be [6]

$$M_{PBH} \approx \frac{c^3 t}{G} \approx 10^{15} \left(\frac{t}{10^{-23} s} \right) g \; . \tag{3.5.1}$$

The above equation gives mass to be 10^{-5}g for $t = 10^{-43}$s and $10^5 M_s$ for $t = 1$s. Thus, there is very wide range of the initial mass of PBHs. Eq. (3.2.9) gives time of evaporation to be $\sim 10^{17}$s for mass 10^{15}g which is of the order of the age of the universe (4.3392×10^{17}s). Thus a PBH of initial mass 10^{15}g would be evaporating today. This implies that there is chance of observing explosion of PBHs at present. It is also suggested that PBHs might have grown in size by accreting matter and radiation with time and have become the massive BHs present in the galaxies. However, as I have mentioned earlier that matter and radiation accretion will be ignored, PBHs evaporating today will be the main focus of my present work. PBHs are divided in four classes on the basis of initial mass range. These mass ranges and their significance is discussed below.

Probe of Early Universe ($M_{PBH} < 10^{15}$g). As mentioned above that BHs of mass 10^{15}g would be evaporating today, PBHs of mass less than this must have evaporated earlier. Their evaporation rate will also be high even at the start as indicated by Eq. (3.2.8). Also, since PBHs are formed before 1s, their evaporation could affect the details of nucleosynthesis [6].

Probe of Gravitational Collapse ($M_{PBH} > 10^{15}$g). It has been suggested that massive BHs that reside at the center of galaxies are the PBHs which grew in size by accreting mass with time. It is also suggested that *clusters* of PBHs condensed to form super-massive BHs. It has also been suggested that PBHs may stop evaporating at Plank mass scale and contribute to the dark matter.

Probe of High Energy Physics ($M_{PBH} \sim 10^{15}$g). Since these are the BHs evaporating today, HR coming from these should contribute in cosmic rays and hence should be detected. Since there are observational limits on the cosmic rays (100MeV), this may lead to the disproving PBHs of this mass.

Probe of Quantum Gravity ($M_{PBH} < 10^{-5}$g). This mass PBHs can be a way of studying quantum gravity because they can reach to Plank mass by Hawking evaporation. Therefore, it may become possible to study quantum gravity effects on TeV scale. This can result in the production of mini BHs in accelerators and cosmic events.

Chapter 4

Accretion of Phantom Energy on Black Holes - a Review

In this chapter, first a paper by Babichev et al [8] on accretion of phantom energy on a Schwarzchild BH will be discussed. In this paper, the rate of change of mass is computed for a Schwarzchild BH and it is found that the mass of a Schwarzchild BH decreases with time by the accretion of phantom energy. Then some part of [9] will be reviewed which deals with the relative significance of Hawking evaporation process and evaporation due to phantom energy accretion as a function of mass of the evaporating BH. Then the work done in [10] to estimate the effect of phantom energy accretion on the initial mass of a PBH evaporating today in addition to HR will be reviewed and a few mistakes will be pointed out. All the analysis given in this chapter is for Schwarzchild BH only.

4.1 Rate of Change of Mass of a Schwarzchild BH due to Phantom Energy Accretion

Using the stress-energy tensor given in Eq. (2.6.1) and the Schwarzchild metric given in Eq. (1.3.15), Babichev et al derived the rate of change of mass of a Schwarzchild BH due to dark energy accretion. The resultant rate of change of mass is [8]

$$\left.\frac{dM}{dt}\right|_p = \frac{4\pi AG^2}{c^5}\,[\rho_\infty c^2 \ + \ p(\rho_\infty)]M^2 \ , \tag{4.1.1}$$

where the subscript p astands for phantom energy and A is a constant of integration which is taken to be 4 for unstable fluid like phantom energy [8] and ρ_∞ is the value of energy density at infinite distance from the BH, i.e., the density of the background. For phantom energy, the quantity in the bracket comes out to be negative. Therefore, the mass is *effectively decreasing* with time. Thus "accretion" of phantom energy is another process by which the BH effectively loses its mass and energy. Also that this evaporation process is significant for higher mass BHs and insignificant for smaller mass BHs. Here I want to clarify that the "accretion" of any form of dark energy does not resemble the accretion of matter. As discussed in Chapter 2, dark energy effectively acts like some kind of anti-gravity force. Neglecting the cosmological evolution of ρ_∞ and using $A = 4$, integration of Eq. (4.1.1) gives

$$-\frac{1}{M} = \frac{16\pi G^2}{c^5}[\rho_\infty c^2 + p(\rho_\infty)]t + B , \tag{4.1.2}$$

where B is the constant of integration. Using the condition that at start when $t = 0$, $M = M_i$ gives $B = -1/M_i$. Thus Eq. (4.1.2) becomes

$$\begin{aligned} -\frac{1}{M} &= \frac{16\pi G^2}{c^5}[\rho_\infty c^2 + p(\rho_\infty)]t - \frac{1}{M_i} , \\ &= -\frac{1}{M_i}\left[1 - \frac{16\pi G^2 M_i(\rho_\infty c^2 + p(\rho_\infty))}{c^5} t\right] . \end{aligned} \tag{4.1.3}$$

Defining $t_x = [16\pi G^2 M_i(\rho_\infty c^2 + p(\rho_\infty))/c^5]^{-1}$, Eq. (4.1.3) can be written as

$$M = M_i\left(1 - \frac{t}{t_x}\right)^{-1} . \tag{4.1.4}$$

Above equation also describes the mass evolution of a BH of some initial mass M_i after time t.

4.2 "Competition" between HR and Phantom Energy Accretion

Till now two different mechanisms of the evaporation of BHs have been discussed: emission of HR and the accretion of phantom energy. The former has its basis in quantum mechanics and the latter is based on GR. However, the effect of both on the mass of a BH is the same, i.e., the evaporation of the BH. But the accretion of matter and radiation increases the BH mass. The evolution of PBHs under the combined effect of HR, accretion of phantom energy and radiation accretion is studied by Guariento et al in 2008 [9]. In 2011 Nayak and Singh discussed the increase in lifetime of PBHs when radiation accretion is taken into account in addition to HR [30]. Also in 2011, Nayak and Jamil have discussed the accretion of matter, radiation and vacuum energy on PBHs [31]. But due to the interesting features associated with BH evaporation and the observed acceleration of the universe, it must be of importance to study the effect of the mass decreasing phenomena only. This has been done a bit by Guariento et al in [9] and by Jamil and Qadir in [10]. In this section, only section 5 of [9] will be discussed.

As we have seen that although the two mechanisms (HR and phantom energy accretion) tend to decrease the mass of a BH, but they have a dependency on the mass of the BH which is opposite to each other in behavior. The former goes as M^2 and the latter as $1/M^2$. Thus for higher masses, when the HR process is negligible, phantom energy accretion is more effective and vice versa. Moreover, accretion of phantom energy can rapidly decrease down the mass of a sufficiently high "initial" mass BH to such an extent that HR becomes significant and continues till the last explosion with the effect of accretion of phantom energy becoming less and less significant with time. Thus there is a sort of "competition" (as it is called in [9]) between the two evaporation processes. Taking the ratio of the two evaporation rates given in Eqs. (4.1.1) and (3.2.8) to be an arbitrary number ξ gives [9]

$$\xi(M) \;=\; \frac{4\pi A G^2[\rho_\infty c^2 \;+\; p(\rho_\infty)]/c^5}{\hbar c^4\alpha/G^2} M^4 \; . \tag{4.2.1}$$

Using $A = 4$ and EoS of phantom energy $p = \rho c^2\omega$ in the above equation gives

$$\xi(M) \;=\; \frac{16\pi G^4\rho_\infty[1 \;+\; \omega]}{\hbar c^4\alpha(M)c^3} M^4 \; . \tag{4.2.2}$$

Defining

$$M_t \;=\; [bc^3/16\pi G^2\rho_\infty(1 \;+\; \omega)]^{1/4} \tag{4.2.3}$$

as "transition mass" and using it in Eq. (4.2.2) gives

$$\xi(M) \;=\; \left(\frac{M}{M_t}\right)^{\frac{1}{4}} \; . \tag{4.2.4}$$

Putting the values of the constants the above equation gives

$$M_t \;\cong\; 5.5\times 10^{17}[(1 \;+\; \omega)\rho_\infty]^{-\frac{1}{4}} g \; , \tag{4.2.5}$$

where ρ_∞ is measured in g/cm^3. The BH mass will decrease mainly due to phantom energy accretion until it reaches M_t. The Hawking evaporation process will be the predominant one after this.

4.3 Evaporation of a Schwarzchild BH due to Hawking Radiation and Phantom Energy Accretion

In 2006, Jamil and Qadir discussed the evaporation of a BH due to the combined effect of accretion of phantom energy and Hawking evaporation in "Primordial Black Holes in Phantom Cosmology" [10]. Here that paper will be reviewed and a few errors will be pointed out. It was corrected by me and A. Qadir in one section of [11] (to be discussed fully in the next chapter). Here the calculations and analysis will be done in the same way as it is done in [10].

Consider the evolution of the scale factor [32]

$$a(t) \;=\; \frac{a(t_o)}{[-\omega + (1+\omega)t/t_o]^{-2/3(1+\omega)}} \;, \qquad for\ t > t_o \tag{4.3.1}$$

where t_o is the time of domination of dark energy [32]. For phantom energy, $a(t) \to \infty$ which implies the denominator of the above equation should be zero. Thus

$$t \equiv t_* \;=\; -\frac{\omega}{1+\omega} t_o \;. \qquad for\ t_* > t_o \tag{4.3.2}$$

Subtracting t_o from both sides of the above equation

$$t_* \;-\; t_o \;=\; \frac{1}{1+\omega} t_o \;. \qquad for\ t_* > t_o \tag{4.3.3}$$

Evolution of energy density of phantom energy is given by [10]

$$\rho_\infty^{-1} \;=\; 6\pi G(1+\omega)^2 (t_* - t_o)^2 \;. \tag{4.3.4}$$

Putting $A = 4$, $p = \omega\rho c^2$ and uising Eqs. (4.3.3) and (4.3.4) in Eq. (4.1.1) gives

$$\frac{dM}{dt} \;=\; \frac{8G}{3c^3}\frac{M^2}{t_o^2}(1+\omega) \;. \tag{4.3.5}$$

The decrease in mass of a BH due to the combined effect of Hawking Radiation and phantom energy can be obtained by [10]

$$\frac{dM}{dt} = \left.\frac{dM}{dt}\right|_{hr} + \left.\frac{dM}{dt}\right|_{p} , \tag{4.3.6}$$

Using Eqs. (3.2.8) and (4.3.5)

$$\frac{dM}{dt} = -\frac{\hbar c^4}{G^2}\frac{\alpha}{M^2} + \frac{8G}{3c^3}\frac{M^2}{t_o^2}(1+\omega) . \tag{4.3.7}$$

Since for phantom energy $\omega < -1$, we may write

$$\omega = -1 - \epsilon , \tag{4.3.8}$$

where $\epsilon > 0$. Rewriting Eq. (4.3.7) as

$$\frac{dM}{dt} = -aM^2 - \frac{b}{M^2} , \tag{4.3.9}$$

where

$$a \equiv \frac{8\epsilon G}{3c^3 t_0^2} , \tag{4.3.10}$$

$$b \equiv \frac{\hbar c^4 \alpha}{G^2} . \tag{4.3.11}$$

Eq. (4.3.9) can be written as

$$-\int dt = \frac{1}{b}\int \frac{M^2}{1+\frac{a}{b}M^4} dM . \tag{4.3.12}$$

To integrate the above equation, we use a new variable x given by

$$x = \left(\frac{a}{b}\right)^{\frac{1}{4}} M , \tag{4.3.13}$$

which yields

$$-\int dt = \frac{1}{(a^3 b)^{\frac{1}{4}}}\int \frac{x^2}{1+x^4} dx . \tag{4.3.14}$$

Consider [33]

$$\int \frac{x^{m-1}}{1+x^{2n}} = -\frac{1}{2n}\sum_{k=1}^{n}\cos\left(\frac{m\pi(2k-1)}{2n}\right) ln\left|1-2x\cos(\frac{2k-1}{2n})\pi + x^2\right| + \frac{1}{n}\sum_{k=1}^{n}\sin\left(\frac{m\pi(2k-1)}{2n}\right)\tan^{-1}\left(\frac{x-\cos(\frac{2k-1}{2n})\pi}{\sin(\frac{2k-1}{2n})\pi}\right) . \tag{4.3.15}$$

In above case $m = 3$ and $n = 2$, hence the above equation yields

$$\int \frac{x^2}{1+x^4}dx = \frac{1}{4\sqrt{2}}\left[\ln|1-\sqrt{2}x+x^2| - \ln|1+\sqrt{2}x+x^2|\right] + \frac{1}{2\sqrt{2}}\left[\tan^{-1}\left(\sqrt{2}x-1\right) + \tan^{-1}\left(\sqrt{2}x+1\right)\right] \quad (4.3.16)$$

In [10], following equation is given by using $\ln(a/b) = \ln a - \ln b$ and an identity for addition of the inverse tangent functions in Eq. (4.3.16)

$$\int \frac{x^2}{1+x^4}dx = \frac{1}{4\sqrt{2}} ln\left|\frac{1-\sqrt{2}x+x^2}{1+\sqrt{2}x+x^2}\right| + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{1+x^2}\right) , \quad (4.3.17)$$

which is incorrect as will be explained in the next chapter. Also, in [10] following equation is given after substituting value of x in the above equation and putting it in Eq. (4.3.20)

$$t = t_o + \frac{1}{4\sqrt{2}} ln\left|\frac{1-\sqrt{2}(\frac{a}{b})^{\frac{1}{4}}M + (\frac{a}{b})^{\frac{1}{2}}M^2}{1+\sqrt{2}(\frac{a}{b})^{\frac{1}{4}}M + (\frac{a}{b})^{\frac{1}{2}}M^2}\right| + \frac{1}{2\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}(\frac{a}{b})^{\frac{1}{4}}M}{1+(\frac{a}{b})^{\frac{1}{2}}M^2}\right) . \quad (4.3.18)$$

But it has a few errors which will be corrected in the next chapter. Proceeding in the same way as is done in [10], define

$$M \equiv mM_i , \quad (4.3.19)$$

where $m = [0, 1]$, Eq. (4.3.9) becomes

$$\frac{d(mM_i)}{dt} = -a(mM_i)^2 - \frac{b}{(mM_i)^2} ,$$

$$\frac{dm}{dt} = -(aM_i)m^2 - \frac{b}{M_i^3}\frac{1}{m^2} ,$$

$$= -a'm^2 - \frac{b'}{m^2} , \quad (4.3.20)$$

where

$$a' \equiv aM_i , \quad (4.3.21)$$

$$b' \equiv \frac{b}{M_i^3} . \quad (4.3.22)$$

For the terms to be of equal strength, it is required that $a' \approx b'$. Thus

$$M_i \;\approx\; \left(\frac{b}{a}\right)^{\frac{1}{4}} . \tag{4.3.23}$$

Now

$$\frac{b}{a} \;=\; \frac{3\hbar c^7 t_o^2 \alpha}{8G^3 \epsilon} \; or \; \epsilon \;=\; \frac{3\hbar c^7 t_o^2 \alpha}{8G^3 M_i^4} . \tag{4.3.24}$$

Using Eqs. (4.3.19,4.3.21,4.3.22) in Eq. (4.3.18)

$$\begin{aligned} t \;&=\; t_o \left[1 - \frac{1}{t_o} \frac{1}{4\sqrt{2}} ln \left| \frac{1 - \sqrt{2}(\frac{a'}{b'})^{\frac{1}{4}} m + (\frac{a'}{b'})^{\frac{1}{2}} m^2}{1 + \sqrt{2}(\frac{a'}{b'})^{\frac{1}{4}} m + (\frac{a'}{b'})^{\frac{1}{2}} m^2} \right| \right. \\ &\left. - \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}(\frac{a'}{b'})^{\frac{1}{4}} m}{1 + (\frac{a'}{b'})^{\frac{1}{2}} m^2} \right) \right] . \end{aligned} \tag{4.3.25}$$

Normalizing above equation gives

$$t = t_o \left[1 - \frac{\frac{1}{4\sqrt{2}} ln \left| \frac{1-\sqrt{2}(\frac{a'}{b'})^{\frac{1}{4}} m + (\frac{a'}{b'})^{\frac{1}{2}} m^2}{1+\sqrt{2}(\frac{a'}{b'})^{\frac{1}{4}} m + (\frac{a'}{b'})^{\frac{1}{2}} m^2} \right| + \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}(\frac{a'}{b'})^{\frac{1}{4}} m}{1+(\frac{a'}{b'})^{\frac{1}{2}} m^2} \right)}{\frac{1}{4\sqrt{2}} ln \left| \frac{1-\sqrt{2}(\frac{a'}{b'})^{\frac{1}{4}} + (\frac{a'}{b'})^{\frac{1}{2}}}{1+\sqrt{2}(\frac{a'}{b'})^{\frac{1}{4}} + (\frac{a'}{b'})^{\frac{1}{2}}} \right| + \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}(\frac{a'}{b'})^{\frac{1}{4}}}{1+(\frac{a'}{b'})^{\frac{1}{2}}} \right)} \right] . \tag{4.3.26}$$

Using

$$P \;\equiv\; \frac{a'}{b'} \;=\; \frac{a}{b} M_i^4 \tag{4.3.27}$$

in above equation gives

$$t \;=\; t_o \left[1 - \frac{\frac{1}{4\sqrt{2}} ln \left| \frac{1-\sqrt{2}P^{\frac{1}{4}} m + P^{\frac{1}{2}} m^2}{1+\sqrt{2}P^{\frac{1}{4}} m + P^{\frac{1}{2}} m^2} \right| + \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}P^{\frac{1}{4}} m}{1+P^{\frac{1}{2}} m^2} \right)}{\frac{1}{4\sqrt{2}} ln \left| \frac{1-\sqrt{2}P^{\frac{1}{4}} + P^{\frac{1}{2}}}{1+\sqrt{2}P^{\frac{1}{4}} + P^{\frac{1}{2}}} \right| + \frac{1}{2\sqrt{2}} \tan^{-1} \left(\frac{\sqrt{2}P^{\frac{1}{4}}}{1+P^{\frac{1}{2}}} \right)} \right] . \tag{4.3.28}$$

Graphs plotted between normalized time $\tau \equiv t/t_o$ and dimensionless parameter m in [10] as shown in Figs. $4.1 - 4.5$ for $p = 0.1, 0.5, 1, 5, 10$ respectively. The curves lie in physically admissible region for $p \leq 1$. For $p > 1$, certain part of the graphs lies in the negative region. This is interpreted as "re-scaling of the initial mass" in [10]. It is said that the initial mass should be only 45 percent of the actual value for $p = 5$; similarly it should be 31.5 percent of that for $p = 10$. Thus upper limit of m is taken to be 0.45 and 0.315 for $p = 5$ and $p = 10$ respectively instead of 1. Quite clearly, by doing so the inadmissible part of the curves is dropped out resulting in curves lying in the physically admissible region only.

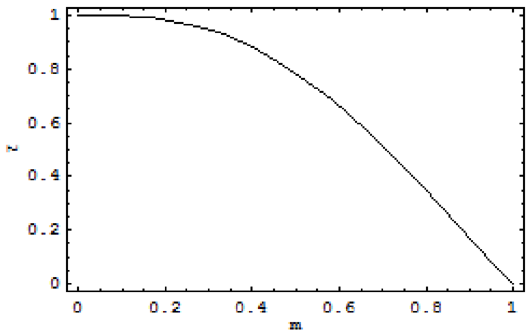

Figure 4.1: τ plotted against m=[0, 1] for P=0.1 .

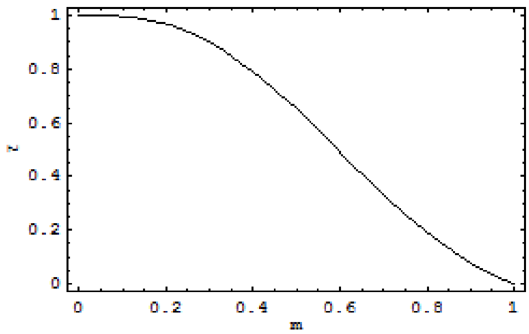

Figure 4.2: τ plotted against m=[0, 1] for P=0.5 .

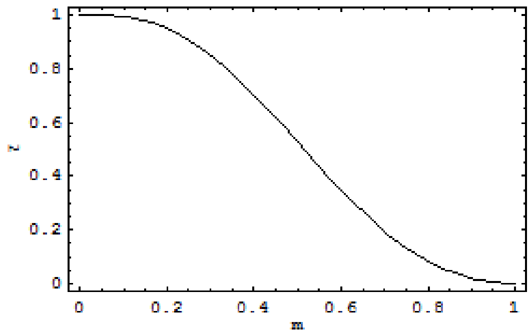

Figure 4.3: τ plotted against m=[0, 1] for P=1 .

These graphs are given in Figs. 4.6 − 4.7. Figs. 4.8 − 4.9 contain the inverted graphs for $p = 5$ and $p = 10$ when initial mass is re-scaled [10].

Also, using Eqs. (4.3.23) and (4.3.24) it was suggested in [10] that $M_i \sim 10^{23}$g to have a BH evaporating today under the combined effect of HR and phantom energy. The concept behind this prediction is also wrong and will be clarified in the next chapter. Also, we shall find the initial mass for this case.

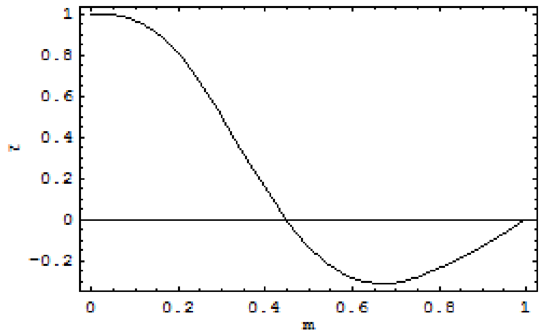

Figure 4.4: τ plotted against m=[0, 1] for P=5 .

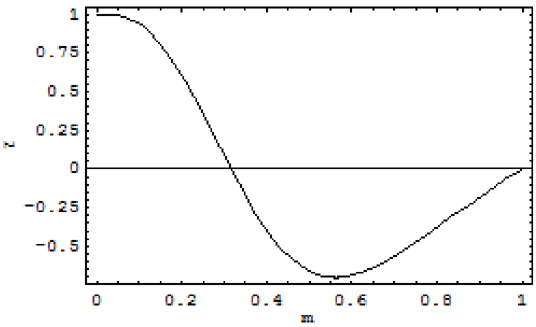

Figure 4.5: τ plotted against m=[0, 1] for P=10 .

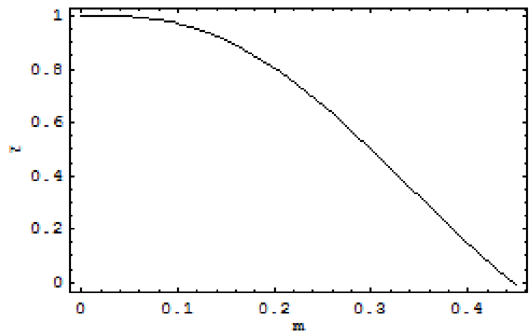

Figure 4.6: τ plotted against m=[0, 0.45] for P=5 .

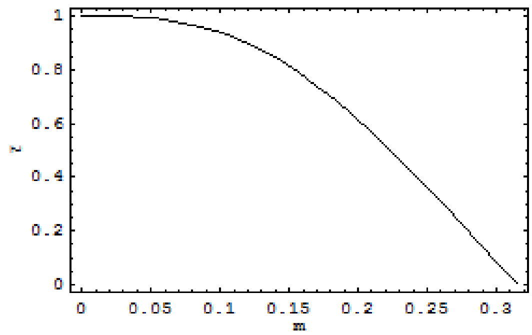

Figure 4.7: τ plotted against m=[0, 0.315] for P=10 .

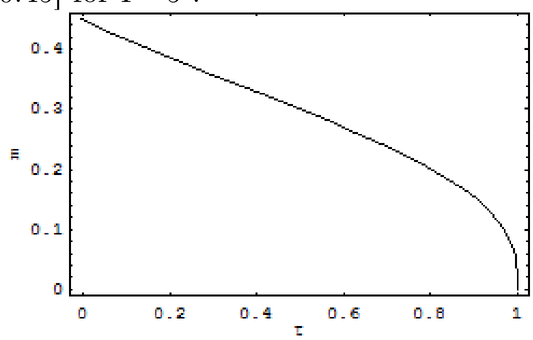

Figure 4.8: m=[0, 0.45] plotted against τ for P=5 .

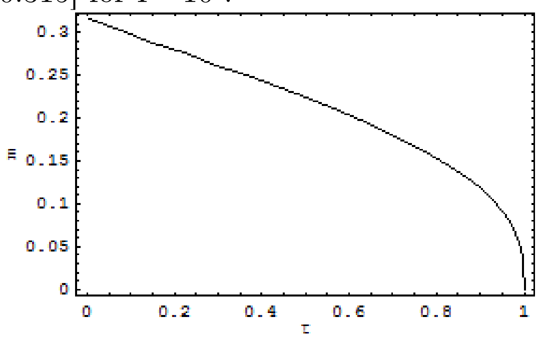

Figure 4.9: m=[0, 0.315] plotted against τ for P=10 .

Chapter 5

Transition Mass of a BH and Initial Mass of a PBH Evaporating Today

This chapter consists of three sections. In the first section, the algebraic errors in [10] will been corrected and the results obtained after these corrections will be given [11]. In the second section, the concept of the transition mass, introduced in the previous chapter, will be discussed referring to [9] and results of our calculations will be provided [11] which will clarify the misconception of taking the time of domination of dark energy as the lifetime of a BH in [10]. Then the initial mass of a PBH evaporating today under the combined effect of HR and phantom energy will be discussed and all the rigorous analysis done to find it in [11] will be given. For completeness, some limits of mass of a BH to exhibit observable effect of phantom energy by its evaporation will be given [12] and evaporation of massive BHs due to phantom energy and HR will also be discussed.

5.1 Correction of Some Formulas

We start with the integration of Eq. (4.3.12) which gives

$$\begin{aligned} -t \quad = \quad & \frac{1}{4\sqrt{2}(a^3 b)^{\frac{1}{4}}} \left[ln \left| \frac{1-\sqrt{2}(\frac{a}{b})^{\frac{1}{4}}M + (\frac{a}{b})^{\frac{1}{2}}M^2}{1+\sqrt{2}(\frac{a}{b})^{\frac{1}{4}}M + (\frac{a}{b})^{\frac{1}{2}}M^2} \right| \right. \\ & \left. + \; 2tan^{-1}\left(1+\sqrt{2}\left(\frac{a}{b}\right)^{\frac{1}{4}}M\right) \; - \; 2tan^{-1}\left(1-\sqrt{2}\left(\frac{a}{b}\right)^{\frac{1}{4}}M\right)\right] \; + A_1 \qquad (5.1.1) \end{aligned}$$

where t =age of universe when mass of black hole is M and A_1 is a constant of integration. Using the initial condition that black hole mass vanishes at t_f, we

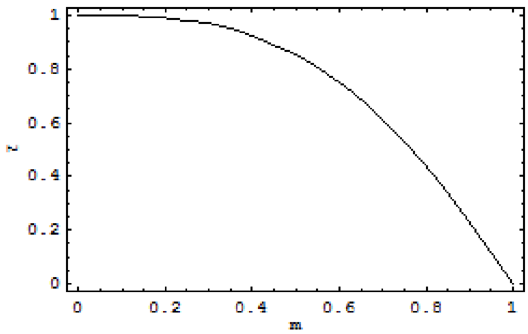

Figure 5.1: τ plotted against m=[0, 1] for P=0.5 .

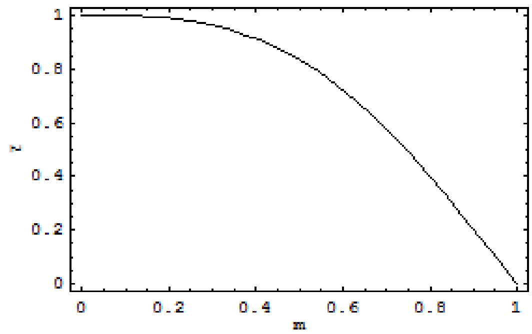

Figure 5.2: τ plotted against m=[0, 1] for P=1 .

get

$$A_1 \;=\; -\; t_f \;. \tag{5.1.2}$$

In [10], t_o was used confusingly instead of t_f. Also the negative sign was ignored. Putting the value of A_1, Eq. (5.1.1) gives

$$\begin{aligned} t_f - t \quad &= \quad \frac{1}{4\sqrt{2}(a^3b)^{\frac{1}{4}}}\left[ln\left|\frac{1-\sqrt{2}(\frac{a}{b})^{\frac{1}{4}}M+(\frac{a}{b})^{\frac{1}{2}}M^2}{1+\sqrt{2}(\frac{a}{b})^{\frac{1}{4}}M+(\frac{a}{b})^{\frac{1}{2}}M^2}\right|\right. \\ &\left. + \; 2\; tan^{-1}\left(1+\sqrt{2}\left(\frac{a}{b}\right)^{\frac{1}{4}}M\right) \;-\; 2\; tan^{-1}\left(1-\sqrt{2}\left(\frac{a}{b}\right)^{\frac{1}{4}}M\right)\right] \end{aligned} \tag{5.1.3}$$

Now by trigonometric identity

$$\tan^{-1}(x) \;-\; \tan^{-1}(y) \;=\; \tan^{-1}\left(\frac{x\;-\;y}{1\;+\;xy}\right) \;+\; n\pi \;, \tag{5.1.4}$$

where n is an integer, we can write Eq. (5.1.3) as

$$\begin{aligned} t_f - t \quad &= \quad \frac{1}{4\sqrt{2}(a^3b)^{\frac{1}{4}}}\left[ln\left|\frac{1-\sqrt{2}(\frac{a}{b})^{\frac{1}{4}}M+(\frac{a}{b})^{\frac{1}{2}}M^2}{1+\sqrt{2}(\frac{a}{b})^{\frac{1}{4}}M+(\frac{a}{b})^{\frac{1}{2}}M^2}\right|\right. \\ &\left. + \;2\;\tan^{-1}\left(\frac{\sqrt{2}(\frac{a}{b})^{\frac{1}{4}}M}{1\;-\;\left(\frac{a}{b}\right)^{\frac{1}{2}}M^2}\right)\;+\;n\pi\right]\;. \end{aligned} \tag{5.1.5}$$

which is different from Eq. (4.3.18) where the constant factor $n\pi$ was ignored and plus sign was written instead of minus sign in the denominator of the argument of inverse tangent function. Now any formula giving evaporation of a BH due to HR and phantom energy must reduce to the corresponding results of the case of HR only when phantom energy is “switched off”. In the above analysis, this can be done be setting $a = 0$. To verify the corrections, we shall check this in Eq. (5.1.3) which should reduce to Eq. (3.2.9) when $a = 0$. we take $t = t_i \equiv$ time of formation of PBH= 0 and corresponding mass will be $M = M_i$. Thus $T \equiv t_f - t_i = t_f$ is the lifetime of a BH evaporating due to HR and phantom energy. To avoid derivatives of fractional powers, we have taken $(a/b)^{1/4}M_i = u$. Thus the Eq. (5.1.3) becomes

$$\begin{aligned} T &= \frac{M_i^3}{4\sqrt{2}u^3 b}\left[ln\left|\frac{1-\sqrt{2}u+u^2}{1+\sqrt{2}u+u^2}\right| \right. \\ &\left. + \ 2\tan^{-1}(1+\sqrt{2}u) \ - \ 2\tan^{-1}(1-\sqrt{2}u)\right] \ . \end{aligned} \tag{5.1.6}$$

For absence of phantom energy, $a \to 0$ implying $u \to 0$. Therefore we can write

$$\begin{aligned} T &= \lim_{u\to 0}\frac{M_i^3}{4\sqrt{2}u^3 b}\left[ln\left|\frac{1-\sqrt{2}u+u^2}{1+\sqrt{2}u+u^2}\right| \ + \ 2\tan^{-1}(1+\sqrt{2}u) \right. \\ &\left. - \ 2\tan^{-1}(1-\sqrt{2}u)\right] \ . \end{aligned} \tag{5.1.7}$$

Setting $u = 0$ in Eq. (5.1.7) gives an undetermined form (0/0). Thus we apply L' Hospital rule and get

$$\begin{aligned} T &= \lim_{u\to 0}\frac{M_i^3}{3b}\frac{1}{4\sqrt{2}u^2}\left[\left(\frac{(1+\sqrt{2}u+u^2)(-\sqrt{2}+2u)-(1-\sqrt{2}u+u^2)(\sqrt{2}+2u)}{(1-\sqrt{2}u+u^2)(1+\sqrt{2}u+u^2)}\right)\right. \\ &\left. + \ 2\left(\frac{\sqrt{2}}{2+2\sqrt{2}u+2u^2} \ - \ \frac{-\sqrt{2}}{2-2\sqrt{2}u+2u^2}\right)\right] \ , \\ &= \lim_{u\to 0}\frac{M_i^3}{3b}\frac{1}{4\sqrt{2}u^2}\left[\frac{-2\sqrt{2}+2\sqrt{2}u^2}{(1+u^2)^2-(\sqrt{2}u)^2} \ + \ \frac{2\sqrt{2}+2\sqrt{2}u^2}{(1+u^2)^2-(\sqrt{2}u)^2}\right] , \\ &= \lim_{u\to 0}\frac{M_i^3}{3b}\frac{1}{2u^2}\left[\frac{2u^2}{1+u^4}\right] \ , \\ &= \lim_{u\to 0}\frac{M_i^3}{3b}\frac{1}{1+u^4} \ , \\ &= \frac{M_i^3}{3b} \ . \end{aligned} \tag{5.1.8}$$

Back substituting value of b from Eq. (4.3.11) we get Eq. (3.2.9). Hence Eq. (5.1.3) reduces to HR formula in the limit when phantom energy effect (in the evaporation of a BH) is absent.

From now we shall use Eq. (5.1.3) to avoid few complications arising by the use of above written trigonometric identity. For our further research work, we need only Eqs. (4.2.2) and (5.1.3) but here we shall proceed with all the calculations done in [10] in order to correct all the mistakes and results of that paper. Now defining $M \equiv mM_i$ and using it in Eq. (5.1.3) gives

$$\begin{aligned} t_f - t &= \frac{1}{4\sqrt{2}(a^3 b)^{\frac{1}{4}}} \left[ln \left| \frac{1-\sqrt{2}(\frac{a}{b})^{\frac{1}{4}} mM_i + (\frac{a}{b})^{\frac{1}{2}} m^2 M_i^2}{1+\sqrt{2}(\frac{a}{b})^{\frac{1}{4}} mM_i + (\frac{a}{b})^{\frac{1}{2}} m^2 M_i^2} \right| \right. \\ &+ 2\tan^{-1}\left(1+\sqrt{2}\left(\frac{a}{b}\right)^{\frac{1}{4}} mM_i\right) \\ &- \left. 2\tan^{-1}\left(1-\sqrt{2}\left(\frac{a}{b}\right)^{\frac{1}{4}} mM_i\right)\right] . \end{aligned} \quad (5.1.9)$$

Using Eqs. (4.3.21) and (4.3.22) in Eq. (5.1.9)

$$\begin{aligned} t &= t_f - \frac{1}{4\sqrt{2}(a'^3 b')^{\frac{1}{4}}} \left[ln \left| \frac{1-\sqrt{2}(\frac{a'}{b'})^{\frac{1}{4}} m + (\frac{a'}{b'})^{\frac{1}{2}} m^2}{1+\sqrt{2}(\frac{a'}{b'})^{\frac{1}{4}} m + (\frac{a'}{b'})^{\frac{1}{2}} m^2} \right| \right. \\ &+ 2tan^{-1}\left(1+\sqrt{2}\left(\frac{a'}{b'}\right)^{\frac{1}{4}} m\right) \\ &- \left. 2tan^{-1}\left(1-\sqrt{2}\left(\frac{a'}{b'}\right)^{\frac{1}{4}} m\right)\right] . \end{aligned} \quad (5.1.10)$$

Using the parameter P defined in Eq. (4.3.27), we can write Eq. (5.1.10) as

$$\begin{aligned} t &= t_f - \frac{1}{4\sqrt{2}P^{\frac{3}{4}} b'} \left[ln \left| \frac{1-\sqrt{2}P^{\frac{1}{4}} m + P^{\frac{1}{2}} m^2}{1+\sqrt{2}P^{\frac{1}{4}} m + P^{\frac{1}{2}} m^2} \right| \right. \\ &+ 2tan^{-1}(1+\sqrt{2}P^{\frac{1}{4}} m) - 2tan^{-1}(1-\sqrt{2}P^{\frac{1}{4}} m)] . \end{aligned} \quad (5.1.11)$$

Dividing by t_f on both sides of Eq. (5.1.11) and defining normalized time $\tau \equiv t/t_f$, we get

$$\begin{aligned} \tau &= 1 - \frac{1}{t_f}\frac{1}{4\sqrt{2}P^{\frac{3}{4}} b'} \left[ln \left| \frac{1-\sqrt{2}P^{\frac{1}{4}} m + P^{\frac{1}{2}} m^2}{1+\sqrt{2}P^{\frac{1}{4}} m + P^{\frac{1}{2}} m^2} \right| \right. \\ &+ 2tan^{-1}(1+\sqrt{2}P^{\frac{1}{4}} m) - \left. 2tan^{-1}(1-\sqrt{2}P^{\frac{1}{4}} m)\right] . \end{aligned} \quad (5.1.12)$$

Now consider that $M = M_i$ at $t = t_i = 0$ which corresponds to $m = 1$. Thus Eq.

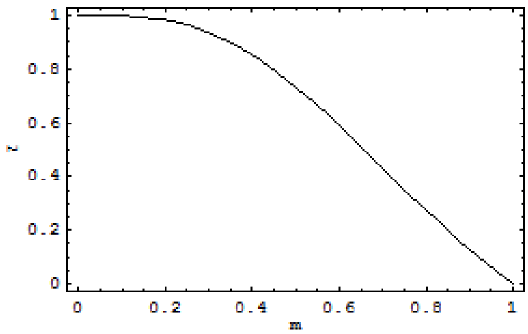

Figure 5.3: τ plotted against m=[0, 1] for P=5 .

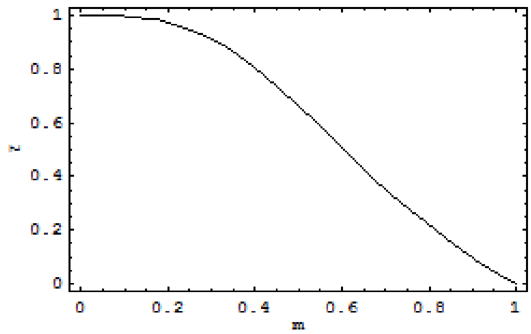

Figure 5.4: τ plotted against m=[0, 1] for P=10 .

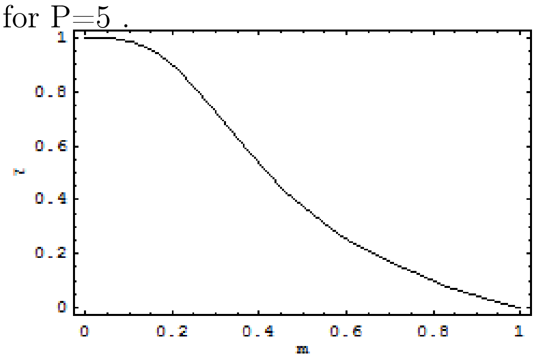

Figure 5.5: m=[0, 1] plotted against τ for P=100 .

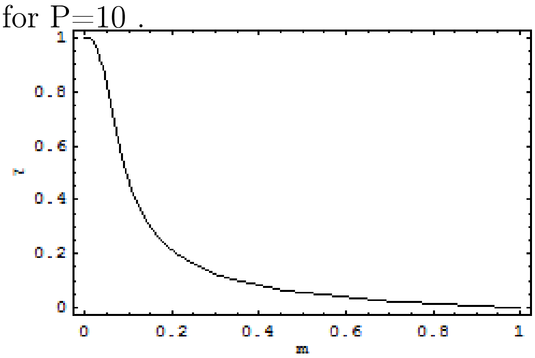

Figure 5.6: m=[0, 1] plotted against τ for P=100000 .

(5.1.11) becomes

$$
\begin{aligned}
t_f \quad &= \quad \frac{1}{4\sqrt{2}P^{\frac{3}{4}}b'}\left[ln\left|\frac{1-\sqrt{2}P^{\frac{1}{4}}+P^{\frac{1}{2}}}{1+\sqrt{2}P^{\frac{1}{4}}+P^{\frac{1}{2}}}\right|\right. \\
&\left. + \ 2\tan^{-1}(1+\sqrt{2}P^{\frac{1}{4}}) - 2\tan^{-1}(1-\sqrt{2}P^{\frac{1}{4}})\right] \ . \qquad (5.1.13)
\end{aligned}
$$

Thus Eq. (5.1.12) can be written as

$$
\tau = 1 - \frac{ln\left|\frac{1-\sqrt{2}P^{\frac{1}{4}}m+P^{\frac{1}{2}}m^2}{1+\sqrt{2}P^{\frac{1}{4}}m+P^{\frac{1}{2}}m^2}\right| + 2\tan^{-1}(1+\sqrt{2}P^{\frac{1}{4}}m) - 2tan^{-1}(1-\sqrt{2}P^{\frac{1}{4}}m)}{ln\left|\frac{1-\sqrt{2}P^{\frac{1}{4}}+P^{\frac{1}{2}}}{1+\sqrt{2}P^{\frac{1}{4}}+P^{\frac{1}{2}}}\right| + 2\tan^{-1}(1+\sqrt{2}P^{\frac{1}{4}}) - 2\tan^{-1}(1-\sqrt{2}P^{\frac{1}{4}})} \qquad (5.1.14)
$$

It is clear that t/t_f has range 0 to 1; starts at former and ends at latter. Also, m starts at 1 when a BH has its maximum (initial) mass and goes to 0 when a BH evaporates completely at $t = t_f$. Now we shall choose different values for P and plot graphs of m vs τ. For $P \leq 1$, we get graphs shown in Fig. 5.1 and 5.2. These are similar in physical interpretation to those given in [10] (Figs. $4.1-4.3$) except for some minor numerical details. But for $P > 1$, graphs given in [10] have a certain part of the curve lying in the region of negative time as shown in Figs. 4.4 – 4.5. Firstly, this is beyond above mentioned range of τ. Secondly, this was not possible physically because there is no concept of time before big bang. We have re-plotted these graphs with the corrected formula given in Eq. (5.1.14) and found that no non-physical part appears now for $P = 5$ and $P = 10$ as shown in Fig. 5.3 and 5.4. To verify the corrections, we have also plotted same graphs for $P = 100$ and $P = 100000$ as shown in Fig. 5.5 and 5.6 and found that the curves lie in the physically admissible region.

5.2 Transition Mass

Consider the ratio of the evaporation rate due to HR to the evaporation rate due to phantom energy accretion as defined in Eq. (4.2.2). For $\xi < 1$, Eq. (4.2.2) gives M at and below which phantom energy is never dominant over HR; $\xi = 1$ gives the value of transition mass and $\xi > 1$ is the case when evaporation rate due to phantom energy is relatively more significant. Considering equal contribution from HR and phantom energy in the evaporation of a BH ($\xi = 1$) and using Eq. (4.3.4) in Eq. (4.2.2) gives

$$M \equiv M_t = \left(\frac{3\hbar c^7 t_o^2 \alpha}{8\epsilon G^3}\right)^{\frac{1}{4}} , \tag{5.2.1}$$

where M_t is the same transition mass defined in [9]. Using the values of constants and $\alpha = 4 \times 10^{-4}$ [34] in Eq. (5.2.1), we get

$$M_t = 3.4637 \times 10^{15} \frac{t_o^{\frac{1}{2}}}{\epsilon^{\frac{1}{4}}} g . \tag{5.2.2}$$

Thus M_t depends on the age of universe after which dark energy domination era started (t_o; measured in seconds) and EoS parameter of phantom energy. When the BH mass is greater than M_t, evaporation due to phantom energy accretion is dominant over the other. Taking M in Eq. (4.2.2) to be the initial mass of a BH we get $\xi = P$

Here we shall explain the physical significance of any fixed value of P. For example, consider $P = 5$. This means that at the very start of our observation of the evaporation, BH's "initial" mass is such that the evaporation due to phantom energy accretion is 5 times the evaporation due to HR and soon after that the ratio ξ starts dropping down from 5 because of the decrease in mass (however P is fix for each curve). The reason is that the BH evaporation rate is directly proportional to the mass-square in case of phantom energy and inversely proportional to the mass-square in case of HR. The decrease in mass with time makes the effect of phantom energy accretion lesser significant than before. Thus the ratio decreases. Thus ξ does not have the same value on the whole curve and P corresponds to only the starting point of evaporation. Near the end in each graph $\xi \to 0$. Thus, we infer from here that all the graphs in Figs. $5.3 - 5.6$ have a part of the curve in the start of the evaporation which represents domination of phantom energy over the HR in the evaporation rate. Infact $P = 100000$ contains the evaporation behavior which is already there in graph for $P = 0.1, 5$ etc. Total lifetime in all graphs is not the same.

We evaluate M_t for the extreme and middle values of EoS parameter (mentioned in section 2.11.2) and then include a much larger value ($\epsilon = 10$) to check the sensitivity of the transition mass to ϵ (in case the claim in [26] is invalid). For t_o to be 13.75 billion years, the results of transition mass ($\xi = 1$) and the mass corresponding to other values of ξ are given in Tab. 5.1 for different values of ϵ.

Table 5.1: Mass of a PBH for Different Contribution from Phantom Energy for Different Values of ϵ for $t_o = 13.75$ billion years .

ξ	M (g) $(\epsilon = 0.05)$	M (g) $(\epsilon = 0.36)$	M (g) $(\epsilon = 0.67)$	M (g) $(\epsilon = 10)$
0.01	1.4832×10^{24}	0.9056×10^{24}	0.7753×10^{24}	0.3944×10^{24}
0.1	2.6377×10^{24}	1.6104×10^{24}	1.3787×10^{24}	0.7014×10^{24}
0.5	3.9443×10^{24}	2.4080×10^{24}	2.0617×10^{24}	1.0488×10^{24}
1.0	4.6906×10^{24}	2.8637×10^{24}	2.4519×10^{24}	1.2473×10^{24}
5	7.0141×10^{24}	4.2821×10^{24}	3.6672×10^{24}	1.8653×10^{24}
10	8.3411×10^{24}	5.0924×10^{24}	4.3600×10^{24}	2.2182×10^{24}

The values of M_t show that HR dominates over phantom energy for a sufficiently higher value of mass. It is important to note that the accretion of phantom energy increases the evaporation rate due to HR at each instant; it may be negligible at some time and may be dominant over HR at some other time but the BH will explode near its end precisely as in the case of only HR. The only way to have the phantom energy effect significant for a PBH is to reduce the value of M_t sufficiently, which is impossible since neither can we take much higher values for ϵ nor can we take much smaller values of t_o. Also, this implies that for a BH of mass less than $\sim 10^{24}$g the dominant evaporation process is always HR.

Table 5.2: Mass of a PBH for Different Contribution from Phantom Energy for Different Values of ϵ for $t_o = 10$ billion years .

ξ	$M\ (g)\ (\epsilon = 0.05)$	$M\ (g)\ (\epsilon = 0.36)$	$M\ (g)\ (\epsilon = 0.67)$	$M\ (g)\ (\epsilon = 10)$
0.01	1.3011×10^{24}	0.7944×10^{24}	0.6801×10^{24}	0.3460×10^{24}
0.1	2.3138×10^{24}	1.4126×10^{24}	1.2094×10^{24}	0.6153×10^{24}
0.5	3.4599×10^{24}	2.1123×10^{24}	1.8085×10^{24}	0.9200×10^{24}
1.0	4.1146×10^{24}	2.5120×10^{24}	2.1508×10^{24}	1.0941×10^{24}
5	6.1527×10^{24}	3.7562×10^{24}	3.2161×10^{24}	1.6362×10^{24}
10	7.3168×10^{24}	4.4670×10^{24}	3.8246×10^{24}	1.9458×10^{24}

To see the increased effect of the phantom energy accretion by taking phantom energy dominated era starting earlier, we take t_o to be 10 and 1 billion years and get the results shown in Tabs. 5.2 and 5.3 respectively. These results have a similar physical interpretation as the results of Tab. 5.1. By comparing the results of the three tables, we find that for a sufficiently early domination of dark energy, the HR will dominate over phantom energy at smaller values of mass (transition mass) but there is not too much difference in the value of transition mass in both cases.

Table 5.3: Mass of a PBH for Different Contribution from Phantom Energy for Different Values of ϵ for $t_o = 1$billion years .

ξ	$M\ (g)\ (\epsilon = 0.05)$	$M\ (g)\ (\epsilon = 0.36)$	$M\ (g)\ (\epsilon = 0.67)$	$M\ (g)\ (\epsilon = 10)$
0.01	0.4141×10^{24}	0.2512×10^{24}	0.2150×10^{24}	0.1094×10^{24}
0.1	0.7317×10^{24}	0.4467×10^{24}	0.3825×10^{24}	0.1946×10^{24}
0.5	1.0941×10^{24}	0.6680×10^{24}	0.5719×10^{24}	0.2910×10^{24}
1.0	1.3012×10^{24}	0.7944×10^{24}	0.6801×10^{24}	0.3460×10^{24}
5	1.9457×10^{24}	1.1878×10^{24}	1.0170×10^{24}	0.5174×10^{24}
10	2.3138×10^{24}	1.4126×10^{24}	1.2095×10^{24}	0.6153×10^{24}

These results show that accretion of phantom energy is of lesser significance in the evaporation of PBHs of mass $\sim 10^{15}$g as compared to Hawking evaporation process.

5.3 Initial Mass of a PBH Evaporating Today due to Hawking Evaporation and Accretion of Phantom Energy

In the previous section, I have concluded that the phantom energy accretion process will have less significance in the evaporation of a PBH of mass $\sim 10^{15}$g as compared to HR. But it is claimed in [10] that the initial mass of a PBH should be $\sim 10^{22}$g if it has to evaporate today under the combined effect of both the evaporation processes. This claim shows that accretion of phantom energy is as much effective for a PBH of initial mass ranging from 10^{15}g to 10^{22}g as is HR. The probability of formation of a PBH is decreased exponentially by increasing the initial mass a bit [6]. Thus, if the claim in [10] is true, then perhaps phantom energy is the reason of PBHs not being observed evaporating today. To find the correct numerical values, we shall proceed with Eq. (5.1.3) and do rigorous calculations to find the answer.

Taking the age of the universe to be 13.75 billion years = 4.9372×10^{17}s, Eq. (3.2.9) gives corresponding mass to be $4.64 \times 10^{14} \equiv M_{hr}$g. Eq. (5.1.3) (or equivalently Eq. (5.1.6)) gives the lifetime of a PBH of some initial mass evaporating under the effect of HR and phantom energy. For $t_o \sim 13.75$ billion years, $a \sim 10^{-75}\ g^{-1}s^{-1}$ ($[a/b]^{\frac{1}{4}} \sim 10^{-25}$ g^{-1}). Infact, even for $t_o \sim 1$ s, $a \sim 10^{-39}\ g^{-1}s^{-1}$ ($[a/b]^{\frac{1}{4}} \sim 10^{-16}$ g^{-1}). As such, we can use a Taylor expansion in Eq. (5.1.6) for small $[a/b]^{\frac{1}{4}} M_i \equiv u$ which imposes an upper limit for the value of M_i for the further analysis to be applicable. First we shall expand each term in Eq. (5.1.6). Consider the expansion formula of logarithmic function

$$ln(1+x) \;=\; \frac{x}{1} - \frac{x^2}{2} + \frac{x^3}{3} - ...... \,. \tag{5.3.1}$$

Also

$$ln\left(\frac{1+y}{1+z}\right) \;=\; ln(1+y) \;-\; ln(1+z)\,. \tag{5.3.2}$$

Using Eq. (5.3.1) in Eq. (5.3.2) gives

$$\begin{aligned} ln\left(\frac{1+y}{1+z}\right) \;&=\; \frac{y}{1} - \frac{y^2}{2} + \frac{y^3}{3} - ...... \\ &\quad - \frac{z}{1} + \frac{z^2}{2} - \frac{z^3}{3} + ... \,. \end{aligned} \tag{5.3.3}$$

Taking $y = -\sqrt{2}u + u^2$ and $z = \sqrt{2}u + u^2$, above equation becomes

$$\begin{aligned} ln\left(\frac{1+(-\sqrt{2}u+u^2)}{1+(\sqrt{2}u+u^2)}\right) &= (-\sqrt{2}u+u^2) - \frac{(-\sqrt{2}u+u^2)^2}{2} \\ &+ \frac{(-\sqrt{2}u+u^2)^3}{3} - ...... - (\sqrt{2}u+u^2) \\ &+ \frac{(\sqrt{2}u+u^2)^2}{2} - \frac{(\sqrt{2}u+u^2)^3}{3} + ... \quad (5.3.4) \end{aligned}$$

Now we shall open up to the fifth order of u and ignore higher order terms. The reason is that the formula of lifetime due to Hawking evaporation only has third power of M_i. In above equation, fourth order terms get canceled. Thus to obtain a first order "correction" in the HR formula due to phantom energy accretion, we need to go to the fifth power. This gives

$$\begin{aligned} ln\left(\frac{1+(-\sqrt{2}u+u^2)}{1+(\sqrt{2}u+u^2)}\right) &= -2\sqrt{2} + \frac{1}{2}[4u^2(\sqrt{2}u)] \\ &+ \frac{1}{3}[-6\sqrt{2}u^5 - 4\sqrt{2}u^3] + \frac{1}{4}[16\sqrt{2}u^5] + \frac{1}{5}[-8\sqrt{2}u^5] , \\ &= -2\sqrt{2}u + 2\sqrt{2}u^3 - 2\sqrt{2}u^5 - \frac{4}{3}\sqrt{2}u^3 + 4\sqrt{2}u^5 - \frac{8}{5}\sqrt{2}u^5 , \\ &= -2\sqrt{2}u + \frac{2}{3}\sqrt{2}u^3 - \frac{2}{5}\sqrt{2}u^5 . \quad (5.3.5) \end{aligned}$$

Now use the Taylor expansion formula for a function $f(x)$ about $x = a$ with steps of size h

$$f(a+h) = f(a) + \frac{h}{1!}f^{(1)}(a) + \frac{h^2}{2!}f^{(2)}(a) + \frac{h^3}{3!}f^{(3)}(a) + ... , \quad (5.3.6)$$

$$f(a-h) = f(a) - \frac{h}{1!}f^{(1)}(a) + \frac{h^2}{2!}f^{(2)}(a) - \frac{h^3}{3!}f^{(3)}(a) + ... , \quad (5.3.7)$$

where $f^{(n)}$ denotes the nth derivative of function f with respect to x. With the same reason as in the case of expansion of ln, we go up to the fifth order (h^5) and get

$$f(a+h) - f(a-h) = 2[hf^{(1)}(a) + \frac{h^3}{6}f^{(3)}(a) + \frac{h^5}{120}f^{(5)}(a)] . \quad (5.3.8)$$

In our formula, $f(x) = tan^{-1}(x)$, $a = 1$ and $h = \sqrt{2}u$. The derivatives are

$$
\begin{aligned}
f^{(1)}(x) &= \frac{1}{1+x^2} && (5.3.9)\\
f^{(2)}(x) &= \frac{-2x}{(1+x^2)^2} && (5.3.10)\\
f^{(3)}(x) &= \frac{-2}{(1+x^2)^2} + \frac{2(2x)^2}{(1+x^2)^3} && (5.3.11)\\
f^{(4)}(x) &= \frac{24x}{(1+x^2)^3} - \frac{48x^3}{(1+x^2)^4} && (5.3.12)\\
f^{(5)}(x) &= \frac{16}{(1+x^2)^3} - \frac{288x^2}{(1+x^2)^4} + \frac{284}{(1+x^2)^5}\ . && (5.3.13)
\end{aligned}
$$

Putting $x = a = 1$ only in the odd degree of derivative in the above equations, we get

$$
f^{(1)}(1) = \frac{1}{2}\ ,\ f^{(3)}(1) = \frac{1}{2}\ ,\ f^{(5)}(1) = -4\ . \qquad (5.3.14)
$$

Thus we can write from Eqs. (5.3.8) and (5.3.14)

$$
\begin{aligned}
\tan^{-1}(a+h) - \tan^{-1}(a-h) &= \tan^{-1}(1+h) - \tan^{-1}(1-h)\\
&= 2\left[hf^{(1)}(1) + \frac{h^3}{6}f^{(3)}(1) + \frac{h^5}{120}f^{(5)}(1)\right]\ ,\\
&= 2h\left(\frac{1}{2}\right) + \frac{h^3}{3}\left(\frac{1}{2}\right) + \frac{h^5}{60}(-4)\ ,\\
&= h + \frac{h^3}{6} - \frac{h^5}{15}\ . \qquad (5.3.15)
\end{aligned}
$$

Substituting the value of h in the above equation, we get

$$
tan^{-1}(1+\sqrt{2}u) - tan^{-1}(1-\sqrt{2}u) = \sqrt{2}u + \frac{2\sqrt{2}u^3}{6} - \frac{4\sqrt{2}u^5}{15}\ . \qquad (5.3.16)
$$

Combining Eqs. (5.3.5) and (5.3.16)

$$\begin{aligned} ln\left(\frac{1+(-\sqrt{2}u+u^2)}{1+(\sqrt{2}u+u^2)}\right) &+ 2[\tan^{-1}(1+\sqrt{2}u)-\tan^{-1}(1-\sqrt{2}u)] \\ &= -2\sqrt{2}u+\frac{2}{3}\sqrt{2}u^3-\frac{2}{5}\sqrt{2}u^5 \\ &\quad + 2\left[\sqrt{2}u + \frac{2\sqrt{2}u^3}{6} - \frac{4\sqrt{2}u^5}{15}\right], \\ &= \frac{4\sqrt{2}}{3}u^3 - \frac{2\sqrt{2}}{15}u^5 . \end{aligned} \tag{5.3.17}$$

Using Eq. (5.3.17) in Eq. (5.1.6), we get

$$\begin{aligned} T_p &= \frac{M_i^3}{4\sqrt{2}bu^3}\left[\frac{4\sqrt{2}}{3}u^3 - \frac{2\sqrt{2}}{15}u^5\right], \\ &= \frac{M_i^3}{3b}\left[1 - \frac{1}{10}u^2\right] . \end{aligned} \tag{5.3.18}$$

Back substituting the value of u, we get

$$\begin{aligned} T_p &= \frac{M_i^3}{3b}\left[1 - \frac{1}{10}\left(\frac{a}{b}\right)^{\frac{1}{2}} M_i^2\right], \\ &= T_{hr}\left[1 - \frac{1}{10}\left(\frac{a}{b}\right)^{\frac{1}{2}} M_i^2\right] . \end{aligned} \tag{5.3.19}$$

Next term in the above expansion would be $O(\frac{a}{b}M_i^4)$ which is neglected. Also, in the limit $a \to 0$, Eq. (5.3.19) reduces to the lifetime due to Hawking evaporation given in Eq. (3.2.9). Thus we have obtained the first order correction term in the formula for lifetime of a Schwarzchild BH in the presence of phantom energy.

We then see that for $t_0 = 13.75$ billion years, $\epsilon = 0.1$ and $M_i = M_{hr}$, the correction factor is $\sim 10^{-22}$ as compared to 1 which corresponds to a decrease of $\sim 10^{-5}$ s in a lifetime of $\sim 10^{17}$ s! Even for $t_o \sim 1$ billion years, the correction factor is $\sim 10^{-21}$ which is also negligible as opposed to the claim in [10].

The lifetime of BHs of initial masses 10^{23}g, 10^{24}g and 10^{25}g are 4×10^{42}s, 4×10^{45}s and 4×10^{48}s respectively if evaporation only due to HR is considered. When the effect of phantom energy accretion is also considered in the evaporation process (taking dark energy domination from now on), Eq. (5.3.19) gives the lifetime to be 4×10^{42}s, 4×10^{45}s and 3×10^{48}s in the same order. These results show that the effect of phantom energy accretion in evaporation process is significant for such higher masses.

Since $M_i \sim 10^{15}$g even when phantom energy effect is also taken in to account, we conclude that for a PBH evaporating today under the combined effect of HR and phantom energy, the dominant evaporation process is HR even for a very early domination of phantom energy.

As 10^{24}g is very close to the limit of approximation, we need to be careful in applying this analysis for such a value of mass. This is done by M. Ahmed and me as follows. We use Eq. (4.3.9) to find the amount of mass that is evaporated in one second if the initial mass of the BH is 10^{24}g and dark energy dominates from now. Using $b \approx 10^{26}\text{g}^3\text{s}^{-1}$ and $a = 10^{-75}\text{g}^{-1}\text{s}^{-1}$, we get $dM/dt \sim 10^{-25}\text{gs}^{-1}$ which shows clearly that such a mass BH can not loose 10^{10}g mass in 10^{-5}s. Thus the result given in [11] is correct. Another way of getting the same result is to take $dM/dt = -1\text{gs}^{-1}$ and find the corresponding mass using Eq. (4.3.9)

$$\begin{aligned} -1 &= -\ 10^{-75}M^2 - \frac{10^{26}}{M^2} , \\ 0 &= 10^{-75}M^4 - M^2 + 10^{26} , \end{aligned} \tag{5.3.20}$$

which is quadratic in M^2. Its roots are

$$M^2 = \frac{1 \pm \sqrt{1 - \eta}}{2 \times 10^{-75}} g^2 , \tag{5.3.21}$$

where $\eta = 4 \times 10^{-49}$. Using binomial expansion and neglecting terms of order higher than one, we get the roots to be

$$M^2 = \frac{2 - \eta/2}{2 \times 10^{-75}} g^2 \ and \ \frac{\eta}{4 \times 10^{-75}} g^2 . \tag{5.3.22}$$

Thus the mass corresponding to evaporation rate -1gs^{-1} is 10^{37}g and 10^{13}g. The value of the transition mass $\sim 10^{24}$g tells that in the former case, the evaporation is due to phantom energy accretion and in the latter case it is due to the HR. This suggests the evaporation rate to be minimum between the two values of mass. Thus the overall evaporation will be considerable only for $M >> 10^{37}$g or $M << 10^{13}$g. If we take $M = M_i$, both values of mass correspond to $P \sim 10^{47}$ and $P \sim 10^{-49}$ respectively which indicates the rate of evaporation or change of mass to be minimum about $P = 1$, i.e., when the mass of the BH is about the value of transition mass. Hence for all values of P considered in this dissertation, the overall evaporation is negligible. This can also be checked as follows.

Differentiating Eq. (4.3.9) with respect to t, we get

$$\frac{d^2M}{dt^2} \;=\; -\,2aM \;+\; \frac{2b}{M^3}\;, \tag{5.3.23}$$

Setting $d^2M/dt^2 = 0$, we get $M = (b/a)^{1/4}$ which is the point of minima and is exactly the value of the transition mass. Therefore there is minimum evaporation at the transition phase.

Now we try to find the limit of the mass of the BH to exhibit observable effect of phantom energy through its evaporation. In case $t_x << t$, we can write Eq. (4.3.2) as [12]

$$M \;\approx\; M_i\left(-\frac{t}{t_x}\right)^{-1}\;. \tag{5.3.24}$$

This can be achieved if the BH is very massive and/or the value of $|\omega|$ is large. Both these conditions correspond to the increased effect of phantom energy. Using the value of t_x in the above equation, we get

$$M \;=\; \frac{c^3}{16\pi G^2}\frac{1}{\rho_\infty t\epsilon}\;, \tag{5.3.25}$$

where we have used $\omega + 1 = -\epsilon$. This shows that in the limit $t_x << t$, the mass of an evaporating BH after certain time t is independent of the initial value of the mass and it depends only on energy density at that time and the value of the EoS parameter. Physically it means that any BH of sufficiently large mass should reach a certain value of mass by evaporation in finite time t due to phantom energy accretion. We find the value of the mass of the BH left after 1s and 10 years using Eq. (5.3.25). Taking $\epsilon = 0.5$ and $\rho_\infty = 10^{-29}$g/cm^3, we get $M = 2.5 \times 10^{73}$g for $t = 1$s and $M = 10^{65}$g for $t = 10$ years (3×10^8s). It should be noted that both these values of the mass are much larger than the total mass of the observable universe ($\sim 10^{57}$g). The mass of the largest observed BH is 10^{43}g. Thus the effect of phantom energy could have been easily observed if this value of the mass (10^{65}g) had been far below 10^{45}g [12].

Now we plot a logarithmic graph between M and dM/dt given in Fig. 5.7 where M varies over 60 orders of magnitude. This graph shows clearly that the behavior of overall evaporation process with change of mass is the same as expected.

Note that Fig. 5.7 does not give mass as a function of time but says how much the rate of decrease of mass would be for a BH of any given mass. Further,

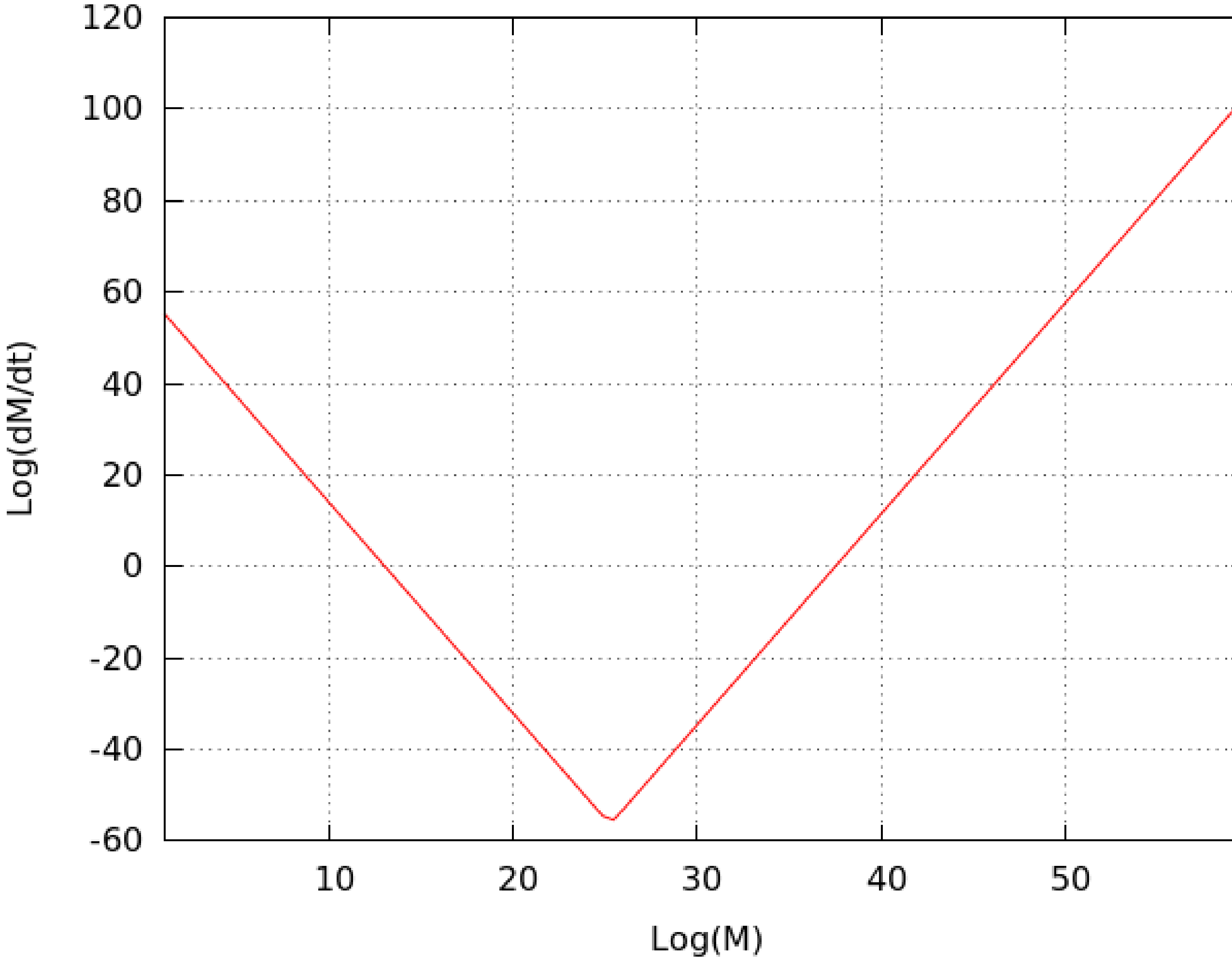

Figure 5.7: Logarithmic plot of M and dM/dt .

note that a BH of 10^{24}g (the transition mass) would decrease in mass most slowly but still have a non-zero ($\sim 10^{-55}$gs^{-1}) rate. This confirms the above statements. Also, there is a unique value of transition mass if all other parameters are held fixed. Most interestingly, note that a BH of 10^{43}g would have the same rate of mass reduction as pure HR at the end of the life of a BH. As Hawking says "near the end of its life the rate of emission would be very high and about 10^{30} erg would be released in the last 0.1s. This is a fairly small explosion by astronomical standards but it is equivalent to about 1 million ton hydrogen bombs" [3]. The evaporation rate of a 10^{43}g BH is $\sim 10^{12}$gs^{-1} i.e. a change in mass per unit mass of $\sim 10^{-31}$s^{-1}.

Chapter 6

Summary

In this dissertation, I have discussed the evaporation of a Schwarzchild BH in a universe filled with phantom energy, i.e., due to Hawking evaporation and the accretion of phantom energy. In particular, the case of the evaporation of Schwarzchild PBHs of the mass $\sim 10^{15}$g which would have been evaporating today due to pure HR has been discussed. The main paper that has been reviewed for this purpose is [10]. There were algebraic errors in [10] that have been discussed and corrected [11]. The correction makes a significant difference. First, there was nonphysical behavior appearing in the graphs for the variation of mass with time that caused severe problems of interpretation. With the correction that nonphysical behavior has disappeared.

Further, I have discussed the concept of the transition mass with reference to [9] and studied its variation with EoS parameter ($\omega < -1$) numerically. This work has determined the boundary value of mass of a Schwarzchild BH where one of the competing phenomena dominates over the other in the over all evaporation. It was found that the transition mass is larger than 10^{15}g by approximately 10 orders of magnitude. This shows that phantom energy may not be more significant in the evaporation process of a Schwarzchild PBH of mass $\sim 10^{15}$g than the HR.

Also, the effect of phantom energy accretion on the life of a Schwarzchild BH has been studied. It has been claimed in [10] that phantom energy accretion decreases the lifetime of a PBH of mass $\sim 10^{15}$g so much that we have to consider the initial mass of a PBH to be 10 orders of magnitude larger than this if we want a PBH to be evaporating today. This was claimed by taking the phantom energy to be 10 times more effective than the HR. But it should be noticed that the relative significance of the two evaporation processes is not fixed; rather it changes continuously with the change in mass of a BH due to continuous evaporation.

In this dissertation, more substantial analytical work has been done to find the first order correction term in the lifetime of a PBH of some initial mass (less than 10^{25}g) evaporating under the combined effect of phantom energy accretion and HR. It has been found that the decrease in the lifetime (which is $\sim 10^{17}$s in case of Hawking evaporation only) of a PBH of mass $\sim 10^{15}$g is just $\sim 10^{-5}$s. This implies that the required increase in initial mass of a PBH evaporating today under the combined effect of phantom energy accretion and HR is insignificant. Then the rate of mass change for a PBH of mass $\sim 10^{22}$g was computed which came out to be negligible. This confirmed the previous result. This analysis highlighted another interesting feature of transition mass, i.e., at this value of mass the evaporation rate is minimum.

In [10], the constant of integration in Eq. (4.3.18) is the lifetime of the BH and is chosen to be t_o (present age of the universe) to study the mass evolution of PBHs evaporating today in a phantom energy environment. Also, α is chosen corresponding to the mass of PBHs evaporating today due to HR only ($\sim 10^{15}$g) given in [34]. But since HR is ignorable for massive BHs, this analysis can be applied for *all* BHs in principle, getting different lifetime of BHs. Also, the value of α can be chosen in accordance with the higher value of mass.

Also, mass limit beyond which the evaporation rate could have been easily observable was found but this limit came out to be much larger than the total mass of the observable universe. However, the study of variation of evaporation rate with constantly changing mass (beyond the mass range of PBHs) showed that the evaporation due to phantom energy accretion is not negligible for massive BHs ($\sim 10^{43}$g) although they may not be the primordial ones.

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