Shear Flow Instability in a Partially-Ionized Plasma Sheath Around Re-entry Vehicle

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Abstract

When a re-entry vehicle enters the earth's atmosphere with speed greater than that of sound a plasma sheath cover the re-entry vehicle which causes communication blackout by either reflecting or absorbing the electromagnetic waves coming to or from the vehicle. In order to lessen this blackout region it important to know then characteristics of the plasma sheath region. Due to shear flow of the plasma sheath, instability within the low-frequency ion acoustic waves take place. As re-entry vehicle enters the earth's atmosphere collision of charged particles with neutrals start to increases. In this thesis the effect of collisions on the growth rate of instability has been studied. It was observed, that by increasing collision growth rate of the instability decreases to a point where the instability is completely damped.

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Chapter 1

Introduction

1.1 How can we define Plasma?

In ordinary conditions, the matter presents itself in three fundamental states: solid, liquid and gas, which are characterized by different level of bonding between them. In general, a phase transition occur if we increase the temperature (i.e. the average kinetic energy of the molecules) , by further increasing the temperature the collisional rate and the degree of ionization of the gas also increases. The ionized gas could then become plasma, if conditions for density, temperature, and characteristic length are met. Plasma is defined as,

"A plasma is a quasineutral gas of charged and neutral particles which exhibits collective behavior."[5]

Quasi-neutrality

The term 'quasi-neutrality', is just a mathematical way of saying that charge densities of free electrons and ions cancel each other in equilibrium. So if the number density of electron and ions is n_e and n_i respectively with charge state Z then

$$n_e \approx Zn_i.$$

It means that in the absence of extrinsic disturbance, the plasma as a whole is neutral, but within plasma there are some places where discrepancies within the number densities of the charge particles takes place.

Collective behavior

The term 'collective behavior ' shows that motions within plasma not only depend on the local conditions but also on the state of plasma in far regions. Consider the case of air which consists of neutral molecules, the force of gravity acting on the air is so less that it can be neglected. The molecules will move undisturbed unless its collision with another molecule takes place. These collisions control the particle's motion within plasma. When a macroscopic force such as sound waves generated from a loudspeaker is applied to the neutral gas, it will be transmitted to an individual atom through collisions. But in the case of plasma, the situation is completely different, as plasma contains charge particles they will move around and create a local concentration of negative and positive charges, which in turn gives rise to an electric field. As these charges move around current within the plasma is generated and a magnetic field is formed. These fields also affect the motion of other charged particles which are present far away. [7]

1.2 Debye shielding/Debye length

This property of plasma provides the measure of distance, over which the electric field of a single particle, is felt by other charged particles present in plasma. If there is an electrostatic field present within the plasma than the charged particles will arrange themselves in such a way to shield out the potential. The distance over which the shielding occurs is of the order of debye length, written as

$$\lambda_{D_e} = \sqrt{\frac{\epsilon_0 K_B T_e}{e^2 n_e}},$$

where T_e is the temperature of electron , and n_e is the number density of electron.A debye sphere is a volume over which the field of the charged particle is felt by other charged particles, it is also known as the sphere of influence, outside the debye sphere the charged particles are electrically screened. Debye sphere has a radius of debye length, and each charge within the debye sphere interact collectively with charges that lie inside the debye sphere. N_D represents the number of electrons that lie inside the debye sphere, given by

$$N_D = \frac{4}{3}\frac{\pi}{n_e^{\frac{1}{2}}}\left(\frac{\epsilon_0 K_B T_e}{e^2}\right)^{\frac{3}{2}}.$$

1.3 Criterion for Plasma

As we have already described the Debye length and plasma oscillations. We can specify the criterion which must be satisfied by an ionized gas for it to have a plasma nature.[16]

1. The first criterion of the plasma is that the physical length "L" of the plasma should be greater than its Debye length λ_D,

   $$\lambda_D << L.$$

   If the above condition is violated then there will not be enough space for collective shielding effect to take place.

2. Due to its collective behavior, shielding effect inside the Debye sphere having radius λ_D within plasma takes place. The number of particles within the Debye sphere is,

   $$N_D = \frac{4\pi}{3} n_e {\lambda_D}^3,$$

   where the plasma parameter is defined as $n_e {\lambda_D}^3 = \Lambda$, and from here the second criterion for plasma arises, which states that the average distance between electrons must be very small compared to the debye length i.e.

   $$\Lambda >>> 1.$$

3. Partially ionized plasmas e.g, the earth's ionosphere has a considerable amount of neutral particles. If there are too many collisions of the charged particles with the neutral particles then the electrons will be compelled to form an equilibrium with the neutrals. Hence the

ionized medium will not behave as plasma but instead will form an electrically conducting neutral gas. In order to make sure that the electrons are not affected by the neutral particles, the following condition is needed to be met,

$$\omega\tau_{en} > 1,$$

where ω is the plasma frequency, and τ_{en} is the mean time between the collision of electrons with the neutrals.

1.4 Plasma in nature

Having a quantity of more than 99% in the universe, plasma is said to be the most abundant state of matter. But within the earth's atmosphere, the quantity of plasma is very less. The few examples in our daily life, to which plasmas are limited to, are the soft glow of Aurora Borealis, the ionization in a rocket exhaust, the gas inside neon signs or fluorescent tubes and, the flash of a lightning bolt. Outside the earth's atmosphere, we can find plasma in solar winds, Van Allen radiation belts, magnetosphere, etc [11]

1.4.1 Solar wind

The solar wind is very conducting plasma which is produced by the sun. It travels at a speed of about 500km/s between the spaces of the planets and is formed due to the supersonic expansion of the aura of plasma surrounding the sun. The solar wind mostly consist of electrons and protons with upto 5% of helium also present in them. When the solar wind hits the earth's magnetic field, it is deflected around it. As it strikes the earth's magnetic field with a speed greater than that of sound a *bow shock* is created, which causes conversion of some of the particles kinetic energy into thermal energy. Behind the bow shock, another region of plasma exists called the *magnetosheath* region. The density and temperature of the magnetosheath region are higher than the solar wind plasma.[1]

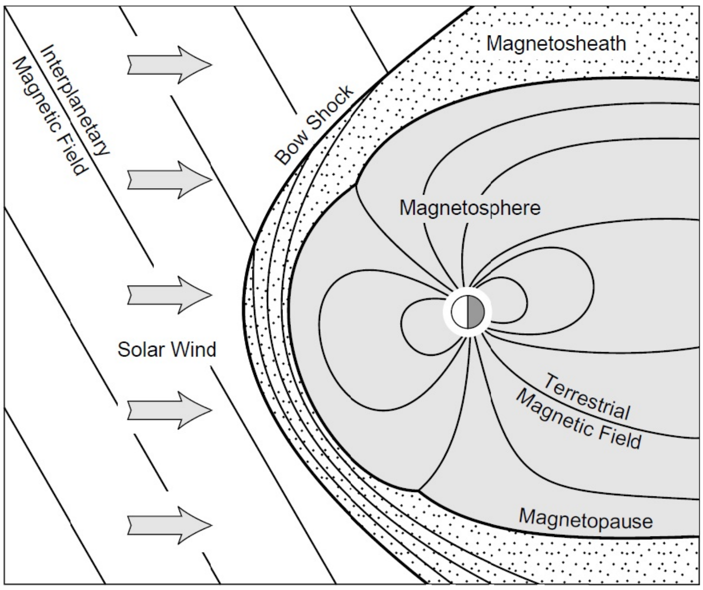

Figure 1.1: Topography of the solar-terrestrial environment

1.4.2 Magnetosphere

A magnetosphere is formed when solar wind interact with the earth's magnetic field. The separation between the shocked solar wind and the magnetosphere is known as *Magnetopause*. The magnetosphere is stretched in a direction opposite to the sun, in such a way that a tail is formed known as *Magnetotail*. The stretching of the magnetosphere is caused due to solar wind's kinetic pressure, due to which the front side of the magnetosphere is compressed while the backside is stretched.

Magnetosphere's plasma mostly have electrons and protons present in it. Due to the solar winds, He^{++} ions are also present. In addition to this, due to the terrestrial ionosphere, magnetosphere also has a fraction of He^{+} and O^{+} ions.The plasma of magnetosphere not evenly distributed but consists of the different region each of which has different plasma temperature and density.[31]

1.4.3 Ionosphere

The interaction of the earth's atmosphere with solar UV light causes ionization of some of the neutral particles which are present there. Collisions which takes place at a height of 80km are very less due to which the ionized particles will not recombine, hence an ionized region known as *Ionosphere* is formed. Electron density at the mid-latitude of the ionosphere is $n_e \approx 10^5 cm^{-3}$ and the temperature is $T_e \approx 10^3 K$.

At high altitudes the electrons from plasma sheet region move in the direction of the earth's terrestrial magnetic field lines, and down to the altitude of the ionosphere, where they collide with the neutral atmosphere and ionizes them, as a result, photons are emitted and these emitted photons form the *aurora* which are also known as the polar lights.[15]

1.5 Collisional frequency and mean free path

Based on collision, plasma can be of two types *collisional* or *collisionless*. In collision less plasma the collisions are infrequent compared to the particle dynamics hence they can be neglected, whereas in the case of collisional plasma collisions are so frequent that they dominate the behavior of the plasma. Collisional plasma can further be classified into two types, partially ionized or fully ionized plasma.[13]

- Partially-ionized plasma consists of charged particles along with a large number of neutral particles. In partially-ionized plasma, most of the collisions that take place, are between the charged and the neutral particles, which affect the motion of the charged particles within the plasma. The number of collision per second is known as the collisional frequency. Where $\nu_{\alpha n}$ is the collisional frequency and is written as

  $$\nu_{\alpha n} = N_{0_n} V_{T_\alpha} \sigma_{\alpha n},$$

  Where N_{0n} is the equilibrium number density of the neutrals, $V_{T\alpha}$ is the average thermal speed of the charged particle, and $\sigma_{\alpha n}$ is the crossection of the given event.

  The particles are considered to move freely between collisions and distance traveled by the particles between collisions is known as the free path. When the distance traveled by the particles between collisions changes in a statistical manner then mean of the distance traveled by the particles is known as mean free path. The formula for the mean free path is

$$\lambda_n = \frac{V_{T_\alpha}}{\nu_{\alpha n}} = \frac{1}{N_{0_n}\sigma_{\alpha n}}$$

- In fully-ionized plasma all the atoms or molecules within the plasma are ionized. In this, the charged particles interact with each other through their Columb electric fields. These fields cause deflection of the particles at an interparticle distance, that is much larger than the atom's radius. Hence the crossection of the colliding particles is increased through Columb's collisions.

1.6 Waves in Plasma

A wave is propagation with periodic motion characterized by, wavelength λ, wave number k, angular frequency ω and amplitude A. The phase velocity $v_{ph} = \frac{\omega}{k}$ characterizes motion of wave crests, and the group velocity $v_g = \frac{d\omega}{dk}$ gives the speed at which the full wave package can propagate. The waves in plasma are an interconnected set of particles and fields that spread periodically throughout the plasma. Waves in plasma can either be electrostatic or electromagnetic depending on the presence of an oscillating magnetic field. The electrostatic waves are purely longitudinal whereas the electromagnetic must have a transverse component but they may also have a longitudinal component. Further classification of the plasma waves is due to the oscillating species. The modes of propagation of the oscillating species are classified by whether they moving ,perpendicular, parallel, or or at and angle to the stationary magnetic field or if their propagation is in the absence of magnetic field[5] [1]

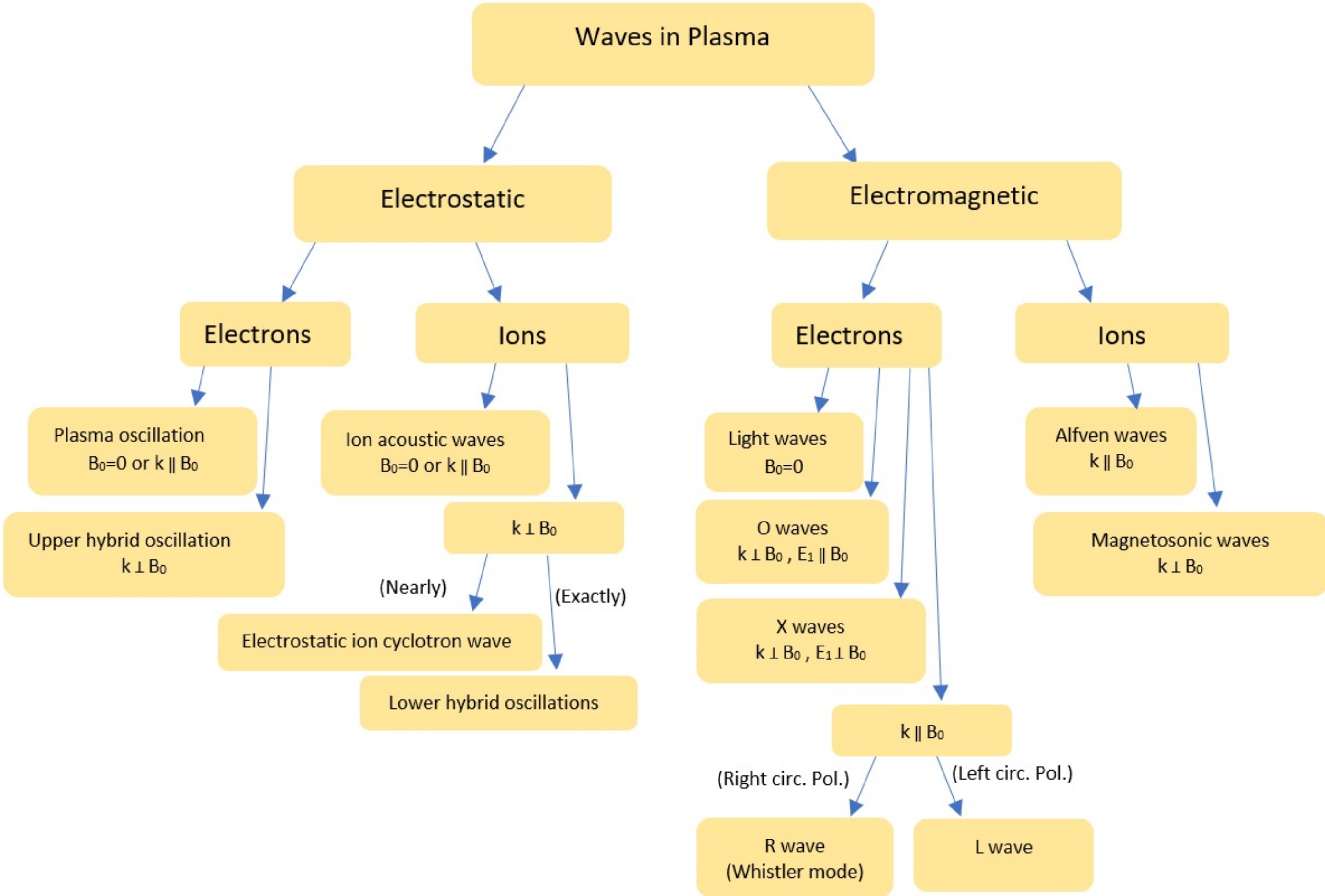

Figure 1.2: Waves in Plasma

1.6.1 Plasma oscillations

In a quasineutral plasma, an electric field is build up when an electron is moved away from its equilibrium position. The electric field's direction will be such that it will pull the electron back to its original position to restore the plasma neutrality. When an electric field acts on electrons, pulling them back to their original position, the inertia of electrons will cause them to overshoot and move to and fro, oscillating about their position of equilibrium at a frequency known as the plasma frequency. The ion being massive in size do not have time to respond to the oscillating electric field due to which they appear fixed at the background. The electron plasma frequency is given as[2]

$$\omega_{p_e} = \left(\frac{n_0 e^2}{m_e \epsilon_0}\right)^2,$$

where n_0 is the equilibrium density, and ϵ_0 is the permittivity in free space.

1.6.2 Electromagnetic waves in an unmagnetized plasma

The dispersion relation for na Electromagnetic wave traveling through plasma in the absence of magnetic field can be written as [27]

$$\omega^2 = \omega_{pe}^2 + k^2 c^2, \qquad (1.1)$$

where c is the speed of light in vacuum, having value $c = \frac{1}{\sqrt{\mu_0 \epsilon_0}}$.

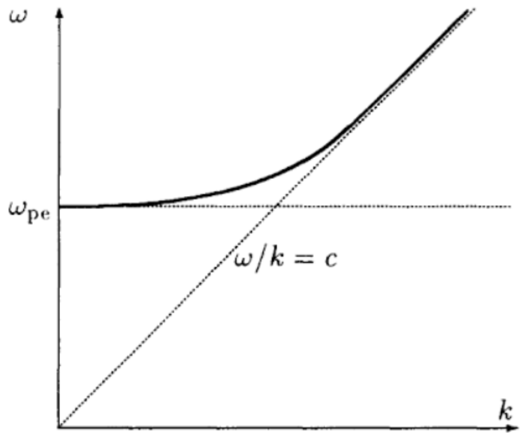

Figure 1.3: Dispersion relation for electromagnetic waves unmagnetized plasma

The fig. 1.3 shows a graph of the dispersion relation for electromagnetic waves in plasma. For a small wavenumber, the group velocity approaches to zero and plasma oscillations take place, whereas for a large wave number both the group velocity and phase velocity converges towards the speed of light. For frequency of waves which are larger than the plasma frequencies we get light waves a from eq. (1.1) $\omega = kc$, the index of refraction for such waves in this case will be $n = \frac{c}{v_{ph}} = \frac{ck}{\omega}$. The index of refraction of a wave having frequency smaller than the plasma frequency is

$$n = \left(1 - \frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}.$$

The waves will propagate through the medium if $n^2 > 0$, hence electromagnetic waves can exist only if $\omega > \omega_{pe}$. When $\omega < \omega_{pe}$ an imaginary refractive index occurs, such waves would not propagate through the medium but they will decay.
Electromagnetic waves can be used in plasma diagnostics in space or ionosphere. We can determine the density of the plasma by sending radio signals to another satellite and detecting the radio wave coming back from them. As plasma absorbs the incident electromagnetic wave when its frequency is equal to that of the plasma frequency , hence if we know frequency of the absorbed wave we can determine the density in the medium.

1.7 Instabilities in Plasma

A Maxwellian distribution is used for the particles in plasma when it is in thermodynamic equilibrium. When plasma deviates from its thermodynamic equilibrium condition, a free source of energy is given to the particles, which under certain conditions gives rise to plasma instabilities. Such a deviation can take place both inhomogeneous plasma and homogeneous plasma systems. In a homogeneous plasma system, the deviation from thermodynamic equilibrium occurs in velocity space. One of the examples of instabilities which occur due to deviation in velocity space is the ion-acoustic instability.[28]

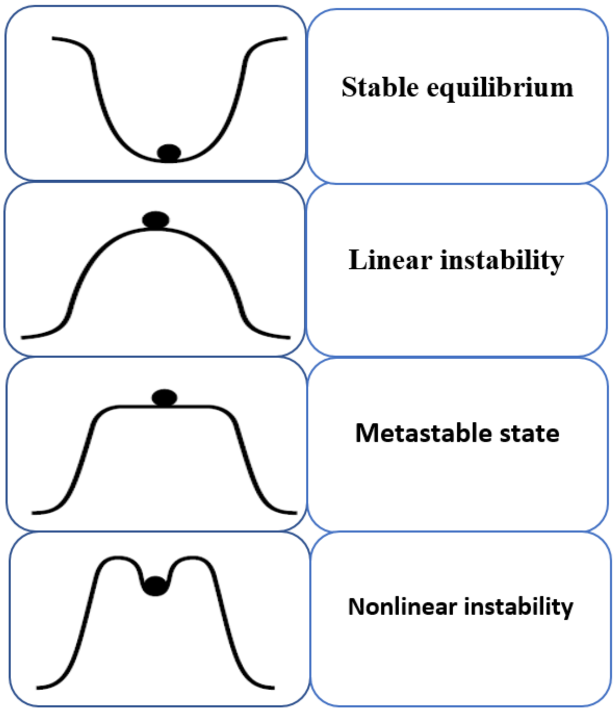

Figure 1.4: Different non equilibrium configurations

The fig. 1.4 shows a simple mechanical analog of how a sphere which is situated in an external potential field can find itself in different stable and unstable situations.

- In *stable equilibrium*,the sphere lies at the lowest point inside the potential trough, and oscillate around its equilibrium position. In the presence of friction, these oscillations will be damped out and the sphere will come to rest at the bottom of the potential trough.
- In the case of *linear instability*, a slight linear distortion will let it roll down the hill, following unstable case sets in spontaneously.
- In the *metastable state*, the sphere lies on the top of a hill's plateau and wander around, until it reaches a point from where it rolls down the plateau.
- In *nonlinear instability* the sphere is stable for a small amplitude of disturbance, but for the case of large amplitudes, it becomes unstable.

In the case of plasma, the potential trough/well corresponds to a free energy source, while the heavy sphere in the potential well corresponds to a wave mode, mostly which are the Eigen-modes of plasma. A plasma is said to be in a state of stable thermodynamic equilibrium if it has a Maxwellian velocity distribution and is homogeneous in space. If plasma is not in a state of thermodynamic equilibrium it means that it has some amount of free energy stored in it, which can be converted into radiation or violent motion of the plasma. In the case of plasma, such processes collectively take place. A plasma can move away from its thermodynamic equilibrium in two ways, first is having velocity distribution other than Maxwellian and the second one is localization in space with a locally higher or lower temperature, pressure density, or other thermodynamic quantity.[18]

The dispersion relation is a complex equation and has several solutions, $\omega = \omega_r + i\gamma$. For complex frequency, the behavior of the wave's amplitude depends on the sign of imaginary part γ of the frequency. For

- $\gamma < 0$, the real part of the frequency decreases exponentially with time and is damped out.
- $\gamma > 0$, the amplitude of the wave grows with time.

Shear flow instability

When there is a local variation of the velocity vector in a given direction, the fluid flow is called shear flow. For example, when the fluid is moving along the x-direction and the magnitude of the fluid velocity is changing along the y-direction, then we can say that the velocity is shear in the fluid flow[30]. The sketch of this simple linear shear flow is given in fig. 1.5.The mathematical expression for the velocity components (v_x, v_y, v_z) of shear flow given in the fig. 1.5 can be written as

$$v_x(y) = Cy, \text{ with } v_y = v_z = 0.$$

Where C is the slope of the profile, it is also called the constant gradient of the velocity.

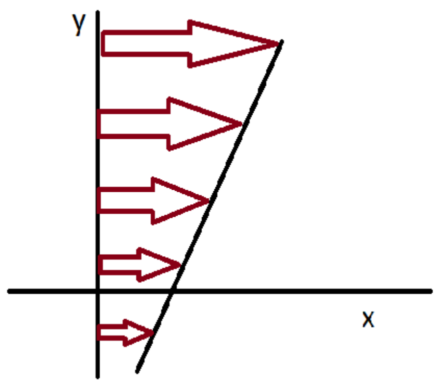

Figure 1.5: A picture of simple shear flow. The fluid is moving along x-direction and velocity is linearly changing with y

The instability which is caused due to a change in the velocity profile of the plasma which takes place due to is called Shear flow instability [33], where the change in velocity profile can occur due to turbulence within Plasma.

1.8 Different theoretical approaches to study the behavior of Plasma

We can describe the dynamics of the plasma by its interaction with external electric and magnetic fields. When the charged particles move within plasma they create a local charge concentration, which give rise to internal electric field. Electric current will be generated by the motion of these particles which also causes internal magnetic fields to be generated. The response of these internal fields to the motion of particles in plasma and external fields makes the study of plasma very difficult. Also, plasma shows different characteristics such that their densities, temperature, and degree of ionization can change. The effect of collisions and electromagnetic forces also has great importance in the study of plasma dynamics. Hence to study different behaviors of plasma, different plasma models are used.[12]

1.8.1 Single particle approach

When an electric or magnetic field is applied to a plasma it affects the motion of the particles. To describe the motion of a single particle, when it comes under the influence of such fields, the single-particle approach is used. While describing the motion using single-particle approach the collective behavior of plasma is neglected. This approach is useful in the study of very-low-density plasma e.g. while studying the energetic particles or the cosmic rays in Van Allen radiation belts.

1.8.2 Plasma as a fluid

As plasma consists of a large number of particles where each particle follows a complicated path, therefore it becomes nearly impossible to deal with each particle separately and observe the behavior of plasma. Therefore to study the behavior of plasma we use other models like,

Magnetohydrodynamics model (MHD)

The study of electrically conducting fluids is known as Magnetohydrodynamics (MHD). In this approach, plasma is taken as a single conducting fluid. It is used to describe equilibrium and Large-scale stability of the magnetized plasma. Drawback of this approach is that the macroscopic properties of each specie within the plasma, such as velocity, density and temperature will be lost, while taking an average of these properties into account. In order to study plasma behavior using the Magnetohydrodynamic (MHD) model, we use the hydrodynamic equations coupled with Maxwell's equations.

Multi-fluid model

In this model, each species of the plasma is treated as a separate fluid element. Advantage of the multi-fluid model over the MHD is that different behavior of different species within the plasma can be taken into account. For example at the same spatial point within the plasma different plasma components can have different velocity, temperature, and pressure.

1.8.3 Kinetic Theory of Plasma

While using the fluid model approach the information regarding the velocity distribution of the particle is lost, as the fluid variables are function of position and time only. Any physical properties of the plasma that depend on this microscopic detail can be discovered only by a description in six-dimensional $(\mathbf{r}, \mathbf{v})$ space. Thus, instead of starting with the density of particles $n(\mathbf{r}, t)$ at position $\mathbf{r}$ and time t, we begin with the so-called distribution function, $f(\mathbf{r}, \mathbf{v}, t)$. The evolution of the distribution function is described by the kinetic theory.

$$\int f(\mathbf{r}, \mathbf{v}, t) dv = n(\mathbf{r}, t)$$

1.9 Shear flow instability in a partially ionized plasma sheath around a re-entry vehicle

Before describing the problem it is important to know about some of the terms, which will be used during discussion,

Shock-wave

When the speed of the source is equal to that of the sound it produces, the sound waves it

produces will pileup at a single point in front of the source. Now when the speed of source is greater than the sound it produces, no wave will be produced in front of it but will pile up behind and will be confined to a cone known as the shock wave cone.
Mach number
The ratio of speed of aircraft to the speed of sound in gas is known as mach number.
Wake region
This region exists behind the re-entry vehicle. In this region, the recombination of electron-ion takes place at a significant rate.[22]

The spacecraft re-entering the Earths atmosphere is traveling very much faster than the speed of sound and is said to be hypersonic. The typical re-entering speed at the lower orbit of the earth is near 175000 mph having Mach number nearly 25. As the spacecraft enters the earth's atmosphere hyper-sonic speed, it possesses a large amount of potential and kinetic energy. The collision of the gas atoms and molecules with the surface of the vehicle causes a shock wave to be produced infront of the vehicle, this shock-wave causes the air around the vehicle to be compressed and heated. This heat increases the air temperature between the surface of the vehicle and the shock wave . Temperature is sufficient to ionize air around the vehicle and as the density of air is less in the upper atmosphere, the formation of plasma around the vehicle takes place. The plasma which envelops the re-entry vehicle is called *plasma sheath* [10]. While studying the plasma sheath we consider following four categories. [19].

- The basic physics i.e. study of ionization and recombination process which takes place during the formation of plasma sheath.
- Calculating how the plasma sheath is flowing, which includes the constituents and geometry of the plasma sheath and wake region.
- Analyzing the interactions between the fields and the plasma sheath.
- Interpreting the radar performance and changes in the data telemetry due to the plasma sheath.

1.9.1 Interaction of electromagnetic wave with plasma sheath

Plasma as a whole is quasi-neutral i.e., it has an equal number of positive ions and free electrons together with several neutral particles. An average equilibrium separation is maintained between the charged particles due to an electrostatic field present between them. If one of the charged particles is kept constant and other is moved from its equilibrium, it will move to and fro about the equilibrium position.[21] The movement of the particle is similar to a mass-spring system, where the particle displaced is similar to the mass attached to the spring and electrostatic restoring force of the neighboring charged particle is the spring, and collision with the neutral particle constitutes the damping. The frequency with which the free charge oscillates in the plasma is known as plasma frequency. The relation for plasma frequency of electron will be

$$\omega_p = \sqrt{\frac{n_0 e^2}{m_e \epsilon_0}}.$$

An analogous equation for the plasma frequency of ion is used where instead of mass of electron mass of the ions is used. As the mass of an ion is four orders of magnitude greater than that of electrons hence the plasma frequency of ions will be less than that of electrons.

When an electromagnetic wave hits the plasma it acts as a periodic driving force on the electron. As already discussed, electron within the plasma oscillates at a natural frequency known as the electron plasma frequency. If the periodic driving force is considerably less than the electron plasma frequency, and damping due to the collision are also small, than the inertial effects of the electron will be small and it will oscillate at the driving frequency. The charge oscillating at driving frequency will act as a dipole radiator, which produces electromagnetic wave traveling both forward and backward direction. The forward traveling wave is out of phase with the driving force and will cancel out the driving signal. The process of canceling out of the driving signal is repeated as the driving signal penetrate the plasma causing an attenuation of the driving signal increases with the thickness of the plasma. The backward traveling electromagnetic wave produced by an oscillating charge will appear as a reflected wave.

When the driving frequency is larger than the natural electron plasma frequency. The electrons will exhibit large inertial effects hence it will weakly oscillate at the driving frequency and if there are no collisions present then damping will not take place and the electromagnetic wave will travel through plasma un-attenuated. In the presence of collision, a slight reflection and attenuation of the electromagnetic waves take place.

When the driving frequency is exactly equal to the electron plasma frequency. The forward and backward traveling wave produced by the oscillating such that the incident electromagnetic wave will not penetrate the plasma and is completely reflected from the surface of the plasma.[23]

Consider an electromagnetic wave having a frequency ω incident upon the plasma medium having a frequency ω_p. In the absence of collision when[9]

- $\omega > \omega_p$, transmission will take place.
- $\omega = \omega_p$, absorption takes place.
- $\omega < \omega_p$, reflection of the incident wave takes place.

Advanced reentry vehicles have an antenna or a sensor present on their heat shield which provides information about the vehicles instantaneous position, gives navigation information and serve communication functions . The plasma formed near the vehicle can interfere with the antenna performance. When there is an instability within the plasma medium, than the electromagnetic wave will be modulated by turbulent plasma which can cause a change in the phase and amplitude of the electromagnetic waves.[17, 4]

Chapter 2

Fluid description of Plasma

There is so much we can do with the single particle approach in plasma. In plasma we come across large number of particles where each particle follow a complex trajectory. Hence it is impossible to follow every single particle and observe the behavior of plasma. Therefore we use the fluid approach to study the behavior of plasma as a whole. In this approach the plasma is assumed to be a conducting fluid and we use the already established equations of fluid mechanics in order to find general properties of the plasma. About 80% or so applications in plasma are sufficiently treated with the fluid approach of plasma, in which electromagnetic forces are taken into account [32]. The plasma fluid equations are modification of the Naiver-Strokes equations, and require conservation of charge and mass. In the case of electromagnetic waves these equations are supplemented by the Maxwell's equation of electromagnetism.

Consider an infinitesimal volume dV surrounding a point r, at time t. We can write the mass density of the fluid is the sum of all the masses of the particles within the volume element dV, divided by the volume dV itself[20]

$$\rho = \frac{\sum m}{dV}.$$

The hydrodynamic velocity of the volume element dV can be written as

$$v = \frac{\sum mv}{\rho dV}.$$

2.1 The fluid equation of motion

The motion of a single particle in a plasma can be describe by [5]

$$m\frac{dv}{dt} = q(E + v \times B). \tag{2.1}$$

In the case of plasma fluid, where the thermal motions of the particles within plasma, and collisions of these particle with other particles within plasma, have not been taken into account, in such a case all the particles in the fluid moves with an average velocity *v(r,t)*. For such case we can write the equation of motion of the fluid by multiplying eq. (2.1) with number density "n". Hence,

$$mn\frac{d\vec{v}}{dt} = qn(\vec{E} + \vec{v} \times \vec{B}). \tag{2.2}$$

In eq. (2.2), $\frac{dv}{dt}$ shows the rate of change of velocity in position and time. To do that we transform the variables of the fluid into a fixed frame (that moves with the fluid element). In order to make the transformation, we consider a function *A(y,t)* which is any property of the fluid in a one dimensional space. The change of function *A(y,t)* in a frame which is moving with the fluid, with respect to time can be written as

$$\frac{dA(y,t)}{dt} = \frac{\partial A}{\partial t} + v_y \frac{\partial A}{\partial y}, \tag{2.3}$$

where

- $\frac{\partial A}{\partial t}$: Change in function A which takes place at a point which is fixed in space,
- $v_y \frac{\partial}{\partial y}$: Change in the property of the fluid as the observer moves with the fluid into the region where A changes.

For the case of three dimensional flow of the fluid, the convective derivative can be written as

$$\frac{dA(r,t)}{dt} = \frac{\partial A}{\partial t} + (\vec{v} \cdot \vec{\nabla})A.$$

For plasma, we take the function A to be the velocity "*v(r,t)*" of the fluid. The equation of momentum for the fluid, in absence of collisions and thermal effects, can be written as

$$mn[\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \vec{\nabla})\vec{v}] = nq(\vec{E} + \vec{v} \times \vec{B}). \tag{2.4}$$

2.1.1 Equation of motion in the presence of thermal effects

If we take the motion of the particles into account, the pressure term will also be added to eq. (2.4). The pressure gradient force does not appear in the momentum equation for a single particle approach, as it takes place due to the random motion of the particles within fluid.[5]

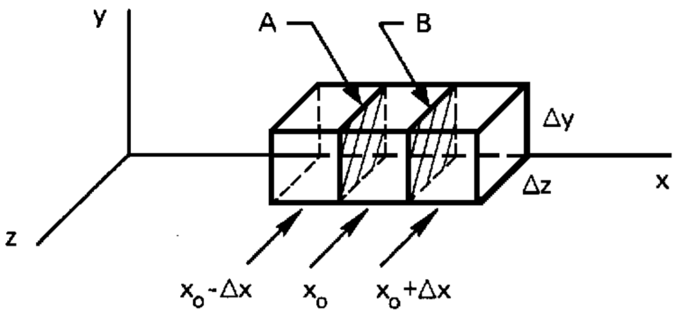

Figure 2.1:

In fig. 2.1, consider the fluid to move through the face A and B, along x-direction, of the fluid element which is centered at $(x, \frac{1}{2}\Delta y, \frac{1}{2}\Delta z)$. The number of particles per second which are passing through the face A, with velocity v_x is

$$\Delta n_v v_x \Delta y \Delta z,$$

where Δn_v is the number of particles per m^3 having velocity v_x, written as

$$\Delta n_v = \Delta v_x \iint f(v_x, v_y, v_z) dv_y dv_z,$$

where $f(v_x, v_y, v_z)$ is the distribution function of the particles in velocity space at a particular spacial location. Each particle within the fluid carries a momentum mv_x. Let P_{A^+} be the momentum of the particle moving into the fluid's volume element at x_0, centered at $(x, \frac{1}{2}\Delta y, \frac{1}{2}\Delta z)$, through the face A. P_{A^+} can be written as

$$P_{A^+} = m\Delta y\Delta z \int_o^\infty {v_x}^2 f dv_x \iint\limits_{-\infty}^{+\infty} dv_y dv_z.$$

In the case of properly normalized function f

$$n = \iiint\limits_{-\infty}^{+\infty} f dv_x dv_y dv_z.$$

The average velocity $< v_x >$ can be written as

$$< v_x >= \frac{\int_o^\infty {v_x}^2 f dv_x \iint\limits_{-\infty}^{+\infty} dv_y dv_z}{\iiint\limits_{-\infty}^{+\infty} f dv_x dv_y dv_z}.$$

The momentum P_{A^+} in terms of the average velocity $< v_x >$ can be written as

$$P_{A^+} = m\Delta y\Delta z\frac{1}{2}[n < {v_x}^2 >]_{x0-\Delta x},$$

where the factor $\frac{1}{2}$ occurs as half of the particles within the cube at $x - \Delta x_0$ are moving toward face A. We can write the momentum of the particles moving out of the cube through face B as

$$P_{B^+} = m\Delta y\Delta z\frac{1}{2}[n < {v_x}^2 >]_{x_0}.$$

The net gain of momentum of the particles is given as

$$P_{A^+} - P_{B^+} = m\Delta y\Delta z\frac{1}{2}[(n < {v_x}^2 >)_{x_0-\Delta x} - (n < {v_x}^2 >)_{x_0}].$$

Hence

$$P_{A^+} - P_{B^+} = -\frac{1}{2}m\Delta y\Delta z\Delta x\frac{\partial}{\partial x}(n < {v_x}^2 >). \tag{2.5}$$

This result obtained in eq. (2.5) will be doubled due to the contribution of the particles moving from the left, as they are moving in the opposite direction with respect to the gradient $n < {v_x}^2 >$. We can therefore write the total change of momentum of the fluid element at point x_0 as

$$\frac{\partial}{\partial t}(nmv_x)\Delta x\Delta y\Delta z == m\frac{\partial}{\partial x}(n < {v_x}^2 >)\Delta x\Delta y\Delta z. \tag{2.6}$$

Let the velocity v_x of the particles within the fluid be decomposed into two parts

$$v_x = u_x + v_{xr},$$

where u_x is the velocity of fluid along x-direction, and v_{xr} is the velocity of random thermal motion of the particles along x-direction, hence

$$< {v_x}^2 >= {u_x}^2 + \frac{K_B T}{m}.$$

By using this value in eq. (2.6), we get

$$\frac{\partial}{\partial t}\Big(nmu_x\Big) = -m\frac{\partial}{\partial x}\Big[n({u_x}^2 + \frac{K_B T}{m})\Big],$$

$$mn\bigg(\frac{\partial}{\partial t}u_x + u_x\frac{\partial}{\partial x}u_x\bigg) + mu_x\bigg(\frac{\partial n}{\partial t} + \frac{\partial(mu_x)}{\partial x}\bigg) = -\frac{\partial}{\partial x}\Big(nK_B T\Big).$$

From equation of continuity $\frac{\partial(n)}{\partial t} + \frac{\partial(nu_x)}{\partial x} = 0$, hence the above equation becomes

$$mn\bigg(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x}\bigg) = -\frac{\partial}{\partial x}\Big(nK_B T\Big),$$

where $P = nK_B T$, so

$$mn\bigg(\frac{\partial u_x}{\partial t} + u_x\frac{\partial u_x}{\partial x}\bigg) = -\frac{\partial P}{\partial x}.$$

For three dimensional case, we get

$$mn\bigg(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\bigg) = -\nabla P. \tag{2.7}$$

The eq. (2.7) gives the pressure gradient force. If we add the electromagnetic force i.e., the Lorentz Force, then the above equation becomes

$$mn\bigg(\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\vec{\nabla})\vec{u}\bigg) = nq(\vec{E} + \vec{u}\times\vec{B}) - \vec{\nabla}P.$$

2.1.2 Equation of motion in presence of collisions

If within plasma collisions occur between like particles then the total momentum which is averaged over all the particles will not change. But for a plasma fluid consisting of two or more species, a collision between them will cause gain or loss of momentum between the species. Consider a plasma fluid consisting of neutral and charged particles. If a charged particle "c" collides with the neutral, an exchange of momentum between the charged particle and neutral will take place. If $\vec{u}_n$ is the velocity of neutral in fluid and $\vec{u}_c$ is the velocity of the charged specie, then the momentum lost per collision will be proportional to $(\vec{u}_c - \vec{u}_n)$. The rate of momentum density lost by the charged specie "c" upon collision with neutral will be[20]

$$\nu_{cn} n_c m_c(\vec{u}_c - \vec{u}_n).$$

The generalized momentum equation for the charged specie can be written as,

$$m_c n_c\bigg[\frac{\partial \vec{u}_c}{\partial t} + (\vec{u}_c\cdot\vec{\nabla})\vec{u}_c\bigg] = n_c q_c(\vec{E} + \vec{u}_c\times\vec{B}) - \vec{\nabla}\cdot\vec{P}_c - \nu_{cn} n_c m_c(\vec{u}_c - \vec{u}_n). \tag{2.8}$$

Collision frequency

The two broad classes of collisions are the inelastic collision and the elastic collision. In order to describe the particle's motion before and after collision, the laws of conservation of mass, momentum and energy is applied for both elastic and inelastic collisions. The difference between both the collisions is that in inelastic collision the internal energy of the particles is changed, whereas in the case of elastic collision the total energies are conserved. Consider a binary encounter of species 1 with 2, the collision frequency between both the species is given by

$$\nu_{12} =< N_2 V_{T_1} \sigma_{12} >,$$

where V_{T_1} is the relative speed, N_2 is the number density of the target specie 2 and σ_{12} is the cross-section of that given event.[13]

2.2 Equation of continuity

Before deriving the equation of continuity it is necessary to know the meaning of mass flux.

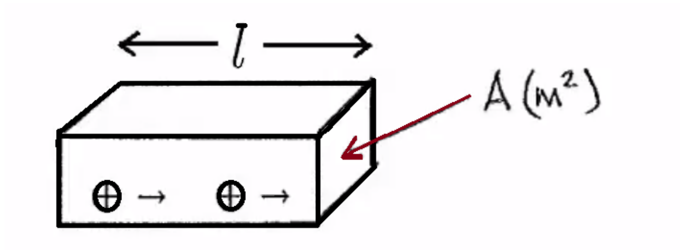

Figure 2.2: Volume element

Consider the volume element shown in fig. 2.2, charge particles are moving to the right through this volume element. The volume element has length "l", and crossection "A". The number N of particles within the box will be

$$N = n \times Al,$$

where n is the number density and Al is the volume of the box. The number of particles that leave the box in time t is

$$\frac{N}{t} = \frac{nAl}{t} = nAv,$$

where $v = \frac{l}{t}$ is the speed of the volume element. Particle flux is the rate of particles exiting the box per unit area. Hence dividing the equation by an area "A" of the volume element, we get

$$Particle\ flux = \frac{N}{At} = nv, \qquad (particles/m^2 s)$$

and

$$Mass\ flux = mnv = \rho v. \qquad (kg/m^2 s)$$

In order to derive the equation of continuity, we consider a fluid which is flowing into and out of an infinitesimally small box as shown in fig. 2.3.

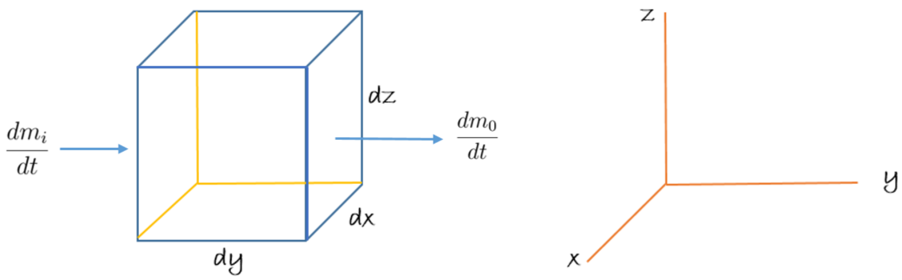

Figure 2.3: Infinitesimal volume

Let the rate of change of mass flowing into the cube be given by $\frac{dm_i}{dt}$, and the rate of change of mass flowing out of the cube be $\frac{dm_0}{dt}$, we can write the equation for mass flow rate through one face of the cube as

$$\frac{dm}{dt} = \frac{dm_0}{dt} - \frac{dm_i}{dt}.$$

The mass flow rate through all the faces of the cube will be

$$\frac{dm}{dt} = (\frac{dm_{x_o}}{dt} + \frac{dm_{y_o}}{dt} + \frac{dm_{z_o}}{dt}) - (\frac{dm_{x_i}}{dt} + \frac{dm_{y_i}}{dt} + \frac{dm_{z_i}}{dt}). \tag{2.9}$$

The mass flow rate was defined as,

$$\frac{dm}{dt} = mass\ flux \times area,$$

$$\frac{dm}{dt} = \rho v \times dxdz.$$

Then eq. (2.9) becomes

$$\frac{d\rho}{dt}dxdydz = (\rho_{x_o}v_{x_o} - \rho_{x_i}v_{x_i})dydz + (\rho_{y_o}v_{y_o} - \rho_{y_i}v_{y_i})dxdz + (\rho_{z_o}v_{z_o} - \rho_{z_i}v_{z_i})dxdy, \tag{2.10}$$

where each element on the right hand side of the equation can be written, in a more compact form, as

$$\rho_{x_o}v_{x_o} - \rho_{x_i}v_{x_i} = -\Delta(\rho_x v_x).$$

The negative sign appears over here because we expect the mass of the particles flowing out to be less or equal to the mass of the particles flowing in. It would be very odd if the rate of flow of mass moving out exceeded the rate of mass flow in, since that would defy the law of conservation of mass. The minus sign will be applicable under those circumstances where the fluid is compressible i.e. where more mass flows into the cube then out of it, which could happen in plasma. Considering the cube to become infinitesimally small then we can rewrite the equation for the change in mass flow rate as

$$\rho_{x_o}v_{x_o} - \rho_{x_i}v_{x_i} = -d(\rho_x v_x).$$

By making similar substitution and dividing both sides of the eq. (2.10) by $dxdydz$, we get

$$\frac{\partial \rho}{\partial t} = -\frac{\partial(\rho_x v_x)}{\partial x} - \frac{\partial(\rho_y v_y)}{\partial y} - \frac{\partial(\rho_z v_z)}{\partial z}.$$

We can write the equation of continuity in a more compact form

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho v) = 0.$$

In terms of number density, the above equation becomes

$$\frac{\partial n}{\partial t} + \nabla.(n v) = 0. \quad (2.11)$$

2.3 Equation of state

In order to describe how pressure "P" changes with time we need to add an additional term called the equation of state, in order to relate the pressure term to the number density "n". Equation of state can be written as[8]

$$P = cn^{\gamma},$$

where c is a constant let it equal to 1, and "γ" is the ratio between specific heat at constant volume and specific heat at constant pressure i.e. $\frac{C_p}{C_v}$. It tells us about the amount of increase in temperature of the plasma as it is compressed. We can write the expression for change in pressure as

$$\nabla P = \nabla n \gamma n^{\gamma-1} = \gamma \ P\frac{\nabla n}{n},$$

hence

$$\frac{\nabla P}{P} = \gamma\frac{\nabla n}{n}.$$

- For an isothermal compression of the plasma, the ratio between specific heat at constant volume and specific heat at constant pressure will be equal to 1 i.e. $\gamma = 1$, hence the gradient in pressure can be written as

$$\nabla P = \nabla n K_B T.$$

- The value of T also changes in the case of adiabatic compression, the value of γ in such case will be

$$\gamma = \frac{(2+N)}{N},$$

  where "N" gives us the number of degree of freedom.

2.4 Maxwell's equations

1. The first is known as the Poisson's equation, which is written as [29]

$$\vec{\nabla} \cdot \vec{D} = \rho,$$

$$\vec{D} = \vec{P} + \epsilon_0 \vec{E} \simeq \epsilon_0 \vec{E}.$$

where above "D" is the displacement vector, ϵ_0 free space permittivity, and "P" is the polarization of atom. In gaseous plasma, the polarization of atom is very small, hence the term "P" can be ignored. So we can write the Poisson's equation for plasma as

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0}. \tag{2.12}$$

For the case where plasma consists of two fluid e.g. electron and ions, the density term can be written

$$\rho = q_e n_e + q_i n_i = -e(n_e - Z n_i).$$

2. Another is the divergence of magnetic field, which is an equation showing the absence of magnetic mono-pole and is written as

$$\vec{\nabla} \cdot \vec{B} = 0. \tag{2.13}$$

3. The third is the equation equivalent of Faraday's law of electromagnetic induction, according to which the variation of the magnetic field in time is accompanied by a spatially-varying electric field and vice versa

$$\vec{\nabla} \times \vec{E} = \frac{\partial \vec{B}}{\partial t}. \tag{2.14}$$

4. The forth Maxwell equation is the generalization of Ampere's law

$$\vec{\nabla} \times \vec{H} = \frac{\partial \vec{D}}{\partial t} + \vec{J},$$

$$\vec{B} = \mu_0(\vec{H} + \vec{M}) \simeq \mu_0 \vec{H},$$

where μ_0 is the free space magnetic permeability , J is the current density of the specie and the magnetization of dipole moment per unit volume is represented by M. As for plasma M is negligible, hence

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t}, \tag{2.15}$$

For a plasma having ions and electrons the current density J, can be written as

$$\vec{J} = q_e n_e \vec{v_e} + q_i n_i \vec{v_i} = -e(n_e \vec{v_e} - Z n_i \vec{v_i}).$$

2.5 Langmuir waves

In warm plasma, we also consider the thermal motion of electrons. Electrons having the thermal motion will stream into the layers of the plasma carrying information about the disturbance occurring in the undisturbed ambient plasma. These disturbances then propagate as a wave known as the electron plasma waves. The dispersion relation for such waves can be derived using the linearized equation of motion, given as

$$m n_0 \frac{\partial v_1}{\partial t} = -e n_0 E_1 - 3 K_B T_e \frac{\partial n_1}{\partial x}.$$

Where v_1, E_1 and n_1 are the perturbed terms and behave sinusoidaly i.e.,

$$\vec{v_1} = v_1 e^{i(kx-\omega t)}\hat{x},$$

$$\vec{E_1} = E_1 e^{i(kx-\omega t)}\hat{x},$$

$$n_1 = n_1 e^{i(kx-\omega t)},$$

hence

$$-im\omega n_0 v_1 = -en_0 E_1 - 3ikn_1 KT_e.$$

$$\omega^2 V_1 = \left(\frac{n_0 e^2}{\epsilon_0 m} + \frac{3KT_e k^2}{m}\right) v_1,$$

$$V_{th}{}^2 = \frac{2KT_e}{m},$$

so

$$\omega^2 = \omega_p{}^2 + \frac{3}{2} k^2 V_{th}{}^2.$$

The dispersion relation of Langmuir waves shows that due to the thermal motion of electrons within the plasma its wave frequency will have a dependence on the wavenumber as well.

2.6 Ion-acoustic waves

Ion waves are low pressure and low-frequency waves that occur in plasma. We get similar dispersion relation for ions wave as we get for the sound waves therefore we call them the Ion acoustic waves (IAW). The difference between sound waves and ion-acoustic waves is that when charges are separated due to ion-acoustic waves, an electric field is induced in the plasma. The electron component of the ion-acoustic wave tends to move faster than its ion component, but the electric field produced by the ion-acoustic retards the motions of the electron, forcing both electrons and ions to propagate together. For ions, the momentum equation in the absence of the magnetic field will be

$$Mn\left[\frac{\partial v_i}{\partial t} + (v_i.\Delta)v_i\right] = -en\Delta\phi - \gamma_i KT_i \Delta n.$$

By linearizing the above equation we get

$$-i\omega M n_0 v_{i_1} = -en_0 ik\phi_1 - \gamma_i KT_i ikn_{i_1}. \tag{2.16}$$

The Boltzmann relation for electrons is given as

$$n_e = n = n_0 exp\left(\frac{e\phi_1}{KT_e}\right) = n_0\left(1 + \frac{e\phi_1}{KT_e} + \dots\right).$$

$$n_e = n_0 + n_{e_1},$$

$$n_{e_1} = n_e - n_0 = n_0 \frac{e\phi_1}{KT_e}.$$

As we are considering low frequency oscillations so we can use the quasi-neutrality condition i.e. $n_{i_1} = n_{e_1}$. Therefore

$$n_{i_1} = n_0 \frac{e\phi_1}{KT_e}. \tag{2.17}$$

After linearizing the equation of continuity, we get

$$i\omega n_{i_1} = n_0 ikv_{i_1}. \tag{2.18}$$

By using values of eq. (2.17) and eq. (2.18) in eq. (2.16), we get

$$\omega^2 = k^2 \left(\frac{K_B T_e}{M} + \frac{\gamma_i K T_i}{M} \right), \tag{2.19}$$

$$\frac{\omega}{k} = \left(\frac{KT_e + \gamma_i KT_i}{M} \right)^{\frac{1}{2}} = v_s. \tag{2.20}$$

The eq. (2.19) gives dispersion relation for ion-acoustic waves, where v_s is the ion-acoustic speed. For $KT_i \approx 0$, the ion-acoustic waves still exist and the ion-acoustic velocity is given by

$$v_s = \frac{KT_e}{M}.$$

The approximation used over here is that of quasi-neutrality. This assumption is not true for higher frequencies, closer to ω_{p_i}, because the electron and ion motion becomes uncorrelated. So in the case of high frequency oscillation we will use the Poisson's equation instead of the quasi-neutrality condition. Therefore we can write

$$\nabla^2 \phi = \frac{e}{\epsilon_0}(n_{i_1} - n_{e_1}),$$

$$\epsilon_0 k^2 \phi = n_{i_1} - n_{e_1}, \tag{2.21}$$

where

$$n_{e_1} = n_0 \frac{e\phi_1}{KT_e},$$

and

$$n_{i_1} = \frac{k}{\omega} n_0 v_{i_1},$$

v_{i_1} can be obtained from eq. (2.16). By putting values of n_{e_1} and n_{i_1} in eq. (2.21) we get

$$\frac{\omega}{k} = \left(\frac{KT_e}{M} \frac{1}{1 + k^2 {\lambda_D}^2} + \frac{\gamma_i KT_i}{M} \right)^{\frac{1}{2}}. \tag{2.22}$$

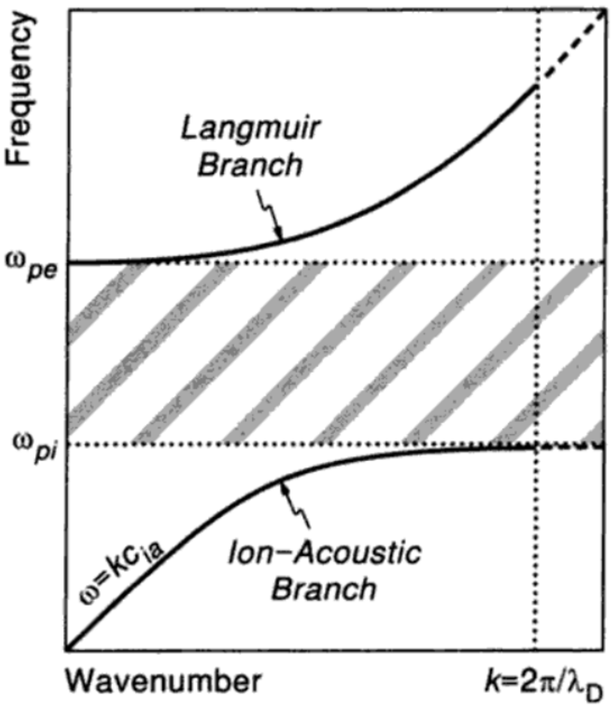

Figure 2.4: Dispersion of Langmuir and ion-acoustic waves

The fig. 2.4 shows a graph of two electrostatic waves that exist in an unmagnetized plasma. The low-frequency ion-acoustic waves start from zero to the ion plasma frequency, whereas the high-frequency waves start at the electron plasma frequency. No electrostatic mode can oscillate between the two plasma frequencies ω_{p_i} and ω_{p_e} in an unmagnetized plasma.

2.7 Electrostatic ion waves $\perp B_0$

Two cases will be considered for electrostatic ion waves having propagation "k" perpendicular to the ambient magnetic field "B_0".

2.7.1 Propagation of the wave nearly perpendicular to the ambient magnetic field "B_0"

In such a case following assumptions are made.

- Considering the plasma to be infinite,
- Unperturbed density and magnetic field n_0 and B_0 are considered to be uniform and have a constant value,
- $v_0 = E_0 = 0$,
- For simplicity let $T_i = 0$,
- $k \times E = 0$ i.e. $E = -\nabla\phi$,

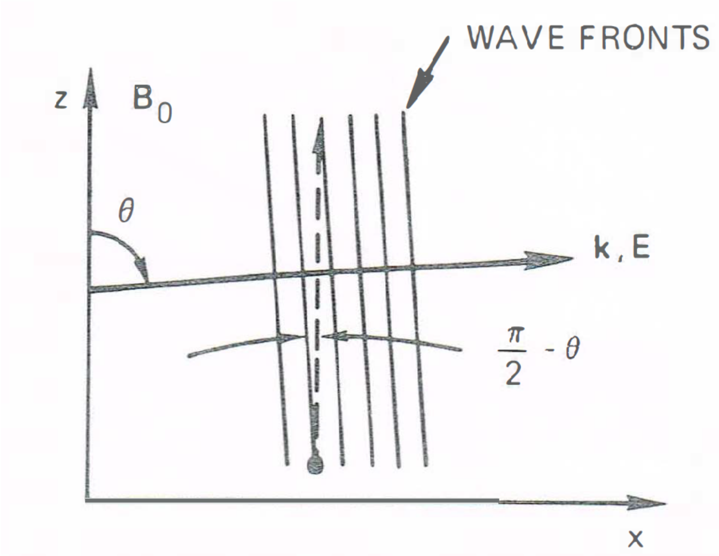

Figure 2.5: Electrostatic ion cyclotron wave nearly perpendicular to magnetic field

The fig. 2.5 gives geometrical description of the low frequency electrostatic waves nearly perpendicular to magnetic field. Due to a small deviation from exact $\frac{\pi}{2}$ the electrons can move along B_0 but ions due to their large inertia will not oscillate along the z-direction i.e. $k_z \approx 0$ for ions. The linearized equation of motion for ions

$$M\frac{\partial \vec{v}_{i_1}}{\partial t} = -e\vec{\nabla}\phi_1 + e\vec{v}_1 \times \vec{B}_0, \tag{2.23}$$

can be written in component form as

$$-i\omega M v_{i_x} = -eik\phi_1 + ev_{i_y}B_0,$$

$$-i\omega M v_{i_y} = -ev_{i_x}B_0.$$

By solving these equations, we get

$$v_{i_x} = \frac{ek}{M\omega}\phi_1\left(\frac{\omega^2}{\omega^2 - {\Omega_c}^2}\right), \tag{2.24}$$

where $\Omega_c = \frac{eB_0}{M}$ is the ion cyclotron frequency. From ion equation of continuity, we can write

$$n_{i_1} = n_0\frac{k}{\omega}v_{i_x}. \tag{2.25}$$

As electron can move along B_0 therefore we can use the Boltzmann relation for electrons

$$\frac{n_{e_1}}{n_0} = \frac{e\phi_1}{KT_e}. \tag{2.26}$$

By using the plasma approximation, we can write

$$\frac{k}{\omega}v_{i_x} = \frac{e\phi_1}{KT_e}.$$

by using the value of v_{i_x} from eq. (2.24), we get

$$\frac{ek^2}{M\omega^2}\phi_1\left(\frac{\omega^2}{\omega^2 - {\Omega_c}^2}\right) = \frac{e\phi_1}{KT_e},$$

$$\omega^2 - \Omega_c^2 = k^2\frac{KT_e}{M}.$$

Hence the dispersion relation for ion cyclotron waves is

$$\omega^2 = \Omega_c^2 + k^2 v_s^2. \tag{2.27}$$

2.7.2 Propagation of the wave perpendicular to the ambient magnetic field "B_0"(Lower hybrid wave)

Now consider the case in which the propagation of the low-frequency electrostatic wave is exactly perpendicular to the ambient magnetic. In this case, the electrons will not obey Boltzmann relation as they will not flow along with the lines force and preserve charge neutrality. So we write the complete equation of motion for electron to get

$$v_{e_x} = \frac{-ek}{m}\left(\frac{\omega}{\omega^2 - \omega_c^2}\right)\phi_1, \tag{2.28}$$

and electron equation of continuity will give us

$$n_{e_1} = n_0 \frac{k}{\omega} v_{e_1}. \tag{2.29}$$

From plasma approximation $n_{i_1} = n_{e_1}$, which for this case can also be written as $v_{e_x} = v_{i_x}$. Using the values of v_{e_x} and v_{i_x} from eq. (2.28) and eq. (2.24) respectively, we get

$$\frac{-1}{m}\left(\frac{1}{\omega^2 - \omega_c^2}\right) = \frac{1}{M}\left(\frac{1}{\omega^2 - \Omega_c^2}\right)$$

$$(m + M)\omega^2 = \omega_c^2 m + \Omega_c^2 M,$$

$$(m + M)\omega^2 = e^2 B^2 \left(\frac{m + M}{Mm}\right),$$

$$\omega^2 = \frac{e^2 B^2}{Mm} = \Omega_c \omega_c = {\omega_l}^2,$$

where ω_l is the lower hybrid frequency, defined as

$$\omega_l = (\Omega_c \omega_c)^{\frac{1}{2}}. \tag{2.30}$$

Instead of using plasma approximation, if we use Poisson's equation then we get the following dispersion relation for lower hybrid wave

$$\frac{1}{\omega_l^2} = \frac{1}{\Omega_c \omega_c} + \frac{1}{\Omega_p^2}. \tag{2.31}$$

2.7.3 Electromagnetic waves in plasma with B_0

For transverse waves traveling through plasma. We have $k \perp E$

1. $\nabla \cdot E = 0.$
   $\nabla \cdot B = 0.$

2. $E_0 = 0.$
   $B_0 = 0.$

3. $\frac{\partial E_0}{\partial t} = \frac{\partial B_0}{\partial t} = 0.$

Using Maxwell's equation,

$$\nabla \times E = -\frac{\partial B}{\partial t}, \tag{2.32}$$

$$\nabla \times B = \frac{1}{c^2}\frac{\partial E}{\partial t} + \mu_0 J. \tag{2.33}$$

By taking cross product of eq. (2.32), we get

$$\nabla \times (\nabla \times E) = -\frac{\partial(\nabla \times B)}{\partial t},$$

$$\nabla(\nabla \cdot E) - \nabla^2 E = -\frac{\partial(\nabla \times B)}{\partial t}.$$

By using the value of $\nabla \times B$ from eq. (2.33), we get

$$\nabla^2 E = \frac{\partial}{\partial t}\left[\frac{1}{c^2}\frac{\partial E}{\partial t} + \mu_0 J\right],$$

$$\nabla^2 E = \left[\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} + \mu_0 \frac{\partial J}{\partial t}\right]. \tag{2.34}$$

By linearizing eq. (2.34), we get

$$\nabla^2 E_1 = \left[\frac{1}{c^2}\frac{\partial^2 E_1}{\partial t}^2 + \mu_0 \frac{\partial J_1}{\partial t}\right].$$

All perturbed quantities behave sinusoidally, therefore replace $\nabla = ik$ and $\frac{\partial}{\partial t} = -i\omega$ in the above equation to get

$$k^2 E_1 = \frac{\omega^2}{c^2} E_1 + i\omega\mu_0 J_1, \tag{2.35}$$

where $J_1 = -n_0 e v_1$. The value of v_1 can be found from electron equation of motion i.e.,

$$v_1 = \frac{eE_1}{im\omega}.$$

Therefore the current density will become

$$J_1 = \frac{in_0 e^2 E_1}{m\omega}. \tag{2.36}$$

Using this value in equation eq. (2.35) we get

$$k^2 E_1 = \frac{\omega^2}{c^2} E_1 - \frac{n_0 e^2 \mu_0 E_1}{m},$$

as $\omega_{pe}^2 = \frac{n_0 e^2}{m\epsilon_0}$, hence

$$k^2 = \frac{\omega^2}{c^2} - \mu_0 \epsilon_0 \omega_{pe}^2.$$

The dispersion relation for electromagnetic waves propagating through the plasma in the absence of magnetic field can be written as

$$\omega^2 = \omega_{pe}^2 + c^2 k^2. \tag{2.37}$$

Chapter 3

Shear flow instability in a partially-ionized plasma sheath around a fast-moving vehicle

When vehicle re-enters the earth's atmosphere a plasma sheath is formed around it, which causes communication blackout either by reflecting or absorbing the Electromagnetic waves coming to or from the vehicle[14]. An increase in turbulent pulsation within the plasma sheath can also affect the on-board sensor. The influence of turbulence of low-frequency waves incompressible plasma in the absence of collisions has been analyzed by Vladimir Ref [25]. As the vehicle moves to lower altitudes of earth's atmosphere the numbers of neutral particles increase, hence collisions with neutral particles increases, therefore it is necessary to include collision while deriving the dispersion relation for instability in plasma sheath region[26]. In this section, the growth rate of low-frequency turbulence (i.e. instability of ion-acoustic wave) due to the hyper-sonic sheared flow of plasma sheath is studied with the inclusion of collisions into the model. This allowed us to correctly evaluate the growth rate of instability and to investigate the role of neutral particles in suppressing such instabilities. This was done by deriving a second-order differential equation for the electrostatic potential of excited ion-acoustic waves in the presence of collisions of charged particles with neutrals. The differential equation was then solved analytically for linear velocity profile of the shear flow, using appropriate boundary conditions for finite thickness of the plasma sheath. An appropriate scaling relationship for the instability in case of the linear velocity profile is obtained from analytical calculations. From different plots of the relationship, the growth rates and eigenfunctions of unstable ion-acoustic modes are obtained. It was also observed that when the density of neutral particles is increased to such a value that ion-neutral collisions rate exceeded the peak dimensionless growth rate, the instability was completely suppressed.

In the present chapter, the typical value of growth rate of the instability of ion-acoustic wave is investigated and the role of neutral particles in suppressing such instability is found. A second-order differential equation for the electrostatic potential of the ion-acoustic wave is derived using a system of nonlinear equations, which include, the momentum equation for ions, electrons and neutral, and the mass conservation equation for these species. These equations are complemented by Poisson equation for electrostatic potential associated with ion-acoustic perturbation, due to the presence of flow shear.[13] In this section, the excitation and turbulent pulsation in a compressible supersonic plasma flow in the two-dimensional case is considered.

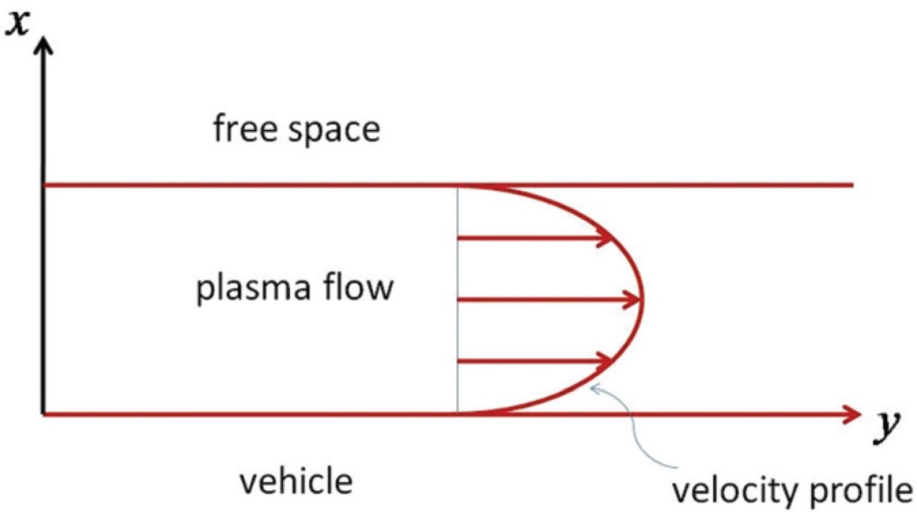

Figure 3.1: Geometry of the sheared plasma flow

fig. 3.1 shows that the plasma sheath is bounded on one side by vehicle's surface, and on the other side by the neutral atmosphere.The plasma is flowing with velocity $V_{0y}(x)$ along y axis, and has velocity shear along x axis.

3.1 Equation of momentum

$$mn[\frac{d\vec{v}}{dt} + (\vec{v} \cdot \vec{\nabla})\vec{v}] = nq(\vec{E} + \vec{v} \times \vec{B}) - \vec{\nabla} P - \frac{mn(\vec{v} - \vec{v_m})}{\tau}. \tag{3.1}$$

As no magnetic field is present therefore $v \times B = 0$.
where in eq. (3.1) $\nabla p = K_B T \nabla n$,
so

$$m[\frac{d\vec{v}}{dt} + (\vec{v} \cdot \vec{\nabla})\vec{v}] = qE - \frac{K_B T \vec{\nabla} n}{n} - \frac{m(\vec{v} - \vec{v_m})}{\tau}. \tag{3.2}$$

Linearizing eq. (3.2) i.e. replace

$$\vec{v} = \vec{v_0} + \vec{v_1},$$

$$\vec{v_m} = \vec{v_{m_0}} + \vec{v_{m_1}},$$

and

$$n = n_0 + n_1.$$

By putting values in eq. (3.2), we get

$$m[\frac{d(\vec{v_0} + \vec{v_1})}{dt} + ((\vec{v_0} + \vec{v_1}) \cdot \vec{\nabla})(\vec{v_0} + \vec{v_1})] = qE - \frac{K_B T \vec{\nabla}(n_0 + n_1)}{(n_0 + n_1)} - \frac{m((\vec{v_0} + \vec{v_1}) - (\vec{v_{m_0}} + \vec{v_{m_1}}))}{\tau}. \tag{3.3}$$

In equation 3.3,

$$((\vec{v_0} + \vec{v_1}) \cdot \vec{\nabla})(\vec{v_0} + \vec{v_1}) = (\vec{v_0} \cdot \vec{\nabla})\vec{v_0} + (\vec{v_0} \cdot \vec{\nabla})\vec{v_1} + (\vec{v_1} \cdot \vec{\nabla})\vec{v_0} + (\vec{v_1} \cdot \vec{\nabla})\vec{v_1}.$$

By ignoring zeroth order and higher order terms, we get

$$((\vec{v_0} + \vec{v_1}) \cdot \vec{\nabla})(\vec{v_0} + \vec{v_1}) = (\vec{v_0} \cdot \vec{\nabla})\vec{v_1} + (\vec{v_1} \cdot \vec{\nabla})\vec{v_0}.$$

where,

$$(\vec{v_0} \cdot \vec{\nabla})\vec{v_1} = (v_0 \frac{d}{dy})\vec{v_1},$$

motion of plasma is along y axis, but change in plasma occurs in x.

$$(\vec{v_1}\cdot\vec{\nabla})\vec{v_0} = (v_{1x}\frac{d}{dx})\vec{v_0} + (v_{1y}\frac{d}{dy})\vec{v_0} + (v_{1z}\frac{d}{dz})\vec{v_0}.$$

As change in unperturbed velocity $v_0(x)$ is along x direction,hence

$$(\vec{v_1}\cdot\vec{\nabla})\vec{v_0} = v_{1x}\frac{d\vec{v_0}}{dx},$$

where

$$\vec{v_0} = v_0\hat{j},$$

so

$$(\vec{v_1}\cdot\vec{\nabla})\vec{v_0} = v_{1x}\frac{dv_0}{dx}\hat{j}.$$

$$((\vec{v_0}+\vec{v_1})\cdot\vec{\nabla})(\vec{v_0}+\vec{v_1}) = (v_0\frac{d\vec{v_1}}{dy}) + v_{1x}\frac{dv_0}{dx}\hat{j}. \tag{3.4}$$

Another term from eq. (3.3) can be written as

$$\frac{K_B T\vec{\nabla}(n_0+n_1)}{(n_0+n_1)} = K_B T\frac{\vec{\nabla}n_0+\vec{\nabla}n_1}{n_0+n_1}.$$

Where $\nabla n_0 = 0$, also $n_0 >> n_1$, so

$$K_B T\frac{\vec{\nabla}n_1}{n_0+n_1} = K_B T\frac{\vec{\nabla}n_1}{n_0(1+\frac{n_1}{n_0})} = K_B T\frac{\vec{\nabla}n_1}{n_0}. \tag{3.5}$$

Given $v_0 = v_{0_m}$,
put eq. (3.4) and eq. (3.5) in equation eq. (3.3), we get

$$m[\frac{d\vec{v_1}}{dt} + v_0\frac{d\vec{v_1}}{dy} + v_{1x}\frac{dv_0}{dx}\hat{j}] = qE - K_B T\frac{\vec{\nabla}n_1}{n_0} + \frac{m(\vec{v_1}-\vec{v_{m_1}})}{\tau}.$$

$$\frac{K_B T}{m} = v_T^2.$$

$$\frac{d\vec{v_1}}{dt} + v_0\frac{d\vec{v_1}}{dy} + v_{1x}\frac{dv_0}{dx}\hat{j} = \frac{qE}{m} - v_T^2\frac{\vec{\nabla}n_1}{n_0} + \frac{(\vec{v_1}-\vec{v_{m_1}})}{\tau}. \tag{3.6}$$

In eq. (3.6), $\frac{1}{\tau} = \nu$ is the collision frequency. The eq. (3.6) gives us general form of momentum equation, writing it for electron,ion and neutral.
Momentum equation for electron is,

$$\frac{d\vec{u_{1_e}}}{dt} + V_{0_e}\frac{d\vec{u_{1_e}}}{dy} + u_{1_{ex}}\frac{dV_{0_e}}{dx}\hat{j} = \frac{-eE}{m_e} - V_{T_e}^2\frac{\vec{\nabla}n_1}{N_{0_e}} + \nu_{en}(\vec{u_{1_e}}-\vec{u_{1_n}}) + \nu_{ei}(\vec{u_{1_e}}-\vec{u_{1_i}}),$$

where $E = -\vec{\nabla}\phi$, so

$$\frac{d\vec{u_{1_e}}}{dt} + V_{0_e}\frac{d\vec{u_{1_e}}}{dy} + u_{1_{ex}}\frac{dV_{0_e}}{dx}\hat{j} = \frac{e\vec{\vec{\nabla}}\phi}{m_e} - V_{T_e}^2\frac{\vec{\nabla}n_1}{N_{0_e}} + \nu_{en}(\vec{u_{1_e}}-\vec{u_{1_n}}) + \nu_{ei}(\vec{u_{1_e}}-\vec{u_{1_i}}). \tag{3.7}$$

Momentum equation for neutral is

$$\frac{d\vec{u_{1_n}}}{dt} + V_{0_n}\frac{d\vec{u_{1_n}}}{dy} + u_{1_{nx}}\frac{dV_{0_n}}{dx}\hat{j} = V_{T_n}^2\frac{\vec{\nabla}n_1}{N_{0_n}} - \nu_{ne}(\vec{u_{1_n}}-\vec{u_{1_e}}) - \nu_{ni}(\vec{u_{1_n}}-\vec{u_{1_i}}). \tag{3.8}$$

Momentum equation for ion is

$$\frac{d\vec{u_{1_i}}}{dt} + V_{0_i}\frac{d\vec{u_{1_i}}}{dy} + u_{1_{ix}}\frac{dV_{0_i}}{dx}\hat{j} = \frac{-e\vec{\vec{\nabla}}\phi}{M_i} - V_{T_i}^2\frac{\vec{\nabla}n_{1_i}}{N_{0_i}} - \nu_{ie}(\vec{u_{1_i}}-\vec{u_{1_e}}) - \nu_{in}(\vec{u_{1_i}}-\vec{u_{1_n}}). \tag{3.9}$$

3.2 Equation of continuity

$$\frac{dn}{dt} + \vec{\nabla} \cdot (n\vec{v}) = 0. \tag{3.10}$$

$$\frac{dn}{dt} + (\vec{\nabla} n) \cdot \vec{v} + (\vec{\nabla} \cdot \vec{v})n = 0.$$

Linearizing eq. (3.10) i.e. replace

$$\vec{v} = \vec{v_0} + \vec{v_1},$$

and

$$n = n_0 + n_1.$$

Hence

$$(\vec{\nabla} n) \cdot \vec{v} = (\vec{\nabla}(n_0 + n_1)) \cdot (\vec{v_0} + \vec{v_1}) = \vec{\nabla} n_1 \cdot \vec{v_0} + \vec{\nabla} n_1 \cdot \vec{v_1}.$$

By ignoring the higher order terms, we get

$$(\vec{\nabla} n) \cdot \vec{v} = \vec{\nabla} n_1 \cdot \vec{v_0} = v_0 \frac{dn_1}{dy}.$$

Similarly

$$(\vec{\nabla} \cdot \vec{v})n = (\vec{\nabla} \cdot (\vec{v_0} + \vec{v_1}))(n_0 + n_1)),$$

$$(\vec{\nabla} \cdot \vec{v})n = n_0(\vec{\nabla} \cdot \vec{v_1}).$$

By putting values in equation eq. (3.10), we get

$$\frac{dn_1}{dt} + v_0 \frac{dn_1}{dy} + n_0(\vec{\nabla} \cdot \vec{v_1}) = 0, \tag{3.11}$$

where eq. (3.11) gives the equation of continuity of plasma when flow is along y direction.

$$\frac{dn_{1_\alpha}}{dt} + V_{0_\alpha} \frac{dn_{1_\alpha}}{dy} + n_{0_\alpha}(\vec{\nabla} \cdot \vec{u_{1_\alpha}}) = 0. \tag{3.12}$$

Where eq. (3.12) gives us the general form of the equation of continuity, the subscript α stands for electron, ions and neutral particles.

3.3 Poisson's equation

From Poisson's euqation

$$\nabla \cdot E = \frac{\rho}{\epsilon_o},$$

where $\rho = \rho_e + \rho_i$, $\rho_e = -en_e$ and $\rho_i = Zen_i$, also in above equation $\nabla \cdot E = \nabla \cdot (-\nabla\phi) = -\nabla^2\phi$. By putting values we get

$$-\nabla^2\phi = \frac{e(Zn_i - n_e)}{\epsilon_o},$$

where $\frac{1}{\epsilon_o} = 4\pi$, so Poisson equation becomes

$$\nabla^2\phi = 4\pi e(n_e - Zn_i). \tag{3.13}$$

3.4 Derivation of second-order differential for electrostatic potential

From electron momentum equation

$$\frac{d\vec{\tilde{u}}_{1_e}}{d\tilde{t}} + \tilde{V}_{0_e}\frac{d\vec{\tilde{u}}_{1_e}}{d\tilde{y}} + \tilde{u}_{1_{ex}}\frac{d\tilde{V}_{0_e}}{d\tilde{x}}\hat{j} = \frac{e\vec{\tilde{\nabla}}\tilde{\phi}}{\tilde{m}_e} - \tilde{V}_{T_e}^2\frac{\vec{\tilde{\nabla}}\tilde{n}_1}{\tilde{N}_{0_e}} - \tilde{\nu}_{en}(\vec{\tilde{u}}_{1_e} - \vec{\tilde{u}}_{1_n}).$$

We will ignore the ion velocity $\vec{\tilde{u}}_{1_n}$ as they appear stationary in front of an electron, hence the equation of motion will becomes

$$\frac{d\vec{\tilde{u}}_{1_e}}{d\tilde{t}} + \tilde{V}_{0_e}\frac{d\vec{\tilde{u}}_{1_e}}{d\tilde{y}} + \tilde{\nu}_{en}\vec{\tilde{u}}_{1_e} = \frac{e\vec{\tilde{\nabla}}\tilde{\phi}}{\tilde{m}_e} - \tilde{V}_{T_e}^2\frac{\vec{\tilde{\nabla}}\tilde{n}_{1_e}}{\tilde{N}_{0_e}} - \tilde{u}_{1_{ex}}\frac{d\tilde{V}_{0_e}}{d\tilde{x}}\hat{j}. \tag{3.14}$$

By dropping subscript 1 in all perturbed velocities and densities we can write eq. (3.14) as

$$\frac{d\vec{\tilde{u}}_e}{d\tilde{t}} + \tilde{V}_{0_e}\frac{d\vec{\tilde{u}}_e}{d\tilde{y}} + \tilde{\nu}_{en}\vec{\tilde{u}}_e = \frac{K_BT_e}{\tilde{m}_e}\left(\frac{e\vec{\tilde{\nabla}}\tilde{\phi}}{K_BT_e} - \frac{\vec{\tilde{\nabla}}\tilde{n}_e}{\tilde{N}_{0_e}}\right) - \tilde{u}_{ex}\frac{d\tilde{V}_{0_e}}{d\tilde{x}}\hat{j}.$$

Normalizing the equation of momentum using normalization conditions:

1. $\phi = \frac{e\tilde{\phi}}{K_BT_e}$,

2. $n_e = \frac{\tilde{n}_e}{\tilde{N}_{0_e}}$,

3. $m = \frac{\tilde{m}_e}{\tilde{M}_i}$, or $\tilde{m}_e = \tilde{M}_i m$,

4. $\tilde{u}_e = \tilde{\omega}_{pi}\tilde{\lambda}_{De}u_e$,

5.$\tilde{x} = \tilde{\lambda}_{De}x$,

6. $\tilde{y} = \tilde{\lambda}_{De}y$,

7. $\tilde{t} = \frac{t}{\tilde{\omega}_{pi}}$,

8. $\vec{\tilde{\nabla}} = \frac{\vec{\nabla}}{\tilde{\lambda}_{De}}$,

9. $\tilde{V}_{0_e} = \tilde{\omega}_{pi}\tilde{\lambda}_{De}V_{0_e}$,

10. $\tilde{\nu}_{en} = \tilde{\omega}_{pi}\nu_{en}$,

$$\frac{d(\tilde{\omega}_{pi}\tilde{\lambda}_{De}\vec{u}_e)}{d(\frac{t}{\tilde{\omega}_{pi}})} + (\tilde{\omega}_{pi}\tilde{\lambda}_{De}V_{0_e})\frac{d(\tilde{\omega}_{pi}\tilde{\lambda}_{De}\vec{u}_e)}{d(\tilde{\lambda}_{De}y)} + (\tilde{\omega}_{pi}\nu_{en})(\tilde{\omega}_{pi}\tilde{\lambda}_{De}\vec{u}_e) =$$

$$\frac{K_BT_e(\frac{\vec{\nabla}}{\tilde{\lambda}_{De}})}{(\tilde{M}_i m)}\left(\frac{e\tilde{\phi}}{K_BT_e} - \frac{\tilde{n}_e}{\tilde{N}_{0_e}}\right) - (\tilde{\omega}_{pi}\tilde{\lambda}_{De}\vec{u}_{ex})\frac{d(\tilde{\omega}_{pi}\tilde{\lambda}_{De}V_{0_e})}{d\tilde{\lambda}_{De}x}\hat{j},$$

where in the equation $\frac{K_BT_e}{\tilde{M}_i} = \tilde{\omega}_{pi}^2\tilde{\lambda}_{De}^2$,

$$\frac{d(\tilde{\omega}_{pi}\tilde{\lambda}_{De}\vec{u}_e)}{d(\frac{t}{\tilde{\omega}_{pi}})} + (\tilde{\omega}_{pi}\tilde{\lambda}_{De}V_{0_e})\frac{d(\tilde{\omega}_{pi}\tilde{\lambda}_{De}\vec{u}_e)}{d(\tilde{\lambda}_{De}y)} + (\tilde{\omega}_{pi}\nu_{en})(\tilde{\omega}_{pi}\tilde{\lambda}_{De}\vec{u}_e) =$$

$$\frac{\tilde{\omega}_{pi}^2\tilde{\lambda}_{De}^2\vec{\nabla}}{m\tilde{\lambda}_{De}}(\frac{e\tilde{\phi}}{K_BT_e} - \frac{\tilde{n}_e}{\tilde{N}_{0_e}}) - (\tilde{\omega}_{pi}\tilde{\lambda}_{De}\vec{u}_{ex})\frac{d(\tilde{\omega}_{pi}\tilde{\lambda}_{De}V_{0_e})}{d\tilde{\lambda}_{De}x}\hat{j}.$$

$$\tilde{\omega}_{pi}^2\tilde{\lambda}_{De}(\frac{d\vec{u_e}}{dt} + V_{0_e}\frac{d\vec{u_e}}{dy} + \nu_{en}\vec{u_e}) = \tilde{\omega}_{pi}^2\tilde{\lambda}_{De}(\frac{1}{m}(\vec{\nabla}\phi - \vec{\nabla}n_e) - u_{ex}\frac{dV_{0_e}}{dx}\hat{j}).$$

$$\frac{d\vec{u_e}}{dt} + V_{0_e}\frac{d\vec{u_e}}{dy} + \nu_{en}\vec{u_e} = \frac{1}{m}(\vec{\nabla}\phi - \vec{\nabla}n_e) - u_{ex}\frac{dV_{0_e}}{dx}\hat{j} \tag{3.15}$$

Separating eq. (3.15) into its x and y component i.e. $\vec{u_e} = u_{ex}\hat{i} + u_{ey}\hat{j}$ and $\vec{\nabla} = \frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j}$

X-component of equation 3.15

$$\frac{du_{ex}}{dt} + V_0\frac{du_{ex}}{dy} + \nu_{en}\vec{u_{ex}} = \frac{1}{m}[\frac{d\phi}{dx} - \frac{dn_e}{dx}].$$

As all perturbed quantities behave sinusoidaly i.e. $f(x, y, t) \sim exp(iky - i\omega t)$ therefore replace $\frac{d}{dt} = -i\omega$ and $\frac{d}{dy} = ik$.

$$[\omega - kV_0 + i\nu_{en}]u_{ex} = \frac{i}{m}\frac{d}{dx}[\phi - n_e], \tag{3.16}$$

where in equation eq. (3.16)

$$\Omega_e = \omega - kV_0 + i\nu_{en},$$

and

$$\psi = \phi - n_e,$$

hence

$$\Omega_e u_{ex} = \frac{i}{m}\frac{d\psi}{dx}. \tag{3.17}$$

Y-component of equation 3.15

$$\frac{du_{ey}}{dt} + V_0\frac{du_{ey}}{dy} + \nu_{en}\vec{u_{ey}} = \frac{1}{m}[\frac{d\phi}{dy} - \frac{dn_e}{dy} - u_{ex}\frac{dV_{0_e}}{dx}].$$

All perturbed quantities behave sinusoidaly, hence

$$[\omega - kV_0 + i\nu_{en}]u_{ey} = -\frac{k}{m}[\phi - n_e] - iu_{ex}\frac{dV_0}{dx}. \tag{3.18}$$

The eq. (3.18) becomes

$$\Omega_e u_{ey} = -\frac{k}{m}\psi - iu_{ex}\frac{dV_0}{dx}. \tag{3.19}$$

From equation of continuity

$$\frac{d\tilde{n}_e}{d\tilde{t}} + \tilde{V}_{0_e}\frac{d\tilde{n}_e}{d\tilde{y}} + \tilde{N}_{0_e}(\vec{\tilde{\nabla}} \cdot \vec{\tilde{u}}_e) = 0.$$

We will normalize the equation of continuity for electron using following normalization conditions.

1. $\tilde{n_e} = n_e \tilde{N}_{0_e}$

2. $\tilde{\vec{u}_e} = \tilde{\omega}_{pi} \tilde{\lambda}_{De} \vec{u}_e$

3. $\tilde{y} = \tilde{\lambda}_{De} y$

4. $\tilde{t} = \frac{t}{\tilde{\omega}_{pi}}$

5. $\tilde{\vec{\nabla}} = \frac{\vec{\nabla}}{\tilde{\lambda}_{De}}$

6. $\tilde{V}_{0_e} = \tilde{\omega}_{pi} \tilde{\lambda}_{De} V_{0_e}$

$$\frac{d(n_e \tilde{N}_{0_e})}{d(\frac{t}{\tilde{\omega}_{pi}})} + (\tilde{\omega}_{pi} \tilde{\lambda}_{De} V_{0_e}) \frac{d(n_e \tilde{N}_{0_e})}{d(\tilde{\lambda}_{De} y)} + \tilde{N}_{0_e} [(\frac{\vec{\nabla}}{\tilde{\lambda}_{De}}) \cdot (\tilde{\omega}_{pi} \tilde{\lambda}_{De} \vec{u_e})] = 0.$$

$$\frac{dn_e}{dt} + V_{0_e} \frac{dn_e}{dy} + N_{0_e} (\vec{\nabla} \cdot \vec{u_e}) = 0.$$

Where above,

$$\nabla \cdot \vec{u_e} = [(\frac{d}{dx} \hat{i} + \frac{d}{dy} \hat{j}) \cdot (u_{ex} \hat{i} + u_{ey} \hat{j})] = (\frac{du_{ex}}{dx} + \frac{du_{ey}}{dy}).$$

All perturbed quantities behave sinusoidaly

$$-i\omega n_e + ikV_{0_e} n_e = -N_{0_e} (iku_{ey} + \frac{du_{ex}}{dx}).$$

$$i(\omega - kV_{0_e}) n_e = N_{0_e} (iku_{ey} + \frac{du_{ex}}{dx}).$$

Considering the unperturbed velocity of electron, ions and neutrals to be same, hence $V_{0_e} = V_{0_n} = V_{0_i} = V_0$,

$$i(\omega - kV_0) n_e = N_{0_e} (iku_{ey} + \frac{du_{ex}}{dx}).$$

In above equation, let $\omega - kV_0 = \Omega$, hence

$$i\Omega \frac{n_e}{N_{0_e}} = (iku_{ey} + \frac{du_{ex}}{dx}).$$

By applying normalization condition i.e. $n_e = \frac{n_e}{N_{0_e}}$, we get

$$i\Omega n_e = iku_{ey} + \frac{du_{ex}}{dx}. \tag{3.20}$$

From eq. (3.17),

$$u_{ex} = \frac{i}{m\Omega_e} \frac{d\psi}{dx}.$$

By taking derivative of the equation with respect to x we get,

$$\frac{du_{ex}}{dx} = \frac{i}{m}\frac{[\frac{d^2\psi}{dx}\Omega_e - \frac{d\psi}{dx}\frac{d\Omega_e}{dx}]}{\Omega_e^2}.$$

As $\Omega_e = \omega - kV_0 + i\nu_{en}$, so

$$\frac{d\Omega_e}{dx} = -k\frac{dV_0}{dx}.$$

By putting values we get

$$\frac{du_{ex}}{dx} = \frac{i}{m}[\frac{1}{\Omega_e}\frac{d^2\psi}{dx} + \frac{k}{\Omega_e^2}\frac{dV_0}{dx}\frac{d\psi}{dx}]. \tag{3.21}$$

From eq. (3.20)

$$\frac{du_{ex}}{dx} = i\Omega n_e - iku_{ey}.$$

By putting value in eq. (3.21) we get,

$$i\Omega n_e = iku_{ey} + \frac{i}{m}[\frac{1}{\Omega_e}\frac{d^2\psi}{dx} + \frac{k}{\Omega_e^2}\frac{dV_0}{dx}\frac{d\psi}{dx}]. \tag{3.22}$$

From eq. (3.19),

$$u_{ey} = -\frac{k}{\Omega_e m}\psi - \frac{iu_{ex}}{\Omega_e}\frac{dV_0}{dx}.$$

By putting value of eq. (3.22) we get,

$$\Omega n_e = -\frac{k^2}{\Omega_e m}\psi - \frac{iku_{ex}}{\Omega_e}\frac{dV_0}{dx} + \frac{1}{m\Omega_e}\frac{d^2\psi}{dx} + \frac{k}{m\Omega_e^2}\frac{dV_0}{dx}\frac{d\psi}{dx},$$

where $u_{ex} = \frac{i}{m\Omega_e}\frac{d\psi}{dx}$, so

$$m\Omega_e\Omega n_e = -k^2\psi + \frac{d^2\psi}{dx} + \frac{2k}{\Omega_e}\frac{dV_0}{dx}\frac{d\psi}{dx}. \tag{3.23}$$

From poisson equation

$$\tilde{\nabla}^2\tilde{\phi} = 4\pi e(\tilde{n}_e - \tilde{n}_i),$$

we have considered $Z = 1$ throughout out calculation. By using normalization conditions:

1. $\tilde{\phi} = \frac{K_B T_e \phi}{e}$,

2. $\tilde{n_e} = n_e \tilde{N}_{0_e}$,

3. $\tilde{n_i} = n_i \tilde{N}_{0_i}$,

4. $\tilde{\vec{\nabla}}^2 = \frac{\vec{\nabla}^2}{\tilde{\lambda}_{De}^2}$,

we get

$$(\frac{\vec{\nabla}^2}{\tilde{\lambda}_{De}^2})(\frac{K_B T_e \phi}{e}) = 4\pi N_{0_e} e(n_e - n_i).$$

As unperturbed number density of electron is equal to that of ion i.e. $N_{0_i} = N_{0_e}$, hence

$$\vec{\nabla}^2\phi = \frac{4\pi N_{0_e} e^2}{\tilde{\lambda}_{De}^2 K_B T_e}(n_e - n_i).$$

where

$$\tilde{\lambda}_{De}^2 = \frac{K_B T_e}{4\pi N_{0_e} e^2},$$

so

$$\vec{\nabla}^2\phi = (n_e - n_i).$$

$\nabla^2 = \frac{d^2}{dx^2} + \frac{d^2}{dy^2}$, where $\frac{d^2}{dy^2} = -k^2$, so

$$n_e = -k^2\phi + \frac{d^2\phi}{dx^2} + n_i. \tag{3.24}$$

As $\psi = \phi - n_e$, so

$$\psi = \phi + k^2\phi - \frac{d^2\phi}{dx^2} - n_i. \tag{3.25}$$

From eq. (3.23),

$$[-k^2 + \frac{d^2}{dx} + \frac{2k}{\Omega_e}\frac{dV_0}{dx}\frac{d}{dx}]\psi = m\Omega_e\Omega n_e,$$

where $-k^2 + \frac{d^2}{dx} + \frac{2k}{\Omega_e}\frac{dV_0}{dx}\frac{d}{dx} = L_e$, so

$$L_e\psi = m\Omega_e\Omega n_e. \tag{3.26}$$

Put value of eq. (3.24) and eq. (3.25) in eq. (3.26) we get

$$L_e[\phi + k^2\phi - \frac{d^2\phi}{dx^2} - n_i] = \frac{m\Omega_e\Omega}{k^2}[k^2(-k^2\phi + \frac{d^2\phi}{dx^2} + n_i)], \tag{3.27}$$

as $\frac{m\Omega_e\Omega}{k^2} << 1$, therefore we we will neglect the left hand side of eq. (3.27), hence

$$L_e[\phi + k^2\phi - \frac{d^2\phi}{dx^2} - n_i] = 0.$$

By decoupling equation, we get

$$\phi + k^2\phi - \frac{d^2\phi}{dx^2} - n_i = 0. \tag{3.28}$$

From neutral momentum equation

$$\frac{d\vec{\tilde{u}}_{1_n}}{d\tilde{t}} + \tilde{V}_{0_n}\frac{d\vec{\tilde{u}}_{1_n}}{d\tilde{y}} + \tilde{u}_{1_{nx}}\frac{d\tilde{V}_{0_n}}{d\tilde{x}}\hat{j} = -\tilde{V}_{T_n}^2\frac{\vec{\tilde{\nabla}}\tilde{n}_{1n}}{\tilde{N}_{0_n}} - \tilde{\nu}_{ne}(\vec{\tilde{u}}_{1_n} - \vec{\tilde{u}}_{1_e}) - \tilde{\nu}_{ni}(\vec{\tilde{u}}_{1_n} - \vec{\tilde{u}}_{1_i}).$$

Normalizing above equation we get,

$$\frac{d\vec{u_{1_n}}}{dt} + V_{0_n}\frac{d\vec{u_{1_n}}}{dy} + u_{1_{nx}}\frac{dV_{0_n}}{dx}\hat{j} = -V_{T_n}^2\frac{\vec{\nabla}n_{1n}}{N_{0_n}} - \nu_{ne}(\vec{u_{1_n}} - \vec{u_{1_e}}) - \nu_{ni}(\vec{u_{1_n}} - \vec{u_{1_i}}).$$

Droping subscript 1 in all perturbed quantities,

$$\frac{d\vec{u_n}}{dt} + V_{0_n}\frac{d\vec{u_n}}{dy} + u_{nx}\frac{dV_{0_n}}{dx}\hat{j} = -V_{T_n}^2\frac{\vec{\nabla}n_n}{N_{0_n}} - \nu_{ne}(\vec{u_n} - \vec{u_e}) - \nu_{ni}(\vec{u_n} - \vec{u_i}).$$

Making following assumptions,

1. $\nu_{ne} << \nu_{ni}$,
2. Ignoring the pressure term in ion and neutral momentum equations.

$$\frac{d\vec{u_n}}{dt} + V_{0_n}\frac{d\vec{u_n}}{dy} + u_{nx}\frac{dV_{0_n}}{dx}\hat{j} = -\nu_{ni}(\vec{u_n} - \vec{u_i}).$$

All perturbed quantities behave sinusoidally, so

$$-i[\omega - kV_{0_n} + i\nu_{ni}]\vec{u}_n = -u_{nx}\frac{dV_{0_n}}{dx}\hat{j} + \nu_{ni}\vec{u_i}.$$

$$-i\Omega_n\vec{u}_n = -u_{nx}\frac{dV_{0_n}}{dx}\hat{j} + \nu_{ni}\vec{v_i}. \tag{3.29}$$

$$\vec{u}_n = \frac{i}{\Omega_n}[-u_{nx}\frac{dV_{0_n}}{dx}\hat{j} + \nu_{ni}\vec{u_i}].$$

By separating the equation into its x and y component, we get
X component

$$u_{n_x} = \frac{i\nu_{ni}u_{i_x}}{\Omega_n}. \tag{3.30}$$

Y component

$$u_{n_y} = \frac{i}{\Omega_n}[-u_{nx}\frac{dV_{0_n}}{dx} + \nu_{ni}u_{i_y}]. \tag{3.31}$$

By putting value of eq. (3.30), in eq. (3.31) we get,

$$u_{n_y} = \frac{\nu_{ni}u_{i_x}}{\Omega_n^2}\frac{dV_{0_n}}{dx} + \frac{i\nu_{ni}u_{i_y}}{\Omega_n}. \tag{3.32}$$

From ion equation of momentum

$$\frac{d\tilde{\vec{u}}_{1_i}}{d\tilde{t}} + \tilde{V}_{0_i}\frac{d\tilde{\vec{u}}_{1_i}}{d\tilde{y}} + \tilde{u}_{1_{ix}}\frac{d\tilde{V}_{0_i}}{d\tilde{x}}\hat{j} = \frac{e\tilde{\vec{\nabla}}\tilde{\phi}}{\tilde{m}_i} - \tilde{V}_{T_i}^2\frac{\tilde{\vec{\nabla}}\tilde{n}_{i1}}{\tilde{N}_{0_i}} - \tilde{\nu}_{in}(\tilde{\vec{u}}_{1_i} - \tilde{\vec{u}}_{1_n}).$$

By normalizing above equation, we get

$$\frac{d\vec{u_i}}{dt} + V_{0_i}\frac{d\vec{u_i}}{dy} + u_{ix}\frac{dV_{0_i}}{dx}\hat{j} = \frac{-e\vec{\nabla}\phi}{M_i} - \nu_{in}(\vec{u_i} - \vec{u_n}). \tag{3.33}$$

Seperating the x and y components of eq. (3.33).
X component

$$\frac{du_{i_x}}{dt} + V_{0_i}\frac{du_{i_x}}{dy} = -\frac{d\phi}{dx} - \nu_{in}[u_{i_x} - u_{n_x}].$$

By putting value eq. (3.30) we get,

$$\frac{du_{i_x}}{dt} + V_{0_i}\frac{du_{i_x}}{dy} + \nu_{in}u_{i_x}[1 - \frac{i\nu_{ni}}{\Omega_n}] = -\frac{d\phi}{dx}.$$

As all perturbed quantities behave sinusoidally, so by replacing $\frac{d}{dt} = -i\omega$ and $\frac{d}{dy} = ik$ we get

$$-i[\omega + kV_0 + i\nu_{in}(1 - \frac{i\nu_{ni}}{\Omega_n})]u_{i_x} = -\frac{d\phi}{dx},$$

where,

$$\Omega_i = \omega - kV_0 + i\nu_{in}(1 - \frac{i\nu_{ni}}{\Omega_n}), \tag{3.34}$$

so

$$u_{i_x} = \frac{-i}{\Omega_i}\frac{d\phi}{dx}. \tag{3.35}$$

Y component

$$\frac{du_{i_y}}{dt} + V_{0_i}\frac{du_{i_y}}{dy} + u_{ix}\frac{dV_0}{dx} = -\frac{d\phi}{dy} - \nu_{in}[u_{i_y} - u_{n_y}]. \tag{3.36}$$

Put value of eq. (3.32), in eq. (3.36) we get,

$$\frac{du_{i_y}}{dt} + V_0\frac{du_{i_y}}{dy} + u_{ix}\frac{dV_0}{dx} = -\frac{d\phi}{dy} - \nu_{in}u_{i_y}[1 - \frac{i\nu_{ni}}{\Omega_n}] + \frac{\nu_{ni}\nu_{in}u_{i_x}}{\Omega_n^2}\frac{dV_{0_n}}{dx}.$$

$$\frac{du_{i_y}}{dt} + V_0\frac{du_{i_y}}{dy} + \nu_{in}u_{i_y}[1 - \frac{i\nu_{ni}}{\Omega_n}] = -\frac{d\phi}{dy} - u_{ix}\frac{dV_0}{dx}[1 - \frac{\nu_{ni}\nu_{in}}{\Omega_n^2}].$$

$$-i[\omega + kV_0 + i\nu_{in}(1 - \frac{i\nu_{ni}}{\Omega_n})]u_{i_y} = -ik\phi - u_{ix}\frac{dV_0}{dx}[1 - \frac{\nu_{ni}\nu_{in}}{\Omega_n^2}].$$

$$[\omega + kV_0 + i\nu_{in}(1 - \frac{i\nu_{ni}}{\Omega_n})]u_{i_y} = ik\phi - iu_{ix}\frac{dV_0}{dx}[1 - \frac{\nu_{ni}\nu_{in}}{\Omega_n^2}].$$

By putting value of eq. (3.34) we get

$$\Omega_i u_{i_y} = k\phi - iu_{ix}\frac{dV_0}{dx}[1 - \frac{\nu_{ni}\nu_{in}}{\Omega_n^2}].$$

$$u_{i_y} = \frac{k\phi}{\Omega_i} - \frac{iu_{ix}}{\Omega_i}\frac{dV_0}{dx}[1 - \frac{\nu_{ni}\nu_{in}}{\Omega_n^2}]. \tag{3.37}$$

By putting value of eq. (3.35) in eq. (3.37), we get

$$u_{i_y} = \frac{k\phi}{\Omega_i} - \frac{i}{\Omega_i^2}\frac{dV_0}{dx}[1 - \frac{\nu_{ni}\nu_{in}}{\Omega_n^2}]\frac{d\phi}{dx}. \tag{3.38}$$

From normalized ion equation of continuity

$$\frac{dn_i}{dt} + V_0\frac{dn_i}{dy} + (\vec{\nabla} \cdot \vec{u_i}) = 0,$$

where, $\vec{\nabla} \cdot \vec{u_i} = \frac{du_{i_x}}{dx} + \frac{du_{i_y}}{dy}$, hence

$$\frac{dn_i}{dt} + V_0\frac{dn_i}{dy} = -\frac{du_{i_x}}{dx} - \frac{du_{i_y}}{dy}.$$

All perturbed quantities behave sinusoidally so replace $\frac{d}{dt} = -i\omega$ and $\frac{d}{dy} = ik$.

$$-i[\omega - kV_0]n_i = -\frac{du_{i_x}}{dx} - iku_{i_y}.$$

$$i\Omega n_i = \frac{du_{i_x}}{dx} + iku_{i_y}.$$

$$n_i = \frac{-i}{\Omega}\frac{du_{i_x}}{dx} + \frac{k}{\Omega}u_{i_y}. \tag{3.39}$$

From eq. (3.35)

$$\frac{du_{i_x}}{dx} = -i\frac{d}{dx}[\frac{1}{\Omega_i}\frac{d\phi}{dx}].$$

$$\frac{du_{i_x}}{dx} = \frac{-i}{\Omega_i^2}[\frac{d^2\phi}{dx^2}\Omega_i - \frac{d\Omega_i}{dx}\frac{d\phi}{dx}].$$

Where

$$\Omega_i = \omega - kV_0 + i\nu_{in}(1 - \frac{i\nu_{ni}}{\Omega_n}),$$

$$\frac{d\Omega_i}{dx} = -k\frac{dV_0}{dx} + \frac{k\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_0}{dx}.$$

$$\frac{du_{i_x}}{dx} = -\frac{i}{\Omega_i^2}[\frac{d^2\phi}{dx^2}\Omega_i - (-k\frac{dV_0}{dx} + \frac{k\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_0}{dx})\frac{d\phi}{dx}].$$

Hence,

$$\frac{du_{i_x}}{dx} = \frac{-i}{\Omega_i}\frac{d^2\phi}{dx^2} + \frac{i}{\Omega_i^2}[-k\frac{dV_0}{dx} + \frac{k\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_0}{dx}]\frac{d\phi}{dx}. \tag{3.40}$$

Put value of eq. (3.40), and eq. (3.38) in eq. (3.39).

$$n_i = \frac{-i}{\Omega}[\frac{-i}{\Omega_i}\frac{d^2\phi}{dx^2} + \frac{i}{\Omega_i^2}(-k\frac{dV_0}{dx} + \frac{k\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_0}{dx})\frac{d\phi}{dx}] + \frac{k}{\Omega}[\frac{k\phi}{\Omega_i} - \frac{1}{\Omega_i^2}\frac{dV_0}{dx}[1 - \frac{\nu_{ni}\nu_{in}}{\Omega_n^2}]\frac{d\phi}{dx}].$$

$$n_i = \frac{-1}{\Omega\Omega_i}\frac{d^2\phi}{dx^2} + \frac{1}{\Omega\Omega_i^2}(-k\frac{dV_0}{dx} + \frac{k\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_0}{dx})\frac{d\phi}{dx} + \frac{k^2\phi}{\Omega\Omega_i} - \frac{k}{\Omega\Omega_i^2 M}\frac{dV_0}{dx}\frac{d\phi}{dx}[1 - \frac{\nu_{ni}\nu_{in}}{\Omega_n^2}].$$

$$n_i = \frac{-1}{\Omega\Omega_i}\frac{d^2\phi}{dx^2} - \frac{k}{\Omega\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx} + \frac{\nu_{in}\nu_{ni}k}{\Omega\Omega_i^2\Omega_n^2}\frac{dV_0}{dx}\frac{d\phi}{dx} + \frac{k^2\phi}{\Omega\Omega_i} - \frac{k}{\Omega\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx} + \frac{\nu_{ni}\nu_{in}}{\Omega_n^2}\frac{k}{\Omega\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx}.$$

$$n_i = -\frac{1}{\Omega\Omega_i}\frac{d^2\phi}{dx^2} - \frac{2k}{\Omega\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx} + \frac{k^2\phi}{\Omega\Omega_i} + \frac{2k\nu_{ni}\nu_{in}}{\Omega_n^2\Omega\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx}. \tag{3.41}$$

By putting value of eq. (3.41) in eq. (3.28) i.e. $\frac{d^2\phi}{dx^2} - \phi - k^2\phi + n_i = 0$, we get

$$\frac{d^2\phi}{dx^2} - \phi - k^2\phi - \frac{1}{\Omega\Omega_i}\frac{d^2\phi}{dx^2} + \frac{2k\nu_{in}\nu_{ni}}{\Omega\Omega_n^2\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx} + \frac{k^2}{\Omega\Omega_i}\phi - \frac{2k}{\Omega\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx} = 0.$$

$$\frac{d^2\phi}{dx^2} - \frac{1}{\Omega\Omega_i}\frac{d^2\phi}{dx^2} - \frac{2k}{\Omega\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx} + \frac{2k\nu_{in}\nu_{ni}}{\Omega\Omega_n^2\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx} + \frac{k^2}{\Omega\Omega_i}\phi - \phi - k^2\phi = 0.$$

$$(1 - \frac{1}{\Omega\Omega_i})\frac{d^2\phi}{dx^2} - \frac{2k}{\Omega\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx}(1 + \frac{\nu_{in}\nu_{ni}}{\Omega_n^2}) - [k^2(1 - \frac{1}{\Omega\Omega_i}) + 1]\phi = 0. \tag{3.42}$$

where above $\varepsilon_i = 1 - \frac{1}{\Omega\Omega_i}$

$$\varepsilon_i\frac{d^2\phi}{dx^2} - \frac{2k}{\Omega\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx}(1 + \frac{\nu_{in}\nu_{ni}}{\Omega_n^2}) - [k^2\varepsilon_i + 1]\phi = 0. \tag{3.43}$$

3.5 Ion-acoustic instability

We are considering low-frequency ion-acoustic oscillation in the presence of velocity shear in the plasma sheath.

$$D_2\frac{d^2\phi}{dx^2} - D_1\frac{d\phi}{dx} - D_0\phi = 0 \tag{3.44}$$

where above,

$D_0 = k^2\varepsilon_i + 1$

$D_2 = \varepsilon_i = 1 - \frac{1}{\Omega\Omega_i}$

$D_1 = \frac{2k}{\Omega\Omega_i^2}\frac{dV_0}{dx}(1 + \frac{\nu_{in}\nu_{ni}}{\Omega_n^2})$

$\Omega = \omega - kV_0$

$\Omega_n = \Omega + i\nu_{ni} = \omega - kV_0 + i\nu_{ni}$

$\Omega_i = \omega - kV_0 + i\nu_{in}(1 - \frac{i\nu_{ni}}{\Omega+i\nu_{ni}})$

From equation 3.43,

$$\frac{d^2\phi}{dx^2} - \frac{2k}{\Omega\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx}(1 + \frac{\nu_{in}\nu_{ni}}{\Omega_n^2}) - [k^2\varepsilon_i + 1]\phi = 0. \tag{3.45}$$

We set x=0 on the conducting surface of the vehicle thus requiring the potential to be equal to zero.

$$\phi(x = 0) = 0. \tag{3.46}$$

On surface beyond the sheath edge(x=L) the potential is

$$\phi(x) = \phi(L)exp[-k(x - L)].$$

On the sheath edge, at $x = L$

$$\frac{d\phi}{dx} = -k\phi,$$

hence

$$[\frac{d\phi}{dx} + k\phi]_{x=L} = 0. \tag{3.47}$$

eq. (3.44), eq. (3.46) and eq. (3.47) constitutes to the initial and boundary values of potential. In order to solve eq. (3.45), consider a linear velocity profile

$$V_0(x) = c_o x.$$

In eq. (3.45) we ignore terms involving $\frac{\nu_{ni}}{\Omega_n}$,also noting that for normalized frequencies such that $|\omega| << 1$ we may assume $\Omega\Omega_i << 1$, so $\varepsilon_i = 1 - \frac{1}{\Omega\Omega_i} \approx -\frac{1}{\Omega\Omega_i}$.

$$-\frac{1}{\Omega\Omega_i}\frac{d^2\phi}{dx^2} - \frac{2k}{\Omega\Omega_i^2}\frac{dV_0}{dx}\frac{d\phi}{dx} - [k^2(-\frac{1}{\Omega\Omega_i}) + 1]\phi = 0.$$

$$-\frac{1}{\Omega\Omega_i}[\frac{d^2\phi}{dx^2} + \frac{2k}{\Omega_i}\frac{dV_0}{dx}\frac{d\phi}{dx} - (k^2 - \Omega\Omega_i)\phi] = 0.$$

Decoupling above equation,

$$\frac{d^2\phi}{dx^2} + \frac{2k}{\Omega_i}\frac{dV_0}{dx}\frac{d\phi}{dx} - (k^2 - \Omega\Omega_i)\phi = 0.$$

$$\frac{d^2\phi}{dx^2} + \frac{2k}{\Omega_i}\frac{dV_0}{dx}\frac{d\phi}{dx} - k^2(1 - \frac{\Omega\Omega_i}{k^2})\phi = 0.$$

Where above $\frac{dV_0}{dx} = c_o$,

$$\frac{d^2\phi}{dx^2} + \frac{2kc_o}{\Omega_i}\frac{d\phi}{dx} - k^2(1 - \frac{\Omega\Omega_i}{k^2})\phi = 0. \tag{3.48}$$

Furthermore making assumptions that for a phase velocity smaller compared to ion speed the last term in eq. (3.48) maybe dropped, also as $\Omega\Omega_i << 1$ so $\frac{\Omega\Omega_i}{k^2} << 1$ therefore we can drop it as well. Hence

$$\frac{d^2\phi}{dx^2} + \frac{2kc_o}{\Omega_i}\frac{d\phi}{dx} - k^2\phi = 0. \tag{3.49}$$

Defining $X = kx$ and $\kappa = \frac{\omega + i\nu_{in}}{c_o}$ also we have dropped term involving $\frac{\nu_n i}{\Omega_n}$, so

$$\Omega_i = \omega - kV_0 + i\nu_{in}.$$

By putting $V_0 = c_o x$ and $X = kx$ in equation we get

$$\Omega_i = \omega - c_o X + i\nu_{in}.$$

$$\Omega_i = c_o[(\frac{\omega + i\nu_{in}}{c_o}) - X].$$

Put $\kappa = \frac{\omega + i\nu_{in}}{c_o}$ in above equation.

$$\Omega_i = c_o(\kappa - X).$$

$$\frac{d^2\phi}{dx^2} + \frac{2kc_o}{c_o(\kappa - X)}\frac{d\phi}{dx} - k^2\phi = 0.$$

$$\frac{d^2\phi}{dX^2} + \frac{2}{(\kappa - X)}\frac{d\phi}{dX} - \phi = 0. \tag{3.50}$$

By solving eq. (3.50) we get

$$\phi(X) = (\kappa - X - 1)C_1 e^{-X} + (\kappa - X + 1)C_2 e^{X}. \tag{3.51}$$

Using boundary conditions

$$\phi(X = 0) = 0, \tag{3.52}$$

$$[\frac{d\phi}{dX} + \phi]_{X=kL} = 0, \tag{3.53}$$

to solve eq. (3.51). From boundary condition eq. (3.52),

$$(\kappa - 1)C_1 + (\kappa + 1)C_2 = 0. \tag{3.54}$$

Similarly from boundary condition eq. (3.53)

$$\frac{d\phi}{dx} = (X-\kappa)C_1e^{-X} + (-X+\kappa)C_2e^{X}.$$

Put $X = kL$ in equation

$$-\phi(kL) = (kL-\kappa)C_1e^{-kL} + (-kL+\kappa)C_2e^{kL}.$$

We know

$$\phi(kL) = (\kappa - kL - 1)C_1e^{-kL} + (\kappa - kL + 1)C_2e^{kL}.$$

So,

$$-(\kappa - kL - 1)C_1e^{-kL} - (\kappa - kL + 1)C_2e^{kL} = (kL-\kappa)C_1e^{-kL} + (-kL+\kappa)C_2e^{kL}.$$

$$C_1e^{-kL} + (-2\kappa + 2kL - 1)C_2e^{kL} = 0.$$
$$-C_1e^{-kL} + 2(\kappa - kL - 0.5)C_2e^{kL} = 0 \qquad (3.55)$$

The boundary conditions give two linear equations for the coefficients C_1 and C_2

$$\begin{bmatrix} (\kappa-1) & (\kappa+1) \\ -e^{-kL} & 2(\kappa - kL - 0.5)e^{kL} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = 0 \qquad (3.56)$$

By setting determinants of the coefficients equal to zero we get

$$\begin{vmatrix} (\kappa-1) & (\kappa+1) \\ -e^{-kL} & 2(\kappa - kL - 0.5)e^{kL} \end{vmatrix} = 0.$$

m

$$e^{kL}\kappa^2 + (0.5e^{kL} - 0.5e^{kL} - kLe^{kL})\kappa + (0.5e^{-kL} - 0.5e^{kL} + kLe^{kL}) = 0.$$

Let

$$a = e^{kL},$$
$$b = 0.5e^{kL} - 0.5e^{kL} - kLe^{kL},$$
$$c = 0.5e^{-kL} - 0.5e^{kL} + kLe^{kL},$$

so

$$a\kappa^2 + b\kappa + c = 0.$$

By solving the quadratic equation for κ we get

$$\kappa = \frac{-b \pm i\sqrt{4ac - b^2}}{2a},$$

w5e have also defined κ as

$$\kappa = \frac{\omega + i\nu_{in}}{c_o},$$

so we can write the dispersion relation as

$$\kappa = \frac{\omega + i\nu_{in}}{c_o} = \frac{-b + i\sqrt{4ac - b^2}}{2a}. \qquad (3.57)$$

Chapter 4

Shear flow instability in a partially-ionized plasma sheath around a fast-moving vehicle in the presence of Earth's magnetic field

4.1 Plasma frequency is greater than the ion cyclotron frequency $\omega_{ci} << \omega << \omega_{ce}$

From electron momentum equation.

$$\frac{d\vec{\tilde{u}}_{1_e}}{d\tilde{t}} + \tilde{V}_{e_y}\frac{d\vec{\tilde{u}}_{1_e}}{d\tilde{y}} + \tilde{V}_{e_z}\frac{d\vec{\tilde{u}}_{1_e}}{d\tilde{z}} + \tilde{u}_{1_{ex}}\frac{d\vec{\tilde{V}}_{0_e}}{d\tilde{x}} = \frac{e\vec{\tilde{\nabla}}\tilde{\phi}}{\tilde{m}_e} - \tilde{V}_{T_e}^2\frac{\vec{\tilde{\nabla}}\tilde{n}_1}{\tilde{N}_{0_e}} - \tilde{\nu}_{en}(\vec{\tilde{u}}_{1_e} - \vec{\tilde{u}}_{1_n}) - \frac{e}{\tilde{m}_e}(\tilde{u}_{1e_y}\tilde{B}\hat{i} - \tilde{u}_{1e_x}\tilde{B}\hat{j}).$$

We will ignore the ion velocity $\vec{\tilde{u}}_{1_n}$ as they appear stationary in front of electron.Above equation become.

$$\frac{d\vec{\tilde{u}}_{1_e}}{d\tilde{t}} + \tilde{V}_{e_y}\frac{d\vec{\tilde{u}}_{1_e}}{d\tilde{y}} + \tilde{V}_{e_z}\frac{d\vec{\tilde{u}}_{1_e}}{d\tilde{z}} + \tilde{\nu}_{en}\vec{\tilde{u}}_{1_e} = \frac{e\vec{\tilde{\nabla}}\tilde{\phi}}{\tilde{m}_e} - \tilde{V}_{T_e}^2\frac{\vec{\tilde{\nabla}}\tilde{n}_{1_e}}{\tilde{N}_{0_e}} - \tilde{u}_{1_{ex}}\frac{d\vec{\tilde{V}}_{0_e}}{d\tilde{x}} - \frac{e}{\tilde{m}_e}(\tilde{u}_{1e_y}\tilde{B}\hat{i} - \tilde{u}_{1e_x}\tilde{B}\hat{j}). \quad (4.1)$$

Dropping subscript 1 in all perturbed velocities and densities.Equation 4.1 becomes,

$$\frac{d\vec{\tilde{u}}_e}{d\tilde{t}} + \tilde{V}_{e_y}\frac{d\vec{\tilde{u}}_e}{d\tilde{y}} + \tilde{V}_{e_z}\frac{d\vec{\tilde{u}}_e}{d\tilde{z}} + \tilde{\nu}_{en}\vec{\tilde{u}}_e = \frac{K_BT_e}{\tilde{m}_e}(\frac{e\vec{\tilde{\nabla}}\tilde{\phi}}{K_BT_e} - \frac{\vec{\tilde{\nabla}}\tilde{n}_e}{\tilde{N}_{0_e}}) - \tilde{u}_{ex}\frac{d\vec{\tilde{V}}_e}{d\tilde{x}} - \frac{e}{\tilde{m}_e}(\tilde{u}_{e_y}\tilde{B}\hat{i} - \tilde{u}_{e_x}\tilde{B}\hat{j}).$$

By using normalization conditions we will normalize the equation of momentum for electron, hence

$$\frac{d\vec{u_e}}{dt} + V_{ey}\frac{d\vec{u_e}}{dy} + V_{ez}\frac{d\vec{u_e}}{dz} + \nu_{en}\vec{u_e} = \frac{1}{m}(\vec{\nabla}\phi - \vec{\nabla}n_e) - u_{ex}\frac{d\vec{V_e}}{dx} - \frac{eB}{m_e}(u_{ey}\hat{i} - u_{ex}\hat{j}). \quad (4.2)$$

where $\frac{eB}{m_i} = \omega_{ce}$, which is the electron cyclotron frequency of electrons,so equation 4.2 becomes

$$\frac{d\vec{u_e}}{dt} + V_{ey}\frac{d\vec{u_e}}{dy} + V_{ez}\frac{d\vec{u_e}}{dz} + \nu_{en}\vec{u_e} = \frac{1}{m}(\vec{\nabla}\phi - \vec{\nabla}n_e) - u_{ex}\frac{d\vec{V_e}}{dx} - \omega_{ce}(u_{ey}\hat{i} - u_{ex}\hat{j}). \quad (4.3)$$

We will separate eq. (4.3) into its x and y component i.e. $\vec{u_e} = u_{ex}\hat{i} + u_{ey}\hat{j}$ and $\vec{\nabla} = \frac{d}{dx}\hat{i} + \frac{d}{dy}\hat{j}$

From X-component of equation 4.3

$$\frac{du_{ex}}{dt} + V_{ey}\frac{du_{ex}}{dy} + V_{ez}\frac{du_{ex}}{dz} + \nu_{en}u_{ex} = \frac{1}{m}[\frac{d\phi}{dx} - \frac{dn_e}{dx}] - \omega_{ce}u_{ey}$$

As all perturbed quantities behave sinusoidaly i.e. $f(x,y,t) \sim exp(ik_y y + ik_z z - i\omega t)$, therefore replace $\frac{d}{dt} = -i\omega, \frac{d}{dy} = ik_y$ and $\frac{d}{dz} = ik_z$,
also let $V_{ey} = V_y$ and $V_{ez} = V_z$,

$$[\omega - k_y V_y - k_z V_z + i\nu_{en}]u_{ex} = \frac{i}{m}\frac{d}{dx}[\phi - n_e] - i\omega_{ce}u_{ey}. \tag{4.4}$$

In eq. (4.4) $\Omega_e = \omega - k_y V_y - k_z V_z + i\nu_{en}$ and $\psi = \phi - n_e$.

$$u_{ex} = \frac{i}{m\Omega_e}\frac{d\psi}{dx} - \frac{i\omega_{ce}}{\Omega_e}u_{ey}. \tag{4.5}$$

From Y-component of equation 4.3

$$\frac{du_{ey}}{dt} + V_y\frac{du_{ey}}{dy} + V_z\frac{du_{ey}}{dz} + \nu_{en}u_{ey} = \frac{1}{m}[\frac{d\phi}{dy} - \frac{dn_e}{dy}] - u_{ex}\frac{dV_y}{dx} + \omega_{ce}u_{ex}.$$

$$[\omega - k_y V_y - k_z V_z + i\nu_{en}]u_{ey} = -\frac{k_y}{m}[\phi - n_e] - u_{ex}\frac{idV_y}{dx} + i\omega_{ce}u_{ex}.$$

$$u_{ey} = -\frac{k_y}{\Omega_e m}\psi + \frac{iu_{ex}}{\Omega_e}[\omega_{ce} - \frac{dV_y}{dx}]. \tag{4.6}$$

By putting value of eq. (4.5) in eq. (4.6), we get

$$u_{ey} = -\frac{k_y}{\Omega_e m}\psi + \frac{i}{\Omega_e}[\frac{i}{m\Omega_e}\frac{d\psi}{dx} + \frac{i\omega_{ce}}{\Omega_e}u_{ey}][\omega_{ce} - \frac{dV_y}{dx}].$$

$$u_{ey}[\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2] = -\frac{k_y\Omega_e}{m}\psi - \frac{1}{m}\frac{d\psi}{dx}[\omega_{ce} - \frac{dV_y}{dx}].$$

$$u_{ey} = \frac{-\Omega_e k_y\psi - \frac{d\psi}{dx}[\omega_{ce} - \frac{dV_y}{dx}]}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)}. \tag{4.7}$$

Similarly by putting value of eq. (4.6) in eq. (4.5) we get

$$u_{ex} = \frac{i}{m\Omega_e}\frac{d\psi}{dx} - \frac{i\omega_{ce}}{\Omega_e}(-\frac{k_y}{\Omega_e m}\psi + \frac{iu_{ex}}{\Omega_e}[\omega_{ce} - \frac{dV_y}{dx}]).$$

$$\frac{u_{ex}}{\Omega_e^2}[\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2] = \frac{i}{m\Omega_e}\frac{d\psi}{dx} + \frac{i\omega_{ce}k_y}{m\Omega_e^2}\psi.$$

$$u_{ex} = \frac{i\Omega_e\frac{d\psi}{dx} + i\omega_{ce}k_y\psi}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)}. \tag{4.8}$$

From Z-component of equation4.3

$$\frac{du_{ez}}{dt} + V_y\frac{du_{ez}}{dy} + V_z\frac{du_{ez}}{dz} + \nu_{en}u_{ez} = \frac{1}{m}[\frac{d\phi}{dz} - \frac{dn_e}{dz}] - u_{ex}\frac{dV_z}{dx}.$$

$$u_{ez} = \frac{-k_z}{m\Omega_e}\psi - \frac{iu_{ex}}{\Omega_e}\frac{dV_z}{dx}. \tag{4.9}$$

By putting value of eq. (4.8) in eq. (4.9) we get

$$u_{ez} = \frac{-k_z}{m\Omega_e}\psi - \frac{i}{\Omega_e}\frac{dV_z}{dx}[\frac{i\Omega_e\frac{d\psi}{dx} + i\omega_{ce}k_y\psi}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)}].$$

$$u_{ez} = \frac{-\Omega_e k_z\psi + \frac{\omega_{ce}}{\Omega_e}\psi[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] + \frac{k_z\omega_{ce}^2}{\Omega_e}\psi + \frac{dV_z}{dx}\frac{d\psi}{dx}}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)}. \tag{4.10}$$

From electron equation of continuity

$$\frac{dn_e}{dt} + V_y\frac{dn_e}{dy} + V_z\frac{dn_e}{dz} = -\frac{du_{ey}}{dy} - \frac{du_{ez}}{dz} - \frac{du_{ex}}{dx}.$$

$$-i(\omega - k_yV_y - k_zV_z)n_e = -ik_yu_{ey} - ik_zu_{ez} - \frac{du_{ex}}{dx}.$$

Let $\omega - k_yV_y - k_zV_z = \Omega$, hence

$$i\Omega n_e = \frac{du_{ex}}{dx} + ik_yu_{ey} + ik_zu_{ez}. \tag{4.11}$$

From eq. (4.8)

$$\frac{du_{ex}}{dx} = \frac{d}{dx}(\frac{i\Omega_e\frac{d\psi}{dx} + i\omega_{ce}k_y\psi}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)})$$

Let

$$A = i\Omega_e\frac{d\psi}{dx} + i\omega_{ce}k_y\psi,$$

and

$$B = m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2).$$

hence

$$\frac{du_{ex}}{dx} = \frac{d}{dx}(\frac{A}{B}) = \frac{B\frac{dA}{dx} - A\frac{dB}{dx}}{B^2}. \tag{4.12}$$

$$\frac{dA}{dx} = \frac{d}{dx}(i\Omega_e\frac{d\psi}{dx} + i\omega_{ce}k_y\psi).$$

Where $\frac{d\Omega_e}{dx} = -k_y\frac{dV_y}{dx} - k_z\frac{dV_z}{dx}$, hence

$$\frac{dA}{dx} = -i\frac{d\psi}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + i\Omega_e\frac{d^2\psi}{dx^2} + i\omega_{ce}k_y\frac{d\psi}{dx}.$$

$$\frac{dB}{dx} = \frac{d}{dx}(m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2).$$

$$\frac{dB}{dx} = -2m\Omega_e(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + m\omega_{ce}\frac{d^2V_y}{dx^2}.$$

By putting values in eq. (4.12), we get

$$\frac{du_{ex}}{dx} = \frac{-i\frac{d\psi}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + i\Omega_e\frac{d^2\psi}{dx^2} + i\omega_{ce}k_y\frac{d\psi}{dx}}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)} - \frac{(i\Omega_e\frac{d\psi}{dx} + i\omega_{ce}k_y\psi)(-2m\Omega_e(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + m\omega_{ce}\frac{d^2V_y}{dx^2})}{m^2(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2} \tag{4.13}$$

$$\begin{aligned}\frac{du_{ex}}{dx} &= \frac{-i\frac{d\psi}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + i\Omega_e\frac{d^2\psi}{dx^2} + i\omega_{ce}k_y\frac{d\psi}{dx}}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)} \\ &\quad - \frac{(i\Omega_e\frac{d\psi}{dx} + i\omega_{ce}k_y\psi)(-2m\Omega_e(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + m\omega_{ce}\frac{d^2V_y}{dx^2})}{m^2(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2}\end{aligned}$$

$$\begin{aligned}\frac{du_{ex}}{dx} &= \frac{-i\frac{d\psi}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + i\Omega_e\frac{d^2\psi}{dx^2} + i\omega_{ce}k_y\frac{d\psi}{dx}}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)} \\ &\quad + \frac{(i\Omega_e\frac{d\psi}{dx} + i\omega_{ce}k_y\psi)(2\Omega_e(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) - \omega_{ce}\frac{d^2V_y}{dx^2})}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2}.\end{aligned}$$

put value of above equation in eq. (4.11)

$$\begin{aligned}i\Omega n_e &= \frac{-i\frac{d\psi}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + i\Omega_e\frac{d^2\psi}{dx^2} + i\omega_{ce}k_y\frac{d\psi}{dx}}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)} \\ &\quad + \frac{(i\Omega_e\frac{d\psi}{dx} + i\omega_{ce}k_y\psi)(2\Omega_e(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) - \omega_{ce}\frac{d^2V_y}{dx^2})}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2} \\ &\quad + ik_y(\frac{-\Omega_e k_y\psi - \frac{d\psi}{dx}[\omega_{ce} - \frac{dV_y}{dx}]}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)}) \\ &\quad + ik_z(\frac{-\Omega_e k_z\psi + \frac{\omega_{ce}}{\Omega_e}\psi[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] + \frac{k_z\omega_{ce}^2}{\Omega_e}\psi + \frac{dV_z}{dx}\frac{d\psi}{dx}}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)}).\end{aligned}$$

$$\begin{aligned}\Omega n_e &= \frac{-\frac{d\psi}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + i\Omega_e\frac{d^2\psi}{dx^2} + \omega_{ce}k_y\frac{d\psi}{dx}}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)} \\ &\quad + \frac{(\Omega_e\frac{d\psi}{dx} + \omega_{ce}k_y\psi)(2\Omega_e(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) - \omega_{ce}\frac{d^2V_y}{dx^2})}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2} \\ &\quad + (\frac{-\Omega_e k_y^2\psi - k_y\frac{d\psi}{dx}[\omega_{ce} - \frac{dV_y}{dx}]}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)}) \\ &\quad + (\frac{-\Omega_e k_z^2\psi + \frac{\omega_{ce}k_z}{\Omega_e}\psi[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] + \frac{k_z^2\omega_{ce}^2}{\Omega_e}\psi + k_z\frac{dV_z}{dx}\frac{d\psi}{dx}}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)}).\end{aligned}$$

$$\Omega n_e = \frac{\Omega_e \frac{d^2\psi}{dx^2} - \Omega_e\psi(k_y^2+k_z^2) + \frac{\omega_{ce}k_z}{\Omega_e}\psi[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] + \frac{k_z^2\omega_{ce}^2}{\Omega_e}\psi}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)} \tag{4.14}$$

$$+\frac{(\Omega_e\frac{d\psi}{dx} + \omega_{ce}k_y\psi)(2\Omega_e(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) - \omega_{ce}\frac{d^2V_y}{dx^2})}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2}. \tag{4.15}$$

A similar result for the density of relectron is obtained in [3]

From normalized neutral equation of momentum

$$\frac{d\vec{u_n}}{dt} + V_y\frac{d\vec{u_n}}{dy} + V_z\frac{d\vec{u_n}}{dz} + \nu_{ni}\vec{u_n} = -u_{nx}\frac{d\vec{V}}{dx} + \nu_{ni}\vec{u_i} \tag{4.16}$$

X-component of eq. (4.16)

$$\frac{du_{nx}}{dt} + V_y\frac{du_{nx}}{dy} + V_z\frac{du_{nx}}{dz} + \nu_{ni}u_{nx} = \nu_{ni}u_{ix}.$$

$$-i[\omega - k_yV_y - k_zV_z + i\nu_{ni}]u_{nx} = \nu_{ni}u_{ix}.$$

$$-i\Omega_n u_{nx} = \nu_{ni}u_{ix}.$$

$$u_{nx} = \frac{i\nu_{ni}u_{ix}}{\Omega_n}. \tag{4.17}$$

Y-component of eq. (4.16)

$$\frac{du_{ny}}{dt} + V_y\frac{du_{ny}}{dy} + V_z\frac{du_{ny}}{dz} + \nu_{ni}u_{ny} = \nu_{ni}u_{iy} - u_{nx}\frac{dV_y}{dx}.$$

$$-i[\omega - k_yV_y - k_zV_z + i\nu_{ni}]u_{ny} = \nu_{ni}u_{iy} - u_{nx}\frac{dV_y}{dx}.$$

$$-i\Omega_n u_{ny} = \nu_{ni}u_{iy} - u_{nx}\frac{dV_y}{dx}.$$

$$u_{ny} = \frac{i\nu_{ni}u_{iy}}{\Omega_n} - \frac{iu_{nx}}{\Omega_n}\frac{dV_y}{dx} \tag{4.18}$$

By putting the value of eq. (4.17) in equation 4.18 we get,

$$u_{ny} = \frac{i\nu_{ni}u_{iy}}{\Omega_n} + \frac{\nu_{ni}u_{ix}}{\Omega_n^2}\frac{dV_y}{dx}. \tag{4.19}$$

Z-component of eq. (4.16)

$$\frac{du_{nz}}{dt} + V_y\frac{du_{nz}}{dy} + V_z\frac{du_{nz}}{dz} + \nu_{ni}u_{nz} = \nu_{ni}u_{iz} - u_{nx}\frac{dV_z}{dx}.$$

$$-i[\omega - k_yV_y - k_zV_z + i\nu_{ni}]u_{nz} = \nu_{ni}u_{iz} - u_{nx}\frac{dV_z}{dx}.$$

$$-i\Omega_n u_{nz} = \nu_{ni}u_{iz} - u_{nx}\frac{dV_z}{dx}.$$

$$u_{nz} = \frac{i\nu_{ni}u_{iz}}{\Omega_n} + \frac{\nu_{ni}u_{ix}}{\Omega_n^2}\frac{dV_z}{dx} \tag{4.20}$$

From Ions equation of momentum

$$\frac{d\vec{u_i}}{dt} + V_y\frac{d\vec{u_i}}{dy} + V_z\frac{d\vec{u_i}}{dz} + \nu_{in}\vec{u_i} - \nu_{in}\vec{u_n} = -\vec{\nabla}\phi - u_{ix}\frac{d\vec{V}}{dx} + \frac{eB}{M_i}(u_{iy}\hat{i} - u_{ix}\hat{j}) \tag{4.21}$$

X-component of eq. (4.21)

$$-i[\omega - k_yV_y - k_zV_z]u_{ix} + \nu_{in}u_{ix} - \nu_{in}u_{nx} = -\frac{d\phi}{dx} + \omega_{ci}u_{iy}.$$

$$-i[\omega - k_yV_y - k_zV_z]u_{ix} + \nu_{in}u_{ix} - \nu_{in}(\frac{i\nu_{ni}u_{ix}}{\Omega_n}) = -\frac{d\phi}{dx} + \omega_{ci}u_{iy}.$$

$$-i[\omega - k_yV_y - k_zV_z + i\nu_{in}(1 - \frac{i\nu_{ni}}{\Omega_n})]u_{ix} = -\frac{d\phi}{dx} + \omega_{ci}u_{iy}.$$

Let $\Omega_i = \omega - k_yV_y - k_zV_z + i\nu_{in}(1 - \frac{i\nu_{ni}}{\Omega_n})$.

$$u_{ix} = -\frac{i}{\Omega_i}\frac{d\phi}{dx} + \frac{i\omega_{ci}}{\Omega_i}u_{iy}. \tag{4.22}$$

Y-component of eq. (4.21)

$$-i[\omega - k_yV_y - k_zV_z]u_{iy} + \nu_{in}u_{iy} - \nu_{in}u_{ny} = -ik_y\phi - \omega_{ci}u_{ix} - u_{ix}\frac{dV_y}{dx}.$$

$$-i[\omega - k_yV_y - k_zV_z]u_{iy} + \nu_{in}u_{iy} - \nu_{in}(\frac{i\nu_{ni}u_{iy}}{\Omega_n} + \frac{\nu_{ni}u_{ix}}{\Omega_n^2}\frac{dV_y}{dx}) = -ik_y\phi - \omega_{ci}u_{ix} - u_{ix}\frac{dV_y}{dx}.$$

$$-i\Omega_iu_{iy} = \frac{\nu_{in}\nu_{ni}u_{ix}}{\Omega_n^2}\frac{dV_y}{dx} - ik_y\phi - \omega_{ci}u_{ix} - u_{ix}\frac{dV_y}{dx}.$$

$$u_{iy} = \frac{k_y\phi}{\Omega_i} + u_{ix}\frac{i}{\Omega_i}[\frac{\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_y}{dx} - \omega_{ci} - \frac{dV_y}{dx}]. \tag{4.23}$$

By putting value of eq. (4.22) in eq. (4.23) we get,

$$\Omega_i^2u_{iy} = k_y\phi\Omega_i + [\frac{d\phi}{dx} - \omega_{ci}u_{iy}][\frac{\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_y}{dx} - \omega_{ci} - \frac{dV_y}{dx}].$$

$$[\Omega_i^2 + \omega_{ci}(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_y}{dx} - \omega_{ci} - \frac{dV_y}{dx})]u_{iy} = k_y\phi\Omega_i + \frac{d\phi}{dx}[\frac{\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_y}{dx} - \omega_{ci} - \frac{dV_y}{dx}].$$

let $C = [\Omega_i^2 + \omega_{ci}(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_y}{dx} - \omega_{ci} - \frac{dV_y}{dx})]$.

$$u_{iy} = \frac{k_y\phi\Omega_i + \frac{d\phi}{dx}[\frac{\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_y}{dx} - \omega_{ci} - \frac{dV_y}{dx}]}{C}. \tag{4.24}$$

By putting value of eq. (4.23) in eq. (4.22) we get,

$$\Omega_i^2 u_{ix} = -i\Omega_i \frac{d\phi}{dx} + i\omega_{ci}[k_y\phi + iu_{ix}[\frac{\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_y}{dx} - \omega_{ci} - \frac{dV_y}{dx}]].$$

$$[\Omega_i^2 + \omega_{ci}(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_y}{dx} - \omega_{ci} - \frac{dV_y}{dx})]u_{ix} = -i\Omega_i \frac{d\phi}{dx} + i\omega_{ci}k_y\phi.$$

$$u_{ix} = \frac{-i\Omega_i \frac{d\phi}{dx} + i\omega_{ci}k_y\phi}{C}. \tag{4.25}$$

From Z-component of eq. (4.21)

$$-i[\omega - k_yV_y - k_zV_z]u_{iz} + \nu_{in}u_{iz} - \nu_{in}u_{nz} = -ik_z\phi - u_{ix}\frac{dV_z}{dx}. \tag{4.26}$$

By putting value of eq. (4.20) in eq. (4.26) we get

$$-i[\omega - k_yV_y - k_zV_z]u_{iz} + \nu_{in}u_{iz} - \frac{i\nu_{in}\nu_{ni}u_{iz}}{\Omega_n} = -ik_z\phi - u_{ix}\frac{dV_z}{dx} + \frac{\nu_{in}\nu_{ni}u_{ix}}{\Omega_n^2}\frac{dV_z}{dx}.$$

$$-i\Omega_i u_{iz} = -ik_z\phi - u_{ix}\frac{dV_z}{dx}[1 - \frac{\nu_{in}\nu_{ni}}{\Omega_n^2}]. \tag{4.27}$$

Put value of eq. (4.25) in eq. (4.27)

$$-i\Omega_i u_{iz} = -ik_z\phi - [\frac{-i\Omega_i \frac{d\phi}{dx} + i\omega_{ci}k_y\phi}{C}]\frac{dV_z}{dx}[1 - \frac{\nu_{in}\nu_{ni}}{\Omega_n^2}].$$

$$u_{iz} = \frac{Ck_z\phi - [\Omega_i\frac{d\phi}{dx} - \omega_{ci}k_y\phi][1 - \frac{\nu_{in}\nu_{ni}}{\Omega_n^2}]\frac{dV_z}{dx}}{C\Omega_i}. \tag{4.28}$$

From ion equation of continuity

$$i\Omega n_i = \frac{du_{ix}}{dx} + ik_yu_{iy} + ik_zu_{iz} \tag{4.29}$$

where

$$\frac{du_{ix}}{dx} = \frac{-i\Omega_i\frac{d^2\phi}{dx^2} - i\frac{d\phi}{dx}[k_y\frac{dV_y}{dx}(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2} - 1) + k_z\frac{dV_z}{dx}(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2} - 1)] + i\omega_{ci}k_y\frac{d\phi}{dx}}{C}$$
$$\frac{[i\Omega_i\frac{d\phi}{dx} - i\omega_{ci}k_y\phi][2\Omega_i(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx})(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2} - 1)]}{C^2}$$
$$+\frac{[i\Omega_i\frac{d\phi}{dx} - i\omega_{ci}k_y\phi][\frac{\omega_{ci}\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{d^2V_y}{dx} + \frac{2\omega_{ci}\nu_{in}\nu_{ni}}{\Omega_n^3}\frac{dV_y}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) - \omega_{ci}\frac{d^2V_y}{dx^2}]}{C^2}.$$

$$
\begin{aligned}
\Omega n_i \;=\;& \frac{-\Omega_i \frac{d^2\phi}{dx^2} - i\frac{d\phi}{dx}[k_y \frac{dV_y}{dx}(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2} - 1) + k_z \frac{dV_z}{dx}(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2} - 1)] + i\omega_{ci}k_y\frac{d\phi}{dx}}{C} \\
& \frac{[\Omega_i \frac{d\phi}{dx} - \omega_{ci}k_y\phi][2\Omega_i(k_y \frac{dV_y}{dx} + k_z \frac{dV_z}{dx})(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2} - 1)]}{C^2} \\
& + \frac{[\Omega_i \frac{d\phi}{dx} - \omega_{ci}k_y\phi][\frac{\omega_{ci}\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{d^2V_y}{dx} + \frac{2\omega_{ci}\nu_{in}\nu_{ni}}{\Omega_n^3}\frac{dV_y}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) - \omega_{ci}\frac{d^2V_y}{dx^2}]}{C^2} \\
& + [\frac{k_y^2\phi\Omega_i + k_y\frac{d\phi}{dx}[\frac{\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_y}{dx} - \omega_{ci} - \frac{dV_y}{dx}]}{C}] \\
& + [\frac{Ck_z^2\phi - k_z[\Omega_i\frac{d\phi}{dx} - \omega_{ci}k_y\phi][1 - \frac{\nu_{in}\nu_{ni}}{\Omega_n^2}]\frac{dV_z}{dx}}{C\Omega_i}].
\end{aligned}
$$

$$
\begin{aligned}
\Omega n_i \;=\;& \frac{(k_y^2 + k_z^2)\Omega_i\phi + \frac{\nu_{in}\nu_{ni}k_z\phi}{\Omega_i\Omega_n^2}\frac{dV_z}{dx}(k_z - k_y) - \frac{\omega_{ci}k_z^2\phi}{\Omega_i}(\omega_{ci} + \frac{dV_y}{dx}) + k_yk_z\omega_{ci}\phi\frac{dV_y}{dx} - \Omega_i\frac{d^2\phi}{dx^2}}{C} \\
& \frac{[\Omega_i \frac{d\phi}{dx} - \omega_{ci}k_y\phi][2\Omega_i(k_y \frac{dV_y}{dx} + k_z \frac{dV_z}{dx})(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2} - 1) + \frac{\omega_{ci}\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{d^2V_y}{dx^2}]}{C^2} \\
& + \frac{[\Omega_i \frac{d\phi}{dx} - \omega_{ci}k_y\phi][\frac{2\omega_{ci}\nu_{in}\nu_{ni}}{\Omega_n^3}\frac{dV_y}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) - \omega_{ci}\frac{d^2V_y}{dx^2}]}{C^2}.
\end{aligned}
$$

$$
\begin{aligned}
n_i \;=\;& \frac{(k_y^2 + k_z^2)\Omega_i\phi + \frac{\nu_{in}\nu_{ni}k_z\omega_{ci}\phi}{\Omega_i\Omega_n^2}\frac{dV_z}{dx}(k_z - k_y) - \frac{\omega_{ci}k_z^2\phi}{\Omega_i}(\omega_{ci} + \frac{dV_y}{dx}) + k_yk_z\omega_{ci}\phi\frac{dV_y}{dx} - \Omega_i\frac{d^2\phi}{dx^2}}{\Omega C} \\
& \frac{[\Omega_i \frac{d\phi}{dx} - \omega_{ci}k_y\phi][2\Omega_i(k_y \frac{dV_y}{dx} + k_z \frac{dV_z}{dx})(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2} - 1) + \frac{\omega_{ci}\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{d^2V_y}{dx^2}]}{\Omega C^2} \\
& + \frac{[\Omega_i \frac{d\phi}{dx} - \omega_{ci}k_y\phi][\frac{2\omega_{ci}\nu_{in}\nu_{ni}}{\Omega_n^3}\frac{dV_y}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) - \omega_{ci}\frac{d^2V_y}{dx^2}]}{\Omega C^2}.
\end{aligned}
$$

Consider ion cyclotron frequency to be to be very leass than the plasma frequency,hence.

$$
\begin{aligned}
n_i \;=\;& \frac{(k_y^2 + k_z^2)\Omega_i\phi + \frac{\nu_{in}\nu_{ni}k_z\omega_{ci}\phi}{\Omega_i\Omega_n^2}\frac{dV_z}{dx}(k_z - k_y) - \frac{\omega_{ci}k_z^2\phi}{\Omega_i}(\omega_{ci} + \frac{dV_y}{dx}) + k_yk_z\omega_{ci}\phi\frac{dV_y}{dx} - \Omega_i\frac{d^2\phi}{dx^2}}{\Omega\Omega_i^2} \\
& \frac{[\Omega_i \frac{d\phi}{dx} - \omega_{ci}k_y\phi][2\Omega_i(k_y \frac{dV_y}{dx} + k_z \frac{dV_z}{dx})(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2} - 1) + \frac{\omega_{ci}\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{d^2V_y}{dx^2}]}{\Omega\Omega_i^4} \\
& + \frac{[\Omega_i \frac{d\phi}{dx} - \omega_{ci}k_y\phi][\frac{2\omega_{ci}\nu_{in}\nu_{ni}}{\Omega_n^3}\frac{dV_y}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) - \omega_{ci}\frac{d^2V_y}{dx^2}]}{\Omega\Omega_i^4}.
\end{aligned}
$$

Ignore terms iovolving $\frac{\nu_{in}\nu_{ni}}{\Omega_n}$, also as we are dealing with linear velocity profile therefore $\frac{d^2V_y}{dx^2} = 0$.

$$
\begin{aligned}
\Omega\Omega_i^4 n_i \;=\;& (k_y^2 + k_z^2)\Omega_i^3\phi - \omega_{ci}\Omega_i k_z^2\phi(\omega_{ci} + \frac{dV_y}{dx}) + k_yk_z\Omega_i^2\omega_{ci}\phi\frac{dV_y}{dx} - \Omega_i^3\frac{d^2\phi}{dx^2} \qquad (4.30) \\
& -2\Omega_i^2\frac{d\phi}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + 2\omega_{ci}k_y\Omega_i\phi(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}). \qquad (4.31)
\end{aligned}
$$

from equation 4.14

$$\Omega n_e = \frac{\Omega_e \frac{d^2\psi}{dx^2} - \Omega_e\psi(k_y^2 + k_z^2) + \frac{\omega_{ce}k_z}{\Omega_e}\psi[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] + \frac{k_z^2\omega_{ce}^2}{\Omega_e}\psi}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)} \quad (4.32)$$

$$+\frac{(\Omega_e\frac{d\psi}{dx} + \omega_{ce}k_y\psi)(2\Omega_e(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) - \omega_{ce}\frac{d^2V_y}{dx^2})}{m(\Omega_e^2 + \omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2} \quad (4.33)$$

as $\Omega_e << \omega_{ce}$ hence above equation becomes

$$n_e = \frac{\Omega_e \frac{d^2\psi}{dx^2} - \Omega_e\psi(k_y^2 + k_z^2) + \frac{\omega_{ce}k_z}{\Omega_e}\psi[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] + \frac{k_z^2\omega_{ce}^2}{\Omega_e}\psi}{m\Omega(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)}$$

$$+\frac{(\Omega_e\frac{d\psi}{dx} + \omega_{ce}k_y\psi)(2\Omega_e(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) - \omega_{ce}\frac{d^2V_y}{dx^2})}{m\Omega(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2}$$

$$\begin{aligned} mn_e\Omega(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2 &= (\Omega_e\frac{d^2\psi}{dx^2} - \Omega_e\psi(k_y^2 + k_z^2) + \frac{\omega_{ce}k_z}{\Omega_e}\psi[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}])(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + \\ &\quad (\frac{k_z^2\omega_{ce}^2}{\Omega_e}\psi)(m\Omega(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)) + (\Omega_e\frac{d\psi}{dx} + \omega_{ce}k_y\psi)(2\Omega_e(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) \\ &\quad -(\Omega_e\frac{d\psi}{dx} + \omega_{ce}k_y\psi)(\omega_{ce}\frac{d^2V_y}{dx^2}) \end{aligned}$$

where $\frac{d^2V_y}{dx^2} = 0$

$$n_e = \frac{\Omega_e \frac{d^2\psi}{dx^2} - \Omega_e\psi(k_y^2 + k_z^2) + \frac{\omega_{ce}k_z}{\Omega_e}\psi[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] + \frac{k_z^2\omega_{ce}^2}{\Omega_e}\psi}{m\Omega(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)} + \frac{2\Omega_e(\Omega_e\frac{d\psi}{dx} + \omega_{ce}k_y\psi)(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx})}{m\Omega(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2} \quad (4.34)$$

$$\begin{aligned} m\Omega(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2 n_e &= (\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)(\Omega_e\frac{d^2\psi}{dx^2} - \Omega_e\psi(k_y^2 + k_z^2) + \frac{\omega_{ce}k_z}{\Omega_e}\psi[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] \\ &\quad + \frac{k_z^2\omega_{ce}^2}{\Omega_e}\psi) + 2\Omega_e(\Omega_e\frac{d\psi}{dx} + \omega_{ce}k_y\psi)(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) \end{aligned} \quad (4.35)$$

$$\begin{aligned} L_e &= (\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)(\Omega_e\frac{d^2}{dx^2} - \Omega_e(k_y^2 + k_z^2) + \frac{\omega_{ce}k_z}{\Omega_e}[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] + \frac{k_z^2\omega_{ce}^2}{\Omega_e}) \\ &\quad +2\Omega_e(\Omega_e\frac{d}{dx} + \omega_{ce}k_y)(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) \end{aligned}$$

$$m\Omega(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2 n_e = L_e\psi$$

$$m\Omega(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2 n_e = L_e(\phi - n_e)$$

$$(m\Omega(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)^2 + L_e)n_e = L_e\phi$$

$$X_e n_e = L_e\phi$$

from LHS of above equation

$$\begin{aligned} X_e &= m\Omega(\omega_{ce}^4 + {\omega_{ce}}^2\frac{dV_y}{dx}^2 - 2{\omega_{ce}}^2\frac{dV_y}{dx}) + (\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)(-\Omega_e(k_y^2 + k_z^2) \\ &+ \frac{\omega_{ce}k_z}{\Omega_e}[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] + \frac{k_z^2\omega_{ce}^2}{\Omega_e}) + 2\Omega_e\omega_{ce}k_y(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + \Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) \\ &+ 2\Omega_e^2\frac{d}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) \end{aligned}$$

$$\begin{aligned} X_e &= m\Omega\omega_{ce}^4 + m\Omega{\omega_{ce}}^2(\frac{dV_y}{dx})^2 - 2m\Omega{\omega_{ce}}^2\frac{dV_y}{dx} - \Omega_e(k_y^2 + k_z^2)(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) \\ &+ \frac{\omega_{ce}^2k_z}{\Omega_e}\frac{dV_y}{dx}[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] + \frac{k_z^2\omega_{ce}^3}{\Omega_e}\frac{dV_y}{dx} - \frac{\omega_{ce}^3k_z}{\Omega_e}[k_y\frac{dV_z}{dx} - k_z\frac{dV_y}{dx}] \\ &- \frac{k_z^2\omega_{ce}^4}{\Omega_e}) + 2\Omega_e\omega_{ce}k_y(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) + \Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2\frac{d}{dx}(k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}) \end{aligned}$$

$$n_e = \frac{L_e\phi}{X_e} \tag{4.36}$$

Considering case where $k_z = 0$
In this following supposition is made that the strength of magnetic field is so less that there is no propagation of plasma along z direction and it will propagate along y axis only with velocity shear along x direction.
From equation equation for the neutral density can be written as.

$$n_i = \frac{k_y^2\phi}{\Omega\Omega_i} - \frac{1}{\Omega_i\Omega}\frac{d^2\phi}{dx^2} + \frac{2k_y}{\Omega_i^2\Omega}\frac{d\phi}{dx}\frac{dV_y}{dx} \tag{4.37}$$

for such case X_e will become.

$$\begin{aligned} X_e &= m\Omega\omega_{ce}^4 + m\Omega{\omega_{ce}}^2(\frac{dV_y}{dx})^2 - 2m\Omega{\omega_{ce}}^2\frac{dV_y}{dx} - \Omega_ek_y^2(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) \\ &+ 2\Omega_e\omega_{ce}k_y^2\frac{dV_y}{dx} + \Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2k_y\frac{d}{dx}\frac{dV_y}{dx} \end{aligned}$$

Similarly L_e will become.

$$L_e = \Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) - \Omega_ek_y^2(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2k_y\frac{dV_y}{dx}\frac{d}{dx} + 2\omega_{ce}\Omega_ek_y^2\frac{dV_y}{dx}$$

From Poisson equation

$$\frac{d^2\phi}{dx^2} - k_y^2\phi = n_e - n_i \tag{4.38}$$

By putting values of equation 4.37 and 4.36 in equation 4.38 we get.

$$\frac{d^2\phi}{dx^2} - k_y^2\phi = \frac{L_e\phi}{X_e} - \frac{k_y^2\phi}{\Omega\Omega_i} + \frac{1}{\Omega_i\Omega}\frac{d^2\phi}{dx^2} + \frac{2k_y}{\Omega_i^2\Omega}\frac{d\phi}{dx}\frac{dV_y}{dx}$$

$$\frac{d^2\phi}{dx^2}(1 - \frac{1}{\Omega_i\Omega}) - k_y^2\phi(1 - \frac{1}{\Omega\Omega_i}) = \frac{L_e\phi}{X_e} + \frac{2k_y}{\Omega_i^2\Omega}\frac{d\phi}{dx}\frac{dV_y}{dx}$$

where $\Omega\Omega_i << 1$ hence above equation becomes

$$-\frac{1}{\Omega_i\Omega}\frac{d^2\phi}{dx^2} + \frac{k_y^2\phi}{\Omega\Omega_i} = \frac{L_e\phi}{X_e} + \frac{2k_y}{\Omega_i^2\Omega}\frac{d\phi}{dx}\frac{dV_y}{dx}$$

$$X_e[-\frac{d^2\phi}{dx^2} + k_y^2\phi - \frac{2k_y}{\Omega_i}\frac{dV_y}{dx}\frac{d\phi}{dx}] = \Omega_i\Omega L_e\phi$$

$$\begin{aligned}&[m\Omega\omega_{ce}^4 + m\Omega{\omega_{ce}}^2(\frac{dV_y}{dx})^2 - 2m\Omega{\omega_{ce}}^2\frac{dV_y}{dx} - \Omega_e k_y^2(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e\omega_{ce}k_y^2\frac{dV_y}{dx} \\ &\quad + \Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2k_y\frac{d}{dx}\frac{dV_y}{dx}][-\frac{d^2\phi}{dx^2} + k_y^2\phi - \frac{2k_y}{\Omega_i}\frac{d\phi}{dx}\frac{dV_y}{dx}] \\ &= \Omega_i\Omega[\Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) - \Omega_e k_y^2(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2k_y\frac{dV_y}{dx}\frac{d}{dx} + 2\omega_{ce}\Omega_e k_y^2\frac{dV_y}{dx}]\phi \quad (4.39)\end{aligned}$$

Solving coefficient of $\frac{d^2\phi}{dx^2}$ in eq. (4.39)

$$\begin{aligned}&-[m\Omega\omega_{ce}^4 + m\Omega{\omega_{ce}}^2(\frac{dV_y}{dx})^2 - 2m\Omega{\omega_{ce}}^2\frac{dV_y}{dx} - \Omega_e k_y^2(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e\omega_{ce}k_y^2\frac{dV_y}{dx} \\ &+\Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2k_y\frac{d}{dx}\frac{dV_y}{dx}] - \Omega_i\Omega\Omega_e[(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) = 0\end{aligned}$$

$$\begin{aligned}&-m\Omega\omega_{ce}^4 - m\Omega{\omega_{ce}}^2(\frac{dV_y}{dx})^2 + 2m\Omega{\omega_{ce}}^2\frac{dV_y}{dx} + \Omega_e k_y^2(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2)(1 - \frac{\Omega_i\Omega}{k_y^2}) - 2\Omega_e\omega_{ce}k_y^2\frac{dV_y}{dx} \\ &-\Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) - 2\Omega_e^2k_y\frac{d}{dx}\frac{dV_y}{dx} = 0\end{aligned}$$

where $\frac{\Omega_i\Omega}{k_y^2} << 1$ hence above equation becomes

$$\begin{aligned}&-[m\Omega\omega_{ce}^4 + m\Omega{\omega_{ce}}^2(\frac{dV_y}{dx})^2 - 2m\Omega{\omega_{ce}}^2\frac{dV_y}{dx} - \Omega_e k_y^2(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e\omega_{ce}k_y^2\frac{dV_y}{dx} \\ &+\Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2k_y\frac{d}{dx}\frac{dV_y}{dx}] = -X_e\end{aligned}$$

Solving coefficient of $\frac{d\phi}{dx}$ in eq. (4.39)

$$\begin{aligned}&[m\Omega\omega_{ce}^4 + m\Omega{\omega_{ce}}^2(\frac{dV_y}{dx})^2 - 2m\Omega{\omega_{ce}}^2\frac{dV_y}{dx} - \Omega_e k_y^2(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e\omega_{ce}k_y^2\frac{dV_y}{dx} \\ &+\Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2k_y\frac{d}{dx}\frac{dV_y}{dx}][\frac{-2k_y}{\Omega_i}\frac{dV_y}{dx}] - 2\Omega_e^2\Omega_i\Omega k_y\frac{dV_y}{dx} = 0.\end{aligned}$$

$$[m\Omega\omega_{ce}^4 + m\Omega{\omega_{ce}}^2(\frac{dV_y}{dx})^2 - 2m\Omega{\omega_{ce}}^2\frac{dV_y}{dx} - \Omega_e k_y^2\omega_{ce}\frac{dV_y}{dx} + 2\Omega_e\omega_{ce}k_y^2\frac{dV_y}{dx}$$
$$+\Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2 k_y\frac{d}{dx}\frac{dV_y}{dx}][\frac{-2k_y}{\Omega_i}\frac{dV_y}{dx}] - \frac{2k_y\Omega_e k_y^2\omega_{ce}^2}{\Omega_i}\frac{dV_y}{dx}[1 - \frac{\Omega_e\Omega_i^2\Omega}{k_y^2\omega_{ce}^2}] = 0.$$

Where the value of $\frac{\Omega_e\Omega_i^2\Omega}{k_y^2\omega_{ce}^2}] << 1$ so it can be ignored, hence

$$[m\Omega\omega_{ce}^4 + m\Omega{\omega_{ce}}^2(\frac{dV_y}{dx})^2 - 2m\Omega{\omega_{ce}}^2\frac{dV_y}{dx} - \Omega_e k_y^2(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e\omega_{ce}k_y^2\frac{dV_y}{dx}$$
$$+\Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2 k_y\frac{d}{dx}\frac{dV_y}{dx}][[\frac{-2k_y}{\Omega_i}\frac{dV_y}{dx}] = \frac{-2k_y X_e}{\Omega_i}\frac{dV_y}{dx}.$$

Solving the coefficient of ϕ in eq. (4.39)

$$[m\Omega\omega_{ce}^4 + m\Omega{\omega_{ce}}^2(\frac{dV_y}{dx})^2 - 2m\Omega{\omega_{ce}}^2\frac{dV_y}{dx} + \Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2 k_y\frac{d}{dx}\frac{dV_y}{dx}][k_y^2]$$
$$+\Omega_e k_y^4\omega_{ce}^2(1 - \frac{\Omega_i\Omega}{k_y^2}) + \Omega_e\omega_{ce}k_y^4\frac{dV_y}{dx}(1 - \frac{\Omega_i\Omega}{k_y^2}) = 0$$

where the value of $\frac{\Omega_i\Omega}{k_y^2} << 1$ therefore it can be ignored, hence

$$[m\Omega\omega_{ce}^4 + m\Omega{\omega_{ce}}^2(\frac{dV_y}{dx})^2 - 2m\Omega{\omega_{ce}}^2\frac{dV_y}{dx} + \Omega_e\frac{d^2}{dx^2}(\omega_{ce}\frac{dV_y}{dx} - \omega_{ce}^2) + 2\Omega_e^2 k_y\frac{d}{dx}\frac{dV_y}{dx}][k_y^2]$$
$$+\Omega_e k_y^4\omega_{ce}^2 + \Omega_e\omega_{ce}k_y^4\frac{dV_y}{dx} = k_y^2 X_e.$$

By putting the values of all the coefficients in eq. (4.39), we get

$$-X_e\frac{d^2\phi}{dx^2} + k_y^2 X_e\phi - \frac{2k_y X_e}{\Omega_i}\frac{dV_y}{dx} = 0. \tag{4.40}$$

By decoupling eq. (4.40) we get

$$\frac{d^2\phi}{dx^2} - k_y^2\phi + \frac{2k_y}{\Omega_i}\frac{dV_y}{dx} = 0. \tag{4.41}$$

The second order differential equation obtained in 4.41 is similar to the differential equation obtained when both electrons and ions are un-magnetized.

4.2 Plasma frequency is less than the ion-cyclotron frequency $\omega << \omega_{ci} << \omega_{ce}$

$$n_i = \frac{(k_y^2 + k_z^2)\Omega_i\phi - \frac{\omega_{ci}k_z^2\phi}{\Omega_i}(\omega_{ci} + \frac{dV_y}{dx}) + k_y k_z\omega_{ci}\phi\frac{dV_y}{dx} - \Omega_i\frac{d^2\phi}{dx^2}}{\Omega C}$$
$$\frac{-2\Omega_i[\Omega_i\frac{d\phi}{dx} - \omega_{ci}k_y\phi](k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}]}{\Omega C^2}$$

where $C = [\Omega_i^2 + \omega_{ci}(\frac{\nu_{in}\nu_{ni}}{\Omega_n^2}\frac{dV_y}{dx} - \omega_{ci} - \frac{dV_y}{dx})]$

hence above equation becomes

$$C = [\Omega_i^2 - \omega_{ci}(\omega_{ci} + \frac{dV_y}{dx})]$$

as $\Omega_i << \omega_{ci}$ [24],hence

$$C = \omega_{ci}(\omega_{ci} + \frac{dV_y}{dx})$$

$$n_i = \frac{(k_y^2 + k_z^2)\Omega_i\phi - \frac{\omega_{ci}k_z^2\phi}{\Omega_i}(\omega_{ci} + \frac{dV_y}{dx}) + k_yk_z\omega_{ci}\phi\frac{dV_y}{dx} - \Omega_i\frac{d^2\phi}{dx^2}}{\Omega\omega_{ci}(\omega_{ci} + \frac{dV_y}{dx})}$$
$$\frac{-2\Omega_i[\Omega_i\frac{d\phi}{dx} - \omega_{ci}k_y\phi](k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}]}{\Omega\omega_{ci}^2(\omega_{ci} + \frac{dV_y}{dx})^2}$$

let

$$X_i = \frac{(k_y^2 + k_z^2)\Omega_i - \frac{\omega_{ci}k_z^2}{\Omega_i}(\omega_{ci} + \frac{dV_y}{dx}) + k_yk_z\omega_{ci}\frac{dV_y}{dx} - \Omega_i\frac{d^2}{dx^2}}{\Omega\omega_{ci}(\omega_{ci} + \frac{dV_y}{dx})}$$
$$\frac{-2\Omega_i[\Omega_i\frac{d}{dx} - \omega_{ci}k_y](k_y\frac{dV_y}{dx} + k_z\frac{dV_z}{dx}]}{\Omega\omega_{ci}^2(\omega_{ci} + \frac{dV_y}{dx})^2}$$

hence

$$n_i = X_i\phi \tag{4.42}$$

put value of n_e and n_i in equation 4.37

$$\frac{d^2\phi}{dx^2} - (k_z^2 + k_y^2)\phi = \frac{L_e\phi}{X_e} - X_i\phi$$

$$X_e(\frac{d^2}{dx^2} - (k_z^2 + k_y^2))\phi = L_e\phi - X_eX_i\phi$$

$$X_e(\frac{d^2}{dx^2} - (k_z^2 + k_y^2) - X_i)\phi = L_e\phi \tag{4.43}$$

We will assume the particles have a sheared drift velocity along the Z direction, which is parallel to the external magnetic field, hence letting $\frac{dV_y}{dx} = 0$

hence

$$X_i = \frac{(k_y^2 + k_z^2)\Omega_i - \frac{\omega_{ci}^2k_z^2}{\Omega_i} - \Omega_i\frac{d^2}{dx^2}}{\Omega\omega_{ci}^2} - \frac{2\Omega_i[\Omega_i\frac{d}{dx} - \omega_{ci}k_y](k_z\frac{dV_z}{dx}]}{\Omega\omega_{ci}^4}$$

$$X_i = \frac{(k_y^2 + k_z^2)\omega_{ci}^2\Omega_i - \frac{\omega_{ci}^4k_z^2}{\Omega_i} - \Omega_i\omega_{ci}^2\frac{d^2}{dx^2} - 2\Omega_i^2k_z\frac{dV_z}{dx}\frac{d}{dx} + 2\Omega_i\omega_{ci}k_yk_z\frac{dV_z}{dx}}{\Omega\omega_{ci}^4}$$

$$X_e = m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^3k_zk_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^4}{\Omega_e} + 2\Omega_e\omega_{ce}k_yk_z\frac{dV_z}{dx}$$

$$L_e = -\omega_{ce}^2\Omega_e\frac{d^2}{dx^2} + \omega_{ce}^2\Omega_e(k_y^2 + k_z^2) - \frac{\omega_{ce}^3k_zk_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^4}{\Omega_e} + 2\Omega_e^2k_z\frac{dV_z}{dx}\frac{d}{dx} + 2\Omega_e\omega_{ce}k_zk_y\frac{dV_z}{dx}$$

from LHS of equation 4.39

$$\frac{d^2}{dx^2} - (k_z^2 + k_y^2) - X_i = \frac{1}{\Omega\omega_{ci}^4}(\Omega\omega_{ci}^4\frac{d^2}{dx^2} - \Omega\omega_{ci}^4(k_z^2 + k_y^2) - (k_y^2 + k_z^2)\omega_{ci}^2\Omega_i - \frac{\omega_{ci}^4 k_z^2}{\Omega_i}$$
$$-\Omega_i\omega_{ci}^2\frac{d^2}{dx^2} - 2\Omega_i^2 k_z\frac{dV_z}{dx}\frac{d}{dx} + 2\Omega_i\omega_{ci}k_yk_z\frac{dV_z}{dx})$$

from equation 4.39

$$X_e(\Omega\omega_{ci}^4\frac{d^2}{dx^2} - \Omega\omega_{ci}^4(k_z^2 + k_y^2) - (k_y^2 + k_z^2)\omega_{ci}^2\Omega_i - \frac{\omega_{ci}^4 k_z^2}{\Omega_i} - \Omega_i\omega_{ci}^2\frac{d^2}{dx^2} - 2\Omega_i^2 k_z\frac{dV_z}{dx}\frac{d}{dx}$$
$$+ 2\Omega_i\omega_{ci}k_yk_z\frac{dV_z}{dx})\phi = \Omega\omega_{ci}^4 L_e\phi \quad (4.44)$$

$$X_e = m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^3 k_z k_y}{\Omega_e}\frac{dV_z}{dx}(1 - \frac{2\Omega_e^2}{\omega_{ce}^2}) - \frac{k_z^2\omega_{ce}^4}{\Omega_e}$$

as $\omega_{ce} > \Omega_e$

$$X_e = m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^3 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^4}{\Omega_e}$$

from of equation 4.40

$$(m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^3 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^4}{\Omega_e})(\Omega\omega_{ci}^4\frac{d^2}{dx^2} - \Omega\omega_{ci}^4(k_z^2 + k_y^2) - (k_y^2 + k_z^2)\omega_{ci}^2\Omega_i$$
$$- \frac{\omega_{ci}^4 k_z^2}{\Omega_i} - \Omega_i\omega_{ci}^2\frac{d^2}{dx^2} - 2\Omega_i^2 k_z\frac{dV_z}{dx}\frac{d}{dx} + 2\Omega_i\omega_{ci}k_yk_z\frac{dV_z}{dx})\phi =$$
$$\Omega\omega_{ci}^4(-\omega_{ce}^2\Omega_e\frac{d^2}{dx^2} + \omega_{ce}^2\Omega_e(k_y^2 + k_z^2) - \frac{\omega_{ce}^3 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^4}{\Omega_e} + 2\Omega_e^2 k_z\frac{dV_z}{dx}\frac{d}{dx} + 2\Omega_e\omega_{ce}k_zk_y\frac{dV_z}{dx})\phi$$
$$(4.45)$$

from coefficient of $\frac{d^2\phi}{dx^2}$

$$(m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^3 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^4}{\Omega_e})(\Omega\omega_{ci}^4 - \Omega_i\omega_{ci}^2) = -\Omega\omega_{ci}^4\omega_{ce}^2\Omega_e$$

$$\omega_{ci}^4[m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^3 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^4}{\Omega_e}][\Omega - \frac{\Omega_i}{\omega_{ci}^2}] = -\Omega\omega_{ce}^2\Omega_e\omega_{ci}^4$$

as $\omega_{ci} >> \Omega_i$ hence

$$\Omega\omega_{ci}^4[m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^3 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^4}{\Omega_e}] + \Omega\omega_{ce}^2\Omega_e\omega_{ci}^4 = I$$

similarly comparing coefficients of $\frac{d\phi}{dx}$

$$(m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^4 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^3}{\Omega_e})(-2\Omega_i^2 k_z\frac{dV_z}{dx}) = 2\omega_{ci}^4\Omega_e^2 k_z\frac{dV_z}{dx}$$

$$(m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^4 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^3}{\Omega_e})(-2\Omega_i^2 k_z\frac{dV_z}{dx}) - 2\omega_{ci}^4\Omega_e^2 k_z\frac{dV_z}{dx} = H$$

from coefficient of ϕ

$$(m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^4 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^3}{\Omega_e})(-\Omega\omega_{ci}^4(k_z^2 + k_y^2) - (k_y^2 + k_z^2)\omega_{ci}^2\Omega_i - \frac{\omega_{ci}^4 k_z^2}{\Omega_i}$$
$$+ 2\Omega_i\omega_{ci}k_y k_z\frac{dV_z}{dx}) =$$
$$\Omega\omega_{ci}^4(\omega_{ce}^2\Omega_e(k_y^2 + k_z^2) - \frac{\omega_{ce}^3 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^4}{\Omega_e} + 2\Omega_e\omega_{ce}k_z k_y\frac{dV_z}{dx}) \quad (4.46)$$

$$(m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^4 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^3}{\Omega_e})(-\Omega\omega_{ci}^4(k_z^2 + k_y^2) - (k_y^2 + k_z^2)\omega_{ci}^2\Omega_i - \frac{\omega_{ci}^4 k_z^2}{\Omega_i}$$
$$+2\Omega_i\omega_{ci}k_y k_z\frac{dV_z}{dx}) = \Omega\omega_{ci}^4(\omega_{ce}^2\Omega_e(k_y^2 + k_z^2) - \frac{\omega_{ce}^3 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^4}{\Omega_e} + 2\Omega_e\omega_{ce}k_z k_y\frac{dV_z}{dx})$$

$$(m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^4 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^3}{\Omega_e})(-\Omega\omega_{ci}^4(k_z^2 + k_y^2)(1 + \frac{\Omega_i}{\Omega\omega_{ci}^2}) - \frac{\omega_{ci}^4 k_z^2}{\Omega_i}$$
$$+2\Omega_i\omega_{ci}k_y k_z\frac{dV_z}{dx}) = \Omega\omega_{ci}^4(\omega_{ce}^2\Omega_e(k_y^2 + k_z^2) - \frac{\omega_{ce}^3 k_z k_y}{\Omega_e}\frac{dV_z}{dx}(1 - \frac{2\Omega_e^2}{\omega_{ce}^2}) + 2\Omega_e\omega_{ce}k_z k_y\frac{dV_z}{dx})$$

$$[m\Omega\omega_{ce}^4 + \Omega_e\omega_{ce}^2(k_y^2 + k_z^2) - \frac{\omega_{ce}^4 k_z k_y}{\Omega_e}\frac{dV_z}{dx} - \frac{k_z^2\omega_{ce}^3}{\Omega_e}][-\Omega\omega_{ci}^4(k_z^2 + k_y^2)(1 + \frac{\Omega_i}{\Omega\omega_{ci}^2}) - \frac{\omega_{ci}^4 k_z^2}{\Omega_i}$$
$$+2\Omega_i\omega_{ci}k_y k_z\frac{dV_z}{dx}] - \Omega\omega_{ci}^4[\omega_{ce}^2\Omega_e(k_y^2 + k_z^2) - \frac{\omega_{ce}^3 k_z k_y}{\Omega_e}\frac{dV_z}{dx}(1 - \frac{2\Omega_e^2}{\omega_{ce}^2}) + 2\Omega_e\omega_{ce}k_z k_y\frac{dV_z}{dx}] = J$$

hence equation 4.42 becomes

$$I\frac{d^2\phi}{dx^2} + H\frac{d\phi}{dx} + J\phi = 0 \quad (4.47)$$

Chapter 5

Results and Discussion

The general form of the dispersion relation for ion acoustic waves wave instability due to hypersonic sheared flow is of form $\omega = \alpha + i\gamma$. Where α and γ are $Re(\omega)$ and $Im(\omega)$ respectively. The real part of frequency corresponds to the propagation of wave, whereas imaginary part of frequency decides the existence or non-existence of the wave in plasma i.e. either the wave is supported by plasma, it grows and can propagate, or the plasma absorbs and damps the wave. This is decided by the sign of the imaginary of frequency, the positive imaginary part of frequency leads into growth of the wave amplitude with time whereas negative sign of the imaginary part leads to damping of the wave.[6]

In order to correctly evaluate the growth rate of instability and the role of neutral particles in suppressing such instabilities, a second order differential equation for electrostatic potential of the ion acoustic waves is derived and solved analytically for linear velocity profile of the sheared flow, using appropriate boundary conditions for finite thickness of plasma sheath. An appropriate scaling relationship for the instability in case of linear velocity profile is obtained through the analytical calculations which is of the form

$$\kappa = \frac{\omega + i\nu_{in}}{c_o} = \frac{-b + i\sqrt{4ac - b^2}}{2a}. \tag{5.1}$$

In eq. (5.1)

$$a = e^{kL},$$
$$b = 0.5e^{kL} - 0.5e^{kL} - kLe^{kL},$$
$$c = 0.5e^{-kL} - 0.5e^{kL} + kLe^{kL}.$$

Where $k = \tilde{k}_y \tilde{\lambda}_{De}$ is a dimensionless number for dimensional wave number $\tilde{k}_y$, $\tilde{\lambda}_{De} = \sqrt{\frac{K_B \tilde{T}_e}{4\pi \tilde{N}_{0_e} e^2}}$ is the electron Debye length, L is the length of sheath normalized to electron Debye length, ω and ν_{in} are the normalized wave frequency and collision frequency with respect to ion plasma frequency i.e. $\tilde{\omega}_{pi} = \sqrt{\frac{4\pi Z^2 e^2 \tilde{N}_{0_{i}max}}{\tilde{M}_i}}$.

The nominal plasma parameters which we will use are $N_{0n} = 10^{10} cm^{-3}$, $N_{0i} = N_{0e} = 10^8 cm^{-3}$, $T_e = 0.5eV$ and $T_i = T_n = 0.1eV$.

In the absence of collision eq. (5.1) can be written as

$$\omega = \frac{-b + i\sqrt{4ac - b^2}}{2a},$$

where

$$\frac{Re(\omega)}{c_0} = \frac{-b}{2a},$$
$$\frac{Im(\omega)}{c_0} = \frac{\sqrt{4ac - b^2}}{2a}.$$

If we plot $\frac{Re(\omega)}{c_0}$ and $\frac{Im(\omega)}{c_0}$ against kL we will get the following results,

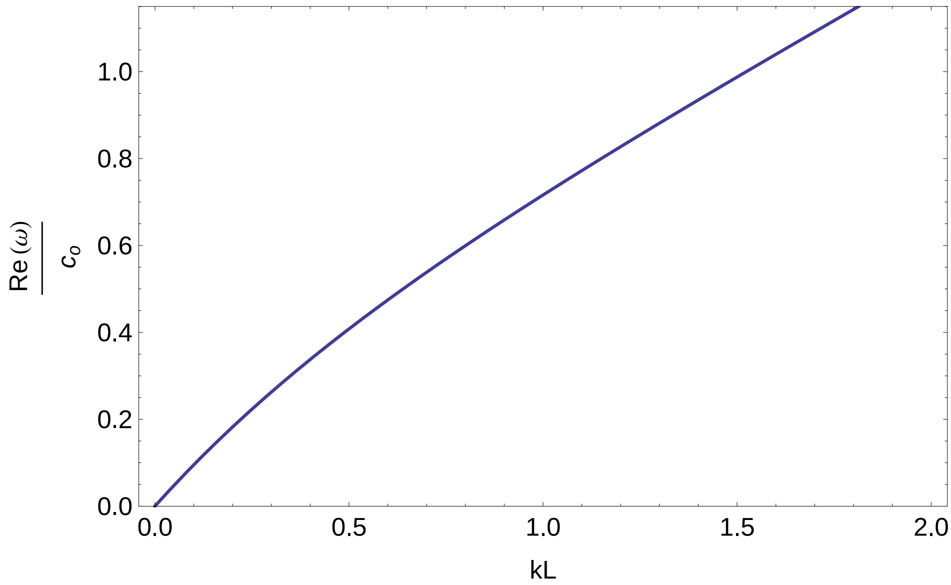

Figure 5.1: Real part of frequency $\frac{Re(\omega)}{c_0}$ as a function kL

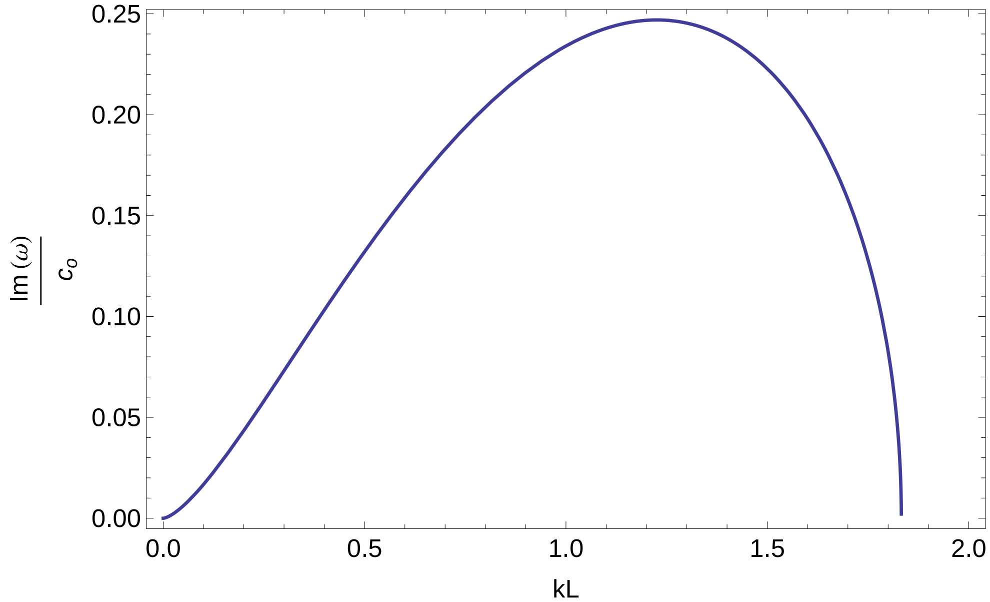

Figure 5.2: The growth rate of frequency $\frac{Im(\omega)}{c_0}$ as a function of kL

Through numerical evaluation of fig. 5.1 and fig. 5.2, the values of maximum growth rate (γ_{max}), the max real frequency ($\omega_{r,max}$) and wavenumber (k_{max}) at peak growth rate, and the wavenumber k_c at cutoff wavelength are obtained, where

$$\gamma_{max} = Max[Im(\omega)] \simeq 0.247c_0,$$

$$\omega_{r,max} \simeq 0.842c_0,$$

$$k_{max} \simeq \frac{1.23}{L},$$

and

$$k_c \simeq \frac{1.83}{L}.$$

We have considered linear velocity profile $V_0(x) = c_0 x$, which is normalized with respect to ion sound speed $C_s = \frac{\tilde{\lambda}_{De}\tilde{\omega}_{pi}}{\sqrt{Z}} = \sqrt{\frac{K_B\tilde{T}_e}{\tilde{M}_i}}$. Considering the value of $L >> 1$ and value normalized of peak sheath velocity at the out edge to be $c_0 L \leq 1$. These scaling relationships are made in order to satisfy our low frequency assumptions.

We will consider two cases, first in which the sheath edge velocity 0.5 times the ion sound speed i.e. $c_0 = \frac{0.5}{L}$ and another case in which the edge velocity is equal to ion sound speed i.e $c_0 = \frac{1}{L}$. For both the cases plots are obtained for the real frequency $Re(\omega)$and growth rate $Im(\omega)$ against normalized wavenumber k. In all cases considered we will take value of $L = 200$.

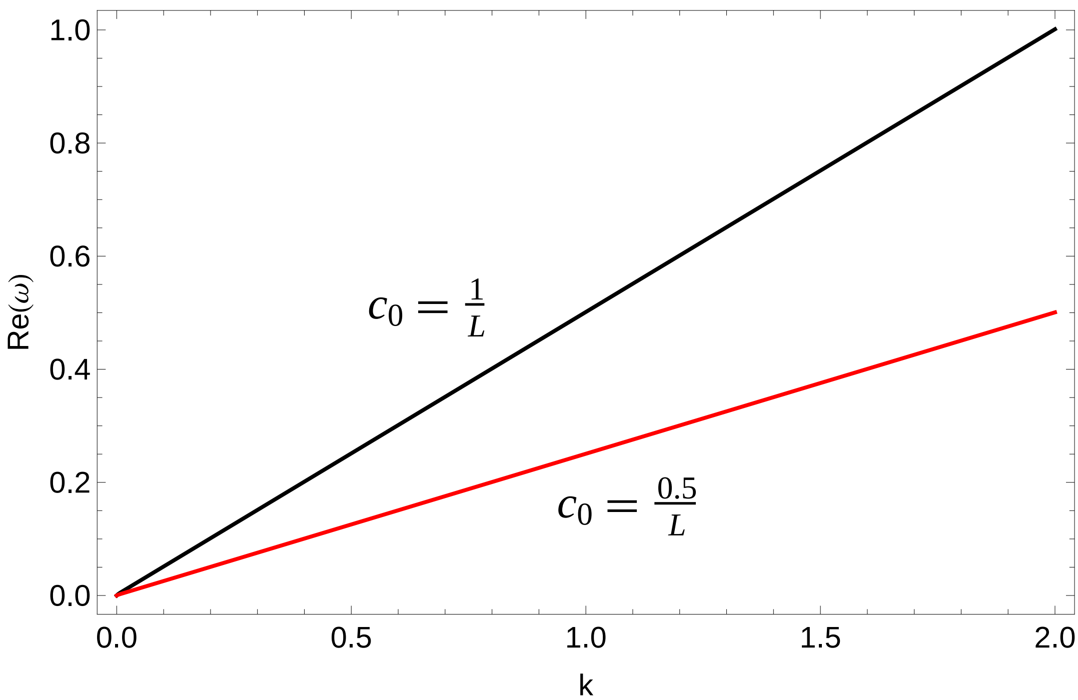

Figure 5.3: Plot of normalized real frequency $Re(\omega)$ as a function of normalized wavenumber k for linear velocity shear with $c_0 = \frac{0.5}{L}$ and $c_0 = \frac{1}{L}$ in the collisionless limit.

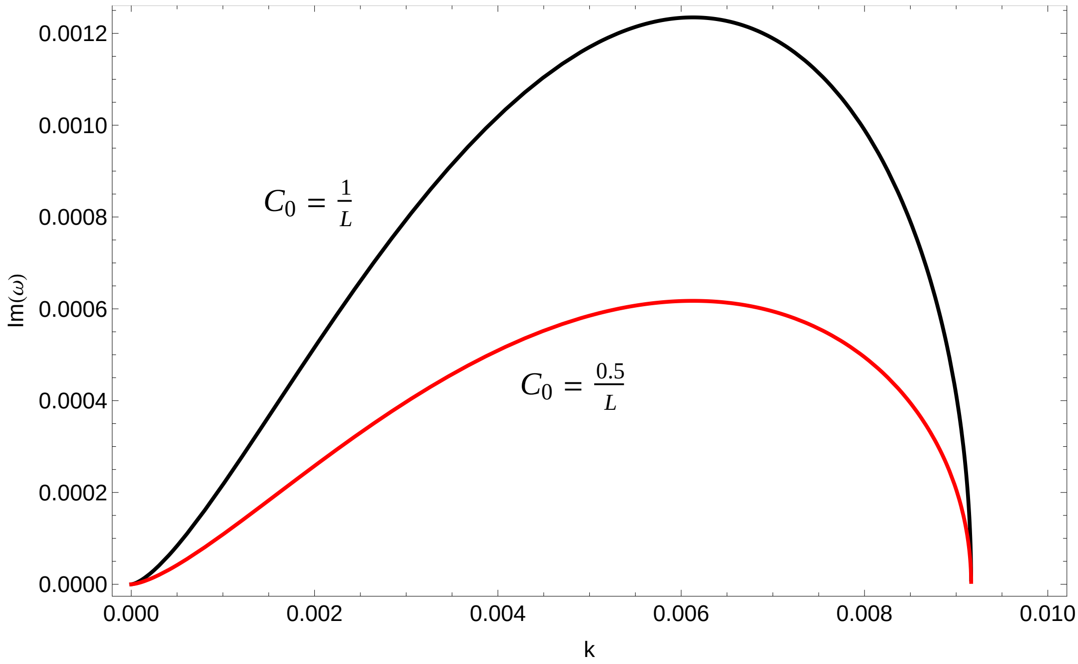

Figure 5.4: Plot of normalized growth rate $Im(\omega)$ as a function of normalized wavenumber k for linear velocity shear with $c_0 = \frac{0.5}{L}$ and $c_0 = \frac{1}{L}$ in the collisionless limit.

From numerical evaluation of fig. 5.4, it was observed that by decreasing the peak sheath velocity at the outer edge by half, a decrease in the maximum growth of instability is observed. The value peak growth rate for $c_0 = \frac{1}{L}$ is $\gamma_{max} \simeq 0.0012348$ and that for $c_0 = \frac{0.5}{L}$ is $\gamma_{max} \simeq 0.0006174$. The value of wavenumber (k_{max}) at peak growth rate, and the wavenumber k_c at cutoff wavelength are, $k_{max} \simeq 0.00615$ and $k_c \simeq 0.00915$ respectively.

Till now the calculations we have done are for collionless limit now if we take collision between ion and neutral into account well get the following dispersion relation

$$\omega = (\frac{-b + i\sqrt{4ac - b^2}}{2a})c_o - i\nu_{in}. \tag{5.2}$$

In eq. (5.2), ν_{in} is the normalized collision frequency which is equal to

$$\nu_{in} = \frac{\tilde{\nu}_{in}}{\tilde{\omega}_{pi}} = \frac{N_{0n} V_{Ti} \sigma_{in}}{\tilde{\omega}_{pi}},$$

where $V_{Ti} = \sqrt{\frac{K_B T_i}{M_i}}$ is the thermal velocity of electron and σ_{in} is the cross section of ions colliding with neutrals having value $\sigma_{in} \simeq 6 \times 10^{-14} cm^2$ (approximating potassium ions,liberated from the outer coating of the vehicle, interacting with nitrogen molecule). By putting values in equation we get $\nu_{in} = 1.4097 \times 10^{-15} N_{0n}$.

We have observed the effect of ion neutral collision on growth rate of instability, by plotting a graph of growth rate as a function of k for different values of N_{0n}. The value of c_0 is taken to be $\frac{1}{L}$, hence

$$Im(\omega) = \frac{\sqrt{4ac - b^2}}{2aL} - \nu_{in}.$$

If we plot the growth rate $Im(\omega)$ of the wave against normalized wave number k for different neutral densities we get the following figures.

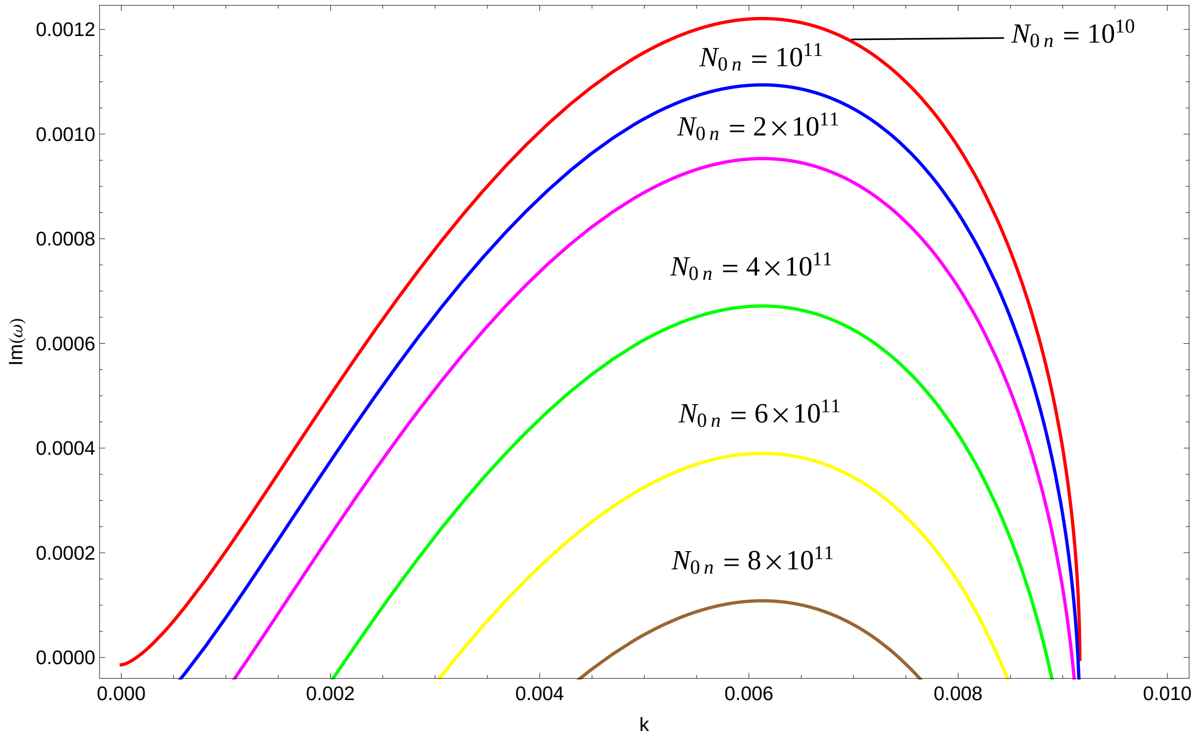

Figure 5.5: Plot of growth rate $Im(\omega)$ as a function of k for different values of the neutral density N_{0n}

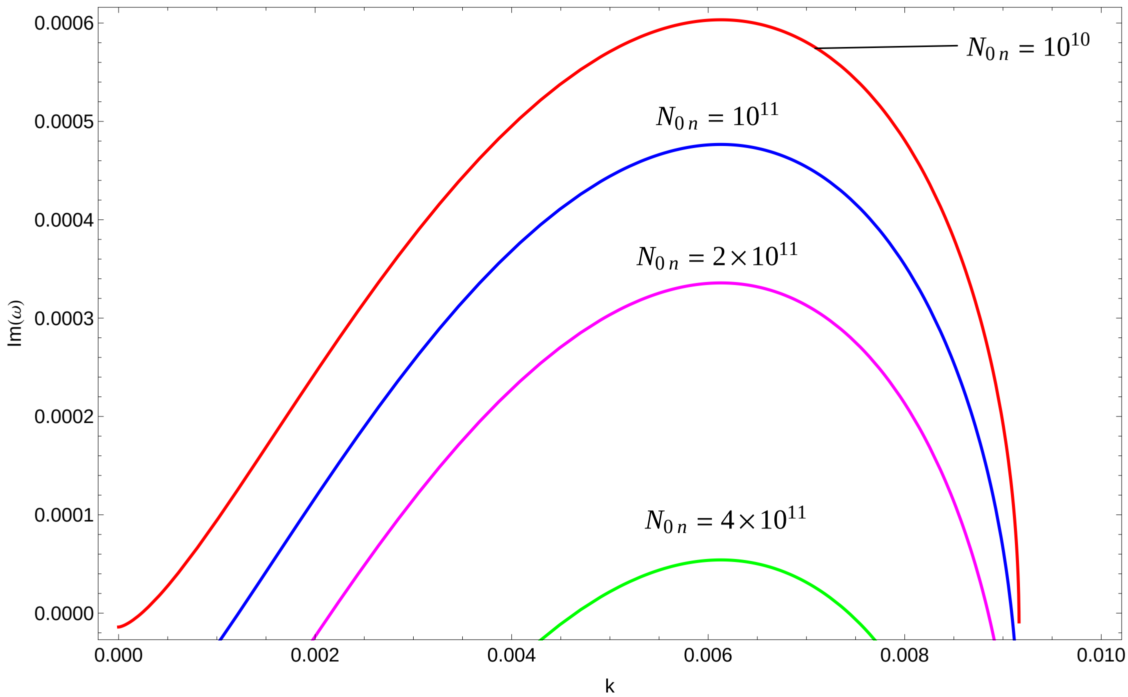

Figure 5.6: Plot of growth rate $Im(\omega)$ as a function of k for different values of the neutral density N_{0n}, when $c_0 = \frac{0.5}{L}$

The fig. 5.5 and fig. 5.6 are plots of growth rate of instability for linear velocity shear as a function of normalized wavenumber k for different values of neutral number density N_{0n}. The values of neural density considered are between the range of $N_{0n} = 10^{10}cm^{-3}$ to $N_{0n} = 10^{12}cm^{-3}$, where substantial damping of ion-acoustic wave instability occurs. It is observed that by increasing the neutral density within the plasma sheath a decrease in the peak growth

rate of instability takes place to a point where it damps out, this occurs due to an increase in the ion-neutral collision frequency.
For $c_0 = \frac{1}{L}$ in collisional limit the peak growth rate was $Im(\omega_{max}) = 1.22 \times 10^{-3}$ a complete damping of wave takes place when neutral density is $N_{0n} = 8.76 \times 10^{11}$ or $\nu_{in} = 1.23 \times 10^{-3}$, which is 1% greater than the peak growth rate. For $c_0 = \frac{0.5}{L}$ in collisional limit the peak growth rate was $Im(\omega_{max}) = 6.03 \times 10^{-4}$, a complete damping of wave takes place when neutral density is $N_{0n} = 4.38 \times 10^{11}$ or $\nu_{in} = 6.17 \times 10^{-4}$, which is 3% greater than the peak growth rate.

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