Study of Bernstein Waves in Different Plasma Environments

by

Waseem Khan

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Supervised by

Dr. Muddasir Ali Shah

School of Natural Sciences (SNS)

National University of Sciences and Technology(NUST)
Islamabad, Pakistan

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Mr. Waseem Khan, Regn No. 00000117260 Titled: Study of Bernstein Waves in Different Plasma Environment be accepted in partial fulfillment of the requirements for the award of **MS** degree.

Examination Committee Members

1. Name: Dr. Aeysha Khalique Signature: Aysha Khalique

2. Name: Dr. Muhammad Asif Farooq Signature:

3. Name:________ Signature:________

External Examiner:: Dr. Gohar Abbas Signature:

Supervisor's Name: Dr. Muddasir Ali Shah Signature:

Head of Department

16-02-18
Date

COUNTERSINGED

Date: 16/02/18

Dean/Principal

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Signature: ______________________

Name of Supervisor: Dr. Muddasir Ali Shah

Date: 16-02-2018

Signature (HoD): ______________________

Date: 16-02-18

Signature (Dean/Principal): ______________________

Date: 16/02/18

Dedicated To
My Parents

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QM

Abstract

By solving the linearized relativistic Vlasov equation along Maxwell equations, a generalized expression for the plasma conductivity tensor is derived. The dispersion relation for Bernstein waves in weakly relativistic plasma is investigated by employing the Maxwell-Boltzmann-Juttner distribution function. The propagation characteristic of electrons Bernstein waves (overlapping, propagation regions, harmonics structure) are examined by using different of η (ratio of rest mass energy to thermal energy) by taking constant ratio of plasma frequency to the cyclotron frequency. We also observed that the propagation characteristics of electron Bernstein waves for different values of the ratio of plasma frequency to the cyclotron frequency by taking constant value of η. Further ,it is observed that due to the relativistic effect harmonics are overlapping with each other, as a result the propagation regions is reduced.

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Chapter 1

Introduction

1.1 What is Plasma

A plasma is a **quasineutral** gas of charged and neutral particles which exhibits **collective behaviour**[1] The quasiineutrality means plasma has some imbalancing between positive and negative charge. The imbalancing of charges create electric fields in plasma medium. This property of plasma medium distinguish from ionized gas[1] [3].
When the charge particles moves in plasma medium create electric current and magnetic field in plasma medium. These fields effects the motion of other charge particles, so collective behaviour mean motions that depends not only on local conditions but on the state of plasma in remote region as well[4][1].

The ionized gas under certain conditions behave like plasma state. These condition are given as,

$$\lambda_D << L$$

$$N_D >>> 1$$

$$\omega\tau > 1$$

Where λ_D is the Debye length and L is the dimensions of the system. N_D is the number of particles in Debye sphere. ω is frequency of plasma oscillation and τ is the mean time of collisions between charge particles and neutral atoms [1].

1.2 Various Plasma Environments

Most of matter in the universe is in plasma state. These matters having different temperature T an density n_0. The properties and environment of plasma change with temperature T and density n_0. The temperature T and density n_0 are called *plasma parameter* [5].

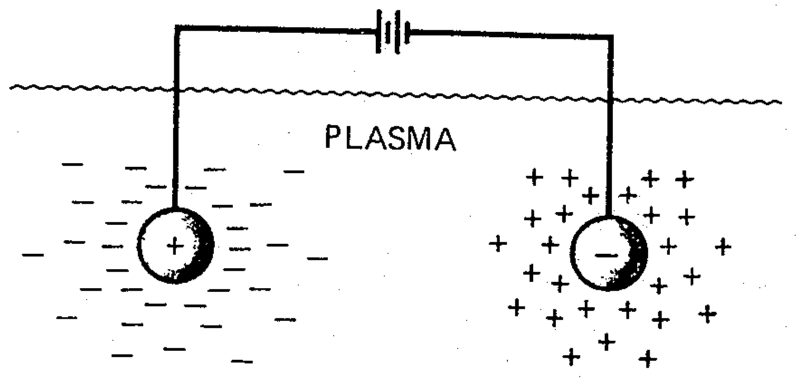

Figure 1.1: Debye shielding [1].

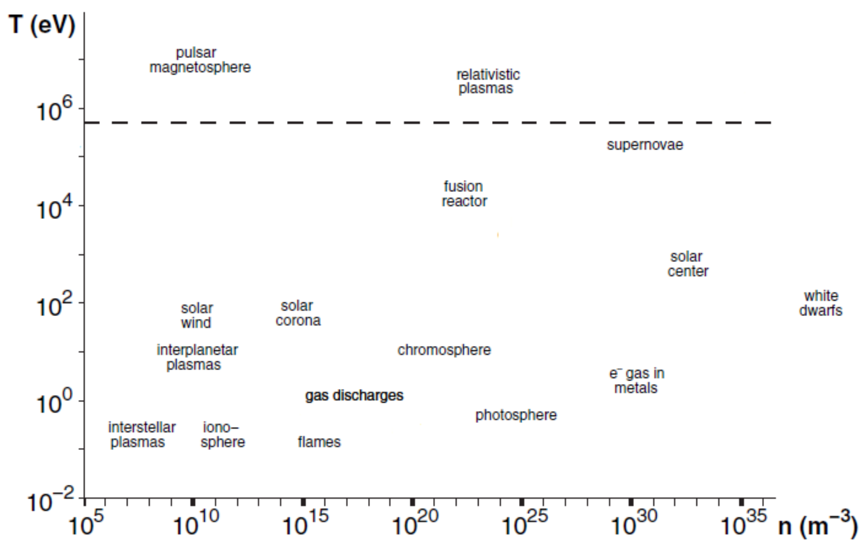

Figure 1.2: Plasma environment with density and temperature

1.2.1 Non-Relativistic and Relativistic Plasma Environment

When the thermal energy of particles is very small as compared to the rest mass energy the plasma environment is known as non-relativistic plasma environment. If the thermal energy of particles is equal or nearly equal to the rest mass energy the relativistic effects are dominant, such a plasma environment is known as relativistic plasma environment [6].

1.2.2 Examples of the non-Relativistic Plasma Environments

Earth Ionosphere

The ionosphere of any planet is the portion of atmosphere where the free electrons and ions having thermal energy exist under the effect of gravity and magnetic field. The upper portion of earth's atmosphere contains large number of free electron and ions having thermal energy. This portion of earth is known as *earth ionosphere.* It affects the propagation of electromagnet waves when they pass through that region. The number of charged particles in the lower layers are less but quite large in higher layers and have maximum value at the altitude of $300 - 500km$. This region of ionosphere is called $F-layer$. Values of plasma parameter for ionosphere, e.g. , F−layer, the density of electron and ions is given as $n_e \approx n_i \approx 10^{12}\ m^{-3}$ and the temperature is rather high of the order of $(0.26$ to $0.43)eV$ [5].

Solar Wind

The external portion of the sun's atmosphere, the solar corona is composed of very hot plasma, a gas having high kinetic energy with free electron and positive ions. When these gases move away from the sun then internal pressure of gases become larger than the weight of upper plasma. These particles flow like a wind in the entire solar system with very high velocity. The flow of these particles with very high velocity is called *solar wind.* The plasma density for solar wind is $n_e \approx n_i \approx 10^{12}\ m^{-3}$ and the temperature is very high of the order of (75 to 100) eV [5].

1.2.3 Example of the Relativistic Plasma Environment

Pulsars

Pulsars are very hot, dense and strongly magnetized rotating stars. Its surface temperature is around 51.72 eV and estimated range of magnetic fields on its surface is of the order of (10^8 to 10^{15}) gauss stronger than the earth's magnetic field. Therefore the rotation of pulsar along with strong magnetic field generates an electric field. This electric field ejects charged particles from the surface of pulsar. In all of these particles electrons get relativistic velocity due to small inertia. Those electrons which move along the curved magnetic field lines radiate γ radiations, when their energy is more then the twice of rest energy of electrons. These γ rays are further converted into electron-positron pair. This pair is also accelerated in the electric field and γ ray photon appear again. In this way the surface of the pulsars are filled with relativistic electron-positron plasma [7].

1.3 Waves in Plasma

There are different types of waves in a plasma, depending upon the direction of propagation with respect to electric and magnetic field.The waves which commonly exist in plasma are perpendicular, parallel, longitudinal, transverse, electrostatics and electro-magnetics[1].
For $\mathbf{E}_0$, $\mathbf{B}_0$ the ambient electric and magnetic fields and $\mathbf{E}_1$, $\mathbf{B}_1$ the perturbed electric and magnetic fields and $\mathbf{k}$ the propagation vector of the wave. Following terminology is usually used in plasma dynamics.
$\mathbf{k} \parallel \mathbf{B}_0 \longrightarrow$ Parallel propagating waves.
$\mathbf{k} \perp \mathbf{B}_0 \longrightarrow$ Perpendicular propagating waves.
$\mathbf{k} \parallel \mathbf{E}_1 \longrightarrow$ Longitudinal waves.
$\mathbf{k} \perp \mathbf{E}_1 \longrightarrow$ Transverse waves.
$\mathbf{B}_1 = 0$ and $\mathbf{k} \parallel \mathbf{E}_1 \longrightarrow$ Electrostatic waves.
$\mathbf{B}_1 = 0$ and $\mathbf{k} \perp \mathbf{E}_1 \longrightarrow$ Electromagnetic waves.
$\mathbf{B}_1 \neq 0 \longrightarrow$ Electromagnetic waves.

1.3.1 Plasma Oscillation

Plasma has a property to restore charge neutrality. If we have a uniform plasma which is made up of electrons and ions, mass of an electron is very small compared to the mass of an ion (approximately 1836 times), so ions can be considered stationary. When electrons are displaced from their mean position by any means, electric field will be developed between the stationary ions and the displaced electrons. Under the influence of this field the electrons will move towards the stationary ions. Due to inertia electrons do not stop at their mean position and start oscillations about the mean position. The frequency with which electrons will oscillate is known as the plasma frequency[1]. The plasma frequency is given by the following relation

$$\omega_{pe} = \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}} \quad , \tag{1.1}$$

where n_0 is the number density of plasma, e is the charge on an electron, m_e is the electron mass and ϵ_0 is the permittivity of free space. The plasma frequency is directly depend on number density of the plasma. Higher the density the greater will be the frequency of

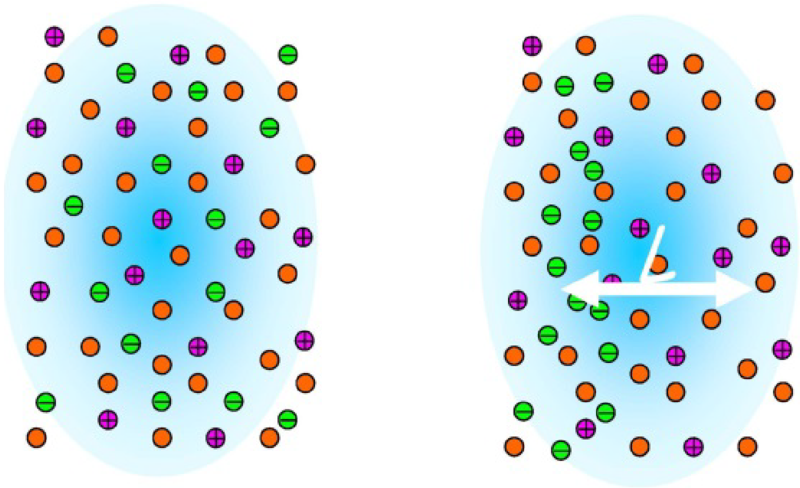

Figure 1.3: Charge separation mechanism[2]

oscillation. The charge separation mechanism is shown in the following figure on next page.

1.3.2 Electron Plasma Waves with $\mathbf{B}_0 = 0$ or $\mathbf{k} \| \mathbf{B}_0$

When we consider the thermal motion of electrons, plasma oscillations propagate with thermal velocity and carry information about oscillating region. These are called *electron plasma waves* with the following dispersion relation,

$$\omega^2 = \omega_{pe}^2 + \frac{3}{2} k^2 v_{the}^2 \tag{1.2}$$

ω is the wave frequency, ω_{pe} is the plasma frequency, k is the wave number and v_{the} is the thermal velocity of electron. When we consider the thermal motion of electron, the wave frequency depends on wave number k, i,e, group velocity $\frac{dw}{dk} \neq 0$. Information carried by wave travel from one region to other region with group velocity, so group velocity v_g can not exceed speed of light c. In figure (1.2) the slope of tangent at any P point on a curve gives us group velocity v_g and slop of any point P on curve drawn from origin gives us phase velocity v_ϕ. From graph we also see that the slope of $\sqrt{\frac{3}{2}} v_{the}$ is also greater than the slope of tangent at any point P. So the above equation holds only when,

$$v_\phi > \sqrt{\frac{3}{2}} v_{the} > v_g.$$

For large k (small λ) $v_g \approx v_{the}$ and for small k (large λ) $v_g < v_{the}$.

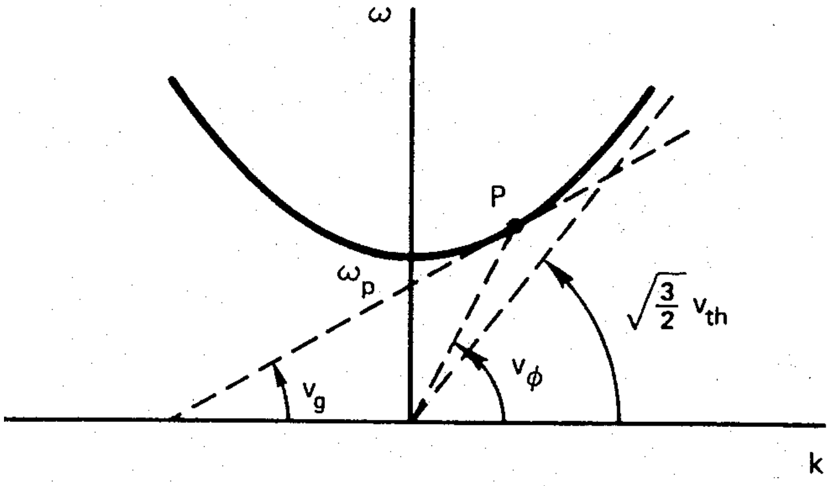

Figure 1.4: Relation among phase group and thermal velocity [1]

1.3.3 Electrostatic Electron Waves Perpendicular to $\mathbf{B}_0$

We are dealing with high frequency response particles like electrons. Ions are massive so they are considered stationary and create a uniform background of positive charge. The electron oscillations perpendicular to $\mathbf{B}_0$ mean direction of propagating vector $\mathbf{k}$ is perpendicular to $\mathbf{B}_0$ and as we are consider electrostatic case so $\mathbf{B}_1 = 0$, and $\mathbf{k}$ parallel to $\mathbf{E}_1$ so they are longitudinal plane wave propagating perpendicular to B_0. Under these assumptions $\mathbf{E}_0 = 0 = \mathbf{v}_0$, $k_B T_e = 0$ (we neglect thermal motion) ,$\nabla n_0 = 0 = \frac{\partial n_0}{\partial t}$ (their is no perturbation in density or uniform plasma) we get the dispersion relation for electrostatic electron waves perpendicular to B_0,

$$w^2 = w_{pe}^2 + w_{ce}^2 = w_h^2, \tag{1.3}$$

where w_{pe} is the plasma frequency of electron, w_{ce} is the cyclotron frequency of electron and define as

$$\omega_{pe} = \sqrt{\frac{n_0 e^2}{\epsilon_0 m_e}},$$

$$\omega_{ce} = \frac{B_0 e}{m_e},$$

where w_h is the upper hybrid frequency. It is 'hybrid' because it is a mixture of the plasma frequency and cyclotron frequency. Electrostatic electrons waves which are moving perpendicular to $\mathbf{B}_0$ having upper hybrid frequency and those waves which are moving along $\mathbf{B}_0$ having only plasma frequency. Magnetic field exerts force on electrostatic electron waves which are propagating perpendicular to magnetic field $\mathbf{B}_0$ and changes their direction into elliptical path, instead of oscillating along a straight line. When the electrons are displaced from their mean position, the electric field will be developed in such direction that it opposes the motion of electrons, but in the initial stage magnetic forces are strong compared to the

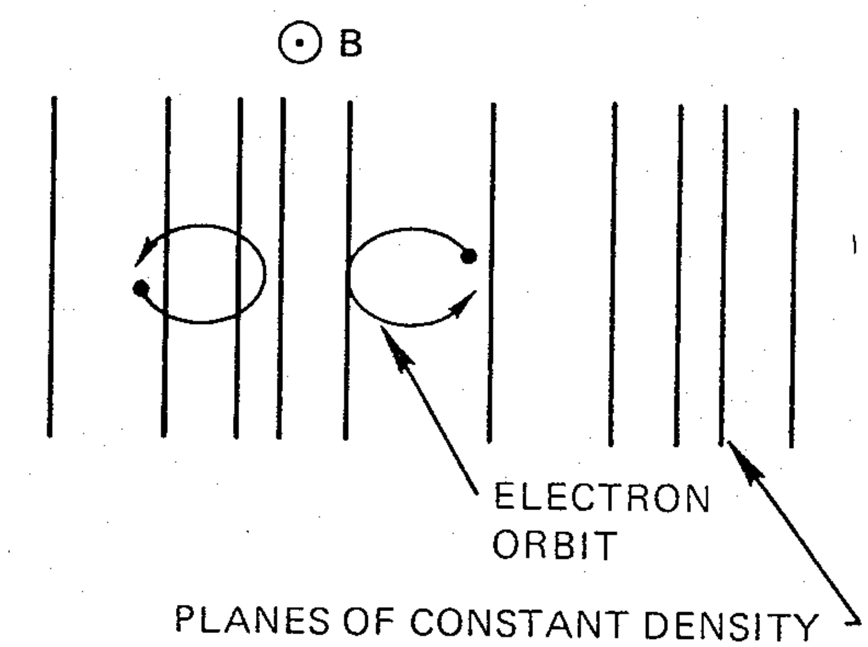

Figure 1.5: Motion of electrons in an upper hybrid oscillation [1].

electric force and motion of particles is governed due to magnetic force. When the speed of particles increases the Lorentz force also increases. As the motion of particles is against the electric field so they lose energy. Their are two type of forces acting on the particles which are propagating perpendicular to magnetic field. One is the electrostatic force and other is the Lorentz force. This additional Lorentz force gives an increase in the frequency.

1.3.4 Electrostatic Ions Wave Perpendicular to $\mathbf{B}_0$

The frequency of electrostatic electron waves is very large as compared to both the plasma and cyclotron frequencies but the response of ions in a field is very small due to large mass. Therefore electrostatic ion waves are the lower frequency case. Here we discuss ions oscillation almost perpendicular to $\mathbf{B}_0$. Mean direction of propagation vector $\mathbf{k}$ is almost perpendicular to $\mathbf{B}_0$ and it is electrostatic case so $\mathbf{B}_1 = 0$. Moreover is parallel to $\mathbf{E}_1$ so longitudinal plane wave is almost perpendicular to $\mathbf{B}_0$. Here we let, $\mathbf{E}_0 = 0 = \mathbf{v}_0$, $k_B T_i = 0$ (we neglect thermal motion of ions) $\nabla n_0 = 0 = \frac{\partial n_0}{\partial t}$ (their is no perturbation in density or uniform plasma). For nearly perpendicular propagation of wave, i,e, for small but non-zero $\mathbf{k}_z$ the electron can oscillate in z-direction under the influence of the electric force due to $\mathbf{E}_{1z}$ and pressure gradient force in z-direction but due to large inertial effect ions can not oscillate along $\mathbf{B}_0$, so we can set $\mathbf{k}_z \approx 0$ for the ions fluid.

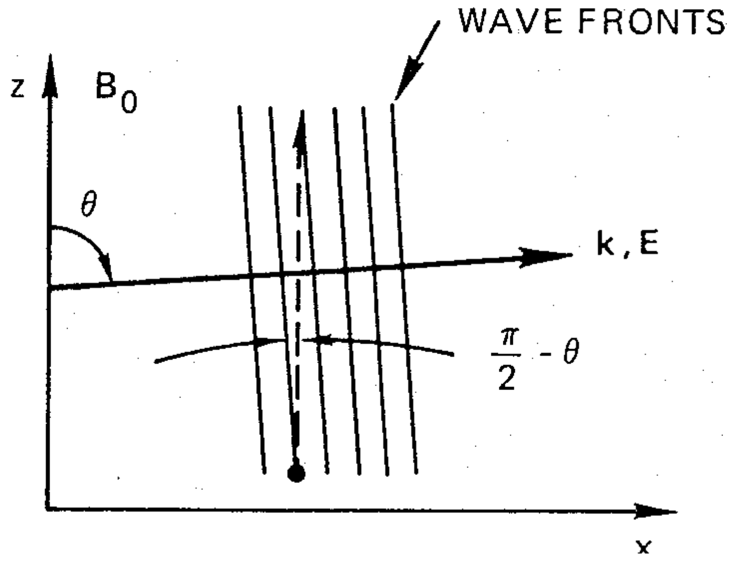

Figure 1.6: Geometry of an electrostatic ion cyclotron wave propagating nearly at right angle to B_0 [1].

The dispersion relation for electrostatic ion waves perpendicular to B_0 is given as,

$$w^2 = w_{ci}^2 + k^2 v_s^2$$

where $w_{ci} = \frac{eB_0}{M}$ and $v_s = \frac{k_B T_e}{M}$. w_{ci} is ions cyclotron frequency and v_s is ions acoustic speed. The ions moving parallel to B_0 having frequency $w^2 = k^2 v_s^2$, it is same as electrostatic ions wave parallel B_0 and the additional cyclotron frequency is due to Lorentz force.

1.4 Different Theoretical Approaches

There are different theories to understand the dynamics of plasma wave depending upon the study of interest and which theory is more efficient to describe the phenomena under study. Here we are discussing two theories of plasma which are more general and most of the researchers use to study the dynamics of plasma. First one is fluid theory and second is the kinetic theory.

1.4.1 Fluid Theory

Plasma is made up of different types of particles, having different masses and charges. It is difficult to study the dynamics of individual particles. To overcome this difficulty, we consider each species as a fluid. Under this approximation the identity of individual particle is neglected and we study the collective behaviour of each species. Actually in fluid theory we

average over velocity, density and temperature. Each species fluid behaves like a continuous medium, so all quantities are function of time, t, and position, r.
We obtain the fluid equation by taking moments of Boltzmann's equation and average over velocity space.

1.4.2 Kinetic Theory

Most of plasma phenomena are accurately describe by using fluid but for some phenomena fluid theory is inadequate, to deal those phenomena we use velocity distribution $f(\mathbf{v})$ for each species, this treatment is called the Kinetic theory. The distribution function depends upon seven independent variable, three for position, three for velocity and one for time $f(\mathbf{r}, \mathbf{v}, t)$. We get more information about plasma, when we use kinetic theory instead of using fluid theory. The distribution function $f(\mathbf{r}, \mathbf{v}, t)$ gives us the information about particles per unit volume at position r and time t with velocity component between v_x and $v_x + dv_x$, v_y and $v_y + dv_y$ and v_z and $v_z + dv_z$ is

$$f(x, y, z, v_x, v_y, v_z, t)dv_x dv_y dv_z,$$

if we integrate over all possible velocities, we get density of particles in given volume [1].

$$n(\mathbf{r}, t) = \int_{-\infty}^{\infty} f(\mathbf{r}, \mathbf{v}, t)d^3v.$$

1.4.3 Examples of Different Distribution Functions

To study the properties of plasma we use different distribution function, it depends upon the the environment of plasma. Here we study two different distribution functions, Maxwellian-Boltzmann distribution and Maxwell-Boltzmann-Jutner distribution.
When we treat with classical plasma we use Maxwellian-Boltzmann distribution because the inter-particles distance become greater than the de-Broglie's wavelength of the charge particles. Therefore the velocity distribution of the charged particles (in thermal equilibrium) is described by the Maxwell Boltzmann velocity distribution function [8].

$$f_{0s} = n_{0s}(\frac{1}{v_{ths}\sqrt{\pi}})^3(\exp -\frac{v_s^2}{v_{ths}^2})$$

We use Maxwellian-Boltzmann distribution for uniform isotropic plasma, it is independent of time. It is a classical distribution function, so any number of particles can be found in any state. The Maxwellian-Boltzmann distribution is applicable for low density or high temperature plasma environment.
When we deal with particles having speed comparable to the speed of light, we can not neglect the relativistic affect on the motion of particles. For those environment the velocity

is comparable to the speed of light we use relativistic distribution function to study their motion[9]

$$f_0(\mathbf{p}) = \frac{1}{4\pi m^3 c^3}\frac{\eta}{K_2(\eta)}\exp(-\eta\gamma)$$

where

$$\eta = \frac{mc^2}{k_B T}$$

is the ratio of the rest mass energy of particles to that of their thermal energy, and K_2 is the modified Bessel function of the second kind and of order two. γ is the relativistic factor given as $\gamma = (1+\frac{p^2}{m^2 * c^2})^{\frac{1}{2}}$

1.5 Bernstein Waves

An important electrostatic mode called Bernstein mode or cyclotron harmonics waves can not predicted by fluid theory, the Bernstein waves depend upon the cyclotron motion of particles about magnetic field line. In fluid theory, we average over larmor orbits, therefore these waves are lost. The Bernstein modes propagate in frequency ranges that lie between harmonics of the cyclotron frequency. These waves are the function of density, temperature and field strength. Majority of oscillations in hot magneto active plasma are strongly damped in time and space, but in some cases damping is weak. The cyclotron waves with frequency near the cyclotrons frequencies of any species $w = nw_{cs}$ are especially interesting, where $n = 1, 2, ...,$ and s for any specie. These collisions -less plasma waves are important in laboratories plasma and used to heat the plasma.

1.6 Thesis outline

In chapter 2, we derived the general expression for hot plasma dielectric tensor in cyclenderical co-ordinates by using Maxwell's equations and Valsov equation. The derived tensor can be used to study any type of plasma waves in non-relativistic regime. The hot plasma dielectric tensor can be used to study for cold plasma waves by using cold plasma limits. In chapter 3, the components of dielectric tensor are used to get the dispersion relation for different type Bernstein waves, like electron, newterlized ions and pure ions Bernstein waves. In this chapter we apply the fluid limits on above Bernstein waves and we get fluid results. In chapter 4, we solve the relativistic Vlasov equation along Maxwell's equations. We get the general expression for conductivity tensor in spherical polar co-ordinate. The derived tensor can be used to study any type of of plasma waves in relativistic regime. Bye using σ_{xx} the components of conductivity tensor we derived the dispersion relation for Bernstein waves in relativistic regime. In chapter 5 we present discussion about results and conclusion of our results.

Chapter 2

Mathematical Model

In section 2.1 we linerized the Vlasov equation and derived the dispersion relation relation for electrons waves using plasma dispersion function. In section 2.2 we find out the dependence of ions damping rates. In section 2.3 we derived the general expression for hot plasma by using linerized Vlasov equation along Maxwell's equations. In section 2.4 we solve the dielectric tensor for isotropic Maxwellian plasma.

2.1 The Plasma Dispersion Function

To study the electromagnetic properties of plasma we need to solve kinetic equations for charge particles. Here we assume that plasma is sufficiently hot, means plasma waves frequency is very high as compare to collisions frequency, so collisions are infrequent and we neglect them. We use kinetic equation with self-consistent field (Valsov equation) for such hot, collision-less plasma. We can study the properties of such a plasma with collision-less approximation by using kinetic equation called Valsov equation.

$$\frac{\partial f}{\partial t} + \mathbf{v}.\nabla f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}).\frac{\partial f}{\partial \mathbf{v}} = 0 \tag{2.1}$$

Since the electric field, $\mathbf{E}$, and $\mathbf{B}$ depend on distribution function, $f(\mathbf{r}, \mathbf{v}, t)$, so the Vlasov equation is first order, non-linear partial differential equation [10, 11, 12]. To linearise the Vlasov equation, we assume that the amplitude of the perturbed quantities is small so we consider only first order perturbation. We consider a uniform plasma with equilibrium distribution function $f_0(\mathbf{v})$ and a small perturbation in it [8]

$$f = f_0 + f_1$$

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$$

and we let $\mathbf{B}_o = 0 = \mathbf{E}_o$ and f_1 is the perturbation in distribution function. Since $\mathbf{v}$ is independent variable and con not be linearized.Know linearizing the other terms.

$$\frac{\partial f}{\partial t} = \frac{\partial f_o}{\partial t} + \frac{\partial f_1}{\partial t}$$

$$\mathbf{v}.\nabla f = \mathbf{v}.\nabla(f_o + f_1)$$

$$\frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}).\frac{\partial f}{\partial \mathbf{v}} = \frac{q}{m}(\mathbf{E}_o + \mathbf{E}_1 + \mathbf{v} \times (\mathbf{B}_o + \mathbf{B}_1)).(\frac{\partial f_o}{\partial \mathbf{v}} + \frac{\partial f_1}{\partial \mathbf{v}})$$

When we neglect the effect of magnetic field, the linearised Vlasov equation is,

$$\frac{\partial f_1}{\partial t} + \mathbf{v}.\nabla f_1 + \frac{q}{m}\mathbf{E}_1.(\frac{\partial f_o}{\partial \mathbf{v}}) = 0 \tag{2.2}$$

For simplicity we consider the wave is propagating in $x-$direction, $f_1 = \exp(\iota(kx - wt))$, where $\mathbf{k}$ is the propagation vector and w is the frequency of the wave. Applying Fourier-Laplace transformation on the equation (2.2).

$$(E, B, f) = \int_0^\infty dt \exp(-st) \int_{-\infty}^\infty \frac{d^3x}{(2\pi)^{\frac{3}{2}}} \exp(-ik.v)(E_1, B_1, f_1),$$

we get,

$$f_1 = -i\frac{q}{m}.\frac{E_1}{(w - kv_x)}.\frac{\partial f_o}{\partial v}$$

For different species we can write

$$f_{1s} = \sum_s -i\frac{q_s}{m_s}.\frac{E_1}{(w - kv_s)}\frac{\partial f_{os}}{\partial v_s}$$

f_{1s} is the perturbation in distribution function of s specie. q_s, m_s and v_s is the charge, mass and velocity of sth specie respectively. The density perturbation of sth specie is given by the following relation,

$$n_{1s} = \int_{-\infty}^\infty f_{1s}(v_s)dv_s \tag{2.3}$$

Let the equilibrium distribution f_{0s} be one dimensional Maxwellian distribution,

$$f_{0s} = (\frac{n_{0s}}{v_{ths}})(\frac{1}{\sqrt{\pi}})(\exp -\frac{v_x^2}{v_{ths}^2})$$

where n_{0s} is the normalizing constant, called number density of sth species and

$$v_{ths} = (\frac{2k_BT_s}{m_s})^{\frac{1}{2}},$$

is the thermal velocity of sth specie. Poisson's equation for electrons,

$$\epsilon_0\nabla.E = -en_{1e} = -e\int_{-\infty}^\infty f_{1e}(v_x)dv_x$$

Applying Fourier-Laplace transformation on above equation. The dispersion relation become,

$$k^2 = \int_{-\infty}^{\infty} \frac{w_{pe}^2}{(v_x - \frac{w}{k})} \frac{\partial f_{oe}}{\partial v_x} \tag{2.4}$$

w_{pe} is the plasma frequency of electrons and define as,

$$w_{pe} = (\frac{e^2 n_{0e}}{\epsilon_0 m_e})^{\frac{1}{2}}$$

Now we calculate the dispersion relation using plasma dispersion function. Introducing the dummy integration variable $S = \frac{v_s}{v_{ths}}$ and substituting value of $f_{1s}(v_s)$

$$n_{1s} = \sum_s i \frac{q_s}{m_s} E \frac{n_{0s}}{k v_{ths}^2} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d}{dS} \exp(-S^2) \frac{1}{(S - \xi)} \tag{2.5}$$

$$\xi = \frac{w}{k v_{ths}}$$

When we study the electrostatic wave in plasma by using kinetic theory we need plasma dispersion function which in principle takes the from [13].

$$Z(\xi) = A \int_{-\infty}^{\infty} \frac{f(S)}{(S - \xi)} \quad (Im(\xi) > 0)$$

where A is the normalization constant, S is the normalized velocity of particles, $f(S)$ is the velocity distribution function and ξ is the normalized phase velocity. For Maxwellian distribution plasma dispersion function is define as

$$Z(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-S^2) \frac{1}{(S - \xi)} \quad (Im(\xi) > 0)$$

This type of integration is not done by simple way due to singularity at $S = \xi$. Landau was the first to treat this type of integration properly. These types of singularities modify the plasma dispersion relation and effect is not predicted by fluid theory. If the perturbation grows or decays ξ will be complex. Above integral is treated as a contour integral in complex plane. To express n_{1i} in term of $Z(\xi)$ we take derivative with respect to ξ and then performing integration by parts we get

$$Z'(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{d}{dS} \exp(-S^2) \frac{1}{(S - \xi)}$$

So equation (2.3) in term of plasma dispersion function

$$n_{1s} = i \frac{q_s}{m_s} E \frac{n_{0s}}{k v_{ths}^2} Z'(\xi) \tag{2.6}$$

Poisson's equation is given as,

$$\epsilon_0 \nabla . \mathbf{E} = \sum_s q_s n_{1s}$$

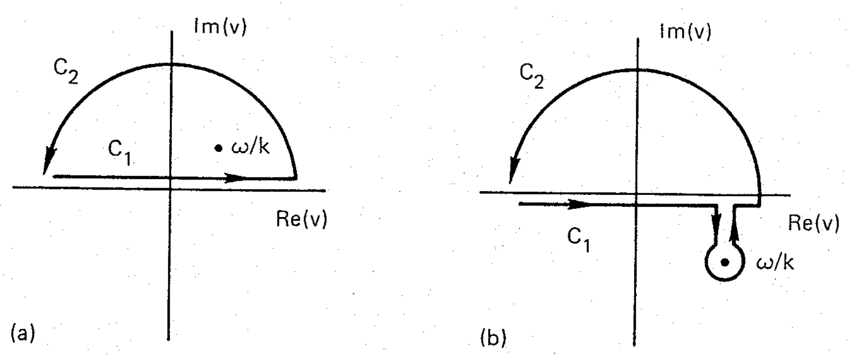

Figure 2.1: (a) Imω>0 an unstable wave (b) Imω<0 a damped wave [1].

After the Fourier-Laplace transformation of Poisson's equation and putting value of n_{1s} we get,

$$k^2 = \sum_s \frac{q_s^2 n_{0s}}{\epsilon_0 m_s} \frac{1}{v_{ths}^2} Z'(\xi)$$

The dispersion relation for electrons in term of plasma dispersion function is given as,

$$k^2 = \frac{e^2 n_{0e}}{\epsilon_0 m_e v_{the}^2} Z'(\xi)$$

$$k^2 = \frac{\omega_{pe}^2}{v_{the}^2} Z'(\xi) \tag{2.7}$$

e is the charge on electron and n_{1e} is the number density of electron. Equation (2.7) is same as (2.4), so from this exercise we see that their no effect on physics of plasma by using plasma dispersion function. If the perturbation grows or decays w will be complex the integral

$$k^2 = +\int_{-\infty}^{\infty} \frac{\omega_{pe}^2}{(v_x - \frac{\omega}{k})} \frac{\partial f_{oe}}{\partial v_x} dv \tag{2.8}$$

must be treated as a contour integral in the complex v plane. Possible contours are shown in above figures. Normally one would evaluate the line integral along the real v-axis by the residue theorem.

$$\int_{c1} G dv + \int_{c2} G dv = 2\Pi i R(\frac{\omega}{k})$$

Where G is the integrand, c_1 is the path along real axis, c_2 is the semicircle at infinity and R($\frac{\omega}{k}$) is the residue at $\frac{\omega}{k}$. The residue theorem work if the integral over c_2 vanished but it become large for Maxwellian when $v \longrightarrow \pm i\infty$. Fried and Conte gives the numerical solution for Maxwellian distribution. We can approximate these integral using Cauchy principal value theorem for large phase velocity and weak damping.

2.2 Ions Wave and Their Damping

The plasma dispersion function

$$Z(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-S^2) \frac{1}{(S-\xi)} dS \quad (Im(\xi) > 0)$$

The Fadeeva function is define as[13, 12]

$$w(\xi) = \exp(-\xi^2)[1 + erf(\iota\xi)]$$

Integral representation of Fadeeva function is given as

$$w(\xi) = -\frac{i}{\pi} \int_{-\infty}^{\infty} \exp(-S^2) \frac{1}{(S-\xi)} dS$$

$$Z(\xi) = i\sqrt{\pi} w(\xi) = i\sqrt{\pi} \exp(-\xi^2)[1 + erf(i\xi)]$$

$$erf(i\xi) = ierfc(\xi)$$

$$Z(\xi) = i\sqrt{\pi} \exp(-\xi^2)[1 + ierfc(\xi)]$$

The above form of plasma dispersion function used in Russian plasma physics literature. Where $\xi = \frac{\omega}{kv_{ths}}$ is ratio of phase velocity to thermal velocity of the specie. Electron has small value of ξ. For small value of ξ we can expand the series of $\exp(-\xi^2)$ and error function. For electron, the plasma dispersion function is given as[14],

$$Z(\xi_e) = i\sqrt{\pi} \exp(-\xi_e^2) - 2\xi_e(1 - \frac{2}{3}\xi_e^2 + ...) \tag{2.9}$$

The imaginary term represent the electron landau damping which was not appeared in fluid theory. For $\xi_e << 1$

$$Z'(\xi_e) \approx -2$$

electrons Landau damping term can be neglect in ions landau damping due to $v_{the} >> v_{thi}$ The dispersion relation for ions is given as,

$$k^2 = \frac{\omega_{pe}^2}{v_{the}^2}(-2) + \sum_i \frac{\omega_{pi}^2}{v_{thi}^2} Z'(\xi_i)$$

Where ω_{pi} is plasma frequency of ions and v_{thi} is thermal velocity of ions.

$$\frac{\omega_{pe}^2}{v_{the}^2}(-2) = -\frac{1}{\lambda_d^2}$$

λ_d is Debye length and it is a measure of shielding distance.

$$\lambda_d^2 \sum_i \frac{\omega_{pi}^2}{v_{thi}^2} Z'(\xi_i) = k^2\lambda_d^2 + 1 \approx 1$$

For a single ion

$$Z'(\xi_i) = \frac{2T_i}{T_e}$$

T_i and T_e temperature of ions and electrons respectively. To obtain the analytical result, we consider the limit $\xi_i >> 1$ and $T_e >> T_i$. The plasma dispersion function is define as

$$Z(\xi) = i\sqrt{\pi}\omega(\xi)$$

For large argument $\xi >> 1$, (means high phase velocity) in this case the position of pole in the integration contour is very close to the real $S - axis$. For large argument we can use, the Plemelj relation to evaluate the plasma dispersion function [15].

$$Z(\xi) = i\sqrt{\pi}\exp(-\xi^2) + \frac{1}{\sqrt{\pi}}P\int_{-\infty}^{\infty}\exp(-S^2)\frac{1}{(S-\xi)}dS$$

$$\frac{1}{(S-\xi)} = \frac{-1}{\xi(1-\frac{S}{\xi})} = \frac{-1}{\xi}[1 + \frac{S}{\xi} + (\frac{S}{\xi})^2 + ...]$$

Putting this power series into above integration and then taking derivative with respect to ξ we get,

$$Z'(\xi) = -2i\sqrt{\pi}\exp(-\xi_i^2) + \frac{1}{2\xi_i^2} + \frac{3}{2\xi_i^4} + ...$$

If the Landau damping is small we can neglect the Landau term $-2i\sqrt{\pi}\exp(-\xi_i^2)$

$$\frac{1}{\xi_i^2}(1 + \frac{3}{2\xi_i^2}) = \frac{2T_i}{T_e}$$

$T_e >> T_i$ so ξ_i^2 is large. We can approximate ξ_i^2 by $\frac{T_e}{2T_i}$ in the second term and final dispersion relation become

$$\frac{\omega^2}{k^2} = \frac{kT_e + 3kT_i}{M}$$

M is the mass of ions.This is the ions wave dispersion relation with $\gamma_i = 3$. Electron behave as isotherm due to high speed and their temperature comes in equilibrium. If we include the landau damping term.

$$\frac{1}{\xi_i^2}(1 + \frac{3}{\frac{T_e}{2T_i}}) - 2i\sqrt{\pi}\xi_i\exp(-\xi_i^2) = \frac{2T_i}{T_e}$$

$$\xi_i = \Re\sqrt{\frac{3 + \frac{T_e}{2T_i}}{2}} - \Im\sqrt{\frac{3 + \frac{T_e}{2T_i}}{2}}\frac{1}{2}\frac{T_e}{2T_i}\xi_i i\sqrt{\pi}\exp(-\xi_i^2)$$

$$\Re\xi_i = \sqrt{\frac{3 + \frac{T_e}{2T_i}}{2}}$$

$$\Im\xi_i = -\frac{T_e}{2T_i}\sqrt{(3 + \frac{T_e}{2T_i})}\sqrt{\frac{\pi}{8}}(\Im\xi_i + \Re\xi_i)\exp-(\Im\xi_i + \Re\xi_i)^2$$

$\Re\xi_i >> \Im\xi_i$ in comparison we drop $\Im\xi_i$ and we can approximate the damping rate

$$-\frac{\Im\xi_i}{\Re\xi_i} = -\frac{\Im\omega}{\mathbb{R}\omega} = \frac{T_e}{2T_i}\sqrt{(3+\frac{T_e}{2T_i})}\sqrt{\frac{\pi}{8}}\exp -\frac{(3+\frac{T_e}{2T_i})}{2}$$

Negative singe indicate damping of the wave. Damping rate depend on ratio of temperature of two species when the temperature of the two species is comparable the the wave is strongly damped and when $T_e >> Ti$ the wave is weakly damped. We can not explain damping effect by using fluid theory.

2.3 Hot Plasma Dielectric Tensor

The linearised Vlasov equation for uniform plasma with ambient magnetic field $\mathbf{B}_0$ is given as [8],

$$\frac{\partial f_1}{\partial t} + \mathbf{v}.\nabla f_1 + \frac{q}{m}(\mathbf{v}\times\mathbf{B}_0).\frac{\partial f_1}{\partial \mathbf{v}} = -\frac{q}{m}(\mathbf{E}_1 + \mathbf{v}\times\mathbf{B}_1).(\frac{\partial f_o}{\partial \mathbf{v}}) \tag{2.10}$$

. Consider all particles of same species are located at an arbitrary point $\mathbf{r}$, $\mathbf{v}$ in phase space at present time. The motion of all these particles is under same Lorentz force so their trajectories are same in both future and past time. All particles having same initial conditions at time t, so they have same $\mathbf{r}$ and $\mathbf{v}$ at any time t. The boundary conditions on the trajectories of are $x(t) = x$ and $v(t) = v$ at time t. The distribution function in phase space is,

$$f_1 = f_1(\mathbf{r}(t), \mathbf{v}(t), t)$$

where $\mathbf{r}$ and $\mathbf{v}$ is same for all particles at any time in phase space. Wave having small amplitude so trajectories of particles are not effected by waves. Since unperturbed particle trajectory equations are

$$\frac{d\mathbf{r}}{dt} = \mathbf{v}$$

,

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}(\mathbf{v}\times\mathbf{B})$$

Taking time derivative of distribution function

$$\frac{d}{dt}f_1(\mathbf{r}(t), \mathbf{v}(t), t) = \frac{\partial f_1}{\partial t} + \nabla f_1.\frac{d\mathbf{r}}{dt} + \frac{\partial f_1}{\partial \mathbf{v}}.\frac{d\mathbf{v}}{dt}$$

$$\frac{d}{dt}f_1(\mathbf{r}(t), \mathbf{v}(t), t) = \frac{\partial f_1}{\partial t} + \nabla f_1.\frac{d\mathbf{r}}{dt} + \frac{\partial f_1}{\partial \mathbf{v}}.\frac{q}{m}(\mathbf{v}'\times\mathbf{B}_0) \tag{2.11}$$

Comparing equation (2.10) and (2.11) and then integrating both side

$$f_1(\mathbf{r}(t), \mathbf{v}(t), t) = -\frac{q}{m}\int_{-\infty}^{t}(\mathbf{E}_1(\mathbf{r}', t') + (\mathbf{v}'\times(\mathbf{B}(\mathbf{r}', t')).\frac{\partial f_0(\mathbf{v}')}{\partial \mathbf{v}}dt' \tag{2.12}$$

At $t' = t$

$$f(\mathbf{r}', \mathbf{v}') = f(\mathbf{r}, \mathbf{v})$$

[1] The right hand side of equation (2.12) is the sum of all forces that modify the distribution function. Unperturbed trajectories are characteristics of homogeneous hyperbolic partial differential equation[16]. The right hand side of equation (2.10) is forcing term that change the homogeneous solution. We can write the velocity in term of Cartesian components.

$$v(t) = (v_\perp \cos\phi, v_\perp \sin\phi, v_z)$$

Let $\mathbf{E} = 0$, $\mathbf{B} = \mathbf{B}_0 z$. Equation of motion of charge particles in a uniform field is given as,

$$\frac{\partial v'}{\partial t'} = \frac{q}{m}(v' \times B_0 z)$$

$$\frac{\partial v'}{\partial t'} = -\omega_{cs} v'_x y + \omega_{cs} v'_y x$$

$\omega_{cs} = \frac{B_0 q_s}{m_s}$ is cyclotron frequency of sth specie. Let $v'_x + \iota v'_y = v''(t')$

$$\frac{dv''}{dt} = -i\omega_{cs} v'' \tag{2.13}$$

equation (2.13) is non exact first order homogeneous equation .To make exact multiplied with integrating factor $\exp(\iota\omega_{cs} t)$ and writing its solution in term of v'_x and v'_y we get,

$$v'_x(t') = v_\perp \cos(-\omega_{cs}(t' - t) + \phi) \tag{2.14}$$

$$v'_y(t') = v_\perp \sin(-\omega_{cs}(t' - t) + \phi) \tag{2.15}$$

$$v'_z(t') = v_\parallel(t) \tag{2.16}$$

integrating above equations with conditions $x = x'$ and $t = t'$ we get

$$x' - x = \frac{v_\perp}{\omega_{cs}}(sin(\omega_{cs}(t' - t) - \phi) + sin\phi))$$

$$y' - y = \frac{v_\perp}{\omega_{cs}}(cos(\omega_{cs}(t' - t) - \phi) - cos\phi))$$

$$z' - z = v_z(t' - t)$$

Maxwell equations are given as,

$$\nabla.\mathbf{E} = \frac{1}{\epsilon_0} \sum_s q_s \int f d^3v \tag{2.17}$$

$$\frac{1}{\mu_0} \nabla\times\mathbf{E} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \sum_s q_s \int \mathbf{v} f d^3v \tag{2.18}$$

$$\nabla\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.19}$$

$$\nabla.\mathbf{E} = 0 \tag{2.20}$$

Know we simplify the equation (2.12). We consider the field is sinusoidal

$$\mathbf{E}_1(\mathbf{r}', t') = E_1 \exp[i(\mathbf{k}.\mathbf{r}' - \omega t')]$$

$$\mathbf{E}_1(\mathbf{r}', t') = E_1 \exp[i(\mathbf{k}.(\mathbf{r}' - \mathbf{r}) - \omega(t' - t)) \exp i(\mathbf{k}.\mathbf{r} - \omega t)]$$

Propagating vector $\mathbf{k}$ is define as

$$\mathbf{k} = k\hat{x} + k\hat{y}$$

$$\mathbf{E}_1(\mathbf{r}', t') = E_1(r, t) \exp(i\frac{k_\perp v_\perp}{\omega_{cs}}[sin(\omega_{cs}(t' - t) - \phi) + sin\phi] + k_\parallel v_\parallel (t' - t) - \omega(t' - t))$$

Using the identity of Bessel function

$$\exp(i\lambda sin(x)) = \sum_{-\infty}^{\infty} J_n(\lambda) \exp(inx)$$

$$\exp(i\frac{k_\perp v_\perp}{\omega_{cs}} sin(\omega_{cs}(t' - t) - \phi)) = \sum_{-\infty}^{\infty} J_n(\frac{k_\perp v_\perp}{\omega_{cs}}) \exp(in(\omega_{cs}(t' - t) - \phi))$$

$$\exp(i\frac{k_\perp v_\perp}{\omega_{cs}} sin(\phi)) = \sum_{-\infty}^{\infty} J_m(\frac{k_\perp v_\perp}{\omega_{cs}}) \exp(im\phi)$$

$$\lambda = \frac{k_\perp v_\perp}{\omega_{cs}}$$

$$\mathbf{E}_1(\mathbf{r}', t') = \mathbf{E}_1(\mathbf{r}, t) \sum_{nm=-\infty}^{\infty} J_n(\lambda) J_n(\lambda) \exp i[(n\omega_{cs} + k_\parallel v_\parallel - \omega)(t' - t) + (m - n)\phi]$$

$$f_1(\mathbf{r}(t), \mathbf{v}(t), t) = -\frac{q}{m}\int_{-\infty}^{t} [(\mathbf{E}_1(\mathbf{r}', t') + (\mathbf{v}' \times \mathbf{B}_1(\mathbf{r}', t')).\frac{\partial f_0(\mathbf{v}')}{\partial \mathbf{v}'}]dt' \tag{2.21}$$

Applying Fourier Laplace transformation on equation (2.19) we get

$$\mathbf{B}_1 = \frac{1}{w}\mathbf{k} \times \mathbf{E}_1$$

Let

$$S = -\frac{q}{m}[\mathbf{E}_1(\mathbf{r}, t) + \frac{1}{w}(\mathbf{v}' \times (\mathbf{k} \times \mathbf{E}_1(\mathbf{r}, t)].\frac{\partial f_0(\mathbf{v}')}{\partial \mathbf{v}}$$

Using identity of vector

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A}.\mathbf{C})\mathbf{B} - (\mathbf{A}.\mathbf{B})\mathbf{C}$$

$$\mathbf{v} \times (\mathbf{k} \times \mathbf{E}_1) = (\mathbf{v}.\mathbf{E}_1)\mathbf{k} - (\mathbf{v}.\mathbf{k})\mathbf{E}_1$$

$$S = -\frac{q}{m}[(1 - \frac{\mathbf{k}.\mathbf{v}'}{w})\mathbf{E}_1(\mathbf{r}, t) + \frac{1}{w}(\mathbf{v}'.\mathbf{E}_1(\mathbf{r}, t))k].\frac{\partial f_0(\mathbf{v}')}{\partial \mathbf{v}'}$$

The velocity in a cylindrical coordinate.

$$\mathbf{v}'(t') = v_\perp cos(-\omega_{cs}(t'-t)+\phi)\hat{x} + v_\perp sin(-\omega_{cs}(t'-t)+\phi)\hat{y} + v_\parallel \hat{z}$$

For simplicity we choice $\mathbf{k}$ is only in x and z direction.

$$\mathbf{k} = k_\perp \hat{x} + k_\parallel \hat{z}$$

$$\mathbf{E}_1 = E_1\hat{x} + E_1\hat{y} + E_1\hat{z}$$

$$\begin{aligned} S = \quad & -\frac{q}{m}[(1 - \frac{k_\perp v_\perp cos((-\omega_{cs})(t'-t)+\phi) + k_\parallel v_\parallel}{\omega})(E_{1x}\frac{\partial f_0}{\partial v'_x} + E_{1y}\frac{\partial f_0}{\partial v'_y} + E_{1z}\frac{\partial f_0}{\partial v'_z}) \\ & + \frac{1}{\omega}((E_{1x}v_\perp cos(-\omega_{cs})(t'-t)+\phi) + (E_{1y}v_\perp sin(-\omega_{cs})(t'-t)+\phi) + v_\parallel E_{1z})(k_\perp \frac{\partial f_0}{\partial v'_x} + k_\parallel \frac{\partial f_0}{\partial v'_z})] \end{aligned}$$

$$\frac{\partial f_0}{\partial v'} = \frac{\partial f_0}{\partial v'_x}x + \frac{\partial f_0}{\partial v'_y} + \frac{\partial f_0}{\partial v'_z}$$

From equations (2.14), (2.15) and (2.16) we see that $v_\perp$ and $v_\parallel$ are constant of motion. This implies that $f_0(v') = f_0(v)$ and $f_0(v)$ is a function of $v_\perp$ and $v_\parallel$. In a cylindrical coordinate we can write,

$$v_\perp^2 = v_x'^2 + v_y'^2$$

$$\frac{\partial f_0}{\partial v'_x} = \frac{\partial v_\perp}{\partial v'_x}\frac{\partial f_0}{\partial v_\perp} = cos[-\omega_{cs}(t'-t)+\phi]\frac{\partial f_0}{\partial v_\perp}$$

$$\frac{\partial f_0}{\partial v'_y} = \frac{\partial v_\perp}{\partial v'_y}\frac{\partial f_0}{\partial v_\perp} = sin[-\omega_{cs}(t'-t)+\phi]\frac{\partial f_0}{\partial v_\perp}$$

$$\frac{\partial f_0}{\partial v'_z} = \frac{\partial f_0}{\partial v_\parallel}$$

$$\begin{aligned} S = \quad & -\frac{q}{\omega m}([(\omega - k_\parallel v_\parallel)\frac{\partial f_0}{\partial v_\perp} + k_\parallel v_\perp \frac{\partial f_0}{\partial v_\parallel}][E_x cos(-\omega_{cs}(t'-t)+\phi) + E_y sin(-w_{cs}(t'-t)+\phi)] \\ & + [\frac{\partial f_0}{\partial v_\parallel}(\omega - k_\perp v_\perp cos(-\omega_{cs}(t'-t)+\phi)) + (v_\parallel k_\perp cos(-\omega_{cs}(t'-t)+\phi)\frac{\partial f_0}{\partial v_\perp})]E_{1z}) \end{aligned}$$

$$f_1(r(t), v(t), t) = \int_{-\infty}^{t} S \sum_{nm=-\infty}^{\infty} J_n(\lambda)J_m(\lambda)\exp i[(n\omega_{cs} + k_\parallel v_\parallel - \omega)(t'-t) + (m-n)\phi]dt'$$

Rearranging the above equation.

$$f_1(r(t), v(t), t) = \int_{-\infty}^{t} S \sum_{nm=-\infty}^{\infty} J_n(\lambda)J_m(\lambda)\exp(-i(n-m))\phi\exp(\iota n\omega_{cs}(t'-t))\exp(i(k_\parallel v_\parallel - \omega)(t'-t))dt'$$

Let $t' - t = \tau$ and $dt' = d\tau$

$$f_1(r(t), v(t), t) = \int_{-\infty}^{0} -\frac{q}{\omega m}([(\omega - k_\parallel v_\parallel)\frac{\partial f_0}{\partial v_\perp} + k_\parallel v_\perp \frac{\partial f_0}{\partial v_\parallel}][E_x cos(-\omega_{cs}(\tau) + \phi) + E_y sin(-\omega_{cs}(\tau) + \phi)]$$
$$+[\frac{\partial f_0}{\partial v_\parallel}(\omega - k_\perp v_\perp cos(-\omega_{cs}(\tau) + \phi)) + v_\parallel k_\perp cos(-\omega_{cs}(\tau) + \phi)\frac{\partial f_0}{\partial v_\perp})]E_{1z}$$
$$\sum_{nm=-\infty}^{\infty} J_n(\lambda) J_m(\lambda) \exp(-\iota(n-m))\phi \exp(in\omega_{cs}(\tau)) \exp(\iota(k_\parallel v_\parallel - \omega)(\tau)) d\tau$$

$$\lambda = \frac{k_\perp v_\perp}{\omega_{cs}}$$

Let

$$P = (\omega - k_\parallel v_\parallel)\frac{\partial f_0}{\partial v_\perp} + k_\parallel v_\perp \frac{\partial f_0}{\partial v_\parallel}$$

$$Q = \frac{n\omega_{cs} v_\parallel}{v_\perp}\frac{\partial f_0}{\partial v_\perp} + (\omega - n\omega_{cs})\frac{\partial f_0}{\partial v_\parallel}$$

$$f_1(\mathbf{r}, \mathbf{v}, t) = \frac{iq}{m\omega}\sum_{nm-\infty}^{\infty} J_m(\lambda) \exp[-\iota(n-m)]\phi \frac{1}{(\omega - k_\parallel v_\parallel - n\omega_{cs})}[\frac{n}{\lambda} P J_n(\lambda) E_x + iPJ_n'(\lambda) E_y$$
$$+QJ_n(\lambda)E_z]$$

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \sum_s q_s \int \mathbf{v} f d^3 v \tag{2.22}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.23}$$

Applying Fourier-Laplace transformation on equation (2.23) we get,

$$\mathbf{B}_1 = \frac{1}{\omega}(\mathbf{k} \times \mathbf{E}_1)$$

Putting value **B** in equation (2.22) and applying Fourier-Laplace transformation we get,

$$\frac{1}{\mu_0 \epsilon_0 \omega}(\mathbf{k} \times \mathbf{k} \times \mathbf{E}_1) = -\omega \overleftrightarrow{\epsilon}.\mathbf{E}_1$$

$$\overleftrightarrow{\epsilon}.\mathbf{E}_1 = \overleftrightarrow{1}.\mathbf{E}_1 + \frac{i}{\epsilon_0 \omega}\sum_s q_s \int \mathbf{v} f(\mathbf{r}, \mathbf{v}, t) d^3 v \tag{2.24}$$

$$(\mathbf{kk} + \mu_0 \epsilon_0 \omega^2 \overleftrightarrow{\epsilon} - k^2 \overleftrightarrow{1}) = 0 \tag{2.25}$$

Know we want to calculate

$$\int \mathbf{v} f(\mathbf{r},\mathbf{v},t) d^3v$$

$$\mathbf{v} = v_\perp cos\phi \hat{x} + v_\perp sin\phi \hat{y} + v_\parallel \hat{z}$$

$$d^3v = v_\perp dv_\perp d\phi dv_\parallel$$

X-component

$$\int v_\perp cos\phi f_1(\mathbf{r},\mathbf{v},t) d^3v \hat{x} = \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty v_\perp dv_\perp \int_0^{2\pi} v_\perp cos\phi d\phi f_1(r,v,t)\hat{x}$$

Identity of delta function

$$\int_0^{2\pi} cos\phi \exp(i[m-n]\phi) d\phi = 2\pi \frac{(\delta_{m,n+1} + \delta_{m,n-1})}{2}$$

identity of Bessel function

$$\frac{J_{n+1}(\lambda) + J_{n-1}(\lambda)}{2} = \frac{n}{\lambda} J_n(\lambda)$$

$$\int v_\perp 2\pi dv_\perp dv_\parallel = dv^3$$

$$\int v_\perp cos\phi f_1(r,v,t) d^3v = -\frac{iq}{mw}\sum_s \int d^3v \frac{1}{(\omega - k_\parallel v_\parallel - n\omega_{cs})}[v_\perp (\frac{nJ_n(\lambda)}{\lambda})^2 PE_x + iv_\perp PJ_n(\lambda)J_n'(\lambda)E_y$$
$$+Qv_\perp \frac{n}{\lambda} J_n^2(\lambda) E_z]\hat{x}$$

Y-component

$$\int v_\perp sin\phi f_1(\mathbf{r},\mathbf{v},t) d^3v \hat{y} = \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty v_\perp dv_\perp \int_0^{2\pi} v_\perp sin\phi d\phi f_1(\mathbf{r},\mathbf{v},t)\hat{y}$$

Identity of delta function

$$\int_0^{2\pi} sin\phi \exp(i[m-n]\phi) d\phi = \frac{2\pi}{2i}(-\delta_{m,n+1} + \delta_{m,n-1})$$

$$\int v_\perp sin\phi f_1(\mathbf{r},\mathbf{v},t) d^3v \hat{y} = -\frac{iq}{m\omega}\sum_s \int d^3v \frac{1}{(\omega - k_\parallel v_\parallel - n\omega_{cs})}[-i\frac{n}{\lambda} v_\perp PJ_n(\lambda)J_n'(\lambda)E_x + v_\perp J_n'^2(\lambda)PE_y$$
$$-iv_\perp QJ_n(\lambda)J_n'(\lambda)E_z]\hat{y}$$

Z-component

$$\int v_\parallel f_1(\mathbf{r},\mathbf{v},t) d^3v \hat{z} = \int_{-\infty}^{\infty} v_\parallel dv_\parallel \int_0^\infty v_\perp dv_\perp \int_0^{2\pi} d\phi f_1(\mathbf{r},\mathbf{v},t)\hat{z}$$

Identity of Bessel function

$$\int_0^{2\pi} \exp(i[m-n]\phi)d\phi = 2\pi(\delta_{m,n})$$

$$\int v_\parallel f_1(\mathbf{r},\mathbf{v},t)d^3v\hat{z} = -\frac{iq}{m\omega}\sum_s\int d^3v\frac{1}{(\omega - k_\parallel v_\parallel - n\omega_{cs})}[v_\parallel(\frac{n}{(\lambda)}J_n(\lambda)PE_x + iv_\parallel PJ_n(\lambda)J_n'(\lambda)E_y$$
$$+Qv_\parallel J_n^2(\lambda)E_z]\hat{z}$$

Know we will comeback to equation

$$\overleftrightarrow{\epsilon}.\mathbf{E}_1 = \overleftrightarrow{1}.\mathbf{E}_1 + \frac{\iota}{\epsilon_0 w}\sum_s q_s\int \mathbf{v}f_1(\mathbf{r},\mathbf{v},t)d^3v \tag{2.26}$$

$$\overleftrightarrow{\epsilon} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2}\sum_{n=-\infty}^{\infty}\frac{1}{n_s}\int\frac{\overleftrightarrow{D}}{(\omega - k_\parallel v_\parallel - n\omega_{cs})}d^3v \tag{2.27}$$

$$\overleftrightarrow{D} = \begin{bmatrix} v_\perp(\frac{nJ_n(\lambda)}{\lambda})^2P & \frac{n}{\lambda}iv_\perp PJ_n(\lambda)J_n'(\lambda) & Qv_\perp\frac{n}{\lambda}J_n^2(\lambda) \\ -i\frac{n}{\lambda}v_\perp PJ_n(\lambda)J_n'(\lambda) & v_\perp J_n'^2(\lambda)P & -iv_\perp QJ_n(\lambda)J_n'(\lambda) \\ v_\parallel(\frac{n}{\lambda})J_n^2(\lambda)P & iv_\parallel PJ_n(\lambda)J_n'(\lambda) & Qv_\parallel J_n^2(\lambda) \end{bmatrix} \tag{2.28}$$

When we use the fluid theory or cold plasma, the dielectric tensor is a function of ω_p and ω_c only. $\overleftrightarrow{D}$ is called hot plasma dispersion tensor. The dielectric tensor is not only a function of ω_p and ω_c. It is also a function of temperature and wave number k. We include the thermal motion of particles also. In cold plasma approximation we neglect thermal motion of particles because of which we lose some important features.

2.4 Dielectric Tensor for an Isotropic Maxwellian Plasma

The Maxwellian distribution is define as[17],

$$f_{0s} = n_{0s}(\frac{1}{v_{ths}\sqrt{\pi}})^3(\exp -\frac{v_s^2}{v_{ths}^2})$$

$$v_{ths} = (\frac{2kT_s}{m_s})^{\frac{1}{2}}$$

v_{ths} is thermal velocity of sth specie.

$$v_s^2 = v_\perp^2 + v_\parallel^2$$

$$P = (\omega - k_\parallel v_\parallel)\frac{\partial f_0}{\partial v_\perp} + k_\parallel v_\perp\frac{\partial f_0}{\partial v_\parallel}$$

$$Q = \frac{n\omega_{cs} v_{\parallel}}{v_{\perp}} \frac{\partial f_0}{\partial v_{\perp}} + (\omega - n\omega_{cs}) \frac{\partial f_0}{\partial v_{\parallel}}$$

Taking partial derivative with respect to $v_{\perp}$ and $v_{\parallel}$ we get

$$\frac{\partial f_0}{\partial v_{\perp}} = A v_{\perp}$$

$$\frac{\partial f_0}{\partial v_{\parallel}} = A v_{\parallel}$$

Where A is define as

$$A = \frac{-2n_{0s}}{v_{ths}^5 \pi^{\frac{3}{2}}} (\exp - \frac{v_{\perp}^2 + v_{\parallel}^2}{v_{ths}^2})$$

Using above results P and Q are simplified as $P = \omega v_{\perp} A$ and $Q = \omega v_{\parallel} A$

2.4.1 Components of Conductivity Tensor

XX-Component

$$\epsilon_{xx} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \frac{1}{n_s} \int \frac{D_{xx}}{(\omega - k_{\parallel} v_{\parallel} - n\omega_{cs})} d^3 v \tag{2.29}$$

we want to calculate

$$I = \int \frac{D_{xx}}{(\omega - k_{\parallel} v_{\parallel} - n\omega_{cs})} d^3 v = \int_{-\infty}^{\infty} dv_{\parallel} \int_0^{\infty} v_{\perp}^3 dv_{\perp} 2\pi \frac{1}{(\omega - k_{\parallel} v_{\parallel} - n\omega_{cs})} (\frac{nJ_n(\lambda)}{\lambda})^2 \omega A$$

We solve integral along parallel and perpendicular component separately.

$$I = I_1 * I_2$$

$$I_1 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{(\omega - k_{\parallel} v_{\parallel} - n\omega_{cs})} \exp(-\frac{v_{\parallel}^2}{v_{ths}^2}) dv_{\parallel}$$

Let $S = \frac{v_{\parallel}}{v_{ths}}$ and $dS = \frac{1}{v_{ths}} dv_{\parallel}$

$$I_1 = -\frac{1}{k_{\parallel}} Z_n(\xi_{ns})$$

$$\xi_n = \frac{\omega - n\omega_{cs}}{k_{\parallel} v_{ths}}$$

Identity of Bessel function[18].

$$\int_{-0}^{\infty} \exp(-p^2 t^2) t J_n^2(at) dt = \frac{1}{2p^2} \exp(-\frac{a^2}{2p^2}) I_n(\frac{a^2}{2p^2})$$

Let $p^2 = \frac{1}{v_{ths}^2}$, $\frac{a^2}{2p^2} = b_s$, $t = v_\perp$ and $\frac{\lambda}{v_\perp} = \frac{k_\perp}{w_{cs}} = a$

$$\int_{-0}^{\infty} \exp\left(-\frac{v_\perp^2}{v_{ths}^2}\right) v_\perp J_n^2(\lambda) dv_\perp = \frac{v_{ths}^2}{2} \exp(-b_s) I_n(b_s)$$

$$I_2 = \frac{-2n_{0s} n^2 \omega_{cs}^2 \omega}{v_{ths}^3 k_\perp^2} \exp(-b_s) I_n(b_s)$$

$I_n(b_s)$ is the modified Bessel function with argument b_s and b_s is define as $b_s = \frac{k_\perp^2 Tk}{\omega_{cs} m}$

$$\epsilon_{xx} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \xi_0 \sum_{n=-\infty}^{\infty} \frac{n^2}{b_s} \exp(-b_s) I_n(b_s) Z(\xi_{ns}) \tag{2.30}$$

Where ξ_0 is define as $\xi_0 = \frac{\omega}{k v_{ths}}$

XY-Component

$$\epsilon_{xy} = 1 + \sum_s \frac{w_{ps}^2}{w^2} \sum_{n=-\infty}^{\infty} \frac{1}{n_s} \int \frac{D_{xy}}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} d^3v \tag{2.31}$$

we want to calculate

$$I = \int \frac{D_{xy}}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} d^3v = \int_{-\infty}^{\infty} dv_\parallel \int_0^{\infty} v_\perp^3 dv_\perp 2\pi \frac{1}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} \left(\frac{n J_n(\lambda) J_n'(\lambda)}{\lambda}\right) \omega A$$

$$I = I_3 * I_4$$

and $I_1 = I_3$. Identity of Bessel function[18].

$$\int_{-0}^{\infty} \exp(-p^2 t^2) t^2 J_n(at) J_n'(at) dt = \frac{a}{4p^4} \exp\left(-\frac{a^2}{2p^2}\right) \left[I_n'\left(\frac{a^2}{2p^2}\right) - I_n\left(\frac{a^2}{2p^2}\right)\right]$$

Let $p^2 = \frac{1}{v_{ths}^2}$, $\frac{a^2}{2p^2} = b_s$, $t = v_\perp$

$$\int_{-0}^{\infty} \exp\left(-\frac{v_\perp^2}{v_{ths}^2}\right) v_\perp^2 J_n(\lambda) J_n'(\lambda) dv_\perp = \frac{k_\perp v_{ths}^2}{4\omega_{cs}} \exp(-b_s) [I_n'(b_s) - I_n(b_s)]$$

$$I_4 = -i n_{0s} \frac{n\omega}{v_{ths}} \exp(-b_s) [I_n'(b_s) - I_n(b_s)]$$

$$\epsilon_{xy} = i \sum_s \frac{\omega_{ps}^2}{\omega^2} \xi_0 \sum_{n=-\infty}^{\infty} n \exp(-b_s) [I_n(b_s) - I_n'(b_s)] Z(\xi_{ns}) \tag{2.32}$$

XZ-Component

$$\epsilon_{xz} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \frac{1}{n_s} \int \frac{D_{xz}}{(w - k_\parallel v_\parallel - nw_{cs})} d^3v \tag{2.33}$$

we want to calculate

$$I = \int \frac{D_{xz}}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} d^3v = \int_{-\infty}^{\infty} v_\parallel dv_\parallel \int_0^\infty v_\perp^2 dv_\perp 2\pi \frac{1}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} (\frac{nJ_n^2(\lambda)}{\lambda})\omega A$$

$$I = I_5 * I_6$$

$$I_5 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} v_\parallel \frac{1}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} \exp{(-\frac{v_\parallel^2}{v_{ths}^2})} dv_\parallel$$

Let $S = \frac{v_\parallel}{v_{ths}}$ and $dS = \frac{1}{v_{ths}} dv_\parallel$

$$I_5 = \frac{v_{ths}}{2k_\parallel} Z_n'(\xi_n)$$

$$I_6 = -4\omega \frac{n\omega_{cs} n_{0s}}{k_\perp v_{ths}^5} \int_0^\infty v_\perp dv_\perp J_n^2(\lambda) \exp{(-\frac{v_\perp^2}{v_{ths}^2})}$$

Identity of Bessel function.

$$\int_{-0}^{\infty} \exp{(-a^2 t^2)} t J_n^2(pt) dt = \frac{1}{2a^2} \exp(-\frac{p^2}{2a^2}) I_n(\frac{p^2}{2a^2})$$

$$\int_{-0}^{\infty} \exp{(-\frac{v_\perp^2}{v_{ths}^2})} v_\perp J_n^2(\lambda) dv_\perp = \frac{v_{ths}^2}{2} \exp(-b_s) I_n(b_s)$$

$$\epsilon_{xz} = -i \sum_s \frac{\omega_{ps}^2}{\omega^2} \xi_0 \sum_{n=-\infty}^{\infty} n \frac{\exp(-b_s)}{\sqrt{2b_s}} I_n(b_s) Z'(\xi_{ns}) \tag{2.34}$$

YY-Component

$$\epsilon_{yy} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \frac{1}{n_s} \int \frac{D_{yy}}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} d^3v \tag{2.35}$$

we want to calculate

$$I = \int \frac{D_{yy}}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} d^3v = \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty v_\perp^3 dv_\perp 2\pi \frac{1}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} J_n'^2(\lambda)\omega A$$

$$I = I_7 * I_8$$

The integral $I_7 = I_1$

$$I_7 = -\frac{1}{k_\parallel} Z_n(\xi_{ns})$$

Identity of Bessel function.

$$\int_{-0}^{\infty} \exp{(-a^2 t^2)} t^3 J_n'^2(pt) dt = \frac{1}{4a^4} \exp(-\frac{p^2}{2a^2})[\frac{n^2}{\frac{p^2}{2a^2}} I_n(\frac{p^2}{2a^2}) + 2(\frac{p^2}{2a^2}) I_n(\frac{p^2}{2a^2}) - 2I_n'(\frac{p^2}{2a^2})]$$

$$I_8 = \frac{-4n_{0s}\omega}{v_{ths}^5} \frac{1}{4} v_{ths}^4 \exp(-b_s)[\frac{n^2}{b_s} I_n(b_s) + 2(b_s) I_n(b_s) - 2I_n'(b_s)]$$

$$\epsilon_{yy} = 1 + i \sum_s \frac{\omega_{ps}^2}{\omega^2} \xi_0 \sum_{n=-\infty}^{\infty} \exp(-b_s)[\frac{n^2}{b_s} I_n(b_s) + 2(b_s) I_n(b_s) - 2I_n'(b_s)] Z(\xi_{ns}) \tag{2.36}$$

YZ-Component

$$\epsilon_{yz} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \frac{1}{n_s} \int \frac{D_{yz}}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} d^3 v \tag{2.37}$$

we want to calculate

$$I = \int \frac{D_{yz}}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} d^3 v = -i \int_{-\infty}^{\infty} v_\parallel dv_\parallel \int_0^{\infty} v_\perp^2 dv_\perp 2\pi \frac{1}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} J_n(\lambda) J_n'(\lambda) \omega A$$

$$I = I_9 * I_{10}$$

$$I_9 = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} v_\parallel \frac{1}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} \exp{(-\frac{v_\parallel^2}{v_{ths}^2})} dv_\parallel$$

Let $S = \frac{v_\parallel}{v_{ths}}$ and $dS = \frac{1}{v_{ths}} dv_\parallel$

$$I_9 = \frac{v_{ths}}{2k_\parallel} Z_n'(\xi_{ns})$$

$$I_{10} = 4iw \frac{n_{0s}}{v_{ths}^5} \int_0^{\infty} v_\perp^2 dv_\perp J_n(\lambda) J_n'(\lambda) \exp{(-\frac{v_\perp^2}{v_{ths}^2})}$$

Identity of Bessel function. Identity of Bessel function. Identity of Bessel function.

$$\int_{-0}^{\infty} \exp{(-a^2 t^2)} t^2 J_n(pt) J_n'(pt) dt = \frac{p}{4a^4} \exp(-\frac{p^2}{2a^2})[I_n'(\frac{p^2}{2a^2}) - I_n(\frac{\partial^2}{2a^2})]$$

$$\int_{-0}^{\infty} \exp{(-\frac{v_\perp^2}{v_{ths}^2})} v_\perp^2 J_n(\lambda) J_n'(\lambda) dv_\perp = \frac{k_\perp v_{ths}^2}{4\omega_{cs}} \exp(-b_s)[I_n'(b_s) - I_n(b_s)]$$

$$I_{10} = in_{0s} \frac{\omega k_\perp}{v_{ths} w_{cs}} \exp(-b_s)[I_n'(b_s) - I_n(b_s)]$$

$$\epsilon_{yz} = -\epsilon_{zy} = -i \sum_s \frac{\omega_{ps}^2}{\omega^2} \sqrt{\frac{b_s}{2}} \xi_0 \sum_{n=-\infty}^{\infty} \exp(-b_s)[I_n'(b_s) - I_n(b_s)] Z'(\xi_{ns}) \tag{2.38}$$

ZZ-Component

$$\epsilon_{zz} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \frac{1}{n_s} \int \frac{D_{zz}}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} d^3v \tag{2.39}$$

we want to calculate

$$I = \int \frac{D_{zz}}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} d^3v = \int_{-\infty}^{\infty} v_\parallel^2 dv_\parallel \int_0^\infty v_\perp dv_\perp 2\pi \frac{1}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} J_n'^2(\lambda)\omega A$$

$$I = I_{11} * I_{12}$$

$$I_{11} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{v_\parallel^2}{(\omega - k_\parallel v_\parallel - n\omega_{cs})} \exp{(-\frac{v_\parallel^2}{v_{ths}^2})} dv_\parallel$$

Let $S = \frac{v_\parallel}{v_{ths}}$ and $dS = \frac{1}{v_{ths}} dv_\parallel$

$$I_{11} = -\frac{v_{ths}^2 \xi_{ns}}{2k_\parallel} Z_n'(\xi_{ns})$$

$$\xi_n = \frac{\omega - n\omega_{cs}}{k_\parallel v_{ths}}$$

Identity of Bessel function.

$$\int_{-0}^{\infty} \exp{(-a^2t^2)} t J_n^2(pt) dt = \frac{1}{2a^2} \exp(-\frac{p^2}{2a^2}) I_n(\frac{p^2}{2a^2})$$

$$\int_{-0}^{\infty} \exp{(-\frac{v_\perp^2}{v_{ths}^2})} v_\perp J_n^2(\lambda) dv_\perp = \frac{v_{ths}^2}{2} \exp(-b_s) I_n(b_s)$$

$$I_{12} = \frac{-2n_{0s}\omega}{v_{ths}^3 k^2} \exp(-b_s) I_n(b_s)$$

$b_s = \frac{k_\perp^2 Tk}{\omega_{cs} m}$

$$\epsilon_{zz} = 1 - \sum_s \frac{w_{ps}^2}{w^2} \xi_0 \sum_{n=-\infty}^{\infty} \exp(-b_s) I_n(b_s) \xi_n Z'(\xi_{ns}) \tag{2.40}$$

We may summarize the results as below,

$$\epsilon_{xx} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \xi_0 \sum_{n=-\infty}^{\infty} \frac{n^2}{b_s} \exp(-b_s) I_n(b_s) Z(\xi_{ns}) \tag{2.41}$$

$$\epsilon_{xy} = i \sum_s \frac{\omega_{ps}^2}{\omega^2} \xi_0 \sum_{n=-\infty}^{\infty} n \exp(-b_s)[I_n(b_s) - I_n'(b_s)] Z(\xi_{ns}) \tag{2.42}$$

$$\epsilon_{yx} = -i\sum_s \frac{\omega_{ps}^2}{\omega^2}\xi_0 \sum_{n=-\infty}^{\infty} n\exp(-b_s)[I_n(b_s) - I_n'(b_s)]Z(\xi_{ns}) \tag{2.43}$$

$$\epsilon_{xz} = -i\sum_s \frac{\omega_{ps}^2}{\omega^2}\xi_0 \sum_{n=-\infty}^{\infty} n\frac{\exp(-b_s)}{\sqrt{2b_s}} I_n(b_s)Z'(\xi_{ns}) \tag{2.44}$$

$$\epsilon_{zx} = -i\sum_s \frac{\omega_{ps}^2}{\omega^2}\xi_0 \sum_{n=-\infty}^{\infty} n\frac{\exp(-b_s)}{\sqrt{2b_s}} I_n(b_s)Z'(\xi_{ns}) \tag{2.45}$$

$$\epsilon_{yy} = 1 + i\sum_s \frac{\omega_{ps}^2}{\omega^2}\xi_0 \sum_{n=-\infty}^{\infty} \exp(-b_s)[\frac{n^2}{b_s} I_n(b_s) + 2(b_s)I_n(b_s) - 2I_n'(b_s)]Z(\xi_{ns}) \tag{2.46}$$

$$\epsilon_{yz} = -i\sum_s \frac{\omega_{ps}^2}{\omega^2}\sqrt{\frac{b_s}{2}}\xi_0 \sum_{n=-\infty}^{\infty} \exp(-b_s)[I_n'(b_) - I_n(b_s)]Z'(\xi ns) \tag{2.47}$$

$$\epsilon_{zy} = i\sum_s \frac{\omega_{ps}^2}{\omega^2}\sqrt{\frac{b_s}{2}}\xi_0 \sum_{n=-\infty}^{\infty} \exp(-b_s)[I_n'(b_s) - I_n(b_s)]Z'(\xi_{ns}) \tag{2.48}$$

$$\epsilon_{zz} = 1 - \sum_s \frac{\omega_{ps}^2}{\omega^2}\xi_0 \sum_{n=-\infty}^{\infty} \exp(-b_s)I_n(b_s)\xi_n Z'(\xi_{ns}) \tag{2.49}$$

The element of ϵ involving along z-axis contain $Z'(\xi)$ which gives rise to landau damping when $n = 0$ and $\frac{\omega}{k_\parallel} \approx v_{ths}$. When $n \neq 0$ then make possible another collision-less damping mechanism, cyclotron damping which occurs when, $\frac{\omega \pm n\omega_{cs}}{k_\parallel} \approx v_{ths}$ [1]

Chapter 3

Bernstein Waves

In section 3.1 we derived the dispersion relation for Bernstein waves. In section 3.2 we derived the dispersion relation for electron Bernstein waves and its subsection 3.2.1 we apply the fluid limit on it we get fluid results from kinetic theory. In section 3.3 we discuss the ions Bernstein waves and in its subsections 3.3.1 we derived the dispersion relation for neutralized ions waves. In subsection 3.3.2 we apply the fluid limit on neutralized ions waves we get fluid results from kinetic theory. In section 3.3.3 we derived the dispersion relation for pure ions Bernstein waves. In sections 3.3.4 we apply fluid limit on pure ions Bernstein waves, we get fluid results from kinetic theory. In section 3.3.5 we use plasma approximation on pure ions Bernstein waves and we get lower hybrid waves. In the last section of chapter 3 we extend this work for electron-positron plasma. In the last section we derived the dispersion relation for electron-positron Bernstein waves. In its subsection we apply the fluid limit on it and we upper hybrid mode for electron-positron.
The Bernstein waves are firstly studied by Bernstein in 1958 and are called Bernstein waves. These waves are also called hot plasma waves because their existence depends on the temperature of plasma. Bernstein waves have been observed in both laboratory experiment and in magneto active stars having finite temperature. Electron Bernstein waves are reported by Crawford in 1965 and Leuterer in 1969. Ions Bernstein waves were also observed in laboratory experiment by Schmitt in 1973. These waves are important in fusion process, to heat the plasma[16].

3.1 Dispersion Relation for Bernstein Waves

Electrostatic wave propagating at right angle to B_0 at harmonics of the cyclotron frequency are called Bernstein wave.
Poisson's equation for electrostatic waves is written as,

$$\nabla.\overleftrightarrow{\epsilon}.\mathbf{E} = 0$$

If we assume electrostatic perturbation such that $\mathbf{E}_1 = -\nabla\phi_1$ and consider the form of $\phi_1 = \phi_1 \exp[\iota(\mathbf{k}.\mathbf{r} - \omega t)]$. Let k is lying in x-z plane, after Fourier transformation the Poisson's equation take the form,

$$(\mathbf{k}.\overleftrightarrow{\epsilon}.\mathbf{k})\phi_1 = 0$$

$\phi_1 \neq 0$ so above equation is written as

$$(\mathbf{k}.\overleftrightarrow{\epsilon}.\mathbf{k}) = \sum_{ij} k_i k_j \epsilon_{ij} = 0$$

, and $\epsilon_{xz} = \epsilon_{zx}$

$$k_\perp^2 \epsilon_{xx} + 2k_\perp k_\parallel \epsilon_{xz} + k_\parallel^2 \epsilon_{zz} = 0. \tag{3.1}$$

putting values of ϵ_{xx}, ϵ_{xz} and ϵ_{zz} in above equation we get,

$$k_\perp^2 + k_\parallel^2 + \sum_s \frac{omega_{ps}^2}{\omega^2} \xi_{0s} \sum_{n=-\infty}^{\infty} \exp(-b_s) I_n(b_s) [\frac{k_\perp^2 n^2 Z(\xi_{ns})}{b_s} - k_\perp k_\parallel \sqrt{\frac{2}{b_s}} n Z'(\xi_{ns}) - k_\parallel^2 \xi_n Z'(\xi_{ns})] = 0$$

using these two identities $Z'(\xi_{ns}) = -2[1 + (\xi_{ns})Z(\xi_{ns})]$, $\sum_{n=-\infty}^{\infty} n I_n(b_s) = 0$ and simplifying the terms in square bracket of above equation we get,

$$k_\perp^2 + k_\parallel^2 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \xi_{0s} \sum_{n=-\infty}^{\infty} \exp(-b_s) I_n(b_s) [2k_z^2(\xi_{ns} + \xi_0^2 Z(\xi_{ns}))] = 0 \tag{3.2}$$

$$\xi_{0s}^2 = \frac{\omega^2}{k_\parallel^2 v_{ths}^2}$$

$$k_{Ds}^2 = \frac{2\omega_{ps}^2 k_\parallel^2 \xi_{0s}^2}{\omega^2} = \frac{2\omega_{ps}^2}{v_{ths}^2}$$

$$k_\perp^2 + k_\parallel^2 + \sum_s k_{Ds}^2 \sum_{n=-\infty}^{\infty} \exp(-b_s) I_n(b_s) [1 + \xi_{0s} Z(\xi_{ns})] = 0$$

Defining $k^2 = k_\perp^2 + k_\parallel^2$

$$1 + \sum_s \frac{k_{Ds}^2}{k^2} \sum_{n=-\infty}^{\infty} \exp(-b_s) I_n(b_s) [1 + \xi_{0s} Z(\xi_{ns})] = 0 \tag{3.3}$$

Above relation is the a general dispersion relation for Bernstein waves[19].

3.2 Electron Bernstein Waves

The electron-cyclotron waves like ordinary waves which are useful to heat the plasma at frequency range of first harmonics and Extra-Ordinary waves are useful at the range of second harmonics. Only for special high temperature, higher heating is possible. Bernstein waves having no such limit due to high cyclotron absorption and use to heat the plasma at the range of higher harmonics[20, 21, 22]. *Those waves are propagating at right angle to the magnetic field and respond at high frequency are called electrons Bernstein waves.* Electron Bernstein waves are the special kind of electrons cyclotron waves, which are high frequency electrostatic waves in magnetized hot plasma. When incoming frequency is high only electron respond to these waves. Ions do not respond high frequency waves due small neutral frequency and consider as stationary. For high frequency waves we may set $k_{\parallel} \approx 0$ because these wave are not sensitive to small deviation from perpendicular propagation. Plasma dispersion function for large ξ_{ne},

$$Z(\xi_{ne}) = i\sqrt{\pi}\exp(-\xi_{ne}^2) - [\frac{1}{\xi_{ne}} + \frac{1}{2\xi_{ne}^3} + ...]$$

for large value of ξ_{ne} the landau term and higher order terms can be neglected, so above equation can be written as,

$$Z(\xi_{ne}) = \frac{-1}{\xi_{ne}}$$

so the dispersion relation become

$$1 + \sum_s \frac{k_{De}^2}{k^2} \sum_{n=-\infty}^{\infty} \exp(-b_e) I_n(b_e)[1 - \frac{\xi_{0e}}{(\xi_{ne})}] = 0 \tag{3.4}$$

when $n = 0$ the term in square bracket is zero. We divide the sum into two sums,

$$k^2 + \sum_s k_{De}^2 \exp(-b_e)[\sum_{n=1}^{\infty} I_n(b_e)(1 - \frac{\xi_{0e}}{(\xi_{ne})}) + \sum_{n=-\infty}^{1} I_n(b_e)(1 - \frac{\xi_{0e}}{(\xi_{ne})})] = 0$$

using the identity of modified Bessel function $I_n(b) = I_{-n}(b)$ and then replacing $n = -n$ the above equation become

$$k^2 + \sum_s k_{De}^2 \exp(-b_e) \sum_{n=1}^{\infty} I_n(b_e)[1 - \frac{\xi_{0e}}{\xi_{ne}} + 1 - \frac{\xi_{0e}}{\xi_{-ne}}] = 0$$

putting values $\xi_{ne} = \frac{\omega - n\omega_{ce}}{k_z v_{the}}$ and $\xi_{-ne} = \frac{\omega + n\omega_{ce}}{k_z v_{the}}$ in above equation. Here we drop sum over species, we consider electron only and we set $k_{\parallel} = 0$ for electrons[15],

$$1 = \frac{k_{De}^2}{k^2} \sum_{n=1}^{\infty} \exp(-b_e) I_n(b_e)[\frac{2n^2\omega_{ce}^2}{(\omega^2 - n^2\omega_{ce}^2)}] \tag{3.5}$$

$$a(\omega, b) = \frac{k_{\perp}^2}{k_{De}^2} = 2\frac{\sum_{n=1}^{\infty} \exp(-b_e) I_n(b_e)}{\frac{\omega^2}{\omega_{ce}^2} - n^2} \tag{3.6}$$

Where k_{De} and b_e are define as $k_{De}^2 = \frac{2\omega_{pe}^2}{v_{the}^2}$, ,$b_e = \frac{k_{\perp}^2 v_{the}^2}{2\omega_{ce}^2} = \frac{1}{2}k_{\perp}^2 r_{Le}^2$, so the above equation

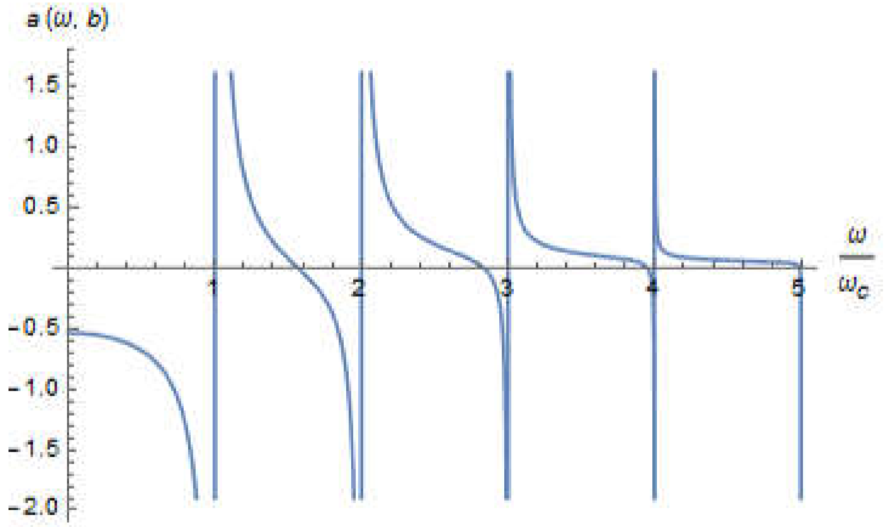

Figure 3.1: The function $a(\omega, b)$ versus ω/ω_{ce} for electrons Bernstein waves.

become,

$$k_{\perp}^2 r_{Le}^2 = \frac{4\omega_{pe}^2}{\omega_{ce}^2} \sum_{n=1}^{\infty} \exp(-\frac{1}{2}k_{\perp}^2 r_{Le}^2) I_n(\frac{1}{2}k_{\perp}^2 r_{Le}^2)[\frac{n^2}{\frac{\omega^2}{\omega_{cs}^2} - n^2}] \tag{3.7}$$

In the limit $k_{\perp} \longrightarrow 0$ and $n = 1$, if $\omega \neq n\omega_c e$ we get upper hybrid frequency only. We can obtain solution for all values of $n > 1$ if $\omega = n\omega_{ce}$.

3.2.1 Fluid Limit on Electron Bernstein Waves

The dispersion function for electron Bernstein wave is,

$$1 = \frac{2\omega_{pe}^2}{b\omega_{ce}^2} \sum_{n=1}^{\infty} \exp(-b_e) I_n(b_e)[\frac{1}{\frac{\omega^2}{\omega_{ce}^2} - n^2}] \tag{3.8}$$

For small value of b_e the modified Bessel function is written as $I_n(b_e) = \frac{1}{n!}(\frac{b_e}{2})^n$ when $b_e \longrightarrow 0$ only $n = 1$ term exist. The equation (3.8) is written as,

$$\omega^2 = \omega_{pe}^2 + \omega_{ce}^2 = \omega_{Uh}^2,$$

which is the upper hybrid oscillation [23]. Electrostatic electron waves across **B** have this frequency and those waves along **B** are oscillating with plasma frequency, $\omega_{ce} = \frac{eB_0}{m}$ when B_0 goes to zero ω_{ce} goes to zero then wave frequency is equal to plasma frequency. $\omega_{pe}^2 = \frac{n_0 e^2}{\epsilon_0 m}$ if plasma density goes to zero, ω_p goes to zero then wave frequency equal to cyclotron frequency.

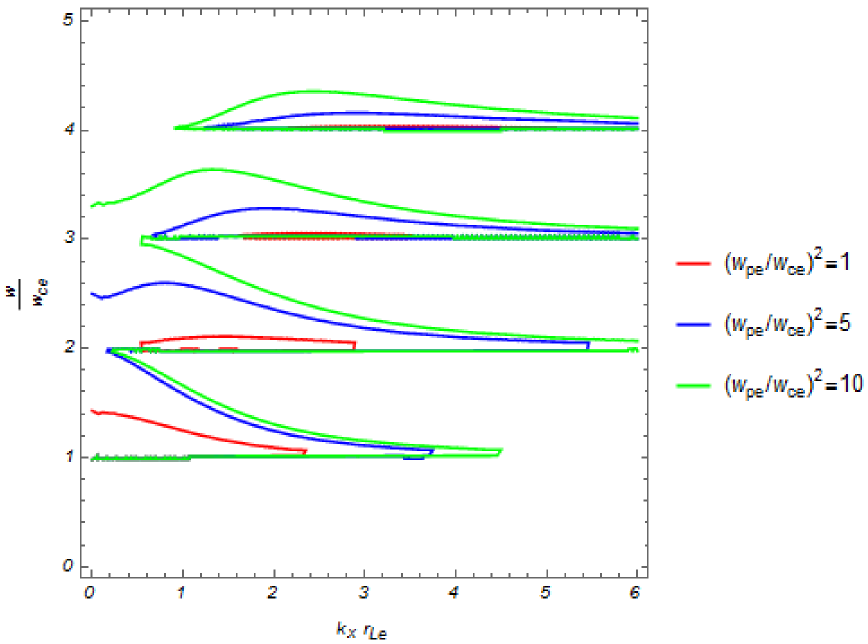

Figure 3.2: Dispersion curve showing solution for EBW.

3.3 Ions Bernstein Waves

Ions Bernstein Waves are similar to electron Bernstein Waves but electrons Bernstein waves are respond at high frequency and ions Bernstein waves responds at low frequency. *Those waves are propagating at right angle to the magnetic field and respond at low frequency are called ions Bernstein waves.* When the incoming waves having low frequency of the order of ions cyclotron frequency. Ions respond these low frequency waves, as a result ions Bernstein waves are produced[24].

$$\omega \approx \omega_{pi} << \omega_{pe}$$

3.3.1 Neutralized Ions Bernstein Waves

General dispersion relation for Bernstein wave.

$$1 + \sum_s \frac{k_{Ds}^2}{k^2} \sum_{n=-\infty}^{\infty} \exp(-b_s) I_n(b_s)[1 + \xi_{0s} Z(\xi_{ns})] = 0 \tag{3.9}$$

Defining $k^2 = k_\perp^2 + k_\parallel^2$ for ions wave we expand the sum over species.

$$\frac{k_{De}^2}{k^2} \sum_{n=-\infty}^{\infty} \exp(-be) I_n(be)[1 + \xi_{0e} Z(\xi_{ne})] + \frac{k_{Di}^2}{k^2} \sum_{n=-\infty}^{\infty} \exp(-bi) I_n(bi)[1 + \xi_{0i} Z(\xi_{ni})] = 1 \tag{3.10}$$

We consider finite k_z such that $\frac{\omega}{k_z} << v_{the}$ then $\xi_{ne} \longrightarrow 0$ and $Z(\xi_{ne}) \approx -2\xi_{ne}$. For perpendicular wavelength of the order of ion gyro radius we further have $b_e << 1$. Hence only $n = 0$ term survives in the first sum. $I_0(b_e << 1) = 1$ so the electron term is written as

$$\frac{k_{De}^2}{k^2}\sum_{n=-\infty}^{\infty}\exp(-be)I_n(be)[1+\xi_{0e}Z(\xi_{ne})] = \frac{k_{De}^2}{k^2}$$

know we solve the ions terms where ξ_{ni} is define as $\xi_{ni} = \frac{\omega - n\omega_{ci}}{k_z v_{thi}}$

$$Z'(\xi_{ni}) = -2i\sqrt{\pi}\xi_{ni}\exp(-\xi_{ni}^2) + \frac{1}{\xi_{ni}^2} + \frac{3}{2\xi_{ni}^4} + ...$$

for large value of ξ_{ni} the landau term and higher terms can be neglected, so above equation can be written as,

$$Z'(\xi_{ni}) = \frac{1}{\xi_{ni}^2}$$

or

$$Z(\xi_{ni}) = \frac{-1}{\xi_{ni}}$$

so the ions term is written as

$$\frac{k_{Di}^2}{k^2}\sum_{n=-\infty}^{\infty}\exp(-bi)I_n(bi)[1-\frac{\xi_{0i}}{(\xi_{ni})}]$$

when $n = 0$ the term in square bracket is zero.So we divide the sum into two sum.

$$\frac{k_{Di}^2}{k^2}\exp(-bi)[\sum_{n=1}^{\infty}I_n(bi)(1-\frac{\xi_{0i}}{(\xi_{ni})}) + \sum_{n=-\infty}^{1}I_n(bi)(1-\frac{\xi_{0i}}{(\xi_{ni})})]$$

using the identity of modified Bessel function $I_n(bi) = I_{-n}(bi)$ and then replacing $n = -n$ the above equation become

$$\frac{k_{Di}^2}{k^2}\exp(-bi)\sum_{n=1}^{\infty}I_n(bi)[1-\frac{\xi_{0i}}{\xi_{ni}}+1-\frac{\xi_{0i}}{\xi_{-ni}}]$$

putting values $\xi_{ni} = \frac{\omega - n\omega_{ci}}{k_z v_{thi}}$ and $\xi_{-ni} = \frac{\omega + n\omega_{ci}}{k_z v_{thi}}$ in above equation and we get

$$-\frac{k_{Di}^2}{k^2}\sum_{n=1}^{\infty}\exp(-bi)I_n(bi)[\frac{2n^2\omega_{ci}^2}{(\omega^2-n^2\omega_{ci}^2)}]$$

so equation (3.11) is written as

$$1+\frac{k_{De}^2}{k^2}-\frac{k_{Di}^2}{k^2}\sum_{n=1}^{\infty}\exp(-bi)I_n(bi)[\frac{2n^2\omega_{ci}^2}{(\omega^2-n^2\omega_{ci}^2)}] = 0$$

$k_{Ds}^2 = \frac{1}{\lambda_{Ds}^2}$ where λ_{Ds} is debye shielding length of species s and $\frac{\lambda_{De}^2}{\lambda_{Di}^2} = \frac{T_e}{T_i}$

$$\sum_{n=1}^{\infty}\exp(-bi)I_n(bi)[\frac{2n^2\omega_{ci}^2}{(\omega^2-n^2\omega_{ci}^2)}] = \frac{T_i}{T_e}(1+k^2\lambda_{De}^2) \tag{3.11}$$

above equation for neutralized Bernstein waves.

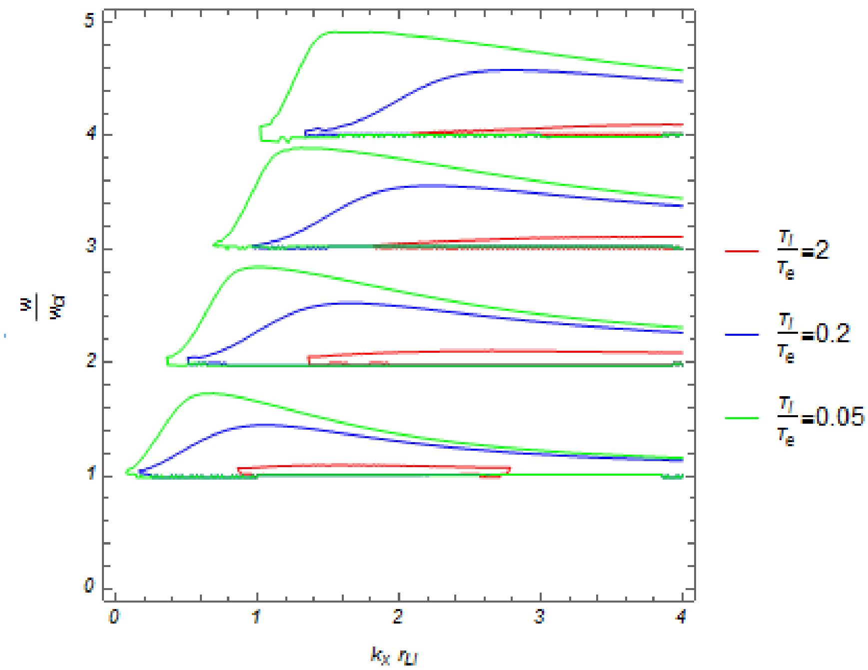

Figure 3.3: Dispersion curve showing solution for Neutralized IBW.

3.3.2 Fluid Limit on the Neutralized Ions Bernstein Waves

The dispersion relation for the neutralized Bernstein waves[25] is given as,

$$1 + \frac{k_{De}^2}{k^2} - \frac{k_{Di}^2}{k^2}\sum_{n=1}^{\infty}\exp(-bi)I_n(bi)[\frac{2n^2\omega_{ci}^2}{(\omega^2 - n^2\omega_{ci}^2)}] = 0$$

When $b_i \longrightarrow 0$ only $n = 1$ term survive, so the dispersion relation is given as,

$$1 + \frac{k_{De}^2}{k^2} - \frac{k_{Di}^2}{k^2}\frac{bi}{2}[\frac{2\omega_{ci}^2}{(\omega^2 - \omega_{ci}^2)}] = 0$$

$$\frac{k_{Di}^2}{k^2}bi[\frac{\omega_{ci}^2}{(\omega^2 - \omega_{ci}^2)}] = 1 + \frac{k_{De}^2}{k^2}$$

where b_i is define as, $b_i = \frac{k_\perp^2 v_{thi}^2}{2\omega_{ci}^2}$ $k_{Ds}^2 = \frac{1}{\lambda_{Ds}^2}$ where λ_{Ds} is debye shielding length of species s and $\frac{\lambda_{De}^2}{\lambda_{Di}^2} = \frac{T_e}{T_i}$

$$\frac{\lambda_{De}^2}{\lambda_{Di}^2}bi[\frac{\omega_{ci}^2}{(\omega^2 - \omega_{ci}^2)}] = 1 + \lambda_{De}^2 k^2 \approx 1$$

$$\frac{T_e}{T_i}k^2 v_{thi}^2[\frac{1}{(\omega^2 - \omega_{ci}^2)}] = 1$$

$$w^2 = w_{ci}^2 + k^2 v_s^2 \tag{3.12}$$

Where $v_s = \frac{k_B T_e}{M}$. From equation (3.12), we conclude that when we apply the fluid limit on Neutralized Ions Bernstein Waves they become ions cyclotron waves.

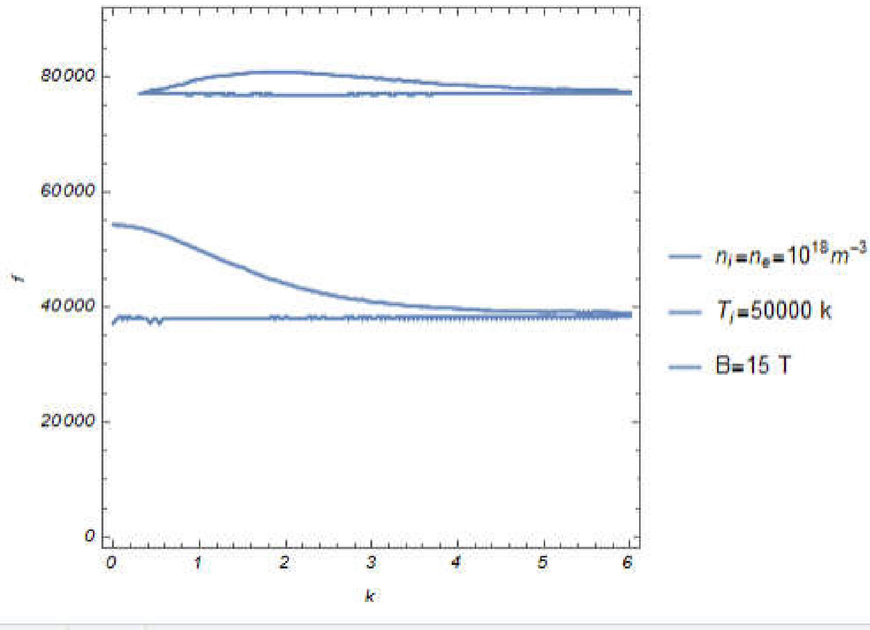

Figure 3.4: Dispersion curve showing solution for pure IBW.

3.3.3 Pure Ions Bernstein Waves

Dispersion function for ions Bernstein waves is written as,

$$1 - \frac{k_{De}^2}{k^2}\sum_{n=1}^{\infty} \exp(-be) I_n(be) [\frac{2n^2\omega_{ce}^2}{(\omega^2 - n^2\omega_{ce}^2)}] - \frac{k_{Di}^2}{k^2}\sum_{n=1}^{\infty} \exp(-bi) I_n(bi) [\frac{2n^2 w_{ci}^2}{(\omega^2 - n^2\omega_{ci}^2)}] = 0$$

In the limit of (almost) exact perpendicular propagation $\frac{\omega}{k_z} >> v_{the}$. We further assume that $b_e << 1$. For small value of b the modified Bessel function is written as $I_{(n)} = \frac{1}{n!}(\frac{b}{2})^n$ when $b \longrightarrow 0$ only $n = 1$ term exist. k_D and b is define as $k_D^2 = \frac{2\omega_p^2}{v_{the}^2}$,$b = \frac{k_\perp^2 v_{the}^2}{2\omega_{ce}^2}$

$$1 - \frac{2\omega_{pe}^2}{\omega_{ce}^2}\frac{1}{b}\frac{b}{2}\frac{\omega_{ce}^2}{\omega^2 - \omega_{ce}^2} - \frac{2\omega_{pi}^2}{k^2 v_{thi}^2}\sum_{n=1}^{\infty} \exp(-bi) I_n(bi) [\frac{2n^2\omega_{ci}^2}{(\omega^2 - n^2\omega_{ci}^2)}] = 0$$

$\omega << \omega_{ce}$

$$1 + \frac{\omega_{pe}^2}{\omega_{ce}^2} - \frac{2\omega_{pi}^2}{k^2 v_{thi}^2}\sum_{n=1}^{\infty} \exp(-bi) I_n(bi) [\frac{2n^2\omega_{ci}^2}{(\omega^2 - n^2\omega_{ci}^2)}] = 0 \quad (3.13)$$

The equation (3.12) represent the dispersion relation for pure ions Bernstein wave.

3.3.4 Fluid Limit on the Pure Ions Bernstein Waves

The dispersion relation for ions Bernstein waves is given as,

$$1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} - \frac{1}{b_i}\sum_{n=1}^{\infty}\exp(-b_i)I_n(b_i)[\frac{2n^2\omega_{pi}^2}{(\omega^2 - n^2\omega_{ci}^2)}] = 0$$

for small value of $b_i << 1$ only $n = 1$ term is survive, the above relation is written as,

$$1 - \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} - \frac{\omega_{pi}^2}{\omega^2 - \omega_{ci}^2} = 0$$

When the incoming frequency $\omega \approx \omega_{pe}$ we neglect the ions dynamics and we get upper hybrid mode. Under the limit $\omega_{ci} << \omega << \omega_{ce}$ and the incoming frequency is $\omega \approx \omega_{pi}$ we get lower hybrid mode. In fluid theory or in clod plasma there no wave between lower and upper hybrid.

3.3.5 Lower Hybrid Waves

The dispersion relation for low frequency response particles, like ions is given as,

$$\frac{\omega_{pi}^2}{b_i}\sum_{n=1}^{\infty}\exp(-b_i)I_n(b_i)[\frac{2n^2}{(\omega^2 - n^2\omega_{ci}^2)}] = 1 + \frac{\omega_{pe}^2}{\omega_{ce}^2}$$

We obtain the lower hybrid waves under the plasma approximation $n_i = n_e$[26].

$$\frac{\omega_{pe}^2}{\omega_{pi}^2} = \frac{\omega_{ce}}{\omega_{ci}}$$

$$\frac{\omega_{Lh}^2}{\omega_{ci}^2}\frac{1}{b_i}\sum_{n=1}^{\infty}\exp(-b_i)I_n(b_i)[\frac{2n^2}{(\frac{\omega^2}{\omega_{ci}^2} - n^2)}] = 1$$

$$\frac{1}{\omega_{Lh}^2} = \frac{1}{\omega_{ci}\omega_{ce}} + \frac{1}{\omega_{pi}^2}$$

Where ω_{Lh} is the lower hybrid frequency.

3.4 Electron Positron Bernstein Waves

When a high energy photon, such as γ-rays photons interact with matter the resulting phenomena is pair production. In pair-production photons energy converted into electron-positron pair. Positron is a particle having mass and charge equal to electron but the charge

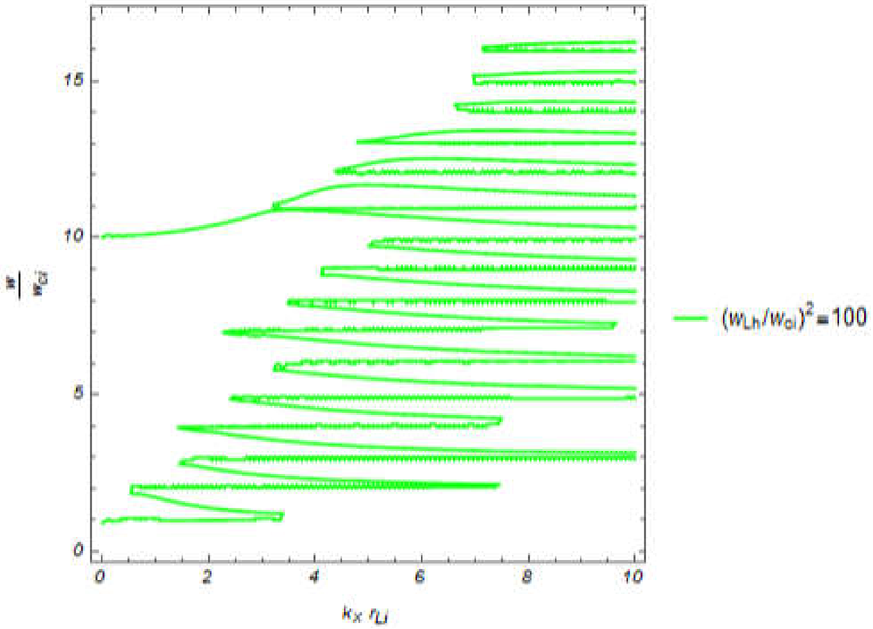

Figure 3.5: Dispersion curve showing solution for Lower hybrid waves.

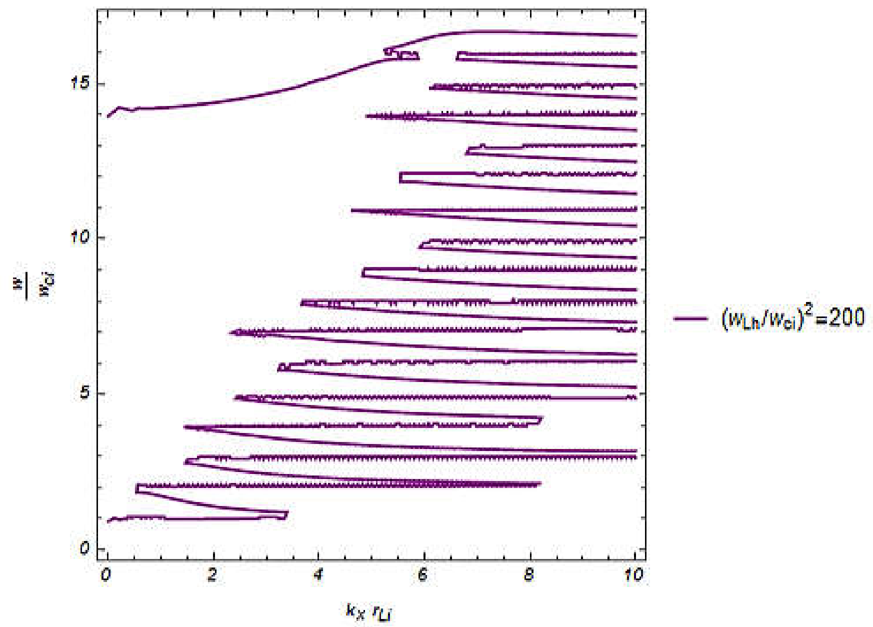

Figure 3.6: Dispersion curve showing solution for Lower hybrid waves.

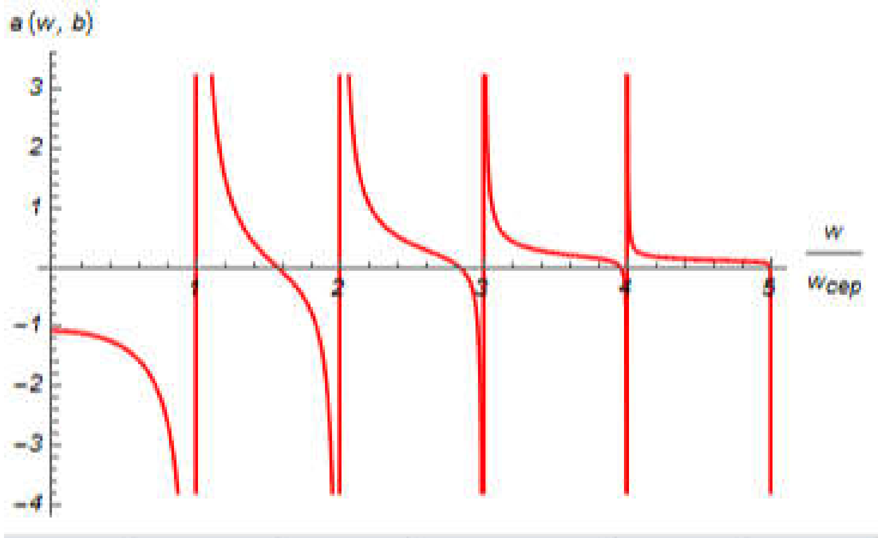

Figure 3.7: The function $a(\omega, b)$ versus ω/ω_{ce} for electrons-positron Bernstein waves.

being of opposite nature. We define the cyclotron frequency for electron and positron as $\omega_{ce} = \frac{eB_0}{m_e}$, $\omega_{cp} = -\frac{eB_0}{m_p}$ Dispersion relation for pure electron positron Bernstein waves

$$1 = \frac{k_{De}^2}{k^2}\sum_{n=1}^{\infty}\exp(-be)I_n(be)[\frac{2n^2\omega_{ce}^2}{(\omega^2 - n^2\omega_{ce}^2)}] + \frac{k_{Dp}^2}{k^2}\sum_{n=1}^{\infty}\exp(-bp)I_n(bp)[\frac{2n^2\omega_{cp}^2}{(\omega^2 - n^2\omega_{cp}^2)}]$$

Here we know that $m_e = m_p$, $q_e = -q_p$ so $k_{De}^2 = k_{Dp}^2$, $b_e = b_p$, $\omega_{ce} = \omega_{cp}$ and dispersion relation is written as,

$$1 = \frac{k_D^2}{k^2}\sum_{n=1}^{\infty}\exp(-b)I_n(b)[\frac{4n^2\omega_c^2}{(\omega^2 - n^2\omega_c^2)}]$$

$k_D^2 = \frac{2\omega_p^2}{v_{th}^2}$,$b = \frac{k_\perp^2 v_{th}^2}{2\omega_c^2}$

$$1 = \frac{\omega_p^2}{\omega_c^2}\sum_{n=1}^{\infty}\frac{\exp(-b)}{b}I_n(b)[\frac{4n^2\omega_c^2}{(\omega^2 - n^2\omega_c^2)}]$$

3.4.1 Fluid Limit on Electron-Positron Bernstein Waves

$$1 = \sum_s \frac{2\omega_p^2}{b\omega_{cs}^2}\sum_{n=1}^{\infty}\exp(-b)I_n(b)[\frac{1}{\frac{\omega^2}{\omega_{cs}^2} - n^2}] \qquad (3.14)$$

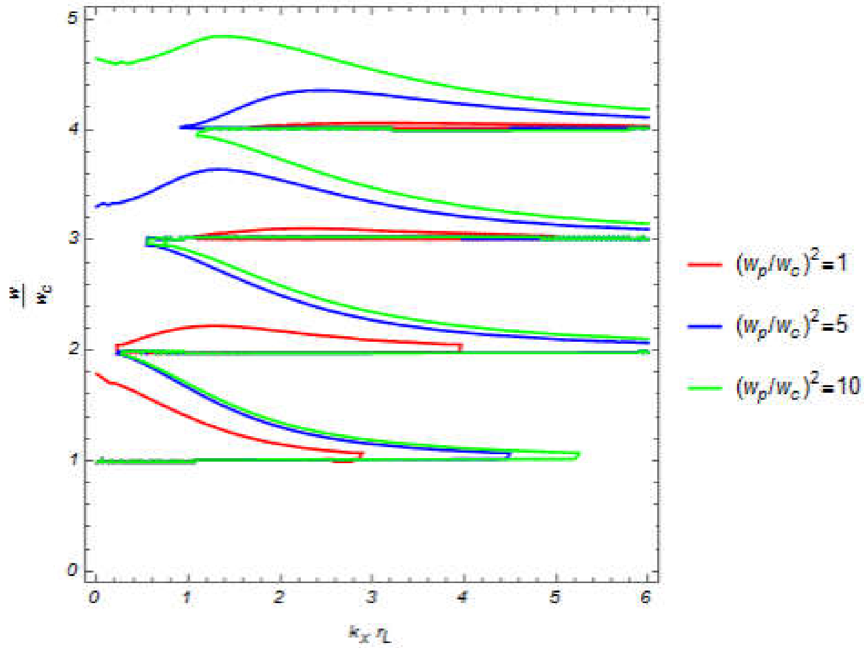

Figure 3.8: Dispersion curve showing solution for Electrons-Positrons Bernstein waves.

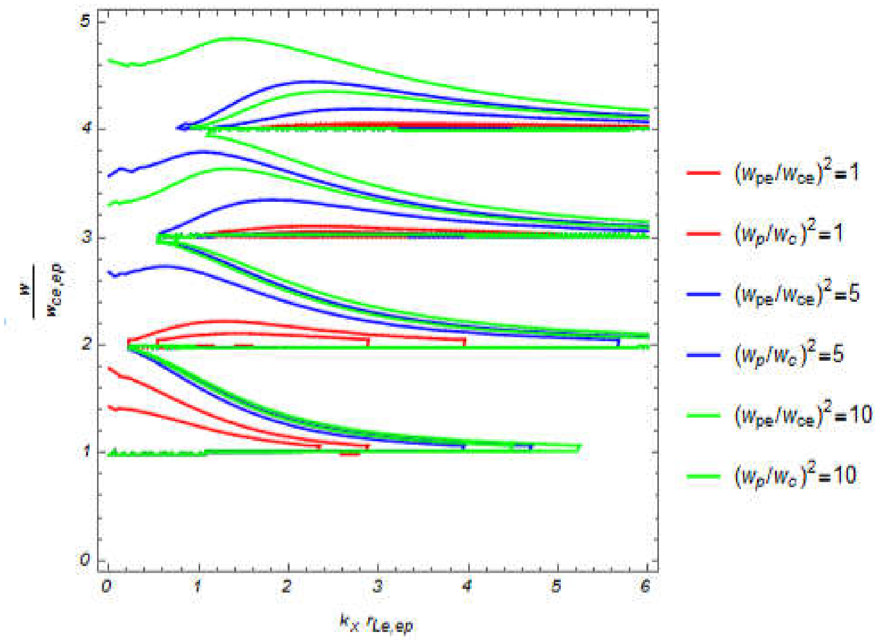

Figure 3.9: Dispersion curve showing comparison between Electrons and Electrons-Positrons Bernstein waves.

for small value of b the modified Bessel function is written as $I_n(b) = \frac{1}{n!}(\frac{b}{2})^n$ when $b \longrightarrow 0$ only $n = 1$ term exist.The equation (3.13) is written as

$$1 = \sum_s \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2}$$

$$1 = \frac{\omega_{pe}^2}{\omega^2 - \omega_{ce}^2} + \frac{\omega_{pp}^2}{\omega^2 - \omega_{cp}^2}$$

where $\omega_{pe}^2 = \omega_{pp}^2$ and $\omega_{ce}^2 = \omega_{cp}^2$

$$\omega^2 = 2\omega_p^2 + \omega_c^2 = \omega_h^2$$

witch is the upper hybrid oscillation. Electrostatic electrons-positrons waves across B have this frequency and those wave along 'B' are oscillating with plasma frequency. $\omega_c = \frac{eB_0}{m}$, when B goes to zero ω_c goes to zero then wave frequency is equal to plasma frequency. $\omega_p^2 = \frac{n_0 e^2}{\epsilon_0 m}$, if plasma density is zero, ω_p goes to zero then wave frequency equal to cyclotron frequency.

Chapter 4

Study of Relativistic Bernstein Waves

In section 4.1 general form of relativistic conductivity tensor in spherical polar co-ordinates is derived. In our problem we need only XX-Component of conductivity tensor, so in subsection 4.1.1, we simplify XX-Component of conductivity tensor. In section 4.2 we check the validity of our relativistic form of conductivity by applying non-relativistic limit. We solve it for non-relativistic electrons Bernstein waves and also we apply fluid limit on it. We get same result as in chapter 3. In section 4.3 we solve the dispersion relation for electrons Bernstein waves in relativistic regime.

4.1 Derivation of Conductivity Tensor

To account the relativistic correction in conductivity tensor, we need to solve the relativistic Vlasov equation along Maxwell's equations. The relativistic Vlasov equation is

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}.\frac{\partial f_\alpha}{\partial \mathbf{x}} + \mathbf{a}.\frac{\partial f_\alpha}{\partial \mathbf{v}} = 0 \tag{4.1}$$

Relativistic momentum $\mathbf{p} = \gamma m\mathbf{v}$, where $\gamma = (1+\frac{p^2}{m^2c^2})^{\frac{1}{2}} = (1-\frac{v^2}{c^2})^{-\frac{1}{2}}$ is the usual relativistic Lorentz factor. Velocity, acceleration and $\frac{\partial f}{\partial \mathbf{v}}$ is define as $\mathbf{v} = \frac{c\mathbf{p}}{(m^2c^2+p^2)^{\frac{1}{2}}}$, $\mathbf{a} = \frac{1}{\gamma m}\frac{\partial \mathbf{p}}{\partial t}$, $\frac{\partial f}{\partial \mathbf{v}} = \gamma m \frac{\partial f}{\partial \mathbf{p}}$ [8].

Relativistic equation of motion of charge particle in $\mathbf{E}$ and $\mathbf{B}$ fields

$$\mathbf{F} = \frac{d\mathbf{p}}{dt} = q[\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}]$$

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v}.\frac{\partial f_\alpha}{\partial \mathbf{x}} + q[\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}].\frac{\partial f_\alpha}{\partial \mathbf{p}} = 0 \tag{4.2}$$

Maxwell equations are given by

$$\nabla.\mathbf{E} = 4\pi \sum_s q_s \int f d^3v$$

$$\nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\sum_s q_s \int \mathbf{v} f d^3v$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

Since the electric field, $\mathbf{E}$, and the magnetic field, $\mathbf{B}$, depend on the distribution function f, so the Vlasov equation is a non-linear equation in. The last term in the Vlasov equation is a non-linear term. To linearise the Vlasov equation we assume that the amplitude of the perturbed quantities is small so we neglect the higher order perturbations. We consider a uniform plasma with an equilibrium distribution function $f_0(\mathbf{v})$ and a small perturbation in it. To linearise the Vlasov equation, we consider

$$f = f_0 + f_1$$

$$\mathbf{E} = \mathbf{E}_0 + \mathbf{E}_1$$

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1$$

The zeroth order term represents the unperturbed part of the variable and the first order term represents the perturbed part of the variable. The linearise Vlasov equation is

$$\frac{\partial f_1}{\partial t} + \mathbf{v}.\frac{\partial f_1}{\partial \mathbf{x}} + q(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0).\frac{\partial f_1}{\partial \mathbf{p}} + q(\mathbf{E}_1 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_1).(\frac{\partial f_o}{\partial \mathbf{v}}) + q(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0).\frac{\partial f_0}{\partial \mathbf{p}} = 0$$

and zeroth order part of equation is

$$q(\mathbf{E}_0 + \frac{1}{c}\mathbf{v}\times\mathbf{B}_0).\frac{\partial f_0}{\partial \mathbf{p}} = 0$$

$\mathbf{E}_0 = 0$, $\mathbf{B} = B_0\hat{z}$

$$\frac{qB_0}{\gamma mc}(\mathbf{p}\times\hat{z}).\frac{\partial f_0}{\partial \mathbf{p}} = 0$$

$$(\mathbf{p}\times\hat{z}).\frac{\partial f_0}{\partial \mathbf{p}} = p_y\frac{\partial f_0}{\partial p_x} - p_x\frac{\partial f_0}{\partial p_y} \tag{4.3}$$

In order to solve the above equation we introduce the spherical coordinate system, and distribution function f_0 is depend on

$$f_0 = f_0(\mathbf{p}, \theta, \phi)$$

$$p_x = psin\theta cos\phi$$

$$p_y = psin\theta sin\phi$$

$$p_z = pcos\theta$$
$$p^2 = p_x^2 + p_y^2 + p_z^2$$
$$\frac{\partial f_0}{\partial p_x} = \frac{\partial f_0}{\partial p}\frac{\partial p}{\partial p_x} + \frac{\partial f_0}{\partial \theta}\frac{\partial \theta}{\partial p_x} + \frac{\partial f_0}{\partial \phi}\frac{\partial \phi}{\partial p_x}$$
$$\frac{\partial f_0}{\partial p_y} = \frac{\partial f_0}{\partial p}\frac{\partial p}{\partial p_y} + \frac{\partial f_0}{\partial \theta}\frac{\partial \theta}{\partial p_y} + \frac{\partial f_0}{\partial \phi}\frac{\partial \phi}{\partial p_y}$$

substituting the value of $\frac{\partial p}{\partial p_x}$, $\frac{\partial \theta}{\partial p_x}$ and $\frac{\partial \phi}{\partial p_x}$ we get

$$\frac{\partial f_0}{\partial p_x} = \frac{\partial f_0}{\partial p}(sin\theta cos\phi) + \frac{\partial f_0}{\partial \theta}(\frac{1}{p}\frac{cos\phi}{cos\theta}) + \frac{\partial f_0}{\partial \phi}(-\frac{sin\phi}{psin\theta})$$

substituting the value of $\frac{\partial p}{\partial p_y}$, $\frac{\partial \theta}{\partial p_y}$ and $\frac{\partial \phi}{\partial p_y}$ we get

$$\frac{\partial f_0}{\partial p_y} = \frac{\partial f_0}{\partial p}(sin\theta sin\phi) + \frac{\partial f_0}{\partial \theta}(\frac{1}{p}\frac{sin\phi}{cos\theta}) + \frac{\partial f_0}{\partial \phi}(\frac{cos\phi}{psin\theta})$$

putting values of $\frac{\partial f_0}{\partial p_x}$ and$\frac{\partial f_0}{\partial p_y}$ and we get

$$\frac{\partial f_0}{\partial \phi} = 0$$

this means that the equilibrium distribution function f_0 does not depend on angle ϕ so

$$f_0 = f_0(\mathbf{p}, \theta)$$

We consider a homogeneous plasma with a uniform magnetic field $\mathbf{B}_0$ and with no electric field present .The distribution function f_0 is choice to any solution of zeroth order equation. $\mathbf{E}_0 = 0$, $\mathbf{B} = B_0\hat{z}$

$$\frac{\partial f_1}{\partial t} + \mathbf{v}.\frac{\partial f_1}{\partial \mathbf{x}} + q(\frac{1}{c}\mathbf{v} \times \mathbf{B}_0).\frac{\partial f_1}{\partial \mathbf{p}} + q(\mathbf{E}_1 + \frac{1}{c}\mathbf{v} \times \mathbf{B}_1).(\frac{\partial f_o}{\partial \mathbf{p}}) = g$$

Applying Fourier and Laplace transformation,

$$(E, B, f) = \int_0^\infty dt \exp(-st)\int_{-\infty}^{\infty} \frac{d^3x}{(2\pi)^{\frac{3}{2}}} \exp(-ik.v)(E_1, B_1, f_1)$$

on above equation we get

$$(s + i\mathbf{k}.\mathbf{v})f + \frac{qB_0}{\gamma mc}(\mathbf{p} \times \hat{z}).\frac{\partial f}{\partial \mathbf{p}} = g - q(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}).(\frac{\partial f_o}{\partial \mathbf{p}})$$

$$(\mathbf{p} \times \hat{z}).\frac{\partial f}{\partial \mathbf{p}} = p_y\frac{\partial f}{\partial p_x} - p_x\frac{\partial f}{\partial p_y} = -\frac{\partial f}{\partial \phi}$$

Let

$$g - q(\mathbf{E} + \frac{1}{c}\mathbf{v} \times \mathbf{B}).(\frac{\partial f_o}{\partial \mathbf{p}}) = -\phi(\phi)$$

and $\Omega = \frac{qB_0}{\gamma mc}$ is the relativistic cyclotron frequency.

$$(s + i\mathbf{k}.\mathbf{v})f - \Omega\frac{\partial f}{\partial \phi} = -\phi(\phi)$$

this is the first order non homogeneous differential equation. First we solve homogeneous part,that is

$$\frac{\partial G(\phi')}{\partial \phi} - \frac{(s + i\mathbf{k}.\mathbf{v})}{\Omega}\phi = 0$$

the solution of above equation is

$$G(\phi') = \exp[\int_{-\phi'}^{\phi} \frac{(s + i\mathbf{k}.\mathbf{v}'')}{\Omega} d\phi'']$$

We can take the wave vector $\mathbf{k}$ as.

$$\mathbf{k} = (k_x, 0, k_z)$$

, and the velocity $\mathbf{v}$ in the spherical coordinate system can be written as

$$\mathbf{v} = (vsin\theta cos\phi, vsin\theta sin\phi, vcos\theta)$$

$$G(\phi') = \exp[\frac{1}{\Omega}((s + ik_z.vcos\theta)(\phi - \phi') + \imath k_x vsin\theta(sin\phi - sin\phi'))]$$

The general solution of the equation is

$$f = \int_{-\infty}^{\phi} \frac{G(\phi')\phi(\phi')}{\Omega} d\phi'$$

As the particles are rotating about the z-axis, so ϕ' will be evolved with the time. The variable ϕ' is related to the time, t' by $\phi' = \frac{\Omega t'}{\gamma}$. As $t' \longrightarrow -\infty$ this means we go back in infinite past where there is no perturbation, so lower integration limit must be $\phi' = \pm\infty$[27]. If $q < 0$ then f_1 will converge at $\phi' \longrightarrow \infty$ and vice versa. As f_1 is periodic in ϕ so its limits of integration should be independent of ϕ. Maxwell equations are given by

$$\nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\sum_s q_s \int \mathbf{v} f d^3v$$

$$\nabla\times\mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \tag{4.4}$$

$$\nabla\times\mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{4.5}$$

Applying Fourier-Laplace transformation on above equations we get,

$$i\mathbf{k} \times \mathbf{B} = \frac{1}{c}[s\mathbf{E} - e] + \frac{4\pi}{c}\mathbf{J}$$

$$i\mathbf{k}\times\mathbf{E} = -\frac{1}{c}(s\mathbf{B} - b)$$

e and b are constants and $s = -i\omega$. Illuminating B from above equations we get,

$$(\omega^2 - c^2k^2)\mathbf{E} + c^2\mathbf{k}(\mathbf{k}.\mathbf{E}) = -4\pi i\omega\mathbf{J}$$

$$(\omega^2 - c^2k^2)E_\alpha + c^2k_\alpha k_\beta E_\beta = -4\pi i\omega J_\alpha$$

Ohm's law is define as

$$J_\alpha = \sigma_{\alpha\beta}E_\beta \tag{4.6}$$

where α,$\beta = x, y, z$. We have used the Einstein summation convention that repeated indices are summed over and are using the Cartesian tensor, so that we do not need to distinguish between covariant and contra variant indices.

$$[(\omega^2 - c^2k^2)\delta_{\alpha\beta} + c^2k_\alpha k_\beta + 4\pi i\omega\sigma_{\alpha\beta}]E_\beta = 0$$

where

$$R_{\alpha\beta} = [(\omega^2 - c^2k^2)\delta_{\alpha\beta} + c^2k_\alpha k_\beta + 4\pi i\omega\sigma_{\alpha\beta}]$$

where the current density is given by

$$\mathbf{J} = \sum_n qn_0 \int \mathbf{v}fd^3p$$

$$J_\alpha = \sum_s q^2n_0 \int \frac{v_\alpha}{\Omega}\int_{-\infty}^{\phi} d\phi' \exp[\frac{1}{\Omega}((s + ik_z.vcos\theta)(\phi - \phi') + ik_xvsin\theta(sin\phi - sin\phi'))]$$
$$(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}).\frac{\partial f_o}{\partial \mathbf{p}}d^3p$$

we simplify the last term first, From Maxwell equation

$$\mathbf{B} = \frac{-c}{s}i\mathbf{k}\times\mathbf{E}$$

$$(\mathbf{E} + \frac{1}{c}\mathbf{v}\times\mathbf{B}).\frac{\partial f_o}{\partial \mathbf{p}} = [\mathbf{E} - \frac{i}{s}\mathbf{v}\times(\mathbf{k}\times\mathbf{E})].\frac{\partial f_o}{\partial \mathbf{p}}$$

Using back cap rule

$$= [\mathbf{E} - \frac{i}{s}((\mathbf{v}.\mathbf{E})\mathbf{k} - (\mathbf{v}.\mathbf{k})\mathbf{E})].\frac{\partial f_o}{\partial \mathbf{p}}$$

$$= [\mathbf{E}.\frac{\partial f_o}{\partial \mathbf{p}} - \frac{i}{s}((\mathbf{v}.\mathbf{E})\mathbf{k}.\frac{\partial f_o}{\partial \mathbf{p}} - (\mathbf{v}.\mathbf{k})\mathbf{E}.\frac{\partial f_o}{\partial \mathbf{p}})]$$

$$= [E_\beta\frac{\partial f_o}{\partial p_\beta} - \frac{i}{s}((E_\beta v_\beta)\mathbf{k}.\frac{\partial f_o}{\partial \mathbf{p}} - (\mathbf{v}.\mathbf{k})E_\beta\frac{\partial f_o}{\partial p_\beta})]$$

$$= [\frac{\partial f_o}{\partial p_\beta} - \frac{i}{s}(\mathbf{v}_\beta(\mathbf{k}.\frac{\partial f_o}{\partial \mathbf{p}}) - (\mathbf{v}.\mathbf{k})\frac{\partial f_o}{\partial p_\beta})]E_\beta$$

$$J_\alpha = \sum_s q^2 n_0 \int \frac{v_\alpha}{\Omega}\int_{-\infty}^{\phi} d\phi' \exp[\frac{1}{\Omega}((s+ik_z.vcos\theta)(\phi-\phi') + ik_x vsin\theta(sin\phi - sin\phi'))]$$
$$[\frac{\partial f_o}{\partial p_\beta} - \frac{i}{s}(v_\beta(k.\frac{\partial f_o}{\partial p}) - (v.k)\frac{\partial f_o}{\partial p_\beta})]E_\beta d^3p$$

comparing above equation with Ohm's law

$$\sigma_{\alpha\beta} = \sum_s q^2 n_0 \int \frac{v_\alpha}{\Omega}\int_{-\infty}^{\phi} d\phi' \exp[\frac{1}{\Omega}((s+ik_z.vcos\theta)(\phi-\phi') + ik_x vsin\theta(sin\phi - sin\phi'))]$$
$$[\frac{\partial f_o}{\partial p_\beta} - \frac{i}{s}(v_\beta(\mathbf{k}.\frac{\partial f_o}{\partial \mathbf{p}}) - (\mathbf{v}.\mathbf{k})\frac{\partial f_o}{\partial p_\beta})]d^3p$$

In spherical polar coordinate

$$d^3p = p^2 sin\theta d\theta dp d\phi$$

and putting $s = -i\omega$

$$\sigma_{\alpha\beta} = \sum_s \frac{q^2 n_0}{\omega}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} p^2 sin\theta d\theta dp d\phi' \frac{v_\alpha}{\Omega}\int_{-\infty}^{\phi} d\phi' \exp[\frac{1}{\Omega}((-i\omega + ik_z.vcos\theta)(\phi-\phi')$$
$$+ik_x vsin\theta(sin\phi - sin\phi'))][(\omega - \mathbf{v}.\mathbf{k})\frac{\partial f_o}{\partial p_\beta} + v_\beta(\mathbf{k}.\frac{\partial f_o}{\partial \mathbf{p}})]$$

This is a general expression for the conductivity tensor for any kind of distribution in a spherical coordinate system.

4.1.1 XX-Component of Conductivity Tensor

In our case we are focussing on the Bernstein waves for which $\mathbf{k} \perp \mathbf{B}_0$. As we are interested in perpendicular prorogation to $\mathbf{B}_0$ so we are left with σ_{xx} component of the conductivity tensor which specifies the dynamics of the Bernstein wave.

$$\sigma_{xx} = \sum_s \frac{q^2 n_0}{\omega}\int_0^{\infty}\int_0^{\pi}\int_0^{2\pi} p^2 sin\theta d\theta dp d\phi' \frac{v_x}{\Omega}\int_{-\infty}^{\phi} d\phi' \exp[\frac{1}{\Omega}((-i\omega + \iota k_z.vcos\theta)(\phi-\phi')$$
$$+ik_x vsin\theta(sin\phi - sin\phi'))][(\omega - \mathbf{v}.\mathbf{k})\frac{\partial f_o}{\partial p_x} + v_x(\mathbf{k}.\frac{\partial f_o}{\partial \mathbf{p}})]$$

$$\frac{\partial p}{\partial p_x} = sin\theta cos\phi$$
$$\frac{\partial p}{\partial p_y} = sin\theta sin\phi$$

$$\frac{\partial p}{\partial p_z} = cos\theta$$

$$[(\omega - \mathbf{v}.\mathbf{k})\frac{\partial f_o}{\partial p_x} + v_x(k.\frac{\partial f_o}{\partial p})] = [(\omega - v_x k_x - v_z k_x z)\frac{\partial f_o}{\partial p_x} + v_x(k_x\frac{\partial f_o}{\partial p_x} + (k_z\frac{\partial f_o}{\partial p_z})]$$

$$= [(\omega - k_x v sin\theta cos\phi - k_z v cos\theta) sin\theta cos\phi\frac{\partial f_o}{\partial p} + v sin\theta cos\phi(k_x sin\theta cos\phi + k_z cos\theta)\frac{\partial f_o}{\partial p}]$$

$$\sigma_{xx} = \sum_s \frac{q^2 n_0}{\omega}\int_0^\infty\int_0^\pi\int_0^{2\pi} p^2 sin\theta d\theta dp d\phi' \frac{v sin\theta cos\phi'}{\Omega}\int_{-\infty}^{\phi} d\phi' \exp[\frac{1}{\Omega}((-i\omega + ik_z.v cos\theta)(\phi - \phi') + ik_x v sin\theta(sin\phi - sin\phi'))]$$
$$[(\omega - k_x v sin\theta cos\phi' - k_z v cos\theta) sin\theta cos\phi'\frac{\partial f_o}{\partial p} + v sin\theta cos\phi'(k_x sin\theta cos\phi' + k_z cos\theta)\frac{\partial f_o}{\partial p}]$$

As f_1 is periodic in ϕ, so we substitute $\phi - \phi' = \alpha$, such that the integration limits become independent of ϕ. Putting

$$\phi - \phi' = \alpha$$

$$d\phi' = -d\alpha$$

$$\sigma_{xx} = -\sum_s q^2 n_0 \int_0^\infty p^2\frac{\partial f_o}{\partial p}dp\int_0^\pi \frac{v sin^3\theta}{\Omega}d\theta\int_0^{2\pi} cos\theta cos(\phi - \alpha)\exp[\frac{1}{\Omega}ik_x v sin\theta(sin\phi - sin(\phi - \alpha))]d\phi$$
$$\int_\infty^0 d\alpha \exp[\frac{1}{\Omega}(-i\omega + ik_z v cos\theta)\alpha]$$

We note that

$$\int_0^{2\pi} cos\theta cos(\phi - \alpha)\exp[\frac{1}{\Omega}ik_x v sin\theta(sin\phi - sin(\phi - \alpha))]d\phi = 2\pi\sum_n \frac{n^2}{\xi'^2}J_n^2(\xi')\exp(in\alpha)$$

where $\xi' = \frac{k_x v sin\theta}{\Omega}$

$$\sigma_{xx} = -2\pi\sum_s q^2 n_0\int_0^\infty p^2\frac{\partial f_o}{\partial p}dp\int_0^\pi\frac{v sin^3\theta}{\Omega}d\theta\sum_n\frac{n^2}{\xi'^2}J_n^2(\xi')\exp(\iota n\alpha)\int_\infty^0 d\alpha\exp[\frac{-i}{\Omega}(\omega - k_z v cos\theta - n\Omega)\alpha]$$

$$\int_\infty^0 d\alpha\exp[\frac{-i}{\Omega}(\omega - k_z v cos\theta - n\Omega)\alpha] = \frac{i\Omega}{(w - k_z v cos\theta - n\Omega)}$$

$$\sigma_{xx} = -2\pi\sum_s q^2 n_0\int_0^\infty p^2\frac{\partial f_o}{\partial p}dp\int_0^\pi\frac{v sin^3\theta}{\Omega}d\theta\sum_n\frac{n^2}{\xi'^2}J_n^2(\xi')\exp(in\alpha)\frac{i\Omega}{(\omega - k_z v cos\theta - n\Omega)}$$

$$\sigma_{xx} = -2\pi i\sum_s\sum_n q^2 n_0\int_0^\infty p^2\frac{\partial f_o}{\partial p}dp\int_0^\pi\frac{sin\theta}{(\omega - k_z v cos\theta - n\Omega)}d\theta\frac{n^2\Omega^2}{k_x^2 v}J_n^2(\xi') \tag{4.7}$$

For Bernstein wave $k_z = 0$

$$\sigma_{xx} = -2\pi i \sum_s \sum_n q^2 n_0 \int_0^\infty p^2 \frac{\partial f_o}{\partial p} dp \int_0^\pi \frac{sin\theta}{(\omega - n\Omega)} d\theta \frac{n^2\Omega^2}{k_x^2 v} J_n^2(\frac{k_x v sin\theta}{\Omega})$$

For $n = 0$ equation become zero so we divide the sum into sum, we get

$$\sigma_{xx} = -2\pi i \sum_s \sum_1^\infty q^2 n_0 \int_0^\infty p^2 \frac{\partial f_o}{\partial p} dp \int_0^\pi sin\theta d\theta \frac{n^2\Omega^2}{k_x^2 v} J_n^2(\frac{k_x v sin\theta}{\Omega})[\frac{1}{(\omega - n\Omega)} + \frac{1}{(\omega + n\Omega)}]$$

$$\int_0^\pi sin\theta d\theta J_n^2(\xi sin\theta) d\theta = \xi^{2n} \Gamma(\frac{1}{2} + n) {}_1F_2[(n + \frac{1}{2}); (n + \frac{3}{2}, 2n+1); -\xi^2]$$

where ${}_pF_q[(a_1, ...a_p), (b_1, ...b_q), x]$ is a generalized Hyper-geometric function[28] and $\xi = \frac{k_x v}{\Omega}$

$$\begin{aligned}\sigma_{xx} = \quad & -2\pi i \sum_s \sum_1^\infty q^2 n_0 \int_0^\infty p^2 \frac{\partial f_o}{\partial p} dp \frac{n^2\Omega^2}{k_x^2 v} (\frac{k_x v}{\Omega})^{2n} \\ & \Gamma(\frac{1}{2} + n) {}_1F_2[(n + \frac{1}{2}); (n + \frac{3}{2}, 2n+1); -\xi^2] \\ & [\frac{1}{(\omega - n\Omega)} + \frac{1}{(\omega + n\Omega)}]\end{aligned}$$

4.2 Non-Relativistic Electron Bernstein Waves

First we solve above for non-relativistic case.

$$f_{0e} = (\frac{1}{m^2 v_{ths}^2 \pi})^{\frac{3}{2}} (\exp - \frac{p_e^2}{m^2 v_{the}^2})$$

$$v_{the} = (\frac{2kT_e}{m_e})^{\frac{1}{2}}$$

v_{the} is thermal velocity of electron and where $\Omega = w_{ce} = \frac{eB_0}{mc}$ is the non-relativistic cyclotron frequency when $\gamma = 1$

$$\frac{\partial f_0(p)}{\partial p} = -\frac{2p}{\pi\sqrt{\pi} m^5 v_{the}^5} \exp(-\frac{p^2}{m^2 v_{the}^2})$$

$$\begin{aligned}\sigma_{xx} = \quad & 8\pi i \sum_1^\infty \frac{q^2 n_0}{m} (\frac{k_x v_0}{\omega_{ce}})^{2n-2} {}_1F_2[(n + \frac{1}{2}); (n + \frac{3}{2}, 2n+1); -(\frac{k_x v_0}{\omega_{ce}})^2] \\ & n^2 \Gamma(\frac{1}{2} + n) \frac{\omega}{(\omega^2 - n^2\omega_{ce}^2)} \int_0^\infty p^3 \frac{\partial f_o}{\partial p} dp\end{aligned}$$

For Bernstein wave k_y and k_z is zero.

$$R_{xx} = \omega^2 + 4\pi i \omega \sigma_{xx} = 0$$

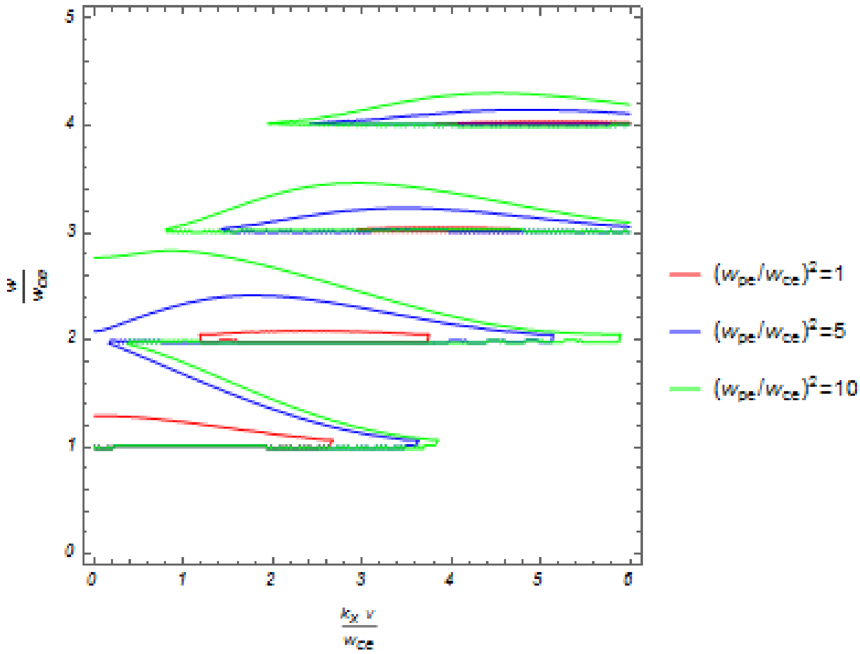

Figure 4.1: Dispersion curve showing solution for EBW.

$$1 = \quad 3\sum_{1}^{\infty}\omega_{pe}^2(\frac{k_x v_0}{\omega_{ce}})^{2n-2}{}_1F_2[(n+\frac{1}{2});(n+\frac{3}{2},2n+1);-(\frac{k_x v_0}{\omega_{ce}})^2]$$
$$n^2\Gamma(\frac{1}{2}+n)[\frac{1}{(\omega^2-n^2\omega_{ce}^2)}$$

4.2.1 Fluid Limit on Non-Relativistic Electron Bernstein Waves

Dispersion relation for non-relativistic electron Bernstein waves is written as,

$$1 = \quad 3\sum_{1}^{\infty}\omega_{pe}^2(\frac{k_x v_0}{\omega_{ce}})^{2n-2}{}_1F_2[(n+\frac{1}{2});(n+\frac{3}{2},2n+1);-(\frac{k_x v_0}{\omega_{ce}})^2]$$
$$n^2\Gamma(\frac{1}{2}+n)[\frac{1}{(\omega^2-n^2\omega_{ce}^2)}$$

For $\frac{k_x v_0}{\omega_{ce}} << 1$ only $n=1$ term is survive. So, for $n=1$ and $\frac{k_x v_0}{\omega_{ce}} << 1$ the above dispersion relation is written as,

$$\omega^2 = \omega_{pe}^2 + \omega_{ce}^2 = \omega_{Uh}^2$$

4.3 Relativistic Electrons Bernstein Waves

In a relativistic case, the cyclotron frequency is a function of momentum, where the relativistic equilibrium distribution function[29, 30].

$$f_0(\mathbf{p}) = \frac{1}{4\pi m^3 c^3} \frac{\eta}{K_2(\eta)} \exp(-\eta\gamma)$$

where

$$\eta = \frac{mc^2}{k_B T}$$

is the ratio of the rest mass energy of particles to that of their thermal energy, and K_2 is the modified Bessel function of the second kind and of order two. Taking derivative of $f_0(\mathbf{p})$ with respect to p we get,

$$\frac{\partial f_0(p)}{\partial p} = -\frac{1}{4\pi m^5 c^5} \frac{\eta^2}{K_2(\eta)} \frac{p}{\gamma} \exp(-\eta\gamma)$$

Where the relativistic cyclotron frequency and relativistic velocity is define as $\Omega = \frac{\Omega_0}{\gamma}$, $\mathbf{v} = \frac{c\mathbf{p}}{(m^2c^2+p^2)^{\frac{1}{2}}}$ Here we easily prove that,

$$\frac{k_x v}{\Omega} = \frac{k_x v_0}{\Omega_0}$$

where v_0 and Ω_0 is non-relativistic velocity and non-relativistic frequency.

$$w^2 - n^2\Omega^2 = w^2 - \frac{n^2 w_{ce}^2}{\gamma^2}$$

$$\begin{aligned}
\sigma_{xx} = \quad & \frac{2i}{m^5 c^5} \frac{\eta^2}{K_2(\eta)} \sum_s \frac{q^2 n_0}{m} \sum_1^{\infty} (\frac{k_x v_0}{\omega_{ce}})^{2n-2} n^2 \\
& \Gamma(\frac{1}{2} + n)_1 F_2[(n + \frac{1}{2}); (n + \frac{3}{2}, 2n+1); -(\frac{k_x v_0}{\omega_{ce}})^2] \\
& \int_0^{\infty} p^4 \exp(-\eta\gamma) \frac{2w}{\gamma^2 \omega^2 - n^2 \omega_{ce}^2} dp
\end{aligned}$$

For electrons Bernstein waves k_y and k_z' is zero.

$$R_{xx} = \omega^2 + 4\pi\iota\omega\sigma_{xx} = 0$$

$$\begin{aligned}
\omega^2 = \quad & \frac{2}{m^5 c^5} \frac{\eta^2}{K_2(\eta)} \sum_s \omega_{ps}^2 \sum_1^{\infty} (\frac{k_x v_0}{\omega_{ce}})^{2n-2} n^2 \\
& \Gamma(\frac{1}{2} + n)_1 F_2[(n + \frac{1}{2}); (n + \frac{3}{2}, 2n+1); -(\frac{k_x v_0}{\omega_{ce}})^2] \\
& \int_0^{\infty} p^4 \exp(-\eta\gamma) \frac{2\omega^2}{\gamma^2 \omega^2 - n^2 \omega_{ce}^2} dp
\end{aligned}$$

This is the general form of fully relativistic Bernstein wave. To solve the above integral we expand the γ in the exponential term, we get weakly relativistic Bernstein wave. Here we consider only electron Bernstein wave so, sum over species is dropped.

$$
\begin{aligned}
w^2 = \quad & \frac{2}{m^5c^5}\frac{\eta^2}{K_2(\eta)}\omega_{pe}^2\sum_1^\infty(\frac{k_xv_0}{\omega_{ce}})^{2n-2}n^2 \\
& \Gamma(\frac{1}{2}+n)_1F_2[(n+\frac{1}{2});(n+\frac{3}{2},2n+1);-(\frac{k_xv_0}{\omega_{ce}})^2] \\
& \int_0^\infty p^4\exp[-\eta(1+\frac{p^2}{2m^2c^2})]\frac{2\omega^2}{(1+\frac{p^2}{m^2c^2})\omega^2-n^2\omega_{ce}^2}dp
\end{aligned}
$$

$$
\begin{aligned}
(\frac{\omega}{\omega_{ce}})^5 = \quad & \frac{4\sqrt{\eta}}{K_2(\eta)}\frac{\omega_{pe}^2}{\omega_{ce}^2}\sum_1^\infty(\frac{k_xv_0}{\omega_{ce}})^{2n-2}n^2 \\
& \Gamma(\frac{1}{2}+n)_1F_2[(n+\frac{1}{2});(n+\frac{3}{2},2n+1);-(\frac{k_xv_0}{\omega_{ce}})^2] \\
& \exp[-\eta]\sqrt{\frac{\pi}{2}}[(\frac{\omega}{\omega_{ce}})^3+\eta\frac{\omega}{\omega_{ce}}(n^2-(\frac{\omega}{\omega_{ce}})^2)-\sqrt{2}\eta^{\frac{3}{2}}(n^2-(\frac{\omega}{\omega_{ce}})^2)^{\frac{3}{2}} \\
& DasonF[\frac{\sqrt{2}\sqrt{n^2-(\frac{\omega}{\omega_{ce}})^2}}{2(\frac{\omega}{\omega_{ce}})}]]
\end{aligned}
$$

ω_{pe} and ω_{ce} are plasma frequency and cyclotron frequency of electron and DawsonF function or DawsonF integral is define as,

$$D_+(x) = \exp[-x^2]\int_0^x \exp[t^2]dt$$

$$D_-(x) = \exp[x^2]\int_0^x \exp[-t^2]dt$$

4.4 Non-Relativistic Electron-Positron Bernstein Waves

First we consider the case of non-relativistic plasma for which we use the non-relativistic Maxwellian distribution function

$$f_{0s} = (\frac{1}{m^2v_{ths}^2\pi})^{\frac{3}{2}}(\exp-\frac{p_s^2}{m^2v_{ths}^2})$$

$$v_{ths} = (\frac{2kT_s}{m_s})^{\frac{1}{2}}$$

in Eq. (3), we get,

$$\sigma_{xx} = \sum_s 8\pi i \sum_1^\infty \frac{q^2 n_{0s}}{m_s}(\frac{k_x v_0}{\omega_{cs}})^{2n-2}{}_1F_2[(n+\frac{1}{2});(n+\frac{3}{2},2n+1);-(\frac{k_x v_0}{\omega_{cs}})^2]$$
$$n^2\Gamma(\frac{1}{2}+n)\frac{\omega}{(\omega^2-n^2\omega_{cs}^2)}\int_0^\infty p^3\frac{\partial f_{os}}{\partial p}dp \qquad (4.8)$$

where v_{ths} is thermal velocity of sth species and $\Omega = \omega_{cs} = \frac{qB_0}{m_s c}$ is the non-relativistic cyclotron frequency.

$$1 = 3\sum_s\sum_1^\infty \omega_{ps}^2(\frac{k_x v_0}{\omega_{cs}})^{2n-2}{}_1F_2[(n+\frac{1}{2});(n+\frac{3}{2},2n+1);-(\frac{k_x v_0}{\omega_{cs}})^2]$$
$$n^2\Gamma(\frac{1}{2}+n)[\frac{1}{(\omega^2-n^2\omega_{cs}^2)} \qquad (4.9)$$

where s for species (s=electron, positron). The above expression is the dispersion relation for the electron-positron Bernstein waves in the non- relativistic limit.

4.4.1 Fluid Limit on Non-Relativistic Electron Positron Bernstein Waves

If we apply the fluid limit i.e., $\frac{k_x v_0}{\omega_{cs}} << 1$ on Eq. 8 we get the fluid result which is the upper hybrid oscillations[31, 32] i.e.,

$$\omega^2 = 2\omega_p^2 + \omega_c^2 = \omega_{Uh}^2$$

.

4.4.2 Relativistic Electron Positron Bernstein Waves

Relativistic electron-positron pair plasmas are found in many astrophysical objects such as neutron star, magnetosphere and white dwarf and many others. Positron is a particle having mass and charge equal to electron but the charge being of opposite nature. The weakly relativistic dispersion relation for Bernstein waves is given as,

$$1 = \sum_s \frac{2}{m^5c^5}\frac{\eta^2}{K_2(\eta)}\omega_{ps}^2\sum_1^\infty(\frac{k_x v_0}{\omega_{cs}})^{2n-2}n^2$$
$$\Gamma(\frac{1}{2}+n){}_1F_2[(n+\frac{1}{2});(n+\frac{3}{2},2n+1);-(\frac{k_x v_0}{\omega_{cs}})^2]$$
$$\int_0^\infty p^4\exp[-\eta(1+\frac{p^2}{2m^2c^2})]\frac{2}{(1+\frac{p^2}{m^2c^2})\omega^2-n^2\omega_{cs}^2}dp \qquad (4.10)$$

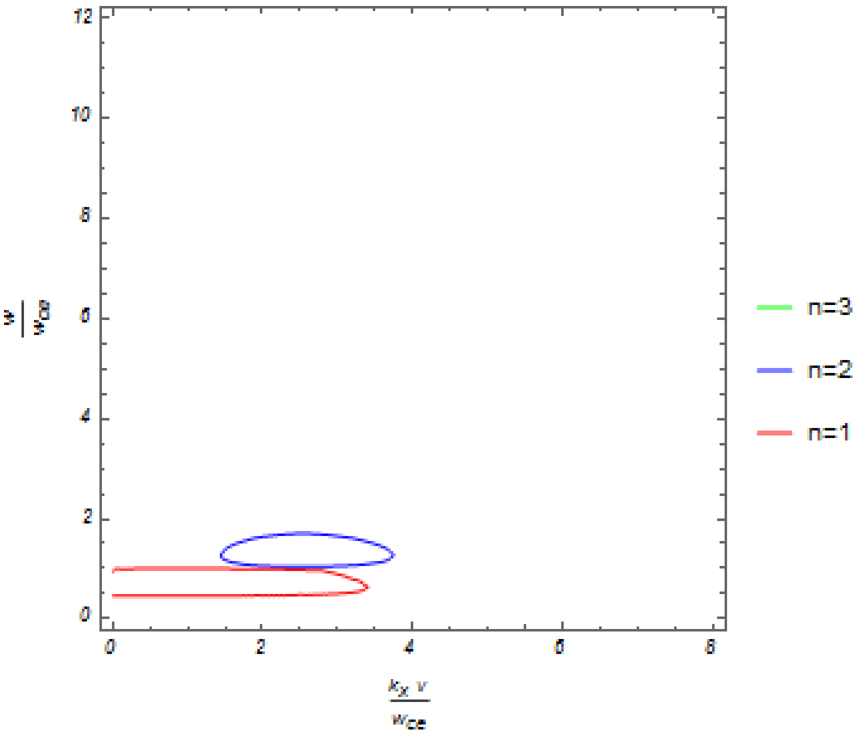

Figure 4.2: Dispersion curves showing solution for $\eta = 1$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 25$.

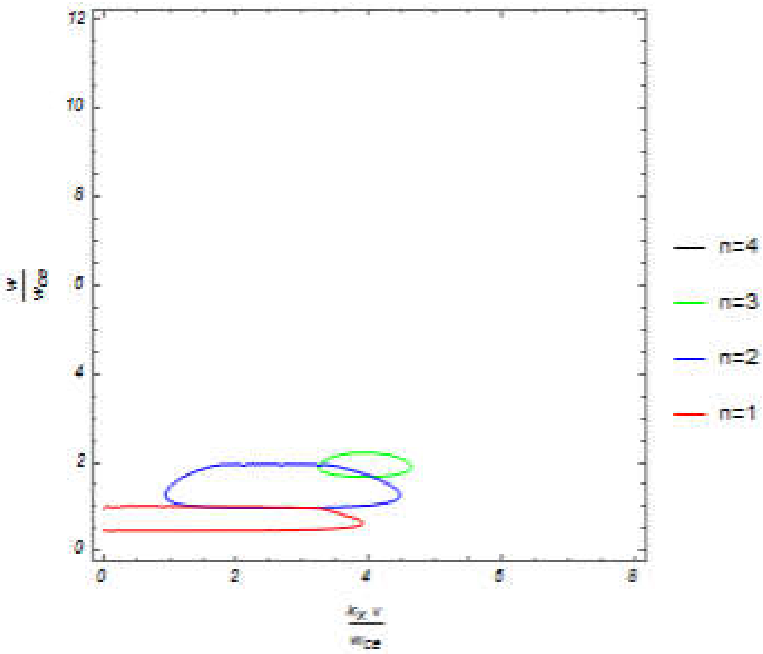

Figure 4.3: Dispersion curves showing solution for $\eta = 1$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 50$.

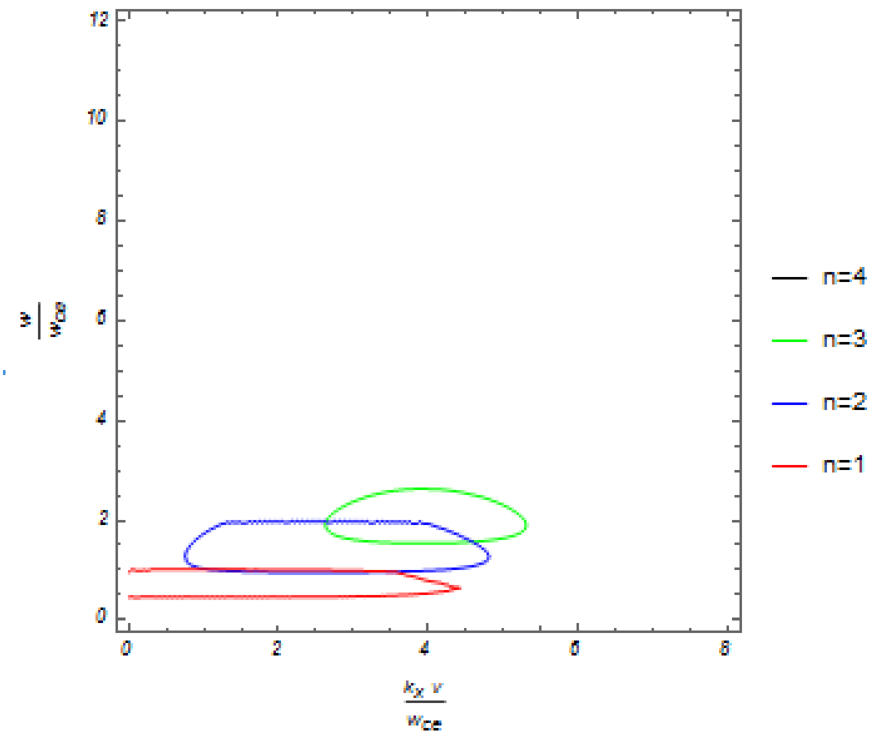

Figure 4.4: Dispersion curves showing solution for $\eta = 1$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 75$.

Where s ($s = e, p$) represent the species. After solving the above equation we get the dispersion relation for weakly relativistic electron-positron Bernstein waves.

$$
\begin{aligned}
(\frac{\omega}{\omega_c})^5 =& \frac{8\sqrt{\eta}}{K_2(\eta)}\frac{\omega_p^2}{\omega_c^2}\sum_1^\infty(\frac{k_x v_0}{\omega_c})^{2n-2}n^2 \\
& \Gamma(\frac{1}{2}+n)_1F_2[(n+\frac{1}{2});(n+\frac{3}{2},2n+1);-(\frac{k_x v_0}{\omega_c})^2] \\
& \exp[-\eta]\sqrt{\frac{\pi}{2}}[(\frac{\omega}{\omega_c})^3+\eta\frac{\omega}{\omega_c}(n^2-(\frac{\omega}{\omega_c})^2)-\sqrt{2}\eta^{\frac{3}{2}}(n^2-(\frac{\omega}{\omega_c})^2)^{\frac{3}{2}} \\
& DasonF[\frac{\sqrt{2}\sqrt{n^2-(\frac{\omega}{\omega_c})^2}}{2(\frac{\omega}{\omega_c})}]]
\end{aligned} \quad (4.11)
$$

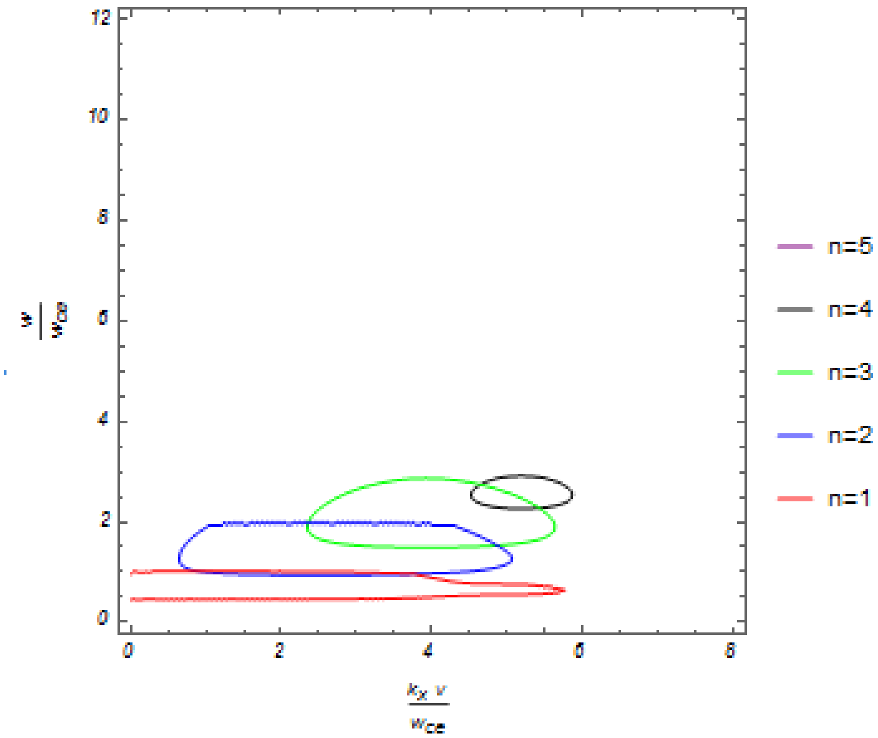

Figure 4.5: Dispersion curves showing solution for $\eta = 1$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 100$.
The dispersion curves in figure 4.2 to 4.5 for $\eta = 1$ are plotted for different value of $(\frac{\omega_{pe}}{\omega_{ce}})$. From these curve we observe that, when we increase the density, (we are increasing the ratio of $(\frac{\omega_{pe}}{\omega_{ce}})$) we get more harmonics.

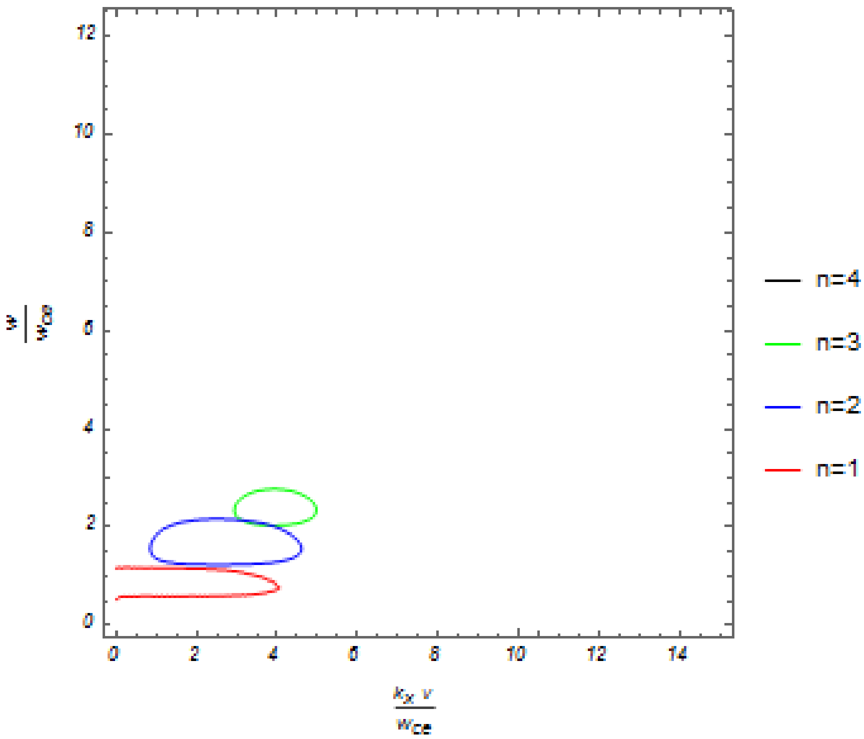

Figure 4.6: Dispersion curves showing solution for $\eta = 2$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 25$.

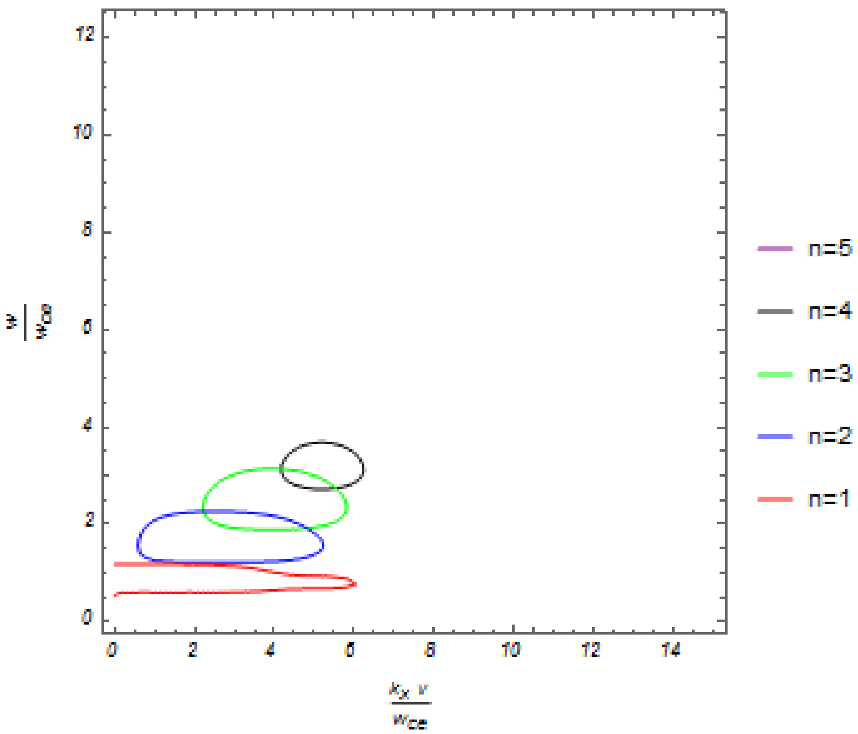

Figure 4.7: Dispersion curves showing solution for $\eta = 2$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 50$.

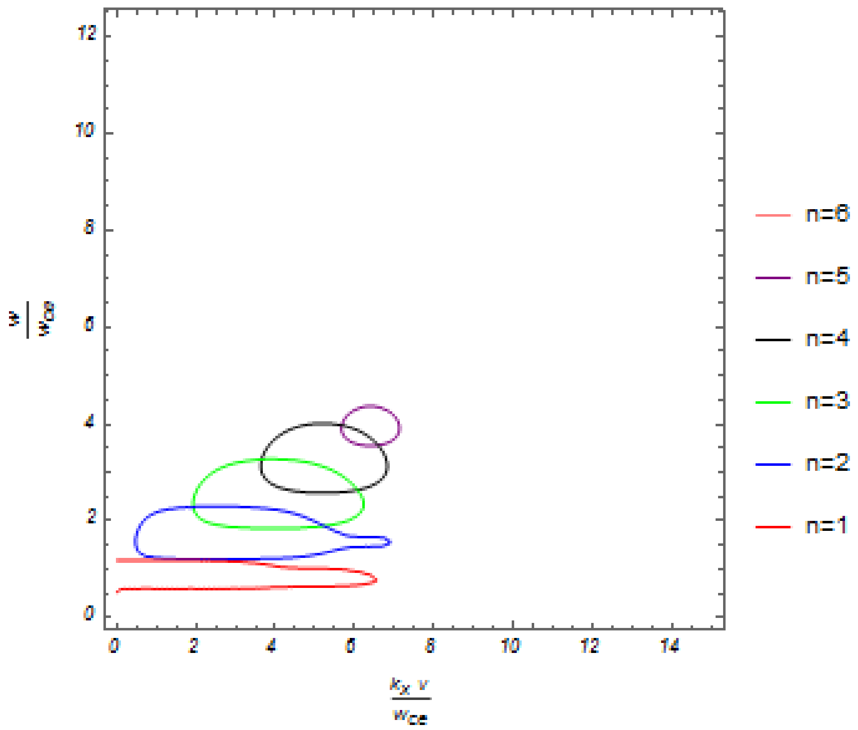

Figure 4.8: Dispersion curves showing solution for $\eta = 2$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 75$.

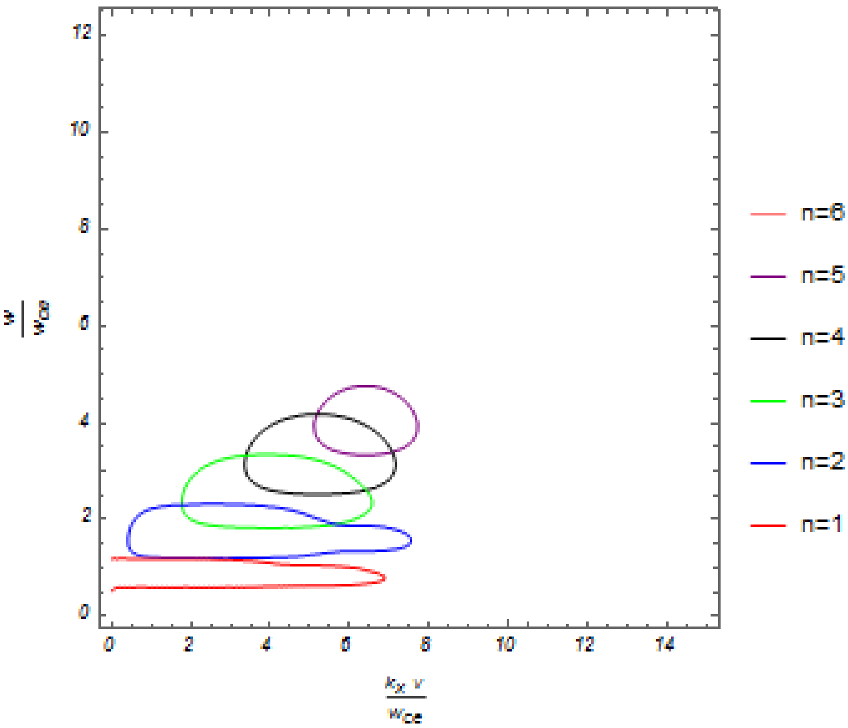

Figure 4.9: Dispersion curves showing solution for $\eta = 2$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 100$.

In figure 4.6 to 4.9, we fix $\eta = 2$ and plot the dispersion curve for different value of $(\frac{\omega_{pe}}{\omega_{ce}})$, we observed that, when we increase the plasma density we get more harmonics. The number of harmonics depend on the value of $(\frac{\omega_{pe}}{\omega_{ce}})$ and η.

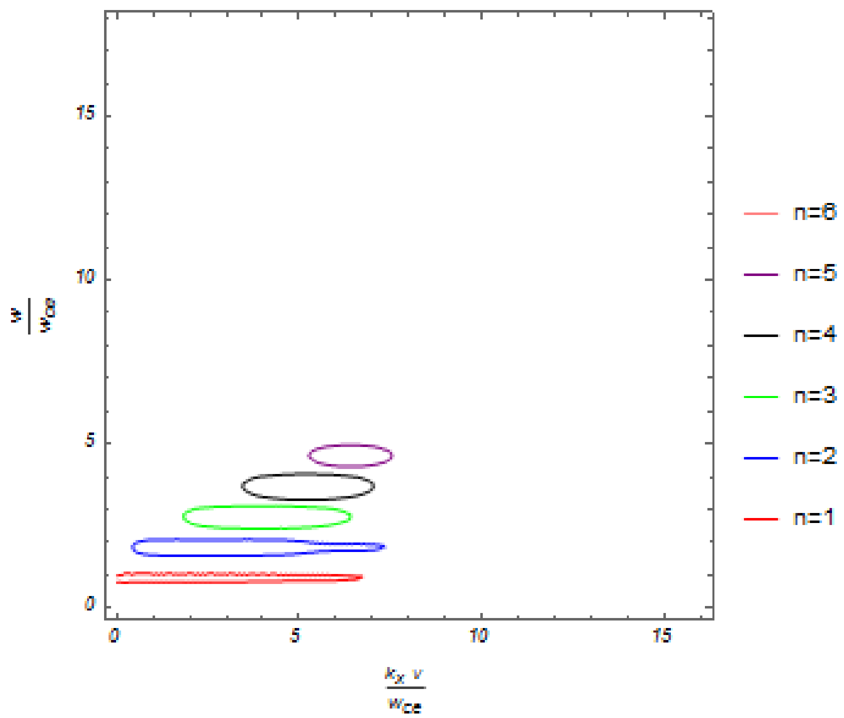

Figure 4.10: Dispersion curves showing solution for $\eta = 6$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 25$.

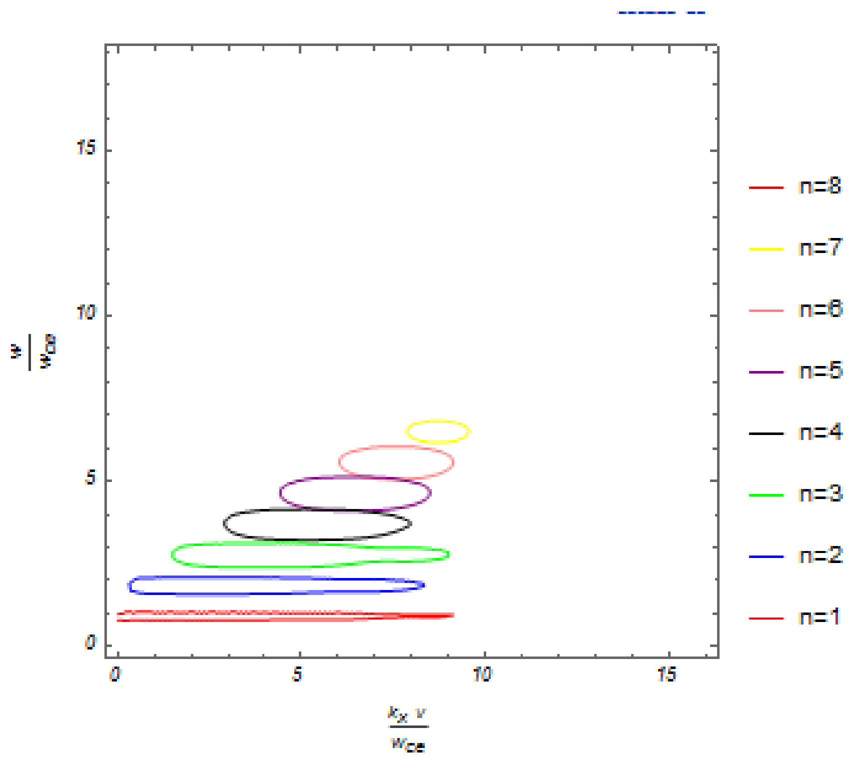

Figure 4.11: Dispersion curves showing solution for $\eta = 6$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 50$.

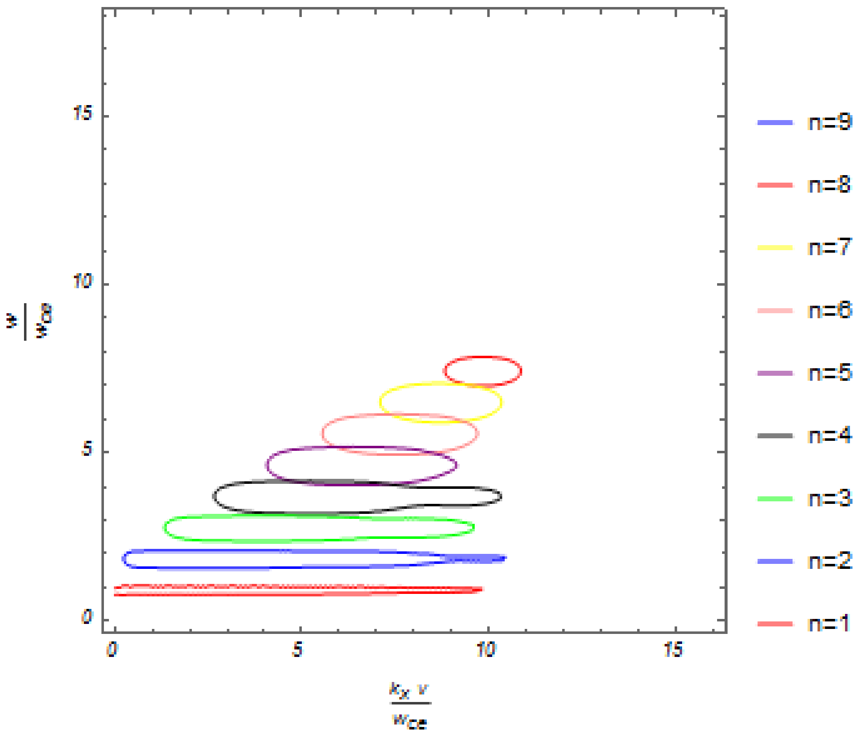

Figure 4.12: Dispersion curves showing solution for $\eta = 6$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 75$.

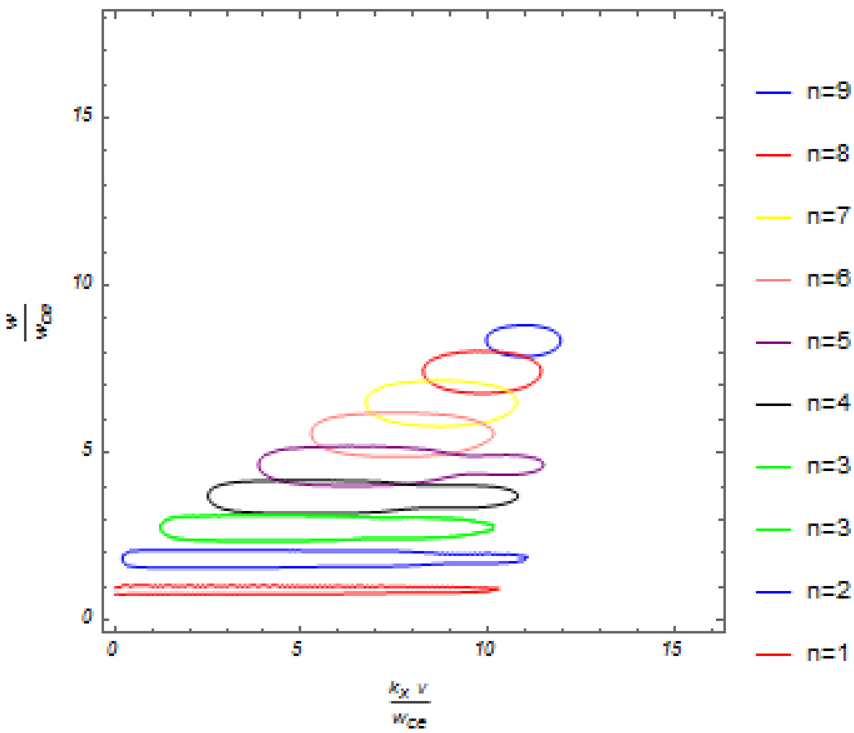

Figure 4.13: Dispersion curves showing solution for $\eta = 6$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 100$.

In figure 4.10 to 4.13 we increase the value of $\eta = 6$, we observe that the number of harmonics are increasing and overlapping is decreasing for same value of $(\frac{\omega_{pe}}{\omega_{ce}})$ as compare to figure 4.2 to 4.5 and 4.6 to 4.9.

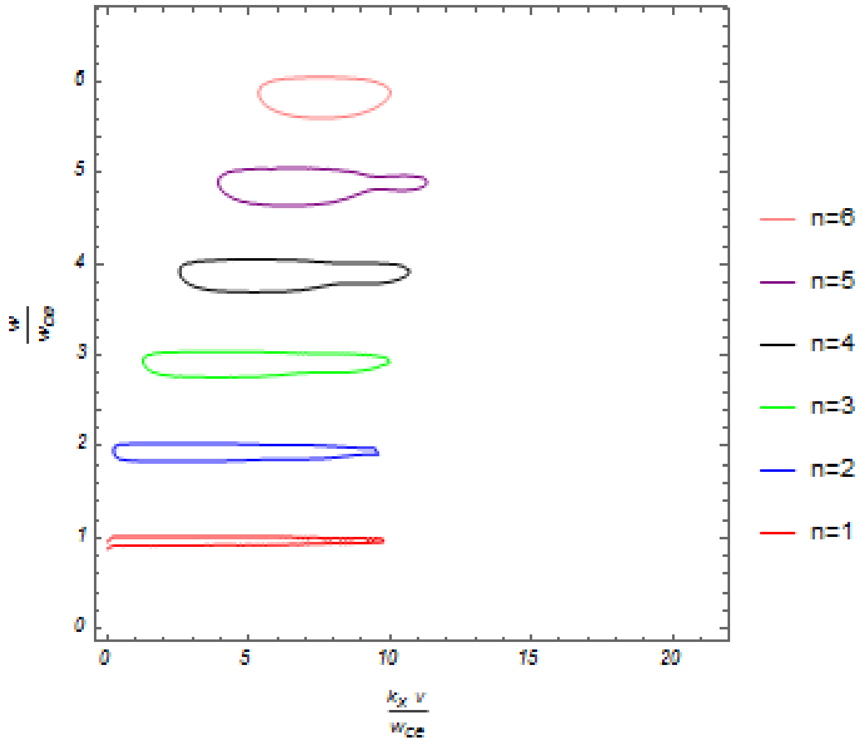

Figure 4.14: Dispersion curves showing solution for $\eta = 20$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 25$.

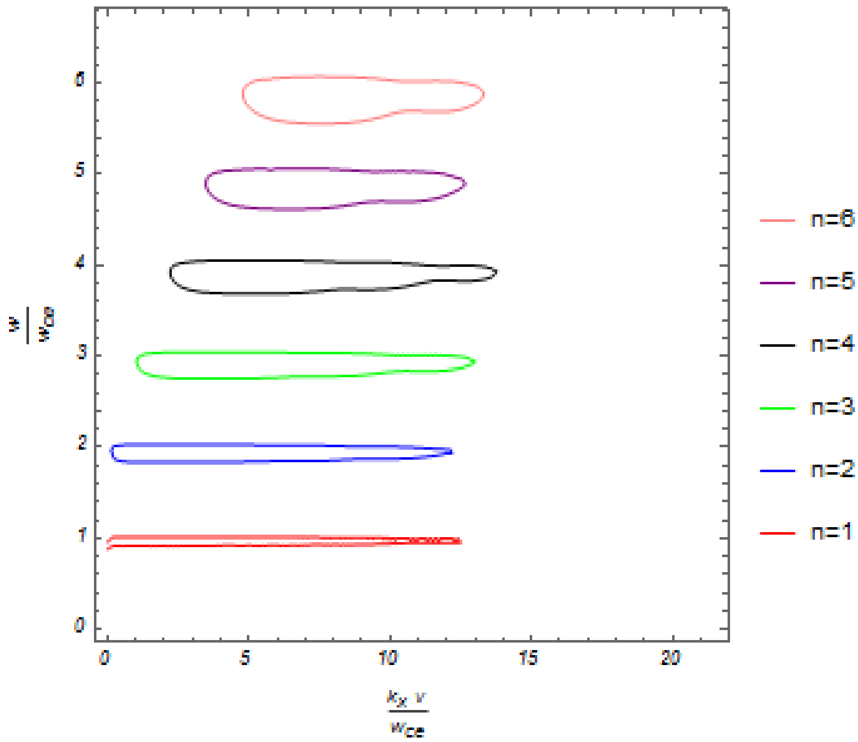

Figure 4.15: Dispersion curves showing solution for $\eta = 20$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 50$.

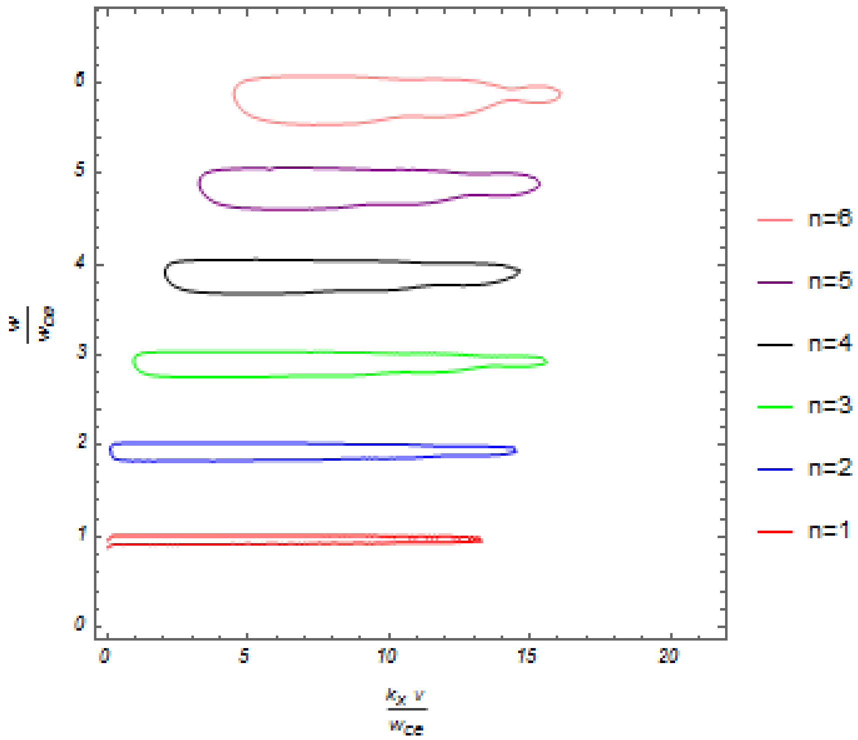

Figure 4.16: Dispersion curves showing solution for $\eta = 20$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 75$.

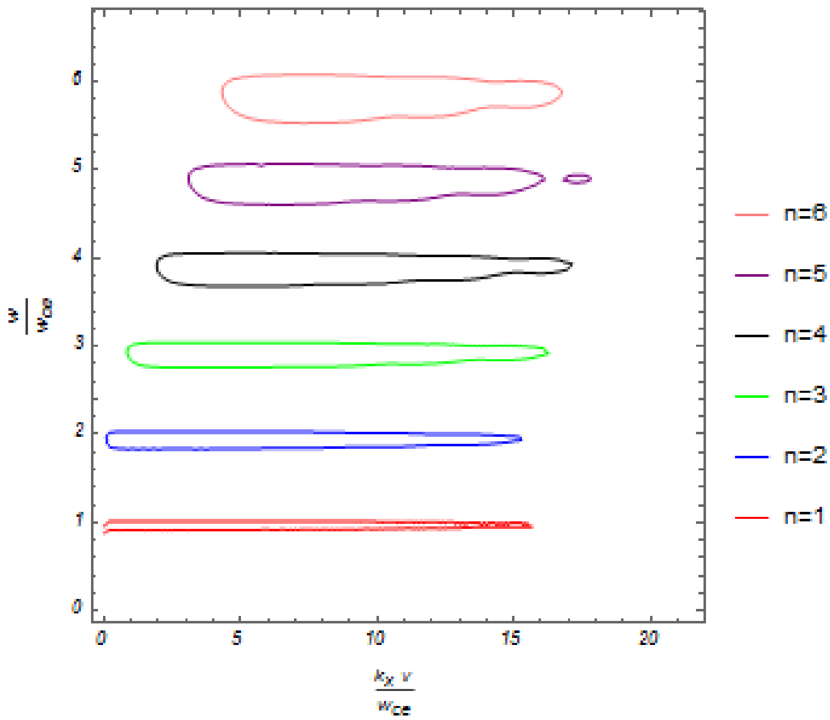

Figure 4.17: Dispersion curves showing solution for $\eta = 20$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 100$.
In figure 4.14 to 4.17 we increase the value of $\eta = 20$(non-relativistic regime). We observed that, we get large number of harmonics with out overlapping for same value of $(\frac{\omega_{pe}}{\omega_{ce}})$ as in above plots.

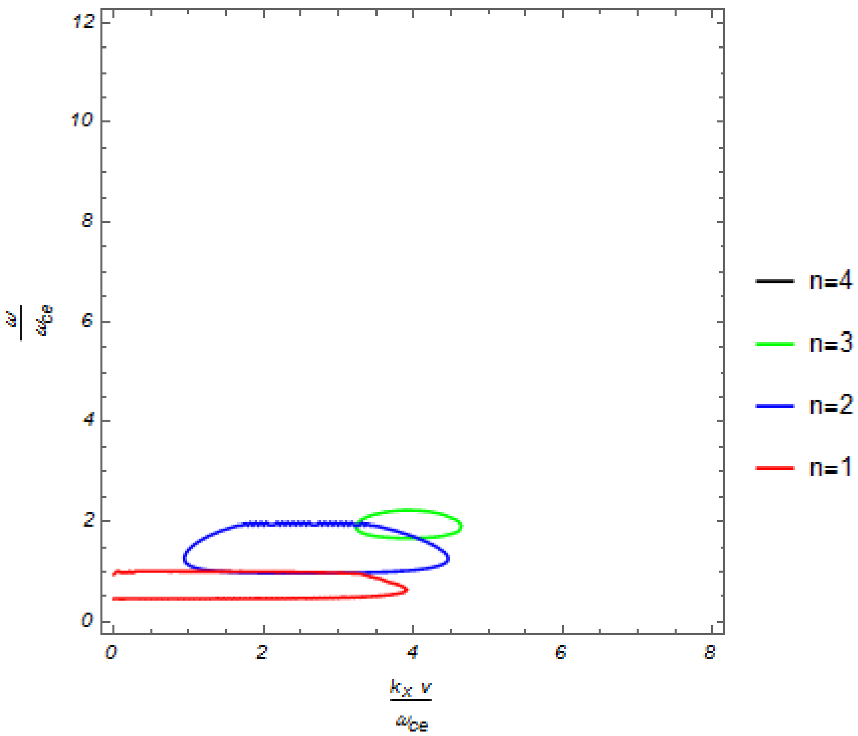

Figure 4.18:

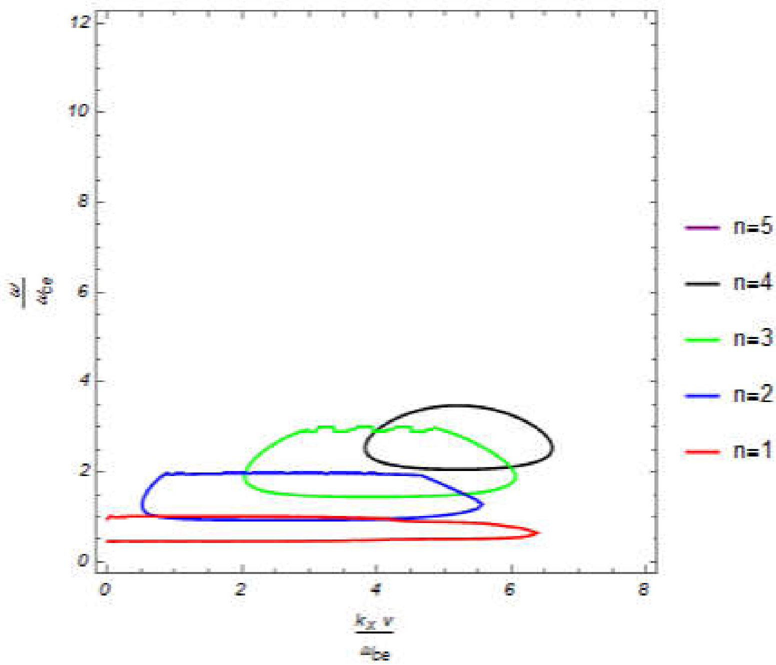

Figure 4.19:

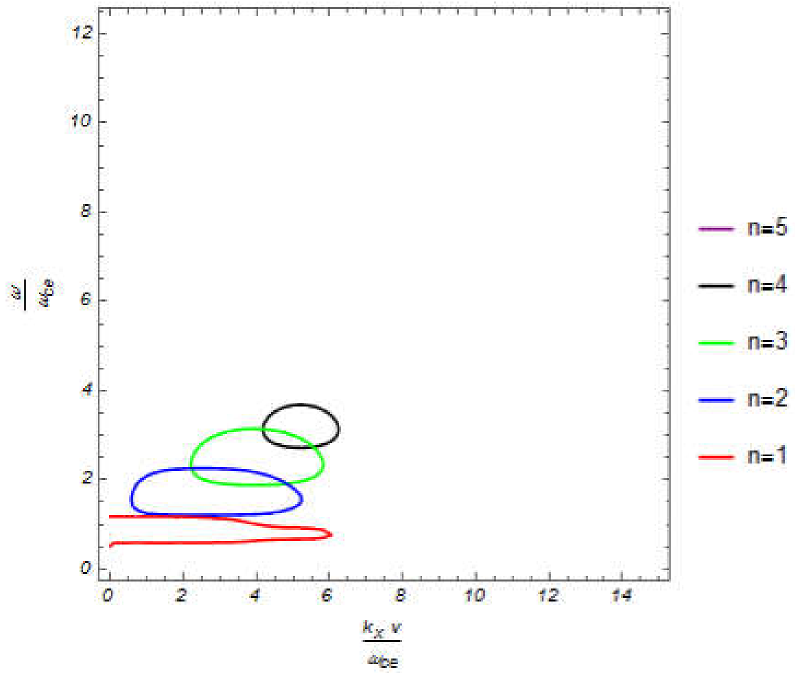

Figure 4.20: Dispersion curves showing solutions for $\eta = 2$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 25$ for the electrons-positrons Bernstein waves

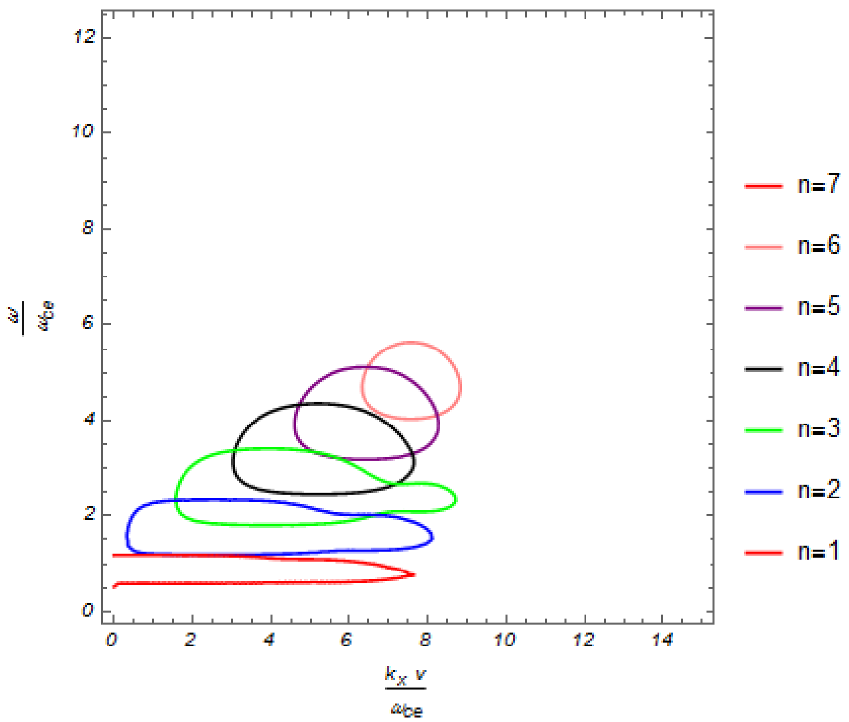

Figure 4.21: Dispersion curves showing solutions for $\eta = 2$ and $(\frac{\omega_{pe}}{\omega_{ce}})^2 = 75$ for the electrons-positrons Bernstein waves

Chapter 5

Discussion and Conclusion

5.1 Discussion

Most of the waves in plasma can be describe by using fluid but some waves can not observed by using fluid theory. In fluid theory we average over the larmor orbits, therefore those waves along larmor orbits are lost. To observe such waves in plasma we kinetic theory of plasma. By using kinetic theory we observe important class of waves. These waves are electron Bernstein waves, neutralized ions Bernstein waves and pure ions Bernstein waves. By applying fluid limits on these kinetic theory predicted waves are converters into the fluid waves. Under fluid limit electrons Bernstein waves are upper hybrid mode, neutralized ions Bernstein waves are converted into ions cyclotrons waves and pure Bernstein waves are converted into lower hybrid mode.
The dispersion curves in the figure 4.2 to 4.17 are plotted for different values of η and $\frac{\omega_{pe}}{\omega_{ce}}$. When we increase η(decrease in thermal energy) the contribution of higher harmonics with the lower harmonics is decreases. Cyclotrons frequency is function of momentum. When the thermal energy of particles is decreases, the momentum of particles also decrease so the dependence cyclotrons frequency on momentum also decrease. Overlapping of higher harmonics with lower harmonics is decreases with increasing η. Higher harmonics contributes with lower harmonics for small value of η and higher harmonics are stay in its own position for lager value of η. In those curves the value of $\eta = 1$ or $\eta = 2$ the relativistic effect are more dominant because the cyclotron frequency frequency strongly depend on momentum. In those curve $\eta = 6$ are intermediate state. The overlapping of the harmonics depend on the density of plasma. Where the density is greater we seen overlapping and where the density is small we have seen no overlapping. In dispersion curves plotted for $\eta = 20$(thermal energy is very small as compare to rest mass energy), we have seen their no overlapping of harmonics with others because dependence of cyclotron frequency on momentum is small and the harmonics of cyclotron stay in their own positions. When we fix η and we increase the density of plasma, the contribution of higher harmonics is due to over all increase in momentum. The particles having higher momentum stay in the higher harmonics.

5.2 Conclusion

On the non-relativistic distribution, although the velocities of particles are different, the cyclotron frequency remains the same. Therefore, non-relativistic particles tend to stay within their corresponding harmonics.
Coming to the relativistic distribution, the mass of particles also vary, depending on their velocities. The faster a particles moves, the more mass it gets. Cyclotron frequency of the heavier particles decrease as a result of which the harmonics of heavier particles tend to come down and interact with the lighter particles. Hence, the overlapping occurs within higher and lower harmonics.
When we increase the density of plasma by taking η(ratio of the rest mass energy to the thermal energy) constant. The more particles having large speed and they get more mass. So the overlapping between higher harmonics and lighter particles is increased in high density plasma. We also observed that the number of harmonics are increasing with density of plasma because particles having low speed do not affects the cyclotron frequency so their harmonics stay at their own positions.
By decreasing the thermal energies of particles($mc^2 = 20k_BT$), we observe that the relativistic effects are vanished, so all harmonics are stay at their own positions.

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