Entanglement Engineering of Field State in Arbitrary Number of High-Q Cavities and its Dynamics in Dissipative Environment

by

Zainub Liaqat

Supervised by

Dr. Ayesha Khalique

Co-supervised by

Dr. Manzoor Ikram

Submitted in the partial fulfillment of the

Degree of Master of Philosophy

In

Physics

School Of Natural Sciences,

National University of Sciences and Technology,

H-12, Islamabad, Pakistan.

FORM TH-4

National University of Sciences & Technology

M.Phil THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: Zainub Liaquat, Regn No. NUST201260320MCAMP78112F Titled: Engineering of N-partite GHZ-State and W-State using arbitrary number of high Q-cavities and their Dynamics be accepted in partial fulfillment of the requirements for the award of **M.Phil** degree.

Examination Committee Members

1. Name: Dr. Shahid Iqbal Signature: ____________
2. Name: Dr. Naeem Shahid Signature: ____________
3. Name: Dr. Rameez ul Islam Signature: ____________
4. Name: Dr. Tasawwar Abbas Signature: ____________

Co-Supervisor's Name: Dr. Manzoor Ikram Signature: ____________

Supervisor's Name: Dr. Aeysha Khalique Signature: ____________

Head of Department

10-09-2015
Date

COUNTERSINGED

Date: 10/9/15

Dean/Principal

This Dissertation is dedicated to

my sister Nida Liaqat

Contents

List of Figures

Acknowledgements

Starting with the name of ALLAH, the most beneficent and the most merciful, Who is entire source of all wisdom and knowledge endowed to mankind. All praise is to ALLAH, Whose favors are the sole cause for enabling me to carry out my research and Who bless me with so many helpful people without whom, it would have been impossible for me to carry out my work.

It is my privilege to express gratitude to my respected supervisor, Dr. Aeysha khalique for her support, guidance, encouragement, care, constant help and for providing all the facilities during my research work. She always understands my problems during my whole research work. I am also thankful to my respected co-supervisor, Prof. Dr. Manzoor Ikram. I really appreciate his kind way of encouraging and directing me when i really stuck somewhere. I am really lucky to be blessed with two the most caring and gorgeous person in the form of supervisor and co-supervisor.
It would always remain incomplete without my thanks to my parents and to my family. All the prays they made for me and stay with me in all my hurdles. I want to thank Syed tahir ameen for such a support for me and for always encouraging me. I want to thank all my respected teachers Dr. Shahid Iqbal, Dr. Qadir, Dr. Faheem Ameen, Dr. Muneer, who worked on me in my course work.

Zainub Liaqat

Abstract

We develop a generalized scheme for engineering of GHZ-state and W-state using arbitrary number of high-Q cavities. We also study about entanglement degradation due to decoherence when the entangled states interact with the environment. GHZ-states and W-state are multipartite and maximally entangled states. They are very useful in quantum networking and quantum information processing. In our model, we use different setups of high-Q cavities for the entangled field generation of the respected state and we have generalized both models. Most important role in the model is of interaction parameters which give successful generation of entangled field state by controlling atom-field interaction using these precalculated interaction times. Further, the subject of dynamics of the initial entangled field states inside the high-Q cavities surrounded by thermal environment is investigated. It is concluded that the decoherence increases with the increase of cavities. Negativity is used as a measure of entanglement of high-Q cavities.

Chapter 1

Introduction

Quantum theory is a milestone in the history of physics. It broughts a revolution in earlier physics. Modern physics based on solid state, lasers, semiconductor and superconducting devices, optics etc is the foundation of quantum mechanics. Quantum theory explains the phenomena which classical theory cannot explain. Black body radiation, Photoelectric effects, Bohr atomic model etc are the basic examples of Quantum Mechanical phenomena. Furthermore, the uncertainty principle in quantum theory implies that non-commuting observable can never have sharply defined value simultaneously and performing measurement on one observable will necessarily influence the outcome of other observable. Another distinction between classical theory and quantum theory is that acquiring information from a quantum system causes a disturbance.

The early views of quantum physics were supported by real experiments. These fundamental phenomena are now used in recent experimentations. Quantum cryptography, for example, is based on an implementation of uncertainty principle. Cryptography is a protocol based on secure key distribution between commutating parties. Cryptographic systems get their security from the hurdle of factorizing large integers. These systems are now insecure with the construction of quantum computer which can factorize large number in much faster time as compared to any classical computer. Quantum

teleportation is another application which relies on quantum entanglement. It is a transfer of quantum state from sender to receiver without transporting particle itself. The main ingredient of these communicational processes is quantum entanglement, that is prepared between communicating parties. The major achievement in the field of quantum mechanics is quantum entanglement. It shows a correlation between two distinct particles and expresses the nonlocality, inherited to quantum mechanics. Quantum entanglement has numerous applications in quantum computational and informational sciences such as teleportation, super dense coding [1] and cryptography. Moreover quantum computers are based on idea of entanglement. Shor's algorithm [2] exploits the entanglement used to solve large integer factorization in polynomial time and Grover's algorithm [3] helps to searches particular state in unorder list using $O(\sqrt{N})$ queries using entanglement instead of the $O(N)$ queries required classically.
In quantum theory there are many analogous terms that come from classical thoery, example of which is discussed in Sec. 1.1. Quantum states are explained thoroughly in Sec. 1.2. In Sec 1.3, we have explained quantum entanglement, whereas various entangled states are discussed in Sec. 1.5. Sec. 1.6 contains dynamics of entangled state.

1.1 Qubit

The basic unit of information in classical world is bit, which can have either of the two possible state 0 or 1, true or false, yes or no. In two states system, for example, a spin-$\frac{1}{2}$ particle system, bit can be in spin "down" state or in a spin "up" state. In quantum world, the elementary unit of information processing is the quantum bit or qubit. Qubit can also be in one of the two states but the difference between the classical bit and quantum bit (qubit) is that the quantum mechanical superposition gives the two-state system another possible state, which is the linear combination of the two states.

These superposition states have no classical analogue. The superposition state in spin-$\frac{1}{2}$, will be written as

$$|\psi\rangle = \alpha|\uparrow\rangle + \beta|\downarrow\rangle. \tag{1.1.1}$$

Here α and β are probabilities amplitudes and are complex in general. The sum of squares of probability amplitude always gives one, i.e.,

$$|\alpha|^2 + |\beta|^2 = 1. \tag{1.1.2}$$

1.2 Quantum States

Quantum state is defined as a linear superposition of all possible basis states in Hilbert space. Hilbert space is an infinite dimentional abstract vector space possessing the structure of an inner product. Quantum state can be in pure state or mixed state. Pure state is represented as,

$$|\psi\rangle = \sum C_{\mathrm{m}}|\psi\rangle, \tag{1.2.1}$$

where C_{m} is the probability amplitudes. Vectors in a Hilbert space (H) represent pure states, which are well suited for isolated systems. In a general, quantum mechanical system needs to describe mixed states. Therefore we define density matrix which is a matrix used when we discuss an ensemble of pure states, or when we describe quantum system of an ensemble of several quantum states . The density matrix of pure state $\rho = |\psi\rangle\langle\psi|$ always gives $\mathrm{Tr}(\rho^2) = 1$, but for mixed state $\mathrm{Tr}(\rho^2) < 1$. Density matrix for mixed state will be

$$|\rho\rangle = \sum k_i|\psi_i\rangle\langle\psi_i|, \tag{1.2.2}$$

where $\sum |k_i|^2 = 1$ and $|\psi_i\rangle$ represents many body system. Quantum systems depict properties such as superposition of quantum states, interference or tunneling, unknown for classical systems, which These are single particle effects. These are not the only difference between quantum systems and classical systems. Other difference manifests quantum object into composite system, described as a system that decomposes into atleast two subsystems. Various methods have been proposed and demonstrated for state preparation and its control.

1.2.1 Multipartite States

Bipartite states are described by the Hilbert space $H= H_1 \otimes H_2$ of two distinct subsystems, whereas Multipartite states are states that consist of more than two subsystems (or composite system). The Hilbert space associated with a multipartite system, is given by the tensor product of the spaces $H_1 \otimes \ldots \otimes H_N$ corresponding to each subsystems.

1.2.2 Cavity Field States

Cavity is a small volume space where we want to hold photons for required time period in order to achieve desired task. This captivated field of photons inside the cavity is called cavity field state. Cavity is composed of arrangement of the mirrors to confine light, which is reflected multiple times from the mirrors, hence producing standing waves of certain resonance frequencies. These reflected cavities are of different types depending on their geometry, which are plane-parallel cavities, spherical cavities, concave convex cavities, hemispherical cavities etc. Cavities are designed to have high Q-factor.

Q-factor

Q-factor is a dimensionless quantity, which describes that under damping how oscillation characterizes its bandwidth (Δf) relative to its center frequency (f_c) as shown in Fig. 1.1. High Q-factor means oscillation dies out slowly due to less energy loss of oscillator.
Various cavity field states are used in various computational tasks according to their specification. Bell's states, NOON states, GHZ-state, W-state etc

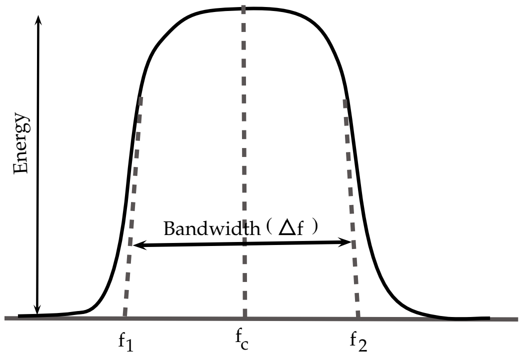

Figure 1.1: Energy verses frequency relation showing Q-factor dependence on the relation $\frac{f_c}{\Delta f}$.

are some of the examples of entangled states which can be realized in field states inside the cavities.

1.3 Quantum Entanglement

One of the most mysterious phenomenon of quantum mechanics is that quantum systems become *entangled*. When pairs or groups of particles are generated or interacted in such a way that each particle cannot be described independently as their physical properties are found to be correlated. Therefore a measurement on one local particle, gives the properties of other nonlocal particle without making measurement on it. This nonlocal property makes entanglement a very useful phenomena. Entanglement is independent of distance of separation between the particles. The most common entangled states are Bell states.

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0_1, 1_2\rangle \pm |1_1, 0_2\rangle), \tag{1.3.1}$$

$$|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2\rangle \pm |1_1, 1_2\rangle), \tag{1.3.2}$$

where 0 and 1 represent two states of two separates particles, for example 0 and 1 photon in cavities 1 and 2, respectively.

The roots of term Entanglement goes long way back when Erwin Shrodinger, called it characteristic trait of quantum mechanics. In 1935, Einstein, Podolsky and Rosen (EPR) [4] gave their famous paper in which they claimed that quantum mechanics is an incomplete theory. EPR and other realist believed that there must be some missing variable which would complete our knowledge and bring us the understanding we seek. They argued that quantum mechanics is "incomplete"in theoretical description of quantum system as the theory follows that non commuting observables can never be simultaneously "element of reality", as one observable can be predicted with certainty and other cannot. The issue that quantum mechanics is an incomplete theory remained unsolved until Bell [30] in 1964 who came with inequalities. These inequalities worked as a boundary between classical and quantum mechanical correlation for long time and are varified experimentally [7–9]. These inequalities are violated in the case of entanglement. Later, Bell and Clauser *et. al.* [10, 30] showed mathemaically that incompatible measurements (i-e non-classical measurements whose simultaneous precision is constrained by uncertainty principle) shows violation and thus entanglement is a non-classical phenomenon. Later, Werner [11] addressed the problem of hidden variables and discussed the classical states that satisfy the Bell's inequalities. There are many schemes for entanglement classification and quantification apart from Bell's inequalities which are discussed in later chapters. Entanglement is applicable in many computational and informational processes.

1.4 Entangled States

Entangled states are very widely studied due to their property of non locality. These states cannot be written in the form of product of individual state.

$$|\psi_{12}\rangle \neq |\psi_1\rangle \otimes |\psi_2\rangle. \tag{1.4.1}$$

One of the example of entangled state is of spin$-\frac{1}{2}$ particle Eq. (1.4.3) Entanglement among many parties is the most essential feature speacialy in long distance communication. Therefore engineering of multi particle state is an important problem. Many studies have been done so far on the generation of multipartite state [12–14]. In this thesis we focus on the engineering of GHZ-state and W-state and afterwards their dynamics.

1.4.1 GHZ-State

GHZ states are atleast three particle or more particle entangled state, named after scientists Greenberger, Horne, Zeilingner [15]. The general form of N-qubit GHZ states is,

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2, \ldots, 0_N\rangle + |1_1, 1_2, \ldots, 1_N\rangle). \tag{1.4.2}$$

GHZ-states are maximally entangled states as they have maximum correlation. They are widely used in qantum computation and quantum communication. One of its important property is that by tracing out one particle, results into complete collapse of entanglement.

1.4.2 W-State

W-states are N-qubit entangled state [16]. W-states between N number of particles is given as

$$|\psi\rangle = \frac{1}{\sqrt{N}}(|1_1, 0_2, \ldots, 0_N\rangle + |0_1, 1_2, \ldots, 0_N\rangle \ldots + |0_1, 0_2, \ldots, 1_N\rangle). \tag{1.4.3}$$

They are also maximally entangled state. They remain entangled even if one particle is traced out. This robustness against particle loss makes W-state used in quantum memories [17], multinodal networks [18] and teleportation [19] etc.

1.5 Decoherence

Quantum entanglement is the key of quantum information processing. But its too fragile to play any role in real world. In real world, quantum system have unwanted interactions with environment leading to entangled state degradation which results into irreversible loss of information. If system is not isolated from the environment i.e., we have open system, it would interact with the environment, and becomes entangled with the environmental degrees of freedom. This imprints a part of the system information into environment and this information can no longer be accessible from the local measurement. This is decoherence. Amplitude decoherence, caused by spontaneous emission of atom and phase decay by random disturbance of relative phase of quantum state, are the typical example of decoherence.

1.6 Entanglement Dynamics

The exposure of entangled systems to the noisy environment leads to entanglement decay. The decay is asymptotic, but not always. An unusual phenomena occurs in which a sudden decay of entanglement takes place, termed as Entanglement Sudden Death(ESD) or Early stage disentanglement. This decay of entanglement in a finite time is against the half-life law where Half-life law says that decay undergoes exponentially. Yu and Eberly [20] were the first who studied ESD for two level system. It is observed that sudden death time increases with increasing number of photons. One can not recover entanglement even through error correction after ESD.

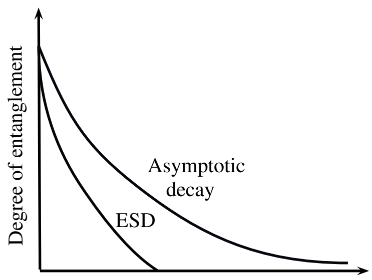

Figure 1.2: Curves shows ESD and asymptotic decay

Amount of entanglement can be measured after decoherence. There are various method to measure entanglement like concurrence [21], von Neumann

Entropy [22, 23] and negativity.

1.7 Quantitative Measurement of Entanglement

This section is about the measures that enable us in determination of the quantitative degree of entanglement. Sec 1.7.1 and Sec 1.7.2 contains concurrence and von neuman entropy which are valid for bipartite system only. However Sec. 1.7.3 covers negativity, valid for higher dimentional system. All these three measures gives same value for separable states and maximally entangled states.

1.7.1 Wooters Concurrence

Concurrence is based on flip and parity operator, defined as

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Concurrence is defined only upto two particle system. For a single particle system concurrence is define as $c = \langle\psi|\tilde{\psi}\rangle$ where $|\tilde{\psi}\rangle = \sigma_y|\psi\rangle$. For two particle system, we define a 4×4 matrix A,

$$A(t) = \rho(t)(\sigma_y \otimes \sigma_y)\rho^*(t)(\sigma_y \otimes \sigma_y), \tag{1.7.1}$$

where ρ is the 2×2 time dependent density matrix. Square roots of eigenvalues of matrix A define concurrence as

$$C = max[0, \wedge(t)], \tag{1.7.2}$$

where $\wedge = \sqrt{\lambda_1(t)} - \sqrt{\lambda_2(t)} - \sqrt{\lambda_3(t)} - \sqrt{\lambda_4(t)}$, and λ_i are the eigenvalues of the density matrix with λ_1 being the largest one. Concurrence, C(t)= 0 when states are separable and C(t)= 1 for maximally entangled states.

1.7.2 von Neumann Entropy

For a quantum-mechanical system described by a density matrix ρ, the von Neumann entropy is

$$S = -tr(\rho ln\rho), \tag{1.7.3}$$

It can be written in terms of eigenvalues (λ_i) of density matrix ρ

$$S = -\sum_i \lambda_i log_2 \lambda_i. \tag{1.7.4}$$

$S = 0$ for separable states and $S = 1$ for a maximally entangled states.

1.7.3 Negativity

For higher dimensional systems of mixed states, it is difficult to quantify the entanglement due to the complex geometry of the state. Negativity is based on Partial Positive Transpose (PPT) criteria. It is the sum of all negative eigenvalues of partial transposed matrix ρ^{PT}. So negativity is defined as

$$N(t) = -2\sum_i \lambda_i, \tag{1.7.5}$$

where λ_i are all the negative eigenvalues of partial transpose matrix. Entanglement can be define in terms of negativity as

$$E(t) = Max[0, N(t)]. \tag{1.7.6}$$

$E(t) = 0$ for states which are seoparate and $E(t) = 1$ for maximally entangled states.

Positive Partial Transpose (PPT)

Partial transpose is Peres-Horodecki criterion, used to find the separability of the system. For two systems, for example, of Hilbert space $H_1 \otimes H_2$, density matrix is written as

$$\rho = \Sigma_{ijkl} A^{ij}_{kl} |i\rangle\langle j| \otimes |k\rangle\langle l|. \tag{1.7.7}$$

Its partial transpose with respect to system-1 is defined as

$$\rho^{T_1} = \Sigma_{ijkl} A^{ij}_{kl} |i\rangle\langle j| \otimes |l\rangle\langle k|. \tag{1.7.8}$$

We worked in higher dimentions so we used negativity as a measure of entanglement, based on Positive Partial Transpose criteria. Negativity is a useful technique to quantify the entanglement among multipartite [24] quantum states.

1.8 Thesis Layout

This thesis is organized as follow. Chapter 2 is based on the generation of entangled state. Starting with the Hamiltonian of the system in the interaction picture, we describe the method of entangled field generation using two cavities having fixed number of photons. Then we have discussed the procedure of generating GHZ-state and W-state using three cavities and their dynamics. Both schemes work only when atom is detected in ground state after passing through the cavities. Atoms interact with each cavity field for precalculated time. Further we have derived master equation for the field inside the cavities surrounding the thermal environment (reservoir). We also discussed the methods of measuring amount of entanglement after interacting with environment.

In chapter 3, we discuss the details of scheme of generating the GHZ-state using four high Q-cavities first and generalize the scheme for arbitrary number of cavities. We select the interaction times of atom-field interaction in such a way that either it would lead to all the cavities having no photon or all the cavities having one photon each. We also examine that how GHZ-state evolved when it comes in contact with the environment. Negativity is used as a separability criteria. At the end, we plot our results.

In Chapter 4, we present a method of generating W-state using four high Q-cavities and generalize the scheme. The interaction times are selected in such a way that they would give one photon in either of the cavity. We examine the behaviour of GHZ-state in dissipative environment as for GHZ-state. Finally we conclude in chapter 5 that we can successfuly engineer GHZ-state and W-state for any number of cavities using our model by avoiding those condition on which these schemes fail.

Chapter 2

Cavity Field Entanglement: A Review

This chapter is based on review of atom field interaction. In this chapter, we discuss system base on atom-field interaction including its Hamiltonian and its evolution. Then we discuss entanglement generation of two separate high Q-cavities by sending N numbers of atom [25]. In the end, we study generation of tripartite GHZ-state and W-state using three high Q-cavities [26] and their interaction times. By controlling these interaction times, we can generate our desired state. We see in first chapter that entanglement is extensively used in quantum information and computational processes such as teleportation, cryptography [27, 28] and super dense coding [29] etc. It is extensively studied nowadays. The very famous example of bipartite entangled state are Bell states [30–32],

$$|\psi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0_1, 1_2\rangle \pm |1_1, 0_2\rangle), \tag{2.0.1}$$

$$|\phi^{\pm}\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2\rangle \pm |1_1, 1_2\rangle), \tag{2.0.2}$$

where 1 and 2 in subscript are the two particles. Bell states are maximally entangled states as these have the maximum correlations among them. Other example of entangled state is tripartite W-state

$$|\psi\rangle_W = \frac{1}{\sqrt{3}}(|1_1, 0_2, 0_3\rangle + |0_1, 1_2, 0_3\rangle + |0_1, 0_2, 1_3\rangle). \tag{2.0.3}$$

Many schemes has been proposed for the generation of entanglement. Since we are dealing with the cavities entangled states so we will discuss schemes involving cavities. The generation of entangled state for one photon in each of two cavities (EPR state) was presented by Davidovich *et. al.* [33]. It was first time when entanglement of two cavities is seen as an intermediate step in the scheme for the teleportation of atomic state. The two-level atom in excited state $|e\rangle$ is sent through two cavities in vacuum state where it is made resonant with the cavities field. The interaction time is set in such a way that atom makes transition ($|e\rangle \longrightarrow |g\rangle$) by having $\pi/2$-pulse (explained in Sec. 2.2) in cavity-1 and then π-pulse in cavity-2. Atom-field state would be

$$|\psi\rangle \xrightarrow{\frac{\pi}{2}-pulse} \frac{1}{\sqrt{2}}(|e, 0_1\rangle + |g, 1_1\rangle) \otimes |0_2\rangle \xrightarrow{\pi-pulse} \frac{1}{\sqrt{2}}(|g, 0_1, 1_2\rangle + |g, 1_1, 0_2\rangle).$$

Atomic state $|g\rangle$ can be traced out leaving the field state inside the two cavities as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0_1, 1_2\rangle + |1_1, 0_2\rangle). \tag{2.0.4}$$

Another scheme, proposed by Bergoue [34] for maximally entanglement generation of two cavities in which either there is no photon (vacuum) in cavities or one photon in all cavities. This scheme begins with two-level atom taken in excited state $|e\rangle$ and the two cavities in vacuum state. Atom is sent first through a region of classical field (Ramsey zone) where they interact with $\epsilon_o gT = \pi/4$. This interaction puts the atom in coherent superposition as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle). \tag{2.0.5}$$

Atom is sent through first cavity by adjusting the interaction time $gT = \pi/2$, giving the atom field state

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0_1\rangle + |1_1\rangle) \otimes |0_2\rangle \otimes |g\rangle. \tag{2.0.6}$$

This becomes initial condition for the second atom. A second atom in excited state is first sent through Ramzey zone. This zone prepares the atom in the state $|+\rangle = \frac{1}{\sqrt{2}}(|g\rangle + |e\rangle)$. Atom is then passed through first cavity where it interacts off resonantly with interaction time $(\Omega^2/\delta)T = \pi$ where δ is the detuning between atomic transition frequency and cavity-1 frequency and Ω is a rabi frequency. This all gives the atom-field state as

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0_1\rangle|+\rangle + |1_1\rangle|-\rangle) \otimes |0_2\rangle, \tag{2.0.7}$$

where $|-\rangle = \frac{1}{\sqrt{2}}(|g\rangle - |e\rangle)$. The atom then passes through the second Ramzey zone with interaction time $\epsilon_o gT/2 = \pi/4$ which has the effect $|-\rangle \longrightarrow -|e\rangle$ and $|+\rangle \longrightarrow |g\rangle$. So Eq. (2.0.7) becomes

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0_1, g\rangle + |1_1, e\rangle) \otimes |0_2\rangle. \tag{2.0.8}$$

Finally the atom passes through the cavity-2 with interaction time $gT = \pi/2$, giving highly entangled state with atom in ground state which can be then discarded. The resulting state we get is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2\rangle + |1_1, 1_2\rangle). \tag{2.0.9}$$

In this chapter we will discuss methods of entangled field state generation, using two cavities for fix number of photons studied by Manzoor Ikram *et. al.* [25] and then tripartite entanglement generation.

2.1 Atom Field Interaction

When single two-level atom interacts with the single mode field, the total Hamiltonian of the system would be given by

$$H = H_o + H_i, \tag{2.1.1}$$

where non interacting part of Hamiltonian is given as

$$H_o = \hbar\omega_f a^\dagger a + \frac{1}{2}\hbar\omega_a\sigma_z, \tag{2.1.2}$$

where ω_f is the field frequency, ω_a is the atom frequency and a($a^\dagger$) is the annihilation(creation) operator. Interacting part of Hamiltonian is given as

$$H_i = \hbar g(\sigma_+ a + a^\dagger \sigma_-), \tag{2.1.3}$$

where σ_+ and σ_- are the raising and lowering operator and g is the coupling constant. In the interaction picture, the Hamiltonian of the system is given as

$$H_{int} = e^{\frac{iH_o t}{\hbar}} H_i e^{-\frac{iH_o t}{\hbar}}. \tag{2.1.4}$$

Using Bakers-Campbell-Hausdorff formula,

$$e^{\alpha B} A e^{-\alpha B} = A + \alpha[B, A] + \frac{\alpha^2}{2!}[B, [B, A]] + \frac{\alpha^3}{3!}[B, [B, [B, A]]] + ... \tag{2.1.5}$$

The Hamiltonian in the Interaction picture becomes

$$H_{int} = \hbar g(\sigma_+ a e^{i\Delta t} + a^\dagger \sigma_- e^{-i\Delta t}), \tag{2.1.6}$$

where $g = -\frac{\hat{\epsilon} e\langle i|r|j\rangle}{\hbar}(\frac{\hbar\omega_f}{2\epsilon_o V})^{\frac{1}{2}}$ is the coupling constant and $\Delta = \omega_f - \omega_a$ is the detunning. For resonance, we have $\omega_f = \omega_a$.

In literature, there are three equivalent methods to solve evolution of atom-field system using their Hamiltonian i.e the probability amplitude method, Heisenberg operator method and unitary time-evolution method. We will use the unitary time evolution operator method [36]. Unitary operator is defined as

$$U(t) = e^{-\frac{iHt}{\hbar}}. \tag{2.1.7}$$

Using it, the unitary operator for the system of Hamiltonian (2.4.22).

$$\begin{aligned} U(T_i) \quad &= \quad \cos(g_i T_i \sqrt{a_i^\dagger a_i + 1})|e\rangle\langle e| + \cos(g_i T_i \sqrt{a_i^\dagger a_i})|g\rangle\langle g| \\ &\quad -i\frac{\sin(g_i T_i \sqrt{a_i^\dagger a_i + 1})}{\sqrt{a_i^\dagger a_i + 1}} a|e\rangle\langle g| - ia^\dagger \frac{\sin(g_i T_i \sqrt{a_i^\dagger a_i + 1}}{\sqrt{a_i^\dagger a_i + 1}})|g\rangle\langle e|, \end{aligned} \tag{2.1.8}$$

where g_i is the coupling constant of ith atom with the field, T_i is the interaction time of an atom with the field, $a_i^\dagger$ and a_i are the creation and annihilation operator of field. Several schemes have been suggested for the preparation of quantum state in cavities.

2.2 $\frac{\pi}{2}$-pulse and π-pulse

Let the the atom in excited state $|e\rangle$, enter the empty cavity. The probability of atom in the same state $|e\rangle$ after the interaction time 't'is then an oscillatory function, known as Rabi oscillations. In this evolution, some specific interaction times show interesting operation.

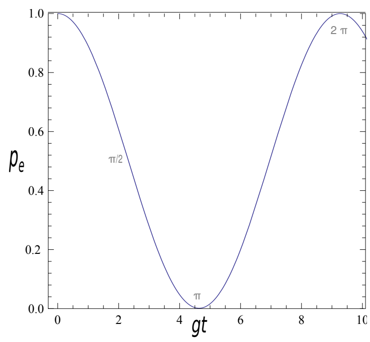

Figure 2.1: The plot shows a single period of Rabi oscillalation.

2.2.1 $\frac{\pi}{2}$-pulse

In Rabi oscillation, an effective interaction time of a quarter of period (as shown in Fig. 2.1) performs the following transformation.

$$|e,0\rangle \longrightarrow \frac{1}{\sqrt{2}}(|e,0\rangle + |g,1\rangle). \tag{2.2.1}$$

Such transformation is named as $\pi/2$-pulse. This pulse creates the coherence among the atom and the field in the cavity.

2.2.2 π-pulse

An effective interaction time to a half period of Rabi oscillation performs the transformation

$$|e,0\rangle \longrightarrow |g,1\rangle,$$
$$|g,1\rangle \longrightarrow -|e,0\rangle.$$

Therefore, it is termed as π-pulse. It flips one state into the other state.

2.3 Entanglement Generation Between Two Separate Cavities

Entanglement generation of two separate cavity fields for arbitrary number of photons is being used in many communication processes (as in teleportation) [25]. Entangled field state for fixed number of photons in two cavities is described as

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{N+1}}\sum_{m=0}^{N}|m_1,(N-m)_2\rangle. \tag{2.3.1}$$

This entangled state shows fixed number of N-photons in cavity-1 and cavity-2. Consider N two-level atoms and two cavities (cavity-1 and cavity-2) initially in vacuum state. These N atoms, sent in excited state $|e\rangle$ through the cavities where the atom and the cavities field become resonant with each other through stark field adjustment [33]. They interact through Jaynes-Cumming Hamiltonian [35]. After interaction, atoms should be detected in ground state, a necessary condition for the proposed entangled state. Atoms are sent one by one from both cavities. Initial atom-field state will be

$$|\psi^{(1)}_{T=0}(1,2)\rangle = |e,0_1,0_2\rangle. \tag{2.3.2}$$

Now first atom is sent through cavity-1 and cavity-2. After passing through both cavities and if it is detected in ground state (a necessary condition), the final atom-field state will be

$$|\psi^{(1)}_{T=0}(1,2)\rangle = E^{(1)}_{0,0}|e,0_1,0_2\rangle + G^{(1)}_{0,1}|g,0_1,1_2\rangle + G^{(1)}_{1,0}|g,1_1,0_2\rangle, \tag{2.3.3}$$

here $E^1_{0,0}, G^1_{0,1}, G^1_{1,0}$ are the probability amplitudes for excited and ground state, define as

$$E^{(1)}_{0,0} = \cos(gT_{11})\cos(gT_{12}), \tag{2.3.4}$$

$$G^{(1)}_{0,1} = -i\cos(gT_{11})\cos(gT_{12}), \tag{2.3.5}$$

$$G^{(1)}_{1,0} = -i\sin(gT_{12}), \tag{2.3.6}$$

where superscript 1 represent the interaction of first atom while T_{ij} represents the interaction time of ith atom with jth cavity. The necessary condition for the generation of required entangled state is detection of atom in ground state $|g\rangle$. A normalized joint state of two cavities

$$|\psi^1_T(1,2)\rangle = S_1[G^{(1)}_{1,0}|1_1,0_2\rangle + G^{(1)}_{0,1}|0_1,1_2\rangle], \tag{2.3.7}$$

here, S_1 is a normalization constant, defined as

$$S_1 = \frac{1}{\sqrt{|G^{(1)}_{1,0}|^2 + |G^{(1)}_{0,1}|^2}}. \tag{2.3.8}$$

Now we want to send second atom to create entangled field state with higher number of photon state. For second atom Eq. (2.3.7) is the initial state condition. For Nth atom, the initial condition would be

$$|\psi^N_{T=0}(1,2)\rangle = S_{N-1}\sum_{m=0}^{N-1} G^{(N-1)}_{(m,N-1-m)}|g,m_1,(N-1-m)_2\rangle. \tag{2.3.9}$$

The evolved state of Nth atom after interaction

$$\begin{aligned}|\psi^N_{T=0}(1,2)\rangle &= S_{N-1}[\sum_{m=0}^{N-1} E^{(N)}_{(m,N-1-m)}|e,m_1,(N-1-m)_2\rangle \\ &\quad + \sum_{m=0}^{N} G^{(N)}_{(m,N-m)}|g,m_1,(N-m)_2\rangle],\end{aligned} \tag{2.3.10}$$

where the probability amplitudes are

$$\begin{aligned} E^{(N)}_{(m,N-1-m)} &= S_{N-1}[G^{(N-1)}_{(m,N-1-m)}\cos(gT_{N1}\sqrt{m+1})\cos(gT_{N2}\sqrt{N-m}) \\ &\quad +G^{(N-1)}_{(m-1,N-m)}\sin(gT_{N1}\sqrt{m})\sin(gT_{N-m}\sqrt{N-m}), \end{aligned} \tag{2.3.11}$$

$$\begin{aligned} G^{(N)}_{(m,N-m)} &= -iS_{N-1}[G^{(N-1)}_{(m,N-1-m)}\cos(gT_{N1}\sqrt{m+1})\sin(gT_{N2}\sqrt{N-m}) \\ &\quad +G^{(N-1)}_{(m-1,N-m)}\sin(gT_{N1}\sqrt{m})\cos(gT_{N-m}\sqrt{N-m}), \end{aligned} \tag{2.3.12}$$

where probability amplitudes of excited states have $m = 0, 1, 2, ..., N$-1 and $m = 0, 1, 2, ..., N$ for the probability amplitudes of ground state.

2.3.1 Calculating Interaction Times

For N atoms we required $2N$ interaction parameters. Using the condition that the probability amplitude of ground state of final atom are same, which gives N equations

$$G^{(N)}_{(0,N)} = G^{(N)}_{(m,N-m)}. \tag{2.3.13}$$

For the rest of N equations, we select

$$E^{(m)}_{(0,m-1)} = 0, \tag{2.3.14}$$

where $m = 1, 2, ..., N$. Probability of detecting Nth atom in ground state decreases on increasing a photon number in the cavities and is defined as

$$P_{g_N} = \sum_{m=0}^{N} |G^N_{m,N-m}|^2. \tag{2.3.15}$$

The probability of detecting all N atom in ground state can be determined

$$P_g = \prod_{m=0}^{N}\sum_{m=0}^{N} |G^N_{m,N-m}|^2. \tag{2.3.16}$$

2.4 Entanglement Generation Of Tripartite State

2.4.1 GHZ-State

Scheme for entanglement generation of GHZ-state was studied by B. Farooq *et. al.* [26]. The tripartite GHZ-state generated is

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2, 0_3\rangle + |1_1, 1_2, 1_3\rangle). \tag{2.4.1}$$

They proposed passage of two atoms initially prepared in ground states $|g\rangle$ through the 3 cavities is shown in Fig. 2.2 such that cavity-1 is in superposition $\frac{1}{2}(|0\rangle + |3\rangle)$ and other two cavities in vacuum. First atom is sent in

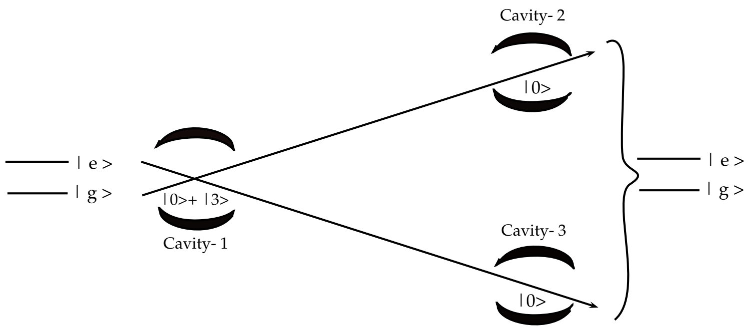

Figure 2.2: A model for Tripartite GHZ-state.

ground state through cavity-1 and then it is passed through cavity-2, interacting with the cavities resonant modes. After passing through the cavities, atom should be detected in ground state, a necessary condition of the scheme otherwise we have to start all over again. The initial atom-field state will be

$$|\psi^{(1)}_{T=0}\rangle = |g_1, g_2, g_3\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle_1 + |3\rangle_1) \otimes |0_2, 0_3\rangle. \tag{2.4.2}$$

After the passage of atom through the cavity-1 and cavity-2, the evolved state using unitary operator described in Eq. (2.1.8), would be

$$|\psi^{(1)}_{T_1,T_2}\rangle = U_2 U_1 |\psi^{(1)}_{T=0}\rangle. \tag{2.4.3}$$

Similarly second atom will be sent but this time through the cavity-1 and cavity-3. For second atom Eq. (2.4.2) would be initial state. The evolved state for second atom is given as

$$|\psi^{(2)}_{T_1,T_3}\rangle = U_3U_1|\psi^{(2)}_{T_1,T_2=0}\rangle. \tag{2.4.4}$$

Final state of the system will be

$$\begin{aligned} |\psi_{T_{11},T_{12},T_{21},T_{23}}\rangle \;=\;& E_{100}|e_1,e_2,1_1,0_2,0_3\rangle + A_{101}|e_1,g_2,1_1,0_2,1_3\rangle \\ & +B_{110}|g_1,e_2,1_1,1_2,0_3\rangle + A_{200}|e_1,g_2,2_1,0_2,0_3\rangle \\ & +B_{200}|g_1,e_2,2_1,0_2,0_3\rangle + G_{000}|g_1,g_2,0_1,0_2,0_3\rangle \\ & +G_{210}|g_1,g_2,2_1,1_2,0_3\rangle + G_{201}|g_1,g_2,2_1,0_2,l_3\rangle \\ & +G_{300}|g_1,g_2,3_1,01_2,0_3\rangle + G_{111}|g_1,g_2,1_1,1_2,1_3\rangle, \end{aligned} \tag{2.4.5}$$

where E's, G's, A's and B's are the probability amplitude given as

$$E_{100} = -\sin(gT_{11}\sqrt{3})\cos(gT_{12})\sin(gT_{21}\sqrt{2})\cos(gT_{23}), \tag{2.4.6}$$

$$A_{101} = i\sin(gT_{11}\sqrt{3})\cos(gT_{12})\sin(gT_{21}\sqrt{2})\sin(gT_{23}), \tag{2.4.7}$$

$$A_{200} = -i\sin(gT_{11}\sqrt{3})\cos(gT_{12})\cos(gT_{21}\sqrt{2}), \tag{2.4.8}$$

$$B_{110} = i\sin(gT_{11}\sqrt{3})\sin_(gT_{12})\sin(gT_{21}\sqrt{2})\cos(gT_{23}), \tag{2.4.9}$$

$$B_{200} = -i\cos(gT_{11}\sqrt{3})\sin(gT_{12}\sqrt{3})\cos(gT_{21}\sqrt{2}), \tag{2.4.10}$$

$$G_{210} = -\sin(gT_{11}\sqrt{3})\sin_(gT_{12})\cos(gT_{21}\sqrt{2}), \tag{2.4.11}$$

$$G_{201} = -\cos(gT_{11}\sqrt{3})\sin(gT_{21}\sqrt{3})\sin(gT_{23}), \tag{2.4.12}$$

$$G_{000} = \cos(gT_{11}\sqrt{3})\cos(gT_2\sqrt{3}), \tag{2.4.13}$$

$$G_{111} = \sin(gT_{11}\sqrt{3})\sin_(gT_{12})\sin(gT_{21}\sqrt{2})\sin(gT_{23}). \tag{2.4.14}$$

2.4.2 Calculating Interaction Times

The necessary condition for generation of required entangled state is detection of both the atoms in ground state. Further by considering the following assumptions, we can calculate interaction times. We set initial condition such that the probability amplitudes of the required state survive and the rest become zero. This leads to the following conditions

$$G_{000} = G_{111}, \tag{2.4.15}$$

$$G_{100} = G_{010} = G_{001} = 0. \tag{2.4.16}$$

Moreover, the probability of detecting atom in the ground state is

$$P_g = 1 - |E_{000}|^2. \tag{2.4.17}$$

Using these condition, we get the interaction times as follow: $gT_{11} = 0.9069$, $gT_{12} = 15708$, $gT_{21} = 1.11072$, $gT_{23} = 1.5708$. Therefore by controlling these interaction durations, we can generate our required GHZ-state with unit probability.

2.4.3 Dynamics in Dissipative Environment

Once entanglement is generated, its dynamics is required for quantum information processing. Therefore, the behaviour of tripartite GHZ-states when it comes in contact with the environment has been studied. The environment behaves as a reservoir for the system. For entanglement dynamics, we need master equation for the cavity fields [38]. Master equation can be derived considering general reservoir theory. The equation of motion for the system-reservoir density matrix ρ_{SR} is given by

$$i\hbar\dot{\rho}_{SR} = [H_{IP}(t), \rho_{SR}(t)]. \tag{2.4.18}$$

This equation is integrated and on substituting back in Eq. (2.4.18), we got

$$\dot{\rho}_{SR} = -\frac{i}{\hbar}[H_{IP}(t), \rho_{SR}(t_i)] - \frac{1}{\hbar^2}\int_{t_i}^{t}[H_{IP}(t), [H_{IP}(t^{'}), \rho_{SR}(t^{'})]]dt^{'}. \tag{2.4.19}$$

Here t_i is the initial time of system-reservoir interaction. We define the solution of $\rho_{SR}(t) = \rho_S(t) \otimes \rho_R(t_i) + \rho_c(t)$ where ρ_c(t) is higher order term in H_{IP}. Assuming reservoir in equilibrium and taking $Tr_R[\rho_c(t)] = 0$, we traced out reservoir modes and arrive at the following reduce density matrix for the system(i-e field)

$$\dot{\rho}_S = -\frac{i}{\hbar}Tr_R[H_{IP}(t), \rho_S(t_i) \otimes \rho_R(t_i)] \tag{2.4.20}$$

$$-\frac{i}{\hbar^2}Tr_R\int_{t_i}^{t}[H_{IP}(t), [H_{IP}(t^{'}), \rho_S(t^{'}) \otimes \rho_R(t_i)]]dt^{'}. \tag{2.4.21}$$

We know Hamiltonian for system-reservoir is defined as

$$H_{int} = \hbar g(b_k^{\dagger}ae^{i\Delta t} + a^{\dagger}b_k e^{-i\Delta t}), \tag{2.4.22}$$

where k represent modes or reservoir. Using the Hamiltonian and Eq. (2.4.20)

$$\dot{\rho} = -\sum_{i=1}^{N}\frac{\kappa_i}{2}\bar{n}(a_i a_i^{\dagger}\rho - 2a_i^{\dagger}\rho a_i + \rho a_i a_i^{\dagger}) - \sum_{i=1}^{N}\frac{\kappa_i}{2}(\bar{n}+1)(a_i^{\dagger}a_i\rho - 2a_i\rho a_i^{\dagger} + \rho a_i^{\dagger}a_i), \tag{2.4.23}$$

where κ_i is the decay rate of ith cavity, $\bar{n}$ is the mean number of photons in the reservoir which is related with the temperature of the environment by the following relation

$$\bar{n} = \frac{1}{exp(\frac{\hbar}{k_B T\nu}) - 1}. \tag{2.4.24}$$

(2.4.25)

2.4.4 W-State

A scheme for tripartite W-state has also been presented by B. Farooq *et. al.* [26]. The tripartite W-state is given as

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|1_1, 0_2, 0_3\rangle + |0_1, 1_2, 0_3\rangle + |0_1, 0_2, 1_3\rangle). \qquad (2.4.26)$$

Two-level atom taken in excited state $|e\rangle$ is sent through the cavities arranged in the order shown in Fig. 2.3

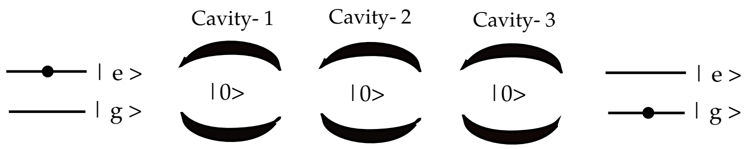

Figure 2.3: Model to generate Tripartite W-state.

Initially all the three cavities are in vacuum state. The two-level atom is taken resonant with the cavities modes. Atom should be detected in ground state $|g\rangle$, after passing through the cavities, a necessary condition of the given scheme. Initial atom-field state would be

$$|\psi_{(T=0)}\rangle = |e, 0_1, 0_2, 0_3\rangle. \qquad (2.4.27)$$

After the passage of atom through cavity-1, cavity-2 and cavity-3, the state evolves as

$$|\psi_{T_1,T_2,T_3}\rangle = U_{T_1} U_{T_2} U_{T_3} |\psi_{T=0}\rangle, \qquad (2.4.28)$$

where U_i is defined in Eq. (2.4.22), giving

$$\begin{aligned} |\psi_{T_1,T_2,T_3}\rangle &= E_{000}|e, 0_1, 0_2, 0_3\rangle + G_{100}|g, 1_1, 0_2, 0_3\rangle \\ &+ G_{010}|g, 0_1, 1_2, 0_3\rangle + G_{001}|g, 0_1, 0_2, 1_3\rangle, \end{aligned} \qquad (2.4.29)$$

where

$$E_{000} = \cos(gT_1)\cos(gT_2)\cos(gT3), \qquad (2.4.30)$$

$$G_{001} = -i\cos(gT_1)\cos(gT_2)\sin(gT_3), \tag{2.4.31}$$

$$G_{010} = -i\cos(gT_1)\sin(gT_2), \tag{2.4.32}$$

$$G_{100} = -i\sin(gT_1), \tag{2.4.33}$$

Probability of atom being detected in ground state is given as

$$P_g = |G_{001}|^2 + |G_{010}|^2 + |G_{100}|^2. \tag{2.4.34}$$

The normalized state of the atom being detected in ground sate $|g\rangle$ becomes

$$\begin{aligned} |\psi_{T_1,T_2,T_3}\rangle &= \mathrm{S}[G_{100}|g,1_1,0_2,0_3\rangle + G_{010}|g,0_1,1_2,0_3\rangle \\ &\quad + G_{001}|g,0_1,0_2,1_3\rangle], \end{aligned} \tag{2.4.35}$$

where S is the normalization constant defined as

$$\mathrm{S} = \frac{1}{\sqrt{|G_{100}|^2 + |G_{010}|^2 + |G_{001}|^2}}. \tag{2.4.36}$$

2.4.5 Interation Times

In order to calculate the interaction time, we keep the probability of atom in ground state maximum, given in Eq. (2.4.34). We take all the probability amplitudes of ground state equal as necessary condition for the generation of required W-state and probability amplitude of excited state to be zero, that is

$$G_{100} = G_{010}, \tag{2.4.37}$$

$$G_{010} = G_{001}, \tag{2.4.38}$$

$$E_{000} = 0. \tag{2.4.39}$$

This will give interaction times with Probability (P_g) maximum to unity. Interaction times are as follows; $gT_1 = 0.61548$, $gT_2 = 0.785398$, $gT_3 = 1.5708$. These interaction times give successful generation of W-state.

In this chapter, we have provided a background of our system, based on atom-field interaction. We have reviewed the entanglement generation and their dynamics in tripartite GHZ-state and W-state. In next chapter, we will present a generalized scheme for GHZ-state using arbitrary number of cavities and will discuss their dynamics.

Chapter 3

Entanglement Engineering of GHZ-state in Arbitrary Number of High-Q Cavities and its Dynamics in Dissipative Environment

The Engineering of entanglement has obvious importance as entanglement is a huge source in Quantum Information Processes (QIP). It has diverse uses specially in long distance communication where entanglement is created in multipartite systems. The novel properties of such multipartite states have been explored extensively in the recent past years concerning quantum communication. In the previous chapter, we considered different model for producing entangled state. In this chapter, our intention is to present a generalize scheme for the production of GHZ-state. Owing to their importance, various experiments have been reported for GHZ-state engineering using the technologies like ion traps, NMR, cavity QED [37]. Our scheme is based on two-level atom interaction with the cavity field for precalculated interaction time. It is shown that different arrangements of cavities provide us different partite GHZ-state. We also discuss its behaviour in dissipative environment.

The general form of maximally GHZ-state is

$$|\psi_{(GHZ)}\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2, ..., 0_N\rangle + |1_1, 1_2, ..., 1_N\rangle). \tag{3.0.1}$$

We consider a system in which N_C-cavities are arranged in the order, as in Fig. 3.1. All cavities are initially in vacuum state except the first cavity

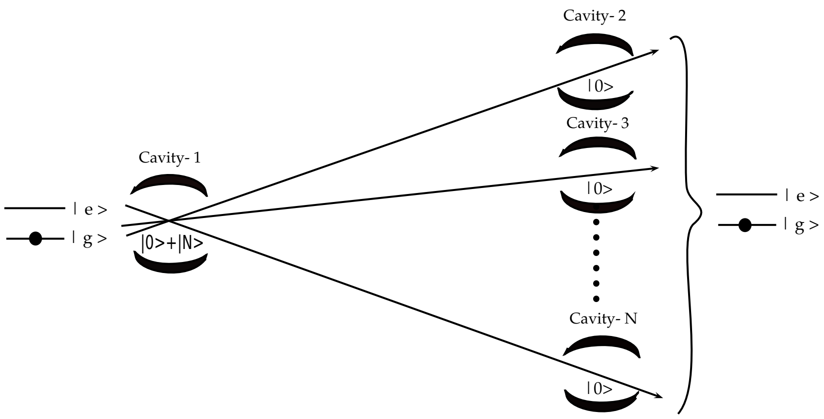

Figure 3.1: Model for N-cavities arranged in such a order to produce GHZ-state.

(cavity-1) which is in a coherent superposition of $|0\rangle$ and $|N_C\rangle$. System has $(N_C - 1)$ number of two level atoms, sent through the cavities, where it interacts with the resonant modes of cavities through the Jaynes Cumming Hamiltonian. However by controlling the interaction time of atoms with the cavities, we can generate GHZ-state. Since for N_C-cavities, we have N_A atoms and $2N_A$ parameters, two for each atom in the respected two cavities. These parameter are chosen in such a way that if any atom (sent in ground) is found in ground state after passing through the cavities, the GHZ-state will be generated. However, if any atom is detected in excited state, we have to vacate the cavities and start all over again. This is necessary condition of our scheme. For three cavities, the scheme is already proposed. In Sec. 3.1, we start with four cavities and then generalize it for N-cavities. Their interaction times are calculated in Sec. 3.2, giving us GHZ-state. At the end of this chapter, we explored the dynamics of GHZ-state given in Sec. 3.3.

3.1 Generation Of GHZ-State In Four Separate Cavities

To generate GHZ-state using four cavities, we can produce the following GHZ-state.

$$|\psi_{1234}\rangle = \frac{1}{\sqrt{2}}(|0_1, 0_2, 0_3, 0_4\rangle + |1_1, 1_2, 1_3, 1_4\rangle). \tag{3.1.1}$$

We pass atom in ground state $|g\rangle$ through the identical cavities for precalculated interaction time. Since we have four cavities ($N_C = 4$), therefore number of atoms would be $N_A = N_C - 1$ i.e., 3. First atom is sent in ground state through cavity-1 and cavity-2 as shown in Fig. 3.3. Therefore initially,

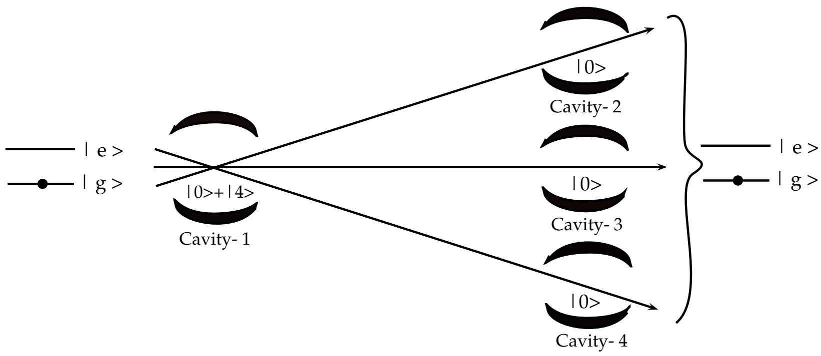

Figure 3.2: A Model to generate four particles GHZ-state.

atom-field state is

$$|\psi^{(1)}_{T=0}\rangle = |g\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle_1 + |4\rangle_1) \otimes |0\rangle_2. \tag{3.1.2}$$

Using Unitary time evolution approach, the initial state evolves to

$$|\psi^{(1)}_T(11, 12)\rangle = U_{T_{12}} U_{T_{11}} |\psi^{(1)}_{(T=0)}\rangle, \tag{3.1.3}$$

where, T_{11} and T_{12} are the interaction time of first atom with cavity-1 and cavity-2 respectively. The first subindex shows number of atom and second subindex shows cavities number whereas $U_{T_{11}}$ and $U_{T_{12}}$ are the unitary operator by Eq. (1.4.3). It evolves as

$$\begin{aligned}
|\psi^{(1)}_{T(11,12)}\rangle &= \frac{1}{\sqrt{2}}(|g_1, 0_1, 0_2\rangle - i\cos(gT_{12})\sin(gT_{11}\sqrt{4})|e_1, 3_1, 0_2\rangle \\
&\quad + \cos(gT_{12}\sqrt{4})|g_1, 4_1, 0_2\rangle - \sin(gT_{12})\sin(gT_{11}\sqrt{4})|g_1, 3_1, 1_2\rangle) \\
&\quad \otimes(|g_2, g_3\rangle \otimes |0_3, 0_4\rangle). \qquad (3.1.4)
\end{aligned}$$

This is the evolved state of first atom after passing through cavity-1 and cavity-2 where the probability amplitudes can be define in terms of E's which is probability amplitude of excited state atom and G's, probability amplitudes of ground state atom.

$$E^{(1)}_{3,0} = -i\frac{1}{\sqrt{2}}\cos(gT_{12}\sin(gT_{11}\sqrt{4}), \qquad (3.1.5)$$

$$G^{(1)}_{0,0} = \frac{1}{\sqrt{2}}, \qquad (3.1.6)$$

$$G^{(1)}_{3,1} = -\frac{1}{\sqrt{2}}\sin(gT_{12})\sin(gT_{11}\sqrt{4}), \qquad (3.1.7)$$

$$G^{(1)}_{4,0} = \frac{1}{\sqrt{2}}\cos(gT_{11}\sqrt{4}), \qquad (3.1.8)$$

where the superscript indicates the passage of first atom. The state after passage of first atom through the cavities-1 and cavity-2 in terms of probability amplitudes as above can be written as

$$\begin{aligned}
|\psi^{(1)}_{T(11,12)}\rangle &= G_{0,0}|g_1, 0_1, 0_2, 0_3\rangle + E_{3,0}|e_1, 3_1, 0_2, 0_3\rangle \\
&\quad + G_{4,0}|g_1, 4_1, 0_2, 0_3\rangle + G_{3,1}|g_1, 3_1, 1_2, 0_3\rangle) \\
&\quad \otimes(|g_2, g_3\rangle \otimes |0_4\rangle), \qquad (3.1.9)
\end{aligned}$$

where $E^{(1)}_{3,0}$ is probability amplitude of first atom detected in excited state, with cavity-1 with three photons and cavity-2 with no photon, $G^{(1)}_{0,0}$ is the probability amplitude of atom detected in ground state, with both of the cavities with no photon, $G^{(1)}_{3,1}$ is the probability amplitude of detecting atom in ground state with cavity-1 having three photons and cavity-2 having one photon and $G^{(1)}_{4,0}$ is the probability amplitude of atom being detected in ground state with four photons in cavity-1 and no photon in cavity-2. Therefore keeping the condition that the atom should be detected in ground state after interacting with the cavities mode, the probabilities of detecting first atom in ground state can be calculated by

$$P^{(1)}_g = |G^{(1)}_{0,0}|^2 + |G^{(1)}_{3,1}|^2 + |G^{(1)}_{4,0}|^2. \tag{3.1.10}$$

The joint atom-field state becomes

$$\begin{aligned} |\psi^{(1)}_{T(11,12)}\rangle &= S_1[G^{(1)}_{0,0}|g_1, 0_1, 0_2, 0_3\rangle + G^{(1)}_{4,0}|g_1, 4_1, 0_2, 0_3\rangle \\ &\quad +G^{(1)}_{3,1}|g_1, 3_1, 1_2, 0_3\rangle \otimes \\ &\quad (|g_2, g_3\rangle \otimes |0_4\rangle)], \end{aligned} \tag{3.1.11}$$

where S_1 is the normalization constant defined as

$$S_1 = \frac{1}{\sqrt{|G^{(1)}_{0,0}|^2 + |G^{(1)}_{3,1}|^2 + |G^{(1)}_{4,0}|^2}}. \tag{3.1.12}$$

Since we want to achieve GHZ-state so we send another atom in ground state but this time through cavity-1 and cavity-3. For second atom, Eq. (3.1.11) would be initial condition. Its evolved state will be written as

$$|\psi^{(2)}_{T(21,23)}\rangle = U_3U_1|\psi^{(1)}_{T(11,12)}\rangle, \tag{3.1.13}$$

$$\begin{aligned} |\psi^{(2)}_{T(21,23)}\rangle &= S_2[G^{(2)}_{0,0}|g1, g_2, 0_1, 0_2, 0_3\rangle + G^{(2)}_{4,0}\cos(gT_{21}\sqrt{4})|g_1, g_2, 4_1, 0_2, 0_3\rangle\rangle \\ &\quad -iG^{(2)}_{4,0}\cos(gT_{23})\sin(gT_{21}\sqrt{4})|g_1, e_2, 3_1, 0_2, 0_3\rangle \\ &\quad -G^{(2)}_{4,0}\sin(gT_{(23)})\sin(gT_{21}\sqrt{4})|g_1, g_2, 3_1, 0_2, 1_3\rangle \\ &\quad -iG^{(2)}_{3,1}\cos(gT_{23})\sin(gT_{23}\sqrt{4})|g_1, e_2, 2_1, 1_2, 0_3\rangle \\ &\quad +G^{(2)}_{3,1}\cos(gT_{21}\sqrt{4})|g_1, g_2, 3_1, 1_2, 0_3\rangle \\ &\quad -iG^{(2)}_{3,1}\sin(gT_{23})\sin(gT_{21}\sqrt{3})|g_1, g_2, 2_1, 1_2, 1_3\rangle \otimes (|g_3\rangle \otimes |0_4\rangle)]. \end{aligned} \tag{3.1.14}$$

Similarly for third atom, Eq. (3.1.14) would be initial state. It will evolve as

$$\begin{aligned}
|\psi^{(3)}_{T(31,34)}\rangle \quad = \quad & S_3[G^{(2)}_{0,0,0}|g_1, g_2, g_3, 0_1, 0_2, 0_3, 0_4\rangle \\
& G^{(2)}_{3,1,0}\cos(gT_{31}\sqrt{3})\cos(gT_{21}\sqrt{3})|g_1, g_2, g_3, 3_1, 1_2, 0_3, 0_4\rangle \\
& +G^{(2)}_{4,0,0}\cos(gT_{31}\sqrt{4})|g_1, g_2, g_3, 4_1, 0_2, 0_3, 0_4\rangle \\
& +G^{(2)}_{3,0,1}\cos(gT_{31}\sqrt{3})|g_1, g_2, g_3, 3_1, 0_2, 1_3, 0_4\rangle \\
& -G^{(2)}_{4,0,0}\sin(gT_{34})\sin(gT_{31}\sqrt{4})|g_1, g_2, g_3, 3_1, 0_2, 0_3, 1_4\rangle \\
& +G^{(2)}_{2,1,1}\cos(gT_{31}\sqrt{2})|g_1, g_2, g_3, 2_1, 1_2, 1_3, 0_4\rangle \\
& -G^{(2)}_{3,1,0}\sin(gT_{34})\sin(gT_{31}\sqrt{3}|g_1, g_2, g_3, 2_1, 1_2, 0_3, 1_4\rangle \\
& -G^{(2)}_{3,0,1}\sin(gT_{34})\sin(gT_{31}\sqrt{3})|g_1, g_2, g_3, 2_1, 0_2, 1_3, 1_4\rangle \\
& -G^{(2)}_{2,1,1}\sin(gT_{34})\sin(gT_{31}\sqrt{2})]|g_1, g_2, g_3, 1_1, 1_2, 1_3, 1_4\rangle],
\end{aligned} \tag{3.1.15}$$

where the probability amplitudes of ground state are

$$G^{(3)}_{4,0,0,0} = G^{(2)}_{4,0,0}\cos(gT_{31}\sqrt{4}), \tag{3.1.16}$$

$$G^{(3)}_{3,1,0,0} = G^{(2)}_{3,1,0}\cos(gT_{31}\sqrt{3})\cos(gT_{21}\sqrt{3}), \tag{3.1.17}$$

$$G^{(3)}_{3,0,1,0} = G^{(2)}_{3,0,1}\cos(gT_{31}\sqrt{3}), \tag{3.1.18}$$

$$G^{(3)}_{3,0,0,1} = -G^{(2)}_{4,0,0}\sin(gT_{34})\sin(gT_{31}\sqrt{4}), \tag{3.1.19}$$

$$G^{(3)}_{2,1,1,0} = G^{(2)}_{2,1,1}\cos(gT_{31}\sqrt{2}), \tag{3.1.20}$$

$$G^{(3)}_{2,1,0,1} = -G^{(2)}_{3,1,0}\sin(gT_{34})\sin(gT_{31}\sqrt{3}), \tag{3.1.21}$$

$$G^{(3)}_{2,0,1,1} = -G^{(2)}_{3,0,1}\sin(gT_{34})\sin(gT_{31}\sqrt{3}), \tag{3.1.22}$$

$$G^{(3)}_{1,1,1,1} = -G^{(2)}_{2,1,1}\sin(gT_{34})\sin(gT_{31}\sqrt{2}), \tag{3.1.23}$$

$$G^{(3)}_{0,0,0,0} = 1. \tag{3.1.24}$$

The normalized state of Eq. (3.1.15)

$$\begin{aligned}|\psi^{(3)}_{T(31,34)}\rangle &= S_3[G^{(3)}_{0,0,0,0} + G^{(3)}_{4,0,0,0} + G^{(3)}_{3,1,0,0} + G^{(3)}_{3,0,1,0} + G^{(3)}_{3,0,0,1} \\ &\quad + G^{(3)}_{2,1,1,0} + G^{(3)}_{2,1,0,1} + G^{(3)}_{2,0,1,1} + G^{(3)}_{1,1,1,1}].\end{aligned} \tag{3.1.25}$$

This is the entangled field state with either no photon in all cavities or one photon in all four cavities. In case of N-cavities, the probability amplitudes of ground state can be generalized as

$$\begin{aligned}G^{N_A}_{i_1,i_2,...,i_{N_C}} &= G^{N_A-1}_{i_1,i_2,...,i_{N_C-1}}[\cos(gT_{N_A,1}\sqrt{i_1})\delta_{i_{N_C},0}] + \\ &\quad G^{N_A-1}_{i_1+1,i_2,...,i_{N_C-1}}[\sin(gT_{N_A,N_C}\sqrt{i_{N_C}}) \times \sin(gT_{N_A,1}\sqrt{i_1+1})],\end{aligned} \tag{3.1.26}$$

where $i_1 = 0, 1, ..., N_C$ corresponds to the number of photons in first cavity. All the other cavities that is from i_2 to i_{N_C}, can only have 0 or 1 photons in it. The generalize normalized state of atom detected in ground state can be defined as

$$|\psi_{N_C}\rangle = S_{N_A}[G_{0,0}|0_1, 0_2..., 0_{N_C}\rangle + \sum_{i_1=1}^{N_C} \sum_{i_2,i_3,...i_{N_C}=0}^{1} G^{N_A}_{i_1,i_2,...,i_{N_C}}|i_1, i_2, ..., i_{N_C}\rangle] \tag{3.1.27}$$

In each $G^{N_A}_{i_1,i_2,...,i_{N_C}}$, the sum of all the cavities photons must be conserved, that is equals to N_C ($\sum_{s=1}^{N_C} i_s = N_C$). Moreover, the number of $G's$ for N_C cavities would be $2^{(N_C-1)} + 1$.

3.2 Interaction Times

To generate four-cavity GHZ-state, eight interaction parameters are required. If we keep our interaction time in such a way that probability amplitude of ground state with zero photon in each cavity is equal to the the probability amplitude of ground state with one photon in each cavity, while all other probability amplitudes of ground states are zero such as,

$$G_{0,0,0,0} = G_{1,1,1,1}, \tag{3.2.1}$$

$$G^{(3)}_{4,0,0,0} = G^{(3)}_{3,1,0,0} = G^{(3)}_{3,1,0,1} = G^{(3)}_{3,0,1,1} = G^{(3)}_{2,1,1,0} = G^{(3)}_{2,1,0,1} = G^{(3)}_{2,0,1,1} = 0. \tag{3.2.2}$$

The necessary condition of atom being detected in ground state is follows as

$$\begin{aligned} P_g &= |G^{(3)}_{4,0,0,0}|^2 + |G^{(3)}_{3,1,0,0}|^2 + |G^{(3)}_{3,0,1,0}|^2 + |G^{(3)}_{3,0,0,1}|^2 + |G^{(3)}_{2,1,1,0}|^2 \\ &\quad + |G^{(3)}_{2,1,0,1}|^2 + |G^{(3)}_{2,0,1,1}|^2 + |G^{(3)}_{1,1,1,1}|^2 + |G^{(3)}_{0,0,0,0}|^2. \end{aligned} \tag{3.2.3}$$

These conditions while keeping the probability of atom in ground state maximum, give interaction times as follow; $gT_{11} = 1.25934$, $gT_{12} = 1.5708$, $gT_{21} = 0.815698$, $gT_{23} = 1.5708$, $gT_{31} = 1.32436$, $gT_{34} = 1.5708$ along with total probability 1. Setting these interaction times of each atom in each cavity, we can yield the GHZ-state given in Eq. (3.1.1). For N-cavities, we require $2N_A$ interaction parameters. For the determination of these interaction parameter, we need the following conditions

$$G_{0_1,0_2,\ldots,0_{N_C}} = G_{1_1,1_2,\ldots,1_{N_C}}, \tag{3.2.4}$$

where N_C is the cavity number. All the rest probabilities amplitudes of ground state are equals to zero i.e.,

$$G_{i_1,i_2,\ldots,i_{N_C}} = 0. \tag{3.2.5}$$

By using Eq. (3.2.4) and Eq. (3.2.5) and also keeping the probability of ground state maximum give the interaction times. The total probability for N-cavities system would be

$$P_g = \sum |G^{N_A}_{i_1,i_2,\ldots,i_{N_C}}|^2. \tag{3.2.6}$$

The probabilities for higher number of cavities does not give unity. As the number of cavities increases, the probability decreases.

3.3 Dynamics

Once the entanglement is created between several cavities, it is used in different quantum processes where it has to interact with the environment. Now question arises that whether entanglement remains stable with time when it interacts with the dissipative environment? If not then how it effects the amount of entanglement and how can one measure that amount of effected entanglement. As far as quantum information processes are concerned, entanglement is required to be steady with time. On the other hand, entanglement is degraded by the non-uniform motion during its dynamical evolution which can even change its characteristics. Moreover this change influences the fidelity of tasks (such as teleportation). Therefore after its degradation, amount of entanglement needs to be calculated.
Different quantitative measures are in literature (as discussed in Sec. 1.7). For quantifying the amount of entanglement in higher dimensional system, negativity is used. To study the dynamics, we have open system of entangled field state, initially defined in terms of arbitrary values of probability amplitudes. For four cavities case, it would be given as

$$|\psi_{(GHZ)}\rangle = C_{0000}|0_1, 0_2, 0_3, 0_4\rangle + C_{1111}|1_1, 1_2, 1_3, 1_4\rangle, \tag{3.3.1}$$

where $|C_{0000}|^2+|C_{1111}|^2 = 1$. System is in contact with the environment (thermal reservoir) having $\bar{n}$ mean photon number. Consider that the correlation time for the reservoir and system is short and the system-reservoir interaction is sufficiently weak. Such assumption is given by Born-Markov approximation. The goal of this assumption is to avoid the loss of information from system into the reservoir. Moreover the master equation for such open system is defined in Eq. (2.4.23) which is used to find the system's dynamics by tracing out the many degrees of freedom of the reservoir. Since for four cavities system, we have 16×16 density matrix. The density matrix element will have the following basis ; $|0000\rangle \longrightarrow |1\rangle, |0001\rangle \longrightarrow |2\rangle, |0010\rangle \longrightarrow |3\rangle$, $|0011\rangle \longrightarrow$

$|4\rangle, |0100\rangle \longrightarrow |5\rangle,|0101\rangle \longrightarrow |6\rangle, |0110\rangle \longrightarrow |7\rangle, |0111\rangle \longrightarrow |8\rangle, |1000\rangle \longrightarrow |9\rangle, |1001\rangle \longrightarrow |10\rangle, |1010\rangle \longrightarrow |11\rangle, |1011\rangle \longrightarrow |12\rangle, |1100\rangle \longrightarrow |13\rangle, |1101\rangle \longrightarrow |14\rangle, |1110\rangle \longrightarrow |15\rangle, |1111\rangle \longrightarrow |16\rangle$

Using master equation, we can find the equation of motion for the system. Since we are using identical cavities. Therefore the decay rate of all the four cavities have to be the same i.e., $\kappa_1 = \kappa_2 = \kappa_3 = \kappa_4 = \kappa$. We also assumed that we have vacuum reservoir i.e., $\bar{n} = 0$. Calculating solution of the master equations, gives density matrix elements. Negativity is a useful technique to quantify the entanglement among multipartite quantum states. The negativity (given in Sec. 1.7) is based on positive partial transpose (PPT) criterion. Since we need to find the partial transpose to measure the entanglement after its dynamical evolution, therefore taking partial transpose over first cavity gives the density matrix ρ^{T1} as

$$\rho^{\mathrm{T1}} = \begin{pmatrix}
\rho_{1,1} & \rho_{1,2} & \rho_{1,3} & \rho_{1,4} & \rho_{1,5} & \rho_{1,6} & \rho_{1,7} & \rho_{1,8} & \rho_{9,1} & \rho_{9,2} & \rho_{9,3} & \rho_{9,4} & \rho_{9,5} & \rho_{9,6} & \rho_{9,7} & \rho_{9,8} \\
\rho_{2,1} & \rho_{2,2} & \rho_{2,3} & \rho_{2,4} & \rho_{2,5} & \rho_{2,6} & \rho_{2,7} & \rho_{2,8} & \rho_{10,1} & \rho_{10,2} & \rho_{10,3} & \rho_{10,4} & \rho_{10,5} & \rho_{10,6} & \rho_{10,7} & \rho_{10,8} \\
\rho_{3,1} & \rho_{3,2} & \rho_{3,3} & \rho_{3,4} & \rho_{3,5} & \rho_{3,6} & \rho_{3,7} & \rho_{3,8} & \rho_{11,1} & \rho_{11,2} & \rho_{11,3} & \rho_{11,4} & \rho_{11,5} & \rho_{11,6} & \rho_{11,7} & \rho_{11,8} \\
\rho_{4,1} & \rho_{4,2} & \rho_{4,3} & \rho_{4,4} & \rho_{4,5} & \rho_{4,6} & \rho_{4,7} & \rho_{4,8} & \rho_{12,1} & \rho_{12,2} & \rho_{12,3} & \rho_{12,4} & \rho_{12,5} & \rho_{12,6} & \rho_{12,7} & \rho_{12,8} \\
\rho_{5,1} & \rho_{5,2} & \rho_{5,3} & \rho_{5,4} & \rho_{5,5} & \rho_{5,6} & \rho_{5,7} & \rho_{5,8} & \rho_{13,1} & \rho_{13,2} & \rho_{13,3} & \rho_{13,4} & \rho_{13,5} & \rho_{13,6} & \rho_{13,7} & \rho_{13,8} \\
\rho_{6,1} & \rho_{6,2} & \rho_{6,3} & \rho_{6,4} & \rho_{6,5} & \rho_{6,6} & \rho_{6,7} & \rho_{6,8} & \rho_{14,1} & \rho_{14,2} & \rho_{14,3} & \rho_{14,4} & \rho_{14,5} & \rho_{14,6} & \rho_{14,7} & \rho_{14,8} \\
\rho_{7,1} & \rho_{7,2} & \rho_{7,3} & \rho_{7,4} & \rho_{7,5} & \rho_{7,6} & \rho_{7,7} & \rho_{7,8} & \rho_{15,1} & \rho_{15,2} & \rho_{15,3} & \rho_{15,4} & \rho_{15,5} & \rho_{15,6} & \rho_{15,7} & \rho_{15,8} \\
\rho_{8,1} & \rho_{8,2} & \rho_{8,3} & \rho_{8,4} & \rho_{8,5} & \rho_{8,6} & \rho_{8,7} & \rho_{8,8} & \rho_{16,1} & \rho_{16,2} & \rho_{16,3} & \rho_{16,4} & \rho_{16,5} & \rho_{16,6} & \rho_{16,7} & \rho_{16,8} \\
\rho_{1,9} & \rho_{1,10} & \rho_{1,11} & \rho_{1,12} & \rho_{1,13} & \rho_{1,14} & \rho_{1,15} & \rho_{1,16} & \rho_{9,9} & \rho_{9,10} & \rho_{9,11} & \rho_{9,12} & \rho_{9,13} & \rho_{9,14} & \rho_{9,15} & \rho_{9,16} \\
\rho_{2,9} & \rho_{2,10} & \rho_{2,11} & \rho_{2,12} & \rho_{2,13} & \rho_{2,14} & \rho_{2,15} & \rho_{2,16} & \rho_{10,9} & \rho_{10,10} & \rho_{10,11} & \rho_{10,12} & \rho_{10,13} & \rho_{10,14} & \rho_{10,15} & \rho_{10,16} \\
\rho_{3,9} & \rho_{3,10} & \rho_{3,11} & \rho_{3,12} & \rho_{3,13} & \rho_{3,14} & \rho_{3,15} & \rho_{3,16} & \rho_{11,9} & \rho_{11,10} & \rho_{11,11} & \rho_{11,12} & \rho_{11,13} & \rho_{11,14} & \rho_{11,15} & \rho_{11,16} \\
\rho_{4,9} & \rho_{4,10} & \rho_{4,11} & \rho_{4,12} & \rho_{4,13} & \rho_{4,14} & \rho_{4,15} & \rho_{4,16} & \rho_{12,9} & \rho_{12,10} & \rho_{12,11} & \rho_{12,12} & \rho_{12,13} & \rho_{12,14} & \rho_{12,15} & \rho_{12,16} \\
\rho_{5,9} & \rho_{5,10} & \rho_{5,11} & \rho_{5,12} & \rho_{5,13} & \rho_{5,14} & \rho_{5,15} & \rho_{5,16} & \rho_{13,9} & \rho_{13,10} & \rho_{13,11} & \rho_{13,12} & \rho_{13,13} & \rho_{13,14} & \rho_{13,15} & \rho_{13,16} \\
\rho_{6,9} & \rho_{6,10} & \rho_{6,11} & \rho_{6,12} & \rho_{6,13} & \rho_{6,14} & \rho_{6,15} & \rho_{6,16} & \rho_{14,9} & \rho_{14,10} & \rho_{14,11} & \rho_{14,12} & \rho_{14,13} & \rho_{14,14} & \rho_{14,15} & \rho_{14,16} \\
\rho_{7,9} & \rho_{7,10} & \rho_{7,11} & \rho_{7,12} & \rho_{7,13} & \rho_{7,14} & \rho_{7,15} & \rho_{7,16} & \rho_{15,9} & \rho_{15,10} & \rho_{15,11} & \rho_{15,12} & \rho_{15,13} & \rho_{15,14} & \rho_{15,15} & \rho_{15,16} \\
\rho_{8,9} & \rho_{8,10} & \rho_{8,11} & \rho_{8,12} & \rho_{8,13} & \rho_{8,14} & \rho_{8,15} & \rho_{8,16} & \rho_{16,9} & \rho_{16,10} & \rho_{16,11} & \rho_{16,12} & \rho_{16,13} & \rho_{16,14} & \rho_{16,15} & \rho_{16,16}
\end{pmatrix}$$

Partial transpose with respect to second, third and forth cavity is given in Appendix-A. Taking eigenvalues of each partial transpose and summing up the negative eigenvalues of respected density matrix give us the negativity with respect to each cavity. This negativity describes decoherence on individual cavity level. A general Mathematica program for calculating the eigenvalues of a partial transpose for any number of cavity (for both GHZ-state and W-state) is given in Appendix-B. This program gives the equation of motion for the four cavity system and the solution of density matrix elements given in Appendix-C. The negativity of the whole system can be calculated using

$$N(t) = (\prod_{i=1}^{N_c} N_i(t))^{1/N_c}, \tag{3.3.2}$$

where i is the number of cavities. If all the eigenvalues are positive, then the system will be unentangled. On the other hand negative eigenvalues suggest that system is entangled. Negativity is necessary and sufficient criteria for system upto $2 \otimes 3$ dimensions. For higher dimensional system, it is not necessary but a sufficient criteria. Negativity for four cavities system is shown

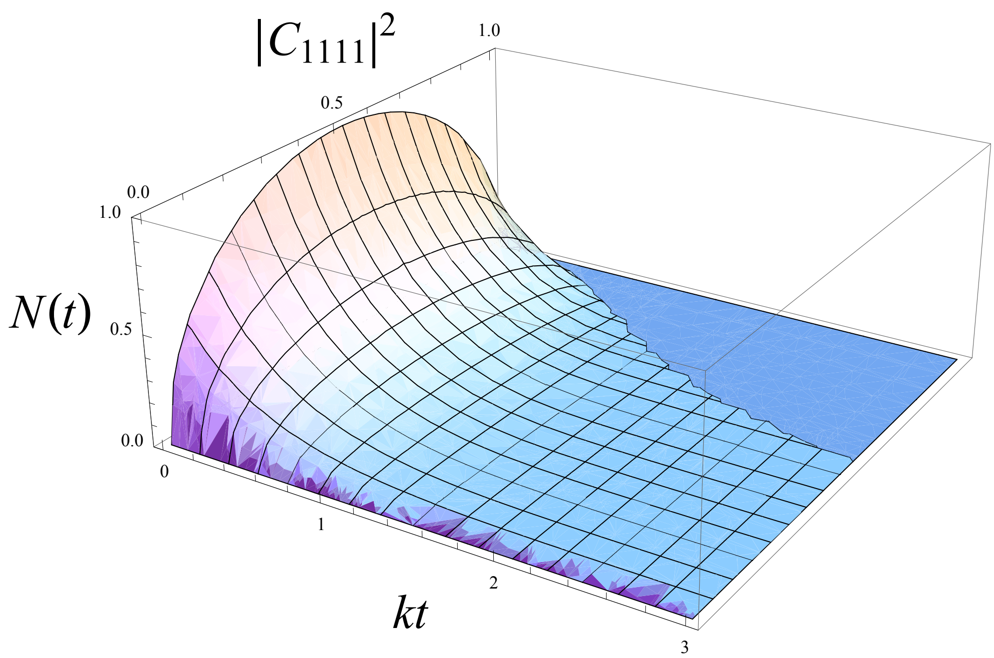

Figure 3.3: The figure shows entanglement dynamics for initially prepared GHZ-state at $\bar{n} = 0$. At all the values of initial probability $|C_{1111}|^2$, it is showing asymptotic behaviour.

in Fig. 3.3, in which the colored region is showing that the system is entangled. As the probability $|C_{1111}|^2$ is small, i.e., $|C_{1111}| < |C_{0000}|$, entanglement dynamics follows the asymptotic behavior. When $|C_{1111}| > |C_{0000}|$, we observe sudden death of entangled (SDE). It shows SDE for this state because it is the most populated state and GHZ states becomes unentangled on the

particle loss.
Thus we have successfully provided a model for the engineering of generalized GHZ-state. We also worked on dynamics of these states in dissipative environment. Next chapter is about W-state engineering. We will also check their dynamical evolution in the environment.

Chapter 4

Entanglement Engineering of W-state in Arbitrary Number of High-Q Cavities and its Dynamics in Dissipative Environment

Last chapter was based on formulating a method for generalization of GHZ-state and also its decoherence due to evolution in dissipative environment. Similar to the GHZ-state, W-state is also an equally important class of multipartite entangled states which is a strong tool for information and computational processes especially in long distance processes because of its robustness against the loss of particle like multinodal networks, quantum memories, dense coding [39] . In this chapter, we suggest a scheme to produce W-state for any number of particle in Sec. 4.1. It is showed that cavities with different set up gives us W-state. The interaction times of atom with each cavity is calculated in Sec. 4.2. Moreover in Sec. 4.3, we investigate its dynamics. The general form of W-state is given in Eq. (1.4.3).

4.1 Model

We have two-level atom in exited state and four cavities, all of them in vacuum state. Excited state atom is sent through the cavities, arranged as shown in Fig. 4.1. This atom interacts with the resonance modes of each cavity. After interacting with the each cavities and contributing photon to either of the cavities, atom must be detected in ground state. Four cavities W-state is defined as

$$\begin{aligned}|\psi_W\rangle &= \frac{1}{\sqrt{4}}(|1_1, 0_2, 0_3, 0_4\rangle + |0_1, 1_2, 0_3, 0_4\rangle \\ &+|0_1, 0_2, 1_3, 0_4\rangle + |0_1, 0_2, 0_3, 1_4\rangle).\end{aligned} \tag{4.1.1}$$

The initial atom and cavities field state, shown in Fig. 4.1 is

$$|\psi_{T=0}\rangle = |e, 0_1, 0_2, 0_3, 0_4\rangle. \tag{4.1.2}$$

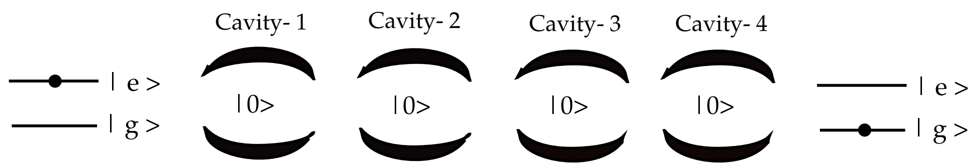

Figure 4.1: A model to produce W-state using Four cavities.

Using Unitary time evolution approach, the initial state evolves to

$$|\psi_T(1234)\rangle = U_{T_4}U_{T_3}U_{T_2}U_{T_1}|\psi_{T=0}\rangle, \tag{4.1.3}$$

where the first subindex shows number of atom and second subindex shows cavities number. However $T_{(1)}$, $T_{(2)}$, $T_{(3)}$, $T_{(4)}$ correspond to the interaction times of the atom in cavity-1, cavity-2, cavity-3 and cavity-4 respectively whereas $U_{T_{(1)}}$, $U_{T_{(2)}}$, $U_{T_{(3)}}$, $U_{T_{(4)}}$. It evolves as,

$$\begin{aligned}|\psi_T(1234)\rangle &= \cos(gT_4)\cos(gT_3)\cos(gT_2)\cos(gT_1)|a, 0_1, 0_2, 0_3, 0_4\rangle \\ &-i\sin(gT_4)\cos(gT_3)\cos(gT_2)\cos(gT_1)|b, 0_1, 0_2, 0_3, 1_4\rangle \\ &-i\sin(gT_3)\cos(gT_2)\cos(gT_1)|b, 0_1, 0_2, 1_3, 0_4\rangle \\ &-i\sin(gT_2)\cos(gT_1)|b, 0_1, 1_2, 0_3, 0_4\rangle - i\sin(gT_1)|b, 1_1, 0_2, 0_3, 0_4\rangle.\end{aligned} \tag{4.1.4}$$

The probability amplitudes of excited state and ground state are,

$$E_{0000} = \cos(gT_4)\cos(gT_3)\cos(gT_2)\cos(gT_1), \tag{4.1.5}$$

$$G_{1000} = -isin(gT_1), \tag{4.1.6}$$

$$G_{0100} = -i\sin(gT_2)\cos(gT_1), \tag{4.1.7}$$

$$G_{0010} = -i\sin(gT_3)\cos(gT_2)\cos(gT_1), \tag{4.1.8}$$

$$G_{0001} = -i\sin(gT_4)\cos(gT_3)\cos(gT_2)\cos(gT_1). \tag{4.1.9}$$

The probability of atom, keeping the condition that it is must be found in ground state is given as,

$$P_g = |G_{1000}|^2 + |G_{0100}|^2 + |G_{0010}|^2 + |G_{0001}|^2. \tag{4.1.10}$$

Considering this necessary condition of the model, the normalized cavity field state becomes

$$\begin{aligned} |\psi_{(T_1,T_2,T_3,T_4)}\rangle \;\; &= \;\; S[G_{1000}|g,1_1,0_2,0_3,0_4\rangle + G_{0100}|g,0_1,1_2,0_3,0_4\rangle \\ &\quad + G_{0010}|g,0_1,0_2,1_3,0_4\rangle + G_{0001}|g,0_1,0_2,0_3,1_4\rangle]. \end{aligned} \tag{4.1.11}$$

Here normalization constant is

$$S = \frac{1}{\sqrt{|G_{1000}|^2 + |G_{0100}|^2 + |G_{0010}|^2 + |G_{0001}|^2}}. \tag{4.1.12}$$

For N-cavities W-state, we have N_C number of probability amplitudes of ground state. For determining $G_{i_1,i_2,\ldots,i_{N_C}}$ in case of W-state, $i_1, i_2, \ldots, i_{N_C}$ form binary string of a number $2^{(N_C-i)} + 1$ where $i = 1, \ldots, N_C$. For example, in the case of five cavities ($N_c = 5$), we have $2^{(N_C-i)}+1 = 2, 3, 5, 9, 17$. Therefore according to this, G's will be

$$\begin{aligned} G_2 &= G_{00001} \\ G_3 &= G_{00010} \\ G_5 &= G_{00100} \\ G_9 &= G_{01000} \\ G_{17} &= G_{10000} \end{aligned} \tag{4.1.13}$$

Probability amplitudes of excited state and ground state can be generalized as

$$E_0 = (\prod_{n=1}^{N_c} \cos(gTn)). \tag{4.1.14}$$

$$G_{2^{(N_C-i)}+1} = (\prod_{n=1}^{p-1} \cos(gTn)) \sin(gTp), \tag{4.1.15}$$

where p shows the cavity that have one photon.

4.2 Calculating Interaction Parameters

For successful production of W-state, we need to find out the interaction durations of atom with each of the four cavities. In order to find interaction durations, we assume all probability amplitudes of ground state to be same that is,

$$G_{1000} = G_{0100} = G_{0010} = G_{0001}. \tag{4.2.1}$$

Further, we also assume the probability amplitude of excited state to be zero that is,

$$E_{0000} = 0. \tag{4.2.2}$$

Therefore using these conditions and keeping the probability of ground state maximum, we have following interaction times; $gT1 = 0.522984$, $gT2 = 0.615253$, $gT3 = 0.783198$, $gT4 = 1.48036$. This gives the total probability 0.99795. By setting these interaction times of atom-field interaction, we can yield the required state given in Eq. (4.1.1). The total probability of detecting atom in ground state will be 0.973078, 0.903899 and 0.459737 for five, six and seven cavities respectively. It decreases with the increase of cavities.

For N-cavities, we have all the cavities in vacuum state. These all cavities are linearly arranged, as shown in Fig. 4.2. Two-level atom in excited state is

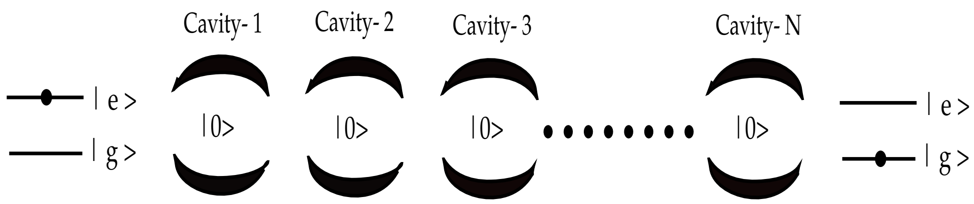

Figure 4.2: Model for N cavities W-state preparation where all the cavities are in vacuum state.

sent through all the cavities as we did for the four cavities. After interacting with all the cavities resonant modes, it must be detected in ground state, which is a required condition of scheme.

4.3 Dynamics

After the successful modeling of generalize scheme for W-state preparation, we are going to examine its dynamics. For higher dimensions we used negativity which is separability criteria. We have open system of entangled field state of W-state, initially defined in terms of arbitrary values of probability amplitudes. For four cavities case, it would be given as a

$$|\psi_W\rangle = C_{0001}|0001\rangle + C_{0010}|0010\rangle + C_{0100}|0100\rangle + C_{1000}|1000\rangle, \quad (4.3.1)$$

where $|C_{0001}|^2 + |C_{0010}|^2 + |C_{0100}|^2 + |C_{1000}|^2 = 1$. The reservoir has $\bar{n}$ mean photon number. Master equation for open system is used to study its dynamics. Under Born-Markov approximation, master equation gives equation of motions as given in Appendix C. The partial transpose (for four cavities) over each cavity is already given in Sec. 3.3. The negative eigenvalues tell that states are non separable. In case of four cavities, the negative eigenvalues after taking partial transpose over the four qubits come out as following,

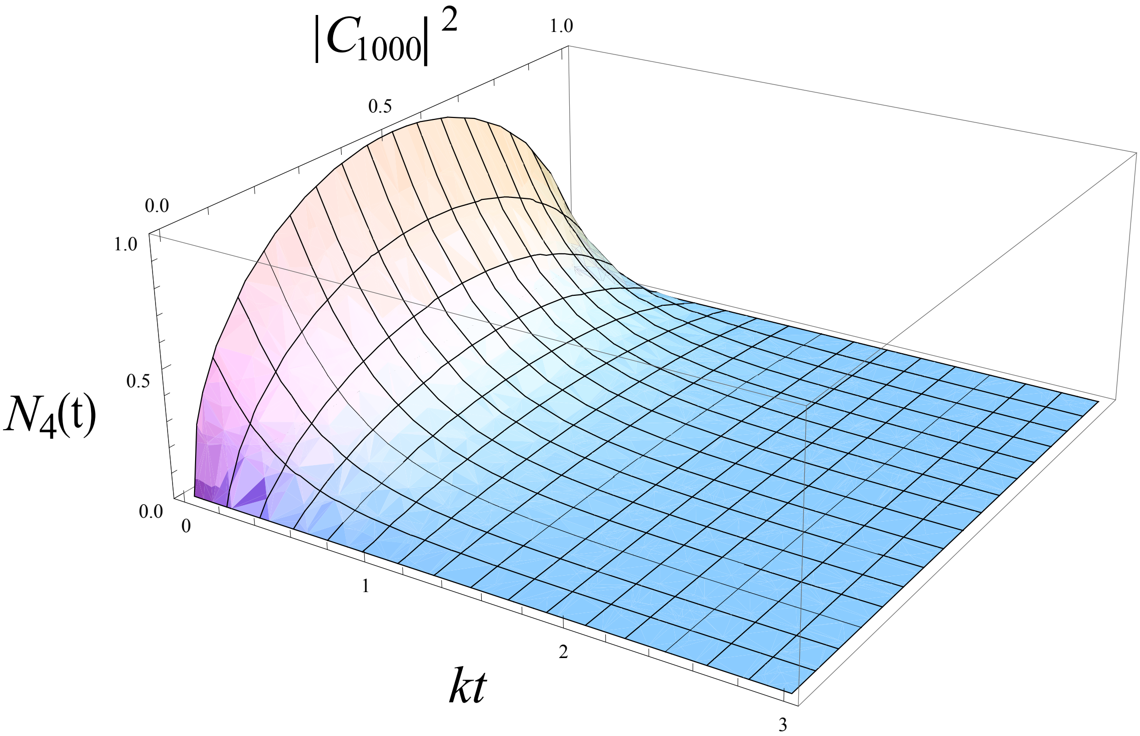

Figure 4.3: The figure shows entanglement dynamics for initially prepared W-state at $\bar{n} = 0$. It is showing asymptotic behaviour.

$$N_1(t) \quad = \quad max[0, (e^{-kt} - 1)(1 + \sqrt{1 + 4e^{-2kt}(e^{kt} - 1)^{-2}|C_{0001}|^2(1 - |C_{0001}|^2)})], \tag{4.3.2}$$

$$N_2(t) \;=\; max[0,(e^{-kt}-1)(1+\sqrt{1+4e^{-2kt}(e^{kt}-1)^{-2}|C_{0010}|^2(1-|C_{0010}|^2)})], \tag{4.3.3}$$

$$N_3(t) \;=\; max[0,(e^{-kt}-1)(1+\sqrt{1+4e^{-2kt}(e^{kt}-1)^{-2}|C_{0100}|^2(1-|C_{0100}|^2)})], \tag{4.3.4}$$

$$N_4(t) \;=\; max[0,(e^{-kt}-1)(1+\sqrt{1+4e^{-2kt}(e^{kt}-1)^{-2}|C_{1000}|^2(1-|C_{1000}|^2)})]. \tag{4.3.5}$$

Negativity against C_{1000} and time is plot in Fig. 4.3.

For N-cavities, negative eigenvalues can be generalized with respect to ith cavity as,

$$N_i(t) \;=\; (e^{-kt}-1)[1+\sqrt{1+4e^{-2kt}(e^{kt}-1)^{-2}|C_{(2^{i-1}+1)}|^2(1-|C_{(2^{i-1}+1)}|^2)}] \;. \tag{4.3.6}$$

where i is the cavity number and $2^{i-1}+1$ is the binary number that gives a binary string. We have provided a generalized scheme for W-state preparation and their dynamical evolution using negativity as a separability criteria. We also generalized their dynamics and showed decoherence of cavity field states in the Plots.

Chapter 5

Conclusion

Quantum theory depicts many concepts that can't be explained classically. Entanglement is one of such phenomena. It is coherent superposition in different parts of composite systems that offers faster computational and enables parallel processing. For the realizable quantum computers there are two crucial points those are largely investigated. One is the generation of coherence or entanglement in different quantum registers and the other is control of quantum logical operations before decoherence.

In this thesis we presented a scheme for the generation of two types of entangled field state i-e W-state and GHZ-state and studied their dynamics. For generalizing the scheme, we extended the method proposed by B. Farooq *et al.* [26]. W-state scheme consists of atom that is prepared in excited state interacts with linearly arranged high Q-cavities, initially in vacuum and contributes one photon to either of the cavities. In the case of GHZ-state, atoms are prepared in ground state and sent one by one through two cavities and after the interaction, either all the cavities should have no photon or one photon. Therefore the main condition for the successful generation of both types of entangled states is that atom must be found in ground state which can be achieved by properly setting the atom's interaction times with each cavity. If this condition is not satisfied, both schemes will be fail and they will be start from beginning. Therefore, scheme is probabilistic one.

For the determination of interaction time of W-state using N cavities, we have single atom while we have N interaction parameters. To find these N parameters we use the condition that all the probability amplitudes of ground state for the states 2^{N_c-i} to be zero where $i = 0, 1..., N$. For another equation, we use a condition that probability amplitude of excited state to be zero. By keeping the probability of atom in ground state maximum, we can get optimum result. The probability decreases with an increase of cavities.
In the case of GHZ-state, the condition used to find the interaction times is that the ground probability amplitudes of 1st state and Nth state to be equal and all the other ground probabilities amplitude to be zero. These conditions gave us the required interaction times. In case of GHZ-state probability of atom in ground state also decreases with the increase of number of cavities.
The schemes presented above is based on the atom interacting with a resonant cavity for a precalculated time. These interaction time can be controlled by stark field adjustment. In stark field adjustment, electric field is applied for stark shift in order to make atom off resonant with the cavity. When we want atom field interaction, we switched off the electric field to make atom resonant with cavity field for the already calculated time. Moreover, we can also control the interaction times by controlling the velocity of atom as we know the length of the cavities. For this purpose, we can use velocity selectors before the cavities. Usually cesium atom and rubidium atom with transition shell $5S_{\frac{1}{2}}$ are used having frequencies 9 GHz and 3.3 GHz respectively, are used
Decoherence is the main hinderance in the implementation of QIP as it ceases the superposition of quantum state. One of the main cause of decoherence in atomic system is spontaneous emission which lead the system to the entanglement degradation. We had open-system dynamics where we had entangled system interacting with reservoir having mean number of photons $\bar{n}$. System undergoes decoherence. The decay of entanglement of generalized N-particle W-state interacting with independent reservoir is asymptotic

but that of GHZ-states shows sudden death of entanglement, after particular values of probability amplitudes of initially prepared state . We investigated the dynamics using negativity, based on partial transpose (PPT) criteria. Negativity is a criteria which tells that state is separable or it is entangled. From the plots of negativity verses time, we find that the decay of entanglement obeys scaling law for W-state and SDE for GHZ-state. However, entanglement becomes arbitrarily small.

Appendix A

Partial Transpose with respect to cavity four

Partial transpose of 16×16 density matrix are given below. Partial transpose with respect to cavity-2, cavity-3 and cavity-4 are

$$\rho^{T2} = \begin{pmatrix}
\rho_{1,1} & \rho_{1,2} & \rho_{1,3} & \rho_{1,4} & \rho_{5,1} & \rho_{5,2} & \rho_{5,3} & \rho_{5,4} & \rho_{1,9} & \rho_{1,10} & \rho_{1,11} & \rho_{1,12} & \rho_{5,9} & \rho_{5,10} & \rho_{5,11} & \rho_{5,12} \\
\rho_{2,1} & \rho_{2,2} & \rho_{2,3} & \rho_{2,4} & \rho_{6,1} & \rho_{6,2} & \rho_{6,3} & \rho_{6,4} & \rho_{2,9} & \rho_{2,10} & \rho_{2,11} & \rho_{2,12} & \rho_{6,9} & \rho_{6,10} & \rho_{6,11} & \rho_{6,12} \\
\rho_{3,1} & \rho_{3,2} & \rho_{3,3} & \rho_{3,4} & \rho_{7,1} & \rho_{7,2} & \rho_{7,3} & \rho_{7,4} & \rho_{3,9} & \rho_{3,10} & \rho_{3,11} & \rho_{3,12} & \rho_{7,9} & \rho_{7,10} & \rho_{7,11} & \rho_{7,12} \\
\rho_{4,1} & \rho_{4,2} & \rho_{4,3} & \rho_{4,4} & \rho_{8,1} & \rho_{8,2} & \rho_{8,3} & \rho_{8,4} & \rho_{4,9} & \rho_{4,10} & \rho_{4,11} & \rho_{4,12} & \rho_{8,9} & \rho_{8,10} & \rho_{8,11} & \rho_{8,12} \\
\rho_{1,5} & \rho_{1,6} & \rho_{1,7} & \rho_{1,8} & \rho_{5,5} & \rho_{5,6} & \rho_{5,7} & \rho_{5,8} & \rho_{1,13} & \rho_{1,14} & \rho_{1,15} & \rho_{1,16} & \rho_{5,13} & \rho_{5,14} & \rho_{5,15} & \rho_{5,16} \\
\rho_{2,5} & \rho_{2,6} & \rho_{2,7} & \rho_{2,8} & \rho_{6,5} & \rho_{6,6} & \rho_{6,7} & \rho_{6,8} & \rho_{2,13} & \rho_{2,14} & \rho_{2,15} & \rho_{2,16} & \rho_{6,13} & \rho_{6,14} & \rho_{6,15} & \rho_{6,16} \\
\rho_{3,5} & \rho_{3,6} & \rho_{3,7} & \rho_{3,8} & \rho_{7,5} & \rho_{7,6} & \rho_{7,7} & \rho_{7,8} & \rho_{3,13} & \rho_{3,14} & \rho_{3,15} & \rho_{3,16} & \rho_{7,13} & \rho_{7,14} & \rho_{7,15} & \rho_{7,16} \\
\rho_{4,5} & \rho_{4,6} & \rho_{4,7} & \rho_{4,8} & \rho_{8,5} & \rho_{8,6} & \rho_{8,7} & \rho_{8,8} & \rho_{4,13} & \rho_{4,14} & \rho_{4,15} & \rho_{4,16} & \rho_{8,13} & \rho_{8,14} & \rho_{8,15} & \rho_{8,16} \\
\rho_{9,1} & \rho_{9,2} & \rho_{9,3} & \rho_{9,4} & \rho_{13,1} & \rho_{13,2} & \rho_{13,3} & \rho_{13,4} & \rho_{9,9} & \rho_{9,10} & \rho_{9,11} & \rho_{9,12} & \rho_{13,9} & \rho_{13,10} & \rho_{13,11} & \rho_{13,12} \\
\rho_{10,1} & \rho_{10,2} & \rho_{10,3} & \rho_{10,4} & \rho_{14,1} & \rho_{14,2} & \rho_{14,3} & \rho_{14,4} & \rho_{10,9} & \rho_{10,10} & \rho_{10,11} & \rho_{10,12} & \rho_{14,9} & \rho_{14,10} & \rho_{14,11} & \rho_{14,12} \\
\rho_{11,1} & \rho_{11,2} & \rho_{11,3} & \rho_{11,4} & \rho_{15,1} & \rho_{15,2} & \rho_{15,3} & \rho_{15,4} & \rho_{11,9} & \rho_{11,10} & \rho_{11,11} & \rho_{11,12} & \rho_{15,9} & \rho_{15,10} & \rho_{15,11} & \rho_{15,12} \\
\rho_{12,1} & \rho_{12,2} & \rho_{12,3} & \rho_{12,4} & \rho_{16,1} & \rho_{16,2} & \rho_{16,3} & \rho_{16,4} & \rho_{12,9} & \rho_{12,10} & \rho_{12,11} & \rho_{12,12} & \rho_{16,9} & \rho_{16,10} & \rho_{16,11} & \rho_{16,12} \\
\rho_{9,5} & \rho_{9,6} & \rho_{9,7} & \rho_{9,8} & \rho_{13,5} & \rho_{13,6} & \rho_{13,7} & \rho_{13,8} & \rho_{9,13} & \rho_{9,14} & \rho_{9,15} & \rho_{9,16} & \rho_{13,13} & \rho_{13,14} & \rho_{13,15} & \rho_{13,16} \\
\rho_{10,5} & \rho_{10,6} & \rho_{10,7} & \rho_{10,8} & \rho_{14,5} & \rho_{14,6} & \rho_{14,7} & \rho_{14,8} & \rho_{10,13} & \rho_{10,14} & \rho_{10,15} & \rho_{10,16} & \rho_{14,13} & \rho_{14,14} & \rho_{14,15} & \rho_{14,16} \\
\rho_{11,5} & \rho_{11,6} & \rho_{11,7} & \rho_{11,8} & \rho_{15,5} & \rho_{15,6} & \rho_{15,7} & \rho_{15,8} & \rho_{11,13} & \rho_{11,14} & \rho_{11,15} & \rho_{11,16} & \rho_{15,13} & \rho_{15,14} & \rho_{15,15} & \rho_{15,16} \\
\rho_{12,5} & \rho_{12,6} & \rho_{12,7} & \rho_{12,8} & \rho_{16,5} & \rho_{16,6} & \rho_{16,7} & \rho_{16,8} & \rho_{12,13} & \rho_{12,14} & \rho_{12,15} & \rho_{12,16} & \rho_{16,13} & \rho_{16,14} & \rho_{16,15} & \rho_{16,16}
\end{pmatrix}$$

$$\rho^{T3} = \begin{pmatrix}
\rho_{1,1} & \rho_{1,2} & \rho_{3,1} & \rho_{3,2} & \rho_{1,5} & \rho_{1,6} & \rho_{3,5} & \rho_{3,6} & \rho_{1,9} & \rho_{1,10} & \rho_{3,9} & \rho_{3,10} & \rho_{1,13} & \rho_{1,14} & \rho_{3,13} & \rho_{3,14} \\
\rho_{2,1} & \rho_{2,2} & \rho_{4,1} & \rho_{4,2} & \rho_{2,5} & \rho_{2,6} & \rho_{4,5} & \rho_{4,6} & \rho_{2,9} & \rho_{2,10} & \rho_{4,9} & \rho_{4,10} & \rho_{2,13} & \rho_{2,14} & \rho_{4,13} & \rho_{4,14} \\
\rho_{1,3} & \rho_{1,4} & \rho_{3,3} & \rho_{3,4} & \rho_{1,7} & \rho_{1,8} & \rho_{3,7} & \rho_{3,8} & \rho_{1,11} & \rho_{1,12} & \rho_{3,11} & \rho_{3,12} & \rho_{1,15} & \rho_{1,16} & \rho_{3,15} & \rho_{3,16} \\
\rho_{2,3} & \rho_{2,4} & \rho_{4,3} & \rho_{4,4} & \rho_{2,7} & \rho_{2,8} & \rho_{4,7} & \rho_{4,8} & \rho_{2,11} & \rho_{2,12} & \rho_{4,11} & \rho_{4,12} & \rho_{2,15} & \rho_{2,16} & \rho_{4,15} & \rho_{4,16} \\
\rho_{5,1} & \rho_{5,2} & \rho_{7,1} & \rho_{7,2} & \rho_{5,5} & \rho_{5,6} & \rho_{7,5} & \rho_{7,6} & \rho_{5,9} & \rho_{5,10} & \rho_{7,9} & \rho_{7,10} & \rho_{5,13} & \rho_{5,14} & \rho_{7,13} & \rho_{7,14} \\
\rho_{6,1} & \rho_{6,2} & \rho_{8,1} & \rho_{8,2} & \rho_{6,5} & \rho_{6,6} & \rho_{8,5} & \rho_{8,6} & \rho_{6,9} & \rho_{6,10} & \rho_{8,9} & \rho_{8,10} & \rho_{6,13} & \rho_{6,14} & \rho_{8,15} & \rho_{8,14} \\
\rho_{5,3} & \rho_{5,4} & \rho_{7,3} & \rho_{7,4} & \rho_{5,7} & \rho_{5,8} & \rho_{7,7} & \rho_{7,8} & \rho_{5,11} & \rho_{5,12} & \rho_{7,11} & \rho_{7,12} & \rho_{5,15} & \rho_{5,16} & \rho_{7,15} & \rho_{7,16} \\
\rho_{6,3} & \rho_{6,4} & \rho_{8,3} & \rho_{8,4} & \rho_{6,7} & \rho_{6,8} & \rho_{8,7} & \rho_{8,8} & \rho_{6,11} & \rho_{6,12} & \rho_{8,11} & \rho_{8,12} & \rho_{6,15} & \rho_{6,16} & \rho_{8,15} & \rho_{8,16} \\
\rho_{9,1} & \rho_{9,2} & \rho_{11,1} & \rho_{11,2} & \rho_{9,5} & \rho_{9,6} & \rho_{11,5} & \rho_{11,6} & \rho_{9,9} & \rho_{9,10} & \rho_{11,9} & \rho_{11,10} & \rho_{9,13} & \rho_{9,14} & \rho_{11,13} & \rho_{11,14} \\
\rho_{10,1} & \rho_{10,2} & \rho_{12,1} & \rho_{12,2} & \rho_{10,5} & \rho_{10,6} & \rho_{12,5} & \rho_{12,6} & \rho_{10,9} & \rho_{10,10} & \rho_{12,9} & \rho_{12,10} & \rho_{10,13} & \rho_{10,14} & \rho_{12,13} & \rho_{12,14} \\
\rho_{9,3} & \rho_{9,4} & \rho_{11,3} & \rho_{11,4} & \rho_{9,7} & \rho_{9,8} & \rho_{11,7} & \rho_{11,8} & \rho_{9,11} & \rho_{9,12} & \rho_{11,11} & \rho_{11,12} & \rho_{9,15} & \rho_{9,16} & \rho_{11,15} & \rho_{11,16} \\
\rho_{10,3} & \rho_{10,4} & \rho_{12,3} & \rho_{12,4} & \rho_{10,7} & \rho_{10,8} & \rho_{12,7} & \rho_{12,8} & \rho_{10,11} & \rho_{10,12} & \rho_{12,11} & \rho_{12,12} & \rho_{10,15} & \rho_{10,16} & \rho_{12,15} & \rho_{12,16} \\
\rho_{13,1} & \rho_{13,2} & \rho_{15,1} & \rho_{15,2} & \rho_{13,5} & \rho_{13,6} & \rho_{15,5} & \rho_{15,6} & \rho_{13,9} & \rho_{13,10} & \rho_{15,9} & \rho_{15,10} & \rho_{13,13} & \rho_{13,14} & \rho_{15,13} & \rho_{15,14} \\
\rho_{14,1} & \rho_{14,2} & \rho_{16,1} & \rho_{16,2} & \rho_{14,5} & \rho_{14,6} & \rho_{16,5} & \rho_{16,6} & \rho_{14,9} & \rho_{14,10} & \rho_{16,9} & \rho_{16,10} & \rho_{14,13} & \rho_{14,14} & \rho_{16,13} & \rho_{16,14} \\
\rho_{13,3} & \rho_{13,4} & \rho_{15,3} & \rho_{15,4} & \rho_{13,7} & \rho_{13,8} & \rho_{15,7} & \rho_{15,8} & \rho_{13,11} & \rho_{13,12} & \rho_{15,11} & \rho_{15,12} & \rho_{13,15} & \rho_{13,16} & \rho_{15,15} & \rho_{15,16} \\
\rho_{14,3} & \rho_{14,4} & \rho_{16,3} & \rho_{16,4} & \rho_{14,7} & \rho_{14,8} & \rho_{16,7} & \rho_{16,8} & \rho_{14,11} & \rho_{14,12} & \rho_{16,11} & \rho_{16,12} & \rho_{14,15} & \rho_{14,16} & \rho_{16,15} & \rho_{16,16}
\end{pmatrix}$$

$$\rho^{T4} = \begin{pmatrix}
\rho_{1,1} & \rho_{2,1} & \rho_{1,3} & \rho_{2,3} & \rho_{1,5} & \rho_{2,5} & \rho_{1,7} & \rho_{1,7} & \rho_{1,9} & \rho_{1,9} & \rho_{1,11} & \rho_{1,11} & \rho_{1,13} & \rho_{2,13} & \rho_{1,15} & \rho_{2,15} \\
\rho_{1,2} & \rho_{2,2} & \rho_{1,4} & \rho_{2,4} & \rho_{1,6} & \rho_{2,6} & \rho_{1,8} & \rho_{2,8} & \rho_{2,10} & \rho_{2,10} & \rho_{2,12} & \rho_{2,12} & \rho_{1,14} & \rho_{2,14} & \rho_{1,16} & \rho_{2,16} \\
\rho_{3,1} & \rho_{4,1} & \rho_{3,3} & \rho_{4,3} & \rho_{3,5} & \rho_{4,5} & \rho_{3,7} & \rho_{3,7} & \rho_{3,9} & \rho_{3,9} & \rho_{3,11} & \rho_{3,11} & \rho_{3,13} & \rho_{4,13} & \rho_{3,15} & \rho_{3,15} \\
\rho_{3,2} & \rho_{4,2} & \rho_{3,4} & \rho_{4,4} & \rho_{3,6} & \rho_{4,6} & \rho_{4,8} & \rho_{4,8} & \rho_{4,10} & \rho_{4,10} & \rho_{4,12} & \rho_{4,12} & \rho_{4,14} & \rho_{4,14} & \rho_{3,16} & \rho_{4,16} \\
\rho_{5,1} & \rho_{6,1} & \rho_{5,3} & \rho_{6,3} & \rho_{5,5} & \rho_{6,5} & \rho_{5,7} & \rho_{5,8} & \rho_{5,9} & \rho_{5,9} & \rho_{5,11} & \rho_{5,11} & \rho_{5,13} & \rho_{6,13} & \rho_{5,15} & \rho_{5,15} \\
\rho_{5,2} & \rho_{6,2} & \rho_{5,4} & \rho_{6,4} & \rho_{5,6} & \rho_{6,6} & \rho_{6,8} & \rho_{6,8} & \rho_{6,10} & \rho_{6,10} & \rho_{6,12} & \rho_{6,12} & \rho_{6,14} & \rho_{6,14} & \rho_{5,16} & \rho_{6,16} \\
\rho_{7,1} & \rho_{8,1} & \rho_{7,3} & \rho_{8,3} & \rho_{7,5} & \rho_{8,5} & \rho_{9,7} & \rho_{7,7} & \rho_{7,9} & \rho_{7,9} & \rho_{7,11} & \rho_{7,11} & \rho_{7,13} & \rho_{8,13} & \rho_{7,15} & \rho_{7,15} \\
\rho_{7,2} & \rho_{8,2} & \rho_{7,4} & \rho_{8,4} & \rho_{7,6} & \rho_{8,6} & \rho_{9,8} & \rho_{8,8} & \rho_{8,10} & \rho_{8,10} & \rho_{8,12} & \rho_{8,12} & \rho_{8,14} & \rho_{8,14} & \rho_{7,16} & \rho_{8,16} \\
\rho_{9,1} & \rho_{10,1} & \rho_{9,3} & \rho_{10,3} & \rho_{9,5} & \rho_{10,5} & \rho_{10,7} & \rho_{9,7} & \rho_{9,9} & \rho_{9,9} & \rho_{9,11} & \rho_{9,11} & \rho_{9,13} & \rho_{10,13} & \rho_{9,15} & \rho_{9,15} \\
\rho_{9,2} & \rho_{10,2} & \rho_{9,4} & \rho_{10,4} & \rho_{9,6} & \rho_{10,6} & \rho_{10,8} & \rho_{10,8} & \rho_{10,10} & \rho_{10,10} & \rho_{10,12} & \rho_{10,12} & \rho_{10,14} & \rho_{10,14} & \rho_{9,16} & \rho_{10,16} \\
\rho_{11,1} & \rho_{12,1} & \rho_{11,3} & \rho_{12,3} & \rho_{11,5} & \rho_{12,5} & \rho_{11,7} & \rho_{11,7} & \rho_{11,9} & \rho_{11,9} & \rho_{11,11} & \rho_{11,11} & \rho_{11,13} & \rho_{12,13} & \rho_{11,15} & \rho_{12,15} \\
\rho_{11,2} & \rho_{12,2} & \rho_{11,4} & \rho_{12,4} & \rho_{11,6} & \rho_{12,6} & \rho_{11,8} & \rho_{12,8} & \rho_{12,10} & \rho_{12,10} & \rho_{12,12} & \rho_{12,12} & \rho_{12,14} & \rho_{12,14} & \rho_{11,16} & \rho_{12,16} \\
\rho_{13,1} & \rho_{14,1} & \rho_{13,3} & \rho_{14,3} & \rho_{13,5} & \rho_{14,5} & \rho_{13,7} & \rho_{13,7} & \rho_{13,9} & \rho_{13,9} & \rho_{13,11} & \rho_{13,11} & \rho_{13,13} & \rho_{14,13} & \rho_{13,15} & \rho_{14,15} \\
\rho_{13,2} & \rho_{14,2} & \rho_{13,4} & \rho_{14,4} & \rho_{13,6} & \rho_{14,6} & \rho_{13,8} & \rho_{14,8} & \rho_{14,10} & \rho_{14,10} & \rho_{14,12} & \rho_{14,12} & \rho_{13,14} & \rho_{14,14} & \rho_{13,16} & \rho_{14,16} \\
\rho_{15,1} & \rho_{16,1} & \rho_{15,3} & \rho_{16,3} & \rho_{15,5} & \rho_{16,5} & \rho_{15,7} & \rho_{15,7} & \rho_{15,9} & \rho_{15,9} & \rho_{15,11} & \rho_{15,11} & \rho_{15,13} & \rho_{16,13} & \rho_{15,15} & \rho_{16,15} \\
\rho_{15,2} & \rho_{16,2} & \rho_{15,4} & \rho_{16,4} & \rho_{15,6} & \rho_{16,6} & \rho_{15,8} & \rho_{16,8} & \rho_{16,10} & \rho_{16,10} & \rho_{16,12} & \rho_{16,12} & \rho_{15,14} & \rho_{16,14} & \rho_{15,16} & \rho_{16,16}
\end{pmatrix}$$

Appendix B

Mathematica Program For N-Cavities

The generalize program for calculating partial transpose of any number of cavities is given below where nmax is the total number of cavities.

$$z[\mathrm{m\_}] := \mathrm{IntegerDigits}[m, 2, \mathrm{nmax}]$$
$$\mathrm{tab} = \mathrm{Table}[z[n], \{n, 0, 2\hat{}\mathrm{nmax} - 1\}];$$
$$x[\mathrm{n\_}, \mathrm{i\_}] := z[n-1][[i]]$$
$$y[\mathrm{m\_}, \mathrm{i\_}] := z[m-1][[i]]$$
$$d = \mathrm{List}[\mathrm{Rho}[n, m][t]];$$
$$\mathrm{Do}\left[\mathrm{AppendTo}[d, \mathrm{Rho}[n, m][t]], \left\{n, 1, 2^{\mathrm{nmax}}\right\}, \left\{m, 1, 2^{\mathrm{nmax}}\right\}\right]; d = \mathrm{Delete}[d, 1];$$
$$\mathrm{o1} = \left(n + 2^{\mathrm{nmax}-i}\right); \mathrm{p1} = \left(m + 2^{\mathrm{nmax}-i}\right); \mathrm{o2} = \left(n - 2^{\mathrm{nmax}-i}\right);$$
$$\mathrm{p2} = \left(m - 2^{\mathrm{nmax}-i}\right);$$
$$\mathrm{Sum}\left[\frac{k_i}{2} * \bar{\eta} * (\mathrm{aadag}\rho[n, m, i][t] - 2 * \mathrm{adag}\rho\mathrm{a}[n, m, i][t] + \rho\mathrm{aadag}[n, m, i][t]), \{i, 1, \mathrm{nmax}\}\right]$$
$$+\mathrm{Sum}\left[\frac{k_i}{2} * (\bar{\eta} + 1) * (\mathrm{adaga}\rho[n, m, i][t] - 2 * \mathrm{a}\rho\mathrm{adag}[n, m, i][t] + \rho\mathrm{adaga}[n, m, i][t]), \{i, 1, \mathrm{nmax}\}\right]$$
$$+\mathrm{Rho}[n, m]'[t] = 0$$
$$.0, \mathrm{True}, \mathrm{Rho}\left[n + 2^{\mathrm{nmax}-i}, m + 2^{\mathrm{nmax}-i}\right][t]$$
$$\mathrm{adag}\rho\mathrm{a}[\mathrm{n\_}, \mathrm{m\_}, \mathrm{i\_}][\mathrm{t\_}] := \mathrm{Which} n > 2\hat{}\mathrm{nmax}, 0, n < 1, 0, m > 2\hat{}\mathrm{nmax}, 0, m < 1, 0,$$
$$\mathrm{o2} > 2\hat{}\mathrm{nmax}, 0, \mathrm{o2} < 1, 0, \mathrm{p2} > 2\hat{}\mathrm{nmax}, 0, \mathrm{p2} < 1, 0, x[n, i]{=}0, 0, y[m, i]{=}0,$$
$$.0, \mathrm{True}, \mathrm{Rho}\left[n - 2^{\mathrm{nmax}-i}, m - 2^{\mathrm{nmax}-i}\right][t]$$
$$\rho\mathrm{adaga}[\mathrm{n\_}, \mathrm{m\_}, \mathrm{i\_}][\mathrm{t\_}] := \mathrm{Which}[n > 2\hat{}\mathrm{nmax}, 0, n < 1, 0, m > 2\hat{}\mathrm{nmax}, 0, m < 1, 0, \mathrm{True}, (y[m, i]) * \mathrm{Rho}[n, m][t]]$$
$$\mathrm{adaga}\rho[\mathrm{n\_}, \mathrm{m\_}, \mathrm{i\_}][\mathrm{t\_}] := \mathrm{Which}[n > 2\hat{}\mathrm{nmax}, 0, n < 1, 0, m > 2\hat{}\mathrm{nmax}, 0, m < 1, 0, \mathrm{True}, (x[n, i]) * \mathrm{Rho}[n, m][t]]$$
$$\mathrm{e1}[\mathrm{n\_}, \mathrm{m\_}, \mathrm{i\_}, \mathrm{j\_}, \mathrm{g\_}, \mathrm{h\_}] := \mathrm{Which}$$
$$\left[n \neq \left(2^{\mathrm{nmax}^i}\right) - j, 0, m \neq \left(2^{\mathrm{nmax}^g}\right) - h, 0, i = j, 0, g = h, 0, \mathrm{True}, z\left[\left(2^{\mathrm{nmax}^i}\right) - j, \left(2^{\mathrm{nmax}^g}\right) - h\right]\right]$$
$$e[\mathrm{n\_}, \mathrm{m\_}] = \mathrm{Sum}[\mathrm{e1}[n, m, i, j, g, h], \{i, 0, 1\}, \{j, 0, 1\}, \{g, 0, 1\}, \{h, 0, 1\}];$$
$$\mathrm{Rhodot1}[\mathrm{n\_}, \mathrm{m\_}][\mathrm{t\_}] :=$$
$$\mathrm{Sum}\left[\frac{k_i}{2} * \bar{\eta} * (\mathrm{aadag}\rho[n, m, i][t] - 2 * \mathrm{adag}\rho\mathrm{a}[n, m, i][t] + \rho\mathrm{aadag}[n, m, i][t]), \{i, 1, \mathrm{nmax}\}\right]$$
$$+\mathrm{Sum}\left[\frac{k_i}{2} * (\bar{\eta} + 1) * (\mathrm{adaga}\rho[n, m, i][t] - 2 * \mathrm{a}\rho\mathrm{adag}[n, m, i][t] + \rho\mathrm{adaga}[n, m, i][t]), \{i, 1, \mathrm{nmax}\}\right]$$
$$+\mathrm{Rho}[n, m]'[t] = 0, \mathrm{d2} = \mathrm{List}[\mathrm{Rhodot1}[n, m][t]];$$
$$\mathrm{Do}\left[\mathrm{AppendTo}[\mathrm{d2}, \mathrm{Rhodot1}[n, m][t]], \left\{n, 1, 2^{\mathrm{nmax}}\right\}, \left\{m, 1, 2^{\mathrm{nmax}}\right\}\right];$$
$$\mathrm{d2} = \mathrm{Delete}[\mathrm{d2}, 1];$$
$$\mathrm{Do}\left[\mathrm{AppendTo}[\mathrm{d2}, \mathrm{Rho}[n, m][0] = e[n, m]], \left\{n, 1, 2^{\mathrm{nmax}}\right\}, \left\{m, 1, 2^{\mathrm{nmax}}\right\}\right];$$
$$\mathrm{d2} = \mathrm{DeleteCases}[\mathrm{d2}, \mathrm{True}];$$
$$\bar{\eta} = 0; k_1 = k; k_2 = k; k_3 = k; k_4 = k; k_5 = k; k_6 = k; k_7 = k;$$
$$r = \mathrm{DSolve}[\mathrm{d2}, d, t]//\mathrm{FullSimplify};$$
$$\mathrm{PT}[\mathrm{n\_}, \mathrm{m\_}, \mathrm{j\_}][\mathrm{t\_}] :=, \mathrm{If}[x[n, j] \neq y[m, j],$$
$$a[l] := \mathrm{Table}\left[\mathrm{PT}[n, m, l][t], \left\{n, 1, 2^{\mathrm{nmax}}\right\}, \left\{m, 1, 2^{\mathrm{nmax}}\right\}\right]; \mathrm{Eigenvalues}[\mathrm{al}]$$
$$\mathrm{RhoSum}\left[2^{\mathrm{nmax}-i} * x[n, i], \{i, 1, j-1\}\right] + 2^{\mathrm{nmax}-j} * y[m, j]$$
$$+\mathrm{Sum}\left[2^{\mathrm{nmax}-i} * x[n, i], \{i, j+1, \mathrm{nmax}\}\right] + 1,$$

where l corresponds to the cavity number. Puting $l = 1, ..., nmax$ gives the eigenvalues of respected lth cavity.

Appendix C

Equations of motion of the density matrix elements of four cavities and their solutions for thermal reservoir

The equations of motion for the four cavities system are given below.

$$
\begin{aligned}
\dot{\rho}_{1,1} &= -\bar{n}(k_1+k_2+k_3+k_4)\rho_{1,1} + (1+\bar{n})(k_4\rho_{2,2}+k_3\rho_{3,3}+k_2\rho_{5,5}+k_1\rho_{9,9}) \\
\dot{\rho}_{1,2} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_4]\rho_{1,2} + (1+\bar{n})(k_3\rho_{3,4}+k_2\rho_{5,6}+k_1\rho_{9,10}) \\
\dot{\rho}_{1,3} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_3]\rho_{1,3} + (1+\bar{n})(k_4\rho_{2,4}+k_2\rho_{5,7}+k_1\rho_{9,11}) \\
\dot{\rho}_{1,4} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_3+\frac{1}{2}k_4]\rho_{1,4} + (1+\bar{n})(k_2\rho_{5,8}+k_1\rho_{9,12}) \\
\dot{\rho}_{1,5} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2]\rho_{1,5} + (1+\bar{n})(k_3\rho_{3,7}+k_1\rho_{9,13}) \\
\dot{\rho}_{1,6} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+\frac{1}{2}k_4]\rho_{1,6} + (1+\bar{n})(k_3\rho_{3,8}+k_1\rho_{9,14}) \\
\dot{\rho}_{1,7} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+\frac{1}{2}k_3]\rho_{1,7} + (1+\bar{n})(k_4\rho_{2,8}+k_1\rho_{9,15}) \\
\dot{\rho}_{1,8} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+\frac{1}{2}k_3+\frac{1}{2}k_4]\rho_{1,8} + (1+\bar{n})(k_3\rho_{9,16}) \\
\dot{\rho}_{1,9} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1]\rho_{1,9} + (1+\bar{n})(k_4\rho_{2,10}+k_3\rho_{3,11}+k_2\rho_{5,13}) \\
\dot{\rho}_{1,10} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_4]\rho_{1,10} + (1+\bar{n})(k_3\rho_{3,12}+k_2\rho_{5,14}) \\
\dot{\rho}_{1,11} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_3]\rho_{1,11} + (1+\bar{n})(k_4\rho_{2,12}+k_2\rho_{5,15}) \\
\dot{\rho}_{1,12} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_3+\frac{1}{2}k_4]\rho_{1,12} + (1+\bar{n})(k_2\rho_{5,16}) \\
\dot{\rho}_{1,13} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_2]\rho_{1,13} + (1+\bar{n})(k_4\rho_{2,14}+k_3\rho_{3,15}) \\
\dot{\rho}_{1,14} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_4]\rho_{1,10} + (1+\bar{n})(k_3\rho_{3,12}+k_2\rho_{5,14})
\end{aligned}
$$

$$
\begin{aligned}
\dot{\rho}_{1,15} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_2+\frac{1}{2}k_3]\rho_{1,14}+(1+\bar{n})(k_4\rho_{2,16})\\
\dot{\rho}_{1,16} &= -[(\bar{n}+\frac{1}{2})(k_1+k_2+k_3+k_4)]\rho_{1,16})\\
\dot{\rho}_{2,2} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_4]\rho_{2,2}+(1+\bar{n})(k_3\rho_{4,4}+k_2\rho_{6,6}+k_1\rho_{10,10})\\
\dot{\rho}_{2,3} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_3+\frac{1}{2}k_4]\rho_{2,3}+(1+\bar{n})(k_2\rho_{6,7}+k_1\rho_{10,11})\\
\dot{\rho}_{2,4} &= \bar{n}k_4\rho_{1,3}-[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+k_4]\rho_{2,4}+(1+\bar{n})(k_2\rho_{6,8}+k_1\rho_{10,12})\\
\dot{\rho}_{2,5} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+\frac{1}{2}k_4]\rho_{2,5}+(1+\bar{n})(k_3\rho_{4,4}+k_3\rho_{4,7}+k_1\rho_{10,13})\\
\dot{\rho}_{2,6} &= \bar{n}k_4\rho_{1,5}-[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+k_4]\rho_{2,6}+(1+\bar{n})(k_3\rho_{4,8}+k_1\rho_{10,14})\\
\dot{\rho}_{2,7} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+\frac{1}{2}k_3+\frac{1}{2}k_4]\rho_{2,7}+(1+\bar{n})(k_1\rho_{10,15})\\
\dot{\rho}_{2,8} &= \bar{n}k_4\rho_{1,7}-[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+\frac{1}{2}k_3+k_4]\rho_{2,8}+(1+\bar{n})(k_4\rho_{2,8}+k_1\rho_{10,16})\\
\dot{\rho}_{2,9} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_4]\rho_{2,9}+(1+\bar{n})(k_3\rho_{4,11}+k_2\rho_{6,13}+k_1\rho_{10,13}\\
\dot{\rho}_{2,10} &= \bar{n}k_4\rho_{1,9}-[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+k_4]\rho_{2,10}+(1+\bar{n})(k_3\rho_{4,12}+k_2\rho_{6,14})\\
\dot{\rho}_{2,11} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_3+\frac{1}{2}k_4]\rho_{2,11}+(1+\bar{n})(k_2\rho_{6,15})\\
\dot{\rho}_{2,12} &= \bar{n}k_4\rho_{1,11}-[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_3+k_4]\rho_{2,12}+(1+\bar{n})(k_2\rho_{6,16})\\
\dot{\rho}_{2,13} &= -[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_2+\frac{1}{2}k_4]\rho_{2,13}+(1+\bar{n})(k_3\rho_{4,15})\\
\dot{\rho}_{2,14} &= \bar{n}k_4\rho_{1,13}-[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{2}{2}k_2+k_4]\rho_{2,14}+(1+\bar{n})(k_3\rho_{4,16})\\
\dot{\rho}_{2,15} &= -[(\bar{n}+\frac{1}{2})(k_1+k_2+k_3+k_4)]\rho_{2,15}\\
\dot{\rho}_{2,16} &= \bar{n}k_4\rho_{1,15}-[\bar{n}(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_2+\frac{1}{2}k_3+k_4]\rho_{2,16}
\end{aligned}
$$

$$
\begin{aligned}
\dot\rho_{3,3} &= \bar n k_3\rho_{1,1} - [\bar n(k_1+k_2+k_3+k_4)+k_3]\rho_{3,3} + (1+\bar n)(k_4\rho_{4,4}+k_2\rho_{7,7}+k_1\rho_{11,11})\\
\dot\rho_{3,4} &= \bar n k_3\rho_{1,2} - [\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_4+k_3]\rho_{3,4} + (1+\bar n)(k_2\rho_{7,8}+k_1\rho_{11,12})\\
\dot\rho_{3,5} &= -[\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+\frac{1}{2}k_3+k_4]\rho_{3,5} + (1+\bar n)(k_4\rho_{4,6}+k_1\rho_{11,13})\\
\dot\rho_{3,6} &= -[\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+\frac{1}{2}k_3+\frac{1}{2}k_4]\rho_{3,6} + (1+\bar n)(k_4\rho_{3,6}+k_1\rho_{11,14})\\
\dot\rho_{3,7} &= \bar n k_3\rho_{1,5} - [\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+k_3]\rho_{3,7} + (1+\bar n)(k_4\rho_{4,8}+k_1\rho_{11,15})\\
\dot\rho_{3,8} &= \bar n k_3\rho_{1,6} - [\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+\frac{1}{2}k_4+k_3]\rho_{3,8} + (1+\bar n)(k_4\rho_{3,8}+k_1\rho_{11,16})\\
\dot\rho_{3,9} &= -[\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_1+k_4]\rho_{2,10} + (1+\bar n)(k_4\rho_{4,10}+k_2\rho_{7,13})\\
\dot\rho_{3,10} &= -[\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_2+\frac{1}{2}k_4]\rho_{3,10} + (1+\bar n)(k_2\rho_{7,14})\\
\dot\rho_{3,11} &= \bar n k_3\rho_{1,9} - [\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+k_3]\rho_{3,11} + (1+\bar n)(k_4\rho_{4,12}+k_2\rho_{7,15})\\
\dot\rho_{3,12} &= \bar n k_3\rho_{1,10} - [\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_4+k_3]\rho_{3,12} + (1+\bar n)(k_2\rho_{7,16})\\
\dot\rho_{3,13} &= -[\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_2+\frac{1}{2}k_3]\rho_{3,13} + (1+\bar n)(k_4\rho_{4,14})\\
\dot\rho_{3,14} &= -[(\bar n+1)(k_1+k_2+k_3+k_4)]\rho_{3,14}\\
\dot\rho_{3,15} &= \bar n k_3\rho_{1,13} - [\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1\frac{1}{2}k_2+k_3]\rho_{3,15} + (1+\bar n)(k_4\rho_{4,16})\\
\dot\rho_{3,16} &= \bar n k_3\rho_{1,14} - [\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_1+\frac{1}{2}k_2+k_3+\frac{1}{2}k_4]\rho_{3,16}\\
\dot\rho_{4,4} &= \bar n k_3\rho_{2,2}+k_4\rho_{3,3}) - [\bar n(k_1+k_2+k_3+k_4)+k_3+k_4]\rho_{4,4} + (1+\bar n)(k_2\rho_{8,8}+k_1\rho_{12,12})\\
\dot\rho_{4,5} &= -[\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+\frac{1}{2}k_3+\frac{1}{2}k_4]\rho_{4,5} + (1+\bar n)(k_1\rho_{12,13})\\
\dot\rho_{4,6} &= \bar n k_4\rho_{3,5} - [\bar n(k_1+k_2+k_3+k_4)+\frac{1}{2}k_2+\frac{1}{2}k_3+k_4]\rho_{4,6} + (1+\bar n)(k_1\rho_{12,14})
\end{aligned}
$$

$$
\begin{aligned}
\dot{\rho}_{4,7} &= \bar{n}(k_3\rho_{2,5}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_2 + \frac{1}{2}k_4 + k_3]\rho_{4,7} + (1+\bar{n})(k_1\rho_{12,15}) \\
\dot{\rho}_{4,8} &= \bar{n}(k_3\rho_{2,6} + k_4\rho_{3,7}) - [\bar{n}(k_1+k_2+k_3+k_4)\frac{1}{2}k_2 + k_3 + k_4]\rho_{4,8} + (1+\bar{n})(k_1\rho_{12,16}) \\
\dot{\rho}_{4,9} &= -[\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + \frac{1}{2}k_3 + \frac{1}{2}k_4]\rho_{4,9} + (1+\bar{n})(k_2\rho_{8,13}) \\
\dot{\rho}_{4,10} &= \bar{n}(k_4\rho_{3,9}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + \frac{1}{2}k_3 + k_4]\rho_{4,10} + (1+\bar{n})(k_2\rho_{8,14}) \\
\dot{\rho}_{4,11} &= \bar{n}(k_3\rho_{2,9}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + \frac{1}{2}k_4 + k_3]\rho_{4,11} + (1+\bar{n})(k_2\rho_{8,15}) \\
\dot{\rho}_{4,12} &= \bar{n}(k_3\rho_{2,10} + k_4\rho_{3,11}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + k_3 + k_4]\rho_{4,12} + (1+\bar{n})(k_2\rho_{8,16}) \\
\dot{\rho}_{4,13} &= -[(\bar{n}+1)(k_1+k_2+k_3+k_4)]\rho_{4,13} \\
\dot{\rho}_{4,14} &= \bar{n}(k_4\rho_{3,13}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + \frac{1}{2}k_2 + \frac{1}{2}k_3 + k_4]\rho_{4,14} \\
\dot{\rho}_{4,15} &= \bar{n}(k_3\rho_{2,13}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + \frac{1}{2}k_2 + \frac{1}{2}k_4 + k_3]\rho_{4,15} \\
\dot{\rho}_{4,16} &= \bar{n}(k_3\rho_{2,14} + k_4\rho_{3,15}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + \frac{1}{2}k_2 + k_3 + k_4]\rho_{4,16} \\
\dot{\rho}_{5,5} &= \bar{n}(k_2\rho_{1,1}) - [\bar{n}(k_1+k_2+k_3+k_4) + k_2]\rho_{5,5} + (1+\bar{n})(k_4\rho_{6,6} + k_3\rho_{7,7} + k_1\rho_{13,13}) \\
\dot{\rho}_{5,6} &= \bar{n}(k_2\rho_{1,2}) - [\bar{n}(k_1+k_2+k_3+k_4) + k_2 + \frac{1}{2}k_4]\rho_{5,6} + (1+\bar{n})(k_3\rho_{7,8} + k_1\rho_{13,14}) \\
\dot{\rho}_{5,7} &= \bar{n}(k_2\rho_{1,3}) - [\bar{n}(k_1+k_2+k_3+k_4) + k_2 + k_4 + \frac{1}{2}k_3]\rho_{5,7} + (1+\bar{n})k_1\rho_{13,14}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\rho}_{5,8} &= \bar{n}(k_2\rho_{1,4}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_4 + \frac{1}{2}k_3 + k_2]\rho_{5,8} + (1+\bar{n})k_1\rho_{13,15} \\
\dot{\rho}_{5,9} &= -[\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_4 + \frac{1}{2}k_3 + k_2]\rho_{5,9} + (1+\bar{n})(k_4\rho_{6,10} + k_3\rho_{7,11}) \\
\dot{\rho}_{5,10} &= -[\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + \frac{1}{2}k_2 + \frac{1}{2}k_4]\rho_{5,10} + (1+\bar{n})k_3\rho_{7,12} \\
\dot{\rho}_{5,11} &= -[\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + \frac{1}{2}k_2 + \frac{1}{2}k_3 + k_4]\rho_{5,11} \\
\dot{\rho}_{5,12} &= -[(\bar{n}+\frac{1}{2})(k_1+k_2+k_3+k_4)]\rho_{5,12} \\
\dot{\rho}_{5,13} &= \bar{n}(k_2\rho_{1,9}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + k_2]\rho_{5,13} + (1+\bar{n})(k_4\rho_{6,14} + k_3\rho_{7,15}) \\
\dot{\rho}_{5,14} &= \bar{n}(k_2\rho_{1,10}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + \frac{1}{2}k_4 + k_2]\rho_{5,14} + (1+\bar{n})k_3\rho_{7,16} \\
\dot{\rho}_{5,15} &= \bar{n}(k_2\rho_{1,11}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + \frac{1}{2}k_3 + k_2 + k_4]\rho_{5,15} + (1+\bar{n})k_4\rho_{6,16} \\
\dot{\rho}_{5,16} &= \bar{n}(k_2\rho_{1,12}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + \frac{1}{2}k_2 + \frac{1}{2}k_4 + k_3]\rho_{5,16} \\
\dot{\rho}_{6,6} &= \bar{n}(k_2\rho_{2,2} + k_4\rho_{5,5}) - [\bar{n}(k_1+k_2+k_3+k_4) + k_2 + k_4]\rho_{6,6} + (1+\bar{n})(k_3\rho_{8,8} + k_1\rho_{14,14} \\
\dot{\rho}_{6,7} &= \bar{n}k_2\rho_{2,3} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_2+k_4) + k_3]\rho_{6,7} + (1+\bar{n})k_1\rho_{14,15} \\
\dot{\rho}_{6,8} &= \bar{n}(k_2\rho_{2,4} + k_4\rho_{5,7}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_3 + k_2 + k_4]\rho_{6,8} + (1+\bar{n})k_1\rho_{14,16} \\
\dot{\rho}_{6,9} &= -[\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_2+k_4)]\rho_{6,9} + (1+\bar{n})k_3\rho_{8,11} \\
\dot{\rho}_{6,10} &= \bar{n}k_4\rho_{5,9} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_2) + k_4]\rho_{6,10} + (1+\bar{n})k_3\rho_{8,12} \\
\dot{\rho}_{6,11} &= -[(\bar{n}+\frac{1}{2})(k_1+k_2+k_3+k_4)]\rho_{6,11} \\
\dot{\rho}_{6,12} &= \bar{n}k_4\rho_{5,11} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_2+k_3) + k_4]\rho_{6,12} \\
\dot{\rho}_{6,13} &= \bar{n}k_2\rho_{2,9} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_4) + k_2]\rho_{6,13} + (1+\bar{n})k_3\rho_{8,15} \\
\dot{\rho}_{6,14} &= \bar{n}(k_2\rho_{2,10} + k_4\rho_{5,13}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + k_2 + k_4]\rho_{6,14} + (1+\bar{n})k_3\rho_{8,16}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\rho}_{6,15} &= \bar{n}k_2\rho_{2,11} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_3+k_4) + k_2]\rho_{6,15} \\
\dot{\rho}_{6,16} &= \bar{n}(k_2\rho_{2,12} + k_4\rho_{5,15}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_3) + k_2 + k_4]\rho_{6,13} \\
\dot{\rho}_{7,7} &= \bar{n}(k_2\rho_{3,3} + k_3\rho_{5,5}) - [\bar{n}(k_1+k_2+k_3+k_4) + k_2 + k_3]\rho_{7,7} + (1+\bar{n})(k_4\rho_{8,8} + k_1\rho_{15,15}) \\
\dot{\rho}_{7,8} &= \bar{n}(k_2\rho_{3,4} + k_3\rho_{5,6}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_4 + k_2 + k_3]\rho_{7,8} + (1+\bar{n})k_1\rho_{15,16} \\
\dot{\rho}_{7,9} &= \bar{n}k_2\rho_{3,4} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_2+k_3)]\rho_{7,8} + (1+\bar{n})k_4\rho_{8,10} \\
\dot{\rho}_{7,10} &= -[(\bar{n}+1)(k_1+k_2+k_3+k_4)\rho_{7,10} \\
\dot{\rho}_{7,11} &= \bar{n}k_3\rho_{5,9} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_2) + k_3]\rho_{7,11} + (1+\bar{n})k_4\rho_{8,12} \\
\dot{\rho}_{7,12} &= \bar{n}k_3\rho_{5,10} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_2+k_4) + k_3]\rho_{7,12} \\
\dot{\rho}_{7,13} &= \bar{n}k_2\rho_{3,9} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_3) + k_2]\rho_{7,13} + (1+\bar{n})k_4\rho_{8,14} \\
\dot{\rho}_{7,14} &= \bar{n}k_2\rho_{3,10} - [\bar{n}(k_1+k_3+k_4) + k_2]\rho_{7,14} \\
\dot{\rho}_{7,15} &= \bar{n}(k_2\rho_{3,11} + k_3\rho_{5,13}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_1 + k_2 + k_3]\rho_{7,15} + (1+\bar{n})k_4\rho_{8,16} \\
\dot{\rho}_{7,16} &= \bar{n}k_2\rho_{3,12} + k_3\rho_{5,14} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_4) + k_2 + k_3]\rho_{7,16} \\
\dot{\rho}_{8,8} &= \bar{n}(k_2\rho_{4,4} + k_3\rho_{6,6}) - [\bar{n}(k_1+k_2+k_3+k_4) + k_2 + k_3 + k_4]\rho_{8,8} + (1+\bar{n})k_1\rho_{15,15}) \\
\dot{\rho}_{8,9} &= -[(\bar{n}+1)(k_1+k_2+k_3+k_4)\rho_{8,9} \\
\dot{\rho}_{8,10} &= \bar{n}k_4\rho_{7,9} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_2+k_3) + k_4]\rho_{8,10} \\
\dot{\rho}_{8,11} &= \bar{n}k_3\rho_{6,9} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_2+k_4) + k_3]\rho_{8,11} \\
\dot{\rho}_{8,12} &= \bar{n}k_3\rho_{6,10} + k_4\rho_{7,11} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_2) + k_3 + k_4]\rho_{8,12} \\
\dot{\rho}_{8,13} &= \bar{n}k_2\rho_{4,9} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_3+k_4) + k_2]\rho_{8,13} \\
\dot{\rho}_{8,14} &= \bar{n}k_2\rho_{4,10} + k_4\rho_{7,13} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_3) + k_2 + k_4]\rho_{8,14} \\
\dot{\rho}_{8,15} &= \bar{n}k_2\rho_{4,11} + k_3\rho_{6,13} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_1+k_4) + k_2 + k_3]\rho_{8,15}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\rho}_{8,16} &= \bar{n}(k_2\rho_{4,12} + k_3\rho_{6,14} + k_4\rho_{7,15}) - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}k_1 + k_2 + k_3 + k_4]\rho_{8,16} \\
&\quad + (1 + \bar{n})k_4\rho_{10,10} \\
\dot{\rho}_{9,9} &= \bar{n}k_1\rho_{1,1} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + k_1]\rho_{9,9} + (1 + \bar{n})(k_3\rho_{11,11} + k_2\rho_{13,13}) \\
\dot{\rho}_{9,10} &= \bar{n}(k_1\rho_{1,2} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}k_4 + k_1]\rho_{9,10} \\
&\quad + (1 + \bar{n})(k_3\rho_{11,12} + k_2\rho_{13,14}) \\
\dot{\rho}_{9,11} &= \bar{n}k_1\rho_{1,3} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}k_3 + k_1 + k_4]\rho_{9,11} + (1 + \bar{n})k_2\rho_{13,15} \\
\dot{\rho}_{9,12} &= \bar{n}k_1\rho_{1,4} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}(k_3 + k_4) + k_1]\rho_{9,12} + (1 + \bar{n})k_2\rho_{13,16} \\
\dot{\rho}_{9,13} &= \bar{n}k_1\rho_{1,5} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}k_2 + k_1 + k_4]\rho_{9,13} + (1 + \bar{n})(k_4\rho_{10,14} + k_3\rho_{11,15}) \\
\dot{\rho}_{9,14} &= \bar{n}k_1\rho_{1,6} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}(k_2 + k_4) + k_1]\rho_{9,14} + (1 + \bar{n})k_3\rho_{11,16} \\
\dot{\rho}_{9,15} &= \bar{n}k_1\rho_{1,7} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}(k_2 + k_3) + k_1 + k_4]\rho_{9,15} + (1 + \bar{n})k_4\rho_{10,16} \\
\dot{\rho}_{9,16} &= \bar{n}k_1\rho_{1,8} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}(k_2 + k_3 + k_4) + k_1]\rho_{8,10} \\
\dot{\rho}_{10,10} &= \bar{n}k_1\rho_{2,2} + k_4\rho_{9,9} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + k_1 + k_4]\rho_{10,10} + (1 + \bar{n})(k_3\rho_{12,12} + k_2\rho_{14,14}) \\
\dot{\rho}_{10,11} &= \bar{n}k_1\rho_{2,3} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}(k_3 + k_4) + k_1]\rho_{10,11} + (1 + \bar{n})k_2\rho_{14,15} \\
\dot{\rho}_{10,12} &= \bar{n}k_1\rho_{2,4} + k_4\rho_{9,11} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}k_3 + k_1 + k_4]\rho_{10,12} \\
&\quad + (1 + \bar{n})k_2\rho_{14,16} \\
\dot{\rho}_{10,13} &= \bar{n}k_1\rho_{2,5} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}(k_2 + k_4) + k_1]\rho_{10,13} + (1 + \bar{n})k_3\rho_{12,15} \\
\dot{\rho}_{10,14} &= \bar{n}(k_1\rho_{2,6} + k_4\rho_{9,13}) - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}k_2 + k_1 + k_4]\rho_{10,11} \\
&\quad + (1 + \bar{n})k_3\rho_{12,16} \\
\dot{\rho}_{10,15} &= \bar{n}k_1\rho_{2,7} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}(k_2 + k_3 + k_4) + k_1]\rho_{10,15} \\
\dot{\rho}_{10,16} &= \bar{n}(k_1\rho_{2,8} + k_4\rho_{9,15}) - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}(k_2 + k_3) + k_1 + k_4]\rho_{10,16} \\
\dot{\rho}_{11,11} &= \bar{n}(k_1\rho_{3,3} + k_3\rho_{9,9}) - [\bar{n}(k_1 + k_2 + k_3 + k_4) + k_1 + k_3]\rho_{11,11} \\
&\quad + (1 + \bar{n})(k_4\rho_{12,12} + k_2\rho_{15,15}) \\
\dot{\rho}_{11,12} &= \bar{n}k_1\rho_{3,4} + k_3\rho_{9,10} - [\bar{n}(k_1 + k_2 + k_3 + k_4) + \frac{1}{2}k_4 + k_1 + k_3]\rho_{11,12} + (1 + \bar{n})k_2\rho_{15,16}
\end{aligned}
$$

$$
\begin{aligned}
\dot{\rho}_{11,13} &= \bar{n}k_1\rho_{3,5} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_2+k_3) + k_1]\rho_{11,13} + (1+\bar{n})k_4\rho_{12,14} \\
\dot{\rho}_{11,14} &= \bar{n}k_1\rho_{3,6} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_2+k_3+k_4) + k_1]\rho_{11,14} \\
\dot{\rho}_{11,15} &= \bar{n}k_1\rho_{3,7} + k_3\rho_{9,13} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_2 + k_1 + k_3]\rho_{11,15} + (1+\bar{n})k_4\rho_{12,16} \\
\dot{\rho}_{11,16} &= \bar{n}k_1\rho_{3,8} + k_3\rho_{9,14} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_2+k_4) + k_1 + k_3]\rho_{11,16} \\
\dot{\rho}_{12,12} &= \bar{n}(k_1\rho_{4,4} + k_3\rho_{10,10} + k_4\rho_{11,11}) - [\bar{n}(k_1+k_2+k_3+k_4) + k_1 + k_3 + k_4]\rho_{12,12} \\
&\quad + (1+\bar{n})k_2\rho_{16,16} \\
\dot{\rho}_{12,13} &= \bar{n}k_1\rho_{4,5} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_2+k_3+k_4) + k_1]\rho_{12,13} \\
\dot{\rho}_{12,14} &= \bar{n}k_1\rho_{4,6} + k_4\rho_{11,13} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_2+k_3) + k_1 + k_4]\rho_{12,14} \\
\dot{\rho}_{12,15} &= \bar{n}k_1\rho_{4,7} + k_3\rho_{10,13} - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_2+k_4) + k_1 + k_3]\rho_{12,15} \\
\dot{\rho}_{12,16} &= \bar{n}(k_1\rho_{4,8} + k_3\rho_{10,14} + k_4\rho_{11,15}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_2 + k_1 + k_3 + k_4]\rho_{12,16} \\
\dot{\rho}_{13,13} &= \bar{n}(k_1\rho_{5,5} + k_2\rho_{9,9}) - [\bar{n}(k_1+k_2+k_3+k_4) + k_1 + k_3 + k_4]\rho_{13,13} \\
&\quad + (1+\bar{n})(k_4\rho_{14,14} + k_3\rho_{15,15}) \\
\dot{\rho}_{13,14} &= \bar{n}(k_1\rho_{5,6} + k_2\rho_{9,10}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_4 + k_1 + k_2]\rho_{13,14} + (1+\bar{n})k_3\rho_{15,16} \\
\dot{\rho}_{13,15} &= \bar{n}(k_1\rho_{5,7} + k_2\rho_{9,11}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_3 + k_1 + k_2]\rho_{13,15} + (1+\bar{n})k_4\rho_{14,16} \\
\dot{\rho}_{13,16} &= \bar{n}(k_1\rho_{5,8} + k_2\rho_{9,12}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_3+k_4) + k_1 + k_2]\rho_{13,16} \\
\dot{\rho}_{14,14} &= \bar{n}(k_1\rho_{6,6} + k_2\rho_{10,10} + k_4\rho_{13,13}) - [\bar{n}(k_1+k_2+k_3+k_4) + k_1 + k_2 + k_4]\rho_{14,14} \\
&\quad + (1+\bar{n})k_3\rho_{16,16} \\
\dot{\rho}_{14,15} &= \bar{n}(k_1\rho_{6,7} + k_2\rho_{10,11}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}(k_3+k_4) + k_1 + k_2]\rho_{14,15} \\
\dot{\rho}_{14,16} &= \bar{n}(k_1\rho_{6,8} + k_3\rho_{10,12} + k_4\rho_{13,15}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_3 + k_1 + k_2 + k_4]\rho_{14,16} \\
\dot{\rho}_{15,15} &= \bar{n}(k_1\rho_{7,7} + k_2\rho_{11,11} + k_3\rho_{13,13}) - [\bar{n}(k_1+k_2+k_3+k_4) + k_1 + k_2 + k_3]\rho_{15,15} \\
&\quad + (1+\bar{n})k_4\rho_{16,16} \\
\dot{\rho}_{15,16} &= \bar{n}(k_1\rho_{7,8} + k_2\rho_{11,12} + k_3\rho_{13,14}) - [\bar{n}(k_1+k_2+k_3+k_4) + \frac{1}{2}k_4 + k_1 + k_2 + k_3]\rho_{15,16} \\
\dot{\rho}_{16,16} &= \bar{n}k_1\rho_{8,8} + k_2\rho_{12,12} + k_3\rho_{14,14} + k_4\rho_{15,15}) - [(\bar{n}+1)(k_1+k_2+k_3+k_4)]\rho_{16,16}
\end{aligned}
$$

For the reservoir in vacuum($n = 0$) and $k_1 = k_2 = k_3 = k_4 = 0$, the solutions for above mentioned density matrix (where $\rho_{ij} = \rho_{ji}$) are given below

$$
\begin{aligned}
\rho_{1,1} &= 1 - e^{-4kt}C_{16,16}e^{1-C_{1,1}} \\
\rho_{1,2} &= -e^{\frac{-7kt}{2}}C_{15,16} + e^{\frac{-5kt}{2}}(C_{7,8} + C_{11,12} + C_{13,14} \\
&\quad +3C_{15,16}) + e^{\frac{-kt}{2}}(C_{1,2} + C_{3,4} + C_{5,6}C_{7,8} + C_{9,10} + C_{11,12} + C_{13,14} + C_{15,16}) \\
&\quad -e^{\frac{-3kt}{2}}(C_{3,4} + C_{5,6} + C_{7,8} + C_{9,10} + 2(C_{11,12} + C_{13,14}) + 3C_{15,16}) \\
\rho_{1,3} &= -e^{\frac{-7kt}{2}}C_{13,16} + e^{\frac{-5kt}{2}}(C_{6,8} + C_{10,12} + C_{13,15} + 3C_{14,16}) \\
&\quad +e^{\frac{-kt}{2}}(C_{1,3} + C_{2,4} + C_{5,7}C_{6,8} + C_{9,11} + C_{10,12} + C_{13,15} + C_{14,16}) \\
&\quad -e^{\frac{-3kt}{2}}(C_{2,4} + C_{5,7} + C_{6,8} + C_{9,11} + 2(C_{10,12} + C_{13,15}) + 3C_{14,16}) \\
\rho_{1,4} &= -e^{-3kt}C_{13,16} + e^{-kt}(C_{1,4} + C_{5,8} + C_{9,12} + C_{13,16}) \\
&\quad +e^{-2kt}(C_{5,8} + C_{9,12} + 2C_{13,16}) + e^{-3kt}(C_{11,16}) + e^{-kt}(C_{1,6} + C_{3,8} \\
&\quad +C_{9,14} + C_{11,16}) - e^{-2kt}(C_{3,8} + C_{9,14} + 2C_{11,16}) \\
\rho_{1,6} &= -e^{-3kt}C_{11,16} + e^{-kt}(C_{1,6} + C_{3,8} + C_{9,14} + C_{11,16}) - e^{-2kt}(C_{3,8} + C_{9,14} + 2C_{11,16}) \\
\rho_{1,7} &= -e^{-3kt}C_{10,16} + e^{-kt}(C_{1,7} + C_{2,8} + C_{9,15} + C_{10,16}) - e^{-2kt}(C_{2,8} + C_{9,15} + 2C_{10,16}) \\
\rho_{1,8} &= -e^{\frac{-5kt}{2}}C_{9,11} + e^{\frac{-3kt}{2}}(C_{1,8} + C_{9,11}) \\
\rho_{1,9} &= -e^{\frac{-7kt}{2}}C_{8,16} + e^{\frac{-5kt}{2}}(C_{4,12} + C_{6,14} + C_{7,15} + 3C_{8,16}) + \\
&\quad e^{\frac{-kt}{2}}(C_{1,9} + C_{2,10} + C_{3,11} + C_{4,12} + C_{6,14} + C_{7,15} + 3C_{8,16}) \\
&\quad -e^{\frac{-3kt}{2}}(C_{2,10} + C_{3,11} + 2C_{4,12} + 2(C_{6,14} + C_{7,15}) + 3C_{8,16}) \\
\rho_{1,10} &= -e^{-3kt}C_{7,16} + e^{-kt}(C_{1,10} + C_{3,12} + C_{5,14} + C_{7,16}) - e^{-2kt}(C_{3,12} + C_{5,14} + 2C_{7,16}) \\
\rho_{1,11} &= -e^{-3kt}C_{6,16} + e^{-kt}(C_{1,11} + C_{2,12} + C_{5,15} + C_{6,16}) - e^{-2kt}(C_{2,12} + C_{5,15} + 2C_{6,16}) \\
\rho_{1,12} &= -e^{\frac{-5kt}{2}}C_{5,16} + e^{\frac{-3kt}{2}}(C_{1,12} + C_{5,16}) \\
\rho_{1,13} &= -e^{-3kt}C_{4,16} + e^{-kt}(C_{1,13} + C_{2,14} + C_{3,15} + C_{4,16}) - e^{-2kt}(C_{2,14} + C_{3,15} + 2C_{4,16}) \\
\rho_{1,14} &= -e^{\frac{-5kt}{2}}C_{3,16} + e^{\frac{-3kt}{2}}(C_{1,14} + C_{3,16}) \\
\rho_{1,12} &= -e^{\frac{-5kt}{2}}C_{2,16} + e^{\frac{-3kt}{2}}(C_{1,15} + C_{2,16}) \\
\rho_{1,16} &= e^{-2kt}C_{16,16} \\
\rho_{2,2} &= -e^{-4kt}C_{16,16} + e^{-3kt}(C_{8,8} + C_{12,12} + C_{14,14} + 3C_{16,16} \\
&\quad +e^{-kt}(C_{2,2} + C_{4,4} + C_{6,6} + C_{8,8} + C_{10,10} + C_{12,12} + C_{14,14} + C_{16,16}) \\
&\quad -e^{-2kt}(C_{4,4} + C_{6,6} + 2C_{8,8} + C_{10,10} + 2(C_{12,12} + C_{14,14}) + 3C_{16,16}) \\
\rho_{2,3} &= e^{-3kt}C_{14,15} + e^{-kt}(C_{2,3} + C_{6,7} + C_{10,11} + C_{14,15}) - e^{-2kt}(C_{6,7} + C_{10,11} + 2C_{14,15})
\end{aligned}
$$

$$
\begin{aligned}
\rho_{2,4} &= e^{\frac{-7kt}{2}}C_{4,16} + e^{\frac{-3kt}{2}}(C_{2,4} + C_{6,8} + C_{10,12} + C_{14,16}) - e^{\frac{-5kt}{2}} \\
&\quad (C_{6,8} + C_{10,12} + 2C_{14,16}) \\
\rho_{2,5} &= e^{-3kt}C_{12,15} + e^{-kt}(C_{2,5} + C_{4,7} + C_{10,13} + C_{12,15}) - \\
&\quad e^{-2kt}(C_{4,7} + C_{10,13} + 2C_{12,15}) \\
\rho_{2,6} &= e^{\frac{-7kt}{2}}C_{12,16} + e^{\frac{-3kt}{2}}(C_{2,6} + C_{4,8} + C_{10,14} + C_{12,16}) \\
&\quad -e^{\frac{-5kt}{2}}(C_{4,8} + C_{10,14} + 2C_{12,16}) \\
\rho_{2,7} &= -e^{\frac{-5kt}{2}}C_{10,15} + e^{\frac{-3kt}{2}}(C_{2,7} + C_{10,15}) \\
\rho_{2,8} &= -e^{-3kt}C_{10,10} + e^{-2kt}(C_{2,8} + C_{10,16}) \\
\rho_{2,9} &= e^{-3kt}C_{8,15} + e^{-kt}(C_{2,9} + C_{4,11} + C_{6,13} + C_{8,15}) - \\
&\quad e^{-2kt}(C_{4,11} + C_{6,13} + 2C_{8,15}) \\
\rho_{2,10} &= e^{\frac{-7kt}{2}}C_{8,16} + e^{\frac{-3kt}{2}}(C_{2,10} + C_{4,12} + C_{6,14} + C_{8,16}) \\
&\quad -e^{\frac{-5kt}{2}}(C_{4,12} + C_{6,14} + 2C_{8,16}) \\
\rho_{2,11} &= -e^{\frac{-5kt}{2}}C_{6,15} + e^{\frac{-3kt}{2}}(C_{2,11} + C_{6,15}) \\
\rho_{2,12} &= -e^{-3kt}C_{6,16} + e^{-2kt}(C_{2,12} + C_{6,16}) \\
\rho_{2,13} &= -e^{\frac{-5kt}{2}}C_{4,15} + e^{\frac{-3kt}{2}}(C_{2,13} + C_{4,15}) \\
\rho_{2,14} &= -e^{-3kt}C_{4,16} + e^{-2kt}(C_{2,14} + C_{4,16}) \\
\rho_{2,15} &= -e^{-2kt}C_{2,15} \\
\rho_{2,16} &= e^{\frac{-5kt}{2}}C_{2,16} \\
\rho_{3,3} &= -e^{-4kt}C_{16,16} + e^{-3kt}(C_{8,8} + C_{12,12} + C_{14,14} + 3C_{16,16} + e^{-kt}(C_{3,3} \\
&\quad +C_{4,4} + C_{7,7} + C_{8,8} + C_{11,11} + C_{12,12}) + C_{15,15} + C_{16,16}) - \\
&\quad e^{-2kt}(C_{4,4} + C_{7,7} + C_{8,8} + C_{11,11} + 2(C_{12,12}) + C_{15,15} + 3C_{16,16}) \\
\rho_{3,4} &= e^{\frac{-7kt}{2}}C_{15,16} + e^{\frac{-3kt}{2}}(C_{3,4} + C_{7,8} + C_{11,12} + C_{15,16}) \\
&\quad -e^{\frac{-5kt}{2}}(C_{7,8} + C_{11,12} + 2C_{15,16}) \\
\rho_{3,5} &= e^{-3kt}C_{12,14} + e^{-kt}(C_{3,5} + C_{4,6} + C_{11,13} + C_{12,14}) - \\
&\quad e^{-2kt}(C_{4,6} + C_{11,13} + 2C_{12,14}) \\
\rho_{3,6} &= -e^{\frac{-5kt}{2}}C_{11,14} + e^{\frac{-3kt}{2}}(C_{3,6} + C_{11,14})
\end{aligned}
$$

$$
\begin{aligned}
\rho_{3,7} &= e^{\frac{-7kt}{2}}C_{12,16} + e^{\frac{-3kt}{2}}(C_{3,7} + C_{4,8} + C_{11,15} + C_{12,16}) \\
&\quad -e^{\frac{-5kt}{2}}(C_{4,8} + C_{11,15} + 2C_{12,16}) \\
\rho_{3,8} &= -e^{-3kt}C_{11,16} + e^{-2kt}(C_{3,8} + C_{11,16}) \\
\rho_{3,9} &= e^{-3kt}C_{8,14} + e^{-kt}(C_{3,9} + C_{4,10} + C_{7,13} + C_{8,14}) - \\
&\quad e^{-2kt}(C_{4,10} + C_{7,13} + 2C_{8,14}) \\
\rho_{3,10} &= -e^{\frac{-5kt}{2}}C_{7,14} + e^{\frac{-3kt}{2}}(C_{3,10} + C_{7,14}) \\
\rho_{3,11} &= e^{\frac{-7kt}{2}}C_{8,10} + e^{\frac{-3kt}{2}}(C_{3,11} + C_{4,12} + C_{7,15} + C_{8,16}) \\
&\quad -e^{\frac{-5kt}{2}}(C_{4,12} + C_{7,15} + 2C_{8,16}) \\
\rho_{3,12} &= -e^{-3kt}C_{7,16} + e^{-2kt}(C_{3,12} + C_{7,16}) \\
\rho_{3,13} &= -e^{\frac{-5kt}{2}}C_{4,14} + e^{\frac{-3kt}{2}}(C_{3,13} + C_{4,14}) \\
\rho_{3,14} &= e^{-2kt}C_{3,14} \\
\rho_{3,15} &= -e^{-3kt}C_{4,16} + e^{-2kt}(C_{3,15} + C_{4,16}) \\
\rho_{3,16} &= e^{-5kt}C_{3,16} \\
\rho_{4,4} &= e^{-4kt}C_{16,16} + e^{-2kt}(C_{4,4} + C_{8,8} + C_{12,12} + C_{16,16} \\
&\quad +e^{-3kt}(C_{8,8} + C_{12,12} + 2C_{16,16}) \\
\rho_{4,5} &= -e^{\frac{-5kt}{2}}C_{12,13} + e^{\frac{-3kt}{2}}(C_{4,5} + C_{12,13}) \\
\rho_{4,6} &= -e^{-3kt}C_{12,14} + e^{-2kt}(C_{4,6} + C_{12,14}) \\
\rho_{4,7} &= -e^{-3kt}C_{12,15} + e^{-2kt}(C_{4,7} + C_{12,15}) \\
\rho_{4,8} &= -e^{\frac{-7kt}{2}}C_{12,16} + e^{\frac{-5kt}{2}}(C_{4,8} + C_{12,16}) \\
\rho_{4,9} &= -e^{\frac{-5kt}{2}}C_{8,13} + e^{\frac{-3kt}{2}}(C_{4,9} + C_{8,13}) \\
\rho_{4,10} &= -e^{-3kt}C_{8,14} + e^{-2kt}(C_{4,10} + C_{8,14}) \\
\rho_{4,11} &= -e^{-3kt}C_{12,15} + e^{-2kt}(C_{4,11} + C_{8,15}) \\
\rho_{4,12} &= -e^{\frac{-7kt}{2}}C_{8,16} + e^{\frac{-5kt}{2}}(C_{4,12} + C_{8,16}) \\
\rho_{4,13} &= e^{-2kt}C_{4,13} \\
\rho_{4,14} &= e^{\frac{-5kt}{2}}C_{4,14} \\
\rho_{4,15} &= e^{\frac{-5kt}{2}}C_{4,15} \\
\rho_{4,16} &= e^{-3kt}C_{4,16}
\end{aligned}
$$

$$
\begin{aligned}
\rho_{5,5} &= -e^{-4kt}C_{16,16} + e^{-3kt}(C_{8,8} + C_{14,14} + C_{15,15} + 3C_{16,16} + e^{-kt}(C_{5,5} + C_{6,6} + C_{7,7} \\
&\quad +C_{8,8} + C_{13,13} + C_{14,14}) + C_{15,15} + C_{16,16}) - e^{-2kt}(C_{6,6} + C_{7,7} + C_{8,8} + C_{13,13} \\
&\quad +2(C_{14,14}) + C_{15,15}) + 3C_{16,16}) \\
\rho_{5,6} &= e^{\frac{-7kt}{2}}C_{15,16} + e^{\frac{-3kt}{2}}(C_{5,6} + C_{7,8} + C_{13,14} + C_{15,16}) \\
&\quad -e^{\frac{-5kt}{2}}(C_{7,8} + C_{13,14} + 2C_{15,16}) \\
\rho_{5,7} &= e^{\frac{-7kt}{2}}C_{14,16} + e^{\frac{-3kt}{2}}(C_{5,7} + C_{6,8} + C_{13,15} + C_{14,16}) \\
&\quad -e^{\frac{-5kt}{2}}(C_{6,8} + C_{13,15} + 2C_{14,16}) \\
\rho_{5,8} &= -e^{-3kt}C_{13,16} + e^{-2kt}(C_{5,8} + C_{13,16}) \\
\rho_{5,9} &= e^{-3kt}C_{6,12} + e^{-kt}(C_{5,9} + C_{6,10} + C_{7,11} + C_{8,12}) - e^{-2kt}(C_{6,10} \\
&\quad +C_{7,11} + 2C_{8,12}) \\
\rho_{5,10} &= -e^{\frac{-5kt}{2}}C_{7,12} + e^{\frac{-3kt}{2}}(C_{5,10} + C_{7,12}) \\
\rho_{5,11} &= -e^{\frac{-5kt}{2}}C_{6,12} + e^{\frac{-3kt}{2}}(C_{5,11} + C_{6,12}) \\
\rho_{5,12} &= e^{-2kt}C_{5,12} \\
\rho_{5,13} &= e^{\frac{-7kt}{2}}C_{8,10} + e^{\frac{-3kt}{2}}(C_{5,13} + C_{6,14} + C_{7,15} + C_{8,16}) \\
&\quad -e^{\frac{-5kt}{2}}(C_{6,14} + C_{7,15} + 2C_{8,16}) \\
\rho_{5,14} &= -e^{-3kt}C_{7,10} + e^{-2kt}(C_{5,14} + C_{7,16}) \\
\rho_{5,15} &= -e^{-3kt}C_{6,16} + e^{-2kt}(C_{5,15} + C_{6,16}) \\
\rho_{5,16} &= e^{\frac{-5kt}{2}}C_{5,16} \\
\rho_{6,6} &= -e^{-4kt}C_{16,16} + e^{-2kt}(C_{6,6} + C_{8,8} + C_{14,14} + C_{16,16} \\
&\quad +e^{-3kt}(C_{8,8} + C_{14,14} + 2C_{16,16}) \\
\rho_{6,7} &= -e^{-3kt}C_{14,15} + e^{-2kt}(C_{6,7} + C_{14,15}) \\
\rho_{6,8} &= e^{\frac{-7kt}{2}}C_{14,16} + e^{\frac{-5kt}{2}}(C_{6,8} + C_{14,16}) \\
\rho_{6,9} &= e^{\frac{-5kt}{2}}C_{8,11} + e^{\frac{-3kt}{2}}(C_{6,9} + C_{8,11}) \\
\rho_{6,10} &= -e^{-3kt}C_{8,12} + e^{-2kt}(C_{6,10} + C_{8,12}) \\
\rho_{6,11} &= e^{-2kt}C_{6,11} \\
\rho_{6,12} &= e^{\frac{-5kt}{2}}C_{6,12} \\
\rho_{6,13} &= -e^{-3kt}C_{8,15} + e^{-2kt}(C_{6,13} + C_{8,15}) \\
\rho_{6,14} &= e^{\frac{-7kt}{2}}C_{8,16} + e^{\frac{-5kt}{2}}(C_{6,14} + C_{8,16}) \\
\rho_{6,15} &= e^{\frac{-5kt}{2}}C_{6,15} \\
\rho_{6,16} &= e^{-3kt}C_{6,16}
\end{aligned}
$$

$$\begin{aligned}
\rho_{7,7} &= e^{-4kt}C_{16,16} + e^{-2kt}(C_{7,7} + C_{8,8} + C_{15,15} + C_{16,16}) - \\
&\quad e^{-3kt}(C_{8,8} + C_{15,15} + 2C_{16,16}) \\
\rho_{7,8} &= -e^{\frac{-7kt}{2}}C_{15,16} + e^{\frac{-5kt}{2}}(C_{7,8} + C_{15,16}) \\
\rho_{7,9} &= -e^{\frac{-5kt}{2}}C_{8,10} + e^{\frac{-3kt}{2}}(C_{7,9} + C_{8,10}) \\
\rho_{7,10} &= e^{-2kt}C_{7,10} \\
\rho_{7,11} &= -e^{-3kt}C_{8,12} + e^{-2kt}(C_{7,11} + C_{8,12}) \\
\rho_{7,12} &= e^{\frac{-5kt}{2}}C_{7,12} \\
\rho_{7,13} &= -e^{-3kt}C_{8,14} + e^{-2kt}(C_{7,13} + C_{8,14}) \\
\rho_{7,14} &= e^{\frac{-5kt}{2}}C_{7,14} \\
\rho_{7,15} &= -e^{\frac{-7kt}{2}}C_{8,10} + e^{\frac{-5kt}{2}}(C_{7,15} + C_{8,10}) \\
\rho_{7,16} &= e^{-3kt}C_{7,16} \\
\rho_{8,8} &= -e^{-4kt}C_{16,16} + e^{-3kt}(C_{8,8} + C_{16,16}) \\
\rho_{8,9} &= e^{-2kt}C_{8,9} \\
\rho_{8,10} &= e^{\frac{-5kt}{2}}C_{8,10} \\
\rho_{8,11} &= e^{\frac{-5kt}{2}}C_{8,11} \\
\rho_{8,12} &= e^{-3kt}C_{8,12} \\
\rho_{8,13} &= e^{\frac{-5kt}{2}}C_{8,13} \\
\rho_{8,14} &= e^{-3kt}C_{8,14} \\
\rho_{8,15} &= e^{-3kt}C_{8,15} \\
\rho_{8,16} &= e^{\frac{-7kt}{2}}C_{8,16} \\
\rho_{9,9} &= -e^{-4kt}C_{16,16} + e^{-3kt}(C_{12,12} + C_{14,14} + C_{15,15} + 3C_{16,16}) + e^{-3kt}(C_{9,9} \\
&\quad +C_{10,10} + C_{11,11} + C_{12,12} + C_{13,13} + C_{14,14} + C_{15,15} + C_{16,16}) - \\
&\quad e^{-2kt}(C_{10,10} + C_{11,11} + 2C_{12,12} + C_{13,13} + 2(C_{14,14} + C_{15,15}) + 3C_{16,16}) \\
\rho_{9,10} &= e^{\frac{-7kt}{2}}C_{15,16} + e^{\frac{-3kt}{2}}(C_{9,10} + C_{11,12} + C_{13,14} + C_{15,16}) \\
&\quad -e^{\frac{-5kt}{2}}(C_{11,12} + C_{13,14} + 2C_{15,16}) \\
\rho_{9,11} &= e^{\frac{-7kt}{2}}C_{14,16} + e^{\frac{-3kt}{2}}(C_{9,11} + C_{10,12} + C_{13,15} + C_{14,16}) \\
&\quad -e^{\frac{-5kt}{2}}(C_{10,12} + C_{13,15} + 2C_{14,16}) \\
\rho_{9,12} &= -e^{-3kt}C_{13,16} + e^{-2kt}(C_{9,12} + C_{13,16}) \\
\rho_{9,13} &= e^{\frac{-7kt}{2}}C_{12,16} + e^{\frac{-3kt}{2}}(C_{9,13} + C_{10,14} + C_{11,15} + C_{12,16}) \\
&\quad -e^{\frac{-5kt}{2}}(C_{10,14} + C_{11,15} + 2C_{12,16})
\end{aligned}$$

$$
\begin{aligned}
\rho_{9,14} &= -e^{-3kt}C_{11,16} + e^{-2kt}(C_{9,14} + C_{11,16}) \\
\rho_{9,15} &= -e^{-3kt}C_{10,16} + e^{-2kt}(C_{9,15} + C_{10,16}) \\
\rho_{9,16} &= e^{\frac{-5kt}{2}}C_{9,16} \\
\rho_{10,10} &= e^{-4kt}C_{16,16} + e^{-2kt}(C_{10,10} + C_{12,12} + C_{14,14} + C_{16,16}) + \\
&\quad e^{-3kt}(C_{12,12} + C_{14,14} + 2C_{16,16}) \\
\rho_{10,11} &= -e^{-3kt}C_{14,15} + e^{-2kt}(C_{10,11} + C_{14,15}) \\
\rho_{10,12} &= -e^{\frac{-7kt}{2}}C_{14,16} + e^{\frac{-5kt}{2}}(C_{10,12} + C_{14,16}) \\
\rho_{10,13} &= -e^{-3kt}C_{12,15} + e^{-2kt}(C_{10,13} + C_{12,15}) \\
\rho_{10,14} &= -e^{\frac{-7kt}{2}}C_{12,16} + e^{\frac{-5kt}{2}}(C_{10,14} + C_{12,16}) \\
\rho_{10,15} &= e^{\frac{-5kt}{2}}C_{10,15} \\
\rho_{10,16} &= e^{-3kt}C_{10,16} \\
\rho_{11,11} &= e^{-4kt}C_{16,16} + e^{-2kt}(C_{11,11} + C_{12,12} + C_{15,15} + C_{16,16}) - \\
&\quad e^{-3kt}(C_{12,12} + C_{15,15} + 2C_{16,16}) \\
\rho_{11,12} &= -e^{\frac{-7kt}{2}}C_{15,16} + e^{\frac{-5kt}{2}}(C_{11,12} + C_{15,16}) \\
\rho_{11,13} &= -e^{-3kt}C_{12,14} + e^{-2kt}(C_{11,13} + C_{12,14}) \\
\rho_{11,14} &= e^{\frac{-5kt}{2}}C_{11,14} \\
\rho_{11,15} &= -e^{\frac{-7kt}{2}}C_{12,16} + e^{\frac{-5kt}{2}}(C_{11,15} + C_{12,16}) \\
\rho_{11,16} &= e^{-3kt}C_{11,16} \\
\rho_{12,12} &= -e^{-4kt}C_{16,16} + e^{-3kt}(C_{12,12} + C_{16,16}) \\
\rho_{12,13} &= e^{\frac{-5kt}{2}}C_{12,13} \\
\rho_{12,14} &= e^{-3kt}C_{12,14} \\
\rho_{12,15} &= e^{-3kt}C_{12,15} \\
\rho_{12,16} &= e^{\frac{-7kt}{2}}C_{12,16} \\
\rho_{13,13} &= e^{-4kt}C_{16,16} + e^{-2kt}(C_{13,13} + C_{14,14} + C_{15,15} + C_{16,16}) - \\
&\quad e^{-3kt}(C_{14,14} + C_{15,15} + 2C_{16,16}) \\
\rho_{13,14} &= -e^{\frac{-7kt}{2}}C_{15,16} + e^{\frac{-5kt}{2}}(C_{13,14} + C_{15,16})
\end{aligned}
$$

$$
\begin{aligned}
\rho_{13,15} &= -e^{\frac{-7kt}{2}}C_{14,16} + e^{\frac{-5kt}{2}}(C_{13,15} + C_{14,16}) \\
\rho_{13,16} &= e^{-3kt}C_{13,16} \\
\rho_{14,14} &= -e^{-4kt}C_{16,16} + e^{-3kt}(C_{14,14} + C_{16,16}) \\
\rho_{14,15} &= e^{-3kt}C_{14,15} \\
\rho_{14,16} &= e^{\frac{-7kt}{2}}C_{14,16} \\
\rho_{15,15} &= -e^{-4kt}C_{16,16} + e^{-3kt}(C_{15,15} + C_{16,16}) \\
\rho_{15,16} &= e^{\frac{-7kt}{2}}C_{15,16} \\
\rho_{16,16} &= e^{-4kt}C_{16,16}
\end{aligned}
$$

Having these equation in hands, partial transpose can be calculated and their respected eigenvalues can be calculated which gives negativity of each four cavities.

Bibliography

[1] C. H. Bennett, S. J. Wiesner, *Communication via one and two-particle operators on Einstein Podolsky Rosen states*, Phys. Rev. Lett. **69**, 2881 (1992).

[2] P. W. Shor, *Polynomial-Time Algorithms for Prime Factorization and Discrete Logarithms on a Quantum Computer*, SIAM J. Comput. **26** (5) 1484 (1997).

[3] L. K. Grover, *Quantum Mechanics Helps in Searching for a Needle in a Haystack*, Phys. Rev. Lett. **79**, 325 (1997).

[4] A. Einstein, B. Podolsky and N. Rosen, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys. Rev. **47**, 777 (1935).

[5] J. S. Bell, *On the Einstein Podolsky Rosen paradox*, Physics **1**, 195 (1964).

[6] N. Bohr, *Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*, Phys. Rev. A **48**, 696 (1935).

[7] E. S. Fry and R. C. Thompson, *Experimental Test of Local Hidden Variable Theories*, Phys. Rev. Lett. **37**, 465 (1976).

[8] A. Aspect, P. Grangier, and G. Roger, *Experimental Tests of Realistic Local Theories via Bellis Theorem*, Phys. Rev. Lett. **47**, 460 (1981).

[9] Z. Y. Ou and L. Mandel, *Violation of Bell's Inequality and Classical Probability in a Two-Photon Correlation Experiment*, Phys. Rev. Lett. **61**, 50 (1988).

[10] J. Clauser, M. Horne, A. Shimony and R. Holt, *Proposed Experiment to Test Local Hidden Variable Theories*, Phys. Rev. Lett. **23**, 880 (1969).

[11] R. F. Werner, *Quantum States with Einstein Podolsky Rosen Correlations admitting a Hidden-Variable Model*, Phys. Rev. A **40**, 4277 (1989).

[12] X. Wang, M. Feng, B. C. Sanders, *Three qubits can be entangled in two inequivalent ways*, Phys. Rev. A **67**, 022302 (2003).

[13] J. Lee, J. Park, S. M. Lee, H. W. Lee, A. H. Khosa, *Generation of Bell, NOON and W states via atom interferometry*, Phys. Rev. A **77**, 032327, (2008).

[14] Q. Guo, C. Zhao, W. Tan, *Multipartite entanglement generation via ultrashort pulse*, J. Opt. Soc. Am. B, **28**, 2240 (2011).

[15] D. M. Greenberger, M. Horn, A. Zeilinger, in:M. Kafatos(Ed.),*Bell's Theorem, Quantum Theory and Conceptions of universe*, ed by M. Kafatos, Kluwer, Dordrecht, p.69, (1989).

[16] W. Dur, G. Vidal, J. I. Cirac, *Three qubits can be entangled in two inequivalent ways*, Phys. Rev. A **62**, 062314 (2000).

[17] M. Fleischhauer, M. D. Lukin, *Quantum memory for photons: Dark-state polaritons*, Phys. Rev. A **65**, 022314 (2002).

[18] X. L. Zhang, K. L. Gao, M. Feng, *Preparation of cluster states and W states with superconducting quantum interference device qubits in cavity QED*, Phys. Rev. A **74**, 02430, (2006).

[19] C. H. Bennett, G. Brassard, C. Crepeau, R. Jozsa, A. Peres, and W. K. Wooters, *Teleporting an unknown quantum state via dual classical and Einstein Podolsky Rosen channels*, Phys. Rev. Lett. **70**, 1895 (1993).

[20] T. Yu and J. H. Eberly, *Finite-Time Disentanglement via Spontaneous Emission*, Phys. Rev. Lett. **93**, 140404 (2004).

[21] W. K. Wootters, *Entanglement of Formation of an Arbitrary State of Two Qubits*, Phys. Rev. Lett **80**, 2245 (1997).

[22] V. Vedral, M. B. Plenio, M. A. Rippin, P.L. Knight, *Quantifying Entanglement*, Phys. Rev. Lett **78**, 2275 (1997).

[23] L. H. Xin, Z. Bo, Chin. Phys **15**, 1914 (2006).

[24] M. Horodecki, P. Horodecki, R. Horodecki, *Separability of mixed states: necessary and sufficient conditions*, Phys. Lett. A **223**, 1 (1996); R. Horodecki, P. Horodecki, M. Horodecki, Phys. Lett. A **210**, 377 (1996); M. Horodecki, P. Horodecki, Phys. Rev. A **59**, 4206 (1996).

[25] M. Ikram, Shi-Yao Zhu, M. S. Zubairy, *Generation of entangled state between two cavities for fixed number of photons*, Optical Communication. **184**, 417 (2000).

[26] B. Farooq, R. Tahira, M. Ikram, *Tripartite entanglement generation using cavity-QED and its dynamics in dissipative environments*, Journal of Modern Optics **59**, 1664 (2012).

[27] C. H. Bennett, F. Bessete, G. Brassard, L. Salvail, J. Smolin, *Experimental quantum cryptography*,J. Cryptogr. **5**, 3 (1992).

[28] J. Kempe, *Multiparticle entanglement and its applications to quantum cryptography*, Phys. Rev. A **60**, 910 (1999).

[29] C. H. Bennett, S. J. Wiesner, *Communication via one- and two-particle operators on Einstein-Podolsky-Rosen states*, Phys. Rev. Lett. **69**, 2881 (1992).

[30] J. S. Bell, *On the Einstein Podolsky Rosen Paradox*, Physics (Long Island City, NY) **1**, 195 (1964).

[31] D. Bohm, *A suggested interpretation of the quantum theory in terms of 'hidden' variables.I*, Phys. Rev. **85**, 166 (1952).

[32] D. Bohm, *A suggested interpretation of the quantum theory in terms of 'hidden' variables. II* Phys. Rev. **85**, 180 (1952).

[33] L. Davidovich, N. Zagury, M. Brune, J. M. Raimond, and S. Haroche, *Teleportation of an atomic state between two cavities using nonlocal microwave fields*, Phys. Rev. A, **50**, R895 (1993).

[34] J. A. Bergou, *Entangles field in multiple cavities as a testing ground for quantum mechanics*, J. Mod. Opt. **44**, 1957 (1997).

[35] C. Gerry, P. Knight, Introductory Quantum Optics; Cambridge University Press: Cambridge, UK, (2005).

[36] M. O. Scully, M. S. Zubairy, *Quantum Optics*, Cambridge University Press, Cambridge, MA, (Section 6.2) (1997).

[37] R. Islam, *Engineering entanglement in cavity quantum electrodynamical systems*, (2008).

[38] M. O. Scully, M. S. Zubairy, *Quantum Optics*, Cambridge University Press: Cambridge, UK, (1997).

[39] P. Agrawal, A. Pati, *Perfect teleportation and superdense coding with W states*, Phys. Rev. A **74**, 062320 (2006).