Bipartite Entanglement of Indistinguishable Particles

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Dedicated to

My Father
Abdul Rasheed, My mother
and
My Taya Jan
(Late) Saghir Ahmed.

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Abstract

Systems made of two or more than two subsystems are called composite systems. Entanglement is a key feature of composite systems that cannot be imitated by classical systems. Initially the phenomenon of entanglement was considered as qualitative feature of quantum theory but later developments, for instance, in the form of Bell's inequalities, made this feature quantitative. In early days, entanglement has been considered as a mysterious phenomenon but nowadays it is regarded as a fundamental resource for quantum information processes, such as, quantum cryptography, quantum teleportation, quantum computing and dense coding. When we talk about entanglement then question arises how to detect and quantify entanglement?

The detection and quantification of entanglement in multipartite systems depend on the particles' identity, i.e, weather they are distinguishable or indistinguishable. There exist, in literature of quantum information and computation, different methods of entanglement quantification for distinguishable particles, for instance, Schmidt decomposition, negativity, concurrence, entanglement of formation etc. Schmidt decomposition is widely used technique to quantify bipartite entanglement of distinguishable pure states. However, indistinguishabiity of particles' identity, that is governed by symmetrization postulate, confuses entanglement with exchange correlations due to permutation symmetry of many particle wave function. Hence the usual entanglement measures, such as Schmidt decomposition, remain controversial for indistinguishable particles. Here we consider a technique of entanglement quantification for indistinguishable particles which do not make use of particle labeling while expressing multiparticle quantum states. This approach is then modified to develop Schmidt decomposition which is suitable to quantify bipartite entanglement of indistinguishable particles. We

explain our results by considering example of two indistinguishable qubits in two separated sites having spin-up and spin-down, and then in the same site with arbitrary spins. It is shown that it yields physically expected results i.e. zero entanglement for product state and maximal entanglement for Bell like states.

Contents

Chapter 1

Introduction and Outline

Introduction

In the early days of quantum mechanics, Einstein and Bohr had debated about nature of reality. According to Einstein our reality is always fixed weather we observe it or not. According to him moon is always there whether we are observing it or not. Infect classical physics suggest that all objects present at their position whether we are observing them or not. But quantum theory doesn't agree with classical physics and accepted fuzzy nature of reality. According to Bohr, our reality does not exist without observation or measurement. In quantum world, we can not tell characteristics of quantum particles unless we measure or observe them. Because before measurement or observation particles will be in all possible available states known as superposition state unless we observe or measure them. After measurement particle wave function collapses as a result we obtain only one state from all possible states and we perceive that particle was present in this state already but this is not true. According to Bohr, particle was in superposition state before measurement and we do'nt know which state will come after measurement from all possible states. This means that reality does not exist before measurement or observation and Copenhagen interpretation work on this theory [1, 2]. Einstein didn't agree with Bohr and argued that quantum theory is incomplete and it should have some local hidden variables and knowing that we can predict anything about nature of reality without using concept of superposition and wave function. A.Einstein, B.Podolosky and N.Rosen write paper on it known as EPR paradox [3] to show that Bohr was wrong. EPR paper was initial step toward quantum entanglement.

Initially entanglement was considered as qualitative feature of quantum theory but the development of Bell's inequalities in 1964 [4], made this distinction quantitative. In early years of the development, entanglement was considered as strange phenomenon but nowadays it is the resource of quantum information processing, enabling tasks like quantum cryptography [5], quantum computation, quantum teleportation [6], dense coding etc. With the development of experimental progress in the field of quantum information the main task is the generation of entanglement. When we talk about the entanglement different questions arises: What is meant by Entangled states? How can one be sure that the entanglement was produced? How can we detect the presence of entanglement? How can we measure the entanglement? The generation of entanglement is purely experimental phenomenon and we will not talk about it in this thesis. However we will answer the question related to entanglement quantification in next few chapters.

To understand the phenomenon of entanglement and entangled states we will consider an experiment in which an unstable particle with spin 0 i.e. photon decays into two spin-$\frac{1}{2}$ particles (electrons), and travel in opposite direction to conserve linear momentum and also have opposite spins to conserve angular momentum i.e if one particle is with spin up then other will be in spin down. Consider two observers A and B having Stern-Gerlach apparatus and wants to measure the spin component of both the particle. Whenever, first observer A measure the spin up component of particle in the given direction then at the same time the other observer B will measure the spin down component along the same direction. The quantum state of this two particle system can be written as $|\Psi\rangle = \frac{1}{\sqrt{2}}(|\uparrow\rangle_1 |\downarrow\rangle_2 - |\downarrow\rangle_1 |\uparrow\rangle_2)$, where, subscripts indicate weather it is first particle or second and product state indicate that we are talking about two-particle system. The probability that particle-1 will have spin up and spin down is 50%, when observer A makes measurement and same will be true for observer B. For large ensemble of decays observer A and B will find random sequence of spin up and spin down components with $50 : 50$ ratio. But because both particles are perfectly correlated, if observer A measures spin down component of particle-1 then at the same time we can predict with 100% probability that observer B will measure spin up and we will say that the given state $|\Psi\rangle$ is entangled [7]. Which means that two particles are linked so that both exists in combined state means we can't tell state of one particles unless we know about state of both particles. Here, we are talking about composite system which can be decomposed into two or more subsystems. When we talk about the composite systems, then those composite systems may be comprise of distinguishable subsystems or indistinguishable subsystems. However, entanglement in distinguishable subsystems has been well studied but remain debated for indistinguishable subsystems. There are well known techniques to check the entanglement between distinguishable-particle systems i.e. Schmidt decomposition [8] for pure state bipartite systems, von Neumann entropy and Concurrence. All of these are for pure states. When we talk about mixed states there are, Entanglement of Formation, Negativity and Convex roofs. All of these have some limitations when we use them for indistinguishable particles.

Schmidt decomposition is widely used for distinguishable particles and also remain debated for indistinguishable particles [9, 10, 11] where it is replaced by Slater decomposition and associated Slater Schmidt rank to quantify entanglement but give different outcomes for bosonic and fermionic particles [11]. For distinguishable particles, Schmidt decomposition reveals entanglement of the system by using von-Neumann entropy of the reduced density matrix. The relationship between Schmidt coefficients and eigenvalues of the reduced density matrix violated in the case of indistinguishable particles [11]. Consequentially the usual concept of partial trace to obtain reduced density matrix has not been considered suitable for the case of indistinguishable particles [10, 11, 24]. When we use this technique i.e. Schmidt decomposition to check the entanglement for indistinguishable particles it gives misleading results i.e. non zero von-Neumann entropy and presence of entanglement for uncorrelated fermions [9]. When we talk about the composite systems then most of the composite systems made of indistinguishable particles. Now the question arises, what is the meaning of indistinuishable particles? or how can we distinguish the indistinguishable particles from distinguishable particles? and how we will quantify entanglement of indistinguishable particles. To answer question 1 and 2, let's consider two particles, one of them is in state $\psi_a(r)$ and the other one is in $\psi_b(r)$. The overall state of the system can be written as $\psi(r_1, r_2){=}\psi_a(r_1)\psi_b(r_2)$. Here we are assuming that we can express the two particles apart, otherwise the claim that particle-1 is in state ψ_a and particle-2 is in state ψ_b makes no sense. In other words we are saying that one of them is in state ψ_a and the other one is in ψ_b but we don't which one is which. In classical mechanics it would be a foolish objection, where you can always tell the particles apart. Quantum mechanics accommodates this situation, i.e. which particle is in which state by expressing the state as $\psi_\pm(r_1, r_2){=}A[\psi_a(r_1)\psi_b(r_2){\pm}\psi_b(r_1)\psi_a(r_2)]$, and this equation is for two types of indistinguishable particles, i.e. for bosons and fermions. For bosonic particles we will use positive sign and for fermionic particles we use negative sign. At this point another interesting question arises, i.e. what are bosons and fermions? All those particles which have integer spin i.e. $(0, 1, 2, 3, ...)$ are bosons and those which have half integer spin i.e. $(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, ...)$ are fermions. If we have two indistinguishable fermions (i.e two electrons) in the same state then $\psi_a{=}\psi_b$ and $\psi_-(r_1, r_2){=}A[\psi_a(r_1)\psi_a(r_2) \pm \psi_a(r_1)\psi_a(r_2)]{=}0$, which means that two that fermions can't occupy the same state and this is renowned Pauli exclusion principle. This principle is not only for electrons but for all indistinguishable fermions [13].

Systems of indistinguishable particles can be used as a resource of quantum information processing i.e. for quantum computation, quantum teleportation and quantum cryptography. So, the understanding of entanglement for system of indistinguishable particles is very important from both fundamental and technological viewpoints. We will investigate the entanglement for indistinguishable particles which is started some time ago but differently from distinguishable particles. This subject i.e. the entanglement between indistinguishable particles remain controversial and this controversy mainly arise the way the indistinguishable particles being treated i.e. by making them artificially distinguishable by assigning them nonobservable labels [13]. The consequences of assigning these labels makes the given state entangled w.r.t labels.

1.1 Thesis Outline

In this work we will talk about entanglement quantification in bipartite system of indistinguishable particles. In chapter 2, we will talk about composite systems and representation of composite systems by using product Hilbert space. We will also discuss about density operator for pure and mixed state and phenomenon of entanglement for pure state and mixed state. When we talk about entanglement then question of entanglement quantification arises. We will review all the techniques of entanglement quantification available in literature in this chapter. In chapter 3, we will talk about distinguishablity and indistinguishability of particles and entanglement of indistinguishable particles by using Slater Schmidt decomposition and Slater rank both for the case of fermions and bosons. Here, we will use standard particle base approach to make indistinguishable particles, distinguishable. We will see that consequence of this labeling to make particles artificially distinguishable which generates phenomenon of entanglement. This entanglement is not physical and can not be used as quantum information processing protocols. In this chapter we will also discuss that Slater Schmidt decomposition yields physically unexpected results i.e. non zero von-Neumann entropy for indistinguishable particles even they are not entangled. In chapter 4, we will again talk about indistinguishable particles and their entanglement by using non standard particle base approach. By using non standard particle base approach we will review all existing problem of entanglement quantification of indistinguishable particles. We will show that this approach will yield physically expected results for indistinguishable particles which was remain debated in previous proposals. And in chapter 5, we will conclude our work.

Chapter 2

Foundations

Interference, superposition of quantum systems and tunneling are fundamental properties used to distinguish between quantum and classical systems. All of these properties are observed in quantum systems comprised of single particle. But this is not only distinction between classical systems and quantum mechanical system. When we talk about composite systems then there will exist correlation between constituent particles. Classical correlation means classical probabilities but quantum mechanical correlation means remote action at distance and such non classical correlation leads us toward entanglement.

Entanglement is the purely quantum mechanical phenomenon and appears when we talk about the composite systems e.g, the system which is comprised by two or more then two subsystems. The composite system which have this very important property plays vital rule in quantum information protocols e.g, quantum cryptography, quantum teleportation, and dense coding. In this chapter we will talk about composite systems which is further divided into bipartite and multipartite subsystems, method of writing their states known as density matrix approach. We will also give the introduction of entanglement theory of bipartite systems, introduce the concept of entanglement and separability, and explain several separability criteria and methods to quantify such non classical correlations.

2.1 Composite Systems

The particles comprised of more then one particles are composite particle. Up to this stage in quantum mechanics we have considered only single particles. But in this chapter we will talk about composite systems made of two or more then two subsystems. The most astonishing feature of composite systems is entanglement which is heart of quantum information processing protocols. Composite systems further divided into multi-partite systems and bipartite systems. Composite system composed of more then two subsystems known as multi-partite systems while composite system consist of only two subsystems are know as Bipartite systems. When we deals with single particles then we write down the state of that systems by using single particle Hilbert state [15]. But the question is that how we will write the state of composite systems? We will discuss the answer of this question in next section.

2.2 Tensor Product

In order to write the state of composite systems we will use the concept of tensor product, in which the individual state of subsystem is represented by single particle Hilbert space. To understand the concept of tensor product we will consider the case of two particle system. Let's consider that one particle state $|\phi\rangle$ is represented by Hilbert space H_1 and the other particle state $|\alpha\rangle$ is represented by Hilbert space H_2. The overall state of composite system is written as

$$|\psi\rangle = |\phi\rangle \otimes |\alpha\rangle \,. \tag{2.1}$$

where $|\phi\rangle \in H_1$, $|\alpha\rangle \in H_2$, $|\psi\rangle \in H$ and symbol $\otimes$ is used for tensor product.
Now, we will construct the basis of Hilbert space of composite systems. Let $\{|a_i\rangle\}$ are basis of system belonging to H_1 and $\{|b_i\rangle\}$ are basis of system belonging to H_2. The basis of composite system belongs to H will be $\{|w_i\rangle\}$ e.g,

$$|w_i\rangle = |a_i\rangle \otimes |b_i\rangle \,. \tag{2.2}$$

Here, we can see that order of tensor product does not matter, meaning

$$|\phi\rangle \otimes |\alpha\rangle = |\alpha\rangle \otimes |\phi\rangle . \tag{2.3}$$

We can also write state $|\phi\rangle \otimes |\alpha\rangle$ simply as $|\phi\rangle\ |\alpha\rangle$ or $|\phi\alpha\rangle$.

2.2.1 Quantifying Inner Product of Larger Hilbert Space

Now, we will quantify inner product of larger Hilbert space H. Let

$$|\psi_1\rangle = |\phi_1\rangle \otimes |\alpha_1\rangle , \tag{2.4}$$

$$|\psi_2\rangle = |\phi_2\rangle \otimes |\alpha_2\rangle . \tag{2.5}$$

Then

$$\langle\psi_1|\psi_2\rangle = (\langle\phi_1| \otimes \langle\alpha_1|) \times (|\phi_2\rangle \otimes |\alpha_2\rangle), \tag{2.6}$$

$$\langle\psi_1|\psi_2\rangle = \langle\phi_1|\phi_2\rangle \langle\alpha_1|\alpha_2\rangle . \tag{2.7}$$

If we wrote $|\psi_1\rangle$ and $|\psi_2\rangle$ in term of column vectors e.g,

$$|\psi_1\rangle = \begin{pmatrix} u \\ v \end{pmatrix} \quad |\psi_2\rangle = \begin{pmatrix} w \\ x \end{pmatrix}, \tag{2.8}$$

Then

$$|\psi_1\rangle \otimes |\psi_2\rangle = \begin{pmatrix} u \\ v \end{pmatrix} \otimes \begin{pmatrix} w \\ x \end{pmatrix} = \begin{pmatrix} uw \\ ux \\ vw \\ vx \end{pmatrix}. \tag{2.9}$$

2.2.2 Action of Operators on Composite Systems

In previous section we have seen that $|\phi\rangle \in H_1$, $|\alpha\rangle \in H_2$ and $|\psi\rangle \in H$. Let C is operator acting on state $|\phi\rangle \in H_1$ and D is operator acting on $|\alpha\rangle \in H_2$. Now we want to find action of $C \otimes D$ on $|\psi\rangle \in H$.

$$C \otimes D\,|\psi\rangle = C \otimes D(|\phi\rangle \otimes |\alpha\rangle) = (C\,|\phi\rangle) \otimes (D\,|\alpha\rangle), \tag{2.10}$$

Suppose $|\phi\rangle$ and $|\alpha\rangle$ are the eigenstates of operators C and D. Thus the action of C and D on states $|\phi\rangle$ and $|\alpha\rangle$ will be

$$C|\phi\rangle = c|\phi\rangle \quad D|\alpha\rangle = d|\alpha\rangle\,, \tag{2.11}$$

where c and d are eigenvalues of operators C and D respectively. By using these relations we can write eq (2.10) as

$$C \otimes D|\psi\rangle = c|\phi\rangle \otimes d|\alpha\rangle = cd(|\phi\rangle \otimes \alpha) = cd|\psi\rangle\,. \tag{2.12}$$

where, cd are eigenvalues of product operator acting on state of composite system.

2.2.3 Kronecker Product

Now, we want to find matrix representation of $C \otimes D$ which is known as Kronecker product. Suppose

$$C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \quad D = \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}, \tag{2.13}$$

then

$$C \otimes D = \begin{pmatrix} C_{11}D & C_{12}D \\ C_{21}D & C_{22}D \end{pmatrix} = \begin{pmatrix} C_{11}D_{11} & C_{21}D_{12} & C_{12}D_{11} & C_{12}D_{1}2 \\ C_{11}D_{21} & C_{11}D_{22} & C_{12}D_{21} & C_{12}D_{22} \\ C_{21}D_{11} & C_{22}D_{12} & C_{22}D_{11} & C_{22}D_{12} \\ C_{21}D_{21} & C_{21}D_{22} & C_{22}D_{21} & C_{22}D_{22} \end{pmatrix}. \tag{2.14}$$

here, we only talk about 2×2 matrices. Similarly we can extend this product to $n \times n$ matrices.

2.2.4 Properties of Tensor Product

Here we will define some important properties of tensor product.

- Let $|\phi\rangle \in H_1$ and $|\alpha\rangle \in H_2$ then

$$z(|\phi\rangle \otimes |\alpha\rangle) = (z|\phi\rangle) \otimes |\alpha\rangle = |\phi\rangle \otimes (z|\alpha\rangle). \tag{2.15}$$

  where, z is any arbitrary scalar.

- For arbitrary $|\phi_1\rangle$ and $|\phi_2\rangle$ in H_1 and $|\alpha\rangle$ in H_2,

$$(|\phi_1\rangle + |\phi_2\rangle) \otimes |\alpha\rangle = |\phi_1\rangle \otimes |\alpha\rangle + |\phi_2\rangle \otimes |\alpha\rangle\,. \tag{2.16}$$

- For arbitrary $|\phi\rangle$ in H_1 and $|\alpha_1\rangle$ and $|\alpha_2\rangle$ in H_2,

$$|\phi\rangle \otimes (|\alpha_1\rangle + |\alpha_2\rangle) = |\phi\rangle \otimes |\alpha_1\rangle + |\phi\rangle \otimes |\alpha_2\rangle\,. \tag{2.17}$$

2.2.5 Local Manipulations

Sometimes we need to know about only one subsystem of the composite system. For that purpose, we have to find the measurement on only one subsystem either c or d without disturbing other known as local measurement. To perform local measurement, we have to separate composite system which is an easier task when two subsystem are already separated by some spatial distance. Suppose we have composite system prepared in product Hilbert space ($H = H_c \otimes H_d$), where H_c is Hilbert space of subsystem c and H_d is Hilbert space of subsystem d. Let U is observable of subsystem c represented as

$$U \otimes I. \tag{2.18}$$

where, I is identity operator. This will only acts on subsystem c without disturbing the other. Similarly for subsystem d we define an operator

$$V \otimes I. \tag{2.19}$$

The observables of type $U \otimes I$ and $V \otimes I$ are called local observables.

Local Operation and Classical Communication

Local operation and classical communication enables us to quantify entanglement. To explain this we will consider two qubits which will not exchange their properties quantum mechanically. We will perform measurement on one qubit (local measurement) and share this information through classical channel e.g, internet or telephone. This process is known as local operation and classical communication, e.g, we perform measurement locally and share this information classically. We can not create entangled state from un-entangled state by performing local operation and classical communication [14].

2.3 Density Operator and Density Matrix

So far, in quantum mechanics we have deal with the system which are completely described by state vector and in such representation state vector contain all the information about the system. In quantum mechanics there is also an alternative and more general approach analogous to the state vector approach known as density operator or density matrix approach. This is more convenient way to thinking for some commonly encountered scenarios in quantum mechanics. In next three sections, we will explain briefly about density operator, general properties of density operator and it's application [16, 17].

2.3.1 Ensembles of Quantum States

When we talk about composite system then we consider more then one system or we study large system or collection of systems called ensembles. In this case, we don't know the state of system completely and use method of density operator to deal with quantum states. Suppose that we have collection of objects, some of which are in quantum state $|\chi_1\rangle$ with probability p_1, some of which are in $|\chi_2\rangle$ with probability p_2 and so on as shown in figure 2.1.

Now if we choose one particle from ensemble, the probability that it is in state $|\chi_j\rangle$ is p_j. We will call $\{p_j, |\chi_j\rangle\}$ an ensemble of pure states [17]. We want to find the expectation value of some observable C in this ensemble. Pick one of the particle from ensembles of particles and measure C, similarly pick another one and do the same. If all the objects are in state $|\chi_j\rangle$ the expectation value of C would be $\langle\chi_j| C |\chi_j\rangle$ with probability p_j. Therefore, the expectation value of C in the ensemble is given by

$$< C >= \sum_j p_j \langle\chi_j| C |\chi_j\rangle = \mathrm{Tr}(C\rho), \tag{2.20}$$

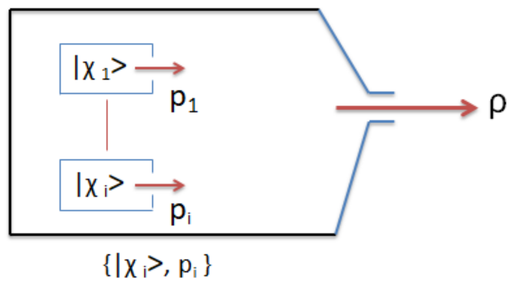

Figure 2.1: Particle prepared in $|\chi_1\rangle$ with probability p_1,$|\chi_2\rangle$ with probability p_2 up n particles written as $\{|\chi_i\rangle, p_i\}$.

where, we have defined the new operator ρ which is known as density operator for mixed states given as

$$\rho = \sum_j p_j |\chi_j\rangle \langle\chi_j| . \tag{2.21}$$

When, we write density operator in the form of matrix it is known as density matrix and both of these terms are interchangeably. We are here using the density matrix approach so all of the postulates which are for the state vectors will be reformulated in terms of density operator language. In this section we will explain how this reformulation happens and will explain when it is useful. The result of both approaches will be same but it is matter of fact that to deal with specific problem one approach will be more easier then other.

Unitary operator U is used to describe the evolution of density operator. Consider an initial state of the closed quantum system is $|\chi_j\rangle$ with probability p_j. After evolution, system will be in the state $U|\chi_j\rangle$ with probability p_j. Thus the evolution of density operator is described by the equation [17]

$$\rho = \sum_j p_j |\chi_j\rangle \langle\chi_j| \xrightarrow{U} \sum_j p_j U |\chi_j\rangle \langle\chi_j| U^\dagger = U\rho U^\dagger. \tag{2.22}$$

In this paragraph we will explain how measurements can be performed by using density operator. Here, we are using the measurement operator N_n to perform measurement on given state. If initial state of the system is χ_j then probability of getting the result n is described by measurement operator N_n which is sandwiched between states χ_j, e.g,

$$p(n|j) = \langle\chi_j| N_n^\dagger N_n |\chi_j\rangle = \mathrm{Tr}\big(N_n^\dagger N_n |\chi_j\rangle \langle\chi_j|\big). \tag{2.23}$$

To obtain the probability of result n we will use the law of total probability which is

$$P(n) = \sum_j p(n|j)p_j. \tag{2.24}$$

Using eq (2.23) in above equation e.g,

$$p(n) = \sum_j p_j \,\mathrm{Tr}\big(N_n^\dagger N_n |\psi_j\rangle \langle\psi_j|\big), \tag{2.25}$$

$$p(n) = \mathrm{Tr}\big(N_n^\dagger N_n \rho\big), \tag{2.26}$$

where, we use $\rho = \sum_j p_j(|\chi_j\rangle \langle\chi_j|)$.

After obtaining the result n, the initial state of the system $|\chi_j\rangle$ will become

$$\left|\chi_j^n\right\rangle = \frac{N_n |\chi_j\rangle}{\langle\chi_j| N_n N_n^\dagger |\chi_j\rangle}. \tag{2.27}$$

So, after measurement we have group of states $\left|\chi_j^n\right\rangle$ with respective probabilities $p(n|j)$. The corresponding density operator for the states $\left|\chi_j^n\right\rangle$ will be

$$\rho_n = \sum_j p(n|i) \left|\chi_j^n\right\rangle \left\langle\chi_j^n\right| = \sum_j p(n|j) \frac{N_n |\chi_j\rangle \langle\chi_j| N_n^\dagger}{\langle\chi_j| N_n N_n^\dagger |\chi_j\rangle}. \tag{2.28}$$

According to the elementary probability theory, $p(j|n) = \frac{p(n,j)}{P(n)} = \frac{p(n|j)p(j)}{p(n)}$. Substituting eq (2.23) and eq (2.26) we obtain

$$\rho_n = \frac{N_n \rho N_n^\dagger}{\mathrm{Tr}\Big(N_n^\dagger N_n \rho\Big)}, \tag{2.29}$$

where, we have again used the definition of density operator e.g $\rho = \sum_j p_i |\chi\rangle \langle\chi_j|)$. In above discussion we reformulated the postulates of quantum mechanics for the density operators.

2.3.2 Density Operator for Pure State

The special case of mixed state is pure state and density operator of pure state can be written by putting $p_j = 1$ in eq (2.21)

$$\rho = 1. \left|\chi\right\rangle \left\langle\chi\right|, \tag{2.30}$$

where, $|\chi\rangle$ is a state of quantum system which can be represented as linear superposition of the basis vector $|n\rangle$ as $|\chi\rangle$=$\sum_i c_i \left|i\right\rangle$. The density operator for this state is

$$\rho = \sum_i \sum_j c_i c_j^* \left|i\right\rangle \left\langle j\right| = \sum_{i,j} \rho_{ij} \left|i\right\rangle \left\langle j\right|, \tag{2.31}$$

where, $\rho_{ij} = \langle i|\chi\rangle \langle\chi|j\rangle = \langle i|\, \rho \,|j\rangle$ are the matrix element of density operator for the pure state. When we perform the measurement on state $|\chi\rangle$ then the probability of getting the state $|i\rangle$ is $|c_i|^2$. This provides the physical meaning to the diagonal elements of the density operator that is diagonal elements are necessarily non negative and hence density operator is positive operator.
The density operator of the pure state have the following properties:

$$\mathrm{Tr}(\rho) = \sum_i \rho_{ii} = \sum_i |c_i^2| = 1. \tag{2.32}$$

Since $\rho^2 = |\chi\rangle \langle\chi|\chi\rangle \langle\chi| = |\chi\rangle \langle\chi| = \rho$, therefore above equation becomes [16]

$$\mathrm{Tr}\left(\rho^2\right) = 1. \tag{2.33}$$

2.3.3 General properties of the density operator

If the given operator satisfying the following given properties then it is said to be the valid density operator.

1. The trace of the given density operator must be one. i.e. $\mathrm{Tr}(\rho) = 1$.
2. The density operator is hermitian operator i.e. $\rho^\dagger = \rho$.

3. For all the given state $|\chi\rangle$, the density matrix is positive i.e. $\langle\chi|\,\rho\,|\chi\rangle \geq 0$. This follows from

$$\langle\chi|\,\rho\,|\chi\rangle = \sum_j p_j \,\langle\chi|\chi_j\rangle^2 \geq 0. \tag{2.34}$$

   In earlier discussion we said that density operator is hermitian, it means that the eigenvalues of the given operators will be greater then or equal to zero, that's why the given density operator is positive.

By finding the trace of the density operator we can define that weather the given sate is pure or not i.e. if $\mathrm{Tr}(\rho^2) = 1$ then the given state is pure and if $\mathrm{Tr}(\rho^2) \leq 1$ then the given state is mixed.

2.4 Bipartite Entanglement

A system which is made of two subsystems is known as bipartite system. In this section we will study about bipartite entangled states and the basic concepts of entanglement detection of bipartite systems.

2.4.1 Entanglement of pure states

Consider two subsystems one is C and the other is D, the physical state of the system C is described by states in a Hilbert space H_C and of the system D in a Hilbert space H_D. The composite system of both subsystem can be written as a direct product of two space $H = H_C \otimes H_D$. Thus any vector in $H = H_C \otimes H_D$ can be written as

$$|\chi\rangle = \sum_{j,k=1}^{d_C,d_D} c_{jk}\,|c_j\rangle \otimes |d_k\rangle \in H = H_C \otimes H_D. \tag{2.35}$$

where, d_C and d_D are dimensions of Hilbert space of subsystems. The direct product $|c\rangle \otimes |d\rangle$ can be written as $|c\rangle\,|d\rangle$ or $|cd\rangle$.
If the given state $|\chi\rangle$ can be written in the form of eq (2.35) then the state $|\chi\rangle$ is said to be product or separable state where the local basis $|c_j\rangle \in H_C$ and $|d_k\rangle \in H_D$.

2.4.2 Entanglement of mixed states

So for we have discussed about pure states but generally the states of quantum system are mixed states and we don't know the exact state of a quantum system. We only know about it with some probability p_j, in one of the states $|\phi_j\rangle \in H$. In this situation we use the notion of density matrix i.e.

$$\rho = \sum_j p_j |\phi_j\rangle \langle\phi_j| , \tag{2.36}$$

with $\sum_j p_j = 1$ and $p_j \geq 0$. If there exits ρ^C for subsystem C and ρ^D for subsystem D, then wee say that ρ is product state and can be written as

$$\rho = \rho^{(C)} \otimes \rho^{(D)}. \tag{2.37}$$

If there exists convex weights p_j and product state $\rho^{(C)} \otimes \rho^{(D)}$ such that

$$\rho = \sum_j p_j \rho^{(C)} \otimes \rho^{(D)}. \tag{2.38}$$

then the state is called separable otherwise entangled. In this case if the states are separable then it means that the states are classically correlated and production of such states is possible with the help of (LOCC) , means some observer say Allice perform measurement on subsystem A locally and share this information with Bob classically for the production of $\rho^{(C)} \otimes \rho^{(D)}$. This process is repeated many times randomly for different states and we will prepare state with relative probabilities p_j. The state prepared by such method is known as classically correlated state.

2.5 Separability and Entanglement

If state of composite system can be written as convex combination of product state i.e.

$$\rho^{(CD)} = \sum_r p_r \rho^{(C)} \otimes \rho^{(D)}, \tag{2.39}$$

then it is called separable or classically correlated state. The states which can not be written as product state are known as entangled state and these states contains non classical correlations known as EPR correlations or quantum correlations.

2.6 Separability Criteria

The definitions given above for separable and entangled states appears simple at first glance but in realistic experiments much more complications arises when we want to find weather given state is entangled or not. Separability is defined via possibility of decomposition of composite system into product state for pure state and convex sum for mixed state. If such decomposition is possible then the given state will be separable. Failure to find such decomposition means that given state is entangled. For pure state such decomposition is possible and known Schmidt decomposition which we will discuss in next section.

2.6.1 Schmidt Decomposition

Among the various useful tools of quantum information for the study of composite systems e.g density operators, partial trace, state purification, negativity, entanglement monotones, concurrence etc, the Schmidt decomposition is one of them. This is very convenient way to approach the problem of entanglement characterization, theory of measurement and state purification. In this section we will discuss that how useful is the Schmidt decomposition for the measurement of bipartite entanglement for pure states [18].

2.6.2 Theorem:(Schmidt decomposition)

If we have pure state $|\psi\rangle$ of a bipartite system $C \otimes D$ there exists orthonormal sets of states $\left|\chi_j^C\right\rangle$ for system C and $\left|\chi_j^D\right\rangle$ for system D such that

$$|\chi\rangle = \sum_j \sqrt{\lambda_j} \left|\chi_j^C\right\rangle \otimes \left|\chi_j^D\right\rangle, \tag{2.40}$$

where, $\left|\chi_j^C\right\rangle$ and $\left|\chi_j^D\right\rangle$ are known as Schmidt basis belonging to system C and D respectively. The expansion coefficients λ_j are non-negative real numbers known as Schmidt coefficients and for normalized state $|\chi\rangle$ we must have $\sum_j \lambda_j^2 = 1$. The expansion (2.40) is known as Schmidt decomposition.

This is very important theorem to detect the entanglement between composite systems. The Schmidt coefficients in equation plays vital role to detect the entanglement. In order to calculate the Schmidt coefficients first we will find density matrix and then perform the partial trace on the one of sub system by fixing the other e.g,[19]

$$\rho^{(C)} = \mathrm{Tr}_D(|\chi\rangle\langle\chi|). \tag{2.41}$$

The above matrix have eigenvalues λ_j^2. The Schmidt number is the number of nonzero eigenvalues λ_j and used as an entanglement witness as following way,

- If a state is separable, then the Schmidt number$= 1$.
- If a state is entangled, then the Schmidt number is > 1 [19].

Schmidt number only tell us that weather the state is entangled or not and unable to answer the strength of the entanglement e.g, weather it is maximally entangled or minimum. That is why Schmidt measure is crude measure of entanglement. The reason that the von Neumann entropy is a better measure of entanglement will be discussed in the next section [16]. The other limitation of Schmidt decomposition is that it is applicable only for distinguishable particles and provide wrong results for indistinguishable particles i.e nonzero Von-Neumann entropy and the existence of entanglement for uncorrelated fermions.

2.7 Entanglement Quantification

In previous section we discuss about entangled and separable states but the question, how we will quantify or measure entanglement remains challenge. Here, we will discuss some requirements for good entanglement measures and some important entanglement quantification techniques both for pure and mixed states.

2.7.1 Requirements for entanglement measurement

Here, we mention some important axioms which are necessary for good entanglement measure E. However, some of entanglement measure given below may not fulfill all the axioms.

- If given state σ is separable then E(σ)=0.
- No increase under LOCC: The entanglement of state σ must not increase under local operation and classical communication (LOCC).
- Addativity: If we have n indistinguishable copies of state σ then it should contain n times entanglement of each copy [20]

$$E(\sigma^{\otimes}) = nE(\sigma). \tag{2.42}$$

- Subaddativity: If we have two states ρ and σ then entanglement of tensor product of ρ and σ should be lesser then sum of entanglement of each state.

$$E(\rho \otimes \sigma) \leq E(\rho) + E(\sigma). \tag{2.43}$$

- Convexity: It should be convex function [21],

$$N(p\rho + (1-\rho)\beta) \leq pN(\rho) + (1-p)N(\beta). \tag{2.44}$$

2.7.2 Some important entanglement measures

In previous section, we have discussed some important requirements for entanglement measure and in this section we will discuss about some important entanglement measure which tells us weather given state is entangled or not.

von Neumann entropy

In thermodynamics the entropy is the measurement of disorder, the greater the disorder the greater the entropy. In statistical mechanics and information theory concept of Shannon entropy used as a quantitative measure of classical information or of ignorance. While in quantum information theory von-Neummann entropy is used for this purpose. In classical information theory classical states and classical probability distribution is used while in quantum mechanics these are replaced by quantum states and density operators. In this section we generalize the definition of the Shannon entropy to the quantum states [16, 17]. The von-Neumann entropy of $\rho^{(1)}$ or $\rho^{(2)}$ which is also known as entanglement entropy can be defined as [22]

$$E(\psi) = S(\rho_r) = -\operatorname{Tr}(\rho_r \ln \rho_r). \tag{2.45}$$

Suppose that the eigenvalues of reduced density matrix are given by λ_j then entanglement entropy in terms of the eigenvalues as [19]

$$S(\rho_r) = \sum_j \lambda_j \ln \lambda_j, \tag{2.46}$$

where λ_j used as a quantifier of entanglement for distinguishable particles. Consider an entangled state of the form

$$|\psi(\alpha)\rangle_{CD} = \cos\alpha\,|00\rangle_{CD} + \sin\alpha\,|11\rangle_{CD}\,. \tag{2.47}$$

The $\rho^{(C)}$ of the subsystem C can be described by

$$\rho^{(C)} = \operatorname{Tr}_D(\rho^{(CD)}) = (\cos^2\alpha\,|0\rangle\langle 0| + \sin^2\alpha\,|1\rangle\langle 1|). \tag{2.48}$$

In matrix form the ρ^C can be written as

$$\rho^C = \begin{pmatrix} \cos^2\alpha & 0 \\ 0 & \sin^2\alpha \end{pmatrix}, \tag{2.49}$$

$\cos^2\alpha$ and the $\sin^2\alpha$ are the eigenvalues of ρ^C and using these eigenvalues in eq (2.46) we can easily compute entanglement entropy which is

$$E(\theta) = -2(\cos^2\alpha \log_2 \cos\alpha + \sin^2\alpha \log_2 \sin\alpha). \tag{2.50}$$

Now, we can easily obtain maxima and minima of $E(\alpha)$ by taking the differential of equation (2.50) and see how it measure entanglement.

$$\frac{dE(\alpha)}{d\alpha} = 2\sin 2\alpha \log_2 \cot\alpha. \tag{2.51}$$

Now maxima and minima on $E(\alpha)$ because $\frac{dE(\alpha)}{d\alpha}$ has set of zeros at different values of $\sin 2\alpha$ and $\cot\alpha$ e.g at $\sin 2\alpha = 0$ or $\alpha = \frac{n\pi}{2}$ entropy is $E(\frac{n\pi}{2}) = 0$. These points are the minima of entropy and at these points over state (2.47) reduces to separable state. For example $|\psi(\alpha)\rangle_{CD} = |00\rangle$ for $\alpha = 0$ and $||\psi(\alpha)\rangle_{CD} = |11\rangle$ for $\alpha = \frac{\pi}{2}$. And for $\log_2 \cot\alpha = 0$, $\cot\alpha = 1$ or $\alpha = \frac{\pi}{4} \pm 2n\pi$ the value of entropy will be $E(\alpha) = 1$. These points are the maxima of entropy and at these points our original state will be maximally entangled. For example, if we consider n=0, i.e $\alpha = \frac{\pi}{4}$ our original state reduces to $|\psi(\alpha)\rangle = \frac{|00\rangle+|11\rangle}{\sqrt{2}}$ which is bell state. By following same procedure we can show that all bell states are maximally entangled.

Now, we can see that eq (2.47) is already in Schmidt decomposed form and if the state is entangled (i.e. if $\alpha \neq 0$ and $\alpha \neq \frac{n\pi}{2}$) then for all values the α Schmidt number is 2 and unable to distinguish between maximal and minimal entangled states. That is why Schmidt measure is crude measure of entanglement and von Neumann Entropy is better than Schmidt decomposition [16].

Concurrence

The other nice and important quantitative measure of amount of entanglement is concurrence and it actually tells us that how much entanglement does a state have. This is the other mathematical tool of entanglement characterization. It is defined as,

$$C(|\chi\rangle) = |\langle\chi|\tilde{\chi}\rangle\,|. \tag{2.52}$$

where, $|\tilde{\chi}\rangle = Y \otimes Y\,|\chi\rangle^*$ and $|\chi\rangle^*$ is the complex conjugate of the state vector $|\chi\rangle$. Here, Y denote the Pauli matrix $\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}$ [8]. The Concurrence C has the following properties:

- The concurrence is bounded by 0 and 1: $0 \leq C \leq 1$.
- If state is separable then $C = 0$.
- For maximally entangled state $C = 1$ [23].

To check the entanglement for mixed state we calculate the concurrence by using density operator $\tilde{\rho} = Z \otimes Z\rho^* Z \otimes Z$. Then we will find the eigenvalues of $\tilde{\rho}$ and by using the following formula we can find out the concurrence for mixed state.e.g, [19]

$$C(\rho) = \max(0, \alpha_1 - \alpha_2 - \alpha_3 - \alpha_4) \tag{2.53}$$

where $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq \alpha_4$ are the eigenvalues of the matrix $R=\sqrt{\rho^{\frac{1}{2}}\tilde{\rho}\rho^{\frac{1}{2}}}$ [16].

Entanglement of formation

Here we will use the concept of concurrence to define entanglement of formation, [16] i.e.

$$E(\rho) = H_2\left(\frac{1+\sqrt{1-C(\rho)^2}}{2}\right), \tag{2.54}$$

where $H_2(y) = -y\log_2 y - (1-y)\log_2(1-y)$ is entropy function and $y = \frac{1+\sqrt{1-C(\rho)^2}}{2}$. $C(\rho)$ is concurrence for mixed state. Entanglement of formation defines amount of resources required to generate particular entangled state (e.g. ρ).

Chapter 3

Bipartite Entanglement: Indistinguishability Transition

Entanglement is most astonishing non classical feature of quantum mechanics used for quantum information processing. Entanglement is well understood for bipartite distinguishable particles and we have discussed it in detail in chapter 2. But for indistinguishable particles it is hardly been investigated and general definition is not given yet. Quantum dots, Bose Einstein condensation, quantum cryptography and parametric down conversion also involves indistinguishable particles that's why it is important to understand the entanglement for indistinguishable particles. Before going to that point it is very important to understand the difference between distinguishable particles and indistinguishable particles. In this chapter we will consider the difference between distinguishable and indistinguishable particles, systematization postulate, consequences of indistinguishability, quantum entanglement for two indistinguishable fermions and for bosons.

3.1 Distinguishable and Indistinguishable Particles

In classical physics, we can distinguish indistinguishable particles by following particle trajectory of each individual particle. We can also distinguish indistinguishable particles in classical physics by coloring them. But in quantum mechanics indistinguishable particles are not distinguishable because we can not tag the particles as we did in classical physics. The other reason is that we are uncertain about the particle trajectory due to Heisenberg uncertainty principle. To illustrate this concept we consider the experiment as shown in figure 3.1 in which two indistinguishable particles are fired from two sources (S_1 and S_2) and both are scattered and detected at detector D_1 and D_2. Because both particles are indistinguishable we can not tell which particle is detected at D_1 and D_2. We can only tell that one particle is detected at D_1 and other at D_2 [25].

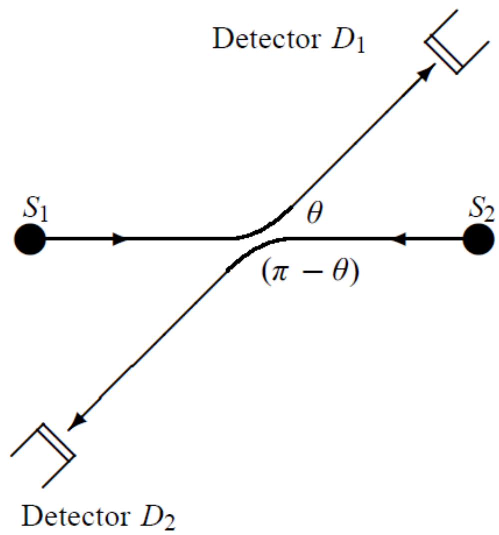

Figure 3.1: Two indistinguishable particles fired from source (S_1 and S_2) and detected by detector (D_1 and D_2).

In above experiment, we talk about two particle system. If there are N-indistinguishable particles having coordinates $\tau_1, \tau_2, \tau_3, ...\tau_N$ and represented by wave function $\psi(\tau_1, \tau_2, \tau_3, ...$ When these N-indistinguishable particles are mixed together, then we can not tell which particle have which coordinate. We can only specify probability of particles having coordinates τ_1 or $\tau_2, ..., \tau_N$ and that probability remain unchanged by interchanging coordinates. i.e,

$$|(\psi(\tau_1, \tau_2, ..., \tau_j, \tau_k, ..., \tau_N))|^2 = |(\psi(\tau_1, \tau_2, ..., \tau_k, \tau_j, ..., \tau_N))|^2, \tag{3.1}$$

and we have

$$|(\psi(\tau_1, \tau_2, ..., \tau_j, \tau_k, ..., \tau_N))| = \pm|(\psi(\tau_1, \tau_2, ..., \tau_k, \tau_j, ..., \tau_N))|. \tag{3.2}$$

which means that particles may have symmetric or anti-symmetric wave-function under the interchange of particles. Positive sign shows that particles have integral spin and wave-function will be symmetric and negative sign shows that particles have half integral spin and wave-function will be anti-symmetric. On the basis of symmetric and anti-symmetric wave-function we can divide the particles into two main classes, i.e.

- **Bosons**: Particles having integral spin $0, 1\hbar, 2\hbar, 3\hbar, ...$
- **Fermions**: Particles having half integral spin $\frac{\hbar}{2}, \frac{3\hbar}{2}, \frac{5\hbar}{2}, ...$

3.2 Symmetrization Postulate

The eq (3.2) is consequence of symmetrization postulate which states that wave-function can be either symmetric or anti-symmetric under the exchange of particles depending upon the nature of particles, i.e. fermions or bosons. Exchange of bosonic particles gives symmetric wave-function while fermionic particles give anti-symmetric wave-function. Fermionic particles obey Fermi-Dirac statistics while Bosonic particles obey Bose Einstein condensation [25].

3.3 Wave-Function of Indistinguishable Particles

We know that wave-function of indistinguishable particles is either totally symmetric or anti-symmetric. Here, we will construct symmetric and anti-symmetric wave-function for system of two and three indistinguishable particles and then we generalize that result to system of N-indistinguishable particles. The symmetric and anti-symmetric wave-function can be written as

$$\psi_s(\tau_1, \tau_2) = \frac{1}{\sqrt{2}} \left[\psi(\tau_1, \tau_2) + \psi(\tau_2, \tau_1) \right]. \tag{3.3}$$

$$\psi_a(\tau_1, \tau_2) = \frac{1}{\sqrt{2}} \left[\psi(\tau_1, \tau_2) - \psi(\tau_2, \tau_1) \right]. \tag{3.4}$$

and for two indistinguishable particle system it can be written as

$$\psi_s(\tau_1, \tau_2) = \frac{1}{\sqrt{2}} \left[\psi_{n1}(\tau_1)\psi_{n2}(\tau_2) + \psi_{n1}(\tau_2)\psi_{n2}(\tau_1)\right], \tag{3.5}$$

$$\psi_s(\tau_1, \tau_2) = \frac{1}{\sqrt{2}} \left[\psi_{n1}(\tau_1)\psi_{n2}(\tau_2) - \psi_{n1}(\tau_2)\psi_{n2}(\tau_1)\right]. \tag{3.6}$$

Generally, we can write symmetric and anti-symmetric wave-function as

$$\psi_s(\tau_1, \tau_2) = \frac{1}{\sqrt{2!}} \sum_P P\psi_{n1}(\tau_1)\psi_{n2}(\tau_2). \tag{3.7}$$

$$\psi_a(\tau_1, \tau_2) = \frac{1}{\sqrt{2!}} \sum_P (-1)^P P\psi_{n1}(\tau_1)\psi_{n2}(\tau_2), \tag{3.8}$$

where, P is permutation operator for all possible permutations. $(-1)^P = +1$ for even permutation and $= -1$ for odd permutation. In determinant form the anti-symmetric wave function can be written as

$$\psi_a(\tau_1, \tau_2) = \frac{1}{\sqrt{2!}} \begin{vmatrix} \psi_{n1}(\tau_1) & \psi_{n1}(\tau_2) \\ \psi_{n2}(\tau_1) & \psi_{n2}(\tau_2) \end{vmatrix}. \tag{3.9}$$

Similarly for three indistinguishable particles it can be written as

$$\psi_a(\tau_1, \tau_2) = \frac{1}{\sqrt{3!}} \begin{vmatrix} \psi_{n1}(\tau_1) & \psi_{n1}(\tau_2) & \psi_{n1}(\tau_3) \\ \psi_{n2}(\tau_1) & \psi_{n2}(\tau_2) & \psi_{n2}(\tau_3) \\ \psi_{n3}(\tau_1) & \psi_{n3}(\tau_2) & \psi_{n3}(\tau_3) \end{vmatrix}, \tag{3.10}$$

And for N-indistinguishable particles it can be written as

$$\psi_a(\tau_1, \tau_2, \tau_3, ..., \tau_N) = \frac{1}{\sqrt{N!}} \begin{vmatrix} \psi_{n1}(\tau_1) & \psi_{n1}(\tau_2) & \dots & \psi_{n1}(\tau_N) \\ \psi_{n2}(\tau_1) & \psi_{n2}(\tau_2) & \dots & \psi_{n2}(\tau_3) \\ \vdots & \vdots & \ddots & \vdots \\ \psi_{nN}(\tau_1) & \psi_{nN}(\tau_2) & \dots & \psi_{nN}(\tau_N) \end{vmatrix}. \tag{3.11}$$

The above $N \times N$ determinant is known as Slater determinant which involves only one particle states. By interchanging particles mean interchanging columns of determinant which induces $(-1)^P = 1$ for even permutation and $(-1)^P = -1$ for odd permutation. In next section we will use the concept of Slater determinant to find out the Slater-Schmidt number which is analogue of Schmidt number used for entanglement detection of indistinguishable particles.

3.4 Consequences of Indistinguishability

Here, we will consider the example of double well potential which consist of qubit having spin degree of freedom represented as $|\uparrow\rangle$ and $|\downarrow\rangle$ to describe the consequences of indistinguishability leads toward entanglement. These qubits are represented by spatial wave-function i.e. $|\phi\rangle$ and $|\chi\rangle$ initially localized in left and right potential well and represented by four dimensional Hilbert space, $\{|\phi\uparrow\rangle, |\phi\downarrow\rangle, |\chi\uparrow\rangle, |\chi\downarrow\rangle\}$. Consider the example in which we have one particle in each well and both wells separated at large distances as shown in figure 3.2, so that both particles behave as distinguishable particles and their complete state can be written as [26]

$$|\psi\rangle = |\phi\rangle_A \otimes |\chi\rangle_B \,. \tag{3.12}$$

where, labels A and B are used for Alice and Bob.

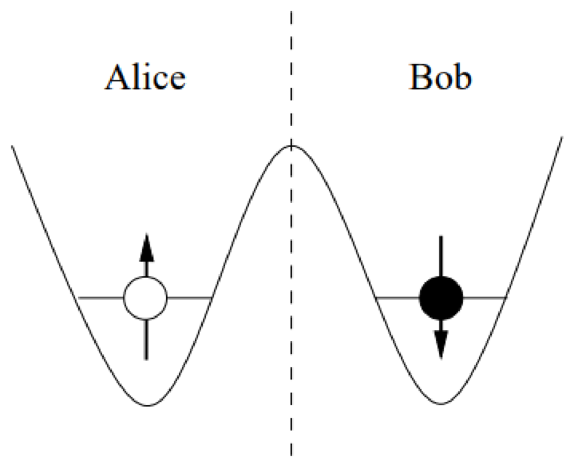

Figure 3.2: Two particles located in two separate wells having spin degrees of freedom

Now, we move two wells close enough so that particle indistinguishability prevails and we can not distinguish which particle having spin up or down is in which well as shown in figure 3.3. Because both particles are indistinguishable state of complete system can be written in term of Slater determinant.

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left[|\phi\uparrow\rangle_1 \otimes |\chi\downarrow\rangle_2 - |\chi\downarrow\rangle_1 \otimes |\phi\uparrow\rangle_2\right]. \tag{3.13}$$

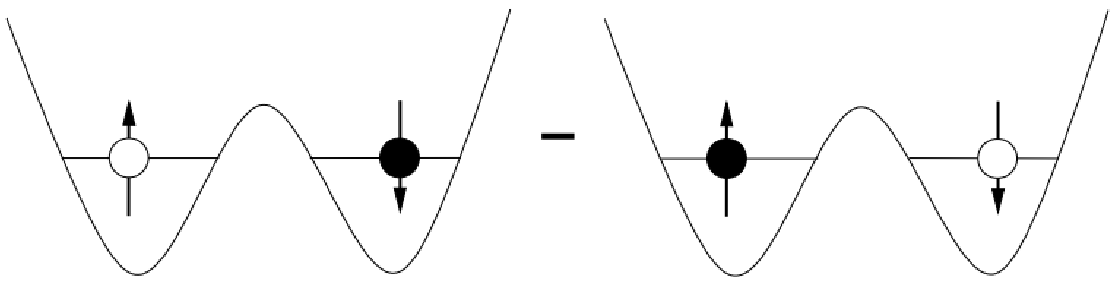

Figure 3.3: Particle localization destroyed by decreasing the distance between two wells

while in term of Slater determinant

$$|\psi\rangle = \frac{1}{\sqrt{2}} \begin{vmatrix} |\phi\uparrow\rangle_1 & |\phi\uparrow\rangle_2 \\ |\chi\downarrow\rangle_1 & |\chi\downarrow\rangle_2 \end{vmatrix}. \tag{3.14}$$

Here, we introduce labels 1 and 2 instead of A and B to emphasize that particle labeling is arbitrary. The negative sign in eq (3.13) is due to anti-symetrization. We can see that labeling of state in eq (3.13) is misleading. Because particle appears entangled w.r.t labels and physically this entanglement has no meaning and can not be used as for quantum information processing. We will reconsider this problem in next chapter and provide possible solution of it.

3.5 Entanglement of Bipartite Distinguishable Particles

Relationship between degree of entanglement and compositness of two bosons can be calculated with the help of Schmidt decomposition. In this section, we will find this relationship by using Schmidt decomposition (discussed in chapter 2) of two distinguishable bosons by considering double Gaussian wave-function. Consider we have composite particle C composed of two particles A and B which are distinguishable and represented by wave-function $\psi(x_A, x_B)$. Schmidt decomposition of $\psi(x_A, x_B)$ will be

$$\psi(x_A, x_B) = \sum_{m}^{\infty} \sqrt{\lambda_m} \varphi_m^{(A)}(x_A) \varphi_m^{(B)}(x_B), \tag{3.15}$$

We can also write Schmidt coefficients λ_m as

$$\kappa = \sum_{m=0}^{\infty} \frac{1}{\lambda_m^2}. \tag{3.16}$$

The particle creation operator known as composite boson operator $c^\dagger$ on vacuum state to create particles in mode $\varphi_m^{(A)}$ and $\varphi_m^{(B)}$. This composite boson operator is

$$c^\dagger = \sum_{m}^{\infty} a_m^\dagger b_m^\dagger, \tag{3.17}$$

where, $a_m^\dagger$ and $b_m^\dagger$ are particle creation operator in mode $\varphi_m^{(A)}$ and $\varphi_m^{(B)}$. The action of composite boson operator on vacuum state to create M particle state is

$$|N\rangle = \chi_N^{-1/2} \frac{c^{\dagger M}}{\sqrt{M}} |0\rangle, \tag{3.18}$$

where, $\chi_M^{-1/2}$ normalization operator and appears because $c^\dagger$ is not an ideal bosonic operator i.e. $[c, c^\dagger] \neq 1$ and show deviation from ideal bosonic character. The commutation relation of c and $c^\dagger$ can be calculated by using eq (3.17) which is

$$[c, c^\dagger] = 1 + f\delta, \tag{3.19}$$

where, $f = +1$ for bosons and -1 for fermions and δ is,

$$\delta = \sum_{m=0}^{\infty} \lambda_m (a_m^\dagger a_m + b_m^\dagger b_m). \tag{3.20}$$

which shows deviation from ideal bosonic character. To find out relationship of entanglement with degree of compositeness we have to calculate $\frac{\chi_{M\pm1}}{\chi_M}$ known composite boson normalization ratio. The action of annihilation composite boson operator on state $|M\rangle$ is

$$c|M\rangle = \beta_M \sqrt{M} |M-1\rangle + |\epsilon_M\rangle. \tag{3.21}$$

where, $|\epsilon_M\rangle$ is correction term and $\beta_M = \frac{\chi_M}{\chi_{M-1}}$. For an ideal composite boson

$$\beta_M \longrightarrow 1 \quad \langle \epsilon_M | \epsilon_M \rangle \longrightarrow 0. \tag{3.22}$$

The inner product of $\langle \epsilon_M | \epsilon_M \rangle$ actually hides composite boson ratio which can be calculated [28] by using eq (3.21) which is,

$$\langle \epsilon_M | \epsilon_M \rangle = 1 + (M-1)\frac{\chi_{M+1}}{\chi_M} - M\frac{\chi_M}{\chi_{M-1}}. \tag{3.23}$$

We will consider double Gaussian wave function of composite bosons to find out relationship between entanglement and degree of compositeness which is,

$$\psi(x_A, x_B) = Ne^{\frac{-(x_A+x_B)^2}{\sigma_c^2}} e^{\frac{(x_A+x_B)^2}{\sigma_r^2}}, \tag{3.24}$$

where, σ_c and σ_r are widths along $(x_A + x_B)$ and $(x_A - x_B)$. Schmidt decomposition of above state is

$$\psi(x_A, x_B) = \sqrt{1-z^2}\sum_{m=0}^{\infty} z^m \varphi_m^A(x_A)\varphi_m^B(x_B). \tag{3.25}$$

By using eq (3.15) and (3.25) we will get

$$\lambda_m = (1 - z^m)z^{2m}, \tag{3.26}$$

where, λ_m are known as Schmidt coefficients which contains information about entanglement. We can write normalization ratio for bosons and fermions as,

$$\chi_M^B = M! \sum_{q_M \geq q_{M-1} \geq q_{M-2} \geq ......\geq q_2 \geq q_1} \lambda_{q_1}\lambda_{q_2}.....\lambda_{q_M}, \tag{3.27}$$

$$\chi_M^F = N! \sum_{q_M > q_{M-1} > q_{M-2} > ......> q_2 > q_1} \lambda_{q_1}\lambda_{q_2}.....\lambda_{q_N}, \tag{3.28}$$

where, subscript q is used to count number of states for bosons and fermions. By using Schmidt coefficients in eq (3.27) and (3.28), we obtain normalization ratio for bosons and fermions [27]

$$\frac{\chi_{M+1}^B}{\chi_M^B} = \frac{(1-z)(M+1)}{(1-z)^{M+1}}, \tag{3.29}$$

$$\frac{\chi_{M+1}^F}{\chi_M^F} = z^M\frac{(1-z)(M+1)}{(1-z)^{M+1}}. \tag{3.30}$$

We can now create its connection with entanglement by using Schmidt eigenvalues λ_m in eq (3.16) as

$$\kappa = \frac{1 - z^2}{(1 - z)^2}. \tag{3.31}$$

We can know relate $\frac{\chi^B_{M+1}}{\chi^B_M}$ and $\frac{\chi^F_{M+1}}{\chi^F_M} = z^M \frac{(M+1)(1-z)}{(1-z)^{M+1}}$ with degree of entanglement by expressing z in term of κ i.e.

$$\frac{\chi_{M+1}}{\chi_M} \approx 1 + f\frac{M}{\kappa}. \tag{3.32}$$

where $f = 1$ for bosons and -1 for fermions. Larger the value of κ means stronger the entanglement. If κ is very very large then $f\frac{M}{\kappa} \to 0$ which means that $f\frac{\chi_{M+1}}{\chi_M} \to 1$ i.e. composite bosons will behave as pure bosons. So we have showed that if particles are strongly correlated then they will also behave as composite particles which means degree of compositness is directly related to entanglement.

3.6 Slater-Schmidt Decomposition

In previous section, we discussed entanglement quantification of bipartite distinguishable particles by using Schmidt decomposition. We showed that schmidt decomposition is excellent criteria of entanglement quantification for distinguishable particles but generates misleading results for indistinguishable particles. Here, we will show that Slater-Schmidt decomposition is alternative approach to deal with indistinguishable particles. In this section we will explain Slater-Schmidt decomposition and its application for bosonic and fermionic case.

Theorem For any $M \otimes M$ matrix there exists unitary transformation $B = UZU^T$ such that [26]

$$Z = diag[Z_0, Z_1, Z_2, ...Z_r], Z_0 = 0, Z_j = \begin{pmatrix} .0 & z_j \\ z_j & 0 \end{pmatrix}, \tag{3.33}$$

where, Z_0 is null matrix and z_j are complex numbers.

3.6.1 Bipartite Fermionic Entanglement

The application of Slater-Schmidt decomposition for two indistinguishable fermions can be explained by considering state $|\psi(1,2)\rangle$ as,

$$|\psi(1,2)\rangle = \sum_{k=1}^{\frac{2s+1}{2}} a_k \frac{1}{\sqrt{2}}[|2k-1\rangle_1 \otimes |2k\rangle_2 - |2k\rangle_1 \otimes |2k-1\rangle_2], \tag{3.34}$$

where $\{|2k-1\rangle, |2k\rangle\}$ are known as orthonormal basis and the number a_k appears in eq (3.34) is known as Slater-Schmidt rank used as an entanglement quantifier for indistinguishable particles. The state $|\psi(1,2)\rangle$ is entangled iff Slater-Schmidt number $a_k > 1$. The relationship between Slater number and entanglement is given in paper [9, 37]. If we have two orthonoraml vectors then state $|\psi(1,2)\rangle$ reduces to,

$$|\psi(1,2)\rangle = \frac{1}{\sqrt{2}} [|1\rangle_1 \otimes |2\rangle_2 - |2\rangle_1 \otimes |1\rangle_1]. \tag{3.35}$$

The above state is un-entangled because $a_k = 1$. We can obtain this state by anti-symetrizing product state $|1\rangle_1 \otimes |2\rangle_2$ but when Slater number > 1 then we can not obtain eq (3.34) by antisymmetrizing product state and we will say that $|\psi(1,2)\rangle$ is entangled state. But contradiction in results arises if we use Von-Neumann entropy to quantify entanglement of state $|\psi(1,2)\rangle$. To obtain Von-Neumann entropy of eq (3.34) we will find reduced density operator $\rho^{(1)}$ or $\rho^{(2)}$ by performing partial trace on either particle 1 or 2, i.e.

$$\rho^{(1)} = \frac{|a_k|^2}{2} [|2k-1\rangle_1 \langle 2k-1|_1 - |2k\rangle_1 \langle 2k|_1]. \tag{3.36}$$

where $\lambda_k = \frac{|a_k|^2}{2}$ are $\rho^{(1)}$. By using eq (2.46) we will get,

$$S(\rho^{(1)}) = 1 - \sum_k |a_k|^2 \log_2 |a_k|^2. \tag{3.37}$$

when Slater rank i.e. $a_k = 1$ then according to eq (3.37) $S(\rho^{(1)}) = 1$ which means states are not entangled [9, 29]. But for the case of Schmidt decomposition when Schmidt number $= 1$ then $S(\rho^{(1)}) = 0$, means non entangled state which is contradiction to above result. So, where is problem? Problem is in interpretation of Von-Neumann entropy because we are not considering real meaning of Von-Neumann entropy e.g, measure of uncertainty in quantum state. For distinguishable particles, we are certain about state of particle but for indistinguishable particle we can not tell which particle is in which state. We will discuss this problem in next chapter where we will define that partial trace is basis dependent to obtain eigenvalues of reduce density operator used to find Von-Neumann entropy. So here is summary of above discussion,

- If Slater rank of state $|\psi(1,2)\rangle = 1$ or Von-Neumann entropy, $\rho^{(1)}$ or $\rho^{(2)} = 1$ then given state will be un-entangled.
- If Slater rank of stae $|\psi(1,2)\rangle > 1$ or Von-Neumann entropy, $\rho^{(1)}$ or $\rho^{(2)} > 1$ then given state will be entangled [30].

3.6.2 Bipartite Bosonic Entanglement

We know that if Schmidt number > 1 then state will be entangled and will not be entangled if Schmidt number $= 1$, but this is not true for two indistinguishable bosons. In this section we will discuss two indistinguishable bosons present in the same site and their entanglement. Let's consider state of two indistinguishable bosons is,

$$|\psi(1,2)\rangle = \sum_{i=1}^{2s+1} c_j \left|j\right\rangle_1 \otimes \left|j\right\rangle_2 , \tag{3.38}$$

where c_j are diagonal elements of matrix Z given in (3.33) and $\{|j\rangle\}$ are orthonormal basis. If Schmidt number of the given state is 1 then state $|\psi(1,2)\rangle$ reduces to

$$|\psi(1,2)\rangle = |j'\rangle \otimes |j'\rangle , \tag{3.39}$$

means two bosons will be in same state and state of that bosons will be un-entangled. But what will happen if Schmidt number > 2 ? If Schmidt number > 2 then state $|\psi(1,2)\rangle$ reduces to,

$$|\psi(1,2)\rangle = c_1 \left|1\right\rangle_1 \otimes \left|1\right\rangle_2 + c_2 \left|2\right\rangle_1 \otimes \left|2\right\rangle_2 . \tag{3.40}$$

where $c_1^2 + c_2^2 = 1$. If $c_1 = c_2 = \frac{1}{\sqrt{2}}$ then state $|\psi(1,2)\rangle$ will be un-entangled because we can obtain state $|\psi(1,2)\rangle$ by symmetrizing product state $(\left|1\right\rangle_1 \otimes \left|2\right\rangle_2)$ of two orthogonal states [30]. If $c_1 \neq c_2$ then state will obtain by symmetrzing orthogonal state and will be entangled according to our criteria given in [30]. Here is summary of above discussion for indistinguishable bosonic particles.

- If Schmidt number of $|\psi(1,2)\rangle = 1$ and $S\rho^{(1)}$ or $S\rho^{(1)} = 0$, then state will be un-entangled.
- If Schmidt number of $|\psi(1,2)\rangle = 2$ and $S\rho^{(1)}$ or $S\rho^{(1)} = 1$, then state will also be un-entangled.
- If Schmidt number of $|\psi(1,2)\rangle > 2$, then state will be entangled.

Chapter 4

Schmidt Decomposition as Entanglement Quantifier for Indistinguishable Particles

Due to indistinguishable nature of particles it is impossible to address the particles individually. Earlier approaches developed in quantum mechanics such as name labeling to make the particles artificially distinguishable have some disadvantages as we have discussed in previous chapter. So here we explain non standard particle base approach which deals with indistinguishability of particles and will be used for entanglement quantification. In this approach we will consider global state described by one particle state vector $|\Phi, \Psi\rangle$, where, $|\Phi\rangle$ represents state of one particle and $|\Psi\rangle$ represent state of other particle. Due to indistinguishable nature of particles we can not tell which particle is in which state. To obtain physical predictions about system, we consider another global two particle state represented as $|\varphi, \zeta\rangle$ and will write probability of these two particles going from state $|\Phi\rangle$ ($|\Psi\rangle$) to $|\varphi\rangle$ ($|\zeta\rangle$) as $\langle\varphi, \zeta|\Phi, \Psi\rangle$. Here, $|\varphi\rangle$ represents probability of finding particle comes from $|\Phi\rangle$ or $|\Psi\rangle$ and similarly $|\zeta\rangle$ represent probability of finding particle comes from $|\Phi\rangle$ or $|\Psi\rangle$ as mentioned in fig 4.1. The probability of finding particle in state $|\varphi\rangle$ or ζ coming from state $|\Phi\rangle$ or $|\Psi\rangle$ can be written as

$$\langle\varphi, \zeta|\Phi, \Psi\rangle = \langle\varphi|\Phi\rangle \langle\zeta|\Psi\rangle + \eta \langle\varphi|\Psi\rangle \langle\zeta|\varphi\rangle . \tag{4.1}$$

Here, η represents spin statistics i.e for bosons $\eta = 1$ and for fermions $\eta = -1$.

We can see that eq (4.1) directly encompasses the symmetrization postulate [31] and

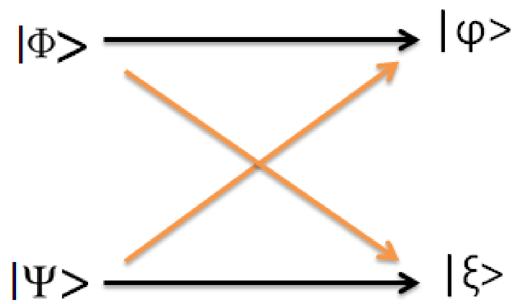

Figure 4.1: Schematic diagram of finding particles probability coming from state $|\Phi\rangle$ or $|\Psi\rangle$ into state $|\varphi\rangle$ or $|\zeta\rangle$.

is the core of non standard particle based approach. The probability of finding the particle in the same state $|\varphi\rangle$ is

$$\langle \varphi, \varphi | \Phi, \Psi \rangle = (1 + \eta) \langle \varphi | \Phi \rangle \langle \zeta | \Phi \rangle . \tag{4.2}$$

and it will be maximum for bosons ($\eta = 1$) and minimum for fermions ($\eta = -1$) means two fermions can not occupy same state (accordance with exclusion principle).
If we have state space of different dimensionality i.e $|\Psi_k\rangle$ then its inner product with state $|\Phi, \Psi\rangle$ will be [24]

$$\langle \Psi_k | \, . \, |\Phi, \psi\rangle = \langle \Psi_k | \Phi, \psi \rangle = \langle \Psi_k | \Phi \rangle |\psi\rangle + \eta \langle \Psi_k | \psi \rangle |\Phi\rangle . \tag{4.3}$$

If we choose two particle orthonormal basis set $|j, k\rangle$, where $|j\rangle$ and $|k\rangle$ being used as single particle basis addresses individual particles. By using these two indistinguishable particle orthonormal basis we can write our state vector as

$$|\psi\rangle = \sum_{jk} c_{jk} |j, k\rangle . \tag{4.4}$$

By using definition of density matrix i.e $\rho = |\psi\rangle \langle\psi|$ and applying partial trace on the basis $|k\rangle$ and using eq (4.3) we will get reduced density matrix e.g. [34]

$$\rho^{(1)} = \frac{1}{2} \sum_{k} \langle k | \psi \rangle \langle \psi | k \rangle = \frac{1}{2} \operatorname{Tr}^{(1)} \rho . \tag{4.5}$$

In above equation we can see that partial trace depends upon particle basis being local or non-local while in the case of distinguishable particles partial trace is not affected by the particle basis being local or non-local because we can address the particles individually in that case. So in next section we will use indistinguishable particle basis and express the state of composite system in that basis. After that we will use eq (4.5) to obtain Schmidt decomposition which will be useful to quantify entanglement for indistinguishable particles. We can know construct following theorem.

4.1 Theorem

If we have two indistinguishable particles described by product Hilbert space then pure state of these two particles can be always as,[18]

$$|\psi\rangle = \frac{1}{\sqrt{2}} \sum_j \sqrt{\lambda_j}\, |j, \bar{j}\rangle \,. \tag{4.6}$$

where, $\sqrt{\lambda_j}$ are the eigenvalues and the states $\{|j\rangle\}$ are eigenstates of reduced density matrix. Here, $\lambda_j > 0$ and $\sum_j \lambda_j = 1$.

Proof: We can prove above theorem by expressing the state $|\psi\rangle$ in the basis $|j,k\rangle$ i.e.

$$|\psi\rangle = \frac{1}{2} \sum_{j,k} |j,k\rangle \langle j,k|\psi\rangle \,, \tag{4.7}$$

where, we have used two particle symmetric identity operator i.e. $\Lambda_2 = \frac{1}{2} \sum_{j,k} |j,k\rangle \langle j,k|$. By defining $|\bar{j}\rangle = \sum_k \langle j,k|\psi\rangle |k\rangle$, we can write eq (4.7) as,

$$|\psi\rangle = \frac{1}{2} \sum_j |j, \bar{j}\rangle \,. \tag{4.8}$$

Generally, the basis $\{|\bar{j}\rangle\}$ are not orthonormal but as for distinguishable particles there will exist orthonormal basis so we write

$$\langle \bar{j}'|\bar{j}\rangle = \sum_{k,k'} \langle \psi|j',k'\rangle \langle j,k|\psi\rangle \langle k'|k\rangle, \tag{4.9}$$

by using $k = k'$, above equation reduces to

$$\langle \bar{j}'|\bar{j}\rangle = \sum_k \langle j,k|\psi\rangle \langle \psi|j',k\rangle \,, \tag{4.10}$$

$$\langle \bar{j}' | \bar{j} \rangle = \sum_{j,j'} \langle j | \times \sum_{k} \langle k|\psi\rangle \langle \psi|k\rangle \times |j'\rangle \,, \tag{4.11}$$

By using the definition of partial trace i.e. $\rho^{(1)} = \frac{1}{2} \sum_k \langle k|\psi\rangle \langle\psi|k\rangle$ we will get

$$\langle \bar{j}' | \bar{j} \rangle = 2 \sum_{j,j'} \langle j | \rho^{(1)} | j' \rangle \,. \tag{4.12}$$

The states $\{|j\rangle\}$ are eigenstates of reduced density operator $\rho^{(1)}$, e.g. $(\rho^{(1)} |j\rangle = \lambda_j |j\rangle)$. The above equation reduces to

$$\langle \bar{j}' | \bar{j} \rangle = 2\lambda_j \sum_{j,j'} \langle j | j' \rangle \,. \tag{4.13}$$

By using condition of orthogonality i.e. $\langle \bar{j}' | \bar{j} \rangle = \delta_{jj'}$, we will get

$$\langle \bar{j} | \bar{j} \rangle = 2\lambda_j. \tag{4.14}$$

So, our required orthonormal state is $|\tilde{j}\rangle = \frac{1}{\sqrt{2\lambda_j}} |\bar{j}\rangle$, and

$$|\bar{j}\rangle = \sqrt{2\lambda_j}\, |\tilde{j}\rangle \,. \tag{4.15}$$

Using the result of $|\bar{j}\rangle$ in equation (4.8), we will get require result i.e.

$$|\psi\rangle = \frac{1}{\sqrt{2}} \sum_{j} \sqrt{\lambda_j}\, |j, \tilde{j}\rangle \,. \tag{4.16}$$

The Schmidt decomposition of state $|\psi\rangle$ defines entanglement in terms of non-separability. Here, we will define Schmidt number (set of non-zero eigenvalues) as entanglement quantifier for the case of distinguishable particles. If 'Schmidt number'$= 1$ then given state will be un-entangled and if 'Schmidt number' > 1 then state will be entangled. The basic difference between this approach and existing one is that in this case partial trace depends upon basis of particles being local or non-local. Here, $\sqrt{\lambda_j}$ are Schmidt coefficients of reduced density matrix which leads toward Von-Neuman entropy i.e.

$$S(\rho^{(1)}) = -\,\mathrm{Tr}^{(1)}(\rho^{(1)} \log_2 \rho^{(1)}) = -\sum_{j} \lambda_j \log_2 \lambda_j. \tag{4.17}$$

which is used as entanglement quantifier for indistinguishable particles.
Here, we will mention some important steps to find Schmidt decomposition of given state and will check that our state is entangled or not for indistinguishable particles.

- First of all find the density matrix of the given state $|\psi\rangle$ by using relation $\rho = |\psi\rangle\langle\psi|$.
- Then we will obtain reduced density matrix $\rho^{(1)}$ by performing partial trace on the chosen single particle basis.
- By using that density matrix we will find eigenvalues λ_j and eigenstaes $\{|j\rangle\}$.
- By using (4.17) we will get Von-Neuman entropy which will provide us the information about the entanglement of the state.
- And finally we will construct eigenbasis $\{|\tilde{j}\rangle\}$ and find SD of given state by using eq (4.6).

Now, we will prove that $|\tilde{j}\rangle$ are the eigenstates of $\rho^{(1)}$ having eigenvalues λ_j, similarly to eigenstates $|j\rangle$. We know that

$$|\tilde{j}\rangle = \sum_k \frac{1}{\sqrt{2\lambda_j}} \langle j, k|\psi\rangle |k\rangle . \tag{4.18}$$

and,

$$\rho^{(1)} = \frac{1}{2} \sum_{j'} \langle j'|\psi\rangle \langle\psi|j'\rangle . \tag{4.19}$$

By applying $\rho^{(1)}$ on $|\tilde{j}\rangle$, we will get

$$\rho^{(1)} |\tilde{j}\rangle = \frac{1}{2\sqrt{2\lambda_j}} \sum_{j',k} \langle j'|\psi\rangle \langle\psi|j'\rangle \langle j, k|\psi\rangle |k\rangle , \tag{4.20}$$

$$\rho^{(1)} |\tilde{j}\rangle = \frac{1}{2\sqrt{2\lambda_j}} \sum_{j'} \langle j'|\psi\rangle \langle j| \times \sum_k \langle k|\psi\rangle \langle\psi|k\rangle \times |j'\rangle . \tag{4.21}$$

and we know that $\sum_k \langle k|\psi\rangle \langle\psi|k\rangle = 2\rho^{(1)}$ and $\langle j|\rho^{(1)}|j'\rangle$=$\lambda_j\delta_{jj'}$, for $j = j'$. Using these values in above equation we will get

$$\rho^{(1)} |\tilde{j}\rangle = \frac{1}{\sqrt{2\lambda_j}} \lambda_j \langle j|\psi\rangle . \tag{4.22}$$

Now we will introduce identity matrix of bipartite system as $\Delta = \frac{1}{2}\sum_{j',k'} |j',k'\rangle\langle j',k'|$ between $\langle j|$ and $|\psi\rangle$.

$$\rho^{(1)}\left|\tilde{j}\right\rangle = \frac{\sqrt{\lambda_j}}{2\sqrt{2}}\sum_{j',k'}\langle j|j',k'\rangle\langle j',k'|\psi\rangle . \tag{4.23}$$

by using eq (4.3) we will get,

$$\langle j|j',k'\rangle = \langle j|j'\rangle|k'\rangle + \eta\langle j|k'\rangle|j'\rangle , \tag{4.24}$$

$$\langle j|j',k'\rangle = \delta_{jj'}|k'\rangle + \eta\delta_{jk'}|j'\rangle . \tag{4.25}$$

By using eq (4.25) in eq (4.23), we will get

$$\rho^{(1)}\left|\tilde{j}\right\rangle = \frac{\sqrt{\lambda_j}}{2\sqrt{2}}\left(\sum_{j',k'}\delta_{jj'}\langle j',k'|\psi\rangle|k'\rangle + \eta\delta_{jk'}\langle j',k'|\psi\rangle|j'\rangle\right). \tag{4.26}$$

By using orthogonality condition i.e.

$$\left.\begin{array}{ll}\delta_{jj'} = 1 & j = j'\\ \delta_{jj'} = 0 & j \neq j'\\ \delta_{jk'} = 1 & j = k'\\ \delta_{jk'} = 0 & j \neq k'\end{array}\right\}, \tag{4.27}$$

using results of eq (4.27) in eq (4.26) we will get

$$\rho^{(1)}\left|\tilde{j}\right\rangle = \frac{\sqrt{\lambda_j}}{2\sqrt{2}}\left(\sum_{j',k'}\langle j,k'|\psi\rangle|k'\rangle + \eta\langle j',j|\psi\rangle|j'\rangle\right). \tag{4.28}$$

and $\eta\langle j',j|\psi\rangle|j\rangle = \eta\left(\eta\langle j,j'|\psi\rangle|j\rangle\right) = \langle j,j'|\psi\rangle|j'\rangle$, where we have used $\eta^2 = 1$. So,

$$\rho^{(1)}\left|\tilde{j}\right\rangle = \frac{\sqrt{\lambda_j}}{2\sqrt{2}}\left(\sum_{k'}\langle j,k'|\psi\rangle|k'\rangle + \sum_{j'}\langle j,j'|\psi\rangle|j'\rangle\right). \tag{4.29}$$

and we know that $\sum_k\langle j,k|\psi\rangle|k\rangle = \sqrt{2\lambda_j}\left|\tilde{j}\right\rangle$, so above equation reduces to

$$\rho^{(1)}\left|\tilde{j}\right\rangle = \frac{\sqrt{\lambda_j}}{2\sqrt{2}}\left(\sqrt{2\lambda_j}\left|\tilde{j}\right\rangle + \sqrt{2\lambda_j}\left|\tilde{j}\right\rangle\right), \tag{4.30}$$

$$\rho^{(1)}\left|\tilde{j}\right\rangle = \lambda_j\left|\tilde{j}\right\rangle . \tag{4.31}$$

We can see that $\left|\tilde{j}\right\rangle$ are the eigenstates of $\rho^{(1)}$ having eigenvalues λ_j, similarly to eigenstates $|j\rangle$. Now we want to find the relationship between $\left|\tilde{j}\right\rangle$ and $|j\rangle$.

4.1.1 Connection between $\left|\tilde{j}\right\rangle$ and $|j\rangle$

To find relationship between $\left|\tilde{j}\right\rangle$ and $|j\rangle$ we have to compute inner product of $\left|\tilde{j}\right\rangle$ and $|j\rangle$. We know that $\left|\tilde{j}\right\rangle = \frac{1}{\sqrt{2\lambda_j}} \sum_k \langle j,k|\psi\rangle\,|k\rangle$. So,

$$\left\langle j\middle|\tilde{j}\right\rangle = \langle j| \sum_k \frac{1}{2\lambda_j} \langle j,k|\psi\rangle\,|k\rangle\,. \tag{4.32}$$

$$\left\langle j\middle|\tilde{j}\right\rangle = \frac{1}{2\sqrt{\lambda_j}} \langle j,j|\psi\rangle\,. \tag{4.33}$$

by expressing state $|\psi\rangle$ in term od Schmidt decomposition i.e. $|\psi\rangle = \sum_k \frac{\sqrt{\lambda_j}}{2}\left|k.\tilde{k}\right\rangle$, above equation becomes,

$$\left\langle j\middle|\tilde{j}\right\rangle = \frac{1}{2\sqrt{\lambda_j}} \langle j,j| \sum_k \frac{\sqrt{\lambda_k}}{2}\left|k,\tilde{k}\right\rangle\,. \tag{4.34}$$

and by using eq (4.3) $\left\langle j,j\middle|k,\tilde{k}\right\rangle = \langle j|k\rangle \left\langle j\middle|\tilde{k}\right\rangle + \eta \left\langle j\middle|\tilde{k}\right\rangle \langle j|k\rangle$ and $\langle j|k\rangle = \delta_{jk}$, above equation reduces to,

$$\left\langle j\middle|\tilde{j}\right\rangle = \frac{1}{2}(1+\eta)\left\langle j\middle|\tilde{j}\right\rangle\,. \tag{4.35}$$

From eq (4.35) it's clear that for fermions i.e. ($\eta = -1$), $\left\langle j\middle|\tilde{j}\right\rangle = 0$, which means that two fermions can't occupy same state (Pauli exclusion principle).

Now we will use this non standard particle based approach to deal with indistinguishable particles for different cases, i.e. when two indistinguishable qubits in separated sites, two qubits in the same site with random spin and two indistinguishable qutrits in the same site, and will show that it yields accurate results and deals with all problems which we have discussed in chapter 2 and 3.

4.2 Two Indistinguishable Particles in Separated Places

In this example, we will consider that we have two indistinguishable particles (qubits) e.g. (bosons or fermions) having opposite spins, e.g, if one particle have spin up in

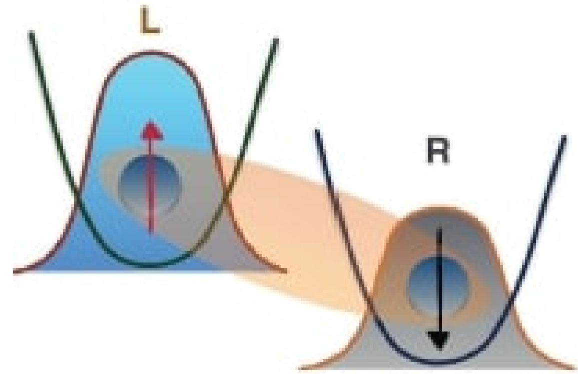

Figure 4.2: Two qubits in two separated sites with orthogonal spins.

one site then the other will have spin down in other site as you can see in figure 4.2. Because both particles are indistinguishable we didn't know about which particle have which spin, so the overall state of the two indistinguishable qubits can be described as

$$|\Psi\rangle = a\,|L\uparrow, R\downarrow\rangle + b\,|L\downarrow, R\uparrow\rangle\,. \tag{4.36}$$

where $a^2 + b^2 = 1$. Here the, site, Left(L) or Right (R) and the spins $(\uparrow,\downarrow)$ are two independent observables and both sites are nonoverlapping. The state $|\Psi\rangle$ is Bell Like states [?, 32] and allow us to discuss the role of local and non-local measurement. When we will perform local measurement e.g. in localized region of space(L) then the partial trace will be local [33]. First of all, we will find the ρ by performing the partial trace in localized region of space, to do so let's calculate density matrix i.e. ρ. We know that

$$\rho = |\Psi\rangle\langle\Psi|\,, \tag{4.37}$$

and from equation (4.36)

$$\langle\Psi| = a^*\langle L\uparrow, R\downarrow| + b^*\langle R\uparrow, L\downarrow|\,, \tag{4.38}$$

So equation (4.37) becomes

$$\rho = (a\,|L\uparrow, R\downarrow\rangle + b\,|L\downarrow, R\uparrow\rangle)(a^*\langle L\uparrow, R\downarrow| + b^*\langle R\uparrow, L\downarrow|), \tag{4.39}$$

$$\begin{aligned}\rho = a^2(|L\uparrow, R\downarrow\rangle\langle L\uparrow, R\downarrow|) + ab^*(|L\uparrow, R\downarrow\rangle\langle R\uparrow, L\downarrow|)\\ + a^*b(|L\downarrow, R\uparrow\rangle\langle L\uparrow, R\downarrow|) + b^2\,|\downarrow, R\uparrow\rangle\langle L\downarrow, R\uparrow|\,.\end{aligned} \tag{4.40}$$

This is the required density matrix of the given state $|\Psi\rangle$. By projecting this density matrix on the local basis e.g. on the subspace $\{|L\uparrow\rangle, |L\downarrow\rangle\}$, we will obtain reduced density matrix e.g.

$$\rho_L = b^2 |R\uparrow\rangle\langle R\downarrow| + a^2 |R\downarrow\rangle\langle R\uparrow|, \tag{4.41}$$

the above equation can be written as,

$$\rho_L = \begin{pmatrix} b^2 & 0 \\ 0 & a^2 \end{pmatrix}. \tag{4.42}$$

The eigenvalues of the above reduced density matrix are, $\lambda_1 = b^2$ and $\lambda_2 = a^2$, and its eigenstates are $|1\rangle$=$|R\uparrow\rangle$, $|\tilde{1}\rangle$=$\eta|L\downarrow\rangle$, $|2\rangle$=$|R\downarrow\rangle$ and $|\tilde{2}\rangle$=$\eta|L\uparrow\rangle$. From these eigenstates, we can see that $\langle j|\tilde{j}\rangle = 0$, (where $j = 1, 2$) which means that both spins of the particles are orthogonal and states of both particles are non overlapping and both particles will behave as distinguishable particles. The Schmidt decomposition of two indistinguishable particles can be written as

$$|\psi\rangle = \frac{1}{\sqrt{2}} \sum_j \sqrt{\lambda_j} |j, \tilde{j}\rangle, \tag{4.43}$$

where, $\lambda_j > 0$ and $\sum_j \lambda_j = 1$. "λ_j" are the eigenvalues of reduced density matrix and $\{|j\rangle\}$ are its eigenstates. By using eq (4.43), Schmidt decomposition of our given state is,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(\sqrt{\lambda_1}|1, \tilde{1}\rangle + \sqrt{\lambda_2}|2, \tilde{2}\rangle), \tag{4.44}$$

where, $\lambda_1 = b^2$ and $\lambda_2 = a^2$ are eigenvalues of reduced density matrix. By using these values we get

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(b|1, \tilde{1}\rangle + a|2, \tilde{2}\rangle). \tag{4.45}$$

4.2.1 Von-Neumann Entropy when Measurement is Local

The Schmidt coefficients of reduced density matrix leads us to find Von-Neumann entropy which is used as entanglement quantifier for indistinguishable particle just like that for distinguishable particles. We know that the general expression of Von-Neumann entropy for distinguishable particles is,

$$S(\rho_L) = -\sum_{j=1,2} \lambda_j \log_2 \lambda_j. \tag{4.46}$$

The above expression can also be used for indistinguishable particles but the fundamental difference is that here we are not using usual idea of partial trace to obtain the reduced density matrix. For indistinguishable particles partial trace depends on the fact that weather single particle basis are local or non local but for distinguishable particles, partial trace is not affected by local or non local nature of the basis because the single particle basis always addresses a given particle. The Von-Neumann entropy for the case of two indistinguishable particles in two separated places is,

$$S(\rho_L) = -\lambda_1 \log_2 \lambda_1 - \lambda_2 \log_2 \lambda_2, \tag{4.47}$$

where, $\lambda_1 = b^2 = 1 - a^2$ where we have used $(a^2+b^2 = 1)$ and $\lambda_2 = a^2$. Using these values in above equation we will get

$$S(\rho_L) = -a^2 \log_2(a^2) - (1 - a^2) \log_2(1 - a^2). \tag{4.48}$$

For Bell like states i.e. for $a = b = \frac{1}{\sqrt{2}}$ we will get maximum entanglement i.e. $(S(\rho_L) = 1)$. This result exactly coincides with the known von-Neumann entropy for distinguishable particles which means that when two indistinguishable particles with orthogonal degrees of freedom in two non overlapping sites will behave like distinguishable particle.

4.2.2 Von-Neumann Entropy when Measurement is Non Local

When we perform measurement on both sites (left(L), Right(R)) simultaneously then the measurement itself and corresponding partial trace will be non-local. The density matrix for the state $|\Psi\rangle$ is given in eq (4.40). We will get reduced density matrix via partial trace as,

$$\rho = \frac{1}{2} \sum_k \langle k|\Psi\rangle \langle\Psi|k\rangle \,, \tag{4.49}$$

here, $\{|k\rangle\}$ are the basis being local or non local. In our case, we will perform partial trace on non-local basis i.e.$\{|L\uparrow\rangle, |L\downarrow\rangle, |R\uparrow, |R\downarrow\rangle\rangle\}$ by using equation (4.49), we will get

$$\rho = \frac{1}{2}(a^2 |L\uparrow\rangle \langle L\uparrow| + b^2 |L\downarrow\rangle \langle L\downarrow| + b^2 |R\uparrow\rangle \langle R\uparrow| + a^2 |R\downarrow\rangle |R\uparrow\rangle). \tag{4.50}$$

In matrix form the above equation can be written as

$$\rho = \begin{pmatrix} \frac{a^2}{2} & 0 & 0 & 0 \\ 0 & \frac{b^2}{2} & 0 & 0 \\ 0 & 0 & \frac{b^2}{2} & 0 \\ 0 & 0 & 0 & \frac{a^2}{2} \end{pmatrix}, \tag{4.51}$$

where, $\lambda_1 = \lambda_4 = \frac{a^2}{2}$ and $\lambda_2 = \lambda_3 = \frac{b^2}{2}$ are the eigenvalues of the above reduced density matrix and its eigenstates are $|1\rangle = \eta\,|\tilde{4}\rangle = |L\uparrow\rangle$, $|\tilde{1}\rangle$=$|4\rangle$=$|R\downarrow\rangle$, $|2\rangle$=$\eta\,|\tilde{3}\rangle$=$L\downarrow$ and $|\tilde{2}\rangle$=$|3\rangle$=$|R\uparrow\rangle$. The Schmidt decomposition of the state $|\Psi\rangle$ (when the measurement and trace are non-local) is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(\sqrt{\lambda_1}\,|1,\tilde{1}\rangle + \sqrt{\lambda_4}\,|4,\tilde{4}\rangle + \sqrt{\lambda_2}\,|2,\tilde{2}\rangle + \sqrt{\lambda_3}\,|3,\tilde{3}\rangle). \tag{4.52}$$

By using the eigenvalues of eq (4.51) in above equation we will get

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(\frac{\alpha}{\sqrt{2}}(|1,\tilde{1}\rangle + |4,\tilde{4}\rangle) + \frac{\beta}{\sqrt{2}}(|2,\tilde{2}\rangle + |3,\tilde{3}\rangle)), \tag{4.53}$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(\frac{\alpha}{\sqrt{2}}(1,\tilde{1} + 4,\tilde{4}) + \frac{\beta}{\sqrt{2}}(2,\tilde{2} + 3,\tilde{3})). \tag{4.54}$$

The Von-Neumann entropy for the non-local measurement is,

$$S(\rho) = -\lambda_1 \log_2 \lambda_1 - \lambda_4 \log_2 \lambda_4 - \lambda_2 \log_2 \lambda_2 - \lambda_3 \log_2 \lambda_3. \tag{4.55}$$

By using eigenvalues λ_1,λ_2,λ_3 and λ_4 in above equation we will get,

$$S(\rho) = -\frac{\alpha^2}{2}\log_2\frac{\alpha^2}{2} - \frac{\alpha^2}{2}\log_2\frac{\alpha^2}{2} - \frac{\beta^2}{2}\log_2\frac{\beta^2}{2} - \frac{\beta^2}{2}\log_2\frac{\beta^2}{2}, \tag{4.56}$$

$$S(\rho) = -\alpha^2 \log_2\frac{\alpha^2}{2} - \beta^2 \log_2\frac{\alpha^2}{2}, \tag{4.57}$$

and from normalization condition we will get $\beta^2 = 1 - \alpha^2$, using this value in above equation $S(\rho)$ reduced to,

$$S(\rho) = -\alpha^2 \log_2 \frac{\alpha^2}{2} - (1 - \alpha^2) \log_2 \frac{\alpha^2}{2}. \tag{4.58}$$

This is the required expression of Von-Neumann entropy for non-local measurement.

4.2.3 Importance of Locality and Non-Locality

The importance of locality and non locality can be seen by considering the special case e.g, when $\alpha = 1$. Due to normalization condition $\beta = 0$ and our state $|\Psi\rangle$ reduces to $|\psi'\rangle$=$|L\uparrow, R\downarrow\rangle$ which is unentangled state [33] because both the particles will be in two different site and behave as distinguishable particles. For this state $(|\psi\rangle)$, $S(\rho) = 1$ and $S(\rho_L) = 0$. The difference between $S(\rho)$ and $S(\rho_L)$ highlights the importance of local and non-local measurement.These expression shows that local single particle measurement provide intrinsic entanglement [33] while non-local measurement produce measurement induced entanglement [11, 35] for the case of indistinguishable particles. These results shows dissimilarity for distinguishable particles where single particle measurement always addresses individual particles. In previous proposals, [36] where, we used normal notion of partial trace where single particle measurement always addresses individual particles. This yields entanglement even for uncorrelated separated particles but in this case, we overcome this issue with the help of local and non-local measurement.

4.3 Two Particles (Qubits) in Same Site with Random Spin

In previous example we talk about the entanglement of two indistinguishable particles when they are in two separated non-overlapping sites with orthogonal degrees of freedom. In this case we will consider the example of two indistinguishable qubits in the same site having arbitrary spin as shown in figure 4.3.

This is only possible for bosons because according to exclusion principle two fermions

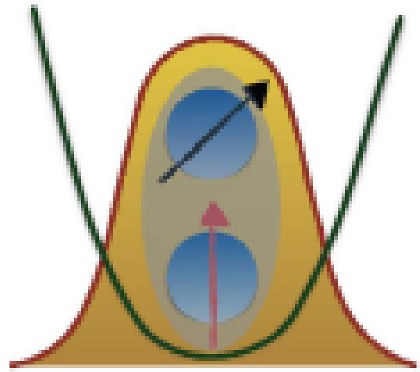

Figure 4.3: Two qubit in same site with arbitrary spin

can only stay in the same site when their spins are orthogonal [33]. We know that the entanglement is name of non-separability [38] of state and when two particles in the same sites with arbitrary spin then their spin creates such non-separability. Consider two indistinguishable qubits (i.e. bosons) one is along z-axis ($\uparrow_z$=$\uparrow$) and the other one along any arbitrary direction $u = (1, \theta, \phi)$ as shown in figure 4.4. The un-normalized form of this state is,

$$|\Psi\rangle = |\uparrow, \uparrow_u\rangle = \cos(\theta/2)\,|\uparrow, \uparrow\rangle + e^{\iota\phi}\sin(\theta/2)\,|\uparrow, \downarrow\rangle\,, \tag{4.59}$$

where, $|\uparrow_u\rangle$=$\cos(\theta/2)\,|\uparrow\rangle$+ $e^{\iota\phi}\sin(\theta/2)\,|\downarrow\rangle$. We can see that the above state is un-normalized. We can find normalized state by finding normalization constant i.e.

$$\begin{aligned}\langle\Psi|\Psi\rangle &= \cos^2\frac{\theta}{2}\,\langle\uparrow,\uparrow|\uparrow,\uparrow\rangle \\ &+ e^{-\iota\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,\langle\downarrow,\uparrow|\uparrow,\uparrow\rangle + e^{\iota\phi}\sin\frac{\theta}{2}2\cos\frac{\theta}{2}\,\langle\uparrow,\uparrow|\uparrow,\downarrow\rangle + \sin^2\frac{\theta}{2}\,\langle\downarrow,\uparrow|\uparrow,\downarrow\rangle\,.\end{aligned} \tag{4.60}$$

By using equation $\langle\varphi,\zeta|\phi,\psi\rangle$=$\langle\varphi|\phi\rangle\,\langle\zeta|\psi\rangle$+$\eta\,\langle\varphi|\psi\rangle\,\langle\zeta|\phi\rangle$,we will find that $\langle\uparrow,\uparrow|\uparrow,\uparrow\rangle$=2, $\langle\downarrow,\uparrow|\uparrow,\uparrow\rangle$=0, $\langle\uparrow,\uparrow|\uparrow,\downarrow\rangle$=0 and $\langle\downarrow,\uparrow|\uparrow,\downarrow\rangle$=1 and using all of these results in equation (4.60) we will get

$$\langle\Psi|\Psi\rangle = 2\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}, \tag{4.61}$$

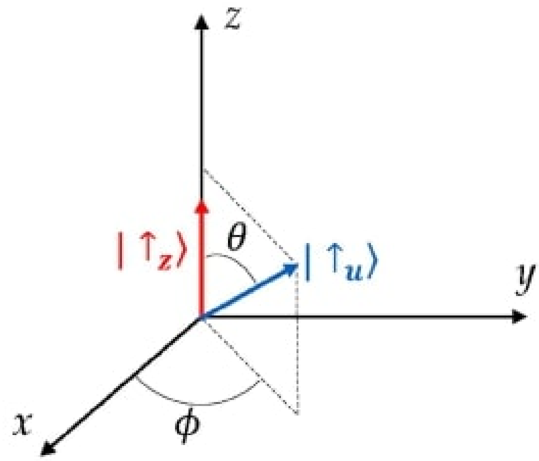

Figure 4.4: Two qubit expressed by state $|\uparrow_u\rangle$ and $|\uparrow_z\rangle$, one is along z-axis and the other one along any arbitrary direction determined by angle θ and ϕ.

$$\langle\Psi|\Psi\rangle = 2\cos^2\frac{\theta}{2} + 1 - \cos^2\frac{\theta}{2,} \tag{4.62}$$

$$\langle\Psi|\Psi\rangle = 1 + \cos^2\frac{\theta}{2} = N. \tag{4.63}$$

This is the normalization constant for the state $|\Psi\rangle$. So now normalized state can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{N}}(\cos\theta/2)\,|\uparrow,\uparrow\rangle + e^{\iota\phi}\sin\theta/2\,|\downarrow,\uparrow\rangle). \tag{4.64}$$

and the density matrix by using $\rho = |\Psi\rangle\langle\Psi|$ is

$$\begin{aligned}\rho = \frac{1}{N}\Bigg(&\cos^2\frac{\theta}{2}\,|\uparrow,\uparrow\rangle\langle\uparrow,\uparrow|) + e^{-\iota\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,|\uparrow,\uparrow\rangle\langle\downarrow,\uparrow| \\ &+ e^{\iota\phi}\sin\frac{\theta}{2}\cos\frac{\theta}{2}\,|\uparrow,\downarrow\rangle\langle\uparrow,\uparrow| + \sin^2\frac{\theta}{2}\,|\uparrow,\downarrow\rangle\langle\downarrow,\uparrow|\Bigg).\end{aligned} \tag{4.65}$$

Now, $\rho^{(1)}$ can be obtained by performing partial trace on basis $\{|\uparrow\rangle, |\downarrow\rangle\}$ by using the relation $\rho^{(1)} = \frac{1}{2}\sum_k \langle k|\Psi\rangle\langle\Psi|k\rangle = \frac{1}{2}\,\mathrm{Tr}^{(1)}\,\rho$, e.g,

$$\begin{aligned}&\rho^{(1)} \\ &= \frac{1}{2N}((4\cos^2\theta/2 + \sin^2\theta/2)\,|\uparrow\rangle\langle\uparrow| + e^{\iota\phi}\sin\theta\,|\downarrow\rangle\langle\uparrow| + e^{-\iota\phi}\sin\theta\,|\uparrow\rangle\langle\downarrow| + \sin^2\theta/2\,|\downarrow\rangle\langle\downarrow|).\end{aligned} \tag{4.66}$$

In matrix form the above equation can be written as

$$\rho^{(1)} = \frac{1}{2N}\begin{pmatrix} 4\cos^2\theta/2+\sin^2\theta/2 & e^{\iota\phi}\sin\theta/2 \\ e^{-\iota\phi}\sin\theta & \sin^2\theta/2 \end{pmatrix}, \tag{4.67}$$

$$\rho^{(1)} = \frac{1}{2N}\begin{pmatrix} a & c \\ c^* & b \end{pmatrix}, \tag{4.68}$$

where, $N = 1+\cos^2\theta/2$, $a = 4\cos^2\theta/2+\sin^2\theta/2$, $c = e^{\iota\phi}\sin\theta/2$ and $b = \sin^2\theta/2$. The eigenvalues of $\rho^{(1)}$ can be determine by solving characteristic equation i.e.

$$|\rho^{(1)} - \lambda I| = 0, \tag{4.69}$$

where, I is identity matrix. By using eq (4.3) in above equation we will get,

$$\begin{vmatrix} \frac{a}{2N}-\lambda & \frac{c}{2N} \\ \frac{c*}{2N} & \frac{b}{2N}-\lambda \end{vmatrix} = 0. \tag{4.70}$$

By solving above determinant we will get,

$$\lambda^2 - \frac{1}{2N}(a+b)\lambda + \frac{1}{4N^2}(ab-cc*) = 0. \tag{4.71}$$

By using values of a, b and c in above equation and solving quadratic formula we will get,

$$\lambda_1 = \frac{2}{N}\cos^4\theta/4\,, \qquad \lambda_2 = 1-\lambda_1 = \frac{2}{N}\sin^4\theta/4. \tag{4.72}$$

The Schmidt decomposition of the state $|\Psi\rangle$ is,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(\sqrt{\lambda_1}\,|1,\tilde{1}\rangle + \sqrt{\lambda_2}\,|2,\tilde{2}\rangle). \tag{4.73}$$

Once again we will use the concept of Von-Neumann entropy to quantify the amount of entanglement by using relation $S(\rho(1) = \sum_{j=1}^{2}\lambda_j\log_2\lambda_1$ where λ_j are eigenvalues of ρ^1, e.g,

$$S(\rho^{(1)}) = -\lambda_1\log_2\lambda_1 - \lambda_2\log_2\lambda_2. \tag{4.74}$$

By using values of λ_1 and λ_2 from eq (4.72) in above equation we will get,

$$S(\rho^{(1)}) = -2/N\cos^4\theta/4\log_2(2/N\cos^4\theta/4) - 2/N\sin^4\theta/4\log_2(2/N\sin^4\theta/4). \tag{4.75}$$

$$S(\rho^{(1)}) = -2/N\left[\cos^4\theta/4\log_2(2/N\cos^4\theta/4) + \sin^4\theta/4\log_2(2/N\sin^4(\theta/4))\right]. \quad (4.76)$$

From above relation we can see the dependence of Von-Neumann entropy on the angle θ i.e.(angle between $|\uparrow_z\rangle$ and $|\uparrow_u\rangle$). If angle θ between $|\uparrow_z\rangle$ and $|\uparrow_u\rangle$ is π (opposite spins) then we will have maximum entanglement and zero for $\theta = 0$ (same spins). The graph between $S(\rho^{(1)})$ and θ is shown in figure 4.5. at different values of θ where entanglement goes from minimum to maximum.

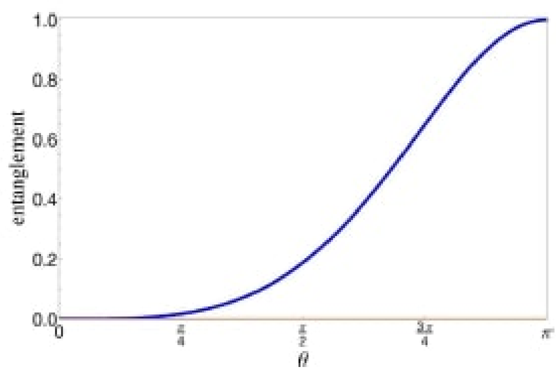

Figure 4.5: Entanglement quantified by the Entanglement entropy is plotted as function of θ.

4.4 Two Indistinguishable Qutrits in the Same Site

In this example we will consider system of three level particles (qutrits) as quantum processor instead of using two level qubits as shown in figure 4.6.

The method we used for system of two indistinguishable qubits also applicable for two indistinguishable qutrits which are placed in same site. These qutrits are characterized by the basis $\{|l_1\rangle, |l_2\rangle, |l_3\rangle\}$. When two indistinguishable qutrits characterized by basis $\{|l_1\rangle, |l_2\rangle, |l_3\rangle\}$ are in same site, then their state $|\Psi\rangle$ can be written as

$$|\Psi\rangle = \sin\theta\cos\phi\,|l_1, l_2\rangle + \sin\theta\sin\phi\,|l_1, l_3\rangle + \cos\theta\,|l_3, l_3\rangle\,. \quad (4.77)$$

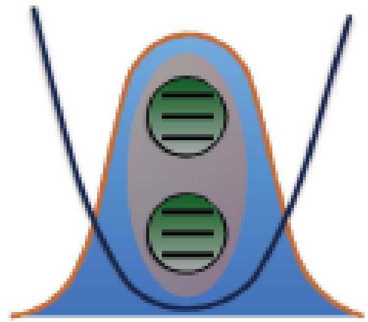

Figure 4.6: Two qutrits in same site with arbitrary spin

We can see that the given state is not normalized so we have to find normalization constant.

$$\begin{aligned}\langle\psi|\psi\rangle &= (\sin\theta\cos\phi\,\langle l_1, l_2| + \sin\theta\sin\phi\,\langle l_1, l_3| \\ &+ \cos\theta\,|l_3, l_3\rangle)(\sin\theta\cos\phi\,|l_1, el_2\rangle + \sin\theta\sin\sin\phi\,|l_1, l_3\rangle + \cos\theta\,|l_3, l_3\rangle),\end{aligned} \tag{4.78}$$

$$\begin{aligned}&\langle\psi|\psi\rangle = \sin^2\theta\cos^2\phi\,\langle l_1, l_2|l_1, l_2\rangle \\ &+\sin^2\theta\sin\phi\cos\phi\,\langle l_1, l_2|l_1, l_3\rangle + \sin\theta\cos\theta\cos\phi\,\langle l_1, l_2|l_3, l_3\rangle + \sin^2\theta\sin\phi\cos\phi\,\langle l_1, l_3|l_1, l_2\rangle \\ &+ \sin^2\theta\sin^2\phi\,\langle l_1, l_3|l_1, l_3\rangle + \sin\theta\sin\phi\cos\theta\,\langle l_1, l_3|l_3, l_3\rangle + \sin\theta\cos\theta\cos\phi\,\langle l_3, l_3|l_1, l_2\rangle \\ &+ \sin\theta\cos\theta\sin\phi\,\langle l_3, l_3|l_1, l_3\rangle + \cos^2\theta\,\langle l_3, l_3|l_3, l_3\rangle\,.\end{aligned} \tag{4.79}$$

By using equation $\langle\varphi,\zeta|\phi,\psi\rangle=\langle\varphi|\phi\rangle\,\langle\zeta|\psi\rangle + \eta\,\langle\varphi|\psi\rangle\,\langle\zeta|\phi\rangle$, we will get,

$$\left.\begin{aligned}\langle l_1, l_2|l_1, l_3\rangle &= 0\\ \langle l_1, l_2|l_3, l_3\rangle &= 0\\ \langle l_1, l_3|l_1, l_2\rangle &= 0\\ \langle l_1, l_3|l_3, l_3\rangle &= 0\\ \langle l_3, l_3|l_1, l_2\rangle &= 0\\ \langle l_3, l_3|l_1, l_3\rangle &= 0\\ \langle l_1, l_2|l_1, l_2\rangle &= 1\\ \langle l_1, l_3|l_1, l_3\rangle &= 1\\ \langle l_3, l_3|l_3, l_3\rangle &= 2\end{aligned}\right\}, \tag{4.80}$$

using all of these equations in eq (4.79) we will get that

$$\langle\psi|\psi\rangle = \sin^2\theta + 2\cos^2\theta = 1 + \cos^2\theta. \tag{4.81}$$

Now normalized state can be written as

$$|\Psi\rangle = \frac{1}{\sqrt{1+\cos^2\theta}}\left(\sin\theta\cos\Phi\,|l_1,l_2\rangle + \sin\theta\sin\Phi\,|l_1,l_3\rangle + \cos\theta\,|l_3,l_3\rangle\right). \tag{4.82}$$

The density matrix of above state is

$$\begin{aligned}\rho &= |\psi\rangle\langle\psi| \\ &= \frac{1}{(1+\cos^2\theta)}(\sin^2\theta\cos^2\theta\,|l_1,l_2\rangle\langle l_1,l_2| + \sin^2\theta\sin\phi\cos\phi\,|l_1,l_2\rangle\langle l_1,l_3| \\ &\qquad + \sin\theta\cos\phi\cos\theta\,|l_1,l_2\rangle\langle l_3,l_3| + \sin^2\theta\sin\phi\cos\phi\,|l_1,l_3\rangle\langle l_1,l_2| \\ &\qquad + \sin^2\theta\sin^2\phi\,|l_1,l_3\rangle\langle l_1,l_3| + \sin\theta\sin\phi\cos\theta\,|l_1,l_3\rangle\langle l_3,l_3| \\ &\qquad + \sin\theta\cos\phi\cos\theta\,|l_3,l_3\rangle\langle l_1,l_2| + \sin\theta\cos\theta\sin\phi\,|l_3,l_3\rangle\langle l_1,l_3| + \cos^2\theta\,|l_3,l_3\rangle\langle l_3,l_3|).\end{aligned} \tag{4.83}$$

By using the eq (4.3) (one particle projective measurement) and performing partial trace on the basis $\{|l_1\rangle,|l_2\rangle,|l_3\rangle\}$ we will get

$$|l_1,l_2\rangle\langle l_1,l_2| = |l_1\rangle\langle l_1,l_2|l_2\rangle = |l_1\rangle\langle l_1|. \tag{4.84}$$

In above equation, we perform the partial trace on the basis $\{|l_2\rangle\}$ and then using eq (4.3) (one particle projective measurement) we get $\langle l_1,l_2|l_2\rangle = \langle l_1|$. Similarly when we will perform the partial trace on the basis $\{|l_1\rangle\}$ we will get

$$|l_1,l_2\rangle\langle l_1,l_2| = |l_2\rangle\langle l_1,l_2|l_1\rangle = |l_2\rangle\langle l_2|. \tag{4.85}$$

Similarly,

$$\left.\begin{aligned}
&|l_1,l_2\rangle\langle l_1,l_3| = |l_1\rangle\langle l_1,l_3|l_2\rangle = 0,\\
&|l_1,l_2\rangle\langle l_3,l_3| = |l_1\rangle\langle l_3,l_3|l_2\rangle = 0,\\
&|l_1,l_2\rangle\langle l_3,l_3| = |l_2\rangle\langle l_3,l_3|l_1\rangle = 0,\\
&|l_1,l_3\rangle\langle l_1,l_2| = |l_1\rangle\langle l_1,l_2|l_3\rangle = 0,\\
&|l_1,l_3\rangle\langle l_3,l_3| = |l_3\rangle\langle l_3,l_3|l_1\rangle = 0,\\
&|l_3,l_3\rangle\langle l_3,l_3| = 4\,|l_3\rangle\langle l_3|\,,\\
&|l_1,l_3\rangle\langle l_1,l_2| = |l_3\rangle\langle l_1,l_2|l_1\rangle = |l_3\rangle\langle l_2|\,,\\
&|l_1,l_2\rangle\langle l_1,l_3| = |l_2\rangle\langle l_1 e l_3|l_1\rangle = |l_2\rangle\langle l_3|\,,\\
&|l_1,l_3\rangle\langle l_1,l_3| = |l_1\rangle\langle l_1,l_3|l_3\rangle = |l_1\rangle\langle l_1|\,,\\
&|l_1,l_3\rangle\langle l_1,l_3| = |l_3\rangle\langle l_1,l_1|l_3\rangle = |l_3\rangle\langle l_3|\,,\\
&|l_1,l_3\rangle\langle l_3,l_3| = |l_1\rangle\langle l_3,l_3|l_3\rangle = 2\,|l_1\rangle\langle l_3|\,,\\
&|l_3,l_3\rangle\langle l_1,l_3| = |l_3\rangle\langle l_1,l_3|l_3\rangle = |l_3\rangle\langle l_1|\,,
\end{aligned}\right\}, \tag{4.86}$$

Using all of these values in eq (4.83) we will obtain reduced density matrix. i.e.

$$\begin{aligned}
\rho = \frac{1}{2}(&\sin^2\theta\cos^2\phi\,|l_1\rangle\langle l_1| + \sin^2\theta\cos^2\phi\,|l_2\rangle\langle l_2| + \sin^2\theta\sin\phi\cos\phi\,|l_2\rangle\langle l_3|\\
&+ \sin^2\theta\sin\phi\cos\phi\,|l_3\rangle\langle l_2| + \sin^2\theta\sin^2\phi\,|l_1\rangle\langle l_1| + \sin^2\theta\sin^2\phi\,|l_3\rangle\langle l_3|\\
&+ 2\sin\theta\sin\phi\cos\theta\,|l_1\rangle\langle l_3| + \sin\theta\cos\theta\sin\phi\,|l_3\rangle\langle l_1| + \sin\theta\cos\theta\sin\phi\,|l_3\rangle\langle l_1|\\
&+ 4\cos^2\theta\,|l_3\rangle\langle l_3|).
\end{aligned} \tag{4.87}$$

and in matrix form the above equation can be written as,

$$\rho = \frac{1}{(1+\cos^2\theta)}\begin{pmatrix} \sin^2\theta & 0 & 2\sin\theta\sin\phi\cos\theta\\ 0 & \sin^2\theta\cos^2\phi & 0\\ 2\sin\theta\cos\theta\sin\phi & \sin^2\theta\sin\phi\cos\phi & \sin^2\theta\sin^2\phi + 4\cos^2\theta \end{pmatrix}, \tag{4.88}$$

where, $(1+\cos^2\theta)$ is normalization constant. To find its eigenvalues and then eigenvectors to construct Schmidt decomposition is much more difficult task. So, here, we took simple case i.e. when $\frac{\theta}{2}$, so above equation reduces to,

$$|\psi_\phi\rangle = \cos\phi\,|l_1,l_2\rangle + \sin\phi\,|l_1,l_3\rangle\,. \tag{4.89}$$

The density matrix of the above state can be written as ,

$$\begin{aligned}\rho &= |\psi_\phi\rangle\langle\psi_\phi|\,, \\ &= \cos^2\phi\,|l_1,l_2\rangle\langle l_1,l_2| + \cos\phi\sin\phi\,|l_1,l_2\rangle\langle l_1,l_3| \\ &\quad + \cos\phi\sin\phi\,|l_1,l_3\rangle\langle l_1,l_2| + \sin^2\phi\,|l_1,l_3\rangle\langle l_1,l_3|\,.\end{aligned} \tag{4.90}$$

Once again using (4.3) and performinng partial trace on th e basis $\{|e_1\rangle, |e_2\rangle, |e_3\rangle\}$ we will get

$$\left.\begin{aligned} |l_1,l_2\rangle\langle l_1,l_2| &= |l_1\rangle\langle l_1,l_2|l_2\rangle = |l_1\rangle\langle l_1|\,, \\ |l_1,l_2\rangle\langle l_1,l_2| &= |l_2\rangle\langle l_1,l_2|l_1\rangle = |l_2\rangle\langle l_2|\,, \\ |l_1,l_2\rangle\langle l_1,l_3| &= |l_2\rangle\langle l_1,l_3|l_1\rangle = |l_2\rangle\langle l_3|\,, \\ |l_1,l_3\rangle\langle l_1,l_2| &= |l_3\rangle\langle l_1,l_2|e_1\rangle = |l_3\rangle\langle l_2|\,, \\ |l_1,l_3\rangle\langle l_1,l_3| &= |l_1\rangle\langle l_1,l_3|l_3\rangle = |l_1\rangle\langle l_1|\,, \\ |l_1,l_3\rangle\langle l_1,l_3| &= |l_3\rangle\langle l_1,l_1|l_3\rangle = |l_3\rangle\langle l_3|\,, \\ |l_1,l_2\rangle\langle l_1,l_3| &= |l_1\rangle\langle l_1,l_3|l_2\rangle = 0, \\ |l_1,l_3\rangle\langle l_1,l_2| &= |l_1\rangle\langle l_1,l_2|l_3\rangle = 0, \end{aligned}\right\}, \tag{4.91}$$

The reduced density matrix can be written as

$$\begin{aligned}\rho^{(1)} = \frac{1}{2}((\cos^2\phi + sin^2\phi)\,|l_1\rangle\langle l_1| + \cos^2\phi\,|l_2\rangle\langle l_2| + \cos\phi\sin\phi\,|l_2\rangle\langle l_3| \\ + \cos\phi\sin\phi\,|l_3\rangle\langle l_2| + \sin^2\phi\,|l_3\rangle\langle l_3|),\end{aligned} \tag{4.92}$$

and in matrix form the above equation can be written as

$$\rho^{(1)} = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & cos^2\phi & sin\phi\cos\phi \\ 0 & \sin\phi\cos\phi & \sin^2\phi \end{pmatrix}. \tag{4.93}$$

The eigenvalues of the $\rho^{(1)}$ can be calculated by using characteristic equation e.g,

$$\begin{vmatrix} \frac{1}{2}-\lambda & 0 & 0 \\ 0 & \frac{cos^2\phi}{2}-\lambda & \frac{\sin\phi\cos\phi}{2} \\ 0 & \frac{\sin\phi\cos\phi}{2} & \frac{\sin^2\phi}{2}-\lambda \end{vmatrix} = 0, \tag{4.94}$$

by solving the above determinant the eigenvalues are $\lambda_1 = \lambda_2 = \frac{1}{2}$ and $\lambda_3 = 0$ and eigenstates are $|1\rangle = |\tilde{2}\rangle = \cos\phi\,|l_2\rangle + \sin\phi\,|l_3\rangle$, $|\tilde{1}\rangle = |2 = |l_1\rangle\rangle$, $|3\rangle = -\sin\phi\,|l_2\rangle + \cos\phi\,|l_3\rangle$, $|\tilde{3}\rangle = 0$. The Schmidt decomposition of the state ψ is

$$|\Psi\rangle = \frac{1}{\sqrt{2}}(\sqrt{\lambda_1}\,|1,\tilde{1}\rangle + \sqrt{\lambda_2}\,|2,\tilde{2}\rangle + \sqrt{\lambda_3}\,|3,\tilde{3}\rangle). \tag{4.95}$$

by using eigenvalues λ_1, λ_2 and λ_3 we will get,

$$|\psi_\phi\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}\,|1,\tilde{1}\rangle + \frac{1}{\sqrt{2}} + |2,\tilde{2}\rangle\right). \tag{4.96}$$

which is required form of Schmidt decomposition when two two indistinguishable qutrits in the same site.

Chapter 5

Summary and Conclusions

In this thesis we have discussed characteristic features of bipartite entanglement in multi-particle systems. Depending on whether the particles are distinguishable or indistinguishable, various techniques have been used to detect and quantify entanglement in multi-particle systems. For example, several methods have been developed in literature to characterize entanglement for distinguishable particle in different scenarios, such as, Schmidt decomposition, negativity, concurrence and entanglement of formation. However, for bipartite systems with pure quantum states, Schmidt decomposition is widely used. On the other hand, the notion of entanglement for indistinguishable particles is confusing due to exchange correlations of many-particle wave function of identical particles governed by symmeterization postulate. In particular, mathematical representation of multi-particle states, for instance in second quantization formalism, the discrimination between entanglement and exchange correlation becomes confusing. The multi-particle states seems to be entangled. However, whether this entanglement can be exploited as a resource remains ambiguous. Hence the usual entanglement measures remain controversial and debatable for indistinguishable particles.

In this work we presented a comparative study of various approaches developed to analyzed the entanglement of indistinguishable particles. In particular, we focus on an approach which do not make use of particle-labels to represent the quantum state of many-particle system. This approach is referred to as non-standard particle-based approach, which encompasses symmetrization postulate and is very useful to quantify entanglement of identical particles. In this approach we write down the state of composite system by expressing it in its own basis. Then we use density matrix approach to find reduced density matrix of that state by performing partial trace on chosen basis. Where we have discussed that partial trace of indistinguishable particles is basis dependent i.e. weather chosen basis of particle are local or non-local. While for the case of distinguishable particles we can address particles individually, so partial trace does not affected by the chosen basis. This is the main difference of this approach and name labeled approach known as standard particle based approach. We apply the concept of Schmidt decomposition to quantify entanglement of indistinguishable particle. But the method of calculating Schmidt coefficients and von Neumann entropy is different from that of distinguishable particles. The main difference between Schmidt decomposition for distinguishable particles and Schmidt decomposition for indistinguishable particles is the choice of basis. To test the validity of Schmidt decomposition for indistinguishable particles we have considered some useful physical situations where two qubits was in two different sites with orthogonal spins, two qubits in the same site with random spin and two qutrits in the same site. In all of these examples we write down state of system, find their reduced density matrices by performing partial trace on chosen basis. Then we find eigenvalues of reduced density matrices by using characteristic equation and using $S(\rho) = -\sum_i \lambda_i \log_2 \lambda_i$ to find von Neumon entropy also known as entanglement entropy. This von Neumon entropy gives us information about state of system weather it is entangled or not. In first example i.e. (Two qubits in different sites) we have seen that loacl single particle measurement induce intrinsic entanglement while non-local measurement yields measurement induced entanglement. This also gives us physically expected results i.e. product state yield zero entanglement and maximal entanglement for Bell like state. In second example (Two qubits in the same site with arbitrary spin) we showed dependence of entanglement entropy on the angle θ between state vector of spin overlap of particles. Where we have seen that entanglement entropy increase when angle between them is increases i.e. is maximum entanglement when $\theta = \pi$ and zero entanglement for $\theta = 0$. Finally we have studied examples of two qutrits in same site which can be used to store quantum information.

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