Some Algebraic Invariants of Squarefree Monomial Ideals

By

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To my beloved

parents,

siblings,

&

husband.

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Bakhtawar Shaukat

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Abstract

Hilbert gave the idea of associating free resolution with finitely generated module to describe the structure of a module. Since then, there has been a lot of progress on the structure and properties of finite free resolutions. The two algebraic invariants namely Castelnuovo-Mumford regularity (or regularity) and projective dimension are associated with minimal graded free resolution of a finitely generated graded module. Regularity measures the complexity of module and projective dimension measures how far a module is from being projective. Projective dimension has a relation with the depth of a module by Auslander–Buchsbaum formula. The depth of a module has been the subject of several studies during the last decades. Let A be a finitely generated multigraded module. In 1982, Stanley conjectured that depth of A is a lower bound for the Stanley depth of A. This conjecture was later disproved by Dual et al. in 2015. However, there still looks to be profound and attractive relationship between the two invariants, which is yet to be understood. Squarefree monomial ideals has been a fascinating area of study in commutative algebra and has a strong connection to combinatorics, which continues to inspire much of current research. The goal of this thesis is to study some algebraic invariants of quotient rings of some squarefree monomial ideals. These algebraic invariants include depth, Stanley depth, regularity, projective dimension, and Krull dimension. We find the precise values of aforementioned invariants of residue class rings of edge ideals of perfect semiregular trees. We find depth, projective dimension and lower bounds for Stanley depth of the quotient rings of edge ideals associated with all cubic circulant graphs. We discuss the said invariants for the quotient rings of the edge ideals associated with some classes of four and five regular circulant graphs.

CHAPTER 0

Introduction

The last three decade has seen a number of exciting developments at the intersection of commutative algebra with combinatorics. Melvin Hochster's [3] study provides the first indication of a connection between combinatorial properties of simplicial complexes and commutative algebra. The face ring of a simplicial complex initially appeared in Gerald Reisner's doctoral thesis (published version in [6]), under the supervision of Hochster, and in two separate articles written by Stanley [4, 5]. Richard Stanley [5] provided a positive answer to the upper bound conjecture for spheres in 1975. By using concepts and techniques from commutative algebra in an orderly way, Stanley was the first to study simplicial complexes by taking into account the Hilbert function of Stanley Reisner rings. Since that time, squarefree monomial ideals have been a fascinating area of study in commutative algebra see [11]. Partitionable complexes and Cohen-Macaulay complexes are two basic types of simplicial complexes in combinatorics. Stanley proposed a conjecture connecting these two concepts: Are all Cohen–Macaulay simplicial complexes partitionable? In the year 1982, Stanley [8] introduced an invariant for finitely generated $\mathbb{Z}^n$-graded modules over the graded commutative ring, known as Stanley depth. According to Stanley's conjecture geometric invariant of a module known as the Stanley depth is an upper bound of an algebraic invariant of the module known as the depth(Stanley's inequality). It is proved in [32] that this conjecture implies his conjecture about partitionable Cohen–Macaulay simplicial complexes. Later on it was proved by Duval et al. [59] in year 2016 that both of these conjectures are generally false. For some recent results regarding Stanley's inequality, see [40, 43, 51, 80]. The primary objective is to investigate the algebraic invariants of squarefree monomial ideals by means of combinatorial structures and graph invariants.

Stanley decompositions have applications in the normal form theory for systems of differential equations see [15, 27, 28]. Herzog et al. [33] provided a method for finding Stanley depth of modules I_1/I_2, where $I_2 \subset I_1$ are the monomial ideals of polynomial ring S. After that, Ichim et al. [62] introduced an algorithm for finding Stanley depth of any finitely generated $\mathbb{Z}^n$-graded S-module. However, it is still a challenging task to compute the Stanley depth even with these algorithms. The study of Stanley depth and depth for modules is a hard problem. Therefore, it is still important to give bounds and values for depth and Stanley depth of a module. For some interesting findings regarding to depth and Stanley depth see [11, 52, 63, 66]. In addition, Krull dimension measures the size of a ring or module. A module or a ring is Cohen-Macaulay if its Krull dimension is equal to its depth. Characterization and construction of Cohen–Macaulay graphs is one of the fundamental problems with rich literature; see for instance [54, 56, 69].

A minimal graded free resolution of a module is a homological tool for studying a module. The minimal free resolution encodes much of the information about the structure of module as well as containing several important numerical invariants of a module. In two well-known articles published in 1890 [1] and 1893 [2], Hilbert proposed the idea of associating a free resolution to a finitely generated module. He proved Hilbert's syzygy theorem, which asserts that there is a finite minimal free resolution for every finitely generated graded module over a polynomial ring. Since then, there has been a lot of progress on properties of finite free resolutions. Free resolutions has many applications in Algebraic Geometry, Computational Algebra and Invariant Theory. For more literature related to resolutions we refer the readers to [14, 21, 34, 61]. There are two important invariants in commutative algebra that measure the size of resolution, regularity and projective dimension. Regularity measures the width of a free resolution whereas the projective dimension measures the length of free resolution. Regularity plays a significant role as one of the keys indicators of a module's complexity and projective dimension measures how far a module is from being projective. Bounds and values for the regularity and projective dimension of edge ideals have been the subject of numerous studies by researchers; see for instance [34, 61, 70, 71, 74]. Moreover, the interplay of algebraic properties of $I(G)$ and graph-theoretic properties of G is also of great interest; see for instance [36, 45, 57, 68]. This research focuses on the aforementioned algebraic invariants of quotient rings of edge ideals associated with some classes of graphs. Our outcomes further demonstrates that the Stanley's inequality holds for the quotient ring of edge ideals associated to the classes of graphs that are being studied.

This thesis comprises five chapters. The first chapter covers some fundamental concepts of ring theory. It also encompasses some basics of module theory includes exact sequences of modules, Krull dimension, grading of a ring and a module. In addition, it includes a brief introduction of graph theory. Further, we discuss depth, Stanley decomposition, Stanley depth of modules and its method of computation. Well known Stanley's conjecture and some results associated to depth and Stanley depth are also stated. This chapter also contains a detailed introduction to graded minimal free resolution of edge ideal and its construction. Moreover, some results related to regularity and projective dimension are also presented here.

In the second chapter, we find the precise formulas for the values of the algebraic invariants depth, projective dimension, Stanley depth, regularity and Krull dimension of edge ideal associated with perfect semiregular tree. The content of this chapter is published in our paper [80].

The third chapter contains values of depth, projective dimension, and lower bounds for Stanley depth of the quotient rings of the edge ideals of all cubic circulant graphs. The work in this chapter is inspired by a recent work of Uribe-Paczka et al. [73], where the authors studied regularity of the edge ideals of cubic circulant graphs. The content of this chapter is available in [81].

Unlike cubic circulant graphs [12], there is no simple characterization or formula to uniquely represent all four and five regular circulant graphs. The classification of all four and five regular circulant graphs is a topic of ongoing research, and many mathematicians and computer scientists are working to gain deeper insights into the properties of these graphs [44, 67]. In practice, researchers often focus on specific subclasses of circulant graphs to make progress in their study. In chapter four, we give the exact values of depth, projective dimension, and bounds for the Stanley depth of edge ideal associated with four regular circulant graphs $C_{2n}(1,n-1)$. We also provide a value for the regularity of the edge ideal associated with $C_{2n}(1,n-1)$ when $n \equiv 0,1 (\bmod 3)$, and sharp bounds when $n \equiv 2 (\bmod 3)$. We also give the exact values of the regularity of the edge ideal associated with four regular circulant graphs $C_{2n}(1,2)$ when n is even and tight bounds when n is odd. Moreover, we provide the exact value for the regularity of edge ideal associated with five regular circulant graphs $C_{2n}(1,n-1,n)$. This work is published in [82]. In the last chapter, we summarize whole research work and give some future directions. We gratefully acknowledge the use of CoCoA [20], Macaulay2 [13] and MATLAB ®.

CHAPTER 1

Preliminaries

In this chapter we state some fundamental concepts of ring theory, module theory and graph theory. Some results related to algebraic invariants including depth, Stanley depth, regularity, projective dimension, and Krull dimension are also presented here. Throughout this thesis, all considered rings are commutative with unity.

1.1 Some elements of ring theory and module theory

Here we recall some definitions and results from [18, 41].

Definition 1.1.1. Let Z be a ring. Under usual addition and multiplication of polynomials, the collection of all polynomials in variable x whose coefficients are in Z forms a ring, this ring is represented as $Z[x]$. The polynomial ring in the variables $x_1, x_2, \dots, x_n$ whose coefficients in Z, denoted by $Z[x_1, \dots, x_n]$, is defined inductively by

$$Z[x_1, \dots, x_n] = Z[x_1, \dots, x_{n-1}][x_n].$$

Definition 1.1.2. A proper ideal Γ of ring Z is said to be a *prime ideal* if for any $z_1, z_2 \in Z$ such that $z_1 z_2 \in \Gamma$ implies $z_1 \in \Gamma$ or $z_2 \in \Gamma$.

Definition 1.1.3. Let Γ be a prime ideal of a ring Z. The *height of* Γ is the supremum of all integers n_i such that a chain of distinct prime ideals of the form

$$\Gamma_0 \subsetneq \Gamma_1 \subsetneq \Gamma_2 \subsetneq \cdots \subsetneq \Gamma_{n_i} = \Gamma$$

exists. The height of Γ is represented by $h(\Gamma)$.

Definition 1.1.4. Let Z represent a ring. The Krull dimension of Z is defined as the supremum of heights of all prime ideals of Z. That is,

$$\dim Z = \sup\{h(\Gamma) : \Gamma \text{ is prime in } Z\}.$$

Definition 1.1.5. An ideal N of Z ($N \neq Z$) is called *maximal* if there exist no other proper ideal containing N.

Definition 1.1.6. A ring is said to be *local* if it contains a unique maximal ideal.

Definition 1.1.7. A ring Z is called Noetherian if it fulfills the ascending chain condition on its ideals that is given any chain:

$$Y_1 \subset Y_2 \subset \cdots \subset Y_k \subset Y_{k+1} \subset \ldots$$

a positive integer n exists such that

$$Y_n = Y_{n+1} = \ldots$$

Monomial ideal and primary decomposition

Consider a polynomial ring $S = K[x_1, \ldots, x_n]$ over a field K, monomials forms the natural K-basis for S. Throughout this work, S represents a polynomial ring over a field K in finite number of variables. Let $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}_+^n$, where $\mathbb{Z}_+$ represents the set of non-negative integers. A monomial is any product of the form $x_1^{b_1} \ldots x_n^{b_n}$. If $w = x_1^{b_1} \ldots x_n^{b_n}$ is a monomial, then we write $w = \mathbf{x}^{\mathbf{b}}$ with $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}_+^n$, and

$$\mathbf{x}^{\mathbf{b_1}}\mathbf{x}^{\mathbf{b_2}} = \mathbf{x}^{\mathbf{b_1}+\mathbf{b_2}}.$$

If the components of $\mathbf{b}$ are 0 and 1, then a monomial $\mathbf{x}^{\mathbf{b}}$ is called squarefree. Mon(S) denotes the set of all monomials in S and it forms a K-basis of S.

Definition 1.1.8. For any polynomial $f \in S$ and for $b_w \in K$

$$f = \sum_{w \in Mon(S)} b_w w,$$

where support of f is defined as

$$\text{supp}(f) = \{w \in Mon(S) : b_w \neq 0\}.$$

Definition 1.1.9. If an ideal of a polynomial ring is generated by monomials, it is referred to as a *monomial ideal*.

Definition 1.1.10. If an ideal is generated by a squarefree monomials then it is called *squarefree monomial ideal*.

Examples 1.1.11. Consider ring $S = K[x_1, x_2, ..., x_6]$ over the field K.

1. The ideals $Y_1 = (x_1^2x_2^4, x_1^3x_2^3, x_1^5x_2)$ and $Y_2 = (x_1^4x_2^5, x_1^6x_2^2)$ are monomial ideals of S.

2. The ideals of the form $Y_3 = (x_1)$, $Y_4 = (x_1x_4, x_2x_5)$ and $Y_5 = (x_1, x_2, ..., x_6)$ are the squarefree monomial ideals in S.

Definition 1.1.12. Let Z be a ring and Y be its ideal. The radical of Y represented by $\sqrt{Y}$ is defined as $\sqrt{Y} = \{z \in Z : z^n \in Y, n > 0\}$.

If $\sqrt{Y} = Y$, then Y is called a radical ideal. All the squarefree ideals are radical ideals.

Definition 1.1.13. Let Y and $Y^{'}$ be the two ideals of a ring Z. Then the quotient ideal (also named as colon ideal) is defined as $(Y : Y^{'}) = \{z \in Z : zY^{'} \subseteq Y\}$.

Definition 1.1.14. An ideal $(0 : Y)$ is called the annihilator of Y represented as $Ann(Y)$ defined as $Ann(Y) = \{z \in Z : zY = 0\}$.

The unique minimal set of monomial generators of monomial ideal L is represented as $\mathbb{G}(L)$ [41].

Proposition 1.1.15. *The colon ideal $B_1 : B_2$ of two monomial ideals B_1 and B_2 of S is a monomial ideal, and*

$$B_1 : B_2 = \bigcap_{g \in \mathbb{G}(B_2)} (B_1 : g).$$

Furthermore, $\{w/gcd(w,g) : w \in \mathbb{G}(B_1)\}$ is the set of generators of $(B_1 : g)$ and $gcd(w,g)$ represents the greatest common divisor of w and g.

Example 1.1.16. *If $S = K[x_1, x_2, x_3, x_4]$, $B_1 = \{x_2x_3^2, x_1x_2^2, x_1^2, x_3x_4^2\}$ and $B_2 = \{x_2x_1^3, x_2x_3, x_3x_4\}$, then*

$$B_1B_2 = \{x_1^2x_3x_4, x_1^2x_2x_3, x_3^2x_4^3, x_1x_2^2x_3x_4, x_2x_3^2x_4, x_2x_3^2x_4, x_1x_2^3x_3, x_2^2x_3^3, x_1^5x_2, x_1^4x_2^3\},$$

$$B_1 + B_2 = \{x_2x_3, x_3x_4, x_1^2, x_1x_2^2\},$$

$$B_1 \cap B_2 = \{x_2x_3^2, x_3x_4^2, x_1^3x_2, x_1^2x_2x_3, x_1^2x_3x_4, x_2x_3x_4^2\},$$

$$B_1 : B_2 = \{x_2x_3, x_1^2, x_4^2, x_1x_2x_4, x_1x_2^2, x_3x_4\}.$$

Definition 1.1.17. A proper ideal Y of a ring Z is called a primary ideal if $z_1z_2 \in Y$, for some $z_1, z_2 \in Z$, then either $z_1 \in Y$ or $z_2^n \in Y$ for some $n \geq 1$.

Definition 1.1.18. A prime ideal $\Gamma \subset S$ is known as associated prime ideal of S/L, if there exist a non-zero element $s \in S/L$ such that $\Gamma = Ann(s)$.

The set of associated prime ideals of S/L is denoted by $Ass(S/L)$. We often write $Ass(L)$ in place of $Ass(S/L)$. For an ideal L, primary decomposition is a way of representing L as an intersection $L = \cap_{j=1}^{m} K_j$, where each K_j is a primary ideal containing L. Let $\{\Gamma_j\} = Ass(K_j)$ if neither of the K_j can be excluded in this intersection and $\Gamma_r \neq \Gamma_s$ for all $r \neq s$, then it is called an irredundant primary decomposition.

Examples 1.1.19. Let $L_1 = (x_2^4, x_3^4, x_2^3x_4^3, x_2x_3x_4^3, x_3^3x_4^3)$ be an ideal of $S = K[x_1, x_2, x_3, x_4]$, then

$$\begin{aligned}
L_1 &= (x_2^4, x_3^4, x_2^3, x_2x_3x_4^3, x_3^3x_4^3) \cap (x_2^4, x_3^4, x_4^3, x_2x_3x_4^3, x_3^3x_4^3) \\
&= (x_2^3, x_3^4, x_2x_3x_4^3, x_3^3x_4^3) \cap (x_2^4, x_3^4, x_4^3) \\
&= (x_2^3, x_3^4, x_2, x_3^3x_4^3) \cap (x_2^3, x_3^4, x_3x_4^3, x_3^3x_4^3) \cap (x_2^4, x_3^4, x_4^3) \\
&= (x_2, x_3^4, x_3^3x_4^3) \cap (x_2^3, x_3^4, x_3x_4^3) \cap (x_2^4, x_3^3, x_4^3) \\
&= (x_2, x_3^4, x_3^3) \cap (x_2, x_3^4, x_4^3) \cap (x_2^3, x_3^4, x_3) \cap (x_2^3, x_3^4, x_4^3) \cap (x_2^4, x_3^3, x_4^3) \\
&= (x_2, x_3^3) \cap (x_2, x_3^4, x_4^3) \cap (x_2^3, x_3) \cap (x_2^4, x_3^4, x_4^3) \\
&= (x_2, x_3^3) \cap (x_2^3, x_3) \cap (x_2^4, x_3^4, x_4^3).
\end{aligned}$$

It is a primary decomposition of L_1 but not irredundant. Here, $\text{Ass}((x_2, x_3^3)) = \text{Ass}((x_2^3, x_3)) = \{(x_2, x_3)\}$. Now for irredundant primary decomposition, take an intersection of (x_2, x_3^3) and (x_2^3, x_3), that is $(x_2, x_3^3) \cap (x_2^3, x_3) = (x_2^3, x_2x_3, x_3^3)$. Hence $L_1 = (x_2^4, x_3^4, x_4^3) \cap (x_2^3, x_2x_3, x_3^3)$. Similarly, if $L_2 = (x_2^3, x_1x_2x_3^2, x_1^3, x_2^2x_3^2, x_1^2x_3^2) \subset S = K[x_1, x_2, x_3]$, then

$$\begin{aligned}
L_2 &= (x_2^3, x_1, x_1^3, x_2^2x_3^2, x_1^2x_3^2) \cap (x_2^3, x_2, x_1^3, x_2^2x_3^2, x_1^2x_3^2) \cap (x_2^3, x_3^2, x_1^3, x_2^2x_3^2, x_1^2x_3^2) \\
&= (x_2^3, x_1, x_2^2x_3^2) \cap (x_2, x_1^3, x_1^2x_3^2) \cap (x_2^3, x_3^2, x_1^3) \\
&= (x_2^3, x_1, x_2^2) \cap (x_2^3, x_1, x_3^2) \cap (x_2, x_1^3, x_1^2) \cap (x_2, x_1^3, x_3^2) \cap (x_3^2, x_1^3, x_2^3) \\
&= (x_1, x_2^2) \cap (x_2^3, x_1, x_3^2) \cap (x_2, x_1^2) \cap (x_2, x_1^3, x_3^2) \cap (x_3^2, x_1^3, x_2^3) \\
&= (x_1, x_2^2) \cap (x_1^3, x_2^3, x_3^2) \cap (x_2, x_1^2).
\end{aligned}$$

We have $\text{Ass}((x_1, x_2^2)) = \text{Ass}((x_2, x_1^2)) = \{(x_1, x_2)\}$. By taking intersection of intersection of (x_1, x_2^2) and (x_2, x_1^2), we get (x_1, x_2)-primary ideal(x_1^2, x_2^2, x_1x_2) and thus the irredundant primary decomposition. Hence $L_2 = (x_1^2, x_2^2, x_1x_2) \cap (x_1^3, x_2^3, x_3^2)$.

Definition 1.1.20. Let A be a finitely generated module A over a Noetherian ring Z. For a Z-module A, $Ann(A) = \cap_{a\in A} Ann(a)$, where $Ann(a) = \{z \in Z : za = 0\}$. The Krull dimension of module A is determined as

$$\dim(A) = \dim(Z/Ann(A)).$$

For the modules of the type S/L,

$$\dim(S/L) = \max\{\dim(S/\Gamma_i) : \Gamma_i \in \mathrm{Ass}(S/L)\},$$

is always finite.

Definition 1.1.21. A sequence of Z-homomorphisms and Z-modules

$$\cdots \longrightarrow A_{i-1} \xrightarrow{\xi_i} A_i \xrightarrow{\xi_{i+1}} A_{i+1} \xrightarrow{\xi_{i+2}} \dots$$

is exact at A_i if $\mathrm{Im}(\xi_i) = \mathrm{Ker}(\xi_{i+1})$. We call the sequence is an exact sequence if it is exact at each A_i.

Proposition 1.1.22. *The sequence*

$$0 \longrightarrow A' \xrightarrow{\xi} A \xrightarrow{\varphi} A'' \longrightarrow 0$$

is said to be an exact if and only if ξ is injective, φ is surjective and Im(ξ) = Ker(φ). This sequence is known as a short exact sequence.

Example 1.1.23. *If B and C are Z-modules, then the sequence of the form*

$$0 \longrightarrow B \xrightarrow{\xi} B \oplus C \xrightarrow{\varphi} C \longrightarrow 0,$$

is a short exact sequence.

Definition 1.1.24. A Z-module A is called Noetherian if each ascending chain of Z-submodules of A is stationary. A ring Z is Noetherian if Z is Noetherian as a Z-module.

Definition 1.1.25. Let $(\Omega, +)$ be an abelian semigroup. An Ω-graded ring is a ring Z together with the following decomposition

$$Z = \bigoplus_{\alpha\in\Omega} Z_\alpha \text{ (as a group)}$$

such that $Z_\alpha Z_\beta \subset Z_{\alpha+\beta}$ for all $\alpha, \beta \in \Omega$. Then for $z \in Z$, we can write a unique expression

$$z = \sum_{\alpha\in\Omega} z_\alpha$$

where $z_a \in Z_a$ and almost all $z_\alpha = 0$. The element z_α is said to be αth homogeneous component and if $z = z_\alpha$, then z is homogeneous of degree α.

Examples 1.1.26. $\mathbb{R}[x]$ and $\mathbb{R}[x,y]$ are $\mathbb{Z}$-graded rings because

- $\mathbb{R}[x] = \mathbb{R} \oplus \mathbb{R}x \oplus \mathbb{R}x^2 \oplus \mathbb{R}x^3 \oplus \mathbb{R}x^4 \oplus \mathbb{R}x^5 \oplus \cdots$

- $\mathbb{R}[x,y] = \mathbb{R} \oplus (\mathbb{R}x + \mathbb{R}y) \oplus (\mathbb{R}x^2 + \mathbb{R}xy + \mathbb{R}y^2) \oplus (\mathbb{R}x^3 + \mathbb{R}x^2y + \mathbb{R}xy^2 + \mathbb{R}y^3) \oplus \cdots$

Definition 1.1.27. If Z is an Ω-graded ring and A is a Z-module with a decomposition

$$A = \bigoplus_{\alpha \in \Omega} A_\alpha \text{ (as a group)}$$

such that $Z_\alpha A_\beta \subset A_{\alpha+\beta}$ for all $\alpha, \beta \in \Omega$, we say that A is an Ω-graded module.

Example 1.1.28. Let $\alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{Z}^n$ and $x^\alpha = x_1^{a_1} x_2^{a_2} \ldots x_n^{a_n}$(a monomial) then $f \in S := K[x_1, \ldots, x_n]$ is known as homogeneous element of degree α if f is of the form cx^α with $c \in K$. The ring of polynomial S is obviously $\mathbb{Z}^n$-graded that is $S = \bigoplus_{\alpha \in \mathbb{Z}^n} S_\alpha$, where

$$S_\alpha = \begin{cases} Kx^\alpha, & \text{if } \alpha \in \mathbb{Z}_+^n; \\ 0, & \text{otherwise.} \end{cases}$$

An S-module A is said to be $\mathbb{Z}^n$-graded if $A = \bigoplus_{\alpha \in \mathbb{Z}^n} A_\alpha$ and $S_\alpha A_\beta \subset A_{\alpha+\beta}$ for all $\alpha, \beta \in \mathbb{Z}^n$.

Remark 1.1.29 ([37])**.** Let A is an Ω-graded module. A non zero element of A_β is called a homogeneous element of degree β. We frequently use the notation $A(\gamma)$, to represent the Ω-graded shift of S-module A that satisfies

$$A(\gamma)_\beta = A_{\gamma+\beta} \tag{1.1.1}$$

for all $\gamma, \beta \in \Omega$.

Definition 1.1.30 ([37])**.** Let A, B be Ω-graded modules over S with $\beta \in \Omega$. An Ω-graded homomorphism of degree β is a homomorphism $\xi : A \longrightarrow B$ such that for all homogeneous $a \in A$

$$deg(\xi(a)) = deg(a) + \beta.$$

If $\beta = 0$, then ξ is referred to as degree-preserving.

Example 1.1.31. Consider the $\mathbb{Z}$-graded ring $S = K[x_0, x_1, x_2]$, then

$$\xi_1 : S \xrightarrow{\cdot x_1} S$$

is a homomorphism of degree 1 but not degree preserving. However a homomorphism,

$$\xi_2 : S(-1) \xrightarrow{\cdot x_1} S$$

is a degree-preserving.

1.2 Finite simple graphs

A graph G consists of a non-empty set of vertices V_G, an edge set E_G and is represented as, $G = (V_G, E_G)$. The size and order of the graph is denoted as $\mid E_G \mid$ and $\mid V_G \mid$, respectively. Now, we recall some definitions gathered from [29, 47]. The graph G is finite if it has finite number of vertices and edges, otherwise infinite. If there are two or more edges between two vertices then the edges are known as multiple (parallel) edges. Similarly, if an edge has the same starting and end vertex it is said to be a loop. A simple graph is a graph with no multiple edges and loops. The vertices $x_1, x_2 \in V_G$ are called adjacent in G if there is an edge between them, which is denoted by x_1x_2 (or x_2x_1). The two edges $e_1, e_2 \in E_G$ are adjacent if e_1 and e_2 have a common vertex. The *degree* of a vertex $x_i \in V(G)$ is the number of adjacent vertices to x_i in graph G and is represented by $\deg_G(x_i)$. For $n \geq 1$, a *path* P_n is a graph on n vertices such that $E(P_n) = \{\{x_j, x_{j+1}\} : 1 \leq j \leq n-1\}$. If a path exists between any two vertices in a graph, then the graph is said to be a *connected*. A graph N_n is said to be a *null graph* on n vertices if $V(N_n) = \{x_1, \ldots, x_n\}$ and $E(N_n) = \emptyset$. Moreover, if $n = 1$ then N_1 is also called a *trivial graph*. A simple and connected graph T_n on n vertices is called a *tree* if a unique path exists between any two vertices of T_n. A vertex of degree one of a graph is called a *pendant vertex* (or *leaf*). A tree is called *semiregular* when all of its non-pendant vertices have the same degree. A *rooted tree* is a tree in which one vertex has been designated the root. The *distance* between two vertices x_j and x_k in a graph is the shortest path between x_j and x_k. Any vertex that is not a leaf is called an *internal vertex*. For $n \geq 1$, a *n-star* is a tree having n leaves and a single vertex with degree n. We denote a n-star by S_n. For a graph G, a subset D of $V(G)$ is known as an *independent set* if no two vertices in D are adjacent. The *independence number* of G is the carnality of the largest independent set of G.

Lemma 1.2.1 ([75, Lemma 1])**.** *Let $I(G)$ be an edge ideal associated with graph G, then*

$$\dim(S/I(G)) = \max\{|D| : \textit{ D is an independent set of } G\}.$$

A *subgraph* W of G, written as $W \subseteq G$, is a graph such that $V(W) \subseteq V(G)$ and $E(W) \subseteq E(G)$. If W is a subgraph of G, then G is called a *supergraph* of W. For a subset $W \subseteq V(G)$, an *induced subgraph* of G is a graph $G' := (W, E(G'))$, such that $E(G') = \{\{x_i, x_j\} \in E(G) : \{x_i, x_j\} \subseteq W\}$. A subset N of $E(G)$ that contains no two adjacent edges is called a *matching* in a graph G. An *induced matching* in G is a matching that forms an induced subgraph of G. An *induced matching number* of G is represented as $\operatorname{indmat}(G)$ and defined as

$$\operatorname{indmat}(G) = \max\{|N| : N \text{ is an induced matching of } G\}.$$

For a graph given in Figure 1.1,

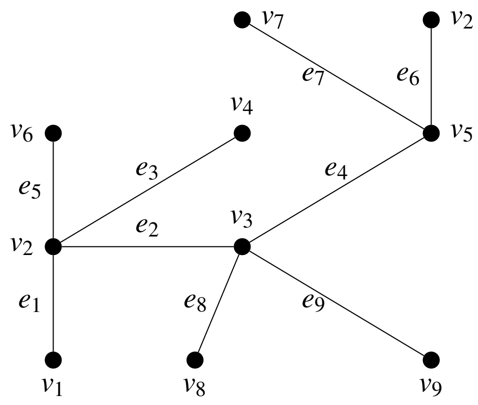


Figure 1.1: Simple graph H

We have $\operatorname{indmat}(H) = 2$. The *neighborhood* $N_G(x_k)$ of a vertex x_k is the set of all neighbors of x_l, that is, $N_G(x_k) := \{x_l \in V(G) \mid \{x_k, x_l\} \in E(G)\}$. If each vertex in graph G has degree q, then graph G is *q-regular*.

Definition 1.2.2 ([41]). Assume G is a graph with edge set $E(G)$ and $V(G) = \{x_1, \dots, x_n\}$. Any ideal $I \subset S := K[V(G)]$ generated by squarefree quadratic monomials can be viewed as the so-called *edge ideal* $I(G)$ of the G whose edges are the sets formed by two variables x_k, x_l such that $x_k x_l$ is a generator of I.

Example 1.2.3. Consider a graph G as given in Figure 1.2 with $V(G) = \{x_1, x_2, x_3, x_4, x_5, x_6\}$. Then edge ideal associated with G is

$$I(G) = (x_1x_3, x_1x_6, x_2x_4, x_2x_5, x_3x_4, x_3x_5, x_4x_6). \tag{1.2.1}$$

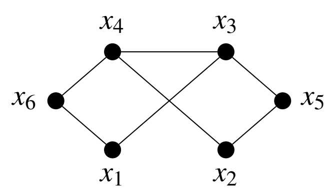


Figure 1.2: Simple Graph G

Definition 1.2.4 ([80])**.** We call a semiregular rooted tree a *perfect semiregular tree* if its all pendent vertices are at the same distance from the root. Let $n \geq 2$ and $k \geq 1$. We denote a perfect semiregular tree by $T_{n,k}$, where k and n represent the distance of the pendent vertices from the root and degree of the non-pendent vertices, respectively.

Definition 1.2.5 ([42])**.** For $n \geq 2$ and $k \geq 1$, a *perfect n-ary tree* is a rooted tree whose root is of degree n, and all other internal vertices (if exist) are of degree $n+1$ and all leaves are at distance k from the root (if $k = 1$, then a perfect n-ary tree is just S_n). A perfect $(n-1)$-ary tree is an induced subtree of $T_{n,k}$, we denote a perfect $(n-1)$-ary tree by $T'_{n,k}$.

Examples 1.2.6. See Figure 1.3 as examples of $T'_{n,k}$ and $T_{n,k}$.

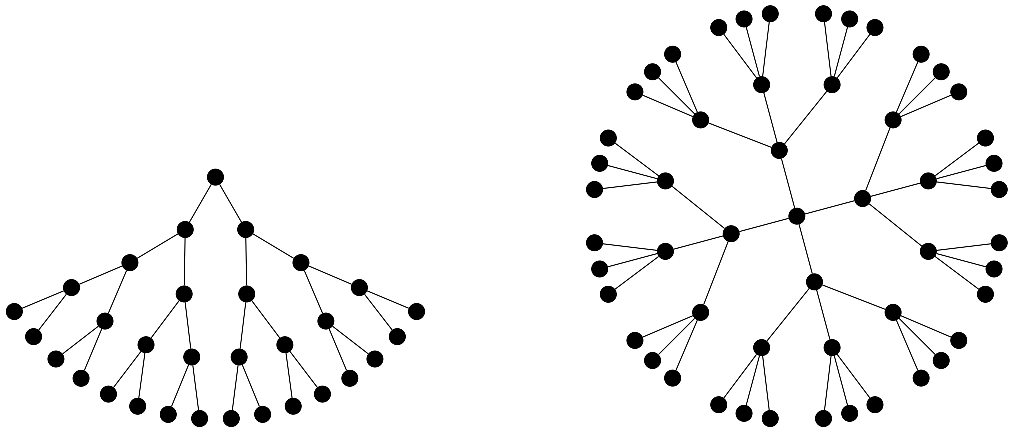


Figure 1.3: From left to right perfect 2-ary tree $T'_{3,4}$ and perfect semiregular tree $T_{4,3}$.

For $n \geq 3$, a *cycle* C_n on n vertices is a graph such that $E(C_n) = \{\{x_j, x_{j+1}\} : 1 \leq j \leq n-1\} \cup \{x_1, x_n\}$. If there is no induced cycle of length strictly greater than three in a graph, then the graph is considered as *chordal*.

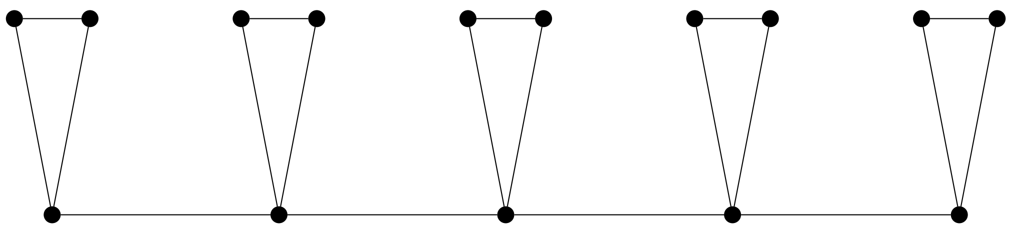


Figure 1.4: Chordal graph

Let $n \geq 1$, a graph is said to be *complete graph* K_n on n vertices if each pair of vertices is connected by an edge. A graph is called *bipartite graph* if its vertex set is splitted into two distinct sets, or partite sets, such that no two vertices of the graph within the same partite set are adjacent. A bipartite graph is called a *complete bipartite graph* if every vertex of one partite set is connected to each vertex of the other partite set. Let $K_{u,v}$ denotes the complete bipartite graph with partite sets $K_u = \{x_1, \dots, x_u\}$ and $K_v = \{x_{u+1}, \dots, x_{u+v}\}$.

Definition 1.2.7 ([65]). Let $n \geq 2$ and $\mathbb{S}$ be a subset of $\{1, \dots, \lfloor \frac{n}{2} \rfloor\}$. A *circulant graph* $C_n(\mathbb{S})$ is defined as a graph with $V(G) = \{x_1, \dots, x_n\}$ such that $\{x_i, x_j\} \in E(C_n(\mathbb{S}))$ if and only if $|i-j|$ or $n - |i-j| \in \mathbb{S}$.

Examples 1.2.8. See Figure 1.5 for examples of circulant graphs.

Figure 1.5: From left to right $C_7(1,3)$ and $C_8(2,4)$.

Since $C_n = C_n(1)$ therefore circulant graphs are sometimes considered as generalized cycles. For convenience the graph $C_n(\{a_1, \dots, a_l\})$ is simply denoted by $C_n(a_1, \dots, a_l)$. A circulant graph $C_n(a_1, \dots, a_l)$ is $2l$-regular, except if $2a_l = n$, in that case, it is $(2l-1)$-regular. As a consequence 3-regular circulant graphs have the form $C_{2n}(a, n)$ with $1 \leq a \leq n$. A 3-regular circulant graph is also named as a *cubic circulant graph*. Circulant graphs were introduced in 1846, and they have a number of applications in computer network design, telecommunication networks, data connection networks, group theory, and others [7, 10, 17, 44]. Several papers have been written on the aforementioned algebraic invariants of edge ideals associated with circulant graphs; see [56, 67, 72].

1.3 Depth and Stanley depth

Definition 1.3.1 ([18]). Let Z be a ring and A be a Z-module. We call a non-zero element $z \in Z$ a zero divisor of a module A if $za = 0$ for some $a \neq 0$ in A.

Definition 1.3.2 ([41]). Let A be an Z-module. A non-zero element z of the ring Z is called A-regular if for any $a \in A$, $za = 0$ implies $a = 0$.

Definition 1.3.3 ([41]). A sequence $z_1,\dots,z_n$ of elements in Z is known as A-regular sequence or, an A-sequence, if it fulfills below conditions:

- z_i is $A/(z_1,\dots,z_{i-1})A$ regular for any i,
- $A \neq (z_1,\dots,z_n)A$.

Definition 1.3.4 ([41]). Let Z be a local Noetherian ring and N be its unique maximal ideal. Let A be a finitely generated Z-module. The common length of all maximal A-sequences in N is known as depth of A and is represented by $\operatorname{depth}(A)$.

Remark 1.3.5 ([11]). Graph G is said to be Cohen-Macaulay if $S/I(G)$ is Cohen-Macaulay.

Stanley Depth and Stanley Decomposition

Definition 1.3.6 ([8]). Let S=$K[x_1,\dots,x_n]$ and A be a finitely generated $\mathbb{Z}^n$-graded S-module. Let $w \in A$ be a homogeneous element and $V \subseteq \{x_1,\dots,x_n\}$. $wK[V]$ represents the K-subspace of A generated by $\{wv : v \in K[V]\}$. Then K-subspace $wK[V]$ is called a Stanley space of dimension $|V|$ if it is a free $K[V]$-module. A Stanley decomposition of A is a finite direct sum of Stanley spaces

$$\mathscr{D} : A = \bigoplus_{j=1}^{s} w_j K[V_j]. \tag{1.3.1}$$

The Stanley depth of a decomposition $\mathscr{D}$ is

$$\operatorname{sdepth} \mathscr{D} = \min\{|V_j| : j = 1,\dots,s\}.$$

The Stanley depth of A is

$$\operatorname{sdepth}_S(A) = \max\{\operatorname{sdepth} \mathscr{D} : \mathscr{D} \text{ is a Stanley decomposition of } A\}.$$

Method of computing Stanley depth for squarefree monomial ideals

In 2009, Herzog et al. [33] provided a method of computing the lower bound for sdepth of squarefree monomial ideals in finite number of steps by using posets. Assume L be a squarefree monomial ideal and let $\mathbb{G}(L) = \{u_1,\dots,u_m\}$ is the minimal generating set of L. The characteristic poset of L with respect to $g = (1,\dots,1)$, written as $\mathscr{P}_L^{(1,\dots,1)}$ is defined as

$$\mathscr{P}_L^{(1,\dots,1)} = \{\gamma \subset [n] \mid \gamma \text{ contains } \operatorname{supp}(u_j) \text{ for some } j\},$$

where $\text{supp}(u_j) = \{i : x_i | u_j\} \subseteq [n] := \{1,\ldots,n\}$. For each $\alpha, \beta \in \mathscr{P}_L^{(1,\ldots,1)}$ where $\alpha \subseteq \beta$, and

$$[\alpha\,,\beta] = \{\gamma \in \mathscr{P}_L^{(1,\ldots,1)} : \alpha \subseteq \gamma \subseteq \beta\}.$$

Let $\mathscr{P} : \mathscr{P}_L^{(1,\ldots,1)} = \cup_{j=1}^{k}[\gamma_j\,,\eta_j]$ be a partition of $\mathscr{P}_L^{(1,\ldots,1)}$, and for every j, suppose $s(j) \in \{0,1\}^n$ is the tuple with $\text{supp}(x^{s(j)}) = \gamma_j$, then the Stanley decomposition $\mathscr{D}(\mathscr{P})$ of L is given by

$$\mathscr{D}(\mathscr{P}) : L = \bigoplus_{j=1}^{r} x^{s(j)} K[\{x_k \,|\, k \in \eta_j\}].$$

Clearly, $\text{sdepth}\,\mathscr{D}(\mathscr{P}) = \min\{|\eta_1|,\ldots,|\eta_r|\}$ and

$$\text{sdepth}(L) = \max\{\text{sdepth}\,\mathscr{D}(\mathscr{P}) \mid \mathscr{P} \text{ is a partition of } \mathscr{P}_L^{(1,\ldots,1)}\}.$$

Example 1.3.7. Let $L = (x_1x_2, x_1x_3, x_1x_4, x_2x_4) \subset K[x_1,x_2,x_3,x_4]$ be a squarefree monomial ideal and $J = 0$. Set $\alpha_1 = (1,1,0,0)$, $\alpha_2 = (1,0,1,0)$, $\alpha_3 = (1,0,0,1)$ and $\alpha_4 = (0,1,0,1)$. Thus L is generated by $x^{\alpha_1}, x^{\alpha_2}, x^{\alpha_3}, x^{\alpha_4}$ and choose $g = (1,1,1,1)$. The poset $Q = P_{L/J}^{g}$ is as follows:

$$\begin{aligned} Q = \{&(1,1,0,0),(1,0,1,0),(1,0,0,1),(0,1,0,1),(1,1,1,0),(1,1,0,1),(1,0,1,1),\\ &(0,1,1,1),(1,1,1,1)\}\end{aligned}$$

Partitions of Q are given by:

$$\begin{aligned}\mathscr{P}_1 : {} &[(1,1,0,0),(1,1,0,0)] \bigcup [(1,0,1,0),(1,0,1,0)] \bigcup [(0,1,0,1),(0,1,0,1)] \bigcup\\ &[(1,0,0,1),(1,0,0,1)] \bigcup [(1,1,1,0),(1,1,1,0)] \bigcup [(1,1,0,1),(1,1,0,1)] \bigcup\\ &[(1,0,1,1),(1,0,1,1)] \bigcup [(0,1,1,1),(0,1,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)]\end{aligned}$$

$$\begin{aligned}\mathscr{P}_2 : {} &[(1,1,0,0),(1,1,1,0)] \bigcup [(1,0,0,1),(1,1,0,1)] \bigcup [(1,0,1,0),(1,0,1,1)] \bigcup\\ &[(0,1,0,1),(0,1,1,1)] \bigcup [(1,1,1,1),(1,1,1,1)]\end{aligned}$$

and the corresponding Stanley decomposition is

$$\begin{aligned}\mathscr{D}(\mathscr{P}_1) \;:=\; &x_1x_2K[x_1,x_2] \oplus x_1x_3K[x_1,x_3] \oplus x_1x_4K[x_1,x_4] \oplus x_2x_4K[x_2,x_4] \oplus\\ &x_1x_2x_3K[x_1,x_2,x_3] \oplus x_1x_2x_4K[x_1,x_2,x_4] \oplus x_1x_3x_4K[x_1x_3x_4] \oplus\\ &x_2x_3x_4K[x_2,x_3,x_4] \oplus x_1x_2x_3x_4K[x_1,x_2,x_3,x_4]\end{aligned}$$

$$\begin{aligned}\mathscr{D}(\mathscr{P}_2) \;:=\; &x_1x_2K[x_1,x_2,x_3] \oplus x_1x_4K[x_1,x_2,x_4] \oplus x_1x_3K[x_1,x_3,x_4] \oplus x_2x_4K[x_2,x_3,x_4] \oplus\\ &x_1x_2x_3x_4K[x_1,x_2,x_3,x_4]\end{aligned}$$

Then

$$\begin{aligned} \text{sdepth}(L) &\geq \max\{\text{sdepth}(\mathscr{D}(\mathscr{P}_1)), \text{sdepth}(\mathscr{D}(\mathscr{P}_1))\} \\ &= \max\{2,3\} \\ &= 3 \end{aligned}$$

Since L is not principal, so $\text{sdepth}(L) = 3$.

Stanley's conjecture

In 1982, Stanley [8] gave a conjecture about an upper bound for the depth of a $\mathbb{Z}^n$-graded S-modules.

$$\text{depth}(A) \leq \text{sdepth}(A).$$

It has been immensely significant as it gave a comparison of two very different invariants of modules. Consider $L \subset S$ be a monomial ideal, then for $n \leq 3$, $n = 4$ and $n = 5$ the conjecture for S/L is proved by Apel [16], Anwar [25] and Popescu [38], respectively. But in 2016, Duval et al. [59] proved that Stanley's conjecture is generally false, by giving a counter example of the module of the type S/L. We will now discuss some fundamental results on depth and Stanley depth.

For monomial ideal $L \subset S$,

$$\text{depth}(L) = \text{depth}(S/L) + 1,$$

whereas this result is not true for Stanley depth. There exists various examples where $\text{sdepth}(L) > \text{sdepth}(S/L)$ but untill now, no such example where $\text{sdepth}(L) < \text{sdepth}(S/L)$ is known. For monomial ideal $L \subset S$, Asia posed a question:

Question 1.3.8 ([40])**.** Does the following inequality holds

$$\text{sdepth}(L) \geq \text{sdepth}(S/L) + 1?$$

Herzog provides the following conjecture for a weaker form of above inequality:

Conjecture 1.3.9 ([51, Conjecture 60])**.** Let L be an ideal generated by monomials then

$$\text{sdepth}(L) \geq \text{sdepth}(S/L).$$

For some special cases, the above conjecture is proved by Popescu and Qureshi in [39] and Asia [40]. Moreover, for any squarefree monimial ideal of $S = K[x_1, \dots, x_7]$ this conjecture is proved by Keller and Young in [64].

Lemma 1.3.10 ([11, 40]). *If $0 \to P \to Q \to R \to 0$ is a short exact sequence of $\mathbb{Z}^n$-graded S-modules, then*

$$\operatorname{depth}(Q) \geq \min\{\operatorname{depth}(R), \operatorname{depth}(P)\},$$

$$\operatorname{sdepth}(Q) \geq \min\{\operatorname{sdepth}(P), \operatorname{sdepth}(Q)\}.$$

Let $1 \leq r < n$. If $L \subset S_1 = K[x_1, \ldots, x_r]$ and $J \subset S_2 = K[x_{r+1}, \ldots, x_n]$ are monomial ideals, then by [68, Proposition 2.2.20] we have $S/(L+J) \cong S_1/L \otimes_K S_2/J$. Thus we get $\operatorname{depth}(S/(L+J)) = \operatorname{depth}(S_1/L \otimes_K S_2/J)$ and $\operatorname{sdepth}(S/(L+J)) = \operatorname{sdepth}(S_1/L \otimes_K S_2/J)$. Now applying [68, Proposition 2.2.21] for depth, and Lemma [40, Theorem 3.1] for Stanley depth, we have the following useful lemma.

Lemma 1.3.11. $\operatorname{depth}_S(S_1/L \otimes_K S_2/J) = \operatorname{depth}_S(S/(L+J)) = \operatorname{depth}_{S_1}(S_1/L) + \operatorname{depth}_{S_2}(S_2/J)$ *and* $\operatorname{sdepth}_S(S_1/L \otimes_K S_2/J) \geq \operatorname{sdepth}_{S_1}(S_1/L) + \operatorname{sdepth}_{S_2}(S_2/J)$.

In the following lemma, we combine two similar results for depth and Stanley depth. For depth the result is proved by Rauf [40, Corollary 1.3], whereas for Stanley depth it is proved by Cimpoeas [43, Proposition 2.7].

Lemma 1.3.12. *Consider a monomial ideal $L \subset S$ and a monomial w in S such that $w \notin L$. Then* $\operatorname{depth}(S/(L:w)) \geq \operatorname{depth}(S/L)$ *and* $\operatorname{sdepth}(S/(L:w)) \geq \operatorname{sdepth}(S/L)$.

Lemma 1.3.13 ([70, Theorem 4.3]). *Let f be an arbitrary monomial in S and $L \subset S$ be a monomial ideal. Then*

$$\operatorname{depth}(S/L) = \operatorname{depth}(S/(L:f)) \text{ if } \operatorname{depth}(S/(L,f)) \geq \operatorname{depth}(S/(L:f)).$$

In the next lemma, we obtain an analogous result for Stanley depth.

Lemma 1.3.14 ([77]). *Let f be a monomial in S and $L \subset S$ is a monomial ideal such that $f \notin L$. Then*

$$\operatorname{sdepth}(S/L) = \operatorname{sdepth}(S/(L:f)) \text{ if } \operatorname{sdepth}(S/(L,f)) \geq \operatorname{sdepth}(S/(L:f)).$$

Proof. Consider the short exact sequence

$$0 \longrightarrow S/(L:f) \xrightarrow{\cdot f} S/L \longrightarrow S/(L,f) \longrightarrow 0.$$

By using Lemma 1.3.10, $\operatorname{sdepth}(S/L) \geq \min\{\operatorname{sdepth}(S/(L:f)), \operatorname{sdepth}(S/(L,f))\}$. If we have $\operatorname{sdepth}(S/(L,f)) \geq \operatorname{sdepth}(S/(L:f))$, then $\operatorname{sdepth}(S/L) \geq \operatorname{sdepth}(S/(L:f))$. By Lemma 1.3.12, $\operatorname{sdepth}(S/L) \leq \operatorname{sdepth}(S/(L:f))$. Thus the desired result follows. □

Lemma 1.3.15 ([30, Theorem 1.4]). *Let A be a $\mathbb{Z}^n$-graded S-module. If* $\operatorname{sdepth}(A)=0$ *then* $\operatorname{depth}(A)=0$. *Conversely, if* $\operatorname{depth}(A)=0$ *and* $\dim_K(A_\alpha)\leq 1$ *for any* $\alpha\in\mathbb{Z}^n$, *then* $\operatorname{sdepth}(A)=0$.

Lemma 1.3.16 ([60, Theorems 2.6 and 2.7]). *If $n\geq 2$, then*

$$\operatorname{depth}(K[V(S_{n-1})]/I(S_{n-1}))=\operatorname{sdepth}(K[V(S_{n-1})]/I(S_{n-1}))=1.$$

We recall the following result proved in [41, Corollary 10.3.7] for depth and for Stanley depth in [48, Theorem 1.1].

Lemma 1.3.17. *Let $n\geq 2$. Then* $\operatorname{depth}(K[V(K_n)]/I(K_n))=\operatorname{sdepth}(K[V(K_n)]/I(K_n))=1$.

Lemma 1.3.18 ([53, Proposition 1.3, Proposition 1.8 and Theorems 1.9]). *If $n\geq 3$, then*

(a) $\operatorname{depth}(K[V(C_n)]/I(C_n))=\left\lceil\frac{n-1}{3}\right\rceil$,

(b) $\operatorname{sdepth}(K[V(C_n)]/I(C_n))=\left\lceil\frac{n-1}{3}\right\rceil$, *for* $n\equiv 0,2(\operatorname{mod}3)$ *and*

$$\left\lceil\frac{n-1}{3}\right\rceil\leq\operatorname{sdepth}(K[V(C_n)]/I(C_n))\leq\left\lceil\frac{n}{3}\right\rceil,\ \text{for } n\equiv 1(\operatorname{mod}3).$$

Lemma 1.3.19 ([38, Theorems 1.4]). *If $u,v\geq 1$, then*

$$\operatorname{depth}(K[V(K_{u,v})]/I(K_{u,v}))=1\leq\operatorname{sdepth}(K[V(K_{u,v})]/I(K_{u,v})).$$

1.4 Invariants associated with minimal graded free resolution

Recall the following definitions taken from [37].

Definition 1.4.1. Let A be a finitely generated S-module. Free resolution of A is an exact sequence, that is $\operatorname{Im}(\xi_i)=\operatorname{Ker}(\xi_{i-1})$ of S-modules

$$0\longrightarrow F_i\xrightarrow{\xi_i}F_{i-1}\xrightarrow{\xi_{i-1}}\dots\xrightarrow{\xi_2}F_1\xrightarrow{\xi_1}F_0\xrightarrow{\xi_0}A\longrightarrow 0,$$

where all F_i are finitely generated free S-modules.

Definition 1.4.2. A graded free resolution for a finitely generated graded module A is a free resolution of A of the type:

$$0\longrightarrow\bigoplus_{j\in H}S(-j)^{\beta_{r,j}(A)}\xrightarrow{\xi_r}\bigoplus_{j\in H}S(-j)^{\beta_{r-1,j}(A)}\longrightarrow\cdots\longrightarrow\bigoplus_{j\in H}S(-j)^{\beta_{0,j}(A)}\xrightarrow{\xi_0}A\longrightarrow 0,$$

in which each map ξ_i is degree preserving. Where $r\leq n$, we write $F_i=\bigoplus_{j\in H}S(-j)^{\beta_{i,j}(A)}$ are finitely generated free S-modules and $\beta_{i,j}(A)$ is the $(i,j)th$ graded Betti number of A; this number equals the number of minimal generators of degree j in the ith syzygy module of A.

Construction

Let $A = A_0$ be a finitely generated graded S-module and $\{a_1, a_2 \ldots a_r\}$ be a set of generators of A. Then a surjective homomorphism of S-modules exists

$$\xi_0 : F_0 = \bigoplus_{i=1}^{r} Se_i \longrightarrow A,$$

with

$$e_i \longrightarrow a_i.$$

If $deg(a_i) = \gamma_i$ for $i = 1, 2, \ldots r$, then we assign to each e_i the degree γ_i and the map

$$\xi_0 : F_0 = \bigoplus_{i=1}^{r} S(-\gamma_i) \longrightarrow A,$$

becomes a homogeneous homomorphism. Therefore its kernel is a graded submodule of F_0. Let $A_1 = Ker(\bigoplus_{j\in H} S(-j)^{\beta_{0,j}(A)} \longrightarrow A)$, where $\beta_{0,j}$ is the number of generators of A of degree j. Thus, we obtain the exact sequence:

$$0 \longrightarrow A_1 \longrightarrow \bigoplus_{j\in H} S(-j)^{\beta_{0,j}(A)} \longrightarrow A \longrightarrow 0,$$

Since A_1 is finitely generated by Hilbert's basis theorem for modules, and hence we again find a surjective S-module homomorphism and by the same method as above $\bigoplus_{j\in H} S(-j)^{\beta_{1,j}(A)} \longrightarrow A_1$. Composing this surjective homomorphism with the inclusion map $A_1 \longrightarrow \bigoplus_{j\in H} S(-j)^{\beta_{0,j}(A))}$, we get the exact sequence

$$\bigoplus_{j\in H} S(-j)^{\beta_{1,j}(A)} \longrightarrow \bigoplus_{j\in H} S(-j)^{\beta_{0,j}(A)} \longrightarrow A \longrightarrow 0,$$

of graded S-modules. Continuing in this manner and composing the homomorphism and the inclusion map, we obtain a long exact sequence:

$$0 \longrightarrow \bigoplus_{j\in H} S(-j)^{\beta_{r,j}(A)} \longrightarrow \cdots \longrightarrow \bigoplus_{j\in H} S(-j)^{\beta_{0,j}(A)} \longrightarrow A \longrightarrow 0,$$

where $r \leq n$.

Example 1.4.3. Let $L = (x_1x_2^2, x_2x_3^2, x_2^3, x_1^3) \subset K[x_1, x_2, x_3]$. We begin our graded free resolution of L with

$$F_o \xrightarrow{d_o} L \xrightarrow{d} 0.$$

The map d is surjective, therefore the generators of L, all of which have degree 3, generate $Ker(d)$. Since $Ker(d) = Im(d_0)$, the generators of L also generate $Im(d_0)$. We therefore set

$F_0 = S(-3) \oplus S(-3) \oplus S(-3) \oplus S(-3)$. Denote f_1, f_2, f_3, f_4 to be the homogeneous generators of the S-modules F_0. Note that, for all i, $deg(f_i) = 3$. We now define d_0 by

$$f_1 \mapsto x_1x_2^2, f_2 \mapsto x_2x_3^2, f_3 \mapsto x_2^3, f_4 \mapsto x_1^3.$$

Our resolution therefore begins

$$F_1 \xrightarrow{d_1} S^4(-3) \xrightarrow{(x_1x_2^2, x_2x_3^2, x_2^3, x_1^3)} L \to 0.$$

Since $Ker(d_0) = Im(d_1)$, we can determine the generating set of $Im(d_1)$ by determining the generating set of $Ker(d_0)$. Let $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 + \alpha_4 f_4 \in Ker(d_0)$ where, for all i, $\alpha_i \in S$. To determine the generators of $Ker(d_0)$, we must solve the equation

$$\alpha_1 x_1x_2^2 + \alpha_2 x_2x_3^2 + \alpha_3 x_2^3 + \alpha_4 x_1^3 = 0,$$

for $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$. The generating solutions to the above equation are $\sigma_1 = (-x_2, 0, x_1, 0)$, $\sigma_2 = (x_1^2, 0, 0, -x_2^2)$, $\sigma_3 = (0, x_1^2, 0, -x_3^2)$, $\sigma_4 = (-x_3^2, x_2^2, 0, 0)$. Thus $-x_2f_1 + x_1f_3$, $x_1^2f_1 - x_2^2f_4$, $x_1^2f_1 - x_3^2f_4$, $-x_3^2f_1 + x_2^2f_2$, generate $Ker(d_0)$. Their degrees are $4, 5, 5, 5$ respectively. We therefore set $F_1 = S(-4) \oplus S(-5) \oplus S(-5) \oplus S(-5)$. Denote g_1, g_2, g_3, g_4 to be the homogeneous generators of the S-modules F_1, respectively. Note that $deg(g_1) = 4$ and $deg(g_i) = 5$ for $2 \leq i \leq 4$. We define d_1 by

$$g_1 \mapsto -x_2f_1 + x_1f_3, g_2 \mapsto x_1^2f_1 - x_2^2f_4, g_3 \mapsto x_1^2f_1 - x_3^2f_4, g_4 \mapsto -x_3^2f_1 + x_2^2f_2$$

with the resulting next step of our resolutions

$$F_2 \xrightarrow{d_2} S(-4) \oplus S^3(-5) \xrightarrow{\begin{bmatrix} -x_2 & x_1^2 & 0 & -x_3^2 \\ 0 & 0 & x_1^2 & x_2^2 \\ x_1 & 0 & 0 & 0 \\ 0 & -x_2^2 & -x_3^2 & 0 \end{bmatrix}} S^4(-3)$$

$$\xrightarrow{(x_1x_2^2, x_2x_3^2, x_2^3, x_1^3)} L \to 0.$$

Recall $\mathrm{Ker}(d_1) = \mathrm{Im}(d_2)$. Let $\beta_1 g_1 + \beta_2 g_2 + \beta_3 g_3 + \beta_4 g_4 \in Ker(d_1)$ where, for all i, $\beta_i \in S$. To determine the generators of $Ker(d_1)$, we must solve the equation

$$\beta_1(-x_2f_1 + x_1f_3) + \beta_2(x_1^2f_1 - x_2^2f_4) + \beta_3(x_1^2f_1 - x_3^2f_4) + \beta_4(-x_3^2f_1 + x_2^2f_2) = 0,$$

for $(\beta_1, \beta_2, \beta_3, \beta_4)$. The generating solution to the above equation is

$$\eta_1 = (0, -x_3^2, -x_2^2, x_1^2).$$

Thus $-x_3^2g_2 - x_2^2g_3 + x_1^2g_4$ generates $Ker(d_1)$. The degree of the generator of Ker(d_1) is 7. We therefore set $F_2 = S(-7)$. Denote h_1 to be the homogeneous generator of $S(-7)$. Note that $deg(h_1) = 7$. We now define d_2 by

$$h_1 \mapsto -x_3^2g_2 - x_2^2g_3 + x_1^2g_4$$

with the resulting next step of our resolutions

$$F_3 \xrightarrow{d_3} S(-7) \xrightarrow{\begin{bmatrix} 0 \\ -x_3^2 \\ -x_2^2 \\ x_1^2 \end{bmatrix}} S(-4) \oplus S^3(-5) \xrightarrow{\begin{bmatrix} -x_2 & x_1^2 & 0 & -x_3^2 \\ 0 & 0 & x_1^2 & x_2^2 \\ x_1 & 0 & 0 & 0 \\ 0 & -x_2^2 & -x_3^2 & 0 \end{bmatrix}}$$

$$S^4(-3) \xrightarrow{(x_1x_2^2, x_2x_3^2, x_2^3, x_1^3)} L \to 0.$$

Recall $Ker(d_2) = Im(d_3)$. Let $\gamma_1 h_1 \in Ker(d_2)$, where $\gamma_1 \in S$. To determine the generators of $Ker(d_2)$, we must solve the equation

$$\gamma_1(-x_3^2g_2 - x_2^2g_3 + x_1^2g_4) = 0.$$

The generating solution to the above equation is $\gamma_1 = 0$. Thus 0 generates $Ker(d_2)$ and $Im(d_3)$. We conclude by setting $F_3 = 0$. We now obtain our complete graded free resolution of L

$$0 \to S(-7) \xrightarrow{\begin{bmatrix} 0 \\ -x_3^2 \\ -x_2^2 \\ x_1^2 \end{bmatrix}} S(-4) \oplus S^3(-5) \xrightarrow{\begin{bmatrix} -x_2 & x_1^2 & 0 & -x_3^2 \\ 0 & 0 & x_1^2 & x_2^2 \\ x_1 & 0 & 0 & 0 \\ 0 & -x_2^2 & -x_3^2 & 0 \end{bmatrix}}$$

$$S^4(-3) \xrightarrow{(x_1x_2^2, x_2x_3^2, x_2^3, x_1^3)} L \to 0.$$

By construction our resolution is graded and free.

Example 1.4.4. Consider $L = (x_0x_1, x_0x_2, x_0x_3) \subset S = K[x_0, x_1, x_2, x_3]$. We begin our graded free resolution of S/L with

$$F_o \xrightarrow{d_o} S/L \xrightarrow{d} 0.$$

The map d is surjective and module S/L is a cyclic module, therefore the generator of S/L have degree 0, generate $Ker(d)$. Since $Ker(d) = Im(d_0)$, the generators of module S/L also generate $Im(d_0)$. We therefore set $F_0 = S$. We define d_0 a natural projection map. Our resolution therefore becomes

$$F_1 \xrightarrow{d_1} S \to S/L \to 0.$$

Since $Ker(d_0) = Im(d_1)$, the generators of L also generate $Im(d_1)$. Therefore, we set $F_1 = S(-2) \oplus S(-2) \oplus S(-2)$. Denote f_1, f_2, f_3 to be the homogeneous generators of the S-modules F_1. Note that, for all i, $deg(f_i) = 2$. We now define d_1 by

$$f_1 \mapsto x_0x_1, f_2 \mapsto x_0x_2, f_3 \mapsto x_0x_3.$$

Our resolution therefore become

$$F_2 \xrightarrow{d_2} S^3(-2) \xrightarrow{(x_0x_1, x_0x_2, x_0x_3)} S \to S/L \to 0.$$

Since $Ker(d_1) = Im(d_2)$, we can determine the generating set of $Im(d_2)$ by determining the generating set of $Ker(d_1)$. Let $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \in Ker(d_1)$ where, for all i, $\alpha_i \in S$. To determine the generators of $Ker(d_1)$, we must solve the equation

$$\alpha_1 x_0x_1 + \alpha_2 x_0x_2 + \alpha_3 x_0x_3 = 0,$$

for $(\alpha_1, \alpha_2, \alpha_3)$. The generating solutions to the above equation are $\sigma_1 = (-x_2, x_1, 0)$, $\sigma_2 = (-x_3, 0, x_1)$, $\sigma_3 = (0, -x_3, x_2)$. Thus $-x_2 f_1 + x_1 f_2$, $-x_3 f_1 + x_1 f_3$, $-x_3 f_2 + x_2 f_3$, generate $Ker(d_1)$. Their degrees are $3, 3, 3$ respectively. We therefore set $F_1 = S(-3) \oplus S(-3) \oplus S(-3)$. Denote g_1, g_2, g_3 to be the homogeneous generators of the S-modules F_2, respectively. Note that $deg(g_i) = 3$ for $1 \leq i \leq 3$. We define d_2 by

$$g_1 \mapsto -x_2 f_1 + x_1 f_2, g_2 \mapsto -x_3 f_1 + x_1 f_3, g_3 \mapsto -x_3 f_2 + x_2 f_3$$

with the resulting next step of our resolutions

$$F_3 \xrightarrow{d_3} S^3(-3) \xrightarrow{\begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix}} S^3(-2) \xrightarrow{(x_0x_1, x_0x_2, x_0x_3)} S \to S/L \to 0$$

Recall $\text{Ker}(d_2) = \text{Im}(d_3)$. Let $\beta_1 g_1 + \beta_2 g_2 + \beta_3 g_3 \in Ker(d_2)$ where, for all i, $\beta_i \in S$. To determine the generators of $Ker(d_2)$, we must solve the equation

$$\beta_1(-x_2 f_1 + x_1 f_2) + \beta_2(-x_3 f_1 + x_1 f_3) + \beta_3(-x_3 f_2 + x_2 f_3) = 0,$$

for $(\beta_1, \beta_2, \beta_3)$. The generating solution to the above equation is

$$\eta_1 = (x_3, -x_2, x_1).$$

Thus $-x_3g_1 - x_2g_2 + x_1g_3$ generates $Ker(d_2)$. The degree of the generator of $\text{Ker}(d_2)$ is 4. We therefore set $F_3 = S(-4)$. Denote h_1 to be the homogeneous generator of $S(-4)$. Note that $deg(h_1) = 4$. We now define d_3 by

$$h_1 \mapsto x_3g_1 - x_2g_2 + x_1g_3$$

with the resulting next step of our resolutions

$$F_4 \xrightarrow{d_4} S(-4) \xrightarrow{\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}} S^3(-3) \xrightarrow{\begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix}} S^3(-2) \xrightarrow{(x_0x_1, x_0x_2, x_0x_3)} S \to S/L \to 0$$

Recall $Ker(d_3) = Im(d_4)$. Let $\gamma_1 h_1 \in Ker(d_3)$, where $\gamma_1 \in S$. To determine the generators of $Ker(d_3)$, we must solve the equation

$$\gamma_1(x_3g_1 - x_2g_2 + x_1g_3) = 0.$$

The generating solution to the above equation is $\gamma_1 = 0$. Thus 0 generates $Ker(d_3)$ and $Im(d_4)$. We conclude by setting $F_4 = 0$. We now obtain our complete $\mathbb{Z}$-graded free resolution for S/L

$$0 \longrightarrow S(-4) \xrightarrow{\begin{bmatrix} x_3 \\ -x_2 \\ x_1 \end{bmatrix}} S^3(-3) \xrightarrow{\begin{bmatrix} -x_2 & -x_3 & 0 \\ x_1 & 0 & -x_3 \\ 0 & x_1 & x_2 \end{bmatrix}} S^3(-2) \xrightarrow{\begin{bmatrix} x_0x_1 & x_0x_2 & x_0x_3 \end{bmatrix}} S \longrightarrow S/L \longrightarrow 0.$$

It is noteworthy that free resolutions of modules, including graded ones, are not unique in general, as the case below illustrates.

Example 1.4.5. Let $S = K[x_1, x_2]$ and let $A = S/(x_1x_2)$. We have

$$0 \longrightarrow S(-2) \xrightarrow{x_1x_2} S \longrightarrow A \longrightarrow 0,$$

with

$$s \longrightarrow (x_1x_2)s$$

is a $\mathbb{Z}$-graded free resolution of A, and

$$0 \longrightarrow S(-2) \bigoplus S \longrightarrow S^2 \longrightarrow A \longrightarrow 0,$$

with

$$(s,u) \longrightarrow ((x_1x_2)s,u)$$

are two $\mathbb{Z}$-graded free resolutions of A.

Restricting to a minimal graded free resolution is necessary for uniqueness (up to isomorphism).

Definition 1.4.6 ([37]). We call graded free resolution of a finitely generated graded S-module A is minimal if

$$\xi_{i+1}(F_{i+1}) \subset (x_1,x_2,\dots x_n)F_i \qquad \text{for all} \quad i \geq 0.$$

The word "minimal" refers to the following two properties:

1. A free resolution is minimal if and only if at each step we make an optimal choice, that is, we select a minimal system of generators of the kernel in order to construct the next differential [37, Theorem 3.4].

2. A minimal free resolution of the module is smallest in the sense that it lies (as a direct summand) inside any other free resolution [37, Theorem 3.5].

The first free resolution in Example 1.4.5, is minimal while the second free resolution of A is not minimal. The properties of the resolution are closely linked to the properties of module A. A core area in Commutative Algebra is devoted to describing the properties of minimal free resolutions and relating them to the structure of the resolved modules. This area has many relations with and applications in other mathematical fields, especially Algebraic Geometry.

Theorem 1.1 ([37]). *(Hilbert's Syzygy Theorem) Every finitely generated module over a polynomial ring $S = K[x_1,x_2,...,x_n]$ has a finite minimal free resolution.*

There are certain invariants associated with a minimal graded free resolution of finitely generated module A. The regularity of an module A is given by

$$reg(A) = max\{j-i|\beta_{i,j}(A) \neq 0\}.$$

The projective dimension of module A is

$$pdim(A) = max\{i|\beta_{i,j}(A) \neq 0\}.$$

Both these two invariants measures the size of resolution. The numbers $\beta_{i,j}$ are named as the graded Betti numbers of A, denoted by $\beta_{i,j}(A)$ and is the number of copies of $S(-j)$ occurring in the free S-module F_i . For a module A generated in degrees ≥ 0 has Betti table of the form given in Figure 1.6.

	0	1	2	…
0	$\beta_{0,0}$	$\beta_{1,1}$	$\beta_{2,2}$	…
1	$\beta_{0,1}$	$\beta_{1,2}$	$\beta_{2,3}$	…
2	$\beta_{0,2}$	$\beta_{1,3}$	$\beta_{2,4}$	…
3	$\beta_{0,3}$	$\beta_{1,4}$	$\beta_{2,5}$	…
⋮	⋮	⋮	⋮	⋮

Figure 1.6: Betti table

Two basic invariants measuring the shape of a Betti table are the regularity and the projective dimension. The projective dimension $pdim(A)$ is the index of the last non-zero column of the Betti table, and thus it measures the length of the table. The width of the table is measured by the index of the last non-zero row of the Betti table, and it is another well-studied numerical invariant the regularity of A.

Remark 1.4.7 ([73])**.** Let L be a homogeneous ideal of S, then

$$\operatorname{reg}(S/L) = \operatorname{reg}(L) - 1,$$

$$\operatorname{pdim}(S/L) = \operatorname{pdim}(L) + 1.$$

Example 1.4.8. Let $S = K[x_1, x_2, x_3]$ and the edge ideal $I(P_2) = (x_1x_2, x_2x_3)$ associated to the graph P_2. The $\mathbb{Z}$-graded minimal free resolution of $I(P_2)$ is as follows

$$0 \longrightarrow S(-3) \xrightarrow{\cdot\begin{pmatrix} x_3 \\ -x_1 \end{pmatrix}} S^2(-2) \xrightarrow{\cdot\begin{pmatrix} x_1x_2 & x_2x_3 \end{pmatrix}} I(P_2) \longrightarrow 0.$$

Then $\operatorname{reg}(I(P_2)) = \max\{2-0, 3-1\} = 2$, and $\operatorname{pdim}(I(P_2)) = \max\{0,1\} = 1$. The Betti table has the form given in Figure 1.7. So, we have $\beta_{0,2}(I(P_2)) = 2$ and $\beta_{1,3}(I(P_2)) = 1$.

Now we recall some well known results related to regularity. Moreover, we will also state certain results of depth and Stanley depth here that relates with the results of regularity.

	0	1
0	–	–
1	–	–
2	2	1

Figure 1.7: Betti table of $I(P_2)$

For finite simple graph G, Katzman proved in [24, Lemma 2.2] that $\operatorname{indmat}(G)$ is a lower bound for the regularity of $S/I(G)$ and then Hà et al. proved in [31, Corollary 6.9] that regularity of $S/I(G)$ is equal to $\operatorname{indmat}(G)$ if G is a chordal graph. We combine these results as follows:

Lemma 1.4.9. *For a finite simple graph G, we have $\operatorname{reg}(S/I(G)) \geq \operatorname{indmat}(G)$. Moreover, if graph G is a chordal, then $\operatorname{reg}(S/I(G)) = \operatorname{indmat}(G)$.*

An interesting property of depth, Stanley depth and regularity is that when we add new variables to the ring then depth and Stanley depth will also increase [33, Lemma 3.6], while regularity will remain the same [45, Lemma 3.5]. The following lemma provides a summary of these findings.

Lemma 1.4.10. *Let $L \subset S$ be a monomial ideal and $\bar{S} = S \otimes_K K[x_{n+1}, x_{n+2}, \dots, x_{n+m}]$ be a polynomial ring in $n+m$ variable. Then $\operatorname{depth}(\bar{S}/L) = \operatorname{depth}(S/L) + m$, $\operatorname{sdepth}(\bar{S}/L) = \operatorname{sdepth}(S/L) + m$ and $\operatorname{reg}(\bar{S}/L) = \operatorname{reg}(S/L)$.*

It is obvious and well known that $\operatorname{depth}(S) = \operatorname{sdepth}(S) = n$ and $\operatorname{reg}(S) = 0$.

Lemma 1.4.11 ([11, Theorems 1.3.3])**.** *(Auslander–Buchsbaum formula) If Z is a commutative Noetherian local ring and A is a non-zero finitely generated Z-module of finite projective dimension, then*

$$\operatorname{pdim}(A) + \operatorname{depth}(A) = \operatorname{depth}(Z).$$

Now, the form in which we state the first lemma is as stated in [70, Theorem 4.7]. Part (a) and (c) of this lemma are immediate consequences of [50, Corollary 20.19 and Proposition 20.20], whereas part (b) is a consequence of [49, Lemma 2.10].

Lemma 1.4.12. *Let $L \subset S$ be a monomial ideal and x be a variable of S. Then*

(a) $\operatorname{reg}(S/L) = \operatorname{reg}(S/(L : x)) + 1$, *if* $\operatorname{reg}(S/(L : x)) > \operatorname{reg}(S/(L, x))$,

(b) $\operatorname{reg}(S/L) \in \{\operatorname{reg}(S/(L, x)) + 1, \operatorname{reg}(S/(L, x))\}$, *if* $\operatorname{reg}(S/(L : x)) = \operatorname{reg}(S/(L, x))$,

(c) $\operatorname{reg}(S/L) = \operatorname{reg}(S/(L,x))$ *if* $\operatorname{reg}(S/(L:x)) < \operatorname{reg}(S/(L,x))$.

The next result is proved by Kalai et al. [23, Theorem 1.4] for squarefree monomial ideals which was then generalized by Herzog in [26, Corollary 3.2] for arbitrary monomial ideals.

Lemma 1.4.13. *If L and J are monomial ideals of S, then* $\operatorname{reg}(S/(L+J)) \leq \operatorname{reg}(S/L) + \operatorname{reg}(S/J)$.

Moreover, if L_1 and L_2 are two edge ideals minimally generated by disjoint sets of variables then Woodroofe proved the following lemma.

Lemma 1.4.14 ([35, Lemma 3.2])**.** *Let $S_1 = K[x_1,\dots,x_r]$ and $S_2 = K[x_{r+1},\dots,x_n]$ be rings of polynomials and L_1 and L_2 be edge ideals of S_1 and S_2, respectively. Then*

$$\operatorname{reg}(S/(L_1S + L_2S)) = \operatorname{reg}(S_1/L_1) + \operatorname{reg}(S_2/L_2).$$

Now we recall the results that were proved in [36, Lemma 2.8], [55, Lemma 4], and [34, Lemma 3.1.1] for depth, Stanley depth and regularity, respectively.

Lemma 1.4.15. *Let $n \geq 2$. Then*

(a) $\operatorname{depth}(K[V(P_n)]/I(P_n)) = \operatorname{sdepth}(K[V(P_n)]/I(P_n)) = \left\lceil \frac{n}{3} \right\rceil$,

(b) $\operatorname{reg}(K[V(P_n)]/I(P_n)) = \left\lceil \frac{n-1}{3} \right\rceil$.

The value of regularity of cycle can be derived from the work of Jacques [19, Theorem 7.6.28] and the required following form is given in [58, Theorem 5.2].

Lemma 1.4.16. *If $n \geq 3$, then*

$$\operatorname{reg}(K[V(C_n)]/I(C_n)) = \begin{cases} \left\lfloor \frac{n}{3} \right\rfloor, & \text{if } n \equiv 0,1 \pmod 3; \\ \left\lfloor \frac{n}{3} \right\rfloor + 1, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

CHAPTER 2

Algebraic invariants of edge ideals of perfect semiregular trees

We start this chapter by discussing a few terms that will be helpful throughout the subsequent sections. First section consists of values of the algebraic invariants depth, projective dimension and Stanley depth of $K[V(T_{n,k})]/I(T_{n,k})$. In the second section, regularity and Krull dimension of cyclic module $K[V(T_{n,k})]/I(T_{n,k})$ are computed. The computations of the said invariants for $K[V(T_{n,k})]/I(T_{n,k})$, the module $K[V(T'_{n,k})]/I(T'_{n,k})$ has a significant importance. At the end, we will discuss the conclusion of our findings.

Let $k \geq 0$ and $L_0, L_1, \ldots, L_k$, be subsets of $V(T_{n,k})$, and for $a \in \{0, 1, \ldots, k\}$, L_a consists of all those vertices whose distance from the root vertex of $T_{n,k}$ is a. Thus $|L_0| = 1$ and for $a \in \{1, 2, \ldots, k\}$, we have $|L_a| = n(n-1)^{a-1}$. It is easy to see that $L_0, L_1, L_2, \ldots, L_k$ partition $V(T_{n,k})$, therefore, $|V(T_{n,k})| = \sum_{a=0}^{k} |L_a| = \frac{n(n-1)^k - 2}{(n-2)}$. We label the root vertex of $T_{n,k}$ by $x_1^{(0)}$, that is, $L_0 = \{x_1^{(0)}\}$ and for $a \geq 1$ the vertices of L_a are labeled as $L_a = \{x_i^{(a)} : 1 \leq i \leq n(n-1)^{a-1}\}$, see Figure 2.1. Using this labeling $T_{n,0} = (\{x_1^{(0)}\}, \emptyset)$ and for $k \geq 1$, $V(T_{n,k}) = \{x_1^{(0)}\} \bigcup_{a=1}^{k} \{x_i^{(a)} : 1 \leq i \leq n(n-1)^{a-1}\}$, $E(T_{n,1}) = \bigcup_{l=1}^{n} \{\{x_1^{(0)}, x_l^{(1)}\}\}$, and for $k \geq 2$, we have

$$E(T_{n,k}) = \bigcup_{l=1}^{n} \{\{x_1^{(0)}, x_l^{(1)}\}\} \bigcup_{a=1}^{k-1} \bigcup_{i=1}^{n(n-1)^{a-1}} \bigcup_{j=(n-1)i-(n-2)}^{(n-1)i} \{\{x_i^{(a)}, x_j^{(a+1)}\}\}.$$

Let $A := \{x_1^{(0)}\} \bigcup_{a=1}^{k} \{x_i^{(a)} : 1 \leq i \leq (n-1)^a\}$ be a subset of $V(T_{n,k})$ and H be an induced subgraph of $T_{n,k}$ on A. It is easy to see that $H = T'_{n,k}$. Let $L'_0, L'_1, L'_2, \ldots, L'_k$, be subsets of $V(T'_{n,k})$, such that for $a \in \{0, 1, 2, \ldots, k\}$, L'_a consists of all those vertices whose distance from the root vertex of $T'_{n,k}$ is a. Thus $|L'_0| = 1$ and for $a \in \{1, 2, \ldots, k\}$ we have $|L'_a| = (n-1)^a$. Since $L'_a \subset L_a$, therefore, $L'_0, L'_1, L'_2, \ldots, L'_k$ partition $V(T'_{n,k})$ and $|V(T'_{n,k})| = \sum_{a=0}^{k} |L'_a| = \frac{(n-1)^{k+1}-1}{(n-2)}$. Clearly, $x_1^{(0)}$ is the root vertex of $T'_{n,k}$, that is, $L'_0 = \{x_1^{(0)}\}$ and for $a \geq 1$, $L'_a = \{x_i^{(a)} : 1 \leq i \leq (n-1)^a\}$ as illustrated in Figure 2.1. Thus $V(T'_{n,k}) = \bigcup\limits_{a=0}^{k} \{x_i^{(a)} : 1 \leq i \leq (n-1)^a\}$, $E(T'_{n,0}) = \emptyset$ and for $k \geq 1$,

$$E(T'_{n,k}) = \bigcup_{a=0}^{k-1} \bigcup_{i=1}^{(n-1)^a} \bigcup_{j=(n-1)i-(n-2)}^{(n-1)i} \{\{x_i^{(a)}, x_j^{(a+1)}\}\}.$$

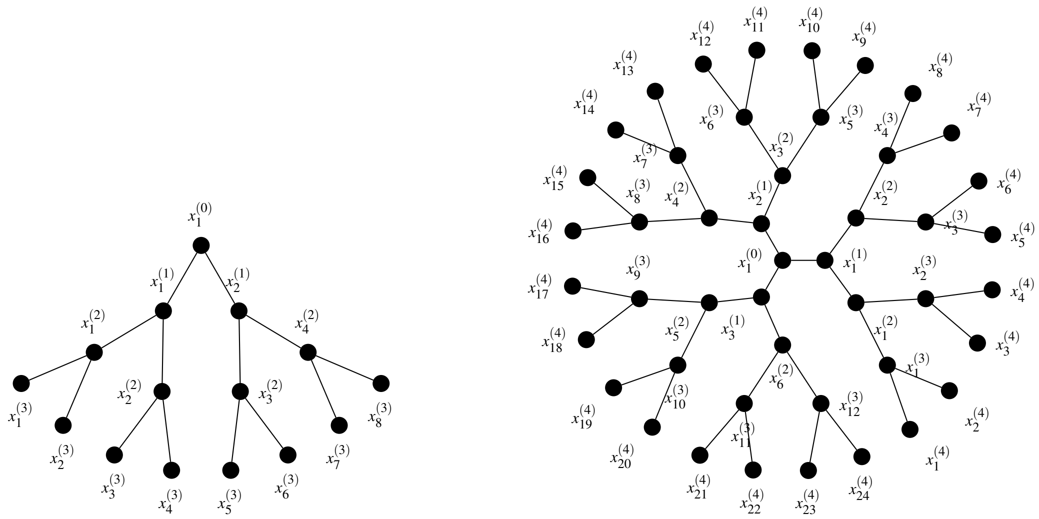


Figure 2.1: From left to right perfect 2-ary tree $T'_{3,3}$ and perfect semiregular tree $T_{3,4}$.

Let $\mathbb{M}_{n,k} := K[V(T'_{n,k})]/I(T'_{n,k})$, and $\mathbb{M}_{n,k}^m$ be a K-algebra which is the tensor product of m copies of $\mathbb{M}_{n,k}$ over K, that is, $\mathbb{M}_{n,k}^m := \bigotimes\limits_{j=1}^{m}{}_K \mathbb{M}_{n,k}$. In the following remark we address some special cases of $\mathbb{M}_{n,k}^m$ that will be encountered in the proofs of our main theorems.

Remark 2.0.1 ([80])**.** We define $\mathbb{M}_{n,k}^0 := K$. If we define $I(T'_{n,0}) = (0)$, then $\mathbb{M}_{n,0} \cong K[x_1^{(0)}]$ and $\mathbb{M}_{n,0}^m \cong \bigotimes\limits_{j=1}^{m}{}_K K[x_1^{(0)}]$. Thus $\operatorname{depth}(\mathbb{M}_{n,0}^m) = \operatorname{sdepth}(\mathbb{M}_{n,0}^m) = m$.

Lemma 2.0.2 ([80])**.** *Let $n \geq 3$ and $k \geq 1$. Then* $\operatorname{depth}(\mathbb{M}_{n,k}^0) = \operatorname{sdepth}(\mathbb{M}_{n,k}^0) = 0$, *and for $m \geq 1$,*

$$\operatorname{depth}(\mathbb{M}_{n,k}^m) = m \cdot \operatorname{depth}(\mathbb{M}_{n,k}),$$

$$\operatorname{sdepth}(\mathbb{M}_{n,k}^m) \geq m \cdot \operatorname{sdepth}(\mathbb{M}_{n,k})$$

and

$$\operatorname{reg}(\mathbb{M}_{n,k}^m) = m \cdot \operatorname{reg}(\mathbb{M}_{n,k}).$$

Proof. The proof follows by using Lemma 1.3.11 and Lemma 1.4.14. □

For a monomial ideal I, $\operatorname{supp}(I) := \{x_j : x_j | w \text{ for some } w \in \mathbb{G}(I)\}$.

Remark 2.0.3 ([80])**.** Let $I \subset S$ be a squarefree monomial ideal minimally generated by monomials of degree at most 2. We associate a graph G_I to the ideal I such that $V(G_I) = \operatorname{supp}(I)$ and $E(G_I) = \{\{x_i, x_j\} : x_i x_j \in \mathbb{G}(I)\}$. Let $x_t \in S$ be a variable such that $x_t \notin I$. Then $(I : x_t)$ and (I, x_t) are the monomial ideals of S such that $G_{(I:x_t)}$ and $G_{(I,x_t)}$ are subgraphs of G_I. See for instance Figure 2.2 and 2.3 as an examples of $G_{(I:x_1^{(0)})}$, $G_{(I,x_1^{(0)})}$ and $G_{(I:x_1^{(3)}x_2^{(3)}x_3^{(3)}x_4^{(3)}x_5^{(3)}x_6^{(3)}x_7^{(3)}x_8^{(3)}x_9^{(3)}x_{10}^{(3)}x_{11}^{(3)}x_{12}^{(3)})}$. It is evident form the Figures 2.2 and 2.3 that we have the following isomorphisms:

$$K[V(T'_{3,3})]/(I(T'_{3,3}) : x_1^{(0)}) \cong \mathbb{M}^4_{3,1} \otimes_K K[x_1^{(0)}],$$

$$K[V(T'_{3,3})]/(I(T'_{3,3}), x_1^{(0)}) \cong \mathbb{M}^2_{3,2},$$

and

$$K[V(T_{3,4})]/\big(I(T_{3,4}) : x_1^{(3)}x_2^{(3)}x_3^{(3)}x_4^{(3)}x_5^{(3)}x_6^{(3)}x_7^{(3)}x_8^{(3)}x_9^{(3)}x_{10}^{(3)}x_{11}^{(3)}x_{12}^{(3)}\big) \cong K[V(T_{3,1})]/I(T_{3,1}) \otimes_K K[L_3].$$

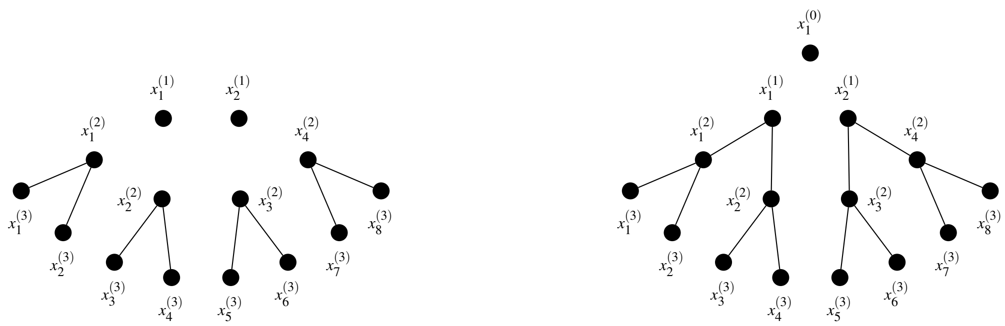


Figure 2.2: From left to right $G_{(I(T'_{3,3}):x_1^{(0)})}$ and $G_{(I(T'_{3,3}),x_1^{(0)})}$.

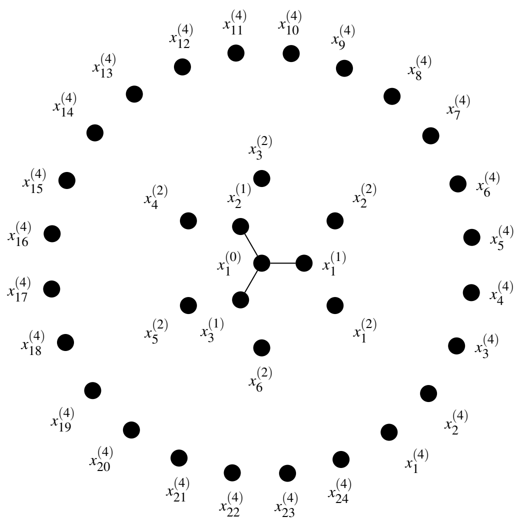


Figure 2.3: $G_{(I(T_{3,4}):x_1^{(3)}x_2^{(3)}x_3^{(3)}x_4^{(3)}x_5^{(3)}x_6^{(3)}x_7^{(3)}x_8^{(3)}x_9^{(3)}x_{10}^{(3)}x_{11}^{(3)}x_{12}^{(3)})}$.

2.1 Depth, Stanley depth and projective dimension

In this section, we find the depth, projective dimension and Stanley depth of the cyclic module $\mathbb{M}_{n,k}$ and using these results we obtain values for the said invariants of the cyclic module $K[V(T_{n,k})]/I(T_{n,k})$.

Lemma 2.1.1 ([80]). *Let $n \geq 3$ and $k \geq 2$. If $k = 2$, then* $\operatorname{depth}(\mathbb{M}_{n,2}), \operatorname{sdepth}(\mathbb{M}_{n,2}) \leq n-1$, *and for $k \geq 3$,*

$$\operatorname{depth}(\mathbb{M}_{n,k}) \leq (n-1)^{k-1} + \operatorname{depth}(\mathbb{M}_{n,k-3}),$$

and

$$\operatorname{sdepth}(\mathbb{M}_{n,k}) \leq (n-1)^{k-1} + \operatorname{sdepth}(\mathbb{M}_{n,k-3}).$$

Proof. Let $S = K[V(T'_{n,k})]$ and $u =: x_1^{(k-1)}x_2^{(k-1)}\cdots x_{(n-1)^{k-1}}^{(k-1)} \notin I(T'_{n,k})$. We have the subsequent K-algebra isomorphisms, $S/(I(T'_{n,2}):u) \cong K[L'_1]$, and for $k \geq 3$,

$$S/(I(T'_{n,k}):u) \cong K[L'_{k-1}] \otimes_K \mathbb{M}_{n,k-3}. \tag{2.1.1}$$

We first present the depth result. By Lemma 1.3.12, we have $\operatorname{depth}(\mathbb{M}_{n,k}) \leq \operatorname{depth}(S/(I(T'_{n,k}):u))$. Here $\operatorname{depth}(S/(I(T'_{n,2}):u)) = \operatorname{depth}(K[L'_1]) = |L'_1| = n-1$ and if $k \geq 3$ then by Lemma 1.4.10 and Eq. 2.1.1, $\operatorname{depth}(S/(I(T'_{n,k}):u)) = |L'_{k-1}| + \operatorname{depth}(\mathbb{M}_{n,k-3}) = (n-1)^{k-1} + \operatorname{depth}(\mathbb{M}_{n,k-3})$. Thus we get the required result for depth. For Stanley depth, by Lemma 1.3.12, $\operatorname{sdepth}(\mathbb{M}_{n,k}) \leq$

$\text{sdepth}(S/(I(T'_{n,k}):u))$. By using the isomorphisms, $\text{sdepth}(S/(I(T'_{n,2}):u)) = \text{sdepth}(K[L'_1]) = |L'_1| = n-1$ and if $k \geq 3$, by applying Lemma 1.4.10 on Eq. 2.1.1, we get $\text{sdepth}(S/(I(T'_{n,k}):u)) = |L'_{k-1}| + \text{sdepth}(\mathbb{M}_{n,k-3}) = (n-1)^{k-1} + \text{sdepth}(\mathbb{M}_{n,k-3})$. This completes the proof. □

Proposition 2.1.2 ([80]). *Let $n \geq 3$ and $k \geq 1$. Then*

$$\text{depth}(\mathbb{M}_{n,k}) = \text{sdepth}(\mathbb{M}_{n,k}) = \begin{cases} \frac{(n-1)^2((n-1)^k-1)}{(n-1)^3-1} + 1, & if \quad k \equiv 0 \pmod 3; \\ \frac{(n-1)^{k+2}-1}{(n-1)^3-1}, & if \quad k \equiv 1 \pmod 3; \\ \frac{(n-1)^{k+2}-n+1}{(n-1)^3-1}, & if \quad k \equiv 2 \pmod 3. \end{cases}$$

Proof. Firstly, we show the result for depth. If $k = 1$, then $\mathbb{M}_{n,1} \cong K[V(S_n)]/I(S_n)$, by Lemma 1.3.16 we have $\text{depth}(\mathbb{M}_{n,1}) = 1$, as required. Let $k \geq 2$ and $S = K[V(T'_{n,k})]$. We have the following short exact sequence

$$0 \longrightarrow S/(I(T'_{n,k}):x_1^{(0)}) \xrightarrow{.x_1^{(0)}} \mathbb{M}_{n,k} \longrightarrow S/(I(T'_{n,k}),x_1^{(0)}) \longrightarrow 0,$$

It is simple to observe that $S/(I(T'_{n,k}):x_1^{(0)}) \cong \mathbb{M}_{n,k-2}^{(n-1)^2} \otimes_K K[x_1^{(0)}]$ and $S/(I(T'_{n,k}),x_1^{(0)}) \cong \mathbb{M}_{n,k-1}^{n-1}$. By Lemma 1.3.11 and Lemma 2.0.2, we have

$$\text{depth}(S/(I(T'_{n,k}):x_1^{(0)})) = (n-1)^2\,\text{depth}(\mathbb{M}_{n,k-2}) + 1,$$

$$\text{depth}(S/(I(T'_{n,k}),x_1^{(0)})) = (n-1)\,\text{depth}(\mathbb{M}_{n,k-1}).$$

Thus by Lemma 1.3.10,

$$\text{depth}(\mathbb{M}_{n,k}) \geq \min\{(n-1)^2\,\text{depth}(\mathbb{M}_{n,k-2}) + 1,\, (n-1)\,\text{depth}(\mathbb{M}_{n,k-1})\}. \tag{2.1.2}$$

If $k = 2$, then by Eq. 2.1.2

$$\begin{aligned} \text{depth}(\mathbb{M}_{n,2}) &\geq \min\{(n-1)^2\,\text{depth}(\mathbb{M}_{n,0}) + 1,\, (n-1)\,\text{depth}(\mathbb{M}_{n,1})\} \\ &= \min\{(n-1)^2 + 1, (n-1)\} \\ &= n-1, \end{aligned}$$

and by Lemma 2.1.1, $\text{depth}(\mathbb{M}_{n,2}) \leq n-1$. Thus $\text{depth}(\mathbb{M}_{n,2}) = n-1$. This prove the result for $k = 2$.

Let $k \geq 3$. For $1 \leq i \leq n-1$, let $A_i := (x_1^{(1)}, x_2^{(1)}, \ldots, x_i^{(1)})$ and $A_0 := (0)$ be prime ideals. Consider the family of following short exact sequences:

$$0 \longrightarrow S/((I(T'_{n,k}),A_{i-1}):x_i^{(1)}) \xrightarrow{.x_i^{(1)}} S/(I(T'_{n,k}),A_{i-1}) \longrightarrow S/(I(T'_{n,k}),A_i) \longrightarrow 0,$$

applying Lemma 1.3.10 on this family of short exact sequences we have the following inequality

$$\operatorname{depth}(\mathbb{M}_{n,k}) = \operatorname{depth}(S/(I(T'_{n,k}),A_0)) \geq$$
$$\min\big\{\min\{\operatorname{depth}(S/((I(T'_{n,k}),A_{i-1}):x_i^{(1)})) : i=1,2,\dots,n-1\}, \operatorname{depth}(S/(I(T'_{n,k}),A_{n-1}))\big\}. \tag{2.1.3}$$

The K-algebras isomorphism:

$$S/((I(T'_{n,k}),A_{i-1}):x_i^{(1)}) \cong \mathbb{M}_{n,k-2}^{(i-1)(n-1)} \otimes_K \mathbb{M}_{n,k-1}^{n-1-i} \otimes_K \mathbb{M}_{n,k-3}^{(n-1)^2} \otimes_K K[x_i^{(1)}],$$

$$S/(I(T'_{n,k}),A_{n-1}) \cong \mathbb{M}_{n,k-2}^{(n-1)^2} \otimes_K K[x_1^{(0)}].$$

By applying Lemma 1.3.11

$$\begin{aligned}&\operatorname{depth}(S/((I(T'_{n,k}),A_{i-1}):x_i^{(1)}))\\ &\qquad= \operatorname{depth}(\mathbb{M}_{n,k-2}^{(i-1)(n-1)}) + \operatorname{depth}(\mathbb{M}_{n,k-1}^{n-1-i}) + \operatorname{depth}(\mathbb{M}_{n,k-3}^{(n-1)^2}) + \operatorname{depth}(K[x_i^{(1)}])\end{aligned}$$

and

$$\operatorname{depth}(S/(I(T'_{n,k}),A_{n-1})) = \operatorname{depth}\left(\mathbb{M}_{n,k-2}^{(n-1)^2}\right) + \operatorname{depth}(K[x_1^{(0)}]).$$

By Lemma 2.0.2

$$\begin{aligned}\operatorname{depth}(S/((I(T'_{n,k}),A_{i-1}):x_i^{(1)})) &= (i-1)(n-1)\operatorname{depth}(\mathbb{M}_{n,k-2}) + (n-1-i)\operatorname{depth}(\mathbb{M}_{n,k-1})\\ &\quad + (n-1)^2\operatorname{depth}(\mathbb{M}_{n,k-3}) + 1\end{aligned}$$

and

$$\operatorname{depth}(S/(I(T'_{n,k}),A_{n-1})) = (n-1)^2\operatorname{depth}(\mathbb{M}_{n,k-2}) + 1.$$

Now by Eq. 2.1.3 we get

$$\begin{aligned}\operatorname{depth}(\mathbb{M}_{n,k}) \geq \min\Big\{\min_{i=1}^{n-1}\Big\{&(i-1)(n-1)\operatorname{depth}(\mathbb{M}_{n,k-2}) + (n-1-i)\operatorname{depth}(\mathbb{M}_{n,k-1})\\ &+ (n-1)^2\operatorname{depth}(\mathbb{M}_{n,k-3}) + 1\Big\}, (n-1)^2\operatorname{depth}(\mathbb{M}_{n,k-2}) + 1\Big\}.\end{aligned} \tag{2.1.4}$$

If $k=3$, then

$$\begin{aligned}\operatorname{depth}(\mathbb{M}_{n,3}) &\geq \min\Big\{\min_{i=1}^{n-1}\{(i-1)(n-1)\operatorname{depth}(\mathbb{M}_{n,1}) + (n-1-i)\operatorname{depth}(\mathbb{M}_{n,2})\\ &\qquad + (n-1)^2\operatorname{depth}(\mathbb{M}_{n,0}) + 1\}, (n-1)^2\operatorname{depth}(\mathbb{M}_{n,1}) + 1\Big\}\\ &= \min\Big\{\min_{i=1}^{n-1}\{(i-1)(n-1) + (n-1-i)(n-1) + (n-1)^2 + 1\}, (n-1)^2 + 1\Big\}.\end{aligned} \tag{2.1.5}$$

Since

$$(i-1)(n-1)+(n-1-i)(n-1)+(n-1)^2+1=(n-1)(2n-3)+1$$
$$=(n-1)^2+1+\big((n-1)^2-(n-1)\big),$$

for all $i\in\{1,2,\dots,n-1\}$. Thus by Eq. 2.1.5 $\operatorname{depth}(\mathbb{M}_{n,k})\geq(n-1)^2+1$. By Lemma 2.1.1 we have $\operatorname{depth}(\mathbb{M}_{n,k})\leq(n-1)^2+1$, we get $\operatorname{depth}(\mathbb{M}_{n,k})=(n-1)^2+1$, as required. Now let $k\geq 4$, we will use mathematical induction on k to get the required result. We consider following cases:

Case 1 Let $k\equiv 1(\mathrm{mod}\,3)$. Recall Eq. 2.1.2

$$\operatorname{depth}(\mathbb{M}_{n,k})\geq\min\{(n-1)^2\operatorname{depth}(\mathbb{M}_{n,k-2})+1,\,(n-1)\operatorname{depth}(\mathbb{M}_{n,k-1})\}.$$

Since $k\equiv 1(\mathrm{mod}\,3)$ so $k-1\equiv 0\,(\mathrm{mod}\ 3)$ and $k-2\equiv 2\,(\mathrm{mod}\ 3)$ thus by induction on k, we get

$$(n-1)^2\operatorname{depth}(\mathbb{M}_{n,k-2})+1=(n-1)^2\left(\frac{(n-1)^{(k-2)+2}-n+1}{(n-1)^3-1}\right)+1$$
$$=\frac{(n-1)^{k+2}-1}{(n-1)^3-1},$$

and

$$(n-1)\operatorname{depth}(\mathbb{M}_{n,k-1})=(n-1)\left(\frac{(n-1)^2((n-1)^{k-1}-1)}{(n-1)^3-1}+1\right)$$
$$=\frac{(n-1)^{k+2}+n^4-5n^3+9n^2-8n+3}{(n-1)^3-1}$$
$$=\frac{(n-1)^{k+2}-1}{(n-1)^3-1}+\frac{n^4-5n^3+9n^2-8n+4}{(n-1)^3-1}.$$

Note that $\frac{n^4-5n^3+9n^2-8n+4}{(n-1)^3-1}>0$ for all $n\geq 3$ by using MATLAB ®. Thus $\operatorname{depth}(\mathbb{M}_{n,k})\geq\frac{(n-1)^{k+2}-1}{(n-1)^3-1}$. By Lemma 2.1.1, $\operatorname{depth}(\mathbb{M}_{n,k})\leq(n-1)^{k-1}+\operatorname{depth}(\mathbb{M}_{n,k-3})$, since $k-3\equiv 1(\mathrm{mod}\,3)$, thus by induction on k, we have $\operatorname{depth}(\mathbb{M}_{n,k})\leq\frac{(n-1)^{k-1}-1}{(n-1)^3-1}+(n-1)^{k-1}=\frac{(n-1)^{k+2}-1}{(n-1)^3-1}$. Hence $\operatorname{depth}(\mathbb{M}_{n,k})=\frac{(n-1)^{k+2}-1}{(n-1)^3-1}$, as required.

Case 2 Let $k\equiv 2(\mathrm{mod}\,3)$. In this case $k-1\equiv 1\,(\mathrm{mod}\ 3)$ and $k-2\equiv 0\,(\mathrm{mod}\ 3)$ thus by induction on k, we have

$$(n-1)^2\operatorname{depth}(\mathbb{M}_{n,k-2})+1=(n-1)^2\left(\frac{(n-1)^2((n-1)^{k-2}-1)}{(n-1)^3-1}+1\right)+1$$
$$=\frac{(n-1)^{k+2}+n^5-6n^4+15n^3-20n^2+14n-5}{(n-1)^3-1}$$
$$=\frac{(n-1)^{k+2}-n+1}{(n-1)^3-1}+\frac{n^5-6n^4+15n^3-20n^2+15n-6}{(n-1)^3-1},$$

and

$$(n-1)\,\mathrm{depth}(\mathbb{M}_{n,k-1}) = (n-1)\left(\frac{(n-1)^{(k-1)+2}-1}{(n-1)^3-1}\right) = \frac{(n-1)^{k+2}-n+1}{(n-1)^3-1}.$$

Note that $\frac{n^5-6n^4+15n^3-20n^2+15n-6}{(n-1)^3-1} > 0$ for all $n \geq 3$ by using MATLAB®. Thus by using Eq. 2.1.2, we have $\mathrm{depth}(\mathbb{M}_{n,k}) \geq \frac{(n-1)^{k+2}-n+1}{(n-1)^3-1}$. For the other inequality we use again Lemma 2.1.1 $\mathrm{depth}(\mathbb{M}_{n,k}) \leq (n-1)^{k-1} + \mathrm{depth}(\mathbb{M}_{n,k-3})$. Since $k-3 \equiv 2(\mathrm{mod}\,3)$, so by using induction on k, $\mathrm{depth}(\mathbb{M}_{n,k}) \leq (n-1)^{k-1} + \frac{(n-1)^{(k-3)+2}-n+1}{(n-1)^3-1} = \frac{(n-1)^{k+2}-n+1}{(n-1)^3-1}$. Hence $\mathrm{depth}(\mathbb{M}_{n,k}) = \frac{(n-1)^{k+2}-n+1}{(n-1)^3-1}$.

Case 3 Let $k \equiv 0(\mathrm{mod}\,3)$. Recall Eq. 2.1.4

$$\mathrm{depth}(\mathbb{M}_{n,k}) \geq \min\Big\{\min_{i=1}^{n-1}\Big\{(i-1)(n-1)\,\mathrm{depth}(\mathbb{M}_{n,k-2}) + (n-1-i)\,\mathrm{depth}(\mathbb{M}_{n,k-1}) + (n-1)^2\,\mathrm{depth}(\mathbb{M}_{n,k-3})+1\Big\}, (n-1)^2\,\mathrm{depth}(\mathbb{M}_{n,k-2})+1\Big\}.$$

Since $k-3 \equiv 0\,(\mathrm{mod}\ 3)$, $k-2 \equiv 1\,(\mathrm{mod}\ 3)$ and $k-1 \equiv 2\,(\mathrm{mod}\ 3)$ and $i \in \{1,2,\ldots,n-1\}$. by induction on k, we have

$$\begin{aligned}
&(i-1)(n-1)\,\mathrm{depth}(\mathbb{M}_{n,k-2}) + (n-1-i)\,\mathrm{depth}(\mathbb{M}_{n,k-1}) + (n-1)^2\,\mathrm{depth}(\mathbb{M}_{n,k-3})+1\\
&\quad= (i-1)(n-1)\left(\frac{(n-1)^{(k-2)+2}-1}{(n-1)^3-1}\right) + (n-1-i)\left(\frac{(n-1)^{(k-1)+2}-n+1}{(n-1)^3-1}\right)\\
&\qquad + (n-1)^2\left(\frac{(n-1)^2((n-1)^{k-3}-1)}{(n-1)^3-1}+1\right)+1\\
&\quad= \frac{(n-1)^{k+2}+n^5-6n^4+15n^3-21n^2+17n-7}{(n-1)^3-1}\\
&\quad= \frac{(n-1)^2((n-1)^k-1)}{(n-1)^3-1}+1+\frac{n^5-6n^4+14n^3-17n^2+12n-4}{(n-1)^3-1}
\end{aligned}$$

and

$$\begin{aligned}
(n-1)^2\,\mathrm{depth}(\mathbb{M}_{n,k-2})+1 &= (n-1)^2\left(\frac{(n-1)^{(k-2)+2}-1}{(n-1)^3-1}\right)+1\\
&= \frac{(n-1)^2((n-1)^k-1)}{(n-1)^3-1}+1.
\end{aligned}$$

Note that $\frac{n^5-6n^4+14n^3-17n^2+12n-4}{(n-1)^3-1} > 0$, for all $n \geq 3$ by using MATLAB®. Thus by Eq. 2.1.4 we have $\mathrm{depth}(\mathbb{M}_{n,k}) \geq \frac{(n-1)^2((n-1)^k-1)}{(n-1)^3-1}+1$. Again by Lemma 2.1.1, $\mathrm{depth}(\mathbb{M}_{n,k}) \leq (n-1)^{k-1} + \mathbb{M}_{n,k-3} = (n-1)^{k-1} + \frac{(n-1)^2((n-1)^{k-3}-1)}{(n-1)^3-1}+1 = \frac{(n-1)^2((n-1)^k-1)}{(n-1)^3-1}+1$.

This ends the proof. Proof for Stanley depth is similar. □

Corollary 2.1.3 ([80]). *Let $n \geq 3$ and $k \geq 1$. Then*

$$\operatorname{pdim}(\mathbb{M}_{n,k}) = \begin{cases} \frac{(n-1)^{k+1}-1}{n-2} - \frac{(n-1)^2((n-1)^k-1)}{(n-1)^3-1} - 1, & if \quad k \equiv 0 \pmod 3; \\ \frac{(n-1)^{k+1}-1}{n-2} - \frac{(n-1)^{k+2}-1}{(n-1)^3-1}, & if \quad k \equiv 1 \pmod 3; \\ \frac{(n-1)^{k+1}-1}{n-2} - \frac{(n-1)^{k+2}-n+1}{(n-1)^3-1}, & if \quad k \equiv 2 \pmod 3. \end{cases}$$

Proof. The result obtain by using Auslander–Buchsbaum formula [11, Theorems 1.3.3] and Proposition 2.1.2. □

Example 2.1.4. For $k \geq 1$, let $T'_{3,k}$ be the graph as given in Figure 2.4.

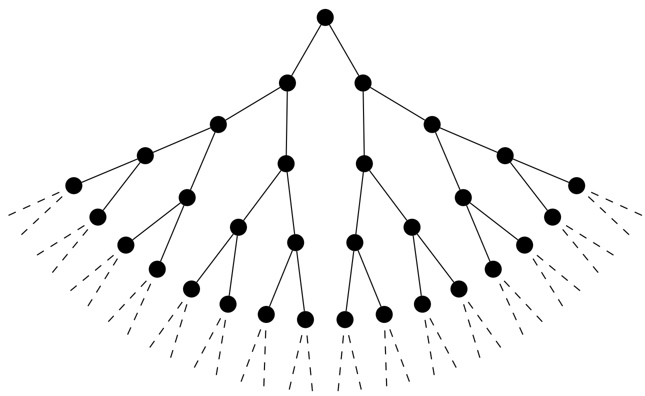


Figure 2.4: $T'_{3,k}$

Then,

$$\text{(a)}\ \operatorname{depth}(\mathbb{M}_{3,k}) = \operatorname{sdepth}(\mathbb{M}_{3,k}) = \begin{cases} \frac{2^{k+2}+3}{7}, & if \quad k \equiv 0 \pmod 3; \\ \frac{2^{k+2}-1}{7}, & if \quad k \equiv 1 \pmod 3; \\ \frac{2^{k+2}-2}{7}, & if \quad k \equiv 2 \pmod 3. \end{cases}$$

$$\text{(b)}\ \operatorname{pdim}(\mathbb{M}_{3,k}) = \begin{cases} \frac{10(2^k-1)}{7}, & if \quad k \equiv 0 \pmod 3; \\ \frac{2(5\cdot 2^k-3)}{7}, & if \quad k \equiv 1 \pmod 3; \\ \frac{5(2^{k+1}-1)}{7}, & if \quad k \equiv 2 \pmod 3. \end{cases}$$

Example 2.1.5. For $k \geq 1$, let $T'_{4,k}$ be the graph as given in Figure 2.5.

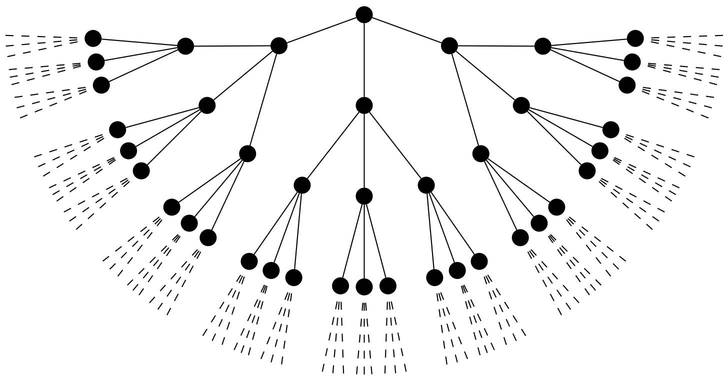


Figure 2.5: $T'_{4,k}$

Then,

$$\text{(a)}\ \ \operatorname{depth}(\mathbb{M}_{4,k}) = \operatorname{sdepth}(\mathbb{M}_{4,k}) = \begin{cases} \frac{3^{k+2}+17}{26}, & if \quad k \equiv 0 \,(\mathrm{mod}\ 3); \\ \frac{3^{k+2}-1}{26}, & if \quad k \equiv 1 \,(\mathrm{mod}\ 3); \\ \frac{3^{k+2}-3}{26}, & if \quad k \equiv 2 \,(\mathrm{mod}\ 3). \end{cases}$$

$$\text{(b)}\ \ \operatorname{pdim}(\mathbb{M}_{4,k}) = \begin{cases} \frac{15(3^{k}-1)}{13}, & if \quad k \equiv 0 \,(\mathrm{mod}\ 3); \\ \frac{3(5\cdot 3^{k}-2)}{13}, & if \quad k \equiv 1 \,(\mathrm{mod}\ 3); \\ \frac{5(3^{k+1}-1)}{13}, & if \quad k \equiv 2 \,(\mathrm{mod}\ 3). \end{cases}$$

Remark 2.1.6 ([80]). Let $n \geq 3$, we define $I(T_{n,0}) = (0)$, thus $K[V(T_{n,0})]/I(T_{n,0}) \cong K[x_1^{(0)}]$. We have $\operatorname{depth}(K[V(T_{n,0})]/I(T_{n,0})) = \operatorname{sdepth}(K[V(T_{n,0})]/I(T_{n,0})) = 1$.

Lemma 2.1.7 ([80]). *Let $n \geq 3$, $k \geq 2$ and $R = K[V(T_{n,k})]$. If $k = 2$, then*

$$\operatorname{depth}(R/I(T_{n,2})), \operatorname{sdepth}(R/I(T_{n,2})) \leq n.$$

If $k \geq 3$, then

$$\operatorname{depth}(R/I(T_{n,k})) \leq n(n-1)^{k-2} + \operatorname{depth}(K[V(T_{n,k-3})]/I(T_{n,k-3})),$$

and

$$\operatorname{sdepth}(R/I(T_{n,k})) \leq n(n-1)^{k-2} + \operatorname{sdepth}(K[V(T_{n,k-3})]/I(T_{n,k-3})).$$

Proof. For $u := x_1^{(k-1)} x_2^{(k-1)} \cdots x_{n(n-1)^{k-2}}^{(k-1)} \notin I(T_{n,k})$, we have K-algebra isomorphisms, $R/(I(T_{n,2}) : u) \cong K[L_1]$, and for $k \geq 3$,

$$R/(I(T_{n,k}) : u) \cong K[L_{k-1}] \otimes_K K[V(T_{n,k-3})]/I(T_{n,k-3}). \tag{2.1.6}$$

To prove the result for depth, we use Lemma 1.3.12, that is $\operatorname{depth}(R/I(T_{n,k})) \leq \operatorname{depth}(R/(I(T_{n,k}) : u))$. Here we have $\operatorname{depth}(R/(I(T_{n,2}) : u)) = \operatorname{depth}(K[L_1]) = |L_1| = n$ and if $k \geq 3$, then by Lemma 1.4.10 and Eq. 2.1.6, $\operatorname{depth}(R/(I(T_{n,k}) : u)) = |L_{k-1}| + \operatorname{depth}(K[V(T_{n,k-3})]/I(T_{n,k-3})) = n(n-1)^{k-2} + \operatorname{depth}(K[V(T_{n,k-3})]/I(T_{n,k-3}))$. Now for Stanley depth, we have $\operatorname{sdepth}(R/(I(T_{n,2}) : u)) = \operatorname{sdepth}(K[L_1]) = |L_1| = n$ and if $k \geq 3$, then by applying Lemma 1.4.10 on Eq. 2.1.6, we get

$$\begin{aligned}\operatorname{sdepth}(R/(I(T_{n,k}) : u)) &= |L_{k-1}| + \operatorname{sdepth}(K[V(T_{n,k-3})]/I(T_{n,k-3})) \\ &= n(n-1)^{k-2} + \operatorname{sdepth}(K[V(T_{n,k-3})]/I(T_{n,k-3})).\end{aligned}$$

This ends the proof. □

Theorem 2.1.8 ([80]). *Let $n \geq 3$ and $k \geq 1$. If $R = K[V(T_{n,k})]$, then*

$$\operatorname{depth}(R/I(T_{n,k})) = \operatorname{sdepth}(R/I(T_{n,k})) = \begin{cases} \frac{n(n-1)^{k+1}+n^3-4n^2+4n-2}{(n-1)^3-1}, & if \quad k \equiv 0 \pmod 3; \\ \frac{n(n-1)^{k+1}-n^2+2n-2}{(n-1)^3-1}, & if \quad k \equiv 1 \pmod 3; \\ \frac{n(n-1)^{k+1}-n}{(n-1)^3-1}, & if \quad k \equiv 2 \pmod 3. \end{cases}$$

Proof. First we provide the result for the computation of depth. If $k = 1$, then $T_{n,1}$ is $(n+1)$-star and by Lemma 1.3.16 we have $\operatorname{depth}(R/I(T_{n,1})) = 1$. Let $k \geq 2$. We have short exact sequence of the form:

$$0 \longrightarrow R/(I(T_{n,k}) : x_1^{(0)}) \xrightarrow{.x_1^{(0)}} R/I(T_{n,k}) \longrightarrow R/(I(T_{n,k}), x_1^{(0)}) \longrightarrow 0.$$

Since

$$R/(I(T_{n,k}) : x_1^{(0)}) \cong \mathbb{M}_{n,k-2}^{n(n-1)} \otimes_K K[x_1^{(0)}], \tag{2.1.7}$$

and

$$R/(I(T_{n,k}), x_1^{(0)}) \cong \mathbb{M}_{n,k-1}^{n}. \tag{2.1.8}$$

By Lemmas 1.3.11 and 2.0.2,

$$\operatorname{depth}(R/(I(T_{n,k}) : x_1^{(0)})) = n(n-1)\operatorname{depth}(\mathbb{M}_{n,k-2}) + 1,$$

and

$$\operatorname{depth}(R/(I(T_{n,k}),x_1^{(0)})) = n\operatorname{depth}(\mathbb{M}_{n,k-1}).$$

Therefore by Lemma 1.3.10,

$$\operatorname{depth}(R/I(T_{n,k})) \geq \min\{n(n-1)\operatorname{depth}(\mathbb{M}_{n,k-2})+1, n\operatorname{depth}(\mathbb{M}_{n,k-1})\}. \tag{2.1.9}$$

If $k=2$ then by using Eq. 2.1.9 and Proposition 2.1.2

$$\operatorname{depth}(R/I(T_{n,2})) \geq \min\{n(n-1)\operatorname{depth}(\mathbb{M}_{n,0})+1, n\operatorname{depth}(\mathbb{M}_{n,1})\} = \min\{n(n-1)+1, n\} = n,$$

and by Lemma 2.1.7, $\operatorname{depth}(R/I(T_{n,2})) \leq n$. Thus $\operatorname{depth}(R/I(T_{n,2})) = n$. This proves the result for $k=2$. Let $k \geq 3$. Consider the short exact sequence

$$0 \longrightarrow R/(I(T_{n,k}):x_1^{(1)}) \xrightarrow{.x_1^{(1)}} R/I(T_{n,k}) \longrightarrow R/(I(T_{n,k}),x_1^{(1)}) \longrightarrow 0.$$

We have

$$R/(I(T_{n,k}):x_1^{(1)}) \cong \mathbb{M}_{n,k-3}^{(n-1)^2} \otimes_K \mathbb{M}_{n,k-1}^{(n-1)} \otimes_K K[x_1^{(1)}],$$

$$R/(I(T_{n,k}),x_1^{(1)}) \cong \mathbb{M}_{n,k} \otimes_K \mathbb{M}_{n,k-2}^{(n-1)}.$$

By using Lemmas 1.3.11 and 2.0.2,

$$\operatorname{depth}(R/(I(T_{n,k}):x_1^{(1)})) = (n-1)^2\operatorname{depth}(\mathbb{M}_{n,k-3}) + (n-1)\operatorname{depth}(\mathbb{M}_{n,k-1}) + 1,$$

and

$$\operatorname{depth}(R/(I(T_{n,k}),x_1^{(1)})) = \operatorname{depth}(\mathbb{M}_{n,k}) + (n-1)\operatorname{depth}(\mathbb{M}_{n,k-2}).$$

Thus by Lemma 1.3.10,

$$\begin{aligned}\operatorname{depth}(R/I(T_{n,k})) \geq \min\Big\{&(n-1)^2\operatorname{depth}(\mathbb{M}_{n,k-3}) + (n-1)\operatorname{depth}(\mathbb{M}_{n,k-1}) + 1,\\ &\operatorname{depth}(\mathbb{M}_{n,k}) + (n-1)\operatorname{depth}(\mathbb{M}_{n,k-2})\Big\}.\end{aligned} \tag{2.1.10}$$

If $k=3$ then by Eq. 2.1.10 and Proposition 2.1.2

$$\begin{aligned}\operatorname{depth}(R/I(T_{n,k})) \geq \min\Big\{&(n-1)^2\operatorname{depth}(\mathbb{M}_{n,0}) + (n-1)\operatorname{depth}(\mathbb{M}_{n,2}) + 1,\\ &\operatorname{depth}(\mathbb{M}_{n,3}) + (n-1)\operatorname{depth}(\mathbb{M}_{n,1})\Big\}\\ &= \min\{(n-1)^2 + (n-1)^2 + 1, (n-1)^2 + 1 + n - 1\}\\ &= \min\{(n-1)^2 + n + (n-1)^2 - n + 1, (n-1)^2 + n\}\\ &= (n-1)^2 + n.\end{aligned}$$

and by using Lemma 2.1.7 we have $\operatorname{depth}(R/I(T_{n,k})) \leq n(n-1)+1 = (n-1)^2+n$. Thus $\operatorname{depth}(R/I(T_{n,k})) = (n-1)^2+n$, as required. Now let $k \geq 4$. We consider the following cases:

Case 1 Let $k \equiv 1 \,(\mathrm{mod}\ 3)$. Recall Eq. 2.1.9

$$\mathrm{depth}(R/I(T_{n,k})) \geq \min\{n(n-1)\,\mathrm{depth}(\mathbb{M}_{n,k-2})+1, n\,\mathrm{depth}(\mathbb{M}_{n,k-1})\}.$$

Since $k \equiv 1 \,(\mathrm{mod}\ 3)$ so $k-1 \equiv 0 \,(\mathrm{mod}\ 3)$ and $k-2 \equiv 2 \,(\mathrm{mod}\ 3)$, thus by using Proposition 2.1.2, we have

$$\begin{aligned} n(n-1)\,\mathrm{depth}(\mathbb{M}_{n,k-2})+1 &= n(n-1)\left(\frac{(n-1)^{(k-2)+2}-n+1}{(n-1)^3-1}\right)+1 \\ &= \frac{n(n-1)^{k+1}-n^2+2n-2}{(n-1)^3-1}, \end{aligned}$$

and

$$\begin{aligned} n\,\mathrm{depth}(\mathbb{M}_{n,k-1}) &= n\left(\frac{(n-1)^2((n-1)^{k-1}-1)}{(n-1)^3-1}+1\right) \\ &= \frac{n(n-1)^{k+1}+n^4-4n^3+5n^2-3n}{(n-1)^3-1} \\ &= \frac{n(n-1)^{k+1}-n^2+2n-2}{(n-1)^3-1}+\frac{n^4-4n^3+6n^2-5n+2}{(n-1)^3-1}. \end{aligned}$$

Note that $\frac{n^4-4n^3+6n^2-5n+2}{(n-1)^3-1} > 0$, for all $n \geq 3$ by using MATLAB®. Thus $\mathrm{depth}(R/I(T_{n,k})) \geq \frac{n(n-1)^{k+1}-n^2+2n-2}{(n-1)^3-1}$. By using Lemma 2.1.7, we have $\mathrm{depth}(R/I(T_{n,k})) \leq n(n-1)^{k-2} + \mathrm{depth}(K[V(T_{n,k-3})]/I(T_{n,k-3}))$, since $k-3 \equiv 1 \,(\mathrm{mod}\ 3)$, thus by induction on k, we have

$$\mathrm{depth}(R/I(T_{n,k})) \leq n(n-1)^{k-2}+\frac{n(n-1)^{(k-3)+1}-n^2+2n-2}{(n-1)^3-1} = \frac{n(n-1)^{k+1}-n^2+2n-2}{(n-1)^3-1}.$$

Hence we get $\mathrm{depth}(R/I(T_{n,k})) = \frac{n(n-1)^{k+1}-n^2+2n-2}{(n-1)^3-1}$, as required.

Case 2 Let $k \equiv 2 \,(\mathrm{mod}\ 3)$. In this case $k-2 \equiv 0 \,(\mathrm{mod}\ 3)$, $k-1 \equiv 1 \,(\mathrm{mod}\ 3)$, by using Proposition 2.1.2, we have

$$\begin{aligned} n(n-1)\,\mathrm{depth}(\mathbb{M}_{n,k-2})+1 &= n(n-1)\left(\frac{(n-1)^2((n-1)^{k-2}-1)}{(n-1)^3-1}+1\right)+1 \\ &= \frac{n(n-1)^{k+1}+n^5-5n^4+10n^3-11n^2+6n-2}{(n-1)^3-1} \\ &= \frac{n(n-1)^{k+1}-n}{(n-1)^3-1}+\frac{n^5-5n^4+10n^3-11n^2+7n-2}{(n-1)^3-1}, \end{aligned}$$

and

$$n\,\mathrm{depth}(\mathbb{M}_{n,k-1}) = n\left(\frac{(n-1)^{(k-1)+2}-1}{(n-1)^3-1}\right) = \frac{n(n-1)^{k+1}-n}{(n-1)^3-1}.$$

Note that $\frac{n^5-5n^4+10n^3-11n^2+7n-2}{(n-1)^3-1} > 0$, for all $n \geq 3$ by using MATLAB®. By Eq. 2.1.9, $\mathrm{depth}(R/I(T_{n,k})) \geq \frac{n(n-1)^{k+1}-n}{(n-1)^3-1}$. For the other inequality, we again use Lemma 2.1.7, that is $\mathrm{depth}(R/I(T_{n,k})) \leq n(n-1)^{k-2} + \mathrm{depth}(K[V(T_{n,k-3})]/I(T_{n,k-3}))$, since $k-3 \equiv 2 \,(\mathrm{mod}\ 3)$, thus by induction on k, we have $\mathrm{depth}(R/I(T_{n,k})) \leq \frac{n(n-1)^{(k-3)+1}-n}{(n-1)^3-1} + n(n-1)^{k-2} = \frac{n(n-1)^{k+1}-n}{(n-1)^3-1}$. Hence $\mathrm{depth}(R/I(T_{n,k})) = \frac{n(n-1)^{k+1}-n}{(n-1)^3-1}$.

Case 3 Let $k \equiv 0 \,(\text{mod } 3)$. Recall Eq. 2.1.10

$$\operatorname{depth}(R/I(T_{n,k})) \geq \min\Big\{(n-1)^2 \operatorname{depth}(\mathbb{M}_{n,k-3}) + (n-1)\operatorname{depth}(\mathbb{M}_{n,k-1}) + 1, \operatorname{depth}(\mathbb{M}_{n,k}) + (n-1)\operatorname{depth}(\mathbb{M}_{n,k-2})\Big\}.$$

Since $k \equiv 0 \,(\text{mod } 3)$ so $k-3 \equiv 0 \,(\text{mod } 3)$, $k-1 \equiv 2 \,(\text{mod } 3)$ and $k-2 \equiv 1 \,(\text{mod } 3)$. By using Proposition 2.1.2, we get

$$\begin{aligned}
&(n-1)^2 \operatorname{depth}(\mathbb{M}_{n,k-3}) + (n-1)\operatorname{depth}(\mathbb{M}_{n,k-1}) + 1 \\
&= (n-1)^2\left(\frac{(n-1)^2((n-1)^{k-3}-1)}{(n-1)^3-1}+1\right) + (n-1)\left(\frac{(n-1)^{(k-1)+2}-n+1}{(n-1)^3-1}\right)+1 \\
&= \frac{n(n-1)^{k+1}+n^5-6n^4+15n^3-21n^2+16n-6}{(n-1)^3-1} \\
&= \frac{n(n-1)^{k+1}+n^3-4n^2+4n-2}{(n-1)^3-1} + \frac{n^5-6n^4+14n^3-17n^2+12n-4}{(n-1)^3-1}
\end{aligned}$$

and

$$\begin{aligned}
\operatorname{depth}(\mathbb{M}_{n,k}) + (n-1)\operatorname{depth}(\mathbb{M}_{n,k-2}) &= \frac{(n-1)^2((n-1)^k-1)}{(n-1)^3-1}+1 \\
&\quad + (n-1)\left(\frac{(n-1)^{(k-2)+2}-1}{(n-1)^3-1}\right) \\
&= \frac{n(n-1)^{k+1}+n^3-4n^2+4n-2}{(n-1)^3-1}.
\end{aligned}$$

Note that $\frac{n^5-6n^4+14n^3-17n^2+12n-4}{(n-1)^3-1} > 0$, for all $n \geq 3$ by using MATLAB®. By Eq. 2.1.10, $\operatorname{depth}(R/I(T_{n,k})) \geq \frac{n(n-1)^{k+1}+n^3-4n^2+4n-2}{(n-1)^3-1}$. Again by Lemma 2.1.7 and induction on k, $\operatorname{depth}(R/I(T_{n,k})) \leq n(n-1)^{k-2} + \operatorname{depth}(K[V(T_{n,k-3})]/I(T_{n,k-3})) = \frac{n(n-1)^{(k-3)+1}+n^3-4n^2+4n-2}{(n-1)^3-1} + n(n-1)^{k-2} = \frac{n(n-1)^{k+1}+n^3-4n^2+4n-2}{(n-1)^3-1}$. This completes the proof for depth. Proof for Stanley depth is similar.

□

Corollary 2.1.9 ([80]). *Let $n \geq 3$ and $k \geq 1$. If $R := K[V(T_{n,k})]$, then*

$$\operatorname{pdim}(R/I(T_{n,k})) = \begin{cases} \frac{n(n-1)^k-2}{n-2} - \frac{n(n-1)^{k+1}+n^3-4n^2+4n-2}{(n-1)^3-1}, & if \quad k \equiv 0 \,(\text{mod } 3); \\[2ex] \frac{n(n-1)^k-2}{n-2} - \frac{n(n-1)^{k+1}-n^2+2n-2}{(n-1)^3-1}, & if \quad k \equiv 1 \,(\text{mod } 3); \\[2ex] \frac{n(n-1)^k-2}{n-2} - \frac{n(n-1)^{k+1}-n}{(n-1)^3-1}, & if \quad k \equiv 2 \,(\text{mod } 3). \end{cases}$$

Proof. By Auslander–Buchsbaum formula [11, Theorems 1.3.3] and Theorem 2.1.8, the required result follows. □

Example 2.1.10. Let $k \geq 1$, and $T_{3,k}$ be a graph as given in Figure 2.6. If $R = K[V(T_{3,k})]$, then

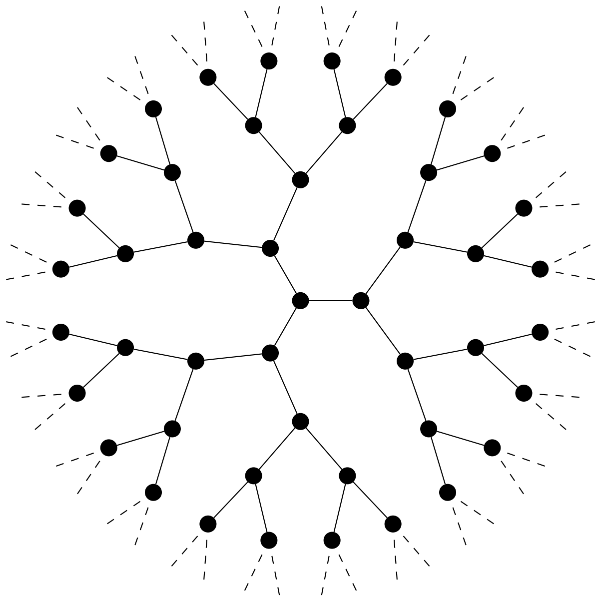


Figure 2.6: $T_{3,k}$

(a) $$\operatorname{depth}(R/I(T_{3,k})) = \operatorname{sdepth}(R/I(T_{3,k})) = \begin{cases} \frac{3\cdot2^{k+1}-1}{7}, & if \quad k \equiv 0 \,(\bmod 3); \\ \frac{3\cdot2^{k+1}-5}{7}, & if \quad k \equiv 1 \,(\bmod 3); \\ \frac{3\cdot2^{k+1}-3}{7}, & if \quad k \equiv 2 \,(\bmod 3). \end{cases}$$

(b) $$\operatorname{pdim}(R/I(T_{3,k})) = \begin{cases} \frac{15\cdot2^{k}-13}{7}, & if \quad k \equiv 0 \,(\bmod 3); \\ \frac{3(5.2^{k}-3)}{7}, & if \quad k \equiv 1 \,(\bmod 3); \\ \frac{15.2^{k}-11}{7}, & if \quad k \equiv 2 \,(\bmod 3). \end{cases}$$

Example 2.1.11. Let $k \geq 1$, and $T_{4,k}$ be a graph as given in Figure 2.7. If $R = K[V(T_{4,k})]$, then

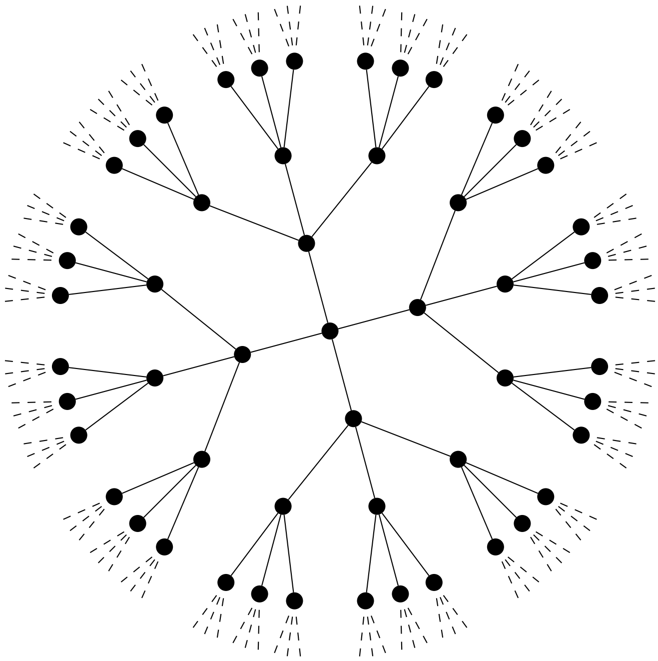


Figure 2.7: $T_{4,k}$

(a) $$\operatorname{depth}(R/I(T_{4,k})) = \operatorname{sdepth}(R/I(T_{4,k})) = \begin{cases} \frac{2.3^{k+1}+7}{13}, & if \quad k \equiv 0 \,(\bmod 3); \\ \frac{2.3^{k+1}-5}{13}, & if \quad k \equiv 1 \,(\bmod 3); \\ \frac{2.3^{k+1}-2}{13}, & if \quad k \equiv 2 \,(\bmod 3). \end{cases}$$

(b) $$\operatorname{pdim}(R/I(T_{4,k})) = \begin{cases} \frac{20(3^{k}-1)}{13}, & if \quad k \equiv 0 \,(\bmod 3); \\ \frac{4(5.3^{k}-2)}{13}, & if \quad k \equiv 1 \,(\bmod 3); \\ \frac{20.3^{k}-11}{13}, & if \quad k \equiv 2 \,(\bmod 3). \end{cases}$$

2.2 Regularity and krull dimension

In this section, we first compute regularity for cyclic module $\mathbb{M}_{n,k}$, after that we find out the regularity of $K[V(T_{n,k})]/I(T_{n,k})$. At the end, we compute the Krull dimension for $K[V(T_{n,k})]/I(T_{n,k})$.

Proposition 2.2.1 ([80]). *Let $n \geq 3$ and $k \geq 1$. Then*

$$\operatorname{reg}(\mathbb{M}_{n,k}) = \begin{cases} \frac{(n-1)^{k+2}-(n-1)^2}{(n-1)^3-1}, & if \quad k \equiv 0 \,(\text{mod } 3); \\ \frac{(n-1)^{k+2}-1}{(n-1)^3-1}, & if \quad k \equiv 1 \,(\text{mod } 3); \\ \frac{(n-1)^{k+2}-n+1}{(n-1)^3-1}, & if \quad k \equiv 2 \,(\text{mod } 3). \end{cases}$$

Proof. We will provide this result by induction on k. If $k=1$, then clearly $\operatorname{indmat}(T'_{n,1}) = 1$, therefore by Lemma 1.4.9, we get $\operatorname{reg}(\mathbb{M}_{n,1}) = 1$. Let $k \geq 2$ and $S = K[V(T'_{n,k})]$. As we have noticed in Proposition 2.1.2, $S/(I(T'_{n,k}) : x_1^{(0)}) \cong \mathbb{M}_{n,k-2}^{(n-1)^2} \otimes_K K[x_1^{(0)}]$ and $S/(I(T'_{n,k}), x_1^{(0)}) \cong \mathbb{M}_{n,k-1}^{n-1}$. By Lemmas 1.4.10 and 2.0.2 , we have

$$\operatorname{reg}(S/(I(T'_{n,k}) : x_1^{(0)})) = \operatorname{reg}(\mathbb{M}_{n,k-2}^{(n-1)^2} \otimes_K K[x_1^{(0)}]) = \operatorname{reg}(\mathbb{M}_{n,k-2}^{(n-1)^2}) = (n-1)^2 \operatorname{reg}(\mathbb{M}_{n,k-2}), \tag{2.2.1}$$

$$\operatorname{reg}(S/(I(T'_{n,k}), x_1^{(0)})) = \operatorname{reg}(\mathbb{M}_{n,k-1}^{n-1}) = (n-1)\operatorname{reg}(\mathbb{M}_{n,k-1}). \tag{2.2.2}$$

If $k=2$, $\operatorname{reg}(S/(I(T'_{n,2}) : x_1^{(0)})) = \operatorname{reg}(K[x_1^{(0)}, x_1^{(2)}, \ldots, x_{(n-1)^2}^{(2)}]) = 0$ and $\operatorname{reg}(S/(I(T'_{n,2}), x_1^{(0)})) = (n-1)\operatorname{reg}(\mathbb{M}_{n,1}) = n-1$. Hence by Lemma 1.4.12(c), we have $\operatorname{reg}(\mathbb{M}_{n,2}) = n-1$. For $k=3$, we have $\operatorname{reg}(S/(I(T'_{n,3}) : x_1^{(0)})) = (n-1)^2 \operatorname{reg}(\mathbb{M}_{n,1}) = (n-1)^2$ and $\operatorname{reg}(S/(I(T'_{n,3}), x_1^{(0)})) = (n-1)\operatorname{reg}(\mathbb{M}_{n,2}) = (n-1)(n-1) = (n-1)^2$. Therefore, by Lemma 1.4.12(b), we have $\operatorname{reg}(\mathbb{M}_{n,3}) \in \{(n-1)^2+1, (n-1)^2\}$. Let us consider the following subsets of $E(T'_{n,3})$.

$$F_1 = \bigcup_{j=1}^{n-1} \{\{x_1^{(2)}, x_j^{(3)}\}\},$$

$$F_2 = \bigcup_{j=(n-1)+1}^{2(n-1)} \{\{x_2^{(2)}, x_j^{(3)}\}\},$$

$$\vdots$$

$$F_{(n-1)^2} = \bigcup_{j=(n-2)(n-1)^2+(n-2)(n-1)+1}^{(n-1)^3} \{\{x_{(n-1)^2}^{(2)}, x_j^{(3)}\}\}.$$

Clearly, $|F_i| = n-1$, for all i. Also each edge of F_i for all i has one vertex in L'_2 and other vertex in L'_3. Let $F' := \{e_1, e_2, \ldots, e_{(n-1)^2} : e_i \in F_i\}$, it is easy to see that F' is an induced matching and $|F'| = (n-1)^2$. If F'' is any induced matching such that it contains an edge between vertices of L'_1 and L'_2 or an edge between vertices of L'_0 and L'_1, then F'' cannot take any edge from $F_1, \ldots, F_{n-1}$. Since, $|L'_3| = (n-1)^3$, $|L'_2| = (n-1)^2$ and $|L'_1| = n-1$, thus we get $|F''| < |F'|$.

Hence F' is a maximal induced matching and by Lemma 1.4.9, we have $\operatorname{reg}(\mathbb{M}_{n,3}) = (n-1)^2$. Thus result follows for $k = 3$. Let $k \geq 4$. We consider three cases:

Case 1 Let $k \equiv 1 (\bmod 3)$. In this case $k-2 \equiv 2 (\bmod 3)$ and $k-1 \equiv 0 (\bmod 3)$, by induction on k and using Eqs. 2.2.1 and 2.2.2 we get

$$\begin{aligned}\operatorname{reg}(S/(I(T'_{n,k}) : x_1^{(0)})) &= (n-1)^2\left(\frac{(n-1)^{(k-2)+2} - n + 1}{(n-1)^3 - 1}\right)\\ &= \frac{(n-1)^{k+2} - (n-1)^3}{(n-1)^3 - 1},\end{aligned}$$

$$\begin{aligned}\operatorname{reg}(S/(I(T'_{n,k}), x_1^{(0)})) &= (n-1)\left(\frac{(n-1)^{(k-1)+2} - (n-1)^2}{(n-1)^3 - 1}\right)\\ &= \frac{(n-1)^{k+2} - (n-1)^3}{(n-1)^3 - 1}.\end{aligned}$$

Since $\operatorname{reg}(S/(I(T'_{n,k}) : x_1^{(0)})) = \operatorname{reg}(S/(I(T'_{n,k}), x_1^{(0)}))$, by Lemma 1.4.12(b), we get $\operatorname{reg}(\mathbb{M}_{n,k}) \leq \operatorname{reg}(S/(I(T'_{n,k}), x_1^{(0)})) + 1 = \frac{(n-1)^{k+2}-1}{(n-1)^3-1}$. For the other inequality, let us define

$$F_{(k-1,k)} = \bigcup_{i=1}^{(n-1)^{k-1}} \{\{x_i^{(k-1)}, x_{(n-1)i}^{(k)}\}\}, \tag{2.2.3}$$

and we have $|F_{(k-1,k)}| = (n-1)^{k-1}$. Consider $F = F_{(k-1,k)} \cup F_{(k-4,k-3)} \cup \cdots \cup F_{(0,1)}$. It is easy to see that F is an induced matching. Therefore, $\operatorname{indmat}(T'_{n,k}) \geq |F|$, where $|F| = (n-1)^{k-1} + (n-1)^{k-4} + \cdots + (n-1)^3 + (n-1)^0 = \frac{(n-1)^{k+2}-1}{(n-1)^3-1}$. By Lemma 1.4.9, we have $\operatorname{reg}(\mathbb{M}_{n,k}) \geq \frac{(n-1)^{k+2}-1}{(n-1)^3-1}$. Therefore, we get the required result.

Case 2 Let $k \equiv 2 (\bmod 3)$. In this case $k-2 \equiv 0 (\bmod 3)$ and $k-1 \equiv 1 (\bmod 3)$. By Eqs. 2.2.1 and 2.2.2 and induction on k we get

$$\begin{aligned}\operatorname{reg}(S/(I(T'_{n,k}) : x_1^{(0)})) &= (n-1)^2\left(\frac{(n-1)^{(k-2)+2} - (n-1)^2}{(n-1)^3 - 1}\right)\\ &= \frac{(n-1)^{k+2} - (n-1)^4}{(n-1)^3 - 1},\end{aligned}$$

and

$$\begin{aligned}\operatorname{reg}(S/(I(T'_{n,k}), x_1^{(0)})) &= (n-1)\left(\frac{(n-1)^{(k-1)+2} - 1}{(n-1)^3 - 1}\right)\\ &= \frac{(n-1)^{k+2} - n + 1}{(n-1)^3 - 1}\\ &= \frac{(n-1)^{k+2} - (n-1)^4}{(n-1)^3 - 1} + (n-1).\end{aligned}$$

Hence by Lemma 1.4.12(c) we have $\operatorname{reg}(\mathbb{M}_{n,k}) = \frac{(n-1)^{k+2}-n+1}{(n-1)^3-1}$, as required.

Case 3 Let $k \equiv 0 \,(\text{mod } 3)$. If $T'_{n,k} = T'_{n,3} \cup \bigcup_{i=1}^{(n-1)^3} B_i$, where $B_i \cong T'_{n,k-3}$, $T'_{n,3} \cap B_i \neq \emptyset$ and $B_i \cap B_j = \emptyset$, for all $i \neq j$. By Lemmas 1.4.13 and 2.0.2, we have

$$\operatorname{reg}(\mathbb{M}_{n,k}) \leq \operatorname{reg}(\mathbb{M}_{n,3}) + \operatorname{reg}(\mathbb{M}_{n,k-3}^{(n-1)^3}) = \operatorname{reg}(\mathbb{M}_{n,3}) + (n-1)^3 \operatorname{reg}(\mathbb{M}_{n,k-3}).$$

In this case $k-3 \equiv 0 \,(\text{mod } 3)$ and by induction on k, we have

$$\begin{aligned}\operatorname{reg}(\mathbb{M}_{n,k}) &\leq (n-1)^2 + (n-1)^3 \left(\frac{(n-1)^{(k-3)+2} - (n-1)^2}{(n-1)^3 - 1} \right) \\ &= \frac{(n-1)^{k+2} - (n-1)^2}{(n-1)^3 - 1}.\end{aligned}$$

For the other inequality, we use Eq. 2.2.3 and define an induced matching $F = F_{(k-1,k)} \cup F_{(k-4,k-3)} \cup \cdots \cup F_{(2,3)}$. As, $\operatorname{indmat}(T'_{n,k}) \geq |F|$, where $|F| = (n-1)^{k-1} + (n-1)^{k-4} + \cdots + (n-1)^5 + (n-1)^2 = \frac{(n-1)^{k+2}-(n-1)^2}{(n-1)^3-1}$. By Proposition 1.4.9(a), we have $\operatorname{reg}(\mathbb{M}_{n,k}) \geq \frac{(n-1)^{k+2}-(n-1)^2}{(n-1)^3-1}$. Hence we get the required result.

□

Theorem 2.2.2 ([80]). *Let $n \geq 3$ and $k \geq 1$. If $R := K[V(T_{n,k})]$, then*

$$\operatorname{reg}(R/I(T_{n,k})) = \begin{cases} \frac{n(n-1)^{k+1}-n(n-1)}{(n-1)^3-1}, & if \quad k \equiv 0 \,(\text{mod } 3); \\ \frac{n(n-1)^{k+1}-(n-1)^2-1}{(n-1)^3-1}, & if \quad k \equiv 1 \,(\text{mod } 3); \\ \frac{n(n-1)^{k+1}-n}{(n-1)^3-1}, & if \quad k \equiv 2 \,(\text{mod } 3). \end{cases}$$

Proof. We will show the result by using Proposition 2.2.1. If $k = 1$, then clearly $\operatorname{indmat}(T_{n,1}) = 1$, therefore by Lemma 1.4.9, we have $\operatorname{reg}(R/I(T_{n,1})) = 1$. Let $k \geq 2$, by using Eqs. 2.1.7 and 2.1.8 we get $\operatorname{reg}(R/(I(T_{n,2}) : x_1^{(0)})) = \operatorname{reg}(K[x_1^{(0)}, x_1^{(2)}, \ldots, x_{n(n-1)}^{(2)}]) = 0$ and by using Lemma 1.4.14, we have $\operatorname{reg}(R/(I(T_{n,2}), x_1^{(0)})) = n \operatorname{reg}(\mathbb{M}_{n,1}) = n$. Hence by Lemma 1.4.12(c), we have $\operatorname{reg}(R/I(T_{n,2})) = n$. For $k = 3$, by using Lemma 1.4.14 on Eqs. 2.1.7 and 2.1.8, we have

$$\operatorname{reg}(R/(I(T_{n,3}) : x_1^{(0)})) = n(n-1) \operatorname{reg}(\mathbb{M}_{n,1}) = n(n-1)$$

and

$$\operatorname{reg}(R/(I(T_{n,3}), x_1^{(0)})) = n \operatorname{reg}(\mathbb{M}_{n,2}) = n(n-1).$$

Thus, by Lemma 1.4.12(b), we get $\operatorname{reg}(R/I(T_{n,3})) \in \{n(n-1)+1, n(n-1)\}$. Let us consider the following subsets of $E(T_{n,3})$

$$G_1 = \bigcup_{j=1}^{n-1} \{\{x_1^{(2)}, x_j^{(3)}\}\},$$

$$G_2 = \bigcup_{j=(n-1)+1}^{2(n-1)} \{\{x_2^{(2)}, x_j^{(3)}\}\},$$

$$\vdots$$

$$G_{n(n-1)} = \bigcup_{j=(n-1)^2+(n-2)n(n-1)+1}^{n(n-1)^2} \{\{x_{n(n-1)}^{(2)}, x_j^{(3)}\}\}.$$

Clearly, $|G_i| = n-1$, for all i. Also each edge of G_i for all i has one vertex in L_2 and other vertex in L_3. Let $G' := \{e_1, e_2, \dots, e_{n(n-1)} : e_i \in G_i\}$, it is easy to observe that G' is an induced matching and $|G'| = n(n-1)$. If G'' is any induced matching such that it contains an edge between vertices of L_1 and L_2 or an edge between vertices of L_0 and L_1, then G'' cannot take any edge from $G_1, \dots, G_{n-1}$. Since, $|L_3| = n(n-1)^2$, $|L_2| = n(n-1)$ and $|L_1| = n$, thus we get $|G''| < |G'|$. Hence G' is a maximal induced matching and by Lemma 1.4.9, we have $\operatorname{reg}(S/I(T_{n,3})) = n(n-1)$. Thus we get the required result for $k=3$. Let $k \geq 4$. Since $R/(I(T_{n,k}) : x_1^{(0)}) \cong \mathbb{M}_{n,k-2}^{n(n-1)} \otimes_K K[x_1^{(0)}]$ and $R/(I(T_{n,k}), x_1^{(0)}) \cong \mathbb{M}_{n,k-1}^{n}$. By Lemma 1.4.14 and 2.0.2, we have

$$\operatorname{reg}(R/(I(T_{n,k}) : x_1^{(0)})) = n(n-1)\operatorname{reg}(\mathbb{M}_{n,k-2}), \tag{2.2.4}$$

$$\operatorname{reg}(R/(I(T_{n,k}), x_1^{(0)})) = n\operatorname{reg}(\mathbb{M}_{n,k-1}). \tag{2.2.5}$$

We consider the following cases:

Case 1 Let $k \equiv 1 \pmod 3$. In this case $k-1 \equiv 0 \pmod 3$, $k-2 \equiv 2 \pmod 3$, by Eqs. 2.2.4 and 2.2.5 and using Proposition 2.2.1, we get

$$\begin{aligned}\operatorname{reg}(R/(I(T_{n,k}) : x_1^{(0)})) &= n(n-1)\left(\frac{(n-1)^{(k-2)+2} - n + 1}{(n-1)^3 - 1}\right)\\ &= \frac{n(n-1)^{k+1} - n(n-1)^2}{(n-1)^3 - 1},\end{aligned}$$

$$\begin{aligned}\operatorname{reg}(R/(I(T_{n,k}), x_1^{(0)})) &= n\left(\frac{(n-1)^{(k-1)+2} - (n-1)^2}{(n-1)^3 - 1}\right)\\ &= \frac{n(n-1)^{k+1} - n(n-1)^2}{(n-1)^3 - 1}.\end{aligned}$$

Since $\operatorname{reg}(R/(I(T_{n,k}) : x_1^{(0)})) = \operatorname{reg}(R/(I(T_{n,k}), x_1^{(0)}))$, by Lemma 1.4.12(b), $\operatorname{reg}(R/I(T_{n,k})) \leq \operatorname{reg}(R/(I(T_{n,k}) : x_1^{(0)})) + 1 = \frac{n(n-1)^{k+1} - n(n-1)^2 + (n-1)^3 - 1}{(n-1)^3 - 1} = \frac{n(n-1)^{k+1} - (n-1)^2 - 1}{(n-1)^3 - 1}$. For the other inequality, let us define

$$F_{(k-1,k)} = \bigcup_{i=1}^{n(n-1)^{k-2}} \{\{x_i^{(k-1)}, x_{(n-1)i}^{(k)}\}\}, \tag{2.2.6}$$

where $|F| = n(n-1)^{k-2}$. Consider $F = F_{(k-1,k)} \cup F_{(k-4,k-3)} \cup \cdots \cup F_{(0,1)}$ be an induced matching. Therefore, $\text{indmat}(T_{n,k}) \geq |F|$, where $|F| = n(n-1)^{k-2} + n(n-1)^{k-5} + \cdots + n(n-1)^2 + 1 = \frac{n(n-1)^{k+1}-(n-1)^2-1}{(n-1)^3-1}$. By Lemma 1.4.9, $\text{reg}(R/I(T_{n,k})) \geq \frac{n(n-1)^{k+1}-(n-1)^2-1}{(n-1)^3-1}$. Thus $\text{reg}(R/I(T_{n,k})) = \frac{n(n-1)^{k+1}-(n-1)^2-1}{(n-1)^3-1}$, as required.

Case 2 Let $k \equiv 2 \pmod 3$. In this case $k-1 \equiv 1 \pmod 3$, $k-2 \equiv 0 \pmod 3$, by using Proposition 2.2.1 and Eqs. 2.2.4 and 2.2.5 we get

$$\begin{aligned}\text{reg}(R/(I(T_{n,k}) : x_1^{(0)})) &= n(n-1)\left(\frac{(n-1)^{(k-2)+2}-(n-1)^2}{(n-1)^3-1}\right)\\ &= \frac{n(n-1)^{k+1}-n(n-1)^3}{(n-1)^3-1},\end{aligned}$$

and

$$\text{reg}(R/(I(T_{n,k}), x_1^{(0)})) = n\left(\frac{(n-1)^{(k-1)+2}-1}{(n-1)^3-1}\right) = \frac{n(n-1)^{k+1}-n(n-1)^3}{(n-1)^3-1} + n.$$

Thus by Lemma 1.4.12(c), $\text{reg}(R/I(T_{n,k})) = \frac{n(n-1)^{k+1}-n}{(n-1)^3-1}$, as required.

Case 3 Let $k \equiv 0 \pmod 3$. Here $T_{n,k} = T_{n,3} \bigcup_{i=1}^{n(n-1)^2} B_i$, where $B_i \cong T'_{n,k-3}$, $B_i \cap B_j = \emptyset$ and $T_{n,3} \cap B_i \neq \emptyset$ for all $i \neq j$. By Lemmas 1.4.13 and 2.0.2, we have

$$\begin{aligned}\text{reg}(R/I(T_{n,k})) &\leq \text{reg}(K[V(T_{n,3})]/I(T_{n,3})) + \text{reg}(\mathbb{M}_{n,k-3}^{n(n-1)^2})\\ &= \text{reg}(K[V(T_{n,3})]/I(T_{n,3})) + n(n-1)^2\,\text{reg}(\mathbb{M}_{n,k-3}).\end{aligned}$$

In this case $k-3 \equiv 0 \pmod 3$. As $\text{reg}(R/I(T_{n,3})) = n(n-1)$, by Proposition 2.2.1 we get

$$\begin{aligned}\text{reg}(R/I(T_{n,k})) &\leq n(n-1) + n(n-1)^2\left(\frac{(n-1)^{(k-3)+2}-(n-1)^2}{(n-1)^3-1}\right)\\ &= \frac{n(n-1)^{k+1}-n(n-1)}{(n-1)^3-1}.\end{aligned}$$

For the other inequality, we use Eq. 2.2.6 and define $F = F_{(k-1,k)} \cup F_{(k-4,k-3)} \cup \cdots \cup F_{(2,3)}$. It is easy to see that F is an induced matching. Therefore, $\text{indmat}(T_{n,k}) \geq |F|$, where $|F| = n(n-1)^{k-2} + n(n-1)^{k-5} + \cdots + n(n-1)^4 + n(n-1) = \frac{n(n-1)^{k+1}-n(n-1)}{(n-1)^3-1}$. By Lemma 1.4.9, we have $\text{reg}(R/I(T_{n,k})) \geq \frac{n(n-1)^{k+1}-n(n-1)}{(n-1)^3-1}$. This completes the proof.

□

Example 2.2.3. *Let* $k \geq 1$.

(a)

$$\operatorname{reg}(\mathbb{M}_{3,k}) = \begin{cases} \frac{2^{k+2}-4}{7}, & if \quad k \equiv 0 \,(\mathrm{mod}\; 3); \\ \frac{2^{k+2}-1}{7} & if \quad k \equiv 1 \,(\mathrm{mod}\; 3); \\ \frac{2^{k+2}-2}{7} & if \quad k \equiv 2 \,(\mathrm{mod}\; 3). \end{cases}$$

(b) If $R := K[V(T_{3,k})]$, then

$$\operatorname{reg}(R/I(T_{3,k})) = \begin{cases} \frac{3.2^{k+1}-6}{7} & if \quad k \equiv 0 \,(\mathrm{mod}\; 3); \\ \frac{3.2^{k+1}-5}{7} & if \quad k \equiv 1 \,(\mathrm{mod}\; 3); \\ \frac{3.2^{k+1}-3}{7} & if \quad k \equiv 2 \,(\mathrm{mod}\; 3). \end{cases}$$

Example 2.2.4. *Let $k \geq 1$.*

(a)

$$\operatorname{reg}(\mathbb{M}_{4,k}) = \begin{cases} \frac{9(3^k-1)}{26}, & if \quad k \equiv 0 \,(\mathrm{mod}\; 3); \\ \frac{3^{k+2}-1}{26}, & if \quad k \equiv 1 \,(\mathrm{mod}\; 3); \\ \frac{3(3^{k+1}-1)}{26}, & if \quad k \equiv 2 \,(\mathrm{mod}\; 3). \end{cases}$$

(b) If $R := K[V(T_{4,k})]$, then

$$\operatorname{reg}(R/I(T_{4,k})) = \begin{cases} \frac{6(3^k-1)}{13}, & if \quad k \equiv 0 \,(\mathrm{mod}\; 3); \\ \frac{2\cdot 3^{k+1}-5}{13}, & if \quad k \equiv 1 \,(\mathrm{mod}\; 3); \\ \frac{2(3^{k+1}-1)}{13}, & if \quad k \equiv 2 \,(\mathrm{mod}\; 3). \end{cases}$$

Lemma 2.2.5 ([80])**.** *Let $n \geq 3$ and $k \geq 1$. If $W \subset V(T_{n,k})$ be an independent set, then W is maximal iff $W \supseteq L_q$, for all q with $k \equiv q \,(\mathrm{mod}\, 2)$.*

Proof. By definition of $T_{n,k}$, $E(T_{n,k}) \cap \{\{u,v\} : u \in L_q, v \in L_{q-1}\} \neq \emptyset$ for all $1 \leq q \leq k$. Also, $|L_0| = 1$, $|L_1| = n$ and for $q \geq 2$, $|L_q| = (n-1)|L_{q-1}|$. Thus $W \subset V(T_{n,k})$ is a maximal independent set, if and only if $W = L_k \cup L_{k-2} \cup \ldots, \cup L_{k-2\lceil \frac{k-1}{2} \rceil}$, that is, if and only if $W \supseteq L_q$, for all q with $k \equiv q \,(\mathrm{mod}\, 2)$. □

Theorem 2.2.6 ([80]). *Let $n \geq 3$ and $k \geq 1$. If $R = K[V(T_{n,k})]$, then* $\dim(R/I(T_{n,k})) = \frac{(n-1)^{k+1}-1}{n-2}$.

Proof. By Lemma 1.2.1, we know that $\dim(R/I(T_{n,k})) = |W|$, where W is a maximal independent set. Now by Lemma 2.2.5, if W is a maximal independent set then

$$W = \begin{cases} L_k \cup L_{k-2} \cup \cdots \cup L_0, & if\ k \text{ is even;} \\ L_k \cup L_{k-2} \cup \cdots \cup L_1, & if\ k \text{ is odd.} \end{cases}$$

Thus

$$|W| = \begin{cases} n(n-1)^{k-1} + n(n-1)^{k-3} + \cdots + n(n-1) + 1, & if\ k \text{ is even;} \\ n(n-1)^{k-1} + n(n-1)^{k-3} + \cdots + n(n-1)^2 + n, & if\ k \text{ is odd.} \end{cases}$$

If k is odd, then

$$n + n(n-1)^2 + \cdots + n(n-1)^{k-5} + n(n-1)^{k-3} + n(n-1)^{k-1} = \frac{(n-1)^{k+1}-1}{n-2},$$

and if k is even, then

$$1 + n(n-1) + \cdots + n(n-1)^{k-5} + n(n-1)^{k-3} + n(n-1)^{k-1} = \frac{(n-1)^{k+1}-1}{n-2}.$$

□

Example 2.2.7. Let $k \geq 1$. Then

$$\dim(K[V(T_{3,k})]/I(T_{3,k})) = 2^{k+1} - 1,$$

and

$$\dim(K[V(T_{4,k})]/I(T_{4,k})) = \frac{3^{k+1}-1}{2}.$$

2.3 Conclusion

This chapter aims to find the precise formulas for the values of the algebraic invariants depth, Stanley depth, regularity, projective dimension and Krull dimension of $K[V(T_{n,k})]/I(T_{n,k})$. It is worth mentioning that for computations of the said invariants for $K[V(T_{n,k})]/I(T_{n,k})$ the module $K[V(T'_{n,k})]/I(T'_{n,k})$ plays a vital role. It is easy to see that $T_{2,k} = P_{2k+1}$ and $T_{n,1} = S_n$, thus our results [80] also complement the previous work on depth, regularity, projective dimension and Stanley depth of $K[V(P_n)]/I(P_n)$ and $K[V(S_{n-1})]/I(S_{n-1})$; see [34, 36, 55, 60]. One can use our results to compute algebraic invariants for other classes of graphs; see for instance [79] as an application of our work.

CHAPTER 3

Invariants of edge ideals of cubic circulant graphs

There are two sections in this chapter. In the first part, the precise values of depth, projective dimension, and Stanley depth of the edge ideals associated with certain supergraphs of ladder graph are given; see for instance Lemma 3.1.5, 3.1.14 and 3.1.17. In the second section, we give values for depth and projective dimension, and lower bounds for Stanley depth of the quotient rings of the edge ideals of all cubic circulant graphs, see Theorem 3.2.11, Corollary 3.2.12 and Theorem 3.2.13. We also give a result in Lemma 3.1.1 that plays a vital role in the computation of depth and a lower bound for Stanley depth in our main findings.

3.1 Invariants of cyclic modules associated to some supergraphs of ladder graph

For $n \geq 2$, the graph A_n as shown in Figure 3.1 is called ladder graph on $2n$ vertices. We introduce some supergraphs of ladder graph namely B_n, Q_n and D_n that play a significant role in our main results. It will be convenient to label the vertices of the aforementioned graphs as shown in Figure 3.1 and Figure 3.2. The vertex sets and edge sets of these graphs are:

- $V(A_n) = \bigcup\limits_{i=1}^{n} \{x_i, y_i\}$, $E(A_n) = \bigcup\limits_{i=1}^{n-1} \{\{x_i, y_i\}, \{x_i, x_{i+1}\}, \{y_i, y_{i+1}\}\} \cup \{x_n, y_n\}$,
- $V(B_n) = V(A_n) \cup \{y_{n+1}\}$, $E(B_n) = E(A_n) \cup \{y_n, y_{n+1}\}$,
- $V(Q_n) = V(A_n) \cup \{y_{n+1}, y_{n+2}\}$, $E(Q_n) = E(A_n) \cup \{\{y_n, y_{n+1}\}, \{y_1, y_{n+2}\}\}$,

- $V(D_n) = V(A_n) \cup \{x_{n+1}, y_{n+1}\}$, $E(D_n) = E(A_n) \cup \{\{y_n, y_{n+1}\}, \{x_1, x_{n+1}\}\}$.

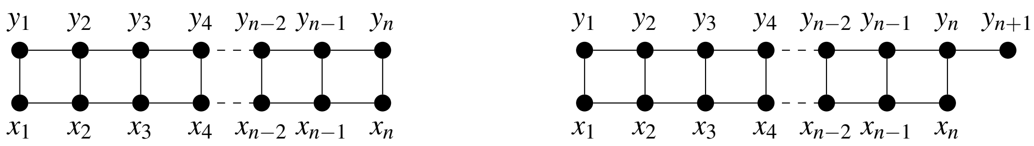


Figure 3.1: From left to right A_n and B_n

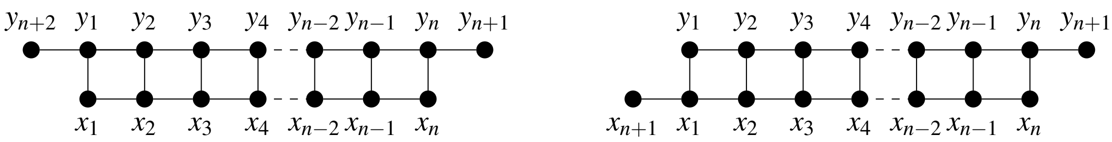


Figure 3.2: From left to right Q_n and D_n

In this section, we compute the values of depth, projective dimension and Stanley depth of the cyclic modules $K[V(B_n)]/I(B_n)$, $K[V(Q_n)]/I(Q_n)$ and $K[V(D_n)]/I(D_n)$. These values play a vital role in our main results in last section. For cyclic module $K[V(A_n)]/I(A_n)$, we give exact value of Stanley depth when $n \equiv 1(\bmod 2)$ and find sharp bounds when $n \equiv 0(\bmod 2)$. First, we prove the following lemma that will help in computing depth and lower bound for Stanley depth, throughout the work. This lemma is inspired by result of Cimpoeas in [53, Proposition 1.3].

Lemma 3.1.1 ([81])**.** *Let G be a connected graph with $V(G) = \{x_1, \dots, x_n\}$, where $n \geq 2$. If $N_G(x_i) = \{x_{i_1}, \dots, x_{i_l}\}$, then*

$$(I(G) : x_i)/I(G) \cong \bigoplus_{t=1}^{l} S_t/J_t[x_{i_t}],$$

where $S_1 = K[V(G) \backslash N_G(x_{i_1})]$ for $t \geq 2$, $S_t = K[V(G) \backslash (N_G(x_{i_t}) \cup \{x_{i_1}, x_{i_2}, \dots, x_{i_{t-1}}\})]$ and for $t \geq 1$, $J_t = (S_t \cap I(G))$.

Proof. If $u \in (I(G) : x_i)$ is a monomial such that $u \notin I(G)$, then it follows that u is divisible by at least one variable from $N_G(x_i) = \{x_{i_1}, \dots, x_{i_l}\}$. Indeed, if u is not divisible by any of the variables from the set of $N_G(x_i)$ then $u \in I(G)$, a contradiction. Without loss of generality we may assume that $x_{i_1}|u$ then $u = x_{i_1}^{\alpha_1} v_1$ with $\alpha_1 \geq 1$. Since $u \notin I(G)$, it follows that $v_1 \in S_1 = K[V(G) \backslash N_G(x_{i_1})]$ and $v_1 \notin J_1 = (S_1 \cap I(G))$. Thus $u \in x_{i_1}(S_1/J_1)[x_{i_1}]$. Now, if $x_{i_2}|u$ and $x_{i_1} \nmid u$, then $u = x_{i_2}^{\alpha_2} v_2$ with $\alpha_2 \geq 1$. It follows that $v_2 \in S_2 = K[V(G) \backslash (N_G(x_{i_2}) \cup \{x_{i_1}\})]$ and

$v_2 \notin J_2 = (S_2 \cap I(G))$. Thus $u \in x_{i_2}(S_2/J_2)[x_{i_2}]$. In a similar manner, for $3 \leq t \leq l$, if $x_{i_t}|u$ and $x_{i_1} \nmid u, x_{i_2} \nmid u, \ldots, x_{i_{t-1}} \nmid u$ then $u = x_{i_t}^{\alpha_t} v_t$ with $\alpha_t \geq 1$. Since $u \notin I(G)$, it follows that $v_t \in S_t = K[V(G) \backslash (N_G(x_{i_t}) \cup \{x_{i_1}, x_{i_2}, \ldots, x_{i_{t-1}}\})]$ and $v_t \notin J_t = (S_t \cap I(G))$. Thus $u \in x_{i_t}(S_t/J_t)[x_{i_t}]$ and we have the following S-module isomorphism

$$(I(G) : x_i)/I(G) \cong \bigoplus_{t=1}^{l} x_{i_t}(S_t/J_t)[x_{i_t}].$$

It is easy to see that x_{i_t} is regular on $S_t/J_t[x_{i_t}]$, therefore we have $x_{i_t}(S_t/J_t)[x_{i_t}] \cong (S_t/J_t)[x_{i_t}]$. This completes the proof. □

Remark 3.1.2 ([81])**.** If $n \leq 1$, then we define the quotient rings associated to A_n, B_n, Q_n and D_n appearing in the proofs of this section as follows:

- $K[V(B_0)]/I(B_0) = K[x]$ and $\text{depth}(K[x]) = \text{sdepth}(K[x]) = 1$.
- $K[V(A_1)]/I(A_1) = K[V(P_2)]/I(P_2)$ and by using Lemma 1.4.15, $\text{depth}(K[V(P_2)]/I(P_2)) = \text{sdepth}(K[V(P_2)]/I(P_2)) = 1$.
- $K[V(B_1)]/I(B_1) = K[V(P_3)]/I(P_3)$, by using Lemma 1.4.15, $\text{depth}(K[V(P_3)]/I(P_3)) = \text{sdepth}(K[V(P_3)]/I(P_3)) = 1$.
- $K[V(Q_1)]/I(Q_1) = K[V(S_4)]/I(S_4)$ and by using Lemma 1.3.16, $\text{depth}(K[V(S_4)]/I(S_4)) = \text{sdepth}(K[V(S_4)]/I(S_4)) = 1$.
- $K[V(D_1)]/I(D_1) = K[V(P_4)]/I(P_4)$ and by using Lemma 1.4.15, $\text{depth}(K[V(P_4)]/I(P_4)) = \text{sdepth}(K[V(P_4)]/I(P_4)) = 2$.

Remark 3.1.3 ([81])**.** Let $x_t, x_l \in S$ such that $x_t, x_l \notin I$. Then $(I : x_t)$, (I, x_t), $((I, x_t), x_l)$ and $((I, x_t) : x_l)$ are the monomial ideals of S such that $G_{(I:x_t)}$, $G_{(I,x_t)}$, $G_{((I,x_t),x_l)}$ and $G_{((I,x_t):x_l)}$ are subgraphs of G_I. By using the labeling of Figure 3.2, see for instance; Figure 3.3 and 3.4 as examples of $G_{(I(Q_7):x_7)}, G_{(I(Q_7),x_7)}, G_{((I(Q_7),x_7),y_8)}$ and $G_{((I(Q_7),x_7):y_8)}$. From Figures 3.3 and 3.4, after suitable renumbering of variables we have the following isomorphisms:

$$K[V(Q_7)]/(I(Q_7) : x_7) \cong K[V(Q_5)]/I(Q_5) \otimes_K K[x_7, y_8],$$

$$K[V(Q_7)]/(I(Q_7), x_7) \cong K[V(Q_6)]/(I(Q_6), y_7 y_8),$$

$$K[V(Q_7)]/((I(Q_7), x_7), y_8) \cong K[V(Q_6)]/I(Q_6),$$

$$K[V(Q_7)]/((I(Q_7), x_7) : y_8) \cong K[V(B_6)]/I(B_6) \otimes_K K[y_8].$$

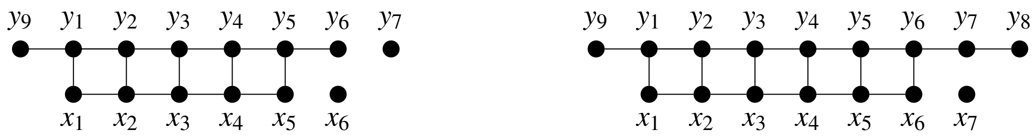


Figure 3.3: From left to right $G_{(I(Q_7):x_7)}$ and $G_{(I(Q_7),x_7)}$.

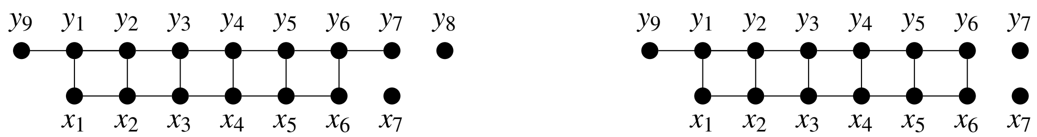


Figure 3.4: From left to right $G_{((I(Q_7),x_7),y_8)}$ and $G_{((I(Q_7),x_7):y_8)}$.

Lemma 3.1.4 (Depth Lemma)**.** *([11, Proposition 1.2.9]) If* $0 \to P \to Q \to R \to 0$ *is a short exact sequence of modules over a local ring S, then*

(a) $\operatorname{depth}(Q) \geq \min\{\operatorname{depth}(R), \operatorname{depth}(P)\}$.

(b) $\operatorname{depth}(P) \geq \min\{\operatorname{depth}(Q), \operatorname{depth}(R)+1\}$.

(c) $\operatorname{depth}(R) \geq \min\{\operatorname{depth}(P)-1, \operatorname{depth}(Q)\}$.

Lemma 3.1.5 ([81])**.** *Let* $n \geq 2$ *and* $S := K[V(B_n)]$. *Then* $\operatorname{depth}(S/I(B_n)) = \operatorname{sdepth}(S/I(B_n)) = \lceil \frac{n+1}{2} \rceil$.

Proof. Firstly, we will first provide the result for depth. Consider the short exact sequence

$$0 \longrightarrow S/(I(B_n):y_n) \xrightarrow{\cdot y_n} S/I(B_n) \longrightarrow S/(I(B_n),y_n) \longrightarrow 0.$$

After a suitable numbering of variables we have the subsequent isomorphisms

$$S/(I(B_n):y_n) \cong K[V(B_{n-2})]/(I(B_{n-2})) \otimes_K K[y_n], \tag{3.1.1}$$

$$S/(I(B_n),y_n) \cong K[V(B_{n-1})]/(I(B_{n-1})) \otimes_K K[y_{n+1}]. \tag{3.1.2}$$

If $n = 2$, then by using Lemma 1.4.10 and Remark 3.1.2, we get $\operatorname{depth}(S/(I(B_2):y_2)) = \operatorname{depth}(K[V(B_0)]/I(B_0))+1 = \operatorname{depth}(K[x_1])+1 = 2$ and

$$\operatorname{depth}(S/(I(B_2),y_2)) = \operatorname{depth}(K[V(B_1)]/I(B_1))+1 = \operatorname{depth}(K[V(P_3)]/I(P_3))+1 = 2.$$

By Lemma 1.3.13, $\operatorname{depth}(K[V(B_2)]/I(B_2)) = 2$. Similarly, the desired result follows when $n = 3$. Let $n \geq 4$. By induction on n and applying Lemma 1.4.10 on Eqs. (3.1.1) and (3.1.2), it follows that

$$\operatorname{depth}(S/(I(B_n):y_n)) = \operatorname{depth}(K[V(B_{n-2})]/I(B_{n-2})) + 1 = \lceil \frac{n-2+1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil,$$

$$\operatorname{depth}(S/(I(B_n), y_n)) = \operatorname{depth}(K[V(B_{n-1})]/I(B_{n-1})) + 1 = \lceil \frac{n-1+1}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil + 1.$$

Clearly, $\operatorname{depth}(S/(I(B_n), y_n)) \geq \operatorname{depth}(S/(I(B_n):y_n))$, therefore by Lemma 1.3.13, we get the required result. For Stanley depth, the proof is similar to depth with the use of Lemma 1.3.14 in place of Lemma 1.3.13. □

Corollary 3.1.6 ([81])**.** *Let $n \geq 2$ and $S := K[V(B_n)]$. Then* $\operatorname{pdim}(S/I(B_n)) = 2n - \lceil \frac{n+1}{2} \rceil + 1$.

Proof. The proof follows by Lemma 1.4.11 and Lemma 3.1.5. □

Example 3.1.7. For $n = 50$, we have

(a) $\operatorname{depth}(K[V(B_{50})]/I(B_{50})) = \operatorname{sdepth}(K[V(B_{50})]/I(B_{50})) = \lceil \frac{50+1}{2} \rceil = 26$.

(b) $\operatorname{pdim}(K[V(B_{50})]/I(B_{50})) = 100 - \lceil \frac{50+1}{2} \rceil + 1 = 73$.

The projective dimension of $K[V(A_n)]/I(A_n)$ has been computed in [78].

Theorem 3.1.8 ([78, Theorem 5.5])**.** *If $n \geq 2$, then* $\operatorname{pdim}(K[V(A_n)]/I(A_n)) = \lfloor \frac{3n}{2} \rfloor$.

Consequently, one can compute its depth by using Auslander–Buchsbaum formula.

Corollary 3.1.9 ([81])**.** *If $n \geq 2$, then* $\operatorname{depth}(K[V(A_n)]/I(A_n)) = 2n - \lfloor \frac{3n}{2} \rfloor = \lceil \frac{n}{2} \rceil$.

Proof. By using Lemma 1.4.11 and Theorem 3.1.8, one can get the required result. □

Here we give an alternative proof for depth by using Lemma 3.1.1 we include this proof because proof for Stanley depth is analogous. In Remark 3.1.10 we explain a situations that arises in special cases in upcoming proofs.

Remark 3.1.10 ([81])**.** Let $i \in \mathbb{Z}^+$, if $k < i$ then we consider $\cup_i^k \{x_i y_i, x_i x_{i+1}, y_i y_{i+1}\} = \emptyset$. Also we take $x_a y_b = 0$, whenever a or b is not positive.

Theorem 3.1.11 ([81])**.** *If $n \geq 2$ and $S = K[V(A_n)]$, then* $\operatorname{sdepth}(S/I(A_n)) \geq \operatorname{depth}(S/I(A_n)) = \lceil \frac{n}{2} \rceil$.

Proof. Firstly, we provide the proof for depth. If $n = 2$, then by the usage of Lemma 1.3.18, $\text{depth}(K[V(A_2)]/I(A_2)) = 1$. Let $n \geq 3$. Consider the short exact sequence

$$0 \longrightarrow (I(A_n) : y_n)/I(A_n) \xrightarrow{\cdot y_n} S/I(A_n) \longrightarrow S/(I(A_n) : y_n) \longrightarrow 0. \tag{3.1.3}$$

We have the below S-module isomorphism

$$S/(I(A_n) : y_n) \cong K[V(B_{n-2})]/I(B_{n-2}) \otimes_K K[y_n]. \tag{3.1.4}$$

Here $N_{A_n}(y_n) = \{y_{n-1}, x_n\}$, $S_1 = K[V(A_n) \backslash N_{A_n}(y_{n-1})]$, $S_2 = K[V(A_n) \backslash (N_{A_n}(x_n) \cup \{y_{n-1}\})]$, $J_1 = (S_1 \cap I(A_n))$, $J_2 = (S_2 \cap I(A_n))$, then by using Lemma 3.1.1, we have

$$\begin{aligned}(I(A_n) : y_n)/I(A_n) &\cong S_1/J_1[y_{n-1}] \oplus S_2/J_2[x_n] \\ &\cong \frac{K[x_1, \ldots, x_{n-2}, x_n, y_1, \ldots, y_{n-3}]}{\left(\cup_{i=1}^{n-4}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} \cup \{x_{n-3}y_{n-3}, x_{n-3}x_{n-2}\}\right)}[y_{n-1}] \\ &\quad \oplus \frac{K[x_1, \ldots, x_{n-2}, y_1, \ldots, y_{n-2}]}{\left(\cup_{i=1}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} \cup \{x_{n-2}y_{n-2}\}\right)}[x_n] \\ &\cong K[V(B_{n-3})]/I(B_{n-3}) \otimes_K K[x_n, y_{n-1}] \oplus K[V(A_{n-2})]/I(A_{n-2}) \otimes_K K[x_n].\end{aligned} \tag{3.1.5}$$

If $n = 3$, then we have

$$K[V(A_3)]/(I(A_3) : y_3) \cong \frac{K[x_1, x_2, y_1]}{(y_1x_1, x_1x_2)}[y_3] \cong K[V(B_1)]/I(B_1) \otimes_K K[y_3], \tag{3.1.6}$$

$$\begin{aligned}(I(A_3) : y_3)/I(A_3) &\cong \frac{K[x_1, x_3]}{(0)}[y_2] \oplus \frac{K[x_1, y_1]}{(x_1y_1)}[x_3] \\ &\cong K[x_1, x_3, y_2] \oplus K[V(A_1)]/I(A_1) \otimes_K K[x_3].\end{aligned} \tag{3.1.7}$$

By applying Lemma 1.4.10 on Eqs. (3.1.6) and (3.1.7) and using Remark 3.1.2 we have

$$\text{depth}(K[V(A_3)]/(I(A_3) : y_3)) = \text{depth}(K[V(B_1)]/I(B_1)) + 1 = \text{depth}(K[V(P_3)]/I(P_3)) + 1 = 2$$

and

$$\begin{aligned}\text{depth}((I(A_3) : y_3)/I(A_3)) &= \min\{\text{depth}(K[x_1, x_3, y_2]), \text{depth}(K[V(A_1)]/I(A_1)) + 1)\} \\ &= \min\{3, \text{depth}(K[V(P_2)]/I(P_2)) + 1)\} \\ &= \min\{3, 2\} = 2.\end{aligned}$$

Thus by applying Depth Lemma on Eq. (3.1.3), we get $\text{depth}(K[V(A_3)]/I(A_3)) = 2$. If $n = 4$, as stated in Remark 3.1.10, we have $\cup_{i=1}^{n-4}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} = \emptyset$ and using the similar strategy and induction on n, one can get the required result. Let $n \geq 5$. By applying Lemma 1.4.10 and Lemma 3.1.5 on Eq. (3.1.4), we get

$$\operatorname{depth}(S/(I(A_n):y_n)) = \operatorname{depth}(K[V(B_{n-2})]/I(B_{n-2}))+1 = \lceil\frac{n-2+1}{2}\rceil+1 = \lceil\frac{n+1}{2}\rceil.$$

By using Eq. (3.1.5) and applying Lemma 1.4.10, Lemma 3.1.5 and induction on n, we have

$$\begin{aligned}\operatorname{depth}((I(A_n):y_n)/I(A_n)) = \min\Big\{&\operatorname{depth}(K[V(B_{n-3})]/I(B_{n-3}))+\operatorname{depth}(K[x_n,y_{n-1}]),\\ &\operatorname{depth}(K[V(A_{n-2})]/I(A_{n-2}))+\operatorname{depth}(K[x_n])\Big\}\\ = \min\Big\{&\lceil\frac{n-3+1}{2}\rceil+2,\lceil\frac{n-2}{2}\rceil+1\Big\} = \lceil\frac{n}{2}\rceil.\end{aligned}$$

We get the desired result by using Depth Lemma on Eq. (3.1.3). Now we provide the proof for Stanley depth. If $n=2$, then by Lemma 1.3.18, we have $\operatorname{sdepth}(K[V(A_2)]/I(A_2)) \geq 1$. If $n=3$, then by using Lemma 1.4.10 and Remark 3.1.2 on Eqs. (3.1.6) and (3.1.7)

$$\operatorname{sdepth}(K[V(A_3)]/(I(A_3):y_3)) = \operatorname{sdepth}(K[V(B_1)]/I(B_1))+1 = \operatorname{sdepth}(K[V(P_3)]/I(P_3))+1 = 2$$

and

$$\begin{aligned}\operatorname{sdepth}((I(A_3):y_3)/I(A_3)) &\geq \min\{\operatorname{sdepth}(K[x_1,x_2,y_2]),\operatorname{sdepth}(K[V(A_1)]/I(A_1))+1)\}\\ &\geq \min\{3,\operatorname{sdepth}(K[V(P_2)]/I(P_2))+1)\}\\ &\geq \min\{3,1+1\} = 2.\end{aligned}$$

Thus by applying Lemma 1.3.10 on Eq. (3.1.3), we get $\operatorname{sdepth}(K[V(A_3)]/I(A_3)) \geq 2$. For $n \geq 4$, we get the required lower bound for Stanley depth by using the similar arguments just by using Lemma 1.3.10 in place of Depth Lemma on the exact sequence (3.1.3). □

Corollary 3.1.12 ([81])**.** *Let $n \geq 2$ and $S = K[V(A_n)]$. If $n \equiv 0 \pmod 2$, then* $\operatorname{sdepth}(S/I(A_n)) \in \{\lceil\frac{n}{2}\rceil,\lceil\frac{n+1}{2}\rceil\}$, *otherwise we have* $\operatorname{sdepth}(S/I(A_n)) = \lceil\frac{n}{2}\rceil$.

Proof. If $n=2$, then one can easily see that the result follows by Theorem 3.1.11 and Lemma 1.3.18. If $n \geq 3$, then by Theorem 3.1.11, we only need to show that $\operatorname{sdepth}(S/I(A_n)) \leq \lceil\frac{n+1}{2}\rceil$. For $y_n \notin I(A_n)$, and by using Lemma 1.3.12, we have $\operatorname{sdepth}(S/I(A_n)) \leq \operatorname{sdepth}(S/(I(A_n):y_n))$. By applying Lemma 1.4.10 and Lemma 3.1.5 on Eq. (3.1.4), $\operatorname{sdepth}(S/(I(A_n):y_n) = \lceil\frac{n-2+1}{2}\rceil+1 = \lceil\frac{n+1}{2}\rceil$ and the desired result follows. □

Example 3.1.13. If $n=101$, then

$$\operatorname{sdepth}(K[V(A_{101})]/I(A_{101})) = \operatorname{depth}(K[V(A_{101})]/I(A_{101})) = 50.$$

If $n=102$, then $\operatorname{depth}(K[V(A_{102})]/I(A_{102})) = 51$ and $51 \leq \operatorname{sdepth}(K[V(A_{102})]/I(A_{102})) \leq 52$.

Lemma 3.1.14 ([81])**.** *Let $n \geq 2$ and $S = K[V(Q_n)]$. Then*

$$\operatorname{depth}(S/I(Q_n)) = \operatorname{sdepth}(S/I(Q_n)) = \begin{cases} \lceil \frac{n}{2} \rceil + 1, & \text{if } n \equiv 0,3 \pmod 4; \\ \lceil \frac{n+1}{2} \rceil, & \text{if } n \equiv 1 \pmod 4; \\ \lceil \frac{n+1}{2} \rceil + 1, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. First, we provide the proof for depth. If $n = 2$, then consider the short exact sequence

$$0 \longrightarrow S/(I(Q_2) : y_2) \xrightarrow{\cdot y_2} S/I(Q_2) \longrightarrow S/(I(Q_2), y_2) \longrightarrow 0. \tag{3.1.8}$$

Here we have $S/(I(Q_2) : y_2) \cong K[x_1, y_2, y_4]$ and $S/(I(Q_2), y_2) \cong K[y_3] \otimes_K K[V(P_4)]/I(P_4)$. By Lemma 1.4.10 and Lemma 1.4.15, we have $\operatorname{depth}(S/(I(Q_2) : y_2)) = \operatorname{depth}(K[x_1, y_2, y_4]) = 3$ and $\operatorname{depth}(S/(I(Q_2), y_2)) = \operatorname{depth}(K[y_3]) + \operatorname{depth}(K[V(P_4)]/I(P_4)) = 3$. By applying Lemma 1.3.13 on Eq. (3.1.8), $\operatorname{depth}(S/(I(Q_2)) = 3 = \lceil \frac{2+1}{2} \rceil + 1$. Let $n \geq 3$ and consider the subsequent short exact sequences

$$0 \longrightarrow S/(I(Q_n) : x_n) \xrightarrow{\cdot x_n} S/I(Q_n) \longrightarrow S/(I(Q_n), x_n) \longrightarrow 0,$$

$$0 \longrightarrow S/\big((I(Q_n), x_n) : y_{n+1}\big) \xrightarrow{\cdot y_{n+1}} S/(I(Q_n), x_n) \longrightarrow S/\big((I(Q_n), x_n), y_{n+1}\big) \longrightarrow 0,$$

and by Depth Lemma

$$\operatorname{depth}(S/I(Q_n) \geq \min\big\{\operatorname{depth}\big(S/(I(Q_n) : x_n)\big), \operatorname{depth}\big(S/(I(Q_n), x_n)\big)\big\}, \tag{3.1.9}$$

$$\operatorname{depth}(S/(I(Q_n), x_n)) \geq \min\big\{\operatorname{depth}\big(S/((I(Q_n), x_n) : y_{n+1})\big), \operatorname{depth}\big(S/((I(Q_n), x_n), y_{n+1})\big)\big\}. \tag{3.1.10}$$

After a suitable numbering of variables, we have the following K-algebra isomorphisms:

$$S/(I(Q_n) : x_n) \cong K[V(Q_{n-2})]/I(Q_{n-2}) \otimes_K K[x_n, y_{n+1}]. \tag{3.1.11}$$

$$S/\big((I(Q_n), x_n), y_{n+1}\big) \cong K[V(Q_{n-1})]/I(Q_{n-1}), \tag{3.1.12}$$

$$S/\big((I(Q_n), x_n) : y_{n+1}\big) \cong K[V(B_{n-1})]/I(B_{n-1}) \otimes_K K[y_{n+1}]. \tag{3.1.13}$$

If $n = 3$, we have by Eq. (3.1.11),

$$S/(I(Q_3) : x_3) \cong K[V(Q_1)]/I(Q_1) \otimes_K K[x_3, y_4]. \tag{3.1.14}$$

By Lemma 1.4.10 and Remark 3.1.2, we have $\operatorname{depth}(S/(I(Q_3) : x_3)) = \operatorname{depth}(K[V(Q_1)]/I(Q_1)) + 2 = \operatorname{depth}(K[V(S_4)]/I(S_4)) + 2 = 3$. By using Eqs. (3.1.12) and (3.1.13), $S/\big((I(Q_3), x_3), y_4\big) \cong K[V(Q_2)]/I(Q_2)$ and $S/\big((I(Q_3), x_3) : y_4\big) \cong K[V(B_2)]/I(B_2) \otimes_K K[y_4]$. With the use of induction on n, $\operatorname{depth}\big(S/((I(Q_3), x_3), y_4)\big) = \operatorname{depth}(K[V(Q_2)]/I(Q_2)) = 3$ and by Lemma 1.4.10 and

Lemma 3.1.5, we get $\operatorname{depth}\big(S/((I(Q_3),x_3):y_4)\big)=\operatorname{depth}(K[V(B_2)]/I(B_2))+1=3$. By Eq. (3.1.10), $\operatorname{depth}(S/(I(Q_3),x_3))\geq 3$ and by Eq. (3.1.9), $\operatorname{depth}(S/(I(Q_3))\geq 3$. For the upper bound, we use Lemma 1.3.12 and Eq. (3.1.14), $\operatorname{depth}(S/(I(Q_3))\leq \operatorname{depth}\big(S/(I(Q_3):x_3)\big)=3$. Thus we get $\operatorname{depth}(S/(I(Q_3))=\lceil\frac{3}{2}\rceil+1=3$. If $n=4$, by using similar strategy as we did when $n=3$, we get the required lower bound that is $\operatorname{depth}(S/I(Q_4))\geq\lceil\frac{4}{2}\rceil+1=3$. For the upper bound, since $x_3y_4\notin I(Q_4)$, we have

$$S/(I(Q_4):x_3y_4)\cong K[V(S_4)]/I(S_4)\otimes_K K[x_3,y_4],$$

and by Lemma 1.4.10 and Lemma 1.3.16, $\operatorname{depth}(S/(I(Q_4):x_3y_4))=\operatorname{depth}(K[V(S_4)]/I(S_4))+2=3$. Therefore, by Lemma 1.3.12, $\operatorname{depth}(S/I(Q_4))\leq\operatorname{depth}(S/(I(Q_4):x_3y_4))=3$. Let $n\geq 5$. We consider the subsequent cases:

Case 1 Let $n\equiv 1(\mathrm{mod}\,4)$. We consider the short exact sequence

$$0\longrightarrow S/(I(Q_n):y_{n+1})\xrightarrow{\cdot y_{n+1}} S/I(Q_n)\longrightarrow S/(I(Q_n),y_{n+1})\longrightarrow 0, \tag{3.1.15}$$

Here $S/(I(Q_n),y_{n+1})\cong K[V(B_n)]/I(B_n)$. By Lemma 3.1.5, $\operatorname{depth}(S/(I(Q_n),y_{n+1}))=\lceil\frac{n+1}{2}\rceil$. Consider another short exact sequence

$$\begin{aligned}0\longrightarrow S/\big((I(Q_n):y_{n+1}):x_n\big)\xrightarrow{\cdot x_n} S/(I(Q_n):y_{n+1})\longrightarrow\\ S/\big((I(Q_n):y_{n+1}),x_n\big)\longrightarrow 0.\end{aligned} \tag{3.1.16}$$

Here $S/\big((I(Q_n):y_{n+1}),x_n\big)\cong K[V(B_{n-1})]/I(B_{n-1})\otimes_K K[y_{n+1}]$. By Lemma 1.4.10 and Lemma 3.1.5, it implies $\operatorname{depth}\big(S/((I(Q_n):y_{n+1}),x_n)\big)=\lceil\frac{n-1+1}{2}\rceil+1=\lceil\frac{n}{2}\rceil+1$. Also we have $S/\big((I(Q_n):y_{n+1}):x_n\big)\cong K[V(Q_{n-2})]/I(Q_{n-2})\otimes_K K[y_{n+1},x_n]$. Since $n-2\equiv 3(\mathrm{mod}\,4)$, by induction on n and Lemma 1.4.10, $\operatorname{depth}(S/((I(Q_n):y_{n+1}):x_n))=\lceil\frac{n-2}{2}\rceil+1+2=\lceil\frac{n}{2}\rceil+2$. By applying Depth Lemma on Eqs. (3.1.15) and (3.1.16)

$$\operatorname{depth}(S/(I(Q_n))\geq\min\big\{\operatorname{depth}(S/(I(Q_n):y_{n+1})),\operatorname{depth}(S/(I(Q_n),y_{n+1}))\big\}, \tag{3.1.17}$$

$$\begin{aligned}\operatorname{depth}(S/(I(Q_n):y_{n+1}))\geq\min\big\{&\operatorname{depth}\big(S/((I(Q_n):y_{n+1}):x_n)\big),\\ &\operatorname{depth}\big(S/((I(Q_n):y_{n+1}),x_n)\big)\big\},\end{aligned} \tag{3.1.18}$$

By using Eq. (3.1.18), $\operatorname{depth}(S/(I(Q_n):y_{n+1}))\geq\lceil\frac{n}{2}\rceil+1$. Since, $\operatorname{depth}(S/(I(Q_n):y_{n+1}))>\operatorname{depth}(S/(I(Q_n),y_{n+1}))$, by Eq. (3.1.17) we obtain the $\operatorname{depth}(S/(I(Q_n))\geq\lceil\frac{n+1}{2}\rceil$. For the other inequality, we have $x_{n-2}y_n\notin I(Q_n)$, and the following K-algebra isomorphism:

$$S/(I(Q_n):x_{n-2}y_n)\cong K[V(Q_{n-4})]/I(Q_{n-4})\otimes_K K[x_{n-2},y_n].$$

Since $n-4\equiv 1(\mathrm{mod}\,4)$, by Remark 3.1.2, Lemma 1.3.12, Lemma 1.4.10 and induction on n, $\operatorname{depth}(S/I(Q_n))\leq\operatorname{depth}(S/(I(Q_n):x_{n-2}y_n))=\lceil\frac{n-4+1}{2}\rceil+2=\lceil\frac{n+1}{2}\rceil$, as required.

Case 2 Let $n \equiv 2(\text{mod}\, 4)$. Consider the following short exact sequences:

$$0 \longrightarrow S/(I(Q_n) : x_{n-1}) \xrightarrow{\cdot x_{n-1}} S/I(Q_n) \longrightarrow S/(I(Q_n), x_{n-1}) \longrightarrow 0,$$

$$0 \longrightarrow S/\big((I(Q_n), x_{n-1}) : y_{n-1}\big) \xrightarrow{\cdot y_{n-1}} S/(I(Q_n), x_{n-1}) \longrightarrow S/\big((I(Q_n), x_{n-1}), y_{n-1}\big) \longrightarrow 0,$$

$$0 \longrightarrow S/\big(((I(Q_n), x_{n-1}) : y_{n-1}) : x_{n-2}\big) \xrightarrow{\cdot x_{n-2}} S/\big((I(Q_n), x_{n-1}) : y_{n-1}\big) \longrightarrow S/\big(((I(Q_n), x_{n-1}) : y_{n-1}), x_{n-2}\big) \longrightarrow 0.$$

We have the subsequent K-algebra isomorphisms:

$$S/(I(Q_n) : x_{n-1}) \cong K[V(Q_{n-3})]/I(Q_{n-3}) \otimes_K K[x_{n-1}] \otimes_K K[V(P_2)]/I(P_2),$$

$$S/\big((I(Q_n), x_{n-1}), y_{n-1}\big) \cong K[V(B_{n-2})]/I(B_{n-2}) \otimes_K K[V(P_3)]/I(P_3),$$

$$S/\big(((I(Q_n), x_{n-1}) : y_{n-1}) : x_{n-2}\big) \cong K[V(Q_{n-4})]/I(Q_{n-4}) \otimes_K K[x_{n-2}, y_{n-1}, x_n, y_{n+1}],$$

$$S/\big(((I(Q_n), x_{n-1}) : y_{n-1}), x_{n-2}\big) \cong K[V(B_{n-3})]/I(B_{n-3}) \otimes_K K[y_{n-1}, x_n, y_{n+1}].$$

Since $n-3 \equiv 3(\text{mod}\, 4)$, by using induction on n, Lemma 1.3.11 and Lemma 1.4.15, we have $\text{depth}(S/I(Q_n) : x_{n-1}) = \text{depth}(K[V(Q_{n-3})]/I(Q_{n-3})) + \text{depth}(K[V(P_2)]/I(P_2)) + 1 = \lceil \frac{n-3}{2} \rceil + 3 = \lceil \frac{n+1}{2} \rceil + 1$. By using Lemma 1.4.15, Lemma 3.1.5 and Lemma 1.3.11, $\text{depth}(S/((I(Q_n), x_{n-1}), y_{n-1})) = \text{depth}(K[V(B_{n-2})]/I(B_{n-2})) + \text{depth}(K[V(P_3)]/I(P_3)) = \lceil \frac{n-2+1}{2} \rceil + 2 = \lceil \frac{n+1}{2} \rceil + 1$. Since $n-4 \equiv 2(\text{mod}\, 4)$ and $n-3 \equiv 3(\text{mod}\, 4)$. By induction on n and Lemma 1.3.12, we have

$$\begin{aligned}\text{depth}\left(S/\big(((I(Q_n), x_{n-1}) : y_{n-1}) : x_{n-2}\big)\right) &= \text{depth}(K[V(Q_{n-4})]/I(Q_{n-4})) + 4 \\ &= \lceil \frac{n-4+1}{2} \rceil + 1 + 4 = \lceil \frac{n+3}{2} \rceil + 2,\end{aligned}$$

and by Lemma 3.1.5,

$$\begin{aligned}\text{depth}\left(S/\big(((I(Q_n), x_{n-1}) : y_{n-1}), x_{n-2}\big)\right) &= \text{depth}(K[V(B_{n-3})]/I(B_{n-3})) + 3 \\ &= \lceil \frac{n-3+1}{2} \rceil + 3 = \lceil \frac{n+1}{2} \rceil + 1.\end{aligned}$$

By Depth Lemma on short exact sequences

$$\text{depth}(S/I(Q_n) \geq \min\left\{\text{depth}(S/(I(Q_n) : x_{n-1})), \text{depth}(S/(I(Q_n), x_{n-1}))\right\}. \tag{3.1.19}$$

$$\text{depth}(S/(I(Q_n), x_{n-1})) \geq \min\left\{\text{depth}\left(S/((I(Q_n), x_{n-1}) : y_{n-1})\right), \text{depth}\left(S/((I(Q_n), x_{n-1}), y_{n-1})\right)\right\}. \tag{3.1.20}$$

$$\operatorname{depth}\big(S/((I(Q_n),x_{n-1}):y_{n-1})\big) \geq \min\Big\{\operatorname{depth}\big(S/(((I(Q_n),x_{n-1}):y_{n-1}):x_{n-2})\big), \operatorname{depth}\big(S/(((I(Q_n),x_{n-1}):y_{n-1}),x_{n-2})\big)\Big\}. \tag{3.1.21}$$

Clearly $\lceil\frac{n+1}{2}\rceil+1 < \lceil\frac{n+3}{2}\rceil+2$, by Eq. (3.1.21) we get $\operatorname{depth}(S/((I(Q_n),x_{n-1}):y_{n-1})) \geq \lceil\frac{n+1}{2}\rceil+1$. By Eq. (3.1.20), we have $\operatorname{depth}(S/(I(Q_n),x_{n-1})) \geq \lceil\frac{n+1}{2}\rceil+1$ and by Eq. (3.1.19), we get $\operatorname{depth}(S/I(Q_n) \geq \lceil\frac{n+1}{2}\rceil+1$. For the other inequality, we have $x_n \notin I(Q_n)$. Since $n-2 \equiv 0(\bmod 4)$, by using induction on n, Lemma 1.3.12 and Lemma 1.4.10 on Eq. (3.1.11), we get $\operatorname{depth}(S/I(Q_n)) \leq \operatorname{depth}(S/(I(Q_n):x_n)) = \lceil\frac{n-2}{2}\rceil+1+2 = \lceil\frac{n}{2}\rceil+2 = \lceil\frac{n+1}{2}\rceil+1$.

Case 3 If $n \equiv 3(\bmod 4)$. Since $n-2 \equiv 1(\bmod 4)$ and $n-1 \equiv 2(\bmod 4)$, by induction on n and Lemma 1.4.10 on Eq. (3.1.11), we get

$$\begin{aligned}\operatorname{depth}(S/(I(Q_n):x_n)) &= \operatorname{depth}(K[V(Q_{n-2})]/I(Q_{n-2}))+2\\ &= \lceil\frac{n-2+1}{2}\rceil+2 = \lceil\frac{n+1}{2}\rceil+1 = \lceil\frac{n}{2}\rceil+1.\end{aligned} \tag{3.1.22}$$

By Eq. (3.1.12), we get $\operatorname{depth}\big(S/((I(Q_n),x_n),y_{n+1})\big) = \operatorname{depth}(K[V(Q_{n-1})]/I(Q_{n-1})) = \lceil\frac{n-1+1}{2}\rceil+1 = \lceil\frac{n}{2}\rceil+1$ and by Lemma 1.4.10 and Lemma 3.1.5 on Eq. (3.1.13), we get $\operatorname{depth}\big(S/((I(Q_n),x_n):y_{n+1})\big) = \operatorname{depth}(K[V(B_{n-1})]/I(B_{n-1}))+1 = \lceil\frac{n}{2}\rceil+1$. Since we have $\operatorname{depth}\big(S/((I(Q_n),x_n),y_{n+1})\big) = \operatorname{depth}\big(S/((I(Q_n),x_n):y_{n+1})\big)$, therefore by Eq. (3.1.10), $\operatorname{depth}(S/(I(Q_n),x_n)) \geq \lceil\frac{n}{2}\rceil+1$. Also $\operatorname{depth}(S/(I(Q_n),x_n)) = \operatorname{depth}(S/(I(Q_n):x_n))$, by Eq. (3.1.9) we get $\operatorname{depth}(S/(I(Q_n)) \geq \lceil\frac{n}{2}\rceil+1$. For the upper bound, we use Lemma 1.3.12 and Eq. (3.1.22), that is $\operatorname{depth}(S/(I(Q_n)) \leq \operatorname{depth}(S/(I(Q_n):x_n)) = \lceil\frac{n}{2}\rceil+1$, the required result.

Case 4 Let $n \equiv 0(\bmod 4)$. In this case $n-2 \equiv 2(\bmod 4)$, by induction on n and Lemma 1.4.10 on Eq. (3.1.11), we get

$$\operatorname{depth}(S/(I(Q_n):x_n)) = \operatorname{depth}(K[V(Q_{n-2})]/I(Q_{n-2}))+2 = \lceil\frac{n-2+1}{2}\rceil+3 = \lceil\frac{n+5}{2}\rceil.$$

As $n-1 \equiv 3(\bmod 4)$, by induction on n and applying Lemma 1.4.10 on Eq. (3.1.12),

$$\operatorname{depth}\big(S/((I(Q_n),x_n),y_{n+1})\big) = \operatorname{depth}(K[V(Q_{n-1})]/I(Q_{n-1})) = \lceil\frac{n-1}{2}\rceil+1 = \lceil\frac{n}{2}\rceil+1.$$

By Lemma 1.4.10 and Lemma 3.1.5 on Eq. (3.1.13), we have

$$\begin{aligned}\operatorname{depth}\big(S/((I(Q_n),x_n):y_{n+1})\big) &= \operatorname{depth}(K[V(B_{n-1})]/I(B_{n-1}))+1\\ &= \lceil\frac{n-1+1}{2}\rceil+1 = \lceil\frac{n}{2}\rceil+1.\end{aligned}$$

By using Eq. (3.1.10), we get $\operatorname{depth}(S/(I(Q_n),x_n)) \geq \operatorname{depth}(S/(I(Q_n),x_n) = \lceil \frac{n}{2} \rceil + 1$. Here we have $\operatorname{depth}(S/(I(Q_n),x_n)) \geq \operatorname{depth}(S/(I(Q_n):x_n))$, thus by Eq. (3.1.9), we get $\operatorname{depth}(S/(I(Q_n)) \geq \lceil \frac{n}{2} \rceil + 1$. For the other inequality, $x_{n-1}y_n \notin I(Q_n)$ and consider

$$S/(I(Q_n):x_{n-1}y_n) \cong K[V(Q_{n-3})]/I(Q_{n-3}) \otimes_K K[x_{n-1},y_n]. \tag{3.1.23}$$

Since $n-3 \equiv 1(\bmod 4)$, by using induction on n, Lemma 1.3.12 and Lemma 1.4.10 on Eq. (3.1.23), $\operatorname{depth}(S/I(Q_n)) \leq \operatorname{depth}(S/(I(Q_n):x_{n-1}y_n)) = \operatorname{depth}(K[V(Q_{n-3})]/I(Q_{n-3})) + 2 = \lceil \frac{n-3+1}{2} \rceil + 2 = \lceil \frac{n}{2} \rceil + 1$.

This ends the proof for depth. Proof for Stanley depth is also similar to depth just by replacing Depth Lemma by Lemma 1.3.10. Also by using Lemma 1.3.14 in place of Lemma 1.3.13. □

Corollary 3.1.15 ([81]). *Let $n \geq 2$ and $S = K[V(Q_n)]$. Then*

$$\operatorname{pdim}(S/I(Q_n)) = \begin{cases} 2n - \lceil \frac{n}{2} \rceil + 1, & \text{if } n \equiv 0,3 \pmod 4; \\ 2n - \lceil \frac{n+1}{2} \rceil + 2, & \text{if } n \equiv 1 \pmod 4; \\ 2n - \lceil \frac{n+1}{2} \rceil + 1, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

Proof. By using Lemma 1.4.11 and Lemma 3.1.14, the result follows. □

Example 3.1.16. *For $n = 103$, we have*

(a) $\operatorname{depth}(K[V(Q_{103})])/I(Q_{103}) = \lceil \frac{103}{2} \rceil + 1 = 53$.

(b) $\operatorname{pdim}(K[V(Q_{103})])/I(Q_{103}) = 206 - \lceil \frac{103}{2} \rceil + 1 = 155$.

Lemma 3.1.17 ([81]). *Let $n \geq 2$ and $S = K[V(D_n)]$. Then*

$$\operatorname{depth}(S/I(D_n)) = \operatorname{sdepth}(S/I(D_n)) = \begin{cases} \lceil \frac{n+1}{2} \rceil + 1, & \text{if } n \equiv 0,1 \pmod 4; \\ \lceil \frac{n+1}{2} \rceil, & \text{if } n \equiv 2,3 \pmod 4. \end{cases}$$

Proof. The following cases are considered:

Case 1 Let $n \equiv 2(\bmod 4)$. Consider the short exact sequence

$$0 \longrightarrow S/(I(D_n):x_{n+1}) \xrightarrow{\cdot x_{n+1}} S/I(D_n) \longrightarrow S/(I(D_n),x_{n+1}) \longrightarrow 0. \tag{3.1.24}$$

We have

$$S/(I(D_n):x_{n+1}) \cong K[V(Q_{n-1})]/I(Q_{n-1}) \otimes_K K[x_{n+1}], \tag{3.1.25}$$

$$S/(I(D_n),x_{n+1}) \cong K[V(B_n)]/I(B_n), \tag{3.1.26}$$

Since $n-1 \equiv 1(\text{mod}\,4)$. By using Lemma 1.4.10, Lemma 3.1.14 and Remark 3.1.2

$$\text{depth}\,(S/(I(D_n):x_{n+1})) = \text{depth}\,(K[V(Q_{n-1})]/I(Q_{n-1}))+1 = \lceil \frac{n+2}{2} \rceil,$$

and by Lemma 3.1.5 and Lemma 1.3.11

$$\text{depth}\,(S/(I(D_n),x_{n+1})) = \text{depth}\,(K[V(B_n)]/I(B_n)) = \lceil \frac{n+1}{2} \rceil. \tag{3.1.27}$$

Here $\lceil \frac{n+1}{2} \rceil = \lceil \frac{n+2}{2} \rceil$, therefore by Lemma 1.3.13, we get the required result. The proof for Stanley depth is the same just by using Lemma 1.3.14 in place of Lemma 1.3.13.

Case 2 Let $n \equiv 3(\text{mod}\,4)$. If we consider Eq. (3.1.24), then by Depth Lemma

$$\text{depth}(S/I(D_n)) \geq \min\{\text{depth}(S/(I(D_n):x_{n+1})), \text{depth}(S/(I(D_n),x_{n+1})\}.$$

In this case $n-1 \equiv 2(\text{mod}\,4)$, thus by using Eq. (3.1.25) and applying Lemma 1.4.10 and Lemma 3.1.14, we get

$$\text{depth}\,(S/(I(D_n):x_{n+1})) = \text{depth}\,(K[V(Q_{n-1})]/I(Q_{n-1}))+1 = \lceil \frac{n}{2} \rceil + 2 = \lceil \frac{n+2}{2} \rceil + 1.$$

Thus by using Eq. (3.1.27) and Depth Lemma, we obtain the $\text{depth}(S/I(D_n)) \geq \lceil \frac{n+1}{2} \rceil$. For the other inequality, since $y_n \notin I(D_n)$, after suitable numbering of the variables, we have the K- algebra isomorphism:

$$S/(I(D_n):y_n) \cong K[V(Q_{n-2})]/I(Q_{n-2}) \otimes_K K[y_n]. \tag{3.1.28}$$

Since $n-2 \equiv 1(\text{mod}\,4)$, by applying Lemma 1.3.12, Lemma 1.4.10, Lemma 3.1.14 and Remark 3.1.2 on Eq. (3.1.28), we have that $\text{depth}(S/I(D_n)) \leq \text{depth}(S/(I(D_n):y_n)) = \text{depth}\,(K[V(Q_{n-2})]/I(Q_{n-2})) + \text{depth}\,(K[y_n]) = \lceil \frac{n-2+1}{2} \rceil + 1 = \lceil \frac{n+1}{2} \rceil$. For Stanley depth, by using a similar strategy for depth, the required result is obtained.

Case 3 Let $n \equiv 0(\text{mod}\,4)$. We consider the short exact sequence

$$0 \longrightarrow S/(I(D_n):y_n) \xrightarrow{\cdot y_n} S/I(D_n) \longrightarrow S/(I(D_n),y_n) \longrightarrow 0,$$

After renumbering the variables, we have

$$S/(I(D_n),y_n) \cong K[V(Q_{n-1})]/I(Q_{n-1}) \otimes_K K[y_{n+1}]. \tag{3.1.29}$$

Since $n-1 \equiv 3(\mathrm{mod}\,4)$ and $n-2 \equiv 2(\mathrm{mod}\,4)$. By using Lemma 1.4.10, Lemma 3.1.14 on Eqs. (3.1.28) and (3.1.29), we get

$$\operatorname{depth}(S/(I(D_n):y_n)) = \operatorname{depth}\left(K[V(Q_{n-2})]/I(Q_{n-2})\right) + \operatorname{depth}\left(K[y_n]\right) = \lceil \frac{n+1}{2} \rceil + 1,$$

and

$$\operatorname{depth}(S/(I(D_n),y_n)) = \operatorname{depth}\left(K[V(Q_{n-1})]/I(Q_{n-1})\right) + \operatorname{depth}\left(K[y_{n+1}]\right) = \lceil \frac{n+1}{2} \rceil + 1.$$

We have $\operatorname{depth}(S/(I(D_n),y_n)) = \operatorname{depth}(S/(I(D_n):y_n))$, thus by using Lemma 1.3.13, $\operatorname{depth}(S/I(D_n)) = \lceil \frac{n+1}{2} \rceil + 1$. The proof for Stanley depth is same as depth just by using Lemma 1.3.14 in place of Lemma 1.3.13.

Case 4 Let $n \equiv 1(\mathrm{mod}\,4)$. In this case $n-1 \equiv 0(\mathrm{mod}\,4)$ and $n-2 \equiv 3(\mathrm{mod}\,4)$. The proof is similar to Case 3.

This completes the proof. □

Corollary 3.1.18 ([81])**.** *Let $n \geq 2$ and $S = K[V(D_n)]$. Then*

$$\operatorname{pdim}(S/I(D_n)) = \begin{cases} 2n - \lceil \frac{n+1}{2} \rceil + 1, & \text{if } n \equiv 0,1 \,(\mathrm{mod}\ 4); \\ 2n - \lceil \frac{n+1}{2} \rceil + 2, & \text{if } n \equiv 2,3 \,(\mathrm{mod}\ 4). \end{cases}$$

Proof. One can get the required result by using Lemma 1.4.11 and Lemma 3.1.17. □

Example 3.1.19. *For $n = 104$, we have*

(a) $\operatorname{depth}(K[V(D_{104})])/I(D_{104}) = \lceil \frac{104+1}{2} \rceil + 1 = 53.$

(b) $\operatorname{pdim}(K[V(D_{104})])/I(D_{104}) = 208 - \lceil \frac{104+1}{2} \rceil + 1 = 156.$

3.2 Invariants of cyclic modules associated to cubic circulant graphs

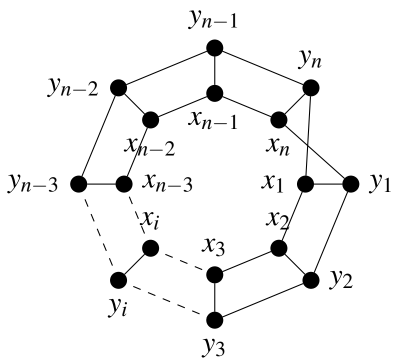
All the cubic circulant graphs has the form $C_{2n}(a,n)$ with integers $1 \leq a \leq n$. Davis and Domke proved the following result:

Theorem 3.2.1 ([12])**.** *Let $1 \leq a < n$ and $t = \gcd(2n,a)$.*

(a) If $\frac{2n}{t}$ is even, then $C_{2n}(a,n)$ is isomorphic to t copies of $C_{\frac{2n}{t}}(1,\frac{n}{t})$.

(b) If $\frac{2n}{t}$ is odd, then $C_{2n}(a,n)$ is isomorphic to $\frac{t}{2}$ copies of $C_{\frac{4n}{t}}(2,\frac{2n}{t})$.

Therefore, the only connected cubic circulant graphs are those circulant graphs that are isomorphic to either $C_{2n}(1,n)$ for $n \geq 2$ or to $C_{2n}(2,n)$ with n is odd and $n \geq 3$ (for the second circulant graph, if n is not odd, then Theorem 3.2.1 implies that this circulant is not connected). See Figure 3.5 for $C_{2n}(1,n)$ and $C_{2n}(2,n)$.



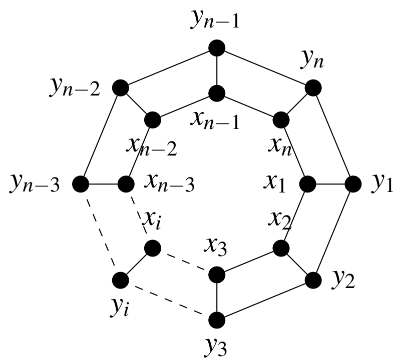


Figure 3.5: From left to right $C_{2n}(1,n)$ and $C_{2n}(2,n)$.

By using Lemma 1.3.11 and Theorem 3.2.1, it suffices to find the depth, projective dimension and lower bound for Stanley depth of the quotient rings of the edge ideals of $C_{2n}(1,n)$ and $C_{2n}(2,n)$ with n odd. Therefore, in this section, we first find the values of depth and projective dimension of cyclic modules $K[V(C_{2n}(1,n)]/I(C_{2n}(1,n))$ and $K[V(C_{2n}(2,n))/I(C_{2n}(2,n))$. We give values and bounds for Stanley depth of such modules. At the end, we compute the values of depth, projective dimension and lower bounds for Stanley depth of all cubic circulant graphs.

The following example will be helpful in understanding the proofs of this section. Using Figure 3.6, it is easy to see that we have the following isomorphism:

$$K[V(C_{16}(1,8))]/(I(C_{16}(1,8)):x_8) \cong K[V(D_5)]/I(D_5) \otimes_K K[x_8].$$

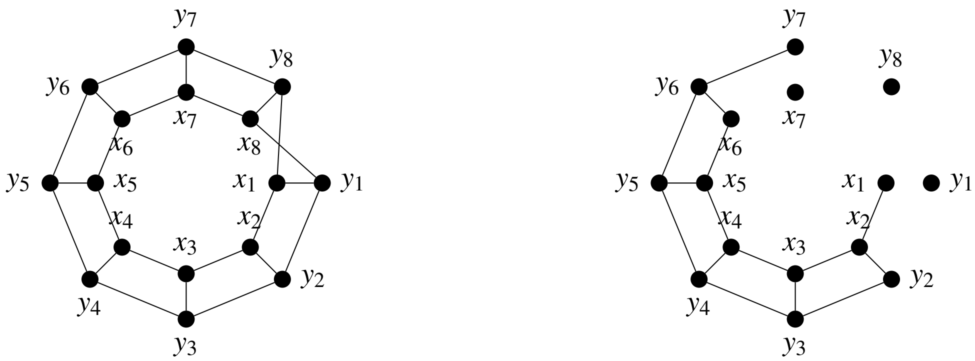


Figure 3.6: From left to right $G_{(I(C_{16}(1,8)))}$ and $G_{(I(C_{16}(1,8)):x_8)}$.

Proposition 3.2.2 ([81])**.** *For $n \geq 2$, let $G = C_{2n}(1,n)$ and $S = K[V(G)]$. Then*

$$\operatorname{depth}(S/I(G)) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 1 \,(\text{mod} 4); \\ \lceil \frac{n-1}{2} \rceil, & \text{otherwise}. \end{cases}$$

Proof. If $n = 2$, we have $C_4(1,2) \cong K_4$, then by Lemma 1.3.17 the required result follows. If $n = 3$, one can see that the required result holds by using Macaulay2 [13]. If $n = 4$, we consider the following short exact sequence

$$0 \longrightarrow (I(G):y_1)/I(G) \xrightarrow{\cdot y_1} S/I(G) \longrightarrow S/(I(G):y_1) \longrightarrow 0. \tag{3.2.1}$$

Here

$$K[V(G)]/(I(G):y_1) \cong \frac{K[x_2,x_3,y_3,y_4]}{(x_2x_3,x_3y_3,y_3y_4)}[y_1] \cong K[V(P_4)]/I(P_4) \otimes_K K[y_1], \tag{3.2.2}$$

and if we have $N_G(y_1) = \{x_4,x_1,y_2\}$, $S_1 = K[V(G)\backslash N_G(x_4)]$, $S_2 = K[V(G)\backslash(N_G(x_1) \cup \{x_4\})]$, $S_3 = K[V(G)\backslash(N_G(y_2) \cup \{x_4,x_1\})]$, $J_1 = (S_1 \cap I(G))$, $J_2 = (S_2 \cap I(G))$, $J_3 = (S_3 \cap I(G))$, then by using Lemma 3.1.1, we get

$$\begin{aligned}(I(G):y_1)/I(G) &\cong S_1/J_1[x_4] \oplus S_2/J_2[x_1] \oplus S_3/J_3[y_2] \\ &\cong \frac{K[x_1,x_2,y_2,y_3]}{(x_1x_2,x_2y_2,y_2y_3)}[x_4] \oplus \frac{K[x_3,y_2,y_3]}{(x_3y_3,y_3y_2)}[x_1] \oplus \frac{K[x_3,y_4]}{(0)}[y_2] \\ &\cong K[V(P_4)]/I(P_4) \otimes_K K[x_4] \oplus \big(K[V(P_3)]/I(P_3) \otimes_K K[x_1]\big) \oplus K[x_3,y_4,y_2].\end{aligned} \tag{3.2.3}$$

By applying Lemma 1.4.10 and Lemma 1.4.15 on Eqs. (3.2.2) and (3.2.3), $\operatorname{depth}(K[V(G)]/(I(G):y_1)) = \operatorname{depth}(K[V(P_4)]/I(P_4)) + \operatorname{depth}(K[y_1]) = 3$ and

$$\operatorname{depth}((I(G):y_1)/I(G)) = \min\{\operatorname{depth}(K[V(P_4)]/I(P_4)) + 1, \operatorname{depth}(K[V(P_3)]/I(P_3)) + 1, \operatorname{depth}(K[x_3,y_4,y_2])\} = 2.$$

By Depth Lemma on Eq. (3.2.1), we have $\operatorname{depth}(S/I(G)) = 2$. Let $n \geq 5$. We have the following K-algebra isomorphisms:

$$S/(I(G):y_1) \cong K[V(D_{n-3})]/I(D_{n-3}) \otimes_K K[y_1], \tag{3.2.4}$$

and if we have $N_G(y_1) = \{x_n, x_1, y_2\}$, $S_1 = K[V(G)\backslash N_G(x_n)]$, $S_2 = K[V(G)\backslash(N_G(x_1) \cup \{x_n\})]$, $S_3 = K[V(G)\backslash(N_G(y_2) \cup \{x_n, x_1\})]$, $J_1 = (S_1 \cap I(G))$, $J_2 = (S_2 \cap I(G))$, $J_3 = (S_3 \cap I(G))$, then by using Lemma 3.1.1, we get

$$\begin{aligned}(I(G):y_1)/I(G) &\cong S_1/J_1[x_n] \oplus S_2/J_2[x_1] \oplus S_3/J_3[y_2] \\ &\cong \frac{K[x_1,\dots,x_{n-2},y_2,\dots,y_{n-1}]}{\left(\cup_{i=2}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} \cup \{x_{n-2}y_{n-2}, x_1x_2, y_{n-2}y_{n-1}\}\right)}[x_n] \\ &\oplus \frac{K[x_3,\dots,x_{n-1},y_2,\dots,y_{n-1}]}{\left(\cup_{i=3}^{n-2}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} \cup \{x_{n-1}y_{n-1}, y_2y_3\}\right)}[x_1] \\ &\oplus \frac{K[x_3,\dots,x_{n-1},y_4,\dots,y_n]}{\left(\cup_{i=4}^{n-2}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} \cup \{x_{n-1}y_{n-1}, x_3x_4, y_{n-1}y_n\}\right)}[y_2] \\ &\cong K[V(D_{n-3})]/I(D_{n-3}) \otimes_K K[x_n] \oplus K[V(B_{n-3})]/I(B_{n-3}) \otimes_K K[x_1] \\ &\oplus K[V(D_{n-4})]/I(D_{n-4}) \otimes_K K[y_2].\end{aligned} \tag{3.2.5}$$

By using Eqs. (3.2.4), (3.2.5) and Lemma 1.4.10, we have

$$\operatorname{depth}(S/(I(G):y_1)) = \operatorname{depth} K[V(D_{n-3})]/I(D_{n-3}) + \operatorname{depth} K[y_1], \tag{3.2.6}$$

$$\begin{aligned}\operatorname{depth}\big((I(G):y_1)/I(G)\big) = \min\Big\{ &\operatorname{depth} K[V(D_{n-3})]/I(D_{n-3}) + 1, \operatorname{depth} K[V(B_{n-3})]/I(B_{n-3}) + 1, \\ &\operatorname{depth} K[V(D_{n-4})]/I(D_{n-4}) + 1\Big\}.\end{aligned} \tag{3.2.7}$$

Now, if $n \equiv 1 \pmod 4$, then $n-3 \equiv 2 \pmod 4$ and $n-4 \equiv 1 \pmod 4$. By using Lemma 3.1.17 in Eq. (3.2.6), we get

$$\operatorname{depth}(S/(I(G):y_1)) = \operatorname{depth} K[V(D_{n-3})]/I(D_{n-3}) + 1 = \lceil \frac{n-3+1}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil.$$

By applying Lemma 3.1.5, Lemma 3.1.17 and Remark 3.1.2 on Eq. (3.2.7), we get

$$\begin{aligned}\operatorname{depth}\big((I(G):y_1)/I(G)\big) &= \min\Big\{\lceil \frac{n-3+1}{2} \rceil + 1, \lceil \frac{n-3+1}{2} \rceil + 1, \lceil \frac{n-4+1}{2} \rceil + 1 + 1\Big\} \\ &= \min\Big\{\lceil \frac{n}{2} \rceil, \lceil \frac{n}{2} \rceil, \lceil \frac{n+1}{2} \rceil\Big\} \\ &= \lceil \frac{n}{2} \rceil.\end{aligned}$$

As $\operatorname{depth}(S/(I(G):y_1)) = \operatorname{depth}\big((I(G):y_1)/I(G)\big) = \lceil \frac{n}{2} \rceil$, therefore by applying Depth Lemma on Eq. (3.2.1), we get $\operatorname{depth}(S/I(G) = \lceil \frac{n}{2} \rceil$. If $n \equiv 2 \pmod 4$, then $n-3 \equiv 3 \pmod 4$ and $n-4 \equiv$

$2 \pmod 4$. To prove the result, we use similar strategy of the previous case and get the desired result that is $\operatorname{depth}(S/I(G)) = \lceil \frac{n-1}{2} \rceil$. If $n \equiv 0, 3 \pmod 4$, the proof is similar. □

Corollary 3.2.3 ([81]). *For $n \geq 2$, let $G = C_{2n}(1,n)$ and $S = K[V(G)]$. Then*

$$\operatorname{pdim}(S/I(G)) = \begin{cases} 2n - \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 1 \pmod 4; \\ 2n - \lceil \frac{n-1}{2} \rceil, & \text{otherwise.} \end{cases}$$

Proof. The required result is the direct consequence by using Lemma 1.4.11 and Proposition 3.2.2. □

Proposition 3.2.4 ([81]). *For $n \geq 2$, let $G = C_{2n}(1,n)$ and $S = K[V(G)]$. Then*

$$\operatorname{sdepth}(S/I(G)) = \begin{cases} \lceil \frac{n}{2} \rceil, & \text{if } n \equiv 1 \pmod 4; \\ \lceil \frac{n-1}{2} \rceil, & \text{if } n \equiv 2 \pmod 4. \end{cases}$$

If $n \equiv 0, 3 \pmod 4$, then

$$\lceil \frac{n-1}{2} \rceil \leq \operatorname{sdepth}(S/I(G)) \leq \lceil \frac{n}{2} \rceil + 1.$$

Proof. If $n = 2$, we have $C_4(1,2) \cong K_4$, then we get the result by Lemma 1.3.17, $\operatorname{sdepth}(S/I(G)) \geq 1$. If $n = 3$, we find the required lower bound by using Lemma 1.3.15. For the upper bound, since $y_1 \notin I(G)$, by Lemma 1.3.12, $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(S/(I(G) : y_1)$. Here

$$K[V(G)]/(I(G) : y_1) \cong \frac{K[x_2, y_3]}{(0)}[y_1],$$

and by Lemma 1.4.10, $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(K[x_1, x_3, y_2]) = 3$. If $n = 4$, one can find a lower bound for Stanley depth in a similar way as depth in Proposition 3.2.2 just by using Lemma 1.3.10 in place of Depth Lemma, that is $\operatorname{sdepth}(S/I(G)) \geq 2$. For the upper bound, by using Eq. (3.2.2), Lemma 1.3.12 and Lemma 1.4.10, $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(S/(I(G) : y_1) = \operatorname{sdepth}(K[V(P_4)]/I(P_4)) + 1 = 3$. Let $n \geq 5$. By using a similar strategy of depth as in Proposition 3.2.2 and applying Lemma 1.3.10 in place of Depth Lemma on Eq. (3.2.1), we get the required lower bound for Stanley depth. For other inequality, by using Lemma 1.3.12, Lemma 1.4.10 and Lemma 3.1.17 on Eq. (3.2.4), we get the required result. This completes the proof. □

Example 3.2.5. Let $n = 11$. If $G = C_{2n}(1,n) = C_{22}(1,11)$, then

(a) $\operatorname{depth}(K[V(C_{22}(1,11))]/I(C_{22}(1,11))) = 5$.

(b) $\operatorname{pdim}(K[V(C_{22}(1,11))]/I(C_{22}(1,11))) = 17$.

(c) $5 \leq \operatorname{sdepth}(K[V(C_{22}(1,11))]/I(C_{22}(1,11))) \leq 7$.

Proposition 3.2.6 ([81]). *For $n \geq 3$, let $G = C_{2n}(2,n)$ and $S = K[V(G)]$. Then*

$$\operatorname{depth}(S/I(G)) = \begin{cases} \lceil \frac{n-1}{2} \rceil, & \text{if } n \equiv 1 \pmod 4; \\ \lceil \frac{n}{2} \rceil, & \text{otherwise.} \end{cases}$$

Proof. If $n = 3$, one can easily see that the result holds by Macaulay2 [13]. If $n = 4$, we consider the short exact sequence

$$0 \longrightarrow (I(G):y_4)/I(G) \xrightarrow{\cdot y_4} S/I(G) \longrightarrow S/(I(G):y_4) \longrightarrow 0. \tag{3.2.8}$$

We have

$$K[V(G)]/(I(G):y_4) \cong \frac{K[x_1,x_2,x_3,y_2]}{(x_1x_2,x_2x_3,x_2y_2)}[y_4] \cong K[V(S_4)]/I(S_4) \otimes_K K[y_4], \tag{3.2.9}$$

and if we have $N_G(y_4) = \{y_3,x_4,y_1\}$, $S_1 = K[V(G) \backslash N_G(y_3)]$, $S_2 = K[V(G) \backslash (N_G(x_4) \cup \{y_3\})]$, $S_3 = K[V(G) \backslash (N_G(y_1) \cup \{y_3,x_4\})]$, $J_1 = (S_1 \cap I(G))$, $J_2 = (S_2 \cap I(G))$, $J_3 = (S_3 \cap I(G))$, then by using Lemma 3.1.1, we get

$$\begin{aligned} (I(G):y_4)/I(G) &\cong S_1/J_1[y_3] \oplus S_2/J_2[x_4] \oplus S_3/J_3[y_1] \\ &\cong \frac{K[x_1,x_2,x_4,y_1]}{(x_1x_2,x_1x_4,x_1y_1)}[y_3] \oplus \frac{K[x_2,y_1,y_2]}{(y_1y_2,y_2x_2)}[x_4] \oplus \frac{K[x_2,x_3]}{(x_2x_3)}[y_1] \\ &\cong K[V(S_4)]/I(S_4) \otimes_K K[y_3] \oplus \big(K[V(P_3)]/I(P_3) \otimes_K K[x_4]\big) \\ &\quad \oplus K[V(P_2)]/I(P_2) \otimes_K K[y_1]. \end{aligned} \tag{3.2.10}$$

We apply Lemma 1.4.10, Lemma 1.4.15 and Lemma 1.3.16 on Eq. (3.2.9), $\operatorname{depth}(K[V(G)]/(I(G):y_4)) = \operatorname{depth}(K[V(S_4)]/I(S_4)) + 1 = 2$ and by Eq.(3.2.10)

$$\begin{aligned} \operatorname{depth}((I(G):y_4)/I(G)) = \min\{&\operatorname{depth}(K[V(S_4)]/I(S_4)) + 1, \operatorname{depth}(K[V(P_3)]/I(P_3)) + 1, \\ &\operatorname{depth}(K[V(P_2)]/I(P_2)) + 1\} \\ = 2.& \end{aligned}$$

By using Depth Lemma on Eq. (3.2.8), $\operatorname{depth}(S/I(G)) = 2$. Let $n \geq 5$. Consider the following short exact sequence

$$0 \longrightarrow (I(G):y_n)/I(G) \xrightarrow{\cdot y_n} S/I(G) \longrightarrow S/(I(G):y_n) \longrightarrow 0. \tag{3.2.11}$$

Here

$$S/(I(G):y_n) \cong K[V(Q_{n-3})]/I(Q_{n-3}) \otimes_K K[y_n], \tag{3.2.12}$$

and if we have $N_G(y_n) = \{y_{n-1}, x_n, y_1\}$, $S_1 = K[V(G) \backslash N_G(y_{n-1})]$, $S_2 = K[V(G) \backslash (N_G(x_n) \cup \{y_{n-1}\})]$, $S_3 = K[V(G) \backslash (N_G(y_1) \cup \{y_{n-1}, x_n\})]$, $J_1 = (S_1 \cap I(G))$, $J_2 = (S_2 \cap I(G))$, $J_3 = (S_3 \cap I(G))$, then by using Lemma 3.1.1, we have

$$\begin{aligned}(I(G):y_n)/I(G) &\cong S_1/J_1[y_{n-1}] \oplus S_2/J_2[x_n] \oplus S_3/J_3[y_1] \\ &\cong \frac{K[x_1,\dots,x_{n-2},x_n,y_1,\dots,y_{n-3}]}{\left(\cup_{i=1}^{n-4}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} \cup \{x_{n-3}y_{n-3}, x_1x_n, x_{n-3}x_{n-2}\}\right)}[y_{n-1}] \\ &\quad \oplus \frac{K[x_2,\dots,x_{n-2},y_1,\dots,y_{n-2}]}{\left(\cup_{i=2}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} \cup \{x_{n-2}y_{n-2}, y_1y_2\}\right)}[x_n] \\ &\quad \oplus \frac{K[x_2,\dots,x_{n-1},y_3,\dots,y_{n-2}]}{\left(\cup_{i=2}^{n-3}\{x_iy_i, x_ix_{i+1}, y_iy_{i+1}\} \cup \{x_{n-2}y_{n-2}, x_{n-2}x_{n-1}, x_2x_3\}\right)}[y_1] \\ &\cong K[V(Q_{n-3})]/I(Q_{n-3}) \otimes_K K[y_{n-1}] \oplus K[V(B_{n-3})]/I(B_{n-3}) \otimes_K K[x_n] \\ &\quad \oplus K[V(Q_{n-4})]/I(Q_{n-4}) \otimes_K K[y_1]\end{aligned} \tag{3.2.13}$$

By Eqs. (3.2.13), (3.2.12) and Lemma 1.4.10, we get

$$\operatorname{depth}(S/(I(G):y_n)) = \operatorname{depth} K[V(Q_{n-3})]/I(Q_{n-3}) + \operatorname{depth} K[y_n], \tag{3.2.14}$$

and

$$\begin{aligned}\operatorname{depth}\big((I(G):y_n)/I(G)\big) = \min\Big\{ &\operatorname{depth} K[V(Q_{n-3})]/I(Q_{n-3}) + 1, \operatorname{depth} K[V(B_{n-3})]/I(B_{n-3}) + 1, \\ &\operatorname{depth} K[V(Q_{n-4})]/I(Q_{n-4}) + 1\Big\}.\end{aligned} \tag{3.2.15}$$

If $n \equiv 1 \pmod 4$, then $n-3 \equiv 2 \pmod 4$ and $n-4 \equiv 1 \pmod 4$. By applying Lemma 3.1.14, Lemma 1.4.10 on Eq. (3.2.14), we get

$$\operatorname{depth}(S/(I(G):y_n)) = \lceil \frac{n-3+1}{2} \rceil + 1 + 1 = \lceil \frac{n}{2} \rceil + 1. \tag{3.2.16}$$

By applying Lemma 3.1.5, Lemma 3.1.14 and Remark 3.1.2 on Eq. (3.2.15), we get

$$\begin{aligned}\operatorname{depth}\big((I(G):y_n)/I(G)\big) &= \min\Big\{\lceil \frac{n-3+1}{2} \rceil + 1 + 1, \lceil \frac{n-3+1}{2} \rceil + 1, \lceil \frac{n-4+1}{2} \rceil + 1\Big\} \\ &= \min\Big\{\lceil \frac{n}{2} \rceil + 1, \lceil \frac{n}{2} \rceil, \lceil \frac{n-1}{2} \rceil\Big\} \\ &= \lceil \frac{n-1}{2} \rceil.\end{aligned} \tag{3.2.17}$$

Since, $\operatorname{depth}\big((I(G):y_n)/I(G)\big) < \operatorname{depth}(S/(I(G):y_n))$, thus the desired result follows by applying Depth Lemma on Eq. (3.2.11). Similarly, if $n \equiv 2 \pmod 4$, then $n-3 \equiv 3 \pmod 4$ and $n-4 \equiv 2 \pmod 4$, then the required results follows by similar strategy that is $\operatorname{depth}(S/I(G)) = \lceil \frac{n}{2} \rceil$. If $n \equiv 0,3 \pmod 4$, the proof follows in a similar way. □

Corollary 3.2.7 ([81]). *For $n \geq 3$, let $G = C_{2n}(2,n)$ and $S = K[V(G)]$. Then*

$$\operatorname{pdim}(S/I(G)) = \begin{cases} 2n - \lceil \frac{n-1}{2} \rceil, & \text{if } n \equiv 1 \pmod 4; \\ 2n - \lceil \frac{n}{2} \rceil, & \text{otherwise.} \end{cases}$$

Proof. The proof follows by Lemma 1.4.11 and Proposition 3.2.6. □

Proposition 3.2.8 ([81]). *For $n \geq 3$, let $G = C_{2n}(2,n)$ and $S = K[V(G)]$. If $n \equiv 0,3 \pmod 4$, then*

$$\operatorname{sdepth}(S/I(G)) = \lceil \frac{n}{2} \rceil.$$

If $n \equiv 1 \pmod 4$, then

$$\lceil \frac{n-1}{2} \rceil \leq \operatorname{sdepth}(S/I(G)) \leq \lceil \frac{n}{2} \rceil + 1,$$

and if $n \equiv 2 \pmod 4$, we have

$$\lceil \frac{n-1}{2} \rceil \leq \operatorname{sdepth}(S/I(G)) \leq \lceil \frac{n-1}{2} \rceil + 1.$$

Proof. If $n = 3$, one can get lower bound by using CoCoA [20] that is $\operatorname{sdepth}(S/I(G)) \geq 2$. For the upper bound, by Lemma 1.3.12, $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(S/(I(G):y_3))$. Here

$$K[V(G)]/(I(G):y_3) \cong \frac{K[x_1,x_2]}{(x_1x_2)}[y_3], \tag{3.2.18}$$

by Lemma 1.4.10 and Lemma 1.4.15, we get $\operatorname{sdepth}(S/I(G)) \leq 2$. Let $n = 4$. For the upper bound, since $y_4 \notin I(G)$, by Lemma 1.3.12, we have $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(S/(I(G):y_4))$. Here

$$K[V(G)]/(I(G):y_4) \cong \frac{K[x_1,x_2,x_3,y_2]}{(x_1x_2,x_2y_2,x_2x_3)}[y_4] \cong K[V(S_4)]/I(S_4) \otimes_K K[y_4],$$

and by Lemma 1.4.10 and Lemma 1.3.16, $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(K[V(S_4)]/I(S_4)) + 1 = 2$. For other inequality, one can find Stanley depth in a similar way as depth in Proposition 3.2.6 just by using Lemma 1.3.10 in place of Depth Lemma, that is $\operatorname{sdepth}(S/I(G)) \geq 2$. Let $n \geq 5$. We get the desired lower bound for Stanley depth by using a similar strategy of depth as in Proposition 3.2.6 and applying Lemma 1.3.10 in-place of Depth Lemma on Eq. (3.2.11). For other inequality, by using Lemma 1.3.12, Lemma 1.4.10 and Lemma 3.1.14 on Eq. (3.2.12), the required result follows. □

Example 3.2.9. For $n = 13$, we have

(a) $\operatorname{depth}(K[V(C_{26}(2,13))]/I(C_{26}(2,13))) = 6$.

(b) $\operatorname{pdim}(K[V(C_{26}(2,13))]/I(C_{26}(2,13))) = 20$.

(c) $6 \leq \operatorname{sdepth}(K[V(C_{26}(2,13))]/I(C_{26}(2,13))) \leq 8$.

Before proving Theorem 3.2.11, we make the following remark.

Remark 3.2.10 ([81])**.** Let $n \geq 2$, $t = \gcd(2n,a)$ and $1 \leq a < n$. Note that by Theorem 3.2.1(b), $C_{2n}(a,n)$ is isomorphic to $\frac{t}{2}$ copies of $C_{\frac{4n}{t}}(2,\frac{2n}{t})$. For $C_{\frac{4n}{t}}(2,\frac{2n}{t})$, we only need to consider the case when $\frac{2n}{t}$ is odd. If $\frac{2n}{t}$ is even, then by Theorem 3.2.1(a), we have t disjoint copies of $C_{\frac{2n}{t}}(1,\frac{n}{t})$.

Theorem 3.2.11 ([81])**.** *Let $n \geq 2$, $t = \gcd(2n,a)$ and $1 \leq a < n$.*

(a) If $\frac{2n}{t}$ is even, then

$$\operatorname{depth}(K[V(C_{2n}(a,n))]/I(C_{2n}(a,n))) = \begin{cases} \lceil \frac{n}{2t} \rceil \cdot t, & \text{if } \frac{n}{t} \equiv 1 \pmod 4; \\ \lceil \frac{n-t}{2t} \rceil \cdot t, & \text{otherwise}. \end{cases}$$

(b) If $\frac{2n}{t}$ is odd, then

$$\operatorname{depth}(K[V(C_{2n}(a,n))]/I(C_{2n}(a,n))) = \begin{cases} \lceil \frac{2n-t}{2t} \rceil \cdot \frac{t}{2}, & \text{if } \frac{2n}{t} \equiv 1 \pmod 4; \\ \lceil \frac{n}{t} \rceil \cdot \frac{t}{2}, & \text{if } \frac{2n}{t} \equiv 3 \pmod 4. \end{cases}$$

Proof. Let $\frac{2n}{t}$ is even. Since $t = \gcd(2n,a)$, therefore $\frac{n}{t} \geq 2$ and a positive integer. Now by using Proposition 3.2.2, we have

$$\operatorname{depth}\big(K[V(C_{\frac{2n}{t}}(1,\frac{n}{t}))]/I(C_{\frac{2n}{t}}(1,\frac{n}{t}))\big) = \begin{cases} \lceil \frac{\frac{n}{t}}{2} \rceil, & \text{if } \frac{n}{t} \equiv 1 \pmod 4; \\ \lceil \frac{\frac{n}{t}-1}{2} \rceil, & \text{otherwise}. \end{cases}$$

By Theorem 3.2.1, $C_{2n}(a,n)$ is isomorphic to t copies of $C_{\frac{2n}{t}}(1,\frac{n}{t})$. Therefore, by Lemma 1.3.11,

$$\operatorname{depth}(K[V(C_{2n}(a,n))]/I(C_{2n}(a,n))) = \begin{cases} \lceil \frac{n}{2t} \rceil \cdot t, & \text{if } \frac{n}{t} \equiv 1 \,(\operatorname{mod} 4); \\ \\ \lceil \frac{\frac{n}{t}-1}{2} \rceil \cdot t, & \text{otherwise.} \end{cases}$$

Now, if $\frac{2n}{t}$ is odd, then $\frac{2n}{t} > 2$ and a positive integer. By using a similar strategy, use Proposition 3.2.6 in place of Proposition 3.2.2 and by Theorem 3.2.1, $C_{2n}(a,n)$ is isomorphic to $\frac{t}{2}$ copies of $C_{\frac{4n}{t}}(2, \frac{2n}{t})$. By Remark 3.2.10, it is enough to consider the cases when $\frac{2n}{t}$ is odd, therefore by Lemma 1.3.11, the required result follows

$$\operatorname{depth}(K[V(C_{2n}(a,n))]/I(C_{2n}(a,n))) = \begin{cases} \lceil \frac{\frac{2n}{t}-1}{2} \rceil \cdot \frac{t}{2}, & \text{if } \frac{2n}{t} \equiv 1 \,(\operatorname{mod} 4); \\ \\ \lceil \frac{\frac{2n}{t}}{2} \rceil \cdot \frac{t}{2}, & \text{if } \frac{2n}{t} \equiv 3 \,(\operatorname{mod} 4). \end{cases}$$

□

Corollary 3.2.12 ([81])**.** *Let $n \geq 2$, $t = \gcd(2n,a)$ and $1 \leq a < n$.*

(a) If $\frac{2n}{t}$ is even, then

$$\operatorname{pdim}(K[V(C_{2n}(a,n))]/I(C_{2n}(a,n))) = \begin{cases} 2n - \lceil \frac{n}{2t} \rceil \cdot t, & \text{if } \frac{n}{t} \equiv 1 \,(\operatorname{mod} 4); \\ \\ 2n - \lceil \frac{n-t}{2t} \rceil \cdot t, & \text{otherwise.} \end{cases}$$

(b) If $\frac{2n}{t}$ is odd, then

$$\operatorname{pdim}(K[V(C_{2n}(a,n))]/I(C_{2n}(a,n))) = \begin{cases} 2n - \lceil \frac{2n-t}{2t} \rceil \cdot \frac{t}{2}, & \text{if } \frac{2n}{t} \equiv 1 \,(\operatorname{mod} 4); \\ \\ 2n - \lceil \frac{n}{t} \rceil \cdot \frac{t}{2}, & \text{if } \frac{2n}{t} \equiv 3 \,(\operatorname{mod} 4). \end{cases}$$

Proof. The proof follows by Lemma 1.4.11 and Theorem 3.2.11. □

Theorem 3.2.13 ([81])**.** *Let $n \geq 2$, $t = \gcd(2n,a)$ and $1 \leq a < n$.*

(a) If $\frac{2n}{t}$ is even, then

$$\operatorname{sdepth}(K[V(C_{2n}(a,n))]/I(C_{2n}(a,n))) \geq \begin{cases} \lceil \frac{n}{2t} \rceil \cdot t, & \text{if } \frac{n}{t} \equiv 1 \,(\operatorname{mod} 4); \\ \\ \lceil \frac{n-t}{2t} \rceil \cdot t, & \text{otherwise.} \end{cases}$$

(b) *If $\frac{2n}{t}$ is odd, then*

$$\operatorname{sdepth}(K[V(C_{2n}(a,n))]/I(C_{2n}(a,n))) \geq \begin{cases} \lceil \frac{2n-t}{2t} \rceil \cdot \frac{t}{2}, & \text{if } \frac{2n}{t} \equiv 1 \pmod 4; \\ \lceil \frac{n}{t} \rceil \cdot \frac{t}{2}, & \text{if } \frac{2n}{t} \equiv 3 \pmod 4. \end{cases}$$

Proof. Since $t = \gcd(2n,a)$, therefore if $\frac{2n}{t}$ is even, then $\frac{n}{t}$ is a positive integer greater than or equals to 2. By Proposition 3.2.4, we have

$$\operatorname{sdepth}(K[V(C_{\frac{2n}{t}}(1,\frac{n}{t}))]/I(C_{\frac{2n}{t}}(1,\frac{n}{t}))) \geq \begin{cases} \lceil \frac{\frac{n}{t}}{2} \rceil, & \text{if } \frac{n}{t} \equiv 1 \pmod 4; \\ \lceil \frac{\frac{n}{t}-1}{2} \rceil, & \text{otherwise.} \end{cases}$$

By using Theorem 3.2.1, $C_{2n}(a,n)$ is isomorphic to t copies of $C_{\frac{2n}{t}}(1,\frac{n}{t})$, therefore by Lemma 1.3.11,

$$\operatorname{sdepth}(K[V(C_{2n}(a,n))]/I(C_{2n}(a,n))) \geq \begin{cases} \lceil \frac{\frac{n}{t}}{2} \rceil \cdot t, & \text{if } \frac{n}{t} \equiv 1 \pmod 4; \\ \lceil \frac{\frac{n}{t}-1}{2} \rceil \cdot t, & \text{otherwise.} \end{cases}$$

Similarly, if $\frac{2n}{t}$ is odd then $\frac{2n}{t}$ is a positive integer strictly greater than 2. We get the required result just by replacing Proposition 3.2.4 with Proposition 3.2.8 and by Theorem 3.2.1, $C_{2n}(a,n)$ is isomorphic to $\frac{t}{2}$ copies of $C_{\frac{4n}{t}}(2,\frac{2n}{t})$. By using Remark 3.2.10, it is enough to cater the cases when $\frac{2n}{t}$ is odd, therefore by Lemma 1.3.11, we get

$$\operatorname{sdepth}(K[V(C_{2n}(a,n))]/I(C_{2n}(a,n))) \geq \begin{cases} \lceil \frac{\frac{2n}{t}-1}{2} \rceil \cdot \frac{t}{2}, & \text{if } \frac{2n}{t} \equiv 1 \pmod 4; \\ \lceil \frac{\frac{2n}{t}}{2} \rceil \cdot \frac{t}{2}, & \text{if } \frac{2n}{t} \equiv 3 \pmod 4. \end{cases}$$

□

Example 3.2.14. Let $n = 5$ and $a = 2$. We have $t = \gcd(10,2) = 2$ and $\frac{2n}{t} = \frac{10}{2} = 5$ is odd, then

(a) $\operatorname{depth}(K[V(C_{10}(2,5))]/I(C_{10}(2,5))) = 2$.

(b) $\operatorname{pdim}(K[V(C_{10}(2,5))]/I(C_{10}(2,5))) = 8$.

(c) $\operatorname{sdepth}(K[V(C_{10}(2,5))]/I(C_{10}(2,5))) \geq 2$.

Example 3.2.15. Let $n = 8$ and $a = 4$. We have $t = \gcd(16,4) = 4$ and $\frac{2n}{t} = \frac{16}{4} = 4$ is even, then

(a) $\operatorname{depth}(K[V(C_{16}(4,8))]/I(C_{16}(4,8))) = 4$.

(b) $\operatorname{pdim}(K[V(C_{16}(4,8))]/I(C_{16}(4,8))) = 12$.

(c) $\operatorname{sdepth}(K[V(C_{16}(4,8))]/I(C_{16}(4,8))) \geq 4$.

Remark 3.2.16 ([81]). Let $n \geq 2$, $t = \gcd(2n, a)$ and $1 \leq a < n$. Then Stanley's inequality holds for $K[V(C_{2n}(a,n))]/I(C_{2n}(a,n))$.

3.3 Conclusion

In this chapter, values of depth, projective dimension, and lower bounds for Stanley depth of the quotient rings of the edge ideals of all cubic circulant graphs are computed. It is worth mentioning that for providing these results, the precise values of the said invariants of the edge ideals associated with certain supergraphs of ladder graph played an important role. For the computation of depth and a lower bound for Stanley depth, Lemma 3.1.1 is very significant in general. This work can be extended for finding the said algebraic invariants of edge ideals associated with other families of circulant graphs, as we did for certain classes of circulant graphs provided in the next chapter.

CHAPTER 4

Algebraic invariants and some circulant graphs

In this chapter, we study some invariants of the edge ideals associated with some families of four and five regular circulant graphs, which include $C_{2n}(1,n-1), C_{2n}(1,2)$, and $C_{2n}(1,n-1,n)$, where $n \geq 3$. These graphs are depicted in Figures 4.1 and 4.2. In the first section of present chapter, we find the algebraic invariants depth, regularity, Stanley depth and projective dimension of cyclic modules associated with certain subgraphs of $C_{2n}(1,n-1), C_{2n}(1,2)$ and $C_{2n}(1,n-1,n)$; see for instance Lemmas 4.1.4 and 4.1.6–4.1.10.

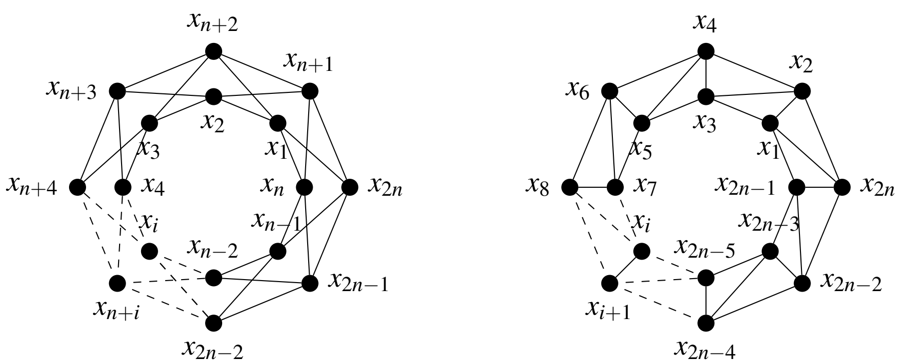


Figure 4.1: From left to right $C_{2n}(1,n-1)$ and $C_{2n}(1,2)$.

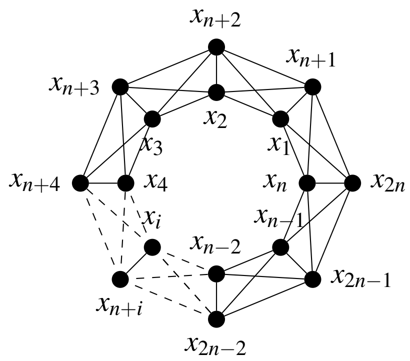


Figure 4.2: $C_{2n}(1,n-1,n)$.

In the next section, we give the exact values of depth, projective dimension, and bounds for the Stanley depth of $K[V(C_{2n}(1,n-1))]/I(C_{2n}(1,n-1))$, see Theorem 4.2.1, Corollary 4.2.2, and Theorem 4.2.3. In Theorem 4.2.5, we give a formula for the regularity of the edge ideal associated with $C_{2n}(1,n-1)$ when $n \equiv 0,1(\bmod 3)$, and sharp bounds when $n \equiv 2(\bmod 3)$. Moreover, we provide the exact values of the regularity of the edge ideal associated with $C_{2n}(1,2)$ when n is even and tight bounds when n is odd, see Theorem 4.2.7. Our result in Theorem 4.2.9 gives the exact value for the regularity of edge ideal associated with $C_{2n}(1,n-1,n)$.

4.1 Invariants of cyclic modules associated with certain subgraphs of $C_{2n}(1,n-1), C_{2n}(1,2)$ and $C_{2n}(1,n-1,n)$

For $n \geq 2$, we introduce some families of subgraphs, namely E_n, F_n and G_n of $C_{2n}(1,n-1), C_{2n}(1,2)$ and $C_{2n}(1,n-1,n)$, respectively as given in Figures 4.3 and 4.4. The vertex sets of these subgraphs are $V(E_n) = V(F_n) = V(G_n) = \bigcup_{i=1}^{n} \{x_i, y_i\}$ and the edge sets are as follows:

- $E(E_n) = \bigcup_{i=1}^{n-1} \Big\{ \{x_i, x_{i+1}\}, \{y_i, y_{i+1}\}, \{x_i, y_{i+1}\}, \{x_{i+1}, y_i\} \Big\}$,
- $E(F_n) = \bigcup_{i=1}^{n-1} \Big\{ \{x_i, y_i\}, \{x_i, x_{i+1}\}, \{y_i, y_{i+1}\}, \{x_i, y_{i+1}\} \Big\} \bigcup \{x_n, y_n\}$,
- $E(G_n) = \bigcup_{i=1}^{n-1} \Big\{ \{x_i, y_i\}, \{x_i, x_{i+1}\}, \{y_i, y_{i+1}\}, \{x_i, y_{i+1}\}, \{x_{i+1}, y_i\} \Big\} \bigcup \{x_n, y_n\}$.

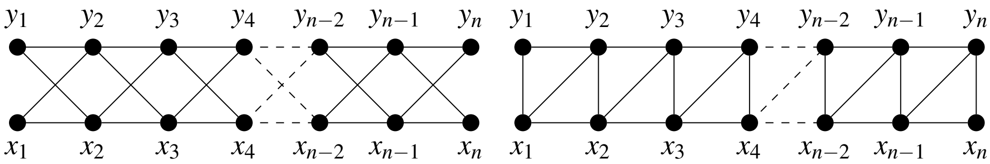


Figure 4.3: From left to right E_n and F_n.

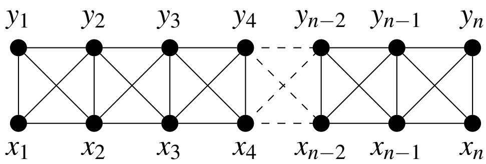


Figure 4.4: G_n.

Here, we give exact values of depth, projective dimension, and regularity of cyclic module $K[V(E_n)]/I(E_n)$. We also give bounds for the Stanley depth of such a module. Moreover, we compute the exact values of regularity of cyclic modules $K[V(F_n)]/I(F_n)$ and $K[V(G_n)]/I(G_n)$. It is worth mentioning that these findings are helpful in the subsequent section for proving our main results.

Remark 4.1.1 ([82])**.** To cater some special cases in the proofs of subsequent results, the quotient rings associated with E_n, G_n and F_n for $n \leq 1$, are described as follows:

- $K[V(E_{-1})]/I(E_{-1}) \cong K[V(E_0)]/I(E_0) \cong K[V(F_0)]/I(F_0) \cong K[V(G_0)]/I(G_0) \cong K$ and $\operatorname{depth}(K) = \operatorname{sdepth}(K) = \operatorname{reg}(K) = 0$;
- $K[V(E_1)]/I(E_1) \cong K[x,y]$, we have $\operatorname{depth}(K[x,y]) = \operatorname{sdepth}(K[x,y]) = 2$ and $\operatorname{reg}(K[x,y]) = 0$;
- $K[V(F_1)]/I(F_1) \cong K[V(G_1)]/I(G_1) \cong K[V(P_2)]/I(P_2)$, then by Lemma 1.4.15, we get $\operatorname{depth}(K[V(P_2)]/I(P_2)) = \operatorname{sdepth}(K[V(P_2)]/I(P_2)) = \operatorname{reg}(K[V(P_2)]/I(P_2)) = 1$.

Remark 4.1.2 ([82])**.** Let $i \in \mathbb{Z}^+$, if $k < i$ then we consider $\cup_i^k \{x_i y_{i+1}, x_i x_{i+1}, y_i y_{i+1}, x_{i+1} y_i\} = \emptyset$. Also we take $x_a y_b = 0$, whenever a or b is not positive.

Remark 4.1.3 ([82])**.** Let $x_t, x_r \in S$ such that $x_t, x_r \notin I$. Then $(I : x_t)$, (I, x_t), $((I, x_t), x_r)$ and $((I, x_t) : x_r)$ are the monomial ideals of S such that $G_{(I:x_t)}$, $G_{(I,x_t)}$, $G_{((I,x_t),x_r)}$ and $G_{((I,x_t):x_r)}$ are subgraphs of G_I.

By using Remark 4.1.3, see Figures 4.5 and 4.6 as examples of $G_{(I(E_7):y_6)}, G_{(I(E_7),y_6)}, G_{((I(E_7),y_6),x_6)}$, and $G_{((I(E_7),y_6):x_6)}$. From Figures 4.5 and 4.6, we have the following isomorphisms:

$$K[V(E_7)]/(I(E_7):y_6) \cong K[V(E_4)]/I(E_4) \otimes_K K[y_6,x_6],$$

$$K[V(E_7)]/(I(E_7),y_6) \cong K[V(E_5)]/(I(E_5),x_5x_6,x_6y_5,x_6y_7,x_6x_7),$$

$$K[V(E_7)]/((I(E_7),y_6),x_6) \cong K[V(E_5)]/I(E_5) \otimes_K K[y_7,x_7],$$

$$K[V(E_7)]/((I(E_7),y_6):x_6) \cong K[V(E_4)]/I(E_4) \otimes_K K[x_6].$$

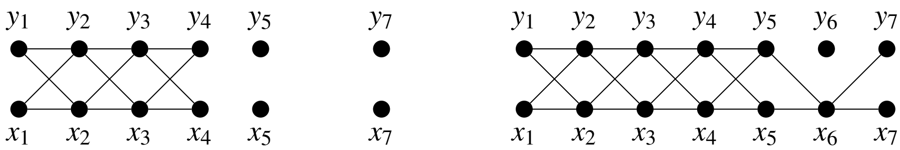


Figure 4.5: From left to right $G_{(I(E_7):y_6)}$ and $G_{(I(E_7),y_6)}$.

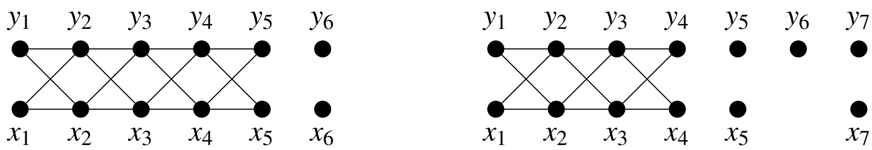


Figure 4.6: From left to right $G_{((I(E_7),y_6),x_6)}$ and $G_{((I(E_7),y_6):x_6)}$.

First, we find the exact value of depth and lower bound of Stanley depth for $K[V(E_n)/I(E_n))$.

Lemma 4.1.4 ([82])**.** *Let $n \geq 2$. If $S = K[V(E_n)]$, then*

$$\operatorname{sdepth}(S/I(E_n)) \geq \operatorname{depth}(S/I(E_n)) = \begin{cases} \lceil \frac{n+4}{3} \rceil, & \text{if } n \equiv 1 \pmod 3; \\ \lceil \frac{n}{3} \rceil, & \text{otherwise.} \end{cases}$$

Proof. We first show the proof for depth. If $n = 2$, then $E_2 \cong C_4$. It is clear that the result holds by using Lemma 1.3.18. If $n = 3$, we have $E_3 \cong K_{4,2}$, then from Lemma 1.3.19, we have $\operatorname{depth}(S/I(E_n)) = 1$. Let $n \geq 4$. We consider the subsequent cases:

Case 1. Let $n \equiv 1 \pmod 3$. Consider the following short exact sequences

$$0 \longrightarrow S/(I(E_n):y_{n-1}) \xrightarrow{\cdot y_{n-1}} S/I(E_n) \longrightarrow S/(I(E_n),y_{n-1}) \longrightarrow 0,$$

$$0 \longrightarrow S/((I(E_n),y_{n-1}):x_{n-1}) \xrightarrow{\cdot x_{n-1}} S/(I(E_n),y_{n-1}) \longrightarrow S/((I(E_n),y_{n-1}),x_{n-1}) \longrightarrow 0.$$

By Lemma 3.1.4,

$$\operatorname{depth}\Big(S/(I(E_n)\Big) \geq \min\Big\{\operatorname{depth}\Big(S/(I(E_n):y_{n-1})\Big), \operatorname{depth}\Big(S/(I(E_n),y_{n-1})\Big)\Big\}, \tag{4.1.1}$$

$$\begin{aligned}&\operatorname{depth}\Big(S/(I(E_n),y_{n-1})\Big)\\ &\geq \min\Big\{\operatorname{depth}\Big(S/((I(E_n),y_{n-1}):x_{n-1})\Big), \operatorname{depth}\Big(S/((I(E_n),y_{n-1}),x_{n-1})\Big)\Big\}.\end{aligned} \tag{4.1.2}$$

We have

$$S/(I(E_n):y_{n-1}) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[y_{n-1},x_{n-1}], \tag{4.1.3}$$

$$S/((I(E_n),y_{n-1}),x_{n-1}) \cong K[V(E_{n-2})]/I(E_{n-2}) \otimes_K K[y_n,x_n], \tag{4.1.4}$$

$$S/((I(E_n),y_{n-1}):x_{n-1}) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_{n-1}]. \tag{4.1.5}$$

As $n-3 \equiv 1 \pmod 3$, by applying Lemma 1.4.10 and Remark 4.1.1 on Eq (4.1.3) and using induction on n, we get

$$\operatorname{depth}\Big(S/(I(E_n):y_{n-1})\Big) = \left\lceil \frac{n-3+4}{3} \right\rceil + 2 = \left\lceil \frac{n+4}{3} \right\rceil + 1.$$

Since $n-2 \equiv 2 \pmod 3$, by using Lemma 1.4.10 on Eq (4.1.4) and induction on n, it follows

$$\operatorname{depth}\Big(S/((I(E_n),y_{n-1}),x_{n-1})\Big) = \left\lceil \frac{n-2}{3} \right\rceil + 2 = \left\lceil \frac{n+4}{2} \right\rceil.$$

Now, by Eq (4.1.5) and applying induction on n, Lemma 1.4.10 and Remark 4.1.1, we get

$$\operatorname{depth}\Big(S/((I(E_n),y_{n-1}):x_{n-1})\Big) = \left\lceil \frac{n-3+4}{3} \right\rceil + 1 = \left\lceil \frac{n+4}{3} \right\rceil.$$

Here

$$\operatorname{depth}\Big(S/((I(E_n),y_{n-1}),x_{n-1})\Big) = \operatorname{depth}\Big(S/((I(E_n),y_{n-1}):x_{n-1})\Big),$$

by using Eq (4.1.2),

$$\operatorname{depth}\Big(S/(I(E_n),y_{n-1})\Big) \geq \left\lceil \frac{n+4}{3} \right\rceil.$$

By Eq (4.1.1), we get

$$\operatorname{depth}(S/(I(E_n)) \geq \left\lceil \frac{n+4}{3} \right\rceil. \tag{4.1.6}$$

For the other inequality, if $y_n \notin I(E_n)$, then

$$S/(I(E_n):y_n) \cong K[V(E_{n-2})]/I(E_{n-2}) \otimes_K K[y_n,x_n].$$

Since $n-2 \equiv 2\,(\text{mod}\,3)$, by Lemmas 1.3.12, 1.4.10 and induction on n,

$$\operatorname{depth}(S/I(E_n)) \leq \operatorname{depth}(S/(I(E_n):y_n)) = \left\lceil \frac{n-2}{3} \right\rceil + 2 = \left\lceil \frac{n+4}{3} \right\rceil. \tag{4.1.7}$$

We get the required result by combining Eqs (4.1.6) and (4.1.7).

Case 2. Let $n \equiv 2(\text{mod}\,3)$. Consider the short exact sequence

$$0 \longrightarrow (I(E_n):y_{n-1})/I(E_n) \xrightarrow{\cdot y_{n-1}} S/I(E_n) \longrightarrow S/(I(E_n):y_{n-1}) \longrightarrow 0. \tag{4.1.8}$$

Note that here we have

$$N_{E_n}(y_{n-1}) = \{y_{n-2}, x_{n-2}, y_n, x_n\},$$

$$S_1 = K[V(E_n) \backslash N_{E_n}(y_{n-2})],$$

$$S_2 = K\big[V(E_n) \backslash (N_{E_n}(x_{n-2}) \cup \{y_{n-2}\})\big],$$

$$S_3 = K\big[V(E_n) \backslash (N_{E_n}(y_n) \cup \{y_{n-2}, x_{n-2}\})\big],$$

$$S_4 = K\big[V(E_n) \backslash (N_{E_n}(x_n) \cup \{y_{n-2}, x_{n-2}, y_n\})\big],$$

$$J_1 = (S_1 \cap I(E_n)), \quad J_2 = (S_2 \cap I(E_n)),$$

$$J_3 = (S_3 \cap I(E_n)), \quad J_4 = (S_4 \cap I(E_n)),$$

then by using Lemma 3.1.1, we get

$$\begin{aligned}
(I(E_n):y_{n-1})/I(E_n) &\cong S_1/J_1[y_{n-2}] \oplus S_2/J_2[x_{n-2}] \oplus S_3/J_3[y_n] \oplus S_4/J_4[x_n] \\
&\cong \frac{K[x_1,\dots,x_{n-4},x_{n-2},x_n,y_1,\dots,y_{n-4},y_n]}{\left(\cup_{i=1}^{n-5}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\}\right)}[y_{n-2}] \\
&\quad \oplus \frac{K[x_1,\dots,x_{n-4},x_n,y_1,\dots,y_{n-4},y_n]}{\left(\cup_{i=1}^{n-5}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\}\right)}[x_{n-2}] \\
&\quad \oplus \frac{K[x_1,\dots,x_{n-3},x_n,y_1,\dots,y_{n-3}]}{\left(\cup_{i=1}^{n-4}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\}\right)}[y_n] \\
&\quad \oplus \frac{K[x_1,\dots,x_{n-3},y_1,\dots,y_{n-3}]}{\left(\cup_{i=1}^{n-4}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\}\right)}[x_n] \\
&\cong \Big(K[V(E_{n-4})]/I(E_{n-4}) \otimes_K K[x_{n-2},x_n,y_n,y_{n-2}]\Big) \\
&\quad \oplus \Big(K[V(E_{n-4})]/I(E_{n-4}) \otimes_K K[x_n,y_n,x_{n-2}]\Big) \\
&\quad \oplus \Big(K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_n,y_n]\Big) \\
&\quad \oplus \Big(K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_n]\Big).
\end{aligned}$$

By Lemma 1.4.10 on Eq (4.1.3),

$$\operatorname{depth}(S/(I(E_n):y_{n-1})) = \operatorname{depth} K[V(E_{n-3})]/I(E_{n-3}) + \operatorname{depth} K[y_{n-1},x_{n-1}]. \tag{4.1.9}$$

Also,

$$\begin{aligned}
&\operatorname{depth}\big((I(E_n):y_{n-1})/I(E_n)\big)\\
&\quad=\min\Big\{\operatorname{depth}(K[V(E_{n-4})]/I(E_{n-4}))+4,\operatorname{depth}(K[V(E_{n-4})]/I(E_{n-4}))+3,\\
&\qquad\qquad\operatorname{depth}(K[V(E_{n-3})]/I(E_{n-3}))+2,\operatorname{depth}(K[V(E_{n-3})]/I(E_{n-3}))+1\Big\}.
\end{aligned}\tag{4.1.10}$$

Here $n-4\equiv 1\,(\bmod 3)$ and $n-3\equiv 2\,(\bmod 3)$. We apply induction on Eq (4.1.9) and get

$$\operatorname{depth}\big(S/(I(E_n):y_{n-1})\big)=\left\lceil\frac{n-3}{3}\right\rceil+2=\left\lceil\frac{n}{3}\right\rceil+1.\tag{4.1.11}$$

Using induction on n and Remark 4.1.1 on Eq (4.1.10),

$$\begin{aligned}
&\operatorname{depth}\big((I(E_n):y_{n-1})/I(E_n)\big)\\
&=\min\left\{\left\lceil\frac{n-4+4}{3}\right\rceil+4,\left\lceil\frac{n-4+4}{3}\right\rceil+3,\left\lceil\frac{n-3}{3}\right\rceil+2,\left\lceil\frac{n-3}{3}\right\rceil+1\right\}=\left\lceil\frac{n}{3}\right\rceil.
\end{aligned}\tag{4.1.12}$$

We get the required result by applying Lemma 3.1.4 on Eq (4.1.8).

Case 3. If $n\equiv 0\,(\bmod 3)$, then $n-4\equiv 2\,(\bmod 3)$ and $n-3\equiv 0\,(\bmod 3)$. By applying induction on Eq (4.1.9),

$$\operatorname{depth}\big(S/(I(E_n):y_{n-1})\big)=\left\lceil\frac{n-3}{3}\right\rceil+2=\left\lceil\frac{n}{3}\right\rceil+1.\tag{4.1.13}$$

By using Eq (4.1.10) and applying induction on n, we get

$$\begin{aligned}
&\operatorname{depth}\big((I(E_n):y_{n-1})/I(E_n)\big)\\
&=\min\left\{\left\lceil\frac{n-4}{3}\right\rceil+4,\left\lceil\frac{n-4}{3}\right\rceil+3,\left\lceil\frac{n-3}{3}\right\rceil+2,\left\lceil\frac{n-3}{3}\right\rceil+1\right\}=\left\lceil\frac{n}{3}\right\rceil.
\end{aligned}\tag{4.1.14}$$

The desired result is obtained by applying Lemma 3.1.4 on Eq (4.1.8).

This ends the proof for depth. Next, we provide the result for the lower bound of Stanley depth. If $n=2$, then $E_2\cong C_4$ and the result holds by Lemma 1.3.18. If $n=3$, we get the required result from Lemma 1.3.19. Let $n\geq 4$. We get the lower bound for Stanley depth in a similar way to the depth just by replacing Lemmas 3.1.4 and 1.3.12 with Lemmas 1.3.10 and 1.3.12, respectively. □

By using the Auslander Buchsbaum formula, we have the following result.

Corollary 4.1.5 ([82])**.** *Let $n \geq 2$ and $S = K[V(E_n)]$. Then*

$$\operatorname{pdim}(S/I(E_n)) = \begin{cases} 2n - \left\lceil \frac{n+4}{3} \right\rceil, & \text{if } n \equiv 1 \pmod 3; \\ 2n - \left\lceil \frac{n}{3} \right\rceil, & \text{otherwise}. \end{cases}$$

Proof. The required result follows from Lemmas 1.4.11 and 4.1.4. □

Now we will find the upper bound for Stanley depth of $K[V(E_n)/I(E_n)]$.

Lemma 4.1.6 ([82])**.** *Let $n \geq 2$ and $S = K[V(E_n)]$. Then*

$$\operatorname{sdepth}(S/I(E_n)) \leq \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 \pmod 3; \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 \pmod 3; \\ \frac{2n+4}{3}, & \text{if } n \equiv 1 \pmod 3. \end{cases}$$

Proof. If $n = 2$, then $E_2 \cong C_4$ and we get the required result by Lemma 1.3.18. If $n = 3$, since $y_2 \notin I(E_3)$ then $S/(I(E_3) : y_2) \cong K[x_2, y_2]/(0)$, we have by Lemma 1.3.12,

$$\operatorname{sdepth}(S/I(E_3)) \leq \operatorname{sdepth}(S/(I(E_3) : y_2)) = \operatorname{sdepth}(K[x_2, y_2]) = 2.$$

Let $n \geq 4$. If $n \equiv 0 \pmod 3$, then $n - 3 \equiv 0 \pmod 3$. Since $x_{n-1}y_{n-1} \notin I(E_n)$, we have

$$S/\big(I(E_n) : x_{n-1}y_{n-1}\big) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_{n-1}, y_{n-1}].$$

By using Lemma 1.4.10 and applying induction on n,

$$\operatorname{sdepth}\big(S/\big(I(E_n) : x_{n-1}y_{n-1}\big)\big) = \operatorname{sdepth}(K[V(E_{n-3})]/I(E_{n-3})) + 2 \leq \frac{2(n-3)}{3} + 2 = \frac{2n}{3}.$$

Therefore, by applying Lemma 1.3.12, we get

$$\operatorname{sdepth}(S/I(E_n)) \leq \operatorname{sdepth}\big(S/\big(I(E_n) : x_{n-1}y_{n-1}\big)\big) \leq \frac{2n}{3}.$$

Let $n \equiv 2 (\text{mod}\, 3)$. Since $y_n \notin I(E_n)$,

$$S/(I(E_n) : y_n) \cong K[V(E_{n-2})]/I(E_{n-2}) \otimes_K K[y_n, x_n].$$

Since $n - 2 \equiv 0 \pmod 3$, by using Lemmas 1.3.12, 1.4.10 and induction on n, we get

$$\begin{aligned} \operatorname{sdepth}(S/I(E_n)) &\leq \operatorname{sdepth}(S/(I(E_n) : y_n)) = \operatorname{sdepth}(K[V(E_{n-2})]/I(E_{n-2})) + 2 \\ &\leq \frac{2(n-2)}{3} + 2 = \frac{2n+2}{3}. \end{aligned}$$

If $n \equiv 1 \pmod 3$, then $n-2 \equiv 2 \pmod 3$. The proof follows a similar strategy and we get

$$\begin{aligned}\operatorname{sdepth}((S/I(E_n))) \leq \operatorname{sdepth}(S/(I(E_n):y_n)) &= \operatorname{sdepth}(K[V(E_{n-2})]/I(E_{n-2}))+2\\ &\leq \frac{2(n-2)+2}{3}+2=\frac{2n+4}{3}.\end{aligned}$$

This ends the proof. □

Example 4.1.7. For $n=226$, we have

(a) $\operatorname{depth}(K[V(E_{226})])/I(E_{226}) = \lceil \frac{226+4}{3} \rceil = 77.$

(b) $77 \leq \operatorname{sdepth}(K[V(E_{226})])/I(E_{226}) \leq \frac{452+4}{3} = 152.$

(c) $\operatorname{pdim}(K[V(E_{226})])/I(E_{226}) = 452 - \lceil \frac{226+4}{3} \rceil = 375.$

In the following lemmas we will find the exact values of the cyclic modules $K[V(E_n)]/I(E_n)$, $K[V(F_n)]/I(F_n)$ and $K[V(G_n)]/I(G_n)$ for regularity.

Lemma 4.1.8 ([82])**.** *Let $n \geq 2$ and $S = K[V(E_n)]$. Then* $\operatorname{reg}(S/I(E_n)) = \lceil \frac{n-1}{3} \rceil$.

Proof. Let $S = K[V(E_n)]$. If $n=2$ then by using Lemma 1.4.16, we have $\operatorname{reg}(K[V(E_2)]/I(E_2)) = \operatorname{reg}(K[V(C_4)]/I(C_4)) = 1$. Let $n \geq 3$, we have the following K-algebra isomorphisms:

$$S/(I(E_n):x_{n-2}) \cong K[V(E_{n-4})]/I(E_{n-4}) \otimes_K K[x_{n-2},y_{n-2},x_n,y_n], \tag{4.1.15}$$

$$S/\big((I(E_n),x_{n-2}),y_{n-2}\big) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[V(E_2)]/I(E_2), \tag{4.1.16}$$

$$S/\big((I(E_n),x_{n-2}):y_{n-2}\big) \cong K[V(E_{n-4})]/I(E_{n-4}) \otimes_K K[y_{n-2},x_n,y_n]. \tag{4.1.17}$$

If $n=3$, by using Eq (4.1.15) we get $S/(I(E_3):x_1) \cong K[V(E_{-1})]/I(E_{-1}) \otimes_K K[x_1,y_1,x_3,y_3]$. Moreover, by Eq (4.1.16), we have $S/((I(E_3),x_1),y_1) \cong K[V(E_0)]/I(E_0) \otimes_K K[V(E_2)]/I(E_2)$, and by Eq (4.1.17), $S/((I(E_3),x_1):y_1) \cong K[V(E_{-1})]/I(E_{-1}) \otimes_K K[y_1,x_3,y_3]$. By Remark 4.1.1 and Lemma 1.4.10, we get

$$\begin{aligned}\operatorname{reg}\Big(S/(I(E_3):x_1)\Big) &= \operatorname{reg}\Big(K[V(E_{-1})]/I(E_{-1})\Big) = \operatorname{reg}(K) = 0,\\ \operatorname{reg}\Big(S/((I(E_3),x_1):y_1)\Big) &= \operatorname{reg}\Big(K[V(E_{-1})]/I(E_{-1})\Big) = \operatorname{reg}(K) = 0,\end{aligned}$$

and

$$\operatorname{reg}\Big(S/((I(E_3),x_1),y_1)\Big) = \operatorname{reg}\Big(K[V(E_0)]/I(E_0)\Big) + \operatorname{reg}\Big(K[V(E_2)]/I(E_2)\Big) = 0+1=1.$$

Since $\operatorname{reg}\Big(S/((I(E_3),x_1):y_1)\Big) < \operatorname{reg}\Big(S/((I(E_3),x_1),y_1)\Big)$, by using Lemma 1.4.12(c), we get $\operatorname{reg}(S/(I(E_3),x_1)) = 1$. Also, $\operatorname{reg}(S/(I(E_3):x_1)) < \operatorname{reg}(S/(I(E_3),x_1)$, again by Lemma 1.4.12(c), $\operatorname{reg}(S/I(E_3)) = 1$. If $n = 4$, by using a similar strategy one can get $\operatorname{reg}(S/I(E_4)) = 1$. Let $n \geq 5$. By using induction on n, Remark 4.1.1, Lemma 1.4.10 and Eqs (4.1.15)–(4.1.17), we get

$$\operatorname{reg}\Big(S/(I(E_n):x_{n-2})\Big) = \operatorname{reg}\Big(K[V(E_{n-4})]/I(E_{n-4})\Big) = \left\lceil \frac{n-5}{3} \right\rceil,$$

$$\operatorname{reg}\Big(S/((I(E_n),x_{n-2}):y_{n-2})\Big) = \operatorname{reg}\Big(K[V(E_{n-4})]/I(E_{n-4})\Big) = \left\lceil \frac{n-5}{3} \right\rceil,$$

and by Lemma 1.4.14,

$$\begin{aligned}\operatorname{reg}\Big(S/((I(E_n),x_{n-2}),y_{n-2})\Big) &= \operatorname{reg}\Big(K[V(E_{n-3})]/I(E_{n-3})\Big) + \operatorname{reg}\Big(K[V(E_2)]/I(E_2)\Big)\\ &= \left\lceil \frac{n-4}{3} \right\rceil + 1 = \left\lceil \frac{n-1}{3} \right\rceil.\end{aligned}$$

Since

$$\operatorname{reg}\Big(S/((I(E_n),x_{n-2}):y_{n-2})\Big) < \operatorname{reg}\Big(S/((I(E_n),x_{n-2}),y_{n-1})\Big),$$

by Lemma 1.4.12(c) we get $\operatorname{reg}\Big(S/(I(E_n),x_{n-2})\Big) = \left\lceil \frac{n-1}{3} \right\rceil$. Also we have

$$\operatorname{reg}\Big(S/(I(E_n):x_{n-2})\Big) < \operatorname{reg}\Big(S/(I(E_n),x_{n-2}\Big).$$

Again by Lemma 1.4.12(c), the desired result follows. □

Lemma 4.1.9 ([82]). *Let $n \geq 2$ and $S = K[V(F_n)]$. Then* $\operatorname{reg}(S/I(F_n)) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. If $n = 2$, then we have $S/(I(F_2):y_1) \cong K[y_1,x_2]$, and $S/(I(F_2),y_1) \cong K[V(C_3)]/I(C_3)$. By Lemmas 1.4.10 and 1.4.16, $\operatorname{reg}\Big(S/(I(F_2):y_1)\Big) = 0$ and $\operatorname{reg}\Big(S/(I(F_2),y_1)\Big) = K[V(C_3)]/I(C_3) = 1$. Since $\operatorname{reg}\Big(S/(I(F_2):y_1)\Big) < \operatorname{reg}\Big(S/(I(F_2),y_1)\Big)$, therefore by Lemma 1.4.12(c), we have $\operatorname{reg}(K[V(F_2)]/I(F_2)) = 1$. Let $n = 3$ and $F_3 = H_1 \cup H_2$, where $H_1 \cong H_2 \cong F_2$ and $H_1 \cap H_2 \neq \emptyset$. By Lemma 1.4.13, we get

$$\operatorname{reg}\Big(S/I(F_3)\Big) \leq \operatorname{reg}\Big(K[V(H_1)]/I(H_1)\Big) + \operatorname{reg}\Big(K[V(H_2)]/I(H_2)\Big) = 2.$$

For the other inequality, let $M = \Big\{\{x_1,y_1\},\{x_3,y_3\}\Big\}$. It is clear that M is an induced matching, therefore, $\operatorname{indmat}(F_n) \geq |M| = 2$. By combining the two inequalities, we get $\operatorname{reg}(S/I(F_3)) = 2$. Let $n \geq 4$. Here we consider the following two cases:

Case 1. If n is even. We have the K-algebra isomorphisms:

$$S/(I(F_n):y_{n-1}) \cong K[V(F_{n-3})]/I(F_{n-3}) \otimes_K K[y_{n-1},x_n], \tag{4.1.18}$$

$$S/\big((I(F_n),y_{n-1}),x_{n-1}\big) \cong K[V(F_{n-2})]/I(F_{n-2}) \otimes_K K[V(P_2)]/I(P_2), \tag{4.1.19}$$

$$S/\big(\big((I(F_n),y_{n-1}):x_{n-1}\big),y_{n-2}\big) \cong K[V(F_{n-3})]/I(F_{n-3}) \otimes_K K[x_{n-1}], \tag{4.1.20}$$

$$S/\big(\big((I(F_n),y_{n-1}):x_{n-1}\big):y_{n-2}\big) \cong K[V(F_{n-4})]/I(F_{n-4}) \otimes_K K[x_{n-1},y_{n-2}]. \tag{4.1.21}$$

If $n=4$, we have

$$S/(I(F_4):y_3) \cong K[V(F_1)]/I(F_1) \otimes_K K[y_3,x_4],$$

$$S/\big((I(F_4),y_3),x_3\big) \cong K[V(F_2)]/I(F_2) \otimes_K K[V(P_2)]/I(P_2),$$

$$S/\big((I(F_4),y_3):x_3\big) \cong K[V(C_3)]/I(C_3) \otimes_K K[x_3].$$

By Lemma 1.4.10, Remark 4.1.1 we have $\operatorname{reg}\Big(S/(I(F_4):y_3)\Big) = \operatorname{reg}\Big(K[V(F_1)]/I(F_1)\Big) = 1$, and by Lemmas 1.4.15 and 1.4.16, we have

$$\operatorname{reg}\Big(S/\big((I(F_4),y_3),x_3\big)\Big) = \operatorname{reg}\Big(K[V(F_2)]/I(F_2)\Big) + \operatorname{reg}\Big(K[V(P_2)]/I(P_2)\Big) = 2$$

and

$$\operatorname{reg}\Big(S/\big((I(F_4),y_3):x_3\big)\Big) = \operatorname{reg}\Big(K[V(C_3)]/I(C_3)\Big) = 1.$$

Since $\operatorname{reg}\Big(S/\big((I(F_4),y_3):x_3\big)\Big) < \operatorname{reg}\Big(S/\big((I(F_4),y_3),x_3\big)\Big)$, by using Lemma 1.4.12(c), we get $\operatorname{reg}\Big(S/(I(F_4),y_3)\Big) = 2$. Moreover, $\operatorname{reg}\Big(S/(I(F_4):y_3)\Big) < \operatorname{reg}\Big(S/(I(F_4),y_3\Big)$, and again by using Lemma 1.4.12(c), we get $\operatorname{reg}\big(S/(I(F_4)\big) = 2$. Let $n \geq 6$. By using induction on n, Lemmas 1.4.10 and 1.4.14 on Eqs (4.1.18)–(4.1.21), we get

$$\operatorname{reg}\Big(S/(I(F_n):y_{n-1})\Big) = \operatorname{reg}\Big(K[V(F_{n-3})]/I(F_{n-3})\Big) = \left\lceil \frac{n-3}{2} \right\rceil,$$

$$\begin{aligned}\operatorname{reg}\Big(S/\big((I(F_n),y_{n-1}),x_{n-1}\big)\Big) &= \operatorname{reg}\Big(K[V(F_{n-2})]/I(F_{n-2})\Big) + \operatorname{reg}\Big(K[V(P_2)]/I(P_2)\Big)\\ &= \left\lceil \frac{n-2}{2} \right\rceil + 1 = \left\lceil \frac{n}{2} \right\rceil,\end{aligned}$$

$$\operatorname{reg}\Big(S/\big(\big((I(F_n),y_{n-1}):x_{n-1}\big),y_{n-2}\big)\Big) = \operatorname{reg}\Big(K[V(F_{n-3})]/I(F_{n-3})\Big) = \left\lceil \frac{n-3}{2} \right\rceil,$$

$$\operatorname{reg}\Big(S/\big(\big((I(F_n),y_{n-1}):x_{n-1}\big):y_{n-2}\big)\Big) = \operatorname{reg}\Big(K[V(F_{n-4})]/I(F_{n-4})\Big) = \left\lceil \frac{n-4}{2} \right\rceil.$$

Since n is even, therefore

$$\operatorname{reg}\Big(S/\big(\big((I(F_n),y_{n-1}):x_{n-1}\big):y_{n-2}\big)\Big) < \operatorname{reg}\Big(S/\big(\big((I(F_n),y_{n-1}):x_{n-1}\big),y_{n-2}\big)\Big),$$

by Lemma 1.4.12(c), we get

$$\operatorname{reg}\Big(S/\big((I(F_n),y_{n-1}):x_{n-1}\big)\Big) = \left\lceil \frac{n-3}{2} \right\rceil.$$

Also, $\operatorname{reg}\Big(S/\big((I(F_n),y_{n-1}):x_{n-1}\big)\Big) < \operatorname{reg}\Big(S/\big((I(F_n),y_{n-1}),x_{n-1}\big)\Big)$, again by Lemma 1.4.12(c), $\operatorname{reg}\Big(S/(I(F_n),y_{n-1})\Big) = \left\lceil \frac{n}{2} \right\rceil$. We have $\operatorname{reg}\Big(S/(I(F_n):y_{n-1})\Big) < \operatorname{reg}\Big(S/(I(F_n),y_{n-1})\Big)$, therefore, by Lemma 1.4.12(c), the required result follows.

Case 2. If n is odd. Here $F_n = F_2 \cup H$, where $H \cong F_{n-1}$ and $F_2 \cap H \neq \emptyset$. By induction on n and Lemma 1.4.13, we get

$$\begin{aligned}\operatorname{reg}(S/I(F_n)) &\leq \operatorname{reg}(K[V(F_2)]/I(F_2)) + \operatorname{reg}(K[V(F_{n-1})]/I(F_{n-1}))\\ &= 1 + \left\lceil \frac{n-1}{2} \right\rceil = \left\lceil \frac{n+1}{2} \right\rceil = \left\lceil \frac{n}{2} \right\rceil.\end{aligned}$$

For the other inequality, we define $M = \Big\{\{x_1,y_1\},\{x_3,y_3\},\ldots,\{x_n,y_n\}\Big\}$. M is clearly an induced matching and $|M| = \left\lceil \frac{n}{2} \right\rceil$, thus, $\operatorname{indmat}(F_n) \geq \left\lceil \frac{n}{2} \right\rceil$. By Lemma 1.4.9, we have $\operatorname{reg}(S/I(F_n)) \geq \left\lceil \frac{n}{2} \right\rceil$.

□

Lemma 4.1.10 ([82]). *If $n \geq 2$ and $S = K[V(G_n)]$, then* $\operatorname{reg}(S/I(G_n)) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. If $n = 2$, then clearly $G_2 \cong K_4$, therefore by Lemma 1.4.9, $\operatorname{indmat}(G_2) = 1$ and we have $\operatorname{reg}(K[V(G_2)]/I(G_2)) = 1$. Let $n \geq 3$, we have the K-algebra isomorphisms:

$$S/(I(G_n):y_{n-1}) \cong K[V(G_{n-3})]/I(G_{n-3}) \otimes_K K[y_{n-1}], \tag{4.1.22}$$

$$S/((I(G_n),y_{n-1}),x_{n-1}) \cong K[V(G_{n-2})]/I(G_{n-2}) \otimes_K K[V(P_2)]/I(P_2), \tag{4.1.23}$$

$$S/((I(G_n),y_{n-1}):x_{n-1}) \cong K[V(G_{n-3})]/I(G_{n-3}) \otimes_K K[x_{n-1}]. \tag{4.1.24}$$

If $n = 3$, we have

$$S/(I(G_3):y_2) \cong K[V(G_0)]/I(G_0) \otimes_K K[y_2],$$

$$S/((I(G_3),y_2),x_2) \cong K[V(G_1)]/I(G_1) \otimes_K K[V(P_2)]/I(P_2),$$

$$S/((I(G_3),y_2):x_2) \cong K[V(G_0)]/I(G_0) \otimes_K K[x_2].$$

By Remark 4.1.1, Lemmas 1.4.10 and 1.4.15, $\operatorname{reg}\Big(S/(I(G_3):y_2)\Big) = 0$, $\operatorname{reg}\Big(S/\big((I(G_3),y_2),x_2\big)\Big) = \operatorname{reg}\Big(K[V(G_1)]/I(G_1)\Big) + \operatorname{reg}\Big(K[V(P_2)]/I(P_2)\Big) = 2$ and $\operatorname{reg}\Big(S/\big((I(G_3),y_2):x_2\big)\Big) = 0$. Since $\operatorname{reg}\Big(S/\big((I(G_3),y_2):x_2\big)\Big) < \operatorname{reg}\Big(S/\big((I(G_3),y_2),x_2\big)\Big)$, by Lemma 1.4.12(c), $\operatorname{reg}\Big(S/(I(G_3),y_2)\Big) = \left\lceil \frac{3}{2} \right\rceil = 2$. Also, we have $\operatorname{reg}\Big(S/(I(G_3):y_2)\Big) < \operatorname{reg}\Big(S/(I(G_3),y_2\Big)$, and again by Lemma 1.4.12(c), we get $\operatorname{reg}(S/I(G_3)) = 2$. Let $n \geq 4$. By induction on n, Lemmas 1.4.10, 1.4.15 and using Eqs (4.1.22)–(4.1.24),

$$\operatorname{reg}\Big(S/(I(G_n):y_{n-1})\Big) = \operatorname{reg}\Big(K[V(G_{n-3})]/I(G_{n-3})\Big) = \left\lceil \frac{n-3}{2} \right\rceil,$$

$$\operatorname{reg}\Big(S/((I(G_n),y_{n-1}):x_{n-1})\Big) = \operatorname{reg}\Big(K[V(G_{n-3})]/I(G_{n-3})\Big) = \left\lceil \frac{n-3}{2} \right\rceil,$$

and by Lemma 1.4.14,

$$\begin{aligned}\operatorname{reg}\Big(S/((I(G_n),y_{n-1}),x_{n-1})\Big) &= \operatorname{reg}\Big(K[V(G_{n-2})]/I(G_{n-2})\Big) + \operatorname{reg}\Big(K[V(P_2)]/I(P_2)\Big)\\ &= \left\lceil \frac{n-2}{2} \right\rceil + 1 = \left\lceil \frac{n}{2} \right\rceil.\end{aligned}$$

Since we have $\operatorname{reg}\Big(S/((I(G_n),y_{n-1}):x_{n-1})\Big) < \operatorname{reg}\Big(S/((I(G_n),y_{n-1}),x_{n-1})\Big)$, by using Lemma 1.4.12(c), $\operatorname{reg}\Big(S/(I(G_n),y_{n-1})\Big) = \lceil \frac{n}{2} \rceil$. Also, $\operatorname{reg}\Big(S/(I(G_n):y_{n-1})\Big) < \operatorname{reg}\Big(S/(I(G_n),y_{n-1}\Big)$, and again by Lemma 1.4.12(c), required result follows. This ends the proof. □

Example 4.1.11. For $n = 1013$, we have

(a) $\operatorname{reg}(K[V(E_{1013})])/I(E_{1013}) = \lceil \frac{1013-1}{3} \rceil = 338.$

(b) $\operatorname{reg}(K[V(F_{1013})])/I(F_{1013}) = \lceil \frac{1013}{2} \rceil = 507.$

(c) $\operatorname{reg}(K[V(G_{1013})])/I(G_{1013}) = \lceil \frac{1013}{2} \rceil = 507.$

4.2 Invariants of cyclic modules associated with $C_{2n}(1,n-1), C_{2n}(1,2)$ and $C_{2n}(1,n-1,n)$

In this section, we find some invariants of the edge ideals of some families of 4-regular and 5-regular circulant graphs. We find depth and projective dimension of the cyclic module $K[V(C_{2n}(1,n-1))]/I(C_{2n}(1,n-1))$. Moreover, bounds for Stanley depth of such module are also given. When $n \equiv 0,1 (\bmod 3)$, we give the exact value for the regularity of such module, otherwise, we have sharp bounds. Zahid et al. gave values and sharp bounds in [66, Corollaries 3.6 and 3.8] for depth and Stanley depth of module $K[V(C_{2n}(1,2))]/I(C_{2n}(1,2))$. For cyclic module $K[V(C_{2n}(1,2))]/I(C_{2n}(1,2))$, we give the exact value of regularity if n is even and sharp bounds if n is odd. Also, the exact values for depth and sharp bounds for Stanley depth of the module $K[V(C_{2n}(1,n-1,n))]/I(C_{2n}(1,n-1,n))$ given by Zahid et al. in [76, Theorem 3.3, and Corollary 3.4]. We find the exact value of the regularity of $K[V(C_{2n}(1,n-1,n))]/I(C_{2n}(1,n-1,n))$.

It will be convenient to use the labeling of the vertices of the graphs as shown in Figures 4.7 and 4.8.

Before proving the main results, we give the following example by using Remark 4.1.3 which will be helpful in understanding the strategy of the proofs. See for instance; Figures 4.9 and 4.10 for subgraphs $G_{(I(C_{16}(1,7)):x_8)}, G_{(I(C_{16}(1,7)),x_8)}, G_{((I(C_{16}(1,7)),x_8),y_8)}$ and $G_{((I(C_{16}(1,7)),x_8):y_8)}$ of circulant graph $G_{I(C_{16}(1,7)}$. It is clear from the Figures 4.9 and 4.10, we have the following isomorphisms:

$$K[V(C_{16}(1,7))]/(I(C_{16}(1,7)):x_8) \cong K[V(E_5)]/I(E_5) \otimes_K K[x_8,y_8],$$

$$K[V(C_{16}(1,7))]/(I(C_{16}(1,7)),x_8) \cong K[V(E_7),y_8]/(I(E_7),x_1y_8,y_1y_8,x_7y_8,y_7y_8),$$

$$K[V(C_{16}(1,7))]/\big((I(C_{16}(1,7)),x_8),y_8\big) \cong K[V(E_7)]/I(E_7),$$

and

$$K[V(C_{16}(1,7))]/\big((I(C_{16}(1,7)),x_8):y_8\big) \cong K[V(E_5)]/I(E_5) \otimes_K K[y_8].$$

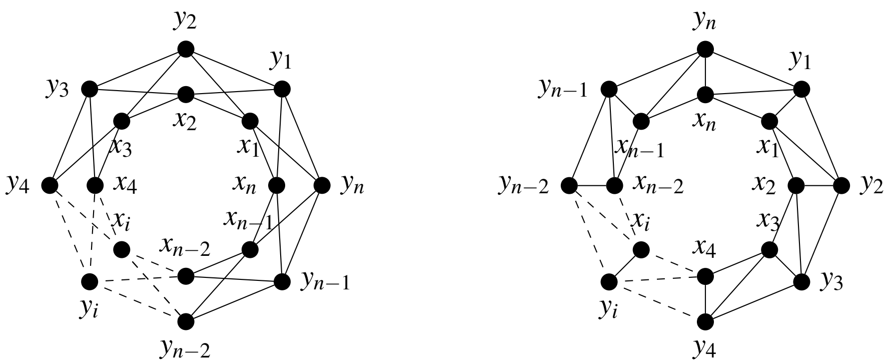


Figure 4.7: From left to right $C_{2n}(1,n-1)$ and $C_{2n}(1,2)$.

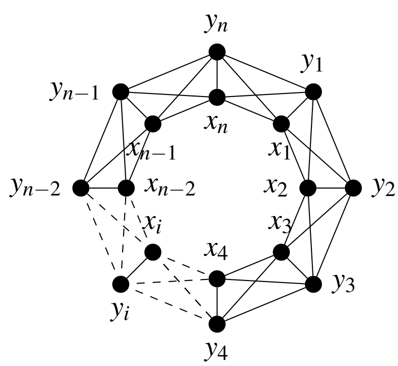


Figure 4.8: $C_{2n}(1,n-1,n)$.

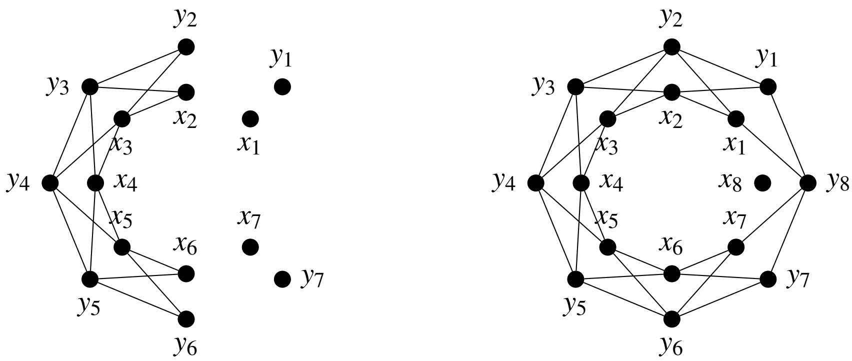


Figure 4.9: From left to right $G_{(I(C_{16}(1,7)):x_8)}$ and $G_{(I(C_{16}(1,7)),x_8)}$.

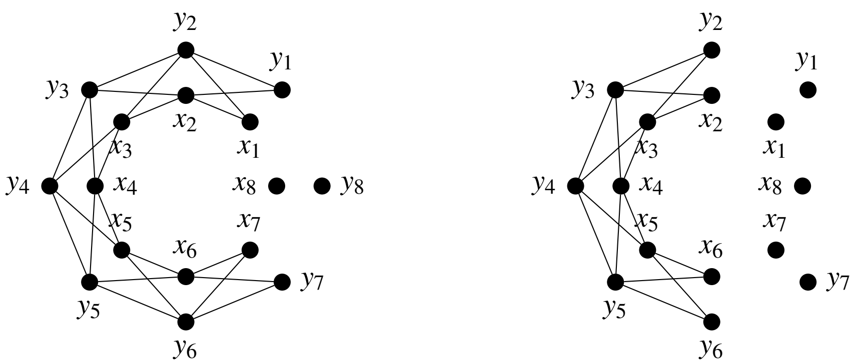


Figure 4.10: From left to right $G_{((I(C_{16}(1,7)),x_8),y_8)}$ and $G_{((I(C_{16}(1,7)),x_8):y_8)}$.

Firstly, we will compute the exact value of depth and lower bound of Stanley depth for cyclic module $K[V(C_{2n}(1,n-1))]/I(C_{2n}(1,n-1))$.

Theorem 4.2.1 ([82]). *Let $n \geq 3$, $G = C_{2n}(1,n-1)$ and $S = K[V(G)]$. Then*

$$\operatorname{sdepth}(S/I(G)) \geq \operatorname{depth}(S/I(G)) = \begin{cases} \left\lceil \frac{n-1}{3} \right\rceil, & \text{if } n \equiv 0,1 \pmod 3; \\ \\ \left\lceil \frac{n}{3} \right\rceil, & \text{otherwise}. \end{cases}$$

Proof. Firstly, we provide the proof for depth. If $n = 3$, we consider the short exact sequence

$$0 \longrightarrow (I(G):x_3)/I(G) \xrightarrow{\cdot x_3} S/I(G) \longrightarrow S/(I(G):x_3) \longrightarrow 0. \tag{4.2.1}$$

We have

$$K[V(G)]/(I(G):x_3) \cong \frac{K[y_3]}{(0)}[x_3], \tag{4.2.2}$$

and

$$N_G(x_3) = \{y_2, x_2, y_1, x_1\},$$

$$S_1 = K[V(G)\backslash N_G(y_2)], \;\; S_2 = K[V(G)\backslash(N_G(x_2)\cup\{y_2\})],$$

$$S_3 = K[V(G)\backslash(N_G(y_1)\cup\{y_2,x_2\})], \;\; S_4 = K[V(G)\backslash(N_G(x_1)\cup\{y_2,x_2,y_1\})],$$

$$J_1 = (S_1 \cap I(G)), \;\; J_2 = (S_2 \cap I(G)),$$

$$J_3 = (S_3 \cap I(G)), \;\; J_4 = (S_4 \cap I(G)),$$

then by using Lemma 3.1.1, we have

$$\begin{aligned}(I(G):x_3)/I(G) &\cong S_1/J_1[y_2]\oplus S_2/J_2[x_2]\oplus S_3/J_3[y_1]\oplus S_4/J_4[x_1] \\ &\cong \frac{K[x_2]}{(0)}[y_2]\oplus\frac{K}{(0)}[x_2]\oplus\frac{K[x_1]}{(0)}[y_1]\oplus\frac{K}{(0)}[x_1].\end{aligned} \tag{4.2.3}$$

We apply Lemma 1.4.10 on Eq (4.2.2), $\operatorname{depth}(K[V(G)]/(I(G):x_3)) = \operatorname{depth}(K[y_3,x_3]) = 2$ and by Eq (4.2.3)

$$\begin{aligned}&\operatorname{depth}((I(G):x_3)/I(G)) \\ &\quad= \min\Big\{\operatorname{depth}(K[x_2])+1, \operatorname{depth}(K[x_2]), \operatorname{depth}(K[x_1])+1, \operatorname{depth}(K[x_1])\Big\} = 1.\end{aligned}$$

By using Lemma 3.1.4 on Eq (4.2.1), $\operatorname{depth}(S/I(G)) = 1$. If $n = 4$, we consider the short exact sequence

$$0 \longrightarrow (I(G):x_4)/I(G) \xrightarrow{\cdot x_4} S/I(G) \longrightarrow S/(I(G):x_4) \longrightarrow 0. \tag{4.2.4}$$

We have

$$K[V(G)]/(I(G):x_4) \cong K[x_2,x_4,y_2,y_4], \tag{4.2.5}$$

and

$$N_G(x_4) = \{y_3, x_3, y_1, x_1\},$$

$$S_1 = K[V(G)\backslash N_G(y_3)], \;\; S_2 = K[V(G)\backslash(N_G(x_3)\cup\{y_3\})],$$

$$S_3 = K[V(G)\backslash(N_G(y_1)\cup\{y_3,x_3\})], \;\; S_4 = K[V(G)\backslash(N_G(x_1)\cup\{y_3,x_3,y_1\})],$$

$$J_1 = (S_1 \cap I(G)), \;\; J_2 = (S_2 \cap I(G)),$$

$$J_3 = (S_3 \cap I(G)), \;\; J_4 = (S_4 \cap I(G)),$$

then by using Lemma 3.1.1,

$$\begin{aligned}(I(G):x_4)/I(G) &\cong S_1/J_1[y_3]\oplus S_2/J_2[x_3]\oplus S_3/J_3[y_1]\oplus S_4/J_4[x_1] \\ &\cong \frac{K[x_1,x_3,y_1]}{(0)}[y_3]\oplus\frac{K[x_1,y_1]}{(0)}[x_3]\oplus\frac{K[x_1]}{(0)}[y_1]\oplus\frac{K}{(0)}[x_1].\end{aligned} \tag{4.2.6}$$

By applying Lemma 1.4.10 on Eq (4.2.5),

$$\operatorname{depth}\Big(K[V(G)]/(I(G):x_4)\Big) = \operatorname{depth}\Big(K[x_2,x_4,y_2,y_4]\Big) = 4$$

and by using Eq (4.2.6) we get

$$\begin{aligned}&\operatorname{depth}\Big((I(G):x_4)/I(G)\Big)\\ &\quad = \min\Big\{\operatorname{depth}(K[x_1,x_3,y_1,y_3]),\operatorname{depth}(K[x_1,y_1,x_3]),\operatorname{depth}(K[x_1,y_1]),\operatorname{depth}(K[x_1])\Big\} = 1.\end{aligned}$$

By using Lemma 3.1.4 on Eq (4.2.4), we get $\operatorname{depth}(S/I(G)) = 1$. Let $n \geq 5$. Consider the short exact sequence

$$0 \longrightarrow (I(G):x_n)/I(G) \xrightarrow{\cdot x_n} S/I(G) \longrightarrow S/(I(G):x_n) \longrightarrow 0. \tag{4.2.7}$$

We have the following K-algebra isomorphisms:

$$S/(I(G):x_n) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[y_n,x_n], \tag{4.2.8}$$

and

$$N_G(x_n) = \{y_{n-1},x_{n-1},y_1,x_1\},$$

$$S_1 = K[V(G)\backslash N_G(y_{n-1})],\ \ S_2 = K[V(G)\backslash(N_G(x_{n-1})\cup\{y_{n-1}\})],$$

$$S_3 = K[V(G)\backslash(N_G(y_1)\cup\{y_{n-1},x_{n-1}\})],\ \ S_4 = K[V(G)\backslash(N_G(x_1)\cup\{y_{n-1},x_{n-1},y_1\})],$$

$$J_1 = (S_1\cap I(G)),\ \ J_2 = (S_2\cap I(G)),$$

$$J_3 = (S_3\cap I(G)),\ \ J_4 = (S_4\cap I(G)),$$

then by Lemma 3.1.1,

$$\begin{aligned}(I(G):x_n)/I(G) &\cong S_1/J_1[y_{n-1}]\oplus S_2/J_2[x_{n-1}]\oplus S_3/J_3[y_1]\oplus S_4/J_4[x_1]\\
&\cong \frac{K[x_1,\dots,x_{n-3},x_{n-1},y_1,\dots,y_{n-3}]}{\big(\cup_{i=1}^{n-4}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\}\big)}[y_{n-1}]\\
&\quad\oplus \frac{K[x_1,\dots,x_{n-3},y_1,\dots,y_{n-3}]}{\big(\cup_{i=1}^{n-4}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\}\big)}[x_{n-1}]\\
&\quad\oplus \frac{K[x_1,x_3,\dots,x_{n-2},y_3,\dots,y_{n-2}]}{\big(\cup_{i=3}^{n-3}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\}\big)}[y_1]\\
&\quad\oplus \frac{K[x_3,\dots,x_{n-2},y_3,\dots,y_{n-2}]}{\big(\cup_{i=3}^{n-3}\{x_iy_{i+1},x_ix_{i+1},y_iy_{i+1},x_{i+1}y_i\}\big)}[x_1]\\
&\cong \Big(K[V(E_{n-3})]/I(E_{n-3})\otimes_K K[x_{n-1},y_{n-1}]\Big)\oplus\Big(K[V(E_{n-3})]/I(E_{n-3})\otimes_K K[x_{n-1}]\Big)\\
&\quad\oplus\Big(K[V(E_{n-4})]/I(E_{n-4})\otimes_K K[x_1,y_1]\Big)\oplus\Big(K[V(E_{n-4})]/I(E_{n-4})\otimes_K K[x_1]\Big).\end{aligned} \tag{4.2.9}$$

By Lemma 1.4.10, we have

$$\operatorname{depth}(S/(I(G):x_n)) = \operatorname{depth} K[V(E_{n-3})]/I(E_{n-3}) + \operatorname{depth} K[y_n, x_n], \tag{4.2.10}$$

$$\begin{aligned}
&\operatorname{depth}\big((I(G):x_n)/I(G)\big) \\
&\quad = \min\Big\{\operatorname{depth}(K[V(E_{n-3})]/I(E_{n-3}))+2, \operatorname{depth}(K[V(E_{n-3})]/I(E_{n-3}))+1, \\
&\qquad \operatorname{depth}(K[V(E_{n-4})]/I(E_{n-4}))+2, \operatorname{depth}(K[V(E_{n-4})]/I(E_{n-4}))+1\Big\}.
\end{aligned} \tag{4.2.11}$$

If $n \equiv 1 \pmod 3$, then $n-3 \equiv 1 \pmod 3$ and $n-4 \equiv 0 \pmod 3$. By using Lemma 4.1.4 in Eq (4.2.10), we get

$$\operatorname{depth}(S/(I(G):x_n)) = \left\lceil \frac{n-3+4}{3} \right\rceil + 2 = \left\lceil \frac{n+4}{3} \right\rceil + 1.$$

By applying Lemma 4.1.4 on Eq (4.2.11), we get

$$\begin{aligned}
\operatorname{depth}\big((I(G):x_n)/I(G)\big) &= \min\left\{ \left\lceil \frac{n-3+4}{3} \right\rceil + 2, \left\lceil \frac{n-3+4}{3} \right\rceil + 1, \left\lceil \frac{n-4}{3} \right\rceil + 2, \left\lceil \frac{n-4}{3} \right\rceil + 1 \right\} \\
&= \min\left\{ \left\lceil \frac{n+4}{3} \right\rceil + 1, \left\lceil \frac{n+4}{3} \right\rceil, \left\lceil \frac{n-1}{3} \right\rceil + 1, \left\lceil \frac{n-1}{3} \right\rceil \right\} \\
&= \left\lceil \frac{n-1}{3} \right\rceil.
\end{aligned}$$

We obtain the desired result by applying Lemma 3.1.4 on Eq (4.2.7). If $n \equiv 0 \pmod 3$, the proof is similar. If $n \equiv 2 \pmod 3$, then $n-3 \equiv 2 \pmod 3$ and $n-4 \equiv 1 \pmod 3$. By using a similar strategy and Remark 4.1.1, we get $\operatorname{depth}(S/I(G)) = \lceil \frac{n}{3} \rceil$. For the lower bound of Stanley depth, the proof is similar to depth one has to replacing the Lemma 3.1.4 with Lemma 1.3.10. This ends the proof. □

Corollary 4.2.2 ([82]). *Let $n \geq 3$, $G = C_{2n}(1, n-1)$ and $S = K[V(G)]$. Then*

$$\operatorname{pdim}(S/I(G)) = \begin{cases} 2n - \lceil \frac{n-1}{3} \rceil, & \text{if } n \equiv 0, 1 \pmod 3; \\[2ex] 2n - \lceil \frac{n}{3} \rceil, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Proof. The required result follows by Lemma 1.4.11 and Theorem 4.2.1. □

Now we give an upper bound for Stanley depth of $K[V(C_{2n}(1, n-1))]/I(C_{2n}(1, n-1))$.

Proposition 4.2.3 ([82]). *Let $n \geq 3$, $G = C_{2n}(1,n-1)$ and $S = K[V(G)]$. Then*

$$\operatorname{sdepth}(S/I(G)) \leq \begin{cases} \frac{2n}{3}, & \text{if } n \equiv 0 \,(\operatorname{mod} 3); \\ \frac{2n+2}{3}, & \text{if } n \equiv 2 \,(\operatorname{mod} 3); \\ \frac{2n+4}{3}, & \text{if } n \equiv 1 \,(\operatorname{mod} 3). \end{cases}$$

Proof. If $n = 3$, by Lemma 1.3.12, $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(S/(I(G):x_3)$. By Eq (4.2.2), Lemma 1.4.10, $\operatorname{sdepth}(S/I(G)) \leq 2$. If $n = 4$ and $x_4 \notin I(G)$, by Lemma 1.3.12, $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(S/(I(G):x_4)$. By Eq (4.2.5), Lemma 1.4.10, $\operatorname{sdepth}(S/I(G)) \leq 4$, let $n \geq 5$. If $n \equiv 1 \,(\operatorname{mod} 3)$, then $n-3 \equiv 1 \,(\operatorname{mod} 3)$. By using Lemmas 1.3.12 and 1.4.10 on Eq (4.2.8), we get $\operatorname{sdepth}(S/I(G)) \leq \operatorname{sdepth}(S/(I(G):x_n)) = \operatorname{sdepth}(K[V(E_{n-3})]/I(E_{n-3})) + 2$. Thus, by using Lemma 4.1.6, $\operatorname{sdepth}(S/(I(G):x_n)) \leq \frac{2(n-3)+4}{3} + 2 = \frac{2n+4}{3}$. The required result follows that is $\operatorname{sdepth}(S/I(G)) \leq \frac{2n+4}{3}$. For $n \equiv 0,2 \,(\operatorname{mod} 3)$, the proof is similar. □

Remark 4.2.4 ([82]). Let $n \geq 3$, then Stanley's inequality for $K[V(C_{2n}(1,n-1))]/I(C_{2n}(1,n-1))$ holds.

The next two results provide the values and bounds for regularity of modules $K[V(C_{2n}(1,n-1))]/I(C_{2n}(1,n-1))$ and $K[V(C_{2n}(1,2))]/I(C_{2n}(1,2))$.

Theorem 4.2.5 ([82]). *Let $n \geq 3$ and $S = K[V(C_{2n}(1,n-1))]$. If $n \equiv 0,1 \,(\operatorname{mod} 3)$, then*

$$\operatorname{reg}\Big(S/I(C_{2n}(1,n-1)\Big) = \left\lceil \frac{n-2}{3} \right\rceil.$$

Otherwise

$$\left\lceil \frac{n-2}{3} \right\rceil \leq \operatorname{reg}\Big(S/I(C_{2n}(1,n-1)\Big) \leq \left\lceil \frac{n-2}{3} \right\rceil + 1.$$

Proof. We have the following K-algebra isomorphisms:

$$S/(I(C_{2n}(1,n-1)):x_n) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[x_n,y_n], \tag{4.2.12}$$

$$S/\Big(\big(I(C_{2n}(1,n-1)),x_n\big),y_n\Big) \cong K[V(E_{n-1})]/I(E_{n-1}), \tag{4.2.13}$$

$$S/\Big(\big(I(C_{2n}(1,n-1)),x_n\big):y_n\Big) \cong K[V(E_{n-3})]/I(E_{n-3}) \otimes_K K[y_n]. \tag{4.2.14}$$

If $n = 3$, we have

$$S/(I(C_6(1,2)):x_3) \cong K[V(E_0)]/I(E_0) \otimes_K K[x_3,y_3],$$

$$S/\big(\big(I(C_6(1,2)),x_3\big),y_3\big) \cong K[V(E_2)]/I(E_2),$$

$$S/\big(\big(I(C_6(1,2)),x_3\big):y_3\big) \cong K[V(E_0)]/I(E_0)\otimes_K K[y_3].$$

By applying Lemmas 1.4.10, 4.1.8 and Remark 4.1.1, we get

$$\operatorname{reg}\Big(S/(I(C_6(1,2)):x_3)\Big) = \operatorname{reg}\Big(K[V(E_0)]/I(E_0)\Big) = 0,$$

$$\operatorname{reg}\Big(S/\big(\big(I(C_6(1,2)),x_3\big):y_3\big)\Big) = \operatorname{reg}\Big(K[V(E_0)]/I(E_0)\Big) = 0,$$

$$\operatorname{reg}\Big(S/\big(\big(I(C_6(1,2)),x_3\big),y_3\big)\Big) = \operatorname{reg}\Big(K[V(E_2)]/I(E_2)\Big) = 1.$$

Since $\operatorname{reg}\Big(S/\big(\big(I(C_6(1,2)),x_3\big):y_3\big)\Big) < \operatorname{reg}\Big(S/\big(\big(I(C_6(1,2)),x_3\big),y_3\big)\Big)$, by Lemma 1.4.12(c), we get $\operatorname{reg}\Big(S/\big(I(C_6(1,2)),x_3\big)\Big) = 1$. Also $\operatorname{reg}\Big(S/(I(C_6(1,2)):x_3)\Big) < \operatorname{reg}\Big(S/\big(I(C_6(1,2)),x_3\big)\Big)$, and by Lemma 1.4.12(c), $\operatorname{reg}(S/I(C_6(1,2))) = 1$. For $n=4$, by using the similar strategy, we get $\operatorname{reg}(S/I(C_8(1,3))) = 1$. Let $n \geq 5$. If $n \equiv 0 \pmod 3$, then $n-3 \equiv 0 \pmod 3$ and $n-1 \equiv 2 \pmod 3$. By applying Lemmas 4.1.8 and 1.4.10 on Eqs (4.2.12)–(4.2.14), we get

$$\operatorname{reg}\Big(S/\big(I(C_{2n}(1,n-1)):x_n\big)\Big) = \operatorname{reg}\Big(K[V(E_{n-3})]/I(E_{n-3})\Big) = \left\lceil \frac{n-4}{3} \right\rceil,$$

$$\operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,n-1)),x_n\big):y_n\big)\Big) = \operatorname{reg}\Big(K[V(E_{n-3})]/I(E_{n-3})\Big) = \left\lceil \frac{n-4}{3} \right\rceil,$$

and

$$\operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,n-1)),x_n\big),y_n\big)\Big) = \operatorname{reg}\Big(K[V(E_{n-1})]/I(E_{n-1})\Big) = \left\lceil \frac{n-2}{3} \right\rceil.$$

Since $\lceil \frac{n-4}{3} \rceil < \lceil \frac{n-2}{3} \rceil$, by Lemma 1.4.12(c) we get $\operatorname{reg}\Big(S/\big(I(C_{2n}(1,n-1)),x_n\big)\Big) = \lceil \frac{n-2}{3} \rceil$. Also we have $\operatorname{reg}\Big(S/(I(C_{2n}(1,n-1)):x_n)\Big) < \operatorname{reg}\Big(S/(I(C_{2n}(1,n-1)),x_n)\Big)$, and again by Lemma 1.4.12(c), we get the required result. If $n \equiv 1 \pmod 3$, then $n-3 \equiv 1 \pmod 3$ and $n-1 \equiv 0 \pmod 3$. By applying the similar strategy, we get the desired result. Let $n \equiv 2 \pmod 3$. Here $C_{2n}(1,n-1) = E_3 \cup H$, where $H \cong E_{n-1}$ and $E_3 \cap H \neq \emptyset$. In this case $n-1 \equiv 1 \pmod 3$ as $\operatorname{reg}(S/I(E_3)) = 1$, by Lemmas 4.1.8 and 1.4.13,

$$\operatorname{reg}(S/I(C_{2n}(1,n-1))) \leq \operatorname{reg}(K[V(E_3)]/I(E_3)) + \operatorname{reg}(K[V(E_{n-1})]/I(E_{n-1})) = 1 + \left\lceil \frac{n-2}{3} \right\rceil.$$

For the other inequality, define $M = \Big\{\{x_1,x_2\},\{x_4,x_5\},\ldots,\{x_{n-3},x_{n-4}\}\Big\}$. Since M is an induced matching and $|M| = \lceil \frac{n-2}{3} \rceil$, then, $\operatorname{indmat}(C_{2n}(1,n-1)) \geq \lceil \frac{n-2}{3} \rceil$. By Lemma 1.4.9, we have $\operatorname{reg}(S/I(C_{2n}(1,n-1))) \geq \lceil \frac{n-2}{3} \rceil$. This ends the proof. □

Example 4.2.6. For $n = 15$ and $G = C_{30}(1,14)$. Then

(a) $\operatorname{sdepth}(K[V(C_{30}(1,14))]/I(C_{30}(1,14))) \geq \operatorname{depth}(K[V(C_{30}(1,14))]/I(C_{30}(1,14))) = 5$.

(b) $\operatorname{pdim}(K[V(C_{30}(1,14))]/I(C_{30}(1,14))) = 25$.

(c) $\operatorname{sdepth}(K[V(C_{30}(1,14))]/I(C_{30}(1,14))) \leq 10$.

(d) $\operatorname{reg}(K[V(C_{30}(1,14))]/I(C_{30}(1,14))) = 5$.

Theorem 4.2.7 ([82]). *Let $n \geq 3$. If n is even, then*

$$\operatorname{reg}\Big(K[V(C_{2n}(1,2))]/I(C_{2n}(1,2))\Big) = \left\lceil \frac{n-1}{2} \right\rceil.$$

If n is odd, we have

$$\frac{n-1}{2} \leq \operatorname{reg}\Big(K[V(C_{2n}(1,2))]/I(C_{2n}(1,2))\Big) \leq \left\lceil \frac{n-1}{2} \right\rceil + 2.$$

Proof. Let $S = K[V(C_{2n}(1,2))]$. If $n = 3$, then $C_6(1,2) = F_3 \cup H$, where $H \cong F_2$ and $F_3 \cap H \neq \emptyset$. By Lemmas 1.4.13 and 4.1.9, we get

$$\operatorname{reg}(K[V(C_6(1,2))]/I(C_6(1,2))) \leq \operatorname{reg}(K[V(F_3)]/I(F_3)) + \operatorname{reg}(K[V(H)]/I(H)) = 3.$$

For the second inequality, let $M = \{\{x_1, y_1\}\}$. Here M is an induced matching, thus we have $\operatorname{indmat}(C_6(1,2)) \geq |M| = 1$ and $1 \leq \operatorname{reg}(K[V(C_6(1,2))]/I(C_6(1,2)) \leq 3$. If $n = 4$,

$$K[V(C_8(1,2))]/(I(C_8(1,2)) : x_3) \cong K[V(C_3)]/I(C_3) \otimes_K K[x_3],$$

$$K[V(C_8(1,2))]/\big((I(C_8(1,2)), x_3), y_3\big) \cong K[V(F_3)]/I(F_3),$$

$$K[V(C_8(1,2))]/\big((I(C_8(1,2)), x_3) : y_3\big) \cong K[V(C_3)]/I(C_3) \otimes_K K[y_3].$$

By using Lemmas 1.4.10, 4.1.9 and 1.4.16, we get

$$\operatorname{reg}\Big(K[V(C_8(1,2))]/(I(C_8(1,2)) : x_3)\Big) = \operatorname{reg}\Big(K[V(C_3)]/I(C_3)\Big) = 1,$$
$$\operatorname{reg}\Big(K[V(C_8(1,2))]/\big((I(C_8(1,2)), x_3), y_3\big)\Big) = \operatorname{reg}\Big(K[V(F_3)]/I(F_3)\Big) = 2,$$
$$\operatorname{reg}\Big(K[V(C_8(1,2))]/\big((I(C_8(1,2)), x_3) : y_3\big)\Big) = \operatorname{reg}\Big(K[V(C_3)]/I(C_3)\Big) = 1,$$

as we have

$$\operatorname{reg}\Big(K[V(C_8(1,2))]/\big((I(C_8(1,2)), x_3) : y_3\big)\Big) < \operatorname{reg}\Big(K[V(C_8(1,2))]/\big((I(C_8(1,2)), x_3), y_3\big)\Big).$$

By Lemma 1.4.12(c),

$$K[V(C_8(1,2))]/(I(C_8(1,2)), x_3) = 2 > K[V(C_8(1,2))]/(I(C_8(1,2)) : x_3),$$

and again by Lemma 1.4.12(c), $\operatorname{reg}\Big(K[V(C_8(1,2))]/I(C_8(1,2)\Big) = 2$. Let $n \geq 5$. Here we consider the following two cases:

Case 1. If n is even. By using Lemma 1.4.12(c), $\operatorname{reg}\Big(S/I(C_{2n}(1,2))\Big) = \operatorname{reg}\Big(S/(I(C_{2n}(1,2)),x_{n-1})\Big)$ if $\operatorname{reg}\Big(S/(I(C_{2n}(1,2)):x_{n-1})\Big) < \operatorname{reg}\Big(S/(I(C_{2n}(1,2)),x_{n-1})\Big)$. We have the following isomorphisms:

$$S/\big(\big(I(C_{2n}(1,2)):x_{n-1}\big):y_{n-2}\big) \cong K[V(F_{n-4})]/I(F_{n-4})\otimes_K K[y_{n-2},x_{n-1}],$$

$$S/\big(\big(I(C_{2n}(1,2)):x_{n-1}\big),y_{n-2}\big) \cong K[V(F_{n-3})]/I(F_{n-3})\otimes_K K[x_{n-1}],$$

$$S/\big(\big(I(C_{2n}(1,2)),x_{n-1}\big),y_{n-1}\big) \cong K[V(F_{n-1})]/I(F_{n-1}),$$

$$S/\big(\big(\big(I(C_{2n}(1,2)),x_{n-1}\big):y_{n-1}\big),x_n\big) \cong K[V(F_{n-3})]/I(F_{n-3})\otimes_K K[y_{n-1}],$$

$$S/\big(\big(\big(I(C_{2n}(1,2)),x_{n-1}\big):y_{n-1}\big):x_n\big) \cong K[V(F_{n-4})]/I(F_{n-4})\otimes_K K[y_{n-1},x_n].$$

By using Lemmas 1.4.10 and 4.1.9 on above isomorphisms, we get

$$\operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,2)):x_{n-1}\big):y_{n-2}\big)\Big) = \operatorname{reg}\Big(K[V(F_{n-4})]/I(F_{n-4})\Big) = \left\lceil\frac{n-4}{2}\right\rceil,$$

$$\operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,2)):x_{n-1}\big),y_{n-2}\big)\Big) = \operatorname{reg}\Big(K[V(F_{n-3})]/I(F_{n-3})\Big) = \left\lceil\frac{n-3}{2}\right\rceil,$$

$$\operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,2)),x_{n-1}\big),y_{n-1}\big)\Big) = \operatorname{reg}\Big(K[V(F_{n-1})]/I(F_{n-1})\Big) = \left\lceil\frac{n-1}{2}\right\rceil,$$

$$\operatorname{reg}\Big(S/\big(\big(\big(I(C_{2n}(1,2)),x_{n-1}\big):y_{n-1}\big),x_n\big)\Big) = \operatorname{reg}\Big(K[V(F_{n-3})]/I(F_{n-3})\Big) = \left\lceil\frac{n-3}{2}\right\rceil,$$

$$\operatorname{reg}\Big(S/\big(\big(\big(I(C_{2n}(1,2)),x_{n-1}\big):y_{n-1}\big):x_n\big)\Big) = \operatorname{reg}\Big(K[V(F_{n-4})]/I(F_{n-4})\Big) = \left\lceil\frac{n-4}{2}\right\rceil.$$

Since $\operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,2)):x_{n-1}\big):y_{n-2}\big)\Big) < \operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,2)):x_{n-1}\big),y_{n-2}\big)\Big)$, by Lemma 1.4.12(c), $\operatorname{reg}\Big(S/(I(C_{2n}(1,2)):x_{n-1})\Big) = \lceil\frac{n-3}{2}\rceil$. Also $\operatorname{reg}\Big(S/\big(\big(\big(I(C_{2n}(1,2)),x_{n-1}\big):y_{n-1}\big):x_n\big)\Big) < \operatorname{reg}\Big(S/\big(\big(\big(I(C_{2n}(1,2)),x_{n-1}\big):y_{n-1}\big),x_n\big)\Big)$, again by Lemma 1.4.12(c) we get $\operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,2)),x_{n-1}\big):y_{n-1}\big)\Big) = \lceil\frac{n-3}{2}\rceil$. This implies $\operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,2)),x_{n-1}\big):y_{n-1}\big)\Big) < \operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,2)),x_{n-1}\big),y_{n-1}\big)\Big)$ by using Lemma 1.4.12(c), we have that $\operatorname{reg}\Big(S/(I(C_{2n}(1,2)),x_{n-1})\Big) = \lceil\frac{n-1}{2}\rceil$. Thus the result follows as

$$\operatorname{reg}\Big(S/(I(C_{2n}(1,2)):x_{n-1})\Big) < \operatorname{reg}\Big(S/(I(C_{2n}(1,2)),x_{n-1})\Big) = \left\lceil\frac{n-1}{2}\right\rceil.$$

Case 2. If n is odd. Here $C_{2n}(1,2) = F_3 \cup H$, where $H \cong F_{n-1}$ and $F_3 \cap H \neq \emptyset$. By Lemmas 1.4.13 and 4.1.9, we get

$$\operatorname{reg}(S/I(C_{2n}(1,2))) \leq \operatorname{reg}(K[V(F_3)]/I(F_3)) + \operatorname{reg}(K[V(F_{n-1})]/I(F_{n-1})) = 2 + \left\lceil\frac{n-1}{2}\right\rceil.$$

In the case of the second inequality, we define $M = \Big\{\{x_1,y_1\},\{x_3,y_3\},\dots,\{x_{n-2},y_{n-2}\}\Big\}$. Clearly M is an induced matching, it follows that $\operatorname{indmat}(C_{2n}(1,2)) \geq |M| = \frac{n-1}{2}$. By Lemma 1.4.9, we have $\operatorname{reg}(S/I(C_{2n}(1,2))) \geq \frac{n-1}{2}$.

□

Example 4.2.8. If $n = 20$ is even, then

$$\operatorname{reg}\Big(K[V(C_{2n}(1,2))]/I(C_{2n}(1,2))\Big) = \operatorname{reg}\Big(K[V(C_{40}(1,2))]/I(C_{40}(1,2))\Big) = 10.$$

If $n = 31$ is odd, we have

$$15 \leq \operatorname{reg}\Big(K[V(C_{2n}(1,2))]/I(C_{2n}(1,2))\Big) = \operatorname{reg}\Big(K[V(C_{62}(1,2))]/I(C_{62}(1,2))\Big) \leq 17.$$

In the following result, we find the exact value for the regularity of cyclic module $K[V(C_{2n}(1,n-1,n))]/I(C_{2n}(1,n-1,n))$.

Theorem 4.2.9 ([82]). *If* $n \geq 3$, *then* $\operatorname{reg}\Big(K[V(C_{2n}(1,n-1,n))]/I(C_{2n}(1,n-1,n))\Big) = \left\lceil \frac{n-1}{2} \right\rceil$.

Proof. Let $S = K[V(C_{2n}(1,n-1,n))]$. We have the following K-algebra isomorphisms:

$$S/(I(C_{2n}(1,n-1,n)) : y_n) \cong K[V(G_{n-3})]/I(G_{n-3}) \otimes_K K[y_n], \tag{4.2.15}$$

$$S/\big(\big(I(C_{2n}(1,n-1,n)),y_n\big),x_n\big) \cong K[V(G_{n-1})]/I(G_{n-1}), \tag{4.2.16}$$

$$S/\big(\big(I(C_{2n}(1,n-1,n)),y_n\big) : x_n\big) \cong K[V(G_{n-3})]/I(G_{n-3}) \otimes_K K[x_n]. \tag{4.2.17}$$

By applying Lemmas 4.1.10, 1.4.10, Remark 4.1.1 on Eqs (4.2.15)–(4.2.17),

$$\operatorname{reg}\Big(S/\left(I(C_{2n}(1,n-1,n)) : y_n\right)\Big) = \operatorname{reg}\Big(K[V(G_{n-3})]/I(G_{n-3})\Big) = \left\lceil \frac{n-3}{2} \right\rceil,$$
$$\operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,n-1,n)),y_n\big) : x_n\big)\Big) = \operatorname{reg}\Big(K[V(G_{n-3})]/I(G_{n-3})\Big) = \left\lceil \frac{n-3}{2} \right\rceil,$$

and

$$\operatorname{reg}\Big(S/\big(\big(I(C_{2n}(1,n-1,n)),y_n\big),x_n\big)\Big) = \operatorname{reg}\Big(K[V(G_{n-1})]/I(G_{n-1})\Big) = \left\lceil \frac{n-1}{2} \right\rceil.$$

Since $\left\lceil \frac{n-3}{2} \right\rceil < \left\lceil \frac{n-1}{2} \right\rceil$, by Lemma 1.4.12(c) we get $\operatorname{reg}\Big(S/\big(I(C_{2n}(1,n-1,n)),y_n\big)\Big) = \left\lceil \frac{n-1}{2} \right\rceil$. Also,

$$\operatorname{reg}\Big(S/\big(I(C_{2n}(1,n-1,n)) : y_n\big)\Big) < \operatorname{reg}\Big(S/\big(I(C_{2n}(1,n-1,n)),y_n\big)\Big),$$

and again by Lemma 1.4.12(c), the required result follows. □

Example 4.2.10. If $n = 50$, then we get

$$\operatorname{reg}\Big(K[V(C_{2n}(1,n-1,n))]/I(C_{2n}(1,n-1,n))\Big) = \operatorname{reg}\Big(K[V(C_{100}(1,49,50))]/I(C_{100}(1,49,50))\Big) = 25.$$

4.3 Conclusion

In this chapter we compute the algebraic invariants namely regularity, projective dimension, depth, and Stanley depth of the edge ideals associated with some classes of circulant graphs. It will be interesting but seems challenging to find these algebraic invariants for the edge ideals of all four and five regular circulant graph.

CHAPTER 5

Summary and future directions

This chapter concludes with a summary of all the important findings and outcomes of this research. Towards the end, some recommendations for further research based on this work will be made.

5.1 Summary

In this dissertation, the precise formulas for the values of the algebraic invariants depth, Stanley depth, regularity, projective dimension and Krull dimension quotient of edge ideals associated with perfect semiregular trees are provided. For the computations of the said invariants of the quotient of the polynomial rings by edge ideals associated to perfect semiregular trees, the exact values regarding the aforementioned invariants for perfect $(n-1)$-ary tree are also given.

Furthermore, values of depth, projective dimension, and lower bounds for Stanley depth of the quotient rings of the edge ideals of all cubic circulant graphs are presented. Lemma 3.1.1 is proved in this thesis, which is inspired by a result of Cimpoeas [53, Proposition 1.3] and it played a vital role in the computation of depth and a lower bound for Stanley depth in our main findings. For proving over main results the precise values of the said invariants of the quotient rings of the edge ideals associated with certain supergraphs are also computed.

In the end, the exact values of depth, projective dimension, and bounds for the Stanley depth of edge ideal associated with four regular circulant graph $C_{2n}(1,n-1)$ are given. Also a formula for the regularity of the edge ideal associated with $C_{2n}(1,n-1)$ when $n \equiv 0,1 (\mathrm{mod}\, 3)$, and sharp bounds when $n \equiv 2 (\mathrm{mod}\, 3)$ are provided. The exact values of the regularity of the edge ideal associated with four regular circulant graph $C_{2n}(1,2)$ when n is even and tight bounds when n

is odd has been established. The exact value for the regularity of edge ideal associated with five regular circulant graph $C_{2n}(1,n-1,n)$ are determined. For computation of the said algebraic invariants for four and five regular circulant graphs, the values/bounds of algebraic invariants associated with certain subgraphs of $C_{2n}(1,n-1), C_{2n}(1,2)$ and $C_{2n}(1,n-1,n)$ are computed.

Future Work

Considering this study, the following can be examined as a future research work:

- Can we find the aforementioned invariants for the quotient rings of edge ideals of all four and five regular circulant graphs?
- We are unable at the moment to find value of Stanley depth of cyclic modules associated with all cubic circulant graphs, one can try to fix the value/upper bound by using some other approach.
- One can explore the Herzog Conjecture for the classes of modules we considered in this study.
- One can find the depth, Stanley depth and regularity of powers of edge ideals considered in this working.

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