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Quantum Electrodynamics In $2+1$ dimensions

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by

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Dedicated to my parents, who valued my education above all else.

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Abstract

This thesis reviews study of the quantum electrodynamics in $2+1$ D. We will start with the discussion of Dirac equation and Maxwell's theory in $3+1$ dimensions and then convert these theories in $2+1$ D. We will show different aspects arising due to reduction of one spatial dimension. Solutions for Dirac equation and their transformation properties are discussed. Basics of Chern-Simon's theory are presented in some detail. Then we will derive Gauge field propagator and from propagator we will derive scalar and vector potentials for a static charge. In last chapter we will perform some initial steps of quantization of the theories in different gauges and we will see that there are a few differences in quantization results between $3+1$ D and $2+1$ D theories. We will also find polarization vectors for photon field in different theories and compare them with usual $3+1$ D theory.

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Notations and conventions

$\mu, \nu, \rho, ...$ spacetime indices

$i, j, k, ...$ spatial indices

We use the minus *timelike*-convention to define spacetime interval as

$$ds^2 := \eta_{\mu\nu} dx^\mu dx^\nu \,, \qquad \mu, \nu = 0, 1, 2$$

such that the squared 4-momentum is positive for real particles (i.e. $p^2 = m^2$, on shell). In three dimensions this implies the metric signature is -1.

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow \eta_{ij} = -\delta_{ij} \,.$$

In natural units

$$\hbar = c = 1.$$

We define

$$x^\mu = (t, x_1, x_2) = (t, \vec{x})$$
$$p^\mu = (E, p_1, p_2) = (E, \vec{p}),$$

where E is the energy of the particle. Space-time derivative is defined as

$$\partial^\mu = \frac{\partial}{\partial x_\mu} = \big(\frac{\partial}{\partial t}, -\nabla\big),$$

and

$$\partial^\mu \partial_\mu = \Box$$

as d'Alembertian. Also the dot represents time derivative and prime denotes space derivative (mentioned other wise). The totally antisymmetric tensor $\varepsilon^{\mu\nu\rho}$ is set as

$$\varepsilon_{012} = 1 \quad \Rightarrow \quad \varepsilon_{102} = -1.$$

All quantities are *weighed* in terms of mass dimensions e.g. the dimension of Compton wavelength $\lambda_c = \frac{\hbar}{mc} = \frac{1}{m}$ becomes $[M^{-1}]$.

Pauli matrices are defined as

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1}$$

For 3-dimensional fourier transform we adopt the convention asymmetric in the factors of 2π such that

$$\begin{aligned}
\phi(x-x') &= \int \frac{d^3p}{(2\pi)^3} e^{-ip(x-x')}\phi(p) \\
\phi(p) &= \int d^3x e^{ip(x-x')}\phi(x-x') \\
\delta(x-x') &= \int \frac{d^3p}{(2\pi)^3} e^{ip(x-x')}.
\end{aligned}$$

Chapter 1

Introduction

In our observable universe there are four basic forces viz. gravity, electromagnetic force, weak force, and strong force. Physicists have been curious about the origin of these forces and have been trying to develop mathematical models in attempt to explain the interaction between different types of particles. Quantum electrodynamics deals with how light and matter interact and is the relativistic quantum field theory. It would be interesting to have a look at brief history of evolution of quantum electrodynamics. After the synthesis of the phenomena of motion, sound and heat, there was the discovery of a number of phenomena that we call electrical and magnetic. In 1873 these phenomena were synthesized with the phenomena of light and optics into a single theory by James Clerk Maxwell, who proposed that light is an electromagnetic wave. A while later electron theory of matter evolved and attempts were made to understand the motion of electron. Newton's laws were initially applied which proved to be wrong in this regime. Then there came the quantum mechanics having concepts beyond the common sense which successfully explained many experimental facts. But still the problem of real understanding of light and matter sustained. Maxwell's theory had to be modified to be in accord with the new principle of quantum mechanics that had been developed. Hence the new theory, the quantum theory of interaction of light and matter evolved which is now called "Quantum Electrodynamics".

On the other side Paul Dirac using the theory of relativity, made a relativistic theory of the electron that did not completely take into account all the effects of the electron's interaction with light [1]. Later on it was observed that we need to combine Maxwell's theory with Dirac's theory to have a full understanding. So there are two basic ingredients of quantum electrodynamics, $1st$ is Maxwell' theory and $2nd$ is Dirac's theory.

The purpose of this study in $2+1$ D is to understand the general behaviour of fields in a plane. Once we learn about the behaviour of particles and fields in a plane we can apply it to different phenomena to get a better understanding. Importance of this study can be understood by looking at wide application of theory for example fractional quantum Hall effect, motion of electrons in graphene and high temperature superconductivity. Another obvious motivation is the theoretical interest of physicists in massive nature of photon. We give a brief introduction to Lagrangian

formalism because all the theories are taken in Lagrangian form.
A system with N degrees of freedom can be described by a set of coordinates $q_i(t)$ with $i = 0, 1, ..., N$, which are denoted often simply by q. The Lagrangian L is a function of q_i's and their first time derivatives $L = L(q, \dot{q})$. Lagrangian[1] is the difference of kinetic energy and potential energy [2].

$$L(q, \dot{q}) = \frac{1}{2}m_i\dot{q}_i^2 - V(q).$$

From a given Lagrangian we find Euler-Lagrange equations

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0.$$

These are also known as equations of motion in Lagrangian formalism. Later in quantization we shall need Hamiltonian which is defined as

$$H(p, q) = p_i\dot{q}_i - L,$$

where p_i is known as conjugate momentum to q_i for each corresponding index, found as

$$p_i = \frac{\partial L}{\partial \dot{q}_i}.$$

Then $\dot{q}_i$ is expressed in terms of p_i or possibly q_i, in Hamiltonian to make it purely a finction of p and q. In field theory the Lagrangian is replaced by Lagrangian density, coordinates with fields and velocities with space time derivatives of fields[2].

$$\begin{aligned} L &\to \mathcal{L} \\ q_i(t) &\to \phi_i(x^\mu) \\ \dot{q}_i(t) &\to \partial_\mu\phi_i(x^\mu). \end{aligned}$$

The importance of Lagrangian formalism was prominent because it was seen that various realistic theories can be cast in the canonical formalism (explained in later chapter). Lagrangian is the starting point for canonical formalism as seen above that Hamiltonian is derived from Lagrangian. We need to choose a suitable Lagrangian for theory under consideration, which may exhibit Lorentz/Poincaré invariance and some other symmetries. Symmetries in turn produce conserved quantities which are of the most interest of physicists [3] [4]. Some symmetries and corresponding conserved quantities are follows

Time translational symmetry implies energy conservation.
Space translational space symmetry implies momentum conservation.
Rotational symmetry implies angular momentum conservation.
Phase angle symmetry of wave-function implies charge conservation.

[1]Einstein summation convention on repeated indices is understood here and after wards.

[2]x^μ in the parenthesis is usually written simply x while temporal and spatial parts are explicitly written when required.

The theoretical frame work we use, is known as Quantum field theory. Quantum field theory is one of the most successful mathematical models to describe physical reality. It has a unifying effect on many different areas of mathematics and physics [5]. It is widely used in condensed matter physics and high energy physics. In one sense it is a field theoretic extension of relativistic quantum mechanics.

We shall have a basics introduction of Maxwell and Dirac theories in $3 + 1$ D and then in 3rd chapter we shall convert these theories in $2 + 1$ D. We shall study some characteristics of separate Dirac and Maxwell's theories and different combined theories like Dirac + Maxwell and Dirac + Maxwell + Proca e.t.c.. In last chapter we shall see how can we quantize the theories in $2 + 1$ D and if there is some difference in quantization results.

Chapter 2

QED in 3+1 Dimensions

We shall have a brief review of conventional $3+1$ D QED so that it becomes easy to make comparison of theories in two different dimensions. As we talked about invariance and symmetries of the theory in previous chapter we need to have a review of Lorentz and Poincaré transformations as these symmetries are fundamental requirements of validity of any theory[1].

2.1 Lorentz and Poincaré Groups

A Lorentz transformation named after Henrik Lorentz was the result of efforts made by Lorentz and many other physicists to explain how the speed of light c was observed to be independent of any arbitrary frame of reference, and also to understand the symmetries of electromagnetic laws. The whole structure of special relativity is based on Lorentz transformations. These transformations describe that how the measurements for an event somewhere in space-time by two observers, in inertial frame[2] are related.

We define Lorentz Group whose elements are Lorentz transformations so that we could study the characteristics of transformations in a compact form. Lorentz group $O(1,3)$ (Orthogonal) is a set of 4×4 real matrices which represent linear coordinate transformations,

$$x^{\mu} \rightarrow x'^{\mu} = \Lambda^{\mu}_{\nu} x^{\nu},$$

which preserve the following quadratic form

$$t^2 - x^2 - y^2 - z^2.$$

Since

$$t^2 - x^2 - y^2 - z^2 = \eta_{\mu\nu} x^{\mu} x^{\nu},$$

[1]Theory must produce consistent results in all frames and it should not be position dependent.

[2]A frame that is at rest or moving with constant velocity with respect to any reference is inertial frame.

the matrix Λ should satisfy following condition to leave above mentioned form invariant

$$\eta_{\mu\nu}x'^{\mu}x'^{\nu} = \eta_{\mu\nu}(\Lambda^{\mu}_{\rho}x^{\rho})(\Lambda^{\nu}_{\sigma}x^{\sigma}) = \eta_{\rho\sigma}x^{\rho}x^{\sigma}.$$

This must be true for any x, so we must have

$$\eta_{\rho\sigma} = \eta_{\mu\nu}\Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}, \tag{2.1}$$

or in matrix notation we write as

$$\Lambda^{T}\eta\,\Lambda = \eta.$$

This implies that *det*$(\Lambda) = \pm 1$. The transformations with $det\Lambda = +1$ are called *Proper Lorentz Transformations.* And the corresponding subgroup is denoted by $SO(1,3)$. Writing the 00 component of eq. (2.1) explicitly we find

$$1 = (\Lambda^{0}_{0})^{2} - \sum_{i=1}^{3}(\Lambda^{i}_{0})^{2}, \tag{2.2}$$

and this implies $(\Lambda^{0}_{0})^{2} \geq 1$. So proper Lorentz group has two components that are disconnected. It is convenient to separate out the transformations corresponding to $\Lambda^{0}_{0} \geq 1$. We take infinitesimal transformation as

$$\Lambda^{\mu}_{\nu} = \delta^{\mu}_{\nu} + \omega^{\mu}_{\nu}, \tag{2.3}$$

then eq. (2.1) gives

$$\omega^{\mu\nu} = -\omega^{\nu\mu}. \tag{2.4}$$

This shows that ω is an antisymmetric matrix which has six independent elements. Hence Lorentz group has six parameters. These six elements are boosts and rotations[3]. To these rotations and boosts correspond six generators which we label with $J^{\mu\nu}$ with the pair of antisymmetric indices μ, ν for convenience. A general Lorentz group element is represented as:

$$\Lambda = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}}. \tag{2.5}$$

A set of objects ϕ^{i} $(i = 1, ..., n)$, transforms under a Lorentz transformation in representation R of dimension n of the Lorentz group as

$$\phi^{i} \rightarrow \left[e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}_{R}}\right]^{i}_{j}\phi^{j}, \tag{2.6}$$

where $e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}_{R}}$ is the n-dimensional matrix representation. $J^{\mu\nu}_{R}$ ($n\times n$ matrices) are the generators in corresponding representation. An infinitesimal transformation gives the variation

$$\delta\phi^{i} = -\frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu}_{R})^{i}_{j}\phi^{j}, \tag{2.7}$$

μ, ν identify the generator while i, j are the indices of matrix in particular representation. We can classify physical quantities according to their transformation properties. A scalar is invariant under

[3]Transformations that leave t invariant are rotations and those which leave $t^2 - x^2$ e.t.c. are boosts.

Lorentz transformation. For a scalar index i can have only one value, so the scalar representation in one dimensional, means $(J_R^{\mu\nu})^i_j$ is a number $(\delta\phi = 0, J^{\mu\nu} = 0)$. A four vector V^μ satisfies transformation law

$$V^\mu \to \Lambda^\mu_\nu V^\nu. \tag{2.8}$$

In four vector representation i, j are also Lorentz indices, each generator $J^{\mu\nu}$ is represented by a 4×4 matrix $(J_R^{\mu\nu})^\rho_\sigma$. The explicit form is

$$(J^{\mu\nu})^\rho_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma). \tag{2.9}$$

Under infinitesimal Lorentz transformation the variation in vector is $\delta V^\mu = \omega^\mu_\nu V^\nu$ or

$$\delta V^\rho = -\frac{i}{2}\omega_{\mu\nu}(J^{\mu\nu})^\rho_\sigma V^\sigma. \tag{2.10}$$

We can use the explicit form to compute the commutators and get

$$[J^{\mu\nu}, J^{\rho\sigma}] = i(J^{\mu\sigma}\eta^{\nu\rho} - J^{\nu\sigma}\eta^{\mu\rho} + J^{\mu\rho}\eta^{\nu\sigma} - J^{\nu\rho}\eta^{\mu\sigma}). \tag{2.11}$$

We rearrange the six components of $J^{\mu\nu}$ into following two spatial vectors for some convenience

$$K^i = J^{i0}, \qquad J^i = \frac{1}{2}\varepsilon^{ijk}J^{jk}. \tag{2.12}$$

In terms of these vectors the commutation relation splits into following

$$[K^i, K^j] = -i\varepsilon^{ijk}J^k$$
$$[J^i, J^j] = i\varepsilon^{ijk}J^k$$
$$[J^i, K^j] = \varepsilon^{ijk}K^k.$$

We also define

$$\eta^i = \omega^{i0}, \qquad \theta^i = \frac{1}{2}\varepsilon^{ijk}\omega^{jk}, \tag{2.13}$$

then

$$\begin{aligned}\frac{1}{2}\omega_{\mu\nu}J^{\mu\nu} &= \sum_{i=1}^{3}\omega_{0i}J^{0i} + \omega_{12}J^{12} + \omega_{23}J^{23} + \omega_{31}J^{31} \\ &= \vec{\theta}\cdot\vec{J} - \vec{\eta}\cdot\vec{K}.\end{aligned} \tag{2.14}$$

Ultimately a Lorentz transformation can be written as

$$\Lambda = e^{-i\vec{\theta}\cdot\vec{J} + i\vec{\eta}\cdot\vec{K}}. \tag{2.15}$$

We shall derive the transformation law for spinor field in detail in next chapter.

Poincaré group extends the Lorentz group by involving space-time translations of the form

$$x^\mu \mapsto x^\mu + a^\mu,$$

a^μ are the parameters of translation. A generic element of the translation group is written as:

$$\Lambda = e^{-ia_\mu P^\mu},$$

where a_μ are the translation parameters, and the 4-momentum P^μ are its generators. Also,

$$[P^\mu, P^\nu] = 0.$$

Note that energy $P^0 = H$ is scalar under rotations but P^i is a vector under rotations,

$$[J^i, H] = 0, \qquad [J^i, P^j] = i\varepsilon^{ijk}P^k,$$
$$[K^i, P^j] = iH\delta^{ij}, \qquad [K^i, H] = iP^i,$$

and on combining we have

$$[P^\mu, J^{\rho\sigma}] = i(\eta^{\mu\rho}P^\sigma - \eta^{\mu\sigma}P^\rho). \tag{2.16}$$

Unlike J^i, the boost generator K^i *do not* commute with the generator of time translation H. Hence the eigenvalues of the operator $\hat{K}$ cannot be used to label the physical states.

Now we look at the brief introduction to the constituents of electrodynamics that are Dirac theory and Maxwell's theory.

2.2 Dirac Theory

The theory of Paul Dirac is an attempt to unify the theories of special relativity and quantum mechanics (A quantum mechanical theory demonstrating Lorentz invariance and hence consistent with special relativity). Dirac started from Klein-Gordon (K.G.) equation. The *Klein-Gordon* equation was an attempt to construct a relativistic version of Schrodinger wave equation, whose solutions are scalar (*Spin 0* particles) fields,

$$\partial^\mu \partial_\mu \psi + m^2\psi = 0. \tag{2.17}$$

This equation gave negative probabilities which was not acceptable in any sense and also negative energy solutions, because it is second order in time. Dirac tried to solve these issues and made factors of Klein-Gordon equation to make equation first order in time. The derivation of Klein-Gordon equation and then Dirac equation is as follows. We start with Schrodinger equation,

$$-\frac{1}{2m}\nabla^2\psi + V\psi = i\frac{\partial\psi}{\partial t}. \tag{2.18}$$

We replace energy and momentum with corresponding operators $p \to -i\nabla, E \to i\frac{\partial}{\partial t}$ and then energy conservation relation $E^2 - p^2 - m^2 = 0$ implies

$$-\frac{\partial^2\psi}{\partial t^2} + \nabla^2\psi - m^2\psi = 0. \tag{2.19}$$

Which in covariant form becomes

$$-\partial^\mu\partial_\mu\psi - m^2\psi = 0. \tag{2.20}$$

For ψ, considering it is a solution to K-G. equation multiply equation 2.19 with $-i\psi^*$, then taking complex conjugate of K-G. equation and multiply with $-i\psi$ we get

$$i\psi^*\frac{\partial^2\psi}{\partial t^2} - i\psi^*\nabla^2\psi + i\psi^* m^2\psi = 0, \tag{2.21}$$

$$i\psi\frac{\partial^2\psi^*}{\partial t^2} - i\psi\nabla^2\psi^* + i\psi m^2\psi^* = 0. \tag{2.22}$$

Subtracting we get

$$\frac{\partial}{\partial t}[i(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t})] + \nabla\cdot[-i(\psi^*\nabla\psi - \psi\nabla\psi^*)] = 0. \tag{2.23}$$

This has the form of equation of continuity

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0, \tag{2.24}$$

with

$$\rho = i(\psi^*\frac{\partial\psi}{\partial t} - \psi\frac{\partial\psi^*}{\partial t}) \tag{2.25}$$

and

$$\vec{j} = i(\psi^*\nabla\psi - \psi\nabla\psi^*). \tag{2.26}$$

Here we can see the problem mentioned before, if we suppose plane wave solutions $\psi = Ne^{-ip_\mu x^\mu}$, we get $\rho = 2E|N|^2$, means for negative energy probability density is negative. So Dirac attempted to solve in the following way.

$$E^2 - |\vec{p}|^2 - m^2 = p^\mu p_\mu - m^2. \tag{2.27}$$

We can write

$$p^\mu p_\mu - m^2 = (\beta^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m), \tag{2.28}$$

or expanding the product

$$(\beta^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m) = \beta^\kappa\gamma^\lambda p_\kappa p_\lambda - m^2 + m\gamma^\lambda p_\lambda - m\beta^\kappa p_\kappa. \tag{2.29}$$

This is equal to $p^\mu p_\mu - m^2$, so we have to eliminate linear terms in p, this can be done by choosing $\beta^\kappa = \gamma^\kappa$. We get

$$p^\mu p_\mu - m^2 = \gamma^\kappa\gamma^\lambda p_\kappa p_\lambda - m^2. \tag{2.30}$$

Expanding right hand side

$$\gamma^\kappa\gamma^\lambda p_\kappa p_\lambda - m^2 = (\gamma^0)^2 p_0^2 + (\gamma^0\gamma^j + \gamma^j\gamma^0)p_0 p_j + \gamma^i\gamma^k p_i p_k - m^2, \tag{2.31}$$

where we should have $p^\mu p_\mu - m^2 = p_0^2 - p_1^2 - p_2^2 - p_3^2 - m^2$, so we see that

$$(\gamma^0)^2 = 1 \tag{2.32}$$

and

$$(\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1. \tag{2.33}$$

We also need

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 0 \text{ if } \mu \neq \nu. \tag{2.34}$$

Dirac realised that γ are 4×4 matrices that satisfy $\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}$. We also define a useful γ^5 matrix

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3, \tag{2.35}$$

which follows

$$(\gamma^5)^2 = 1, \{\gamma^5, \gamma^\mu\} = 0. \tag{2.36}$$

Some further properties are

$$\gamma^{0\dagger} = \gamma^0, \gamma^{5\dagger} = \gamma^5, \gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 = -\gamma^{\mu\dagger} \text{ for } \mu \neq 0. \tag{2.37}$$

If the matrices satisfy above algebra we can make factors of energy momentum conservation equation as already suggested

$$p^\mu p_\mu - m^2 = (\gamma^\kappa p_\kappa + m)(\gamma^\lambda p_\lambda - m) = 0. \tag{2.38}$$

Dirac equation contains one of the factors, conventionally

$$(\gamma^\lambda p_\lambda - m) = 0. \tag{2.39}$$

Using $p_\mu \to i\partial_\mu$ we get covariant form of Dirac equation

$$(i\gamma^\mu\partial_\mu - m)\psi = 0 \tag{2.40}$$

or using Feynman notation $\gamma^\mu\partial_\mu = \not{\partial}$ we finally get

$$(i\not{\partial} - m)\psi = 0. \tag{2.41}$$

Where ψ is a four component spinor and not a four vector, we shall skip further detailed description of this field in $3{+}1$ D but its characteristics are discussed in $2{+}1$ D in later chapters. Dirac equation is the whole story about free spin half massive particles for which parity symmetry holds and is the first theory which accounts fully in context with quantum mechanical principles [6] [5].
Same equation can be derived from Dirac Lagrangian given as

$$\mathcal{L} = i\bar{\psi}\gamma^\mu\partial_\mu\psi - m\bar{\psi}\psi. \tag{2.42}$$

Where $\bar{\psi} = \psi^\dagger\gamma^0$ is Dirac adjoint, required for Lorentz invariance of the theory [2] [7]. Electromagnetic interactions can be introduced in the lagrangian but first we shall see some basic aspects of electromagnetic field.

2.3 Maxwell's Theory

The four source less equations contain basic laws of electricity and magnetism [8] are as follows

$$\nabla \cdot E = 0 \tag{2.43}$$

$$\nabla \cdot B = 0 \tag{2.44}$$

$$\nabla \times E = -\frac{\partial B}{\partial t} \tag{2.45}$$

$$\nabla \times B = \frac{\partial E}{\partial t}, \tag{2.46}$$

known after the name James Clerk Maxwell represent the state of electromagnetic theory. The physical contents of these equations were known earlier but deriving such a compact form was the contribution of Maxwell [9]. These equations can be presented in a more compact, elegant and explicitly covariant form as

$$\partial_\mu F^{\mu\nu} = 0 \tag{2.47}$$

$$\partial_{[\alpha} F_{\beta\gamma]} = \partial_\alpha F_{\beta\gamma} + \partial_\gamma F_{\alpha\beta} + \partial_\beta F_{\gamma\alpha} = 0 \quad \text{where} \quad F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu, \tag{2.48}$$

by introducing gauge field A^μ.

The explicit form of Maxwell's equations mentioned above can be obtained from a field Lagrangian. The simplest gauge invariant lagrangian for any gauge field A_μ can be written as

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{2.49}$$

Where gauge transformation[4] is defined as [10]

$$A_\mu \rightarrow A'_\mu = A_\mu - \partial_\mu \Lambda(x). \tag{2.50}$$

Gauge invariance is a leading principle in constructing the theories of fundamental interactions, and these theories are called gauge theories [2] [10]. Concept of transformations and corresponding symmetries[5] plays a fundamental role in modern physics. Over the development of past years it has become clear that the structure of elementary particles can be organized in terms of symmetry principles [4].

This free field lagrangian gives $\partial_\mu F^{\mu\nu} = 0$ which gives $1st$ and 4th of Maxwell's equations for source free case. Also $F_{\mu\nu}$ obeys Bianchi identity $\partial_{[\alpha} F_{\beta\gamma]} = 0$ wich gives rest of the Maxwell's equations. Note that in source less Maxwell's equations if we replace $E \rightarrow -B$ and $B \rightarrow E$. we get same equations again. Such a transformation is known as Dual Transformation. As in classical electrodynamics we have a current source term which presents the presence of electric charges, we put a source term in the lagrangian and it becomes

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - J^\mu A_\mu. \tag{2.51}$$

[4] This is a local gauge transformation as Λ is an arbitrary function of x [2]

[5] Transformations under which the equations are invariant are referred to as symmetries.

Euler Lagrange equation $\partial_\mu F^{\mu\nu} = J^\nu$ for this lagrangian gives Maxwell's equations with source as $\nabla \cdot \mathbf{E} = \rho, \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}$.
Gauge invariance does not allow to write mass term for vector field so we do not have that term.
If we want to study the motion of fermions in external field we shall have to couple the external field with fermionic field. First note that the Dirac Lagrangian is invariant under global phase transformations[6]

$$\psi \rightarrow \psi' = e^{ie\theta}\psi, \tag{2.52}$$

but is non-invariant under local transformation when θ is a function of x. In order to make Lagrangian guage invariant, we define covariant derivative of ψ as

$$D_\mu \psi = \partial_\mu \psi + ieA_\mu \psi. \tag{2.53}$$

Under the combined transformations (2.50) and (2.52) we see that

$$D_\mu \psi \rightarrow e^{ie\theta} D_\mu(x)\psi, \tag{2.54}$$

$D_\mu \psi$ transforms the same way as ψ transforms even for θ being a function of x. It becomes easy to construct Lagrangian with local U(1) invariance. All we need to do is to replace all the derivatives with covariant derivatives. This process is called minimal coupling [2].
Then the Lagrangian becomes

$$\mathcal{L} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi. \tag{2.55}$$

We have introduced the electromagnetic interaction in Lagrangian. While writing a Lagrangian we include all possible terms that do not spoil the required symmetries, containing the involved fields. So to present complete picture of QED we add the kinetic term for electromagnetic field and get

$$\mathcal{L} = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi. \tag{2.56}$$

This is a complete and most successful theory for spin half particles and electromagnetic interactions. This is an abelian gauge theory with symmetry group U(1). The beauty of theory lies in complete correspondence with quantum mechanics and special relativity.
Finally we perform a brief dimensional analysis in $3+1$ D to show the dimensions of above defined Dirac and vector fields because when we switch to $2+1$ D, we will see that dimensions of fields are also reduced.
Lagrangian density has the dimensions $[M]^4$. We have $[x^\mu] = [M^{-1}]$ and $[\partial^\mu] = [M]$. So

$$\begin{aligned} [\partial^2 A^2] &= [M^4] \text{ gives } [A] = [M^{+1}] \\ [m\bar{\psi}\psi] &= [M^4] \text{ gives } [\psi] = [M^{\frac{+3}{2}}]. \end{aligned} \tag{2.57}$$

Contents so far discussed are considered in $3+1$ dimensions but if we confine our theories to plane, different aspects arise which practically have very nice applications and are theoretically important. In the next chapter we will see that in $2+1$ D both the theories have considerably different behaviours. We will study the basic characteristics of different theories and then in 4th chapter we will see that how to proceed to quantization of different theories.

[6] "e" is coupling constant usually known as charge of the particle.

Chapter 3

QED in 2+1 Dimensions

Behaviour of fermions and gauge fields is different than the usual behaviour in $3+1$ D and specially a new gauge theory arises in $2+1$ dimensions which is different from Maxwell's theory, known as Chern-Simon's theory. In these dimensions Lorentz and Poincaré algebras are obviously different from ordinary $3+1$ D algebras. We will have a look at these algebras and then we will study the $2+1$ D theories.

3.1 Lorentz and Poincaré Algebras

Apparent form of element of Lorentz group is same as before (2.5),

$$\Lambda = e^{-\frac{i}{2}w_{\mu\nu}J^{\mu\nu}}. \tag{3.1}$$

As stated previously, general set of objects $\phi^i, i = 1.2.3...n$ transforms under representation R of dimension n as

$$\phi^i \to [e^{-\frac{i}{2}w_{\mu\nu}J_R^{\mu\nu}}]^i_j \phi^j. \tag{3.2}$$

In our case $n = 2$ so $i, j = 1, 2$ while $\mu, \nu = 0, 1, 2$. For a scalar in space time, $[J_R^{\mu\nu}]^i_j$ is a number while for a space time vector

$$(J^{\mu\nu})^\rho_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma). \tag{3.3}$$

As this is an antisymmetric matrix 3×3 it has 3 independent elements which are renamed as

$$J = \frac{1}{2}\varepsilon^{ij}J^{ij} \text{ and } K^i = J^{i0}. \tag{3.4}$$

We have two generators K^i of boost and only one generator J of rotation in plane so all the rotations in a plane commute we can write the commutation relations as

$$\begin{aligned} [J, K^i] &= i\varepsilon^{ij}K^j \\ [K^i, K^j] &= -i\varepsilon^{ij}J, \end{aligned} \tag{3.5}$$

where $\varepsilon^i j$ is two dimensinal levi-civita tensor. It is convenient to rename the infinitesimal parameter ω as

$$\theta = \frac{1}{2}\varepsilon^{ij}\omega^{ij} \text{ and } \eta^i = \omega^{i0}. \tag{3.6}$$

So a general element of Lorentz group is now given by

$$\Lambda = exp[-i\theta J + i\vec{\eta}.\vec{K}]. \tag{3.7}$$

The Lorentz generators $J^{\mu\nu}$ and the generators of translation P^μ satisfy the standard Poincare algebra commutation relations, which can be expressed in a more compact form as

$$\begin{aligned} &[J^\mu, J^\nu] = -i\varepsilon^{\mu\nu\rho}J_\rho \\ &[J^\mu, P^\nu] = -i\varepsilon^{\mu\nu\rho}P_\rho \\ &[P^\mu, P^\nu] = 0, \end{aligned} \tag{3.8}$$

where J^μ is the pseudovector generator defined as

$$J^\mu = \frac{1}{2}\varepsilon^{\mu\nu\rho}J_{\nu\rho}. \tag{3.9}$$

The above form of J^μ is simply a more compact form of the three generators with no new physics in it. Using P^μ and J^μ we can make Casimir operators like

$$P^2 = P^\mu P_\mu \quad , \qquad W = P^\mu J_\mu. \tag{3.10}$$

Irreducible representations of the algebra can be characterized by the eigenvalues of these Casimir operators [2]. We will not discuss Casimir operators further.

3.2 Dirac Theory in 2+1 D

The explicit form of Dirac equation is not changed in $2 + 1$ D but we will see that solutions of Dirac equation have very different form and different aspects arise here. The obvious difference will be the irreducible set of Dirac matrices as the Dirac matrices are always irreducible in odd dimensions [11] [12]. Here these matrices are 2×2 matrices, rather then usual 4×4 matrices [11]. Fermion fields are 2-component spinors that are irreducible and dirac gamma matrices are

$$\gamma^0 = \sigma^3, \quad \gamma^1 = i\sigma^1, \quad \gamma^2 = i\sigma^2, \tag{3.11}$$

and obey

$$\gamma^\mu\gamma^\nu = \eta^{\mu\nu} - i\varepsilon^{\mu\nu\alpha}\gamma_\alpha. \tag{3.12}$$

$\sigma's$ are Pauli matrices. As the Dirac equation is

$$(i\partial\!\!\!/ - m)\psi = 0. \tag{3.13}$$

We take the plane wave solutions with constant $u(p)$ of the form

$$\psi(x) = u(p)e^{-ip^\mu x_\mu}. \tag{3.14}$$

Putting in (3.13) we get

$$(\gamma^\mu p_\mu - m)u(p) = 0 \text{ where } \mu = 0, 1, 2. \tag{3.15}$$

Using the Dirac representation (say representation A) of Gamma matrices and writing $u(p)$ as

$$u(p) = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{3.16}$$

We get

$$\begin{pmatrix} E - m & -ip_x - p_y \\ -ip_x + p_y & -E - m \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0. \tag{3.17}$$

This gives two equations out of which we get

$$\begin{aligned} u_1 &= \frac{ip_x + p_y}{E - m} u_2 \\ u_2 &= \frac{-ip_x + p_y}{E + m} u_1. \end{aligned} \tag{3.18}$$

So we can write solutions for $u_1 = 1$ and $u_2 = 1$ as

$$\begin{aligned} \psi_A &= \begin{pmatrix} 1 \\ \frac{-ip_x + p_y}{E+m} \end{pmatrix} e^{-ip^\mu x_\mu} = u_1^+ e^{-ip^\mu x_\mu} \\ \psi_A &= \begin{pmatrix} \frac{ip_x + p_y}{E-m} \\ 1 \end{pmatrix} e^{-ip^\mu x_\mu} = u_2^+ e^{-ip^\mu x_\mu}. \end{aligned} \tag{3.19}$$

It is customary to re-define the solutions for particles and antiparticles as

$$\begin{aligned} u_A(E, p) &= u_1^+(E, p) = \begin{pmatrix} 1 \\ \frac{-ip_x + p_y}{E+m} \end{pmatrix} \\ v_A(E, p) &= u_2^+(-E, -p) = \begin{pmatrix} \frac{ip_x + p_y}{E+m} \\ 1 \end{pmatrix}. \end{aligned} \tag{3.20}$$

So the solutions are

$$\begin{aligned} \psi_A^P(E, p) &= u_A(p) e^{-ip^\mu x_\mu} \\ \psi_A^N(-E, -p) &= v_A(p) e^{ip^\mu x_\mu}. \end{aligned} \tag{3.21}$$

The first one corresponds to the particle and second is for antiparticle. The spin operator corresponds to the generator of rotation $i\varepsilon_{ij}B^{ij}$ where B^{ij} generator of rotation in spinor space, defined in section (3.2.2). We define S as

$$S = iB^{12} = \frac{i}{2}\gamma^1\gamma^2 = \frac{\gamma^0}{2}. \tag{3.22}$$

We take the solutions at rest $p = 0$ because for $p \neq 0$ the states are not eigen states of spin operator. Spin operator acts as

$$\begin{aligned} \frac{\gamma^0}{2} u(E, 0) &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ \frac{\gamma^0}{2} v(E, 0) &= \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -\frac{1}{2} \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned} \tag{3.23}$$

We see that the solutions are eigenstates of spin operator only at rest and u has spin up while v has spin down. So particle has spin up and antiparticle has spin down. Under parity transformation spin changes the sign so particle has spin down and antiparticle has spin up after parity transformation while these two particles are not the solution of equation. So to make the theory parity invariant we need at the same time particles with spin down and antiparticles with spin up. These particles are brought into the spectrum using another inequivalent Dirac representation. The particle spectrum is incomplete but luckily we have another inequivalent representation (say representation B) of Gamma matrices given as

$$\gamma^0 = \sigma^3 \,, \gamma^1 = i\sigma^1 \,, \gamma^{2\prime} = -i\sigma^2 = -\gamma^2. \tag{3.24}$$

Under this representation we find solutions ϕ_B^P and ϕ_B^N and re-define transformed[1] solutions [13] [14]

$$\begin{aligned} \psi_B^P = i\gamma^2\phi_B^P &= \begin{pmatrix} \frac{ip_x+p_y}{E+m} \\ 1 \end{pmatrix} e^{-ip^\mu x_\mu} = u_B(p)e^{-ip^\mu x_\mu} \\ \psi_B^N = i\gamma^2\phi_B^N &= \begin{pmatrix} 1 \\ \frac{-ip_x+p_y}{E+m} \end{pmatrix} e^{ip^\mu x_\mu} = v_B(p)e^{ip^\mu x_\mu}. \end{aligned} \tag{3.25}$$

In this way we get the particles with spin down and anti particles with spin up and hence our particle spectrum is complete. The extended Dirac Lagrangian[2] can be written as

$$\mathcal{L} = \bar{\psi}_A(i\partial\!\!\!/ - m)\psi_A + \bar{\psi}_B(i\partial\!\!\!/ + m)\psi_B. \tag{3.26}$$

This lagrangian is parity invariant [12]. So this way we reach a theory that is CPT invariant (as we shall see further in transformation properties) and depicts complete physical reality as its particle spectrum is complete.

The completeness property shows that the u's and v's can be taken as basis and any state can be expanded in terms of us and vs [5]. The completeness relations in this case are

$$\begin{aligned} u_A\bar{u}_A &= p\!\!\!/ + m \\ v_A\bar{v}_A &= p\!\!\!/ - m. \end{aligned} \tag{3.27}$$

Completeness relations are not changed although we have one particle and one anti-particle spinor. We can define projection operators using above definations,

$$\Lambda_\pm = \frac{\pm p\!\!\!/ + m}{2m}, \tag{3.28}$$

which project out the particle and anti-particle spinors respectively.

3.2.1 Conserved currents and Chirality in 2+1 D

Conserved current j^μ is a current which satisfies the equation of continuity $\partial_\mu j^\mu = 0$ [15]. A continuity equation expresses a conservation law. In other words the flow of the canonical conjugate of

[1] The solutions corresponding to representation B are expanded in basis corresponding to representation A.
[2] Only one of the representations is used, say representation A.

a quantity that posses some continuous translational symmetry is the conserved current. Conserved currents play consequential role in theoretical physics. Noether's theorem connects the existence of a symmetry to existence of conserved current [16], thats why sometimes called Noether currents.
An object is chiral if it can not be mapped by mare rotations and translations to its mirror image. Chirality of a particle determines that how the particle state transforms under right or left handed[3] representation of Poincaré group [2]. For massless particle, chirality and helicity are same but for massive particle they should be distinguished. Dirac Lagrangian has chiral symmetry if it is massless [17]. Regardless of being massive or massless of Dirac field $\psi(x)$ in $3+1D$, it is very useful to decompose Dirac filed in to right and left handed fields,

$$\psi(x) = \psi_L(x) + \psi_R(x). \tag{3.29}$$

Chirality for Dirac field is defined by operator $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ already defined, with eigen values ± 1. So above relation can be written as

$$\psi(x) = \psi_L(x) + \psi_R(x) \equiv \frac{1}{2}(1-\gamma^5)\psi(x) + \frac{1}{2}(1+\gamma^5)\psi(x). \tag{3.30}$$

We can write Dirac Lagrangian as

$$\mathcal{L} = \psi_L^\dagger i\bar{\sigma}^\mu \partial_\mu \psi_L + \psi_R^\dagger i\sigma^\mu \partial_\mu \psi_R - m(\psi_L^\dagger \psi_R + \psi_R^\dagger \psi_L). \tag{3.31}$$

We can see that kinetic energy term connects left-left and right-right fields but mass term connects left-right fields [18].

Transformation $\psi \to e^{i\theta}\psi$ leaves the above Lagrangian invariant. Corresponding Noether's current is $j^\mu = \bar{\psi}\gamma^\mu\psi$. When we project the Dirac field into left and right handed fields, we see that they also transform in this manner i.e. $\psi_L \to e^{i\theta}\psi_L$ and $\psi_R \to e^{i\theta}\psi_R$.
If $m = 0$, Lagrangian exhibits an additional symmetry called chiral symmetry. Chiral transformation[4] is defined as $\psi \to e^{i\phi\gamma^5}\psi$. Noether's theorem says, the conserved current is $j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$. The left handed and right handed fields transform in opposite manner, $\psi_L \to e^{-i\phi}\psi_L$ and $\psi_R \to e^{i\phi}\psi_R$ [18] [7]. Chiral transformation is a continuous transformation. Note that mass term breaks chiral symmetry.

As in odd dimension we do not have an a gamma matrix like γ^5 that anti commutes with all other gamma matrices [11], we can not define chiral transformations in usual way. We look at the symmetries of the extended lagrangian. We can see that we have an exchange symmetry $\psi_A \leftrightarrow \psi_B$

[3]Helicity or handedness of a particle is determined by the direction of component of angular momentum in the direction of motion of particle [3], right-handed means they are in same direction and left-handed means they are in opposite direction.

[4]Generally any symmetry that transforms the right and left handed fields differently is reffered to as chiral transformation [17].

for massless case. This is a discrete symmetry and can not be a candidate for chiral symmetry which is continuous. So we go for some simultaneous continuous transformations like

$$\begin{aligned}\psi_A \rightarrow \psi'_A = \psi_A + \alpha\psi_B \\ \psi_B \rightarrow \psi'_B = \psi_B - \alpha\psi_A,\end{aligned} \tag{3.32}$$

and

$$\begin{aligned}\psi_A \rightarrow \psi'_A = \psi_A + i\alpha\psi_B \\ \psi_B \rightarrow \psi'_B = \psi_B + i\alpha\psi_A,\end{aligned} \tag{3.33}$$

where α is some real constant. The corresponding transformation in Lagrangian is

$$\mathcal{L}'_1 = \mathcal{L} - 2m\alpha(\bar{\psi}_A\psi_B + \bar{\psi}_B\psi_A), \tag{3.34}$$

$$\mathcal{L}'_2 = \mathcal{L} - 2im\alpha(\bar{\psi}_A\psi_B - \bar{\psi}_B\psi_A). \tag{3.35}$$

So we see that Lagrangian is invariant only in massless case. Using

$$j^\mu = \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_A}\delta\psi_A + \frac{\partial\mathcal{L}}{\partial\partial_\mu\psi_B}\delta\psi_B \tag{3.36}$$

for massless case we can find corresponding conserved currents which are

$$j_1^\mu = (\bar{\psi}_A\gamma^\mu\psi_B - \bar{\psi}_B\gamma^\mu\psi_A), \tag{3.37}$$

$$j_2^\mu = (\bar{\psi}_A\gamma^\mu\psi_B + \bar{\psi}_B\gamma^\mu\psi_A). \tag{3.38}$$

To check whether these are really chiral transformations we compare this with reducible $3+1$ D Dirac Lagrangian and its chiral transformations

$$\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi. \tag{3.39}$$

Take the 4×4 representation of gamma matrices

$$\begin{aligned}\gamma^0 = \begin{bmatrix} \sigma_3 & 0 \\ 0 & -\sigma_3 \end{bmatrix} \\ \gamma^i = \begin{bmatrix} i\sigma^i & o \\ 0 & -i\sigma^i \end{bmatrix}.\end{aligned} \tag{3.40}$$

Where $i = 1, 2$. Note that this is not usual 4 D gamma matrix representation which is irreducible but a 3 D reducible 4×4 representation. Here we can define two matrices which anti-commute with other matrices,

$$\begin{aligned}\gamma^5 = i\begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix} \\ \gamma^3 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}.\end{aligned} \tag{3.41}$$

We can define following transformations

$$\psi \to \psi' = e^{i\alpha\gamma^5}\psi, \tag{3.42}$$

$$\psi \to \psi' = e^{i\alpha\gamma^3}\psi. \tag{3.43}$$

The two corresponding chiral currents in massless limit are

$$j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi, \tag{3.44}$$

$$j_3^\mu = \bar{\psi}\gamma^\mu\gamma^3\psi. \tag{3.45}$$

If we write ψ as

$$\psi = \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \tag{3.46}$$

the Lagrangian $\mathcal{L} = \bar{\psi}(i\partial\!\!\!/ - m)\psi$ reduces to the form given in (3.26). So ψ_A, ψ_B satisfy the Dirac equation in the 2-D representation of gamma matrices. In this form the chiral currents can be written as

$$j_5^\mu = \begin{pmatrix} \bar{\psi}_A & \bar{\psi}_B \end{pmatrix} \gamma^\mu \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix}, \tag{3.47}$$

Which ultimately gives

$$j_5^\mu = (\bar{\psi}_A\gamma^\mu\psi_B - \bar{\psi}_B\gamma^\mu\psi_A), \tag{3.48}$$

similarly

$$j_3^\mu = (\bar{\psi}_A\gamma^\mu\psi_B + \bar{\psi}_B\gamma^\mu\psi_A). \tag{3.49}$$

On comparison of (3.37,3.38) with (3.49,3.48) we see that currents obtained from chiral transformation in 4-D representation and continuous transformations given in 2-D representation are identical. Therefore transformations (3.32) and (3.33) are chiral transformations indeed [19].

3.2.2 Transformations Properties of Spinor Field

We will now derive the general transformation law for spinor field and then we can find all the transformations from it. Then we will use it in discrete transformations of spinor field in next chapter. Let field transforms as

$$\psi'(x') = S(\Lambda)\psi(x) \text{ where } x' = \Lambda x. \tag{3.50}$$

Lorentz covariance of the Dirac equation suggests that

$$(i\partial\!\!\!/' - m)\psi'(x') = 0. \tag{3.51}$$

Since gamma matrices are same in all frames so we only need to find $S(\Lambda)$ which satisfies above relation

$$(i\gamma^\mu\Lambda_\mu^\alpha\partial_\alpha - m)S(\Lambda)\psi(x) = 0. \tag{3.52}$$

Apply S^{-1}.(∂ and S commute because S is a constant matrix and Λ is a number for given indices). We get

$$iS^{-1}\gamma^{\mu}S\Lambda^{\alpha}_{\mu}\partial_{\alpha}\psi(x) - mS^{-1}S\psi(x) = 0. \tag{3.53}$$

So we require

$$S^{-1}\gamma^{\mu}S\Lambda^{\alpha}_{\mu} = \gamma^{\alpha}, \tag{3.54}$$

Apply Λ^{ν}_{α} from right and get

$$S^{-1}\gamma^{\nu}S = \Lambda^{\nu}_{\alpha}\gamma^{\alpha}. \tag{3.55}$$

This is the condition that S has to obey. For S to be a spinor representation of Lorentz group we should have

$$\begin{aligned} S(\Lambda_1) &\leftrightarrow \Lambda_1 \\ S(\Lambda_2) &\leftrightarrow \Lambda_2 \\ S(\Lambda_1\Lambda_2) &\leftrightarrow \Lambda_1\Lambda_2 \\ S(\Lambda_1)S(\Lambda_2) &\leftrightarrow \Lambda_1\Lambda_2. \end{aligned} \tag{3.56}$$

For transformations continuously connected to the identity S and Λ can be written in exponential form using generators and the generators for S, Λ have same structure constants. So to find S we need to know the generators. Let the generators of Λ are $J^{\mu\nu}$ and that of S are $B^{\mu\nu}$ and common structure constants are $a_{\mu\nu}$. Then

$$S(\Lambda) = e^{-\frac{i}{2}a_{\mu\nu}B^{\mu\nu}} \leftrightarrow \Lambda = e^{-\frac{i}{2}a_{\mu\nu}J^{\mu\nu}}. \tag{3.57}$$

Expanding left and right hand side of equation (3.55) we get

$$S^{-1}\gamma^{\mu}S = \gamma^{\mu} - \frac{i}{2}a_{\alpha\beta}[\gamma^{\mu}, B^{\alpha\beta}] + ..., \tag{3.58}$$

$$\Lambda^{\nu}_{\alpha}\gamma^{\alpha} = \gamma^{\mu} - \frac{i}{2}a_{\alpha\beta}(J^{\alpha\beta})^{\mu}_{\nu}\gamma^{\nu} + ... \tag{3.59}$$

Comparing above two equations

$$[\gamma^{\mu}, B^{\alpha\beta}] = (J^{\alpha\beta})^{\mu}_{\nu}\gamma^{\nu}. \tag{3.60}$$

The solution is

$$B^{\alpha\beta} = \frac{i}{4}[\gamma^{\alpha}, \gamma^{\beta}]. \tag{3.61}$$

or

$$B^{\alpha\beta} = \frac{i}{2}\gamma^{\alpha}\gamma^{\beta} \qquad \text{where} \qquad \alpha \neq \beta. \tag{3.62}$$

So the generators of boost and rotation can be written in terms of gamma matrices.

Parity, charge conjugation and time reversal are discrete transformations. Here we look at the transformation characteristics of field in $2+1$ D. As we have two different representations, the transformations will give some interesting results.

In odd dimensional spaces if we reverse the signs of all the space coordinates it becomes similar to a rotation so we reverse the signs of all space coordinates but one of them[5]. So in our case parity

[5]Parity transformation matrix should have determinant -1 as being an improper Lorentz transformation. The usual definition of parity with all spatial coordinates flipped, corresponds to rotation.

matrix is (orthochronous, but improper)

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.63}$$

Spinor representation of Parity transformation must satisfy equation (3.55) as

$$S_P^{-1}\gamma^\nu S_P = P^\nu_\alpha \gamma^\alpha. \tag{3.64}$$

The form of (3.63) suggests that S_P should be such that

$$\begin{aligned} S_P^{-1}\gamma^0 S_P &= P^0_\alpha \gamma^\alpha = \gamma^0 \\ S_P^{-1}\gamma^1 S_P &= P^1_\alpha \gamma^\alpha = -\gamma^1 \\ S_P^{-1}\gamma^2 S_P &= P^2_\alpha \gamma^\alpha = \gamma^2. \end{aligned} \tag{3.65}$$

In this representation we have

$$S_P = \gamma^1, \tag{3.66}$$

such that ψ transfroms as

$$\begin{aligned} &S_P\psi = \gamma^1\psi \\ &S_P\psi^\dagger = \psi^\dagger \gamma^{1\dagger} \\ &\bar{\psi}\psi \to -\bar{\psi}\psi. \end{aligned} \tag{3.67}$$

Note that fermionic mass term breaks the parity symmetry. Parity transformation mixes the spinors in two inequivalent representations and converts the particle of one spin to the particle of opposite spin, and the same is true for the antiparticle. It is demonstrated as follows

$$(\psi_A^P)^{\mathcal{P}} = \begin{pmatrix} 1 \\ \frac{ip_x+p_y}{E+m} \end{pmatrix} e^{-ip^\mu x_\mu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \frac{ip_x+p_y}{E+m} \\ 1 \end{pmatrix} e^{-ip^\mu x_\mu} = -i\gamma^1 \psi_B^P. \tag{3.68}$$

This shows that parity transformation converts solution of one representation to solution of other representation with some extra factor.

The charge conjugation in general is defined as $\psi_C = S_C \psi^*$ where S_C must satisfy in our case

$$(\gamma^\mu)^* = -S_C^{-1}\gamma^\mu S_C. \tag{3.69}$$

In representation A, $S_C = \gamma^2$ and spinors transform as

$$S_C\psi = \gamma^2\gamma^0\psi^* \tag{3.70}$$

$$(\psi_A^P)^{\mathcal{C}} = \gamma^2\gamma^0\psi_A^{*P} = \gamma^2\gamma^0 \begin{pmatrix} 1 \\ \frac{ip_x+p_y}{E+m} \end{pmatrix} e^{ip^\mu x_\mu} = \gamma^2(\bar{\psi}_A^N)^T. \tag{3.71}$$

The charge conjugation operation relates the particle of a given spin to the antiparticle of the same representation. This particle is not there in the spectrum. Note that charge conjugation does not mix the two representations. Also it is obvious that

$$(A^\mu)^{\mathcal{C}} = -A^\mu. \tag{3.72}$$

Where A^μ is the electromagnetic potential field which will be described in Maxwell's theory. Time reversal operation is defined as nonorthochronous and improper matrix

$$T = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{3.73}$$

Under this transformation $x_0 \to -x_0$, $p_0 \to p_0$ and $p \to -p$. Time reversal operator is different from other transformation operators being anti-unitary. It can be written as product of a unitary operator and an anti-unitary operator[6].

$$S_T = U_T k. \tag{3.74}$$

Where U_T is a unitary operator and k is anti-unitary such that $U_T^{-1}U_T = 1 = k^{-1}k$. k is in fact complex conjugation operator defined as $k^{-1}ik = -i$. S_T, like other discrete transformations must satisfy (3.55). So

$$S_T^{-1}\gamma^\mu S_T = T^\mu_\nu \gamma^\nu. \tag{3.75}$$

Gamma matrices transform as

$$\begin{aligned} S_T^{-1}\gamma^0 S_T &= T^0_\alpha \gamma^\alpha = -\gamma^0 \\ S_T^{-1}\gamma^1 S_T &= T^1_\alpha \gamma^\alpha = \gamma^1 \\ S_T^{-1}\gamma^2 S_T &= T^2_\alpha \gamma^\alpha = \gamma^2. \end{aligned} \tag{3.76}$$

We find that

$$U_T = \sigma^2 = -i\gamma^2. \tag{3.77}$$

Spinor transforms as

$$S_T\psi = \gamma^2\psi, \tag{3.78}$$

or

$$(\psi_A^P)^{\mathcal{T}} = \begin{pmatrix} 1 \\ \frac{-ip_x - p_y}{E+m} \end{pmatrix} e^{-ip^\mu x_\mu} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{ip_x + p_y}{E+m} \\ 1 \end{pmatrix} e^{-ip^\mu x_\mu} = \gamma^2 \psi_B^P. \tag{3.79}$$

This shows that time reversal converts the particle of spin up in one representation into particle of spin down of other representation. Also it can be seen that under time reversal $A_0 \to A_0$ and $\vec{A} \to -\vec{A}$.

3.3 Maxwell's Theory in 2+1 D

Maxwell's theory holds in any dimensions all we need to do is to change the range of space-time index. Maxwell lagrangian 2.49 and corresponding equations of motion do not change their apparent form. All that changes is number of independent fields contained in antisymmetric tensor

[6] An anti-unitary operator can always be written as a product of a unitary and an anti-unitary operator.

$F^{\mu\nu}$. $F^{\mu\nu}$ is a $d \times d$ antisymmetric matrix, number of fields is $\frac{1}{2}d(d-1)$. Some comparison of $3+1$ and $2+1$ dimensions is as follows

$$\text{for } \mu = 0,1,2,3 \quad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$
$$\text{for } \mu = 0,1,2 \qquad F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y \\ E_x & 0 & -B \\ E_y & B & 0 \end{pmatrix}. \tag{3.80}$$

In $2+1$ dimensions we have two electric field components $\mathbf{E} = -\nabla\phi - \frac{\partial \mathbf{A}}{\partial t}$ (vector potential $\mathbf{A}$ is a two-dimensional vector) that lie in a plane, $F^{0i} = -E^i$ which is usual but we have

$$B = \epsilon^{ij}\partial_i A^j. \tag{3.81}$$

which shows that B here is a *pseudo scalar*[7] instead of being a pseudo vector as in general case.

As we stated earlier that to describe a physical phenomena we write a proper Lagrangian which includes all the possible term relevant to involved field. The thing so interesting in $2+1$ dimensions is the different gauge theory that has all the required properties (it is Lorentz invariant, gauge invariant, and local[8]) but is different from Maxwell's theory and is referred to as Chern-Simon's theory. Chern-Simon's lagrangian is

$$\mathcal{L} = \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha, \tag{3.82}$$

and Chern-Simon's lagrangian with source term is

$$\mathcal{L} = \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha - J^\mu A_\mu \quad \text{where } J^\mu = -e\bar{\psi}\gamma^\mu\psi. \tag{3.83}$$

Where μ is the mass parameter and indices run here from 0 to 2.
Several comments are to be made here that we shall discuss briefly.

This theory seems to be gauge non-invariant as the lagrangian (3.83) has explicit dependence on gauge field but when we apply following gauge transformations

$$\begin{aligned} A_\mu &\to A_\mu + \frac{1}{e}\partial_\mu\Omega \\ \psi &\to e^{i\Omega}\psi. \end{aligned} \tag{3.84}$$

The lagrangian changes by a total derivative

$$\partial_\alpha\left(\frac{\mu}{4e}\varepsilon^{\alpha\mu\nu}F_{\mu\nu}\Omega\right), \tag{3.85}$$

[7] A scalar quantity that changes sign under parity transformation.

[8] A local theory is the one in which no coupling term like $\phi(\vec{x})\phi(\vec{y})$ where $\vec{x} \neq \vec{y}$ is present [20].

and finally while writing Chern-Simon's action we can neglect boundary terms so action is invariant. Also $F_{\mu\nu}$ is manifestly invariant in all dimensions. Bianchi identity

$$\varepsilon^{\mu\nu\rho}\partial_\mu F_{\nu\rho} = 0, \tag{3.86}$$

is compatible with current conservation $\partial_\mu J^\mu = 0$. Chern-Simon term is particular to $2+1$ dimensions, gauge invariance does not allow such a term in $3+1$ dimensions. It is possible to write down a Chern-Simon's theory in any odd space-time dimension, but it is only in 2 + 1 dimensions that the Lagrangian is quadratic in the gauge field. For example the Lagrangian in five-dimensional space-time is

$$\mathcal{L} = \varepsilon^{\mu\nu\rho\sigma\tau} A_\mu \partial_\nu A_\rho \partial_\sigma A_\tau. \tag{3.87}$$

The equation of motion from (3.83) is

$$\frac{\mu}{2}\varepsilon^{\nu\alpha\beta} F_{\alpha\beta} = J^\nu \text{ or equivalently } F_{\mu\nu} = \frac{1}{\mu}\varepsilon_{\mu\nu\alpha} J^\alpha. \tag{3.88}$$

Pure Chern-Simon's theory looks trivial for source free case as above equation reduces to $F_{\mu\nu} = 0$, which gives pure gauges and nothing else. On contrary pure Maxwell's theory gives physical solution even when source free (plane waves).

Chern-Simon's theory can be made interesting by coupling it to matter current or combining with reduced form of Maxwell's theory given above. Lets first see effects of source $J^\mu = (\rho, \mathbf{J})$. Equation (3.88) gives interesting results, in components form we see

$$\mu B = \rho \quad \text{and} \quad \mu\varepsilon^{ij} E_j = J^j. \tag{3.89}$$

This gives strange physical picture that the scalar magnetic field is associated with local charge density with mass parameter as proportionality constant. So B varies with the charge density locally and two dimensional current in plane are associated with electric field with proportionality constant again the mass parameter μ.

Continuity equation $\dot{\rho} + \partial_i J^i = 0$ implies

$$\mu\dot{B} + \mu\varepsilon^{ij}\partial_i E_j = 0. \tag{3.90}$$

From above relation we can say the matter fields have their own dynamics and the effect of Chern-Simon's coupling is to attach magnetic flux to the matter charge density in such a way that it follows the charge density wherever it moves.

3.3.1 Complete Gauge Field Theory of 2+1 D

Why not to see now the Maxwell and Chern-Simon coupling with a source

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha - J^\mu A_\mu. \tag{3.91}$$

This is complete gauge invariant theory in $2+1$ D. The equation of motion it gives is

$$\partial_\mu F^{\mu\nu} + \frac{\mu}{2}\varepsilon^{\nu\alpha\beta}F_{\alpha\beta} = J^\nu. \tag{3.92}$$

It gives modified Maxwell's equations

$$\vec{\nabla}\cdot\vec{E} - \mu B = \rho \tag{3.93}$$

$$\partial_0 E^i + 2\varepsilon^{ji}\partial_j B + \mu\varepsilon^{ij}E^j = -J^i. \tag{3.94}$$

First of all the mass μ associated with gauge field, arises only in $2+1$ dimensions. It is a surprising fact that gauge fields are massive in these dimensions. To make sure that μ is mass we may write the equation in terms of dual field tensor

$$^*F^\mu = \frac{1}{2}\varepsilon^{\mu\alpha\beta}F_{\alpha\beta}, \tag{3.95}$$

and then take divergence. Equation (3.92) can be written as

$$(\eta^{\mu\nu} + \varepsilon^{\mu\nu\alpha}\frac{\partial_\alpha}{\mu})^* F_\nu = J^\mu/\mu. \tag{3.96}$$

We can operate on it with

$$(\eta_{\nu\mu} - \varepsilon_{\nu\mu\rho}\frac{\partial^\rho}{\mu}). \tag{3.97}$$

First for $J^\nu = 0$

$$(\Box + \mu^2)^* F^\mu = 0. \tag{3.98}$$

This clearly demonstrates that fields are massive. Now for source we have

$$(\Box + \mu^2)^* F_\nu = \mu^2(\eta_{\nu\mu} - \varepsilon_{\nu\mu\rho}\frac{\partial^\rho}{\mu})J^\mu. \tag{3.99}$$

Moreover dual tensor is compatible with Bianchi identity $\partial^*_\mu F^\mu = 0$. It gives the third modified Maxwell's equation

$$\vec{\nabla}\times\vec{E} = -\partial_0 B. \tag{3.100}$$

Note that a Maxwell's equation $\vec{\nabla}\cdot\vec{B} = 0$ is missing because B is a scalar in these dimensions.

We may add the gauge non-invariant massive term $m^2 A^\mu A_\mu$ for gauge field as for the photon being a massive particle in $2+1$ D is not a new issue related to this mass term[9]. This lagrangian is now

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha + \frac{1}{2}m^2 A^\mu A_\mu, \tag{3.101}$$

where m is the mass arising due to gauge field mass term and is different from Topological mass. This theory if taken without topological term is a parity conserving theory. However the topological term violates the parity.

The equation of motion satisfying Lorentz condition $\partial_\mu A^\mu = 0$ is given as

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu + \frac{1}{2}\mu\varepsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0. \tag{3.102}$$

[9]But the theory obviously becomes ugly as being gauge non-invariant

Let us first look at the massive properties of the gauge field. Using the notation of self dual field already used we can write the above equation of motion as

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu + \mu M A^\nu = 0. \tag{3.103}$$

or

$$\partial_\mu F^{\mu\nu} + (m^2 + \mu M) A^\nu = 0. \tag{3.104}$$

Massive guage field A^μ should satisfy an equation like

$$(\Box + M^2) A^\nu = 0, \tag{3.105}$$

comparing equations (3.105) and (3.104) we get

$$M_\pm = \frac{\mu}{2} \pm (m^2 + \frac{\mu^2}{4})^{\frac{1}{2}}. \tag{3.106}$$

This suggests that field may have two distinct masses $M_\pm$ [21]. A more rigourous detail about these masses and degrees of freedom will be viewed in discussion under quantization.
One very odd thing about the Lagrangian we took is it is gauge non-invariant due to mass term. The massive photon has one longitudinal polarization and one transverse polarization. We can construct the gauge invariant Lagrangian by Stueckelberg prescription. Stueckelberg approach consists of nice trick, introducing an extra physical scalar field θ in addition to three components of gauge field we get total of four fields to describe the covariant polarization of massive field. We shall see the covariant quantization using this field in later chapter. The gauge invariance of the theory is proved [22] [23] and we shall skip this derivation and shall continue to quantization in coming chapter.
Here again we perform a dimensional analysis to see how the dimensions of fields are changed in $2+1$ D. Lagrangian density has the dimensions $[M]^3$ so we need to recheck the dimensions of fields involved. We have $[\partial^\mu] = [M]$

$$\begin{aligned} &[\partial^2 A^2] = [M^3] \text{ gives } [A] = [M^{\frac{1}{2}}] \\ &[m\bar{\psi}\psi] = [M^3] \text{ gives } [\psi] = [M] \\ &\text{and } [\mu] = [M] \text{ in Chern-Simon's term .} \end{aligned} \tag{3.107}$$

We can see that reducing one spatial dimension has effects on physical dimensions of the fields involved.

3.4 Propagators in 2+1 D

In QFT and QM, a propagator represents the probability amplitude for a particle traveling from one point to another in a given time. In the 1940s, Feynman emphasized on the importance of propagators, considering their fundamental importance for quantum physics [5]. Feynman diagrams calculate theoretically the rate of collisions of particles in quantum field theory, virtual particles contribute their propagators to the rate of scattering of particles described by the diagram. They are

also viewed as the inverse of the wave operator corresponding to the particle, and thats why they are called Green's Functions some times. Another physical explanation about propagators is that they are the factors that represent the transfer or propagation of momentum in a scattering process from one particle to another [24].
Mathematically propagators come from quadratic part of the Lagrangian [20]. We wish to compare the gauge field propagators in $3+1$ and $2+1$ dimensions. The propagator in $3+1$ dimensions is given as [2]

$$\tilde{D}_{\mu\nu}(p) = \frac{-i}{p^2+i\epsilon}(\eta_{\mu\nu} - (1-\xi)\frac{p_\mu p_\nu}{p^2}). \tag{3.108}$$

We write (3.91) with generic ξ[10] parameter[11]

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha - J^\mu A_\mu - \frac{1}{2\xi}(\partial^\mu A_\mu)^2. \tag{3.109}$$

ξ plays role of a Lagrange multiplier. Euler-Lagrange equation is

$$(\eta^{\sigma\rho}\Box - (1-\frac{1}{\xi})\partial^\sigma\partial^\rho + \mu\varepsilon^{\mu\sigma\rho}\partial_\mu)A_\sigma = J^\rho. \tag{3.110}$$

We define $G_{\sigma\nu}(x,x')$ such that

$$(\eta^{\sigma\rho}\Box - (1-\frac{1}{\xi})\partial^\sigma\partial^\rho + \mu\varepsilon^{\mu\sigma\rho}\partial_\mu)G_{\sigma\nu}(x,x') = \delta^\rho_\nu\delta^3(x,x'). \tag{3.111}$$

In Fourier space it reads as

$$(\eta^{\sigma\rho}\Box - (1-\frac{1}{\xi})\partial^\sigma\partial^\rho + \mu\varepsilon^{\mu\sigma\rho}\partial_\mu)\int\frac{d^3p}{(2\pi)^3}\tilde{G}_{\sigma\nu}(p)e^{-ip^\mu(x_\mu-x'_\mu)} = \delta^\rho_\nu\int\frac{d^3p}{(2\pi)^3}e^{-ip^\mu(x_\mu-x'_\mu)}. \tag{3.112}$$

Operator gives us

$$(-\eta^{\sigma\rho}p^2 + (1-\frac{1}{\xi})p^\sigma p^\rho - i\mu\varepsilon^{\mu\sigma\rho}p_\mu)\tilde{G}_{\sigma\nu}(p) = \delta^\rho_\nu, \tag{3.113}$$

$\tilde{G}_{\sigma\nu}(p)$ must be of the form $(a\eta_{\sigma\nu} + bp_\sigma p_\nu + c\varepsilon_{\sigma\nu\gamma}p^\gamma)$ so that,

$$(-\eta^{\sigma\rho}p^2 + (1-\frac{1}{\xi})p^\sigma p^\rho - i\mu\varepsilon^{\mu\sigma\rho}p_\mu)(a\eta_{\sigma\nu} + bp_\sigma p_\nu + c\varepsilon_{\sigma\nu\gamma}p^\gamma) = \delta^\rho_\nu. \tag{3.114}$$

a,b and c are to be found by comparison on both sides. We get

$$\tilde{G}_{\sigma\nu} = \Big(\frac{\eta_{\sigma\nu}}{\mu^2-p^2} + \frac{p^2(\xi-1)-\xi\mu^2}{p^4(\mu^2-p^2)}p_\sigma p_\nu + \frac{i\mu}{p^2(\mu^2-p^2)}\varepsilon_{\sigma\nu\gamma}p^\gamma\Big). \tag{3.115}$$

$A^\mu(p)$ can be found by taking Fourier transform of equation (3.110).

[10]The choice parameter determines the gauge. Most of the quantum field theory calculations are simple in Feynman gauge $\xi = 1$.
[11]Propagator would not exist without the term associated with generic parameter [20], further aspects of this term will be seen later in quantization.

$$(\eta^{\sigma\rho}\Box - (1-\frac{1}{\xi})\partial^\sigma\partial^\rho + \mu\varepsilon^{\mu\sigma\rho}\partial_\mu)\int \frac{d^3p}{(2\pi)^3}A_\sigma(p)e^{-ip^\mu x_\mu} = \int \frac{d^3p}{(2\pi)^3}J^\rho(p)e^{-ip^\mu x_\mu}. \quad (3.116)$$

Applying the operator and then applying the $\tilde{G}_{\rho\mu}$ on both sides

$$\tilde{G}_{\rho\mu}(-\eta^{\sigma\rho}p^2 + (1-\frac{1}{\xi})p^\sigma p^\rho - i\mu\varepsilon^{\mu\sigma\rho}p_\mu)\int \frac{d^3p}{(2\pi)^3}A_\sigma(p)e^{-ip^\mu x_\mu} = \tilde{G}_{\rho\mu}\int \frac{d^3p}{(2\pi)^3}J^\rho(p)e^{-ip^\mu x_\mu}. \quad (3.117)$$

We get

$$\delta^\sigma_\mu A_\sigma(p) = \big(\frac{\eta_{\rho\mu}}{\mu^2-p^2} + \frac{p^2(\xi-1)-\xi\mu^2}{p^4(\mu^2-p^2)}p_\rho p_\mu + \frac{i\mu}{p^2(\mu^2-p^2)}\varepsilon_{\rho\mu\gamma}p^\gamma\big)J^\rho(p), \quad (3.118)$$

or

$$A_\mu(p) = \frac{1}{\mu^2-p^2}\big(J_\mu(p) + \frac{p^2(\xi-1)-\xi\mu^2}{p^4}p_\rho p_\mu J^\rho(p) + \frac{i\mu}{p^2}\varepsilon_{\rho\mu\gamma}p^\gamma J^\rho(p)\big). \quad (3.119)$$

Now we look at the fermionic propagator. As the equation of motion from Dirac Lagrangian is

$$(i\partial\!\!\!/ - m)\psi = 0. \quad (3.120)$$

Repeating steps as above

$$(i\partial\!\!\!/ - m)\int \frac{d^3p}{(2\pi)^3}\tilde{S}(p)e^{-ip^\mu x_\mu} = \int \frac{d^3p}{(2\pi)^3}e^{-ip^\mu x_\mu}, \quad (3.121)$$

or

$$\tilde{S}(p) = \frac{1}{p\!\!\!/ - m} = \frac{p\!\!\!/ + m}{p^2-m^2}. \quad (3.122)$$

This propagator is same as in $3+1$ D [20] [2]. So we see that apparent form of Dirac field remains same in $2+1$ D and so is the case of Dirac field propagator.

3.4.1 Scalar and vector potential in 2+1 D

As the current is conserved the second term in (3.119) $p_\rho J^\rho$ vanishes so we are left with

$$A_\mu(p) = \frac{1}{\mu^2-p^2}\big(J_\mu(p) + \frac{i\mu}{p^2}\varepsilon_{\rho\mu\gamma}p^\gamma J^\rho(p)\big). \quad (3.123)$$

Lets take a static charge at position x''

$$\begin{aligned}&\rho(\vec{x}') = e\delta^2(\vec{x}' - \vec{x}'')\\ &\mathbf{J} = 0.\end{aligned} \quad (3.124)$$

We have potential at observation point x as

$$A_\mu(x) = \int_0^\infty G_{\mu\nu}(x-x')J^\nu(x')d^3x'. \quad (3.125)$$

So scalar potential is

$$A_0(x) = \int_0^\infty G_{00}(x - x')J^0(x')d^3x', \tag{3.126}$$

$$A_0(x) = \int_0^\infty \int_{-\infty}^\infty \frac{e}{\mu^2 - p^2} e^{-ip^\mu(x_\mu - x'_\mu)} \delta^2(\vec{x}' - \vec{x}'')d^3x' \frac{d^3p}{(2\pi)^3}. \tag{3.127}$$

Using delta function and then integrating $dx^{0\prime}$

$$= \int_{-\infty}^\infty \frac{e}{\mu^2 - p^2} e^{-ip^0 x_0} e^{-i\vec{p}.(\vec{x} - \vec{x}'')} \delta(p^0) \frac{d^3p}{2(2\pi)^3}. \tag{3.128}$$

Solving $\delta(p^0)$ and going to spherical polar coordinates we reach

$$= \frac{e}{2(2\pi)^3} \int_0^\infty \int_0^{2\pi} \frac{1}{\vec{p}^2 + \mu^2} e^{i|\vec{p}||(\vec{x} - \vec{x}'')|cos\theta} |\vec{p}|d|\vec{p}|d\theta. \tag{3.129}$$

On solving this integration [25] we get[12]

$$= \frac{e}{2(2\pi)^2} K_0\left(\mu|(\vec{x} - \vec{x}'')|\right). \tag{3.130}$$

We set $x'' = 0$, means that charge is at origin. Then $|\vec{x}|$ is just the distance from charge at the origin. We can plot the function to see how the field changes with distance.

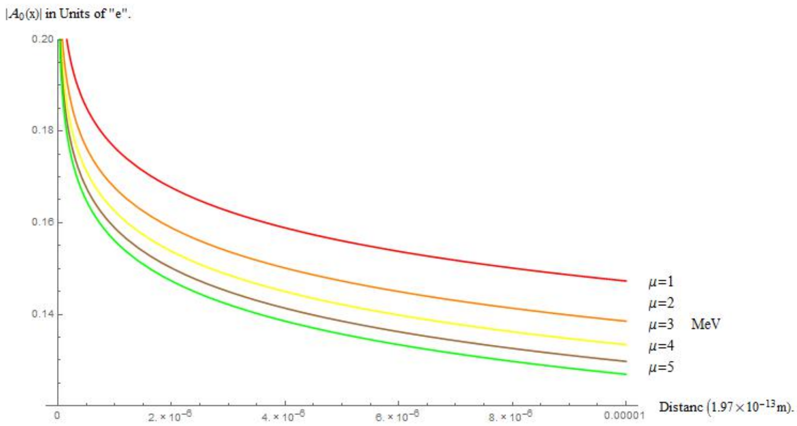

Figure 1: Scalar Potential v/s Distance from Charge.

Further the vector potential is

$$A_i(x) = \int_0^\infty G_{i0}(x - x')J^0(x')d^3x', \tag{3.131}$$

[12] $K_0(x)$ is the modified Bessel function of the second kind which is the solution of equation of type $x^2y'' + xy' - (n^2 + x^2)\, y = 0$.

or

$$A_i(x) = \int_0^\infty \int_{-\infty}^\infty \frac{i\mu e \varepsilon_{i0j} p^j}{p^2(\mu^2 - p^2)} e^{-ip^\mu(x_\mu - x'_\mu)} \delta^2(x' - x'') d^3x' \frac{d^3p}{(2\pi)^3}. \tag{3.132}$$

Following the steps as before we get

$$A_i(x) = -\int_0^\infty \int_0^{2\pi} \frac{i\mu e \varepsilon_{ij} p^j}{\vec{p}^2(\vec{p}^2 + \mu^2)} e^{-i|\vec{p}||(\vec{x} - \vec{x}'')|cos\theta} |\vec{p}| d|\vec{p}| \frac{d\theta}{2(2\pi)^3}, \tag{3.133}$$

or

$$\vec{A}(x) = -\int_0^\infty \int_0^{2\pi} \frac{i\mu e |\vec{p}| \hat{p}}{\vec{p}^2(\vec{p}^2 + \mu^2)} e^{-i|\vec{p}||(\vec{x} - \vec{x}'')|cos\theta} |\vec{p}| d|\vec{p}| \frac{d\theta}{2(2\pi)^3}, \tag{3.134}$$

where $\hat{p}$ is unit vector in the direction of p.

$$= \int_0^\infty \int_0^{2\pi} \frac{i\mu e \hat{p}}{(\vec{p}^2 + \mu^2)} e^{-i|\vec{p}||(\vec{x} - \vec{x}'')|cos\theta} d|\vec{p}| \frac{d\theta}{2(2\pi)^3}. \tag{3.135}$$

By integration [25] we get[13] [14]

$$= i\frac{e}{8\pi}[I_0(\mu|(\vec{x} - \vec{x}'')|) - \boldsymbol{L}_0(\mu|(\vec{x} - \vec{x}'')|)]. \tag{3.136}$$

We again set $x'' = 0$ and then $|\vec{x}| = x$ simply. The plots of these functions are

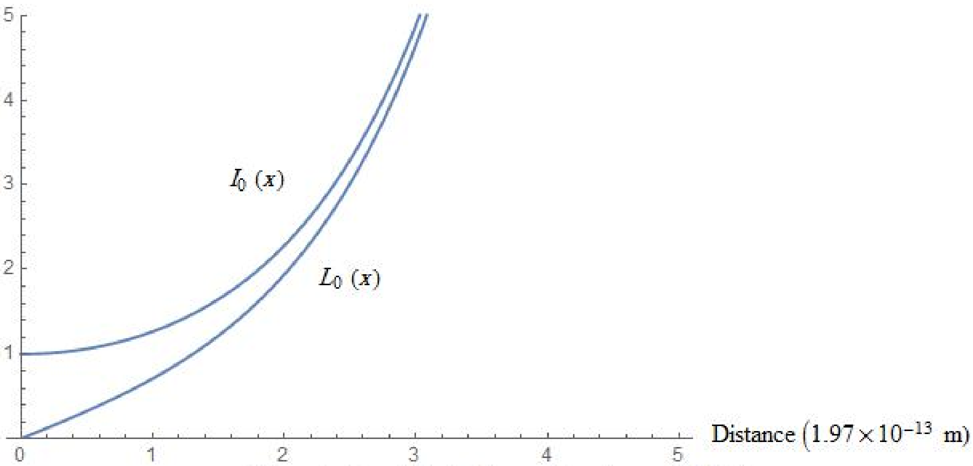

Figure 2: I_0 and L_0 v/s distance keeping μ=1 MeV.

We plot the magnitude of vector potential by combining both functions as

[13] $I[0, x]$ is modified Bessel function of first kind which satisfies the equation $x^2y'' + xy' - (x^2 + n^2) = 0$.

[14] $L[0, x]$ is modified Struve function, Struve functions $H_\alpha(x)$ are the solutions of inhomogeneous Bessel's differential equation $x^2y'' + xy' - (x^2 + \alpha^2) = \frac{4(\frac{x}{2})^{\alpha+1}}{\sqrt{\pi}\Gamma(\alpha+\frac{1}{2})}$ and $L[\alpha, x] = -ie^{\frac{i\alpha\pi}{2}} H_\alpha(ix)$.

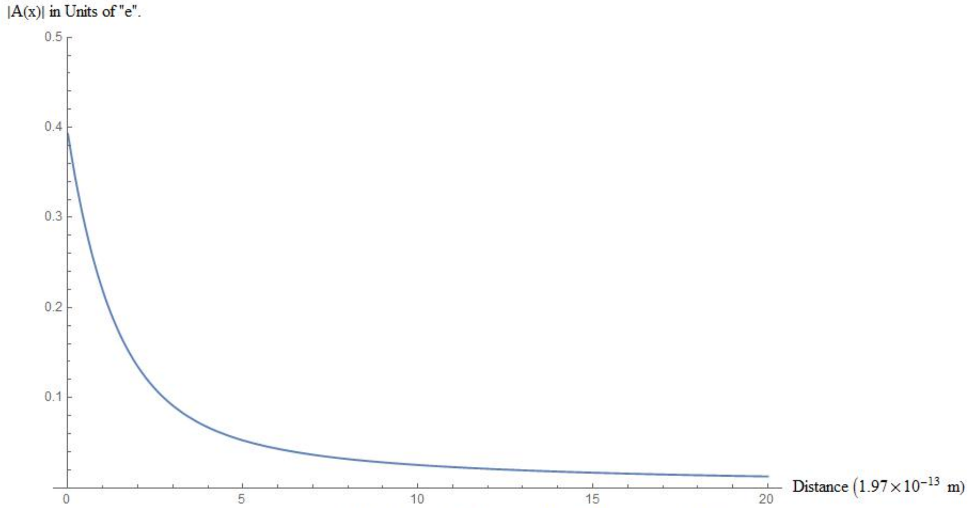

Figure 3: Magnitude of vector potential v/s "x" keeping μ=1 MeV.

In above figure we can see how the vector potential changes with distance from charge. This behaviour is different from that in $3 + 1$ D where scalar potential and vector potential change with inverse of the distance from changes [8].
In next chapter we shall see how can we quantize the theories in different gauges and then proceed to our conclusion.

Chapter 4

Quantization In $2 + 1$ D

Quantization is a process by which we evolve our understanding of any physical phenomena from classical to quantum mechanical understanding. A first quantization is the semi classical treatment of any problem in quantum mechanics. In first quantization particles are represented by wave functions but the interacting atmosphere like potential well or external fields are taken as classical. Our mathematical ingredients in this process are quantum states represented by bra's and kets, observables represented by Hermitian operators acting on Hilbert space [26]. In some cases first quantization fails as for Klein-Gordon and free Dirac equation[1]. The proper interpretation of these theories comes when we quantize the interacting atmosphere that are fields [2]. This is called second quantization. This is the procedure to construct a quantum field theory starting from classical field theory. For $2nd$ quantization we convert classical fields into operators acting on quantized states of field theory. We define the lowest possible energy states called vacuum states. The operators act on space and produce quanta of the fields that are usually the particles and antiparticles [2] [3].

For quantization purposes we have different approaches like Canonical Formalism and Path Integral Formalism [2] [3] [6]. In our work we shall use canonical approach. Ever since the arrival of Quantum Field theory, its development has been linked to canonical formalism and it is so much that it seems natural to begin any treatment of some physical phenomena by writing a suitable Lagrangian and then applying to it the rules of canonical quantization [3]. Most of the book writers have adopted same formalism in their texts. Canonical Quantization is similar as the construction of quantum mechanics from classical mechanics. Fields are taken as dynamical variables and when combined with their conjugate momenta we get a phase space. The step wise procedure is given as [27]

Classical mechanics (Lagrangian, equations of motions e.t.c.).

$\downarrow$

[1] As discussed earlier, we get the negative energy solutions.

Convert classical mechanics in the Hamiltonian formalism, write Poisson brackets.

$\downarrow\downarrow$

Replace canonical variables by linear operators, replace $\{f, g\} \rightarrow \frac{i}{\hbar}[\hat{f}, \hat{g}]$.

$\downarrow\downarrow\downarrow$

Explicitly construct Hilbert space and operators acting on it.

$\downarrow\downarrow\downarrow\downarrow$

Construct quantum mechanics in Heisenberg picture.

In this chapter we have worked up to the third step for different theories. Before moving to quantization we need to introduce some terms that are required to understand the adopted procedure.
To quantize an unconstrained theory we simply impose the Poisson brackets to fields and their conjugate momenta, for any two functions f and g Poisson bracket is defined as

$$\{f, g\}_{PB} = \sum_{i=1}^{N} \left[\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i} \frac{\partial g}{\partial q_i} \right]. \tag{4.1}$$

But when we have a constrained[2] system we can not directly impose the **Poisson brackets** on phase space variables. We need Poisson brackets that hold constraints and do not contradict. There are different types of constraints, if system has **holonomic**[3] constraints, one generally adds Lagrange multipliers to Lagrangian [16]. The same procedure is adopted in Hamiltonian formalism [4]. To understand types of holonomic constraints we need to know ***weak equality*** and ***strong equality***. Two functions on phase space A and B are called weakly equal if they are equal only when all the constraints are satisfied. Weak equality is represented as $A \approx B$. If A and B are equal to each other independently of any constraints then these are called strongly equal. Strong equality is represented by common equality sign, $A = B$ [28].
As far as the constraints are concerned, these can be imposed on the system or some times they appear naturally in system when we find canonical momenta and equations of motion, constraints derived this way or imposed on the system are called ***primary constraints***. These constraints must vanish weakly $\phi_i(p, q) \approx 0$. These constraints must be stable with time, means their time-derivatives must vanish. Some times these stability conditions result in new constraints which are called ***secondary constraints*** [29] [28] (detailed description is avoided here).

[2]Constraint is a relation between field coordinates and their conjugate momenta or simply it is the restriction on motion of particles in a system.

[3]Constraint is Holonomic provided it is given by equality on a set of functions which may depend on positions, explicitly on time but not on velocities. All other constraints are non-holonomic [4] [16].

Sometimes primary and secondry constraints are indiscriminately used and depend on Lagrangian we are considering but there are two discriminating characteristics (***first class*** and ***second class***) that should be counted. If the Poisson bracket of any constraint weakly vanishes with all other constraints it is called first class constraint and if the Poisson bracket is non-zero with at least one of other constraints it is called second class [29] [28] [30].
Some times a theory is quantized by imposing the commutation relations of creation and annihilation operator and then finding the commutation/anti-commutation relations of involved fields but this process is not the standard one and seems to be a reverse method [3]. We shall use this method just once, only for quantization of pure Maxwell's theory in Coulomb gauge because most of the authors have quantized the $3+1$ D theory in same fashion, it will make us realize that same approach holds in $2+1$ D. Then for other theories we shall use standard method of quantization.

4.1 Maxwell's Theory

We start from non interacting Maxwell lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{4.2}$$

As this theory holds in all space-time dimensions, the hamiltonian formalism is standard for it. The equation of motion is

$$\partial_\mu F^{\mu\nu} = 0. \tag{4.3}$$

In this lagrangian, field A_0 has no kinetic term so it is not dynamical. The equation of motion can be written as $\nabla.\vec{E} = 0$ which on expanding becomes

$$\nabla^2 A_0 + \nabla \cdot \frac{\partial \vec{A}}{\partial t} = 0. \tag{4.4}$$

So A_0 is not independent of other components. It seems A_μ has 2 degrees of freedom unless we see any other constraint. The equation of motion may be written as

$$[\eta_{\mu\nu}(\partial^\rho \partial_\rho) - \partial_\mu \partial_\nu] A^\nu = 0. \tag{4.5}$$

The conjugate momenta are as

$$\begin{aligned} \pi^0 &= \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0 \\ \pi^i &= \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = F^{i0} = E^i. \end{aligned} \tag{4.6}$$

Here we see a primary constraint $\pi^0 = 0$ and also from $F^{i0} = E^i$

$$\begin{aligned} &\partial^i A^0 - \partial^0 A^i = E^i \\ &or \\ &\dot{A}^i = -E^i + \partial^i A^0. \end{aligned} \tag{4.7}$$

The hamiltonian can be written as

$$\begin{aligned} H &= \int d^2x[\pi^i \dot{A}_i - \mathcal{L}] \\ &= \int d^2x \frac{1}{2}(E^2 + B^2 - 2A^0(\nabla \cdot \vec{E})). \end{aligned} \tag{4.8}$$

If we write lagrangian

$$\begin{aligned} L &= \int d^2x \pi^i \dot{A}_i - H \\ L &= \int d^2x E^i \dot{A}_i - \int d^2x \frac{1}{2}(E^2 + B^2) - \int d^2x A^0(\nabla \cdot \vec{E}), \end{aligned} \tag{4.9}$$

the first term shows that E_i and A_j are conjugate and but we can not define Poison bracket as

$$\{E_i(\vec{x}), A_j(\vec{y})\} = \delta_{ij}\delta^2(\vec{x} - \vec{y}), \tag{4.10}$$

because we have to deal with constraint in the system. A_0 can be seen as lagrange multiplier explicitly which imposes Gauss' law

$$\nabla \cdot \vec{E} = 0. \tag{4.11}$$

This is another constraint on our supposed system. Let's see how can we use different gauge fixing schemes.

4.1.1 Coulomb Gauge

Coulomb Gauge implies $A_0 = 0$ and is useful to show explicitly the degrees of freedom of gauge field. Equation of motion in this gauge is

$$\partial_\mu \partial^\mu \vec{A} = 0. \tag{4.12}$$

This has solutions of the form

$$\vec{A}(x) = \int \frac{d^2p}{(2\pi)^2} \vec{\epsilon}(\vec{p}) e^{ipx}, \tag{4.13}$$

at $p_0^2 = |\vec{p}|^2$. As we have constraint $\nabla \cdot \vec{A} = 0$, $\vec{\epsilon}(\vec{p})$ must satisfy,

$$\vec{p} \cdot \vec{\epsilon} = 0. \tag{4.14}$$

Here we have only one dimension orthogonal to momentum in a plane, so there is only one polarization direction which makes sense because electric field is confined to one plane only. We can write $\vec{A}$ as

$$\vec{A}(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2|\vec{p}|}} \vec{\epsilon}(\vec{p})[a_{\vec{p}} e^{-ipx} + a^\dagger_{\vec{p}} e^{ipx}]. \tag{4.15}$$

Electric field or conjugate momentum is

$$\vec{E}(x) = -i \int \frac{d^2p}{(2\pi)^2} \sqrt{\frac{|\vec{p}|}{2}} \vec{\epsilon}(\vec{p})[a_{\vec{p}} e^{-ipx} - a^\dagger_{\vec{p}} e^{ipx}]. \tag{4.16}$$

We impose

$$\begin{aligned}
&[a_{\vec{p}\lambda}, a^\dagger_{\vec{q}\lambda'}] = (2\pi)^2\delta^2(\vec{p}-\vec{q})\delta_{\lambda\lambda'} \\
&[a_{\vec{p}\lambda}, a_{\vec{q}\lambda'}] = 0 \\
&[a^\dagger_{\vec{p}\lambda}, a^\dagger_{\vec{q}\lambda'}] = 0,
\end{aligned} \tag{4.17}$$

to find equal time commutation relations as these are the defining quantization rules in radiation gauge [2],

$$\begin{aligned}
&[A^i(\vec{x}), E^j(\vec{y})] \\
&= \frac{-i}{2}\int \frac{d^2pd^2q}{(2\pi)^4}\epsilon^i(\vec{p})\epsilon^j(\vec{q})\big[[a_{\vec{p}}e^{-i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}e^{i\vec{p}\cdot\vec{x}}], [-a_{\vec{q}}e^{-i\vec{q}\cdot\vec{y}} + a^\dagger_{\vec{q}}e^{i\vec{q}\cdot\vec{y}}]\big] \\
&= -i\int \frac{d^2pd^2q}{(2\pi)^2}\frac{1}{2}\big(\epsilon^i(\vec{p})\epsilon^j(\vec{p}) + \epsilon^i(-\vec{p})\epsilon^j(-\vec{p})\big)e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \\
&= -i\int \frac{d^2p}{(2\pi)^2}\big(\delta^{ij} - \frac{p^ip^j}{p^2}\big)e^{-i\vec{p}\cdot(\vec{x}-\vec{y})} \\
&= -i\delta^{ij}_{tr}(\vec{x}-\vec{y}).
\end{aligned} \tag{4.18}$$

Where we have used completeness relation

$$\frac{1}{2}\big(\epsilon^i(\vec{p})\epsilon^j(\vec{q}) + \epsilon^i(-\vec{p})\epsilon^j(-\vec{q})\big) = \big(\delta^{ij} - \frac{p^ip^j}{p^2}\big). \tag{4.19}$$

This can be trivially checked taking $\vec{p} = (0, p)$ and $\vec{\epsilon}(\vec{p}) = (1, 0)$.

4.1.2 Lorenz Gauge

We can work in Lorenz Gauge starting from a gauge non-invariant Lagrangian. This approach is different from previous (Coulomb Gauge) method. Here we take the Lagrangian such that proper equation of motion arises directly from it,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A_\mu)^2. \tag{4.20}$$

with arbitrary constant ξ. Equation of motion is

$$\partial_\mu\partial^\mu A^\nu = 0. \tag{4.21}$$

The quantization is independent of this parameter. Different choices of ξ are referred to different gauges,

$$\begin{aligned}
&\xi = 1 \quad \text{Feynman gauge} \\
&\xi = 0 \quad \text{Landau gauge.}
\end{aligned} \tag{4.22}$$

We take the Feynman gauge for simplicity of calculation. In this approach we quantize the theory first and then impose the constraint on Hilbert space in a suitable manner. First of all we will see that by taking this form of Lagrangian we get both π^0 and π^i dynamical which enables us to impose

the poison brackets without any hurdle.
Conjugate momenta are

$$\begin{aligned}\pi^0 &= \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\partial_\mu A^\mu \\ \pi^i &= \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = F^{i0} = \partial^i A^0 - \dot{A}^i.\end{aligned} \tag{4.23}$$

We have defined the lagrangian so that conjugate momenta are non-vanishing. So we can impose equal time commutation relations

$$[A^i(\vec{x}), \pi^j(\vec{y})] = -i\delta^{ij}\delta^2(\vec{x}-\vec{y}), \tag{4.24}$$

$$[A^0(\vec{x}), \pi^0(\vec{y})] = i\delta^2(\vec{x}-\vec{y}), \tag{4.25}$$

$$[A^i(\vec{x}), A^j(\vec{y})] = 0. \tag{4.26}$$

These relations have covariant generalizations [2]

$$[A^\mu(\vec{x}), \pi^\nu(\vec{y})] = i\eta^{\mu\nu}\delta^2(\vec{x}-\vec{y}), \tag{4.27}$$

$$[A^\mu(\vec{x}), A^\nu(\vec{y})] = 0. \tag{4.28}$$

As the equation of motion is simply

$$\Box A_\mu = 0, \tag{4.29}$$

we can expand A_μ as usual creation and annihilation operators with 3 polarization vectors $\epsilon_\mu(\vec{p},\lambda)$ with $\lambda = 0, 1, 2$ so A_μ can be written as

$$A_\mu(x) = \int \frac{d^2p}{(2\pi)^2\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{2} \epsilon_\mu(\vec{p},\lambda)[a_{\vec{p},\lambda}e^{-ipx} + a^\dagger_{\vec{p},\lambda}e^{ipx}], \tag{4.30}$$

for $p^2 = 0$. Note that there is no constraint on ϵ^μ because lagrangian is not gauge invariant and hence there is no constraint on A^μ. Therefore there are three independent solutions for $\epsilon^\mu(\vec{p},\lambda)$ where $\lambda = 0, 1, 2$. In the frame $p^\mu = (p^0, 0, p)$ we can have the basis as

$$\begin{aligned}\mu &= (0,1,2) \\ \epsilon^\mu(\vec{p},0) &= (1,0,0) \\ \epsilon^\mu(\vec{p},1) &= (0,1,0) \\ \epsilon^\mu(\vec{p},2) &= (0,0,1).\end{aligned} \tag{4.31}$$

These basis vectors satisfy following relation

$$\frac{1}{2}\big(\epsilon^i(\vec{p},\lambda)\epsilon^j(\vec{q},\lambda) + \epsilon^i(-\vec{p},\lambda)\epsilon^j(-\vec{q},\lambda)\big) = \delta^{ij}. \tag{4.32}$$

Again we see that only $\epsilon^\mu(\vec{p},1) = (0,1,0)$ is transverse to p_μ that is $\epsilon^\mu p_\mu = 0$ and for $\epsilon^\mu(\vec{p},0), \epsilon^\mu(\vec{p},2)$ we get $\epsilon^\mu p_\mu \neq 0$. Conjugate momentum is given by

$$\pi^\nu(x) = i\int \frac{d^2p}{(2\pi)^2}\sqrt{\frac{|\vec{p}|}{2}}\sum_{\lambda=0}^{2}\epsilon_\nu(\vec{p},\lambda)[a_{\vec{p},\lambda}e^{-ip\cdot x} - a^\dagger_{\vec{p},\lambda}e^{ip\cdot x}]. \tag{4.33}$$

Note that momentum π^μ appears with factor $+i$ instead of usual $-i$ factor. This can be traced from eq. (4.23). We can find the commutation relations for creation and annihilation operators using the commutator of field and conjugate momenta

$$\begin{aligned}
&[A_\mu(\vec{x}), \pi_\nu(\vec{y})] = \\
&\frac{i}{2}\int \frac{d^2pd^2q}{(2\pi)^4}\sum_{\lambda,\lambda'=0}^{2}\epsilon_\mu(\vec{p},\lambda)\epsilon_\nu(\vec{q},\lambda')\big[[a_{\vec{p},\lambda}e^{-i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p},\lambda}e^{i\vec{p}\cdot\vec{x}}], [-a_{\vec{q},\lambda'}e^{-i\vec{q}\cdot\vec{y}} + a^\dagger_{\vec{q},\lambda'}e^{i\vec{q}\cdot\vec{y}}]\big] \\
&= \frac{i}{2}\int \frac{d^2pd^2q}{(2\pi)^4}\sum_{\lambda,\lambda'=0}^{2}\epsilon_\mu(\vec{p},\lambda)\epsilon_\nu(\vec{q},\lambda')\big[[a_{\vec{q},\lambda'}, a_{\vec{p},\lambda}]e^{-i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})} + \\
&[a_{\vec{p},\lambda}, a^\dagger_{\vec{q},\lambda'}]e^{-i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + [a_{\vec{q},\lambda'}, a^\dagger_{\vec{p},\lambda}]e^{i(\vec{p}\cdot\vec{x}-\vec{q}\cdot\vec{y})} + [a^\dagger_{\vec{p},\lambda}, a^\dagger_{\vec{q},\lambda'}]e^{i(\vec{p}\cdot\vec{x}+\vec{q}\cdot\vec{y})}\big],
\end{aligned} \tag{4.34}$$

is equal to

$$i\eta_{\mu\nu}\delta^3(\vec{x}-\vec{y}), \tag{4.35}$$

only if

$$\begin{aligned}
&[a_{\vec{p}\lambda}, a^\dagger_{\vec{q}\lambda'}] = -(2\pi)^2\delta^2(\vec{p}-\vec{q})\eta_{\lambda\lambda'} \\
&[a_{\vec{p}\lambda}, a_{\vec{q}\lambda'}] = 0 \\
&[a^\dagger_{\vec{p}\lambda}, a^\dagger_{\vec{q}\lambda'}] = 0.
\end{aligned} \tag{4.36}$$

We have achieved the commutation relations but when we explore further we find some issues that are not acceptable and must be solved in order to make the theory valid.
Every thing looks fine for space like $\lambda = 1, 2$,

$$[a_{\vec{p}\lambda}, a^\dagger_{\vec{q}\lambda'}] = (2\pi)^2\delta^2(\vec{p}-\vec{q})\delta_{\lambda\lambda'}, \tag{4.37}$$

but for time like part we get,

$$[a_{\vec{p}0}, a^\dagger_{\vec{q}0}] = -(2\pi)^2\delta^2(\vec{p}-\vec{q}). \tag{4.38}$$

This is odd. And why is this odd?, can be seen as follows.
If we define vacuum $|0\rangle$ by

$$a_{\vec{p}\lambda}|0\rangle = 0, \tag{4.39}$$

we can create particle states in usual way

$$a^\dagger_{\vec{p}\lambda}|0\rangle = |\vec{p},\lambda\rangle. \tag{4.40}$$

For space like $\lambda = 1, 2$ we can calculate the norm as

$$\langle\vec{p},\lambda|\vec{q},\lambda\rangle = \langle 0|a_{\vec{p}\lambda}a^\dagger_{\vec{q}\lambda}|0\rangle = (2\pi)^2\delta^2(\vec{p}-\vec{q}), \tag{4.41}$$

it seems well but for time like part

$$\langle\vec{p},0|\vec{q},0\rangle = \langle 0|a_{\vec{p}0}a^\dagger_{\vec{q}0}|0\rangle = -(2\pi)^2\delta^2(\vec{p}-\vec{q}). \tag{4.42}$$

Negative norm is not really sensible. As earlier we mentioned that we shall quantize the theory first and then apply the constraint on Hilbert space. This constraint is going to be our rescue. We shall see that the constraint will remove the negative norm states. Also it will separate the physical polarizations and non-physical one. And hence the problem of extra degrees of freedom will be solved.
We can write $A_\mu(x)$ as

$$A_\mu(x) = A_\mu^+(x) + A_\mu^-(x), \tag{4.43}$$

where,

$$\begin{aligned} A_\mu^+(x) &= \int \frac{d^2p}{(2\pi)^2\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{2} \epsilon_\mu(\vec{p},\lambda) a_{\vec{p},\lambda} e^{-ipx} \\ A_\mu^-(x) &= \int \frac{d^2p}{(2\pi)^2\sqrt{2|\vec{p}|}} \sum_{\lambda=0}^{2} \epsilon_\mu(\vec{p},\lambda) a^\dagger_{\vec{p},\lambda} e^{ipx}. \end{aligned} \tag{4.44}$$

We take $\partial^\mu A_\mu$ as an operator on Hilbert space. Let $|\Psi\rangle$ be any state, we can define physical states as

$$\partial^\mu A_\mu^+(x)|\Psi\rangle = 0, \tag{4.45}$$

what this ensures is,

$$\langle\Psi|\partial^\mu A_\mu|\Psi\rangle = 0. \tag{4.46}$$

So operator $\partial^\mu A_\mu$ has elements that are vanishing between physical states.
What we have done above is, we have imposed constraint on Hilbert space instead of field operators. In other words we have split Hilbert states into two parts one containing physical states that we require and other non-physical states that we do not like to keep with us. This method was proposed by Gupta-Bleuler and is know after their name [2] [31] [20]. The equation 4.45 is called Gupta-Bleuler condition. We shall not discuss the further consequences of this procedure as our purpose was to quantize the theory.

4.2 Chern-Simons Theory

Consider now the pure Chern-Simon theory i.e. without Maxwell term

$$\mathcal{L} = \frac{\mu}{4}\varepsilon^{\mu\nu\alpha} F_{\mu\nu} A_\alpha, \tag{4.47}$$

or

$$\mathcal{L} = \frac{\mu}{2}\epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{4.48}$$

The canonical form is

$$\mathcal{L} = \frac{\mu}{2}\varepsilon^{0ij}\partial_0 A_i A_j + \frac{\mu}{2}\varepsilon^{i0j}\partial_i A_0 A_j + \frac{\mu}{2}\varepsilon^{ij0}\partial_i A_j A_0, \tag{4.49}$$

writing the second term as total divergence and using

$$B = -\varepsilon^{ij}\partial_i A_j, \tag{4.50}$$

we get

$$\mathcal{L} = \frac{\mu}{2}\varepsilon^{ij}\dot{A}_i A_j - \mu A_0 B. \tag{4.51}$$

A_0 again appears as lagrange multiplier and imposes the Gauss law $B = 0$. The above lagrangian is $1st$ order in time so it is already in Legendre transformed form

$$L = p\dot{q} - H, \tag{4.52}$$

with $H = 0$. So there is no dynamics found in the system. Dynamics would come from coupling to some dynamical (matter) fields. The momentum conjugate to A_i is

$$\pi^i = \frac{\mu}{2}\varepsilon^{ij} A_j. \tag{4.53}$$

The first term in lagrangian and above equation shows that components of gauge fields are canonically conjugate and would imply

$$[A_i(\vec{x}), \pi^j(\vec{y})] = [A_i(\vec{x}), \frac{\mu}{2}\varepsilon^{jk} A_k(\vec{y})] = i\delta_i^j \delta^2(\vec{x} - \vec{y}), \tag{4.54}$$

or

$$[A_i(\vec{x}), A_j(\vec{y})] = \frac{2i}{\mu}\epsilon_{ij}\delta^2(\vec{x} - \vec{y}). \tag{4.55}$$

This behaviour is different from Maxwell's theory. In Maxwell theory the components of gauge field commute and As and Es are canonically conjugate. So pure Chern-Simon theory has this different behaviour of having components of gauge field not commuting.

4.3 Maxwell-Chern-Simon's Theory

Now we consider the combined theories, the lagrangian can be written as

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{2}\epsilon^{\mu\nu\rho}A_\mu \partial_\nu A_\rho. \tag{4.56}$$

Canonical form can be written as

$$\mathcal{L} = \frac{1}{2}E_i^2 - \frac{1}{2}B^2 + \frac{\mu}{2}\varepsilon^{ij}\dot{A}_i A_j + \mu A_0 \varepsilon^{ij}\partial_i A_j. \tag{4.57}$$

Different gauges can be used for quantization as shown in following subsections.

4.3.1 Weyl Gauge:

In $A_0 = 0$ gauge, the momentum corresponding to A_i is

$$\pi^i = \dot{A}_i + \frac{\mu}{2}\varepsilon^{ij} A_j, \tag{4.58}$$

and Hamiltonian is

$$\begin{aligned}\mathcal{H} &= \pi^i \dot{A}_i - \mathcal{L} \\ &= \frac{1}{2}(\pi^i - \frac{\mu}{2}\varepsilon^{ij}A_j)^2 + \frac{1}{2}B^2 - A_0(\partial_i \pi^i - \frac{\mu}{2}B).\end{aligned} \tag{4.59}$$

As before A_0 is again non-dynamical and appears as lagrange multiplier. It implies extended form of Gauss law

$$\nabla \cdot \pi - \frac{\mu}{2}B = 0. \tag{4.60}$$

For $A_0 = 0$

$$\mathcal{H} = \frac{1}{2}(\pi^i - \frac{\mu}{2}\varepsilon^{ij}A_j)^2 + \frac{1}{2}B^2. \tag{4.61}$$

The equation of motion (3.110) (in source-less and without arbitrary gauge case) is

$$(\eta^{\nu\beta}\Box - \eta^{\mu\beta}\partial_\mu\partial^\nu + \mu\varepsilon^{\nu\alpha\beta}\partial_\alpha)A_\beta = 0. \tag{4.62}$$

In Weyl gauge we have

$$(\eta^{\nu i}\Box - \eta^{\mu i}\partial_\mu\partial^\nu + \mu\varepsilon^{\nu\alpha i}\partial_\alpha)A_i = 0. \tag{4.63}$$

The fields and their conjugate momenta satisfy canonical equal time Poisson brackets which become equal time canonical commutation relations in quantum theory.

$$[A_i(\vec{x}), \pi^j(\vec{y})] = i\delta_i^j\delta^2(\vec{x} - \vec{y}). \tag{4.64}$$

This implies

$$[E^i(\vec{x}), E^j(\vec{y})] = -i\mu\varepsilon^{ij}\delta^2(\vec{x} - \vec{y}), \tag{4.65}$$

which shows that electric fields do not commute in massive case. Further we have

$$[E^i(\vec{x}), B(\vec{y})] = i\mu\varepsilon^{ij}\partial_j\delta^2(\vec{x} - \vec{y}), \tag{4.66}$$

$$[B(\vec{x}), B(\vec{y})] = 0. \tag{4.67}$$

The Hamiltonian equations of motion are

$$\dot{A}_i = i[\mathcal{H}, A_i] = \pi^i - \frac{\mu}{2}\varepsilon^{ij}A_j, \tag{4.68}$$

$$\dot{\pi}^i = i[\mathcal{H}, \pi^i] = -\frac{\mu}{2}\varepsilon^{ij}\pi^j + \frac{\mu^2}{4}A^i - \frac{1}{2}\varepsilon^{ij}\partial_j B, \tag{4.69}$$

which give the spatial components of the field equation. As for time component we got the extended form of Gauss law

$$G \equiv \nabla \cdot \pi - \frac{\mu}{2}B = 0. \tag{4.70}$$

This constraint must be imposed as a condition on physical states and hence we can construct physical states of the theory [32].

4.3.2 Lorenz Gauge

In $\partial_\mu A^\mu = 0$ gauge we have equation of motion

$$(\eta^{\nu\beta}\Box + \mu\varepsilon^{\nu\alpha\beta}\partial_\alpha)A_\beta = 0. \tag{4.71}$$

As we know that in Lorenz gauge we have physical and non-physical degrees of freedom as well. We look at the polarization vector and its characteristics first of all. We take the general form of (on shell) solution to equation of motion,

$$A_\beta(p) = \epsilon_\beta(\vec{p})e^{ipx}. \tag{4.72}$$

$\epsilon_\beta(p)$ is constrained as

$$p^\beta \epsilon_\beta(\vec{p}) = 0. \tag{4.73}$$

By putting the solution in equation of motion we get

$$(-p^2\eta^{\nu\beta} + i\mu\varepsilon^{\nu\alpha\beta}p_\alpha)\epsilon_\beta(p) = 0. \tag{4.74}$$

Let us define a matrix $M^{\nu\beta}$ as

$$M^{\nu\beta} = -p^2\eta^{\nu\beta} + i\mu\varepsilon^{\nu\alpha\beta}p_\alpha. \tag{4.75}$$

The necessary and sufficient condition for

$$M^{\nu\beta}\epsilon_\beta(p) = 0, \tag{4.76}$$

to have non trivial solutions $\epsilon_\beta(p) = 0$ is

$$detM^{\nu\beta} = 0. \tag{4.77}$$

The matrix is a 3×3 matrix whose determinant is

$$detM^{\nu\beta} = p^4(p^2 - \mu^2) = 0. \tag{4.78}$$

We shall have two types of solutions corresponding to $p^2 = 0$ and $(p^2 - \mu^2) = 0$. For $p^2 = 0$ we solve

$$\begin{aligned} M^{\nu\beta}\epsilon_\beta(\vec{p}) &= 0 \\ p^\beta \epsilon_\beta(\vec{p}) &= 0, \end{aligned} \tag{4.79}$$

we get

$$\varepsilon^{\nu\alpha\beta}p_\alpha\epsilon_\beta(\vec{p}) = 0. \tag{4.80}$$

Any function of the type

$$\epsilon_\beta(\vec{p}) = p_\beta\chi(\vec{p}), \tag{4.81}$$

is the solution of above equation where $\chi(p)$ is some arbitrary function of p. This shows that massless solutions are pure gauge artifacts that is they have redundancy in their nature. These modes can be ignored, this is somewhat same as gauge fixing eliminates some of the spurious degrees of freedom. Now we consider the case $(p^2 - \mu^2) = 0$, which implies the excitations of mass $|\mu|$. The massive mode enables us to take rest frame for the sake of simplicity of calculations. So we take

$$p^\mu = (|\mu|, 0, 0). \tag{4.82}$$

In this frame, equations (4.79) gives

$$\begin{aligned}\epsilon^0(0) &= 0\\ \epsilon^2(0) &= -i\frac{\mu}{|\mu|}\epsilon^1(0).\end{aligned} \tag{4.83}$$

So the polarization vector in rest frame is

$$\begin{aligned}\epsilon^\mu(0) &= \big(\epsilon^0(0), \epsilon^1(0), \epsilon^2(0)\big)\\ &= \big(0, \epsilon^1(0), -i\frac{\mu}{|\mu|}\epsilon^1(0)\big).\end{aligned} \tag{4.84}$$

We need to fix $\epsilon^1(0)$ which we do by considering normalization and following condition for space-like nature of vector ϵ^μ.

$$\epsilon^\mu(0)\epsilon^*_\mu(0) = -1. \tag{4.85}$$

Normalization condition fixes

$$|\epsilon^1(0)|^2 = \frac{1}{2}. \tag{4.86}$$

Hence we get normalized polarization vector in rest frame

$$\epsilon^\mu(0) = \frac{1}{\sqrt{2}}\big(0, 1, -i\frac{\mu}{|\mu|}\big). \tag{4.87}$$

Here we see an important distinction from Maxwell case that polarization vector has complex entries so that the norm is real. Further more this vector has $U(1)$ invariance that means if ϵ^μ is a solution then $e^{i\theta}\epsilon^\mu$ is also a solution [33].

We can perform an equivalent gauge independent derivation in which without imposing the gauge we shall reach above result. Consider again the equation of motion

$$(\eta^{\nu\beta}\Box - \eta^{\mu\beta}\partial_\mu\partial^\nu + \mu\varepsilon^{\nu\alpha\beta}\partial_\alpha)A_\beta = 0. \tag{4.88}$$

When we put the solution (4.72) in this equation, we get

$$-p^2\epsilon^\nu + \epsilon^\mu p_\mu p^\nu + i\mu\varepsilon^{\nu\alpha\beta}\epsilon_\beta p_\alpha = 0. \tag{4.89}$$

We solve it for ϵ for both massive and massless cases. For $p^2 = 0$

$$\epsilon^\mu p_\mu p^\nu = -i\mu\varepsilon^{\nu\alpha\beta}\epsilon_\beta p_\alpha. \tag{4.90}$$

Multiplying both sides with $\varepsilon_{\nu\rho\sigma}p^\rho$

$$0 = i\mu p_\sigma \epsilon_\rho p^\rho, \tag{4.91}$$

which gives the condition

$$\epsilon_\rho p^\rho = 0. \tag{4.92}$$

We see this condition arising from the system itself, when we solve the equation of motion for massless case with this condition we get again solution of type

$$\epsilon^{\mu} = \Xi(\vec{p})p^{\mu}. \tag{4.93}$$

$f(p)$ is arbitrary function so massless excitations are pure gauge artifacts and should be ignored. Next we consider massive case and as before take rest frame $p^{\mu} = (m, 0, 0)$. Equation becomes

$$\epsilon^{\nu} = \frac{1}{p^2}\big[\epsilon^{\mu}p_{\mu}p^{\nu} + i\mu\varepsilon^{\nu\alpha\beta}\epsilon_{\beta}p_{\alpha}\big]. \tag{4.94}$$

Suppose the polarization vector is

$$\epsilon^{\mu}(0) = \big(\epsilon^{0}(0), \epsilon^{1}(0), \epsilon^{2}(0)\big). \tag{4.95}$$

Then (4.94) yields

$$\epsilon^{1}(0) = i\frac{\mu}{m}\epsilon^{2}(0) \tag{4.96}$$

$$\epsilon^{2}(0) = -i\frac{\mu}{m}\epsilon^{1}(0). \tag{4.97}$$

Put (4.97) back into (4.96)

$$\mu^2 = m^2. \tag{4.98}$$

So we again get

$$m = |\mu|. \tag{4.99}$$

$\epsilon^{0}(0)$ can be set equal to zero because of gauge invariance of the model [33]. We end up with the same earlier result. This result is compatible with covariance although the covariant condition was not imposed explicitly on the system. This is an important aspect of this theory which is different from that of Maxwell theory where polarization vector does not satisfy this. Now it is straight forward to calculate polarization vector in moving frame, all we need is the boost matrix

$$\begin{pmatrix} \epsilon^{0}(p) \\ \epsilon^{1}(p) \\ \epsilon^{2}(p) \end{pmatrix} = \begin{pmatrix} \gamma & \gamma\beta^{1} & \gamma\beta^{2} \\ \gamma\beta^{1} & 1 + \frac{(\gamma-1)(\beta^{1})^2}{(\vec{\beta})^2} & \frac{(\gamma-1)\beta^{1}\beta^{2}}{(\vec{\beta})^2} \\ \gamma\beta^{2} & \frac{(\gamma-1)\beta^{1}\beta^{2}}{(\vec{\beta})^2} & 1 + \frac{(\gamma-1)(\beta^{2})^2}{(\vec{\beta})^2} \end{pmatrix} \begin{pmatrix} \epsilon^{0}(0) \\ \epsilon^{1}(0) \\ \epsilon^{2}(0) \end{pmatrix}, \tag{4.100}$$

where $\vec{\beta} = \frac{\vec{p}}{p^0}$ and $\gamma = \frac{p^0}{|\mu|}$. Ultimately the vector in boosted frame is

$$\epsilon^{\mu}(p) = \big(\frac{\vec{\epsilon}(0)\cdot\vec{p}}{|\mu|}, \vec{\epsilon}(0), \frac{\vec{\epsilon}(0)\cdot\vec{p}}{(p^0 + |\mu|)|\mu|}\vec{p}\big), \tag{4.101}$$

where $\vec{\epsilon}(0)$ stands for space part of the polarization vector.
For quantization we take the lagrangian with arbitrary gauge

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}A_{\alpha} - \frac{1}{2\xi}(\partial^{\mu}A_{\mu})^2 \tag{4.102}$$

and then different gauges can be use for quantization.

4.3.3 Feynman Gauge

In Feynman gauge $\xi = 1$ the lagrangian can be rewritten as follows

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{1}{2}\partial_\mu\big[A_\nu \partial^\nu A^\mu - A^\mu \partial_\nu A^\mu\big] + \frac{\mu}{2}\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{4.103}$$

We can ignore the total divergence term and write as

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{\mu}{2}\varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho, \tag{4.104}$$

conjugate momenta are

$$\begin{aligned} \pi^0 &= \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = -\dot{A}^0 \\ \pi^i &= \frac{\partial \mathcal{L}}{\partial \dot{A}_i} = F^{i0} = -\dot{A}^i + \frac{\mu}{2}\varepsilon^{ij} A_i. \end{aligned} \tag{4.105}$$

The corresponding Hamiltonian is

$$\begin{aligned} H =& \int d^2x \big[-\frac{1}{2}\pi^\mu \pi_\mu + \frac{1}{2}\partial_k A_\nu \partial^k A^\nu \big] \\ &+ \int d^2x \big[-\frac{\mu}{2}\varepsilon^{ij}(A_i \pi_j + A_0 \partial_i A_j + A_i \partial_j A_0) + \frac{1}{8}\mu^2 A_i^2 \big]. \end{aligned} \tag{4.106}$$

We can simply write the Poisson bracket as there are no unsolved constraints.

$$\{A^\mu(x), \pi^\nu(y)\} = \eta^{\mu\nu}\delta(x-y). \tag{4.107}$$

The Hamiltonian equations are

$$\dot{A}^\mu = \{A^\mu, H\} \tag{4.108}$$

and

$$\dot{\pi}^\mu = \{A^\mu, H\}. \tag{4.109}$$

We have a more general approach to perform explicitly covariant quantization through Nakanishi-Lautrup auxiliary field. Since we know that gauge transformations act as symmetries on the theory and gauge degrees of freedom are irrelevant to the final outcome of the theory. Due to gauge transformations we have redundancy in our theory. So most often in gauge invariant theories, one usually deals with local fields that has exceeding number of degrees of freedom then the physical degrees of freedom. As we see that in electrodynamics in order to maintain manifest Lorentz invariance we use the four component vector potential $A_\mu(x)$, while photon has only two degrees of freedom in $3+1$ D (Polarizations). Thus we need a suitable mechanism to get rid of the unphysical degrees. In order to deal with the issue of redundancy , Fadeev and Popov introduced the concept of Ghost-Fields [34]. These are fictitious fields and this is an other way of achieving our goal. Nakanishi-Laurtup field is also a ghost scalar field. We can introduce an auxiliary field say $\mathcal{B}$ which linearizes the gauge fixing term as

$$\mathcal{L}_a = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha + \mathcal{B}\partial^\mu A_\mu + \frac{\xi}{2}\mathcal{B}^2. \tag{4.110}$$

We can see that lagrangian does not contain $\dot{\mathcal{B}}$ and also it is linear in $\dot{A}_0$. It is an auxiliary field as its quadratic term did not appear with derivatives, hence this field appears without it's own dynamics [35]. Introducing the auxiliary field plays an important role in imposing subsidiary conditions that are constraint relations [36]. We will skip further details about this auxiliary field and continue to quantization.
The Euler-Lagrange equations for photon field and auxiliary field are

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu + \mathcal{B}) + \mu\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho = 0 \tag{4.111}$$

and

$$\mathcal{B} = -\frac{1}{\xi}\partial_\nu A^\nu. \tag{4.112}$$

With choice $\xi = 1$ we can eliminate $\mathcal{B}$ and rewrite

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu + \frac{\mu}{2}\varepsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho, \tag{4.113}$$

which is same as we got earlier in Feynman case. Conjugate momenta to the fields A^0, A^i and $\mathcal{B}$ are respectively given as

$$\pi^0 = \mathcal{B}, \tag{4.114}$$

$$\pi^i = \partial^i A^0 - \dot{A}^i + \frac{\mu}{2}\varepsilon^{ij}A_j, \tag{4.115}$$

$$\pi_{\mathcal{B}} = 0. \tag{4.116}$$

The Hamiltonian that we can get from $\mathcal{L}_a$

$$\begin{aligned} H_a = &\frac{1}{2}\int d^2x\big[\pi^{i2} + \mu\varepsilon^{ij}\pi^i A^j + \frac{\mu^2 A^{i2}}{4} + \frac{1}{2}F^{ij}F_{ij}\big] \\ &+ \int d^2x\big[A^0(\partial^i\pi^i - \frac{\mu}{2}\varepsilon^{ij}\partial^i A_i) - \mathcal{B}\partial^i A^j - \frac{1}{\xi}\mathcal{B}^2\big]. \end{aligned} \tag{4.117}$$

We have seen the constraints from equations of motion as

$$C_1 = \pi_0 - \mathcal{B} \approx 0 \tag{4.118}$$

and

$$C_2 = \pi_{\mathcal{B}} \approx 0. \tag{4.119}$$

By setting the first constraint strongly equal to zero we get $\pi_0 = \mathcal{B}$ and we get rid of auxiliary field so Hamiltonian becomes

$$\begin{aligned} H_a = &\frac{1}{2}\int d^2x\big[\pi^{i2} + \mu\varepsilon^{ij}\pi^i A^j + \frac{\mu^2 A^{i2}}{4} + \frac{1}{2}F^{ij}F_{ij}\big] \\ &+ \int d^2x\big[A^0(\partial^i\pi^i - \frac{\mu}{2}\varepsilon^{ij}\partial^i A_i) - \pi^0\partial^i A^j - \frac{1}{\xi}(\pi^0)^2\big]. \end{aligned} \tag{4.120}$$

We have eliminated the auxiliary field and also we have non-zero conjugate momenta so we can write the brackets between fields and their conjugate momenta as

$$\{A^\mu(\vec{x}), \pi^\nu(\vec{y})\} = \eta^{\mu\nu}\delta(\vec{x} - \vec{y}). \tag{4.121}$$

This is the same result as we had in previous section. The procedure adopted here is the most general procedure for manifest covariant quantization. Procedure in previous section was a special case of this procedure [33].
Now the Hamilton's equations can be written as

$$\dot{A}_\mu = \{A_\mu, H_a\}, \tag{4.122}$$

$$\dot{\pi}^\mu = \{\pi^\mu, H_a\}. \tag{4.123}$$

We know the benefit of Hamilton's equations i.e. they are first order in time derivative.

4.3.4 Coulomb Gauge

As the equation of motion from MCS Lagrangian is

$$\Box A^\mu - \partial^\mu(\partial_\nu A^\nu) + \mu\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho = 0. \tag{4.124}$$

The gauge condition is

$$\partial_i A^i = 0. \tag{4.125}$$

In this gauge above equation reduces to

$$\Box A^\mu - \partial^\mu(\partial_0 A^0) + \mu\varepsilon^{\mu\nu\rho}\partial_\nu A_\rho = 0. \tag{4.126}$$

This case, in Hamiltonian language, possesses two first-class constraints so we need two subsidiary conditions to fix the gauge completely [37]. Thus we shall be left with only two (independent) variables in phase space, one momentum and one coordinate. In other words we say that theory exhibits one degree of freedom in configuration space. For this reason it should be possible to write the plane wave solution of reduced equation in terms of a single polarization vector,

$$A_\beta = \epsilon_\beta(\vec{p})e^{ip\cdot x}. \tag{4.127}$$

Same as we did before but this time the polarization vector is constrained by

$$p^i\epsilon_i(\vec{p}) = 0. \tag{4.128}$$

We define a matrix $M^{\nu\beta}$

$$M^{\beta\alpha} = -p^2\eta^{\beta\alpha} + p^\beta p^0\eta^{0\alpha} + i\mu\varepsilon^{\beta\rho\alpha}p_\rho, \tag{4.129}$$

so

$$M^{\beta\alpha}\epsilon_\beta(\vec{p}) = 0. \tag{4.130}$$

It is necessary and sufficient condition for the the determinant of the matrix to vanish to have non-trivial solutions $\epsilon_i(p) = 0$,

$$detM = |\vec{p}^2|p^2(p^2 - \mu^2) = 0. \tag{4.131}$$

This gives three independent solutions for the system while we should have only one degree of freedom. This obviously contradicts our previously conclusion. Polarization vector associated with massless mode $p^2 = 0$ is possible only if $p^\mu = 0$ which in turn implies that the gauge field is just a constant. Same argument can be made for $\vec{p}^2 = 0$. So only one of the solutions is dynamical which is associated with massive mode $p^2 = \mu^2$. Gauge condition is satisfied if we take the solution as

$$\epsilon^i(\vec{p}) = \varepsilon^{ij} p^j f(\vec{p}). \tag{4.132}$$

If we replace this solution in equation for $\alpha = 0$ we obtain

$$\epsilon^0(\vec{p}) = i\mu f(\vec{p}) \tag{4.133}$$

and for $\alpha = i$

$$(-p^2 + \mu^2)\varepsilon^{ij} p^j f(\vec{p}) = 0. \tag{4.134}$$

We can not find $f(\vec{p})$ from any of above equations, even normalization condition of space-like polarization vector can not be used to find this function. However through this device one can only find the modulus of $f(\vec{p})$. This difficulty is arising due to massive nature of the theory. The reason behind this is the existence of rest frame of reference for these particles and in such frame the Coulomb condition $\partial^i A_i = 0$ is ambiguous. This is obviously a new situation as compared with ordinary massless Coulomb gauge formulation. We can adopt a different strategy which consists of reaching Coulomb gauge through Lorenz gauge.

For this purpose we should have some transformation $\Lambda(x)$ which links the Lorenz gauge and Coulomb gauge. It must be such that

$$\epsilon^\mu(\vec{p}) = \epsilon^\mu_L(\vec{p}) + ip^\mu \chi(\vec{p}). \tag{4.135}$$

We have already found the polarization vector in Lorenz gauge which from here on will be denoted by $\epsilon^\mu_L(p)$. The transformation is given as

$$\Lambda(x) = \chi(\vec{p}) e^{ipx}. \tag{4.136}$$

Whether we are dealing with massive of massless case we can find from (4.135)

$$\chi(p) = i\frac{\vec{p}\cdot\vec{\epsilon}_L(\vec{p})}{\vec{p}^2}. \tag{4.137}$$

According to the relation $\epsilon^\mu_L(\vec{p}) = p^\mu f(\vec{p})$, $\chi(\vec{p})$ in massless case reduces to

$$\chi(\vec{p}) = if(\vec{p}). \tag{4.138}$$

When we put this result back into (4.135) we get $\epsilon^\mu = 0$, which indicates that massless solutions are pure gauges and are not present in Coulomb gauge. It is well known that Coulomb gauge is faithful gauge as it explicitly shows only physical solutions, same result should we get here. In massive case we put (4.101) into (4.137) and obtain

$$\chi(p) = i\frac{\vec{p}\cdot\vec{\epsilon}_L(0)}{p^2} + i\frac{\vec{p}\cdot\vec{\epsilon}_L(0)}{(p^0 + |\mu|)|\mu|}. \tag{4.139}$$

From equations (4.101),(4.135) and (4.139) we find

$$\epsilon^0(p) = -\mu \frac{p^j \epsilon_L^i(0)}{p^2} \tag{4.140}$$

and

$$\epsilon^i(p) = \big(\delta^{ij} - \frac{p^i p^j}{p^2}\big)\epsilon_L^j(0). \tag{4.141}$$

Now Free MCS theory (without arbitrary gauge term) in Hamiltonian frame work is given as

$$H_a = \frac{1}{2}\int d^2x \big[\pi^{i2} - \mu\varepsilon^{ij}\pi^i A^j + \frac{\mu^2}{4}A^{i2} + \frac{1}{2}F^{ij}F_{ij}\big]. \tag{4.142}$$

Primary $1st$ class constraint is

$$C_1 = \pi^0 \approx 0 \tag{4.143}$$

and secondary $1st$ class constraint is

$$C_2 = \partial^i \pi_i + \frac{\mu}{2}\varepsilon^{ij}\partial_i A_j \approx 0. \tag{4.144}$$

A_0 and π_0 as usual can be eliminated from phase space, π_0 is fixed by the first constraint condition (4.143) by changing it to strong equality while A_0 is a lagrange multiplier of C_2 and can be determined after the gauge fixing as a function of remaining canonical variables. So we can get rid of both A_0 and π_0.
In Coulomb gauge we take the final constraint as already mentioned $C_3 = \partial_i A^i = 0$. The set of constraint $C_2 \approx 0$ and $C_3 \approx 0$ is second class by its construction and when we have second class constraints we can use Dirac brackets for quantization in usual manner. Then we can promote the phase space variables A_i, π_i to operators which obey set of equal time commutation relations that are abstracted from the corresponding Dirac brackets.
The Dirac bracket quantization procedure yields

$$[A^i(\vec{x}), A^j(\vec{y})] = 0, \tag{4.145}$$

$$[A^i(\vec{x}), \pi^j(\vec{y})] = i\delta_T^{ij}\delta(\vec{x} - \vec{y}), \tag{4.146}$$

$$[\pi^i(\vec{x}), \pi^j(\vec{y})] = -i\frac{\mu}{2}\varepsilon^{ij}\delta(\vec{x} - \vec{y}). \tag{4.147}$$

Where transverse delta function δ_T^{ij} in this case is defined as

$$\delta_T^{ij} = -\mu^2\eta^{ij} + p^i p^j. \tag{4.148}$$

in momentum space.

4.4 Maxwell-Chern-Simon-Proca Theory

As discussed earlier the equation of motion for this theory is

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu + \frac{1}{2}\mu\varepsilon^{\nu\alpha\beta}F_{\alpha\beta} = 0 \tag{4.149}$$

or

$$\partial_\mu F^{\mu\nu} + m^2 A^\nu + \mu \varepsilon^{\nu\alpha\beta}\partial_\alpha A_\beta = 0. \tag{4.150}$$

In Lorenz gauge it becomes

$$\big[(\Box + m^2)\eta^{\mu\nu} + \mu\varepsilon^{\mu\alpha\nu}\partial_\alpha\big] A_\nu = 0. \tag{4.151}$$

We let the solution as before $A_\beta = \epsilon_\beta(\vec{p})e^{ipx}$ which when plugged in, gives

$$\big[(-p^2 + m^2)\eta^{\mu\nu} + i\mu\varepsilon^{\mu\alpha\nu}p_\alpha\big]\epsilon_\nu = 0 \tag{4.152}$$

and from Lorenz or transversality condition we get

$$p^\nu \epsilon_\nu = 0. \tag{4.153}$$

Let us define $M^{\mu\nu}$

$$M^{\mu\nu} = (-p^2 + m^2)\eta^{\mu\nu} + i\mu\varepsilon^{\mu\alpha\nu}p_\alpha. \tag{4.154}$$

Equation of motion becomes

$$M^{\mu\nu}\epsilon_\nu = 0, \tag{4.155}$$

for ϵ_ν to have non-trivial solutions we have the condition

$$detM = 0 \tag{4.156}$$

or

$$(-p^2 + m^2)\big[(-p^2 + m^2)^2 - \mu^2 p^2\big] = 0. \tag{4.157}$$

This implies, either we have

$$-p^2 + m^2 = 0 \tag{4.158}$$

or

$$(-p^2 + m^2)^2 - \mu^2 p^2 = 0 \tag{4.159}$$

If we put (4.158) in (4.152) we get solution of the form $\epsilon_\nu(\vec{p}) = p^\mu f(\vec{p})$. Where $f(\vec{p})$ is an arbitrary function but we see that this solution does not satisfy Lorenz condition as

$$p^\mu p_\mu f(\vec{p}) \neq 0 \tag{4.160}$$

hence this solution must be ignored. The second possibility leads to

$$p^2 = m_\pm^2 = [(m^2 + \frac{\mu^2}{4})^{\frac{1}{2}} \pm \frac{\mu}{2}]^2 \tag{4.161}$$

or we can write

$$m_\pm = (m^2 + \frac{\mu^2}{4})^{\frac{1}{2}} \pm \frac{\mu}{2} \tag{4.162}$$

Some nice characteristics of above solutions are

$$\mu = m_+ - m_- \tag{4.163}$$

and

$$m^2 = m_+ m_- \tag{4.164}$$

Let us introduce different notation, we say $\epsilon_\pm(\vec{p}_\pm)$ are the polarization vectors corresponding to $p^2 = m_\pm^2$. As we did earlier, we can find polarization vectors $\epsilon_\pm(0)$ in rest frame $p^\mu = (p^0, 0, 0) = (|m_\pm|, 0, 0)$ and then boost to get a general result. So in rest frame equation of motion (4.152) gives

$$\begin{aligned}(m^2 - m_\pm^2)\epsilon_\pm^1(0) &= 0 \\ -(m^2 - m_\pm^2)\epsilon_\pm^1(0) - imm_\pm\epsilon_\pm^2(0) &= 0 \\ -(m^2 - m_\pm^2)\epsilon_\pm^2(0) + imm_\pm\epsilon_\pm^1(0) &= 0\end{aligned} \tag{4.165}$$

The result derived from above equations is

$$\epsilon_\pm^0(0) = 0 \tag{4.166}$$

$$\epsilon_\pm^2(0) = \frac{imm_\pm}{m^2 - m_\pm^2}\epsilon_\pm^1(0) = \mp i\epsilon_\pm^1(0) \tag{4.167}$$

using the normalization condition $\epsilon_{\pm\mu}(0)\epsilon_\pm^{*\mu}(0) = -1$ and fixing $\epsilon_\pm^1(0)$

$$|\epsilon_\pm^1(0)|^2 = \frac{1}{2} \tag{4.168}$$

finally we reach

$$\epsilon_\pm^\mu(0) = \frac{1}{\sqrt{2}}(0, 1, \mp i) \tag{4.169}$$

This solution can be checked to verify the gauge condition $p^\nu\epsilon_\nu = 0$. Polarization vectors correspond to two massive modes $m_\pm$ in boosted frame are

$$\epsilon_\pm^\mu(p_\pm) = \Big(\frac{p^1 \mp ip^2}{\sqrt{2}m_\pm}, \frac{1}{\sqrt{2}} + \frac{p^1 \mp ip^2}{\sqrt{2}(p_\pm^0 + m_\pm)m_\pm}p^1, \mp\frac{i}{\sqrt{2}} + \frac{p^1 \mp ip^2}{\sqrt{2}(p_\pm^0 + m_\pm)m_\pm}p^2\Big) \tag{4.170}$$

These vectors show some very nice properties. First of all both polarization vectors are related with parity transformation with condition $p_-^0 \to p_+^0$ which implies $m_- \to m_+$.

$$\begin{aligned}\epsilon_+^0(p_+^0, p^1, p^2) &= \epsilon_-^0(p_-^0 \to p_+^0, p^1 \to -p^1, p^2 \to p^2) \\ \epsilon_+^1(p_+^0, p^1, p^2) &= -\epsilon_-^1(p_-^0 \to p_+^0, p^1 \to -p^1, p^2 \to p^2) \\ \epsilon_+^2(p_+^0, p^1, p^2) &= \epsilon_-^2(p_-^0 \to p_+^0, p^1 \to -p^1, p^2 \to p^2)\end{aligned} \tag{4.171}$$

Also they are related with complex conjugation

$$\epsilon_+^\mu(p_+) = \epsilon_-^{*\mu}(p_-) \tag{4.172}$$

Polarization vectors satisfy

$$\epsilon_{\pm\mu}(0)\epsilon_\pm^\mu(0) = 0 \tag{4.173}$$

and

$$\epsilon_{-\mu}(0)\epsilon_+^{*\mu}(0) = 0 \tag{4.174}$$

orthogonality relations. Now we move towards the quantization of the theory.

4.4.1 Quantization

Canonical momenta corresponding to MCSP Lagrangian are defined as

$$\pi^0 \approx 0 \tag{4.175}$$

$$\pi^i = -\big(F^{i0} + \frac{\mu}{2}\epsilon^{ij}A_j\big) \tag{4.176}$$

Equation (4.175) is primary constraint. Canonical Hamiltonian can be written as

$$\begin{aligned} H = \frac{1}{2}\int d^2x\big[\pi^{i2} - \mu\varepsilon^{ij}\pi^i A^j + \big(\frac{\mu^2}{4} + m^2\big)A^{i2} + \frac{1}{2}F^{ij}F_{ij} + m^2 A^{02}\big] \\ + \int d^2x\big[A^0(\partial^i\pi^i - \frac{\mu}{2}\varepsilon^{ij}\partial^i A_j - m^2 A^0)\big] \end{aligned} \tag{4.177}$$

The Lagrange multiplier A_0 implies another constraint

$$\partial^i\pi^i - \frac{\mu}{2}\varepsilon^{ij}\partial^i A_j - m^2 A^0 \approx 0 \tag{4.178}$$

We use eq. (4.178) to eliminate A_0 from (4.177) and obtain a reduced hamiltonian say H_R

$$\begin{aligned} H_R = \frac{1}{2}\int d^2x\big[\pi^{i2} - \mu\varepsilon^{ij}\pi^i A^j + \big(\frac{\mu^2}{4} + m^2\big)A^{i2} + \big(\frac{\mu^2}{8m^2} + \frac{1}{2}\big)F^{ij}F_{ij}\big] \\ + \frac{1}{2m^2}\int d^2x\big[(\partial^i\pi^i)^2 - \mu\partial^i\pi^i\varepsilon^{ij}\partial^i A_j\big] \end{aligned} \tag{4.179}$$

Now we can write the only non vanishing bracket between phase space variables [33]

$$\{A^i(\vec{x}), \pi^i(\vec{y})\} = -\delta^{ij}\delta(\vec{x} - \vec{y}) \tag{4.180}$$

Hamilton's equations are as

$$\dot{A}^i = \{A^i, H_R\} \tag{4.181}$$

$$\dot{\pi}^i = \{\pi^i, H_R\} \tag{4.182}$$

Let us discuss now the gauge invariant MSCP theory. As discussed earlier, we need to introduce Stueckelberg scalar field θ and define the following extended Lagrangian as

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{\mu}{4}\varepsilon^{\mu\nu\alpha}F_{\mu\nu}A_\alpha + \frac{m^2}{2}(A^\mu - \partial^\mu\theta)(A_\mu - \partial_\mu\theta) \tag{4.183}$$

We calculate conjugate momenta

$$\pi^0 = m^2\theta \tag{4.184}$$

$$\pi^i = -\dot{A}^i + \partial^i A_0 - \frac{\mu}{2}\varepsilon^{ij}A_j \tag{4.185}$$

$$\pi_\theta = \frac{\partial\mathcal{L}}{\partial\dot{\theta}} = m^2\theta \tag{4.186}$$

We can impose Poisson brackets

$$\{A_\mu(x), \pi_\nu(y)\} = -\eta_{\mu\nu}\delta(x-y) \tag{4.187}$$

$$\{\theta(x), \pi_\theta(y)\} = \delta(x-y) \tag{4.188}$$

A nice aspect of Stueckelburg's formalism is that the derivatives are absent from commutation relations. On the other hand we saw that number of degrees of freedom has increased to four, instead of required two for massive planer vector field. This issue can be solved by taking a subsidiary condition on Hilbert space (same as we did in Gupta-Bleuler formalism) which separates the physical and non physical states [23]. Also we may take the theory to some other gauges by gauge transformations as we did earlier that we shall not discuss here [22] [38].

Chapter 5

Conclusion

We learned that the planer QED is different from $3 + 1$ D QED in many aspects. There was no difference found in apparent forms of Dirac and Maxwell's theory from their usual forms but indeed there was difference in the structure of involved fields. Dirac equation was solved for two representations and it was found that we need to combine both types of solutions to complete the particle spectrum of the theory. It was shown that extended Dirac Lagrangian has two continuous symmetries that were proved to be chiral symmetries of Lagrangian. The transformation rules for two component spinor field differ as they relate together the solutions of two different representations. In planer Maxwell's theory we saw that magnetic field is pseudo scalar and Chern-Simon's theory as a different theory that made the $2 + 1$ D more interesting. We saw that a local magnetic field was attached to charge with mass as a proportionality constant. The photon field is massive and propagator has a different form from that in $3 + 1$ D. As a consequence the behaviour of scalar and vector potential is also different.

Lastly in final chapter we performed initial steps of quantization of different theories in Feynman, Coulomb, Weyl and Lorenz gauges e.t.c. and saw that there were a few differences in quantization results that appeared in quantization of Chern-Simon's theory. The main difference found was, field components were canonically conjugate and did not commute in massive case. All other theories had same results of quantization as in $3 + 1$ D. For covariant quantization of Maxwell's and Proca theory we used ghost fields suggested by Nakanishi and Stueckelburg respectively. Finally we can say, the results of quantization in both $2 + 1$ and $3 + 1$ dimensions are almost same.

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