Bose-Einstein Condensation of Composite Bosons: Quantum Information Approach

Tooba Bibi
Regn.00000173008

A thesis submitted in partial fulfillment of the requirements for the degree of **Master of Science**
in
Physics

Supervised by: Dr. Shahid Iqbal

Department of Physics
School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan

Year 2019

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: TOOBA BIBI, Regn No. 00000173008 Titled: BOSE – ENSTIEN CONDENSATION OF COMPOSITE BOSONS: QUANTUM INFORMATION APPROACH be accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: DR. MUHAMMAD ALI PARACHA Signature:

2. Name: DR. NAILA AMIR Signature:

External Examiner: DR. RAMEEZ UL ISLAM Signature:

Supervisor's Name DR. SHAHID IQBAL Signature:

Head of Department

25/01/2019.
Date

COUNTERSINGED

Date: 25/01/19

Dean/Principal

THESIS ACCEPTANCE CERTIFICATE

Certified that final copy of MS thesis written by **Ms. Tooba Bibi**, Registration No. **00000173008**), of **School of Natural Sciences** has been vetted by undersigned, found complete in all respects as per NUST statutes/regulations, is free of plagiarism, errors, and mistakes and is accepted as partial fulfillment for award of MS/M.Phil degree. It is further certified that necessary amendments as pointed out by GEC members and external examiner of the scholar have also been incorporated in the said thesis.

Signature: ______________________

Name of Supervisor: Dr. Shahid Iqbal

Date: 25/01/19

Signature (HoD): ______________________

Date: 25/01/19

Signature (Dean/Principal): ______________________

Date: 25/01/19

Dedicated to

My mother
for her love, encouragement and support.

Acknowledgements

My limitless thanks to Allah Almighty who blessed me to complete this research work.

I am grateful to my supervisor Dr. Shahid Iqbal for his priceless guidance, encouragement and thoughtful kindness. He has always expressed his support and concern about my research work. Without his guidance, this work would have never become truth.

My deepest thanks go to respectable GEC members, Dr. Naila Amir and Dr. Ali Paracha for their guidance in the improvement of my research work. They were always there for me as kind supporters and helping me out at every step to the completion of my work.

I would like to take this opportunity to say warm thanks to my mother, my sisters, my brothers and my dear friends who were always there to inspire and motivate me with their love and care during all my stay at NUST.

Tooba Bibi

Abstract

When a bosonic gas is cooled to a temperature very close to absolute zero, large number of bosons suddenly condense into the lowest energy state. This phenomenon is known as Bose-Einstein condensation (BEC). In this thesis, we investigate the phenomenon of BEC for identical composite bosons (cobosons) composed of either two distinguishable fermions or bosons. We characterize the BEC by counting the effective number $\langle n \rangle$ of cobosons in ground state. In this regard, first we study the internal structure of coboson and investigate the bosonic quality of coboson. We show that the bosonic behavior of coboson is correlated to the degree of entanglement between its constituent particles. More the constituents of coboson are entangled, the better coboson behave like a pure boson. Furthermore, we derive boundaries on a quantity that governs the bosonic character of a coboson. These boundaries depend on the purity of the single-particle density matrix and manifest that if the entanglement is sufficiently strong, the quantity approaches its ideal bosonic value. We see that the effective number of cobosons is also depending on the degree of entanglement. Thus, first we analyze the phenomenon of BEC for cobosons, made of two fermions, by means of entanglement between the constituents. We found that for the pair of fermions, the effective number of cobosons in lowest energy state increases with an increase in the degree of entanglement between constituent fermions. Secondly we have also discussed cobosons comprised of a pair of bosons briefly. We see that pair of bosons in BEC behaves oppositely, that is, with the increase in entanglement between constituent bosons the effective number of ground state cobosons decreases. At maximum entanglement, the effective number

of composite bosons is equal to the total number of cobosons in the system which constitute the BEC as expected in the case of ideal bosons.

Contents

Chapter 1

Introduction

All particles that exist in nature can be classified into two categories. This classification is based on their spin difference, particles with half-integral spin are known as fermions and particles with integral spin are bosons. Fermions repel each other, while bosons prefer to stay in the same state. For this reason, when a gas consist of bosonic particles cooled to extremely low temperature nearly absolute zero, all bosons condense into the lowest energy state, a Bose-Einstein condensate (BEC). In Bose-Einstein condensation, the phase transition in gas do not require interaction between particles, this is totally a statistical phenomena.

The idea of BEC was first given by Bose in 1924 [1]. He published an article on quantum statistics of light quanta (now we call them photons). Later Einstein modified it for material particles. In these articles, it was suggested that if we have diluted bosonic gas that is trapped in some potential, at very low temperature a large number of bosons will fall into the lowest energy quantum state and the quantum effects become visible on a macroscopic level. First gaseous condensate were produced by Eric Cornell and Carl Wieman at University of Colorado Boulder, on June 5, 1995 [2-4]. They cooled rubidium atoms to 170 nanoKelvin (nK). That was a significant achievement in physics, scientists were amazed to see that how fundamental laws of quantum mechanics could be proven to be correct experimentally. Some other great achievements were the observation of giant matter-wave interference [5], and Mott insulator transition [6].

BCS Theory and Bi-fermionic Condensation

Bose-Einstein condensation is not specified for bosons only. A pair of fermions can also form BEC when they act as a boson. In 1957, three scientists Bardeen, Cooper and Schrieffer proposed a theory known as BCS theory [7]. According to this theory, effect of superconductivity at a microscopic level is due to the condensation of electron's pair which behave like bosons. BCS theory says that electrons become inter-correlated to each other and below critical temperature they form Cooper pair and thus behave like a BEC. In 1972, it was also shown that helium-3 fermion reveals super-fluidity below 0.0027 K [8]. BCS theory gave a hint that strongly correlated fermions can act as bosons and they condense to form the state of BEC under the right experimental condition.

With the help of the concepts of quantum information, it was proved in 2005 by C.K. Law that the inter-correlation of fermions can be explained by the entanglement between fermions [9].

In this thesis, we have studied the Bose-Einstein condensation of composite bosons. We have seen that how much BEC of cobosons differ from ideal BEC and what are the factors on which behavior of cobosons depends. It has also been observed that the correlation between constituent particles of the composite boson is playing fundamental role. Therefore, before discussing in detail about BEC of cobosons, one should know the basics about bosons and the difference between elementary particles and composite particles. Also understanding the concept of entanglement is mandatory. The following sections have detailed discussion about these concepts.

1.1 Elementry Particles

In every day life, most of the particles that we deal with are composite in nature. The term composite means that particle is made up of two or more than two sub-particles. In the Periodic Table its member elements, for example, are not strictly elemental. These are actually lists of atoms, and each atom is composed of further smaller and even more basic particles. Thus the atoms are composite in nature. As molecules are made up of atoms, so molecules are composites of a composite. In our daily life, every object we deal with is in turn combination of molecules, and so each object is composite in nature too. Thus understanding the properties of composite systems is an important aspect. It is the basic inspiration behind the interest to study composite particles. A question that may then be raised is what are the effects this composite nature of particles adds to the physics of the system. In this thesis, we follow the following ideas and discuss their connection with each other. That is when we introduce constituent particles of a system, we must have to introduce quantum correlations between them. And understanding of these non-classical correlations is mandatory to understand composite particles. As in this thesis mainly we will deal with pure states, for which we need to consider only one type of correlation: Entanglement. Before introducing appropriately the topic of composite particles and correlations, it is beneficial to discuss about the basic objects that these particles are made of, that are bosons and fermions.

1.2 Bosons and Fermions

1.2.1 Symmetrical and Anti-symmetrical States

Consider the simple example of two identical particles a and b. We consider that for a single particle, there exist a complete set of orthogonal basis. Let us denote these states using Dirac notation $|n\rangle$ where n is quantum number such that $n = 1, 2, 3, ...$ The quantum state of a system (that is made up of two particles in this example) is some superposition of the state $|n\rangle_a \otimes |m\rangle_b$. In case of two-particle system, we drop the subscript for simplicity and the order of quantum mode will show the label of particle, $|n\rangle_a \otimes |m\rangle_b \equiv |n\rangle \otimes |m\rangle \equiv |n, m\rangle$. For the system of two identical particles, as the particles are indistinguishable, every possible superposition of $|n, m\rangle$ is not allowed. For example, if we consider state $|n, m\rangle$ where $n \neq m$, after a measurement if we get the resultant state $|n\rangle$, then it is for sure related to particle a and if the resultant state is $|m\rangle$ then it is related to particle b, therefore it allows both particles to be distinguished from one another. Thus measurement will differentiate between two particles for such type of states. If the two particles get exchanged with each other then we get $|\langle n,m|m,n\rangle|^2 = 0$. But it is not possible as it contradicts the concept of indistinguishability. In other words, reasoning forces us to arrive at a result that all possible superposition of $|n,m\rangle$ for indistinguishable particles are not allowed. Following are the states that do not contradict the properties of indistinguishable particles:

$$|n,n\rangle, \tag{1.1}$$

$$\frac{1}{\sqrt{2}}(|n,m\rangle + |m,n\rangle), \tag{1.2}$$

$$\frac{1}{\sqrt{2}}(|n,m\rangle - |m,n\rangle). \tag{1.3}$$

where $m \neq n$. In Eq. (1.1) and Eq. (1.2), we can see that if we exchange quantum numbers of particles with each other, the state will remain invariant. While in case of Eq. (1.3), if we replace n with m, there will be a change and resulting state is anti-symmetric in nature. It is important to note here that overall, physical state is preserved when particles get exchange. For example, for the state $(|n,m\rangle + e^{\iota\theta}|m,n\rangle)\frac{1}{\sqrt{2}}$ where $0 \leq \theta \leq \pi$, by exchanging particle we obtain $(|n,m\rangle + e^{-\iota\theta}|m,n\rangle)\frac{1}{\sqrt{2}}$. For both states, before and after swapping the particle, inner product is not equal to one except when $\theta = 0$ or π. When $\theta = 0$, the state is symmetric and for $\theta = \pi$ we get an anti-symmetric state. The classification of elementary particles is based on these symmetric and anti-symmetric states

1.2.2 Quantum Statistics followed by Indistinguishable Particles

Superposition principle has an important place in quantum mechanics. It states that two quantum states can be superposed to form a new state that is also valid in the system we are observing. An important question arises that " Is it possible to describe some system (here we are considering a system of two indistinguishable particles) with both symmetric and antisymmetric states?". This means that we need to check that both states $|\psi\rangle = (|n,m\rangle + |m,n\rangle)\frac{1}{\sqrt{2}}$ and $|\phi\rangle = (|n,m\rangle - |m,n\rangle)\frac{1}{\sqrt{2}}$ are valid for the system or not. The superposition of both states is given as:

$$\frac{(|\psi\rangle + |\phi\rangle)}{\sqrt{2}} = |n,m\rangle.$$

We already know that the state $|n,m\rangle$ is not valid for identical particles. Thus, to preserve the superposition principle we conclude that there is no system for which both states are valid simultaneously. This means that identical particles are either symmetric or anti-symmetric in nature and can be described by only one type of state. Conventionally, we name the particles having a symmetric state as **bosons** and particles with anti-symmetric state are known as **fermions**. Bosons are integral spin particles and comply with Bose-Einstein statistics while fermions are half-integral spin particles with Fermi-Dirac statistics. Now for fermions, antisymmetric property also leads us to Pauli exclusion principle. For example, in Eq. (1.3), when $n = m$, we have

$$\frac{(|n,n\rangle - |n,n\rangle)}{\sqrt{2}} = \mathbf{0}. \tag{1.4}$$

Presence of null vector on the right hand side of above equation indicates non-existence of two fermions in a single state. This is known as Pauli Exclusion Principle. For more details about symmetrical and anti-symmetrical states for the system made up of more than two particles, one can refer to books on quantum mechanics, for example [10].

1.3 Composite Particle

A composite particle can be defined as a particle that is composed of two or more than two elementary particles. The composite particle can be either boson or fermion depending on its constituent particles. For example, proton and neutron both are composite particle. Proton is composed of three fermions (one up quark, two down quarks). It is well known by Spin-Statistic theorem that fermions have half-integral spin while bosons have integral spin. In case of composite particle we need to extend the Pauli exclusion principle. If a particle is composed of two sub-particles having half-integral spin, we get either zero spin or one for a whole particle. In either case, the particle

we get is a boson known as composite boson. Now if we combine an odd number of fermions we get half-integer spin and more precisely a composite fermion, for instance a proton or a neutron.
The study of composite particle is owned by the field of many-body theories [11]. An adequate amount of literature is available related to this subject but unfortunately complexity in a system arises when we increase the number of the particles of the system. There are various ways to deal with this problem but among these bosonization is considered to be the best technique. Bosonization is a process to transform interacting fermions into low dimensional non-interacting bosons. This is a useful method to simplify the problem and have applications in particle physics and condensed matter physics. For the present work, our interest is to explore that how the connection between constituent particles is responsible for several physical properties of the system. Moreover, comparison of the pure elementary particles (specifically composite boson) is part of this thesis.
In order to study the properties of composite boson we need to understand the quantum correlation between constituent particles.

1.4 Non-Classical Correlation

Correlation is a statistic's term. If two systems are correlated that means they are connected in such a way that we can predict a result of the second system if we knew the result of the first one with some uncertainty. If a measurement of one system gives the result A then a measurement on the second system will give the result B with some probability.

For example, if we have a bag filled of pieces of paper, with 00 or 11 printed on each paper having equal probability of occurence. If we randomly pick a piece of paper and only look at one of the two numbers, then automatically we will know the other number entirely. While if we don't look at any number printed at that piece of paper, there is only fifty percent chance that we guessed correctly. That is how knowing part of the system helps to understand the other one. Above example is an example of classical correlation. The only property we were interested was a number. Let us take another example, consider we have two balls, one is blue and other is red. We give these balls to two different observers A and B with closed eyes, let say red to observer A and blue ball to observer B. The observers only know the probability of having either color but are unaware of the exact state (color) of their balls. Now If the observer A looks at his ball and notice that his ball is red, immediately he get to know that the color of the second ball is blue. Concerning the spin of particles consider the products of the spin state of the particles 1 and 2 of the form

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 |\downarrow\rangle_2 + |\downarrow\rangle_1 |\uparrow\rangle_2).$$

When spin of the first particle is up then spin of the second particle will be down and vice-versa. Suppose we want to find the spin of a particle along x-direction, from measurement of the spin of particle 1 along S_x we can also determine the x-component of the spin of a second particle. Thus, the state of a particle is correlated and this is a classically correlated state with the probability of fifty percent outcome in this case. But what if we want to measure the spin along two directions like along x and y direction both. Suppose we have two observers Alice and Bob and the spin of a particle along x-direction is measured by Alice and that is up. Bob measures spin of that particle along y-direction and obtains spin down. Alice can never say that she has managed to measure two complementary properties simultaneously because of Bob's measurement. Imagine her surprise, then, when she tries to confirm her conclusion by measuring spin along y-direction and obtain herself spin up. Thus each of these properties is individually anti-correlated with each other. But the Bell states formed for quantum system have quantum correlation between them. This is a quantum phenomena which we cannot explain classically.

Quantum entanglement is a special kind of correlation in which connection between two systems cannot be explained by local cause.

1.4.1 Entanglement

Quantum entanglement is a physical phenomenon in which we describe quantum state of two or more particles with reference to each other. For entangled particles, quantum state associated with each particle cannot be expressed independently but, we can define the quantum state for a whole system that contains complete information of that system. These particles are connected in such a way that action operated on any one of them influences the whole system and this connection can be explained in terms of entanglement. In terms of Schrödinger's famous paradox, entangled state can be written as:

$|\psi_{atom-cat}\rangle = \frac{1}{\sqrt{2}}[|atom\ not\ decayed\rangle|cat\ alive\rangle + |atom\ decayed\rangle|cat\ dead\rangle]$.

Preparation of Entangled States:

Let us we consider a simple example to become familiar with the production of entangle states. Entangled state is produced when a single photon passes through a calcite crystal. There are two degrees of freedom related to a photon, location of the track that photon follows and its polarization. In spite of the fact that both degrees of freedom are associated with the same photon, we can treat them as two constituents of a composite system. Because of the crystal orientation photon can be polarize either along x-direction or y-direction denoted by x and y, respectively. Let u denote the location of an ordinary path (rays) that photon follows and v denote extraordinary rays. Thus, for a state of a photon, complete basis could be: $x \otimes v$, $x \otimes u$, $y \otimes v$, and $y \otimes u$. For example, if we say that a photon state is $y \otimes u$, the meaning of having this state is, we can guess that photon will pass the test if it is subject to test polarization y, and if that test is located in the ordinary ray u. Besides that photon will not pass a test for polarization x (for any location) or excite a detector located in v (for any polarization). Suppose that before passing through calcite crystal, initial state of the photon is $(ax + by) \otimes w$, where a and b are complex numbers satisfying the condition $| a |^2 + | b |^2 = 1$ and w denotes the location of the path that photon follows before passing through crystal. As our crystal only tests the polarization along x and y axis,

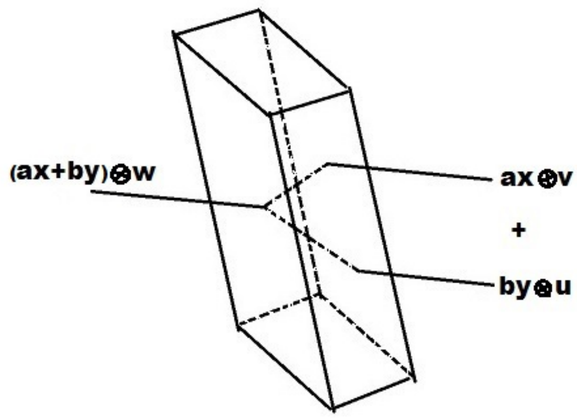

Figure 1.1: Preparation of a photon in an entangled state [10].

therefore we can only predict the state of photon statistically. When the photon pass through a crystal there is probability $\mid a \mid^2$ and $\mid b \mid^2$ of finding the photon in ordinary ray v with polarization x, or in track u with polarization y, respectively. Thus, after passing through the calcite crystal, the state of the photon can be written as

$$\mid \psi\rangle = a(x \otimes v) + b(y \otimes u)$$

which is an entangled state. The sketch of the corresponding process is given in Fig. (1.1).

Historical Background of Entanglement

Interest in entanglement began because of the famous experiment known as Einsten-Podolsky-Rosen (EPR) paradox. In their research paper [12], Einstein *et al* gave an intuitive argument using the theory of quantum mechanics and theory of special relativity. They exploited the unique properties of an "EPR pair" which nowadays is known as entangled states and raised a question on the completeness of quantum mechanics. At that time, Einstein, Podolsky, and Rosen were arguing for an objective reality that was about local-realism, which quantum mechanics with its postulate of uncertainty appear to contradict. John S. Bell was the one who worked on the EPR argument further in 1960s and showed that local-realism based theories are out of scope for the correlations between measurements of entangle state predicted by quantum mechanics [13, 14]. The inequalities derived by Bell and others were tested for entangled photons and these experiments proved the predictions of quantum mechanics [15, 17] Although explanations given by Einstein and his fellows were not satisfactory and their conclusion is now proven to be invalid. However, it drew attention towards most important phenomena of entanglement and raised the possibility that there exist special kind of particles (entangled particles) in quantum mechanics. Even though, entangled states were identified since the beginning of quantum mechanics but recent concept of entanglement has modified and our understanding is very different from what Einstein and his fellows had in mind. Most of the present day entanglement theory is motivated by discoveries in the 1990s that use the strangeness of entanglement in various applications like in quantum teleportation [18], quantum cryptography [19] and quantum dense coding [20]. All these discoveries are experimentally demonstrated, that shows entanglement is completely quantum mechanical phenomena that have no classical replacement.

1.4.2 Entanglement and Composite nature of particles

We relate entanglement and compositeness of a particle with each other and study the properties of a composite particle. Specifically, we study composite boson that is made up of a pair of distinguishable fermions(or bosons). With the help of entanglement, we can find out compositeness of particle and can tell how much the behavior of composite boson is close to pure boson.

1.5 Outline

The organization of this thesis is as under:
Chapter 2 deals with preliminary concepts which provide us the necessary background for the later work. This chapter includes basics of Bose-Einstein condensation, techniques we can use to achieve BEC experimentally. Also, we discuss the factors on

which the phenomena of BEC depends.

In Chapter 3, We built a formalism for a composite boson to understand its internal structure and composite behavior. We discuss the brief description of entanglement in a composite system in terms of Schmidt decomposition and relate the composite behavior of boson to quantum entanglement. We find out that bosonic character of coboson depends on the strength of entanglement of its constituent particles.

Chapter 4 is dedicated to the detailed description of Bose-Einstein condensation of indistinguishable cobosons and its properties. We present a systematic analysis of Bose-Einstein condensation of composite bosons where each coboson is made up of two distinguishable particles. With the help of effective number operator of composite boson, we study, in what way variation in entanglement between constituent particles can effect Bose-Einstein condensation.

We close this discussion by presenting concluding remarks in Chapter 5.

Chapter 2

Bose-Einstein Condensation

The range of temperature we are most familiar with is the room temperature which is 300 to 400K. If we observe the ideal gas of identical particles at this temperature, we see that particles behave classically and follow the Maxwell-Boltzmann distribution. When the gas is cooled down, particles no longer act like billiard balls. The world is entirely strange place at extremely low temperature where our everyday rules do not work. At that level, rules of quantum physics dominates its mysterious laws. One of the important aspects of quantum mechanics is wave-particle duality. Wave nature of particles gets apparent at low temperature (range of few nano-Kelvin) and atoms behave as waves. We can detect matter wave associated with particles by average thermal de Broglie wavelength λ_{th} that is given by

$$\lambda_{th} = \sqrt{\frac{2\pi\hbar^2}{mk_BT}},$$

where T is a temperature of an ideal gas. When a cooled gas reaches to transition temperature, thermal de Broglie wavelength become comparable to inter-spacing distance between particles and quantum nature of particles become dominant. At the quantum degeneracy, fermionic and bosonic nature of particles gets apparent. This phenomena is illustrated in Fig 2.1. Fermions start to fill the lowest vacant state of the trap but only single fermion exists in a single energy state. As there is no restriction for bosons, they quickly gets condensed into the lowest energy state of the trap, forming a Bose-Einstein condensation.

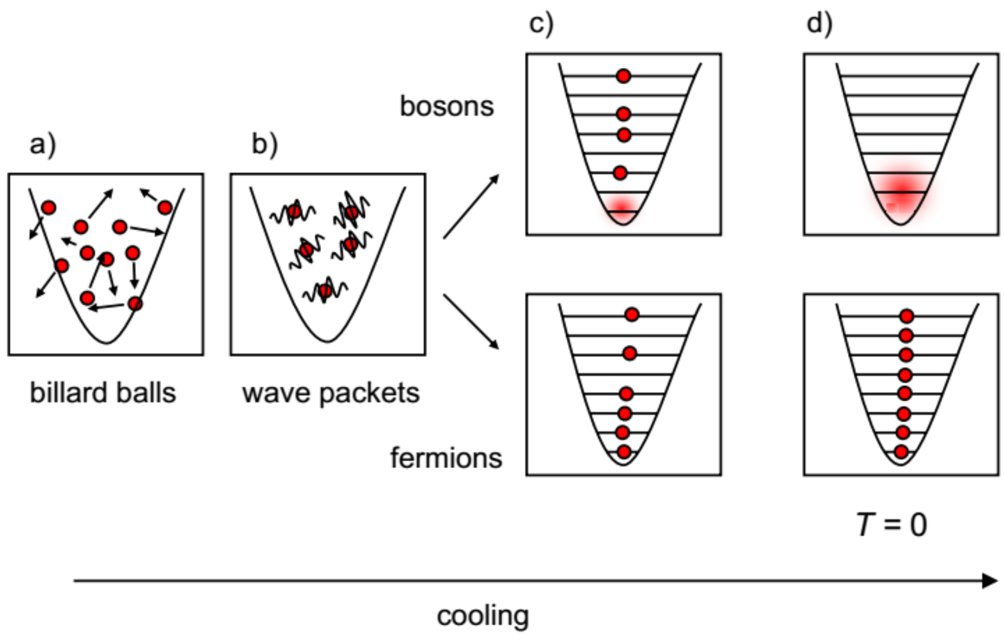

Figure 2.1: A gas of identical particles at different temperatures. a) At high temperatures, the particles behave classically . b) At low temperature, the wave nature of the particles gets apparent. c) When $\lambda_{th} \approx$ inter-particle spacing, bosons start to condense into lowest energy state of the trap, while fermions start to fill the lowest vacant state of the trap but only single fermion exist in single energy state. d) At absolute zero temperature, the bosons are fully condensed, fermions form a Fermi sea [21].

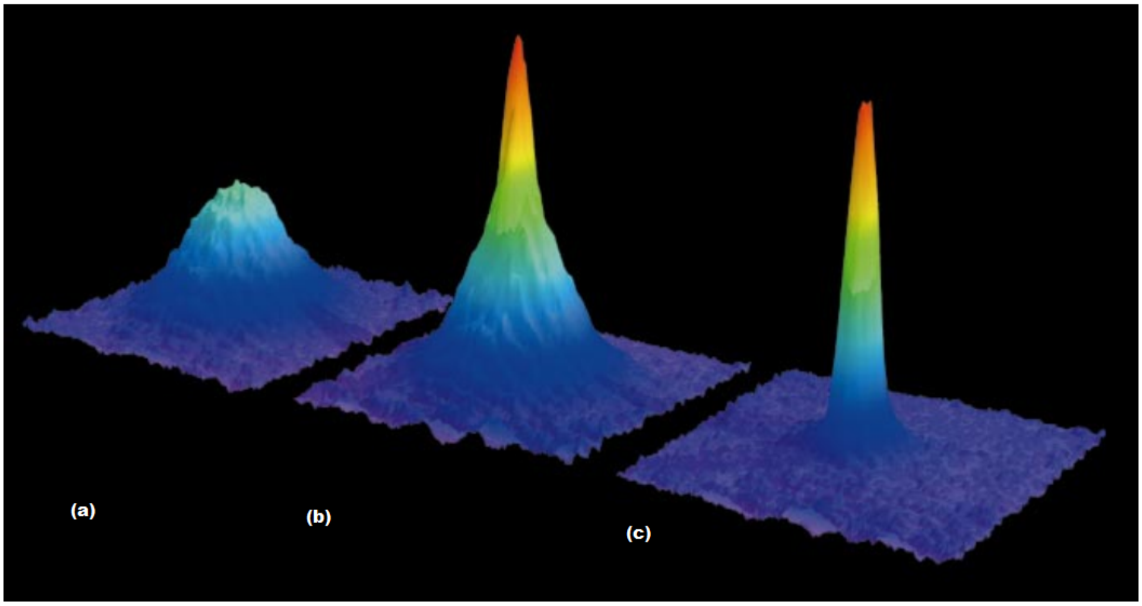

Figure 2.2: An illustration of the transition of ultra-cold particles from an ordinary gas to a BEC. i) The plot (a) shows broad velocity distribution of particles above the transition temperature. ii) The plot (b) illustrates a change in the velocity distribution of particles when the temperature reaches the transition temperature. (iii) The plot (c) corresponds to the absolute zero temperature where all particles condensate themselves in the ground state [22].

Bose-Einstein condensation (BEC) is a phenomenon in which gas of particles with integral spin is cooled to nearly absolute zero temperature will suddenly condense into the lowest energy state. The temperature at which particles start to condense is known as critical temperature.

Figure (2.2) illustrates the transition of a gas particles from an ordinary gas to a Bose-Einstein condensate when temperature is decreased. These figures are showing the velocity distribution of particles along two dimensions, at three different temperatures. The blue color is showing that there are very few particles having corresponding low velocity and the red color is representing that comparatively large number of particles have the corresponding high velocity. The color in the center is representing the particles with zero velocity. The plot (a) corresponds to the temperature higher than

the transition temperature. Here particles have broad velocity distribution from maximum to zero velocity. The middle plot illustrates the change in velocity distribution of particles when the temperature reaches the transition temperature. The velocity distribution comprises now two different contributions, sharply peaked one and a broad one. That corresponds to the point when particles start to condense in the lowest energy state. The plot (c) corresponds to the absolute zero temperature where all particles condensate themselves in the ground state.
BEC is usually achieved by laser cooling method. We use the counter propagating diode lasers with a frequency tuned to the desired resonance frequency of the atomic levels of optical lattice. The atoms in the vapor absorb the photons from the laser and get recoil momentum. Hence the velocity of the atoms drop down resulting in the decrease of kinetic energy. Since the energy is proportional to the temperature of the atomic vapor, therefore, as the energy drops down the corresponding vapor phase temperature also drops down. Eventually, the temperature drops down nearly to absolute zero Kelvin, where the BEC state exists.

Another successful technique used for cooling mechanisms is evaporative cooling. In this technique, we reduce the depth of the trap gradually. This process is illustrated in Figure (2.3). In this phenomena high velocity particles escape from the magnetic traps, leaving the mean squared velocity of the remaining system low. This results in the decrease of temperature of having sample as well of the remaining sample also cools down. The primary applications of atomic BEC system are in basic research areas presently. One of the current research area these days are simulation of condensed matter systems by using Bose-Einstein condensate. Optical lattice systems have significant advantage over real condensed matter systems because they are more flexible. We can easily vary the space of lattice, the strength of the interaction between atoms, and the number density of atoms in the lattice. It allows us to look into a range of various parameters with essentially the same sample. While for real condensed matter systems it is very difficult because for every new set of values we want to inquire, we need to grow all new samples [23]. Another latest area of research is the use of BEC in quantum information processing and precision measurement.

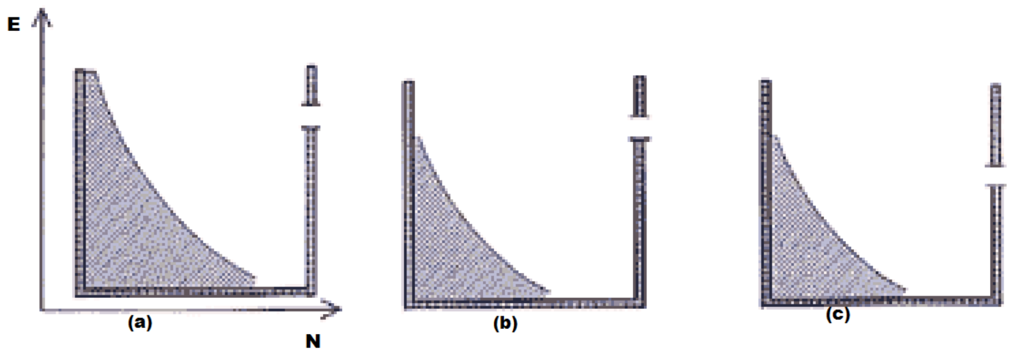

Figure 2.3: Schematic diagram of evaporative cooling. a) Releasing the high energy particles. b) Collision thermalize remaining particle at the new temperature. c) Again releasing high energy particles by lowering the hole [24].

2.1 Bose-Einstein Distribution

Generally, we use Boltzmann distribution to find a distribution of non-interacting particles of gas among various states. But at sufficiently low temperature of a gas, the behavior of molecules get change and Boltzmann distribution becomes inappropriate. At very low temperature, we replace Boltzmann statistics by quantum statistics.

Suppose we have a system of N particles and these particles follows two different statistics depending on their nature. The two categories of particles are based on their difference in a wave function. Bose-Einstein statistics apply to the particles with symmetrical wave-function (bosons). While particles with anti-symmetrical wave function (fermions) obey Fermi-Dirac statistic. Since our concern is specifically with bosons, we discuss Bose-Einstein distribution in detail here.

For bosons there is no restriction and more than one particle can occupy quantum states. Mean occupation number of bosons in any quantum state m is define as

$$\langle n_m \rangle = \frac{1}{e^{(\varepsilon_m - \mu)/kT} - 1}, \tag{2.1}$$

where ε_m is the energy of mth state while μ is the chemical potential. Chemical potential is the function of temperature T and the number of particles present in the system. Total number of particles in a system is the sum of mean occupation number of individual states, mathematically written as

$$N = \sum_m \langle n_m \rangle = \sum_m \frac{1}{e^{(\varepsilon_m - \mu)/kT} - 1}. \tag{2.2}$$

2.1.1 Ideal Bose Gas

When the number of particles in excited state are less than total number of particles N remaining particles must be condensed in a ground state. Thus the occupation number of ground state has larger values and the system under observation has Bose-Einstein condensate. The temperature at which particles condensate is known as transition temperature. Generally we describe Bose gas in grand canonical ensemble where formalism becomes simple. In quantum mechanics, statistical ensemble is represented by density operator ρ, defined as

$$\rho = \sum_m P_m |n_m\rangle\langle n_m|. \tag{2.3}$$

In grand canonical ensemble, the probability of occupation of N particles in m_{th} state with energy ε_m is

$$P_m = \frac{e^{(\mu N - \varepsilon_m)/kT}}{Z(T, \mu)}. \tag{2.4}$$

Here $Z(T, \mu)$ is a partition function and it is equal to $\sum\limits_m e^{(\mu N - \varepsilon_m)/kT}$. As here we are considering non-interacting particles so in this case $N = \sum\limits_i n_i$ and $\varepsilon_m = \sum\limits_i \varepsilon_i n_i$ where ϵ_i is the energy of single-particle. Thus the grand partition function can be written as

$$Z = \sum_{n_0} \left(e^{(\mu - \varepsilon_0)n_0}\right) \sum_{n_1} \left(e^{(\mu - \varepsilon_1)n_1}\right) ..... \tag{2.5}$$

Chemical potential "μ" is always less than ground state energy at high temperature. When temperature decreases $T \longrightarrow 0$, chemical potential increases and tends to zero $\mu \longrightarrow 0$. But μ can never increase more than lowest state energy ε_{min} because for $\mu \gg \varepsilon_{min}$, Bose-Einstein distribution function gives the negative value of energy for a ground state which will be unphysical. Consequently, for any excited single-particle state, its mean occupation number cannot be greater than the mean occupation number of its lowest energy state, i.e.

$$\frac{1}{e^{(\varepsilon_m-\varepsilon_{min})/kT}-1}.$$

2.2 Gas trapped in Harmonic Potential

Let us consider a gas in harmonic trap, where the harmonic potential is

$$V=\frac{1}{2}{\omega_1}^2x^2+\frac{1}{2}{\omega_2}^2y^2+\frac{1}{2}{\omega_3}^2z^2,$$

where ω_1, ω_2 and ω_3 are the three oscillator frequencies in three directions. Hamiltonian of the single particle of mass m is

$$H_{sp}=\frac{p^2}{2m}+V.$$

Energy eigenvalues of single particle are

$$\varepsilon_{(n_1,n_2,n_3)}=\left(n_1+\frac{1}{2}\right)\hbar\omega_1+\left(n_2+\frac{1}{2}\right)\hbar\omega_2+\left(n_3+\frac{1}{2}\right)\hbar\omega_3, \tag{2.6}$$

where n_1, n_2, n_3 have non-negative integral values. We can find lowest energy state of a system for bosons trapped in harmonic potential by taking values of n_1, n_2, n_3 equal to zero. Thus the energy of ground state will be $\varepsilon=\hbar\left(\omega_1+\omega_2+\omega_3\right)/2$.

2.2.1 Density of States for Bosons

Let us first discuss the number of states $G(\varepsilon)$ and density of states for a free particle in peculiar internal state. In three dimensional case, on average there exist a single quantum state per volume of phase space. In momentum space, particle of momentum p has energy ε equals $\varepsilon = p^2/2m$. And volume of the region where magnitude of momentum is less than p is $4\pi p^3/3$ (that is equal to the volume of sphere of radius p).

Total number of states G within the energy less than ε is

$$G(\varepsilon) = \frac{V}{(2\pi\hbar)^3}\frac{4\pi p^3}{3}, \tag{2.7}$$

$$= \frac{4\pi(2m\varepsilon)^{3/2}V}{3(2\pi\hbar)^3},$$

$$= \frac{\sqrt{2(m\varepsilon)^3}V}{3\pi^2\hbar^3}, \tag{2.8}$$

where V is the volume of the system under observation.
Total number of states exist between energy ε and $\varepsilon + d\varepsilon$ is $g(\varepsilon)d\varepsilon$ defined as

$$g(\varepsilon) = \frac{dG}{d\varepsilon},$$

where $g(\varepsilon)$ is the density of states. By taking derivative of Eq. (2.8), we get a density of state as

$$g(\varepsilon) = \frac{m^{3/2}V\varepsilon^{1/2}}{\sqrt{2}\pi^2\hbar^3}. \tag{2.9}$$

In general, for N-dimensions we can write density of states as $g(\varepsilon) \propto \varepsilon^{(n/2-1)}$. As density of state $g(\varepsilon)$ changes with the variation of power of energy, we can write $g(\varepsilon)$ generally in terms of $C_\alpha\varepsilon^{(\alpha-1)}$,

$$g(\varepsilon) = C_\alpha\varepsilon^{(\alpha-1)}. \tag{2.10}$$

This generalized formula for density of state helps us to find thermodynamic properties for different systems like for gas confined in $3D$ rigid walls, or the particle trapped in harmonic oscillator potential.
Now we determine number of states $G(\varepsilon)$ for the particle trapped in harmonic potential. For large values of energy we treat n_i as continuous variable and neglect zero-point motion. Therefore we can consider coordinate system having three variables $\varepsilon_i = \hbar\omega_i n_i$, for which energy in Eq. (2.6) is the plane $\varepsilon = \varepsilon_1 + \varepsilon_2 + \varepsilon_3$. Thus the number of states $G(\varepsilon)$ is proportional to the volume in first octant bounded by the plane,

$$G = \frac{1}{\hbar^3\omega_1\omega_2\omega_3}\int_0^{\varepsilon} d\varepsilon_1 \int_0^{\varepsilon-\varepsilon_1} d\varepsilon_2 \int_0^{\varepsilon-\varepsilon_1-\varepsilon_2} d\varepsilon_3. \tag{2.11}$$

As density of state is $g(\varepsilon) = dG/d\varepsilon$, we obtain

$$g(\varepsilon) = \frac{\varepsilon^2}{2\hbar^3\omega_1\omega_2\omega_3}. \tag{2.12}$$

From Eq. (2.10) we can see that, C_α in this case is

$$C_3 = \frac{1}{2\hbar^3\omega_1\omega_2\omega_3}.$$

2.2.2 Transition Temperature for Ideal Bose Gas to Condensate

The maximum temperature at which N number of particles in a system settles down at lowest energy level and large occupation of particles in ground state appears is known as transition temperature T_c.

Number of particles in phase space in volume element $dp_x dp_y dp_z dV$ can be find out by multiplying distribution number with Eq. (2.1). These are the particles that are in excited state,

$$N_{ex} = \int_0^\infty f(\varepsilon)g(\varepsilon)d\varepsilon. \tag{2.13}$$

Putting values from Eq. (2.1) and Eq. (2.10) in above equation, we get

$$N_{ex} = C_\alpha \int_0^\infty \frac{(\varepsilon)^{\alpha-1}}{\exp^{(\varepsilon-\mu)/kT_c} - 1} d\varepsilon.$$

Greatest value of N can be achieve at chemical potential $\mu = 0$. We find transition temperature T_c by assuming the condition that all particles are present in excited state. So the total number of particles N at transition temperature T_c and chemical potential $\mu = 0$ are

$$N = N_{ex} = C_\alpha \int_0^\infty \frac{(\varepsilon)^{\alpha-1}}{\exp^{(\varepsilon)/kT_c} - 1} d\varepsilon. \tag{2.14}$$

By substituting $y = \varepsilon/T$ in above equation, we get

$$N = C_\alpha (kT_c)^\alpha \int_0^\infty \frac{y^{\alpha-1}}{\exp^{(y)} - 1} dy. \tag{2.15}$$

Integral in above equation is equal to

$$\int_0^\infty \frac{y^{\alpha-1}}{\exp^{(y)}-1}dy = \Gamma(\alpha)\zeta(\alpha). \tag{2.16}$$

Here $\Gamma(\alpha)$ is gamma function while $\zeta(\alpha)$ is Riemann zeta function. Putting Eq. (2.16) in above Eq. (2.15), we find N equal to

$$N = C_\alpha (kT_c)^\alpha \Gamma(\alpha)\zeta(\alpha), \tag{2.17}$$

or

$$kT_c = \frac{N^{1/\alpha}}{[C_\alpha \Gamma(\alpha)\zeta(\alpha)]^{1/\alpha}}. \tag{2.18}$$

In case of 3D harmonic oscillator potential, putting $\alpha = 3$ in Eq. (2.18), we can calculate transition temperature as

$$kT_c = \frac{N^{1/3}}{[C_3 \Gamma(3)\zeta(3)]^{1/3}}. \tag{2.19}$$

Putting the value of C_3 in above equation,

$$kT_c = \frac{\hbar(\omega_1\omega_2\omega_3)^{1/3} N^{\frac{1}{3}}}{[\zeta(3)]^{\frac{1}{3}}}. \tag{2.20}$$

Thus the transition temperature T_c for three dimension harmonic potential is equal to

$$T_c = \left(0.94\hbar(\omega_1\omega_2\omega_3)^{1/3} N^{1/3}\right)/k. \tag{2.21}$$

2.2.3 Condensate Fraction for Ideal Bose Gas

Condensate fraction is the ratio of number of particles in lowest energy state (condensate) to the total number of particles [25]. At temperature less than transition temperature T_c, number of particles in excited state are

$$N_{ex} = C_\alpha (kT)^\alpha \Gamma(\alpha)\zeta(\alpha). \tag{2.22}$$

Putting the value from Eq. (2.18) in Eq. (2.22), we get

$$N_{ex} = N\left(\frac{T}{T_c}\right)^\alpha. \tag{2.23}$$

The number of particles that condensate to ground level can be found by following equation:

$$N_0 = N - N_{ex}. \tag{2.24}$$

Putting the value of N_{ex} from Eq. (2.23) in above equation, we find that

$$N_0 = N \left[1 - \left(\frac{T}{T_c} \right)^{\alpha} \right]. \tag{2.25}$$

For a particle trapped in harmonic potential number of particles that condensates are given by

$$N_0 = N \left[1 - \left(\frac{T}{T_c} \right)^{3} \right]. \tag{2.26}$$

Thus for T even a little less than T_c a large number of particles are in the ground state, whereas for $T > T_c$ there are practically no particles in the ground state. We call T_c the degeneracy temperature or the condensation temperature.

Chapter 3

Composite Boson: Quantum Information Approach

3.1 Introduction

Present chapter is dedicated to the discussion of composite particles particularly a bi-constituent boson. We will use the tools of quantum information to develop the formalism that helps to understand the internal structure of composite boson. We will mainly focus on the coboson made up of pair of fermions in detail, however, we also discuss coboson comprised of pair of bosons briefly. In this chapter we will see that with the help of entanglement we can get all information about the composite behavior of composite boson . It is being observed that measurement of the degree of entanglement between the sub-particles explains the deviation of the composite character of a composite particle from a pure bosonic character. In other words it explains, how closely composite boson is behaving like a pure boson. This phenomena entails some interesting ideas about the constituent particles that these particles are somehow bound by quantum entanglement. For the discussion of a composite particle and its behavior in a bipartite system the mechanical binding forces are actually not necessary when we try to apply the quantum correlations. Since the representation of a composite system is not confined to position or momentum space, therefore, the correlations between the constituent particles can be find out in several ways. In rest of the chapter, the underlying role of entanglement will be discussed by using the concept of second quantization, on the basis of properties of ladder operators associated with composite particles.

3.2 Fundamental Concepts of Quantum Information

Quantum information is encoded in terms of the state of any quantum system. Quantum mechanics allows a quantum system to be in superposition. Superposition is the property of a quantum system to be in two states at the same time. However, when we observe a system, the system has to decide where to be, and we can only see it in one of those two states. Quantum information is the effort to both understand and use the properties of the quantum world. We use the concepts of quantum information to understand the internal structure of composite system and its properties. In this section, we describe the basic concepts and mathematical methods that will help us to understand a later subject. We discuss the state of a composite quantum system, specifically, bipartite quantum system and construct the formalism to express the states of the composite quantum system in terms of states of subsystems. We also explain the concept of entanglement and relate it with Schmidt decomposition.

3.2.1 Multipartite Quantum System

A composite quantum system is one that includes a number of quantum objects. A composite quantum system can decompose naturally into its subsystems, where every subsystem is proper quantum system. Usually, we distinguish the subsystems from each other on the basis of the distance between them which must be larger than the individual subsystem's size. For example, hydrogen atom is composite in nature since it consists of an electron and proton. Another common example of a composite system is the string of ions in which every ion acts as a subsystem.
In quantum mechanics, we often associate a Hilbert space denoted by H, with a physical system. If we have some system that is made up of two or more than two subsystems (known as multipartite system), Hilbert space H associated with that system is given

by tensor product of all Hilbert spaces of subsystems.

$$H = H_1 \otimes H_2 \otimes H_3 \otimes H_4 ...$$

Here H_1 is the Hilbert space associated with first subsystem, H_2 with second subsystem and so on. For a bipartite quantum system (system which have only two subsystems), the associated Hilbert space H of a system is the tensor product of a Hilbert space of individual subsystems. Let A and B represent the subsystems then the tensor product of corresponding Hilbert spaces is given as

$$H = H_A \otimes H_B. \tag{3.1}$$

The physical state of a quantum system is represented by a state vector in a Hilbert space. This state vector contains all the information about that physical state of the system. Suppose the system has state ψ, then the state vector is denoted by $|\psi\rangle$. There are two types of states of a quantum system based on either there is enough information to specify the state $|\psi\rangle$ of a system or not. These two types are classified as:

Pure State: The state of a quantum system $|\psi\rangle$ is said to be pure when the system is in defined state. In other words, system is in known state $|\psi\rangle$. We always get the same result when we measure the state by some well defined observable.

Mixed State: Mix state is define as a linear combination of different pure states. In real life experiments, instead of single quantum system often there are collection of quantum systems (ensemble). Also each member of ensemble can be found in more than one quantum states. Thus most of the time we encounter mix states instead of pure states. For example, let us consider $2D$ Hilbert space with the basis $\{|a\rangle$ and $|b\rangle\}$. Suppose we have an ensemble containing N number of quantum systems, where each individual system is prepared in one of two state vectors $|x\rangle$ and $|y\rangle$.

$$|x\rangle = \delta|a\rangle + \gamma|b\rangle, \tag{3.2}$$

$$|y\rangle = \alpha|a\rangle + \beta|b\rangle. \tag{3.3}$$

Suppose n systems are prepared in state $|x\rangle$ and m systems are prepared in state $|y\rangle$. As total number of systems are N, then $n+m$ should be equal to N, i.e,

$$\frac{n}{N} + \frac{m}{N} = 1.$$

The above equation shows that, for a random selected system in ensemble, probability that system will be in state $|x\rangle$ is $p = n/N$ and probability of finding system in state $|y\rangle$ is $1-p$. Thus we can write state $|\psi\rangle$ as

$$|\psi\rangle = p|x\rangle + (1-p)|y\rangle. \tag{3.4}$$

This is an example of a mixed state. To deal with a statistical mixture of quantum states like the example we have discussed above, we have to find the probability of different pure states inside ensemble. Convenient method to explain pure state and specially mix state is the density matrix formalism.

3.2.2 Density Matrix Formalism

"Density matrix" is a very powerful formalism in which we describe quantum state by its density matrix. It is the alternative formalism to describe a quantum state by Dirac notations (bra-ket notation). Density operator is an average operator and basically useful in describing statistical mixture. It is denoted by ρ.

For pure states where the state of a system is definite, ρ can be constructed by the outer product of state $|\psi\rangle$. To see this, let us consider some operator $\hat{Q}$ and we find the average value (expectation value) of this operator. The expectation value $\langle\hat{Q}\rangle$ can be written as

$$\langle\hat{Q}\rangle = \langle\psi|Q|\psi\rangle.$$

Expanding $|\psi\rangle$ in its orthonormal basis, we obtain:

$$\begin{aligned}
\langle\hat{Q}\rangle &= (\gamma_1^*\langle u_1| + \gamma_2^*\langle u_2| + \gamma_3^*\langle u_3|.........\gamma_m^*\langle u_m|)Q(\gamma_1|u_1\rangle + \gamma_2|u_2\rangle + \gamma_3|u_3\rangle.........\gamma_m|u_m\rangle) \\
&= \sum_{k,l=1}^{n} \gamma_k^*\gamma_l\langle u_k|Q|u_l\rangle \\
&= \sum_{k,l=1}^{n} \gamma_k^*\gamma_l Q_{k,l}. \qquad (3.5)
\end{aligned}$$

As the expansion coefficient can be written as

$$\gamma_l = \langle u_l|\psi\rangle,$$

and the complex conjugate is

$$\gamma_k^* = \langle\psi|u_k\rangle,$$

this means that $\gamma_k^*\gamma_l = \langle u_l(|\psi\rangle\langle\psi|)u_k\rangle$. Thus the average value of operator $\hat{Q}$ becomes

$$\langle\hat{Q}\rangle = \sum_{k,l=1}^{n}\langle u_l|\psi\rangle\langle\psi|u_k\rangle Q_{kl}.$$

We call this outer product $|\psi\rangle\langle\psi|$, a density operator ρ. And the expectation value of operator $\hat{Q}$ with respect to the state $|\psi\rangle$ is

$$\langle\hat{Q}\rangle = \sum_{k,l=1}^{n}\langle u_l|\rho|u_k\rangle Q_{kl}.$$

In terms of trace, we can write expectation value as

$$\langle\hat{Q}\rangle = Tr\left(\rho Q\right).$$

For pure states, we see that

$$\rho^2 = (|\psi\rangle\langle\psi|)(|\psi\rangle\langle\psi|) = (|\psi\rangle\langle\psi|) = \rho.$$

Since $Tr(\rho) = 1$, this means

$$Tr(\rho^2) = 1,$$

indicating that if trace of a square of density operator is equal to one then state of a system is pure.

For mixed states, let us suppose that there are N number of possible states. For state $|\psi_n\rangle$, density operator can be written as $\rho_n = |\psi_n\rangle\langle\psi_n|$. If probability that system in ensemble has been prepared in state $|\psi_n\rangle$ is p_n, then the density operator for ensemble is

$$\rho = \sum_{n=1}^{N} p_n |\psi_n\rangle|\langle\psi_n|.$$

We can characterize that given state of system is a mixed state, if square of density operator $Tr(\rho^2) < 1$.

3.2.3 Separable States and Entangled States

For composite quantum systems, we can further divide the pure and mixed state into separable and entangled state.

Pure Product State

A state is known as separable state or product state if we can write it in terms of product of subsystem's state. For a bipartite system, state of composite system, $|\psi\rangle \in H_A \otimes H_B$ is a separable state if

$$|\psi_s\rangle = |\psi_A\rangle \otimes |\psi_B\rangle. \tag{3.6}$$

where $|\psi_A\rangle \in H_A$, $|\psi_B\rangle \in H_B$ are states of subsystems A, B respectively and these are prepared in pure states. In terms of the density operator, it can be written as

$$\rho_s = \rho_A \otimes \rho_B.$$

A simple example of a separable state is

$$|\psi\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle).$$

We can easily see that it is the tensor product of the following states,

$$\frac{|0\rangle + |1\rangle}{\sqrt{2}} \otimes \frac{|0\rangle + |1\rangle}{\sqrt{2}}. \tag{3.7}$$

There is a simple test to check that given state is separable or not [26]. We write a state in the form of column vector,

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix},$$

If $ad = bc$ then the state is separable. In the above example Eq. (3.7), column vector of a state is

$$\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}.$$

In this example, we can see ad is equal to bc , showing that the state is separable. But this method has its own limitations. This is useful only for the states in four-dimensional vector space. We will discuss general criteria to distinguish separable state from the one which is not separable. But first we check a property of state when some observable acts on it.

Consider a product state

$$|\psi_s\rangle = |\psi_A\rangle \otimes |\psi_B\rangle.$$

In terms of a density operator, it will be

$$\rho = |\psi_s\rangle\langle\psi_s|.$$

Suppose an observable $\hat{A} \otimes \hat{I}$ can execute any local measurement. Here $\hat{A}$ is an Hermitian operator acts on states in H_A and $\hat{I}$ is an identity operator operating on states in H_B. After measurements the state of the first subsystem stands out in terms of the eigenstate of operator $\hat{A}$ but there will be no change in the state of the second subsystem as identity operator brings no change. Mathematically, it is given as

$$\begin{aligned} \langle\hat{A}\rangle &= \langle\psi_s|\hat{A} \otimes \hat{I}|\psi_s\rangle, \\ &= Tr(\hat{A} \otimes \hat{I}|\psi_s\rangle\langle\psi_s|), \\ &= Tr_A(\hat{A}Tr_B|\psi_s\rangle\langle\psi_s|), \\ &= Tr_A(\hat{A}\rho_A). \end{aligned} \tag{3.8}$$

where Tr_A is a partial trace associated with subsystem A and ρ_A is equal to $|\psi_A\rangle\langle\psi_A|$. Similarly, if we execute another local measurement on the second subsystem, its outcome appears independent to the result of the other measurement. And we can find $\rho_B = |\psi_B\rangle\langle\psi_B|$.

Therefore, we can conclude that the results after measurements for the different subsystems are lacking the mutual correlation, it only depends on the states of the respective subsystem. From density matrix ρ_A and ρ_B, we can find density matrix ρ of state by tensor product i.e $\rho = \rho_A \otimes \rho_B$.

Pure Entangle states

As from the above discussion of pure product states, one can assume that states that can be written as a product of pure states, as in Eq. (3.6), are called separable or product states. On the other hand, if no local states $|\psi_A\rangle$ and $|\psi_B\rangle$ belonging to H_A and H_B exists, then we cannot write the state of a system $|\psi\rangle$ as a product of both,

$$\nexists \quad |\psi_A\rangle \in H_A \ , \quad |\psi_B\rangle \in H_B,$$

such that

$$|\psi\rangle = |\psi_A\rangle \otimes |\psi_B\rangle,$$

then $|\psi\rangle$ is said to be an entangled state.

Generally, a pure state in the Hilbert space is the superposition of a pure states of the form Eq. (3.6).

$$|\psi_e\rangle = \frac{1}{\sqrt{2}}(|\phi_A\rangle \otimes |\phi_B\rangle + |\varphi_A\rangle \otimes |\varphi_B\rangle), \tag{3.9}$$

where $|\phi_i\rangle \neq |\varphi_i\rangle$ while $(i = A, B)$.

A simple example of an entangled state are:

$$|\alpha\rangle = \frac{|00\rangle + |11\rangle}{\sqrt{2}},$$

$$|\beta\rangle = \frac{|01\rangle + |10\rangle}{\sqrt{2}},$$

$$|\gamma\rangle = \frac{|00\rangle - |11\rangle}{\sqrt{2}},$$

$$|\varphi\rangle = \frac{|01\rangle - |10\rangle}{\sqrt{2}},$$

Let us write the state $|\varphi\rangle$ as

$$\rho = |\varphi\rangle\langle\varphi| = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Reduced density matrix of subsystem A can be written as

$$\rho_A = Tr_B(\rho) = \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Since the trace of a density matrix $Tr({\rho_A}^2) < 1$, thus the state of subsystems are mixed. But as the state of a whole system was pure, this is quite strange. This shows that for the entangled states, states of a subsystems can only be described with respect to one another.

In both examples, if we test them by the method we discussed above, we can see that $ad \neq bc$. These are entangled states. We can also check how the state of the composite system $|\psi_e\rangle$ looks like when we try to measure its subsystems individually. Therefore applying local measurement $\hat{A} \otimes \hat{I}$ where $\hat{A}$ is the operator related to the first subsystem, then the result of expectation value in this experiment appears as

$$\langle\hat{A}\rangle = \langle\psi_e|\hat{A} \otimes 1|\psi_e\rangle.$$

We can write above equation in terms of a trace as

$$\langle\hat{A}\rangle \quad = \quad Tr_A(\hat{A}\rho_A), \tag{3.10}$$

where Tr_A is the partial trace associated to subsystem A. The density operator of a subsystem A is $\rho_A = Tr_B|\psi_e\rangle\langle\psi_e|$. The state of the subsystems independently can easily be given in terms of reduce density operators ρ_A and ρ_B for subsystems A and B, respectively. However, we cannot say that the state representing the composite system is equal to the tensor product of the states of two subsystems, i.e,

$$\rho = |\psi_e\rangle\langle\psi_e| \neq \rho_A \otimes \rho_B.$$

Also if we execute any local measurement on any of the subsystems individually, the state of the overall composite system gets reduced completely. Thus, the outcome possibly we get as a result of measurement on any subsystem get affected by earlier measurements which have been done on the other subsystem. This shows that the measurement for the non-interacting and possibly distant subsystems are completely correlated.

Criteria to distinguish Separable and Entangled States

The above criteria to distinguish separable states and entangled states seems very simple on first sight. But if we check different states, we observe that in some cases checking separability of state gets complicated. As for pure states, we have defined the criteria of separability which is the existence of the decomposition of a state into product states, or for mixed states by a convex sum of tensor products. Therefore, when we look at the given state to check the separability, we have to find such decomposition. Once we find the decomposition, it gets clear that the state is separable. But in case of failure, there are two possible reasons: either the state is actually separable but reasonable decomposition could not be identified, or the state is entangled so there is no decomposition.

Due to this reason, there is a need for a standard but straightforward criterion to distinguish separable and entangled states which do not require an explicit search. In the following section, we discuss the criterion to differentiate separable and entangled states unambiguously.

Schmidt Decomposition

Let us consider a bipartite system consist of two subsystems A and B. Let the system has a pure state $|\psi\rangle$ in the Hilbert space H which is given by the direct product of the Hilbert spaces of subsystems as mentioned in the Eq. (3.1), that is

$$H = H_A \otimes H_B,$$

where H_A and H_B are the Hilbert spaces belonged to subsystems. For each subsystem, there exist orthonormal basis $|ß\rangle_A$ and $|æ\rangle_B$, respectively.
In terms of above mentioned basis, the state $|\psi\rangle$ of a system can be expressed as

$$|\psi\rangle = \sum_{i,j} \alpha_{ij}(|i\rangle_A \otimes |j\rangle_B), \tag{3.11}$$

where α_{ij} is the expansion coefficient, which represents the overlap of a state of system with the basis vectors,

$$\begin{aligned} \alpha_{i,j} &= \langle i_A| \otimes \langle j_B|\psi\rangle, \\ &= \langle i_A|\psi\rangle\langle j_B|\psi\rangle. \end{aligned} \tag{3.12}$$

Now , let us write matrix formed by expansion coefficient α_{ij} as $d_A \times d_B$ matrix C where d_A and d_B are equal to dimensions of Hilbert space H_A and H_B, respectively.

$$[C]_{i,j} = \alpha_{i,j}. \tag{3.13}$$

As every matrix has singular value decomposition (SVD), with the help SVD we will solve this matrix to find Schmidt eigenvalues.

Singular Value Decomposition

Singular value decomposition (SVD) is the factorization of any $m \times n$ matrix (let say we name it matrix A) into three matrices UDV^T. Where U and V are the orthogonal matrices of size $m \times m$ and $n \times n$, respectively. While D is diagonal matrix of size $m \times n$ and these diagonal entries are called singular values of matrix A.
To understand this concept physically, for any vector, when matrix A multiply with a column vector, it rotates the vector and also stretch it. In case of circle (two dimensional case of sphere), when the matrix A apply on sphere, it rotates the circle and also stretch it, so that it becomes ellipse. Here let us denote the orthogonal vectors of circle by v_1 and v_2, while major and minor axis of ellipse are denoted by u_1 and u_2,

respectively. So, when matrix A applies to vector v_1, it gives

$$Av_1 = \sigma_1 u_1$$

Similarly, when matrix A applies to vector v_2, it gives

$$Av_2 = \sigma_2 u_2,$$

where σ_1 and σ_2 are stretching factors. In case of N-dimensional sphere (hyper sphere), after the operation we get a hyper ellipse,

$$Av_j = \sigma_j u_j. \tag{3.14}$$

We can see Eq. (3.14) is like a eigen-value problem. In the matrix form, we can write it as

$$[A]\begin{bmatrix} v_1 & v_2 & . & . & v_n \end{bmatrix} = \begin{bmatrix} u_1 & u_2 & . & . & u_n \end{bmatrix}\begin{bmatrix} \sigma_1 & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & 0 & 0 & 0 \\ 0 & 0 & . & 0 & 0 \\ 0 & 0 & 0 & . & 0 \\ 0 & 0 & 0 & 0 & \sigma_n \end{bmatrix},$$

Here A is a matrix of order $m \times n$. Generally, it can be written as, $AV = UD$, where D is a diagonal matrix while V and U are the unitary matrices as the vectors belong to them are orthonormal.

$$A = UDV^T.$$

This is a singular value decomposition.

Now writing a matrix C from Eq. (3.13) in SVD as $U\Lambda V$, where U is a unitary matrix of dimensions $d_A \times d_A$, V is unitary matrix of dimensions $d_B \times d_B$ and Λ is a $d_A \times d_B$ diagonal matrix. The diagonal matrix has strictly positive numbers c_n along the diagonal which are known as singular values. Let us write the matrix elements of U as $u_{i,n}$ and V as $v_{n,j}$. Then the above matrix C is equal to

$$\sum_{i,j,n} \alpha_{i,j} = \sum_{n} u_{i,n} c_n v_{n,j}. \tag{3.15}$$

Substituting equation (3.15) in (3.11), we will get

$$
\begin{aligned}
|\psi\rangle &= \sum_{i,j,n} u_{i,n} c_n v_{n,j} (|i\rangle_A \otimes |j\rangle_B), \\
|\psi\rangle &= \sum_{n} c_n (|n\rangle_A \otimes |n\rangle_B), \\
|\psi\rangle &= \sum_{n} \sqrt{\lambda_n} (|n\rangle_A \otimes |n\rangle_B).
\end{aligned} \tag{3.16}
$$

We have defined the orthonormal bases on the system A as $|n\rangle_A = \sum_i u_{i,n}|i\rangle_A$, on system B as $|n\rangle_B = \sum_j v_{n,j}|j\rangle_B$. These orthonormal bases are known as Schmidt bases while $\lambda_n = c_n^2$ is known as Schmidt coefficient. The Schmidt coefficients λ_n are like eigenvalues of a matrix and unique for any state $|\psi\rangle$. We can extract the information related to the entanglement of state in quantum system from the factor of Schmidt coefficient.

The standard criteria to check separability of any state $|\psi\rangle$ is that if decomposed state contains one non-zero Schmidt coefficient, then we can say that the state must be separable. On the other hand, if there exists more than one non-zero Schmidt coefficients, then the state $|\psi\rangle$ is not separable and we cannot write it in terms of Eq. (3.6). Hence, we can conclude that the pure state is separable if there exists only one non-zero Schmidt coefficient.

As discussed above the Schmidt coefficients are very helpful in differentiating between entangled and separable states, therefore our main focus is how we can evaluate them. We can do this with the help of reduce density matrices, so then reduced density matrices are explicitly useful.

3.3 Purity

Schmidt decomposition can help us to find out whether the state of a system is separable or entangled [27, 30]. For checking the degree of entanglement of a state (how much state is entangled) we use the concept of purity. We can characterize the degree of entanglement by the degree of purity of either of the subsystems. Purity of any normalized quantum state can be defined as the trace of the squared value of its density operator.

$$P = Tr(\rho^2), \tag{3.17}$$

where the range of purity is

$$0 < P \leq 1.$$

For a subsystem, let's say for a subsystem A, purity is defined as: $P = Tr(\rho_A^2) = \sum_n \lambda_n^2$. When purity is equal to one, this means that our state of system is separable, and when purity is less than one, the state of a system is entangled.
Measurement of the entanglement can be define in terms of the Schmidt number κ as

$$\kappa \equiv \frac{1}{P_A} = \sum_{n=0}^{\infty} \frac{1}{\lambda_n^2}. \tag{3.18}$$

If $\kappa = 1$, this means that the state will be separable. For all other values of κ the state of a composite system will be entangled.

3.4 Composite Boson

In this section, we comprehensively take a look at the bipartite coboson from the perspective of quantum information. We take a look at the fermionic and bosonic algebra for ideal fermions and bosons and modify it for the composite boson.
Let us consider a composite boson C in Hilbert space H comprised of two fermions (or bosons). Let us denote individual fermions with A and B. Hilbert space H_A be the Hilbert space associated with particle A and H_B be the Hilbert space related with particle B. The Hilbert space H of the composite boson C is given by the tensor product of Hilbert spaces of subsystems.

$$H = H_A \otimes H_B. \tag{3.19}$$

Both constituent particles are distinguishable and can be either bosons or fermions. Collectively, both particles behave as coboson. We will find that how much the nature of coboson is deviating from ideal boson. The state of the composite system can be written in terms of Schmidt decomposition and we can write it in terms of the basis of its constituent particles as

$$|\psi\rangle_C = \sum_{i,j=0}^{\infty} \alpha_{ij} |i\rangle_A \otimes |j\rangle_B. \tag{3.20}$$

where $|i\rangle_A$ and $|j\rangle_B$ are the states of the subsystems.

3.4.1 Fock Space

Fock Space is important because we use it to study many particle system, as well as the system where number of particles may not be conserved. For example, in non-ideal optical cavity there is possibility of leakage of photons. Another example is excited atoms, these atoms emit a photon. Formalism that allows us to describe such type of systems in a good way is explained below. This formalism is actually build up using concept of creation and annihilation operators that we use to describe harmonic oscillator in quantum mechanics. Let us consider a system having single particle. First we describe this basic system and later we can easily generalize it for more than one particle. As we are observing quantum mechanically, we cannot forget importance of vacuum state. In a system, there must be state that represent zero of particle. We denote vacuum state by $|0\rangle$ and its inner product with itself $\langle 0|0\rangle$ equal to one. Now just like harmonic oscillator, we define the creation and annihilation operator as $\hat{a}_n^\dagger$ and $\hat{a}_m$, respectively. It important to keep in mind that here harmonic oscillator is not involved, its just that we are defining operators in *ad hoc*. Now the state of single particle is $|1\rangle = \hat{a}^\dagger|0\rangle$, thus creation operator is adding one particle in system. And annihilation operator just remove a particle from the system.

$$\langle 0|\hat{a}\hat{a}^\dagger|0\rangle = \langle 0|0\rangle = 1.$$

In Fock space, orthonormal basis consists of the vacuum state $|0\rangle$, the complete set of single particle state $\{|\varphi_\alpha\rangle : \alpha = 1, 2, 3, 4.....\}$, the complete set of bipartite system's state, the complete set of tripartite system's state and so on. The formalism for Fock Space is different for bosons and fermions and thus we discuss them separately here.

Fermionic Algebra

The creation and annihilation operator of fermions has following properties :

$$\hat{a}_n^\dagger|0\rangle = |\varphi_n\rangle, \tag{3.21}$$

$$\hat{a}_m^\dagger|\varphi_n\rangle = \hat{a}_m^\dagger\hat{a}_n^\dagger|0\rangle = |\varphi_m\varphi_n\rangle = -|\varphi_n\varphi_m\rangle, \tag{3.22}$$

$$\hat{a}_n|\varphi_n\rangle = |0\rangle, \tag{3.23}$$

$$\hat{a}_n|0\rangle = 0. \tag{3.24}$$

Here n is quantum number of fermions. From Eq. (3.22), we can define that vector states are anti-symmetric when two fermions gets interchange. Also the fermionic operators follows the anti-commutation relations :

$$\{\hat{a}_n^\dagger, \hat{a}_m^\dagger\} = \{\hat{a}_n, \hat{a}_m\} = 0, \{\hat{a}_n, \hat{a}_m^\dagger\} = \delta_{nm}, \tag{3.25}$$

$$\{\hat{a}_n^\dagger, \hat{b}_m^\dagger\} = \{\hat{a}_n, \hat{b}_m\} = \{\hat{a}_n^\dagger, \hat{b}_m\} = \{\hat{a}_n, \hat{b}_m^\dagger\} = 0. \tag{3.26}$$

In Eq. (3.26), these are the anti-commutation relation while $\hat{a}$ and $\hat{b}$ are operators belongs to two distinguishable fermions [31].

Bosonic Algebra

The creation and annihilation operator for bosons in mode β are $\hat{c}_\beta^\dagger$ and $\hat{c}_\beta$, respectively. Following are the properties that creation and annihilation operator have:

$$\hat{c}_\beta^\dagger = |0,0,0....,m_\beta = 0,....\rangle = |\varphi_\beta\rangle = |0,0,.....,m_\beta = 1,0,.....\rangle, \tag{3.27}$$

$$\hat{c}_\beta|m_1,m_2,....,m_\beta = 0,....\rangle = 0, \tag{3.28}$$

$$\hat{c}_\beta^\dagger|m_1,m_2,....,m_\beta,....\rangle = \sqrt{(m_\beta+1)}|m_1,m_2,.....,m_\beta+1,...\rangle, \tag{3.29}$$

$$\hat{c}_\beta|m_1,m_2,....,m_\beta,....\rangle = \sqrt{(m_\beta)}|m_1,m_2,.....,m_\beta-1,...\rangle. \tag{3.30}$$

Bosons follow the commutation relation,

$$[\hat{c}_n, \hat{c}_m^\dagger] = \delta_{nm}. \tag{3.31}$$

If all bosons are present in the same state then above relations can be written as

$$\hat{c}^{\dagger}|0\rangle = 0, \tag{3.32}$$

$$\hat{c}|0\rangle = 0, \tag{3.33}$$

$$\hat{c}^{\dagger}|m\rangle = \sqrt{(m+1)}|m+1\rangle, \tag{3.34}$$

$$\hat{c}|m\rangle = \sqrt{m}|m-1\rangle, \tag{3.35}$$

$$\left[\hat{c}, \hat{c}^{\dagger}\right] = 1. \tag{3.36}$$

3.5 Composite Bosonic Operator and its Properties

Now for the composite two-particle system, using the vision of second quantization we can write the state in terms of ladder operators as

$$|\psi_c\rangle = \hat{c}^{\dagger}|0\rangle. \tag{3.37}$$

Hence, by comparing it with the equation (3.20) we can say that this creation operator which is creating a particle in a composite system can also be the combination of two other creation operators which can create sub-particle in the relevant subsystem therefore,

$$\hat{c}^{\dagger} = \sum_{ij}^{\infty} \alpha_{ij}\hat{a}_i^{\dagger}\hat{b}_j^{\dagger}, \tag{3.38}$$

where α_{ij} is the probability amplitude of having particle A in $|i\rangle$ basis and particle B in $|j\rangle$ basis. $\hat{a}_i^{\dagger}$ and $\hat{b}_j^{\dagger}$ are the creation operators of particle A and particle B in the mode of $|i\rangle$ and $|j\rangle$. In the perspective of entanglement theory we use the process of decomposition to calculate the probability amplitude therefore we can rewrite the state expressed above as

$$|\psi_c\rangle = \hat{c}^{\dagger}|0\rangle = \sum_{n=0}^{\infty} \sqrt{\lambda_n}\hat{a}_n^{\dagger}\hat{b}_n^{\dagger}|0\rangle, \tag{3.39}$$

where basis n is the superposition of i and j and $\sqrt{n}$ is the Schmidt coefficient which tells us about the probability of having both particles in the same basis n. The value of λ_n also provides the measure of entanglement as we have discussed previously. In terms of the Schmidt number κ it is

$$\kappa = \frac{1}{\sum\limits_{n=0}^{\infty} \lambda_n^2}. \tag{3.40}$$

Thus the operator for composite particle in terms of the Schmidt coefficient is written as

$$\hat{c}^\dagger = \sum_{n=0}^{\infty} \sqrt{\lambda_n} \hat{a}_n^\dagger \hat{b}_n^\dagger. \tag{3.41}$$

The operator $c^\dagger$ can be treated as the ladder operator for the composite particle and we can discuss its properties as well [32, 35].

Being ladder operator $\hat{c}$ and $\hat{c}^\dagger$ satisfies the Bosonic algebra. If constituent particles A and B both are bosons then the commutation relation results as

$$[\hat{c}, \hat{c}^\dagger] = 1 + \sum_{n=0}^{\infty} \lambda_n (\hat{a}_n^\dagger \hat{a}_n + \hat{b}_n^\dagger \hat{b}_n). \tag{3.42}$$

If constituent particles are fermions than

$$[\hat{c}, \hat{c}^\dagger] = 1 - \sum_{n=0}^{\infty} [\lambda_n (\hat{a}_n^\dagger \hat{a}_n + \hat{b}_n^\dagger \hat{b}_n)]. \tag{3.43}$$

Collectively, we can write the above relation as

$$[\hat{c}, \hat{c}^\dagger] = 1 + s\Delta, \tag{3.44}$$

where $s = +1$ when both A and B are bosonic particles and $s = -1$ if both are fermionc.

The operator Δ is defined as

$$\Delta = \sum_{n=0}^{\infty} [\lambda_n (\hat{a}_n^\dagger \hat{a}_n + \hat{b}_n^\dagger \hat{b}_n)]. \tag{3.45}$$

Here the operator Δ appears as it shows how much composite bosonic operator deviates from pure bosonic operator [36, 38]. It should be minimum so that $\hat{c}$ and $\hat{c}^\dagger$ will operate similar as pure bosonic operator. Therefore N particle state for composite particle is

$$|N\rangle = \frac{1}{\sqrt{\chi_N}} \frac{(\hat{c}^\dagger)^N}{\sqrt{N!}} |0\rangle, \tag{3.46}$$

$|N\rangle$ is a normalized state. χ_N is the normalization constant and it is must as $\hat{c}^\dagger$ is not a perfect bosonic creation operator [39]. Also χ_N measures the quality of bosonic character for entire system of N composite bosons. When $\chi_N = 1$ this means that our composite system is like a pure bosonic system while when $\chi_N = 0$ means system is least bosonic and any intermediate value represents sub-bosonic quality. We can calculate this normalization constant by considering $\langle N|N\rangle = 1$. By taking projection of state $\langle N|$ with itself, Eq. (3.46), we can write χ_N in the following form:

$$\langle 0|\hat{c}^N(\hat{c}^\dagger)^N|0\rangle = N!\chi_N. \tag{3.47}$$

In order to understand that how well $\hat{c}$ behaves as a bosonic annihilation operator, we check its action on the composite particle state $|N\rangle$. This is defined as

$$\hat{c}|N\rangle = \alpha_N\sqrt{N}|N-1\rangle + |\xi_N\rangle, \tag{3.48}$$

where α_N is constant and ξ_N is another term which appears to be orthogonal to $|N-1\rangle$. It is basically correction term which should appear here because the state of composite particle $|N\rangle$ is only subset itself of the whole Hilbert space associated with composite system. The value of α_N can be find out by using following equation:

$$\begin{aligned}
\langle N-1|\hat{c}|N\rangle &= \alpha_N\sqrt{N}\langle N-1|N-1\rangle + \langle N-1|\xi_N\rangle, && (3.49)\\
\langle N-1|\hat{c}|N\rangle &= \alpha_N\sqrt{N}. && (3.50)
\end{aligned}$$

Also;

$$\langle N-1|\hat{c}|N\rangle = \sqrt{N}\frac{\langle 0|\hat{c}^N(\hat{c}^\dagger)^N|0\rangle}{\sqrt{\chi_N}\sqrt{\chi_{N-1}}N!}. \tag{3.51}$$

Putting the value from Eq. (3.47) in Eq. (3.51),

$$\langle N-1|\hat{c}|N\rangle = \frac{\sqrt{\chi_N}\sqrt{N}}{\sqrt{\chi_{N-1}}}. \tag{3.52}$$

Comparing Eq. (3.50) and Eq. (3.52), we find the value of α,

$$\alpha_N = \sqrt{\frac{\chi_N}{\chi_{N-1}}}. \tag{3.53}$$

In Eq. (3.48), we can see, bosonic operator will be pure bosonic, if it satisfied the following two conditions:

$$\alpha_N \longrightarrow 1,$$

$$\langle \xi_N | \xi_N \rangle \longrightarrow 0,$$

where $\langle \varepsilon_N | \varepsilon_N \rangle$ can be derive using Eq. (3.48). We can write Eq. (3.48) as

$$|\xi_N\rangle = \hat{c}|N\rangle - \alpha_N \sqrt{N}|N-1\rangle.$$

Also;

$$\langle \xi_N| = \langle N|\hat{c}^\dagger - \alpha_N \sqrt{N}\langle N-1|.$$

Thus we get

$$\langle \xi_N|\xi_N\rangle = \langle N|\hat{c}^\dagger \hat{c}|N\rangle + {\alpha_N}^2 N - \alpha_N\sqrt{N}\langle N|\hat{c}^\dagger|N-1\rangle - \alpha_N\sqrt{N}\langle N-1|\hat{c}|N\rangle. \tag{3.54}$$

By solving Eq. (3.54), we get (for detailed calculation, see Appendix A),

$$\begin{aligned}\langle \xi_N|\xi_N\rangle &= N-(N-1)\left(1-\frac{\chi_{N+1}}{\chi_N}\right) + N\frac{\chi_N}{\chi_{N-1}} - N + (N)\left(1-\frac{\chi_N}{\chi_{N-1}}\right) - N\left(\frac{\chi_N}{\chi_{N-1}}\right).\\ \langle \xi_N|\xi_N\rangle &= 1+(N-1)\left(\frac{\chi_{N+1}}{\chi_N}\right) - N\left(\frac{\chi_N}{\chi_{N-1}}\right). \qquad (3.55)\end{aligned}$$

Thus from Eq. (3.53) and Eq. (3.55), we can see that both conditions depends on ratio of normalization constant. Composite bosonic operator will act like a pure bosonic operator when $\frac{\chi_{N\pm1}}{\chi_N} \to 1$.

3.6 Bosonic Quality of Single Coboson

As normalization constant is given as

$$\chi_N = \frac{1}{N!}\langle 0|\hat{c}^N(\hat{c}^\dagger)^N|0\rangle. \tag{3.56}$$

χ_N is derived to be (see Appendix B)

$$\chi_N^F = N! \sum_{p_1 < p_2 ... < p_N} \lambda_{p_1} \lambda_{p_2} \lambda_{p_3} .... \lambda_{p_N}, \tag{3.57}$$

and

$$\chi_N^B = N! \sum_{p_1 \leq p_2 ... \leq p_N} \lambda_{p_1} \lambda_{p_2} \lambda_{p_3} .... \lambda_{p_N}, \tag{3.58}$$

Here χ_N^F refers to the normalization constant, when our composite particle is made up of pair of fermions. While χ_N^B is the normalization constant, when constituent particles are pair of bosons.

For the case of two particle wave function we can consider χ_N in terms of some specified Schmidt eigenvalues, which allows the very close and exact form to our system. Therefore we choose Schmidt eigenvalue

$$\lambda_n = (1 - z) z^n, n = 0, 1, 2, 3, 4.... \tag{3.59}$$

Here z has defined in the range of $0 < z < 1$. To find normalization constant, we make some assumptions. let us take

$$p_1 = r_N,$$

$$p_2 = r_N + r_{N-1},$$

$$p_3 = r_N + r_{N-1} + r_{N-2},$$

$$.$$

$$.$$

$$.$$

$$p_N = r_N + r_{N-1} + ... + r_1.$$

Substituting above values in Eq. (3.58), normalization constant for pair of bosons become

$$\chi^B{}_N = N!(1-z)^N \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\sum_{r_3=0}^{\infty} ......\sum_{r_N=0}^{\infty} z^{r_1+2r_2+3r_3+.....Nr_N}. \tag{3.60}$$

$$\Rightarrow \quad \chi^B_N = N!(1-z)^N \sum_{r_1=0}^{\infty} z^{r_1} \sum_{r_2=0}^{\infty} (z^2)^{r_2} \sum_{r_3=0}^{\infty} (z^3)^{r_3} + ... \tag{3.61}$$

For Geometric series when it converges for $|r| < 1$

$$s = \sum_{k=0}^{\infty} r^k = 1/(1-r).$$

By solving series in Eq. (3.61), equation become

$$\chi^B_N = \frac{N!(1-z)^N}{(1-z^1)(1-z^2)(1-z^3)......(1-z^N)}. \tag{3.62}$$

Similarly, for normalization constant of composite boson made of pair of fermions Eq.(3.57) become

$$\chi_N{}^F = N! \sum_{p_N>p>....p_2>p_1} \lambda_{p_1}\lambda_{p_2}....\lambda_{p_N}. \tag{3.63}$$

After few mathematical steps we reaches at equation below, that is

$$\chi^F{}_N = N!(1-z)^N \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\sum_{r_3=1}^{\infty} ......\sum_{r_N=1}^{\infty} z^{r_1+2r_2+3r_3+.....Nr_N}, \tag{3.64}$$

$$\chi^F_N = \frac{N!(1-z)^N}{(1-z^1)(1-z^2)(1-z^3)......(1-z^N)} z^{N(N-1)/2}. \tag{3.65}$$

Now we can find normalization ratio ,that are

$$\frac{\chi^B_{N+1}}{\chi^B_N} = \frac{(N+1)(1-z)}{1-z^{N+1}}, \tag{3.66}$$

$$\frac{\chi^F_{N+1}}{\chi^F_N} = \frac{z^N(N+1)(1-z)}{1-z^{N+1}}. \tag{3.67}$$

This normalization ratio shows modification in a system when we add new composite particle to N-particle state. Result shows that the ratio of normalization constant for

pair of boson is $\frac{\chi^B_{N+1}}{\chi^B_N} > 1$ and for pair of fermions it is $\frac{\chi^F_{N+1}}{\chi^F_N} < 1$. Here the difference seen between the normalization ratios of fermions and bosons is because of their difference in nature. Bosons are the particles that can stay together in the same state but fermions behave opposite to them as fermions follows Pauli- exclusion principle.
As the value of integer z is between zero and one. We can see from our result that when z approaches to one, ratio of normalization constants approaches to one and composite boson will behave like pure boson. For z less than one, composite will show deviation depending on the value of normalization ratio. In other words χ_{N+1}/χ_N is interconnected to the strength of correlation in composite boson. As particle behaves as a pure boson when z approaches to one, thus the quantum statistics associated to the constituent particles appears to be less important at that point.

3.6.1 Entanglement and Degree of Compositeness of a Composite Boson

Now we can relate quantum entanglement with the normalization constant by using the definition of quantum number κ. As for the Schmidt eigenvalues given by equation (3.59), Schmidt number κ defined in equation (3.40) becomes

$$\kappa = \frac{1+z}{1-z}$$

κ increases monotonically in the range of $0 < z < 1$. Degree of entanglement can be related to bose enhancement factor $\frac{\chi^B_{N+1}}{\chi^B_N}$ and $\frac{\chi^F_{N+1}}{\chi^F_N}$ when we express them in terms of z because of the relation $\kappa = \frac{1+z}{1-z}$. We notice that by increasing Schmidt number κ, $\frac{\chi^B_{N+1}}{\chi^B_N}$ and $\frac{\chi^F_{N+1}}{\chi^F_N}$ approaches to one. Specifically, we can show for $\kappa \gg N$

$$\frac{\chi_{N+1}}{\chi_N} \approx 1 + \frac{sN}{\kappa}, \tag{3.68}$$

where $s = 1$ for pair of bosons and $s = -1$ for pair of fermions inside composite particle.

In this section, we have discussed two-particle wave function and provided the basic information about the composite system. It tells us that the composite character is directly related to the correlation between the constituent element. Therefore, we can apply a composite representation to those particles which are strongly entangled. The ratio χ_{N+1}/χ_N is consider as the quantifier of the bosonic character where χ_N is basically a normalization factor which appears here due to the presence of composite behavior which is different from the ideal case. Ideally for pure bosons, $\chi_N = 1$ for all N.

3.7 Boundry for Bosonic Quality in Composite Boson

For further understanding of the bosonic nature of composite particle, we can compare how much the resultant $N + 1$ particle state will deviate from an ideal bosonic state of $N + 1$ particles when we add one composite boson in N state. When composite creation operator acts on N state of ideal bosons, we get $|N + 1\rangle$ state.

$$\hat{c}^{\dagger}|N\rangle = \alpha_{N+1}\sqrt{N+1}|N+1\rangle, \tag{3.69}$$

where $\alpha_{N+1} = \sqrt{\frac{\chi_{N+1}}{\chi_N}}$.
In Eq. (3.69), α_{N+1} or in other words the normalization ratio tells the bosonic quality of created composite boson. In research article [40], it is shown that the normalization ratio $\frac{\chi_N}{\chi_{N-1}}$ is non-increasing as number of composite bosons N increases. That simply means $\frac{\chi_{N+2}}{\chi_{N+1}} \leq \frac{\chi_{N+1}}{\chi_N}$ where $N \in 1, 2, 3, 4, ...$
Besides, in [41, 42], it is also shown that ratio $\frac{\chi_N}{\chi_{N-1}}$ is bounded by the purity from above and below. As purity of reduced density matrix $\rho_{A(B)}$ for a particle $A(B)$ is defined as $P = Tr\{\rho^2_{A(B)}\}$. As the state of our composite system is pure state, the purity of both constituent particles are guaranteed to be equal. The relationship between purity and composite particles was first proven as

$$1 - NP \leq \frac{\chi_{N+1}}{\chi_N} \leq 1 - P \tag{3.70}$$

As we know purity P physically quantifies the amount of entanglement between the pair of constituent particles. As range of purity P is 0 to 1, when P increases from zero to one, the strength of entanglement between pair of fermions (bosons) decreases

from infinity to zero.
From previous discussion, as we already know that, $\frac{\chi_{N+1}}{\chi_N}$ quantifies how much the annihilation operator and creation operator of the composite particle is similar to ideal one. Thus closer the normalization ratio is to one, the more creation and annihilation operators behave like ideal bosonic operator. So when purity goes to zero, $\frac{\chi_{N+1}}{\chi_N} \to 1$ and the composite boson behaves like an ideal boson. As we have seen that, for composite bosons, quantity that how much it is bosonic, is related to the entanglement. Afterwards, we present the proof of the inequality, but first we need to find out the amount of entanglement for single composite particle, that is

$$\hat{c}^\dagger|0\rangle = \sum_n \sqrt{\lambda_n}\hat{a}_n^\dagger \hat{b}_n^\dagger|0\rangle. \tag{3.71}$$

The reduced density matrix of the subsystem B is

$$\begin{aligned} \rho_B &= Tr_A(\hat{c}^\dagger|0\rangle\langle 0|\hat{c}), \\ &= \sum_n \lambda_n |n\rangle\langle n|. \end{aligned} \tag{3.72}$$

Thus the purity of subsystem B is

$$Tr(\rho_B^2) = \sum_n \lambda_n^2. \tag{3.73}$$

Now as normalization constant is

$$\begin{aligned} \chi_N &= \frac{1}{N!}\langle 0|\hat{c}^N(\hat{c}^\dagger)^N|0,\rangle \\ \chi_N &= \frac{1}{N!}(N!)^2 \sum_{p_1<p_2...<p_N} \lambda_{p_1}\lambda_{p_2}\lambda_{p_3}....\lambda_{p_N}, \\ \chi_N &= N! \sum_{p_1<p_2...<p_N} \lambda_{p_1}\lambda_{p_2}\lambda_{p_3}....\lambda_{p_N}, \\ \chi_N &= \sum_{p_1\neq p_2...\neq p_N} \lambda_{p_1}\lambda_{p_2}\lambda_{p_3}....\lambda_{p_N}, \\ \chi_N &= \sum_{p_1\neq p_2...\neq p_N} \prod_{j=1}^{N} \lambda_{p_j}. \end{aligned} \tag{3.74}$$

And after some steps of calculation, required inequality is proven to be

$$\chi_{N+1} - \chi_N(1 - NP) = \sum_{p_1 \neq p_2 \ldots \neq p_{N+1}} \prod_{j=1}^{N+1} \lambda_{p_j} - (1 - N\sum_p \lambda_p^2) \sum_{p_1 \neq p_2 \ldots \neq p_N} \prod_{j=1}^{N} \lambda_{p_j}. \quad (3.75)$$

In above equation many terms are identical, by canceling similar terms with each other we have

$$\chi_{N+1} - \chi_N(1 - NP) = \frac{N(N-1)}{2} \sum_{p_1 \neq p_2 \ldots \neq p_N} \prod_{j=1}^{N} \lambda_{p_j}(\lambda_{p_1} - \lambda_{p_2})^2 \geqslant 0. \quad (3.76)$$

This proves lower bound as $\frac{\chi_{N+1}}{\chi_N} \geqslant 1 - NP$. Similarly, we can prove upper bound, that is

$$(1 - P)\chi_N - \chi_{N+1} \geqslant 0.$$

Thus, the required inequality is $1 - NP \leq \frac{\chi_N+1}{\chi^N} \leq 1 - P$. These relations show the connection between an entanglement and compositeness of coboson.

Chapter 4

BEC of Identical Cobosons

The concept of Bose-Einstein condensation was initially presented for a gas of pure bosons. But experimentalists often use to observe Bose-Einstein condensation for composite particles, like atomic hydrogen or alkali atoms. These are the composite bosons made up of even number of fermions. While studying the composite particles often we ignore the internal structure of a particle but sometimes the internal structure plays an important role.

In this chapter, we consider a simple model of identical composite bosons and observe the phenomena of Bose-Einstein condensation. We consider the internal structure of coboson and study how the internal structure of composite bosonic particle could affect the phenomena of BEC. Also, we observe that how much the BEC of coboson has variations or similarities with respect to the BEC of elementary bosons. We assume that cobosonic particles are not interacting with each other, also their constituent particles are stable and temperature independent. These simplifications helps us to focus that how Bose-Einstein condensation depends on the internal structure of composite boson.

4.1 Effective Number of Cobosons

We observe the phenomenon of BEC by counting the number of particles in lowest energy state, larger the number of particles in ground state as compared to excited states of the system more the system experiencing BEC phenomenon. Here we use the cobosonic number operator to count the number of cobosons in a ground state [43]. Specifically, we take an effective mean number of $\langle \hat{N}_m \rangle$ cobosons on specific state $|N\rangle$ which is

$$\langle N|\hat{N}_m|N\rangle = 1 + (N-1)\,\frac{\chi_{N+1}}{\chi_N}. \tag{4.1}$$

The normalization ratio χ_{N+1}/χ_N shows the variation in properties of composite boson with respect to ideal boson and it depends on a degree of entanglement of constituent particles. Therefore, effective number is related to the internal structure of a composite particle and it helps to study the behavior of cobosons in BEC. In the following sections, we deal with the two main cases, when the coboson is comprised of a pair of fermions and when coboson is comprised of a pair of bosons and observe the phenomenon of BEC.

4.2 Bi-fermionic Cobosons

Let us consider a composite boson made up of two distinguishable fermions. Normalization ratio for a pair of fermions $[\chi_{N+1}/\chi_N]_F$ from Eq. (3.67) is

$$\frac{\chi^F_{N+1}}{\chi^F_N} = \frac{z^N(N+1)(1-z)}{1-z^{N+1}}. \tag{4.2}$$

We use this expression to study the effects on BEC due to the variation in the degree of entanglement. First, we consider a simple case of a two-level system and discuss the effective number of cobosons at different degrees of entanglement. Later we consider a more realistic model and discuss the behavior of ultracold cobosns at a minimum and a maximum value of entanglement.

4.2.1 Two-Level System

Let suppose we have N number of composite bosons in a two level-system shown in figure (4.1). If n number of cobosons are in one state of energy, then $N-n$ particles will be in the second state. Thus the state of a system is $|n, N-n\rangle$, that can be written as

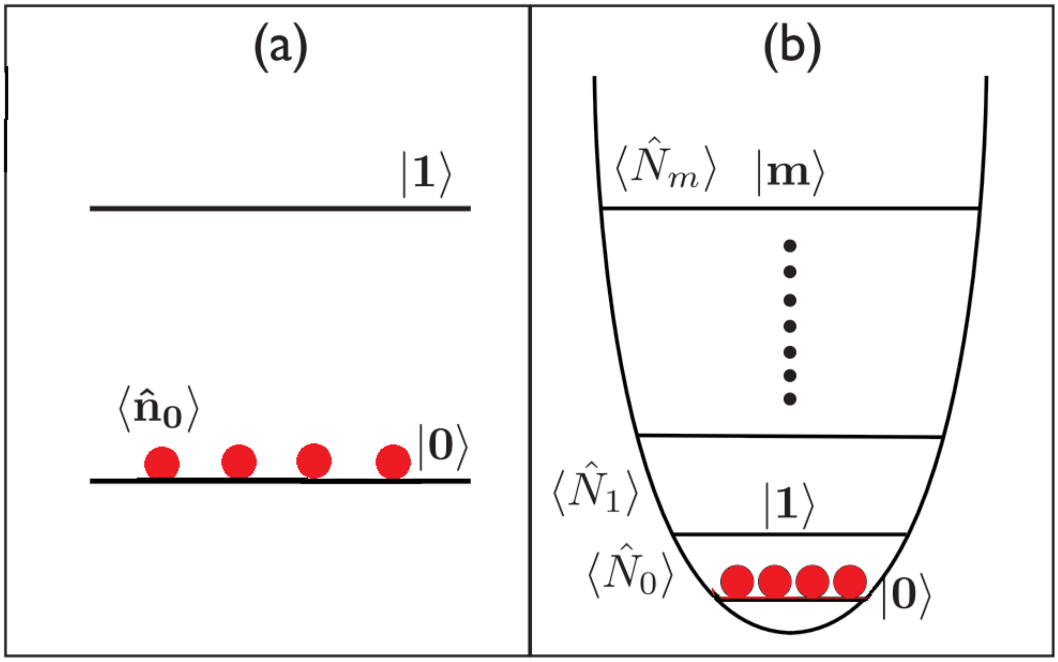

Figure 4.1: BEC using indistinguishable cobosons in (a) a two-level system and (b) a multi-level system [48].

$$|n, N-n\rangle = \frac{1}{\sqrt{\chi_n \chi_{N-n}}} \frac{(\hat{c}^\dagger)^N}{\sqrt{n!}} \frac{(\hat{c}^\dagger)^{N-n}}{\sqrt{(N-n)!}} |0,0\rangle. \tag{4.3}$$

Density operator of state $|n, N-n\rangle$ can be written as

$$\hat{\rho} = \frac{1}{Z} \left(\sum_{n=0}^{N} e^{-n\varepsilon_0/kT} e^{-(N-n)\varepsilon_1/kT} \right) |n, N-n\rangle\langle n, N-n|, \tag{4.4}$$

where Z is the partition function that is equal to

$$Z = \sum_{n=0}^{N} e^{-n\varepsilon_0/kT} e^{-(N-n)\varepsilon_1/kT}. \tag{4.5}$$

In the above equation, ε_0 and ε_1 are representing two energy levels in a system, k is Boltzmann constant and T is the temperature of the system.

The general expression for the effective number of composite bosons in any state is

$$\langle \hat{n} \rangle = Tr\left[\hat{c}^\dagger \hat{c} \rho\right], \tag{4.6}$$

The effective number of coboson in ground state can be derived as

$$\langle \hat{n}_0 \rangle = \sum_{n=0}^{N} P_n \langle \hat{c}_0^{\dagger} \hat{c}_0 \rangle. \tag{4.7}$$

As we know that expectation value of N cobosons in v_{th} energy level is

$$\langle N | \hat{c}_v^{\dagger} \hat{c}_v | N \rangle = \left[1 + (N-1) \frac{\chi_{N+1}}{\chi_N} \right]. \tag{4.8}$$

Putting the values from Eq. (4.8) in Eq. (4.7),

$$\langle \hat{n}_0 \rangle = \frac{1}{Z} \sum_{n=0}^{N} e^{-n\varepsilon_0/kT} e^{-(N-n)\varepsilon_1/kT} \left[1 + (n-1) \frac{\chi_{n+1}}{\chi_n} \right]. \tag{4.9}$$

We assume that energy of ground state ε_0 is equal to zero and $\varepsilon_1 = 1$, then Eq. (4.9) becomes

$$\langle \hat{n}_0 \rangle = \frac{1}{Z} \sum_{n=0}^{N} e^{-(N-n)/kT} \left[1 + (n-1) \frac{\chi_{n+1}}{\chi_n} \right]. \tag{4.10}$$

The partition function Z at $\varepsilon_0 = 0$ ans $\varepsilon_1 = 1$ becomes

$$Z = \frac{1 - e^{-(N+1)/kT}}{1 - e^{-1/kT}}.$$

Now we consider the role of constituent particles of cobosons and see what happens when constituent particles are maximally entangled, nearly maximal entangled and not entangled at all. In the case of composite boson made up of a pair of fermions, we already know the ratio of the normalization constant for a pair of fermions $[\chi_{n+1}/\chi_n]_F$, thus the effective number of cobosons at ground state can be written as

$$\langle \hat{n}_0 \rangle = \frac{1}{Z} \sum_{n=0}^{N} e^{-(N-n)/kT} \left[1 + (n-1) \frac{z^n (n+1)(1-z)}{1 - z^{n+1}} \right].$$

Minimum Entanglement

When the pair of fermions is not entangled at all (the parameter $z = 0$), the normalization ratio approach to zero. The effective number of cobosons in a ground state can be found by putting $z = 0$ in Eq. (3.67), thus $\langle \hat{n}_0 \rangle$ will be

$$\langle \hat{n}_0 \rangle_{z=0} = \frac{1}{Z} \sum_{n=0}^{N} e^{-(N-n)/kT}. \tag{4.11}$$

The effective number $\langle \hat{n}_0 \rangle_{z=0}$ becomes equal to one. Here it is important to notice that our result is not depending on the temperature. Physically, we can understand this as when the pair of fermions are separable their fermionic nature becomes apparent. Thus only two pairs of fermions can exist in two levels even at absolute zero temperature, as fermions obey the Pauli exclusion principle [44, 47].

Maximum Entanglement

When the constituent fermions of a particle are maximally entangled (the parameter z approach to 1), the normalization ratio of fermions also approach to one. By putting $z = 1$ in Eq. (3.67), the effective number of composite boson in the ground state becomes

$$\langle \hat{n}_0 \rangle_{z=1} = \frac{1}{Z} \sum_{n=0}^{N} e^{-(N-n)/kT} \left[1 + (n-1)\right], \tag{4.12}$$

$$\langle \hat{n}_0 \rangle_{z=1} = \frac{\left(1 - e^{-1/kT}\right)}{1 - e^{-(N+1)/kT}} \sum_{n=0}^{N} n e^{-(N-n)/kT}. \tag{4.13}$$

Solving the series in the above equation, the end result of $\langle \hat{n}_0 \rangle_{z=1}$ is

$$\langle \hat{n}_0 \rangle_{z=1} = \frac{1}{1 - e^{-(N+1)/kT}} \left[N - \frac{e^{-1/kT}\left(1 - e^{-N/kT}\right)}{1 - e^{-1/kT}} \right]. \tag{4.14}$$

From above equation, we can see that when temperature T goes to zero, number of composite bosons in ground state approach to N. Physical meaning of the Eq. (4.14) is that all cobosons are behaving like ideal bosons and at absolute zero temperature, all condensed in a ground state.

Moderate entanglement

The case when constituent fermions have entanglement slightly less than maximum entanglement ($\kappa \gg N$), we use the approximation $\chi_{n+1}/\chi_n \approx 1 - (n/\kappa)$ to derive the result. Effective number of cobosons in the ground state at nearly maximal entanglement is

$$\langle \hat{n}_0 \rangle = \frac{1}{Z} \sum_{n=0}^{N} n e^{-(N-n)/kT} \left[n - \frac{n(n-1)}{\kappa} \right], \tag{4.15}$$

$$\langle \hat{n}_0 \rangle = \langle n_0 \rangle_{z=1} - \frac{1}{Z} \sum_{n=0}^{N} \frac{n(n-1)}{\kappa} e^{-(N-n)/kT}, \tag{4.16}$$

By solving the series we find that

$$\langle \hat{n}_0 \rangle = \langle \hat{n}_0 \rangle_{z=1} - \frac{N}{\kappa \left(1 - e^{-(1+N)/kT}\right)} \left[N - 1 - \frac{2e^{-1/kT}}{1 - e^{-1/kT}} + \frac{2e^{-1/kT}\left(1 - e^{-N/kT}\right)}{N \left(1 - e^{-1/kT}\right)^2} \right]. \tag{4.17}$$

At absolute zero temperature $T \longrightarrow 0$, above expression become

$$\langle \hat{n}_0 \rangle = N - \frac{N(N-1)}{\kappa} \tag{4.18}$$

From the above equation, we can understand that at absolute zero temperature, condensation of cobosons are depending on the degree of entanglement. With the increase in value of κ, Eq. (4.18) shows that the effective value of $\langle n_0 \rangle$ will also increase. When Schmidt number κ is infinitely large then normalization ratio goes to one. This means that cobosons will behave like pure bosons and all composite bosons will condense in lowest energy level ε_0.

4.2.2 Realistic Model for Bi-fermions

Let us examine more realistic problem by considering N number of identical cobosons trapped in three-dimensional harmonic trap [48]. We fix the average number of cobosons in system equal to total number of cobosons considering the system in grand canonical ensemble. This makes a problem simple to solve. The effective number of particles in any particular state are defined as

$$\langle \hat{N}_m \rangle = \sum_{n=0}^{\infty} P_m \langle \hat{N}_m \rangle. \tag{4.19}$$

Here P_m is the probability of n number of particles in some particular state m. Number operator $\hat{N}_m = \hat{c}_m^\dagger \hat{c}_m$ is

$$\langle \hat{N}_m \rangle = \frac{1}{Z_m} \sum_{n=0}^{\infty} e^{-(\varepsilon_m - \mu)n/kT} \left[1 + (n-1) \frac{\chi_{n+1}}{\chi_n} \right], \tag{4.20}$$

where Z_m is partition function of that specific state and it is equal to $\left(1 - e^{-(\varepsilon_m - \mu)n/kT}\right)^{-1}$. We fix an average number of particles equal to the total number of particles $\langle \hat{N} \rangle = N$, such that

$$N = \sum_{m=0}^{\infty} \langle \hat{N}_m \rangle.$$

Minimum Entanglement

Now for the case when a pair of fermions in coboson are not entangled, we see how cobosons will behave inside a harmonic trap. For harmonic oscillator general expression of energy is $E_m = \hbar\omega(m_x + m_y + m_z + 3/2)$, where $m_x, m_y, m_z = 0, 1, \ldots$. We consider the energy of a ground state ε_0 equal to zero and find out the number of cobosons in the ground state. By substituting the values of normalization ratio from Eq. (3.67) in Eq. (4.20) and solving further, we find the effective number of cobosons $\langle \hat{N}_0 \rangle$ equal to one . This shows that only one coboson can occupy energy level ε_0 when constituent fermions are disentangled.

Maximum Entanglement

When we consider a pair of fermions maximally entangled by putting the value of z in Eq. (3.67) equal to one, normalization ratio also goes to one. Effective number of cobosons at ground state gives the value

$$\langle \hat{N}_0 \rangle = \frac{1}{e^{-\mu/kT} - 1}. \tag{4.21}$$

This is Bose-Einstein distribution for the ground state, thus the composite bosons are also behaving like ideal bosons, when their constituents have maximum entanglement.

Moderate Entanglement

Let us consider that pair of fermions have slightly less entanglement ($z = (1-\delta), \delta \ll 1$). We approximate normalization ratio of fermion to $\chi_{n+1}/\chi_n \sim z^n$, thus $\chi_{n+1}/\chi_n \approx$

$(1-n\delta)$. The effective number for any specific energy level m is

$$\langle\hat{N}_m\rangle = \frac{1}{Z_m}\sum_{n=0}^{\infty} e^{-(\varepsilon_m-\mu)n/kT}\left[n - n(n-1)\delta\right],$$

$$\langle\hat{N}_m\rangle = \frac{1}{e^{(\varepsilon_m-\mu)/kT}-1} + 2\delta\left[\frac{1}{e^{(\varepsilon_m-\mu)/kT}-1} - \frac{e^{(\varepsilon_m-\mu)/kT}}{\left(e^{(\varepsilon_m-\mu)/kT}-1\right)^2}\right]. \tag{4.22}$$

Thus the effective number of cobosons at lowest energy state will be

$$\langle\hat{N}_0\rangle = \frac{1+2\delta}{e^{-\mu/kT}-1} - \frac{2\delta e^{(-\mu)/kT}}{\left(e^{-\mu/kT}-1\right)^2}. \tag{4.23}$$

The total effective number of composite bosons is $N = \sum_m \langle\hat{N}_m\rangle$, therefore

$$N = \sum_m^{\infty} \frac{1}{e^{(\varepsilon_m-\mu)/kT}-1} + 2\delta\left[\frac{1}{e^{(\varepsilon_m-\mu)/kT}-1} - \frac{e^{(\varepsilon_m-\mu)/kT}}{\left(e^{(\varepsilon_m-\mu)/kT}-1\right)^2}\right],$$

where $m = m_x, m_y, m_z$ and $\varepsilon_m = \hbar\omega\,(m_x+m_y+m_z+3/2)$ or $\varepsilon_m = \hbar\omega\,(p+3/2)$ by supposing a parameter $p = m_x+m_y+m_z$. The reference temperature for $3D$ potential is $\hbar\omega/kT = T_0/(TN^{1/3})$. Thus we get

$$N = \sum_{p=0}^{\infty}\left[(1+2\delta)\frac{\frac{1}{2}p^2+\frac{3}{2}p+1}{e^{(T_0/TN^{1/3})p+\alpha}-1} - 2\delta\frac{\left(\frac{1}{2}p^2+\frac{3}{2}p+1\right)e^{(T_0/TN^{1/3})p+\alpha}}{(e^{(T_0/TN^{1/3})p+\alpha}-1)^2}\right],$$

where $\alpha = 3T_0/2TN^{1/3} - \mu/kT$. By solving this equation, the end result we get is the sum of cobosons present in the ground state plus the number of cobosons present in all other excited states [48] . This generalized equation depends on two main things, temperature and entanglement. By variation in the temperature T with respect to transition temperature T_0, the number of cobosons in the ground state will vary. Also the variation in the degree of entanglement effects the behavior of cobosons in the phenomenon of BEC.

4.3 Bi-fermionic Cobosons

We consider a composite boson made up of two distinguishable bosons. Normalization ratio for a pair of bosons $[\chi_{N+1}/\chi_N]_B$ from Eq. (3.66) is

$$\frac{\chi_{N+1}^B}{\chi_N^B} = \frac{(N+1)(1-z)}{1-z^{N+1}}. \tag{4.24}$$

First we consider a simple case of a two-level system and then multi-level system and we check the effective number of cobosons at different degrees of entanglement.

4.3.1 Two-Level System

Let us consider that we have N number of cobosons in a two-level system. If one level of a system let say ground level has n number of particles then the other one will have $N-n$ cobosons. Density operator of state $|n, N-n\rangle$ can be written as

$$\hat{\rho} = \frac{1}{Z}\left(\sum_{n=0}^{N} e^{-n\varepsilon_0/kT} e^{-(N-n)\varepsilon_1/kT}\right) |n, N-n\rangle\langle n, N-n|, \tag{4.25}$$

where Z is the partition function that is equal to

$$Z = \sum_{n=0}^{N} e^{-n\varepsilon_0/kT} e^{-(N-n)\varepsilon_1/kT}. \tag{4.26}$$

The effective number of bosons in the ground state $\langle \hat{n}_0 \rangle$ is

$$\langle \hat{n}_0 \rangle = \frac{1}{Z}\sum_{n=0}^{N} e^{-n\varepsilon_0/kT} e^{-(N-n)\varepsilon_1/kT} \left[1 + (n-1)\left[\frac{\chi_{n+1}}{\chi_n}\right]_B\right]. \tag{4.27}$$

Minimum Entanglement

When the pair of bosons have minimum entanglement (the parameter $z = 0$), normalization ratio of bosons approach to $n+1$. We consider that ground state energy ε_0 is equal to zero and energy of first level ε_1 equal to one. Effective number $\langle \hat{n}_0 \rangle$ is

$$\langle \hat{n}_0 \rangle = \frac{1}{Z}\sum_{n=0}^{N} n^2 e^{-(N-n)/kT}. \tag{4.28}$$

After solving the above series , the result we obtain is

$$\langle \hat{n}_0 \rangle = \frac{N}{1-e^{-(1+N)/kT}}\left[N - \frac{2e^{-1/kT}}{1-e^{-1/kT}} + \frac{e^{-1/kT}\left(1+e^{-1/kT}\right)\left(1-e^{-N/kT}\right)}{N\left(1-e^{-1/kT}\right)^2}\right]. \tag{4.29}$$

At temperature $T \to 0$ effective number $\langle \hat{n}_0 \rangle_{z=0} \to N^2$. This result shows that, at minimum entanglement each constituent boson started to show bosonic behavior independently. Thus the effective number of cobosons at ground state increased with increase in degree of entanglement.

We can check it directly from the coboson number operator. At minimum entanglement, $\hat{c}^\dagger$ can be represented by direct product of $\hat{a}^\dagger\hat{b}^\dagger$. At temperature $T \to 0$, the state of composite bosons at ground state can be described by $|N\rangle$, where N is the total number of cobosons. The effective number of cobosons in lowest energy state is given by

$$\langle N|\hat{c}^\dagger\hat{c}|N\rangle = \langle N_a, N_b|\hat{a}^\dagger\hat{a}\hat{b}^\dagger\hat{b}|N_a, N_b\rangle = N^2. \tag{4.30}$$

Here a and b are representing different modes.

Maximum Entanglement

When the constituent bosons are maximally entangled ($z = 1$), the effective number $\langle\hat{n}_0\rangle$ is given by

$$\langle\hat{n}_0\rangle_{z=1} = \frac{\left(1 - e^{-1/kT}\right)}{1 - e^{-(N+1)/kT}} \sum_{n=0}^{N} n e^{-(N-n)\varepsilon_1/kT}. \tag{4.31}$$

Solving the series in the above equation, the end result of $\langle n_0\rangle_{z=1}$ is

$$\langle\hat{n}_0\rangle_{z=1} = \frac{1}{1 - e^{-(N+1)/kT}} \left[N - \frac{e^{-1/kT}\left(1 - e^{-N/kT}\right)}{1 - e^{-1/kT}}\right]. \tag{4.32}$$

At absolute zero temperature ($T \longrightarrow 0$), the effective number of cobosons in ground state approach to N. At maximum entanglement, all cobosons are behaving like pure boson and all condensed in a ground state.

Moderate Entanglement

When the constituent bosons of coboson are entangled slightly less than the maximum entanglement, we approximate normalization ratio of bosons to $\chi_{n+1}/\chi_n \approx 1 + n/\kappa$. Effective number $\langle\hat{n}_0\rangle$ is given by

$$\langle\hat{n}_0\rangle = \langle\hat{n}_0\rangle_{z=1} + \frac{N}{\kappa(1 - e^{-(1+N)/kT})} \left[N + 1 - \frac{2e^{-1/kT}}{1 - e^{-1/kT}} + \frac{2e^{-2/kT}\left(1 - e^{-N/kT}\right)}{N\left(1 - e^{-1/kT}\right)^2}\right]. \tag{4.33}$$

When temperature $T \to 0$, we obtain

$$\langle \hat{n}_0 \rangle = N + N(N+1)/\kappa \geqslant N.$$

At zero temperature, $\langle \hat{n}_0 \rangle$ is depending on Schmidt number κ. If Schmidt number approaches infinity this means that the entanglement between constituent bosons is increasing. Thus the effective number of cobosons $\langle \hat{n}_0 \rangle$ decrease and approach to N.

4.3.2 Realistic Model for Bi-bosons

We consider N number of identical cobosons trapped in three-dimensional harmonic trap [48]. When constituent bosons are not correlated ($z = 0$), by using Eq. (4.20), the effective number of cobosons for ground state is

$$\langle \hat{N}_0 \rangle = \frac{1}{Z_0} \sum_{n=0}^{\infty} e^{-(\varepsilon_0 - \mu)n/kT} n^2. \tag{4.34}$$

Putting the value of a partition function and solving Eq. (4.34), we find $\langle \hat{N}_0 \rangle$ which is

$$\langle \hat{N}_0 \rangle = \frac{e^{\mu/kT}\left(1 + e^{\mu/kT}\right)}{\left(1 - e^{\mu/kt}\right)^2}. \tag{4.35}$$

The term $e^{\mu/kT}$ is fugacity and we have considered ground level energy equal to zero. As Bose-Einstein distribution for ideal bosons in the ground state is $1/e^{-\mu/kT} - 1$, $\langle \hat{N}_0 \rangle$ is always larger than N. When a pair of bosons is maximally entangled $z = 1$, the effective number of bosons is same as Bose-Einstein distribution i.e, $1/e^{-\mu/kT} - 1$. Thus at absolute zero temperature $\langle \hat{N}_0 \rangle$ approach to N.

Chapter 5

Summary and Conclusion

In this thesis, we have discussed some exciting features about the bosonic nature of composite bosons. Earlier, in the content of Bose-Einstein condensation, it was not clear that why strongly correlated bunch of fermions show bosonic behavior. Later on, it was found, in the content of of quantum information, that such a connection between a pair of fermions maybe expressible by quantum entanglement. There is a possibility that entanglement is not the only reason for a group of fermions to behave like a boson, but it provides a convincing reason to do so. As entangled particles are independent of local correlation, we can entangle two fermions that are spatially separated and together they can behave like a boson. Describing the correlations of fermions with the help of entanglement is something new and highly interesting.

In this thesis, the primary focus of our work is on the composite particles made up of two distinguishable particles, both either bosons or fermions e.g., a hydrogen atom. The main reason behind it is that relation between entanglement and bipartite composites is better developed and understood than tripartite or more system. In this regard, we have used second quantization formalism to represent composite particles by means of their annihilation and creation operators and constructed the quantum mechanical states (number states) for cobosons. Using these commutation relations of bosonic annihilation and creation operator, we derive some conditions for the composite character of particles and relate it with entanglement through Schmidt number. Schmidt numbers are the parameters that can be used to determine the extent of entanglement.

Finally, we studied deviation of coboson Bose-Einstein condensation from the behavior of Bose-Einstein condensation compose of pure bosons. We have seen that effective number of composite bosons are related to the degree of correlation between paired particles. For the maximum level of entanglement, the effective number of composite bosons is equal to the total number of cobosons in a system. When the degree of entanglement between constituents is weak, the effective number of composite bosons is smaller (larger)than the total number of cobosons composed of fermions (bosons).

Appendix A

As from Eq. (3.48)

$$\begin{aligned}
|\xi_N\rangle &= \hat{c}|N\rangle - \alpha_N\sqrt{N}|N-1\rangle \\
\langle\xi_N| &= \langle N|\hat{c}^\dagger - \alpha_N\sqrt{N}\langle N-1| \\
\langle\xi_N|\xi_N\rangle &= \langle N|\hat{c}^\dagger\hat{c}|N\rangle + {\alpha_N}^2 N - \alpha_N\sqrt{N}\langle N|\hat{c}^\dagger|N-1\rangle - \alpha_N\sqrt{N}\langle N-1|\hat{c}|N\rangle
\end{aligned} \tag{5.1}$$

Now solving first term in above equation.

$$\begin{aligned}
\langle N|\hat{c}^\dagger\hat{c}|N\rangle &= \frac{1}{N!}\langle 0|\hat{c}^N\hat{c}^\dagger\hat{c}(\hat{c}^\dagger)^N|0\rangle \\
&= \frac{1}{N!}(\langle 0|\hat{c}^N\hat{c}^\dagger(N(\hat{c}^\dagger)^{N-1} - N(N-1)(\hat{c}^\dagger)^{N-2}b^\dagger)|0\rangle) \\
&= \frac{N}{N!}\langle 0|\hat{c}^N\hat{c}^\dagger(\hat{c}^\dagger)^{N-1}|0\rangle - \frac{N(N-1)}{N!}\langle 0|\hat{c}^N\hat{c}^\dagger(\hat{c}^\dagger)^{N-2}b^\dagger|0\rangle \\
&= N - \frac{N(N-1)}{N!}\langle 0|\hat{c}^N(\hat{c}^\dagger)^{N-1}b^\dagger|0\rangle
\end{aligned} \tag{5.2}$$

Since

$$[\Delta, (\hat{c}^\dagger)^N] = \Delta(\hat{c}^\dagger)^N - (\hat{c}^\dagger)^N\Delta$$

That means,

$$\Delta(\hat{c}^\dagger)^N|0\rangle = (\hat{c}^\dagger)^{N-1}[\triangle, \hat{c}^\dagger]|0\rangle + [\Delta, (\hat{c}^\dagger)^{N-1}]\hat{c}^\dagger|0\rangle + (\hat{c}^\dagger)^N\Delta|0\rangle$$

As $[\Delta, \hat{c}^\dagger] = 2b^\dagger$, by putting it in above equation, we obtain

$$\begin{aligned}
\Delta(\hat{c}^\dagger)^N|0\rangle &= 2(\hat{c}^\dagger)^{N-1}b^\dagger|0\rangle + [\Delta, (\hat{c}^\dagger)^{N-1}]\hat{c}^\dagger|0\rangle \\
&= 2(\hat{c}^\dagger)^{N-1}b^\dagger|0\rangle + ((\hat{c}^\dagger)^{N-2}[\Delta, \hat{c}^\dagger]\hat{c}^\dagger + [\Delta, (\hat{c}^\dagger)^{N-2}](\hat{c}^\dagger)^2)|0\rangle \\
\Delta(\hat{c}^\dagger)^N|0\rangle &= 2(\hat{c}^\dagger)^{N-1}b^\dagger|0\rangle + 2(\hat{c}^\dagger)^{N-1}b^\dagger|0\rangle + [\Delta, (\hat{c}^\dagger)^{N-2}]\hat{c}^\dagger|0\rangle
\end{aligned}$$

Taking $b^\dagger$ common from above equation, we get

$$\begin{aligned}
\Delta {c^\dagger}^N|0\rangle &= 2[(\hat{c}^\dagger)^{N-1} + (\hat{c}^\dagger)^{N-1} + ...]b^\dagger|0\rangle \\
\Delta(\hat{c}^\dagger)^N|0\rangle &= 2N(\hat{c}^\dagger)^{N-1}b^\dagger|0\rangle \qquad (5.3)
\end{aligned}$$

Putting the value from Eq. (5.3) in Eq. (5.2), we get

$$\langle N|\hat{c}^\dagger\hat{c}|N\rangle = N - \frac{N-1}{2}\langle\Delta\rangle_N$$

As $\langle\Delta\rangle_N = 2(1 - \frac{\chi_{N+1}}{\chi_N})$

$$\implies \qquad \langle N|\hat{c}^\dagger\hat{c}|N\rangle = N - \frac{N-1}{2}\{2(1 - \frac{\chi_{N+1}}{\chi_N})\}$$

or

$$\langle N|c^\dagger c|N\rangle = N - (N-1)(1 - \frac{\chi_{N+1}}{\chi_N}) \qquad (5.4)$$

Now we solve the third term in Eq. (5.1).

$$\begin{aligned}
\alpha_N\sqrt{N}\langle N|\hat{c}^\dagger|N-1\rangle &= \frac{\alpha_N\sqrt{N}}{\sqrt{N!}\sqrt{(N-1)!}\sqrt{\chi_N}\sqrt{\chi_{N-1}}}\langle 0|\hat{c}^N\hat{c}^\dagger(\hat{c}^\dagger)^{N-1}|0\rangle \\
&= \frac{\alpha_N\sqrt{N}}{\sqrt{N!}\sqrt{(N-1)!}\sqrt{\chi_N}\sqrt{\chi_{N-1}}}\langle 0|\hat{c}^N(\hat{c}^\dagger)^{N}|0\rangle \\
&= \frac{\alpha_N\sqrt{N}}{\sqrt{N!}\sqrt{(N-1)!}\sqrt{\chi_N}\sqrt{\chi_{N-1}}}N!\chi_N \\
&= \frac{\alpha_N(\sqrt{N})^2}{N!}N!\sqrt{\frac{\chi_N}{\chi_{N-1}}} \\
&= N\frac{\chi_N}{\chi_{N-1}} \qquad (5.5)
\end{aligned}$$

Now we solve the fourth term in Eq. (5.1).

$$\alpha_N\sqrt{N}\langle N-1|\hat{c}|N\rangle = \frac{\alpha_N\sqrt{N}}{\sqrt{N!}\sqrt{(N-1)!}\sqrt{\chi_N}\sqrt{\chi_{N-1}}}\langle 0|\hat{c}^{N-1}\hat{c}(\hat{c}^\dagger)^N|0\rangle$$

$$\alpha_N\sqrt{N}\langle N-1|\hat{c}|N\rangle = \frac{\alpha_N\sqrt{N}}{\sqrt{N!}\sqrt{(N-1)!}\sqrt{\chi_N}\sqrt{\chi_{N-1}}}\langle 0|c^{N-1}(N(\hat{c}^\dagger)^{N-1} - N(N-1)$$

$$(\hat{c}^\dagger)^{N-2}b^\dagger|0\rangle)$$

$$= N - \frac{\alpha_N(N-1)}{(N-1)!\sqrt{\chi_N}\sqrt{\chi_{N-1}}}\langle 0|\hat{c}^{N-1}(\hat{c}^\dagger)^{N-2}b^\dagger|0\rangle$$

$$= N - \frac{\alpha_N(N)}{2(N-1)!\sqrt{\chi_N}\sqrt{\chi_{N-1}}} < 0|\hat{c}^{N-1}\Delta(\hat{c}^\dagger)^N|0\rangle$$

$$= N - \frac{N}{2}\langle N-1|\Delta|N-1\rangle$$

$$= N - \frac{1}{2}(N)2(1-\frac{\chi_N}{\chi_{N-1}})$$

$$= N-(N)(1-\frac{\chi_N}{\chi_{N-1}}) \tag{5.6}$$

Putting values from Eq. (5.4) , Eq. (5.5) and Eq. (5.6) in Eq. (5.1), we find $\langle\varepsilon_N|\varepsilon_N\rangle$

$$\langle\varepsilon_N|\varepsilon_N\rangle = N-(N-1)(1-\frac{\chi_{N+1}}{\chi_N}) + N\frac{\chi_N}{\chi_{N-1}} - N + (N)(1-\frac{\chi_N}{\chi_{N-1}}) - N\frac{\chi_N}{\chi_{N-1}}$$

$$\langle\varepsilon_N|\varepsilon_N\rangle = 1+(N-1)(\frac{\chi_{N+1}}{\chi_N}) - N\frac{\chi_N}{\chi_{N-1}} \tag{5.7}$$

Appendix B

As $\hat{c}^\dagger$ is defined in Eq. (3.41) as $\hat{c}^\dagger = \sum_{n=0}^{\infty} \sqrt{\lambda_n} \hat{a}_n^\dagger \hat{b}_n^\dagger.$, then

$$\langle 0|c^N {c^\dagger}^N|0\rangle = \langle 0| \sum_{p_1 p_2 \ldots p_N} \sum_{q_1 q_2 \ldots q_N} \sqrt{\lambda_{p_1}\lambda_{p_2\ldots}\lambda_{p_N}} \sqrt{\lambda_{q_1}\lambda_{q_2\ldots}\lambda_{q_N}} (a_{p_N} b_{p_N} a_{p_{N-1}} b_{p_{N-1}} \ldots\ldots . a_{p_2} b_{p_2}$$

$$a_{p_1} b_{p_1})(a^\dagger_{q_N} b^\dagger_{q_N} a^\dagger_{q_{N-1}} b^\dagger_{q} {}_{N-1} \ldots\ldots . a^\dagger_{q_1} b^\dagger_{q_1})|0\rangle$$

$$\langle 0|c^N {c^\dagger}^N|0\rangle = \langle 0| \sum_{p_1 p_2 \ldots p_N} \sum_{q_1 q_2 \ldots q_N} \sqrt{\lambda_{p_1}\lambda_{p_2\ldots}\lambda_{p_N}} \sqrt{\lambda_{q_1}\lambda_{q_2\ldots}\lambda_{q_N} a_{p_N}} a_{p_{N-1}} \ldots\ldots . a_{p_2} a_{p_1}$$

$$a^\dagger_{q_N} a^\dagger_{q_{N-1}} \ldots\ldots . a^\dagger_{q_1} b_{p_n} b_{p_{N-1}} \ldots\ldots . b_{p_2} b_{p_1} b^\dagger_{q_N} b^\dagger_{q_{N-1}} \ldots\ldots . b^\dagger_{q_1}|0\rangle$$

$$\begin{aligned}
\langle 0|c^N {c^\dagger}^N|0\rangle &= N!\langle 0| \sum_{p_1<p_2<p_3\ldots .<p_N} \lambda_{p_1}\lambda_{p_2}\ldots . . \lambda p_N \quad a_{p_N} a_{p_{N-1}} \ldots\ldots a_{p_1} \;\; a^\dagger_{p_1} a^\dagger_{p_2} \ldots . . a^\dagger_{p_N}|0\rangle \\
&= N! \sum_{p_1\langle p_2<p_3\ldots<p_N} \lambda_{p_1}\lambda_{p_2}\ldots . \lambda p_N \quad \langle 0|a_{p_N} a_{p_{N-1}} \ldots\ldots a_{p_1} a^\dagger_{p_1} a^\dagger_{p_2} \ldots . . a^\dagger_{p_N}|0\rangle
\end{aligned}$$

$$\langle 0|c^N c^{\dagger N}|0\rangle = (N!)^2 \sum_{p_1<p_2<p_3....<p_N} \lambda_{p_1}\lambda_{p_2}\lambda_{p_3}.....\lambda_{p_N}$$

As we already knew that

$$\langle 0|c^N c^{\dagger N}|0\rangle = N!\chi_N$$

This means that

$$\chi_N^F = N! \sum_{p_1<p_2...<p_N} \lambda_{p_1}\lambda_{p_2}\lambda_{p_3}....\lambda_{p_N}$$

Bibliography

[1] S. N. Bose, *Planck's law and the light quantum hypothesis*, Journal of Astrophysics and Astronomy, **15** (1994) 3.

[2] E. A. Cornell, and C. E. Wieman, *Nobel Lecture: Bose-Einstein condensation in a dilute gas, the first 70 years and some recent experiments*, Reviews of Modern Physics, **74** (2002) 875.

[3] M. H. Anderson, J. R. Ensher, M. R. Matthews, C. E. Wieman, and E.A Cornell, *Observation of Bose-Einstein Condensation in a Dilute Atomic Vapor*, Science, **269** (1995) 198.

[4] C. C. Bradley, C.A Sackett, J.J Tollett, and R.G Hulet, *Evidence of Bose-Einstein Condensation in an Atomic Gas with Attractive Interactions*, Physical Review Letters, **75** (1995) 1687.

[5] M. R. Andrews, C. G. Townsend, H. J. Miesner, D. S. Durfee, D. M. Kurn, and W. Ketterle, *Observation of Interference Between Two Bose-Einstein Condensates*, Science, **275** (1997) 637.

[6] M. Greiner, O. Mandel, T. Esslinger, T. W Hänsch, and I. Bloch, *Quantum phase transition from a superfluid to a Mott insulator in a gas of ultracold atoms*, Nature, **415** (2002) 39.

[7] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Theory of superconductivity*, Physical Review, **108** (1957) 1175.

[8] D. D. Osheroff, R. C. Richardson, and D. M. Lee, *Evidence for a new phase of solid He 3*, Physical Review Letters, **28** (1972) 885.

[9] C. K. Law, *Quantum entanglement as an interpretation of bosonic character in composite two-particle systems*, Physical Review A, **71** (2005) 034306.

[10] L. E. Ballentine, *Quantum Mechanics*, A Modern Development Second Edition: World Scientific Publishing Company, (2014).

[11] M. Combescot, O. Betbeder-Matibet, and F. Dubin, *The many-body physics of composite bosons*, Physics Reports, **463** (2008) 215.

[12] A. Einstein, B. Podolsky, and N. Rosen, *Can quantum-mechanical description of physical reality be considered complete?*, Physical review, **47** (1935) 777.

[13] J. S. Bell, *On The Einstein Podolsky Rosen Paradox*, Physics, **1** (1964) 195.

[14] D. M. Greenberger, M. A. Horne, A. Shimony, and A. Zeilinger, *Bell's theorem without inequalities*, American Journal of Physics, **58** (1990) 1131.

[15] J. F. Clauser, M. A. Horne, A. Shimony, and R. A. Holt, *Proposed experiment to test local hidden-variable theories*, Physical review letters, **23** (1969) 880.

[16] A. Aspect, P. Grangier, and G. Roger, *Experimental realization of Einstein-Podolsky-Rosen-Bohm Gedankenexperiment: a new violation of Bell's inequalities*, Physical review letters, **49** (1982) 91.

[17] Z. Y. Ou, and L. Mandel, *Violation of Bell's inequality and classical probability in a two-photon correlation experiment*, Physical Review Letters, **61** (1988) 50.

[18] C. H. Bennett, G. Brassard, C. Crépeau, R. Jozsa, A. Peres, and W. K. Wootters, *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels*, Physical review letters, **70** (1993) 1895.

[19] A. K. Ekert, *Quantum cryptography based on Bell's theorem*, Physical review letters, **67** (1991) 661.

[20] C. H. Bennett, and S. J. Wiesner, *Communication via one-and two-particle operators on Einstein-Podolsky-Rosen states*, Physical review letters, **69** (1992) 2881.

[21] S. Jochim, M. Bartenstein, A. Altmeyer, G. Hendl, S. Riedl, C. Chin, J. H. Denschlag, and R. Grimm, *Bose-Einstein condensation of molecules*, Science, **302** (2003), 2101.

[22] W. Ketterle, *When Atoms Behave as Waves: Bose–Einstein Condensation and the Atom Laser (Nobel Lecture)*, ChemPhysChem, **3** (2002) 736.

[23] I. Bloch, J. Dalibard, and W. Zwerger, *Many-body physics with ultracold gases*, Reviews of modern physics, **80** (2008) 885.

[24] P. Meystre, *Atom optics*, **33** (2001), Springer Science and Business Media.

[25] S. Giorgini, L. P. Pitaevskii, and S. Stringari, *Condensate Fraction And Critical Temperature Of A Trapped Interacting Bose Gas*, Physical Review A, **54** (1996) R4633.

[26] M. A. Nielsen, and I. Chuang, *Quantum computation and quantum information*, New York: Cambridge University Press, (2000).

[27] R. Grobe, K. Razazewski, and J. H. Eberly, *Measure of electron-electron correlation in atomic physics*, Journal of Physics B: Atomic, Molecular and Optical Physics, **27** (1994) L503.

[28] A. Ekert, and P. L. Knight, *Entangled quantum systems and the Schmidt decomposition*, American Journal of Physics, **63** (1995) 415.

[29] J.H. Eberly, K. W. Chan, and C. K. Law, *Schmidt-mode analysis of entanglement for quantum information studies*, In Quantum Communication and Information Technologies (2003) 1-12, Springer, Dordrecht.

[30] M. V. Fedorov and N. I. Miklin, *Schmidt modes and entanglement*, Contemporary Physics, **55** (2014) 94.

[31] S. Y. Lee, C. Noh, and D. Kaszlikowski, *Identifying quantumness via addition-then-subtraction operation*, Physical Review A, **88** (2013) 013806.

[32] M. Combescot, *"Commutator formalism" for pairs correlated through Schmidt decomposition as used in Quantum Information*, EPL (Europhysics Letters), **96** (2011) 60002.

[33] P. Kurzyński, R. Ramanathan, A. Soeda, T. K. Chuan, and D. Kaszlikowski, *Particle addition and subtraction channels and the behavior of composite particles*, New Journal of Physics, **14** (2012) 093047.

[34] M. Combescot, and O. Betbeder-Matibet, *General Many-Body Formalism for Composite Quantum Particles*, Physical review letters, **104** (2010) 206404.

[35] Z. Lasmar, D. Kaszlikowski, and P. Kurzyński, *Nonlocal bunching of composite bosons*, Physical Review A, **96** (2017) 032325.

[36] M. Combescot, and C. Tanguy, *New criteria for bosonic behavior of excitons*, EPL (Europhysics Letters), **55** (2001) 390.

[37] M. C. Tichy, P. A. Bouvrie, and K. Mølmer, *Bosons Made of Bosons: Condensation of the Whole or Condensation of the Parts?*, nuclear physics, **21** (2013) 23.

[38] M. C. Tichy, P. A. Bouvrie, and K. Mølmer, *Two-boson composites*, Physical Review A, **88** (2013) 061602.

[39] M. Combescot, X. Leyronas, and C. Tanguy, *On the N-exciton normalization factor*, The European Physical Journal B-Condensed Matter and Complex Systems, **31** (2003) 17.

[40] C. Chudzicki, O. Oke, and W. K. Wootters, *Entanglement and composite bosons*, Physical review letters, **104** (2010) 070402.

[41] M. C. Tichy, P. A. Bouvrie, and K. Mølmer, *Bosonic behavior of entangled fermions*, Physical Review A, **86** (2012) 042317.

[42] M. C. Tichy, P. A. Bouvrie, and K. Mølmer, *How bosonic is a pair of fermions?*, Applied Physics B, **117** (2014) 785.

[43] A. Thilagam, *Influence of the Pauli exclusion principle on scattering properties of cobosons*, Physica B: Condensed Matter, **457** (2015) 232.

[44] M. Combescot, F. Dubin, and M. A. Dupertuis, *Role of fermion exchanges in statistical signatures of composite bosons*, Physical Review A, **80** (2009) 013612.

[45] S. Rombouts, D. Van Neck, K. Peirs, and L. Pollet, *Maximum occupation number for composite boson states*, Modern Physics Letters A, **17** (2002) 1899.

[46] S. S. Avancini, J. R. Marinelli, and G. Krein, *Compositeness effects in the Bose–Einstein condensation*, Journal of Physics A: Mathematical and General, **36** (2003) 9045.

[47] P. Sancho, *Compositeness effects, Pauli's principle and entanglement*, Journal of Physics A: Mathematical and General, **39** (2006) 12525.

[48] S.Y. Lee, J. Thomsan, S. Raeisi, P. Kurzyński and D. Kaszlikowski, *Quantum information approach to Bose–Einstein condensation of composite bosons*, New Journal of Physics, **17** (2015) 113015.