Optimal System and Invariant Solutions of the Hyperbolic Heat Equations

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Dedication

In loving memory of my dearest sister, **Sidra Javed**, whose light enlightened my life and whose spirit continues to inspire me every day. Though she is no longer with us in the physical realm, her memory lives on in the pages of this thesis.

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"In the name of Allah, the Most Gracious, the Most Merciful"

"So, surely with hardship comes ease" ***(Quran 94:6)***

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Abstract

In this research study, we use the Lie symmetry analysis and the optimal systems of subalgebras that underlie it to study the invariant solutions to the nonlinear hyperbolic heat equation. Optimal systems of two specific cases for the equation are obtained. We apply an invariance method to determine the optimal set of non-similar symmetry generators for the nonlinear hyperbolic heat equation and present the results in a convenient tree leaf diagram. Complete symmetry reductions and the invariant solutions corresponding to each case are computed. Subsequently, a thorough analysis is provided, leading to a graphical representation of the solutions of non linear hyperbolic heat equation.

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Chapter 1

Introduction

Differential equations have a significant role in the framework of our universe because they demonstrate connections between mathematical ideas and complex physical processes. As we can see from the past several centuries, differential equations have made it possible for us to comprehend the motion of celestial or heavenly bodies, model fluid flow, understand electrical circuit behavior, and disclose the mysteries of quantum mechanics etc. Differential equations have emerged as essential tools for explaining the relationships between quantities and their rates of change from the conceptual underpinnings of calculus. Numerous interconnected elements in our cosmos demonstrate ongoing progression. Differential equations can be classified into two categories: One type of differential equations is called ordinary differential equations (ODE), which only has one independent variable and their derivatives with respect to the independent variable. The other type of differential equations is called partial differential equations (PDE), which includes an unknown function of two or more variables and its partial derivatives with respect to those variables. Furthermore, both of these types can be categorized into two distinct forms: a linear differential equation, in which the unknown function and its derivatives only appear with a power of one and are combined linearly, without multiplication or raising to a power; and a nonlinear differential equation, in which the unknown function and its derivatives appear in any nonlinear form or be multiplied together, raised to a power, or in any other nonlinear manner. PDEs have their origins in the mathematical modeling of physical processes. Solutions to a broad spectrum of physical, technical, and scientific events are described in terms of PDEs. PDEs are crucial for comprehending and forecasting the behavior of complex systems because they emerge often in several fields, including physics, engineering, economics, biology, and computer science.

In the 17th and 18th centuries, physicists and mathematicians like Isaac Newton, Joseph Fourier, and Jean-Baptiste le Rond d'Alembert produced the first substantial contributions in the field of DEs. Scientists from the 18th century who introduced the PDEs were Euler, Lagrange, and Laplace [1]. However, it was not until the 19th century that it truly gained popularity, mostly as a result of Reimann's influence on certain fields of mathematics [1]. The study of PDEs has benefited significantly from the contributions of James Clerk Maxwell, a prominent physicist and mathematician of the 19th century. Maxwell's equations [1] briefly capture the essence of the entire theory of electricity and magnetism through four fundamental differential equations.

1.1 Symmetry

One of the useful techniques for solving DEs is symmetry method. Symmetry can be thought of as a transformation that preserves the properties of a given structure when applied to it. Scientists use symmetry as a tool for understanding current problems. The classical symmetry method is used to determine exact solutions to differential equations, sometimes referred to as group analysis.

A variety of methods, including the separation of variables, the superposition principle, the Laplace transform, the Fourier transform, etc, can be used to solve linear partial differential equations. Non-linear partial differential equations, however, are difficult to solve analytically. Solving non-linear PDEs analytically is challenging. These types of equations often show up in engineering and science problems, and they are generally more complex and harder to understand compared to linear PDEs. To be able to study and solve PDEs, symmetry approaches are of the utmost importance because they simplify equations by reducing variables, recognizing patterns, applying group transformations (e.g., Lie groups), derive accurate solutions, and identify conservation laws. A method based on symmetry is now regarded as one of the finest ways to solve PDEs. Due to the inherent complexity of nonlinear DEs, a complete classification of such equations is unattainable. Symmetry methods for the solution of DEs, whether linear or nonlinear, may be considered as valuable tools, providing an adaptable approach for analyzing and solving DEs [2–14]

The theory of symmetry methods was first established by a Norwegian mathematician, Marius Sophus Lie. He worked in the area of continuous symmetries, which he put to use in the study of DEs and geometry and was inspired by Galois' theory. Evariste Galois (1811-1832) made fundamental mathematical contributions, especially in Integrating the theory of equations into the field of symmetry. Galois developed the concept of Galois groups, which provide a thorough understanding of the algebraic equations' symmetries. Evariste Galois used the theoretical method of groups to solve algebraic problems such as quadratic, cubic, and quartic during in the 19th century. Based on the concept of comparison, Lie proposed the idea that infinite groups, which consistently depend on at least one real or complex variable, are likely to play a crucial role in addressing ODEs and PDEs, similar to how finite groups are essential for determining the solvability of finite-degree polynomial equations [10, 12, 15, 16].

The significance of continuous symmetries in the study of DEs was recognized by Lie. Lie groups are mathematical structures that represent the idea of continuous symmetry, were also introduced by him. The transformations that establish invariance of a specific differential equation can be understood within the approach of Lie groups. The modern concept of transformation groups is based on his work. Lie formulated the notion of Lie algebra, serving as a linear approximation of Lie group. The examination of infinitesimal symmetries in DEs can be carried out via the analysis of Lie algebras. Local symmetries and conservation laws of a differential equation can be determined by employing the Lie algebra connected to a Lie group. Powerful techniques for analyzing and solving differential equations have been made possible due to Lie's work on Lie algebras. To construct the symmetries of DEs, Lie developed the basic concepts and techniques that are necessary for transformation group theory. He developed the theory of infinitesimal transformations, which serves as a tool to derive Lie's symmetries. The study of differential equations and modern group theory are both based on Lie's work on transformation groups.

Symmetries can be categorized mainly into two categories: discrete symmetries and continuous or Lie point symmetries. Discrete symmetries, as the term suggests, are non-continuous symmetries that fall outside the realm of Lie groups. These symmetries are characterized by finite or countable collections of distinct transformations. Unlike continuous symmetries, which include smooth and continuous transformations, discrete symmetries involve well-defined transformations that result in identical or similar configurations of an object or system. Discrete symmetries of differential conditions have certain significant applications, which are discussed in [14, 17–19].

It is indeed important to have efficient methods for determining the discrete symmetries, especially when dealing with equations that possess a finite-dimensional Lie algebra of infinitesimal generators, which corresponds to the Lie group of point symmetries. Peter E. Hydon is a renowned mathematician who has made significant contributions to the field of symmetry methods for differential equations. His work has focused on developing systematic approaches for finding and analyzing symmetries of differential equations, including discrete symmetries [14, 17, 18, 20, 21]. The foundation of his strategy is the notion that any point symmetry generates an automorphism for the Lie algebra of the Lie point symmetry generators.

1.2 Optimal Systems

In the late 1950s, a renowned Soviet scientist, Lev Vasilyevich Ovsyannikov (1919–2014), revived the application of Lie group theory [2], [6]. Ovsyannikov's work was primarily concerned with applying group-theoretical methods to the study of differential equations. Lie's work has been firmly established, involving the exploration of various topics and ongoing research since the rebirth of applying Lie theory to DEs. These topics include the linearization of ordinary and partial differential equations, generating new solutions based on existing ones, the development of an equivalence group, the reduction of PDEs through similarity or invariant solutions, consideration of approximate symmetries, and investigation of classification problems related to groups involving the development of generalized local and nonlocal symmetries. Significant progress has been made by identifying symmetries in stochastic differential equations (SDEs) [22], as well as in the context of conservation laws, integro-differential equations, difference equations, algebraic equations, geodesic equations, functional differential equations, and other related areas [23]. Employing Lie group analysis, Ovsyannikov [2] examine the realm of differential equations in 1958. His outstanding research and theory are based on the concept of continuous transformation groups. These classifications offer a unique examination of solutions for particular differential equations. He presented partially invariant and invariant solutions, along with an efficient algorithm for their development.

Solutions invariant influenced by a subgroup within a symmetry group are referred to as invariant solutions. Symmetries are employed to construct these solutions, aiding in the simplification of differential equations or systems of differential equations. In the context of partial differential equations, simplification can occur in two ways: either in relation to the order of the partial differential equation or system of partial differential equations, as well as the count of independent variables.

One of Ovsyannikov's major contributions was the formulation of the notion of an optimal system. He created a general framework to construct optimal systems of differential equations that capture the symmetries in a given problem. Ovsyannikov's method allowed for the systematic reduction of differential equation complexity by identifying transformations that maintained their form. The optimal system is determined by systematically constructing non-equivalent classes of invariant solutions [2, 4, 6, 9, 10]. It's essential to derive a single invariant solution from each category; subsequently, by using symmetries, the entire class can be constructed. This approach reduces the effort needed to acquire invariant solutions.

Ovsyannikov [6] employed the concept of optimal systems of subalgebras within a specific Lie algebra to elucidate fundamentally distinct invariant solutions. This concept proves valuable when addressing mathematical models. Ovsyannikov [6] explored the similarity between two subalgebras, investigating whether a symmetry group transformation could align both subalgebras in the same class. If such alignment occurs, the same transformation connects the associated invariant solutions. Consequently, to attain an optimal system, one must construct conjugacy classes for these subalgebras. A considerable body of literature is dedicated to group-invariant solutions and optimal systems [24–26].

1.3 Background of Hyperbolic Heat Equation

Heat flows through a material due to a temperature difference when it is heated unevenly because of the varying temperature within the material. The parabolic heat conduction equation, which is the classical model of heat conduction, has been widely used to study this heat conduction process. Furthermore, the second law of thermodynamics is incompatible with the parabolic model. The telegrapher's equation, which represents a hyperbolic model of heat conduction

$$w_{tt} + w_t = (K(w)w_u)_u, K(w) \neq const,$$

is considered more physically accurate because it predicts the finite speed at which heat diffuses through a material. The damped wave equation, which serves as a model for hyperbolic heat conduction, was initially introduced by Carlo Cattaneo using kinetic theory. It is also possible to obtain this equation through alternative physical principles, such as the second law of thermodynamics. When the thermal diffusivity remains constant, the speed at which temperature disturbances propagate is $\sqrt{K}$ which addresses the unphysical infinite propagation speed.

Chapter 2

Preliminaries of Lie Point Symmetries of Differential Equations

This section aims to provide a comprehensive ellaboration of fundamental concepts associated to Lie point symmetries of DEs. We intend to gain a thorough insight of these notions and their significance in the field of mathematical analysis by studying the available literature. The fundamental definitions and notations are introduced. Furthermore, criteria that serve as guidelines for establishing the Lie point symmetries of DEs are presented. Relevant references are provided throughout this chapter for more detailed proofs and insights. All of the theorems are stated without proof. Each theorem and its significance will be explained concisely.

2.1 One-Parameter Lie Group of Point Transformation

This section delves into the theory of one-parameter Lie groups of point transformations. One-parameter Lie groups are important in understanding the transformations that preserve the form and behavior of differential equations. A point transformation refers to a change of variables that maps one set of independent and dependent variables to another set in the context of DEs, it is defined as the transformation of independent and dependent variables, that is t and w respectively. A point transformation maps points (t, w) into points $(\tilde{t}, \widetilde{w})$ [7]

$$\begin{aligned} \tilde{t} &= \tilde{t}(t, w), \\ \widetilde{w} &= \widetilde{w}(t, w), \end{aligned} \tag{2.1}$$

where $\tilde{t}$ and $\widetilde{w}$ are continuous functions. Furthermore, in some cases the transformation may depend on at least one continuous parameter say ε, that is

$$\begin{aligned} \tilde{t} &= \tilde{t}(t, w, \varepsilon), \\ \widetilde{w} &= \widetilde{w}(t, w, \varepsilon), \end{aligned} \tag{2.2}$$

where $\tilde{t}$ and $\widetilde{w}$ are infinitely differentiable with respect to (t and w).

This section provides the basic definitions required for the point transformations within the one-parameter Lie group [10].

Definition 2.1.1. *If the group operations*

$$\mathfrak{f} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}, \quad \mathfrak{f}(p, q) = p \cdot q, \quad p, q \in G,$$

and

$$\tilde{\mathfrak{f}} : \mathcal{G} \to \mathcal{G}, \quad \tilde{\mathfrak{f}}(p) = p^{-1}, \quad p \in \mathcal{G},$$

of a group $\mathcal{G}$ are r-parameter Lie groups, the group is described as an r-parameter Lie group. The smooth maps between the manifolds act on the r-dimensional C^{∞}-manifold.

Definition 2.1.2. *Let $\boldsymbol{M}$ be a C^{∞}-manifold, and there is an r-parameter Lie group G, in the presence of a smooth map*

$$\varphi : G \times \boldsymbol{M} \to \boldsymbol{M}, \quad \varphi(p, m) = pm,$$

then the Lie group G is stated as the Lie group of transformation satisfying the following two conditions:

1. *$(p_1 \cdot p_2)m = p_1(p_2 m)$, $\forall\ p_1, p_2 \in G$ and $m \in \boldsymbol{M}$.*
2. *Let I be the identity element of G then Im=m $\forall\ m \in \boldsymbol{M}$.*

Now if

$$\widetilde{\boldsymbol{\upsilon}} = \boldsymbol{\varphi}(\boldsymbol{\upsilon}, \varepsilon), \tag{2.3}$$

and

$$\widetilde{\widetilde{\boldsymbol{\upsilon}}} = \boldsymbol{\varphi}(\boldsymbol{\varphi}(\boldsymbol{\upsilon}, \varepsilon), \nu) = \boldsymbol{\varphi}(\widetilde{\boldsymbol{\upsilon}}, \nu), \tag{2.4}$$

where $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, ..., \upsilon_n)$, $\widetilde{\boldsymbol{\upsilon}} = (\widetilde{\upsilon_1}, \widetilde{\upsilon_2}, ..., \widetilde{\upsilon_n})$, $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, ..., \varphi_n)$ *and* $\Phi(\varepsilon, \nu)$ *be the law of composition of parameters* $\varepsilon, \nu \in \boldsymbol{V}$, *if the following conditions hold* [27], *in region* $\mathcal{D}$, *a one-parameter group of transformations is established.*

1. $\boldsymbol{V}$ *constitutes a group under the law of composition* Φ.

2. *For each* $\boldsymbol{\upsilon}$ *in the region* $\mathcal{D}$, *we have* $\widetilde{\boldsymbol{\upsilon}} = \boldsymbol{\upsilon}$, *for* $\varepsilon = \varepsilon_0$, *which corresponds to an identity element.*

3. *For* $\widetilde{\boldsymbol{\upsilon}} \in \mathcal{D}$ *the transformation in* $\mathcal{D}$ *for each* $\varepsilon \in \boldsymbol{V}$ *must be injective.*

4. *We can deduce from Eqs.* (2.3) *and* (2.4), *that*

$$\widetilde{\widetilde{\boldsymbol{\upsilon}}} = \boldsymbol{\varphi}(\boldsymbol{\upsilon}, \Phi(\varepsilon, \nu)), \tag{2.5}$$

   where $\widetilde{\boldsymbol{\upsilon}}, \widetilde{\widetilde{\boldsymbol{\upsilon}}} \in \mathcal{D}$.

Definition 2.1.3. *Suppose* $\mathcal{G}$ *be a Lie group and* $\boldsymbol{M}$ *be the* C^{∞}*-manifold, where* $\Phi(\varepsilon, \nu)$ *is a composition function. Then a Lie group is said to be a one-parameter Lie group of transformation if it meets the subsequent criteria*

i. *For the parameter* $\varepsilon = 0$, *it relates to the identity transformation, and for* $\varepsilon = -\varepsilon$ *or* ε^{-1}, *it corresponds to the inverse transformation group. Here,* ε *is a continuous parameter with* $\varepsilon \in \boldsymbol{V} \subset \mathbb{R}$.

ii. *Let* $\boldsymbol{x}$ *and* $\boldsymbol{u}$ *be any points in the region* $\mathcal{D} \subset \mathbb{R}$, *then the functions* $\widetilde{\boldsymbol{x}}$ *and* $\widetilde{\boldsymbol{u}}$ *are continuously differentiable with respect to* $\boldsymbol{x}$ *&* $\boldsymbol{u}$. *Moreover, these functions are analytic in* $\varepsilon \subset \boldsymbol{V}$.

iii. *The composition function* $\Phi(\varepsilon, \nu)$ *is an analytic function with respect to both parameters* ε *and* ν, *where* $\varepsilon, \nu \in \boldsymbol{V}$.

2.2 Infinitesimal Transformation and the Symmetry Generators

Infinitesimal transformations and their generators are essential mathematical concepts used to describe infinitesimally small changes in dynamic systems. We now define the infinitesimal transformations and their corresponding generators. Consider Eq. (2.3)

$$\widetilde{\boldsymbol{v}} = \boldsymbol{\varphi}(\boldsymbol{v}, \varepsilon),$$

then Taylor expansion at $\varepsilon = 0$, gives us

$$\widetilde{\boldsymbol{v}} = \boldsymbol{v} + \varepsilon \frac{\partial}{\partial \varepsilon} \boldsymbol{\varphi}(\boldsymbol{v}, \varepsilon) \bigg|_{\varepsilon=0} + \frac{\varepsilon^2}{2} \frac{\partial^2}{\partial \varepsilon^2} \boldsymbol{\varphi}(\boldsymbol{v}, \varepsilon) \bigg|_{\varepsilon=0} + \mathcal{O}(\varepsilon^3).$$

Now we will consider

$$\frac{\partial}{\partial \varepsilon} \boldsymbol{\varphi}(\boldsymbol{v}, \varepsilon) \bigg|_{\varepsilon=0} = \boldsymbol{\alpha}(\boldsymbol{v}). \tag{2.6}$$

Subsequently, the Lie group's infinitesimal transformation can be expressed as

$$\widetilde{\boldsymbol{v}} = \boldsymbol{v} + \varepsilon \boldsymbol{\alpha}\left(\boldsymbol{v}\right). \tag{2.7}$$

The application of Eq. (2.6) forms the basis for **Lie's first fundamental theorem**, offering a method to re-parameterize a one-parameter group of transformations with a well-defined structure.

Theorem 2.2.1. *To establish the equivalence between the Lie group of transformations* (2.3) *and the solution of an initial value problem for the autonomous system of first-order ODEs, a parametrization function* $\zeta(\varepsilon)$ *is presented, as*

$$\frac{\partial \widetilde{\boldsymbol{v}}}{\partial \zeta} = \alpha\left(\boldsymbol{v}\right), \tag{2.8}$$

with the given condition that $\widetilde{\boldsymbol{v}} = \boldsymbol{v}$ *at* $\zeta = 0$ [16]*. Specifically,*

$$\zeta\left(\varepsilon\right) = \int_0^{\varepsilon} \Lambda\left(\varepsilon'\right) d\varepsilon', \tag{2.9}$$

where

$$\Lambda\left(\varepsilon\right) = \frac{\partial}{\partial j} \Phi\left(i, j\right) \bigg|_{(i,j)=\left(\varepsilon, \varepsilon'\right)}, \qquad \Lambda(0) = 1. \tag{2.10}$$

In the following definition, we incorporate a representation of a one-parameter Lie group of transformations in the form of a group generator [10, 16].

Definition 2.2.1. *The infinitesimal generator for a one-parameter Lie group of transformations can be defined by a linear differential operator*

$$\Gamma = \boldsymbol{\alpha}(\boldsymbol{\upsilon}).\nabla = \sum_{k=1}^{n} \alpha_k(\boldsymbol{\upsilon}) \frac{\partial}{\partial \upsilon^k}, \tag{2.11}$$

where $\boldsymbol{\alpha}(\boldsymbol{\upsilon}) = (\alpha_1(\boldsymbol{\upsilon}), \alpha_2(\boldsymbol{\upsilon}), ..., \alpha_n(\boldsymbol{\upsilon}))$ *and* ∇ *is the gradient operator.*

For any given differential equation

$$F(\mathbf{t}) = F(t_1, t_2, ..., t_n), \tag{2.12}$$

we can express

$$\boldsymbol{\Gamma} F(\mathbf{t}) = \boldsymbol{\alpha}(\boldsymbol{\upsilon}).\nabla F(t) = \sum_{k=1}^{n} \alpha_k(\boldsymbol{\upsilon}).\frac{\partial F(\mathbf{t})}{\partial \upsilon^k}. \tag{2.13}$$

Theorem 2.2.2. *Let us define the linear operator* $\boldsymbol{\Gamma}$ *using the expression in Eq.* (2.13) *and consider Eq.* (2.3) *given by*

$$\widetilde{\boldsymbol{\upsilon}} = \boldsymbol{\varphi}(\boldsymbol{\upsilon}, \varepsilon),$$

then the associated generators for the one-parameter Lie group of transformations are

$$\begin{aligned} \widetilde{\boldsymbol{\upsilon}} = \boldsymbol{\varphi}(\boldsymbol{\upsilon}, \varepsilon) = e^{\varepsilon \boldsymbol{\Gamma}} \boldsymbol{\upsilon} &= \boldsymbol{\upsilon} + \varepsilon \boldsymbol{\Gamma} \boldsymbol{\upsilon} + \frac{\varepsilon^2}{2} \boldsymbol{\Gamma}^2 \boldsymbol{\upsilon} + \mathcal{O}\left(\varepsilon^3\right), \\ &= \sum_{n=0}^{\infty} \frac{\varepsilon^n}{n!} \boldsymbol{\Gamma}^n \boldsymbol{\upsilon}, \end{aligned}$$

and $\boldsymbol{\Gamma}^n = \boldsymbol{\Gamma}\boldsymbol{\Gamma}^{n-1}$ [10, 16].

Furthermore, considering a one-parameter Lie group of transformation described by Eq. (2.3) and its corresponding infinitesimal generator Eq. (2.13), the extension of Theorem 2.2.2 for any analytic function $\mathcal{V}$ is provided by the generalization [10, 16], which can be expressed as

$$\mathcal{V}(\widetilde{\boldsymbol{\upsilon}}) = \mathcal{V}\left(e^{\varepsilon \boldsymbol{\Gamma}} \boldsymbol{\upsilon}\right) = e^{\varepsilon \boldsymbol{\Gamma}} \mathcal{V}(\boldsymbol{\upsilon}).$$

2.3 Prolongation of Lie Group of Point Transformations and Their Symmetry Generators

The expression of Lie's first fundamental theorem in Eq. (2.6) for an ODE involving one dependent and one independent variable is expressed as

$$\alpha(t,w) = \frac{\partial \tilde{t}}{\partial \varepsilon}(t,w,\varepsilon)\bigg|_{\varepsilon=0}, \qquad \beta(t,w) = \frac{\partial \widetilde{w}}{\partial \varepsilon}(t,w,\varepsilon)\bigg|_{\varepsilon=0}, \tag{2.14}$$

respectively. Now if we intend to employ Eq. (2.2) to an ordinary differential equation [7]

$$F\left(t, w, \dot{w}, \ddot{w}, ..., w^{(n)}\right) = 0, \tag{2.15}$$

where dot represents derivative with respect to t, then the initial step involves extending the point transformation to include the m^{th} order derivative of $w^{(n)}$, $n = 1, 2, ..., m$. Consequently, through a recursive relation, we obtain

$$\widetilde{w}^{(n)} \equiv \frac{D_t \widetilde{w}^{(n-1)}}{D_t \tilde{t}}, \tag{2.16}$$

using $\widetilde{w}^{(0)} \equiv \widetilde{w}$, and D_t representing the total derivative with respect to t provided by

$$D_t = \frac{\partial}{\partial t} + \dot{w}\frac{\partial}{\partial w} + \ddot{w}\frac{\partial}{\partial \dot{w}} + \cdots .$$

As a result, we can express it as follows

$$\tilde{t} = t + \varepsilon\alpha\,(t,w) + \cdots = t + \varepsilon\mathbf{\Gamma} t + \cdots , \tag{2.17}$$

$$\widetilde{w} = w + \varepsilon\beta(t,w) + \cdots = w + \varepsilon\mathbf{\Gamma} w + \cdots , \tag{2.18}$$

$$\widetilde{\dot{w}} = \dot{w} + \varepsilon\dot{\beta}(t,w) + \cdots = \dot{w} + \varepsilon\mathbf{\Gamma}\dot{w} + \cdots , \tag{2.19}$$

$$\vdots$$

$$\widetilde{w}^{(m)} = w^{(m)} + \varepsilon\beta^{(m)}(t,w) + \cdots = w^{(m)} + \varepsilon\mathbf{\Gamma} w^{(m)} + \cdots , \tag{2.20}$$

where β, $\dot{\beta}$, $\ddot{\beta}$,$\cdots$, $\beta^{(m)}$ are defined by

$$\beta = \frac{d\widetilde{w}}{d\varepsilon}, \qquad \dot{\beta} = \frac{d\widetilde{\dot{w}}}{d\varepsilon}, \qquad \ddot{\beta} = \frac{d\widetilde{\ddot{w}}}{d\varepsilon}, \cdots, \beta^{(m)} = \frac{d\widetilde{w}^{m}}{d\varepsilon}, \quad \text{at } \varepsilon = 0. \tag{2.21}$$

Now, upon comparing Eqs. (2.16) and (2.20), we can deduce that

$$\widetilde{w}^{(m)} = w^{(m)} + \varepsilon \left(D_t \beta^{m-1} - w^{(m)} D_t \alpha \right), \tag{2.22}$$

with $\beta^{(0)} \equiv \beta$.

In addition, the computation of β, $\dot{\beta}$, $\ddot{\beta}$, and other successive derivatives up to $\beta^{(m)}$ can be achieved by

$$\beta^{(s)} = D_t \beta^{s-1} - w^{(s)} D_t \alpha. \tag{2.23}$$

Likewise, the extension of generator $\mathbf{\Gamma}$ can be obtained using Eqs. (2.17)-(2.20)

$$\mathbf{\Gamma}^{(s)} = \alpha \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial w} + \dot{\beta} \frac{\partial}{\partial \dot{w}} + \cdots + \beta^{(s)} \frac{\partial}{\partial w^{(s)}}. \tag{2.24}$$

2.4 Multi-Parameter Lie Group of Point Transformation and Their Infinitesimal Symmetry Generators

Lie groups of point transformations with multiple-parameters involve transformations along with more than one parameter, and their infinitesimal generators are vector fields representing small changes induced by each parameter. This section focuses on extending the notion of one-parameter Lie groups of point transformations to Lie groups with r-parameters [10, 16]. Now considering the transformation

$$\widetilde{\boldsymbol{\upsilon}} = \boldsymbol{\varphi}(\boldsymbol{\upsilon}, \boldsymbol{\varepsilon}),$$

where $\widetilde{\boldsymbol{\upsilon}} = (\widetilde{\upsilon_1}, \widetilde{\upsilon_2}, \cdots, \widetilde{\upsilon_n})$, and $\boldsymbol{\upsilon} = (\upsilon_1, \upsilon_2, \cdots, \upsilon_n)$ belong to the region $\mathcal{D} \subset \mathbb{R}^{\mathbf{n}}$ with $\boldsymbol{\varphi} = (\varphi_1, \varphi_2, \cdots, \varphi_n)$. It relies on more then one parameter, i.e., r-parameters ε_N, that is $\boldsymbol{\varepsilon} = (\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_r) \in \mathbf{V} \subset \mathbb{R}^{\mathbf{n}}$ satisfying all the properties of a group. The group operation is defined as $\mathbf{\Phi}\,(\boldsymbol{\varepsilon}, \boldsymbol{\nu})$. The Lie group of transformation, which depends on the r-parameter is given by

$$\widetilde{\boldsymbol{\upsilon}} = \boldsymbol{\varphi}\left(\boldsymbol{\upsilon}, \boldsymbol{\varepsilon}\right) = \prod_{N=1}^{r} \exp\left(\varepsilon_N \mathbf{\Gamma}_N\right) \boldsymbol{\upsilon}. \tag{2.25}$$

Additionally, the corresponding general infinitesimal transformation [7] for a system with one dependent and one independent variable, as described in Eq. (2.3) can be stated in the form

$$\mathbf{\Gamma}_N = \alpha_N(t,w)\frac{\partial}{\partial t} + \beta_N(t,w)\frac{\partial}{\partial w}, \tag{2.26}$$

with

$$\alpha_N(t,w) = \left.\frac{\partial \tilde{t}}{\partial \varepsilon_N}\right|_{\varepsilon=0}, \quad \text{and} \quad \beta_N(t,w) = \left.\frac{\partial \widetilde{w}}{\partial \varepsilon_N}\right|_{\varepsilon=0}. \tag{2.27}$$

In case of r-parameter group, the vector $\boldsymbol{\alpha}\,(\boldsymbol{\upsilon})$ takes the form of a matrix $\alpha_{Nj}(\mathbf{\Gamma})$, where $\varepsilon = 1,2,\cdots,r$ and $j = 1,2,\cdots,n$. Then, the associated generator $\mathbf{\Gamma}_N$ in accordance with the parameter ε_N of the r-parameter Lie group of transformation is defined as

$$\mathbf{\Gamma}_N = \sum_{j=1}^{n} \alpha_{Nj}(\boldsymbol{\upsilon})\frac{\partial}{\partial \upsilon^j}, \ \ N = 1,2,\cdots,r. \tag{2.28}$$

2.5 Lie Algebra of Infinitesimal Symmetry Generators

The Lie algebra of infinitesimal generators is a crucial concept linking continuous group elements and the algebraic structure of Lie groups. This section begins by presenting the definition of a Lie algebra, which serves as an algebraic structure [10].

Definition 2.5.1. *Consider a vector space $\mathcal{K}$ over a field F, where a commutator product $[\,,\,]$ is defined. In such case, $\mathcal{K}$ is known as a Lie algebra if it meets the following conditions.*

i. $[\mathbf{\Gamma}_u, \mathbf{\Gamma}_v] \in \mathcal{K}$, $\forall\ \mathbf{\Gamma}_u, \mathbf{\Gamma}_v \in \mathcal{K}$.

ii. $[\mathbf{\Gamma}_u, \mathbf{\Gamma}_v] = -[\mathbf{\Gamma}_v, \mathbf{\Gamma}_u]$, $\forall\ \mathbf{\Gamma}_u, \mathbf{\Gamma}_v \in \mathcal{K}$.

iii. $[\mathbf{\Gamma}_u, a_1\mathbf{\Gamma}_v + a_2\mathbf{\Gamma}_s] = [\mathbf{\Gamma}_u, a_1\mathbf{\Gamma}_v] + [\mathbf{\Gamma}_u, a_2\mathbf{\Gamma}_s]$, $\quad \forall\ \mathbf{\Gamma}_u, \mathbf{\Gamma}_v, \mathbf{\Gamma}_s \in \mathcal{K}$ *and for all* $a_1, a_2 \in F$.

iv. $[\mathbf{\Gamma}_u, [\mathbf{\Gamma}_v, \mathbf{\Gamma}_s]] + [\mathbf{\Gamma}_s, [\mathbf{\Gamma}_u, \mathbf{\Gamma}_v]] + [\mathbf{\Gamma}_v, [\mathbf{\Gamma}_s, \mathbf{\Gamma}_u]] = 0$, $\forall\ \mathbf{\Gamma}_u, \mathbf{\Gamma}_v, \mathbf{\Gamma}_s \in \mathcal{K}$.

As a consequence of the second property it follows that $[\mathbf{\Gamma}_u, \mathbf{\Gamma}_v] = 0$, as a result of this property, it yields the following definition of an abelian Lie algebra [7].

Definition 2.5.2. *A Lie algebra $\mathcal{K}$ is known as abelian if and only if for all $\mathbf{\Gamma}_u, \mathbf{\Gamma}_v \in \mathcal{K}$, the following condition holds*

$$[\mathbf{\Gamma}_u, \mathbf{\Gamma}_v] = 0.$$

The definition of commutators for two generators, $\mathbf{\Gamma}_u$ and $\mathbf{\Gamma}_v$ is as follows

$$[\mathbf{\Gamma}_u, \mathbf{\Gamma}_v] = \mathbf{\Gamma}_u \mathbf{\Gamma}_v - \mathbf{\Gamma}_v \mathbf{\Gamma}_u. \tag{2.29}$$

As Eq. (2.29) satisfies all the properties of a Lie algebra, it implies that the set of all $\{\mathbf{\Gamma}_u\}$, combined with the commutator, forms a Lie algebra within the group. The subsequent two theorems illustrate how a Lie algebra can be represented as a linear combination of r basic generators, commonly referred to as **Lie's second and third fundamental theorems** [16], respectively.

Theorem 2.5.1. *Consider $\mathbf{\Gamma}_u$ and $\mathbf{\Gamma}_v$ as any two infinitesimal generators within an r-parameter Lie group of point transformations. In such a case, the commutator $[\mathbf{\Gamma}_u, \mathbf{\Gamma}_v]$ is again an infinitesimal generator*

$$[\mathbf{\Gamma}_u, \mathbf{\Gamma}_v] = \sum_{e=1}^{r} C_{uv}^{e} \mathbf{\Gamma}_e, \tag{2.30}$$

where the coefficients C_{uv}^{e}, $u,v=1,2,\cdots,r$ are called structure constants.

Theorem 2.5.2. *According to Eq. (2.30), the structure constants demonstrate following two key properties.*

i. The lower two indices of the structure constants exhibit antisymmetry.

$$C_{uv}^{e} = -C_{vu}^{e}.$$

ii. The structure constants are required to adhere to Lie's identity, that is

$$\sum_{f=1}^{r} \left[C_{uv}^{f} C_{fg}^{h} + C_{vg}^{f} C_{fu}^{h} + C_{gu}^{f} C_{fv}^{h} \right] = 0.$$

2.6 Symmetry Condition for Ordinary Differential Equations

Having laid down the foundational mathematical concepts, we are now in a position to present a fundamental theorem that allows us to ascertain the Lie point symmetries of a DE.

Theorem 2.6.1. *An ordinary differential equation*

$$F\left(t, w, \dot{w}, \ddot{w}, \cdots, w^{(n)}\right) = 0.$$

A diffeomorphism that transforms the solution set of the ODE into itself is a symmetry. Any diffeomorphism,

$$\Gamma : (t, w) \to (\tilde{t}, \widetilde{w}),$$

connects smooth planar curves to one another. To address the influence of Lie symmetries on nth-order derivatives, we introduce the extended infinitesimal generator.

$$\Gamma^{(n)} = \alpha\partial_t + \beta\partial_w + \beta^{(1)}\partial_{\dot{w}} + \cdots + \beta^{(n)}\partial_{w^{(n)}}.$$

An ODE possesses a group of symmetries represented by generator Γ *if and only if*

$$\mathbf{\Gamma}^{(n)}F\big|_{F=0} = 0, \tag{2.31}$$

holds [7]*.*

2.7 Lie Point Symmetries of Partial Differential Equations

Taking into account the system of non-linear PDEs of pth order, involving both R-independent and S-dependent variables as

$$F_m\left(\mathbf{t}, \mathbf{w}, \mathbf{w^{(1)}}, \mathbf{w^{(2)}}, \cdots, \mathbf{w^{(p)}}\right) = 0, \qquad m = 1, 2, 3, \cdots, l, \tag{2.32}$$

where $\mathbf{t} = \left(t^1, t^2, \cdots, t^R\right) \in \mathbf{T} \subset \mathbb{R}^{\mathbf{R}}$ and $\mathbf{w} = \left(w^1, w^2, \cdots, w^S\right) \in \mathbf{W} \subset \mathbb{R}^{\mathbf{S}}$ are the associated R-independent and S-dependent variables [7]. Furthermore, w^p signifies all the nth order partial derivatives of w with respect to $\mathbf{t}$ with the corresponding coordinate for $w^{(n)}$ is $\frac{\partial^p w}{(\partial t^{r_1} \partial t^{r_2} \cdots \partial t^{r_n})}$ given by $w^p_{r_1 r_2 \cdots r_j}$, $r = 1, 2, 3, \cdots, R$ for $n = 1, 2, 3, \cdots, S$. Considering the coordinates $\mathbf{t}, \mathbf{w^1}, \mathbf{w^2}, \cdots, \mathbf{w^p}$, Eq. (2.32) can be expressed as an algebraic equation that represents a hypersurface in $(\mathbf{t}, \mathbf{w}, \mathbf{w^1}, \mathbf{w^2}, \cdots, \mathbf{w^p})$-space.

The point transformation represented by Eq. (2.1) is applied to the independent variables $\tilde{t}^r$, $r = 1, 2, 3, \cdots, R$ and the dependent variables $\widetilde{w}^s$, $s = 1, 2, 3, \cdots, S$ of the p^{th} order system of PDE is

$$\tilde{t}^r = \tilde{t}^r \left(t^f, w^g\right), \qquad \widetilde{w}^s = \widetilde{w}^s \left(t^f, w^g\right), \tag{2.33}$$

where $f, r = 1, 2, 3, \cdots, R$, and $g, s = 1, 2, 3, \cdots, S$. Similarly, for any specific parameter denoted as $\varepsilon \in \mathbf{V} \subset \mathbb{R}$, Eq. (2.33) can be expressed as follows

$$\tilde{t}^r = \tilde{t}^r \left(t^f, w^g; \varepsilon\right), \qquad \widetilde{w}^s = \widetilde{w}^s \left(t^r, w^g; \varepsilon\right). \tag{2.34}$$

Then the infinitesimal generator of the one-parameter Lie group of point transformations can be described as follows

$$\mathbf{\Gamma} = \alpha^r \left(t^f, w^g\right) \frac{\partial}{\partial t^r} + \beta^s \left(t^f, w^g\right) \frac{\partial}{\partial w^s}. \tag{2.35}$$

The corresponding infinitesimal transformation can be described as

$$\alpha^r \equiv \left.\frac{\partial \tilde{t}^r}{\partial \varepsilon}\right|_{\varepsilon=0}, \qquad \beta^s \equiv \left.\frac{\partial \widetilde{w}^s}{\partial \varepsilon}\right|_{\varepsilon=0}. \tag{2.36}$$

Furthermore, the extension of the infinitesimal generator, as presented in Eq. (2.35) for arbitrary order derivatives [7], can be expressed as

$$\mathbf{\Gamma} = \alpha^r \frac{\partial}{\partial t^r} + \beta^s \frac{\partial}{\partial w^s} + \beta^s_r \frac{\partial}{\partial w^s_r} + \beta^s_{rc} \frac{\partial}{\partial w^s_{rc}} + \beta^s_{rcd} \frac{\partial}{\partial w^s_{rcd}} + \cdots, \tag{2.37}$$

where

$$\beta^s_r = \frac{D\beta^s}{Dt^r} - w^s_f \frac{D\alpha^f}{Dt^r}, \tag{2.38}$$

$$\beta^s_{rc} = \frac{D\beta^s_r}{Dt^c} - w^s_{rf} \frac{D\alpha^f}{Dt^c}, \tag{2.39}$$

with the total derivative $\frac{D}{Dt^r}$ can be define as

$$\frac{D}{Dt^r} = \frac{\partial}{\partial t^r} + w^s_r \frac{\partial}{\partial w^s} + w^s_{rc} \frac{\partial}{\partial w^s_c} + \cdots. \tag{2.40}$$

The subsequent theorem represents the symmetry condition applicable to a partial differential equation [7].

Theorem 2.7.1. *Let*

$$\mathbf{\Gamma}^{(P)} = \alpha^r(t,w)\frac{\partial}{\partial t^r} + \beta(t,w)\frac{\partial}{\partial w} + \beta_r^{(1)}\left(t,w,w^{(1)}\right)\frac{\partial}{\partial w^1} + \cdots$$
$$\cdots + \beta^{(P)}_{r_1,r_2,\cdots,r_j}\left(t,w,w^{(1)},w^{(2)},\cdots,w^{(P)}\right)\frac{\partial}{\partial w^{r_1,r_2,\cdots,r_n}}, \tag{2.41}$$

be the P^{th} order prolonged infinitesimal generator Eq. (2.35) *of the corresponding one-parameter Lie group of transformation*

$$\tilde{t} = T\left(t,w;\varepsilon\right), \tag{2.42}$$
$$\widetilde{w} = W\left(t,w;\varepsilon\right), \tag{2.43}$$

with

$$\beta_r^1 = D_r\beta - \left(D_r\alpha_r\right)w_n, \quad r = 1,2,3,\cdots,R, \tag{2.44}$$
$$\beta^n_{r_1,r_2,\cdots,r_P} = D_{r_P}\beta^{(P-1)}_{r_1,r_2,\cdots,r_{P-1}} - \left(D_{r_P}\alpha_n\right)w_{r_1,r_2,\cdots,r_{(P-1)}n}, \tag{2.45}$$

where $r_n = 1,2,3,\cdots,R$ for $n = 1,2,3,\cdots,P$ with $P = 1,2,3,\cdots$. Then a partial differential Eq. (2.32) *admits one-parameter Lie group of transformations Eqs.* (2.41)-(2.43) *iff*

$$\mathbf{\Gamma}^{(P)}F\left(\boldsymbol{t},w,w^{(1)},w^{(2)},\cdots,w^{(P)}\right)\big|_{F=0} = 0, \tag{2.46}$$

holds.

Specifically, when dealing with two independent variables (t,y) and one dependent variable w, the expression for Eq. (2.41) with $P = 2$ can be represented as

$$\begin{aligned}\mathbf{\Gamma}^{(2)} = {} & \alpha(t,y,w)\frac{\partial}{\partial t} + \zeta(t,y,w)\frac{\partial}{\partial y} + \beta(t,y,w)\frac{\partial}{\partial w} + \beta_t\left(t,y,w,w_t\right)\frac{\partial}{\partial w_t} \\ & + \beta_y\left(t,y,w,w_t,w_y\right)\frac{\partial}{\partial w_y} + \beta_{tt}\left(t,y,w,w_t,w_y,w_{tt}\right)\frac{\partial}{\partial w_{tt}} \\ & + \beta_{ty}\left(t,y,w,w_t,w_y,w_{tt},w_{ty}\right)\frac{\partial}{\partial w_{ty}} + \beta_{yy}\left(t,y,w,w_t,w_y,w_{tt},w_{ty},w_{yy}\right)\frac{\partial}{\partial w_{yy}}.\end{aligned}$$

Eqs. (2.44) and (2.45) give

$$\begin{aligned}\beta_t &= D_t\left(\beta\right) - w_y D_t(\zeta) - w_t D_t\left(\alpha\right),\\ &= \beta_t + \left(\beta_w - \alpha_t\right) w_t - \zeta_t w_y - \alpha_w \left(w_t\right)^2 - \zeta_w w_y w_t. \end{aligned}\tag{2.47}$$

$$\begin{aligned}\beta_y &= D_y\left(\beta\right) - w_y D_y(\zeta) - w_y D_y\left(\alpha\right),\\ &= \beta_y + \left(\beta_w - \zeta_y\right) w_y - \alpha_y w_t - \zeta_w \left(w_y\right)^2 - \alpha_w w_y w_t. \end{aligned}\tag{2.48}$$

$$\begin{aligned}\beta_{tt} &= D_t\left(\beta_t\right) - w_{yt} D_t(\zeta) - w_{tt} D_t\left(\alpha\right),\\ &= \beta_{tt} + \left(2\beta_{tw} - \alpha_{tt}\right) w_t - \zeta_{tt} w_y + \left(\beta_w - 2\alpha_t\right) w_{tt} - 2\zeta_t w_{yt}\\ &\quad + \left(\beta_{ww} - 2\alpha_{tw}\right)\left(w_t\right)^2 - 2\zeta_{tw} w_y w_t - \alpha_{ww}\left(w_t\right)^3 - \zeta_{ww} w_y \left(w_t\right)^2\\ &\quad - 3\alpha_w w_t w_{tt} - \zeta_w w_y w_{tt} - 2\zeta_w w_t w_{yt}. \end{aligned}\tag{2.49}$$

$$\begin{aligned}\beta_{ty} &= D_t\left(\beta_y\right) - w_{yy} D_t(\zeta) - w_{yt} D_t\left(\alpha\right),\\ &= \beta_{yt} + \left(2\beta_{yw} - \alpha_{yt}\right) w_t + \left(\beta_{wt} - \zeta_{yt}\right) w_y + \left(\beta_{ww} - \zeta_{yw} - \alpha_{wt}\right) w_y w_t\\ &\quad + \left(\beta_w - \zeta_y - \alpha_t\right) w_{yt} - \zeta_{wt}\left(w_y\right)^2 - \zeta_{ww} w_t \left(w_y\right)^2 - \alpha_w w_y w_{tt} - \alpha_{yw}\left(w_t\right)^2\\ &\quad - \alpha_y w_{tt} - \zeta_t w_{yy} - \alpha_{ww} w_y \left(w_t\right)^2 - 2\alpha_w w_t w_{yt} - \zeta_w w_t w_{ty} - 2\zeta_w w_y w_{ty}. \end{aligned}\tag{2.50}$$

$$\begin{aligned}\beta_{yy} &= D_y\left(\beta_y\right) - w_{yy} D_y(\zeta) - w_{yt} D_y\left(\alpha\right),\\ &= \beta_{yy} + \left(2\beta_{yw} - \zeta_{yy}\right) w_y + \left(\beta_{ww} - 2\zeta_{yw}\right)\left(w_y\right)^2 - \zeta_{ww}\left(w_y\right)^3 - 3\zeta_w w_y w_{yy}\\ &\quad - \alpha_{yy} w_t - 2\alpha_{wy} w_y w_t - 2\alpha_y w_{yt} - \alpha_{ww} w_t \left(w_y\right)^2 - \alpha_w w_t w_{yy} - 2\alpha_w w_y w_{ty}. \end{aligned}\tag{2.51}$$

Where

$$D_t = \frac{\partial}{\partial t} + w_t \frac{\partial}{\partial w} + w_{ty}\frac{\partial}{\partial w_y} + w_{tt}\frac{\partial}{\partial w_t} + \cdots,$$

and,

$$D_y = \frac{\partial}{\partial y} + w_y \frac{\partial}{\partial w} + w_{yt}\frac{\partial}{\partial w_t} + w_{yy}\frac{\partial}{\partial w_y} + \cdots.$$

In general, when considering one independent variable t and one dependent variable w, the symmetry condition described in Eq. (2.46) results in a non-linear partial differential equation involving the functions $(\alpha(t,w), \beta(t,w))$. By examining the coefficients of the derivatives of w, this equation gives rise to a system of PDEs. For every infinitesimal generator, a solution to the system of PDEs can be obtained by expressing it in terms of α and β. This process results in the formation of a Lie algebra.

Now, let's provide an illustration to understand the process.

Example 2.7.1. *Consider the Pavlov equation* [28]

$$w_{vv} = w_{tu} + w_v w_{uu} - w_u w_{uv}, \tag{2.52}$$

Pavlov equation has 3 independent variables and 1 dependent variable, where $w(u,v,t)$ represents the magnitude of the relevant wave, which varies based on the spatial factors u, v and time t.

The Pavlov equation is nonlinear PDE, where α^u, α^v, α^t and β are the infinitesimals. Hence the associated vector field is

$$\Gamma = \alpha^u \frac{\partial}{\partial u} + \alpha^v \frac{\partial}{\partial v} + \alpha^t \frac{\partial}{\partial t} + \beta \frac{\partial}{\partial w}.$$

Now, the first prolongation is

$$\Gamma^1 = \Gamma + \beta^u \frac{\partial}{\partial w_u} + \beta^v \frac{\partial}{\partial w_v} + \beta^t \frac{\partial}{\partial w_t}.$$

As Eq. (2.52) is a second-order equation, we use the following second-order prolongation which is required for this infinitesimal operators

$$\Gamma^2 = \Gamma + \beta^u \frac{\partial}{\partial w_u} + \beta^v \frac{\partial}{\partial w_v} + \beta^t \frac{\partial}{\partial w_t} + \beta^{uu} \frac{\partial}{\partial w_{uu}} + \beta^{ut} \frac{\partial}{\partial w_{ut}} + \beta^{uv} \frac{\partial}{\partial w_{uv}} + \beta^{vv} \frac{\partial}{\partial w_{vv}}$$

with the coefficients given in Eqs. (2.47)-(2.52). To employ the condition of Lie point symmetry for partial differential equations, let us take into account the following

$$P = w_{vv} - w_{tu} - w_v w_{uu} + w_u w_{uv} \tag{2.53}$$

then by Theorem (2.7.1), we have

$$\Gamma^2 P\big|_{P=0} = 0. \tag{2.54}$$

Eq. (2.54) yields

$$\beta^{vv} = \beta^{tu} + \beta^{v} w_{uu} + w_{v} \beta^{uu} - \beta^{u} w_{uv} - w_{u} \beta^{uv}. \tag{2.55}$$

With coefficients

$$\begin{aligned}
\beta^{u} &= D_{u}\left(\beta\right) - w_{u} D_{u}(\xi^{u}) - w_{v} D_{u}\left(\xi^{v}\right) - w_{t} D_{u}\left(\alpha^{t}\right), \\
\beta^{v} &= D_{v}\left(\beta\right) - w_{u} D_{v}(\xi^{u}) - w_{v} D_{v}\left(\xi^{v}\right) - w_{t} D_{v}\left(\alpha^{t}\right), \\
\beta^{uu} &= D_{u}\left(\beta^{u}\right) - w_{uu} D_{u}(\xi^{u}) - w_{uv} D_{u}\left(\xi^{v}\right) - w_{ut} D_{u}\left(\alpha^{t}\right), \\
\beta^{uv} &= D_{v}\left(\beta^{u}\right) - w_{uu} D_{v}(\xi^{u}) - w_{uv} D_{v}\left(\xi^{v}\right) - w_{ut} D_{v}\left(\alpha^{t}\right), \\
\beta^{ut} &= D_{t}\left(\beta^{u}\right) - w_{uu} D_{t}(\xi^{u}) - w_{uv} D_{t}\left(\xi^{v}\right) - w_{ut} D_{t}\left(\alpha^{t}\right).
\end{aligned}$$

Now, substituting the respective values of β^{vv}, β^{tu}, β^{v}, β^{uu}, β^{u} and β^{uv} and comparing the powers of dependent variable w and its partial derivatives, we obtain the following determining equations

$$\beta_{tu} = \alpha^{v}_{tt}, \tag{2.56}$$

$$\beta_{uu} = 0, \tag{2.57}$$

$$\beta_{uv} = \alpha^{t}_{tt}, \tag{2.58}$$

$$\beta_{vv} = \alpha^{v}_{tt}, \tag{2.59}$$

$$\alpha^{v}_{tv} = \alpha^{t}_{tt}, \tag{2.60}$$

$$\alpha^{v}_{vv} = 0, \tag{2.61}$$

$$\beta_{w} = -2\alpha^{t}_{t} + 3\alpha^{v}_{v}, \tag{2.62}$$

$$\alpha^{t}_{w} = 0, \tag{2.63}$$

$$\alpha^{t}_{u} = 0, \tag{2.64}$$

$$\alpha^{t}_{v} = 0, \tag{2.65}$$

$$\alpha^{u}_{t} = \beta_{v}, \tag{2.66}$$

$$\alpha^{u}_{w} = 0, \tag{2.67}$$

$$\alpha^{u}_{u} = -\alpha^{t}_{t} + 2\alpha^{v}_{v}, \tag{2.68}$$

$$\alpha^{u}_{v} = \frac{1}{2}\beta_{u} + \frac{1}{2}\alpha^{v}_{t}, \tag{2.69}$$

$$\alpha^v_w = 0, \tag{2.70}$$

$$\alpha^v_u = 0. \tag{2.71}$$

As a result, the solutions to the aforementioned equations lead to the following infinitesimals:

$$\alpha^u = (\dot{g}_1(t) + 2c_1)u + \frac{1}{2}\ddot{g}_1(t)v^2 + (\dot{g}_2(t) + \frac{1}{2}c_2)v + g_3(t) + c_3, \tag{2.72}$$

$$\alpha^v = (\dot{g}_1(t) + c_1)v + g_2(t), \tag{2.73}$$

$$\alpha^t = g_1(t), \tag{2.74}$$

$$\beta = (\dot{g}_1(t) + 3c_1)w + (\ddot{g}_1(t)v + \dot{g}_2(t) + c_2)u + \frac{1}{6}\dddot{g}_1(t)v^3 + \frac{1}{2}\ddot{g}_2(t)v^2 + \dot{g}_3(t)v + g_4(t). \tag{2.75}$$

The infinitesimals undergo a transformation when considering $g_1(t) = c_4 + tc_6$, $g_2(t) = c_5$, $g_3(t) = 0$, and $g_4(t) = c_7$

$$\alpha^u = c_6 u + 2c_1 u + \frac{1}{2}vc_2 + c_3, \tag{2.76}$$

$$\alpha^v = c_6 v + c_1 v + c_5, \tag{2.77}$$

$$\alpha^t = c_4 + tc_6, \tag{2.78}$$

$$\beta = (c_6 + 3c_1 w) + c_2 u + c_7. \tag{2.79}$$

Hence, the Pavlov equation possesses a seven-dimensional Lie algebra, which is generated by the following vector fields

$$\mathbf{\Gamma}_1 = \frac{\partial}{\partial u}, \tag{2.80}$$

$$\mathbf{\Gamma}_2 = \frac{\partial}{\partial v}, \tag{2.81}$$

$$\mathbf{\Gamma}_3 = \frac{\partial}{\partial t}, \tag{2.82}$$

$$\mathbf{\Gamma}_4 = \frac{\partial}{\partial w}, \tag{2.83}$$

$$\mathbf{\Gamma}_5 = \frac{1}{2}v\frac{\partial}{\partial u} + u\frac{\partial}{\partial w}, \tag{2.84}$$

$$\mathbf{\Gamma}_6 = 2u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + 3w\frac{\partial}{\partial w}, \tag{2.85}$$

$$\mathbf{\Gamma}_7 = u\frac{\partial}{\partial u} + v\frac{\partial}{\partial v} + t\frac{\partial}{\partial t} + w\frac{\partial}{\partial w}. \tag{2.86}$$

The commutator relations among these vector fields are provided in Table 2.1,

	Γ_1	Γ_2	Γ_3	Γ_4	Γ_5	Γ_6	Γ_7
Γ_1	0	0	0	0	Γ_4	$2\Gamma_1$	Γ_1
Γ_2	0	0	0	0	$\frac{1}{2}\Gamma_1$	Γ_2	Γ_2
Γ_3	0	0	0	0	0	0	Γ_3
Γ_4	0	0	0	0	0	$3\Gamma_4$	Γ_4
Γ_5	-Γ_4	-$\frac{1}{2}\Gamma_1$	0	0	0	Γ_5	0
Γ_6	-$2\Gamma_1$	-Γ_2	0	-$3\Gamma_4$	$-\Gamma_5$	0	0
Γ_7	-Γ_1	-Γ_2	-Γ_3	-Γ_4	0	0	0

Table 2.1: Commutator Table for Symmetries of Pavlov Equation

Chapter 3

Optimal Systems of Differential Equations

The primary emphasis of this chapter is to offer a thorough presentation of the core principles related to optimal systems governed by differential equations. The study of optimal systems has evolved as a powerful and adaptable tool for optimizing and controlling complex systems within a wide range of differential equations. Our study begins with an examination of the fundamental principles and the historical background surrounding optimal systems, laying the basis for the subsequent chapters of this thesis.

Sophus Lie developed an extremely useful approach for finding solutions that is applicable to all types of DEs. This approach is based on the set of transformations inherent in a given DE. Every set of transformations corresponds to a distinct set of solutions that remain invariant under these transformations. An infinite number of such groups can be constructed by examining the group of transformations of a DE, leading to an infinite number of group invariant solutions. These invariant solutions can be classified into equivalence classes. A collection that includes precisely one generator from each equivalence class is referred to as an optimal set of generators. In other words, it represents group invariant solutions. These solutions serve as the foundational basis from which all other solutions can be derived [10]. Several techniques for obtaining optimal systems are available in the literature [29–41].

Our main goal is to understand the structure of optimal system of algebra. Before looking into the exact mathematical definition of an optimal system, it's essential to provide an overview of crucial findings and key terminologies.

3.1 Optimal Systems and Group-Invarient Solutions

By applying the proposition below, the arrangement of subgroups within the group $\mathcal{G}$ of symmetries through the binary operation of conjugation could lead to the group-invariant solutions.
Therefore, the mapping $h \mapsto ghg^{-1}$ within a Lie group carries substantial significance.

Proposition 1. *Let $\mathcal{G}$ be the symmetry group of a system of DEs* (2.15)*, and let $\mathcal{H}$ be an r-parameter subgroup of $\mathcal{G}$. If there exists a solution $w = \mathfrak{f}(t)$ that is invariant under the subgroup $\mathcal{H}$ in Eq.* (2.15)*, then the transformed function $w = \tilde{\mathfrak{f}}(t) = g.\mathfrak{f}(t)$, where g is a member of the group $\mathcal{G}$ and $\widetilde{\mathcal{H}} = g\mathcal{H}g^{-1}$ represents the subgroup conjugate to $\mathcal{H}$ under $\mathcal{G}$. The transformed function $\tilde{\mathfrak{f}}(t)$ is also an $\widetilde{\mathcal{H}}$-invariant solution* [10].

3.1.1 Adjoint Representation

Given knowledge of the adjoint action Ad $\mathcal{L}$ within the Lie group $\mathcal{G}$, reconstructing the adjoint representation of the underlying group, Ad $\mathcal{G}$, becomes possible.

Definition 3.1.1. *Consider a Lie group $\mathcal{G}$ and its associated Lie algebra $\mathcal{L}$. The adjoint representation of a group $\mathcal{G}$ is differential related to the conjugtation of the group or the conjugacy mapping* [10]

$$\mathcal{O}_g : \mathsf{k} \mapsto g\mathsf{k}g^{-1};\ \mathsf{k}, g \in \mathcal{G}. \tag{3.1}$$

This representation operates on the group $\mathcal{G}$ itself.

$$Ad_g(\Gamma) \equiv d\mathcal{O}_g(v);\ \forall g \in \mathcal{G},\ \Gamma \in \mathcal{L}. \tag{3.2}$$

Remark 1. *If $\Gamma \in \mathcal{L}$ produces a single-parameter subgroup $\mathcal{H} = e^{\varepsilon\Gamma}$, where $\varepsilon \in \mathbb{R}$, then the element $Ad_g(\Gamma)$ forms the single-parameter subgroup obtained by conjugating $\mathcal{H}$ by g. This is expressed as $\mathcal{O}_g(\mathcal{H}) = g\mathcal{H}g^{-1}$.*

In a systematic context, the adjoint representation can be described as a function that maps the group $\mathcal{G}$ to the set of linear operators acting on the tangent space at the identity, denoted as $\mathcal{T}e(\mathcal{G})$. This mapping satisfies the property $\mathcal{O}g(hh') = \mathcal{O}g(h)\mathcal{O}g(h')$, where $\mathcal{O}_g$ denotes the adjoint representation. This homomorphism ensures the preservation of the identity element by mapping it to the identity element for each element $g \in \mathcal{G}$:

$$\mathcal{O}_g(e) = geg^{-1} = gg^{-1} = e.$$

This characteristic implies that every curve traversing the identity element e on the manifold $\mathcal{G}$ will be transformed by this homomorphism into a distinct curve, which may not necessarily be the same as the original one. As a result, the adjoint representation transforms any tangent vector (associated with a curve on $\mathcal{G}$) within the tangent space $\mathcal{T}_e(\mathcal{G})$ into another tangent vector within the same tangent space $\mathcal{T}_e(\mathcal{G})$.

Currently, we are demonstrating that the Lie algebra's adjoint action on itself is achieved through the commutator (Lie bracket). Examine the manifold $\mathcal{G}$ and a curve $\mathfrak{c}(t)$ with $\mathfrak{c}(0) = e \in \mathcal{G}$ and the tangent vector $\mathfrak{c}'(0) = \Gamma \in \mathcal{T}e(\mathcal{G})$. Moreover, suppose that the curve traverses any element g within the set $\mathcal{G}$. By employing this curve, we can represent the adjoint action $Adg(\Gamma) = g\Gamma g^{-1}$ as:

$$Ad_g(\mathcal{Y}) = Ad_{\mathfrak{c}(t)}(\mathcal{Y}) = \mathfrak{c}(t)\Gamma\mathfrak{c}(t)^{-1}.$$

Taking the derivative of this function at the identity $t = 0$ results in

$$\begin{aligned} \frac{d}{dt}Ad_{\mathfrak{c}(t)}(\mathcal{Y})\bigg|_{t=0} &= \mathfrak{c}(t)\mathcal{Y}\mathfrak{c}(t)^{-1}\bigg|_{t=0}, \\ &= \mathfrak{c}(0)\mathcal{Y}\mathfrak{c}(0)^{-1} + \mathfrak{c}(t)\mathcal{Y}\frac{d}{dt}\mathfrak{c}(t)^{-1}\bigg|_{t=0}. \end{aligned} \tag{3.3}$$

For the computation of $\frac{d}{dt}\mathfrak{c}(t)^{-1}\Big|_{t=0}$, we will make use of the following identity

$$\frac{d}{dt}B^{-1}(t) = -B^{-1}(t)\left(\frac{d}{dt}B(t)\right)B^{-1}(t). \tag{3.4}$$

This equation originates from

$$\frac{d}{dt}\left(B(t)B^{-1}(t)\right) = \frac{d}{dt}(e) = 0. \tag{3.5}$$

Applying the product rule and left-multiplying with $B^{-1}(t)$ results in

$$\begin{aligned} B^{-1}(t)\frac{d}{dt}(B(t))B^{-1}(t) + B^{-1}(t)B(t)\frac{d}{dt}(B^{-1}(t)) = 0, \\ \frac{d}{dt}(B^{-1}(t)) = -B^{-1}(t)\frac{d}{dt}(B(t))B^{-1}(t). \end{aligned}$$

It follows that,

$$\begin{aligned} \frac{d}{dt}Ad_{\mathfrak{c}(t)}(\mathcal{Y})\Big|_{t=0} &= \mathfrak{c}(0)\mathcal{Y}\mathfrak{c}(0)^{-1} + \mathfrak{c}(t)\mathcal{Y}\frac{d}{dt}\mathfrak{c}(t)^{-1}\Big|_{t=0}, \\ &= \mathfrak{c}(0)\mathcal{Y}\mathfrak{c}(0)^{-1} + \mathfrak{c}(0)\mathcal{Y}(-\mathfrak{c}(0)^{-1}\mathfrak{c}'(0)\mathfrak{c}(0)^{-1}), \\ &= \Gamma\mathcal{Y}e + e\mathcal{Y}(-e\Gamma e), \\ &= [\Gamma, \mathcal{Y}] = ad\mathcal{Y}\big|_{\mathfrak{c}(t)}. \end{aligned}$$

3.2 Adjoint Representation Table.

If details about the adjoint action $ad\mathcal{L}$ concerning a Lie algebra $\mathcal{L}$ acting on itself are available, the reconstruction of the adjoint representation $Ad\mathcal{G}$ pertaining to the underlying Lie group can be accomplished, either through the integration of a system of linear ordinary differential equations.

$$\frac{d\mathcal{Y}}{d\varepsilon} = ad\Gamma|_{\mathcal{Y}}, \ \mathcal{Y}(0) = \mathcal{Y}_0, \tag{3.6}$$

with solution $\mathcal{Y}(\varepsilon) = Ad(\exp(\varepsilon\Gamma))\mathcal{Y}_0$. Alternatively, we can define this solution by employing a Lie series in cojugation with a table of commutation relation.

$$\begin{aligned} Ad(\exp(\varepsilon\Gamma))\mathcal{Y}_0 &= \sum_{n=0}^{\infty}\frac{\varepsilon^n}{n!}(ad\Gamma)^n(\mathcal{Y}_0) \\ &= \mathcal{Y}_0 - \varepsilon[\Gamma, \mathcal{Y}_0] + \frac{\varepsilon^2}{2}[\Gamma, [\Gamma, \mathcal{Y}_0]] - \cdots, \end{aligned}$$

where $[\Gamma, \mathcal{Y}_0]$ denote the commutator operation applied to the generators Γ and $\mathcal{Y}_0$. Adjoint representation tables are employed to depict the relationships between conjugacy mappings within a specific Lie algebra. It is suitable to represent the conjugacy relationships between every subalgebra and each subsequent subalgebra presented in tabular form. In the context of an n-dimensional Lie algebra denoted as $\mathcal{L}$, the adjoint representation table can be described as an $n \times n$ matrix. In this matrix, each entry in the (i, j) position signifies the adjoint action of Γ_i on Γ_j as $Ad(\exp(\varepsilon\Gamma_i))\Gamma_j$. It's important to note that the adjoint representation associates a linear operator with a group element rather than a matrix.

The result of the following proposition emphasizes that the process of computing an optimal system for a given subgroup is intricately linked to the challenge of determining the optimal system for subalgebras.

Proposition 2. *Consider two connected, r-dimensional Lie subgroups, $\mathcal{H}_1$ and $\mathcal{H}_2$, of the Lie group $\mathcal{G}$. These subgroups are themselves Lie groups and have associated Lie subalgebras, $\mathfrak{h}_1$ and $\mathfrak{h}_2$ of the Lie algebra $\mathcal{L}$ of $\mathcal{G}$. Then $\mathcal{H}_2 = g\mathcal{H}_1 g^{-1}$ are conjugate subgroups if and only if $\mathfrak{h}_2 = Adg(\mathfrak{h}_1)$ are conjugate subalgebras* [10].

3.2.1 Optimal Systems

In the realm of a Lie group $\mathcal{G}$, the term **optimal system** of r-parameter subgroups refers to a set of r-parameter subgroups that are distinct under conjugation, and any additional subgroup is conjugate to precisely one subgroup within this designated grouping.

In a similar manner, an optimal system is formed by a group of r-parameter subalgebras if each r-parameter subalgebra within $\mathcal{L}$ can be associated with a single member from that collection under some element of the adjoint representation: $\mathfrak{h}_2 = Adg(\mathfrak{h}_1)$, $g \in \mathcal{G}$.

There are essentially two methods for generating the one-dimensional optimal system of subalgebras. One approach, as outlined by Ovsiannikov [2, 6], involves the calculation of the inner automorphism matrix that corresponds to the operators in the adjoint group associated with a specified Lie algebra. The other approach, as presented by Olver [10], involves simplifying the generator extensively to the fullest extent by applying the selected adjoint transformation to it. In this context, we choose to adopt and provide a concise overview of Olver's approach for further discussion.

3.2.2 Algorithm of 1-dimensional Optimal System

Examine the n-dimensional Lie algebra $\mathcal{L}_n$, associated with a Lie group $\mathcal{G}$ in a differential equation (or system of differential equations), generated by Γi, where $i = 1, \ldots, n$.

Step 1: Commutator Table. The commutator table, displaying the structure constants of $\mathcal{G}$, provides a comprehensive representation of the group's structure, offering a complete description of its properties and relationships, often up to isomorphisms. This is accomplished by creating a table that outlines the Lie brackets of the generators Γ_i.

Step 2: Adjoint Representation Table. The adjoint operator, also known as the adjoint transformation or Hermitian adjoint, can be described as

$$Ad(e^{\varepsilon\Gamma})\mathcal{Y} \equiv e^{-\varepsilon\Gamma}\mathcal{Y}e^{\varepsilon\Gamma}.$$

Here $\Gamma \in \mathcal{L}$ and $\mathcal{Y} \in \mathcal{K}$ ($\mathcal{K} \subset \mathcal{L}$). By using the well-known Campbell-Hausdorff theorem (as referenced in [3]), the operation can be expressed as

$$Ad(e^{\varepsilon\Gamma})\mathcal{Y} = \mathcal{Y} - \varepsilon[\Gamma, \mathcal{Y}] + \frac{\varepsilon^2}{2}[\Gamma, [\Gamma, \mathcal{Y}]] - \cdots . \tag{3.7}$$

The adjoint representation table is then employed to produce the adjoint transformations in subsequent steps.

3.3 Invariant Solutions

Once we have identified an optimal system of generators, the method of characteristics can be used to calculate the corresponding invariant solutions. An optimal system of invariant solutions refers to a complete collection of these solutions, providing a basis for deriving all other invariant solutions. Two challenges may arise: The initial challenge is that a generator in the optimal system may not necessarily lead to any invariant solutions. The second challenge lies in the potential difficulty of analytically solving the reduced equation(s) that establish one or more invariant solutions. Even when we do not achieve an optimal system, there is still a possibility of discovering certain invariant solutions [14].

Definition 3.3.1. *Consider two manifolds, M and N. Let $\mathcal{G}$ represents the local group of transformations that operate on M. A function $\mathfrak{f} : M \longrightarrow N$ is considered $\mathcal{G}$-invariant function when the subsequent equality* [10]

$$\mathfrak{f}(g.x) = \mathfrak{f}(x), \tag{3.8}$$

holds for every $x \in M$ and $g \in \mathcal{G}$. An invariant of $\mathcal{G}$ is a real-valued function that remains unchanged under $\mathcal{G}$-invariant transformations. A similar definition can be applied in situations where $N = \mathcal{R}^n$. In this scenario, $\mathfrak{f}$ is considered $\mathcal{G}$-invariant when each component $\mathfrak{f}i$ is itself an invariant of $\mathcal{G}$. Here, $\mathfrak{f} = \mathfrak{f}_1, \mathfrak{f}_2, \cdots, \mathfrak{f}_n$.

Infinitesimal Invariance. The infinitesimal criteria will be essential in identifying the symmetry group of the system of differential equations. The conditions for infinitesimal invariance [10] can be derived directly by considering the formula that describes how a function changes under the influence of the flow induced by a vector field.

3.4 Method for Constructing Invariants

Now, let's outline the systematic procedure for creating the invariants associated with a group action, providing a clear step by step approach. Initially, consider $\mathcal{G}$ as a group of transformations operating on M, where $\mathcal{G}$ is characterized by one parameter for a system of ODEs and has a generator

$$\Gamma = \alpha(t, w^i)\partial t + \beta^j(t, w^i)\partial w^j, \ (i, j = 1, \cdots, k)$$

A local invariant χ is a mathematical function that satisfies the given PDEs

$$\Gamma\chi = \alpha(t, w^i)\partial t\chi + \beta^j(t, w^i)\partial w^j\chi = 0. \tag{3.9}$$

The solution of Eq. (3.9) is derived by solving the characteristic system of ODEs,

$$\frac{dt}{\alpha(t, w^i)} = \frac{dw^1}{\beta^1(t, w^i)} = \cdots = \frac{dw^k}{\beta^k(t, w^i)}. \tag{3.10}$$

The solution to Eq. (3.10) can be expressed in a more general form as

$$\chi^1(t, w^i) = \mathcal{I}_1, \cdots, \chi^{k-1}(t, w^i) = \mathcal{I}_{k-1},$$

the constants of integration are $\mathcal{I}_1, \cdots, \mathcal{I}_{k-1}$. The functions $\chi^1, \cdots, \chi^{k-1}$ are solutions to Eq. (3.9) that are functionally independent solutions.

This approach entails identifying invariants from the prolongations of given symmetries. In simpler terms, if presented with a second-order ordinary differential equation, reducing the order of this equation by one can be achieved by deriving invariants from the first extension of the provided symmetry generator.

3.5 Review of One-dimensional Optimal System of Subalgebras

The objective of seeking the optimal set of Lie symmetries is to discern unique, non-equivalent classes referred as the optimal system of subalgebras. Every element from the optimal set illustrates a general class of symmetry and contributes to the construction of a general class of invariant solutions. To achieve this objective, the methodology outlined by Olver [10], as illustrated in [42], is adopted. The adjoint representation can be expressed as

$$Ad(\exp(\varepsilon\Gamma_i))\Gamma_j = \Gamma_j - \varepsilon[\Gamma_i, \Gamma_j] + \frac{\varepsilon^2}{2!}[\Gamma_i, [\Gamma_i, \Gamma_j]] - \cdots, \tag{3.11}$$

where $\varepsilon \in \mathbb{R}$ and $[\Gamma_i, \Gamma_j]$ signifies the Lie product defined by

$$[\Gamma_i, \Gamma_j] = \Gamma_i\Gamma_j - \Gamma_j\Gamma_i. \tag{3.12}$$

3.6 Optimal System of Generalized Fisher Equation in Cylindrical Coordinates

In this section, we delve into a comprehensive review of the paper by Ali Raza *et al.*, titled "Optimal System and Conservation Laws for the Generalized Fisher Equation in Cylindrical Coordinates" [43]. Fisher proposed the Fisher equation for population dynamics in 1937. The Fisher equation can be expressed in cylindrical coordinates as

$$w_t - \frac{1}{u}.(ul(w)w_u)_u = m(w), \tag{3.13}$$

where $l(w)$ and $m(w)$ are arbitrary functions. The optimal system for all of the cases are presented here:

3.6.1 Principal Case

The optimal set of subalgebras for the principal algebra is provided here.

$$\Gamma^1 = \Gamma_1. \tag{3.14}$$

3.6.2 Case-1

For this case, $l(w) = w$ and $m(w) = w(1-w)$:
It has two-dimensional algebra and is expressed as outlined,

$$\Gamma_1 = e^{-t}\frac{\partial}{\partial t} + we^{-t}\frac{\partial}{\partial w},$$
$$\Gamma_2 = \frac{\partial}{\partial t},$$

With a non-zero commutator as specified by,

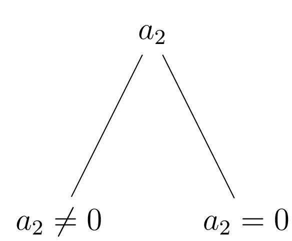
$$[\Gamma_1, \Gamma_2] = \Gamma_1. \tag{3.15}$$

The adjoint representations correspond to Case 1, as detailed in Table3.1.

Ad	Γ_1	Γ_2
Γ_1	Γ_1	$\Gamma_2 - \varepsilon\Gamma_1$
Γ_2	$e^{\varepsilon}\Gamma_1$	Γ_2

Table 3.1: Adjoint Representation Table for Case-1

When we apply the adjoint representation to a general element Γ, represented as a linear combination of elements $\Gamma = a_1\Gamma_1 + a_2\Gamma_2 \in \mathcal{L}_2$, we derive an optimal system of one-dimension that includes every subalgebra in Case-1, represented by



The comprehensive details of each leaf are given below.

Case i. $a_2 \neq 0$

In this case, $\Gamma = a_1\Gamma_1 + a_2\Gamma_2 \in \mathcal{L}_2$, we apply the adjoint action to Γ_1, and we have obtained

$$\begin{aligned}
\Gamma' = Ad(e^{a\Gamma_1})\Gamma &= \Gamma - a[\Gamma_1, \Gamma] + \frac{a^2}{2!}[\Gamma_1, [\Gamma_1, \Gamma]] - \cdots, \\
&= \Gamma - a[\Gamma_1, a_2\Gamma_2] + \frac{a^2}{2}[\Gamma_1, [\Gamma_1, a_2\Gamma_2]] - \cdots, \\
&= \Gamma - aa_2\Gamma_1 + \frac{a^2}{2}[\Gamma_1, a_2\Gamma_1]] - \cdots, \\
&= \Gamma - aa_2\Gamma_1 \\
&= a_1\Gamma_1 + a_2\Gamma_2 - aa_2\Gamma_1; \;\; a = \frac{a_1}{a_2}, \\
&= a_2\Gamma_2; \;\; a_2 = 1 \\
&= \Gamma_2
\end{aligned}$$

Hence on adjoint action on Γ_1, we possess

$$\Gamma' = \Gamma_2. \tag{3.16}$$

Case ii. $a_2 = 0$

In this case, $\Gamma = a_1\Gamma_1$. After applying the adjoint action to Γ_2 and examining the outcomes,

$$\begin{aligned}
\Gamma' = Ad(e^{a\Gamma_2})\Gamma &= \Gamma - a[\Gamma_2, \Gamma] + \frac{a^2}{2!}[\Gamma_2, [\Gamma_2, \Gamma]] - \cdots, \\
&= \Gamma - a[\Gamma_2, a_1\Gamma_1] + \frac{a^2}{2}[\Gamma_2, [\Gamma_2, a_1\Gamma_1]] - \cdots, \\
&= \Gamma - aa_1(-\Gamma_1) + \frac{a^2}{2}[\Gamma_2, a_1(-\Gamma_1)] - \cdots, \\
&= \Gamma + aa_1\Gamma_1 + \frac{a^2}{2}a_1\Gamma_1 - \cdots, \\
&= a_1\Gamma_1 + aa_1\Gamma_1 + \frac{a^2}{2}a_1\Gamma_1 - \cdots, \\
&= a_1\Gamma_1(1 + a + \frac{a^2}{2} + \cdots), \\
&= a_1\Gamma_1 e^a; \;\; a_1 e^a = \pm 1; \;\; a = ln\left|\pm\frac{1}{a_1}\right|, \\
&= \pm\Gamma_1
\end{aligned}$$

Non-similar symmetry generators for Case-1:

$$\Gamma^1 = \Gamma_2$$
$$\Gamma^2 = \pm\Gamma_1$$

3.6.3 Case-2

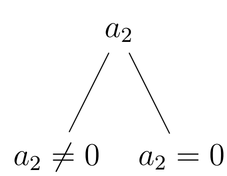
In this case, $l(w) = mw$ and $m(w) = pw^2$:
The two-dimensional algebra is written as

$$\Gamma_1 = \frac{\partial}{\partial t},$$
$$\Gamma_2 = t\frac{\partial}{\partial t} - w\frac{\partial}{\partial w}.$$

The non-zero commutator is given by:

$$[\Gamma_1, \Gamma_2] = \Gamma_1.$$

In this case, the one-dimensional optimal system is the similar to the one described in Case-1 from Table-3.1 because they share the same algebraic structure. Non-equivalent symmetry generators for this case are also the same as the previous ones, as shown in the tree leaf diagram.



Non-equivalent symmetry generators for Case-2:

$$\Gamma^1 = \Gamma_2$$
$$\Gamma^2 = \pm\Gamma_1$$

3.6.4 Case-3

We consider $l(w) = mw^n$ and $m(w) = pw^q$:
The non-zero commutator is presented as,

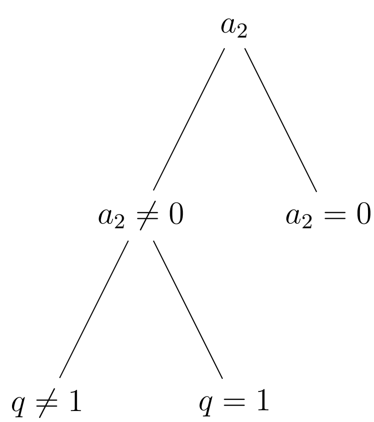
$$[\Gamma_1, \Gamma_2] = 2(q-1)\Gamma_1. \tag{3.17}$$

The adjoint representations associated with Case-3 are outlined in the Table3.2.

Ad	Γ_1	Γ_2
Γ_1	Γ_1	$\Gamma_2 - 2\varepsilon(q-1)\Gamma_1$
Γ_2	$e^{2\varepsilon(q-1)}\Gamma_1$	Γ_2

Table 3.2: Adjoint Representation Table for Case-3

When the adjoint representation is applied to a general element, Γ, which is represented as a linear combination of components, $\Gamma = a_1\Gamma_1 + a_2\Gamma_2 \in \mathcal{L}_2$, we establish a one-dimensional optimal system including every subalgebras in Case-3 provided by



All the specific information for each individual leaf is provided here.
Case-i. $a_2 \neq 0$ **and** $q \neq 1$**.**

In this case $\Gamma = a_1\Gamma_1 + a_2\Gamma_2$. Through the adjoint action applied to Γ_1, we acquire

$$\begin{aligned}
\Gamma' = Ad(e^{a\Gamma_1})\Gamma &= \Gamma - a[\Gamma_1, \Gamma] + \frac{a^2}{2!}[\Gamma_1, [\Gamma_1, \Gamma]] - \cdots , \\
&= \Gamma - a[\Gamma_1, a_2\Gamma_2] + \frac{a^2}{2}[\Gamma_1, [\Gamma_1, a_2\Gamma_2]] - \cdots , \\
&= \Gamma - aa_2(2(q-1))\Gamma_1 + \frac{a^2}{2}[\Gamma_1, 2a_2(q-1)\Gamma_1]] - \cdots , \\
&= \Gamma - aa_2(2(q-1))\Gamma_1, \\
&= a_1\Gamma_1 + a_2\Gamma_2 - 2aa_2(q-1)\Gamma_1; \; a = \frac{a_1}{a_2}, \\
&= a_1\Gamma_1 + a_2\Gamma_2 - 2a_1(q-1)\Gamma_1; \; as \; q \neq 1, \; q = \frac{3}{2}, \\
&= a_1\Gamma_1 + a_2\Gamma_2 - a_1\Gamma_1, \\
&= a_2\Gamma_2, \\
&= \Gamma_2.
\end{aligned}$$

Case-ii. $a_2 \neq 0$ **and** $q = 1$**.**

$\Gamma = a_1\Gamma_1 + a_2\Gamma_2$ in this case. By using the adjoint operation on Γ_1, we obtain

$$\begin{aligned}
\Gamma' = Ad(e^{a\Gamma_1})\Gamma &= \Gamma - a[\Gamma_1, \Gamma] + \frac{a^2}{2!}[\Gamma_1, [\Gamma_1, \Gamma]] - \cdots , \\
&= \Gamma - a[\Gamma_1, a_2\Gamma_2] + \frac{a^2}{2}[\Gamma_1, [\Gamma_1, a_2\Gamma_2]] - \cdots , \\
&= \Gamma - aa_2(2(q-1))\Gamma_1 + \frac{a^2}{2}[\Gamma_1, 2a_2(q-1)\Gamma_1]] - \cdots ; \; q = 1, \\
&= \Gamma, \\
&= a_1\Gamma_1 + a_2\Gamma_2, \\
&= a_1\Gamma_1 + \Gamma_2; \; a_2 \neq 0, \; a_2 = 1.
\end{aligned}$$

Case-ii. $a_2 = 0$**.**

$\Gamma = a_1\Gamma_1$ in this case. When we consider the adjoint action applied to any element $\mathcal{Y}$ and for any given arbitrary value of ε, we get the following result

$$\Gamma' = Ad(e^{\varepsilon\mathcal{Y}})\Gamma = \Gamma_1. \tag{3.18}$$

Non-similar symmetry generators for Case-3:

$$\Gamma^1 = \Gamma_2,$$
$$\Gamma^2 = a_1\Gamma_1 + \Gamma_2,$$
$$\Gamma^3 = \Gamma_1.$$

3.6.5 Case-4

We consider $l(w) = aw^n$ and $m(w) = w$:
The algebra is three dimensional, which is written as,

$$\Gamma_1 = \frac{\partial}{\partial t},$$
$$\Gamma_2 = u\frac{\partial}{\partial u} + \frac{2}{n}w\frac{\partial}{\partial w},$$
$$\Gamma_3 = e^{-nt}\frac{\partial}{\partial t} + we^{-nt}\frac{\partial}{\partial w}.$$

The non-zero commutator can be expressed as,

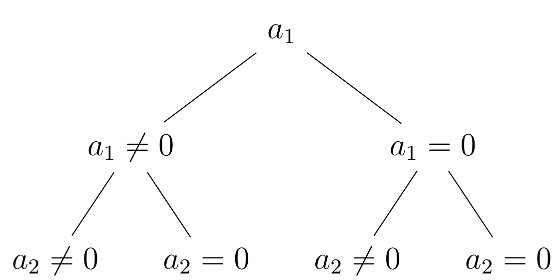
$$[\Gamma_1, \Gamma_3] = -n\Gamma_3, \ n \neq 0. \tag{3.19}$$

The adjoint representations are presented in Table 3.3.

Ad	Γ_1	Γ_2	Γ_3
Γ_1	Γ_1	Γ_2	$e^{n\varepsilon}\Gamma_3$
Γ_2	Γ_1	Γ_2	Γ_3
Γ_3	$\Gamma_1 - n\varepsilon\Gamma_3$	Γ_2	Γ_3

Table 3.3: Adjoint Representation Table for Case-4

The adjoint representation, when applied to a general element $\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 \in \mathcal{L}_3$, establishes the one-dimensional optimal system that encompasses all subalgebras in this particular case, given by



Each leaf is described in detail, with its characteristics and features explained thoroughly.

Case-i. $a_1 \neq 0$ and $a_2 \neq 0$.
We determine the basis for the optimal system by applying the adjoint action on Γ_3 for $\varepsilon = a$, using the procedure mentioned in earlier cases, we obtain

$$\begin{aligned}\Gamma' &= \Gamma - aa_1(n\Gamma_3) + \frac{a^2}{2}[\Gamma_3, a_1(n\Gamma_3)] - \cdots ,\\ &= \Gamma - aa_1(n\Gamma_3),\\ &= a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 - aa_1 n\Gamma_3;\ a = \frac{a_3}{a_1},\ n = 1,\\ &= a_1\Gamma_1 + a_2\Gamma_2.\end{aligned}$$

Hence,

$$\Gamma' = Ad(e^{a\Gamma_3}\Gamma) = \Gamma_1 + \Gamma_2. \tag{3.20}$$

Case-ii. $a_1 \neq 0$ and $a_2 = 0$.
In this situation, the adjoint action on Γ_3 yields the basis for the optimal system for $\varepsilon = a$ and $\Gamma = a_1\Gamma_1 + a_3\Gamma_3$,

$$\begin{aligned}\Gamma' &= Ad(e^{a\Gamma_3}\Gamma) = a_1\Gamma_1 + a_3\Gamma_3 - aa_1 n\Gamma_3;\ a = \frac{a_3}{a_1},\ n = 1,\\ &= a_1\Gamma_1,\\ &= \Gamma_1.\end{aligned}$$

Therefore,

$$\Gamma' = Ad(e^{a\Gamma_3}\Gamma) = \Gamma_1. \tag{3.21}$$

Case-iii. $a_1 = 0$ and $a_2 \neq 0$.
$\Gamma = a_2\Gamma_2 + a_3\Gamma_3$ in this case. By employing the same method as in previous cases and employing the adjoint action to Γ_1 with $\varepsilon = a$, we obtain,

$$\begin{aligned}\Gamma' &= Ad(e^{a\Gamma_1}\Gamma) = \Gamma + aa_3(n\Gamma_3) + \frac{a^2}{2}a_3 n^2\Gamma_3 - \cdots ,\\ &= a_2\Gamma_2 + a_3\Gamma_3 + aa_3 n\Gamma_3 + \frac{a^2}{2}a_3 n^2\Gamma_3 - \cdots ,\\ &= a_2\Gamma_2 + a_3\Gamma_3 e^{an},\\ &= a_2\Gamma_2 \pm \Gamma_3.\end{aligned}$$

Hence,

$$\Gamma' = Ad(e^{a\Gamma_1}\Gamma) = a_2\Gamma_2 \pm \Gamma_3, \tag{3.22}$$

where $a = \frac{1}{n} ln \left| \pm \frac{1}{a_3} \right|$.

Case-iv. $a_1 = 0$ **and** $a_2 = 0$**.**

In this instance, with $\Gamma = a_3\Gamma_3$, we follow the same approach used in earlier cases and apply the adjoint action to Γ_1 for $\varepsilon = a$, we obtain the following result,

$$\Gamma' = Ad(e^{a\Gamma_1}\Gamma) = \pm\Gamma_3, \tag{3.23}$$

Non-equivalent symmetry generators for Case-4:

$$\begin{aligned}
\Gamma^1 &= \Gamma_1 + \Gamma_2, \\
\Gamma^2 &= \Gamma_1, \\
\Gamma^3 &= a_2\Gamma_2 \pm \Gamma_3, \\
\Gamma^4 &= \pm\Gamma_3.
\end{aligned}$$

3.6.6 Case-5

$l(w) = \sqrt[n]{w}$ and $m(w) = w$

The algebra is three-dimensional, and Γ_1, Γ_2 and Γ_3 are as follows,

$$\begin{aligned}
\Gamma_1 &= \frac{\partial}{\partial t}, \\
\Gamma_2 &= e^{-\frac{1}{n}t}\frac{\partial}{\partial t} + we^{-\frac{1}{n}t}\frac{\partial}{\partial w}, \\
\Gamma_3 &= u\frac{\partial}{\partial u} + 2nw\frac{\partial}{\partial w},
\end{aligned}$$

and the non-zero commutator is,

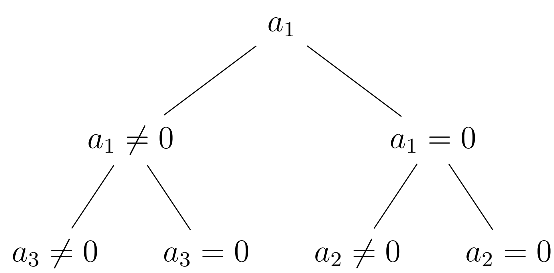
$$[\Gamma_1, \Gamma_2] = -\frac{1}{n}\Gamma_2,\ n \neq 0. \tag{3.24}$$

Table-3.4 provides information about the adjoint representations for the functions $l(w) = \sqrt[n]{w}$ and $m(w) = w$.

Ad	Γ_1	Γ_2	Γ_3
Γ_1	Γ_1	$e^{\frac{1}{n}\varepsilon}\Gamma_2$	Γ_3
Γ_2	$\Gamma_1 - \frac{1}{n}\varepsilon\Gamma_2$	Γ_2	Γ_3
Γ_3	Γ_1	Γ_2	Γ_3

Table 3.4: Adjoint Representation Table for Case-5

The adjoint representation, when applied to the general element $\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 \in \mathcal{L}_3$, determines the one-dimensional optimal system of all subalgebras.



Each leaf is thoroughly examined below

Case-i. $a_1 \neq 0$ **and** $a_3 \neq 0$**.**

Through the adjoint action applied to Γ_2 and by using the same procedure as in earlier cases, we have

$$\begin{aligned}\Gamma' = Ad(e^{a\Gamma_2})\Gamma &= \Gamma - a[\Gamma_2, a_1\Gamma_1] + \frac{a^2}{2!}[\Gamma_2, [\Gamma_2, a_1\Gamma_1]] - \cdots,\\ &= \Gamma - aa_1(\frac{1}{n})\Gamma_2,\\ &= a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 - \frac{1}{n}aa_1\Gamma_2,\\ &= a_1\Gamma_1 + a_3\Gamma_3.\end{aligned}$$

Therefore,

$$\Gamma' = Ad(e^{a\Gamma_2})\Gamma = \Gamma_1 + \Gamma_3, \tag{3.25}$$

where $a = \dfrac{a_2}{a_1}$, and $n = 1$.

Case-ii. $a_1 \neq 0$ **and** $a_3 = 0$**.**

When we apply the adjoint action on Γ_2, we get the similar result as described in case-i.

$$\Gamma' = Ad(e^{a\Gamma_2})\Gamma = \Gamma_1. \tag{3.26}$$

Case-iii. $a_1 = 0$ **and** $a_2 \neq 0$**.**
When we perform an adjoint action on Γ_1, the outcome is as follows,

$$\begin{aligned}\Gamma' = Ad(e^{a\Gamma_1})\Gamma &= \Gamma - aa_2(-\frac{1}{n})\Gamma_2 + \frac{a^2}{2!}[\Gamma_1, a_2(-\frac{1}{n})\Gamma_2] - \cdots ,\\ &= a_2\Gamma_2 + a_3\Gamma_3 + aa_2(\frac{1}{n})\Gamma_2 + \frac{a^2}{2}a_2(\frac{1}{n^2})\Gamma_2 - \cdots ,\\ &= a_2\Gamma_2 e^{\frac{a}{n}} + a_3\Gamma_3,\end{aligned}$$

hence,

$$\Gamma' = Ad(e^{a\Gamma_1})\Gamma = \pm\Gamma_2 + a_3\Gamma_3, \tag{3.27}$$

where, $a = (n)ln\left|\pm\frac{1}{a_2}\right|$.
Case-iv. $a_1 = 0$ **and** $a_2 = 0$**.**
the adjoint action on Γ_1 provides

$$\Gamma' = Ad(e^{a\Gamma_1})\Gamma = \Gamma_3. \tag{3.28}$$

Non-equivalent symmetry generators for Case-5:

$$\begin{aligned}\Gamma^1 &= \Gamma_1 + \Gamma_3,\\ \Gamma^2 &= \Gamma_1,\\ \Gamma^3 &= \pm\Gamma_2 + a_3\Gamma_3,\\ \Gamma^4 &= \Gamma_3.\end{aligned}$$

3.6.7 Case-6

$l(w) = aw^n$ and $m(w) = \sqrt[m]{w}$
The algebra under consideration is a two-dimensional algebra, and the basis elements Γ_1 and Γ_2 can be represented as follows,

$$\begin{aligned}\Gamma_1 &= \frac{\partial}{\partial t},\\ \Gamma_2 &= 2t(m-1)\frac{\partial}{\partial t} + u[(n+1)m-1]\frac{\partial}{\partial u} + 2mw\frac{\partial}{\partial w}.\end{aligned}$$

With a commutator that is not equal to zero,

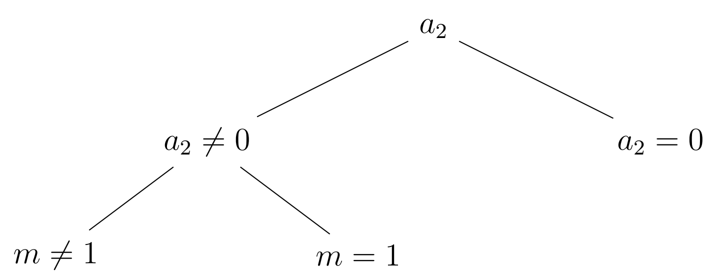
$$[\Gamma_1, \Gamma_2] = 2(m-1)\Gamma_1. \tag{3.29}$$

The adjoint representations for the given functions $l(w) = aw^n$ and $m(w) = \sqrt[m]{w}$ can be found in Table 3.5.

Ad	Γ_1	Γ_2
Γ_1	Γ_1	$\Gamma_2 - 2\varepsilon(m-1)\Gamma_1$
Γ_2	$e^{2\varepsilon(m-1)}\Gamma_1$	Γ_2

Table 3.5: Adjoint Representation Table for Case-6

The one-dimensional optimal system of subalgebras for this instance is provided below when applying the adjoint representation to a general element $\Gamma = a_1\Gamma_1 + a_2\Gamma_2 \in \mathcal{L}_2$.



Below are the particulars for each leaf:

Case-i. $a_2 \neq 0$ **and** $m \neq 1$.

We achieve this by applying the adjoint action on Γ_1, using a similar approach as above for $\varepsilon = a$,

$$\begin{aligned}\Gamma' = Ad(e^{a\Gamma_1})\Gamma &= \Gamma - 2aa_2(m-1)\Gamma_1,\\ &= a_1\Gamma_1 + a_2\Gamma_2 - 2aa_2(m-1)\Gamma_1,\\ &= a_2\Gamma_2,\end{aligned}$$

therefore,

$$\Gamma' = Ad(e^{a\Gamma_1})\Gamma = \Gamma_2, \tag{3.30}$$

where, $a = \frac{a_1}{a_2}$ and $m = \frac{3}{2}$.

Case-ii. $a_2 \neq 0$ **and** $m = 1$.

Through the adjoint action on Γ_1 and $\varepsilon = a$, we have acquired

$$\Gamma' = Ad(e^{a\Gamma_1})\Gamma = a_1\Gamma_1 + \Gamma_2. \tag{3.31}$$

Case-iii. $a_2 = 0$.

The outcome of the adjoint action for any $\mathcal{Y}$ and for any ε is as follows,

$$\Gamma' = Ad(e^{\varepsilon\mathcal{Y}})\Gamma = \Gamma_1. \tag{3.32}$$

Non-similar symmetry generators for Case-6:

$$\begin{aligned}
\Gamma^1 &= \Gamma_2, \\
\Gamma^2 &= a_1\Gamma_1 + \Gamma_2, \\
\Gamma^3 &= \Gamma_1.
\end{aligned}$$

While reviewing Ali Raza *et al.*'s paper on the "Generalized Fisher Equation in Cylindrical Coordinates," we came across the absence of Γ_2 in Cases-4 and Case-5, and adding it will make a complete set of non similar symmetry generators for Case-4 and Case-5.

Chapter 4

Optimal System and Exact Soutions of the Hyperbolic Heat Equations

In the study of heat transfer, the solutions of hyperbolic heat equations [44] provide an essential framework to understand how temperature distribution varies dynamically over time. The primary objective of this chapter is to determine the optimal systems of the hyperbolic heat equations and try to find their exact solutions. These equations

$$w_{tt} + w_t = (K(w)w_u)_u, \quad K(w) \neq const, \tag{4.1}$$

were proposed by Alexander Oron and Philip Rosenau in 1986.

4.1 Lie Point Symmetry Generators of the Hyperbolic Heat Equations

This segment examines the group properties of Eq. (4.1), taking into account Lie symmetries, and investigates various cases by considering values of the arbitrary function $K(w)$. As Eq. (4.1) represents a second-order PDE, it is necessary to employ the second-order prolongation Γ^2 to get the symmetry generators. This involves using the associated coefficients provided in Eqs. (2.47)-(2.51). To apply the infinitesimal criterion of invariance, let us introduce

$$H = w_{tt} + w_t - (K(w)w_u)_u, \tag{4.2}$$

hence, using Theorem (2.7.1), we have

$$\Gamma^2 H|_{H=0} \equiv 0, \tag{4.3}$$

that can be simplified to

$$\beta_{tt} + \beta_t - K(w)\beta_{uu} - \dot{K}(w)w_u = 0. \tag{4.4}$$

Eq. (4.4) yields the results that follow using the values of β_t, β_{tt} and β_{uu} from Eqs. (2.47)-(2.51).

$$\begin{aligned}
&(-w_u^2K(w) + w_t^2)(\beta_{w,w}) + (w_u^3K(w) - w_t^2w_u)(\alpha^u)_{w,w} \\
&+ (w_tw_u^2K(w) - w_t^3)(\alpha^t)_{w,w} + 2w_t(\beta_w)_{t,w} - 2(\beta)_{w,u}K(w)w_u \\
&- 2(\alpha^u)_{t,w}w_uw_t + 2(\alpha^u)_{w,u}K(w)w_u^2 - 2(\alpha^t)_{t,w}w_t^2 + 2(\alpha^t)_{w,u}K(w)w_tw_u \\
&+ (\beta)_{t,t} - (\beta)_{u,u}K(w) - (\alpha^u)_{t,t}w_u + (\alpha^u)_{u,u}K(w)w_u \\
&- (\alpha^t)_{t,t}w_t + (\alpha^t)_{u,u}K(w)w_t + ((-2w_tw_{u,u} + 2w_uw_{u,t})K(w) + 2w_t^2(\alpha^t)_w \\
&+ (2w_uw_{u,u}K(w) - 2w_{u,t}w_t)(\alpha^u)_w + (-w_u - 2w_{u,t})(\alpha^u)_t \\
&+ (-2K(w)w_{u,u} + w_t)(\alpha^t)_t - \beta(t,u,w)(K_w)w_{u,u} + 2(\alpha^u)_uK(w)w_{u,u} \\
&+ 2(\alpha^t)_uK(w)w_{u,t} + \beta_t = 0.
\end{aligned} \tag{4.5}$$

The following determining equations are now obtained by comparing different powers of partial derivatives of dependent variable w.

$$(\alpha^t)_t = 0, \tag{4.6}$$

$$(\alpha^t)_w = 0, \tag{4.7}$$

$$(\alpha^t)_u = 0, \tag{4.8}$$

$$(\alpha^u)_t = 0, \tag{4.9}$$

$$(\alpha^u)_w = 0, \tag{4.10}$$

$$(\alpha^u)_u = 0, \tag{4.11}$$

$$\beta(t,u,w) = 0. \tag{4.12}$$

Simplifying the given set of determining equations allows us to get the values of infinitesimals. Consequently, for the arbitrary function $K(w)$, we have

$$\beta(t,u,w) = 0, \quad \alpha^t(t,u,w) = C_1, \quad \alpha^u(t,u,w) = C_2,$$

The obtained symmetry generator comprises the principal Lie algebra given by

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial u}.$$

Here, we explore the group properties by examining different $K(w)$ values. There is one additional symmetry in each of the following cases [44].

Case-1: $K = \Lambda e^{\rho w},\ \Lambda \neq 0,\ \rho \neq 0:$

In this case, the symmetry algebra is expanded by

$$\Gamma_3 = \rho u \frac{\partial}{\partial u} + 2\frac{\partial}{\partial w}.$$

Case-2: $K = \Lambda(w+\mu)^{\rho},\ \Lambda \neq 0,\ \rho \neq 0, -4/3, -2:$

In this case, the symmetry algebra is expanded by

$$\Gamma_3 = \rho u \frac{\partial}{\partial u} + 2(w+\mu)\frac{\partial}{\partial w}.$$

In the next section the one dimensional optimal system of both algebras are obtained.

4.2 One-dimensional Optimal System of Subalgebras

4.2.1 Optimal System for Principal Case

For principal case,

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial u}.$$

The algebraic structure in this case is two-dimensional, and it exhibits no non-zero commutators.

$$[\Gamma_1, \Gamma_2] = 0. \tag{4.13}$$

The adjoint table that aids in calculating the optimal system of one-dimensional subalgebras for principal case is shown in table 4.1.

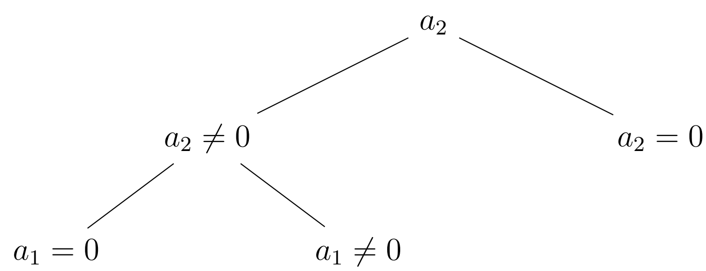
Ad	Γ_1	Γ_2
Γ_1	Γ_1	Γ_2
Γ_2	Γ_1	Γ_2

Table 4.1: Adjoint Representation Table for Principal Algebra

Take an arbitrary element $\Gamma \in \mathcal{L}_2$. We have

$$\Gamma = a_1\Gamma_1 + a_2\Gamma_2. \tag{4.14}$$

Thus, using the adjoint Table 4.1 and the adjoint operator in Eq. (3.10) on a general element Γ given in Eq. (4.14), we establish an optimal system of one-dimensional subalgebras for the principal case, given below.



Below are the specifics for each leaf:

Case-i. $a_2 \neq 0$ **and** $a_1 = 0$:
When the adjoint action of the element $\varepsilon = a$ is applied to Γ, we observe the following:

$$\begin{aligned}
\Gamma' &= Ad(e^{a\Gamma_1})\Gamma = \Gamma - a[\Gamma_1, \Gamma] + \frac{a^2}{2!}[\Gamma_1, [\Gamma_1, \Gamma]] - \cdots , \\
&= \Gamma - a[\Gamma_1, a_2\Gamma_2] + \frac{a^2}{2}[\Gamma_1, [\Gamma_1, a_2\Gamma_2]] - \cdots , \\
&= a_2\Gamma_2,\ \ a_2 \neq 0, \\
&= \Gamma_2,
\end{aligned}$$

we get

$$\Gamma^1 = \Gamma_2. \tag{4.15}$$

Case-ii. $a_2 \neq 0$ **and** $a_1 \neq 0$:
The following is evident when we apply the adjoint action of the element $\varepsilon = a$ on Γ

$$\begin{aligned}
\Gamma' &= Ad(e^{a\Gamma_1})\Gamma = \Gamma - a[\Gamma_1, \Gamma] + \frac{a^2}{2!}[\Gamma_1, [\Gamma_1, \Gamma]] - \cdots , \\
&= \Gamma - a[\Gamma_1, a_2\Gamma_2] + \frac{a^2}{2}[\Gamma_1, [\Gamma_1, a_2\Gamma_2]] - \cdots , \\
&= a_1\Gamma_1 + a_2\Gamma_2.
\end{aligned}$$

We deduce that

$$\Gamma^2 = a_1\Gamma_1 + a_2\Gamma_2. \tag{4.16}$$

Case-iii. $a_2 = 0$:

The following is evident when we apply the element $\varepsilon = a$ on Γ,

$$\begin{aligned}\Gamma' &= Ad(e^{a\Gamma_2})\Gamma = \Gamma - a[\Gamma_2, \Gamma] + \frac{a^2}{2!}[\Gamma_2, [\Gamma_2, \Gamma]] - \cdots ,\\ &= \Gamma,\\ &= a_1\Gamma_1,\ a_1 \neq 0,\\ &= \Gamma_1,\end{aligned}$$

It follows that,

$$\Gamma^3 = \Gamma_1. \tag{4.17}$$

The following are the non-similar symmetry generators for the principal case that can be determined using Eqs. (4.15), (4.16) and (4.17).

	Generators	Conditions
Case-i:	$\Gamma^1 = \Gamma_2$	$a_2 \neq 0$ and $a_1 = 0$
Case-ii:	$\Gamma^2 = a_1\Gamma_1 + a_2\Gamma_2$	$a_2 \neq 0$ and $a_1 \neq 0$
Case-iii:	$\Gamma^3 = \Gamma_1$	$a_2 = 0$

Table 4.2: Symmetry Generators for Principal Case

4.2.2 Optimal System for Case-1

For Case-1,

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial u}, \quad \Gamma_3 = \rho u\frac{\partial}{\partial u} + 2\frac{\partial}{\partial w}.$$

The algebraic structure is three-dimensional with non-zero commutators,

$$[\Gamma_2, \Gamma_3] = \rho\Gamma_2. \tag{4.18}$$

Table 4.2 displays the adjoint table for Case-1, which facilitates the computation of the optimal system of one-dimensional subalgebras.

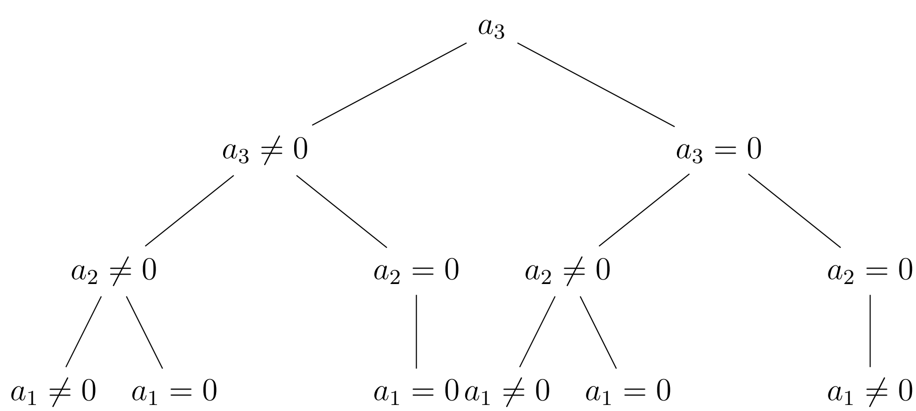
Ad	Γ_1	Γ_2	Γ_3
Γ_1	Γ_1	Γ_2	Γ_3
Γ_2	Γ_1	Γ_2	$\Gamma_3 - \varepsilon\rho\Gamma_2$
Γ_3	Γ_1	$e^{\varepsilon\rho}\Gamma_2$	Γ_3

Table 4.3: Adjoint Representation Table for Case-1

Let $\Gamma \in \mathcal{L}_3$. Consequently, we have

$$\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3. \tag{4.19}$$

Similarly, the adjoint representation governs the determination of the one-dimensional optimal system for all subalgebras in this case.



Each leaf is illustrated in detail below.

Case-i. $a_3 \neq 0$**,** $a_2 \neq 0$ **and** $a_1 \neq 0$:
The following is observed when Γ is subjected to the adjoint action of the element $\varepsilon = a$.

$$\begin{aligned}
\Gamma' = Ad(e^{a\Gamma_2})\Gamma &= \Gamma - a[\Gamma_2, \Gamma] + \frac{a^2}{2!}[\Gamma_2, [\Gamma_2, \Gamma]] - \cdots, \\
&= \Gamma - a[\Gamma_2, a_3\Gamma_3] + \frac{a^2}{2}[\Gamma_2, [\Gamma_2, a_3\Gamma_3]] - \cdots, \\
&= \Gamma - aa_3\rho\Gamma_2, \\
&= a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 - aa_3\rho\Gamma_2, \\
&= a_1\Gamma_1 + a_3\Gamma_3, \ a = \frac{a_2}{a_3}, \ \rho = 1.
\end{aligned}$$

Hence, we can conclude that

$$\Gamma^1 = a_1\Gamma_1 + a_3\Gamma_3. \tag{4.20}$$

Case-ii. $a_3 \neq 0$**,** $a_2 \neq 0$ **and** $a_1 = 0$:
In the presence of the adjoint action on Γ induced by $\varepsilon = a$, we have

$$\begin{aligned}
\Gamma' = Ad(e^{a\Gamma_1})\Gamma &= \Gamma - a[\Gamma_1, \Gamma] + \frac{a^2}{2!}[\Gamma_1, [\Gamma_1, \Gamma]] - \cdots , \\
&= \Gamma - a[\Gamma_1, a_2\Gamma_2 + a_3\Gamma_3] + \frac{a^2}{2}[\Gamma_1, [\Gamma_1, a_2\Gamma_2 + a_3\Gamma_3]] - \cdots , \\
&= \Gamma, \\
&= a_2\Gamma_2 + a_3\Gamma_3.
\end{aligned}$$

From this we get,

$$\Gamma^2 = a_2\Gamma_2 + a_3\Gamma_3. \tag{4.21}$$

Case-iii. $a_3 \neq 0$**,** $a_2 = 0$ **and** $a_1 = 0$:
Under the influence of the adjoint action on Γ with $\varepsilon = a$, we have the following outcome,

$$\begin{aligned}
\Gamma' = Ad(e^{a\Gamma_1})\Gamma &= \Gamma - a[\Gamma_1, \Gamma] + \frac{a^2}{2!}[\Gamma_1, [\Gamma_1, \Gamma]] - \cdots , \\
&= \Gamma - a[\Gamma_1, a_3\Gamma_3] + \frac{a^2}{2}[\Gamma_1, [\Gamma_1, a_3\Gamma_3]] - \cdots , \\
&= \Gamma, \\
&= a_3\Gamma_3, \ \ a_3 \neq 0,
\end{aligned}$$

hence, we have

$$\Gamma^3 = \Gamma_3. \tag{4.22}$$

Case-iv. $a_3 = 0$**,** $a_2 \neq 0$ **and** $a_1 \neq 0$:

Through the adjoint operation on Γ for $\varepsilon = a$, we have

$$\begin{aligned}
\Gamma' = Ad(e^{a\Gamma_3})\Gamma &= \Gamma - a[\Gamma_3, \Gamma] + \frac{a^2}{2!}[\Gamma_3, [\Gamma_3, \Gamma]] - \cdots, \\
&= \Gamma - a[\Gamma_3, a_2\Gamma_2] + \frac{a^2}{2}[\Gamma_3, [\Gamma_3, a_2\Gamma_2]] - \cdots, \\
&= \Gamma - aa_2(-\rho\Gamma_2) + \frac{a^2}{2}[\Gamma_3, a_2(-\rho\Gamma_2) - \cdots, \\
&= \Gamma + aa_2\rho\Gamma_2 + \frac{a^2}{2}a_2\rho^2\Gamma_2 - \cdots, \\
&= a_1\Gamma_1 + a_2\Gamma_2 + aa_2\rho\Gamma_2 + \frac{a^2}{2}a_2\rho^2\Gamma_2 - \cdots, \\
&= a_1\Gamma_1 + a_2\Gamma_2 e^{a\rho}, \\
&= a_1\Gamma_1 \pm \Gamma_2, \ a\rho = \pm\ln(\frac{1}{a_2}), \\
&= \Gamma_1 \pm \Gamma_2, \ a_1 \neq 0,
\end{aligned}$$

we get the result as,

$$\Gamma^4 = \Gamma_1 \pm \Gamma_2. \tag{4.23}$$

Case-v. $a_3 = 0$**,** $a_2 \neq 0$ **and** $a_1 = 0$:
By employing the adjoint operation on Γ with $\varepsilon = a$, we derive

$$\begin{aligned}
\Gamma' = Ad(e^{a\Gamma_3})\Gamma &= \Gamma - a[\Gamma_3, \Gamma] + \frac{a^2}{2!}[\Gamma_3, [\Gamma_3, \Gamma]] - \cdots, \\
&= \Gamma - a[\Gamma_3, a_2\Gamma_2] + \frac{a^2}{2}[\Gamma_3, [\Gamma_3, a_2\Gamma_2]] - \cdots, \\
&= \Gamma - aa_2(-\rho\Gamma_2) + \frac{a^2}{2}[\Gamma_3, a_2(-\rho\Gamma_2) - \cdots, \\
&= \Gamma + aa_2\rho\Gamma_2 + \frac{a^2}{2}a_2\rho^2\Gamma_2 - \cdots, \\
&= a_2\Gamma_2 + aa_2\rho\Gamma_2 + \frac{a^2}{2}a_2\rho^2\Gamma_2 - \cdots, \\
&= a_2\Gamma_2 e^{a\rho}, \\
&= \pm\Gamma_2, \ a\rho = \pm\ln(\frac{1}{a_2}).
\end{aligned}$$

Consequently, we obtain

$$\Gamma^5 = \pm\Gamma_2. \tag{4.24}$$

Case-vi. $a_3 = 0$**,** $a_2 = 0$ **and** $a_1 \neq 0$:

$$\begin{aligned}\Gamma' = Ad(e^{a\Gamma_2})\Gamma &= \Gamma - a[\Gamma_2, \Gamma] + \frac{a^2}{2!}[\Gamma_2, [\Gamma_2, \Gamma]] - \cdots, \\ &= \Gamma - a[\Gamma_2, a_1\Gamma_1] + \frac{a^2}{2}[\Gamma_2, [\Gamma_2, a_1\Gamma_1]] - \cdots, \\ &= \Gamma, \\ &= a_1\Gamma_1, \\ &= \Gamma_1, \ a_1 \neq 0.\end{aligned}$$

Thus, we get

$$\Gamma^6 = \Gamma_1. \tag{4.25}$$

The non-equivalent symmetry generators for Case-1 can be identified using Eqs. (4.20)-(4.25).

	Generators	Conditions
Case-i:	$\Gamma^1 = a_1\Gamma_1 + a_3\Gamma_3$	$a_3 \neq 0$, $a_2 \neq 0$ and $a_1 \neq 0$
Case-ii:	$\Gamma^2 = a_2\Gamma_2 + a_3\Gamma_3$	$a_3 \neq 0$, $a_2 \neq 0$ and $a_1 = 0$
Case-iii:	$\Gamma^3 = \Gamma_3$	$a_3 \neq 0$, $a_2 = 0$ and $a_1 = 0$
Case-iv:	$\Gamma^4 = \Gamma_1 \pm \Gamma_2$	$a_3 = 0$, $a_2 \neq 0$ and $a_1 \neq 0$
Case-v:	$\Gamma^5 = \pm\Gamma_2$	$a_3 = 0$, $a_2 \neq 0$ and $a_1 = 0$
Case-vi:	$\Gamma^6 = \Gamma_1$	$a_3 = 0$, $a_2 = 0$ and $a_1 \neq 0$

Table 4.4: Symmetry Generators for Case-1

4.2.3 Optimal System for Case-2

For Case-2 following is the is three dimensional algebra.

$$\Gamma_1 = \frac{\partial}{\partial t}, \quad \Gamma_2 = \frac{\partial}{\partial u}, \quad \Gamma_3 = \rho u\frac{\partial}{\partial u} + 2(w + \mu)\frac{\partial}{\partial w}.$$

The only non-zero commutator is.

$$[\Gamma_2, \Gamma_3] = \rho\Gamma_2. \tag{4.26}$$

Table4.3 shows the adjoint table for Case-2.

Ad	Γ_1	Γ_2	Γ_3
Γ_1	Γ_1	Γ_2	Γ_3
Γ_2	Γ_1	Γ_2	$\Gamma_3 - \varepsilon\rho\Gamma_2$
Γ_3	Γ_1	$e^{\varepsilon\rho}\Gamma_2$	Γ_3

Table 4.5: Adjoint Representation Table for Case-2

Using the adjoint representation on the general element, $\Gamma = a_1\Gamma_1 + a_2\Gamma_2 + a_3\Gamma_3 \in \mathcal{L}_3$, the optimal system of one-dimensional algebra for Case-2 is obtained. The obtained optimal system is identical to the optimal system derived for Case-1. Hence, the non-equivalent symmetry generators for Case-2 are as follows:

	Generators	Conditions
Case-i:	$\Gamma^1 = a_1\Gamma_1 + a_3\Gamma_3$	$a_3 \neq 0$, $a_2 \neq 0$ and $a_1 \neq 0$
Case-ii:	$\Gamma^2 = a_2\Gamma_2 + a_3\Gamma_3$	$a_3 \neq 0$, $a_2 \neq 0$ and $a_1 = 0$
Case-iii:	$\Gamma^3 = \Gamma_3$	$a_3 \neq 0$, $a_2 = 0$ and $a_1 = 0$
Case-iv:	$\Gamma^4 = \Gamma_1 \pm \Gamma_2$	$a_3 = 0$, $a_2 \neq 0$ and $a_1 \neq 0$
Case-v:	$\Gamma^5 = \pm\Gamma_2$	$a_3 = 0$, $a_2 \neq 0$ and $a_1 = 0$
Case-vi:	$\Gamma^6 = \Gamma_1$	$a_3 = 0$, $a_2 = 0$ and $a_1 \neq 0$

Table 4.6: Symmetry Generators for Case-2

The procedure for constructing optimal systems of Lie algebras allowed by Eq. (4.1) has been illustrated above, where each example corresponds to a specific value of $K(w)$. Using the symmetry generators identified in the optimal system, the associated class of invariant solutions is established in the following segment. We also observe a reduction in order of Eq. (4.1) under the optimal system of symmetry generators that has been computed.

4.3 Reduction and Similarity Solutions

Using the one-dimensional optimal systems of algebras from the preceding section, we now reduce the Eq. (4.1). By using symmetry reduction, it is possible to generate exact solutions to the corresponding equations using the equivalence classes of symmetry generators. Authoritative texts on the subject explain the well-known and very algorithmic process of symmetry reduction with respect to algebras of Lie invariance algebras [10, 27]. The concept of seeking group-invariant solutions extends naturally to partial differential equations involving any number of independent and dependent variables. One can reduce the number of independent variables by one by employing a one-parameter group that has a nontrivial impact on one or more of the independent variables. A one-parameter group that has a nontrivial impact on one or more independent variables allows for a reduction of the number of independent variables by one. In this part, our emphasis is on the invariant form method, where we explicitly solve the conditions for surface invariance by addressing the associated characteristic equation as provided by

$$\frac{dt}{\alpha_t(t,u,w)} = \frac{du}{\alpha_u(t,u,w)} = \frac{dw}{\beta(t,u,w)}. \tag{4.27}$$

Additionally, we have performed all possible reductions for Eq. (4.1) for every optimal system of obtained subalgebras in our study. From the optimal systems (section 4.2), we obtain reductions of Eq. (4.1) for the following nontrivial symmetry generators.

4.3.1 Symmetry Reductions for Principal Case

Case-i. Consider $\Gamma^1 = \Gamma_2$. The corresponding characteristic equation is

$$\frac{du}{1} = \frac{dt}{0} = \frac{dw}{0}.$$

The similarity variables are obtained as follows:

$$w = F(T), T = t,$$

now, differentiating w with respect to t and u, we get

$$\begin{aligned} w_t &= \frac{\partial w}{\partial t} = \frac{\partial}{\partial t}(F(T)) = \dot{F}, \\ w_{tt} &= \ddot{F}, \\ w_u &= \frac{\partial}{\partial u}(F(T)) = 0. \end{aligned}$$

With the help of this transformation, the simplified form of Eq. (4.1) can be expressed as follows:

$$\begin{aligned} \dot{F} + \ddot{F} &= 0, \\ F &= c_1 + c_2 e^{-T}. \end{aligned}$$

Consequently, the solution to Eq. (4.1) that is invariant under Γ_2 is

$$w = c_1 + c_2 e^{-t}. \tag{4.28}$$

We now plot the 2D graph of this solution of the ODE with different values of c_1 and c_2.

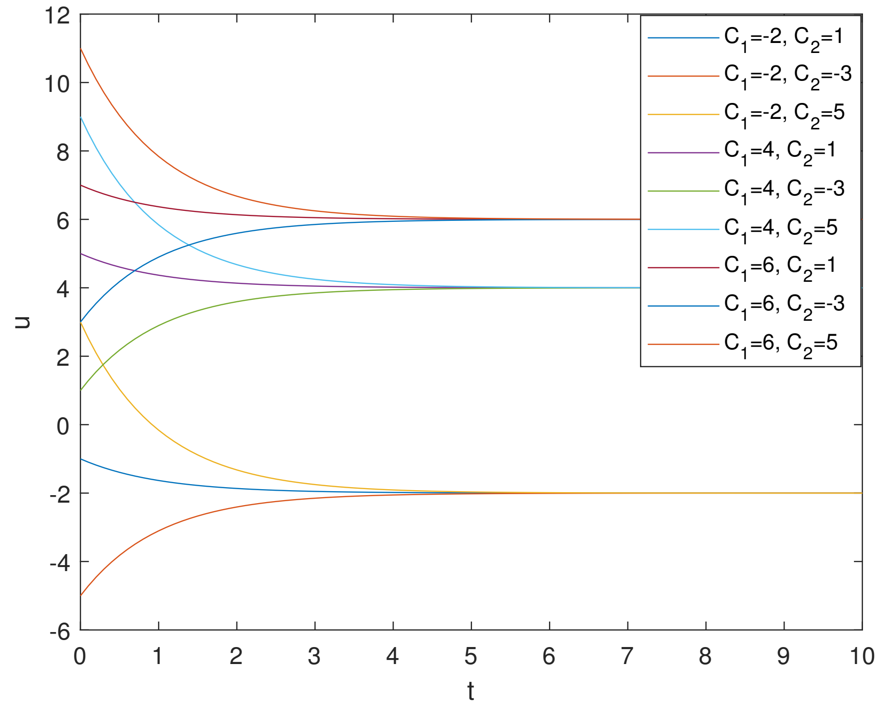


Figure 4.1: 2D plot of $w = c_1 + c_2e^{-t}$

The function $w(t, c_1, c_2)$ is expressed as the sum of a constant term c_1 and an exponentially decreasing term c_2e^{-t}. Each curve illustrated on the plot corresponds to a unique combination of c_1 and c_2. The graphical examination is intended to offer insights into how changes in the constants c_1 and c_2 influence the overall behavior of the function.

Case-ii. For this case, we have $\Gamma^2 = a_1\Gamma_1 + a_2\Gamma_2$. The associated characteristic equation can be expressed as

$$\frac{du}{a_2} = \frac{dt}{a_1} = \frac{dw}{0}.$$

We derive the similarity variables as

$$w = F(g), \quad \text{where} \;\; g = u - ct, \;\; c = \frac{a_2}{a_1}.$$

By differentiating w with respect to both t and u, we get the subsequent expressions,

$$\begin{aligned}
w_t &= \frac{\partial}{\partial t}(F(g)) = \frac{\partial F}{\partial g}\frac{\partial g}{\partial t} = -c\dot{F},\\
w_{tt} &= c^2\ddot{F},\\
w_u &= \frac{\partial}{\partial u}(F(g)) = \frac{\partial F}{\partial g}\frac{\partial g}{\partial u} = \dot{F},\\
w_{uu} &= \ddot{F}.
\end{aligned}$$

Therefore, Eq. (4.1) becomes.

$$c^2\ddot{F} - c\dot{F} - \dot{K}(F)(\dot{F})^2 - K(F)\ddot{F} = 0, \tag{4.29}$$

$$F = c_1, \int^F \frac{e^{Db}(bK-c^2)^{\frac{Dc^2}{K}}}{\int -(bK-c^2)^{\frac{Dc^2-K}{K}} ce^{Db} db + c_1} db - F - c_2 = 0. \tag{4.30}$$

Hence, the following equation represents a solution to Eq. (4.1).

$$w = c_1, \int^w \frac{e^{Db}(bK-c^2)^{\frac{Dc^2}{K}}}{\int -(bK-c^2)^{\frac{Dc^2-K}{K}} ce^{Db} db + c_1} db - w - c_2 = 0 \tag{4.31}$$

Case-iii. Consider $\Gamma^3 = \Gamma_1$. The corresponding characteristic equation is

$$\frac{du}{0} = \frac{dt}{1} = \frac{dw}{0}.$$

The invariant variables can be obtained in this manner

$$let \quad u = U, w = F(U),$$

The results of differentiating w with respect to t and u are as follows;

$$\begin{aligned}
w_t &= \frac{\partial}{\partial t}(F(U)) = 0, & w_{tt} &= 0,\\
w_u &= \frac{\partial}{\partial u}(F(U)) = \frac{\partial F}{\partial U}\frac{\partial U}{\partial u} = \dot{F}, & w_{uu} &= \ddot{F}.
\end{aligned}$$

$$\dot{K}(F)(\dot{F})^2 - K(F)\ddot{F} = 0. \tag{4.32}$$

Consequently, the form of an invariant solution is

$$F = c_1, \ F = \frac{K\ln\left(\frac{\dot{K}(c_1U+c_2)}{K}\right)}{\dot{K}}. \tag{4.33}$$

Hence,

$$w = c_1, \ w = \frac{K\ln\left(\frac{\dot{K}(c_1u+c_2)}{K}\right)}{\dot{K}}. \tag{4.34}$$

Eq. (4.34) represent solutions to Eq. (4.1).

4.3.2 Symmetry Reductions for Case-1

Case-i. Now we examine $\Gamma^1 = a_1\Gamma_1 + a_3\Gamma_3$. The characteristic equation associated with this is

$$\frac{du}{\rho a_3 u} = \frac{dt}{a_1} = \frac{dw}{2a_3}.$$

The invariant variables can be obtained this way

$$\ln u = (a_3/a_1)\rho t - \xi, \quad \xi = (a_3/a_1)\rho t - \ln u, \quad \textit{and}$$
$$U = 2\ln u - \rho w, \quad w = \frac{2\ln u - U}{\rho}.$$

After w differentiates with respect to t and u, the following occurs

$$w_t = \frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial t} = -(a_3/a_1)\dot{U},$$
$$w_{tt} = -(a_3/a_1)\ddot{U},$$
$$w_u = \frac{\partial w}{\partial U}\frac{\partial U}{\partial u} = -\frac{2}{\rho u},$$
$$w_{uu} = \frac{2}{\rho u^2}.$$

This transformation can be used to express the simplified version of Eq. (4.1) as follows:

$$\ddot{U} + \dot{U} + 6\Lambda\left(\frac{a_1}{a_3 e^U}\right) = 0. \tag{4.35}$$

with symmetry generator,

$$\Gamma = \frac{\partial}{\partial \xi}.$$

We consider the transformations $\xi = s$ and $U = r$. Now,

$$\frac{dr}{ds} = \dot{r} = \dot{s}^{-1},$$
$$\ddot{r} = \frac{-\ddot{s}}{\dot{s}^3}.$$

By using this transformation in Eq. (4.35), we obtain

$$-\ddot{s} + \dot{s}^2 + 6\Lambda\dot{s}^3\left(\frac{a_1}{a_3 e^r}\right) = 0. \tag{4.36}$$

Now, let $\dot{s} = S$ and $\ddot{s} = \dot{S}$. Using this in Eq. (4.36), we can transform the ODE into a reduced form as given.

$$\dot{S} - S^2 - (6\Lambda\frac{a_1}{a_3})\frac{S^3}{e^r} = 0. \tag{4.37}$$

Case-ii. In this case, consider $\Gamma^2 = a_2\Gamma_2 + a_3\Gamma_3$. The characteristic equation that corresponds to this is as follows,

$$\frac{du}{(a_2 + a_3\rho u)} = \frac{dt}{0} = \frac{dw}{2a_3}.$$

In the following manner, one can acquire the variables that remain invariant

$$\begin{aligned} w &= \frac{2}{\rho}\ln(\rho u + 1) + \xi, \ a_3 \neq 0, \\ \xi &= w - \frac{2}{\rho}\ln(\rho u + 1), \ \textit{and} \\ t &= T, \end{aligned}$$

By calculating the derivatives of w for both t and u, the following takes place

$$\begin{aligned} w_t &= -\dot{F}, \\ w_{tt} &= -\ddot{F}, \\ w_u &= \frac{2}{\rho u + 1}, \\ w_{uu} &= -\frac{2\rho}{(\rho u + 1)^2}, \end{aligned}$$

the application of this transformation enables the representation of the simplified form of Eq. (4.1) as,

$$\ddot{F} + \dot{F} + \Lambda(4\rho - 1)e^{-\rho F} = 0. \tag{4.38}$$

With symmetry generator,

$$\Gamma = \frac{\partial}{\partial t}.$$

We consider the transformations $t = s$ and $F = r$. Now,

$$\begin{aligned} \frac{dr}{ds} &= \dot{r} = \dot{s}^{-1}, \\ \ddot{r} &= \frac{-\ddot{s}}{\dot{s}^3}. \end{aligned}$$

By using this transformation in Eq. (4.38), we obtain

$$-\ddot{s} + \dot{s}^2 + \frac{3\Lambda \dot{s}^3}{e^r} = 0, \ \ \rho = 1. \tag{4.39}$$

Now, let $\dot{s} = S$ and $\ddot{s} = \dot{S}$. Using this in Eq. (4.39), we can transform the ODE into a reduced form as given.

$$\dot{S} - S^2 - \frac{3\Lambda S^3}{e^r} = 0. \tag{4.40}$$

Case-iii. Consider $\Gamma^3 = \Gamma_3$. The following is an expression for the corresponding characteristic equation.

$$\frac{du}{\rho u} = \frac{dt}{0} = \frac{dw}{2}.$$

The invariant variables can be obtained with this approach,

$$\begin{aligned} w &= \ln(u^{\frac{2}{\rho}}) - \xi, \\ \xi &= \ln(u^{\frac{2}{\rho}}) - w, \ \ and \\ t &= T. \end{aligned}$$

By computing the derivatives of w with respect to both t and u, the following unfolds,

$$\begin{aligned} w_t &= -\dot{F}, \\ w_{tt} &= \ddot{F}, \\ w_u &= \frac{2}{\rho u}, \\ w_{uu} &= -\frac{2}{\rho(u)^2}, \end{aligned}$$

The implementation of this transformation allows for the expression of the simplified form of Eq. (4.1) as follows:

$$\ddot{F} - \dot{F} - \frac{2\Lambda}{\rho} e^{-\rho F} = 0. \tag{4.41}$$

With symmetry generator,

$$\Gamma = \frac{\partial}{\partial t}.$$

We consider the transformations $t = s$ and $F = r$. Now,

$$\frac{dr}{ds} = \dot{r} = \dot{s}^{-1},$$
$$\ddot{r} = \frac{-\ddot{s}}{\dot{s}^3}.$$

By using this transformation in Eq. (4.41), we obtain

$$-\ddot{s} - \dot{s}^2 - \frac{2\Lambda\dot{s}^3}{e^r} = 0, \;\; \rho = 1. \tag{4.42}$$

Now, let $\dot{s} = S$ and $\ddot{s} = \dot{S}$. Using this in Eq. (4.42), we can transform the ODE into a reduced form as given.

$$\dot{S} + S^2 + \frac{2\Lambda S^3}{e^r} = 0. \tag{4.43}$$

Case-iv. Consider the following, $\Gamma^4 = \Gamma_1 \pm \Gamma_2$. The characteristic equation for this has the following expression

$$\frac{du}{1} = \frac{dt}{1} = \frac{dw}{0}.$$

We can obtain the variables that are invariant in the following way:

$$w = F(\xi),$$
$$u = t - \xi, \;\; and$$
$$\xi = t - u.$$

The following can be observed by calculating the derivatives of w with respect to both t and u.

$$w_t = \dot{F},$$
$$w_{tt} = \ddot{F},$$
$$w_u = -\dot{F},$$
$$w_{uu} = \ddot{F}.$$

The implementation of this transformation enables the expression of the simplified form of Eq. (4.1) as follows:

$$\ddot{F} - \dot{F} - \Lambda e^{\rho F}(\rho \dot{F}^2 + \ddot{F}) = 0. \tag{4.44}$$

Therefore, the exact solutions of Eq. (4.44) is.

$$w = c_1, \mathrm{e}^{RootOf(\Lambda\, \mathrm{Ei}_1(-\rho \mathrm{e}^{Z}) + c_2 \mathrm{e}^{\rho c_1} + Z\mathrm{e}^{\rho c_1} + \xi \mathrm{e}^{\rho c_1})} - c_1, \tag{4.45}$$

is the solution to Eq. (4.1).

Case-v. The exact solution for this case are the same as we already did for Case-i of the principal case.

Case-vi. We consider $\Gamma^6 = \Gamma_1$. The characteristic equation associated with this is as follows:

$$\frac{du}{0} = \frac{dt}{1} = \frac{dw}{0}.$$

We determine the similarity variables in the following manner:

$$w = F(U), u = U,$$

after w is differentiated with respect to t and u, the resulting expressions are obtained.

$$\begin{aligned} w_t &= 0, \\ w_{tt} &= 0, \\ w_u &= \dot{F}, \\ w_{uu} &= \ddot{F}. \end{aligned}$$

By applying this transformation, we can express the simplified form of Eq. (4.1) as follows:

$$\rho \dot{F}^2 + \ddot{F} = 0, \tag{4.46}$$

$$F = \frac{\ln(U c_1 \rho + c_2 \rho)}{\rho}. \tag{4.47}$$

Consequently, the exact solution to Eq. (4.1) is,

$$w = \frac{\ln(u c_1 \rho + c_2 \rho)}{\rho}. \tag{4.48}$$

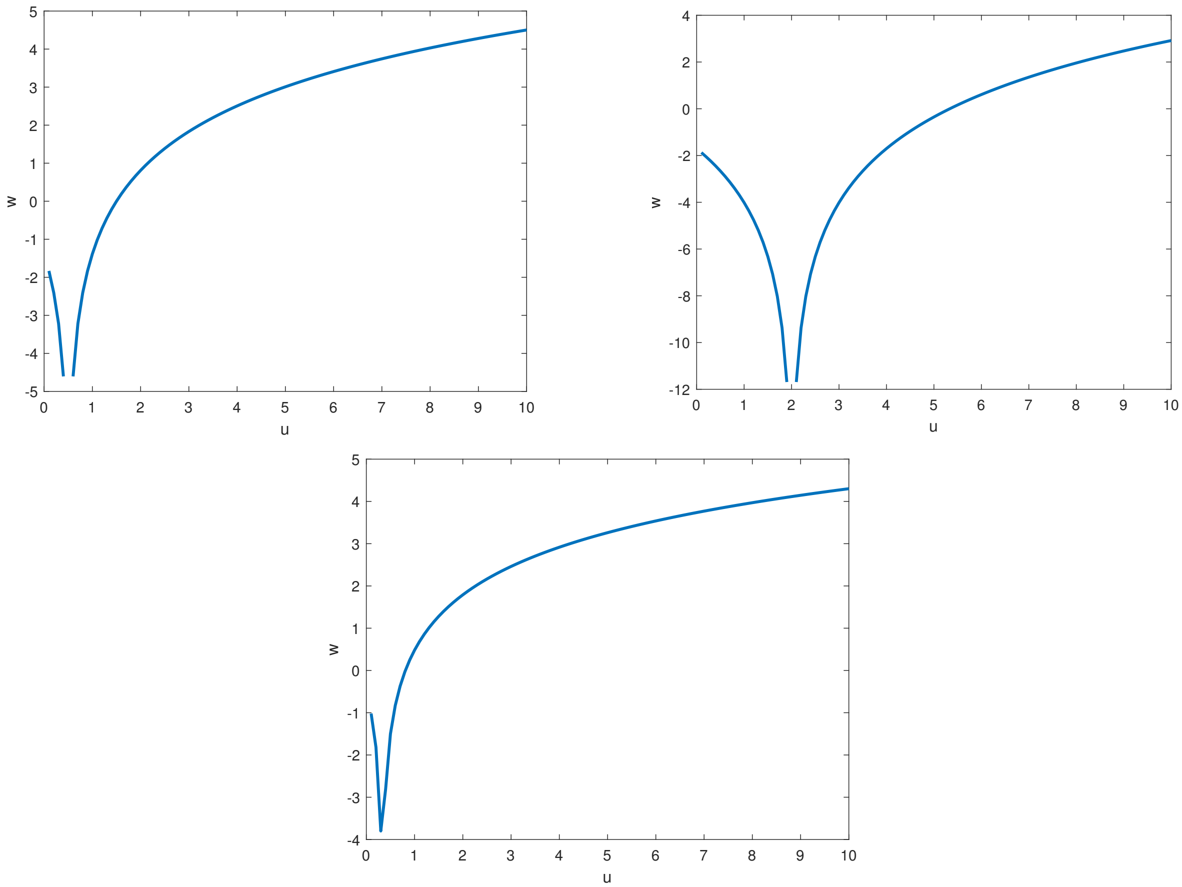


Figure 4.2: Graphical representation of the function $w = \frac{\ln(-uc_1\rho - c_2\rho c_1)}{\rho}$

4.3.3 Symmetry Reductions for Case-2

Case-i. Consider $\Gamma^1 = a_1\Gamma_1 + a_3\Gamma_3$. The characteristic equation is given as

$$\frac{dt}{a_1} = \frac{du}{a_3\rho u} = \frac{dw}{2a_3(w+\mu)}.$$

The similarity variables are,

$$\xi = (a_3/a_1)\rho t - \ln u, \quad and$$
$$U = \frac{2}{\rho}\ln u - \ln(w+\mu).$$

Case-ii. Consider $\Gamma^2 = a_2\Gamma_2 + a_3\Gamma_3$. The corresponding characteristic equation takes the following form

$$\frac{du}{a_2 + a_3\rho u} = \frac{dt}{0} = \frac{dw}{2a_3(w+\mu)},\ a_3 \neq 0.$$

We now derive the invariant variables as,

$$t = T,$$
$$w = F(T)(\rho u + 1)^{\frac{2}{\rho}} - \mu, \quad F(T) = (w + \mu)(\rho u + 1)^{-\frac{2}{\rho}},$$

The following can be observed by calculating the derivatives of w with respect to both t and u.

$$w_t = \dot{F}(\rho u + 1)^{\frac{2}{\rho}},$$
$$w_{tt} = \ddot{F}(\rho u + 1)^{\frac{2}{\rho}},$$
$$w_u = 2(\rho u + 1)^{\frac{2}{\rho}-1}F,$$
$$w_{uu} = 2(2 - \rho)(\rho u + 1)^{\frac{2}{\rho}-2}F.$$

This transformation allows Eq. (4.1) to be expressed in a simplified form as,

$$\ddot{F} + \dot{F} - 6\Lambda F^2 = 0. \tag{4.49}$$

With symmetry generator,

$$\Gamma = \frac{\partial}{\partial t}.$$

We consider the transformations $t = s$ and $F = r$. Now,

$$\frac{dr}{ds} = \dot{r} = \dot{s}^{-1},$$
$$\ddot{r} = \frac{-\ddot{s}}{\dot{s}^3}.$$

By using this transformation in Eq. (4.49), we obtain

$$-\ddot{s} + \dot{s}^2 - 6\Lambda r^2 \dot{s}^3 = 0, \;\; \rho = 1. \tag{4.50}$$

Now, let $\dot{s} = S$ and $\ddot{s} = \dot{S}$. Using this in Eq. (4.50), we can transform the ODE into a reduced form as given.

$$\dot{S} - S^2 + 6\Lambda r^3 S^3 = 0. \tag{4.51}$$

Case-iii. The solution in this case are the same as employed in case-ii above.

Case-iv. In this instance, evaluate $\Gamma^4 = \Gamma_1 \pm \Gamma_2$. The following is the characteristic equation associated with this

$$\frac{du}{1} = \frac{dt}{1} = \frac{dw}{0}.$$

The similarity variables are,

$$u = t - \xi, \ \textit{and}$$
$$w = F(\xi).$$

Determining the derivatives of w with respect to both t and u will result in

$$w_t = \dot{F},$$
$$w_{tt} = \ddot{F},$$
$$w_u = -\dot{F},$$
$$w_{uu} = -\ddot{F}.$$

After putting this transformation into practice, Eq. (4.1) can be expressed in the following simplified form

$$\ddot{F}(1 + \Lambda F + \Lambda\mu) - \dot{F}(\Lambda\dot{F} - 1) = 0. \tag{4.52}$$

As a result, the exact solution for Eq. (4.52) is

$$F = \frac{-c_1\Lambda^2\mu + \Lambda e^{\Lambda c_1(c_2+\xi)} - \Lambda c_1 - 1}{\Lambda^2 c_1}. \tag{4.53}$$

Now, the solution to PDE (4.1) by putting $\xi = t - u$ is

$$w = \frac{\Lambda e^{\Lambda c_1(c_2+t-u)} - \Lambda c_1(\Lambda\mu + 1) - 1}{\Lambda^2 c_1}. \tag{4.54}$$

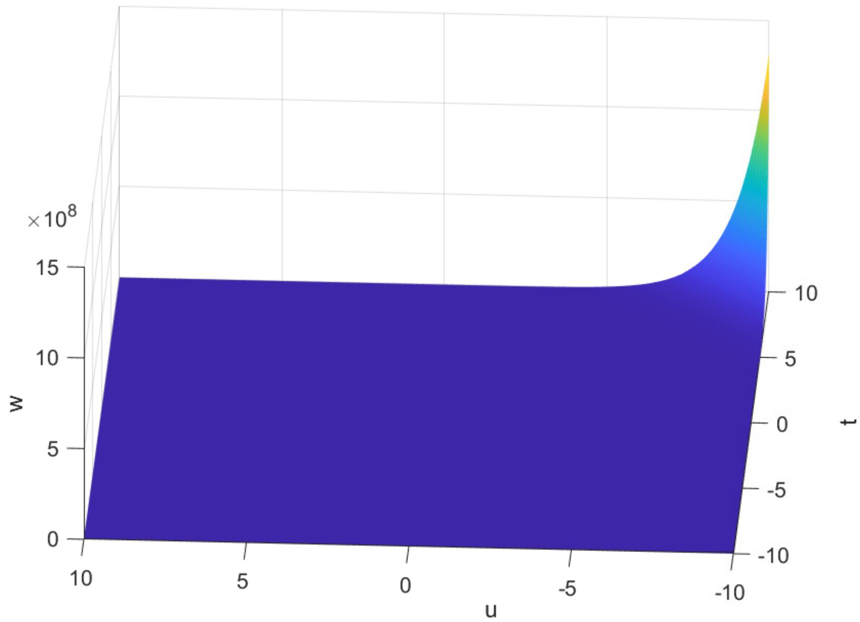


Figure 4.3: 3D plot of $w = \frac{\Lambda e^{\Lambda c_1 (c_2+t-u)} - \Lambda c_1(\Lambda\mu+1)-1}{\Lambda^2 c_1}$

Case-v. Here, analyze $\Gamma^5 = \pm\Gamma_2$. The characteristic equation linked to this is as follows:

$$\frac{du}{1} = \frac{dt}{0} = \frac{dw}{0}.$$

We proceed to deduce the invariant variables as follows:

$$w = F(T), T = t.$$

Now, differentiating w with respect to t and u, we get

$$w_t = \dot{F},$$
$$w_{tt} = \ddot{F},$$
$$w_u = 0.$$

This transformation allows us to describe the simpler form of Eq. (4.1) as

$$\dot{F} + \ddot{F} = 0,$$
$$F = c_1 + c_2 e^{-T}.$$

the exact solution is,

$$w = c_1 + c_2 e^{-t}. \tag{4.55}$$

Case-vi. Here, examine $\Gamma^6 = \Gamma_1$. The characteristic equation associated with this is

$$\frac{du}{0} = \frac{dt}{1} = \frac{dw}{0}.$$

The invariant variables are then determined as

$$u = U, \ and$$
$$w = F(U).$$

Calculating the derivatives of w with respect to t and u yield,

$$w_t = 0,$$
$$w_{tt} = 0,$$
$$w_u = \dot{F},$$
$$w_{uu} = \ddot{F}.$$

This conversion enables us to express the simplified version of Eq. (4.1) as,

$$\Lambda(\dot{F})^2 + \Lambda(F + \mu)\ddot{F} = 0. \tag{4.56}$$

Therefore, the solution to Eq. (4.56) is

$$F = -\mu \pm \sqrt{2c_1 U + \mu^2 + 2c_2}. \tag{4.57}$$

The exact solution of Eq. (4.1) is

$$w = -\mu \pm \sqrt{2c_1 u + \mu^2 + 2c_2}. \tag{4.58}$$

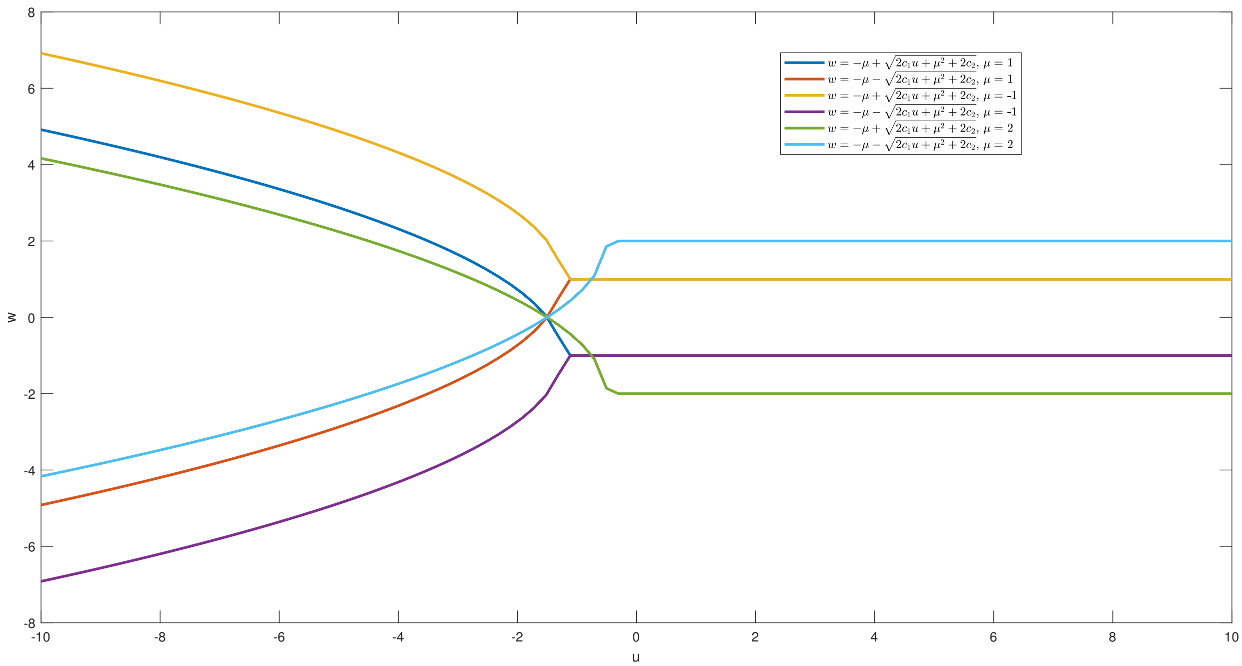


Figure 4.4: 2D plot of $w = -\mu \pm \sqrt{2c_1u + \mu^2 + 2c_2}$

Figure 4.4 displays multiple curves, each corresponding to a different value of μ. For each μ, there are two curves indicating the positive and negative solutions of the function. As μ undergoes changes, the complete set of curves vertically shifts. This adjustment occurs because of the term $-\mu$ in the function, which causes a vertical translation of the graph.

The primary form of each curve is parabolic, representing the quadratic term $\sqrt{2c_1u + \mu^2 + 2c_2}$. The quadratic equation's solutions are shown by the places where the positive and negative curves overlap. The intersection of the function's two branches is at these points.

Chapter 5

Summary

We study the basic ideas that have shaped the field of our research in the introductory chapter. We start by looking at symmetry and optimal systems, which lay the groundwork for our study. Lastly, we dig into the hyperbolic heat equation, a way to understand how heat transfers over time.

The key concepts for understanding Lie point symmetries of differential equations is covered in the second chapter. The first part of the chapter introduces the basic ideas and fundamental concept related to Lie point symmetries. A thorough examination of the essential theoretical aspects is included, such asa definition of Lie point symmetries and their significance in solving differential equations. To illustrate the application of these concepts, an example of the Pavlov equation is presented towards the end of the chapter.

Chapter 3 provides a detailed study of optimal systems, the adjoint representation, and one-dimensional optimal system algebras within the scope of Lie point symmetries. To further improve the reader's understanding, the chapter includes a review of the paper by Ali Raza et al., titled "Optimal System and Conservation Laws for the Generalized Fisher Equation in Cylindrical Coordinates." The study of six cases in this paper highlights the identification of a missing generator in Case-4 and Case-5, as emphasized in the paper's review.

In the final chapter, we study the invariant solutions for the nonlinear hyperbolic heat equation using Lie symmetry analysis and optimal systems of subalgebras. The focus lies on categorizing Lie generators through optimal systems, with special attention given to two specific cases identified in the study. Using an invariance method, we determine the optimal set of non-similar symmetry generators for the nonlinear hyperbolic heat equation, which is visually represented in a convenient tree leaf diagram. The study further involves computing complete symmetry reductions and corresponding invariant solutions for each case. In certain cases, we were able to find the exact solutions, however, in the remaining cases, we have non-linear ODEs whose solutions can be obtained using some other techniques. That solution, if exists, will lead to the exact solution of Eq. (4.1). A detailed analysis is then provided, offering insights into the characteristics of the nonlinear hyperbolic heat equation solutions, all presented in a graphical format for enhanced understanding.

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