Algebraic Invariants of Edge Ideals of Some Bristled Circulant Graphs

By

Ibad Ur Rehman

Supervised by

Dr. Muhammad Ishaq

Department of Mathematics

School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan
2024

THESIS ACCEPTANCE CERTIFICATE

Certified that final copy of MS thesis written by **Ibad Ur Rehman** (Registration No **00000402958)**, of **School of Natural Sciences** has been vetted by undersigned, found complete in all respects as per NUST statutes/regulations, is free of plagiarism, errors, and mistakes and is accepted as partial fulfillment for award of MS/M.Phil degree. It is further certified that necessary amendments as pointed out by GEC members and external examiner of the scholar have also been incorporated in the said thesis.

Signature: ______________

Name of Supervisor: Dr. Muhammad Ishaq

Date: 23-7-2024

Signature (HoD): ______________

Date: 23-7-2024

Signature (Dean/Principal): ______________

Date: 23.7.2024

FORM TH-4

National University of Sciences & Technology

MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by Ibad Ur Rehman, Regn No. 00000402958 Titled Some Algebraic Invariants of Edge Ideals of Some Bristled Circulant Graphs be Accepted in partial fulfillment of the requirements for the award of MS degree.

Examination Committee Members

1. Name: PROF. MATLOOB ANWAR ____________ Signature: ____________

2. Name: DR. HAFIZ MUHAMMAD FAHAD ____________ Signature: ____________

Supervisor's Name: DR. MUHAMMAD ISHAQ ____________ Signature: ____________

____
Head of Department

23/7/2024
Date

COUNTERSINGED

Date: 23.7.2024

____
Dean/Principal

I dedicate this thesis to my loving parents, venerable supervisor, respectable teachers and fellows for their limitless support and encouragement.

Acknowledgement

I begin by expressing my sincere gratitude to Allah Almighty for bestowing upon me countless blessings, granting me the ability and strength to undertake and complete this research work.

I wish to convey my gratitude to supervisor, Dr. Muhammad Ishaq, for his exceptional guidance, encouragement, support, and wholehearted dedication throughout my project. It is due to his mentorship that I was able to successfully complete this work. Driven by his profound wisdom and discipline, he generously shared his knowledge, which will undoubtedly continue to benefit me in the future.

I extend my heartfelt gratitude to my parents and brothers for the invaluable gift of their prayers. Their constant encouragement and appreciation have been instrumental in helping me pursue my life's goals. I am also deeply thankful to my friends for their guidance and unwavering moral support..

Abstract

The purpose of this dissertation is to analyze various algebraic invariants, including Stanley depth, depth, regularity, and the projective dimension of the quotient rings obtained from the bristled of some four and five regular circulent graph. The study establishes exact values for the depth, regularity, and the projective dimension of these quotient rings. Moreover we obtain a lower bound for stanley depth.

Contents

List of Figures

Introduction

Abstract algebra is a branch of mathematics that concerns itself with the study of algebraic structures, particularly rings and modules. Within this field, algebraic and geometric invariants play a pivotal role in the characterization of rings and modules. These invariants, such as Stanley depth, projective dimension, depth, and regularity, provide valuable insights into the properties and characteristics of these mathematical structures. They serve as useful tools for describing and analyzing modules and rings, revealing their structural and algebraic properties.

Stanley depth, initially introduced by Stanley in 1982, is a significant invariant used in the study of finitely generated $\mathbb{Z}^n$-graded modules over commutative rings. Stanley also proposed a conjecture, known as Stanley's conjecture, which relates the Stanley depth to the depth of a module. This conjecture was presented in the reference [25]. However, in a subsequent work by Duval et al. [9], it was demonstrated that Stanley's conjecture does not hold for modules $P/\mathcal{J}$, where P represents the ring of polynomials over a field and $\mathcal{J}$ denotes a monomial ideal. Consequently, the task of identifying classes of modules that satisfy Stanley's inequality remains a challenging problem. Several results related to Stanley depth can be found in the references [6, 10, 12, 15, 16]. The regularity of edge ideals (REIs) and projective dimension (pdim) have been extensively investigated by various researchers. To explore these two invariants further, one can refer to the references [1, 2, 7, 4]. These works provide additional insights into the values and bounds associated with the regularity of edge ideals and pdim.

In this thesis exact values of Depth and projective dimension are computed moreover we also compute bounds for the Stanley depth of the quotient ring corresponding edge ideals of bristled of some four and five regular circulent graphes . Also exact values of regularity for such structures are calculated.

Chapter 1 of this thesis serves as an introductory chapter, providing an overview of the fundamental concepts in Ring and Module Theory. It covers essential definitions, examples, and results pertaining to these algebraic structures. Moreover, the chapter includes a concise explanation of graph theory, along with relevant examples, which will be employed in the subsequent analysis.

Chapter 2 of this thesis offers a comprehensive overview of Stanley depth, depth, regularity, and projective dimension. It explores various results pertaining to these algebraic invariants and includes illustrative examples. The chapter also introduces Stanley's conjecture and outlines the methodology for calculating Stanley depth for squarefree monomial ideals, as well as for quotient rings associated with these ideals.

In Chapter 3, exact values for the depth, projective dimension and bound Stanley depth of the quotient ring corresponding to some special subgraphs of $Br_t(C_{2n}(1, n-1))$, $Br_t(C_{2n}(1, 2))$ and $Br_t(C_{2n}(1, n-1, n))$ graphs are computed.

In Chapter 4, exact values for the depth, projective dimension and bound Stanley depth of the quotient ring corresponding cyclic modules associated with bristles of some four and five regular circulant Graphs are computed.

Chapter 1

Preliminaries

In chapter 1, we discuss some concepts related to rings, modules, and graph theory, along with providing relevant definitions and examples.

Ring Theory is a field of study that originated in the 1800s. Emmy Noether played a significant role in introducing the general concept of commutative rings. Initially, the focus was primarily on commutative rings, but later the study expanded to include non-commutative rings.

Module Theory deals with the study of modules. A module is an algebraic structure that is associated with a ring, where the ring acts on the module.

Graph Theory involves the examination of graphs and encompasses various operations performed on them. In this chapter, we introduce fundamental terminologies used in Graph Theory and provide illustrative examples of graphs that will be utilized in subsequent chapters of the study.

1.1 Ring Theory

This section delves into the properties of rings, offering examples to illustrate these concepts. It explores different definitions that characterize various types of rings and their associated properties. Additionally, the discussion covers the concept of ideals and explores several related operations.

Definition 1.1.1. A ring P with binary operations namely addition " $+$ " and multiplication " $\times$ " is a set such that the following hold:

- with respect to addition P is a commutative group,
- P is semigroup w.r.t. multiplication, i.e
  $a \times (b \times c) = (a \times b) \times c)$ and also distributive over addition i.e
  1. $(a \times b) + (a \times c) = a \times (b + c)$,
  2. $(b \times a) + (c \times a) = (b + c) \times a, \; \forall \; a, b, c \in P$.

Definition 1.1.2. Let P be a ring. Then,

- P is known as a ring having multiplicative identity if $\exists \; e$, s.t. $\forall \; b \in P, \; b \times e = e \times b = b$.
- If $b \times a = a \times b, \; \forall \; a, b \in P$, then P is commutative ring.

Throughout this thesis we will consider commutative rings having multiplicative identity.

Example 1.1.1.
- $\mathbb{R}, \mathbb{Z}$ and $\mathbb{Q}$ are commutative rings with identity.
- For $n \geq 2$ the ring $n\mathbb{Z}$ is a commutative ring without identity.

1.1.1 Polynomial Ring

Definition 1.1.3. Let P be a commutative ring with unity. For $n \geq 0$ and $s_i \in P$, $P(Z) = s_0 + s_1 Z + s_2 Z^2 + \cdots + s_{n-1} Z^{n-1} + s_n Z^n$ is termed as a polynomial in indeterminate Z having co-efficients s_i.

Example 1.1.2. $\mathbb{R}[Y]$, $\mathbb{Q}[Y]$ and $\mathbb{Z}[Y]$ are few examples of polynomial rings.

1.1.2 Noetherian Ring

Definition 1.1.4. *(Chain conditions)* Let $\mathcal{R}$ be a poset w.r.t $\leq$. Then the following are equivalent:

- Every increasing sequence $y_1 \leq \ldots \leq y_p \leq \ldots$ in $\mathcal{R}$ is stationary i.e $\exists \; q$ such that $x_m = x_q \; \forall \; m \geq q$,
- A maximal element is present in every non-empty subset of $\mathcal{R}$.

If $\mathcal{R}$ represents the set containing all the ideals of the ring P which is ordered by the relation $\subseteq$ then the first chain condition is called the ascending chain condition.

Definition 1.1.5. A ring P is termed to be Noetherian if it has an ascending chain condition on its ideal.

Example 1.1.3. $\mathbb{Z}$, which is an example Noetherian ring.

1.1.3 Ring Homomorphisms and Quotient Rings

Definition 1.1.6. For rings $\mathcal{U}$ and $\mathcal{V}$, a map $\alpha : \mathcal{U} \longrightarrow \mathcal{V}$ is a homomorphism that satisfies the following:

- $\alpha(x + y) = \alpha(x) + \alpha(y)$,
- $\alpha(xy) = \alpha(x)\alpha(y)$, for all $x \in \mathcal{U}$ and $y \in \mathcal{U}$.

Definition 1.1.7. Suppose $\mathcal{U}$ and $\mathcal{V}$ are rings and if $\alpha : \mathcal{U} \longrightarrow \mathcal{V}$ is a homomorphism then the kernal of α is given as $\ker \alpha = \{x \in \mathcal{U} : \alpha(x) = 0\}$ and the image of α is defined as $\alpha(\mathcal{U}) = \{y \in \mathcal{V} : y = \alpha(x)\}$.

Example 1.1.4. The map $\alpha : \mathbb{Z} \longrightarrow \mathbb{Z}_n$ defined as $\alpha(z) = z(\text{mod } n)$ is a ring homomorphism with $\ker \alpha = n\mathbb{Z}$ and $\operatorname{Im} \alpha = \mathbb{Z}_n$.

Definition 1.1.8. Let P be a ring with ideal A, then the quotient ring P/A is the set of cosets with addition and multiplication defined as $(a_1 + A) + (a_2 + A) = (a_1 + a_2) + A$ and $(a_1 + A)(a_2 + A) = a_1a_2 + A$.

1.1.4 Ideals and Their Properties

Definition 1.1.9. Let $A \subseteq P$, then A is said to be an ideal of P if

- A is an additive abelian subgroup of P,
- If $a \in A$ and $d \in P$, then the product $da \in A$, i.e $dA \subseteq A$ and also $ad \in A$, i.e $Ad \subseteq A$.

Example 1.1.5. For the ring $\mathbb{Z}$ its ideals are of the type $n\mathbb{Z}$, where $n \in \mathbb{Z}$.

Every ideal is a subring but the converse doesnot necessarily hold. For example $\mathbb{Z}$ is a subring of $\mathbb{Q}$ but not an ideal of $\mathbb{Q}$.

Definition 1.1.10. Let C and D be the ideals of P. Then

- $C+D$ is the sum of ideals C and D given by $C+D=\{i+j : i\in C, j\in D\}$,
- CD is the product of C and D given by $CD=\{i_1j_1+i_2j_2+\cdots+i_rj_r : i_1,\ldots,i_r\in C, j_1,\ldots,j_r\in D\}$,
- The intersection $C\cap D$ is defined as $C\cap D=\{a\in P : a\in C \text{ and } a\in D\}$.

Example 1.1.6. Let $k,l\in\mathbb{Z}^+$. If $C=(k)=k\mathbb{Z}$ and $D=(l)=l\mathbb{Z}$ be ideals of $P=\mathbb{Z}$. Then

- $C+D=gcd(k,l)\mathbb{Z}$,
- $C\cap D=lcm(k,l)\mathbb{Z}$,
- $CD=(kl)=kl\mathbb{Z}$.

Example 1.1.7. Let $C=3\mathbb{Z}$ and $D=9\mathbb{Z}$ then, $C+D=3\mathbb{Z}$, $C\cap D=9\mathbb{Z}$ and $CD=27\mathbb{Z}$.

Definition 1.1.11. Let C and D be two ideals of P. The ideal quotient of C and D, also known as the colon ideal, is defined as $(C:D)=\{c\in P : cD\subseteq C\}$.

Definition 1.1.12. In a ring P, a maximal ideal $\mathcal{M}$ is a proper ideal that has no other proper ideal lying between it and the entire ring P.

1.1.5 Monomial Ideals

Definition 1.1.13. Let $P=\Re[x_1,\ldots,x_d]$ where $\Re$ represents a field. The product $x_1^{b_1}x_2^{b_2}\ldots x_n^{b_d}$, with $b_i\in\{0,1,2,\ldots\}$ is called a monomial of P. A monomial is square free when $b_i\in\{0,1\}$.

Definition 1.1.14. An ideal $B \subseteq P = \Re[x_1, \ldots, x_d]$ if generated by monomials is classified as a monomial ideal. Furthermore, we denote by $G(B)$ the set of monomials in B that are minimal w.r.t divisibility.

Example 1.1.8. Let $P = \Re[x_1, \ldots, x_5]$. Then $A = (x_4^2 x_3, x_1^2 x_2, x_3, x_5)$ is a monomial ideal and $B = (x_2 x_3, x_1 x_5, x_2)$ is a monomial ideal which is square free.

Definition 1.1.15. Let $P = \Re[x_1, \ldots, x_n], C = (a_1, \ldots, a_n)$ and $D = (b_1, \ldots, b_m)$ are monomial ideals of P. Then

- $C + D = (a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m)$,
- $CD = (a_1 b_1, \ldots, a_1 b_m, a_2 b_1, \ldots, a_2 b_m, a_n b_1, \ldots, a_n b_m)$,
- $C^2 = (a_1^2, a_1 a_2, \ldots, a_2 a_1, \ldots, a_2 a_n, \ldots, a_n a_1, \ldots, a_n^2)$,
- $C \cap D = \{lcm(a, b) : a \in G(C), b \in G(D)\}$,
- $C : D = \cap_{b \in G(D)} (C : b) = \{\frac{a}{gcd(b,b)} : a \in G(C)\}$.

Example 1.1.9. Let C and D be monomial ideals of $P = \Re[a_1, a_2, a_3]$ such that $C = (a_1 a_2, a_1^2 a_3, a_2 a_3^2, a_1 a_3^2)$ and $D = (a_1 a_3, a_2^2 a_3, a_1^2)$. Then

- $C + D = (a_1 a_2, a_1^2 a_3, a_2 a_3^2, a_1 a_3, a_2^2 a_3, a_1^2) = (a_1 a_2, a_2 a_3^2, a_1 a_3, a_2^2 a_3, a_1^2)$,
- $CD = (a_1^2 a_2 a_3, a_1 a_2^3 a_3, a_1^3 a_2, a_1^3 a_3^2, a_1^2 a_2^2 a_3^2, a_1^4 a_3, a_1 a_2 a_3^3, a_2^3 a_3^3, a_1^2 a_2 a_3^2, a_1^2 a_3^3, a_1 a_2^2 a_3^3,$
  $a_1^3 a_3^2) = (a_1^2 a_2 a_3, a_1 a_2^3 a_3, a_1^3 a_2, a_1^3 a_3^2, a_1^4 a_3, a_1 a_2 a_3^3, a_2^3 a_3^3, a_1^2 a_3^3)$,
- $C \cap D = (a_1 a_2 a_3, a_1 a_2^2 a_3, a_1^2 a_2, a_1^2 a_3, a_1^2 a_2^2 a_3, a_1^2 a_3, a_1 a_2 a_3^2, a_2^2 a_3^2, a_1^2 a_2 a_3^2, a_1 a_3^2$
  $a_1 a_2^2 a_3^2, a_1^2 a_3^2) = (a_1 a_2 a_3, a_1^2 a_2, a_1^2 a_3, a_2 a_3^2, a_1 a_3^2)$,
- $C : D = (C : a_1 a_3) \cap (C : a_2^2 a_3) \cap (C : a_1^2)$
  $= (\frac{a_1 a_2}{gcd(a_1 a_2, a_1 a_3)}, \frac{a_1^2 a_3}{gcd(a_1^2 a_3, a_1 a_3)}, \frac{a_2 a_3^2}{gcd(a_2 a_3^2, a_2^2 a_3)}, \frac{a_1 a_3^2}{gcd(a_1 a_3^2, a_1 a_3)}) \cap (\frac{a_1 a_2}{gcd(a_1 a_2, a_2^2 a_3)},$
  $\frac{a_1^2 a_3}{gcd(a_1^2 a_3, a_2^2 a_3)}, \frac{a_2 a_3^2}{gcd(a_2 a_3^2, a_2^2 a_3)}, \frac{a_1 a_3^2}{gcd(a_1 a_3^2, a_2^2 a_3)}) \cap \frac{a_1 a_2}{gcd(a_1 a_2, a_1^2)}, \frac{a_1^2 a_3}{gcd(a_1^2 a_3, a_1^2)},$
  $\frac{a_2 a_3^2}{gcd(a_2 a_3^2, a_1^2)}, \frac{a_1 a_3^2}{gcd(a_1 a_3^2, a_1^2)})$

$= (a_2, a_1, a_3, a_3) \cap (a_1, a_1^2, a_3, a_1a_3) \cap (a_2, a_3, a_2a_3^2, a_3^2)$

$= (a_1, a_2, a_3) \cap (a_1, a_3) \cap (a_2, a_3) = (a_1a_2, a_1a_3, a_2a_3, a_3).$

Example 1.1.10. Let C and D be monomial ideals of $P = \Re[a_1, a_2, a_3, a_4]$ such that $C = (a_1a_3, a_3a_4, a_2a_3)$ and $D = (a_1a_4, a_2a_4, a_1a_3)$. Then

- $C + D = (a_1a_3, a_3a_4, a_2a_3, a_1a_4, a_2a_4, a_1a_3) = (a_1a_3, a_3a_4, a_2a_3, a_1a_4, a_2a_4)$,
- $CD = (a_1^2a_3a_4, a_1a_2a_3a_4, a_1^2a_3^2, a_1a_3a_4^2, a_2a_3a_4^2, a_1a_3^2a_4, a_2^2a_3a_4, a_1a_2a_3a_4, a_2^2a_3a_4,$
  $a_1a_2a_3^2) = (a_1^2a_3a_4, a_1^2a_3^2, a_1a_3a_4^2, a_2a_3a_4^2, a_1^4a_3, a_1a_3^2a_4, a_1a_2a_3a_4, a_2^2a_3a_4, a_1a_2a_3^2)$,
- $C \ \cap \ D = (a_1a_3a_4, a_1a_2a_3a_4, a_1a_3, a_1a_3a_4, a_2a_3a_4, a_1a_3a_4, a_1a_2a_3a_4, a_2^2a_3a_4,$
  $a_1a_2a_3^2) = (a_1a_3, a_2a_3a_4)$,

Example 1.1.11. *Let C and D be monomial ideals of $P = \Re[a_1, a_2, a_3, a_4, a_5]$ such that $C = (a_1a_3, a_2^2a_4, a_5^2a_1)$ and $D = (a_1a_4, a_2a_3, a_4a_5)$. Then*

- $C + D = (a_1a_3, a_2^2a_4, a_1a_5^2, a_1a_4, a_2a_3, a_4a_5)$,
- $CD = (a_1^2a_3a_4, a_1a_2a_3^2, a_1a_3a_4a_5, a_1a_2^2a_4^2, a_2^3a_3a_4, a_2^2a_4^2a_5, a_1^2a_4a_5^2, a_1a_2a_3a_5^2,$
  $, a_1a_4a_5^3) = (a_1^2a_3a_4, a_1^2a_3^2, a_1a_3a_4^2, a_2a_3a_4^2, a_1^4a_3, a_1a_3^2a_4, a_1a_2a_3a_4, a_2^2a_3a_4, a_1a_2a_3^2, a_1a_4a_5^3)$,
- $C^2 = (a_1^2a_3^2, a_1a_2^2a_3a_4, a_1^2a_3a_5^2, a_1a_2^2a_4^2, a_2^3a_3a_4, a_2^2a_4^2a_5, a_1^2a_4a_5^2, a_1a_2a_3a_5^2,$
  $, a_1a_4a_5^3) = (a_1^2a_3^2, a_1^2a_3a_5^2, a_1a_2^2a_4^2, a_2^3a_3a_4, a_2^2a_4^2a_5, a_1^2a_4a_5^2, a_1a_2a_3a_5^2, a_1a_4a_5^3)$,
- $D^2 = (a_1^2a_4^2, a_1a_2a_3a_4, a_1a_4^2a_5, a_1a_2a_3a_4, a_2^2a_3^2, a_2a_3a_4a_5, a_1a_4^2a_5, a_2a_3a_4a_5,$
  $, a_4^2a_5^2) = (a_1^2a_4^2, a_1a_4^2a_5, a_1a_2a_3a_4, a_2^2a_3^2, a_2a_3a_4a_5, a_1a_4^2a_5, a_2a_3a_4a_5, a_4^2a_5^2)$,
- $C \ \cap \ D = (a_1a_3a_4, a_1a_2a_3, a_1a_3a_4a_5, a_1a_2^2a_4, a_2a_3a_4, a_2^2a_4a_5, a_5^2a_1a_4, a_5^2a_1a_2a_3,$
  $a_1a_4a_5) = (a_1a_3a_4, a_1a_2a_3, a_1a_3a_4a_5, a_1a_2^2a_4, a_2a_3a_4, a_2^2a_4a_5, a_5^2a_1a_4, a_5^2a_1a_2a_3, a_1a_4a_5)$,
- $C : D = (C : a_1a_4) \cap (C : a_2^2a_3) \cap (C : a_4a_5)$
  $= (\frac{a_1a_3}{gcd(a_1a_3, a_1a_4)}, \frac{a_2^2a_4}{gcd(a_2^2a_4, a_1a_4)}, \frac{a_1a_5^2}{gcd(a_1a_5^2, a_1a_4)}) \ \cap \ (\frac{a_1a_3}{gcd(a_1a_3, a_2a_3)},$

$$\frac{a_2^2a_4}{gcd(a_2^2a_4,a_2a_3)}, \frac{a_5^2a_1}{gcd(a_5^2a_1,a_2a_3)}), \cap \frac{a_1a_3}{gcd(a_1a_3,a_4a_5)}, \frac{a_2^2a_4}{gcd(a_2^2a_4,a_4a_5)}, \frac{a_1a_5^2}{gcd(a_1a_5^2,a_4a_5)})$$

$$= (a_3, a_2^2, a_5^2) \cap (a_1, a_1^2, a_3, a_1a_3) \cap (a_2, a_3, a_2a_3^2, a_3^2)$$

$$= (a_1, a_2, a_3) \cap (a_1, a_3) \cap (a_2, a_3) = (a_1a_2, a_1a_3, a_2a_3, a_3).$$

1.2 Application of Ring Theory in Commutative Algebra

The field of Commutative Algebra has greatly benefited from the application of ring theory, a fundamental branch of abstract algebra. Commutative algebra, which focuses on the study of commutative rings and modules, often utilizes the concepts and techniques of ring theory to gain deeper insights and develop powerful tools for problem-solving.

One of the primary ways in which ring theory is applied in commutative algebra is through the study of the structure of commutative rings. Ring theorists have developed a rich set of tools and theorems for understanding the properties of rings, such as ideals, homomorphisms, and various classes of rings (e.g., principal ideal domains, Noetherian rings, and Artinian rings). These concepts are extensively used in commutative algebra to characterize the structure of commutative rings and to investigate their fundamental properties.

Another important application of ring theory in commutative algebra is the study of module theory. Modules, which are generalizations of vector spaces over commutative rings, are essential objects of study in commutative algebra. Ring theoretic concepts, such as module homomorphisms, exact sequences, and projective and injective modules, are crucial for understanding the structure and behavior of modules over commutative rings. These insights have led to the development of important results.

Additionally, ring theory provides a powerful framework for the study of polynomial rings and their ideals, which are central objects in commutative algebra. The theory of Noetherian rings, which guarantees the existence of certain ideal-theoretic properties, has been instrumental in the study of polynomial rings and their applications in areas such as algebraic geometry and computational algebra.

Furthermore, ring theory has applications in the study of commutative algebra over non-traditional number systems, such as finite fields or modular arithmetic. In these contexts, the understanding of the ring-theoretic properties of the underlying number system can provide valuable insights into the structure and behavior of the corresponding commutative rings.

The synergy between ring theory and commutative algebra has been a fruitful and ongoing area of research, leading to the development of powerful tools and techniques for understanding the structure and properties of commutative rings and modules. By leveraging the insights and methods of ring theory, researchers in commutative algebra have been able to make significant advancements in this field.

1.3 Module Theory

This section explores the various properties of modules, provides examples to illustrate their characteristics, and discusses important results in the field of Module Theory.

Definition 1.3.1. A *module* $\mathcal{M}$ over a commutative ring P is an abelian group $(\mathcal{M}, +)$ together with a scalar multiplication operation $P \times \mathcal{M} \to \mathcal{M}$, denoted as $(r, m) \to r \cdot m$, satisfying the aforementioned conditions: If $m \in \mathcal{M}$ and $\forall r, s \in P$ then

1. Closure under scalar multiplication: The product $r \cdot m$ is also an element of $\mathcal{M}$.

2. Distributivity of scalar multiplication:

$$r \cdot m + s \cdot m = (r + s) \cdot m$$
$$r \cdot m + r \cdot n = r \cdot (m + n)$$

3. Associativity of scalar multiplication:

$$(r \cdot s) \cdot m = r \cdot (s \cdot m)$$

4. Compatibility with the ring operations:

$$(r \cdot s) \cdot m = r \cdot (s \cdot m)$$

The elements of a module $\mathcal{M}$ are called *module elements* or simply *elements of* $\mathcal{M}$. The commutative ring P is often referred to as the *base ring* or the *coefficient ring* of the module $\mathcal{M}$. If the ring contains unity then we have one more axiom that is,

- $1m = m \ \forall \ m \in \mathcal{M}$.

Example 1.3.1. If P represents a ring and $B \subseteq P$ is the ideal of P, then P/B is an P-module under the scalar multiplication $a(b + B) = ab + B \ \forall \ a \in P, b + B \in P/B$.

Definition 1.3.2. For a ring P and P-module $\mathcal{M}$, a subset $\mathcal{O} \subseteq \mathcal{M}$ which is non empty is a submodule of $\mathcal{M}$ if $\mathcal{O}$ is a subgroup of $\mathcal{M}$ which is an additive group which also satisfies the module axioms using the scalar multiplication on $\mathcal{M}$.

Criteria for a Submodule

A subset $\mathcal{N}$ of $\mathcal{M}$ is considered a submodule of $\mathcal{M}$ if it satisfies the following criteria:

1. **Closed under Addition**: The subset $\mathcal{N}$ must be closed under the addition operation inherited from the module $\mathcal{M}$. This means that for any $n_1, n_2 \in \mathcal{N}$, their sum $n_1 + n_2$ is also an element of $\mathcal{N}$. Additionally, $\mathcal{N}$ must be closed under negation, so that for any $n \in \mathcal{N}$, the additive inverse $-n$ is also an element of $\mathcal{N}$. These properties ensure that $\mathcal{N}$ forms an abelian group under the addition operation.

2. **Closed under Scalar Multiplication**: For any element r in the commutative ring P and any element n in the subset $\mathcal{N}$, the scalar product $r \cdot n$ must also be an element of $\mathcal{N}$. This property ensures that $\mathcal{N}$ inherits the module structure from $\mathcal{M}$ and can be considered a module in its own right.

These two conditions, collectively, ensure that a subset $\mathcal{N}$ of a module $\mathcal{M}$ over a commutative ring P can be regarded as a submodule of $\mathcal{M}$. In other words, $\mathcal{N}$ must be both an abelian group and closed under the scalar multiplication operation defined by the ring P in order to be considered a submodule of $\mathcal{M}$.

1.3.1 Noetherian Modules

Definition 1.3.3. For a ring P, an P-module $\mathcal{M}$ is said to be Noetherian if every increasing sequence of P-submodules of $\mathcal{M}$ eventually becomes stationary. In other words, for any sequence $\mathcal{N}_1 \subseteq \mathcal{N}_2 \subseteq \mathcal{N}_3 \subseteq \cdots$ of P-submodules of $\mathcal{M}$, $\exists\ N$ s.t. $\mathcal{N}_n = \mathcal{N}_N$ for all $n \geq N$.

Example 1.3.2. A finite additive abelain group G, which is a $\mathbb{Z}$-module is Noetherian.

1.3.2 Exact Sequences

Definition 1.3.4. A sequence of P-modules and P-homomorphisms given by

$$\ldots \longrightarrow \mathcal{M}_{i-1} \xrightarrow{k_i} \mathcal{M}_i \xrightarrow{k_{i+1}} \mathcal{M}_{i+1} \xrightarrow{k_{i+2}} \ldots$$

is termed as exact at $\mathcal{M}_i$ if the condition $Im(k_i) = ker(k_{i+1})$ is satisfied. If it is exact at every $\mathcal{M}_i$ then it is called exact sequence.

Definition 1.3.5. The sequence

$$0 \longrightarrow \mathcal{M}' \xrightarrow{k} \mathcal{M} \xrightarrow{l} \mathcal{M}'' \longrightarrow 0$$

is defined to be an exact sequence iff the map k is injective (one-to-one), the map l is surjective (onto), and the condition $\mathrm{Im}(k) = \ker(l)$ is satisfied. Such a sequence is referred to as a short exact sequence.

Example 1.3.3. Let $n \geq 2$. Then the sequence

$$0 \longrightarrow \mathbb{Z} \xrightarrow{k} \mathbb{Z} \xrightarrow{l} \mathbb{Z}/n\mathbb{Z} \longrightarrow 0$$

is said to be a short exact sequence where $k : \mathbb{Z} \longrightarrow \mathbb{Z}$ is defined as $h(x) = nx$ and $l : \mathbb{Z} \longrightarrow \mathbb{Z}/n\mathbb{Z}$ is defined as $k(x) = x + n\mathbb{Z}$.

1.3.3 Module Homomorphisms and Quotient Modules

Definition 1.3.6. For a ring P and P-modules $\mathcal{M}$ and $\mathcal{N}$, a map $\alpha : \mathcal{M} \longrightarrow \mathcal{N}$ is defined to be an P-module homomorphism if it preserves the P-module structure of $\mathcal{M}$ and $\mathcal{N}$. In other words, α satisfies the following conditions:

- $\alpha(n + p) = \alpha(n) + \alpha(p)$ for all $n, p \in \mathcal{M}$, and
- $\alpha(dn) = d\alpha(n)$ for all $n \in \mathcal{M}$ and $d \in \mathcal{M}$.

Definition 1.3.7. Let P be a ring, and consider an P-module homomorphism $\beta : \mathcal{M} \longrightarrow \mathcal{N}$. The kernel of β, denoted as $\ker(\beta)$, is defined as the set $\{m \in \mathcal{M} : \beta(m) = 0\}$, consisting of all elements in $\mathcal{M}$ that are mapped to zero under β. Similarly, the image of β is defined as $\beta(\mathcal{M}) = \{n \in \mathcal{N} : n = \beta(m) \text{ for some } m \in \mathcal{M}\}$, which represents the set of all elements in $\mathcal{N}$ that are obtained as the image of some element in $\mathcal{M}$ under β.

1.4 Application of Module Theory in Commutative Algebra

The field of commutative algebra has greatly benefited from the application of module theory, a fundamental branch of abstract algebra. Commutative algebra, which focuses on the study of commutative rings and their associated modules, relies heavily on the concepts and techniques of module theory to gain deeper insights and develop powerful tools for problem-solving.

One of the primary ways in which module theory is applied in commutative algebra is through the study of module structures over commutative rings. Modules, which are generalizations of vector spaces over CRs, are essential objects of study in commutative algebra. Concepts from module theory, such as module homomorphisms, exact sequences, and projective and injective modules, are crucial for understanding the structure and behavior of modules over CRs.

For instance, the classification of modules over a commutative ring often relies on the understanding of module structures and the decomposition of modules into direct sums of simpler modules. This approach has led to the development of important results. These insights have had significant implications in areas like algebraic geometry, where the study of sheaves and their modules is central.

Another application of module theory in commutative algebra is the investigation of the ideal structure of commutative rings. Ideals, which are special subsets of commutative rings, can be studied through the lens of module theory. By considering ideals as modules over the ring itself, researchers can gain valuable information about the properties and behavior of these ideals, such as their generation, primary decomposition, and associated primes.

Furthermore, module theory provides a powerful framework for the study of localization and completion of commutative rings. These operations, which are essential in many areas of commutative algebra, rely heavily on the understanding of module structures and the behavior of modules under localization and completion.

Additionally, module theory has applications in the study of commutative algebra over non-traditional number systems, such as finite fields or modular arithmetic. In these contexts, the understanding of module structures and their properties can provide valuable insights into the structure and behavior of the corresponding commutative rings.

The integration of module theory and commutative algebra has been a fruitful and ongoing area of research, leading to the development of powerful tools and techniques for understanding the structure and properties of commutative rings and their associated modules. By leveraging the insights and methods of module theory, researchers in commutative algebra have been able to make significant advancements in this field.

1.5 Application of Group Theory in Commutative Algebra

The field of commutative algebra has greatly benefited from the application of group theory, a fundamental branch of abstract algebra. Commutative algebra, which focuses on the study of commutative rings and modules, often utilizes group theory to gain deeper insights and develop powerful tools for problem-solving.

One of the primary ways in which group theory is applied in commutative algebra is through the study of group actions on commutative rings. By understanding how a group acts on a commutative ring, researchers can gain valuable information about the structure and properties of the ring. The concept of group invariants, which are the elements of a ring that are fixed under the action of a group, is an important tool in this context. These invariants can be used to characterize the structure of the ring and to study various ideals and modules within it.

Another significant application of group theory in commutative algebra is the investigation of Galois groups. In the context of commutative algebra, Galois groups are used to study the structure of field extensions, which are closely related to the study of polynomial equations. The Galois group of a field extension encodes important information about the roots of the associated polynomial, and this information can be used to develop powerful theorems and algorithms in commutative algebra.

Furthermore, group theory plays a crucial role to examined module structures over commutative rings. The classification of modules over a CR often relies on the understanding of group actions and the decomposition of modules into direct sums of simpler modules.

Additionally, group theory has applications in the study of commutative algebra over non-traditional number systems, such as finite fields or modular arithmetic. In these contexts, the understanding of group actions and group invariants can provide valuable insights into the structure and properties of the corresponding commutative rings.

The interplay between group theory and commutative algebra has been a fruitful and ongoing area of research, leading to the development of powerful tools and techniques for understanding the structure and properties of commutative rings and modules. By combining the insights of group theory with the specific properties of commutative algebra, researchers have been able to make significant advancements in this field.

1.6 Graph Theory

We discuss some definations related to Graph Theory in this section.

Definition 1.6.1. Let $G(V, E)$ denotes the graph where V and E represented the vertex and edge set, respectively. Furthermore, each edge is associated with a relation that connects two vertices, referred to as its endpoints, which can be either identical or distinct.

Definition 1.6.2. In graph theory, the simple graphs has no loop and no multiple edges. On the other hand, multiple edges are edges that have the same pair of endpoints.

Definition 1.6.3. If, for every pair of vertices in a graph G, there exists a unique path that connects these vertices, the graph is considered connected. A component of a connected graph G is a connected subgraph that is not a part of any larger subgraph.

Definition 1.6.4. A graph having n vertices such that there is an edge which connects the adjacent vertices is said to be a path graph denoted as P_n. A graph having n vertices such that there is an edge which connects the adjacent vertices and also the first and the last vertex is known as a cycle graph and is denoted by C_n.

(a) Path (b) Cycle

Figure 1.1

Definition 1.6.5. A circulant graph $C_p(S)$ is a graph with vertex set $\{e_1, \ldots, e_p\}$. In this graph, an edge $\{e_i, e_j\}$ belongs to the edge set $E(C_p(S))$ whenever the absolute difference $|i - j|$ or m-$|i - j|$ is an element of S. Examples of circulant graphs can be seen in $Figure1.2$. It is worth noting that C_p can be considered a generalized cycle since it represents $C_p(1)$. For simplicity, the graph $C_p(\{e_1, \ldots, e_m\})$, is denoted as $C_p(e_1, \ldots, e_m)$. A circulant graph $C_p(a_1, \ldots, a_m)$ is $2p$-regular, except in the case where $2a_p$, equals p, in which case it is $(2p-1)$-regular .

Figure 1.2: From left to right $C_9(1,3)$ and $C_6(2,4)$.

Figure 1.3: From left to right $Br_2(C_9(1,3))$ and $Br_2(C_6(2,4))$.

Definition 1.6.6. A graph is called t-fold bristled graph if we attach t pandents on each vartex.

Definition 1.6.7. A caterpillar is a connected graph consisting of the path that has p vertices and t pendants attached to each vertex along the path, which is represented as $S_{p,t}$.

Definition 1.6.8. Consider $t \geq 1$. A tree with one internal vertex and t leaves incident on it is known as a $t - star$, denoted as S_t.

Chapter 2

Stanley Depth, Depth, Regularity and Projective Dimension and Some of Its Applications

In this section, the aforementioned invariants of a P-module $\mathcal{M}$ and a few related results are discussed. The ring considered throughout this section is the ring of polynomials in d variables, denoted as $P = \Re[u_1, \ldots, u_d]$.

2.1 Depth

In this subsection, the concept of depth in the context of graded modules is discussed. A few preliminary definitions are provided, followed by the presentation of well-established results that set the groundwork for subsequent discussions.

Definition 2.1.1. For an P-module $\mathcal{M}$, a non-zero element $d \in P$ is a regular element on $\mathcal{M}$ if whenever the product $dn = 0$ where $n \in \mathcal{M}$ implies that $n = 0$.

Definition 2.1.2. Let $\mathcal{M}$ be an P-module and $u_1, u_2, \ldots, u_n$ be a sequence of elements of the ring P. This sequence is defined as $\mathcal{M}$-regular if:

- u_j is regular on $\mathcal{M}/(u_1, u_2, \ldots, u_n)$ for any j and
- $\mathcal{M} = \mathcal{M}/(u_1, u_2, ..., u_n)\mathcal{M}$.

Example 2.1.1. Consider $P = \Re[u_1, u_2, u_3, u_4]$ as a module over itself. Then as u_1 is regular on $P/(0)P$, u_2 is regular on $P/(u_1)P$, u_3 is regular on $P/(u_1, u_2)P$ and u_4 is regular on $P/(u_1, u_2, u_3)P$, so u_1, u_2, u_3, u_4 is the $\mathcal{M}$-regular sequence in P.

Definition 2.1.3. The depth of the P-module $\mathcal{M}$ finitely generated over the Noetherian ring P, w.r.t the unique maximal ideal $\mathbf{n}$, is defined to be the common length of all the maximal regular sequences in the ideal $\mathbf{n}$ when considered on P.

2.1.1 Applications of Depth in Commutative Algebra,Algebraic Geometry and Homological Algebra

In commutative algebra, depth is a measure of the complexity of a module over a commutative ring. The depth of a module captures its structural properties and provides important information about the ring and the module. It helps in understanding the behavior of prime ideals, regular sequences, and the Cohen-Macaulay property of rings. Depth is used to study properties like the Hilbert function, depth stratification, and depth formulas.

Depth has significant applications in algebraic geometry, particularly in the study of singularities. The depth of a point on an algebraic variety measures the singularity of the variety at that point. It helps in distinguishing between smooth points and singular points, and it plays a crucial role in the classification and resolution of singularities. Depth is related to the dimension of local rings and the computation of the local cohomology of algebraic varieties. The depth of a module is related to the depth of its associated homology modules, and it provides information about the complexity and connectivity of the algebraic structure. Depth is used to analyze the behavior of derived functors, projective resolutions, and injective resolutions in homological algebra.

2.1.2 Few Results Related to Depth

Lemma 2.1.4 ([3, Proposition 1.2.9])**.** *For a given short exact sequence* $0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$ *of modules which is considered over a local ring or a graded Noetherian ring having local ring* P_0*, we have*

- $\operatorname{depth}(\mathcal{M}_2) \geq \min\{\operatorname{depth}(\mathcal{M}_3), \operatorname{depth}(\mathcal{M}_1)\}$,
- $\operatorname{depth}(\mathcal{M}_1) \geq \min\{\operatorname{depth}(\mathcal{M}_2), \operatorname{depth}(\mathcal{M}_3) + 1\}$,
- $\operatorname{depth}(\mathcal{M}_3) \geq \min\{\operatorname{depth}(\mathcal{M}_1) - 1, \operatorname{depth}(\mathcal{M}_2)\}$.

Lemma 2.1.5 ([12, Lemma 3.6]). *Let I be the monomial ideal of a ring P. If $P' = P[u]$ is a ring of polynomials over P in the variable u. Then* $\operatorname{depth}(P'/IP') = \operatorname{depth}(P/I) + 1$.

Lemma 2.1.6 ([8, Lemma 2.12]). *Let $I_1 \subset P_1 = \Re[u_1, \ldots, u_r]$ and $I_2 \subset P_2 = \Re[u_{r+1}, u_{r+2}, \ldots, u_d]$ be monomial ideals where $1 \leq r \leq d$. If $P = P_1 \otimes_{\Re} P_2$. Then* $\operatorname{depth}_P(P_1/I_1 \otimes_{\Re} P_2/I_2) = \operatorname{depth}_P(P/(I_1P + I_2P)) = \operatorname{depth}_{P_1}(P_1/I_1) + \operatorname{depth}_{P_2}(P_2/I_2)$.

Proposition 2.1.2 ([23, Corollary 1.3]). *For $I \subset P$. and $u \notin I$ then* $\operatorname{depth}(P/I) \leq \operatorname{depth}(P/(I : u))$.

Corollary 2.1.3 ([5, Corollary 1.6]). *Let $I \subseteq P$ be a monomial ideal. Then* $\operatorname{sdepth}(P/I) = 0$ *iff* $\operatorname{depth}(P/I) = 0$.

Theorem 2.1.4 ([11, Proposition 4.3]). *Consider a graph G having c connected components and let $A = A(G)$ and $d = d(G)$ represent the diameter of the graph G. Then for $t \geq 1$, we have* $\operatorname{depth} \geq c - t$.

Lemma 2.1.5 ([19, Lemma 2.8]). *Let $B = B(P_n)$ be an edge ideal of a path graph with $n \geq 2$. Then* $\operatorname{depth}(P/B) = \lceil \frac{n}{3} \rceil$.

2.2 Stanley Depth

This subsection delves into the concept of Stanley depth, which is a combinatorial property associated with modules. It provides a concise overview of Stanley's conjecture and outlines a methodology for determining the Stanley depth. Additionally, a few results related to Stanley depth are discussed.

Definition 2.2.1. Consider a $\mathbb{Z}^l$-graded P-module $\mathcal{M}$ which is finitely generated. Let $n\Re[Z]$ be the subspace of $\mathcal{M}$ which is generated by the elements of the type nx, where n is an element in $\mathcal{M}$ and is homogenous, x is a monomial in the ring of polynomial $\Re[Z]$, and Z is a subset of $\{u_1, \ldots, u_l\}$. If say $u\Re[Z]$ is a free $\Re[Z]$-module, then it is referred to as a Stanley space having dimension equal to $|Z|$. For the module $\mathcal{M}$ its Stanley decomposition is its representation as a finite direct sum of these Stanley spaces. The Stanley depth of

$$\mathcal{T} : \mathcal{M} = \bigoplus_{k=1}^{s} u_k \Re[Z_k],$$

is sdepth$(\mathcal{D})$ = min$\{|Z_k|$: $k = 1, \ldots, s\}$ and that of $\mathcal{N}$ is the number sdepth$(\mathcal{M})$ = max$\{$sdepth$(\mathcal{T})$: $\mathcal{T}$ is a Stanley decomposition of $\mathcal{M}\}$.

2.2.1 Application Of Stanley Depth in Commutative Algebra, Algebraic Geometry and Homological Algebra

Stanley depth is primarily studied in the context of commutative algebra. It provides refined information about the depth of modules over commutative rings. Depth is a fundamental invariant in commutative algebra that measures the complexity of modules and rings. Stanley depth extends this notion and provides more detailed information, allowing for a deeper analysis of the algebraic structure.

Stanley depth has connections to algebraic geometry, particularly in the study of singularities and the geometry of algebraic varieties. It provides information about the singular locus of varieties, the depth of singular points, and the behavior of local cohomology modules. Stanley depth helps in understanding the geometry and algebraic properties of varieties, such as smoothness, irreducibility, and the intersection theory.

Stanley depth has applications in homological algebra, which studies algebraic structures through their chain complexes and homology groups. It is used to analyze the depth of homology modules and the behavior of derived functors. Stanley depth provides insights into the complexity and connectivity of algebraic structures and helps in understanding the derived category and its properties.

2.2.2 Stanley's Conjecture

Stanley in 1982 [25] gave a conjecture which stated that for a Z^l-graded P-module $\mathcal{M}$ which is finitely generated, we have that $\operatorname{sdepth}(\mathcal{M}) \geq \operatorname{depth}(\mathcal{M})$. Duval et al in [9] disproved this conjecture later with the help of a counterexample for the module of the type P/M. An algorithm for the computation of Stanley depth is provided in [14] by Ichim et al..

2.2.3 Method for The Computation of Stanley Depth for Squarefree Monomial Ideals

In 2009, Herzog et al. in [12] introduced an innovative approach for computing the lower bound of the Stanley depth of monomial ideals. This method involves utilizing posets and is designed to achieve this computation in a fixed number of steps. Let A be a squarefree monomial ideal having generating set as $G(A) = \{e_1, \dots, e_m\}$. Now the characteristic poset of A w.r.t $g = (1, \dots, 1)$, written as $\mathcal{Q}_I^{(1,\dots,1)}$ is defined to be

$$\mathcal{Q}_I^{(1,\dots,1)} = \{\gamma \subset [n] \mid \gamma \text{ contains } \operatorname{supp}(e_j) \text{ for, some } j\},$$

where $\operatorname{supp}(e_j) = \{i \,:\, x_i | e_j\} \subseteq [n] := \{1, \dots, n\}$. For each $\rho, \sigma \in \mathcal{Q}_I^{(1,\dots,1)}$ where $\rho \subseteq \sigma$, and

$$[\rho\,,\,\sigma] = \{\gamma \in \mathcal{Q}_I^{(1,\dots,1)} \,:\, \rho \subseteq \gamma \subseteq \sigma\}.$$

Let $\mathcal{Q} \,:\, \mathcal{Q}_I^{(1,\dots,1)} = \cup_{j=1}^{k} [\gamma_j\,,\,\eta_j]$ be a partition of $\mathcal{Q}_I^{(1,\dots,1)}$, and for every j, suppose $s(j) \in \{0,1\}^n$ is the tuple with $\operatorname{supp}(x^{s(j)}) = \gamma_j$, then the Stanley decomposition $\mathcal{D}(\mathcal{Q})$ of A is given by

$$\mathcal{D}(\mathcal{Q}) \,:\, A = \bigoplus_{j=1}^{r} x^{s(j)} \Re[\{x_k \mid k \in \eta_j\}].$$

Clearly, $\operatorname{sdepth} \mathcal{D}(\mathcal{Q}) = \min\{|\eta_1|, \dots, |\eta_r|\}$ and

$$\operatorname{sdepth}(A) = \max\{\operatorname{sdepth} \mathcal{D}(\mathcal{Q}) \mid \mathcal{Q} \text{ is a partition of } \mathcal{Q}_I^{(1,\dots,1)}\}.$$

Example 2.2.1. Let $P = \Re[x_1, x_2, x_3, x_4, x_5, x_6]$, consider $A = (x_1x_3, x_2x_5, x_4x_6, x_1x_4x_6)$. Then select $g = (1,1,1,1,1,1)$ and the poset $\mathcal{Q} = \mathcal{Q}^g_{\mathcal{P}/A}$ is given by:

$$
\begin{aligned}
\mathcal{Q} = \{&(0,0,0,0,0,0),(1,0,0,0,0,0),(0,1,0,0,0,0),(0,0,1,0,0,0),(0,0,0,1,0,0),\\
&(0,0,0,0,1,0),(0,0,0,0,0,1),(1,1,0,0,0,0),(1,0,0,1,0,0),(1,0,0,0,1,0),\\
&(1,0,0,0,0,1),(0,1,1,0,0.0),(0,1,0,1,0.0),(0,0,1,1,0,0),(0,0,0,1,1,0),\\
&(0,0,1,0,0,1),(0,0,0,1,1,0),(0,0,0,1,0.1).(0,0,0,0,1,1),(1,1,0,1,0,0),\\
&(1,0,0,1,1,0),(1,0,0,0,1,1),(0,1,1,1,0,0),(0,0,1,1,1,0),(0,0,1,1,0,1),\\
&(0,0,1,0,1,1),(0,0,0,1,1,1),(0,0,1,1,1,1)\}.
\end{aligned}
$$

The partitions of $\mathcal{Q}$ can be written as :

$$
\begin{aligned}
\mathcal{Q}_1 :\ &[(0,0,0,0,0,0),(1,0,0,0,0,0)] \bigcup [(0,1,0,0,0,0),(0,1,0,0,0,0)] \bigcup\\
&[(0,0,1,0,0,0),(0,0,1,0,0,0)] \bigcup [(0,0,0,1,0,0),(0,0,0,1,0,0)] \bigcup\\
&[(0,0,0,0,1,0),(0,0,0,0,1,0)] \bigcup [(0,0,0,0,0,1),(0,0,0,0,0,1)] \bigcup\\
&[(1,1,0,0,0,0),(1,1,0,0,0,0)] \bigcup [(1,0,0,1,0,0),(1,0,0,1,0,0)] \bigcup\\
&[(0,1,0,1,0,0),(0,1,0,1,0,0)] \bigcup [(0,0,1,1,0,0),(0,0,1,1,0,0)] \bigcup\\
&[(0,0,0,1,1,0),(0,0,0,1,1,0)] \bigcup [(0,0,0,1,0,1),(0,0,0,1,0,1)] \bigcup\\
&[(0,0,0,0,1,1),(0,0,0,0,1,1)] \bigcup [(1,1,0,1,0.0),(1,1,0,1,0,0)] \bigcup\\
&[(1,0,0,1,1,0),(1,0,0,1,1,0)] \bigcup [(1,0,0,0,1,1),(1,0,0,0,1,1)] \bigcup\\
&[(0,0,0,1,1,0),(0,0,0,1,1,0)] \bigcup [(0,0,0,0,1,1),(0,0,0,0,1,1)] \bigcup\\
&[(1,1,0,1,0,0),(1,1,0,1,0,0)] \bigcup [(1,0,0,1,1,0),(1,0,0,1,1,0)] \bigcup\\
&[(1,0,0,0,1,1),(1,0,0,0,1,1)] \bigcup [(0,1,1,1,0,0),(0,1,1,1,0,0)] \bigcup\\
&[(0,0,1,1,1,0),(0,0,1,1,1,0)] \bigcup [(0,0,1,1,0,1),(0,0,1,1,0,1,)] \bigcup\\
&[(0,0,0,1,1,1),(0,0,0,1,1,1)] \bigcup [(0,0,1,1,1,1),(0,0,1,1,1,1,)],
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{Q}_2 :\ &[(0,0,0,0,0,0),(1,1,0,1,0,0)] \bigcup [(0,0,1,0,0,0),(0,1,1,1,0,0)] \bigcup\\
&[(0,0,0,0,1,0),(1,0,0,1,1,0)] \bigcup [(0,0,0,0,0,1),(1,0,0,0,1,1)] \bigcup\\
&[(0,0,0,1,1,0),(0,0,1,1,1,0)] \bigcup [(0,0,1,0,0,1),(0,0,0,1,0,1)] \bigcup\\
&[(0,0,0,1,0,1),(0,0,0,1,1,1)].
\end{aligned}
$$

So the corresponding Stanley decomposition is of the partitions will be

$$\begin{aligned}\mathcal{D}(\mathcal{Q}_1) := {} & \Re[x_1] \oplus x_2\Re[x_2] \oplus x_3\Re[x_3] \oplus x_4\Re[x_4] \oplus x_5\Re[x_5] \oplus x_6\Re[x_6] \oplus \\ & x_1x_2\Re[x_1,x_2] \oplus x_1x_4\Re[x_1,x_4] \oplus x_2x_4\Re[x_2,x_4] \oplus x_3x_4\Re[x_3,x_4] \\ & \oplus x_4x_5\Re[x_4,x_5] \oplus x_4x_6\Re[x_4,x_6] \oplus x_5x_6\Re[x_5,x_6] \oplus x_1x_2x_4\Re[x_1,x_2,x_4] \\ & \oplus x_1x_4x_5\Re[x_1,x_4,x_5] \oplus x_1x_5x_6\Re[x_1,x_5,x_6] \oplus x_4x_5\Re[x_4,x_5] \\ & \oplus x_4x_5\Re[x_4,x_5] \oplus x_5x_6\Re[x_5,x_6] \oplus x_1x_2x_4\Re[x_1,x_2,x_4] \oplus x_1x_4x_5\Re[x_1,x_4,x_5] \\ & \oplus x_1x_5x_6\Re[x_1,x_5,x_6] \oplus x_2x_3x_4\Re[x_2,x_3,x_4] \oplus x_3x_4x_5\Re[x_3,x_4,x_5] \\ & \oplus x_3x_4x_6\Re[x_3,x_4,x_6] \oplus x_4x_5x_6\Re[x_4,x_5,x_6] \oplus x_3x_4x_5x_6\Re[x_3,x_4,x_5,x_6], \\ \mathcal{D}(\mathcal{Q}_2) := {} & \Re[x_1,x_2,x_4] \oplus x_3\Re[x_2,x_3,x_4] \oplus x_5\Re[x_1,x_4,x_5] \oplus x_6\Re[x_1,x_5,x_6] \oplus \\ & x_4x_5\Re[x_3,x_4,x_5] \oplus x_3x_6\Re[x_4,x_6] \oplus x_4x_6\Re[x_4,x_5,x_6].\end{aligned}$$

Then

$$\begin{aligned}\operatorname{sdepth}(\mathcal{Q}/A) & \geq \max\{\operatorname{sdepth}(\mathcal{D}(\mathcal{Q}_1)), \operatorname{sdepth}(\mathcal{D}(\mathcal{Q}_2))\} \\ & = \max\{1,3\} \\ & = 3.\end{aligned}$$

2.2.4 Few Results Related to Sdepth

Lemma 2.2.2 ([23, Lemma 2.2]). *Consider the sequence* $0 \longrightarrow \mathcal{M}_1 \longrightarrow \mathcal{M}_2 \longrightarrow \mathcal{M}_3 \longrightarrow 0$ *of* P*- modules which are* $\mathbb{Z}^l$*-graded we have* $\operatorname{sdepth}(\mathcal{M}_2) \geq \min\{\operatorname{sdepth}(\mathcal{M}_1), \operatorname{sdepth}(\mathcal{M}_3)\}$.

Lemma 2.2.3 ([12, Lemma 3.6]). *For a monomial ideal* I *of a ring* P*. If* $P' = P[y]$ *is a ring of polynomials over* P *in the variable* y*. Then* $\operatorname{sdepth}(P'/IP') = \operatorname{sdepth}(P/I) + 1$.

Lemma 2.2.4 ([8, Lemma 2.13]). *Let* $I_1 \subset P_1 = \Re[u_1, ..., u_r]$ *and* $I_2 \subset P_2 = \Re[u_{r+1}, u_{r+2}, \dots, u_d]$ *be monomial ideals where* $1 \leq r \leq d$*. If* $P = P_1 \otimes_{\Re} P_2$*. Then* $\operatorname{sdepth}_P(P_1/I_1 \otimes_{\Re} P_2/I_2) \geq \operatorname{sdepth}_{\mathcal{P}_1}(P_1/I_1) + \operatorname{sdepth}_{P_2}(P_2/I_2)$.

Proposition 2.2.2 ([6, Proposition 2.7]). *Consider a monomial ideal* I *of a ring* P*. Then for any monomial* $u \notin I, \operatorname{sdepth}(P/I) \leq \operatorname{sdepth}(P/(I:u))$.

Theorem 2.2.3 ([22, Theorem 2.3]). *Let* $I \subseteq P$ *be monomil ideal which is generated by a minimal set of* u *elements. Then, we have* $\operatorname{sdepth}(I) \geq \max\{1, d - \lfloor \frac{u}{2} \rfloor\}$.

2.3 Regularity

Definition 2.3.1. For a field $\Re$ let $P = \Re[u_1, \ldots, u_d]$ be polynomials in d variables over $\Re$. Then for a Z-graded P-module $\mathcal{M}$ which is finitely generated having a minimal free resolution

$$0 \longrightarrow \bigoplus_{j\in Z} P(-j)^{\beta_{r,j}(\mathcal{M})} \longrightarrow \bigoplus_{j\in Z} P(-j)^{\beta_{r-1,j}(\mathcal{M})} \longrightarrow .....\longrightarrow \bigoplus_{j\in Z} P(-j)^{\beta_{0,j}(\mathcal{M})} \longrightarrow 0$$

the regularity of $\mathcal{M}$ is $\text{reg}(\mathcal{M}) = \max\{j - k : \beta_{k,j}\ (\mathcal{M}) \neq 0\}$.

2.3.1 Application of Regularity in Commutative Algebra,Algebraic Geometry And Homological Algebra

Regularity plays a central role in commutative algebra. It is used to study the algebraic and geometric properties of rings, modules, and ideals. Regularity provides information about the complexity and depth of modules, the behavior of prime ideals, and the Cohen-Macaulay property of rings. It is a key ingredient in the study of homological properties, such as the vanishing of Ext modules and the structure of resolutions.

Regularity has significant applications in algebraic geometry, where it is closely related to the geometric properties of algebraic varieties. Regularity of a projective variety is linked to the embedding of the variety in projective space and the behavior of its homogeneous coordinate ring. It helps in understanding the birational geometry, rational maps, and the classification of varieties. Regularity also plays a role in the study of singularities and the resolution of singularities.

Regularity is employed in homological algebra, which studies algebraic structures through their chain complexes and homology groups. It helps in analyzing the complexity and connectivity of algebraic structures. Regularity provides information about the regularity of homology modules, the behavior of derived functors, and the regularity of resolutions.

2.3.2 Few Results Related to Regularity

Lemma 2.3.1 ([4, Theorem 4.7]). *Consider a monomial ideal I in the ring P, and let u_i be one of the variables in P. Then*

(a) $\operatorname{reg}(P/I) = reg(P/(I : u_i)) + 1$, *if* $\operatorname{reg}(P/(I : u_i)) \geq \operatorname{reg}(P/(I, u_i))$.

(b) $\operatorname{reg}(P/I) = \operatorname{reg}(P/(I, u_i))$ *if* $\operatorname{reg}(P/(I : u_i)) \leq \operatorname{reg}(P/(I, u_i))$.

(c) $\operatorname{reg}(P/I) \in \operatorname{reg}(P/(I, u_i)) + 1, \operatorname{reg}(P/(I, u_i))$, *if* $\operatorname{reg}(P/(I : u_i)) = \operatorname{reg}(P/(I, u_i))$.

Lemma 2.3.2 ([20, Lemma 3.6]). *Consider a monomial ideal I in the ring P. If $P' = P[y]$ is a ring of polynomials over P in the variable y. Then* $\operatorname{reg}(P'/I) = \operatorname{reg}(P/I)$.

Lemma 2.3.3 ([27, lemma 8]). *Let $P_1 = \Re[u_1, \ldots, u_r]$ and $P_2 = \Re[u_{r+1}, \ldots, u_d]$ be rings of polynomials and I_1 and I_2 be edge ideals of P_1 and P_2, respectively. Then* $\operatorname{reg}(P/(I_1P + I_2P)) = \operatorname{reg}(P_1/I_1) + \operatorname{reg}(P_2/I_2)$.

Proposition 2.3.4 ([17, Lemma 2.2]). *For a simple graph G which is finite we have,* $\operatorname{reg}(P/I(G)) \geq \operatorname{indmat}(G)$.

Lemma 2.3.5 ([18, Theorem 1.4]). *Let I_1 and I_2 are the monomial ideals of P, then* $\operatorname{reg}(P/(I_1 + I_2)) \leq \operatorname{reg}(P/I_1) + \operatorname{reg}(P/I_2)$.

2.4 Projective Dimension

Definition 2.4.1. For a field $\Re$ let $P = \Re[u_1, \ldots, u_l]$ be the ring of polynomials in l variables over the given field. Then for a $\mathbb{Z}$-graded P-module $\mathcal{M}$ which is finitely generated having a minimal free resolution

$$0 \longrightarrow \bigoplus_{j \in Z} P(-j)^{\beta_{r,j}(\mathcal{M})} \longrightarrow \bigoplus_{j \in Z} P(-j)^{\beta_{r-1,j}(\mathcal{M})} \longrightarrow \ldots.. \longrightarrow \bigoplus_{j \in Z} P(-j)^{\beta_{0,j}(\mathcal{M})} \longrightarrow 0$$

the projective dimension of $\mathcal{M}$ is given by $\operatorname{pdim}(\mathcal{M}) = \max\{k : \beta_{k,j}\ (\mathcal{M}) \neq 0\}$.

2.4.1 Application of Projective Dimension in Commutative Algebra,Algebric Geometry and Representation Theory

Projective dimension plays a significant role in commutative algebra, where it helps in understanding the algebraic and geometric properties of rings and modules. It is used to study properties like the depth, regularity, and Cohen-Macaulayness of modules and rings. Projective dimension is employed in the classification of modules and the computation of local cohomology modules.

Projective dimension has applications in algebraic geometry, particularly in the study of projective varieties and coherent sheaves. It is used to measure the complexity and sheaf cohomology of coherent sheaves on projective varieties. Projective dimension helps in understanding the geometry and algebraic properties of projective spaces, projective curves, and higher-dimensional projective varieties.

Projective dimension is relevant in the study of representations of algebraic structures, such as groups, algebras, and Lie algebras. It provides information about the complexity and structure of modules and representations. Projective dimension is used to characterize projective and injective modules and to analyze the behavior of homological functors in representation theory.

2.4.2 Few Results Related to Projective Dimension

Lemma 2.4.1 ([3, Theorem 1.3.3])**.** *(Auslander–Buchsbaum formula) Suppose P is a local Noetherian ring and is also commutative, and $\mathcal{M}$ is a non-zero P-module which is finitely generated and has finite projective dimension. Then, the sum of the projective dimension and the depth of the module $\mathcal{M}$ is equal to the dimension of the ring P this is* $\text{pdim}(\mathcal{M}) + \text{depth}(\mathcal{M}) = \text{depth}(P)$.

Lemma 2.4.2 ([7, Lemma 5.1])**.** *For I a square-free monomial ideal, and any subset of the variables R relabeled as $R = \{u_1, \ldots, u_l\}$, either $\exists$ a l with $1 \leq l \leq j$ such that* $\text{pdim}(P/B) = \text{pdim}(P/((B, u_1, \ldots, u_{l-1}) : u_d))$ *or* $\text{pdim}(P/I) = \text{pdim}(P/((S, u_1, \ldots, u_j)))$.

Lemma 2.4.3 ([2, Proposition 3.1.1])**.** *Let $A = A(P_n) \subset \mathcal{X}$, where P_n is a path of length n. Then* $\text{pdim}(P/A(P_n)) = \lceil \frac{2n}{3} \rceil$.

Chapter 3

Algebraic Invariants of cyclic modules associated to some special subgraphs of $Br_t(C_{2n}(1, n-1))$, $Br_t(C_{2n}(1, 2))$ and $Br_t(C_{2n}(1, n-1, n))$ graphs

In this chapter a Circulant graph is defined as in Definition 1.3.6. we discuss the values for the regularity of the cyclic modules $\Re[\mathsf{V}(\Upsilon_{n,t})]/I(\Upsilon_{n,t})$, $\Re[\mathsf{V}(\Phi_{n,t})]/I(\Phi_{n,t})$, $\Re[\mathsf{V}(\Psi_{n,t})]/I(\Psi_{n,t})$. We also compute depth, Stanley depth, and projective dimension of $\Re[\mathsf{V}(\Upsilon_{n,t})]/I(\Upsilon_{n,t})$. These computations will be crucial in the subsequent section as we proceed to calculate our key findings.

3.1 Preliminaries

We discuss a few preliminaries first which aid in proving the theorems given in this chapter. These basic results include a few Lemma's which help in simplifying the structures under study and a few other basic theorems related to bipartite graphs and trees in general are given.

Lemma 3.1.1 ([26, Lemma 2.2 and Lemma 2.26])**.** *Let $L = I(S_t)$. Then,*

$$\operatorname{depth}(P/L) = \operatorname{reg}(P/L) = \operatorname{sdepth}(P/L) = 1.$$

Lemma 3.1.2 ([26, Theorem 2.28])**.** *Let $L = I(S_{p,t})$ Then,* *(a)* $\operatorname{reg}(P/L) = \lceil \frac{n}{2} \rceil$.

(b) $\operatorname{depth}(P/L) = \operatorname{sdepth}(P/L) = \lceil \frac{n}{2} \rceil + \lceil \frac{n-1}{2} \rceil t$.

Lemma 3.1.3 ([26, Theorem 2.30 and Theorm 2.9]). *Let $L = I(C_{n,t})$ Then,*

(a) $\operatorname{reg}(P/L) = \lceil \frac{n-1}{2} \rceil$.

(b) $\operatorname{depth}(P/L) = \operatorname{sdepth}(P/L) = \lceil \frac{n}{2} \rceil + \lceil \frac{n-1}{2} \rceil t$.

Lemma 3.1.4 ([12, Lemma 3.6]). *Let $I_1 \subset P$ be a monomial ideal. If $P' = P \otimes_{\Re} \Re[z_{l+1}] \cong P[z_{l+1}]$, then*

(a) $\operatorname{depth}(P'/I_1P') = \operatorname{depth}(P/I_1) + 1$.

(b) $\operatorname{sdepth}(P'/I_1P') = \operatorname{sdepth}(P/I_1) - 1$.

(c) $\operatorname{reg}(P'/I_1P') = \operatorname{reg}(P/I_1)$.

Lemma 3.1.5 ([13, Lemma 3.2]). *If $I_1 \subset \mathcal{P}' = \Re[z_1, \dots, u_l]$ and $I_2 \subset \mathcal{P}'' = \Re[z_{l+1}, \dots, u_p]$ are non-zero homogeneous ideals of $\mathcal{P}'$ and $\mathcal{P}''$ and regard $I_1 + I_2$ as a homogeneous ideal of $P = \mathcal{P}' \otimes_{\Re} \mathcal{P}''$, then*

$$\operatorname{reg}(P/(I_1 + I_2) = \operatorname{reg}(\mathcal{P}'/I_1) + \operatorname{reg}(\mathcal{P}''/I_2).$$

Remark 3.1.6. We have some special cases of $\Upsilon_{n,t}$, $\Phi_{n,t}$, and $\Psi_{n,t}$, for n=0. That can be defined as follows

- $\Re[\mathsf{V}(\Upsilon_{0,t}/I(\Upsilon_{0,t}))] \cong \Re$. Thus, $\operatorname{reg}(\Re) = \operatorname{depth}(\Re) = \operatorname{sdepth}(\Re) = 0$.
- $\Re[\mathsf{V}(\Phi_{0,t}/I(\Phi_{0,t}))] \cong \Re$. Thus, $\operatorname{reg}(\Re) = \operatorname{depth}(\Re) = \operatorname{sdepth}(\Re) = 0$.
- $\Re[\mathsf{V}(\Psi_{0,t}/I(\Psi_{0,t}))] \cong \Re$. Thus, $\operatorname{reg}(\Re) = \operatorname{depth}(\Re) = \operatorname{sdepth}(\Re) = 0$.

If $n = 1$, then we have $\Re[\mathsf{V}(\Upsilon_{1,t}/I(\Upsilon_{1,t}))] \cong \Re[\mathsf{V}(S_t)]/I(S_t) \otimes_{\Re} \Re[\mathsf{V}(S_t)]/I(S_t)$. By Lemma 3.1.1 and Lemma 3.1.5, $\operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{1,t}/I(\Upsilon_{1,t}))]) = 2$.

When $n \geq 1$, we consider several types of graphs denoted as Υ_n, Φ_n,Ψ_n, and these are the subsets of $C_{2n}(1, n-1)$ and $C_{2n}(1, 2)$ and $C_{2n}(1, n-1, n)$.

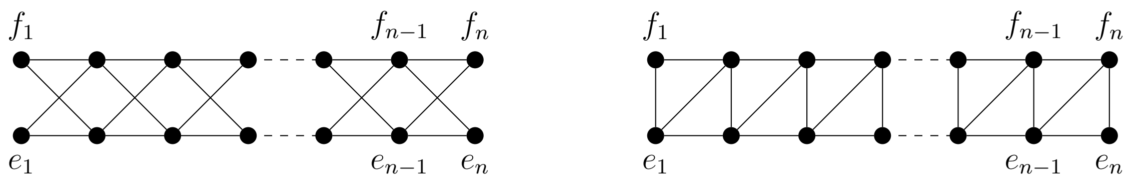

Figure 3.1: From left to right, Υ_n and Φ_n .

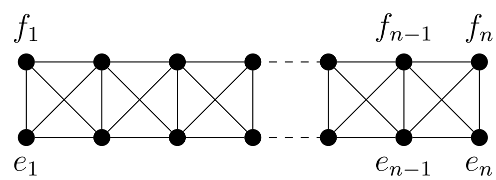

Figure 3.2: Ψ_n.

Now we define the pendents attached to different vertices with specific variables. That is $C_i := \{e_{i,1}, \dots, e_{i,t}\}$ and $D_i := \{f_{i,1}, \dots, f_{i,t}\}$ where $1 \leq i \leq n$.

Let $\Upsilon_{n,t} := Br_t(\Upsilon_n)$, $\Phi_{n,t} := Br_t(\Phi_n)$, $\Psi_{n,t} := Br_t(\Psi_n)$. Now we discuss the edge set of $\Upsilon_{n,t}$, $\Phi_{n,t}$, $\Psi_{n,t}$. Here $\mathcal{G}(I)$ denotes the minimal generating set of monomial ideal I. If $n = 2$, then

$$\mathcal{G}(\Upsilon_{2,t}) := \{e_1e_2, e_1e_{1,1}, \dots, e_1e_{1,t}, e_2e_{2,1}, e_2e_{2,2}, \dots, e_2e_{2,t}, f_1f_2, f_1f_{1,1}, \dots, f_1f_{1,t}, f_2f_{2,1}, f_2f_{2,2}, \dots, f_2f_{2,t}\}.$$

For $n \geq 3$,

$$\mathcal{G}(\Psi_{n,t}) := \cup_{j=1}^{n-1}\{e_je_{j+1}, f_jf_{j+1}\}\cup_{j=1}^{n}\{e_jf_j\}\cup_{j=1}^{n-1}\{e_jf_{j+1}, e_{j+1}f_j\}\cup_{j=1}^{t}\{e_1e_{1j}, \dots, e_ne_{nj}, f_1f_{1j}, \dots, f_nf_{nj}\},$$

$$\mathcal{G}(\Phi_{n,t}) := \cup_{j=1}^{n-1}\{e_je_{j+1}, f_jf_{j+1}\} \cup_{j=1}^{n} \{e_jf_j\} \cup_{j=1}^{n-1} \{e_jf_{j+1}\} \cup_{j=1}^{t} \{e_1e_{1j}, \dots, e_ne_{nj}, f_1f_{1j}, \dots, f_nf_{nj}\},$$

$$\mathcal{G}(\Upsilon_{n,t}) := \cup_{j=1}^{n-1}\{e_je_{j+1}, f_jf_{j+1}\} \cup_{j=1}^{n-1} \{e_jf_{j+1}, e_{j+1}f_j\} \cup_{j=1}^{t} \{e_1e_{1j}, \dots, e_ne_{nj}, f_1f_{1j}, \dots, f_nf_{nj}\}.$$

In this part, we discuss the values for the regularity of the cyclic modules $\Re[\mathsf{V}(\Upsilon_{n,t})]/I(\Upsilon_{n,t})$, $\Re[\mathsf{V}(\Phi_{n,t})]/I(\Phi_{n,t})$, $\Re[\mathsf{V}(\Psi_{n,t})]/I(\Psi_{n,t})$. We also compute depth, Stanley depth, and projective dimension of $\Re[\mathbb{V}(\Upsilon_{n,t})]/I(\Upsilon_{n,t})$. These computations will be crucial in the subsequent section as we proceed to calculate our key findings. Let $I \subset P$ be a squarefree monomial ideal that is minimally generated by a monomial of degree at most 2. We define a graph G_I associated with the ideal I, where $\mathsf{V}(G_I)$ represents the *support* of I and $\mathsf{E}(G_I)$ consists of pairs $\{\{u_i, u_j\} : u_iu_j \in G(I)\}$. Within the polynomial ring P, let e_t and e_l be variables such that e_t and e_l are not elements of I. The ideals $(I : e_l)$ and (I, e_t) are monomial ideals of P, and their corresponding graphs $G_{(I:e_l)}$ and $G_{(I,e_t)}$ are subgraphs of G_I.

3.2 Algebraic Invariants of cyclic modules associated to some special subgraphs of $Br_t(C_{2n}(1, n-1))$, $Br_t(C_{2n}(1,2))$ and $Br_t(C_{2n}(1, n-1, n))$ graphs

Lemma 3.2.1. *Let $n, q \geq 1$ and $P = \Re[\mathsf{V}(\Upsilon_{n,t})]$. Then*

$$\operatorname{reg}(P/I(\Upsilon_{n,t})) = \begin{cases} n+1, & \text{if } n \text{ is odd;} \\ n, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $n = 2$. Then we have $\Re[\mathsf{V}(\Upsilon_{2,t}/(I(\Upsilon_2, q) : f_2) \cong \Re[\mathsf{V}(S_t)]/I(S_t)$. By Lemma 3.1.1 , $\operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{2,t})]/I(\Upsilon_{2,t} : f_2)) = 1$. Also $\Re[\mathsf{V}(\Upsilon_{2,t})/I(\Upsilon_{2,t}, f_2) \cong \Re[\mathsf{V}(S_{3,t})]/I(S_{3,t})$, then by using Lemma 3.1.2, $\operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{2,t}, f_2)]/I(\Upsilon_{2,t}, f_2)) = 2$. Now finally by Lemma 2.3.1(b),
$\operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{2,t})/(I(\Upsilon_2, q))) = 2$. Now, if $n = 3$, then $\Upsilon_{3,t} \cong \Upsilon_{2,t} \cup \mathcal{U}_{2,t}$, here

$$\mathcal{U}_{2,t} := \{a_1a_2, b_1b_2, a_1b_1, a_2b_2, a_1a_{11}, \dots, a_1a_{1t}, a_2a_{21}, \dots, a_2a_{2t}, b_1b_{11}, \dots, b_1b_{1t}, b_2b_{21}, \dots, b_2b_{2t}\}.$$

Also $\Upsilon_{2,t} \cap \mathcal{U}_{2,t} \neq \emptyset$. By applying Lemma 2.3.5,

$$\operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{3,t})/I(\Upsilon_{3,t})) \leq \operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{2,t})]/I(\Upsilon_{2,t})) + \operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{2,t})]/I(\Upsilon_2, q)).$$

Thus by induction on n, we obtain $\operatorname{reg}(P/I(\Upsilon_{3,t})) \leq 2 + 2 = 4$. As far as another inequality is concerned, let $\mathcal{M} = \Big\{\{f_1, f_{1,1}\}, \{e_1, e_{1,1}\}, \{f_3, f_{3,1}\}, \{e_3, e_{3,1}\}\Big\}$. Where $\mathcal{M}$ is induced matching, thus, $\operatorname{indmat}(\Upsilon_{3,t}) \geq 4$. By Lemma 2.3.4, $\operatorname{reg}(P/I(\Upsilon_{3,t})) \geq 4$. Thus, $\operatorname{reg}(P/I(\Upsilon_{3,t})) = 4$. For $n \geq 4$, we have;

Case 1 : Let n is even. Then

$$P/(I(\Upsilon_{n,t}) : f_n) \cong \Re[\mathsf{V}(\Upsilon_{n-2,t})]/I(\Upsilon_{n-2,t}) \otimes_{\Re} \Re[\mathsf{V}(S_t)]/I(S_t) \otimes_{\Re} \Re[f_n, D_{n-1}, C_{n-1}], \tag{3.1}$$

$$P/((I(\Upsilon_{n,t}), f_n), e_n) \cong \Re[\mathsf{V}(\Upsilon_{n-1,t})]/I(\Upsilon_{n-1,t}) \otimes_{\Re} \Re[D_n, C_n], \tag{3.2}$$

$$P/((I(\Upsilon_{n,t}), f_n) : e_n) \cong \Re[\mathsf{V}(\Upsilon_{n-2,t})]/I(\Upsilon_{n-2,t}) \otimes_{\Re} \Re[e_n, D_n, D_{n-1}, C_{n-1}], \tag{3.3}$$

By mathematical induction on n, along with the use of Lemma 3.1.5 and Lemma 3.1.4 on Eqs. (4.25)- (4.27) respectively, also apply Lemma 3.1.1 on Eq (4.25),

$$\begin{aligned}\operatorname{reg}(P/(I(\Upsilon_{n,t}) : f_n) &= \operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{n-2,t})]/I(\Upsilon_{n-2,t})) + \operatorname{reg}(\Re[\mathsf{V}(S_t)]/I(S_t)) \\ &= n - 2 + 1 = n - 1,\end{aligned}$$

$$\operatorname{reg}\left(P/((I(\Upsilon_{n,t}), f_n), e_n)\right) = \operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{n-1,t})]/I(\Upsilon_{n-1,t})) = n - 1 + 1 = n,$$

$$\operatorname{reg}\left(P/((I(\Upsilon_{n,t}), f_n) : e_n)\right) = \operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{n-2,t})]/I(\Upsilon_{n-2,t})) = n - 2.$$

As $\operatorname{reg}\left(P/((I(\Upsilon_{n,t}), f_n) : e_n))\right) < \operatorname{reg}\left(P/((I(\Upsilon_{n,t}), f_n), e_n)\right)$, by applying of Lemma 2.3.1(b), $\operatorname{reg}(P/(I(\Upsilon_{n,t}), f_n)) = n$. Also,

$$\operatorname{reg}\left(P/((I(\Upsilon_{n,t}) : f_n)\right) < \operatorname{reg}\left(P/((I(\Upsilon_{n,t}), f_n)\right).$$

Finally by Lemma 2.3.1(b), $\operatorname{reg}(P/(I(\Upsilon_{n,t})) = n$.

Case 2 Let n is odd. Then $\Upsilon_{n,t} \cong \Upsilon_{n-1,t} \cup H$, where $H \cong \Upsilon_{2,t}$ also $\Upsilon_{n-1,t} \cap H \neq \emptyset$.

By induction on n and using Lemma 2.3.5,

$$\operatorname{reg}(P/I(\Upsilon_{n,t}) \leq \operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{n-1,t})]/I(\Upsilon_{n-1,t})) + \operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{2,t})]/I(\Upsilon_{2,t})) = n+1.$$

To obtain the second inequality, let $\mathcal{M} = \Big\{\{f_1, f_{1,1}\}, \{e_1, e_{1,1}\}, \{e_3, e_{3,1}\}, \{f_3, f_{3,1}\} \ldots,$ $\{e_{n-2}, e_{n-2,1}\}, \{f_{n-2}, f_{n-2,1}\}, \{e_n, e_{n,1}\}, \{f_n, f_{n,1}\}\Big\}$ it is obvious that $\mathcal{M}$ forms an induced matching, so, $\operatorname{indmat}(\Upsilon_{n,t}) \geq n+1$. By Lemma 2.3.4, $\operatorname{reg}(P/I(\Upsilon_{n,t})) \geq n+1$. Therefore, $\operatorname{reg}(P/I(\Upsilon_{n,t})) = n+1$.

□

Lemma 3.2.2. *Let $n, t \geq 1$ and $P = \Re[\mathsf{V}(\Phi_{n,t})]$. Then*

$$\operatorname{reg}(P/I(\Phi_{n,t})) = \begin{cases} 2\lceil \frac{n-1}{3} \rceil + 1, & \text{if } n = 1 \bmod (3); \\ 2\lceil \frac{n}{3} \rceil, & \text{if } n = 0, 2 \bmod (3). \end{cases}$$

Proof. Let $n = 2$. Then

$$P/(I(\Phi_{2,t}) : f_2) \cong \Re[\mathsf{V}(\Phi_{0,t})]/I(\Phi_{0,t}) \otimes_{\Re} \Re[f_2, D_1, C_1, C_2],$$

$$P/(I(\Phi_{2,t}), f_2) \cong \Re[\mathsf{V}(S_{3,t})]/I(S_{3,t}) \otimes_{\Re} \Re[D_2].$$

By Lemma 3.1.4, $\operatorname{reg}(P/(I(\Phi_{2,t}) : f_2) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{0,t})]/I(\Phi_{0,t}))$ and $\operatorname{reg}(P/(I(\Phi_{2,t}), f_2) =$

$\operatorname{reg}(\Re[\mathsf{V}(S_{3,t})]/I(S_{3,t}))$. By Remark 3.1.6 and Lemma 3.1.2, $\operatorname{reg}(P/(I(\Phi_{2,t}) : f_2) = 0$ and

$\operatorname{reg}(P/(I(\Phi_{2,t}), f_2) = 2$. As $\operatorname{reg}(P/((I(\Phi_{2,t}) : f_2) < \operatorname{reg}(P/((I(\Phi_{2,t}), f_2))$. Finally by Lemma 2.3.1(b), $\operatorname{reg}(P/((I(\Phi_{2,t})) = 2$. Now let $n = 3$. Then

$$P/(I(\Phi_{3,t}) : f_3) \cong \Re[\mathsf{V}(S_{2,t})]/I(S_{2,t}) \otimes_{\Re} \Re[f_3, D_2, C_3, C_2],$$

$$P/(I(\Phi_{3,t}), f_3), e_3) \cong \Re[\mathsf{V}(\Phi_{2,t})]/I(\Phi_{2,t}) \otimes_{\Re} \Re[f_3, e_3, D_3, C_3],$$

$$P/(I(\Phi_{3,t}), f_3) : e_3) \cong \Re[\mathsf{V}(C_{3,t})]/I(C_{3,t}) \otimes_{\Re} \Re[e_3, C_2].$$

By applying Lemma 3.1.4 we get,

$$\operatorname{reg}(P/(I(\Phi_{3,t}) : f_3) = \operatorname{reg}(\Re[\mathsf{V}(S_{2,t})]/I(S_{2,t})),$$

$$\mathrm{reg}(P/(I(\Phi_{3,t}),f_3),e_3)=\mathrm{reg}(\Re[\mathsf{V}(\Phi_{2,t})]/I(\Phi_{2,t})),$$

$$\mathrm{reg}(P/(I(\Phi_{3,t}),f_3):e_3)=\mathrm{reg}(\Re[\mathsf{V}(C_{3,t})]/I(C_{3,t})).$$

By induction and applying Lemmas 3.1.2 and 3.1.3 we have,

$$\mathrm{reg}(P/(I(\Phi_{3,t}):f_3)=1,$$

$$\mathrm{reg}(P/(I(\Phi_{3,t}),f_3),e_3)=2,$$

and

$$\mathrm{reg}(P/(I(\Phi_{3,t}),f_3):e_3)=1.$$

As $\mathrm{reg}(P/(I(\Phi_{3,t}),f_3),e_3)>\mathrm{reg}(P/(I(\Phi_{3,t}),f_3):e_3)$ So by Lemma 2.3.1(b), we have

$$\mathrm{reg}(P/(I(\Phi_{3,t}),f_3)=2.$$

Also $\mathrm{reg}(P/(I(\Phi_{3,t}),f_3)>\mathrm{reg}(P/(I(\Phi_{3,t}):f_3))$ so again by Lemma 2.3.1(b), we have

$$\mathrm{reg}(P/(I(\Phi_{3,t}))=2.$$

For $n=4$. we have

$$P/(I(\Phi_{4,t}):f_3)\cong\Re[\mathsf{V}(S_{2,t})]/I(S_{2,t})\otimes_{\Re}\Re[\mathsf{V}(S_t)]/I(S_t)\otimes_{\Re}\Re[f_3,D_2,C_2,C_3,D_4],$$

$$P/(I(\Phi_{4,t}),f_3),e_3)\cong\Re[\mathsf{V}(\Phi_{2,t})]/I(\Phi_{2,t})\otimes_{\Re}\Re[\mathsf{V}(S_{2,t})]/I(S_{2,t})\otimes_{\Re}\Re[f_3,e_3,D_3,C_3],$$

$$P/(I(\Phi_{4,t}),f_3):e_3)\cong\Phi[\mathsf{V}(C_{3,t})]/I(C_{3,t})\otimes_{\Re}\Re[e_3,C_2,D_4].$$

By applying Lemma 3.1.4 we get,

$$\mathrm{reg}(P/(I(\Phi_{4,t}):f_3)=\mathrm{reg}(\Re[\mathsf{V}(S_{2,t})]/I(S_{2,t}))+\mathrm{reg}(\Re[\mathsf{V}(S_t)]/I(S_t)),$$

$$\mathrm{reg}(P/(I(\Phi_{4,t}),f_3),e_3)=\mathrm{reg}(\Re[\mathsf{V}(\Phi_{2,t})]/I(\Phi_{2,t}))+\mathrm{reg}(\Re[\mathsf{V}(S_{2,t})]/I(S_{2,t})),$$

$$\mathrm{reg}(P/(I(\Phi_{3,t}),f_3):e_3)=\mathrm{reg}(\Re[\mathsf{V}(C_{3,t})]/I(C_{3,t})),$$

By applying Lemmas (3.1.1)and (3.1.3) we have,

$$\mathrm{reg}(P/(I(\Phi_{4,t}):f_3)=2,$$

$$\mathrm{reg}(P/(I(\Phi_{4,t}), f_3), e_3) = 3,$$

and

$$\mathrm{reg}(P/(I(\Phi_{4,t}), f_3) : e_3) = 2.$$

As $\mathrm{reg}(P/(I(\Phi_{4,t}), f_3), e_3) > \mathrm{reg}(P/(I(\Phi_{4,t}), f_3) : e_3)$ So by Lemma 2.3.1(b), we have

$$\mathrm{reg}(P/(I(\Phi_{3,t}), f_3) = 3$$

Also $\mathrm{reg}(P/(I(\Phi_{4,t}), f_3) > \mathrm{reg}(P/(I(\Phi_{4,t}) : f_3))$ so again by Lemma 2.3.1(b), we have

$$\mathrm{reg}(P/(I(\Phi_{4,t})) = 3.$$

Now for $n \geq 5$ we have the following cases,

Case 1: Let $n \equiv 0 \bmod(3)$. Then

$$P/(I(\Phi_{n,t}) : f_{n-1}) \cong \Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t}) \otimes_{\Re} \Re[\mathsf{V}(S_t)]/I(S_t) \otimes_{\Re} \Re[f_{n-1}, D_{n-2}, C_{n-2}, C_{n-1}, D_n], \quad (3.4)$$

$$P/((I(\Phi_{n,t}), f_{n-1}), e_{n-1})) \cong \Re[\mathsf{V}(\Phi_{n-2,t})]/I(\Phi_{n-2,t}) \otimes_{\Re} \Re[\mathsf{V}(S_{2,t})]/I(S_{2,t}) \otimes_{\Re} \Re[D_{n-1}, C_{n-1}], \quad (3.5)$$

$$P/((I(\Phi_{n,t}), f_{n-1}) : e_{n-1}), f_{n-2})) \cong \Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t}) \otimes_{\Re} \Re[e_{n-1}, D_{n-1}, C_{n-2}, D_n, C_n, D_{n-2}], \quad (3.6)$$

$$P/((I(\Phi_{n,t}), f_{n-1}) : e_{n-1}) : f_{n-2})) \cong \Re[\mathsf{V}(\Phi_{n-4,t})]/I(\Phi_{n-4,t}) \otimes_{\Re} \Re[e_{n-1}, f_{n-2}, D_{n-1}, C_{n-2}, C_n, D_n, D_{n-3}, C_{n-3}]. \quad (3.7)$$

By applying Lemma 3.1.4 and Lemma 3.1.5 on Eqs. (3.4) - (3.7), respectively we obtain,

$$\mathrm{reg}(P/(I(\Phi_{n,t}) : f_{n-1})) = \mathrm{reg}(\Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t})) + \mathrm{reg}(\Re[\mathsf{V}(S_t)]/I(S_t)), \quad (3.8)$$

$$\operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-1}), e_{n-1}))) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-2,t})]/I(\Phi_{n-2,t})) + \operatorname{reg}(\Re[\mathsf{V}(S_{2,t})]/I(S_{2,t})), \quad (3.9)$$

$$\operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-1}) : e_{n-1}), f_{n-2}))) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t})), \quad (3.10)$$

$$\operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-1}) : e_{n-1}) : f_{n-2}))) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-4,t})]/I(\Phi_{n-4,t})). \quad (3.11)$$

By using induction on n along with the use of Lemma 3.1.1 and Lemma 3.1.2 on Eqs. (3.8) - (3.11), respectively we obtain,

$$\operatorname{reg}(P/(I(\Phi_{n,t}) : f_{n-1})) = 2\lceil\frac{n-3}{3}\rceil + 1,$$

$$\operatorname{reg}(P/I(\Phi_{n,t}), f_{n-1}), e_{n-1})) = 2\lceil\frac{n}{3}\rceil,$$

$$\operatorname{reg}(P/(I(\Phi_{n,t}), f_{n-1}) : e_{n-1}), f_{n-2})) = 2\lceil\frac{n-3}{3}\rceil,$$

$$\operatorname{reg}(P/(I(\Phi_{n,t}), f_{n-1}) : e_{n-1}) : f_{n-2})) = 2\lceil\frac{n-4}{3}\rceil.$$

As $\operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-1}) : e_{n-1}) : f_{n-2})) < \operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-1}) : e_{n-1}), f_{n-2})))$, so by Lemma 2.3.1(b), $\operatorname{reg}(P/I(\Phi_{n,t}, f_{n-1}) : e_{n-1})) = 2\lceil\frac{n-3}{3}\rceil$. Also,

$$\operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-1}) : e_{n-1}))) < \operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-1}), e_{n-1}))).$$

Thus by Lemma 2.3.1(b),

$$\operatorname{reg}(P/I(\Phi_{n,t}, f_{n-1})) = 2\lceil\frac{n}{3}\rceil.$$

Similarly $\operatorname{reg}(P/((I(\Phi_{n,t}) : f_{n-1}))) < \operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-1})))$. Therefore by Lemma 2.3.1(b), $\operatorname{reg}(P/I(\Phi_{n,t}) = 2\lceil\frac{n}{3}\rceil$.

Case 2: Let $n \equiv 2 \bmod(3)$. Then $\Phi_{n,t} \cong \Phi_{n-2,t} \cup \Phi_{3,t}$, where $\Phi_{n-2,t} \cap \Phi_{3,t} \neq \emptyset$. By induction on n and using Lemma 2.3.5,

$$\operatorname{reg}(P/I(\Phi_{n,t}) \leq \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-2,t})]/I(\Phi_{n-2,t})) + \operatorname{reg}(\Re[\mathsf{V}(\Phi_{3,t})]/I(\Phi_{3,t})) = 2\lceil\frac{n-1}{3}\rceil + 2 = 2\lceil\frac{n}{3}\rceil.$$

To obtain the second inequality, let $\mathcal{M} = \Big\{\{f_1, f_{1,1}\}, \{e_2, e_{2,1}\}, \{f_3, f_{4,1}\}, \{e_5, e_{5,1}\} \dots,$ $\{f_{n-4}, f_{n-4,1}\}, \{e_{n-3}, e_{n-3,1}\}, \{f_{n-1}, f_{n-1,1}\}, \{e_n, e_{n,1}\}\Big\}$ clearly $\mathcal{M}$ forms an induced matching. Therefore, $\operatorname{indmat}(\Phi_{n,t}) \geq 2\lceil \frac{n}{3} \rceil$. By applying Lemma 2.3.4, $\operatorname{reg}(P/I(\Phi_{n,t})) \geq 2\lceil \frac{n}{3} \rceil$. Thus $\operatorname{reg}(P/I(\Phi_{n,t})) = 2\lceil \frac{n}{3} \rceil$.

Case 3: Let $n \equiv 1 \pmod 3$. Then

$$P/(I(\Phi_{n,t}) : f_{n-2}) \cong \Re(\Phi_{n-4,t})]/I(\Phi_{n-4,t}) \otimes_{\Re} \Re[\mathsf{V}(C_{3,t})]/I(C_{3,t}) \otimes_{\Re} \Re[f_{n-2}, D_{n-1}, C_{n-2}, C_{n-3}, D_{n-3}], \quad (3.12)$$

$$P/((I(\Phi_{n,t}), f_{n-2}), e_{n-2}) \cong \Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t}) \otimes_{\Re} \Re[\mathsf{V}(\Phi_{2,t})]/I(\Phi_{2,t}) \otimes_{\Re} \Re[D_{n-2}, C_{n-2}], \quad (3.13)$$

$$P/((I(\Phi_{n,t}), f_{n-2}) : e_{n-2}), f_{n-3})) \cong \Re[\mathsf{V}(\Phi_{n-4,t})]/I(\Phi_{n-4,t}) \otimes_{\Re} \Re[\mathsf{V}(S_{2,t})]/I(S_{2,t}) \otimes_{\Re} \Re[e_{n-2}, C_{n-1}, D_{n-1}, D_{n-2}, C_{n-3}, D_{n-2}, D_{n-3}], \quad (3.14)$$

$$P/((I(\Phi_{n,t}), f_{n-2}) : e_{n-2} : f_{n-3})) \cong \Re[\mathsf{V}(\Phi_{n-5,q})]/I(\Phi_{n-5,q}) \otimes_{\Re} \Re[\mathsf{V}(S_{2,t})]/I(S_{2,t}) \otimes_{\Re} \Re[e_{n-2}, f_{n-3}, C_{n-3}, C_{n-1}, D_{n-2}, D_{n-1}, D_{n-4}, C_{n-3}, C_{n-4}, D_{n-2}]. \quad (3.15)$$

By applying Lemma 3.1.4 and Lemma 3.1.5 on Eqs. (3.12) - (3.15) we get,

$$\operatorname{reg}(P/(I(\Phi_{n,t}) : f_{n-2}) = \operatorname{reg}(\Re(\Phi_{n-4,t})]/I(L_{n-4,t})) + \operatorname{reg}(\Re[\mathsf{V}(C_{3,t})]/I(C_{3,t})), \quad (3.16)$$

$$\operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-2}), e_{n-2}) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t})) + \operatorname{reg}(\Re[\mathsf{V}(\Phi_{2,t})]/I(\Phi_{2,t})), \quad (3.17)$$

$$\operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-2}) : e_{n-2}), f_{n-3})) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-4,t})]/I(\Phi_{n-4,t})) + \operatorname{reg}(\Re[\mathsf{V}(S_{2,t})]/I(S_{2,t})), \quad (3.18)$$

$$\operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-2}) : e_{n-2} : f_{n-3})) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-5,q})]/I(\Phi_{n-5,q})) + \Re[\mathsf{V}(S_{2,t})]/I(S_{2,t}). \quad (3.19)$$

By using induction on n along with the use of Lemma 3.1.2 and Lemma 3.1.3 on Eqs. (3.16) - (3.19) ,

$$\operatorname{reg}(P/(I(\Phi_{n,t}) : f_{n-2})) = 2\lceil\frac{n-4}{3}\rceil + 1,$$

$$\operatorname{reg}(P/(I(\Phi_{n,t}), f_{n-2}), e_{n-2})) = 2\lceil\frac{n-1}{3}\rceil + 1,$$

$$\operatorname{reg}(P/(I(\Phi_{n,t}), f_{n-2}) : e_{n-2}), f_{n-3})) = 2\lceil\frac{n-4}{3}\rceil + 1,$$

$$\operatorname{reg}(P/(I(\Phi_{n,t}), f_{n-2}) : e_{n-2}) : f_{n-3})) = 2\lceil\frac{n-5}{3}\rceil + 1.$$

As $\operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-2}) : e_{n-2}) : f_{n-3})) < \operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-2}) : e_{n-2}), f_{n-3}))),$ by Lemma 2.3.1(b), $\operatorname{reg}(P/I(\Phi_{n,t}, f_{n-2}) : e_{n-2})) = 2\lceil\frac{n-4}{3}\rceil + 1$. Also

$$\operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-2}) : e_{n-2}))) < \operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-2}), e_{n-2}))),$$

by Lemma 2.3.1(b),

$$\operatorname{reg}(P/I(\Phi_{n,t}, f_{n-2})) = 2\lceil\frac{n-1}{3}\rceil + 1.$$

Furthermore, $\operatorname{reg}(P/((I(\Phi_{n,t}) : f_{n-2})))) < \operatorname{reg}(P/((I(\Phi_{n,t}), f_{n-2}))),$ by Lemma 2.3.1(b),

$$\operatorname{reg}(P/I(\Phi_{n,t}) = 2\lceil\frac{n-1}{3}\rceil + 1.$$

□

Lemma 3.2.3. *Let $n, q \geq 1$ and $P = \Re[\mathsf{V}(\Psi_{n,t})]$, then* $\operatorname{reg}(P/I(\Psi_{n,t})) = \lceil\frac{n}{2}\rceil$.

Proof. Let $n = 2$. We have

$$P/(I(\Psi_{2,t}) : f_1) \cong \Re[\mathsf{V}(\Psi_{0,t})]/\mathcal{I}(\Psi_{0,t}) \otimes_{\Re} \Re[f_1, D_2, C_1, C_2]. \quad (3.20)$$

Moreover, we have

$$P/((I(\Psi_{2,t}), f_1, e_1)) \cong \Re[\mathsf{V}(S_{2,t})]/I(S_{2,t}) \otimes_{\Re} \Re[D_1, C_1], \quad (3.21)$$

also

$$P/(I(\Psi_{2,t}), f_1 : e_1)) \cong (\Re/\mathsf{V}(\Psi_{0,t})/I(\Psi_{0,t})) \otimes_{\Re} \Re[e_1, D_1, C_2, D_2]. \tag{3.22}$$

By applying Lemma 3.1.4 on Eqs. (4.4) - (4.6), respectively we get,

$$\operatorname{reg}(P/(I(\Psi_{2,t}) : f_1) = \operatorname{reg}(\Re[\mathsf{V}(\Psi_{0,t})]/\mathcal{I}(\Psi_{0,t})), \tag{3.23}$$

$$\operatorname{reg}(P/((I(\Psi_{2,t}), f_1, e_1)) = \operatorname{reg}(\Re[\mathsf{V}(S_{2,t})]/I(S_{2,t})), \tag{3.24}$$

$$\operatorname{reg}(P/(I(\Psi_{2,t}), f_1) : e_1)) = \operatorname{reg}(\Re[\mathsf{V}(\Psi_{0,t})/I(\Psi_{0,t})). \tag{3.25}$$

By using Lemma 3.1.2 and Remark 3.1.6, we have
$\operatorname{reg}(P/((I(\Psi_{2,t}) : f_1)) = 0$, and $\operatorname{reg}(P/(I(\Psi_{2,t}), f_1, e_1)) = 1$. also

$$\operatorname{reg}(P/((I(\Psi_{2,t}), f_1 : e_1)) = 0.$$

As $\operatorname{reg}(P/((I(\Psi_{2,t}), f_1 : e_1)) < \operatorname{reg}(P/((I(\Psi_{2,t}), f_1, e_1))$, by applying Lemma 2.3.1(b), $\operatorname{reg}(P/(I(\Psi_{2,t}, f_1))) = 1$. Also $\operatorname{reg}(P/((I(\Psi_{2,t}) : f_1)) < \operatorname{reg}(P/((I(\Psi_{2,t}), f_1))$, so by Lemma 2.3.1(b) $\operatorname{reg}(P/((I(\Psi_{2,t})) = 1$. For $n \geq 3$ we have:

$$P/((I(\Psi_{n,t}) : f_{n-1})) \cong \Re[\mathsf{V}(\Psi_{n-3,t})]/I(\Psi_{n-3,t}) \otimes_{\Re} \Re[f_{n-1}, D_n, D_{n-2}, C_n, C_{n-2}], \tag{3.26}$$

$$P/((I(\Psi_{n,t}), f_{n-1}), e_{n-1})) \cong \Re[\mathsf{V}(\Psi_{n-2,t})]/I(\Psi_{n-2,t}) \otimes_{\Re} \Re[\mathsf{V}(S_{2,t})]/I(S_{2,t}) \otimes_{\Re} \Re[D_{n-1}, C_{n-1}], \tag{3.27}$$

$$P/((I(\Psi_{n,t}), f_{n-1}) : e_{n-1})) \cong \Re[\mathsf{V}(\Psi_{n-3,t})]/I(\Psi_{n-3,t}) \otimes_{\Re} \Re[e_{n-1}, D_{n-1}, C_n, C_{n-2}, D_{n-1}, D_n, D_{n-2}]. \tag{3.28}$$

By using Lemma 3.1.4 and Lemma 3.1.5 on Eqs.(3.4) - (3.28) we have,

$$\operatorname{reg}(P/((I(\Psi_{n,t}) : f_{n-1})) = \operatorname{reg}(\Re[\mathsf{V}(\Psi_{n-3,t})]/I(\Psi_{n-3,t})), \tag{3.29}$$

$$\operatorname{reg}(P/((I(\Psi_{n,t}), f_{n-1}), e_{n-1})) = \operatorname{reg}(\Re[\mathsf{V}(\Psi_{n-2,t})]/I(\Psi_{n-2,t}) + \operatorname{reg}(\Re[\mathsf{V}(S_{2,t})]/I(S_{2,t})), \tag{3.30}$$

$$\operatorname{reg}(P/((I(\Psi_{n,t}), f_{n-1}) : e_{n-1})) = \operatorname{reg}(\Re[\mathsf{V}(\Psi_{n-3,t})]/I(\Psi_{n-3,t})). \tag{3.31}$$

By induction on n and applying Lemma 3.1.1 on Eqs. (3.29) - (3.31), respectively we have ,

$$\operatorname{reg}(P/(I(\Psi_{n,t}) : f_{n-1})) = \lceil \frac{n-3}{2} \rceil,$$

$$\operatorname{reg}(P/(I(\Psi_{n,t}), f_{n-1}), e_{n-1})) = \lceil \frac{n-2}{2} \rceil + 1 = \lceil \frac{n}{2} \rceil,$$

$$\operatorname{reg}(P/(I(\Psi_{n,t}), f_{n-1}) : e_{n-1}))) = \lceil \frac{n-3}{2} \rceil.$$

As, $\operatorname{reg}(P/((I(\Psi_{n,t}), f_{n-1}) : e_{n-1}))) < \operatorname{reg}(P/((I(\Psi_{n,t}), f_{n-1}), e_{n-1})))$, by Lemma 2.3.1(b),

$$\operatorname{reg}(P/I(\Psi_{n,t}, f_{n-1})) = \lceil \frac{n}{2} \rceil.$$

Also, $\operatorname{reg}(P/((I(\Psi_{n,t}) : f_{n-1}))) < \operatorname{reg}(P/((I(\Psi_{n,t}), f_{n-1})))$, by Lemma 2.3.1(b),

$$\operatorname{reg}(P/I(\Psi_{n,t})) = \lceil \frac{n}{2} \rceil.$$

□

Lemma 3.2.4. *Let $n \geq 2$, $t \geq 1$ and $P = \Re[\mathsf{V}(\Upsilon_{n,t})]$. Then*

$$\operatorname{sdepth}(P/I(\Upsilon_{n,t})) \geq \operatorname{depth}(P/I(\Upsilon_{n,t})) = \begin{cases} n(t+1), & \text{if } n \text{ is even;} \\ n(1+t) - t + 1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $n = 2$. We have,

$$0 \longrightarrow P/I(\Upsilon_{2,t} : f_2) \xrightarrow{\cdot f_2} P/I(\Upsilon_{2,t}) \longrightarrow P/I(\Upsilon_{2,t}, f_2) \longrightarrow 0, \tag{3.32}$$

and

$$P/(I(\Upsilon_{2,t}) : f_2) \cong \Re[\mathsf{V}(S_t)]/I(S_t) \otimes_{\Re} \Re[f_2, D_1, C_1],$$

$$P/((I(\Upsilon_{2,t}), f_2) \cong \Re[\mathsf{V}(S_{3,t})]/I(S_{3,t}) \otimes_{\Re} \Re[D_2].$$

By Lemma 2.1.6 and Lemma 3.1.4, $\operatorname{depth}(P/(I(\Upsilon_{2,t}) : f_2)) = \operatorname{depth}(\mathsf{V}(S_t)]/I(S_t)) + 1 + t + t$, and $\operatorname{depth}(P/((I(\Upsilon_{2,t}), f_2))) = \operatorname{depth}(\Re[\mathsf{V}(S_{3,t})]/I(S_{3,t})) + t$. By applying Lemma 3.1.1 and Lemma 3.1.2, $\operatorname{depth}(P/(I(\Upsilon_{2,t}) : f_2)) = 2t + 2 = 2(1 + t)$, and $\operatorname{depth}(P/((I(\Upsilon_{2,t}), f_2)) = 2 + t$. By Lemma 2.1.4 and Lemma 2.1.2 along with the use of short exact sequence 3.32, $\operatorname{depth}(P/(I(\Upsilon_{2,t}))) = 2t + 2$. Let $n \geq 3$. Then

$$0 \longrightarrow P/I(\Upsilon_{n,t} : f_n) \xrightarrow{\cdot f_n} P/I(\Upsilon_{n,t}) \longrightarrow P/I(\Upsilon_{n,t}, f_n) \longrightarrow 0,$$

$$P/(I(\Upsilon_{n,t}) : f_n) \cong \Re[\mathsf{V}(\Upsilon_{n-2})]/I(\Upsilon_{n-2}) \otimes_{\Re} \Re[\mathsf{V}(S_t)]/I(S_t)] \otimes_{\Re} \Re[f_n, D_{n-1}, C_{n-1}]. \tag{3.33}$$

Let

$$L := (I(\Upsilon_{n,t}), f_n) = (I(\Upsilon_{n-1,t}), e_n, f_n f_{n+1}, f_{n+1} f_{n+1,1}, f_{n+1} f_{n+1,2}, \dots, f_{n+1,t}).$$

Again,

$$0 \longrightarrow P/I(L : e_n) \xrightarrow{\cdot e_n} S/I(L) \longrightarrow P/I(L, e_n) \longrightarrow 0.$$

$$P/I(L : e_n) \cong \Re[\mathsf{V}(\Upsilon_{n-2,t})]/I(\Upsilon_{n-2,t}) \otimes_{\Re} \Re[e_n, D_{n-1}, C_{n-1}], \tag{3.34}$$

$$P/I(L, e_n) \cong \Re[\mathsf{V}(\Upsilon_{n-1,t})]/I(\Upsilon_{n-1,t}) \otimes_{\Re} \Re[C_n, D_n]. \tag{3.35}$$

Case 1 If n is odd, then by Lemma 2.1.6 and Lemma 3.1.4 on Eqs. (3.33) - (3.35), respectively,

$$\text{depth}(P/(I(\Upsilon_{n,t}) : f_n)) = \text{depth}(\Re[\mathsf{V}(\Upsilon_{n-2,t})]/I(\Upsilon_{n-2,t})) + \text{depth}[(\Re(\mathsf{V}(S_t))]/I(S_t)) + 1 + 2t,$$

$$\text{depth}(P/(L, e_n)) = \text{depth}(\Re[\mathsf{V}(\Upsilon_{n-1,t})]/I(\Upsilon_{n-1,t})) + t + t,$$

$$\text{depth}(P/(L : e_n)) = \text{depth}(\Re[\mathsf{V}(\Upsilon_{n-2,t})]/I(\Upsilon_{n-2,t})) + t + t + t.$$

By mathematical induction on n, along with the use of Lemma 3.1.1 ,

$$\text{depth}(P/(I(\Upsilon_{n,t}) : f_n)) = (n-2)(t+1) - t + 1 + 1 + 1 + 2t = n(t+1) - t + 1, \tag{3.36}$$

$$\text{depth}(P/(L, e_n)) = (n-1)(t+1) + t + t = n(1+t) + t - 1, \tag{3.37}$$

$$\text{depth}(P/(L : e_n)) = (n-2)(t+1) - t + 1 + t + 1 + 2t = n(1+t). \tag{3.38}$$

By applying Lemma 2.1.4 and Lemma 2.1.2 on Eqs. 3.37 and 3.38, respectively,

$$\text{depth}(P/(I(\Upsilon_{n,t}), f_n)) = n(t+1). \tag{3.39}$$

By applying Lemma 2.1.4 and Lemma 2.1.2 on Eqs. 3.36 and 3.39, respectively,

$$\text{depth}(P/(I(\Upsilon_{n,t})) = n(t+1) - t + 1.$$

Case 2 If n is even, then by applying Lemma 2.1.6 and Lemma 3.1.4 on Eqs. (3.33) - (3.35), respectively,

$$\operatorname{depth}(P/(I(\Upsilon_{n,t}):f_n)) = \operatorname{depth}(\Re[\mathsf{V}(\Upsilon_{n-2,t})]/I(\Upsilon_{n-2,t}))+\operatorname{depth}(\mathsf{V}(S_t)]/I(S_t))+1+t+t,$$

$$\operatorname{depth}(P/(L:e_n)) = \operatorname{depth}(\Re[\mathsf{V}(\Upsilon_{n-2,t})]/I(\Upsilon_{n-2,t})) + 1 + t + t + t = n(1+t) + t - 1,$$

$$\operatorname{depth}(P/(L,e_n)) = \operatorname{depth}(\Re[\mathsf{V}(\Upsilon_{n-1,t})]/I(\Upsilon_{n-1,t})) + t + t.$$

By mathematical induction on n, along with the use of Lemma 3.1.1,

$$\operatorname{depth}(P/(I(\Upsilon_{n,t}):f_n)) = (n-2)(t+1)+1+1+2t = n(t+1), \tag{3.40}$$

$$\operatorname{depth}(P/(L:e_n)) = (n-2)(t+1) - t + 1 + 2t = n(1+t), \tag{3.41}$$

$$\operatorname{depth}(P/(L,e_n)) = (n-2)(t+1) + t + 1 + 2t = n(1+t) + t - 1. \tag{3.42}$$

By applying Lemma 2.1.4 and Lemma 2.1.2 on Eqs. 3.41 and 3.42, respectively,

$$\operatorname{depth}(P/(I(\Upsilon_{n,t}),f_n)) \geq n(1+t). \tag{3.43}$$

Since $e_{n-1} \notin L$ and

$$\operatorname{depth}(P/I(L:e_{n-1})) \cong \Re[\mathsf{V}(\Upsilon_{n-3})]/I(\Upsilon_{n-3}) \otimes_{\Re} \Re[\mathsf{V}(S_t)]/I(S_t) \otimes_{\Re} \Re[e_{n-1}, e_{n-2,1}, \dots, e_{n-2,t}, f_{n-2,1}, \dots, f_{n-2,t}, e_{n,1}, \dots, e_{n,t}]. \tag{3.44}$$

By using induction on n and applying Lemma 3.1.1, we get

$$\operatorname{depth}(P/(I(L:e_{n-1})) = (n-3)(t+1) - t + 1 + 1 + 1 + 3t + t = n(t+1). \tag{3.45}$$

By applying Lemma 2.1.2 on eq. 3.45 we have

$$\operatorname{depth}(P/(I(\Upsilon_{n,t}),f_n))) \leq n(1+t). \tag{3.46}$$

From Eqs (3.43) and (3.46), we get

$$\operatorname{depth}(P/(I(\Upsilon_{n,t},f_n)) = n(t+1). \tag{3.47}$$

By applying depth Lemma on Eqs (3.40) and (3.47) we get

$$\operatorname{depth}(P/(I(\Upsilon_{n,t}) = n(t+1). \tag{3.48}$$

To find the values of Stanley depth, the proof is similar, just by replacing Lemma 2.1.4 with Lemma 2.2.2. □

Lemma 3.2.5. *Let $t, n \geq 1$ and $P = \Re[\mathsf{V}(\Upsilon_{n,t})]$. Then*

$$\operatorname{pdim}(P/\mathcal{I}(\Upsilon_{n,t})) = \begin{cases} n(1+t), & \text{if } n \text{ is even;} \\ n(1+t)+t-1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. The desired outcome can be obtained by applying Lemma 2.4.1 and Lemma 3.2.4. □

Chapter 4

Algebraic Invariants of Cyclic Modules Associated with Bristles of Some Circulant Graphs

In this Chapter, we investigate the properties of edge ideals of certain families of four and five-bristled circulant graphs. We analyze the depth and projective dimension of the cyclic module $\Re[\mathsf{V}(\mathcal{D}_{n,t})]/I(\mathcal{D}_{n,t})$. Additionally, we provide a lower bound for the Stanley depth of this module.Furthermore, the exact value of regularity of $\Re[\mathsf{V}(\mathcal{D}_{n,t})]/I(\mathcal{D}_{n,t})$, where $\mathcal{D}_{n,t} =: Br_t[(C_{2n}(1, n-1))]$ and $\Re[\mathsf{V}(\mathcal{E}_{n,t})]/I(\mathcal{E}_{n,t})$, where $\mathcal{E}_{n,t} =: Br_t[(C_{2n}(1,2))]$ also $\Re[\mathsf{V}(\mathcal{F}_{n,t})]/I(\mathcal{F}_{n,t})$, where
$\mathcal{F}_{n,t} := Br_t[(C_{2n}(1, n-1, n))]$. It is worth noting that the labeling of the graphs corresponds to the conventions shown in Figure 4.1 and Figure 4.2.

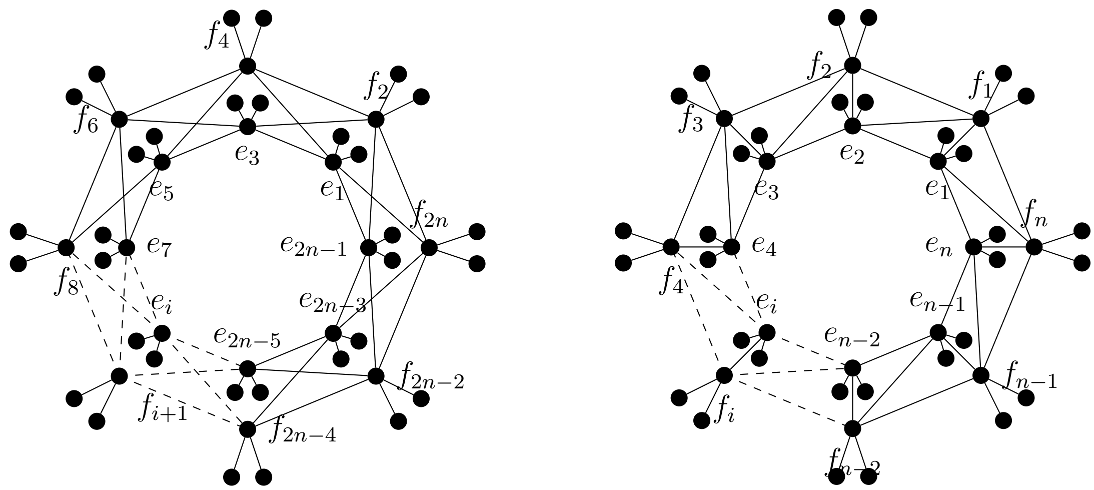

Figure 4.1: From left to right, $\mathcal{D}_{n,2}$ and $\mathcal{E}_{n,2}$.

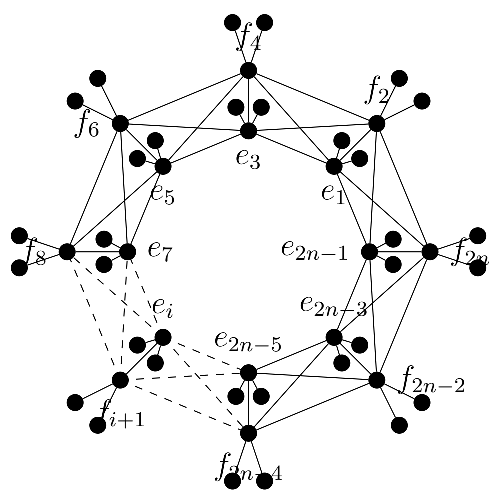

Figure 4.2: $\mathcal{F}_{n,2}$.

Theorem 4.0.1. *Let* $n \geq 3$ *and* $q \geq 1$, $\mathcal{D}_{n,t} = Br_t[C(1, n-1)]$ *and* $P = Br_t[\Re[\mathsf{V}(\mathcal{D}_{n,t})]$. *Then*

$$\operatorname{reg}(P/I(\mathcal{D}_{n,t})) = \begin{cases} n, & \text{if } n \text{ is even;} \\ n-1, & \text{if } n \text{ is odd.} \end{cases}$$

Proof. Let $n = 3$. Then

$$P/(I(\mathcal{D}_{3,t}):e_3)\cong \Re[\mathsf{V}(\Upsilon_{0,t})]/I(\Upsilon_{0,t})\otimes_{\Re}\Re[\mathsf{V}(S_q)]/I(S_q)\otimes_{\Re}\Re[e_3,C_1,C_2,D_1,D_2], \quad (4.1)$$

$$P/((I(\mathcal{D}_{3,t}),e_3):f_3)\cong \Re[\mathsf{V}(\Upsilon_{0,t})]/I(\Upsilon_{0,t})\otimes_{\Re}\Re[f_3,D_1,D_2,C_1,C_2,C_3], \quad (4.2)$$

$$P/((I(\mathcal{D}_{3,t}),e_3),f_3)\cong \Re[\mathsf{V}(\Upsilon_{2,t})]/I(\Upsilon_{2,t})\otimes_{\Re}\Re[C_3,D_3]. \quad (4.3)$$

By using Lemmas 3.1.4 and 3.1.5 on Eqs.(4.1) - (4.3) we obtain,

$$\operatorname{reg}(P/(I(\mathcal{D}_{3,t}):e_3)=\operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{0,t})]/I(\Upsilon_{0,t})+\operatorname{reg}(\Re[\mathsf{V}(S_q)]/I(S_q)), \quad (4.4)$$

$$\operatorname{reg}(P/((I(\mathcal{D}_{3,t}),e_3):f_3)=\operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{0,t})]/I(\Upsilon_{0,t})), \quad (4.5)$$

$$\operatorname{reg}(P/((I(\mathcal{D}_{3,t}),e_3),f_3)=\operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{2,t})]/I(\Upsilon_{2,t})). \quad (4.6)$$

By applying induction and applying Lemma 3.1.1 and Remark 3.1.6 we obtain,

$\operatorname{reg}(P/(I(\mathcal{D}_{3,t}):e_3))=1$, and $\operatorname{reg}(P/((I(\mathcal{D}_{3,t}),e_3):f_3))=0$, moreover $\operatorname{reg}(P/((I(\mathcal{D}_{3,t}),e_3),f_3))=2$. Since

$$\operatorname{reg}(P/((I(\mathcal{D}_{3,t}),e_3):f_3))<\operatorname{reg}(P/((I(\mathcal{D}_{3,t}),e_3),f_3)).$$

By Lemma 2.3.1(b), $\operatorname{reg}(P/(I(\mathcal{D}_{3,t}),e_3))=2>\operatorname{reg}(P/(I(\mathcal{D}_{3,t}):e_3))$.

Again by applying Lemma 2.3.1(b), $\operatorname{reg}(P/I(\mathcal{D}_{3,t})=2$. If $n=4$, then

$$P/(I(\mathcal{D}_{4,q}):e_4)\cong \Re[\mathbb{V}(\Upsilon_{1,t})]/I(\Upsilon_{1,t})\otimes_{\Re}\Re[\mathbb{V}(S_q)]/I(S_q)\otimes_{\Re}\Re[e_4,C_1,C_3,D_1,D_3], \quad (4.7)$$

$$P/((I(\mathcal{D}_{4,t},e_4):f_4)\cong \Re[\mathsf{V}(\Upsilon_{1,t})]/I(\Upsilon_{1,t})\otimes_{\Re}\Re[f_4,D_1,D_3,C_1,C_3,C_4], \quad (4.8)$$

$$P/((I(\mathcal{D}_{4,t},e_4),f_4)\cong \Re[\mathsf{V}(\Upsilon_{3,t})]/I(\Upsilon_{3,t})\otimes_{\Re}\Re[C_4,D_4]. \quad (4.9)$$

By applying Lemma 3.1.4 on Eqs. (4.7) - (4.9) we get

$$\operatorname{reg}(P/(I(\mathcal{D}_{4,t}):e_4)=\operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{1,t})]/I(\Upsilon_{1,t})+\operatorname{reg}(\Re[\mathbb{V}(S_q)]/I(S_q)), \quad (4.10)$$

$$\operatorname{reg}(P/((I(\mathcal{D}_{4,t},e_4):f_4)=\operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{1,t})]/I(\Upsilon_{1,t})), \quad (4.11)$$

$$\operatorname{reg}(P/((I(\mathcal{D}_{4,t},e_4),f_4)=\operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{3,t})]/I(\Upsilon_{3,t})), \quad (4.12)$$

By applying Remark 3.1.6 and Lemma 3.1.1 we have,

$$\text{reg}(P/(I(\mathcal{D}_{4,t}) : e_4)) = 3,$$

$\text{reg}(P/((I(\mathcal{D}_{3,t}), e_4) : f_4)) = 2.$ and, $\text{reg}(P/((I(\mathcal{D}_{4,t}), e_4), f_4)) = 4$. Since

$$\text{reg}(P/((I(\mathcal{D}_{4,t}), e_4) : f_4)) < \text{reg}\,(\Re[\mathsf{V}(\mathcal{D}_{4,t})]/((I(\mathcal{D}_{4,t}), e_4), f_4)).$$

By Lemma 2.3.1(b),

$$\text{reg}(P/(I(\mathcal{D}_{4,t}), e_4)) = 4 > \text{reg}(\Re[\mathsf{V}(\mathcal{D}_{4,t})]/(I(\mathcal{D}_{4,t}) : e_4)).$$

Again by applying Lemma 2.3.1(b),

$$\text{reg}(P/I(\mathcal{D}_{4,t}) = 4.$$

Now we have the following cases:

Case 1 In the case where n is an even number. By applying Lemma 2.3.1(b), $\text{reg}(P/I(\mathcal{D}_{n,t})) = \text{reg}(P/(I(\mathcal{D}_{n,t}), e_n))$, if,

$$\text{reg}(P/(I(\mathcal{D}_{n,t}) : e_n)) < \text{reg}(P/(I(\mathcal{D}_{n,t}), e_n)).$$

Since

$$P/((I(\mathcal{D}_{n,t}) : e_n)) \cong \Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t})] \otimes_{\Re} \Re[\mathsf{V}(S_q)]/I(S_q) \otimes_{\Re} \Re[e_n, C_1, C_{n-1}, D_1, D_{n-1}], \quad (4.13)$$

$$P/((I(\mathcal{D}_{n,t}), e_n) : f_n) \cong \Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t}) \otimes_{\Re} \Re[f_n, D_1, D_{n-1}, C_1, C_{n-1}, C_n], \quad (4.14)$$

$$P/((I(\mathcal{D}_{n,t}), e_n), f_n) \cong \Re[\mathsf{V}(\Upsilon_{n-1,t})]/I(\Upsilon_{n-1,t})] \otimes_{\Re} \Re[C_n, D_n]. \quad (4.15)$$

By applying Lemma 3.1.4, Lemma 3.1.5 on Eqs. (4.13) - (4.15),

$$\text{reg}\,(P/((I(\mathcal{D}_{n,t}) : e_n)) = \text{reg}(\Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t})) + \Re[\mathsf{V}(S_q)]/I(S_q), \quad (4.16)$$

$$\operatorname{reg}(P/((I(\mathcal{D}_{n,t}),e_n):f_n)) = \operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t})), \tag{4.17}$$

$$\operatorname{reg}(P/((I(\mathcal{D}_{n,t}),e_n),f_n)) = \operatorname{reg}(\Re[\mathbb{V}(\Upsilon_{n-1,t})]/I(\Upsilon_{n-1,t})). \tag{4.18}$$

By using induction on n and applying Lemma 3.2.1 and 3.1.1 on Eqs.(4.16) - (4.18) we have,

$$\operatorname{reg}(P/((I(\mathcal{D}_{n,t}):e_n))) = n-3+1+1 = n-1,$$

$$\operatorname{reg}(P/((I(\mathcal{D}_{n,t}),e_n:f_n))) = n-3+1 = n-2,$$

$$\operatorname{reg}(P/((I(\mathcal{D}_{n,t}),e_n,f_n))) = n-1+1 = n.$$

As, $\operatorname{reg}(P/((I(\mathcal{D}_{n,t}),e_n):f_n)) < \operatorname{reg}(P/((I(\mathcal{D}_{n,t}),e_n),f_n))$,

by applying Lemma 2.3.1(b),

$$\operatorname{reg}(P/(I(R_{n,t}),e_n)) = n.$$

Also, $\operatorname{reg}(P/(((I(\mathcal{D}_{n,t}):e_n))) < \operatorname{reg}(P/(((I(\mathcal{D}_{n,t}),e_n)))$,

again by applying Lemma 2.3.1(b), $\operatorname{reg}(P/((I(\mathcal{D}_{n,t}))) = n$.

Case 2 When n is odd.

$$P/((I(\mathcal{D}_{n,t}):e_n) \cong \Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t})] \otimes_{\Re} \Re[\mathsf{V}(S_q)]/I(S_q) \otimes_{\Re} \Re[e_n, C_1, C_{n-1}, D_{1,1}, D_{n-1}]., \tag{4.19}$$

$$P/((I(\mathcal{D}_{n,t}),e_n):f_n) \cong \Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t}) \otimes_{\Re} \Re[f_n, D_1, D_{n-1}, C_1, C_{n-1}, C_n], \tag{4.20}$$

$$P/((I(\mathcal{D}_{n,t}),e_n),f_n) \cong \Re[\mathsf{V}(\Upsilon_{n-1,t})]/I(\Upsilon_{n-1,t})] \otimes_{\Re} \Re[C_n, D_n], \tag{4.21}$$

By Lemma 3.1.4 and Lemma 3.1.5 on Eqs.(4.19) - (4.21) we get,

$$\operatorname{reg}(P/((I(\mathcal{D}_{n,t}):e_n)) = \operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t})) + \Re[\mathsf{V}(S_q)]/I(S_q), \tag{4.22}$$

$$\operatorname{reg}(P/((I(\mathcal{D}_{n,t}),e_n):f_n)) = \operatorname{reg}(\Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t})), \tag{4.23}$$

$$\text{reg}\,(P/((I(\mathcal{D}_{n,t}), e_n), f_n)) = \text{reg}(\Re[\mathsf{V}(\Upsilon_{n-1,t})]/I(\Upsilon_{n-1,t})). \tag{4.24}$$

By applying induction on n along with the use of Lemma 3.2.1 and 3.1.1 on Eqs. (4.22) - (4.24) we have,

$$\text{reg}(P/((I(\mathcal{D}_{n,t}) : e_n)) = n - 3 + 1 = n - 2,$$

$$\text{reg}(P/((I(\mathcal{D}_{n,t}), e_n : f_n)) = n - 3,$$

$$\text{reg}(P/((I(\mathcal{D}_{n,t}), e_n, f_n)) = n - 1.$$

As, $\text{reg}\,(P/((I(\mathcal{D}_{n,t}), e_n) : f_n)) < \text{reg}(P/((I(\mathcal{D}_{n,t}), e_n), f_n))$, by applying Lemma 2.3.1(b),

$$\text{reg}(P/(I(\mathcal{D}_{n,t}), e_n)) = n - 1.$$

Also $\text{reg}\,(P/(((I(\mathcal{D}_{n,t}) : e_n))) < \text{reg}\,(P/(((I(\mathcal{D}_{n,t}), e_n)))$, again by applying Lemma 2.3.1(b), $\text{reg}\,(P/((I(\mathcal{D}_{n,t}))) = n - 1$.

□

Theorem 4.0.2. *Let $n \geq 3$, $t \geq 1$ and $\mathcal{E}_{n,t} = Br_t[(C_{2n}(1,2))]$. and $P = \Re[\mathsf{V}(\mathcal{E}_{n,t})]$ Then*

$$\text{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/I(\mathcal{E}_{n,t})) = \begin{cases} 2\lceil \frac{n-1}{3} \rceil, & \text{if } n \equiv 1 \pmod 3; \\ 2\lceil \frac{n-1}{3} \rceil - 1, & \text{if } n \equiv 2 \pmod 3; \end{cases}$$

if $n \equiv 0 \pmod 3$.

$$\lceil \frac{n}{2} \rceil \leq \text{reg}(\mathbb{K}[\mathsf{V}(\mathcal{E}_{n,t})]/I(\mathcal{E}_{n,t})) \leq 2\lceil \frac{n-1}{3} \rceil + 2$$

Proof. **Case 1** In the case where $n \equiv 2 \pmod 3$. By applying Lemma 2.3.1(b),

$$\text{reg}(P/I(\mathcal{E}_{n,t})) = \text{reg}(P/(I(\mathcal{E}_{n,t}), e_n)),$$

if, $\text{reg}(P/(I(\mathcal{E}_{n,t}) : e_n)) < \text{reg}(P/(I(\mathcal{E}_{n,t}), e_n))$. We have the following K-algebric isomorphism:

$$P/((I(\mathcal{E}_{n,t}) : e_{n-1}) : f_{n-2}) \cong \Re[\mathsf{V}(\Phi_{n-4,t})]/I(\Phi_{n-4,t})] \otimes_{\Re} \Re[e_{n-1}, f_{n-2}, C_n, C_{n-2}, D_{n-1}, D_n, C_{n-3}, D_{n-3}], \tag{4.25}$$

$$P/((I(\mathcal{E}_{n,t}) : e_{n-1}), f_{n-2}) \cong \Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t}) \otimes_{\Re} \Re[e_{n-1}, C_n, C_{n-2}, D_{n-1}, D_{n-2}], \quad (4.26)$$

$$P/((I(\mathcal{E}_{n,t}), e_{n-1}), f_{n-1}) \cong \Re[\mathsf{V}(\Phi_{n-1,t})]/I(\Phi_{n-1,t}) \otimes_{\Re} \Re[C_{n-1}, D_{n-1}], \quad (4.27)$$

$$P/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}), e_n) \cong \Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t}) \otimes_{\Re} \Re[f_{n-1}, C_n, D_n, D_{n-2}, C_{n-1}, C_{n-2}], \quad (4.28)$$

$$P/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}) : e_n) \cong (\Re[\mathbb{V}(\Phi_{n-4,t})]/I(\Phi_{n-4,t})) \otimes_{\Re} \Re[f_{n-1}, e_n, C_{n-1}, C_1, D_1, D_n, C_{n-2}, D_{n-2}]. \quad (4.29)$$

By applying induction on n and using Lemma 3.2.2 and 3.1.4 we get,

$$\operatorname{reg}(P/((I(\mathcal{E}_{n,t}) : e_{n-1}) : f_{n-2})) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-4,t})]/I(\Phi_{n-4,t})) = 2\lceil \frac{n-1}{3} \rceil - 3,$$

since $n-1 \equiv 1 \pmod 3$.

$$\operatorname{reg}(P/((I(\mathcal{E}_{n,t}) : e_{n-1}), f_{n-2})) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t})) = 2\lceil \frac{n-1}{3} \rceil - 2,$$

since $n-3 \equiv 0 \pmod 3$.

$$\operatorname{reg}(P/((I(\mathcal{E}_{n,t}), e_{n-1}), f_{n-1})) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-1,t})]/I(\Phi_{n-1,t})) = 2\lceil \frac{n-1}{3} \rceil - 1,$$

as we have $n-1 \equiv 0 \pmod 3$.

$$\operatorname{reg}(P/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}), e_n) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t})) = 2\lceil \frac{n-1}{3} \rceil - 2,$$

since $n-3 \equiv 0 \pmod 3$.

$$\operatorname{reg}(P/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1} : e_n)) \cong \Re[\mathsf{V}(\Phi_{n-1,t})]/I(\Phi_{n-1,t}) = 2\lceil \frac{n-1}{3} \rceil - 3.$$

Since

$$\operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/((I(\mathcal{E}_{n,t}) : e_{n-1}) : f_{n-2})) < \operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/((I(\mathcal{E}_{n,t}) : e_{n-1}), f_{n-2})).$$

By using Lemma 2.3.1(b),

$$\operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/(I(\mathcal{E}_{n,t}):e_{n-1}))=2\lceil\frac{n-1}{3}\rceil-2.$$

Also $\operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/(((I(\mathcal{E}_{n,t}):e_{n-1}):f_{n-1}):e_n)) < \operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/(((I(\mathcal{E}_{n,t}):e_{n-1}),f_{n-1}),e_n)$. By using Lemma 2.3.1(b),

$$\operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/(I(\mathcal{E}_{n,t}),e_{n-1}):f_{n-1}))=2\lceil\frac{n-1}{3}\rceil-2.$$

And $\operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/((I(\mathcal{E}_{n,t}),e_{n-1}):f_{n-1}))) < \operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/((I(\mathcal{E}_{n,t}):e_{n-1}),f_{n-1}))$. By using Lemma 2.3.1(b),

$$\operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/(I(\mathcal{E}_{n,t}),e_{n-1}))=2\lceil\frac{n-1}{3}\rceil-1.$$

Also $\operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/((I(\mathcal{E}_{n,t}):e_{n-1}) < \operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/((I(\mathcal{E}_{n,t}),e_{n-1})$. By using Lemma 2.3.1(b),

$$\operatorname{reg}(\Re[\mathsf{V}(\mathcal{E}_{n,t})]/(I(\mathcal{E}_{n,t}))=2\lceil\frac{n-1}{3}\rceil-1.$$

Case 2 When $n\equiv 1 \pmod 3$ We have the following isomorphism:

$$P/((I(\mathcal{E}_{n,t}):e_{n-1}):f_{n-2}))\cong \Re[\mathbb{V}(\Phi_{n-4,t})]/I(\Phi_{n-4,t})]\otimes_{\Re}\Re[e_{n-1},f_{n-2},C_n, C_{n-2},D_{n-1},D_n,C_{n-3},D_{n-3}], \quad (4.30)$$

$$P/((I(\mathcal{E}_{n,t}):e_{n-1}),f_{n-2})\cong \Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t})\otimes_{\Re}\Re[e_{n-1},C_n, C_{n-2},D_{n-1},D_{n-2}], \quad (4.31)$$

$$P/((I(\mathcal{E}_{n,t}),e_{n-1}),f_{n-1}) \;\cong\; \Re[\mathsf{V}(\Phi_{n-1,t})]/I(\Phi_{n-1,t})]\otimes_{\Re}\Re[C_{n-1},D_{n-1}], \quad (4.32)$$

$$P/((I(\mathcal{E}_{n,t}),e_{n-1}):f_{n-1}),e_n)\cong \Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t})]\otimes_{\Re}\Re[f_{n-1},C_n,D_n, D_{n-2},C_{n-1},C_{n-2}], \quad (4.33)$$

$$P/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}) : e_n) \cong \Re[\mathsf{V}(\Phi_{n-4,t})]/I(\Phi_{n-4,t}) \otimes_{\Re} \Re[f_{n-1}, e_n, C_{n-1}, C_1, D_1, D_n, C_{n-2}, D_{n-2}]. \quad (4.34)$$

By using induction on n and using Lemmas 3.2.2 and 3.1.4 on Eqs.(4.30) - (4.34), we have

$$\begin{aligned} \operatorname{reg}(P/((I(\mathcal{E}_{n,t}) : e_{n-1}) : f_{n-2}) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-4,t})]/I(\Phi_{n-4,t})) &= 2\lceil \frac{n-4}{3} \rceil \\ &= 2\lceil \frac{n-1}{3} \rceil - 2, \end{aligned}$$

since $n-4 \equiv 0 \pmod 3$. Furthermore

$$\begin{aligned} \operatorname{reg}(P/((I(\mathcal{E}_{n,t}) : e_n), f_{n-2})) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t})) &= 2\lceil \frac{n-3-1}{3} \rceil + 1 \\ &= 2\lceil \frac{n-1}{3} \rceil - 1, \end{aligned}$$

as we have $n-3 \equiv 1 \pmod 3$.

$$\operatorname{reg}(P/((I(\mathcal{E}_{n,t}), e_{n-1}), f_{n-1})) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-1,t})]/I(\Phi_{n-1,t})) = 2\lceil \frac{n-1}{3} \rceil.$$
$$\text{as } n-1 \equiv 0 \pmod 3.$$

And

$$\operatorname{reg}(P/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}), e_n) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-3,t})]/I(\Phi_{n-3,t})) = 2\lceil \frac{n-1}{3} \rceil - 1,$$

since $n-3 \equiv 1 \pmod 3$. Moreover

$$\operatorname{reg}(P/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}) : e_n) = \operatorname{reg}(\Re[\mathsf{V}(\Phi_{n-4,t})]/I(\Phi_{n-4,t})) = 2\lceil \frac{n-1}{3} \rceil - 2.$$

As $n-4 \equiv 0 \pmod 3$. As, $\operatorname{reg}(P/((I(\mathcal{E}_{n,t}) : e_{n-1}) : f_{n-2})) < \operatorname{reg}(P/((I(\mathcal{E}_{n,t}) : e_{n-1}), f_{n-2}))$, by applying Lemma 2.3.1(b), $\operatorname{reg}(P/(I(\mathcal{E}_{n,t}) : e_{n-1})) = 2\lceil \frac{n-1}{3} \rceil - 1$. Also, $\operatorname{reg}(P/(((I(\mathcal{E}_{n,t}), e_{n-1})) : f_{n-1}) : e_n) < \operatorname{reg}(P/(((I(\mathcal{E}_{n,t}), e_{n-1})) : f_{n-1}), e_n)$, by applying Lemma 2.3.1(b), $\operatorname{reg}(P/((I(\mathcal{E}_{n,t}), e_{n-1}) : f_{n-1}) = 2\lceil \frac{n-1}{3} \rceil - 1$. Similarly, $\operatorname{reg}(P/(((I(\mathcal{E}_{n,t}), e_{n-1})) : f_{n-1}) < \operatorname{reg}(P/(((I(\mathcal{E}_{n,t}), e_{n-1})) : f_{n-1})$, by applying Lemma 2.3.1(b), $\operatorname{reg}(P/((I(\mathsf{E}_{n,t}), e_{n-1}) = 2\lceil \frac{n-1}{3} \rceil$. Also we have,

$$\operatorname{reg}(P/(((I(\mathcal{E}_{n,t}) : e_{n-1})) < \operatorname{reg}(P/(((I(\mathcal{E}_{n,t}), e_{n-1})),$$

by applying Lemma 2.3.1(b), $\operatorname{reg}(P/((I(\mathcal{E}_{n,t}) = 2\lceil \frac{n-1}{3} \rceil$.

Case 3 :As $\mathcal{E}_{n,t} = \Phi_{n-1,t} \cup H$ where $H \cong \Phi_{2,t}$ and $\Phi_{n-1,t} \cap H \neq \emptyset$ by applying Lemma 2.3.4 and 3.2.2

$$\text{reg}(P/I(\mathcal{E}_{n,t}) \leq \text{reg}(\Re[\mathsf{V}(\Phi_{n-1,t})]/I(\Phi_{n-1,t})) + \text{reg}(\Re[\mathsf{V}(\Phi_{2,t})]/I(\Phi_{2,t})) = 2\lceil \frac{n-1}{3} \rceil + 2.$$

To obtain the second inequality, let $\mathcal{M} = \Big\{\{f_1, f_{1,1}\}, \{e_3, e_{3,1}\}, \{f_5, f_{5,1}\}, \{e_7, e_{7,1}\} \ldots,$ $\{f_{n-2}, f_{n-2,1}\}, \{e_n, e_{n,1}\}$ clearly $\mathcal{M}$ forms an induced matching. Therefore, $\text{indmat}(\mathcal{E}_{n,t}) \geq \lceil \frac{n}{2} \rceil$. By applying Lemma 2.3.4, $\text{reg}(P/I(\mathcal{E}_{n,t})) \geq \lceil \frac{n}{2} \rceil$.

□

Theorem 4.0.3. *Let $n \geq 3$, $t \geq 1$, $\mathcal{F}_{n,t}{=}Br_t[(C_{2n}(1, n-1, n))]$, and $P = \Re[\mathsf{V}(\mathcal{F}_{n,t})]$. Then*

$$\text{reg}\,(P/I(\mathcal{F}_{n,t})) = \lceil \frac{n-1}{2} \rceil.$$

Proof. We have the following isomorphism

$$P/((I(\mathcal{F}_{n,t}) : f_n) \cong \Re[\mathsf{V}(\Psi_{n-3,t})]/I(\Psi_{n-3,t}) \otimes_{\Re} \Re[f_n, D_{n-1}, D_1, C_n, C_{n-1}, C_1], \quad (4.35)$$

$$P/((I(\mathcal{F}_{n,t}), f_n), e_n) \cong \Re[\mathsf{V}(\Psi_{n-1,t})]/I(\Psi_{n-1,t}) \otimes_{\Re} \Re[D_n, C_n], \quad (4.36)$$

$$P/((I(\mathcal{F}_{n,t}), f_n) \; : \; e_n) \;\; \cong \;\; \Re[\mathsf{V}(\Psi_{n-3,t})]/I(\Psi_{n-3,t}) \;\; \otimes_{\Re} \; \Re[e_n, D_n, C_{n-1}, C_1], \quad (4.37)$$

Applying Lemma 3.2.3 and Lemma 3.1.4 on (4.35) - (4.37) we have:

$$\text{reg}\,(P/((I(\mathcal{F}_{n,t}) : f_n) = \text{reg}(\Re[\mathsf{V}(\Psi_{n-3,t})]/I(\Psi_{n-3,t})) = \lceil \frac{n-3}{2} \rceil,$$

$$\text{reg}\,(P/((I(\mathcal{F}_{n,t}), f_n), e_n) = \text{reg}(\Re[\mathsf{V}(\Psi_{n-1,t})]/I(\Psi_{n-1,t})) = \lceil \frac{n-1}{2} \rceil,$$

$$\text{reg}\,(P/((I(\mathcal{F}_{n,t}), f_n) : e_n) = \text{reg}(\Re[\mathsf{V}(\Psi_{n-3,t})]/I(\Psi_{n-3,t})) = \lceil \frac{n-3}{2} \rceil.$$

Since, $\text{reg}\,(P/(((I(\mathcal{F}_{n,t}), f_n)) : e_n) < \text{reg}\,(P/(((I(\mathcal{F}_{n,t}), e_n)) : f_n)$, by applying Lemma 2.3.1(b),

$$\text{reg}\,(P/((I(\mathcal{F}_{n,t}), f_n) = \lceil \frac{n-1}{2} \rceil.$$

Also, $\text{reg}\,(P/(((I(\mathcal{F}_{n,t}) : f_n)) < \text{reg}\,(P/(((I(\mathcal{F}_{n,t}), f_n)))$, by applying Lemma 2.3.1(b), $\text{reg}\,(P/((I(\mathcal{F}_{n,t}) = \lceil \frac{n-1}{2} \rceil$.

Theorem 4.0.4. *Let $n \geq 3$, $t \geq 1$, $\mathcal{D}_{n,t}$=$Br_t[(C_{2n}(1, n-1))]$, and $P = \Re[\mathsf{V}(\mathcal{D}_{n,t})]$. Then*

$$\operatorname{sdepth}(P/I(\mathcal{D}_{n,t})) \geq \operatorname{depth}(P/I(\mathcal{D}_{n,t})) = \begin{cases} n(q+1), & \text{if } n \text{ is odd;} \\ n(q+1)+q-1, & \text{if } n \text{ is even.} \end{cases}$$

Proof. Let $n = 3$, then we have the following short exact sequence,

$$0 \longrightarrow P/I(\mathcal{D}_{3,t} : e_3) \xrightarrow{\cdot e_3} P/I(\mathcal{D}_{3,t}) \longrightarrow P/I(\mathcal{D}_{3,t}, e_3) \longrightarrow 0.$$

And,

$$P/(I(\mathcal{D}_{3,t}) : e_3) \cong \Re[\mathbb{V}(S_t)]/I(S_t) \otimes_{\Re} \Re[e_3, C_2, C_1, D_2], \tag{4.38}$$

and,

$$P/(I(\mathcal{D}_{3,t}), e_3) \cong \Re[\mathsf{V}(\Upsilon_{2,t})]/I(\Upsilon_{2,t}) \otimes_{\Re} \Re[D_3], \tag{4.39}$$

By applyin Lemma 3.1.4 we have,

$$\operatorname{depth}(P/(I(\mathcal{D}_{3,t}) : e_3)) = \operatorname{depth}(\Re[\mathsf{V}(S_t)]/I(S_t)) + 1 + t + t + t + t, \tag{4.40}$$

$$\operatorname{depth}(P/(I(\mathcal{D}_{3,t}), e_3)) = \operatorname{depth}(\Re[\mathsf{V}(\Upsilon_{2,t})]/I(\Upsilon_{2,t})) + t + t, \tag{4.41}$$

Now by applying Lemma 3.1.1 and 3.2.4 on Eq. 4.40 and 4.41 we get

$$\operatorname{depth}(P/(I(\mathcal{D}_{3,t}) : e_3)) = 2 + 2t. \tag{4.42}$$

And

$$\operatorname{depth}(P/(I(\mathcal{D}_{3,t}), e_3)) = 2(1+t) + 2t. \tag{4.43}$$

by using 2.1.4 we get

$$\operatorname{depth}(P/(I(\mathcal{D}_{3,t})) \geq 2 + 2t. \tag{4.44}$$

Also $x_3 \notin I$ so By applying Lemma 2.1.2 we get

$$\operatorname{depth}(P/(I(\mathcal{D}_{3,t})) \leq \operatorname{depth}(P/(I(\mathcal{D}_{3,t} : e_3)) = 2 + 2t, \tag{4.45}$$

From Eqs. 4.44 and 4.45 we get

$$\operatorname{depth}(P/(\mathcal{D}_{3,t})) = 2 + 2t. \tag{4.46}$$

Suppose $n \geq 4$,

$$0 \longrightarrow P/I(\mathcal{D}_{n,t} : e_n) \xrightarrow{\cdot e_n} P/I(\mathcal{D}_{n,t}) \longrightarrow P/I(\mathcal{D}_{n,t}, e_n) \longrightarrow 0.$$

And

$$0 \longrightarrow P/I(L : f_n) \xrightarrow{\cdot f_n} P/I(L) \longrightarrow P/I(L, f_n) \longrightarrow 0.$$

Where

$$L := (I(\mathcal{D}_{n,t}), e_n) = (I(\mathcal{D}_{n-1,t}), e_n, e_{n,1}, ..., e_{n,t}),$$

then,

$$P/(I(\mathcal{D}_{n,t}) : e_n) \cong \Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t}) \otimes_{\Re} \Re[\mathbb{V}(S_t)]/I(S_t)) \otimes_{\Re} \Re[e_n, C_{n-1}, C_1, D_{n-1}, D_1], \quad (4.47)$$

$$P/(L : f_n) \cong \Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t}) \otimes_{\Re} \Re[f_n, D_{n-1}, D_1, C_{n-1}, C_1], \quad (4.48)$$

$$P/(L, f_n) \cong \Re[\mathsf{V}(\Upsilon_{n-1})]/I(\Upsilon_{n-1}) \otimes_{\Re} \Re[C_n, D_n]. \quad (4.49)$$

Now we have two cases:

Case 1 If n is odd, then by applying Lemma 2.1.6, and Lemma 3.1.4 on Eq. 4.47,

$$\text{depth}(P/(I(\mathcal{D}_{n,t}) : e_n)) = \text{depth}(\Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t}))+ \text{depth}(\Re[\mathbb{V}(S_t)]/I(S_t)) + 1 + 4t, \quad (4.50)$$

$$\text{depth}(P/(L : f_n)) = \text{depth}(\Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t})) + 1 + t + t + t + t + t, \quad (4.51)$$

$$\text{depth}(P/(L, f_n)) = \text{depth}(\Re[\mathsf{V}(\Upsilon_{n-1})]/I(\Upsilon_{n-1})) + t + t. \quad (4.52)$$

Now by induction and applying Lemma 3.1.1 and Lemma 3.2.4 on Eqs.(4.50) - (4.52) respectively,

$$\text{depth}(P/(I(\mathcal{D}_{n,t}) : e_n)) = n(1 + t) + t - 1, \quad (4.53)$$

$$\text{depth}(P/(L : f_n)) = n(1 + t) + 2t - 2, \quad (4.54)$$

$$\operatorname{depth}(P/(L,f_n)) = n(1+t)+t-1. \tag{4.55}$$

By applying Lemma 2.1.4 we have ,

$$\operatorname{depth}(P/(I(L)) \geq n(t+1)+t-1. \tag{4.56}$$

Also since $x_1 \notin L$ and depth $(P/(L:x_1)) = n(1+t)+t-1$,

so By applying Lemma 2.1.2 ,

$$\operatorname{depth}(P/(I(L)) \leq n(t+1)+t-1.$$

So

$$\operatorname{depth}(P/(I(L)) = n(t+1)+t-1. \tag{4.57}$$

Now by applying depth lemma on Eqs. 4.53 and 4.58 we get,

$$\operatorname{depth}(P/(I(\mathcal{D}_{n,t})) = n(t+1)+t-1.$$

Case 2 If n is even, applying Lemma 2.1.6 and 3.1.4 on Eqs. (4.47) - (4.49),

$$\begin{aligned}\operatorname{depth}(P/(I(\mathcal{D}_{n,t}):e_n)) = \operatorname{depth}(&\Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t})) \\ &+ \operatorname{depth}(\Re[\mathsf{V}(S_t)]/I(S_t)) + 1 + 4t + 1,\end{aligned} \tag{4.58}$$

$$\operatorname{depth}(P/(L:f_n)) = \operatorname{depth}(\Re[\mathsf{V}(\Upsilon_{n-3,t})]/I(\Upsilon_{n-3,t})) + 1 + 5t, \tag{4.59}$$

$$\operatorname{depth}(P/(L,f_n)) = \operatorname{depth}(\Re[\mathsf{V}(\Upsilon_{n-1})]/I(\Upsilon_{n-1})) + 2t, \tag{4.60}$$

By using Lemma 3.1.1 and Lemma 3.2.4 on Eqs. (4.58) - (4.60) we get,

$$\operatorname{depth}(P/(I(\mathcal{D}_{n,t}):e_n)) = n(1+t), \tag{4.61}$$

$$\operatorname{depth}(P/(L:f_n)) = n(1+t)+t-1, \tag{4.62}$$

$$\operatorname{depth}(P/(L,f_n)) = n(1+t). \tag{4.63}$$

By using Lemma 2.1.4,

$$\operatorname{depth}(P/I(L)) \geq n(t+1). \tag{4.64}$$

Also since $x_1 \notin L$ and depth $(P/(L : e_1)) = n(1 + t)$,

so By using Lemma 2.1.2,

$$\text{depth}(P/(I(L)) \leq n(t + 1).$$

So

$$\text{depth}(P/(I(L)) = n(t + 1). \tag{4.65}$$

By using the depth lemma on Eqs. 4.61 and 4.65,

$$\text{depth}(P/(I(\mathcal{D}_{n,t})) = n(t + 1).$$

For Stanley depth, we use lemma 2.2.2 instead of lemma 2.1.4.

□

Bibliography

[1] Beyarslan, S., Hà, H. T., Trung, T. N. (2015). Regularity of powers of forests and cycles. Journal of Algebraic Combinatorics, 42, 1077-1095.

[2] Bouchat, R. R. (2010). Free resolutions of some edge ideals of simple graphs. Journal of Commutative Algebra, 2(1), 1-35.

[3] Bruns, W., Herzog, H. J. (1998). Cohen-Macaulay rings. Cambridge University Press.

[4] Caviglia, G., Hà, H. T., Herzog, J., Kummini, M., Terai, N., Trung, N. V. (2019). Depth and regularity modulo a principal ideal. Journal of Algebraic Combinatorics, 49(1), 1-20.

[5] Cimpoeas, M., (2008). Stanley depth of monomial ideals in three variables. arXiv preprint arXiv:0807.2166.

[6] Cimpoeas, M. (2012). Several inequalities regarding Stanley depth. Romanian Journal of Math. and Computer Science, 2(1), 28-40.

[7] Dao, H., Huneke, C., Schweig, J. (2013). Bounds on the regularity and projective dimension of ideals associated to graphs. Journal of Algebraic Combinatorics, 38(1), 37-55.

[8] Din, N. U., Ishaq, M., Sajid, Z. (2021). Values and bounds for depth and Stanley depth of some classes of edge ideals. AIMS Mathematics, 6(8), 8544-8566.

[9] Duval, A. M., Goeckner, B., Klivans, C. J., Martin, J. L. (2016). A non-partitionable Cohen–Macaulay simplicial complex. Advances in Mathematics, 299, 381-395.

[10] Fakhari, S. S. (2017). On the Stanley depth of powers of edge ideals. Journal of Algebra, 489, 463-474.

[11] Fouli, L., Morey, S. (2015). A lower bound for depths of powers of edge ideals. Journal of Algebraic Combinatorics, 42, 829-848.

[12] Herzog, J., Vladoiu, M., Zheng, X. (2009). How to compute the Stanley depth of a monomial ideal. Journal of Algebra, 322(9), 3151-3169.

[13] Hoa, L. T., Tam, N. D. (2010). On some invariants of a mixed product of ideals. Archiv der Mathematik, 94(4), 327-337.

[14] Ichim, B., Katthän, L., Moyano-Fernández, J. J. (2017). How to compute the Stanley depth of a module. Mathematics of Computation, 86(303), 455-472.

[15] Ishaq, M., Qureshi, M. I. (2013). Upper and lower bounds for the Stanley depth of certain classes of monomial ideals and their residue class rings. Communications in Algebra, 41(3), 1107-1116.

[16] Ishaq, M. (2012). Upper bounds for the Stanley depth. Communications in Algebra, 40(1), 87-97.

[17] Katzman, M., 2006. Characteristic independence of Betti numbers of graph ideals. Journal of Combinatorial Theory, Series A, 113(3), 435-454.

[18] Kalai, G., Meshulam, R. (2006). Intersections of Leray complexes and regularity of monomial ideals. Journal of Combinatorial Theory, Series A, 113, 1586-1592.

[19] Morey, S. (2010). Depths of powers of the edge ideal of a tree. Communications in Algebra, 38(11), 4042-4055.

[20] Morey, S., Villarreal, R. H. (2012). Edge ideals: algebraic and combinatorial properties. Progress in commutative algebra, 1, 85-126.

[21] N. Terai, (1998). Alexander duality theorem and Stanley-Reisner rings. Free resolutions of coordinate rings of projective varieties and related topics (Japanese) (Kyoto 1998) Surikaisekikenkyusho Kokyuroku No. 1078 (1999), 174–184

[22] Okazaki, R., 2011. A lower bound of Stanley depth of monomial ideals. Journal of Commutative Algebra, 3(1), 83-88.

[23] Rauf, A. (2010). Depth and Stanley depth of multigraded modules. Communications in Algebra, 38(2), 773-784.

[24] Shaukat, B., Ishaq, M., Haq, A. U., Iqbal, Z. (2022). On some algebraic invariants and Cohen-Macaulay graphs. arXiv preprint arXiv:2211.05721.

[25] Stanley, R. P. (1982). Linear Diophantine equations and local cohomology. Inventiones mathematicae, 68(2), 175-193.

[26] Shaukat, B., Ishaq, M., Haq, A. U., Iqbal, Z. (2022). Algebraic properties of edge ideals of corona product of certain graphs. arXiv:2211.05721.

[27] Woodroofe, R., 2014. Matchings, coverings, and Castelnuovo Mumford regularity. Journal of Commutative Algebra, 6(2), 287-304.