Quantum Fisher Information of Atomic System in the Presence of Dissipative Cavity and Kerr Medium

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This thesis is dedicated to my family.

Abstract

The study examines the Quantum Fisher Information (QFI) of an atomic system with a dissipation cavity and kerr-like medium present. We first develop the QFI formulae for two different initial atomic states interacting with a dissipation cavity after developing the basic ideas of quantum information theory, atom field interactions, and quantum fisher information. The effects of the Markovian and non-Markovian environments with weak and strong atom-cavity coupling on the dynamical nature of QFI have been studied. Regarding the Kerr medium parameter of two atoms interacting with a single cavity mode, the second scenario talks about QFI. The results indicate that there is a difference in the peak values for the separable and partially entangled initial states for the Kerr medium QFI. Additionally, the presence or lack of photons within the cavity has an impact on it.

Keywords: *Quantum Fisher Information, Markovian, non-Markovian, atom field interactions, kerr medium, dissipation cavity*

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Abbreviations

CIT	Classical Information Theory
QIT	Quantum Information Theory
JCM	Jaynes Cummings Model
PDF	Probability Distribution Function
CFI	Classical Fisher Information
QFI	Quantum Fisher Information

Chapter 1

Fundamentals of Quantum Information

Between the 1980s and the 1990s, the theory of quantum information developed as a field [1]. Since actual physical goods are utilized to process and transmit information, we could state that "information is physical." However, reality is essentially quantum mechanical; quantum mechanics plays a crucial role in information theory.

Information is traditionally defined as everything that can be sent in the form of symbols, numbers, or alphabets from a sender to a recipient [2]. No matter what kind of physical system is used, information can be transmitted without loss using electrical currents, laser pulses in optical fibers, or symbols on paper. Even in the event that losses occur, the reasons for those losses are well known, and solutions are available. This viewpoint falters when considering quantum information, because it depends on the microscopic particles that are created and seen in an experiment involving quantum mechanics. It becomes crucial to distinguish between the carriers of quantum and classical information.

A classical bit is intended to occupy discrete areas inside its parameter space to designate ’0’ and ’1’. It is often composed of many atoms and represented by continuous signals, such as voltages [3]. Classical computers store and process binary data using bit memory that can exist in the logical states of 0’s and 1’s. Boolean operations like ’NOT, ’AND, and ’OR’ act on individual bits to produce deterministic outputs.

Quantum bits, or "qubits" for short, are the units of information used in quantum information theory. These are essentially tiny systems, like polarized photons or atomic spins. The states $|0\rangle$ and $|1\rangle$ represent the distinct states of these systems, and a qubit is formed by their superposition.

The necessity for information processing and communication has always been a driving force behind the advancement of information technology. Due to this, extremely strong supercomputers have been created, and with the development of quantum technology, we are now on the verge of creating quantum computers, which might answer problems tenfold more quickly than conventional computers.

1.1 What is a Qubit?

To define a quantum bit, or "qubit," the basic unit of information in quantum information theory (QIT), the state vectors from a 2 dimensional Hilbert space $\mathcal{H}$ are used. These state vectors are written as a superposition (linear combination) of two states that are called the "basis" states in $\mathcal{H}$[1]:

$$|q\rangle = c_1|0\rangle + c_2|1\rangle. \tag{1.1}$$

The state $|q\rangle$ is called a qubit, $|0\rangle$ and $|1\rangle$ are the basis states; and c_1 and c_2 are their corresponding probability amplitudes, which satisfy the condition $|c_1|^2 + |c_2|^2 = 1$. Any measurement on a quantum system, described by the state (1.1), will project the state onto $|0\rangle$ and $|1\rangle$ depending upon their probabilities $|c_1|^2$ and $|c_2|^2$ respectively. The coefficients c_1 and c_2 also provide information about the relative phase between the basis states. Accordingly, one way to express the qubit generally is as follows: [4]

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{\iota\phi}|1\rangle, \tag{1.2}$$

where the relative phase is indicated by the exponential factor $e^{\iota\phi}$, which results from a unitary transformation between the states.

1.1.1 Symmetry Transformation

We must examine the idea of symmetry in order to comprehend how we get to the general state of the qubit described in equation (1.2).

Any action on the state of any *object* that leaves it unaltered is called a symmetry transformation. This item is known as the quantum state vector in quantum mechanics, and the symmetry transformation is represented by unitary operators operating on that state. The corresponding unitary operator for every symmetry operation R is $U(R)$ [1]. In dynamical systems, the evolution of the quantum states with respect to time is given by the time evolution operator $U(t, t_0)$:

$$U(t, t_0) = \exp(-\iota \Delta t H), \tag{1.3}$$

where, $\Delta t = t - t_0$ and H is the Hamiltonian that governs the dynamics of the quantum system. Thus, $U(t, t_0)$ generates a time translation symmetry.

Let us now create the unitary operator of a transformation, say *rotation.* By taking into account an infinitesimal rotation θ, we may achieve that. The transformation R performs this rotation and is given for the infinitesimal rotation as

$$R = 1 + \epsilon T, \tag{1.4}$$

where T is some rotation generator and $\epsilon << 1$. For R, the unitary operator is:

$$U(R) = \mathcal{I} + \iota\epsilon Q + Q(\epsilon^2). \tag{1.5}$$

The higher-order contributions are included in the third expression. Q is an operator for an observable. The following criteria must be satisfied in order for us to describe the symmetry operations by an operator like a unitary operator:

$$U(R_1)U(R_2) = U(R_1 o R_2). \tag{1.6}$$

Let ϵ be the outcome of $1/N$ times the rotation θ, thus $\epsilon = \frac{\theta}{N}$, to provide a finite rotation.This indicates that there are N infinitesimal divisions of length ϵ in the rotation θ. If we continue this process N times, we obtain

$$R = (\mathcal{I} + \frac{\theta}{N} T)^N, \tag{1.7}$$

and the unitary operator for R will be

$$U(R) = (\mathcal{I} + \frac{\theta}{N} Q)^N, \tag{1.8}$$

for $N \to \infty$, we can write $U(R)$ in the exponential form,

$$U(R) = \exp(\iota\theta Q). \tag{1.9}$$

The fact that the sequence of operations is irrelevant is a significant outcome of symmetry transformations. A system that experiences rotation after time evolution $(U(t - t_0))$ will arrive at the same end state as a system that experiences rotation after time evolution. The following commutator yields this outcome.

$$[U(t-t_0), U(R)] = 0. \tag{1.10}$$

Given that the angular momentum serves as the generator of rotational symmetry and that the operator Q is observable, the operation R for an infinitesimal rotation $d\theta$ about the unit vector $\hat{n}$ is

$$R(\hat{n}, d\theta) = \mathcal{I} + \iota d\theta \hat{n}.J, \tag{1.11}$$

and for a finite rotation θ

$$U(\hat{n}, \theta) = \exp{(\iota\theta\hat{n}.J)}, \tag{1.12}$$

in this case, the angular momentum operator is denoted by J, and its components (J_x, J_y, J_z) adhere to the following commutation relation:

$$[J_k, J_l] = \iota\epsilon_{klm}J_m, \qquad \text{k=l=m = x, y, z} \tag{1.13}$$

where $J_k = \frac{1}{2}\sigma_k$ ($\hbar$ has been set as 1). The Pauli matrices are denoted by σ_k. Now, equation (1.12) becomes

$$U(\hat{n}, \theta) = \exp{(\iota\frac{\theta}{2}\hat{n}.\sigma)}. \tag{1.14}$$

Expanding the above equation upto first order term

$$U(\hat{n}, \theta) = \mathcal{I} + \iota\frac{\theta}{2}(\hat{n}.\sigma). \tag{1.15}$$

By making use of the condition, $\theta << 1$, we can apply the following approximations

$$\begin{aligned} \sin\frac{\theta}{2} &\approx \frac{\theta}{2}, \\ \cos\frac{\theta}{2} &\approx 1. \end{aligned} \tag{1.16}$$

Thus, unitary operator $U(\hat{n}, \theta)$ becomes

$$U(\hat{n}, \theta) = \mathcal{I}\cos\frac{\theta}{2} + \iota\sin\frac{\theta}{2}(\hat{n}.\sigma), \tag{1.17}$$

where the dot product $\hat{n}.\sigma = n_x\sigma_x + n_y\sigma_y + n_z\sigma_z$. This indicates that a qubit may conduct a total of 4 rotation operations $(\mathcal{I}, \sigma_x, \sigma_x, \sigma_y, \sigma_z)$, which can be about the x, y, or z axis. All of space is covered by these rotation operation.

An arbitrary unit vector $\hat{n}$ oriented in space has the components $(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ and lets say we have a state, $|\psi\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, aligned along the z-axis. Now we want to rotate $\hat{n}$ such that it coincides with the z-axis. This can be done by considering an axis of rotation with the components such as $\hat{n}^{'} = (\sin\phi, -\cos\phi, 0)$ and substitute this $\hat{n}^{'}$ in place of $\hat{n}$ in equation (1.17), we will get the following $U(\hat{n}, \theta)$

$$U(\hat{n}^{'}, \theta) = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-\iota\phi}\sin\frac{\theta}{2} \\ e^{\iota\phi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}. \tag{1.18}$$

Applying this operator $U(\hat{n}^{'}, \theta)$ on the state $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ we get the general qubit state as

$$|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + \sin\frac{\theta}{2}e^{\iota\phi}|1\rangle. \tag{1.19}$$

The state given in equation (1.19) is geometrically represented as vector on a unit sphere known as *Bloch Sphere* [4] and any operation on the state is given by a rotation on this sphere.

1.1.2 Operations on a Qubit

Now we look at some specific examples for understanding these rotation operations on a qubit. For this, the generators of rotation are the three Pauli matrices σ_x, σ_y, σ_z and the identity matrix I. Consider the qubit state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{1.20}$$

and lets see some of the operations done on it. The Identity operator will leave the state unchanged. For σ_x

$$\sigma_x|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle\langle 1| + |1\rangle\langle 0|)[|0\rangle + |1\rangle], \tag{1.21}$$

$$= \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{1.22}$$

here, σ_x changed the basis vectors from $|0\rangle$ to $|1\rangle$ and $|1\rangle$ to $|0\rangle$ thus it acts like a bit flip operator but since it is a symmetric state the overall action resulted in the original state. The scenario would be different for an anti-symmetric state like

$$\sigma_x|\psi\rangle = \sigma_x(\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)), \tag{1.23}$$

$$= \frac{1}{\sqrt{2}}(|1\rangle - |0\rangle) = -|\psi\rangle, \tag{1.24}$$

Next we have the σ_z operator

$$\sigma_z|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle\langle 0| - |1\rangle\langle 1|)[|0\rangle + |1\rangle], \tag{1.25}$$

$$= \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle). \tag{1.26}$$

The action of σ_z changed the relative phase between the basis states. Finally, we have σ_y

$$\sigma_y|\psi\rangle = \frac{1}{\sqrt{2}}(-\iota|0\rangle\langle 1| + \iota|1\rangle\langle 0|)[|0\rangle + |1\rangle], \tag{1.27}$$

$$= \frac{1}{\sqrt{2}}(\iota|1\rangle - \iota|0\rangle), \tag{1.28}$$

$$= -\frac{\iota}{\sqrt{2}}(|0\rangle - |1\rangle), \tag{1.29}$$

we can see that σ_y functions as the bit flip operator as well as changes the relative phase.

Thus, we have shown some of the basic operators that can be applied to a qubit, however many more of such operations are possible as you dive deeper into the field of quantum information.

1.2 Density Matrix

According to quantum mechanics, all available information on a system's state is presented in the form of a state vector, such as equation (1.1), assuming that α and β are known [5]. Pure state is the term used to describe such a condition. However, in practice, quantum systems are not isolated; rather, they can be interacting with an *ensemble*, which is a reservoir containing an abundance of states.The density matrix is crucial for writing down these mixtures of states as well as for distinguishing between pure and mixed states.

Definition:

The density matrix $\mathcal{D}$ for a pure state $|q\rangle$ is defined as a projection operator of that state

$$\mathcal{D} = |q\rangle\langle q|. \tag{1.30}$$

Lets take an example, Consider the state

$$|q\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{1.31}$$

the density matrix will be

$$\begin{aligned}\mathcal{D} &= |q\rangle\langle q|,\\ &= \frac{1}{2}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|),\end{aligned} \tag{1.32}$$

and in matrix form we can write,

$$\mathcal{D} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{1.33}$$

Now, the importance of the density matrix is to conveniently express a mixture of states, which otherwise cannot be done in the form of a single state vector. Again, consider a mixture of the following three pure states:

$$|q_1\rangle = |1\rangle, \tag{1.34}$$

$$|q_2\rangle = |0\rangle, \tag{1.35}$$

$$|q_3\rangle = \frac{1}{\sqrt{2}}(|1\rangle + |0\rangle), \tag{1.36}$$

such that each of the states is a $\frac{1}{3}$ of the mixture, so a mixed state density matrix will be the sum of the density matrices of the individual pure states times their weighted contribution in the mixture

$$\mathcal{D} = \frac{1}{3}\mathcal{D}_{q_1} + \frac{1}{3}\mathcal{D}_{q_2} + \frac{1}{3}\mathcal{D}_{q_3}, \tag{1.37}$$

$$\mathcal{D} = \frac{1}{3}|q_1\rangle\langle q_1| + \frac{1}{3}|q_2\rangle\langle q_2| + \frac{1}{3}|q_3\rangle\langle q_3|, \tag{1.38}$$

$$\mathcal{D} = \frac{1}{3}|1\rangle\langle 1| + \frac{1}{3}|0\rangle\langle 0| + \frac{1}{6}(|0\rangle\langle 0| + |0\rangle\langle 1| + |1\rangle\langle 0| + |1\rangle\langle 1|), \tag{1.39}$$

$$\mathcal{D} = \frac{1}{2}|1\rangle\langle 1| + \frac{1}{6}|1\rangle\langle 0| + \frac{1}{6}|0\rangle\langle 1| + \frac{1}{2}|0\rangle\langle 0|, \tag{1.40}$$

and in matrix form, we will have,

$$\mathcal{D} = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix}. \tag{1.41}$$

If we observe the equations (1.33) and (1.41), for pure state we have

$$\mathcal{D}_{nn}\mathcal{D}_{mm} = \mathcal{D}_{nm}\mathcal{D}_{mn}, \tag{1.42}$$

where as for the mixed state,

$$\mathcal{D}_{nn}\mathcal{D}_{mm} \neq \mathcal{D}_{nm}\mathcal{D}_{mn}. \tag{1.43}$$

This is one of the ways to differentiate between a pure and a mixed state if we are given any density matrix. A completely mixed state would have the off-diagonal terms zero and it will just be a statistical mixture, which can also be dealt with classically.
Thus, we can now define the density matrix for mixed state in a more general way as

$$\mathcal{D} = \sum_m p_m |q_m\rangle\langle q_m|, \tag{1.44}$$

where p_m are the probabilities associated to $|q_m\rangle$.

1.2.1 Properties of Density Matrix

Now we will see some of the characteristic properties of the density matrix [5] [6],

1) $\mathcal{D}$ is Hermitian ($\mathcal{D}^\dagger = \mathcal{D}$):

$$\mathcal{D} = \sum_m p_m |q_m\rangle\langle q_m|, \tag{1.45}$$

$$\mathcal{D}^\dagger = \sum_m p_m^* |q_m\rangle\langle q_m|, \tag{1.46}$$

$$p_m^* = p_m,$$

$$\mathcal{D}^\dagger = \sum_m p_m |q_m\rangle\langle q_m| = \mathcal{D}. \tag{1.47}$$

2) $\mathcal{D}$ is positive semi - definite operator:

This means that $\forall |q\rangle$ the condition, $\langle q|\mathcal{D}|q\rangle \geq 0$ is true.

Proof:

$$\langle q|\mathcal{D}|q\rangle = \langle q| \sum_m p_m |q_m\rangle\langle q_m|q\rangle, \tag{1.48}$$

$$= \sum_i p_m \langle q|q_m\rangle\langle q_m|q\rangle, \tag{1.49}$$

$$= \sum_m p_m \langle q|q_m\rangle\langle q|q_m\rangle^*, \tag{1.50}$$

$$= \sum_m p_m |\langle q|q_m\rangle|^2. \tag{1.51}$$

It will always be positive except when $|q\rangle$ and $|q_m\rangle$ are orthogonal; only then will it be zero.

3) $\mathbf{Tr}[\mathcal{D}] = 1$:

Proof:

$$\mathrm{Tr}[\mathcal{D}] = \mathrm{Tr}(\sum_m p_m |q_m\rangle\langle q_m|), \tag{1.52}$$

$$= \sum_m p_m \mathrm{Tr}(|q_m\rangle\langle q_m|), \tag{1.53}$$

$$= \sum_m p_m = 1. \tag{1.54}$$

4) Tr$[\mathcal{D}^2]$**:**

This property has different results for pure and mixed states.

a) For pure state, $\mathrm{Tr}[\mathcal{D}^2] = 1$.

Proof:

$$\mathrm{Tr}[\mathcal{D}^2] = \mathrm{Tr}(|q\rangle\langle q|q\rangle\langle q|), \tag{1.55}$$

$$= \mathrm{Tr}(|q\rangle\langle q|) = 1. \tag{1.56}$$

b) For a mixed state $\mathrm{Tr}[\mathcal{D}]^2 < 1$.

Proof: It will be easier to understand this if we use matrix example from equation (1.41). So,

$$\mathcal{D}^2 = \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{2} & \frac{1}{6} \\ \\ \frac{1}{6} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} \frac{5}{18} & \frac{1}{6} \\ \\ \frac{1}{6} & \frac{5}{18} \end{pmatrix}. \tag{1.57}$$

The trace of this matrix will be sum of the diagonal elements

$$\mathrm{Tr}[\mathcal{D}^2] = \mathrm{Tr}\begin{pmatrix} \frac{5}{18} & \frac{1}{6} \\ \\ \frac{1}{6} & \frac{5}{18} \end{pmatrix}, \tag{1.58}$$

$$= \frac{5}{18} + \frac{5}{18} = \frac{5}{9} \approx 0.5 < 1. \tag{1.59}$$

5) All eigen values of $\mathcal{D}$ lie within the interval $[0, 1]$

Other than these properties, it is worth mentioning here that the expectation value for any operator **A** can be expressed as follows

$$\langle A\rangle = \mathrm{Tr}(\mathcal{D}A). \tag{1.60}$$

This result will be helpful in later sections.

1.2.2 Time Evolution of Density Matrix

If we are given the state $|q(t_0)\rangle$ of the system at initial time t_0, then in quantum mechanics, the state $|q(t)\rangle$ at some later time t can be evaluated using the Schrodinger equation[7].

$$H(t)|q(t)\rangle = \imath\hbar\frac{d}{dt}|q(t)\rangle. \tag{1.61}$$

Using the time evolution operator, we can write

$$|q(t)\rangle = U(t,t_0)|q(t_0)\rangle. \tag{1.62}$$

By substituting this in the Schrodinger equation above, we will arrive to the following equation:

$$\imath\hbar\frac{d}{dt}U(t,t_0) = H(t)U(t,t_0). \tag{1.63}$$

Here $U(t,t_0)$ is the unitary operator, which means that,

$$U(t,t_0)U^\dagger(t,t_0) = U^\dagger(t,t_0)U(t,t_0) = \mathcal{I} \tag{1.64}$$

In case of a closed or isolated physical system, the Hamiltonian becomes time-independent and we can write

$$U(t,t_0) = \exp\left(-\imath\hbar(t-t_0)H\right) \tag{1.65}$$

To find the equation which describes the evolution of density matrices, let us assume that the state of the system at some time t is given by the density matrix

$$\mathcal{D}(t) = \sum_k p_m|q_m(t)\rangle\langle q_m(t)|, \tag{1.66}$$

again, by applying the substitution from equation (1.62),

$$\mathcal{D}(t) = \sum_k p_m U(t,t_0)|q_k(t_0)\rangle\langle q_k(t_0)|U^\dagger(t,t_0). \tag{1.67}$$

This equation reduces to,

$$\mathcal{D}(t) = U(t,t_0)\mathcal{D}(t_0)U^\dagger(t,t_0), \tag{1.68}$$

Finally, after differentiating with respect to time, we will get the equation of motion for the density matrix [5][8]

$$\frac{d}{dt}\mathcal{D}(t) = -\frac{\imath}{\hbar}[H(t),\mathcal{D}(t)]. \tag{1.69}$$

This is called the *von Neumann* equation and it governs the dynamics of the state of the closed system in density matrix formalism.

1.2.3 Open Quantum system and Markovian Master Equation

The dynamical equation (1.67) is used when the system is closed and isolated from its surroundings and the evolutions is unitary. Thus, there is no exchange of energy and information between the system and its surroundings. However, in general quantum systems are not isolated and it thus requires modifications to be made in the dynamical equation of the density matrix. These modifications will account for the interactions of the system and its environment. These dynamical equations of the density matrix are also called quantum master equations and there are several different approaches to acquire such an equation and the one particular we are interested in is called Microscopic approach to a Markovian quantum master equation.

We start by writing the Hamiltonian for the total system belonging to the Hilbert space $\mathcal{H}_{sys} + \mathcal{H}_{env}$ [7],

$$H = H_{sys} + H_{env} + H_I, \tag{1.70}$$

The free Hamiltonians of the system of interest and environment are denoted by H_{sys} and H_{env} in this case, while H_I represents the interaction Hamiltonian. Since we are dealing with the composite systems (system + environment), the initial state density matrix will be a tensor product of the initial states of the individual sub-systems

$$\mathcal{D}_{sys-env}(0) = \mathcal{D}_{sys}(0) \otimes \mathcal{D}_{env}. \tag{1.71}$$

The unitary transformation that follows provides the time evolution of $\mathcal{D}$:

$$\mathcal{D}_{sys-env}(t) = U(t)\mathcal{D}_{sys-env}(0)U^{\dagger}(t), \tag{1.72}$$

and $U = \exp(-\iota H t)$. Tracing out the environmental degrees of freedom, we get the state of the system only

$$\begin{aligned} \mathcal{D}_{sys} &= Tr_{env}[U(t)\mathcal{D}_{sys-env}(0)U^{\dagger}(t)], && (1.73)\\ &= Tr_{env}[U(t)\mathcal{D}_{sys}(0) \otimes \mathcal{D}_{env}U^{\dagger}(t)], && (1.74)\\ &\equiv \mathcal{V}_t\mathcal{D}_{sys}(0), && (1.75) \end{aligned}$$

where $\mathcal{V}_t$ is an operator that maps $\mathcal{D}_{sys}(0)$ to $\mathcal{D}_{sys}(t)$, called a dynamical mapping. The master equation is derived while making use of a number of approximations. In the interaction picture, we will start with the von-Neumann equation for the total density matrix $\mathcal{D}_{sys-env}(t)$.

$$\frac{\partial}{\partial t}\mathcal{D}_{sys-env}(t) = -\iota[H_I(t), \mathcal{D}_{sys-env}(t)], \tag{1.76}$$

here we have taken $\hbar = 1$. The integral form is given as

$$\mathcal{D}_{sys-env}(t) = \mathcal{D}_{sys-env}(0) - \iota\int_0^t [H_I(s), \mathcal{D}_{sys-env}(s)]ds, \tag{1.77}$$

inserting this equation (1.77) into (1.76)

$$\frac{\partial}{\partial t}\mathcal{D}_{sys-env}(t) = -\iota[H_I(t), \mathcal{D}_{sys-env}(0)] - \int_0^t [H_I(t), [H_I(s), \mathcal{D}_{sys-env}(s)]]ds. \tag{1.78}$$

tracing over the states of the environment, leaves us with the state of the system

$$\frac{\partial}{\partial t}\mathcal{D}_{sys}(t) = -\int_0^t \mathrm{Tr}_{env}\{[H_I(t), [H_I(s), \mathcal{D}_{sys-env}(s)]]\}ds, \tag{1.79}$$

here we assumed that[7]

$$\mathrm{Tr}_{env}\{[H_I(t), \mathcal{D}_{sys-env}(0)]\} = 0. \tag{1.80}$$

It is now that we wish to make our first approximation, the *Born Approximation*, according to this the system is weakly coupled to its environment, the interaction will have a negligible impact on the system and the final density matrix of the composite system can be written as:

$$\mathcal{D}_{sys-env} \approx \mathcal{D}_{sys} \otimes \mathcal{D}_{env}, \tag{1.81}$$

with this we get the closed integro-differential equation for $\rho_{sys}(t)$

$$\frac{\partial}{\partial t}\mathcal{D}_{sys}(t) = -\int_0^t \mathrm{Tr}_{env}\{[H_I(t), [H_I(s), \mathcal{D}_{sys}(s) \otimes \mathcal{D}_{env}]]\}ds. \tag{1.82}$$

It should be noted that the system's evolution at time t in the previously mentioned equation depends on the previous states of $\mathcal{D}_{sys}(s)$ as determined by the integral for values $s < t$. Stated differently, the states $\mathcal{D}_{sys}(s)$ at all times $s < t$ determine the change in $\mathcal{D}_{sys}(t)$ at time t. The *Markovian* approximation is the second approximation that we shall use. Under this approximation we want to find the time local equation for the evolution of the state of the system. To accomplish this, substitute $\mathcal{D}_{sys}(t)$ in place of $\mathcal{D}_{sys}(s)$.

$$\frac{\partial}{\partial t}\mathcal{D}_{sys}(t) = -\int_0^t \mathrm{Tr}_{env}\{[H_I(t), [H_I(s), \mathcal{D}_{sys}(t) \otimes \mathcal{D}_{env}]]\}ds. \tag{1.83}$$

If there is no significant change in the density matrix $\mathcal{D}_{sys}(s)$ over the interval $[0, t]$, we can make this modification. We refer to this time-local equation (1.83) as the **Redfield** equation. The upper limit of the integral approaches infinity and s is substituted with $(t-s)$. The difference between the current state's time t and a previous state's time s is represented by the $t-s$. Setting the upper limit to infinity indicates that the system's evolution time τ_R (relaxation time) is significantly longer than the environment's correlation time τ_{env}.

$$\frac{\partial}{\partial t}\mathcal{D}_{sys}(t) = -\int_0^{\infty} \mathrm{Tr}_{env}\{[H_I(t),[H_I(t-s),\mathcal{D}_{sys}(t)\otimes\mathcal{D}_{env}]]\}ds. \tag{1.84}$$

This is the Markovian master equation in the interaction picture, which describes the evolution of the state of the system such that the previous states have no influence on the present state. This is also known as *memory-less* behaviour because the system does not have any memory of its past.

Let us now decompose the Schrodinger picture interaction hamiltonian H_I as follows [9]

$$H_I = \sum_l A_l \otimes E_l, \tag{1.85}$$

where A_l and E_l are the corresponding system and environment operators.These operators are hermitian i.e. $A_l = A_l^l$ and $E_l = E_l^l$. We are now supposing that the spectrum of H_{sys} is discrete, this may be achieved by writing the system operators as

$$A_l(\omega) = \sum_{\epsilon'-\epsilon=\omega} \prod(\epsilon) A_l \prod(\epsilon'), \tag{1.86}$$

ϵ are the eigenvalues of H_{sys} and $\prod(\epsilon)$ are the projection operators of the eigenspace of H_{sys}. The system operators (A_l) satisfy the following commutation relations:

$$[H_{sys}, A_l(\omega)] = -\omega A_l(\omega), \tag{1.87}$$

$$[H_{sys}, A_l(\omega)] = +\omega A_l(\omega), \tag{1.88}$$

where $A_l(\omega)$ and $A_l^\dagger(\omega)$ are said to be eigen operators of H_S belonging to the frequencies $\pm\omega$. The interaction picture operators for the system and environment are:

$$A_l(\omega,t) = \exp[-\iota\omega t]A_l(\omega), \qquad A_l^\dagger(\omega,t) = \exp[+\iota\omega t]A^{\iota}(\omega) \tag{1.89}$$

$$E_l(t) = U^\dagger E_l U. \tag{1.90}$$

By using these conversions and doing some algebra, one arrives at the following master equation for the system: [9] [7]

$$\begin{aligned}\frac{\partial}{\partial t}\mathcal{D}_{sys}(t) =& -\iota[H_{sys},\mathcal{D}_{sys}(t)]\\ &+\{\sum_{l}\sum_{\omega>0}\gamma(\omega)[A_l(\omega)\mathcal{D}_{sys}(t)A^{\dagger}_{l\,\omega}-\frac{1}{2}\{A^{\dagger}_l(\omega)A_l(\omega),\mathcal{D}_{sys}(t)\}]+(\text{h.c})\},\end{aligned} \tag{1.91}$$

where $A_l(-\omega) = A^{\dagger}_l(\omega)$ has been used. In the later sections we shall apply this procedure on the Jaynes Cummings Model under the initial condition that the total number of excitation in the atom-cavity is one.

1.3 Bipartite Quantum System

Suppose we have two particles and we label them as A and B. Each particle can be a separate system and described by state vectors. These state vectors are linear combinations of some basis vectors that belong to a Hilbert space. Since the two particles are separate systems, we define two sets of basis vectors: $\{|\phi_{Ai}\rangle\}$ and $\{|\phi_{Bj}\rangle\}$ that belong to their own Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively.

Let us write the state vectors corresponding to the particles A and B as

$$|\psi_A\rangle = \sum_k a_k|\phi_{Ak}\rangle, \tag{1.92}$$

$$|\psi_B\rangle = \sum_l b_l|\phi_{Bl}\rangle, \tag{1.93}$$

where $|\phi_{Ak}\rangle \in \mathcal{H}_A$ and $|\phi_{Bl}\rangle \in \mathcal{H}_B$.

To define a system for more than one particle, we combine the two Hilbert spaces by using the tensor product,

$$\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B. \tag{1.94}$$

The state vector in this combined Hilbert space will also be the tensor product of $|\psi_A\rangle$ and $|\psi_B\rangle$, [6]

$$|\psi_{AB}\rangle = |\psi_A\rangle \otimes |\psi_B\rangle, \tag{1.95}$$

$$= \sum_{k,l} a_k b_l|\phi_{Ak}\rangle \otimes |\phi_{Bl}\rangle, \tag{1.96}$$

$$= \sum_{k,l} \mathcal{C}_{kl}|\phi_{Ak}\rangle \otimes |\phi_{Bl}\rangle, \tag{1.97}$$

here $\mathcal{C}_{kl} = a_k b_l$. Let us consider an example:

Say we have two qubits of the form,

$$|q_A\rangle = c_1|0\rangle + c_2|1\rangle, \tag{1.98}$$

$$|q_B\rangle = c_3|0\rangle + c_4|1\rangle, \tag{1.99}$$

their product state is then

$$\begin{aligned}
|q_{AB}\rangle &= (c_1|0\rangle + c_2|1\rangle) \otimes (c_3|0\rangle + c_4|1\rangle), && (1.100)\\
&= c_1c_3|0\rangle \otimes |0\rangle + c_1c_4|0\rangle \otimes |1\rangle + c_2c_3|1\rangle \otimes |0\rangle + c_2c_4|1\rangle \otimes |1\rangle, && (1.101)\\
&= c_1c_3|00\rangle + c_1c_4|01\rangle + c_2c_3|10\rangle + c_2c_4|11\rangle. && (1.102)
\end{aligned}$$

We can see that, this product state can easily be separated into two states, which is a distinguishing feature of a separable product state from an entangled state, which is not separable.

1.3.1 Measurements in Combined System

We mentioned before in equation (1.60) that for any operator, its expectation value is related to the density matrix. As the expectation value gives the information about the measurement outcome and we are focusing on the density matrix formalism, it is, therefore important to write the product state in terms of density matrix. [6].

$$\begin{aligned}
\mathcal{D}_{AB} &= |\psi_{AB}\rangle\langle\psi_{AB}| && (1.103)\\
&= \sum_{i,j} a_i b_j a_i^* b_j^* |\phi_{Ai}\rangle \otimes |\phi_{Bj}\rangle\langle\phi_{Ai}| \otimes \langle\phi_{Bj}| && (1.104)\\
&= \sum_{i,j} a_i a_i^* |\phi_{Ai}\rangle\langle\phi_{Ai}| \otimes b_j b_j^* |\phi_{Bj}\rangle\langle\phi_{Bj}| && (1.105)\\
&= \sum_{i,j} |a_i|^2 |\phi_{Ai}\rangle\langle\phi_{Ai}| \otimes |b_j|^2 |\phi_{Bj}\rangle\langle\phi_{Bj}| && (1.106)\\
&= |\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B| && (1.107)\\
\mathcal{D}_{AB} &= \mathcal{D}_A \otimes \mathcal{D}_B && (1.108)
\end{aligned}$$

Thus, the tensor product of density matrices of the states $|\psi_A\rangle$ and $|\psi_B\rangle$ gives the density matrix of the product state $|\psi_{AB}\rangle$.

Now, if O is any operator acting on $|\psi_{AB}\rangle$, the expectation value is given by

$$\langle O\rangle = \mathrm{Tr}([O_A \otimes O_B]\mathcal{D}_{AB}) \tag{1.109}$$

it shows that the operator O is acting on both Hilbert spaces A and B. It is also possible that the operator O just acts on one of the states in the combined system say the state corresponding to particle A, in that case

$$\langle O_A \rangle = \mathrm{Tr}([O_A \otimes I_B]\mathcal{D}_{AB}) \tag{1.110}$$

The identity I_B means that the $\mathcal{H}_B$ is unaffected by the action of O_A. Another aspect of the product state density matrix is that trace over one Hilbert space leaves us with the state of the other Hilbert space,

$$\mathcal{D}_A = Tr_B[\mathcal{D}_{AB}] \tag{1.111}$$

The expectation value of O_A will be [6]

$$\begin{aligned}
\langle O_A \rangle &= \mathrm{Tr}([O_A \otimes I_B]\mathcal{D}_{AB}) && (1.112)\\
&= \sum_{i,j} \langle \phi_{Bj}| \otimes \langle \phi_{Ai}|([O_A \otimes I_B]\mathcal{D}_{AB})|\phi_{Ai}\rangle \otimes |\phi_{Bj}\rangle && (1.113)\\
&= \sum_{i,j} \langle \phi_{Ai}|O_A \otimes \langle \phi_{Bj}|I_B\mathcal{D}_{AB}|\phi_{Bj}\rangle|\phi_{Ai}\rangle && (1.114)\\
&= \sum_{i,j} \langle \phi_{Ai}|O_A \otimes \langle \phi_{Bj}|\mathcal{D}_{AB}|\phi_{Bj}\rangle|\phi_{Ai}\rangle && (1.115)
\end{aligned}$$

In the above equation, the expression $\langle \phi_{Bj}|\mathcal{D}_{AB}|\phi_{Bj}\rangle$ is the trace of $\mathcal{D}_{AB}$ over the Hilbert space B, which, according to equation (1.111), is equal to $\mathcal{D}_A$. Thus,

$$\begin{aligned}
\langle O_A \rangle &= \sum_{i} \langle \phi_{Ai}|O_A\mathcal{D}_A|\phi_{Ai}\rangle && (1.116)\\
&= Tr_A[O_A\mathcal{D}_A] && (1.117)
\end{aligned}$$

Hence, we have just seen that how the product state of a combined system can be reduced to any one of the individual states. However, there exist such states that are non-separable. These states are called **Entangled** states.

1.3.2 Entanglement

In quantum mechanics, the concept of entanglement arises in composite systems (combination of many sub-systems) [6]. We have already seen that the Hilbert space of a composite system is the tensor product of the Hilbert spaces of individual sub-systems and in density matrix formalism, we write it as,

$$\mathcal{D}_{AB} = \mathcal{D}_A \otimes \mathcal{D}_B \tag{1.118}$$

where $\mathcal{D}_A$ and $\mathcal{D}_B$ belong to the sub-systems A and B. A pure state of the composite system is said to be entangled if it cannot be written in the form of a product state; in other words,

$$\mathcal{D}_{AB} \neq \mathcal{D}_A \otimes \mathcal{D}_B \tag{1.119}$$

One of the examples of entangled states are the Bell states:

$$|\psi_+^{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{1.120}$$

$$|\psi_-^{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{1.121}$$

$$|\psi_+^{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{1.122}$$

$$|\psi_-^{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{1.123}$$

These are also the maximally entangled states.

So far we have looked at some of the basic concepts of quantum information theory. We developed the idea of qubit, the difference of pure and mixed states, density matrix formalism and finally entanglement. These concepts are essential in many fields such as Quantum Computing, Quantum Metrology and Quantum Optics. In later sections we will be dealing with topics that require a good understanding of these concepts.

Chapter 2

Atomic Interaction with Fields and Fisher Information

2.1 Dipole Approximation

Consider an electron inside the potential of an atom. The Hamiltonian for such an electron can be written as [10]

$$H_0 = \frac{1}{2m}\mathbf{P}^2 + \mathcal{V}(r). \tag{2.1}$$

Here, $\mathcal{V}(r)$ gives the Coulomb potential of the electron. Initially, there is no external electromagnetic field acting on the atom. In configuration space representation, we define the momentum operator as $\mathbf{P} = -\iota\nabla$ and $\hat{r}|\overrightarrow{r}\rangle = \overrightarrow{r}|\overrightarrow{r}\rangle$. The wave function that depends upon the position is written as $q(\overrightarrow{r}) = \langle\overrightarrow{r}|q\rangle$. Let us define $|k\rangle$ as the eigen states of the Hamiltonian H_0. These states satisfy the following eigenvalue equation:

$$H_0 q_k^{(0)}(\overrightarrow{r}) = E_k q_k^{(0)}(\overrightarrow{r}), \tag{2.2}$$

where, $q_k^{(0)}(\overrightarrow{r}) = \langle\overrightarrow{r}|k\rangle$ and E_k are the eigen values of H_0. By applying an external electromagnetic field, the modified Hamiltonian becomes:

$$H(\overrightarrow{r}, t) = \frac{[\mathbf{P} + e\mathcal{A}(\overrightarrow{r}, t)]^2}{2m} - e\Phi(\overrightarrow{r}, t) + \mathcal{V}(r). \tag{2.3}$$

Where $\Phi(\overrightarrow{r}, t)$ is the scalar potential, $\mathcal{A}(\overrightarrow{r}, t)$ is the vector potential, and $-e$ is the charge of the electron. The electric and magnetic fields can be written in terms of these potentials as: [10][11]

$$\begin{aligned} \mathbf{E} &= -\nabla\Phi(\overrightarrow{r},t) - \frac{\partial}{\partial t}\mathcal{A}(\overrightarrow{r},t), \\ \mathbf{B} &= \nabla \times \mathcal{A}. \end{aligned} \tag{2.4}$$

Under the following gauge transformations for the potentials, the electric and magnetic fields are invariant

$$\begin{aligned} \Phi^{'}(\overrightarrow{r},t) &= \Phi(\overrightarrow{r},t) - \frac{\partial}{\partial t}\zeta(\overrightarrow{r},t), \\ \mathcal{A}^{'}(\overrightarrow{r},t) &= \mathcal{A}(\overrightarrow{r},t) + \nabla\zeta(\overrightarrow{r},t). \end{aligned} \tag{2.5}$$

The time-dependent Schrodinger equation is

$$H\psi(\overrightarrow{r},t) = \iota\hbar\frac{\partial}{\partial t}\psi(\overrightarrow{r},t). \tag{2.6}$$

The Hamiltonian of the atom-field interaction must be simplified. To do this, we apply the gauge transformations from equation (2.5) and the unitary transformations $\psi_{(new)}(\overrightarrow{r},t) = W\psi(\overrightarrow{r},t)$, $(W = exp[-\iota\zeta(\overrightarrow{r},t)/\hbar])$. The Schrodinger equation is going to be

$$H_{(new)}\psi_{(new)}(\overrightarrow{r},t) = \iota\hbar\frac{\partial}{\partial t}\psi_{(new)}(\overrightarrow{r},t). \tag{2.7}$$

The Hamiltonian under gauge transformation becomes

$$\begin{aligned} H_{(new)}(\overrightarrow{r},t) &= \frac{1}{2m}[\mathbf{P} + e\mathcal{A}^{'}(\overrightarrow{r},t)]^2 - e\Phi^{'}(\overrightarrow{r},t) + \mathcal{V}(r) \\ &= \frac{1}{2m}[\mathbf{P} + e(\mathcal{A}(\overrightarrow{r},t) + \nabla\zeta)]^2 - e(\Phi(\overrightarrow{r},t) - \frac{\partial}{\partial t}\zeta) + \mathcal{V}(r). \end{aligned} \tag{2.8}$$

The Coulomb gauge provides two conditions: $\nabla.\mathcal{A} = 0$ and $\Phi = 0$. With these two conditions, we get the advantage that the external fields depend only upon the vector potential $\mathcal{A}$ and we get $H_{(new)}$ as

$$H_{(new)}(\overrightarrow{r},t) = \frac{1}{2m}[\mathbf{P} + e(\mathcal{A}(\overrightarrow{r},t) + \nabla\zeta)]^2 - e(-\frac{\partial}{\partial t}\zeta) + \mathcal{V}(r). \tag{2.9}$$

The vector potential $\mathcal{A}$, far from any source charge (or absence of source charge), satisfies the wave equation

$$\nabla^2\mathcal{A} - \frac{1}{c^2}\frac{\partial^2\mathcal{A}}{\partial t^2} = 0. \tag{2.10}$$

The solution to this wave equation is

$$\mathcal{A}(\overrightarrow{r},t) = \mathcal{A}_0 e^{[\iota(\overrightarrow{k}.\overrightarrow{r}-\omega t)]} + \mathcal{A}_0 e^{[-\iota(\overrightarrow{k}.\overrightarrow{r}-\omega t)]}, \tag{2.11}$$

where $|\overrightarrow{k}| = \frac{2\pi}{\lambda}$ is the wave vector. The $|\overrightarrow{r}|$ is of the order of 10^{-10} (atomic dimension) and λ of the order of a few hundred nanometers (optical wavelength, 400nm–700nm). This makes $\overrightarrow{k}.\overrightarrow{r} << 1$ so we can approximate the vector potential as

$$\mathcal{A}(\overrightarrow{r}, t) \rightarrow \mathcal{A}(t). \tag{2.12}$$

This is the dipole approximation. Now, we make a choice for the gauge function as $\zeta(\overrightarrow{r}, t) = -\overrightarrow{r}.\mathcal{A}(t)$, which gives,

$$\nabla\zeta = \frac{\partial}{\partial r}(-\overrightarrow{r}.\mathcal{A}(t)) = -\mathcal{A}(t), \tag{2.13}$$

$$\frac{\partial\zeta}{\partial t} = \frac{\partial}{\partial t}(-\overrightarrow{r}.\mathcal{A}(t)) = -\overrightarrow{r}.\frac{\partial\mathcal{A}}{\partial t}. \tag{2.14}$$

Because $\Phi = 0$, from equation (2.4) we get $\mathbf{E} = -\frac{\partial\mathcal{A}}{\partial t}$. So,

$$\frac{\partial\zeta}{\partial t} = \overrightarrow{r}.\mathbf{E} \tag{2.15}$$

After substituting these terms in the equation (2.9), the atom-field interaction Hamiltonian now becomes

$$\begin{aligned} H_{(new)}(\overrightarrow{r}, t) &= \frac{[\mathbf{P} + e(\mathcal{A}(t) - \mathcal{A}(t))]^2}{2m} - e(-\overrightarrow{r}.\mathbf{E}) + \mathcal{V}(r), \\ &= \frac{1}{2m}\mathbf{P}^2 + \mathcal{V}(r) - \mathbf{d}.\mathbf{E}. \end{aligned} \tag{2.16}$$

The total Hamiltonian is divided into two parts, H_0 and the interaction part H_1

$$H = H_0 + H_1, \tag{2.17}$$

where $H_1 = -\mathbf{d}.\mathbf{E}$ and $d = -e\hat{r}.\mathbf{E}$ is the dipole operator $\mathbf{d}$.

2.2 Jaynes Cummings Model (JCM)

The Jaynes Cummings model provides us with a picture of a physical model where we suppose an atom consisting of two energy levels, a ground state $|0\rangle$ and an excited state $|1\rangle$, as shown in figure (2.1). This atom is placed inside a cavity where it interacts with an electromagnetic field having a single mode of frequency, [10]:

$$E(z, t) = \hat{e}[\frac{\hbar\omega}{\epsilon_0 V}]^{1/2}(a + a^\dagger)sin(kz). \tag{2.18}$$

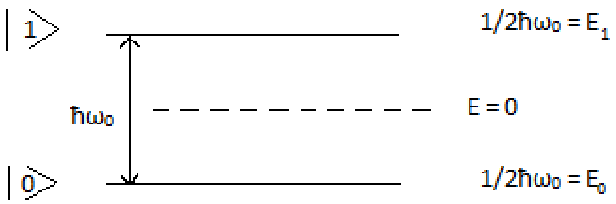


Figure 2.1: Energy levels of atomic system where $E = 0$ is taken between the two levels.

Here, $\hat{e}$ is an arbitrary polarization vector of the electric field. The Jaynes Cummings Hamiltonian is

$$H = H_{field} + H_{atom} + H_{atom-field}. \tag{2.19}$$

The free field Hamiltonian without the the zero point energy term is

$$H_{field} = \hbar\omega_f a^\dagger a. \tag{2.20}$$

The Hamiltonian for the two-level atom can be written as

$$H_{atom} = \sum_{j=0,1} E_j |j\rangle\langle j|. \tag{2.21}$$

If we consider the energy level to be zero in between the two energy states, then we can write the H_{atom} in the form [10]

$$H_{atom} = \frac{(E_1 - E_0)}{2}(|1\rangle\langle 1| - |0\rangle\langle 0|) + \frac{(E_1 + E_0)}{2}(|1\rangle\langle 1| + |0\rangle\langle 0|). \tag{2.22}$$

The second term vanishes under rotating wave approximation (RWA) and the final expression for the atomic Hamiltonian is

$$H_{atom} = \frac{1}{2}\hbar\omega_0\sigma_3, \tag{2.23}$$

here, $\sigma_3 = |1\rangle\langle 1| - |0\rangle\langle 0|$ (inversion operator).

The final part is the atom-field interaction, $H_{atom-field}$. To derive this, we will use the dipole interaction Hamiltonian from equation (2.16). Again, consider the single mode cavity field from equation (2.18) and the interaction Hamiltonian, $H_1 = H_{atom-field} = -d.E$

$$\begin{aligned} H_{atom-field} &= -\hat{d}.\hat{e}[\frac{\hbar\omega}{\epsilon_0 V}]^{1/2}(\hat{a} + \hat{a}^\dagger)sin(kz), \\ &= dg(a + a^\dagger), \end{aligned} \tag{2.24}$$

where $d = \hat{d}.\hat{e}$ and $g = -[\frac{\hbar\omega}{\epsilon_0 V}]^{1/2} sin(kz)$. Now, we define the atomic transition operators as

$$\begin{aligned}\sigma_+ &= |1\rangle\langle 0|,\\ \sigma_- &= |0\rangle\langle 1|.\end{aligned} \tag{2.25}$$

We can write the dipole operator $\hat{d}$ in the form of matrix elements

$$d \equiv d_{ij} = \langle i|d|j\rangle. \tag{2.26}$$

The diagonal terms in the dipole operator, d_{11} and d_{00} will be zero, the reason for this is that the atomic system under consideration here can be taken as a dipole and the separation between the states acts like the separation between the two ends of a dipole, which makes $\langle 1|\hat{d}|1\rangle = \langle 0|\hat{d}|0\rangle = 0$. So only the off-diagonal terms survive, $d_{10} = d_{01} = d$ and we can write

$$d = d[|1\rangle\langle 0| + |0\rangle\langle 1|], \tag{2.27}$$

or we can use equation (2.25) and re-write $\hat{d}$ as

$$d = d[\sigma_+ + \sigma_-]. \tag{2.28}$$

Going back to the interaction Hamiltonian (2.24)

$$H_{atom-field} = dg[\sigma_+ + \sigma_-](a + a^\dagger), \tag{2.29}$$

$$= dg(\sigma_+ a + \sigma_+ a^\dagger + \sigma_- a + \sigma_- a^\dagger). \tag{2.30}$$

According to energy conservation, the terms $\hat{\sigma_+}\hat{a}^\dagger$ and $\hat{\sigma_-}\hat{a}$ will vanish as they are not conserved and finally we will have the atom field interaction Hamiltonian as

$$H_{atom-field} = \hbar\lambda(\sigma_+ a + \sigma_- a^\dagger), \tag{2.31}$$

where $\lambda = dg/\hbar$. We now have all the pieces of the equation (2.19) and we can write our full JCM Hamiltonian

$$H = \hbar\omega_f a^\dagger a + \frac{1}{2}\hbar\omega_0\sigma_3 + \hbar\lambda(\sigma_+ a + \sigma_- a^\dagger). \tag{2.32}$$

2.2.1 Modifications in JCM Hamiltonian

As we now have the basic structure of the Jaynes Cummings Hamiltonian, we can modify it for different physical situations. For example, we can have a case of atom-field interaction where there are two atoms interacting with the single-mode field. For this case, the Hamiltonian will be

$$H = \hbar\omega_f a^\dagger a + \frac{1}{2}\hbar\omega_0 \sum_{j=A,B} \sigma_3^j + \hbar\lambda \sum_{j=A,B} (\sigma_+^j a + \sigma_-^j a^\dagger), \tag{2.33}$$

Where the index $j = A, B$ represents for the two atoms A and B. Similarly, we can have another case where there is a multi-mode field interacting with the atom(s). So many possibilities exist with this model.

2.3 Non-linear Optics; Induced Dipole and Polarization

The atom as a whole is electrically neutral. The positive (nucleus) and negative (electrons) charges within an atom are still influenced by external electromagnetic fields. And if the field is strong enough the electrons can be pulled out of the atom making it ionized. However, if the field is not that strong, the electrons are displaced slightly causing an imbalance in the charge distribution and as a result a dipole if induced which depends on the E-field applied to the atom. For a single dipole $\overrightarrow{p}$ we have [11]

$$\overrightarrow{p} = \alpha\overrightarrow{E}. \tag{2.34}$$

In this case, atomic polarizability is denoted by α. The dipole moment per unit volume is the definition of the polarization $\overrightarrow{P}$. Similar to an atomic dipole moment, the effects of an optical field on material systems (dielectric materials) are expressed in terms of polarization induced throughout the material. The total polarization in a material is directly dependent on the applied external field

$$\overrightarrow{P} = \epsilon_0\chi_e\overrightarrow{E}. \tag{2.35}$$

The electrical susceptibility χ_e is the structural property of the material, and ϵ_0 is the permittivity of free space.

2.3.1 Non-Linear Polarization and Kerr Effect

The equation (2.35) is used to describe "linear optics". By expanding this equation in the power series of $\hat{E}$, we may generalize the concept to non-linear polarization, where polarization relies non-linearly upon the strength of the electric field.[12]

$$\mathbf{P} = \chi_e\mathbf{E} + \chi_e^{(1)}\mathbf{E}^1 + \chi_e^{(2)}\mathbf{E}^2 + \chi_e^{(3)}\mathbf{E}^3 + ..., \tag{2.36}$$

here, the polarization is shown in the form of field operators in quantum mechanics. We are interested in the Kerr effect, in which an external field induces a change in the refractive index of the material. The Kerr effect is described by the third-order term in the equation (2.36). Using the Hamiltonian density $\mathcal{H} = \epsilon_0 \mathbf{E}_i \mathbf{P}_i$ for the polarization in the material [10] and the third order polarization term $\mathbf{P}_i = \chi_e^{(3)} \mathbf{E}_i^3$ we can write the Hamiltonian as the integral over the volume

$$H = \epsilon_0 \int_a^b d^3 r E_i P_i. \tag{2.37}$$

Now, let

$$E_i = E_x$$

then E_x is defined as

$$E_x(z,t) = x\sqrt{\frac{\hbar\omega_f}{\epsilon_0 V}}(a + a^\dagger)sin(kz), \tag{2.38}$$

putting these values in the Hamiltonian above

$$\begin{aligned} H &= \epsilon_0 \int_a^b d^3 r E_i (\chi_e^{(3)} E_i^3), & (2.39)\\ &= \epsilon_0 \chi^{(3)} \int_a^b d^3 r E_x^4. & (2.40)\end{aligned}$$

Let solve for E_x^4 first

$$\begin{aligned} E_x^4 &= [\hat{x}\sqrt{\frac{\hbar\omega_f}{\epsilon_0 V}}(a + a^\dagger)sin(kz)]^4, & (2.41)\\ &= (\frac{\hbar\omega_f}{\epsilon_0 V})^2 [a + a^\dagger]^4 sin^4(kz). & (2.42)\end{aligned}$$

The term $[a + a^\dagger]^4$ is expanded as

$$\begin{aligned} (a + a^\dagger)^4 = {} & aaaa + aaaa^\dagger + aaa^\dagger a + aaa^\dagger a^\dagger \\ & + aa^\dagger aa + aa^\dagger aa^\dagger + aa^\dagger a^\dagger a + aa^\dagger a^\dagger a^\dagger \\ & + a^\dagger aaa + a^\dagger aaa^\dagger + a^\dagger aa^\dagger a + a^\dagger aa^\dagger a^\dagger \\ & + a^\dagger a^\dagger aa + a^\dagger a^\dagger aa^\dagger + a^\dagger a^\dagger a^\dagger a + a^\dagger a^\dagger a^\dagger a^\dagger. \end{aligned} \tag{2.43}$$

From the above equation only 6 terms are conserving energy

$$\begin{aligned} (a + a^\dagger)^4 \approx {} & aaa^\dagger a^\dagger + aa^\dagger aa^\dagger + aa^\dagger a^\dagger a \\ & + a^\dagger aaa^\dagger + a^\dagger aa^\dagger a + a^\dagger a^\dagger aa. \end{aligned} \tag{2.44}$$

Using the commutation relations of the creation and annihilation operators

$$\begin{aligned} [a, a^\dagger] &= 1, & (2.45)\\ [a^\dagger, a] &= -1, & (2.46)\end{aligned}$$

we can further simplify the equation (2.44) as

$$(a+a^{\dagger})^4 \approx 3(a^{\dagger}aa^{\dagger}a + aa^{\dagger}aa^{\dagger}), \tag{2.47}$$

$$(a+a^{\dagger})^4 \approx 2a^{\dagger}aa^{\dagger}a + 2a^{\dagger}a + 1. \tag{2.48}$$

Keeping only the non-linear term ($1st$ term) for the kerr medium:

$$E_x^4 = (\frac{\hbar\omega_f}{\epsilon_0 V})^2 sin^4(kz)[a^{\dagger}aa^{\dagger}a]. \tag{2.49}$$

Now we substitute this value of E_x^4 in the equation (**??**) and solve the integral

$$\begin{aligned} H &= \epsilon_0\chi^{(3)}\int_a^b d^3r(\frac{\hbar\omega_f}{\epsilon_0 V})^2 sin^4(kz)[a^{\dagger}aa^{\dagger}a], \\ &= \epsilon_0\chi^{(3)}(\frac{\hbar\omega_f}{\epsilon_0 V})^2\int_S d^2S\int_a^b sin^4(kz)dz[a^{\dagger}aa^{\dagger}a]. \end{aligned} \tag{2.50}$$

Consider the figure below, where the kerr medium is placed in the cavity field of mode ω_f. The material is centered at $z = 0$ so we can take the limits of z to be $-l/2 \leq z \leq +l/2$

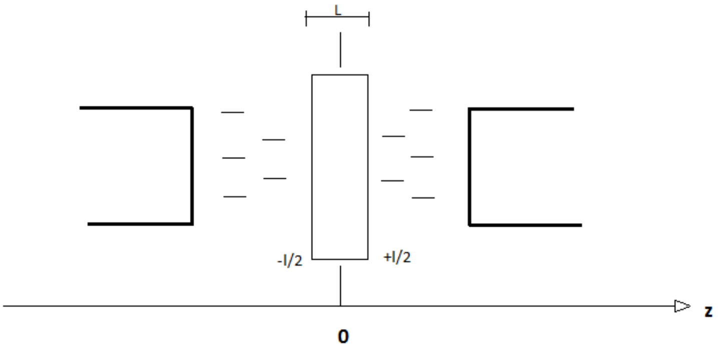


Figure 2.2: Kerr medium placed in cavity field of mode ω_f

where l is the total length of the material along the z-axis. Solving the integral in equation (2.50) under these limits for z

$$\int_{-l/2}^{+l/2} sin^4(kz)dz = \frac{6kl - 8sin(kl) + sin(2kl)}{16k} \tag{2.51}$$

$$= \frac{3kl}{8k} + \frac{1}{2}\frac{sin(kl)}{k} + \frac{1}{16}\frac{sin(2kl)}{k} \tag{2.52}$$

$$= \frac{3l}{8} + \frac{1}{2}\frac{sin(kl)}{k} + \frac{1}{16}\frac{sin(2kl)}{k} \tag{2.53}$$

Now, lets say that the wavelength of light is of the order of $100nm(\lambda = 400nm - 700nm$, optical wavelength) which gives $k = \frac{2\pi}{\lambda} \rightarrow 10^7$ and the length l of the material could be in micro-meter (10^{-6}) range. Thus, the impact of the 2nd and 3rd term in the above equation is negligible an so we have

$$\int_{-l/2}^{+l/2} sin^4(kz)dz \approx \frac{3}{8}l \tag{2.54}$$

Finally, we can write the complete Hamiltonian for the 3rd order polarization term as

$$\begin{aligned} H &= \epsilon_0\chi^{(3)}(\frac{\hbar\omega_f}{\epsilon_0 V})^2\int_S d^2 S(\frac{3}{8}l)[a^\dagger a a^\dagger a], && (2.55)\\ &= \epsilon_0\chi^{(3)}(\frac{\hbar\omega_f}{\epsilon_0 V})^2(\frac{3}{8}l)S[a^\dagger a a^\dagger a], && (2.56)\\ &= \hbar\epsilon_0\hbar\chi^{(3)}\frac{3}{8}l(\frac{\omega_f}{\epsilon_0 V})^2 S[a^\dagger a a^\dagger a], && (2.57)\\ H &= \hbar\chi(a^\dagger a a^\dagger a). && (2.58)\end{aligned}$$

The integral $\int_S d^2$ is the total surface area (S) of the material in the cavity field and $\chi = \frac{3}{8}\epsilon_0\hbar\chi^{(3)}l(\frac{\omega_f}{\epsilon_0 V})^2 S$ denotes the coupling strength between the field and the kerr medium.

2.4 Parameter Estimation and Fisher Information

Suppose we have a data sample denoted by the random variable X [13]. We can model N random experiments as $X_1, X_1, X_1, ..., X_N$. The experimental results are denoted as $x_1, x_2, x_3, ..., x_N$ and they are represented in the form a density function called as the probability distribution function $P_f(x)$ (PDF). In an *Estimation* process, we want to know how much information about an unknown parameter θ resides within the collected data sample.

2.4.1 Estimation Process

Consider that a system has a parameter θ and we want to estimate that parameter. Let there are N data values $(y_1, y_2, y_3, ..., y_N) \equiv \mathbf{y}$

The system has an associated conditional probability $P(\mathbf{y}|\theta)$. Each data sample is a measurement of the parameter value along with some intrinsic noise in the system [14]

$$\mathbf{y} = \theta + \mathbf{x}, \tag{2.59}$$

here, $\mathbf{x} \equiv (x_1, x_2, x_3, ..., x_N)$ are the noise in the measurement. Thus θ is the exact value and $\mathbf{x}$ causes the measured value $\mathbf{y}$ to deviate form the actual value. To acquire the actual value or get close to actual value for θ we make an *Estimation Function* or *Estimator* $\hat{\theta}(\mathbf{y})$ which inputs the data values and estimates a much better value for θ. That is why this process is called a *smart measurement* process. For example, this estimator might be an average of some data sample

$$Average = \sum_{n=1}^{N} \frac{y_n}{N}. \tag{2.60}$$

This average value is much closer to the actual value than all the other measurements in the data sample.

2.4.2 Fisher Information

Fisher Information is a way of measuring the information content in a set of measurements that we might make about a parameter. This information arises as measure of the expected error in a smart measurement. This means that the information is related to the expected error. In any measurement process there is always some level of uncertainty or error associated with the results obtained. The expected error is a measure of the anticipated level of uncertainty or error in the results based on the techniques used. Considering the unbiased estimators, which is a class of estimators whose expectation value is equal to the true parameter value [14]

$$\langle \hat{\mathcal{Y}}(\mathbf{y}) \rangle = \mathcal{Y}. \tag{2.61}$$

If a^2 is the error, squared, for such an estimator, then the following relation holds:

$$a^2 I \geq 1, \tag{2.62}$$

where, I is the Fisher information. This is called the Cramér-Rao inequality. It represents an inverse relation between mean squared error and fisher information. So, when a^2 decreases the estimation quality gets better and the information I increases. This inequality thus sets a lower bound on the error.

Derivation

Again consider the class of unbiased estimators

$$\langle \hat{\mathcal{Y}}(\mathbf{y}) - \mathcal{Y} \rangle = 0, \tag{2.63}$$

which can be written in integral form as

$$\int dy[\hat{\mathcal{Y}}(\mathbf{y}) - \mathcal{Y}]P(\mathbf{y}|\mathcal{Y}) = 0. \tag{2.64}$$

The PDF $P(\mathbf{y}|\mathcal{Y})$ represents how the measured values $\mathbf{y}$ are expected to fluctuate or vary in the for a particular parameter value $\mathcal{Y}$. Differentiating the integrand in equation (2.64) w.r.t $\mathcal{Y}$

$$\int dy \frac{\partial}{\partial \mathcal{Y}}[(\hat{\mathcal{Y}}(\mathbf{y}) - \mathcal{Y})P(\mathbf{y}|\mathcal{Y})] = 0, \tag{2.65}$$

$$\int dy(\hat{\mathcal{Y}}(\mathbf{y}) - \mathcal{Y})\frac{\partial P}{\partial \mathcal{Y}} - \int dyP = 0. \tag{2.66}$$

By using the identity $\frac{\partial P}{\partial \mathcal{Y}} = P\frac{\partial lnP}{\partial \mathcal{Y}}$, we get

$$\int dy(\hat{\mathcal{Y}} - \mathcal{Y})\frac{\partial lnP}{\partial \mathcal{Y}}P = 1. \tag{2.67}$$

We can express the above equation in the form

$$\int dy(\frac{\partial lnP}{\partial \mathcal{Y}}\sqrt{P})((\hat{\mathcal{Y}} - \mathcal{Y})\sqrt{P}) = 1. \tag{2.68}$$

After squaring this equation and applying the Schwarz Inequality,

$$[\int dy(\frac{\partial lnP}{\partial \mathcal{Y}})^2P][\int dy(\hat{\mathcal{Y}} - \mathcal{Y})^2P] \geq 1. \tag{2.69}$$

This is the Cramér-Rao inequality given in equation (3.4). The first factor on the left hand side is the fisher information

$$I = \int dy(\frac{\partial lnP}{\partial \mathcal{Y}})^2P, \tag{2.70}$$

and the second factor is the mean squared error

$$a^2 = \int dy(\hat{\mathcal{Y}} - \mathcal{Y})^2P. \tag{2.71}$$

We can see that the expression of I is an expectation value function containing the derivative of the logarithmic function of probability.

2.4.3 Fisher Information in Quantum Mechanics

The idea in quantum information theory that quantifies our ability to estimate parameters using quantum systems is called quantum fisher information. Starting with the following equation, the expression for QFI will be developed for the case of unbiased estimators. [15] [16]

$$F_\theta = \mathrm{Tr}(\mathcal{D}_\theta L^2). \tag{2.72}$$

Here θ is the parameter. For an N-dimensional system (N can be infinite),

$$\hat{\mathcal{D}}_\theta = \sum_{i=1}^{s} P_i |V_i\rangle\langle V_i|, \tag{2.73}$$

where s is the index for support (set of eigenvectors corresponding to non-zero eigenvalues) of the density matrix and $\sum_{i=1}^{s} P_i = 1$ .The term L^2 is called the Symmetric Logarithmic Derivative (SLD) and it can be determined from the following equation:

$$\partial_\theta \mathcal{D}_\theta = \frac{1}{2}[L\mathcal{D}_\theta + \mathcal{D}_\theta L]. \tag{2.74}$$

In the eigenbasis of $\mathcal{D}_\theta$ we can write

$$\langle V_i|\partial_\theta \mathcal{D}_\theta|V_j\rangle = \frac{1}{2}\langle V_i|[L\mathcal{D}_\theta + \mathcal{D}_\theta L]|V_j\rangle, \tag{2.75}$$

$$= \frac{1}{2}[\langle V_i|L\mathcal{D}_\theta|V_j\rangle] + [\langle V_i|\mathcal{D}_\theta L|\psi_j\rangle]. \tag{2.76}$$

Using equation (2.73)

$$\langle V_i|\partial_\theta \mathcal{D}_\theta|V_j\rangle = \frac{1}{2}[\langle V_i|L\sum_{j=1}^{N} P_j|V_j\rangle\langle V_j|V_j\rangle] + [\langle V_i|\sum_{i=1}^{s} P_i|V_i\rangle\langle V_i|L|V_j\rangle], \tag{2.77}$$

$$= \frac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{N}(P_i + P_j)\langle V_i|L|V_j\rangle, \tag{2.78}$$

$$= \frac{1}{2}\sum_{i=1}^{s}\sum_{j=1}^{N}(P_i + P_j)L_{ij}. \tag{2.79}$$

Going back to the equation (2.72),

$$F_\theta = \mathrm{Tr}(\mathcal{D}_\theta L^2) = \mathrm{Tr}(\sum_{i=1}^{s} P_i|V_i\rangle\langle V_i|L.L), \tag{2.80}$$

$$= \sum_{i=1}^{s} P_i\langle V_i|V_i\rangle\langle V_i|L.L|V_i\rangle. \tag{2.81}$$

Using the completeness of basis, $1 = \sum_{j=1}^{N} |V_j\rangle\langle V_j|$,

$$F_\theta = \sum_{i=1}^{s}\sum_{j=1}^{N} P_i\langle V_i|L|V_j\rangle\langle V_j|L|V_i\rangle, \tag{2.82}$$

$$= \sum_{i=1}^{s}\sum_{j=1}^{N} P_i L_{ij} L_{ji}. \tag{2.83}$$

The value of L_{ij} can be taken from equation (2.82),

$$L_{ij} = \frac{2(\partial_\theta \mathcal{D}_\theta)_{ij}}{P_i + P_j}, \tag{2.84}$$

similarly,

$$L_{ji} = \frac{2(\partial_\theta \mathcal{D}_\theta)_{ji}}{P_i + P_j}, \tag{2.85}$$

where $(\partial_\theta \mathcal{D}_\theta)_{ij} = \langle V_i|\partial_\theta \mathcal{D}_\theta|V_j\rangle$. Now substituting these expressions back into equation (2.86), F_θ will become

$$F_\theta = \sum_{i=1}^{s}\sum_{j=1}^{N} P_i[\frac{2(\partial_\theta \mathcal{D}_\theta)_{ij}}{P_i + P_j}][\frac{2(\partial_\theta \mathcal{D}_\theta)_{ji}}{P_i + P_j}], \tag{2.86}$$

$$F_\theta = \sum_{i=1}^{s}\sum_{j=1}^{N} \frac{4P_i}{(P_i + P_j)^2}|(\partial_\theta \mathcal{D}_\theta)_{ij}|^2. \tag{2.87}$$

If we expand $(\partial_\theta \mathcal{D}_\theta)_{ij}$ and use the diagonalization of $\mathcal{D}_\theta$

$$\begin{aligned}(\partial_\theta \mathcal{D}_\theta)_{ij} &= \langle V_i|\partial_\theta \mathcal{D}_\theta|V_j\rangle, & (2.88)\\ &= \langle V_i|\partial_\theta(P_i|V_i\rangle\langle V_i|)|\psi_j\rangle, & (2.89)\\ &= \langle V_i|\partial P_i|V_i\rangle\langle V_i| + P_i|\partial_\theta V_i\rangle\langle V_i| + P_i|V_i\rangle\langle\partial_\theta V_i|\psi_j\rangle, & (2.90)\\ &= \partial_\theta P_i\delta_{ij} + P_i\langle V_i|\partial_\theta V_i\rangle\delta_{ij} + P_i\langle\partial_\theta V_i|V_j\rangle, & (2.91)\end{aligned}$$

From the orthogonality condition, $\langle V_i|V_j\rangle = \delta_{ij}$, we get the equation

$$-\langle V_i|\partial_\theta V_j\rangle = \langle\partial_\theta V_i|V_j\rangle. \tag{2.92}$$

By using the above equation, we get

$$(\partial_\theta \mathcal{D}_\theta)_{ij} = \partial_\theta P_i\delta_{ij} + (P_j - P_i)\langle V_i|\partial_\theta V_j\rangle. \tag{2.93}$$

consequently, the mod square of it will be

$$|(\partial_\theta \mathcal{D}_\theta)_{ij}|^2 = (\partial_\theta P_i)^2 + (P_i - P_j)^2|\langle V_i|\partial_\theta V_j\rangle|^2. \tag{2.94}$$

Using equation (3.36) in equation (3.29),

$$F_\theta = \sum_{i=1}^{s}\sum_{j=1}^{N} \frac{4P_i}{(P_i + P_j)^2}(\partial_\theta P_i)^2 + \sum_{i=1}^{s}\sum_{j=1}^{N} \frac{4P_i(P_i - P_j)^2}{(P_i + P_j)^2}|\langle V_i|\partial_\theta V_j\rangle|^2. \tag{2.95}$$

We can split the summation over j in the second term as,

$$\sum_{j=1}^{N} = \sum_{j=1}^{s} + \sum_{j=s+1}^{N}. \tag{2.96}$$

This separates the terms of the index j that are equal to the terms of index i from those that are not equal,

$$\begin{aligned} F_\theta =& \sum_{i=1}^{s}\sum_{j=1}^{N}\frac{4P_i}{(P_i+P_j)^2}(\partial_\theta P_i)^2 \\ &+\sum_{i=1}^{s}\sum_{j=1}^{s}\frac{4P_i(P_i-P_j)^2}{(P_i+P_j)^2}|\langle V_i|\partial_\theta V_j\rangle|^2 \\ &+\sum_{i=1}^{s}\sum_{j=s+1}^{N}\frac{4P_i(P_i-P_j)^2}{(P_i+P_j)^2}|\langle V_i|\partial_\theta V_j\rangle|^2. \end{aligned} \tag{2.97}$$

Lets say that the second and third terms in equation (3.39) are called F_1 and F_2 respectively. As we know that $\sum_{i=1}^{s} P_i = 1$ then $\sum_{i=1}^{s} P_j = 1$ and $\sum_{s+1=1}^{N} P_j = 0$ because the total summation $\sum_{j=1}^{N}$ must be equal to 1. Thus, we can write

$$F_1 = \sum_{i=1}^{s}\sum_{j=1}^{s}\frac{4P_i(P_i-P_j)^2}{(P_i+P_j)^2}|\langle V_i|\partial_\theta V_j\rangle|^2, \tag{2.98}$$

$$F_2 = \sum_{i=1}^{s}\sum_{j=s+1}^{N}\frac{4P_i(P_i-P_j)^2}{(P_i+P_j)^2}|\langle V_i|\partial_\theta V_j\rangle|^2. \tag{2.99}$$

Since F_2 has the sum over j from $s+1$ to N,P_j would be zero and we can rewrite it as,

$$F_2 = \sum_{i=1}^{s}\sum_{j=s+1}^{N} 4P_i|\langle V_i|\partial_\theta V_j\rangle|^2. \tag{2.100}$$

From the normalization relation, we can write

$$\sum_{j=s+1}^{N}|V_j\rangle\langle V_j| = 1-\sum_{j=1}^{s}|V_j\rangle\langle V_j|. \tag{2.101}$$

After utilizing the above condition in equation (2.100), F_2 can be written as,

$$F_2 = \sum_{i=1}^{s}4P_i\langle\partial_\theta V_i|\partial_\theta V_i\rangle - \sum_{i=1}^{s}\sum_{j=1}^{s}4P_i|\langle V_i|\partial_\theta V_j\rangle|^2. \tag{2.102}$$

We finally return to the expression of QFI in equation (3.39),

$$F_\theta = \sum_{i=1}^{s}\sum_{j=1}^{N}\frac{4P_i}{(P_i+P_j)^2}(\partial_\theta P_i)^2 + F_1 + F_2, \tag{2.103}$$

$$\begin{aligned} =& \sum_{i=1}^{s}\sum_{j=1}^{N}\frac{4P_i}{(P_i+P_j)^2}(\partial_\theta P_i)^2 + \sum_{i=1}^{s}\sum_{j=1}^{s}\frac{4P_i(P_i-P_j)^2}{(P_i+P_j)^2}|\langle V_i|\partial_\theta V_j\rangle|^2 \\ &+\sum_{i=1}^{s}4P_i\langle\partial_\theta V_i|\partial_\theta V_i\rangle - \sum_{i=1}^{s}\sum_{j=1}^{s}4P_i|\langle V_i|\partial_\theta V_j\rangle|^2, \end{aligned} \tag{2.104}$$

$$= \sum_{i=1}^{s}\frac{1}{P_i}(\partial_\theta P_i)^2 + \sum_{i=1}^{s}4P_i\langle\partial_\theta V_i|\partial_\theta V_i\rangle - \sum_{i,j=1}^{s}\frac{8P_iP_j}{P_i+P_j}|\langle V_i|\partial_\theta V_j\rangle|^2. \tag{2.105}$$

In equation (3.45), the first term corresponds to $i = j$ which reduces to $\sum_{i=1}^{s} \frac{1}{P_i}(\partial_\theta P_i)^2$. From the third term of equation (2.105) we separate out the $i = j$ terms and we will arrive at the final expression for the QFI,

$$
\begin{aligned}
F_\theta = & \sum_{i=1}^{s} \frac{1}{P_i}(\partial_\theta P_i) + 4\sum_{i=1}^{s} P_i(\langle \partial_\theta V_i|\partial_\theta V_i\rangle - |\langle V_i|\partial_\theta V_i\rangle|^2) \\
& - 8\sum_{i \neq j=1}^{s} \frac{P_i P_j}{P_i + P_j} |\langle V_i|\partial_\theta V_j\rangle|^2 .
\end{aligned} \tag{2.106}
$$

This equation can be written as a sum of two parts; namely the classical and quantum contributions:

$$
F_\theta = F_{classical} + F_{quantum}. \tag{2.107}
$$

The first term, $\sum_{i=1}^{s} \frac{1}{P_i}(\partial_\theta P_i)$ is analogous to the expression of classical fisher information. The values of P_i would give us a probability distribution as is required in CFI. The rest of the terms in F_θ correspond to the quantum part. The second term refers to the fisher information related to the eigenstates of the density matrix, and the third term is the information for the mixture of states. That is why, for a pure state, the third term would have no part in the equation, and we could say from this argument that the QFI for a pure state is greater than that of the mixed state.

The quantum fisher information estimates the parameter of interest within the quantum states, and also quantifies the sensitivity of the states to the changes in parameter. In other words, it tells about how sensitive the changes in the quantum states are with respect to changes in the parameter.

Chapter 3

Modulating Quantum Fisher Information of an Atomic System in a Dissipative Cavity by Coupling Strength

3.1 Physical Model

This model presents a picture of a two-level atom that is placed inside a cavity, which consists of a single-mode electromagnetic field in resonance with the atom. A bosonic environment surrounds the cavity. Inside the cavity, the atom interacts with the field, and due to the dissipative nature of the cavity, the field interacts with the environment. The Hamiltonian describing this system is [17]

$$H = \frac{1}{2}\omega_o\sigma_z + \omega_o a^\dagger a + \Omega(\sigma_+ a + \sigma_- a^\dagger) + \sum_k \omega_k c_k^\dagger c_k + (a^\dagger + a)\sum_k g_k(c_k^\dagger + c_k), \qquad (3.1)$$

where ω_0 is the Bohr frequency of the atom, a and $a^\dagger$ are the annihilation and creation operators for the field inside the cavity, Ω is the coupling strength of the atom-cavity interaction, c and $c^\dagger$ are the annihilation and creation operators of the different modes of environment, and g_k is the coupling in the cavity-environment interaction Hamiltonian. In the next section, we will look into the development of time-evolved state of the atomic system

3.1.1 Time Evolution in the Dressed State Basis

The Hamiltonian in equation (3.1) is made up of three parts

$$H = H_{jc} + H_E + H_I, \tag{3.2}$$

where the first term is the Jaynes-Cummings Hamiltonian corresponding to the first three terms in equation (3.1), this is the Hamiltonian of our system of interest. H_E (fourth term) is the environment Hamiltonian, and H_I (fifth term) is for the interaction of the cavity and the environment. The total excitation number $\mathcal{N}$, which is a constant of motion, for the atom-cavity system is given by[9]

$$\mathcal{N} = a^\dagger a + \sigma_z + \frac{1}{2}. \tag{3.3}$$

Using this quantity, we can write H_{jc} as

$$H_{jc} = \omega_0(\mathcal{N} - \frac{1}{2}) - \frac{\omega_0}{2}\sigma_z + \Omega(\sigma_+ a + \sigma_- a^\dagger), \tag{3.4}$$

Now, with N number of photons each of mode ω_0, we can choose our basis as $|N, g\rangle$ and $|N-1, e\rangle$. Then we can find the following eigenvalues and eigenvectors for H_{jc}

$$\begin{aligned} E_{N,+} &= (N - \frac{1}{2})\omega_0 + \Omega\sqrt{N}, \\ E_{N,-} &= (N - \frac{1}{2})\omega_0 - \Omega\sqrt{N}. \end{aligned} \tag{3.5}$$

$$\begin{aligned} |E_{N,+}\rangle &= \frac{1}{\sqrt{2}}(|N-1, e\rangle + |N, g\rangle), \\ |E_{N,-}\rangle &= \frac{1}{\sqrt{2}}(|N-1, e\rangle - |N, g\rangle), \end{aligned} \tag{3.6}$$

along with the following ground state of the atom-cavity system

$$|E_0\rangle = |0, g\rangle, \qquad E_0 = -\frac{\omega_0}{2}. \tag{3.7}$$

The initial conditions that we want to impose on our system are that the cavity has one photon ($n = 1$) of mode ω_0, the number of total excitation in the system is 1, and the environment is at temperature $T = 0$. The eigenstates, along with their energy eigenvalues, will be modified as

$$E_{1,+} = \frac{\omega_0}{2} + \Omega \qquad , E_{1,-} \quad = \frac{\omega_0}{2} - \Omega. \tag{3.8}$$

$$\begin{aligned}|E_{1,+}\rangle &= \frac{1}{\sqrt{2}}(|0,e\rangle + |1,g\rangle),\\ |E_{1,-}\rangle &= \frac{1}{\sqrt{2}}(|0,e\rangle - |1,g\rangle).\end{aligned} \tag{3.9}$$

The equations (3.7) and (3.9) form the set of dressed state basis $\{|E_{1,+}\rangle, |E_{1,-}\rangle, |E_0\rangle\}$ for this system. By following the procedures to derive the master equation for open systems from chapter 1, we can write our master equation for the atom-cavity system as[17][18]

$$\begin{aligned}\frac{\partial \mathcal{D}(t)}{\partial t} =& -\iota[H_{jc}, \mathcal{D}(t)]\\ &+ \gamma_{1,+}(\omega_0 + \Omega)[\frac{1}{2}|E_0\rangle\langle E_{1,+}|\mathcal{D}(t)|E_{1,+}\rangle\langle E_0| - \frac{1}{4}\{|E_{1,+}\rangle\langle E_{1,+}|, \mathcal{D}(t)\}]\\ &+ \gamma_{1,-}(\omega_0 - \Omega)[\frac{1}{2}|E_0\rangle\langle E_{1,-}|\mathcal{D}(t)|E_{1,-}\rangle\langle E_0| - \frac{1}{4}\{|E_{1,-}\rangle\langle E_{1,-}|, \mathcal{D}(t)\}],\end{aligned} \tag{3.10}$$

where we have taken $\hbar = 1$ and $\gamma_{1,\pm}(\omega_0 \pm \Omega)$ are the rates of relaxation for the states $|E_{1,\pm}\rangle$. To solve the above master equation, we follow the procedure as described below. Given the master equation of the form

$$\frac{\partial \mathcal{D}(t)}{\partial t} = \mathcal{L}\mathcal{D}(t), \tag{3.11}$$

where $\mathcal{L}$ is the super operator, called Liouvillian, acting on $\mathcal{D}(t)$, consider the following eigen value equation

$$\mathcal{L}\mathcal{D}_\alpha = \alpha\mathcal{D}_\alpha. \tag{3.12}$$

According to the above eigenvalue equation we developed a sub-space of the eigen operators $\mathcal{D}_\alpha$ of the super operator $\mathcal{L}$ and the factor α is the eigen value. If this set of eigen operators $\{\mathcal{D}_\alpha\}$ form basis operators that act on the system, we can write any operator as a linear combination of these eigen operators, so

$$\mathcal{D}(0) = \sum_\alpha c_\alpha \mathcal{D}_\alpha, \tag{3.13}$$

where the coefficients c_α are given as

$$c_\alpha = Tr[\mathcal{D}_\alpha \mathcal{D}(0)]. \tag{3.14}$$

Now, the general solution to equation (3.11) is of the form

$$\mathcal{D}(t) = \exp{[\mathcal{L}t]}\mathcal{D}(0), \tag{3.15}$$

since $\mathcal{D}(0)$ is given in terms of the eigen operators of $\mathcal{L}$ we can write $\mathcal{D}(t)$ as

$$\mathcal{D}(t) = \sum_{\alpha} c_{\alpha} e^{\alpha t} \mathcal{D}_{\alpha}. \tag{3.16}$$

Applying this procedure to the equation (3.10), we get the elements of time evolved density matrix of the combined atom-cavity system as

$$\begin{aligned}
&\mathcal{D}_{11}(t) = M_{11}^{11}\mathcal{D}_{11}(0), \quad \mathcal{D}_{12}(t) = M_{12}^{12}\mathcal{D}_{12}(0), \quad \mathcal{D}_{13}(t) = M_{13}^{13}\mathcal{D}_{13}(0),\\
&\mathcal{D}_{22}(t) = M_{22}^{22}\mathcal{D}_{22}(0), \quad \mathcal{D}_{23}(t) = M_{23}^{23}\mathcal{D}_{23}(0), \quad \mathcal{D}_{31}(t) = M_{31}^{31}\mathcal{D}_{31}(0),\\
&\mathcal{D}_{32}(t) = M_{32}^{32}\mathcal{D}_{32}(0), \quad \mathcal{D}_{33}(t) = M_{33}^{11}\mathcal{D}_{11}(0) + M_{33}^{22}\mathcal{D}_{22}(0) + M_{33}^{33}\mathcal{D}_{33}(0)
\end{aligned} \tag{3.17}$$

and

$$\begin{aligned}
&M_{11}^{11} = e^{-\frac{1}{2}I_{+}}, \quad M_{12}^{12} = e^{-\iota 2\Omega t} e^{-\frac{1}{4}(I_{+}+I_{-})}, \quad M_{13}^{13} = e^{-\iota(\omega_0+\Omega)t} e^{-\frac{1}{4}I_{+}},\\
&M_{22}^{22} = e^{-\frac{1}{2}I_{-}}, \quad M_{13}^{13} = e^{-\iota(\omega_0-\Omega)t} e^{-\frac{1}{4}I_{-}}, \quad M_{33}^{11} = 1 - M_{11}^{11},\\
&M_{33}^{22} = 1 - M_{22}^{22}, \quad M_{33}^{33} = 1.
\end{aligned} \tag{3.18}$$

The quantities I_{+} and I_{-} are given by the following integrals: [19]

$$I_{+} = \int_{0}^{t} \gamma_{1,+}(\omega_0 + \Omega, t^{'})dt^{'}, \quad I_{-} = \int_{0}^{t} \gamma_{1,-}(\omega_0 - \Omega, t^{'})dt^{'}. \tag{3.19}$$

To find the decay rates $\gamma_{1,\pm}(\omega_0 \pm \Omega, t^{'})$ the distribution of the environment given by the Lorentz distribution $J(\omega)$[19][17]:

$$\mathcal{J}(\omega) = \frac{1}{2\pi} \frac{\gamma_0 \lambda^2}{(\omega_1 - \omega)^2 + \lambda^2}. \tag{3.20}$$

The spread of the distribution is represented by λ. The environment has time dependent correlations which are affected by the strength of the coupling. The time these correlations last, τ_R is related to the parameter λ as $\tau_R = 1/\lambda$. And γ_0 is defined as the inverse of the relaxation time as $\tau_S = 1/\gamma_0$.

Let us take the case where the distribution of the environment has a peak value on the frequencies of the state $|E_{-}\rangle$, i.e., $\omega_1 = \omega_0 - \Omega$. In general, the relaxation rate $\gamma(\omega_B, t)$ with a Bohr frequency ω_B is equal to

$$\gamma(\omega_B, t) = 2Re\{\Gamma(\omega_B, t)\}, \tag{3.21}$$

where $\Gamma(\omega_B, t)$ depends upon the density function of the environment by the integral

$$\Gamma(\omega_B, t) = \int_{0}^{t} d\tau \int_{-\infty}^{+\infty} d\omega^{'} e^{\iota(\omega_B - \omega^{'})\tau} J(\omega^{'}). \tag{3.22}$$

After solving the above integral using the method of residues, the relaxation rate rate $\gamma(\omega_B, t)$ is given as

$$\gamma(\omega_B, t) = \frac{\gamma_0\lambda^2}{(\omega_1 - \omega_B)^2 + \lambda^2}(1 + [\frac{\omega_1 - \omega_B}{\lambda}\sin{(\omega_1 - \omega_B)t} - \cos{(\omega_1 - \omega_B)t}]e^{-\lambda t}). \quad (3.23)$$

Substituting the values $\omega_1 = \omega_0 - \Omega$ and $\omega_B = \omega_0 \pm \Omega$ we get the rates of decay for the two eigenstates $|E_{1,\pm}\rangle$:

$$\gamma_{1,+}(\omega_0 + \Omega, t) = \frac{\gamma_0\lambda^2(1 + [\frac{2\Omega}{\lambda}\sin 2\Omega t - \cos 2\Omega t]e^{-\lambda t})}{(2\Omega)^2 + \lambda^2}, \quad (3.24)$$

$$\gamma_{1,-}(\omega_0 - \Omega, t) = \gamma_0 - \gamma_0 e^{-\lambda t}. \quad (3.25)$$

Now that we return to the equation (3.19), the quantities $I_\pm$ can be shown to have the following results[17]:

$$I_+ = \frac{\gamma_0\lambda^2}{4\Omega^2 + \lambda^2}\{t - \frac{4\Omega e^{-\lambda t}\sin 2\Omega t}{4\Omega^2 + \lambda^2} + \frac{(\lambda^2 - 4\Omega^2)(e^{-\lambda t}\cos 2\Omega t - 1)}{\lambda(4\Omega^2 + \lambda^2)}\}, \quad (3.26)$$

$$I_- = \gamma_0 t + \frac{\gamma_0}{\lambda}(e^{-\lambda t} - 1). \quad (3.27)$$

The density matrix in the equation (3.17) represents the combined system of atom and cavity so to acquire the atomic state, we need to trace over the cavity,

$$\mathcal{D}_{atom}(t) = \begin{pmatrix} \mathcal{D}^{11}_{atom}(t) & \mathcal{D}^{12}_{atom}(t) \\ \mathcal{D}^{21}_{atom}(t) & \mathcal{D}^{22}_{atom}(t) \end{pmatrix}, \quad (3.28)$$

here

$$\begin{aligned} \mathcal{D}^{11}_{atom} &= \frac{1}{2}(\mathcal{D}_{11}(t) + \mathcal{D}_{12}(t) + \mathcal{D}_{21}(t) + \mathcal{D}_{22}(t) + 2\mathcal{D}_{33}(t)), \\ \mathcal{D}^{12}_{atom} &= \frac{1}{\sqrt{2}}(\mathcal{D}_{13}(t) - \mathcal{D}_{23}(t)), \\ \mathcal{D}^{21}_{atom} &= \frac{1}{\sqrt{2}}(\mathcal{D}_{31}(t) - \mathcal{D}_{32}(t)), \\ \mathcal{D}^{22}_{atom} &= \frac{1}{2}(\mathcal{D}_{11}(t) - \mathcal{D}_{12}(t) - \mathcal{D}_{21}(t) + \mathcal{D}_{22}(t)). \end{aligned} \quad (3.29)$$

In the following sections, we will look at two examples; one will be in the dressed state basis while the other will be an arbitrary qubit state and we shall calculate the QFI for these two examples. Furthermore, we can study the dynamical nature of QFI by using the quantity called the QFI flow defined as:

$$I_\phi = \partial_t F_\phi \quad (3.30)$$

where F_ϕ is the QFI and the subscript ϕ is the parameter of interest.

3.1.2 QFI for a quantum state in dressed state basis

Lets say that we constructed a three state system given in terms of the dressed state basis $\{\ |E_+\rangle, |E_-\rangle, |E_0\rangle\ \}$ as[17]

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} e^{\iota\phi} \sin\frac{\theta}{2} |E_{1,+}\rangle - \frac{1}{\sqrt{2}} e^{\iota\phi} \sin\frac{\theta}{2} |E_{1,-}\rangle + \cos\frac{\theta}{2} |E_0\rangle. \tag{3.31}$$

The parameters $\theta \in [0, \pi)$ and $\phi \in [0, 2\pi)$ correspond to the information about the amplitude and the phase, respectively. Using this state, the initial density matrix is:

$$\mathcal{D}(0) = |\psi_1\rangle\langle\psi_1|, \tag{3.32}$$

the matrix elements are

$$\begin{aligned}
&\mathcal{D}_{11}(0) = \frac{1}{2}\sin^2\frac{\theta}{2}, \quad \mathcal{D}_{12}(0) = -\frac{1}{2}\sin^2\frac{\theta}{2}, \quad \mathcal{D}_{13}(0) = \frac{1}{\sqrt{2}} e^{\iota\phi} \sin\frac{\theta}{2}\cos\frac{\theta}{2}, \\
&\mathcal{D}_{21}(0) = -\frac{1}{2}\sin^2\frac{\theta}{2}, \quad \mathcal{D}_{22}(0) = \frac{1}{2}\sin^2\frac{\theta}{2}, \quad \mathcal{D}_{23}(0) = -\frac{1}{\sqrt{2}} e^{\iota\phi} \sin\frac{\theta}{2}\cos\frac{\theta}{2}, \\
&\mathcal{D}_{31}(0) = \frac{1}{\sqrt{2}} e^{-\iota\phi} \sin\frac{\theta}{2}\cos\frac{\theta}{2}, \quad \mathcal{D}_{32}(0) = -\frac{1}{\sqrt{2}} e^{-\iota\phi} \sin\frac{\theta}{2}\cos\frac{\theta}{2}, \\
&\mathcal{D}_{33}(0) = \cos^2\frac{\theta}{2}.
\end{aligned} \tag{3.33}$$

Now we can get the time-evolved reduced density matrix directly from equation (3.29). For this, we first substitute the time-dependent elements from equation (3.17), after that, using the initial state in equation (3.33), we have the following result:

$$\rho'_{11}(t) = [1 - \frac{1}{4}(M^{11}_{11} + M^{12}_{12} + M^{21}_{21} + M^{22}_{22})]\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2}, \tag{3.34}$$

$$\rho'_{22}(t) = \frac{1}{4}(M^{11}_{11} + M^{12}_{12} + M^{21}_{21} + M^{22}_{22})\sin^2\frac{\theta}{2}, \tag{3.35}$$

$$\rho'_{12}(t) = \frac{1}{2}(M^{13}_{13} + M^{23}_{23}) e^{\iota\phi} \sin\frac{\theta}{2}\cos\frac{\theta}{2}, \tag{3.36}$$

$$\rho'_{21}(t) = \frac{1}{2}(M^{31}_{31} + M^{32}_{32}) e^{-\iota\phi} \sin\frac{\theta}{2}\cos\frac{\theta}{2}, \tag{3.37}$$

$$\rho'(t) = \begin{pmatrix} \rho'_{11}(t) & \rho'_{12}(t) \\ \rho'_{21}(t) & \rho'_{22}(t) \end{pmatrix}. \tag{3.38}$$

Quantum fisher information has already been discussed in detail in the previous chapter, we are just going to mention the general form (3.14) used here to calculate the QFI:

$$F_\phi = Tr\{\rho'_\phi L^2_\phi\} = Tr\{\partial_\phi \rho'_\phi L_\phi\}, \tag{3.39}$$

where L_ϕ is the SLD operator, determined by

$$\partial_\phi \rho_\phi^{'} = L\rho_\phi^{'} + \rho_\phi^{'} L, \tag{3.40}$$

we can modify equation (3.39) as

$$F_\phi = 2\frac{\langle\psi_i|\partial_\phi \rho_\phi^{'}|\psi_j\rangle\langle\psi_j|\partial_\phi \rho_\phi^{'}|\psi_i\rangle}{P_i + P_j}. \tag{3.41}$$

In the denominator, we have the sum of probabilities of the states which is given by the diagonal elements of $\rho_\phi^{'}$ and for a two level system $\sum_{i=1}^{2} P_i = 1$. Using $\rho^{'}(t)$ in equation (3.41) we get the QFI for state $|\psi_1\rangle$

$$F_\phi^1 = \frac{1}{4}(M_{13}^{13} + M_{23}^{23})(M_{31}^{31} + M_{32}^{32})\sin^2\theta, \tag{3.42}$$

here the superscript in F^1 indicates for the state $|\psi_1\rangle$.

3.1.3 QFI for an arbitrary qubit

Consider the qubit state [17]

$$|\psi_2\rangle = \cos\frac{\theta}{2}|e\rangle + e^{\iota\phi}\sin\frac{\theta}{2}|g\rangle, \tag{3.43}$$

according to Bloch sphere θ is the polar angle and ϕ is the azimuth angle. Since the right hand side of the equation (3.29) has the elements of $\mathcal{D}(t)$ which is in terms of dressed state basis, one way we can approach this is that we make a product state with the initial atomic state $|\psi_2\rangle$ and initial field state $|0\rangle$

$$|\psi_2^{'}\rangle = |0\rangle \otimes |\psi_2\rangle. \tag{3.44}$$

After converting it in the dressed state basis, we get the initial state as,

$$|\psi_2^{'}\rangle = \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}|E_{1,+}\rangle + \frac{1}{\sqrt{2}}\cos\frac{\theta}{2}|E_{1,-}\rangle + e^{\iota\phi}\sin\frac{\theta}{2}|E_0\rangle. \tag{3.45}$$

The procedure to find F_ϕ^2 is same as used for the state $|\psi_1\rangle$,

$$F_\phi^2 = \frac{1}{4}(M_{13}^{13} - M_{23}^{23})(M_{31}^{31} - M_{32}^{32})\sin^2\theta, \tag{3.46}$$

here the superscript in F^2 indicates for the state $|\psi_2\rangle$.

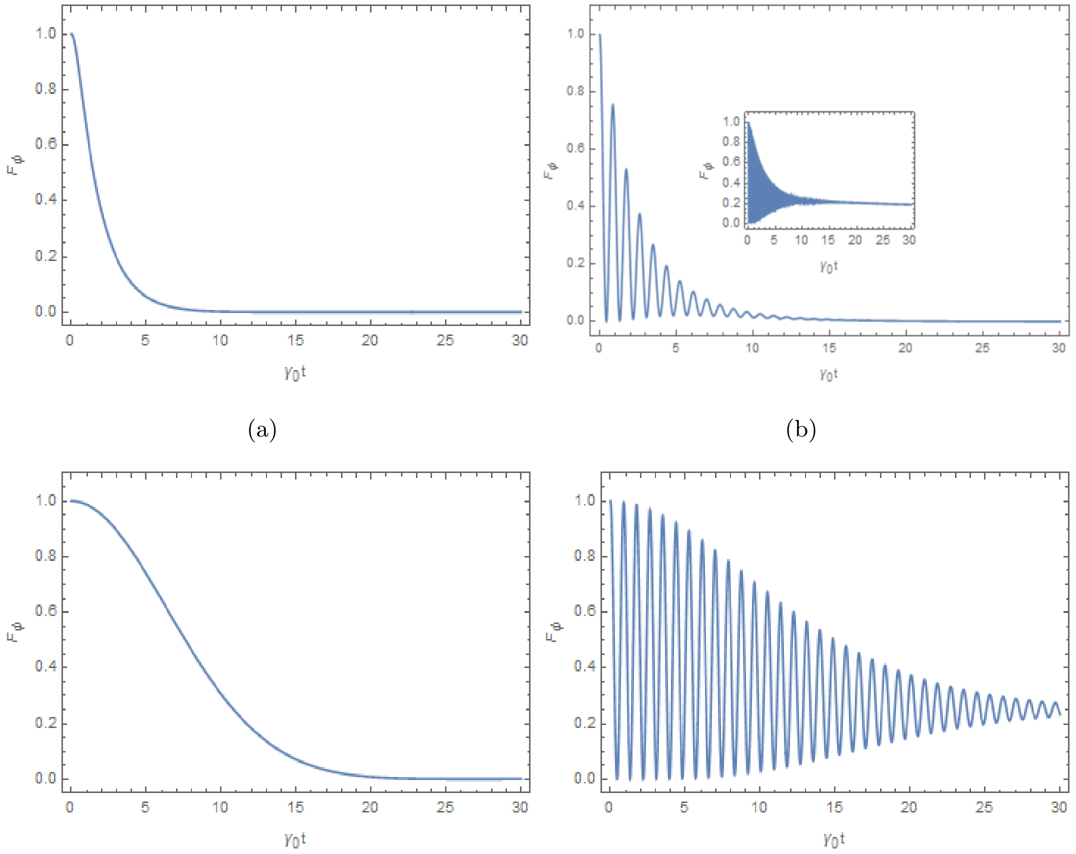
3.2 Results and Discussion

The QFI for the states $|\psi_1\rangle$ and $|\psi_2\rangle$ is provided by equations (3.42) and (3.46), respectively. We will now talk about these equations' numerical solutions. Since the atom and the dissipation cavity are coupled, as the physical model indicates, QFI and its dynamic nature are influenced by the atom-cavity interaction as well as the interaction of the cavity and reservoir. We investigate the effects of variations in the different coupling strengths between the atom, cavity, and reservoir on the QFI under both non-Markovian and Markovian conditions. We want to observe what effects weak and strong coupling strengths have on the QFI.

3.2.1 Numerical Results of QFI F_ϕ^1

In figure (3.1) we have the results for the dynamics of QFI of $|\psi_1\rangle$ for Markovian ($\lambda = 5\gamma_0$) and non-Markovian ($\lambda = 0.05\gamma_0$) cases.

Let us break it down: when $\lambda > 2\gamma_0$, we have a Markovian reservoir, while $\lambda < 2\gamma_0$ is for the non-Markovian setup. Similarly, $\Omega > 2\gamma_0$ represents a strong atom-cavity coupling and $\Omega < 3\gamma_0$ gives a weak coupling of atom and cavity. In figure (3.1 a), with a Markovian reservoir ($\lambda = 5\gamma_0$) and weak atom-cavity coupling ($\Omega = 0.05\gamma_0$), F_ϕ steadily diminishes and quickly approaches zero. In contrast, figure (3.1 b) still holds a Markovian reservoir ($\lambda = 5\gamma_0$) but with a strong atom cavity coupling ($\Omega = 3\gamma_0$); here, F_ϕ oscillates, and rises to 0.75 before eventually decaying to zero. Comparing (3.1 a) and (3.1 b), as energy and information can be exchanged between the atom and cavity when ($\Omega = 3\gamma_0$) and ($\lambda = 5\gamma_0$). Thus, due to strong coupling F_ϕ shows significant oscillations and the Markovian nature of reservoir causes dissipation and decay of F_ϕ to zero. Meanwhile, (3.1 c) illustrates F_ϕ's steady reduction over time in the non-Markovian setup ($\Omega = 0.05\gamma_0$) and weak atom-cavity coupling ($\lambda = 5\gamma_0$), vanishing only in the asymptotic limit. Comparing (3.1 a) and (3.1 c), the decay rate of F_ϕ is notably faster in the former. Lastly, (3.1 d) depicts the scenario with $\lambda = 0.05\gamma_0$ (non-Markovian) and $\Omega = 3\gamma_0$ (strong atom-cavity), resulting in F_ϕ approaching a stable value instead of zero after oscillating for multiple cycles. Comparing (3.1 b) and (3.1 d), the survival time of the oscillations increases in (3.1 d). The cause of this increase is the memory and feedback effects of the non-Markovian reservoir; and the stable value in (3.1 d) is



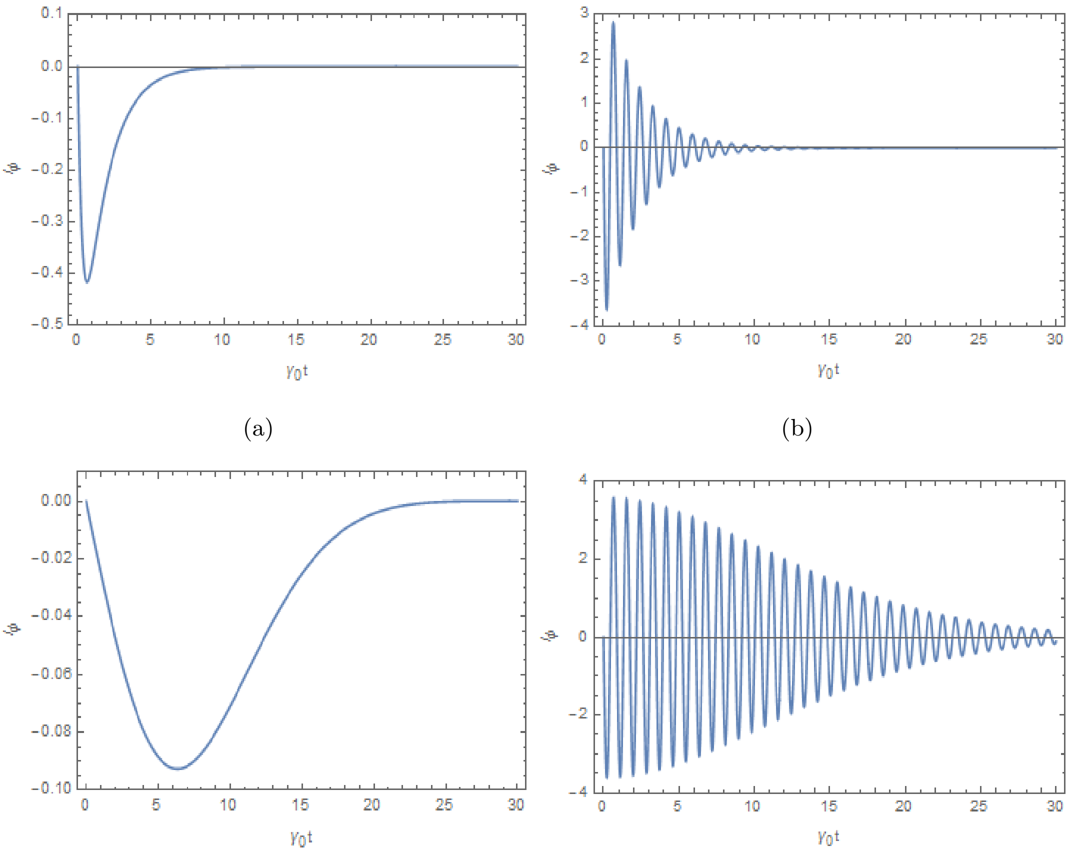
(a) (b)

(c) (d)

Figure 3.1: F_ϕ as a function of $(\gamma_0 t)$ for the the initial state $|\psi_1\rangle$.Here $\theta = \frac{\pi]}{2}$.**(a)**$\lambda = 5\gamma_0$, $\Omega = 0.05\gamma_0$; **(b)**$\lambda = 5\gamma_0$, $\Omega = 3\gamma_0$; **(c)**$\lambda = 0.05\gamma_0$, $\Omega = 0.05\gamma_0$; **(d)**$\lambda = 0.05\gamma_0$, $\Omega = 3\gamma_0$. The inset in **(b)** is for $\Omega = 20\gamma_0$.

greater than in (3.1 b).

The QFI flow I_ϕ for the initial state $|\psi_1\rangle$ is plotted in figure (3.2) as a function of $\gamma_0 t$ for Markovian and non-Markovian regime. In figure (3.2 a), I_ϕ experiences a rapid shift from 0 to -0.4, followed by a return to 0 for $\lambda = 5\gamma_0$ and $\Omega = 0.05\gamma_0$, consistently remaining negative. This indicates swift energy and information transfer from the atom to the reservoir, causing a rapid decline of F_ϕ to zero, consistent with the trend in figure (3.1 a). In figure (3.2 b), I_ϕ drops rapidly to 3.0 from 0, then rises swiftly to +2.3, eventually damping to 0. The transitions in I_ϕ from negative to positive values, mirror energy and information flows out of and back into the atom, leading to swift declines in



(a) (b)

(c) (d)

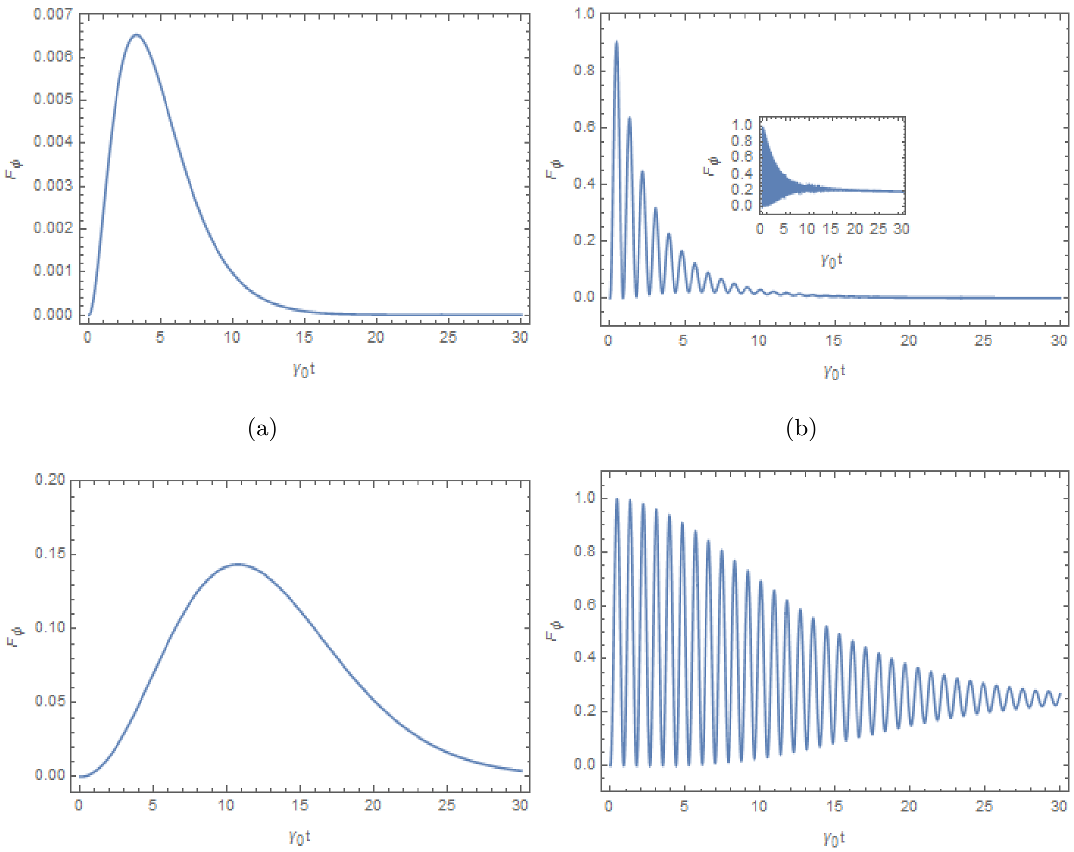
Figure 3.2: the effect of Ω and λ on I_ϕ as a function of ($\gamma_0 t$) for the the initial state $|\psi_1\rangle$ is shown in this figure.Here $\theta = \frac{\pi]}{2}$.**(a)**$\lambda = 5\gamma_0$, $\Omega = 0.05\gamma_0$; **(b)**$\lambda = 5\gamma_0$, $\Omega = 3\gamma_0$; **(c)**$\lambda = 0.05\gamma_0$, $\Omega = 0.05\gamma_0$; **(d)**$\lambda = 0.05\gamma_0$, $\Omega = 3\gamma_0$. The inset in **(b)** is for $\Omega = 20\gamma_0$.

F_ϕ ($0 \to -0.5 \to 0$) and subsequent rapid rises to a specific value ($0 \to 2.3 \to 0$), akin to the pattern observed in figure (3.1 b). Additionally, comparing figures (3.2 b) and (3.2 d) showcases similar QFI flow patterns, differing mainly in the λ value. Smaller λ values accentuate non-Markovian effects, slowing the decay of I_ϕ. By comparing figures (3.2 a) and (3.2 b), as well as figures (3.2 c) and (3.2 d), higher Ω values indicative of stronger cavity-reservoir coupling, accelerate QFI flow oscillations and recovery. These patterns demonstrate a reverse flow of information and energy, i.e., from the reservoir to the system, emphasizing the inherent relationship between QFI and information interchange between the system and its surroundings.

3.2.2 Numerical Results of QFI F_ϕ^2

Now we move to the second example, where we considered the arbitrary qubit state $|\psi_2\rangle$ from equation (**??**) and figure (3.3) shows the effect of atom-cavity coupling on the evolution of F_ϕ, in the Markovian ($\lambda = 5\gamma_0$) and non-Markovian ($\lambda = 0.05\gamma_0$) regimes. Here $\theta = \frac{\pi}{2}$. In figure (3.3 a), F_ϕ increases from zero to 0.006 then decays back to zero for the later values of $\gamma_0 t$. This is the case of $\lambda = 5\gamma_0$ (Markovian) and $\Omega = 0.05\gamma_0$ (weak atom-field coupling). The considerable drop of the F_ϕ implies a higher accuracy of the estimate. In the figure (3.3 b), f_ϕ first increases from zero to close to 1.0 and then oscillates damply to zero.This is the case of ($\lambda = 5\gamma_0$) and ($\Omega = 3\gamma_0$) .In the inset of the figure (3.3 b) it is shown that for $\Omega = 20\gamma_0$, F_ϕ does not decreases to zero, but rather it undergoes damping oscillation until it reaches a stable value of 0.25. Comparing Figure (3.3 a) to Figure (3.3 c) reveals similar F_ϕ dynamics, primarily differing in their maximum values, with the latter taking longer to approach zero. Meanwhile, Figure (3.3 b) and (3.3 d) showcase analogous F_ϕ trends, where the former stabilizes at 0.25 while the latter diminishes at lower Ω, yet converges to 0.25 at $\Omega = 20\gamma_0$. Increasing Ω across Figure (3.3) accelerates F_ϕ oscillation frequency and decelerates F_ϕ decay. Weak coupling scenarios (figure (3.3 a) and (3.3 c)) result in rapid decline of all QFIs to zero, while strong coupling (Figure (3.3 b) and (3.3 d)) induces rapid oscillations before stabilizing at 0.25. .

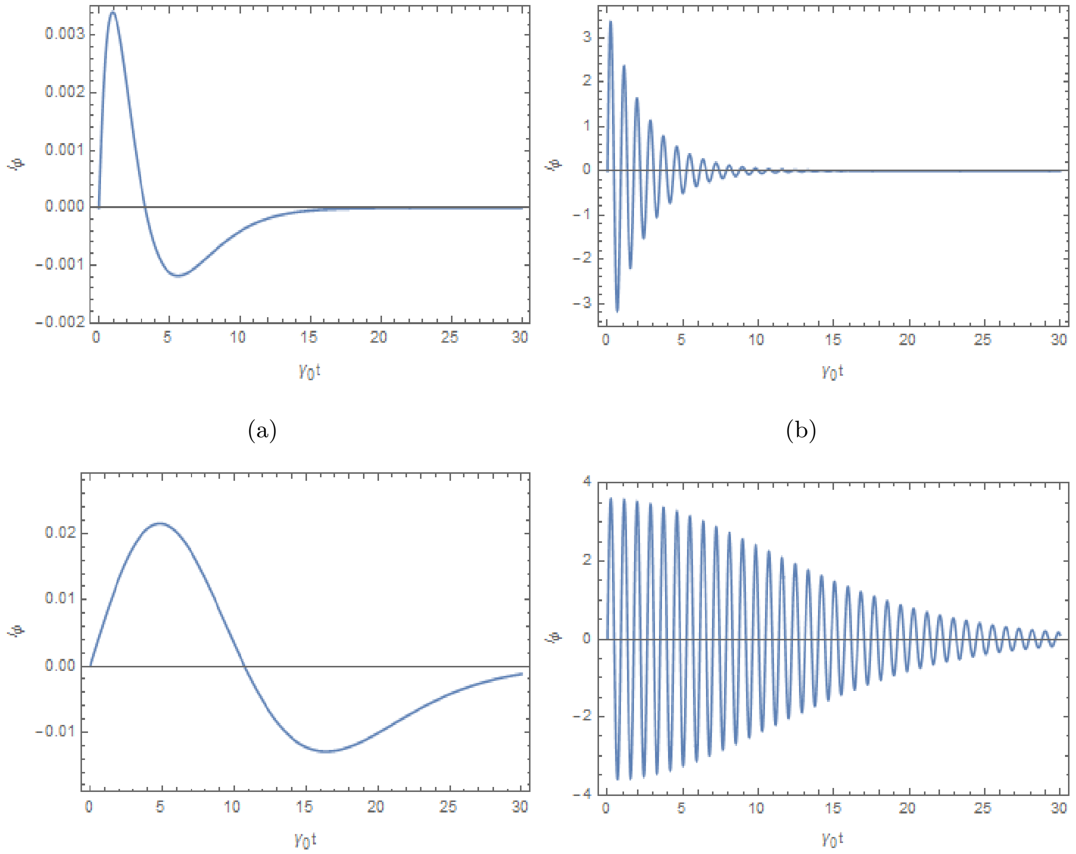
The atom-cavity coupling, Ω, affects the dynamics of I_ϕ in both the non-Markovian ($\lambda = 5\gamma_0$) and Markovian ($\lambda = 0.05\gamma_0$) regimes, as shown in Figure (3.4). In the Markovian setting ($\lambda = 0.05\gamma_0$) with weak atom-cavity coupling ($\Omega = 3\gamma_0$), as shown in figure (3.4 a), I_ϕ increases with increasing time t from zero to 0.0025, then oscillates back to zero. Similar to the F_ϕ oscillation in figure (3.4 a), this shows the atom first obtaining energy and information from the cavity as a result of atom-cavity interaction, followed by dissipation that causes this energy and information to vanish. Moreover, in figure (3.4 b), with strong coupling, I_ϕ rapidly oscillates to zero, corresponding to the F_ϕ oscillation in figure (3.3 b). The environment's memory effect allows some lost information to return to the cavity system. In figure (3.4 c), I_ϕ gradually oscillates to zero, indicating slower decay as Ω increases, observed by comparing figure (3.4 a) and figure (3.4 c). Figure (3.4 d) demonstrates rapid oscillation and decay of I_ϕ, reflecting swift interflow of F_ϕ from the system to the environment and then back to system. .



(a) (b)

(c) (d)

Figure 3.3: F_ϕ as a function of $(\gamma_0 t)$ for the the initial state $|\psi_2\rangle$.Here $\theta = \frac{\pi]}{2}$.**(a)**$\lambda = 5\gamma_0$, $\Omega = 0.05\gamma_0$; **(b)**$\lambda = 5\gamma_0$, $\Omega = 3\gamma_0$; **(c)**$\lambda = 0.05\gamma_0$, $\Omega = 0.05\gamma_0$; **(d)**$\lambda = 0.05\gamma_0$, $\Omega = 3\gamma_0$. The inset in **(b)** is for $\Omega = 20\gamma_0$.



(a)

(b)

(c)

(d)

Figure 3.4: the effect of Ω and λ on I_ϕ as a function of $(\gamma_0 t)$ for the the initial state $|\psi_2\rangle$ is shown in this figure.Here $\theta = \frac{\pi]}{2}$.**(a)**$\lambda = 5\gamma_0$, $\Omega = 0.05\gamma_0$; **(b)**$\lambda = 5\gamma_0$, $\Omega = 3\gamma_0$; **(c)**$\lambda = 0.05\gamma_0$, $\Omega = 0.05\gamma_0$; **(d)**$\lambda = 0.05\gamma_0$, $\Omega = 3\gamma_0$. The inset in **(b)** is for $\Omega = 20\gamma_0$.

3.2.3 Physical Analysis of QFI F_ϕ and QFI flow I_ϕ

The following points give the overall analysis of QFI and QFI flow for the states $|\psi_1\rangle$ and $|\psi_2\rangle$:

- If we look at figures (3.1) and (3.3), the F_ϕ decreases and decays to zero, for weak atom-cavity coupling ($\Omega < 2\gamma_0$) and both cases of Markovian and non-Markovian environments. However, the decay is a bit delayed for the non-Markovian case.
- When $\Omega > 2\gamma_0$ (strong atom-cavity coupling), F_ϕ shows damped oscillations, resulting in a stable value at later times. This is shown in figures (3.1 b, 3.1 d) and (3.3 b, 3.3 d).
- For the state $|\psi_2\rangle$ there is a slight difference in the behavior of F_ϕ. In figure (3.3), the QFI starts from zero, increases to a maximum value and then decays back to zero. The reason for this difference is the memory and feedback effect of the non-Markovian environment.
- In the figures (3.2) and (3.4), we can see that I_ϕ can be positive or negative as it increases and decreases. This positive and negative behavior corresponds to inward and outward flow of information F_ϕ of the atom.
- When the coupling Ω is weak, then I_ϕ is shown to be negative which is an outward flow for both Markovian and non-Markovian regimes in figures (3.2 a, 3.2 c) and (3.4 a, 3.4 c). The initial rise in I_ϕ is due to interaction with the environment.
- Finally when the atom-cavity coupling is strong ($\Omega > 2\gamma_0$) I_ϕ is undergoing damped oscillations which corresponds to inward and outward flow (figures 3.2 b, 3.2 d and 3.4 b, 3.4 d). The variations in I_ϕ for the states $|\psi_1\rangle$ and $|\psi_2\rangle$ is in accordance with the changes in F_ϕ for the two states.

Chapter 4

Application of Jaynes Cummings Model to the Kerr Medium

4.1 Physical Model

Let us consider a system in which 2 two-level atoms are placed inside a cavity. Within the cavity is an electromagnetic field of single-mode frequency. The atomic system is interacting with the electromagnetic field in the presence of a Kerr medium. As introduced in Chapter 2, the Kerr medium is a medium in which the Kerr effect is induced, which is the change in the refractive index of the material due to an external field. Its a non-linear effect where we have to consider higher-order perturbations in the polarization of the medium, more specifically third-order contributions. We shall begin by writing the Hamiltonian for this system, the ingredients of which have already been discussed in the Chapter 2. So we have [20] [15]

$$H = \omega_f a^\dagger a + \frac{1}{2}\omega_0 \sum_{j=A,B} \sigma_3^j + \lambda \sum_{j=A,B} (\sigma_+^j a + \sigma_-^j a^\dagger) + \chi(a^\dagger a a^\dagger a), \tag{4.1}$$

where $\hbar = 1$. Let the two energy levels of the atoms be $|0\rangle_j$ (ground state) and $|1\rangle_j$ (excited state). The index j refers to the two atoms A and B. Also $\sigma_+^j = |1\rangle_{jj}\langle 0|$, $\sigma_-^j = |0\rangle_{jj}\langle 1|$ and $\sigma_3^j = |1\rangle_{jj}\langle 1| - |0\rangle_{jj}\langle 0|$.

The field states are defined by the Fock state $|N\rangle$ and it is prepared initially in the state $|n+1\rangle$. This combined system of atoms and field is represented by the product state,

$$|\psi\rangle = |field\rangle \otimes |atoms\rangle_{AB}. \tag{4.2}$$

Now we will define the basis vectors for our system as

$$|N, 11\rangle, \tag{4.3}$$

$$|N+1, 10\rangle, \tag{4.4}$$

$$|N+1, 01\rangle, \tag{4.5}$$

$$|N+2, 00\rangle. \tag{4.6}$$

There is a conserved quantity G [20] for the Hamiltonian describing this system,

$$G = a^\dagger a + 1 + \frac{1}{2}(\sigma_3^A + \sigma_3^B). \tag{4.7}$$

This operator G gives the value $n+2$ for all the basis vectors above. Using this quantity, we can split our Hamiltonian into two parts:

$$H = H_I + H_{II}. \tag{4.8}$$

In this, H_I will contain all the conserved terms and H_{II} will have all the dynamical terms. From the expression of G we take $a^\dagger a$

$$a^\dagger a = G - 1 - \frac{1}{2}(\sigma_3^A + \sigma_3^B). \tag{4.9}$$

Next, we take the square of the equation (4.7) to get

$$\begin{aligned} G^2 &= [a^\dagger a + 1 + \frac{1}{2}(\sigma_3^A + \sigma_3^B)]^2, && (4.10)\\ &= (a^\dagger a + 1)^2 + \frac{1}{4}(\sigma_3^A + \sigma_3^B)^2 + (a^\dagger a + 1)(\sigma_3^A + \sigma_3^B), && (4.11)\\ &= a^\dagger a a^\dagger a + 2a^\dagger a + 1 + \frac{1}{4}(\sigma_3^A + \sigma_3^B)^2 + a^\dagger a(\sigma_3^A + \sigma_3^B) + (\sigma_3^A + \sigma_3^B). && (4.12) \end{aligned}$$

Using the equation (4.9), we make a substitution in the 2nd term as

$$\begin{aligned} G^2 = {} & a^\dagger a a^\dagger a + (2G - 2 - (\sigma_3^A + \sigma_3^B) + 1)\\ & + \frac{1}{4}(\sigma_3^A + \sigma_3^B)^2 + a^\dagger a(\sigma_3^A + \sigma_3^B) + (\sigma_3^A + \sigma_3^B). \end{aligned} \tag{4.13}$$

After simplifying, we get

$$G^2 = a^\dagger a a^\dagger a + 2G - 2 + \frac{1}{4}(\sigma_3^A + \sigma_3^B)^2 + a^\dagger a(\sigma_3^A + \sigma_3^B) + 1. \tag{4.14}$$

We can rearrange this equation to get $a^\dagger a a^\dagger a$ as

$$a^\dagger a a^\dagger a = (G^2 - 2G + 2) - a^\dagger a(\sigma_3^A + \sigma_3^B) - \frac{1}{4}(\sigma_3^A + \sigma_3^B)^2 - 1. \tag{4.15}$$

Now we modify the Hamiltonian in equation (4.1) by substituting the equations (4.9) and (4.15)

$$\begin{aligned}H = \omega_f(G - 1 - \frac{1}{2}(\sigma_3^A + \sigma_3^B)) + \frac{1}{2}\omega_0 \sum_{j=A,B} \sigma_3^j \\ + \lambda \sum_{j=A,B} (\sigma_+^j a + \sigma_-^j a^\dagger) + \chi((G^2 - 2G + 2)) \\ + \chi(-a^\dagger a(\sigma_3^A + \sigma_3^B) - \frac{1}{4}(\sigma_3^A + \sigma_3^B)^2 - 1).\end{aligned} \tag{4.16}$$

After simplification

$$\begin{aligned}H = \omega_f(G - 1) + \chi(G^2 - 2G + 2) \\ + \frac{\Delta}{2} \sum_{j=A,B} \sigma_3^j + \lambda \sum_{j=A,B} (\sigma_+^j a + \sigma_-^j a^\dagger) \\ - \chi(a^\dagger a(\sigma_3^A + \sigma_3^B) + \frac{1}{4}(\sigma_3^A + \sigma_3^B)^2 + 1),\end{aligned} \tag{4.17}$$

Where

$$H_I = \omega_f(G - 1) + \chi(G^2 - 2G + 2) \tag{4.18}$$

$$H_{II} = \frac{\Delta}{2} \sum_{j=A,B} \sigma_3^j + \lambda \sum_{j=A,B} (\sigma_+^j a + \sigma_-^j a^\dagger) - \chi(a^\dagger a(\sigma_3^A + \sigma_3^B) + \frac{1}{4}(\sigma_3^A + \sigma_3^B)^2 + 1). \tag{4.19}$$

$\Delta = \omega_0 - \omega_f$ is the detuning. Using the following eigenvalue equation, we can find the eigenvalues and eigenvectors of H_{II}

$$H_{II}|\psi\rangle = E|\psi\rangle. \tag{4.20}$$

E is the eigenvalue and the state $|\psi\rangle = \sum_j c_j|u_j\rangle$ is the eigenvector. $|u_j\rangle$ are the basis vectors from equations (4.3) - (4.6),

$$\langle u_i|H_{II}|\psi\rangle = E\langle u_i|\psi\rangle. \tag{4.21}$$

Using the completeness relation $\sum_j |u_j\rangle\langle u_j| = 1$, we can write

$$\sum_j \langle u_i|H_{II}|u_j\rangle\langle u_j|\psi\rangle = E\langle u_i|\psi\rangle. \tag{4.22}$$

The quantity, $\langle u_i|H_{II}|u_j\rangle = H_{ij}$ forms the matrix elements of H_{II}. Thus, we have the eigenvalue equation in the form

$$\sum_{ij}[H_{ij} - E\delta_{ij}]c_j = 0. \tag{4.23}$$

To get the matrix form of the interaction Hamiltonian we need to solve for each of the matrix elements $\langle u_i|H_{II}|u_j\rangle$ [20], then

$$\hat{H}_{II} = \begin{pmatrix} \Delta - \kappa(2N+2) & \lambda\sqrt{N+1} & \lambda\sqrt{N+1} & 0 \\ \lambda\sqrt{N+1} & -\chi & 0 & \lambda\sqrt{N+2} \\ \lambda\sqrt{N+1} & 0 & -\chi & \lambda\sqrt{N+2} \\ 0 & \lambda\sqrt{N+2} & \lambda\sqrt{N+2} & \Delta - \kappa(2N+2) \end{pmatrix}. \tag{4.24}$$

Using the characteristic polynomial $\mathrm{Det}[H_{ij} - E\delta_{ij}]$ we can find the eigenvalues of $\hat{H}_{II}$ as [15]

$$\begin{array}{cc} E_1 = \Delta - \chi(2N+2), & E_2 = -\chi, \\ E_3 = \frac{1}{2}[\Delta - (2N+3)\chi - \delta], & E_4 = \frac{1}{2}[\Delta - (2N+3)\chi + \delta] \end{array}, \tag{4.25}$$

where $\delta = \sqrt{(\Delta + 2N\chi + \chi)^2 + 8(2N+3)\lambda^2}$. The normalized eigenvectors corresponding to these eigenvalues are

$$\begin{aligned} |\psi_1\rangle &= -\sqrt{\frac{N+2}{2N+3}}|N,11\rangle + \sqrt{\frac{N+1}{2N+3}}|N+2,00\rangle, \\ |\psi_2\rangle &= \frac{1}{\sqrt{2}}(-|N+1,10\rangle + |N+1,01\rangle), \\ |\psi_3\rangle &= \frac{2\lambda\sqrt{N+1}}{\sqrt{\eta_1}}|N,11\rangle - \frac{\eta_1}{2\delta}|N+1,10\rangle \\ &\quad - \frac{\eta_1}{2\delta}|N+1,01\rangle + \frac{2\lambda\sqrt{N+1}}{\sqrt{\eta_1}}|N+2,00\rangle, \\ |\psi_4\rangle &= \frac{2\lambda\sqrt{N+1}}{\sqrt{\eta_2}}|N,11\rangle - \frac{\eta_2}{2\delta}|N+1,10\rangle \\ &\quad - \frac{\eta_2}{2\delta}|N+1,01\rangle + \frac{2\lambda\sqrt{N+1}}{\sqrt{\eta_2}}|N+2,00\rangle. \end{aligned} \tag{4.26}$$

Here, $\eta_1 = \delta(\delta + \Delta - (2N+1)\chi)$ and $\eta_2 = \delta(\delta - \Delta + (2N+1)\chi)$. Next, we shall apply the density matrix approach to solve for the evolution of the atomic system.

4.2 Density Matrix Approach

To solve for the time-evolved state of the atomic system,let the initial state of the composite system be defined by the density matrix

$$\mathcal{D}(0) = |\psi_{sys}(0)\rangle\langle\psi_{sys}(0)|, \tag{4.27}$$

where $|\psi_{sys}(0)\rangle = |\psi_f(0)\rangle \otimes |\psi_{AB}(0)\rangle$. The initial state of the field and the atoms [15] is $|\psi_f(0)\rangle = |n+1\rangle$ and $|\psi_{AB}(0)\rangle = \cos\phi|10\rangle + \sin\phi|01\rangle$, respectively. Expanding $\mathcal{D}(0)$ as

$$\begin{aligned}\mathcal{D}(0) = \cos^2\phi|N+1,10\rangle\langle N+1,10| + \cos\phi\sin\phi|N+1,10\rangle\langle N+1,01| \\ + \sin\phi\cos\phi|N+1,01\rangle\langle N+1,10| + \sin^2\phi|N+1,01\rangle\langle N+1,01|.\end{aligned} \tag{4.28}$$

The dynamics of this density matrix under Markov approximation is governed by the quantum Master Equation [15] [21],

$$\frac{\partial}{\partial t}\mathcal{D}(t) = -\iota[H,\mathcal{D}(t)] - \frac{1}{2\gamma}[H,[H,\mathcal{D}(t)]], \tag{4.29}$$

the general solution to this equation is of the form:

$$\mathcal{D}(t) = \sum_{\nu=0}^{\infty}(\frac{t}{\gamma})^{\nu}\frac{1}{\nu!}Q^{\nu}\mathcal{D}(0)Q^{\dagger\nu}, \tag{4.30}$$

where

$$Q^{\nu} = H^{\nu}\exp(-\iota tH)\exp(-\frac{t}{2\gamma}H^2). \tag{4.31}$$

The objective is to find the time-evolved state of the atomic system $\mathcal{D}_{AB}$ of the initial state. To do this, the equation (4.30) is to be solved in the eigenbasis of the Hamiltonian, and the expansion of $\mathcal{D}(t)$ is carried out by using the following formula:

$$e^{Z}Y e^{-Z} = Y + [Z,Y] + \frac{1}{2!}[Z,[Z,Y]] + ... \tag{4.32}$$

The expansion of $\mathcal{D}(t)$ will be upto the order of $1/\gamma$ according to the equation (4.29). Then finally tracing over the field will provide us with $\mathcal{D}_{AB}(t)$ of the form:

$$\begin{aligned}\mathcal{D}_{AB}(t) =& \mathcal{D}_{11}|11\rangle\langle 11| + \mathcal{D}_{22}|10\rangle\langle 10| + \mathcal{D}_{33}|01\rangle\langle 01| \\ &+ \mathcal{D}_{44}|00\rangle\langle 00| + \mathcal{D}_{23}|10\rangle\langle 10| + \mathcal{D}_{32}|01\rangle\langle 01|,\end{aligned} \tag{4.33}$$

where, the matrix elements are

$$\begin{aligned}
\mathcal{D}_{11} &= \frac{2\lambda^2(N+1)}{\delta^2}X, \\
\mathcal{D}_{44} &= \frac{2\lambda^2(N+2)}{\delta^2}X, \\
\mathcal{D}_{22} &= \frac{1}{2} - \frac{\lambda^2(2N+3)}{\delta^2}X + \frac{\cos(2\phi)}{4\delta^2}[Y_1\cos(\frac{t\eta_1}{2\delta}) + Y_2\cos(\frac{t\eta_2}{2\delta})], \\
\mathcal{D}_{33} &= \frac{1}{2} - \frac{\lambda^2(2N+3)}{\delta^2}X - \frac{\cos(2\phi)}{4\delta^2}[Y_1\cos(\frac{t\eta_1}{2\delta}) + Y_2\cos(\frac{t\eta_2}{2\delta})], \\
\mathcal{D}_{23} &= \frac{\sin(2\phi)}{2} - \frac{\lambda^2(2N+3)}{\delta^2}X - \frac{\iota\cos(2\phi)}{4\delta^2}[Y_1\sin(\frac{t\eta_1}{2\delta}) - Y_2\sin(\frac{t\eta_2}{2\delta})], \\
\mathcal{D}_{32} &= \frac{\sin(2\phi)}{2} - \frac{\lambda^2(2N+3)}{\delta^2}X + \frac{\iota\cos(2\phi)}{4\delta^2}[Y_1\sin(\frac{t\eta_1}{2\delta}) - Y_2\sin(\frac{t\eta_2}{2\delta})],
\end{aligned}$$

and the values of X, Y_1 and Y_2 are given as

$$X = (1+\sin(2\phi))[1-\exp(-\frac{\gamma t\delta^2}{2})]\cos(t\delta),$$
$$Y_1 = \eta_1 \exp[-\frac{\gamma t}{2}(\frac{\eta_1}{2\delta})^2],$$
$$Y_2 = \eta_2 \exp[-\frac{\gamma t}{2}(\frac{\eta_2}{2\delta})^2].$$

We see that $\rho_{AB}(t)$ is depending upon the several parameters like, phase decoherence γ,mean number of photons n, kerr medium coupling κ and the detuning Δ. The primary subject of the system under study is the kerr medium parameter, which we want to estimate and for that, a technique called quantum fisher information will be applied in the following section.

4.3 QFI for Kerr Medium Parameter

To solve the expression of QFI in equation (**??**) we must diagonalize the density matrix of our atomic system.The most frequently used method for solving the quantum fisher information is the diagonalization of the density matrix $\mathcal{D}_\theta$. In our case the density matrix is given by the equation (4.33) and after we diagoanlize $\mathcal{D}_{AB}$ we get

$$\mathcal{D}_{AB} = \sum_{i=1}^{s} P_i|\psi_i\rangle\langle\psi_i|, \tag{4.34}$$

where P_i are the eigenvalues and $|\psi_i\rangle$ are the eigenvectors,which are given as follows [15]

$$\begin{gathered} P_1 = \mathcal{D}_{11}, P_2 = \mathcal{D}_{44}, \\ P_{3,4} = \tfrac{1}{2}(\mathcal{D}_{22} + \mathcal{D}_{33} \pm \sqrt{(\mathcal{D}_{22}-\mathcal{D}_{33})^2 + 4|\mathcal{D}_{23}|^2}). \end{gathered} \tag{4.35}$$

The corresponding eigenvectors are

$$|\psi_1\rangle = (1,0,0,0), \tag{4.36}$$
$$|\psi_2\rangle = (0,0,0,1), \tag{4.37}$$
$$|\psi_3\rangle = \frac{1}{|\mathcal{D}_{23}|\sqrt{|\mathcal{D}_{23}|^2 + (P_3-\mathcal{D}_{33})^2}}(0,(P_3-\mathcal{D}_{33})\mathcal{D}_{23},|\mathcal{D}_{23}|^2,0), \tag{4.38}$$
$$|\psi_4\rangle = \frac{1}{|\mathcal{D}_{23}|\sqrt{|\mathcal{D}_{23}|^2 + (P_4-\mathcal{D}_{33})^2}}(0,(P_4-\mathcal{D}_{33})\mathcal{D}_{23},|\mathcal{D}_{23}|^2,0). \tag{4.39}$$

The expression for F_θ becomes too complicated after substituting the eigenvalues and eigenvectors, thus it is solved numerically and the following section provides the numerical analysis for the results obtained.

4.3.1 Numerical Results

To estimate the parameter χ, we solve F_χ in this section numerically. In the situation of resonance, the effect of change in the photon number N and the degree of entanglement are explored. The behavior of the quantum fisher information for a Kerr-like medium parameter is observed in figure (4.1) under two distinct scenarios. The atom-field system is first brought into resonance. The atomic system is partly entangled and separable, and the cavity field can be in the vacuum state or in the single photon Fock state, initially. As soon as the interaction begins, F_χ spikes immediately, reaching its maximum value before oscillating within the upper and lower limits. The upper limits of oscillations gradually fall and eventually vanish over time. It is also noteworthy that these upper bounds reduce with increasing weight angle, indicating that these bounds are dependent on the initial atomic state. Thus, if classical information is encoded in the initial atomic arrangement, then higher estimates of the parameter χ may be possible.

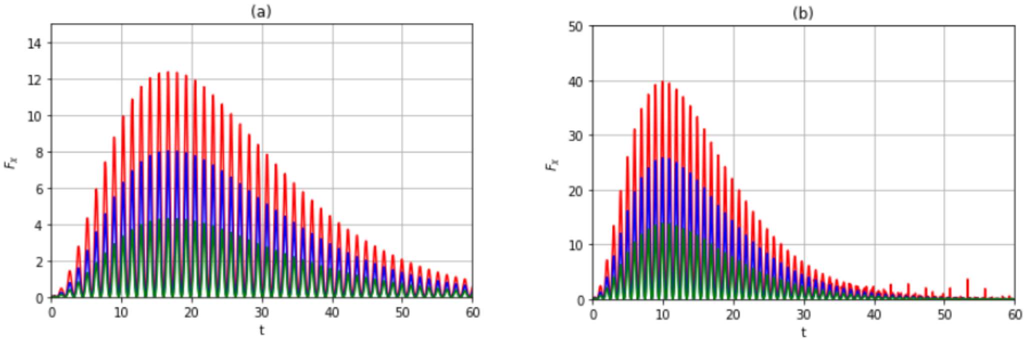


Figure 4.1: F_χ plotted as function of time t with $\lambda = 1$, $\gamma = 0.02$, $\chi = 0.1$ and different initial state settings. The red, blue and green curves are for the three weight angles $0, \pi/8, \pi/3$, respectively. The sub-figures (a) and (b) are for the initial conditions: **(a)** $N = 0$, $\Delta = 0$ and **(b)** $N = 1$, $\Delta = 0$

By comparing the two plots (a) and (b) in figure (4.1), the change in N has a prominent effect. The upper limit for $N = 1$ are higher than those of the $N = 0$.

Chapter 5

Conclusion

In this work, we started with the fundamental concepts of quantum information theory. After that, we delved into the details of the atom field interactions, primarily focusing on the Jaynes Cummings model. Then we discussed briefly the estimation process and classical fisher information, followed by a detailed derivation for quantum fisher information.

Quantum fisher information is studied for two different scenarios. The first scenario deals with the modulation of QFI in a dissipation cavity for Markovian and non-Markovian environments with both weak and strong atom-cavity coupling. Two examples of the initial states were considered; state 1 was the dressed state, and state 2 was an arbitrary qubit. For both states, the QFI showed a decreasing behavior, but for state 2 QFI initially increased from zero to a maximum value and then decayed to zero. The damped oscillations in the behavior of QFI are the result of the strong atom-cavity coupling. We have also seen that the stronger the coupling, the higher the resulting stable value of the QFI. Then comes the quantity of QFI flow, which is the derivative of QFI w.r.t. time. While QFI is only positive, QFI flow can be both positive and negative. The positive and negative behavior of QFI flow indicates the flow of information and energy in and out of the system, which can be seen in the corresponding changes in the QFI (increase and decrease of QFI). Finally, the difference between the Markovian and non-Markovian environments can be seen as QFI completely decays for Markovian regime but retains a stable non-zero value for the non-Markovian regime.

The second scenario deals with the QFI for an atom-cavity interacting system in a Kerr medium environment. The initial states we considered were separable states and partially entangled states. The QFI of the Kerr medium parameter is studied for the case of resonance with zero and one photon Fock state. The QFI shows an initial increase, which indicates that the effect of parameters (Kerr medium, resonance, photon number) arises only after the interaction starts. The QFI reaches a peak value and then decays to zero. This increasing and decreasing behavior is displayed in the form of rapid oscillations. As we go from a separable state to partial entanglements of the initial state, the upper bounds of the QFI decrease. Also, when comparing the results of the zero photon and one photon cases, the QFI shows a significant increase in the peak values. However, the QFI diminishes much quicker for the one photon Fock state.

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