Quantum Dense Coding Through an Optical Model Using Continuous Modes of Photons

by

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Dedicated to my parents,

who valued my education above all else

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Abstract

Quantum Dense Coding (QDC) is one of the primary and basic protocol used in quantum information. It can be performed through photons having polarization and spatial degree of freedom. The optical model which uses linear optics can performs this task by producing entaglement in the polarization degree of freedom. The spectral effects can be seen in QDC taking frequency distribution of photons. Hence detectors can be treated in continuous modes and their result can be compared to other models, performing QDC. The frequency contribution is used to bring a time delay in one of photon modes. We also introduced the theoretical model of detectors in continuous modes and develop tools for QDC in these modes.

Contents

Chapter 1

Introduction

Quantum Information is relatively younger branch of quantum physics. It implements the concept of quantum physics in information theory. The quantum behavior and its effects in different forms become very useful for many information processes and communication protocols. There are many protocols which use different types of quantum system but the most common and useful one is light. Information traveling with the speed of light, makes it an ideal system for communication both in quantum and classical domain. Here, in our work, we have used the spatial and polarization degrees of freedom of photon as a source of quantum communication, considering the spectral effects of light in frequency phase space [1].

1.1 Linear Algebra in Quantum Domains

The behavior of Quantum system is studeid with the help of a Hilbert space. The physical properties of the system reperesent states in a Hilbert space. e.g in case of an electron there are two states of spin, i.e. "Up" and "down". Hence the degree of freedom for an electron system in case is two. So the dimentionality of a Hilbert space is decided by the way quantem system behaves. The two states $|\alpha\rangle, |\beta\rangle$ of finite dimentionality 'm' belongs to a Hilbert space ‘H’ such that the inner product of thses two vectors

$$|\alpha\rangle = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ . \\ . \\ . \end{pmatrix}, |\beta\rangle = \begin{pmatrix} \beta_1 \\ \beta_2 \\ . \\ . \\ . \end{pmatrix}, |\alpha\rangle = (|\alpha\rangle)^\dagger = (\bar{\alpha}_1 \bar{\alpha}_2 .... \bar{\alpha}_m), \tag{1.1}$$

becomes

$$\langle\alpha|\beta\rangle = \bar{\alpha}_1\beta_1 + \bar{\alpha}_2\beta_2 + ... + \bar{\alpha}_m\beta_m := \sum_{j=1}^{m} \bar{\alpha}_j\beta_j \in \mathcal{C},$$

Where $\bar{z}$ is a complex number. Normally a vector is normalized by dividing it by its norm i.e. ||z||=$\sqrt{\langle z|z\rangle}$. The inner product must satisfy the Cauchy-Schwarz inequality($||\alpha||\beta|| \geq || \langle\alpha|\beta\rangle||$). Consider the following matrix X

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} & . & . & x_{1,m} \\ x_{2,1} & x_{2,2} & . & . & x_{2,m} \\ & . & . & & . \\ & . & & . & . \\ & . & & & . \\ x_{m,1} & x_{m,2} & . & . & x_{m,m} \end{pmatrix} \tag{1.2}$$

The transpose(X^T) and the complex conjugate ($\bar{X}$) are given below

$$X^T = \begin{pmatrix} x_{1,1} & x_{2,1} & . & . & x_{m,1} \\ x_{1,2} & x_{2,2} & . & . & x_{m,2} \\ & . & . & & . \\ & . & & . & . \\ & . & & & . \\ x_{1,m} & x_{2,m} & . & . & x_{m,m} \end{pmatrix}, \bar{X} = \begin{pmatrix} \bar{x}_{1,1} & \bar{x}_{1,2} & . & . & \bar{x}_{1,m} \\ \bar{x}_{2,1} & \bar{x}_{2,2} & . & . & \bar{x}_{2,m} \\ & . & . & & . \\ & . & & . & . \\ & . & & & . \\ \bar{x}_{m,1} & \bar{x}_{m,2} & . & . & \bar{x}_{m,m} \end{pmatrix} \tag{1.3}$$

Now the condition for Hermitian operator or matrix is

$$X^{\dagger} = \begin{pmatrix} \bar{x}_{1,1} & \bar{x}_{2,1} & . & . & \bar{x}_{m,1} \\ \bar{x}_{1,2} & \bar{x}_{2,2} & . & . & \bar{x}_{m,2} \\ & . & . & & . \\ & . & & . & . \\ & . & & & . \\ \bar{x}_{1,m} & \bar{x}_{2,m} & . & . & \bar{x}_{m,m} \end{pmatrix}, \tag{1.4}$$

which is $X^{\dagger} = \bar{X}^T$. Also the Hermitian matrix X is called positive semidefinite if $\langle c|Xc\rangle \geq 0$, where c $\in$ H and X $\geq$ 0. For c $\neq$ 0, X is positive definite if $\langle c|Xc\rangle > 0$. The eigenvalues of matrix X are either zero or positive if X is diagonalized and it obeys the condition discussed above.

In a bra-ket notation for any matrix X, $|Xc\rangle = X\,|c\rangle\,, \langle Xc| = \langle c|\,X^{\dagger}$. For Hermitian matrix X, $\langle x|Xy\rangle = \langle Xx|y\rangle$ which also equal to Tr $|y\rangle\langle x|$X. The commutator relation for two matrices M and N is

$$\begin{aligned} \{M, N\} &= \frac{1}{2}(MN + NM) \\ \left[M, N\right] &= \frac{1}{2}(MN - NM). \end{aligned} \tag{1.5}$$

If a vector space obey's the above two relation, then it is called as Jorden or Lie algebra [2].

1.2 Composite Systems and Tensor Products

Composite systems come under discussion when more than one quantum system are treated in quantum mechaanical problems. In order to understand it, consider two Hilbert spaces H_X and H_Y which represents two quantum systems with orthogonal basis $x_1, x_2, ..., x_d$ and $y_1, y_2, ..., y_d$ respectively. The joint system of H_X and H_Y is called the composite system. This system then belongs to a Hilbert space H $= H_X \otimes H_Y$, with the tensor product of H_X and H_Y having orthogonal basis $\{x_1 \otimes y_1, x_1 \otimes y_2, x_1 \otimes y_3, ..., x_1 \otimes y_d, x_2 \otimes y_1, x_2 \otimes y_2, x_2 \otimes y_3 ...., x_2 \otimes y_d, x_d \otimes y_1, x_d \otimes y_2, x_d \otimes y_3 ...., x_d \otimes y_d$. So the system is in a dimention of d×d [3]. The tensor peoduct of two vectors

$$a \equiv \sum_i c^i |x_i\rangle \tag{1.6}$$

and

$$b \equiv \sum_j \acute{c}^j |y_j\rangle , \tag{1.7}$$

is given as below

$$|a \otimes b\rangle = \sum_{i,j} c^i \acute{c}^j |x_i\rangle \otimes |y_j\rangle , \tag{1.8}$$

or in a simple way

$$|ab\rangle = \sum_{i,j} c^i \acute{c}^j |x_i y_j\rangle \tag{1.9}$$

The operator in form of a matrix for H_X is A^X and for H_Y is B^Y. The tensor product of $A^X \otimes B^Y$ is defined as

$$A^X \otimes B^Y(|x_i\rangle \otimes |y_j\rangle) = A^X |x_i\rangle \otimes B^Y |y_j\rangle , \tag{1.10}$$

In addition, the trace of the tensor product can be written as

$$Tr(A^X \otimes B^Y) = TrA^X TrA^Y. \tag{1.11}$$

In general for any Hermitian operators A^X, B^X and $C^X \in H_X$, and A^Y, B^Y and $C^Y \in H_Y$ has the following property

$$(A^X \otimes A^Y)(B^X \otimes B^Y)(C^X \otimes C^Y) = (A^X B^X C^X) \otimes (A^Y B^Y C^Y) \tag{1.12}$$

So the trace of above equation becomes

$$Tr(A^X \otimes A^Y)(B^X \otimes B^Y)(C^X \otimes C^Y) = Tr(A^X B^X C^X)Tr(A^Y B^Y C^Y). \tag{1.13}$$

In this section we studied the opertion of traces on states of composite systems.

1.3 Entangled States

A thought experiment normally known as EPR paradox was proposed in 1935 by Einstein, Padolsky and Rosen, which concluded that the laws of quantum mechanics are incomplete [4]. It was explained through a non local property of QM, later on called as entanglement. The entangled particle shows the phenomena of two particles, far apart, which effects their physical properties by interacting at a long distance with one another. This phenomena was also observed through many experiments [5].

The Entanglement introduces correlation which cannot be measured through classical approach [6] [7]. To explain this concept mathematically, consider two qubits X and Y, which coresponds to the states of two electrons i.e. spin "up" and spin "doen". The state $|1\rangle$ represents spin "up" state whilw $|0\rangle$ spin "down" for each electron system. The qubits of electron X and Y are written as

$$\phi_X = x_0 |0\rangle + x_1 |1\rangle , \phi_Y = y_0 |0\rangle + y_1 |1\rangle . \tag{1.14}$$

The composite state of two qubits are as under

$$\phi_X \otimes \phi_Y = (x_0 |0\rangle + x_1 |1\rangle) \otimes (y_0 |0\rangle + y_1 |1\rangle) \tag{1.15}$$

$$= x_0 y_0 |00\rangle + x_1 y_0 |10\rangle + x_0 y_1 |01\rangle + x_1 y_1 |11\rangle . \tag{1.16}$$

which is simply the tensor product. So the states become entangled, if it can not be expressed in the tensor product. Now to write the entangle state formed by the superposition of $|00\rangle$ and $|11\rangle$ is shown below

$$\left|\phi^+\right\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{XY} \tag{1.17}$$

The above state is an example of entangled state [8]. It can be proved by taking a condradiction, consider that the obove equation can be written in tensor product of individual quantum systems. Then

$$\left|\phi^+\right\rangle_{XY} = (x_0 |0\rangle + x_1 |1\rangle)_X (y_0 |0\rangle + y_1 |1\rangle)_Y , \tag{1.18}$$

$$= x_0 y_0 |00\rangle + x_1 y_0 |10\rangle + x_0 y_1 |01\rangle + x_1 y_1 |11\rangle . \tag{1.19}$$

By equating the coefficients of Eq.(1.17) and Eq.(1.19),

$$x_0y_0 = \frac{1}{\sqrt{2}}, x_1y_0 = 0, x_0y_1 = 0, x_1y_1 = \frac{1}{\sqrt{2}}, \tag{1.20}$$

which seems impractical , due to the fact that if $x_1y_0 = 0$, then it reveals either $x_1 = 0$ or $y_0 = 0$ or both are zero. By taking $x_1 = 0$ and $y_0 \neq 0$, above equation suggests that x_1y_1 must be zero, but it is assigned the valaue i.e. $x_1y_1 = \frac{1}{2}$ in Eq. (1.20), So this shows clear contradiction with the above supposition. In addition, the first and the fourth term of Eq.(1.20) shows x_0, x_1, y_0 and $y_1 \neq 0$, while the second and third term pionts out that either all the amplitudes i.e. x_0, x_1, y_0, $y_1 = 0$ or at least some of them, which seems unfeasible. So the state of Eq.1.17 is unfactorizable and it exhibit the property of entanglement. The best example is the Bell's states given below

$$\left|z^1\right\rangle_{XY} = \left|\psi^+\right\rangle_{XY} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)_{XY}, \tag{1.21}$$

$$\left|z^2\right\rangle_{XY} = \left|\psi^-\right\rangle_{XY} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)_{XY}, \tag{1.22}$$

$$\left|z^3\right\rangle_{XY} = \left|\phi^+\right\rangle_{XY} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)_{XY}, \tag{1.23}$$

$$\left|z^2\right\rangle_{XY} = \left|\phi^-\right\rangle_{XY} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)_{XY}, \tag{1.24}$$

So all of these states are maximally entangled states and they are orthonormal

$$\left\langle z^i \middle| z^j \right\rangle_{XY} = \delta_{ij}. \tag{1.25}$$

Such states can be expressed in $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ basis or $\{\left|z^1\right\rangle, \left|z^2\right\rangle, \left|z^3\right\rangle, \left|z^4\right\rangle\}$ basis respectively.

1.4 Quantum Gates

Since quantum mechanics depicts all the physical processes that take place in the universe, therfore its study involves the effect of operators in quantum states. This process is known as measurement in quantum mechanics which yield a number or a constant. The real value demands that the operator must be unitary. This unitary operation on quantum states are called quantum gates [9].

Figure 1.1: Basic operation of Unitary operator

Figure1.1 shows a basic operation of any unitary operation 'U' on quantum state $|\phi\rangle$. The initial states $|\phi\rangle$ changes to a new state U$|\phi\rangle$ as a result of single gate operation.
A qubit is basically the superposition of $|0\rangle$ and $|1\rangle$ which corresponds to number '0' and '1' in classical domain. This qubit contains large number of superposed states which is the elementary principal of quantum information. Figure1.2 introduces various quantum gates including the Hadamard, Pauli gates X, Y, and Z and Phase gate along with their transformation matrices. In Dirac Notation these gates are written as

$$
\begin{aligned}
&H = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)\langle 0| + \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)\langle 1|,\\
&I = |0\rangle\langle 0| + |1\rangle\langle 1|,\\
&X = |1\rangle\langle 0| + |0\rangle\langle 1|,\\
&Y = \iota|1\rangle\langle 0| - \iota|0\rangle\langle 1|,\\
&Z = |0\rangle\langle 0| - |1\rangle\langle 1|.
\end{aligned}
$$

The Paul(X) gate changes the state quantum state $|0\rangle$ and $|1\rangle$ into $|1\rangle$ and $|0\rangle$ respectively, therfore it named as flip or Not gate. In a similar way the operation of CNot gate in Dirac notation is shown below

$$CNot = |00\rangle\langle 00| + |01\rangle\langle 01| + |11\rangle\langle 10| + |10\rangle\langle 11|. \tag{1.26}$$

The above action of CNot emphasizes that there must be two particles, so that the first qubit is treated as control to make changes in the second qubit. So when the condition that the first qubit must be $|1\rangle$ is met, only then the second qubit would flip or change. Figure1.2 represents various quantum gates.

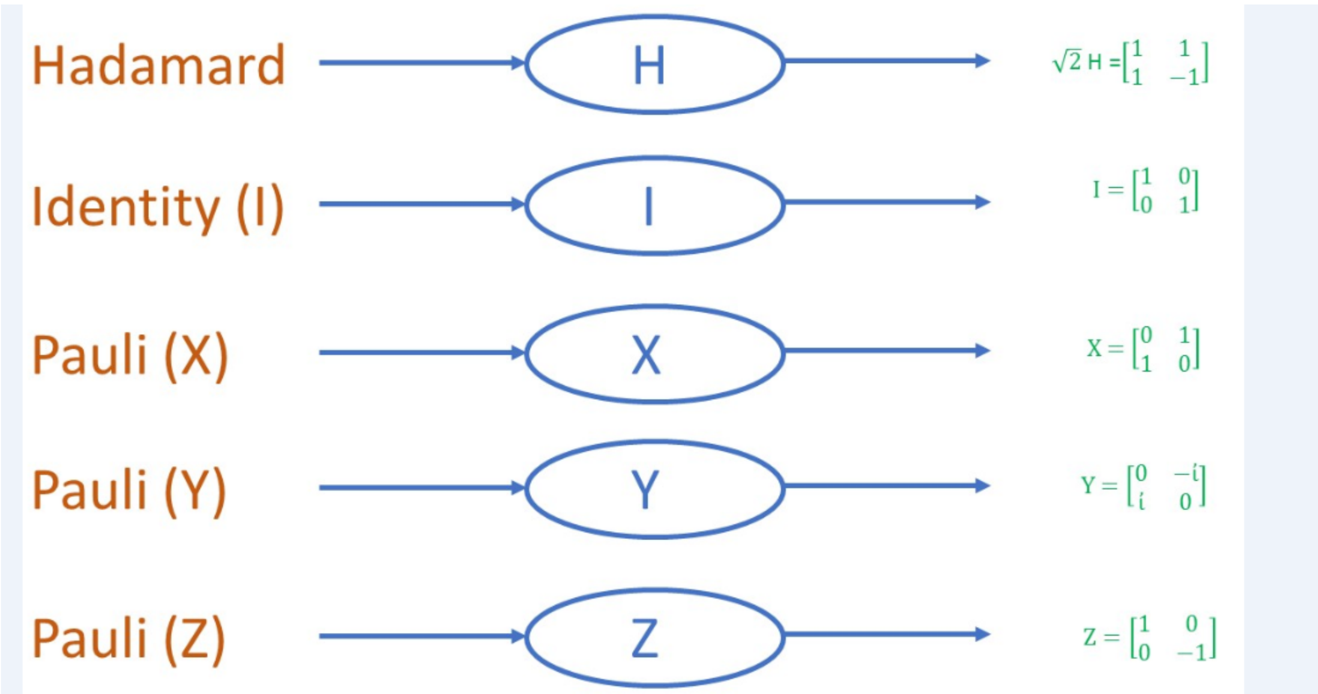

Figure 1.2: Quantum Gates

1.5 Formulism of Entanglement

The formation of entangled states, in particular the Bell's states is achieved using the Hadamard gate and CNot gate. Consider the the general state $|ab\rangle_{XY}$ where a,b $\in 0, 1$.

To produce $\left|\psi^{+}\right\rangle_{XY}$ state, put a = 0 and b = 1, and apply the respective gates in an order mentioned above. The whole process is shown in figure1.3

$$|01\rangle_{AB} \xrightarrow{H_A} \frac{1}{\sqrt{2}}(|01\rangle + |11\rangle)_{AB} \xrightarrow{Control=A} \frac{1}{2}(|10\rangle + |10\rangle)_{AB} = \left|\psi^{+}\right\rangle \tag{1.27}$$

For $\left|\psi^{-}\right\rangle$ $a = 1$ and $b = 1$

$$|11\rangle_{AB} \xrightarrow{H_A} \frac{1}{2}(|01\rangle - |11\rangle)_{AB} \xrightarrow{Control=A} \frac{1}{2}(|01\rangle - |10\rangle)_{AB} = \left|\psi^{-}\right\rangle \tag{1.28}$$

To produce $\left|\phi^{+}\right\rangle$ state then $m = n = 0$

$$|00\rangle_{AB} \xrightarrow{H_A} \frac{1}{\sqrt{2}}(|00\rangle + |10\rangle)_{AB} \xrightarrow{Control=A} \frac{1}{2}(|00\rangle + |11\rangle)_{AB} = \left|\phi^{+}\right\rangle \tag{1.29}$$

and for $\left|\phi^{-}\right\rangle$ state $m = 1$ and $n = 0$

$$|10\rangle_{AB} \xrightarrow{H_A} \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)_{AB} \xrightarrow{Control=A} \frac{1}{2}(|00\rangle - |11\rangle)_{AB} = \left|\phi^{-}\right\rangle \tag{1.30}$$

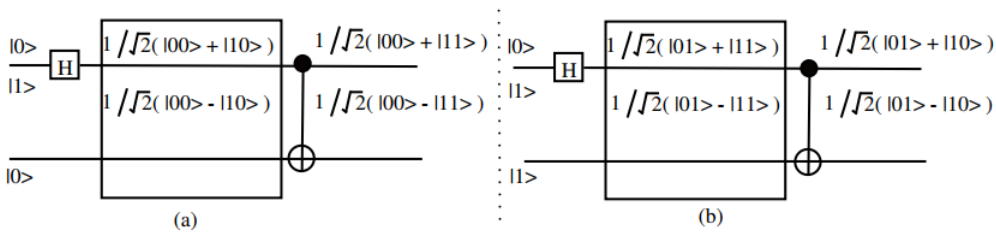

Figure 1.3: Formation of Bell's States

Rotational Invariance of Bell states One of the most important feauteres of Bell states are that, their effect remains the same irrespective of the basis. Which means that they are rotationary invarient under every basis. In order to prove it consider two sets of basis $|0\rangle, |1\rangle$

and $\left|+\right\rangle, \left|-\right\rangle$. First write $\left|\phi^{+}\right\rangle$ state in $\left|0\right\rangle, \left|1\right\rangle$ basis,

$$\left|\phi\right\rangle = \frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle). \tag{1.31}$$

Now to introduce a change of 45° anglular rotation in it i.e.

$$\begin{aligned} \left|0\right\rangle &= \frac{1}{\sqrt{2}}(\left|+\right\rangle + \left|-\right\rangle) \\ \left|1\right\rangle &= \frac{1}{\sqrt{2}}(\left|+\right\rangle - \left|-\right\rangle) \end{aligned} \tag{1.32}$$

So by substituting the values of $\left|0\right\rangle$ and $\left|1\right\rangle$ in Eq. (1.31)

$$\begin{aligned} \left|\phi^{+}\right\rangle &= \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(\left|+\right\rangle + \left|-\right\rangle)\frac{1}{\sqrt{2}}(\left|+\right\rangle + \left|-\right\rangle) + \frac{1}{\sqrt{2}}(\left|+\right\rangle - \left|-\right\rangle)\frac{1}{\sqrt{2}}(\left|+\right\rangle - \left|-\right\rangle)), \\ \left|\phi^{+}\right\rangle &= \frac{1}{\sqrt{2}}(\frac{1}{2}(\left|++\right\rangle + \left|-+\right\rangle + \left|+-\right\rangle + \left|--\right\rangle) + \frac{1}{2}(\left|++\right\rangle - \left|-+\right\rangle + \left|--\right\rangle - \left|+-\right\rangle)), \\ \left|\phi^{+}\right\rangle &= \frac{1}{\sqrt{2}}(\left|++\right\rangle + \left|--\right\rangle). \end{aligned} \tag{1.33}$$

which shows identical behavior compared to Eq. (1.31). This can be varified through any general orthonormal basis. For example

$$\left|\omega\right\rangle = x_0\left|0\right\rangle + x_1\left|1\right\rangle \text{ and } \left|\acute{\omega}\right\rangle = x_1\left|0\right\rangle - x_0\left|1\right\rangle, \text{with condition that} |x_0|^2 + |x_1|^2 = 1. \tag{1.34}$$

writing $\left|\phi^{+}\right\rangle$ state in $\{\left|\omega\right\rangle, \left|\acute{\omega}\right\rangle\}$ basis

$$\left|\phi^{+}\right\rangle = \frac{1}{\sqrt{2}}(\left|\omega\omega\right\rangle + \left|\acute{\omega}\acute{\omega}\right\rangle). \tag{1.35}$$

By putting values of $|\omega\rangle$ and $|\acute{\omega}\rangle$ from Eq. (1.34) into the above equation.

$$\begin{aligned}
\left|\phi^+\right\rangle &= \frac{1}{\sqrt{2}}((x_0\left|0\right\rangle + x_1\left|1\right\rangle)(x_0\left|0\right\rangle + x_1\left|1\right\rangle) + (x_1\left|0\right\rangle - x_0\left|1\right\rangle)(x_1\left|0\right\rangle - x_0\left|1\right\rangle)), \\
\left|\phi^+\right\rangle &= \frac{1}{\sqrt{2}}(\left|x_0\right|^2\left|00\right\rangle + x_0x_1\left|10\right\rangle + x_0x_1\left|01\right\rangle + \left|x_1\right|^2\left|11\right\rangle \\
&\quad + \left|x_1\right|^2\left|00\right\rangle - x_0x_1\left|10\right\rangle - x_0x_1\left|01\right\rangle + \left|x_0\right|^2\left|11\right\rangle), \\
\left|\phi^+\right\rangle &= \frac{1}{\sqrt{2}}((\left|x_0\right|^2 + \left|x_1\right|^2)\left|00\right\rangle + (\left|x_0\right|^2 + \left|x_1\right|^2)\left|11\right\rangle). \qquad (1.36)
\end{aligned}$$

From the condition of Eq. (1.34)

$$\left|\phi^+\right\rangle = \frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle). \qquad (1.37)$$

Which suggests that the Bell's states can be formed using any othonormal basis in two dimentional Hilbert space i.e.

$$\left|\phi^+\right\rangle = \frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle) = \frac{1}{\sqrt{2}}(\left|{++}\right\rangle + \left|{--}\right\rangle) = \frac{1}{\sqrt{2}}(\left|\omega\omega\right\rangle + \left|\acute{\omega}\acute{\omega}\right\rangle). \qquad (1.38)$$

The non classical correlation is explained through the rotational invariance of the Bell's states.

1.6 No-Cloning Theorem

In 1982 Wooters and Zurek [10] publishhed a theorem which states that non-orthonormal quantum states can not be completely cloned or copied. In this context consider a unitary operator 'U', which acts an copying machine on some state, say $|\beta\rangle$ by copying the orthogonal states $|0\rangle$ to $|1\rangle$ in the following manner

$$U|0\rangle|\alpha\rangle = |0\rangle|0\rangle$$
$$U|1\rangle|\alpha\rangle = |1\rangle|1\rangle$$

Now to copy a superposed state i.e. $\lambda|0\rangle + \gamma|1\rangle$ is given below

$$\begin{aligned} U(\lambda|0\rangle + \gamma|1\rangle)|\alpha\rangle &= \lambda U|0\rangle|\alpha\rangle + \gamma U|1\rangle|\alpha\rangle \\ &= \lambda|0\rangle|0\rangle + \gamma|1\rangle|1\rangle \\ &\neq (\lambda|0\rangle + \gamma|1\rangle)(\beta|0\rangle + \gamma|1\rangle) \end{aligned}$$

So it is concluded above that to clone a non-orthonormal quantum state is ssimply unachievable.

1.7 Quantum Dense Coding (QDC)

In quantum information, the technique used to send two classical bits using quantum channel or qubits is known as quantum dense coding. Like quantum teleportation [11] [12], the first step to achieve QDC, is the formation of entangled states. The entaglement permits the spatial and polarization of photons to link together and send information between two parties, irrespective of their partition.

The first step towards QDC protocol is the formation of an entangled state, which is shared to both parties i.e Alia and Bob. Each recieved a qubit of shared entangled state. Alia encodes her classical message on her qubit by applying some local operation on it. She has a choice of four possible messages but one at a time. The classical bit encoding is shown in the table given below

Encoding classical bits		
$I\vert\Phi^{+}\rangle$	$\vert\Phi^{+}\rangle$	00
$X\vert\Phi^{+}\rangle$	$\vert\Psi^{+}\rangle$	01
$Z\vert\Phi^{+}\rangle$	$\vert\Phi^{-}\rangle$	10
$-XZ\vert\Phi^{+}\rangle$	$\vert\Psi^{-}\rangle$	11

Table 1.1: Encoding process of QDC

Each classical two bits with Alia corresponds to one of four Bell states. If she has recieved $|\phi^+\rangle$ bell state then she can perform local operation on her qubit to send four different messages in form of four Bell states. After encoding process, she sent it to Bob who already has his qubit. Bob then apply decoding process on the Bell state in his station and retrieve the classical information.

The capacity of bits that can be sent through a single qubit is two which occurs due to the entanglement between Alia and Bob. So it basically doubles the speed from one bit to two classical bits per qubit [13].

It should be noted that though two qubits are used in the process, one part of entangled pair already with Bob and other sent by Alice, but at the time of sharing the message only one qubit is sent. Additionally the message is completely secured as information can not be ritrieved by one qubit sent by Alice alone. Classically two classical bits can not be sent securely by two bits alone. This makes this coding scheme a quantum dense coding one.

Chapter 2

Basics Tools for Quantum Dense Coding

The main ingredient for various quantum information protocols is quantum entanglement. Its the first step towards the formulism of quantum cryptography, quantum teleportation, quantum dense coding and quantum computaion using linear optics [14]. The efficient, controlled and bright formation of entangled photons is the prerequisite requirment of these protocols. Moreover the detection mechanism requires a theoretical background to model the detectors keeping practical efficeincies in mind. The main topics discussed in this chapter are: entangled phtons sources in real world in Sec.2.1, theoratical treatment of sources in Sec.2.2, practical sources for quantum dense coding in Sec.2.3, Bell states in Sec.2.4 and dectectors of our model is been in Sec.2.5.

2.1 Spontaneous Parametric Down Conversion(SPDC)

The first established experiment regarding entanglement of positron annihilation was on spatiallly seperated quantum states. This opposed prediction was established after the observation of EPR proposal for spin half systems by Bohm [15] and the discovery of the Bell. Various experimental observations on polarization entangled photons emitted from calcium are in a visible region, using standard linear optical instruments. But the photons on emission do not conserve momentum because of the random orientation of the momentum of two photons from the atom. So this makes it impossible to study experimental setup. The alternate method of producing more efficient photon pair, that can be easily handled experimently too, is the process named as Parametric Down Conversion (PDC) [16].

The process of Spontaneous Parametric Down Conversion (SPDC) is used to produce entangled photon pair, when a non-linear material is subjected to a single photon. This property of material permits the convesrsion of a single photon into two i.e. a pair, obeying the both law of conservation of momentum and energy. Each photon of the manufactured pair are identified by naming them, the one as idler with frequncy 'ω_i'and the other partner as signal with frequncy 'ω_s'. The photon that passes through a non-linear object is named as pump photon with frequency 'ω_p'. The material used for the SPDC process to occur is Beta Barium Oxide (BBO) by obeying the following two fundamental laws

$$\omega_p = \omega_s + \omega_i, \qquad \text{(Law of Conservation of energy)} \tag{2.1}$$

$$K_p = K_s + K_i, \qquad \text{(Law of conservation of momentum)} \tag{2.2}$$

$$\tag{2.3}$$

Figure 2.1 shows the process of SPDC, where the single photon laser passes through a non-linear medium i.e. BBO and emitted on the other side as a pair of two cones each. Each cone represents the idler photon and the singnal. The overlap region of cones specifies the presence of two photons in effect of one another. The frequencies of the pair of photon satisfies the condition of mismatch, which corresponds to the conservation of momentum. The efficiency of SPDC is usually very low and it approaches to 10^{-6} [17]. The main criteria for the whole process is that, the detection of signal photon predicts the presence of idler photon, when measured. Following are the two types of SPDC

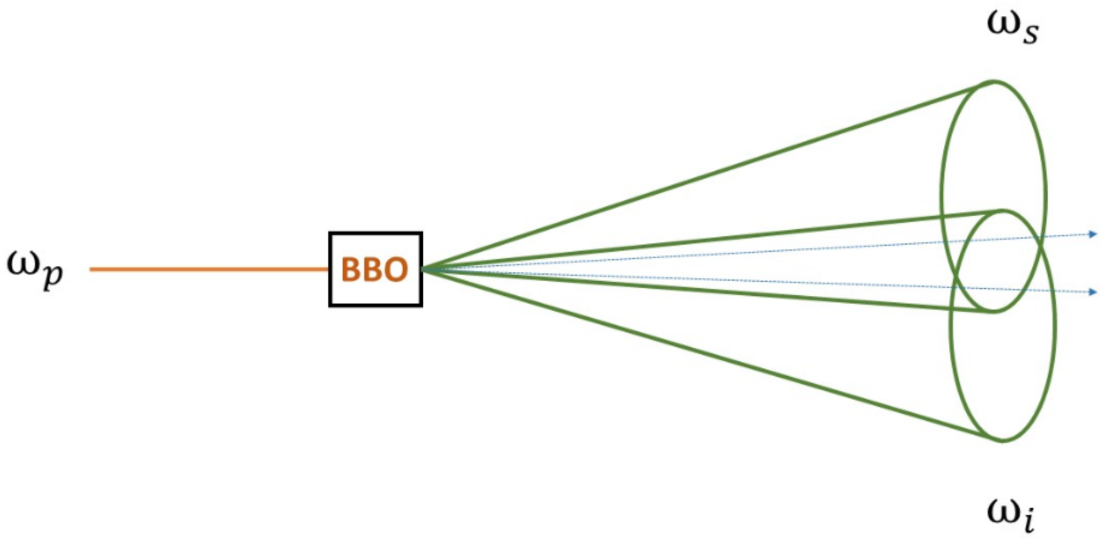

Figure 2.1: The two photons of frequency ω_i and ω_s, formed as a result of a laser (pump) photon passes through a BBO crystal with freqeucy ω_p. The dotted lines depicts the region, where two entangled photons effect each other.

2.1.1 SPDC Type-I

This type offers the production of two photons with identical polarization. Consider for instance that the emitted entangled pair of photon have ordinary polarization than the pump pulse would have the extraordinary polarization [18]. The Bell's states production by such type of SPDC is as under

$$|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|HH\rangle \pm |VV\rangle). \tag{2.4}$$

Linear optical techniques are used to examine the coreespondent Bell's state of two photons.

2.1.2 SPDC Type-II

This type of SPDC assigns non-identical polarization to the spatial degree of photons. In addition, there are two refractive indexes in various directions. So, if the pumping photon carries ordinary polarization then the output entangled photon pair must have extraordinary polarization. The shape of Bell's states folowing type-II takes the form

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|HV\rangle \pm |VH\rangle). \tag{2.5}$$

The real experimental processes manufactures less entangled states as compared to the ideal Bell's states.

2.2 Theoretical Setup for Practical sources

In any quantum communication domain specifically, the protocols of quantum information demands greatly the practical sources for the generation of entanglement. Efforts are been made to make models for the real formation of entanglement. The protocls for which such models works are quantum teleportation, enataglement swapping and quantum dense coding as in our case. All these protocols are in a big need of the real sources for entaglement generation in correspondence to the Bell's states. SPDC process is not a deterministic one so, it produces either one pair, two pairs or even multiple pairs at any instant of time in a random fashion. The prediction of number pairs is completely equivocal. Such process obeys a direct relation of property of non-linear medium $\kappa^{(2)}$ with the strength of pump pulse laser and the time which incident photon(pump) takes to pass through that medium. The mathematical structure of SPDC follows Lie Algebra SU(1,1) having basis $\{M_x, M_y, M_z\}$ and makes use of following conditions

$$[Mx, My] = -\iota M_z, [M_y, M_z] = \iota M_x, [M_z, M_x] = \iota M_y. \tag{2.6}$$

The SPDC Type-I follows SU(1,1) algebra by the following generators

$$M_x^{(i)} = \frac{1}{4}(d_i^\dagger d_i^\dagger + d_i d_i), \tag{2.7}$$

$$M_y^{(i)} = \frac{1}{4}\iota(d_i^\dagger d_i^\dagger - d_i d_i), \tag{2.8}$$

$$M_z^{(i)} = \frac{1}{4}(d_i^\dagger d_i + d_i d_i^\dagger), \tag{2.9}$$

where 'd_i' represents any of the operators i.e. a_H, a_V, b_H and b_V. In addition SPDC type-II uses the same algebra using the following generators

$$M_x^{(ik)} = \frac{1}{2}(d_i^\dagger d_k^\dagger + d_i d_k), \tag{2.10}$$

$$M_x^{(ik)} = \frac{1}{2}\iota(d_i^\dagger d_k^\dagger - d_i d_k), \tag{2.11}$$

$$M_x^{(ik)} = \frac{1}{2}(d_i^\dagger d_k + d_i d_k^\dagger). \tag{2.12}$$

In case of SPDC Type-I, generators illustrates that $d_i = a_H$ and $d_k = b_H$, which shows identical polarization. while SPDC Type-II exhibit distinct polarization and the generators are repersented in such way i.e. $d_i = a_H$ and $d_k = b_V$. Mathematically SU(1,1) transformation involves the treatment of operators on a vacuum state[18]

$$Z(\gamma) = \exp(\iota\gamma M_x) \left|\text{vac}\right\rangle, \gamma \in \mathcal{R}, \left|\text{vac}\right\rangle = \left|0_H 0_V 0_H 0_V\right\rangle. \tag{2.13}$$

Furthermore, generators must satisfy the following conditions of commutators

$$[M_X, M_y] = -\iota M_Z, [M_y, M_z] = \iota M_x, [M_z, M_x] = \iota M_y. \tag{2.14}$$

The genarators obeying the above conditions for SPDC Type-I, takes the form

$$M_x = \frac{1}{2}(a_H^\dagger b_H^\dagger + a_H b_H), \tag{2.15}$$

where the generators make use of annihilation and creation operators as spatial mode with their respective degree of polarization i.e. Horizental (H) and Vertical (V). The ultimate result of this process includes, production of quantum state $\left|K(\kappa)\right\rangle$ which is a superposotion of fock state and a vacuum state. The state also contain single pair, double and even multiple pairs of high order. To an approximate small value of (γ), state $\mathscr{K}(\gamma)$ turn into following form

$$K(\gamma) = |\text{vac}\rangle + \frac{1}{2}\iota\gamma\,|1010\rangle\,, \gamma \in \mathcal{R} \tag{2.16}$$

Polarization corresponds to order (HVHV) and state $|\text{ijkl}\rangle$ shows that i, j, k and l photons are in spatial mode 'a' and 'b'. The presence of vacuum state suggests high probabilty of not even a single photon creation.

2.3 Modeling Practical Sources For Practical Quantum Dense Coding

As it is obvious that many quantum information protocols require entanglement is the first priority of the problem. In a similar way quantum dense coding also need such state, similar to Bell's state. But the main task is to prodeuce practically entangled sources. Here in this problem, SPDC Type-II source is used to accomplish this job. The aim is to produce state, which carreis identical behavior to that of $\left|\psi^+\right\rangle$ Bell state. The spatial mode are chosen to be 'd' and 'e' with orthogonal polarization of Horizental(H) and vertical(V). Now starting from the $\left|\psi^+\right\rangle$ Bell state

$$\left|\psi^+\right\rangle_{de} = \frac{1}{\sqrt{2}}(|HV\rangle + |VH\rangle), \tag{2.17}$$

$$= \frac{1}{\sqrt{2}}(|1001\rangle + |0110\rangle)_{d_H d_V e_H e_V} \tag{2.18}$$

For the formation of Bell state as written above, following generators play the role as

$$M_x = \frac{1}{2}(d_H^\dagger e_V^\dagger - d_V^\dagger e_H^\dagger + d_H e_V - d_V e_H), \tag{2.19}$$

$$M_y = \frac{1}{2\iota}(d_H^\dagger e_V^\dagger - d_V^\dagger e_H^\dagger - d_H e_V + d_V e_H), \tag{2.20}$$

$$M_z = \frac{1}{2}(d_H^\dagger d_H - e_V e_V^\dagger - d_V^\dagger d_V + e_H e_H^\dagger). \tag{2.21}$$

where the $\{M_x, M_y, M_Z\}$ obeys the condition of Eq(2.6) imposed by SU(1,1) algebra. The ultimate state produced by such SPDC is given below

$$|\chi\rangle = \exp[\iota\chi(d_H^\dagger e_V^\dagger - d_V^\dagger e_H^\dagger + d_H e_V - d_V e_H)]\,|\text{vac}\rangle\,, \tag{2.22}$$

where χ is the efficiency parameter of the source with condition $\chi = \frac{1}{2}\gamma \in$ R. The condition of commutator relation for operator is already discussed in section 2.2, But these relation for operators are as under

$$[d_i, d_j^\dagger] = \delta_{ij}, \qquad [e_i, e_j^\dagger] = \delta_{ij}, \tag{2.23}$$

$$[d_i, e_j^\dagger] = 0, \qquad [e_i, d_j] = 0, \tag{2.24}$$

where the indices i,j $\in$ polarization $\{H, V\}$. Making use of above commutations relation, Eq(2.22) beomes

$$|\chi\rangle = \exp[\iota\chi(d_H^\dagger e_V^\dagger + d_H e_V)]\exp[-\iota\chi(d_H^\dagger e_V^\dagger + d_V e_H)]\,|\text{vac}\rangle\,, \tag{2.25}$$

Now to futher solve the above equation, it will be better to transform the operators to normal order. For this purpose some generator in terms of opertaors used in above equation are defined as

$$J_+ = d^\dagger e^\dagger, \qquad J_- = de, \qquad and \qquad J_0 = \frac{1}{2}(d^\dagger d + e^\dagger e + 1). \tag{2.26}$$

where new generators are $\{J_+, J_-, J_0\}$ of SU(1,1) algebra which satisfies the following commutation relation

$$[J_-, J_+] = 2J_0, \qquad [J_0, J_\pm] = \pm J_\pm. \tag{2.27}$$

These generators in normal order are given by [19].

$$\exp[\alpha_+ J_+ + \alpha_0 J_0 + \alpha_- J_-] = \exp[D_+ J_+]\exp[\ln(D_0)J_0]\exp[D_- J_-], \tag{2.28}$$

Now $D_\mp, D_0$ are defined as

$$D_\mp = \frac{\alpha_\mp / V \sinh V}{\cosh V - (\frac{\alpha_0}{2V})\sinh V}, \tag{2.29}$$

$$D_0 = [\cosh V - (\frac{\alpha_0}{2V})\sinh V]^{-1}, \tag{2.30}$$

$$V = [(\frac{\alpha_0}{2})^2 - \alpha_+ \alpha_-]^{\frac{1}{2}}. \tag{2.31}$$

So applying the transformation defined in Eq.(2.28) to first part of Eq.(2.25)

$$\exp[\iota\chi(d_H^\dagger e_V^\dagger + d_H e_V)] = \exp[\iota\chi(d_H^\dagger e_V^\dagger + \iota\chi d_H e_V)]$$
$$= \exp[\phi(\chi) d_H^\dagger e_V^\dagger]\exp[\omega(\chi)(d_H^\dagger d_H + e_V^\dagger e_V + 1)]\exp[(\phi(\chi) d_H e_V)] \tag{2.32}$$

In above scenario $\alpha_+ = \alpha_- = \iota\chi$ and $\alpha_0 = 0$, and using Eq.(2.31), the values of $\phi(\chi)$ and $\omega(\chi)$ can be calculated as

$$\phi(\chi) \equiv \iota \tanh(\chi), \qquad \omega(\chi) \equiv -\ln[\cosh(\chi)], \qquad V = \chi. \tag{2.33}$$

In a similar way the second part of Eq.(2.25) changes to

$$\exp[-\iota\chi(d_H^\dagger e_V^\dagger + d_V e_H)] = \exp[-\iota\chi d_H^\dagger e_V^\dagger + -\iota\chi d_V e_H], \tag{2.34}$$

$$= \exp[\acute{\acute{\phi}}(\chi)d_H^\dagger e_V^\dagger]\exp[\acute{\omega}(\chi)(d_V^\dagger d_V + e_H^\dagger e_H + 1)]\exp[\acute{\acute{\phi}}(\chi)d_V e_H], \tag{2.35}$$

with parameters

$$\acute{\acute{\phi}}(\chi) := \text{itanh}(\chi), \qquad \acute{\omega}(\chi) := -ln[\text{Cosh}(\chi)], \qquad \acute{V} := \chi \tag{2.36}$$

The normal state is obtained from SPDC Type-II by putting Eq. (2.32) and Eq. (2.35 in Eq.(2.25), we get

$$\begin{aligned} |\chi\rangle &= \exp[\phi(\chi)(d_H^\dagger c_V^\dagger + d_V^\dagger c_H^\dagger)]\exp[\omega(\chi)(d_H^\dagger d_H + d_V^\dagger d_V + c_H^\dagger c_H + c_V^\dagger c_V + 2)] \\ &\quad \exp[\phi(\chi)(d_H c_V - d_V c_H)]\,|\text{Vac}\rangle\,. \end{aligned} \tag{2.37}$$

where $|\text{Vac}\rangle = \left|0_{d_H}0_{d_V}0_{c_H}0_{c_V}\right\rangle$. In above equation the last two exponential vanishes away due to the action of annihilation operator on vacuum state, therefore the state we obtain from SPDC becomes

$$|\chi\rangle = \exp[2\omega(\chi)]\exp[\phi(\chi)(d_H^\dagger c_V^\dagger + d_v^\dagger c_H^\dagger)]\,|\text{Vac}\rangle \tag{2.38}$$

This is the final state which we use in our work for QDC.

2.4 Bell state Measurement

The quantum system usually usually represents quantum information protocols through a quantum state. Th quantum state keeps this information unless it exposed to measurement. The whole system is controlled by the Hamiltonian of that system. To recover this information, the system usually interacts with detectors and the state vanishes. In this section, the measurement of Bell states will be discussed.

Bell state keep information in a particular order, depending on the way ,they are constructed. To recover this information and use it in different quantum information processes, Bell state measurement plays its role and gives result in classical bits [16]. The Bell state measuremet process consists of a 50 : 50 beam splitter (BS) followed by a two polarization beam splitters (PBS). The detectors which click specify the presence of respective bell state in those modes. Here we take two spatail modes "b" and "c" which first paseses through beam splitter (BS) and passes through polarization beam splitter (PBS), which only allows to transmit horizontal polarized photon and reflects the vertical polarized photon mode. The photons finally strike with detectors spatially apart form on another. As a result different spatial mode photons are detected with the help of photo detectors with coreesponding polarizations. Those detectors which click shows that photon has been detected, while no click means no photon has been detected by detector.

The four Bell states are given below

$$\left|\psi^{\pm}\right\rangle_{bc} = \frac{1}{\sqrt{2}}(|1001\rangle \pm |0110\rangle)_{b_H b_V c_H c_V}, \tag{2.39}$$

$$\left|\phi^{\pm}\right\rangle_{bc} = \frac{1}{\sqrt{2}}(|1010\rangle \pm |0101\rangle)_{b_H b_V c_H c_V}, \tag{2.40}$$

$$\left|\psi^{\pm}\right\rangle_{bc} = \frac{1}{\sqrt{2}}(b^{\dagger}_{H_b} c^{\dagger}_{V_c} |0000\rangle \pm b^{\dagger}_{V_b} c^{\dagger}_{H_c} |0000\rangle), \tag{2.41}$$

$$\left|\phi^{\pm}\right\rangle_{bc} = \frac{1}{\sqrt{2}}(a^{\dagger}_{H_b} c^{\dagger}_{H_c} |0000\rangle \pm b^{\dagger}_{V_b} c^{\dagger}_{V_c} |0000\rangle), \tag{2.42}$$

So the modes "b" and "c" when pass through beam splitter use the following transformations i.e

$$b^{\dagger}_{H} \longrightarrow \frac{1}{\sqrt{2}}(b^{\dagger}_{H_1} + ic^{\dagger}_{H_4}),\ b^{\dagger}_{V} \longrightarrow \frac{1}{\sqrt{2}}(b^{\dagger}_{V_2} + ic^{\dagger}_{V_3}) \tag{2.43}$$

$$b^{\dagger}_{H} \longrightarrow \frac{1}{\sqrt{2}}(ib^{\dagger}_{H_1} + c^{\dagger}_{H_4}),\ c^{\dagger}_{V} \longrightarrow \frac{1}{\sqrt{2}}(ib^{\dagger}_{V_2} + c^{\dagger}_{V_3}) \tag{2.44}$$

So the above transformations on Eq.(2.42) and Eq.(2.42) leads to

$$\begin{aligned}|\psi\rangle_{bc}^{-}\longrightarrow\quad & \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(b_{H_1}^{\dagger}+ic_{H_4}^{\dagger}).\frac{1}{\sqrt{2}}(ib_{V_2}^{\dagger}+c_{V_3}^{\dagger})\left|0000\right\rangle \\ & -\frac{1}{\sqrt{2}}(ib_{V_2}^{\dagger}+c_{V_3}^{\dagger}).\frac{1}{\sqrt{2}}(ib_{H_1}^{\dagger}+c_{H_4}^{\dagger})\left|0000\right\rangle)\end{aligned}$$

$$\begin{aligned}|\psi\rangle_{bc}^{-}\longrightarrow\quad & \frac{1}{2\sqrt{2}}((b_{H_1}^{\dagger}b_{V_2}^{\dagger}-c_{H_4}^{\dagger}b_{V_2}^{\dagger}+b_{H_1}^{\dagger}c_{V_3}^{\dagger}+ic_{H_4}^{\dagger}c_{V_3}^{\dagger} \\ & -(ib_{V_2}^{\dagger}b_{H_2}^{\dagger}-c_{V_3}^{\dagger}b_{H_1}^{\dagger}+b_{V_2}^{\dagger}c_{H_4}^{\dagger}+ic_{V_3}^{\dagger}c_{H_4}^{\dagger})\left|0000\right\rangle)\end{aligned}$$

$$\begin{aligned}|\psi\rangle_{bc}^{-} &= \frac{1}{2\sqrt{2}}(i\left|1100\right\rangle-\left|0110\right\rangle+\left|1001\right\rangle+i\left|0011\right\rangle \\ &\quad -i\left|1100\right\rangle+\left|1001\right\rangle-\left|0110\right\rangle-i\left|0011\right\rangle)_{H_1V_2H_4V_3} \\ &= \frac{1}{2\sqrt{2}}(2\left|1001\right\rangle-2\left|0110\right\rangle)_{H_1V_2H_4V_3} \\ |\psi\rangle_{bc}^{-} &= \frac{1}{\sqrt{2}}(\left|1001\right\rangle-\left|0110\right\rangle)_{H_1V_2H_4V_3}\end{aligned} \tag{2.45}$$

In a similar way $\left|\psi^{+}\right\rangle$ changes in the following way due to above transformation

$$\begin{aligned}|\psi\rangle_{bc}^{+}\longrightarrow\quad & \frac{1}{\sqrt{2}}(\frac{1}{\sqrt{2}}(b_{H_1}^{\dagger}+ic_{H_4}^{\dagger}).\frac{1}{\sqrt{2}}(ib_{V_2}^{\dagger}+c_{V_3}^{\dagger})\left|0000\right\rangle \\ & +\frac{1}{\sqrt{2}}(ib_{V_2}^{\dagger}+c_{V_3}^{\dagger}).\frac{1}{\sqrt{2}}(ib_{H_1}^{\dagger}+c_{H_4}^{\dagger})\left|0000\right\rangle)\end{aligned}$$

$$\begin{aligned}|\psi\rangle_{bc}^{+}\longrightarrow\quad & \frac{1}{2\sqrt{2}}((b_{H_1}^{\dagger}b_{V_2}^{\dagger}-c_{H_4}^{\dagger}b_{V_2}^{\dagger}+b_{H_1}^{\dagger}c_{V_3}^{\dagger}+ic_{H_4}^{\dagger}c_{V_3}^{\dagger} \\ & +(ib_{V_2}^{\dagger}b_{H_2}^{\dagger}-c_{V_3}^{\dagger}b_{H_1}^{\dagger}+b_{V_2}^{\dagger}c_{H_4}^{\dagger}+ic_{V_3}^{\dagger}c_{H_4}^{\dagger})\left|0000\right\rangle)\end{aligned}$$

$$
\begin{aligned}
|\psi\rangle_{bc}^{+} &= \frac{1}{2\sqrt{2}}(i\,|1100\rangle - |0110\rangle - |1001\rangle - i\,|0011\rangle \\
&\quad +i\,|1100\rangle - |1001\rangle + |0110\rangle + i\,|0011\rangle)_{H_1V_2H_4V_3} \\
&= \frac{1}{2\sqrt{2}}(2\,|1001\rangle + 2\,|0110\rangle)_{H_1V_2H_4V_3} \\
|\psi\rangle_{bc}^{+} &= \frac{1}{\sqrt{2}}(|1001\rangle + |0110\rangle)_{H_1V_2H_4V_3} \qquad (2.46)
\end{aligned}
$$

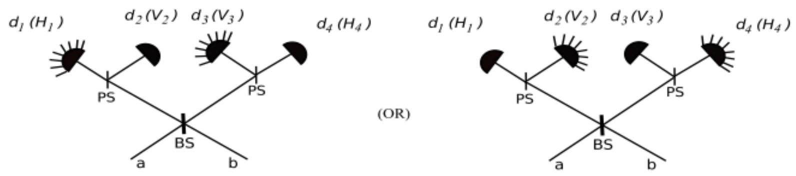

Figure 2.2: The $\left|\psi^{-}\right\rangle$ state is detected when d_1 and d_3 OR d_2 and d_4 clicks.

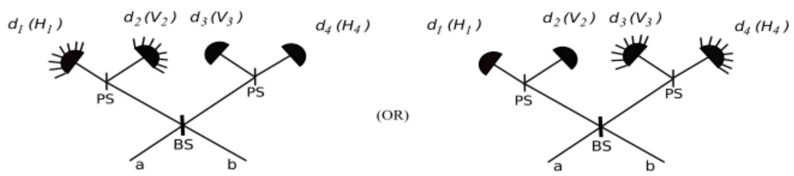

Figure 2.3: The $\left|\psi^{+}\right\rangle$ state is detected when d_1 and d_2 OR d_3 and d_4 clicks.

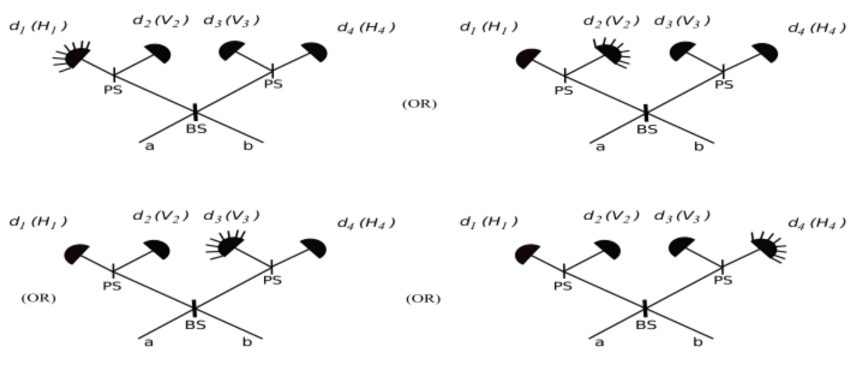

Figure 2.4: The $\left|\psi^{-}\right\rangle$, $\left|\psi^{+}\right\rangle$ states are detected when d_1 OR d_3 OR d_2 OR d_4 clicks.

2.5 Detectors

Detectors plays a very important role in quantum information processes. Many of quantum information protocols uses detectors such as quantum teleportation, quantum dense coding and entaglement swapping. In each protocols, detectors are made through specific mechanism in order to study the behavior of quantum mechanical model. In this section we will discuss different types of detectors with their uinque limitations to detect photons.

The detectors used for an optical system are usually photo diodes. They work on the principle of photoelecric effect. One of the primary dtectors used are semi conductor detector such as Indium Galium Arsenide (InGaAs), which usually works at room temperature[[19]]. In such detector the photon presence is detected through avalanch of electrons, which symbolized as a "click". However these detectors are not able to discriminate the photon number. There are some detectors which have the ability to discriminate between photons number, one of them is named as superconducting transition edge sensors (TES) which has efficiency upto 88 %. They work at 1550 nm and also account for dark counts rates [20]. The presence of dark counts refers to event "click", even in the absence of single photon. These drak counts lower the efficiency of detectors. On practical basis, it is impossible to construct an ideal detector, therefore we have used a theoretical model to deal with such issue, discussed in next section.

2.5.1 Ideal Photon Number Discriminating Detectors

These detectors register a "click", when photons are present in certain modes and shows "no click" in the absence of even single photon. These detectors carry the information of number of photons striking, which is measured through the strength of the click. They are known as projective valued measurement (PVM). Mathematically they are represented using projection operators i.e

$$\Pi_n = |n\rangle\langle n| \qquad \text{where n} = 0, 1, 2, ...$$

Here $|n\rangle$ is a fock state and ‘n’ belongs to natural nubers (N) of certain modes. If we have four modes, then $|n\rangle = |lkmn\rangle$ and the PVM would become

$$\Pi_{lkmn} = |l\rangle\langle l| \otimes |k\rangle\langle k| \otimes |m\rangle\langle m| \otimes |n\rangle\langle n|, \text{where l, k, m, n} = \text{0, 1, 2, 3,....} \tag{2.47}$$

2.5.2 Inefficient Photon Number Discriminating Detector with no Dark Counts

These detectors are considerd to be inefficient because of dark counts. In our model , the efficiency of detector is represented by "η", and $0 < \eta < 1$. The inefficiency demands that there is possibility of "click" of a detector, even when no single photon present. Our theoratical model tackle such issue by introdusing a beam splitter with transmittance "η". So the photon which passes through such beam splitter are only allowed to strike the detector and get measured, while the reflected photons vanishes by taking trace on reflected photons. Due to absence of dark counts, we take a vacuum state at one of the input port of beam splitter as shown in figure(2.5). Hence the probability that the detector measured n photons when input state "ρ_{input}" reaches to other port of beam splitter is given below

$$P(\eta/\rho_{\text{input}}) = \text{Tr}_{tran}[\Pi_n \text{Tr}_{ref}(B_\eta \rho_{\text{input}} \otimes |Vac\rangle\langle Vac|) B_\eta^\dagger \Pi_n] \tag{2.48}$$

The beam spltter action is done through unitary matrix "B_η", while the "ρ_{input}" state are basically fock statesas discussed above i.e $\rho_{\text{input}} = |l\rangle\langle l|$. So by putting $\rho_{input} = |l\rangle\langle l|$ in above equation, we get the final expression for probability

$$P(n/l) = \mathrm{Tr}_{tran}[\Pi_n \mathrm{Tr}_{ref} \mathrm{B}_\eta(|l\rangle\langle l| \otimes |\mathrm{Vac}\rangle\langle vac|)\mathrm{B}_\eta^\dagger \Pi_n] \tag{2.49}$$

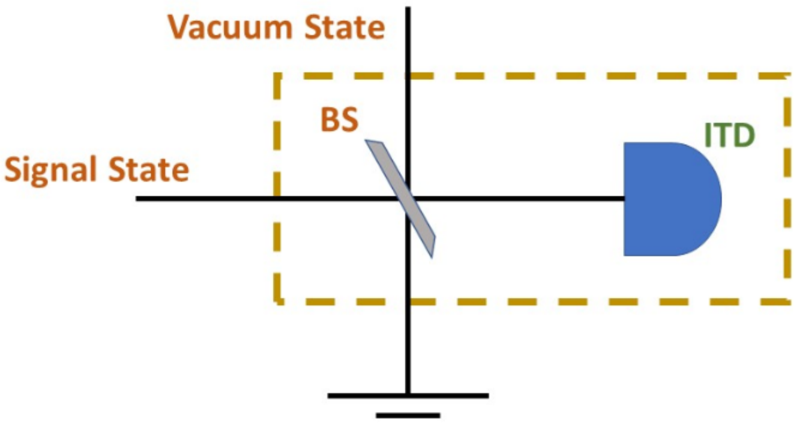

Figure 2.5: The model for an imperfect threshold detector.

The above equation basically suggests that in order to detect "n" photons, $l > n$ unless dark counts are taken into account.

2.5.3 Threshold Detectors

These detectors have two outcomes i.e either a "click" or "no click". These detectors are easily available and understandable than the other detectors discussed so for. They does not have the ability to disciminate photon numbers. The expressions for these outcomes are as under

$$\text{No click} = \Pi_o = |0\rangle\langle 0|$$

and

$$\text{click} = \Pi_o^{\perp} = \mathrm{I} - |0\rangle\langle 0|$$

For threshold detectors with dark count probability of no click is

$$\Pi_o^{dc} = (1 - P_{dc})|0\rangle\langle 0|.$$

while probability of click is

$$\Pi_1^{dc} = \mathrm{I} - \Pi_o^{dc} = \mathrm{I} - (1 - P_{dc})|0\rangle\langle 0|.$$

we used above operators for detection of photons in our model of quantum dense coding.

In order to proceed further, we will transport two classical bits through quantum dense coding in the next chapter. We would use the prctical sources for entanglement and will use threshold detectors for our theoratical model of QDC.

Chapter 3

Tools for Quantum Dense Coding through continuous Modes

Dense Coding protocol enables communication by sending classical information of two classical bits, through quantum bit or qubit. The first step to acomplish this task is the formation of enatangled state and sending it to both parties. The sharing of this entangled state provides the basis for QDC. In order to produce shared entangled state using practical sources, we have used SPDC type-II process.

Hence the state which we obtain from PDC source becomes

$$\left|\psi^{+}\right\rangle = \exp\left[2\omega(\chi)\right]\exp\left[\phi(\chi)(a^{\dagger}_{H_a}(\omega) + b^{\dagger}_{V_b}(\acute{\omega}) + a^{\dagger}_{V_a}\omega + b^{\dagger}_{H_b}\acute{\omega})\left|vac\right\rangle\right] \tag{3.1}$$

So now we convert the state obtained from SPDC type-II in Eq. (2.35) to continous mode frequency distribution in Sec.3.1. In Sec.3.2 we see how two photons can be made distinguishable. In Sec.3.3 we present detector model in continuous mode and in Sec.3.4 we develop quantum state after passing through beam splitter.

3.1 Continuous Mode (Frequency Space)

We have taken the state in frequency distribution of photon modes. The state in frequency distribution can be written as

$$|1\rangle \rightarrow \int_{-\infty}^{\infty} f(\omega)\left|1(\omega)\right\rangle d\omega \rightarrow \int_{-\infty}^{\infty} f(\omega)a^{\dagger}(\omega)\left|0\right\rangle d\omega \tag{3.2}$$

where $f(\omega)$ is the distribution function and is normally a gaussian [21]. Hence above $\left|\psi^{+}\right\rangle$ state in frequency distribution takes the form

$$\left|\psi^{+}\right\rangle = e^{[2\omega(\chi)]}\exp\left[\phi(\chi)\int\int[f(\omega,\acute{\omega})a_{H}^{\dagger}(\omega)b_{V}^{\dagger}(\acute{\omega}) + g(\omega,\acute{\omega})a_{V}^{\dagger}(\omega)b_{H}^{\dagger}(\acute{\omega})d\omega d\acute{\omega}]\left|vac\right\rangle\right] \tag{3.3}$$

In order to simplify the exponential function, we used the power series expansion. Using Maclaurin's formula, the exponential function can be written as

$$\exp(x) = \sum_{n=0}^{\infty}\frac{x^{n}}{n!}$$

Now using above formula in Eq.(3.3), the stste $\left|\psi^{+}\right\rangle$ becomes

$$\begin{aligned}
\left|\psi^{+}\right\rangle &= \exp\left[\phi(\chi)\int\int[f(\omega,\acute{\omega})a_{H}^{\dagger}(\omega)b_{V}^{\dagger}(\acute{\omega}) + g(\omega,\acute{\omega})a_{V}^{\dagger}(\omega)b_{H}^{\dagger}(\acute{\omega})d\omega d\acute{\omega}]\left|vac\right\rangle\right] \\
&= \sum_{n=0}^{\infty}\frac{[\phi(\chi)\int\int[f(\omega,\acute{\omega})a_{H}^{\dagger}(\omega)b_{V}^{\dagger}(\acute{\omega}) + g(\omega,\acute{\omega})a_{V}^{\dagger}(\omega)b_{H}^{\dagger}(\acute{\omega})d\omega d\acute{\omega}]\left|vac\right\rangle]^{n}}{n!} \\
&= \sum_{n=0}^{\infty}\frac{[\phi(\chi)]^{n}}{n!}\Big[\int\int\Big(f(\omega,\acute{\omega})a_{H}^{\dagger}(\omega)b_{V}^{\dagger}(\acute{\omega}) + g(\omega,\acute{\omega})a_{V}^{\dagger}(\omega)b_{H}^{\dagger}(\acute{\omega})d\omega d\acute{\omega}\Big)\left|vac\right\rangle\Big]^{n}
\end{aligned}$$

so $\sum_{n=0}^{\infty}\frac{[\phi(\chi)]^{n}}{n!} = e^{[\phi(\chi)]}$. Subsituting it in above equation, we get

$$\left|\psi^{+}\right\rangle = e^{[\phi(\chi)]}\sum_{n=0}^{\infty}\Big[\int\int\Big(f(\omega,\acute{\omega})a_{H}^{\dagger}(\omega)b_{V}^{\dagger}(\acute{\omega}) + g(\omega,\acute{\omega})a_{V}^{\dagger}(\omega)b_{H}^{\dagger}(\acute{\omega})d\omega d\acute{\omega}\Big)\left|vac\right\rangle\Big]^{n}$$

since $(a+b)^n = \prod_{i=1}^{i=n}(a_i + b_i)$. Above equation becomes

$$\left|\psi^+\right\rangle = e^{[\phi(\chi)]}\sum_{n=0}^{\infty}\prod_{i=1}^{i=n}\Big[\int\int f(\omega_i,\acute{\omega_i})a_H^\dagger(\omega_i)b_V^\dagger(\acute{\omega_i}) + g(\omega_i,\acute{\omega_i})a_V^\dagger(\omega_i)b_H^\dagger(\acute{\omega_i})d\omega_i d\acute{\omega_i}\Big]\left|vac\right\rangle$$

where ω_i and $\acute{\omega_i}$ represents two photons with different frequency. So substituting it in Eq.(3.3), we get

$$\left|\psi^+\right\rangle = \exp\left[2\omega(\chi)+\phi(\chi)\right]\sum_{n=0}^{\infty}\prod_{i=1}^{i=n}\Big[\int\int[f(\omega_i,\acute{\omega_i})a_H^\dagger(\omega_i)b_V^\dagger(\acute{\omega_i}) + g(\omega_i,\acute{\omega_i})a_V^\dagger(\omega_i)b_H^\dagger(\acute{\omega_i})d\omega_i d\acute{\omega_i}]\Big]\left|vac\right\rangle$$

Above equation gives the $\left|\psi^+\right\rangle$ state in continuous modes.

3.2 Time Delay in one mode

In our problem the photon mode having frequency (ω_i) reaches the beam splitter with a time delay of '$\frac{\Delta}{c}$'. Therefore, we introduce here a phase shift factor i.e. $\exp^{[\iota\omega_i\Delta/c]}$. Mathematically to acomplish this job we used the fourier transformation i.e

$$f(t) = \int \acute{f}(\omega)e^{\omega t}d\omega$$

The above fourier transform for time delay can be written in the following way.

$$f(t+\frac{\Delta}{c}) = \int \acute{f}(\omega)e^{\iota\omega(t+\frac{\Delta}{c})}d\omega = \int \acute{f}(\omega)e^{\iota\omega t}e^{\iota\omega\frac{\Delta}{c}}d\omega$$

The above equation shows that the exponential factor "$e^{\iota\omega\frac{\Delta}{c}}$" appears due to time delay in one of mode of photons in frequency domain. In a similar context the above $|\psi^+\rangle$ becomes

$$\begin{aligned} |\psi^+\rangle &= \exp\left[2\omega(\chi) + \phi(\chi)\right] \sum_{n=0}^{\infty} \prod_{i=1}^{i=n} [\int\int e^{(\iota\omega_i\Delta/c)} [f(\omega_i, \acute{\omega_i}) a_H^\dagger(\omega_i) b_V^\dagger(\acute{\omega_i}) \\ &\quad + g(\omega_i, \acute{\omega_i}) a_V^\dagger(\omega_i) b_H^\dagger(\acute{\omega_i}) d\omega_i d\acute{\omega_i}] \, |vac\rangle] \end{aligned} \tag{3.4}$$

The action of annihilation and creation operators on vacuum state '$|vac\rangle$' are as under

$$\begin{aligned} |\psi^+\rangle &= \exp\left[2\omega(\chi) + \phi(\chi)\right] \sum_{n=0}^{\infty} \prod_{i=1}^{i=n} [\int\int e^{[\iota\omega_i\Delta/c]} [f(\omega_i, \acute{\omega_i}) \, |H_a(\omega_i)\rangle \, |V_b\acute{\omega_i}\rangle \\ &\quad + g(\omega_i, \acute{\omega_i}) \, |V_a(\omega_i)\rangle \, |H_b(\acute{\omega_i})\rangle \, d\omega_i d\acute{\omega_i}]] \end{aligned}$$

In next step we will perform Bell state measurement in continuous modes.

3.3 Bell State Measurement in Continuous Mode(BSM)

Usually an entangled state which contains qubit is from one out of four Bell states. The Bell state measurement refers to the investigation, which verify the holding of one out of four possible Bell state that an entangled particals exist. Normally this process is done through linear optical device such as beam splitter. So in order to perform BSM in our case we use beam splitter which works on the principle of equivalent reflectance and transmittance i.e $|r| = |t|$, known as $50 : 50$ beam splitter. The transformation that we used for such action are given below.

$$\begin{pmatrix} \hat{a}_{H,V} \\ \hat{b}_{H,V} \end{pmatrix} \rightarrow U_{BS} \begin{pmatrix} \hat{a}_{H,V} \\ \hat{b}_{H,V} \end{pmatrix}$$

Thus after passing through beam splitter, we obtained the $\left|\acute{\psi}\right\rangle$ state i.e

$$\left|\acute{\psi}\right\rangle = \frac{1}{2}\exp\left[2\omega(\chi)+\phi(\chi)\right]\sum_{n=0}^{\infty}\prod_{i=1}^{n}\Big[\int\int\Big(e^{(\iota\omega_i\Delta/c)}f(\omega_i,\acute{\omega_i})[\left|H_b(\omega_i)V_b(\acute{\omega_i})\right\rangle - \iota\left|H_b(\omega_i)V_a(\acute{\omega_i})\right\rangle$$
$$+\iota\left|H_a(\omega_i)V_b(\acute{\omega_i})\right\rangle + \left|H_a(\omega_i)V_a(\acute{\omega_i})\right\rangle] + g(\omega_i,\acute{\omega_i})[\left|V_b(\omega_i)H_b(\acute{\omega_i})\right\rangle - \iota\left|V_b(\omega_i)H_a(\acute{\omega_i})\right\rangle$$
$$+\iota\left|V_a(\omega)H_b(\acute{\omega_i})\right\rangle + \left|V_a(\omega_i)H_a(\acute{\omega_i})\right\rangle d\omega_i d\acute{\omega_i}]\Big)\Big] \quad (3.5)$$

After BSM, the resultant state strikes the detectors. Since the photons are in continous mode, so we would discuss the detectors for continous frequency distribution in the next section.

3.4 Detector Model for continuous Modes

Now we explain our theory of detectors for practical QDC. Since ideal photon number discrimination detectors rarely exist in practical laboratory therfore, we haved used threshold detectors. These detectors are used to investigate the existance of no photon or presence of at least one photon. Hence they detects two states i.e "no click" or "click" which represents no photon and prsence of at least one photon. So the ideal threshold detector can be represented mathematically as

$$\Pi_o = |0\rangle\langle 0|$$

But for practical purpose , we have used inefficient threshold dectectors wich have dark counts. These dark counts refers to such state of detector when it display "click" even in the presence of no photon. Such drak counts accounts for the inefficiency of dectector which in our case is 'η'. The theroratical model for such detector has been shown in the figure (2.5). This figure explains our theoretical model for inefficent threshold detector having an efficiency of η. The beam splitter has transmittance η and have a signal state on one input port and a vacuum state on the second. A perfect threshold detector with unit efficiency is placed on one exits port while second port have trash to discard the reflected photons. For the no click of a detector with unit efficiency and have dark counts is given by operator

$$\Pi_o = (1 - P_{dc})|0\rangle\langle 0|$$

and the "click" event is calculated using operator obtained by substracting the "no click" operator from identity 'I'. i.e

$$\begin{aligned} \text{probability of "click"} &= \text{I} - \text{"probability of no click"} \\ &= \text{I} - \Pi_o \\ &= \text{I} - (1 - P_{dc})|0\rangle\langle 0| \end{aligned}$$

3.5 Continuous mode Bell state after action of beam splitter

The $\left|\acute{\psi}\right\rangle$ state after passing through beam splitter reaches to the detectors. For an imperfect threshold detector with efficiency "η", in our model, the quantum state passes through beam splitter with transmitivity"η" as in figure 3.1, with following transformations

$$\begin{pmatrix} \hat{a} \\ \hat{0} \end{pmatrix} \longrightarrow B_\eta^\dagger \begin{pmatrix} \hat{a} \\ \hat{r_a} \end{pmatrix}, \quad \begin{pmatrix} \hat{b} \\ \hat{0} \end{pmatrix} \longrightarrow B_\eta^\dagger \begin{pmatrix} \hat{b} \\ \hat{r_b} \end{pmatrix} \tag{3.6}$$

While the transformations for operators are as under

$$a^\dagger \longrightarrow \sqrt{\eta}a^\dagger - \sqrt{1-\eta}r_a^\dagger, \qquad b^\dagger \longrightarrow \sqrt{1-\eta}b^\dagger + \sqrt{\eta}r_b^\dagger \tag{3.7}$$

Where B_η is the unitary matrix of the beam splitter.

$$B_\eta = \begin{pmatrix} \sqrt{\eta} & \sqrt{1-\eta} \\ -\sqrt{1-\eta} & \sqrt{\eta} \end{pmatrix} \tag{3.8}$$

and r_a and r_b are the reflected modes from detector beam splitter B_η After passing through a beam splitter with above tranformation, the $\left|\acute{\psi}\right\rangle$ becomes

$$\begin{aligned}
\left|\acute{\acute{\psi}}\right\rangle &= B_\eta \left|\acute{\psi}\right\rangle \\
\left|\acute{\acute{\psi}}\right\rangle &= \frac{1}{2} e^{[2\omega(\chi)+\phi(\chi)]} \sum_{n=0}^{\infty} \prod_{i=1}^{n} \Big[\int\int \Big([e^{[\iota\omega_i\Delta/c]} f(\omega_i, \acute{\omega_i})[[\sqrt{1-\eta} r^\dagger_{H,a}(\omega_i) + \sqrt{\eta} b^\dagger_H(\omega_i)][\sqrt{1-\eta} r^\dagger_{V,a}(\acute{\omega_i}) \\
&\quad + \sqrt{\eta} b^\dagger_V(\acute{\omega_i})] - \iota[\sqrt{1-\eta} r^\dagger_{H,a}(\omega_i) + \sqrt{\eta} b^\dagger_H(\omega_i)][\sqrt{\eta} a^\dagger_V(\acute{\omega_i}) - \sqrt{1-\eta} r^\dagger_{V,b}(\acute{\omega_i})] + \iota[\sqrt{\eta} a^\dagger_H(\omega_i) \\
&\quad - \sqrt{1-\eta} r^\dagger_{H,b}(\omega_i)][\sqrt{1-\eta} r^\dagger_{V,a}(\acute{\omega_i}) + \sqrt{\eta} b^\dagger_V(\acute{\omega_i})] + [\sqrt{\eta} a^\dagger_V(\omega_i) - \sqrt{1-\eta} r^\dagger_{H,b}(\omega_i)][\sqrt{\eta} a^\dagger_V(\acute{\omega_i}) \\
&\quad - \sqrt{1-\eta} r^\dagger_{V,b}(\acute{\omega_i})]] \left|vac\right\rangle d\omega_i d\acute{\omega_i} + g(\omega_i, \acute{\omega_i})[[\sqrt{1-\eta} r^\dagger_{V,a}(\omega_i) + \sqrt{\eta} b^\dagger_V(\omega_i)][\sqrt{1-\eta} r^\dagger_{H,a}(\acute{\omega_i}) \\
&\quad + \sqrt{\eta} b^\dagger_H(\acute{\omega_i})] - \iota[\sqrt{1-\eta} r^\dagger_{V,a}(\omega_i) + \sqrt{\eta} b^\dagger_V(\omega_i)][\sqrt{\eta} a^\dagger_H(\acute{\omega_i}) - \sqrt{1-\eta} r^\dagger_{H,b}(\acute{\omega_i})] + \iota[\sqrt{\eta} a^\dagger_V(\omega_i) \\
&\quad - \sqrt{1-\eta} r^\dagger_{V,b}][\sqrt{1-\eta} r^\dagger_{H,a}(\acute{\omega_i}) + \sqrt{\eta} b^\dagger_H(\acute{\omega_i})] + [\sqrt{\eta} a^\dagger_V(\omega_i) - \sqrt{1-\eta} r^\dagger_{V,b}(\omega_i)][\sqrt{\eta} a^\dagger_H(\acute{\omega_i}) \\
&\quad - \sqrt{1-\eta} r^\dagger_{H,b}(\acute{\omega_i})]] \left|vac\right\rangle] d\omega_i d\acute{\omega_i} \Big) \Big]
\end{aligned} \tag{3.9}$$

where probability that the detector gives a "click"

$$P = \mathrm{Tr}_{\mathrm{tran}} \Big[(\mathrm{I} - (1 - P_{dc}) |0,0\rangle\langle 0,0|) \mathrm{Tr}_{\mathrm{ref}} \left|\acute{\acute{\psi}}\right\rangle \left\langle \acute{\acute{\psi}}\right| \Big] \tag{3.10}$$

Using our detector model theory, the event "no click" becomes

$$\text{no click} = (1 - P_{dc}) |0,0\rangle\langle 0,0|) \mathrm{Tr}_{\mathrm{ref}} \left|\acute{\acute{\psi}}\right\rangle \left\langle \acute{\acute{\psi}}\right|,$$

Where in our case $|0,0\rangle\langle 0,0|$ represents the coincidence of $\left|\psi^{+}\right\rangle$ state which is given below

$$|0,0\rangle\langle 0,0| = \left|0_{a_H}, 0_{a_V}\right\rangle\left\langle 0_{a_H}, 0_V\right| \tag{3.11}$$

Hence the detector will register a "click" as

$$\text{click} = [(\mathrm{I} - (1 - P_{dc})|0,0\rangle\langle 0,0|)\mathrm{Tr}_{\mathrm{ref}}[B_\eta \rho B_\eta^\dagger]] \tag{3.12}$$

which obviously will be traced out and gives the probability of an event given by Eq.(3.10). we have developed certain tool for continous mode quantum dense coding including thee Bell state, Bell state measurement and detectors.

Chapter 4

Conclution

Quantum Entanglement is the prime requisite of every quantum information protocol. In our work we used the PDC process to generate the $|\psi^+\rangle$ Bell state. We come to know that the PDC process is completely a random process i.e it either produce single pair photons or multi-pair, and taking small value of non linearity $\chi^{(2)}$ of BBO crystal. Hence the non linearity plays a vital role in the formation of an entangled state. We bring a path diference between two modes of photon by taking a delay of time on one of the modes. This time delay appears as a fourier tranform in mathematical form which is represented by a phase factor. In next step we perform Bell state measurement in continuous modes of frequency distribution. We use linear optics for this process. Further, we develop a theory of detectors in continuous modes in frequency domian. In order to construct a theory for practical purposes, we take into account the inefficiency of dectectors and dark counts. Hence we obtain a theoretical model for detectors, which detects the photons close to the practical scale. At the end we calulate the density matrix of theoratical model of QDC. In future we will calculate the final probability of QDC process for above optical model and will observe its variation with the path difference "Δ". It lead us to compare the variation of probabilty and "Δ" with the experimental quanutm dense coding.

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