Entanglement Transfer via Continuous Time Quantum Walk

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Dedicated to

My father, my mom, my supervisor, and my teachers, whose endless difficult paintings, sacrifices, and encouragement have been the foundation of my accomplishments. Your belief in my potential has been my steady motivation.

Abstract

In this thesis, we review the theory and applications of various types of Quantum Walks (QW) along with their properties. In particular we studied about the behaviour of Entanglement and entanglement transfer. Here, we studied the entanglement transfer on a triangle chain using the continuous-time quantum walk (CTQW). In particular, we considered two site-entangled situations of the Bell type. We take spatially entagled state and examine the behaviour of state by finding occupation probability and concurrence.

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Chapter 1

Introduction:

Quantum information is the field that increases the power of quantum mechanics and information sciences to develop technologies including quantum sensors, networks, and computers that have new speed and accuracy [Tripathy and Singh, 2023].

Quantum mechanics is a basic theory of physics, which explains how matter and energy behave at the atomic and subatomic level. It is very different, compared to classical mechanics, which describes the behavior of macroscopic things.

In quantum mechanics, entanglement is a phenomena in which two or more particles form a unique bond. Regardless of that how much distance between the particles, they are correlated, when they are entangled. A quantum walk is a kind of computing model that takes inspiration from the random walk theory found in classical physics. A particle randomly moves on a grid and take different steps in various directions in a classical random walk. A particle can be in a superposition state during a quantum walk, which allows it to discribe several paths across the grid. In quantum mechanics, entanglement is a phenomena in which two or more particles form a unique bond. Regardless of that how much distance between the particles, they are correlated, when they are entangled. A quantum walk is a kind of computing model that takes inspiration from the random walk theory found in classical physics. Entanglement

and quantum walks are similar in that they can investigate several options. Particles can be correlated and share information through entanglement, a single particle can be in numerous states at a time through superposition. Quantum walks are integral part of quantum communication.Here we explain why studying quantum walks is important.

Quantum walk (QW) has shown to be a helpful tool in numerous study fields. For example, quantum walking is widely used in many computer algorithms,making the algorithm exponentially faster than other classical computer algorithms. QWs have been used to simulate quantum state transfer in superconducting qubits and energy transfer in photosynthetic complexes from the standpoint of comprehending nature.

A quantum random walk is the quantum counterpart of a classical random walk. The concept of the classical random walk has long been utilized as a computational basis for developing classical approaches for complex situations. Quantum analogs of random walks can speed up the computational capacity of several algorithms, including element distinctness, spatial search, graph correctness, etc. Over the last ten years, quantum walks have become a widely used model for computing. When using quantum walk formulas in graph theory, graph traversal speeds are quadratic or polynomial, but when using conventional methods, they are exponential. Quantum computers have also been developed using quantum walk models.

Quantum walks are the quantum counterparts of classical random walks. There are three ways that randomness might appear in quantum walk.

1. quantum state superposition;
2. reversible, non-random unitary evolution; and
3. wave function collapse as a result of state measurements [Venegas-Andraca, 2012].

In contrast, randomization arises from random transitions between states in the

traditional random walk, where the walker traverses between discrete states.
Quantum walks have many features that are not present in classical random walks. Specifically, they spread either faster or slower than the initial distribution and do not approach the limiting distribution due to the quantum interference's strength. Just as classical random walks inspire formulations in both discrete and continuous time, so do quantum walks.

The discrete quantum walk is the first model with two quantum generators: the walker and the coin, and the evolutionary operator is used for both. This machine consists of separate steps only. The evolution of unitary operators forms the mathematical basis of this approach.

$$|\psi_t^2 = \hat{U}|\psi_t^1| \tag{1.1}$$

Another model is "continuous quantum walks." It is composed of an always-usable Hamiltonian system evolution operator and a walker [Whitfield et al., 2010]. The walker is free to move whenever they like. This model's mathematical structure is evolution through the Schrodinger equation.
Quantum walks are executed on discrete graph topologies, and both discrete and continuous models predict their features. This is primarily because computer science and development in this sector, developing quantum algorithms based on quantum walks has been given priority.

1.1 Thesis outline:

The thesis is organized as follows.
In chapter 2, we give the fundamentals of quantum information. We thoroughly give an overview of quantum bits, entanglement, entanglement measurement and transfer.

Chapter 3 comprises the detail of quantum walks. First we explain what a graph is. Then an introduction to classical random walk is given. This is followed by details of both discrete and continuous time quantum walks.We also give the evolution of continuous time quantum walks.

In chapter 4, we review the entanglement transfer on a triangular chain in continuous time quantum walk. We show how degree of entanglement transfer depends on initial state.

Finally we conclude in chapter 5.

Chapter 2

Entanglement theory

We examine the fundamentals of quantum information science in this part. Let's begin with the qubit, which is the fundamental unit of measurement in quantum information. We study its characteristics and how its usage and functionality vary from that of conventional bits. Next, we consider the concept of entanglement and examine its mathematical representation, considering Bell states as canonical instances that are based on maximality or non-maximality.To be able to measure entanglement, we introduce concepts like concurrence that help us make sense of entangled systems through a quantitative approach.

2.1 Quantum Bits:

A bit is said to be quantum if it is made up of a quantum system, like an electron or a photon. Just like a classical bit, a quantum bit requires two distinct states: one for "0" and one for "1." In contrast to classical bits, quantum bits are susceptible to incompatible measurements, can exist in superposition states, and can even entangle with other quantum bits. The fact that qubits can regulate the forces of superposition, interference, and entanglement sets them apart from classical bits and makes them significantly more powerful.

In mathematics, a qubit state is represented by a two-dimensional vector. For quantum mechanical linear algebra, physicists usually utilize Dirac notation.

The state of qubit can be represented as $|\phi\rangle = \alpha|0\rangle + \beta|1\rangle$, where the standard basis states are represented by the variables $|0\rangle$ and $\|1\rangle$ and the probability amplitudes, which are typically complex numbers, are represented by the variables α and β. The qubit functions as a conventional bit if one of the two variables α or β is zero; when both are nonzero, the qubit is in superposition.

A classical bit is the outcome of measuring a qubit in the standard basis. When measuring a qubit , the norm-squared connection between amplitudes and probabilities is described by the Born rule. The state collapses to either $|0\rangle$ with probability $|\alpha|^2$, or to $|1\rangle$ with probability $|\beta|^2$. The coefficients α and β of each valid qubit state are such that $|\alpha|^2 + |\beta|^2 = 1$. For instance, measuring the qubit $\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$ would yield, with equal probability, either $|0\rangle$ or $|1\rangle$.

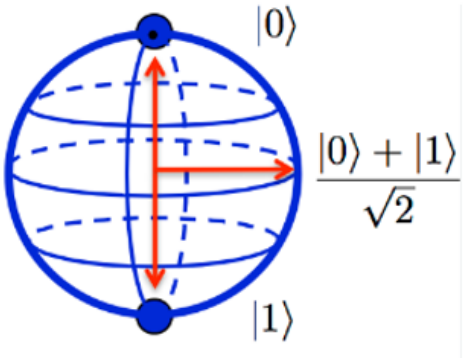


Figure 2.1: Qubit
[Mutiara et al., 2013]:

2.2 Entanglement:

One of the key characteristics of quantum systems is entanglement. This was first noted by Nathan Rosen, Boris Podolsky, and Albert Einstein in a paper outlining the theory's illogical predictions. [Einstein et al., 1935]. According to their research,

the imperfect wave function is the source of the quantum mechanical explanation of physical reality.

The individual qubit readings could yield a result of either 0 or 1. Still, there will always be a correlation between the results of the measurement on one qubit and the measurement on the other. The features shared by the quantum correlations of entangled states are essential to many quantum information (QI) protocols, including quantum cryptography, quantum key distribution (QKD), quantum random number generation, and teleportation.

Entanglement is a phenomenon that manifests at subatomic scales, much like other facets of quantum science. When two particles entangle, as a pair of electrons or photons, they remain connected even if they are far away. This change could occur instantly, even when objects were very far away [Bell, 1964]. Bell theorem is an important idea in physics, despite being inconsistent with other ideas of the discipline. While before Bell, Albert Einstein showed that information cannot travel faster than the speed of light . Confused, Einstein called this effect "spooky action at a distance".

2.2.1 Entangled States:

If a pure state of two quantum systems cannot be factorized, it is referred to as entangled. In other words, if every distinct system lacks a pure state of its own.

Now, let us consider pure states of two particles $|\psi_1\rangle$ and $|\psi_2\rangle$. The combine state of two uncorelated particles can be written as [Azzini et al., 2020]

$$|\psi\rangle = |\psi_1\rangle \otimes |\psi_2\rangle. \tag{2.1}$$

Pure states which can not be written like above are called entangled states. A common example of entangled states are the Bell states. For a two-qubit system, the

one of the Bell states is:

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

If there exists a bipartite mixed state ρ that can be expressed as a convex combination, then it is considered separable and remains unentangled.

$$\rho_{AB} = \sum_k p_k \rho_A^k \otimes \rho_B^k \tag{2.2}$$

2.2.2 Non-Maximally Entangled States:

Quantum states that display entanglement, but not to the greatest extent feasible, are known as non-maximally entangled states. The degree of correlation between the component subsystems in these states is lower than in maximally entangled states, such as Bell states. A general two-qubit state can be expressed as:

$$|\psi\rangle = \alpha|00\rangle + \beta|01\rangle + \gamma|10\rangle + \delta|11\rangle \tag{2.3}$$

where the complex coefficients α, β, γ, and δ fulfill the normalizing requirement $|\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1$.

If the reduced density matrix of either subsystem is maximally mixed, meaning that it is proportional to the identity matrix, then the state is maximally entangled. The reduced density matrix will be somewhat pure for non-maximally entangled states. The entanglement measurements of non-maximally entangled states are smaller than those of maximally entangled ones. Non-maximally entangled states may occasionally be more resilient to noise and decoherence than maximally entangled states. Quantum meteorology, quantum cryptography, and quantum error correction could all benefit from the use of these states. Essentially, non-maximally entangled states are a larger

class of quantum states that show subsystem correlations, but not to their greatest extent.

2.2.3 Bell states:

Four maximally entangled two-qubit states, or Bell states, are the simplest example of entanglement. They are as follows.

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{2.4}$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{2.5}$$

$$|\Psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{2.6}$$

$$|\Psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{2.7}$$

The most basic type of entanglement that are utilized in both theoretical and experimental study are the Bell states.

2.2.4 Properties of Bell states:

- Bell states form an orthogonal basis in Hilbert space. That is, every two-qubit pure state is written as a superposition of Bell states.
- Bell states are classified as either symmetric or anti-symmetric. A symmetric Bell state remains the same under the exchange of two particles, while anti-symmetric states accumulate a negative sign. From the equations above, $|\Phi^+\rangle$ (Eq. (2.4)) and $|\Psi^+\rangle$ (Eq. (2.6)) are symmetric states, while $|\Phi^-\rangle$ (Eq. (2.5)) and $|\Psi^-\rangle$ (Eq. (2.7)) are anti-symmetric.
- The measurement outcomes of Bell states are correlated. If we measure one qubit in a specific basis, the other qubit is guaranteed to have the same value

in the corresponding basis (for symmetric states) or the opposite value (for anti-symmetric states).

2.3 Measurement of Entanglement through concurrence:

Entanglement of formation and concurrence are the quantities to measure degree of entanglement.

Entanglement of formation is an increasing function of concurrence. Let us consider a density matrix ρ for two quantum systems, A and B [Wootters, 1998].

$$\rho_{AB} = \sum_i p_i \psi_i\rangle\langle\psi_i|, \tag{2.8}$$

where p_i is the probability that A and B are in pure state $|\psi_i\rangle$. At this point, the entropy of one of the two systems of A and B is what defines entanglement for a pure state.

$$E(\psi) = -\mathrm{Tr}(\rho_1 \log_2 \rho_1) \tag{2.9}$$

$$E(\psi) = -\mathrm{Tr}(\rho_2 \log_2 \rho_2), \tag{2.10}$$

where ρ_1 is the partial trace of ρ over subsystem B and ρ_2 is the partial trace of ρ over subsystem A. Now for the mixed state the entanglement is defined as:

$$E = \min \sum_i p_i E(\psi_i), \tag{2.11}$$

This equation describes that the average entanglement of pure state decomposition is minimized over all decomposition of ρ. To find entanglement of formation (EOF) we use the "spin-flip" transformation. For a pure state of single qubits, the spin-flip

is given by.

$$|\psi_i'| = \sigma_y |\psi_i^*| \tag{2.12}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{2.13}$$

Now for the two-qubit system, the $\tilde{\rho}$ is given by

$$\tilde{\rho} = (\sigma_y \otimes \sigma_y)\rho^*(\sigma_y \otimes \sigma_y) \tag{2.14}$$

Now entanglement defined in Eq 2.11 for two qubit system takes the form,

$$E(\psi) = E(C(\psi)) \tag{2.15}$$

Where we have

$$C(\psi) = | < \psi | \tilde{\psi} > | \tag{2.16}$$

where $\tilde{\psi}$ denote transformed state.

$$|\tilde{\psi}\rangle = (\sigma_y \otimes \sigma_y)|\psi^*\rangle \tag{2.17}$$

while the function

$$E(C) = h(1 + \sqrt{1 - C^2}/2) \tag{2.18}$$

where $h(x) = -xlog_2(x) - (1-x)log_2(1-x)$

The range of E(C) depend on specific system but generally its range is always positive while concurrence lies between $0 < C < 1$. So we can take concurrence as a measure of entanglement on its own right. If the state is unchanged after spin-flipped state, then the concurrence of that state is 1, while if the state changes itself and converts into an orthogonal state of that state then its concurrence is zero. Now let

us write the state into a standard basis $|\Phi\rangle = a|00\rangle + b|01\rangle + c|10\rangle + d|11\rangle$.

It is clear that $|\Phi\rangle$ is factorizable if ad = bc . So we can take the difference between ad and bc as the measurement of entanglement and concurrence is also defined as [Hill and Wootters, 1997].

$$C(\Phi) = 2|ad - bc| \tag{2.19}$$

After defining the spin-flip state and the function E(C), we can now able to write the formulae for the mixed state of two qubits.

$$E(p) = E(C(p)) \tag{2.20}$$

Where C(p) is concurrence for mixed state and defined as follows $C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$, where λ_i's are the eigenvalues of the Hermitian matrix $R = (\rho^{1/2}\tilde{\rho}\rho^{1/2})^{1/2}$. So we can say that λ_i are the square root of the eigen values of R . After taking the square root of the eigen values of R we can write

[Yu and Eberly, 2007]

$C(\rho) = \max\{0, (\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)\}$.

2.4 Entanglement transfer:

In quantum physics, entanglement is transferred from one pair of particles (source particles) to another pair (target particles) by a process known as entanglement transfer [Man et al., 2014]. Consider that we have two ordinary coins (target) and two entangled coins (source). We can move the entanglement from one coin (source coins) to the other coin (target coins) by using entanglement transfer. During the transfer, the source coins lose their entanglement while the target coins gains the entanglement . Different methods are used for transferring entanglement, depending on the

particular system (photons, atoms, etc.). We prepare the entangled source particles by manipulating the source particles until they become entangled, then we interact source and target particles. For entanglement transfer, there should be a precise interaction between the source and target particles [Giordani et al., 2021]. Depending on the method, entanglement in the target pair may be established by measuring the source particles or by manipulating control signals.

Massive quantum networks cannot be built without the concept of entanglement transfer. A few potential applications are as follows. Secure long-distance communication channels can be implemented in quantum communication determined by use of entanglement transmission.

Chapter 3

Quantum Walks

This chapter discusses the basic concepts of graph theory like graph structures, adjacency matrices and sub graph. Next, it introduces discrete and continuous quantum walks and shows how they relate to graph topology.

3.1 Graphs:

As shown in Fig 3.1, a graph is made up of vertices V (also known as nodes or sites; we'll use these terms interchangeably) connected by edges E. There is an adjacency matrix on the graph, connected to it, which indicates how two vertices are connected (see Fig 3.1). $|G|$ represents the total number of vertices. If there is an edge between two vertices, we say that they are neighboring [Neogy and Mer, 2021]. An adjacency matrix, denoted by $\mathbf{A}$ (a square matrix of size $n \times n$), exists for a graph with $|G| = n$ vertices. The element at row i and column j of the matrix, A_{ij}, represents the edge (i, j) in the graph. Here's how the value of A_{ij} is determined: If vertices i and j are connected by an edge, then $A_{ij} = 1$. If there's no edge connecting them, $A_{ij} = 0$. In cases with multiple edges between vertices, A_{ij} can take the value of the number of edges connecting them (e.g., $A_{ij} = m$ for m edges). Real or complex numbers can also be used as weights for edges. If there's a weight of w on edge (i, j), then $A_{ij} = w$.

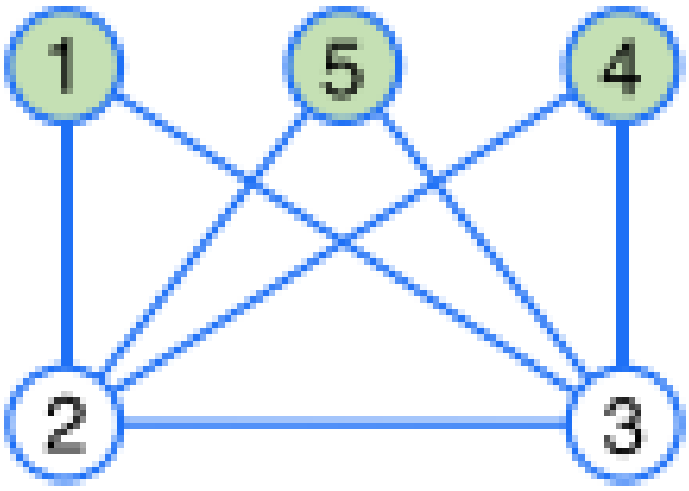


Figure 3.1: A graph with $|G| = 5$
[Qu et al., 2022]

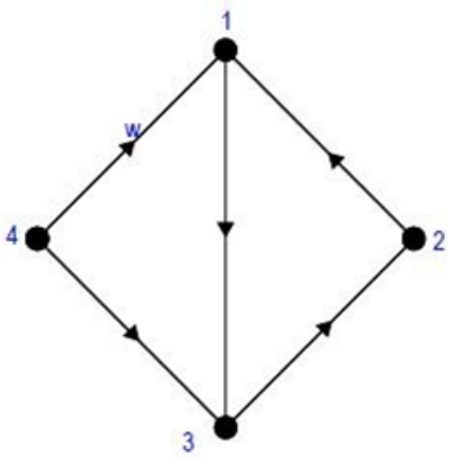
$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

Figure 3.2: Adjacency matrix of Graph
[Qu et al., 2022]

A sub-graph of a graph refers to a subset that fulfills the following conditions:

$V(\Gamma) \subseteq V(G)$ (all vertices in the subgraph, Γ, belong to the original graph, G)

$E(\Gamma) \subseteq E(G)$ (all edges in the subgraph belong to the original graph).



$$A = \begin{pmatrix} 0 & 1 & 0 & w \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure 3.3: Adjacancy matrix with weight
[Qu et al., 2022]

3.2 Discrete Time Quantum Walks:

Having defined what graphs are, let's move on to see what a walk is. A walk is defined using an imaginary walker allowed to move along the edges to different vertices on the graph with some specified rules. This movement from one vertex to another is known as a walk. In classical discrete random walk (CDRW), the imaginary walker starts at some particular vertex, v_0 and moves to adjacent vertex v_i with probability $1/d_0$, where d_0 is the number of edges connected to vertex v_0. Then the walker moves to a vertex v_j adjacent to v_i with probability $1/d_i$, where d_i is the number of edges connected to vertex v_i . The walker keeps moving like this creating a probability distribution (the probability of being at vertex k after n steps). The simplest example of classical discrete random walk (CDRW) is a walk on infinite line which creates the binomial probability distribution. In discrete time quantum walk (DTQW) the walker can be in a superposition of different vertices. This property of simultaneously being on different vertices is what makes DTQW unique as compared to its classical counterpart.

Firstly, let us briefly recapitulate the discrete temporal features of the classical random walk. The expression of the coin operator takes the form of a discrete-time

classical random walk in the Hilbert space H_p. The direction in which the walker should walk is essentially determined by the toss of a coin, so there is a balanced position in place. To travel on a line, a coin has two sides (head or tail) designed for left and right movement respectively. This kind of walk can be better understood by looking at a gadget known as the Galton's Board (Fig. 3.4).

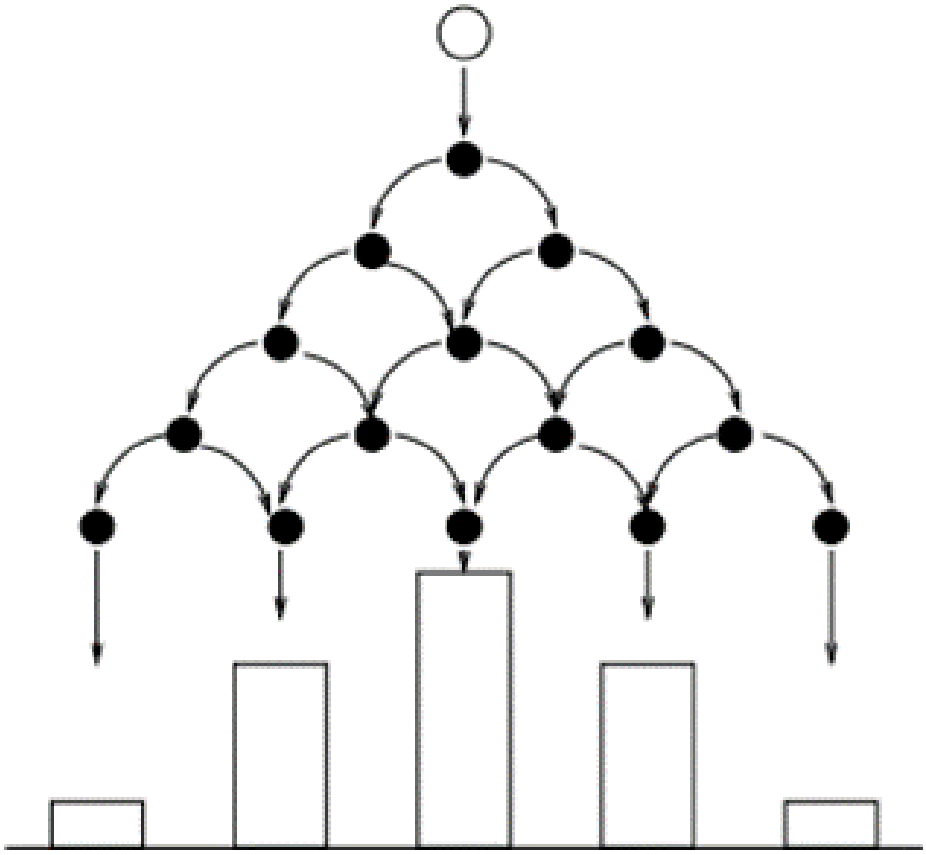


Figure 3.4: Galton's Board
[Kempe, 2003]:

Galtons Board (Quincunx) in Figure 3.4: It is a mechanism that involves dropping a bead from top to bottom into a board that has a variety of pins embedded in it. The rows of pins are evenly spaced apart. Similar to a regular coin, there is an equal chance that the bead will fall to either side when it strikes a pin. Until the bead passes through every row and gathers in a container at the bottom, the procedure is repeated.

The classical-quantum walk and its discrete-time counterpart share a remarkably similar structure. An analog of a classical coin flip is given by a quantum coin operation, which specifies the superposition of direction in which the walker moves concurrently.

A discrete-time quantum walk (DTQW) can be developed on graphs with greater degrees and higher dimensions. A line's H_p is spanned by the states $\{|i\rangle : i \in \mathbb{Z}\}$; for a circle of size N, H_p is $\{|i\rangle : i = 0, \ldots, N-1\}$, where $|i\rangle$ denotes the vertex. Here we have two Hilbert spaces, unlike the classical case. A coin Hilbert space H_C, spanned by two basis states in the previous situation, which correspond to the two sides of an ordinary coin, augments the position Hilbert H_p space. After that, $H = H_C \otimes H_p$ becomes the complete space. The formula for a discrete-time quantum walk on a line in Hilbert space is

$$H = H_C \otimes H_p \tag{3.1}$$

The coin Hilbert space is represented by H_C, whereas the position Hilbert space is represented by H_p. The basic states span H_C for DTQW are $|1\rangle$ and $|0\rangle$, while H_p is spanned by the basis states of position $|\psi_j\rangle$. Quantum walk of initial state objects in discrete time [Chandrashekar, 2010]:

$$|\Psi_{ini}\rangle = (\cos(\delta)|0\rangle + e^{i\eta}\sin(\delta)|1\rangle) \otimes |\psi_0\rangle \tag{3.2}$$

The operation of the quantum coin toss

$$B(\zeta, \alpha, \beta, \gamma) = e^{i\zeta}e^{i\alpha\sigma_x}e^{i\beta\sigma_y}e^{i\gamma\sigma_z} \tag{3.3}$$

is applied. The particle is evolved from its original position to the left and right using

the quantum coin operation.

$$S = e^{-i(|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes P_l} \tag{3.4}$$

Here P is the momentum operator while l is the step length. At $l = 1$, S can be written as

$$S = e^{-i(|0\rangle\langle 0| - |1\rangle\langle 1|) \otimes P_l} \tag{3.5}$$

The Hadamard walk, in which the quantum coin operation is the Hadamard operation H, is the most prevalent example of a discrete-time quantum walk. A particle initially in basis state $|0\rangle$ or $|1\rangle$ of H_C can evolve into a superposition over H_C states with equal probability, through the Hadamard walk.

The matrix H is given by:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \tag{3.6}$$

such that

$$(H \otimes I)(|0\rangle \otimes |\psi_0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \otimes |\psi_0\rangle \tag{3.7}$$

$$(H \otimes I)(|1\rangle \otimes |\psi_0\rangle) = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \otimes |\psi_0\rangle \tag{3.8}$$

3.3 Continuous time quantum walk:

The continuous-time quantum walk is primarily characterized by the Markov process and is governed by a graph-defined Hamiltonian. In a continuous-time quantum walk, a coin operator is not necessary. Figure 3.5 allows us to define the Markov process.

When describing a series of events, a Markov process is a kind of mathematical model in which the probability of the subsequent occurrence is only dependent on its

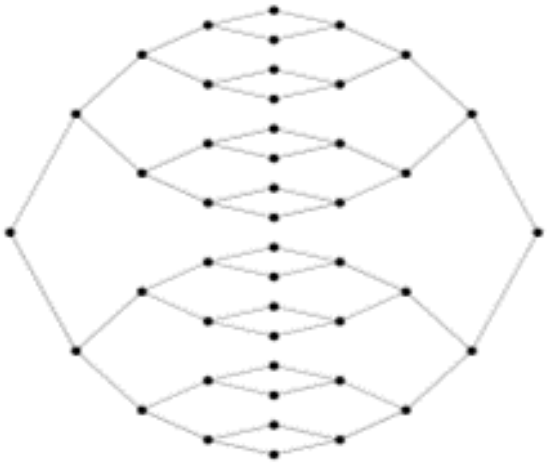


Figure 3.5: Markov process
[Johnson, 2015]:

current state and not on its past.

Let us suppose that a graph has n vertices, where these vertices are connected through edges. Let γ denote the transfer rate per unit time between vertices. We define a stochastic generator matrix M such that walker only travel between nodes that are connected by an edge.

$$M_{ab} = \begin{cases} -\gamma & \text{for } a \neq b, \text{ and } a \text{ and } b \text{ connected by an edge} \\ 0 & \text{for } a \neq b, \text{ and } a \text{ and } b \text{ not connected} \\ k\gamma & \text{for } a = b, \text{ where } k \text{ is the valence of vertex } a \end{cases} \tag{3.9}$$

It follows that the diagonals are empty and only the matrix elements closest to each other have non-zero entries. The probability of walker at vertex a at a given time t is given by:

$$\frac{dp_a(t)}{dt} = -\sum_b M_{ab} p_b(t) \tag{3.10}$$

where

$$\sum_a p_a(t) = 1, \tag{3.11}$$

where $p_a(t)$ is the probability that is attached to a 'a' event or result at time

't'. The classical continuous-time walk is converted into a continuous-time quantum walk by creating a n-dimensional Hilbert space with the $|\alpha\rangle$ vectors as its constituent values.The basis states are typically denoted as $|\alpha\rangle$. The Schrödinger equation is derived in this instance.

$$\frac{d}{dt}\langle\psi(t)|H|b\rangle = \sum_b \langle\psi(t)|H|b\rangle\langle b|\psi(t)\rangle \tag{3.12}$$

Probability is preserved in quantum information, so sum of probabilities of all vertices is

$$\sum_a |\langle a|\psi(t)\rangle|^2 = 1. \tag{3.13}$$

So, in quantum information, we can say that the sum of the squares of the overlap of all vertices is equal to one. By comparing Eq 3.12 and Eq 3.10 we found that

$$M_{ab} = \langle a|H|b\rangle \tag{3.14}$$

The aforementioned formula demonstrates how the graph adjacency matrix affects the Hamiltonian of a continuous-time quantum walk. If the adjacency matrix represents the Hamiltonian, the unitary evolution operator can be found by

$$U = U^\dagger = e^{-iHt} \tag{3.15}$$

where $U^\dagger$ denotes the conjugate transpose of U. Therefore, we can define a continuous-time quantum walk as follows by [Johnson, 2015]:

Definition: The unitary transformation U defines a continuous-time quantum walk over a graph G, such that the evolution of the state vector at time t in the Hilbert space is

$$|\psi(t)\rangle = U|\psi(0)\rangle \tag{3.16}$$

3.4 CTQW evolution:

Considering $|\psi(t)\rangle$ to be the state vector for the CTQW, the evolution of this vector with the left-most initial state follows as below.

$$|\psi(t)\rangle = e^{-iHt}|1\rangle$$

$$= \sum_{n=0}^{\infty} \frac{(-iHt)^n}{n!}|1\rangle \tag{3.17}$$

$$= |1\rangle - it|2\rangle - \frac{t^2}{2}(|1\rangle + |3\rangle) + \frac{it^3}{6}(2|2\rangle + |4\rangle) + \frac{t^4}{24}(2|1\rangle + 3|3\rangle + |5\rangle) + ... \tag{3.18}$$

$$= \sum_{j=1}^{n} \alpha_j(t)|j\rangle$$

In equation 3.18, the exponential is expanded and the Hamiltonian is iterated on the initial state successively. For example, the first term corresponds to the zeroth order term of the exponential; hence, the Hamiltonian is not applied. Consequently, the initial state remains the same. For the second term, the Hamiltonian is applied once and we see the walk evolving from node 1 to node 2. This continues for further terms and by collecting the non-complex coefficients on each vertex, we see the quantumness of the walk at play.

Chapter 4

Entanglemement Transfer Using CTQW

We will now use Continuous-Time Quantum Walks (CTQWs) to describe our results on entanglement transfer in a triangle chain. We examine how the dynamics of entanglement behave in this system. In such systems, the entanglement dynamics are studied through the use of concurrence. We quantify the efficiency of entanglement transfer through concurrence.

4.1 Techniques:

To better test understanding of the subject, I reproduced some of the results in the paper [Sağlam et al., 2023]. Reproduced these results using three techniques (i) using the state vector evolution formalism, (ii) using the density matrix formalism, and (iii) by solving the time-dependent equation for entanglement transfer. By state vector formalism we mean that we represent the state of the walker using $\psi(t) = |e^{-iHt}|\psi(0)|$ to get the state vector for any a time t given we know the initial state vector, $\psi(0)|$. By density matrix formalism we mean that we represent the state of the walker. using a density matrix

4.2 Introduction to Model:

We examine the phenomena of CTQW on a linear chain that forms a triangle, as illustrated in Fig 4.1, the most basic graph that allows the entanglement transfer. On a triangular chain, walker can move to the nearest node. Quantum walks define the spatial entanglement in the site basis. The starting condition of the system is assumed to exist as a spatially entangled qubit. This leads to the injection of this spatially entangled qubit state from one end into the chain. To investigate the transmission of spatially entangled Bell states which are important for quantum computation and communication, we take into account two site spatial entanglement. Completion of the density matrix and concurrence tests the quality of the transfer. The structure of this document is as follows.

4.3 CTQW on a triangular chain:

A triangular chain is a graph where we study the phenomena of entanglement transfer via continuous-time quantum walk. To determine which site is occupied, we will utilize the site basis $|i\rangle\}$, where $i = [1, \ldots, 5]$ [Sağlam et al., 2023].

The edges $e = (i, j)$ of a graph are determined by the set of connected sites and are represented by the so-called adjacency matrix $\mathbf{A}$. For a triangular chain of $N = 5$ sites, matrix $\mathbf{A}$ is given by:

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \tag{4.1}$$

In CTQW, where every J_{nm} is real-valued. No complex value allowed in CTQW.

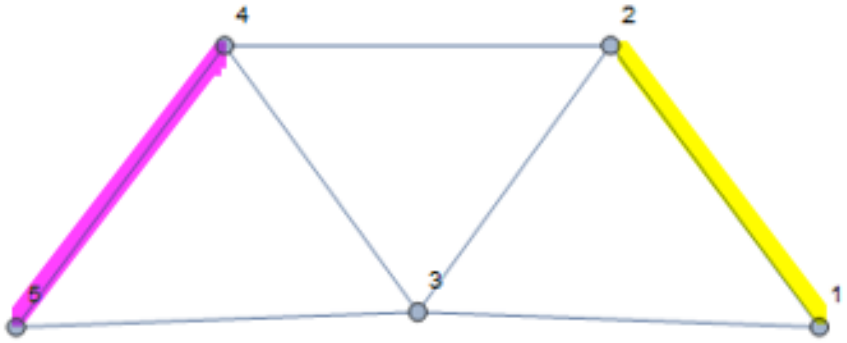


Figure 4.1: Graph of linear chain with 5 vertices

Firstly we examine the triangular chain. Complex edge weights are generally not required. We introduce complex edges in the initial state. We want to transfer entanglement to the end of the chain that is to sites 4 and 5 (denoted by pink in Fig 4.1) from site 1 and 2 (denoted by yellow in Fig 4.1). In particular, we examine a qubit that exhibits spatial entanglement at two locations inside the site basis.

$$|\psi_{\text{spatial}}(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - e^{i\phi}|2\rangle) \tag{4.2}$$

4.4 Entanglement transfer in CTQW on a triangular chain:

Let's start with the initial state $|\varphi(0)\rangle = |1\rangle$ to understand the phenomena of state transfer on CTQW. The probability distribution graph for state $|\varphi(0)\rangle = |1\rangle$ is shown in Fig 4.2. The sites $i = 1$ (Fig. 4.3) and i = 5 (Fig. 4.4) have occupancy probabilities. We have seen that the probability of i=1 is much greater than the probability of i=5 as shown in Fig 4.3 and Fig 4.4. By Fig it is clear that transferring from initial occupied site $|1\rangle$ to site $|5\rangle$ is found to be very weak which is almost 45 percent less.

The combined graph for probabilities having initial equation $|\varphi(0)\rangle = |1\rangle$ (by blue, and red curves for P_1 and P_5 respectively) shown in Fig 4.5.

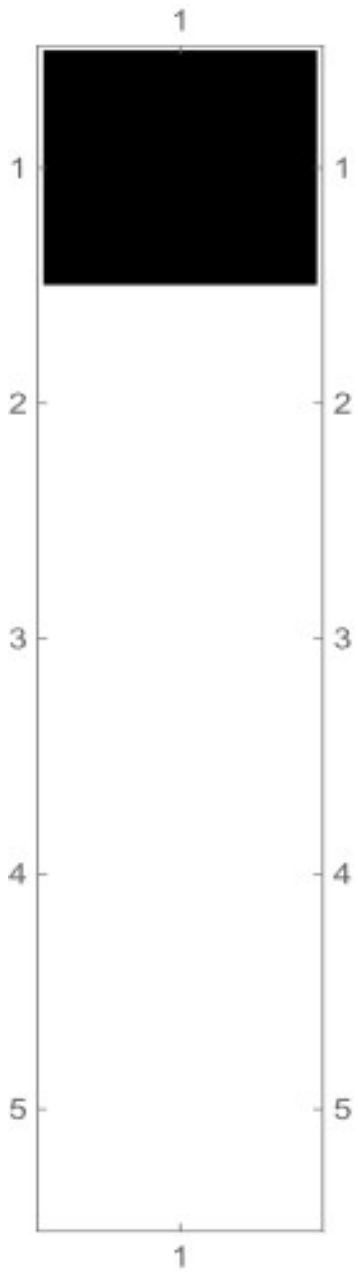


Figure 4.2: Probabilty distribution for initial state $|\varphi(0)\rangle = |1\rangle$

Now let us take the initial state from Eq 4.2.

$$|\psi_{\text{spatial}}(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - e^{i\phi}|2\rangle) \tag{4.3}$$

Probability distribution graph for Eq 4.3 is shown in Fig 4.6. Depending on the initial phase φ, we got different graphs for occupation probabilities. This allows us

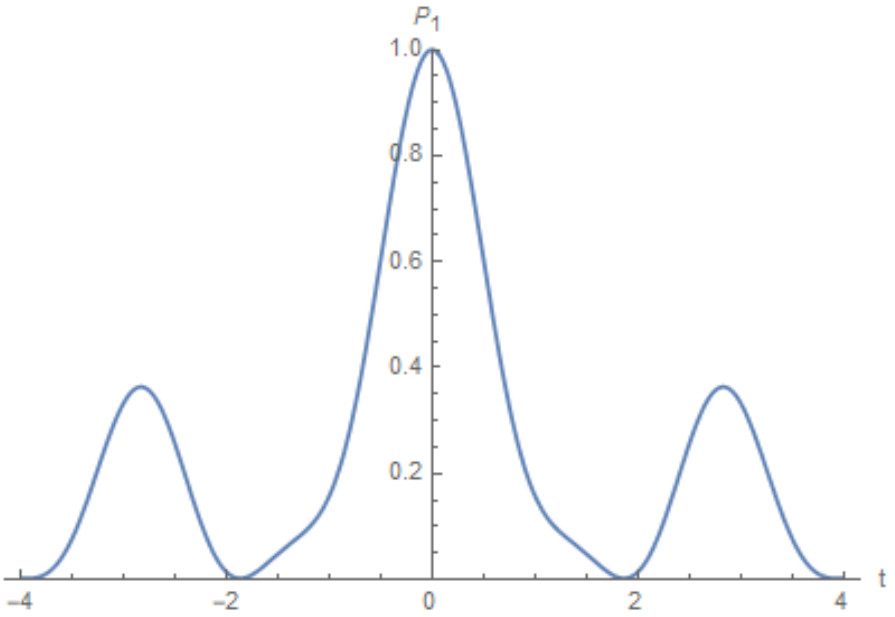


Figure 4.3: Occupation probability at i=1 for state $|\varphi(0)\rangle = |1\rangle$

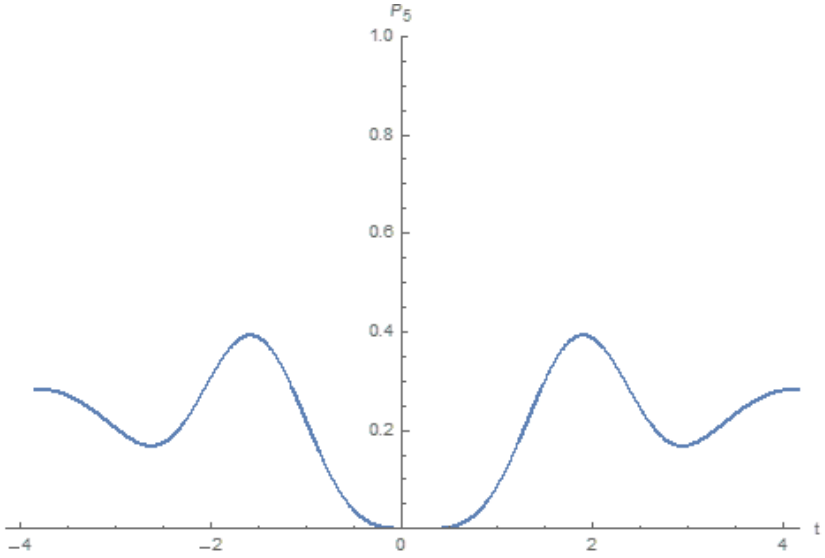


Figure 4.4: Occupation probability at i=5 for state $|\varphi(0)\rangle = |1\rangle$

to manage path interference as well. After comparing P_5 (t) numerically for various φ, we discovered that $\varphi = \frac{3\pi}{4}$ results is the maximum in occupancy of site i=5. The graph for P_1 (t) is shown in Fig. 4.6 and for P_5 (t) is shown in Fig. 4.7 at $\varphi = \frac{3\pi}{4}$.

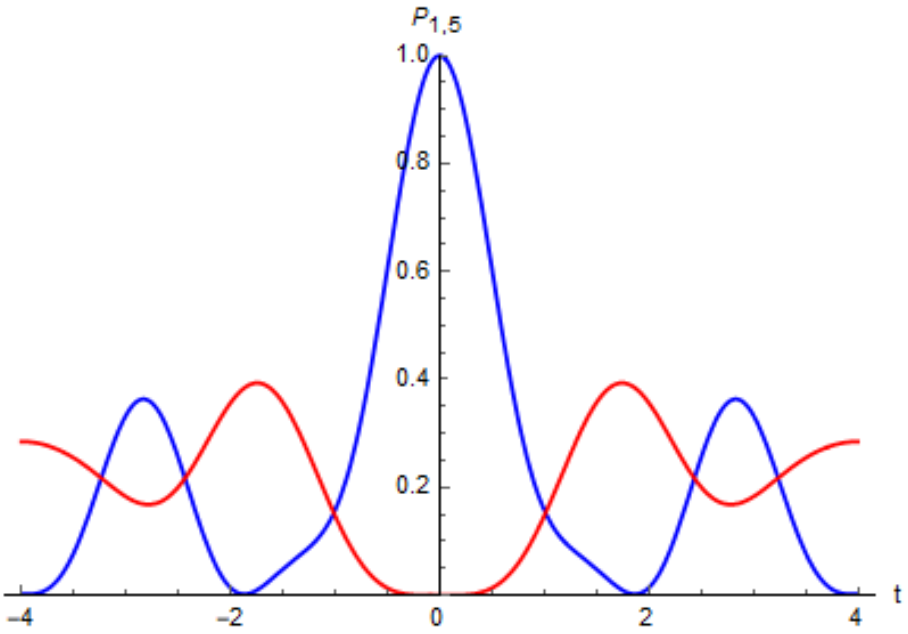


Figure 4.5: Combined graph of probabilities for state $|\varphi(0)\rangle = |1\rangle$, where blue curve for i=1 and red for i=5.

From the Figures (4.6 and 4.7), it is clear that the transfer rate increases when we use the initial state in the form of quantum superposition. We determine that injecting the particle into the chain will result in a more successful transfer of the particle from the one end to the other end site, in comparison to initiating a well-localized particle at a single site, concurrently at two sites with sufficient quantum coherence is more successful.

The combined graph for probabilities having initial equation as mentioned above in Eq 4.2 (by green and purple curve for P_1 and P_5 respectively) at $\varphi = \frac{3\pi}{4}$ is shown in Fig 4.9.

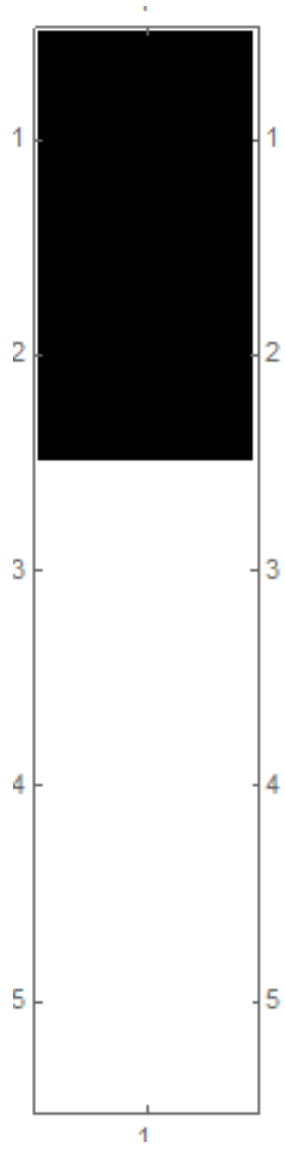


Figure 4.6: Probabilty distribution for initial state $|\psi_{\text{spatial}}(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - e^{i\phi}|2\rangle)$

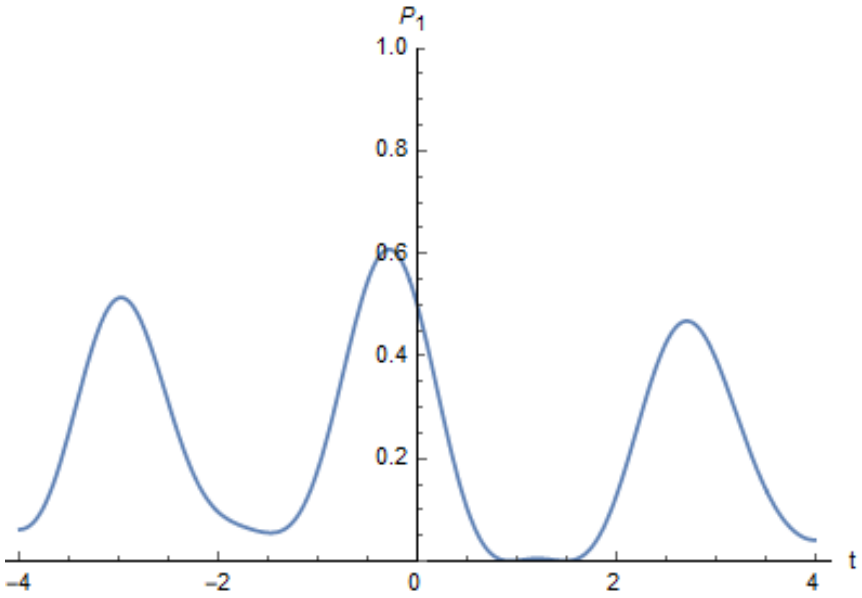


Figure 4.7: Occupation probability at i=1 for state $|\psi_{\text{spatial}}(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - e^{i\phi}|2\rangle)$ at $\varphi = \frac{3\pi}{4}$.

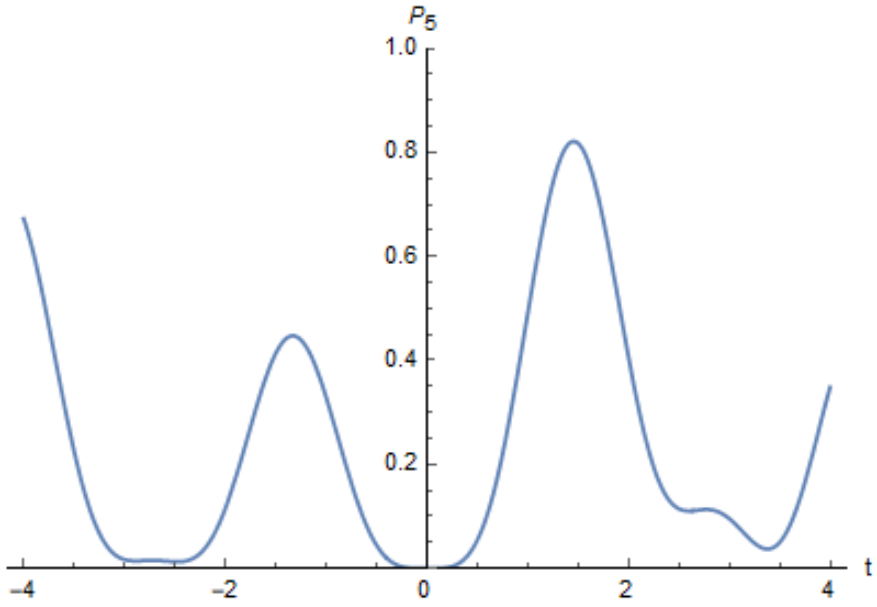


Figure 4.8: Occupation probability at i=5 for state $|\psi_{\text{spatial}}(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - e^{i\phi}|2\rangle)$ at $\varphi = \frac{3\pi}{4}$.

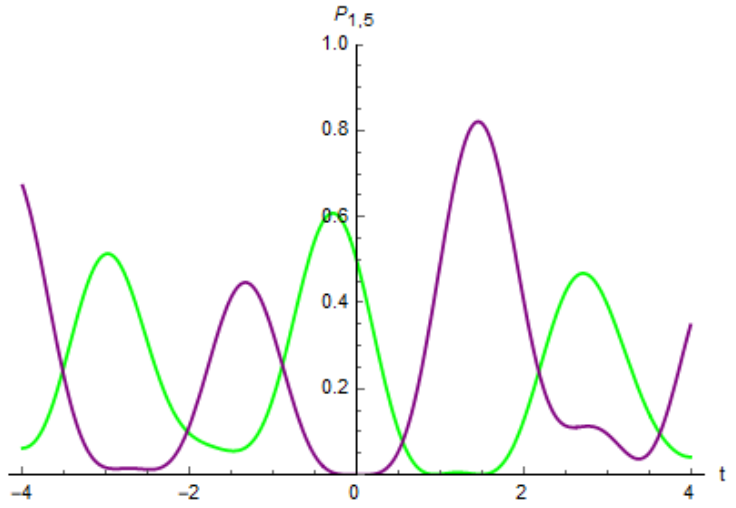


Figure 4.9: Combined graph of probabilities for stat $|\psi_{\text{spatial}}(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - e^{i\phi}|2\rangle)$ at $\varphi = \frac{3\pi}{4}$, where green curve is for i=1 and purple is for i=5.

4.5 Entanglement transfer evaluation through concurrence:

Concurrence plays an important role in describing quantum phase transition Recall the equation 4.2

$$|\psi_{\text{spatial}}(0)\rangle = \frac{1}{\sqrt{2}}(|1\rangle - e^{i\phi}|2\rangle) \tag{4.4}$$

Now after applying the unitary operation, equation 4.2 evolves into the state in the form

$$|\psi(t)\rangle = (A_1|100\rangle_{123}+A_2|010\rangle_{123}+A_3|001\rangle)|00\rangle_{45}+|000\rangle_{123}(A_4|10\rangle_{45}+A_5|01\rangle_{45}) \tag{4.5}$$

where are the time-dependent coefficients depending on the eigenvalues of the Hamiltonion. Now we need a reduced density matrix for concurrence. So for this purpose, we take a trace on sites 1,2 and 3 of the equation. After taking the trace we get the reduced density matrix for sites 4 and 5.

$$\rho_{4,5} = \begin{pmatrix} 1-a_{44}-a_{55} & 0 & 0 & 0 \\ 0 & a_{44} & a_{45} & 0 \\ 0 & a_{45}^* & a_{55} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.6}$$

where

$$a_{ij} = A_i A_j^*.$$

As we defined earlier, concurrence is given by

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4).$$

We have plotted the concurrence versus time graph for different values of φ. Fig. 4.10 shows the concurrence at $\varphi = \pm\frac{3\pi}{4}$. For time $t = 1.12$, concurrence is found to be maximum, that is, 0.8.

For $\varphi = \pm\frac{\pi}{4}$, the graph between concurrence and time is shown in (Fig. 4.11). Concurrence is also plot for $\varphi = \pm\frac{\pi}{2}$ and $\varphi = \pm\frac{\pi}{3}$ respectively in (Fig 4.12) and (Fig 4.13) respectively.

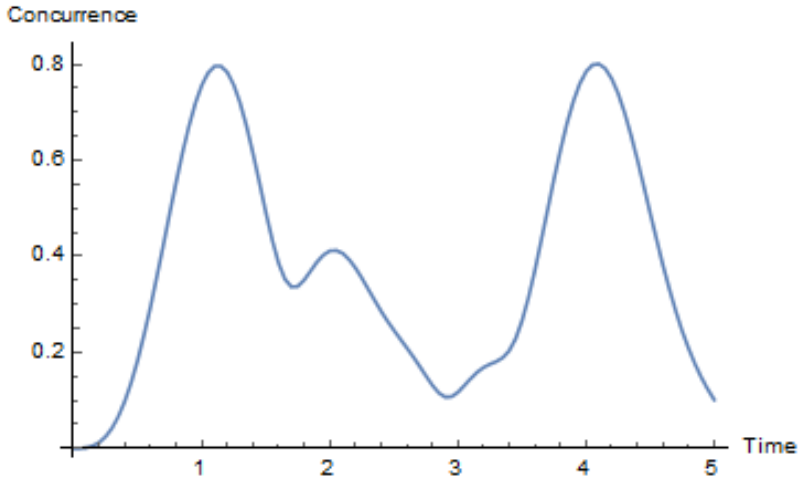


Figure 4.10: Concurrence at $\varphi = \pm\frac{3\pi}{4}$

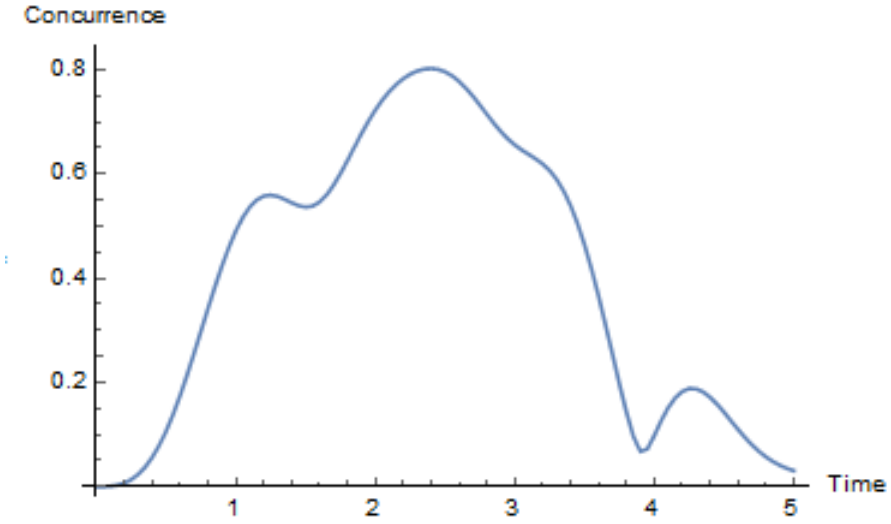


Figure 4.11: Concurrence at $\varphi = \pm\frac{\pi}{4}$

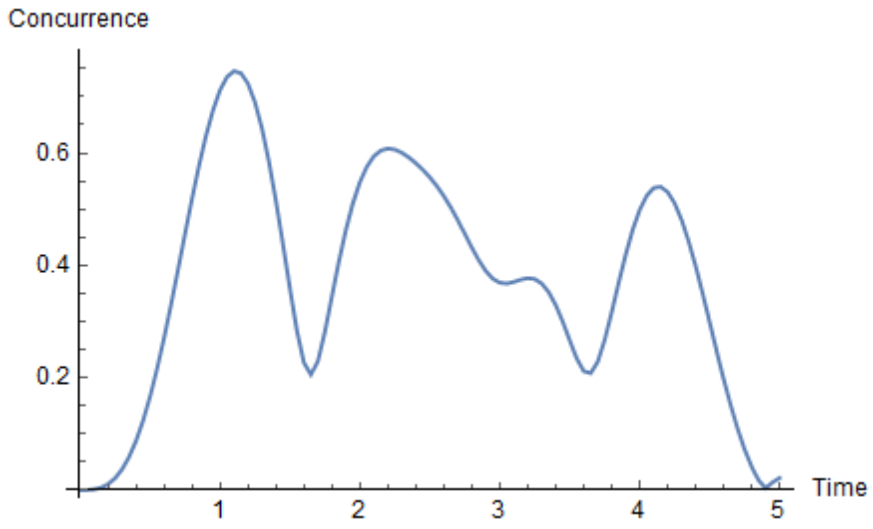


Figure 4.12: Concurrence at $\varphi = \pm\frac{\pi}{2}$

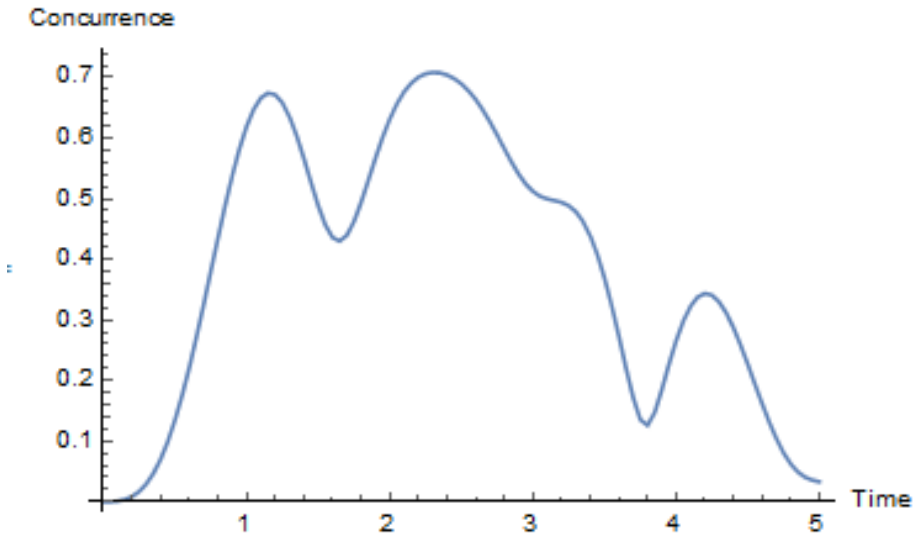


Figure 4.13: Concurrence at $\varphi = \pm\frac{\pi}{3}$

Chapter 5

Conclusion:

In this work, we reviewed the dynamics of quantum walks and how they differ more than their classical counterparts, making them valuable tools for modeling natural systems and quantum information technology. We first established the foundation for quantum walks and then studied entanglement transfer. On a triangular chain, we investigated the transfer of entanglement of a single particle undergoing continuous-time quantum walk (CTQW). We discovered that when the particle is concurrently injected from one corner of sites in a certain quantum (Bell-type) superposition state, the transfer rate is more successful. We found that success in occupation probability depends on the phase ϕ of the initial quantum coherent superposition. Notably, for the same ϕ but at a different time, the most successful transfer also involves the maximally entangled state.

Bibliography

[Azzini et al., 2020] Azzini, S., Mazzucchi, S., Moretti, V., Pastorello, D., and Pavesi, L. (2020). Single-particle entanglement. *Advanced Quantum Technologies*, 3(1):100008.

[Bell, 1964] Bell, J. S. (1964). On the einstein podolsky rosen paradox. *Physics Physique* , 1(3):195–200.

[Chandrashekar, 2010] Chandrashekar, C. (2010). Fractional recurrence in discrete-time quantum walk. *Open Physics*, 8(3):579–583.

[Einstein et al., 1935] Einstein, A., Podolsky, B., and Rosen, N. (1935). Can quantum-mechanical description of physical reality be considered complete? *Physical Review*, 47(10):777–780.

[Giordani et al., 2021] Giordani, T., Innocenti, L., Suprano, A., Polino, E., and et al. (2021). Entanglement transfer, accumulation and retrieval via quantum-walk-based qubit–qudit dynamics. *New Journal of Physics*, 23(11).

[Hill and Wootters, 1997] Hill, S. and Wootters, W. K. (1997). Preservation of entanglement by local unitary operations. *Physical Review Letters*, 70(25):5022–5025.

[Johnson, 2015] Johnson, M. (2015). Continuous time quantum walks (ctqw).

[Kempe, 2003] Kempe, J. (2003). Quantum random walks: An introductory overview. *Contemporary Physics*, 44(4):307–327.

[Mutiara et al., 2013] Mutiara, A., Refianti, R., and Karamoy, J. (2013). On a testing and implementation of quantum gate and measurement emulator (qgame). *International Journal of Engineering and Technology (IJET)*, 5:2186–.

[Neogy and Mer, 2021] Neogy, S. K. and Mer, V. N. (2021). *Chapter 4 Copositive Optimization and Its Applications in Graph Theory*. Springer Science and Business Media LLC.

[Qu et al., 2022] Qu, D., Xiao, L., Wang, K., Zhan, X., and Xue, P. (2022). Experimental investigation of equivalent laplacian and adjacency quantum walks on irregular graphs. *Physical Review A*, 105(6):062448.

[Sağlam et al., 2023] Sağlam, U., Paternostro, M., and Müstecaplıoğlu, E. (2023). Entanglement transfer via chiral and continuous-time quantum walks on a triangular chain. *Physica A: Statistical Mechanics and its Applications*.

[Tripathy and Singh, 2023] Tripathy, B. K. and Singh, A. (2023). *Chapter 14 Role of Quantum Computing in a Data Analytics Environment*. IGI Global.

[Venegas-Andraca, 2012] Venegas-Andraca, S. E. (2012). Quantum walks: a comprehensive review. *Quantum Information Processing*, 11(5):1015–1106.

[Whitfield et al., 2010] Whitfield, J. D., Rodríguez-Rosario, C. A., and Aspuru-Guzik, A. (2010). Quantum stochastic walks: A generalization of classical random walks and quantum walks. *Physical Review A*, 82(032334).

[Wootters, 1998] Wootters, W. K. (1998). Entanglement. *Physical Review Letters*, 80(10):2245–2248.

[Yu and Eberly, 2007] Yu, T. and Eberly, J. H. (2007). Entanglement and teleportation with a composite photonic system. *Quantum Information and Computation*. in press.