Weighted Fractional Calculus with respect to Functions

By

Tazeen Zahra

Registration No: 00000402962

Department of Mathematics

School of Natural Sciences

National University of Sciences and Technology (NUST)

Islamabad, Pakistan

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Tazeen Zahra

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Supervisor: Prof. Mujeeb ur Rehman

Co Supervisor: Dr. Hafiz Muhammad Fahad

School of Natural Sciences

National University of Sciences and Technology (NUST)

Islamabad, Pakistan

(2024)

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Signature: ______________________

Name of Supervisor: Prof. Mujeeb ur Rehman

Date: 21-8-2024

Signature (HoD): ______________________

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1. Name: PROF. MATLOOB ANWAR Signature: ______________

2. Name: DR. AHMAD JAVID Signature: ______________

Supervisor's Name: PROF. MUJEEB UR REHMAN Signature: ______________

Co-Supervisor's Name: DR. HAFIZ M. FAHAD Signature: ______________

____
Head of Department

21-8-2024
Date

COUNTERSINGED

Date: 21.8.2024

____
Dean/Principal

DEDICATION

This work is dedicated to my supervisor, Dr. Mujeeb ur Rehman, my co-supervisor, Dr. Hafiz Muhammad Fahad, my beloved parents, Nusrat Ali and Ishrat Batool, and my supportive husband, Sarshad Hussain

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Contents

LIST OF SYMBOLS AND ABBREVIATIONS

FC	Fractional calculus
M–L	Mittag–Leffler
RL	Riemann–Liouville
CFD	Caputo fractional derivatives
WFC	Weighted fractional calculus
WFO	Weighted fractional operators
MVT	Mean value theorem
IVP	Initial value problem

Abstract

This thesis explores the domain of fractional calculus, with a particular focus on generalized fractional calculus and weighted fractional calculus with respect to functions.

The core research focuses on further exploring the existing theory of weighted fractional calculus with respect to functions. This theory seeks to prove important results for fractional differential equations of weighted Caputo fractional derivatives with respect to functions that have not yet been explored. By investigating the fundamental principles of these operators, we establish mean value theorems, Taylor's theorems, and integration by parts formulas. Our study extends to the Leibniz rule for weighted Riemann–Liouville derivatives with respect to functions. Additionally, we establish the existence and uniqueness theorems for a class of initial value problems involving weighted Caputo fractional derivatives with respect to functions in the Sobolev space.

Chapter 1

Introduction

Fractional calculus (FC) has grown to be a subject of considerable attention by scientists over the last few decades due to its deep theoretical base and its continuously expanding applicability [1, 2]. It generalizes classical calculus by introducing fractional order of differ-integration, allowing for a more comprehensive understanding of complex systems exhibiting non-integer order behaviour. FC is widely applicable in various fields such as pure and applied sciences, engineering, and biology [3–7].

Unlike classical calculus, FC offers many different possible ways of defining operators of non-integer order. Since its establishment in 1695, several definitions of fractional differintegral operators have been developed. Detailed studies in this field can be found in the works of well-known authors such as Samko et al. [8], Kiryakova [11], Diethelm [9], Podlubny [10], Kilbas et al. [12], Hilfer [6], and Tarasov [13].

Over time, definitions of fractional operators have evolved significantly, leading to numerous ways to generalize $\frac{d^n}{dx^n} f(x)$ for non-integer n, starting with two main types, the RL and the CFD. The RL derivative, introduced in the early 19th century, is historically the first and has a well-established mathematical foundation. However, it poses certain challenges when applied to real-world problems. To address these challenges, the Caputo derivatives was developed. Although it is related to the RL concept, it includes modifications which make it more suitable for some applied problems. More recent studies have further expanded the scope of fractional operators, including Hadamard, tempered, and weighted fractional calculus (WFC), to name just a few [14–16]. A broad class of fractional operators in this field is the class of FC with analytic kernels proposed by Fernandez et al. [17]. Recent advancements in the field included the introduction of general fractional derivatives with Sonine kernels by Luchko in 2021 [18], the development of operational calculus for general fractional derivatives of any order by Al-Kandari et al. [19] in 2022, the generalization of FC with Sonine kernels through conjugation relations by Al-Refai and Fernandez [20] in 2023, and the study of operational calculus for general conjugated fractional derivatives by Fernandez in the same year [21].

Furthermore, an important class usually called FC with respect to functions, represents a broad and significant area within the field of FC. The concept was first introduced by Liouville in 1835, who suggested fractional integration with respect to functions [22]. Later, in 1865, Holmgren formally proposed the notion of fractional integrals in this context [23]. Gradually, the idea of this generalized class in FC gained significant attention in several subsequent papers. Notably, Erdélyi made significant contributions in 1964 [24] and 1970 [25], followed by Talenti in 1965 [26]. In 1971, Chrysovergis examined some fundamental properties of integral operators of this generalized class [27]. Subsequently, many renowned researchers, such as Osler in 1970 [28, 29] and Samko et al. [8], further advanced the study of this topic. Recent work by Fahad et al. [30] focuses on algebraic expressions of conjugation relations between the original classical operators and their corresponding generalized versions, offering valuable perspectives for both theoretical and practical developments in FC.

WFC with respect to functions, is another important class of FC. This concept was initially addressed by Agrawal in 2012 [31, 32]. In his work, he introduced a type of operator called weighted (also reffered as scaled) FC with respect to functions. He also discussed how these operators can be applied in probabilistic modeling and variational calculus [32]. Recent studies have thoroughly examined the theoretical properties and corresponding fractional differential equations of these weighted fractional operators (WFO) [33–36].

Our work in this article focuses on WFC with respect to functions, extending the work by Fernandez and Fahad [16], and providing a foundation for future research in the area of WFC. Fahad and Fernandez examined conjugation relations, provided illustrative examples, and investigated Laplace transforms and convolutions in the context of a general class of WFC with respect to functions. This general class includes special cases such as Hadamard-type and tempered operators with respect to functions, which have also been extensively researched for their detailed properties [15, 37]. Motivated by the work of researchers such as Diethelm [38], who studied the mean value theorem (MVT) for fractional operators, Ricardo in 2017 [39], who examined the MVT and integration by parts formulae for CFD with respect to functions, and Osler's formulation of the Leibniz rule for RL fractional derivatives with respect to functions in 1970 [28], our contribution in this paper lies in generalizing these concepts within the framework of WFC with respect to functions, which has not been previously addressed in the literature. Furthermore, Trujillo et al. [40] derived the generalized Taylor's formula in the setting of RL fractional operators, and Mali et al. [37] recently discussed Taylor theorems for ψ–tempered fractional derivatives. Many researchers have extensively investigated Taylor's theorem, its generalizations, and their diverse applications [41–43]. Drawing inspiration from these contributions, we explore the generalized Taylor's formulae tailored for WFO with respect to functions.

In recent years, significant research has been done into the initial value problems (IVPs) of FDEs. Various fixed-point theorems have been utilized to derive interesting results related to the existence, uniqueness, and stability of IVPs of Caputo FDEs with respect to functions [37, 44, 45]. In contrast, our article focuses on establishing the existence and uniqueness of the IVP for weighted Caputo FDEs with respect to functions within the framework of Sobolev spaces. Inspired by the work of [45], our approach provides a distinct perspective in determining the existence and uniqueness solutions for such IVPs of weighted Caputo FDE without requiring the continuity of the function.

In this thesis, the content is organized as follows: Chapter 2 is related to the historical background of FC, providing essential preliminaries. This chapter reviews key milestones and contributions of mathematicians who have shaped the field. It also introduces essential concepts, including function spaces and foundational principles that are essential for understanding the definitions of fractional operators. In the Chapter 3, the focus shifts to FC with respect to functions along with their properties, offering a thorough understanding of the operator.

Chapter 4, the main chapter of this thesis, is divided into several sections. Section 4.1 introduces weighted FC, detailing its unique properties. Section 4.2 provides detailed information about WFC with respect to functions. Section 4.3 presents the MVT for WFO with respect to functions, including the MVT for generalized RL fractional operators using conjugation relations. Section 4.4 includes Taylor's theorem for weighted Caputo and RL fractional derivatives with respect to functions. Section 4.5 establishes the integration by parts formulae in the context of this operator. Section 4.6 covers the Leibniz rule formulae for RL fractional derivatives with respect to functions, weighted calculus, and weighted RL fractional derivatives with respect to functions using conjugation relations. Section 4.7 addresses the IVP of weighted Caputo FDEs with respect to functions, focusing on the existence and uniqueness within a Sobolev space framework. The final chapter provides a summary of the main results and concluding statements.

Chapter 2

Fractional calculus

This chapter focuses on the historical development of FC and provides detailed information about special functions, such as the gamma, beta, and Mittag–Leffler (M–L) function, along with their properties. Following this, we discuss the function spaces associated with FC, which help us understand the behaviour of fractional operators. We then define these operators and examine their key properties, such as linearity, the semigroup property, and composition relations.

2.1 Historical development of fractional calculus

FC originated in correspondence between Leibniz and L'Hospital in 1695. In a letter from 1695, L'Hospital asked Leibniz about the meaning of a non-integer order derivative. Leibniz responded, describing it as an apparent paradox that might yield valuable results someday. This exchange sparked interest among many renowned mathematicians, including Lacroix, Laplace, Euler, Fourier, Abel, Riemann, and Liouville, all of whom contributed to the development of FC [46]

In 1819, Lacroix developed the formula for the n^{th} order derivatives. He started with the function $\mathrm{u}(x) = x^k$ where k is a positive integer, and derived the k^{th} derivative [46].

$$\mathcal{D}^n \mathrm{u}(x) = \frac{k! x^{k-n}}{(k-n)!}, \quad n \leq k,$$

$$\mathcal{D}^n \mathrm{u}(x) = \frac{\Gamma(k+1) x^{k-n}}{\Gamma(k-n+1)}.$$

where, $\mathcal{D} = \frac{d}{dx}$. Since the above expression can be defined for non-integer numbers by replacing integers k and m by σ and ς respectively.

$$\mathcal{D}^{\sigma}\mathrm{u}(x) = \frac{\Gamma(\varsigma+1)x^{\varsigma-\sigma}}{\Gamma(\varsigma-\sigma+1)}.$$

By setting $n = 1$ and $\sigma = \frac{1}{2}$, Lacroix obtained

$$\mathcal{D}^{\frac{1}{2}}x = \frac{\Gamma(2)}{\Gamma\left(\frac{3}{2}\right)}x^{1/2} = 2\sqrt{\frac{x}{\pi}}.$$

In 1823, Abel used FC to solve the tautochrone problem, demonstrating the practical utility of these concepts. Later, Liouville made significant advances by defining fractional derivatives for exponential and power functions, as shown in his two formulas [46].

$$\mathcal{D}^{\sigma}e^{ax} = a^{\sigma}e^{ax},$$
$$\mathcal{D}^{\sigma}x^{-a} = \frac{(-1)^{\sigma}\Gamma(a+\sigma)}{\Gamma(a)x^{a+\sigma}}.$$

The RL fractional integral, developed in the late 19th century, provided a unifying definition that encompassed earlier work by Lacroix and Liouville. This definition remains one of the most widely used in FC today.

Throughout the 19th and 20th centuries, FC expanded rapidly, with contributions from mathematicians such as Grunwald, Letnikov, and Riesz. For detailed history and additional references, see [8, 46, 47].

Although FC was just recently formalized as a mathematical field, its foundational ideas date back thousands of years. FC concepts are somewhat similar to Archimedes' work on the geometric methods he used to calculate areas and volumes. Significant progress was achieved in the study of non-integer derivatives in the seventeenth century. Fractional order derivatives were studied by the Swiss mathematician Johann Bernoulli, mainly in relation to infinite series and differential equations [47].

Gottfried Wilhelm Leibniz, a German mathematician, and philosopher best recognized for his contributions to the invention of calculus, attempted in the beginning to expand the calculus framework to encompass non-integer orders. The concept of "transcendental calculus" proposed by Leibniz suggested that fractional derivatives may be handled, although his work was still mostly theoretical [48].

In the mid-19th century, mathematicians such as Joseph Liouville and Bernhard Riemann played significant roles in formalizing FC. Riemann introduced the RL fractional integral operator, which provided a rigorous foundation for fractional integration. Building on Riemann's work, Liouville defined fractional derivatives, laying the groundwork for the field now known as FC [47].

The late 20th century saw a rise in interest in FC due to developments in mathematical analysis and its applications in various scientific fields. Scientists realized how useful FC is for simulating intricate phenomena with memory, non-locality, and aberrant behaviour [49]. Applications for FC have emerged in several disciplines in recent decades, including signal processing, physics, engineering, biology, and finance [3, 5–7]. Theoretical advances, computational methods, and real-world applications of FC in complicated problem-solving are still being investigated in ongoing research [49].

2.1.1 Distinction from integer-order calculus

FC distinguishes itself from classical integer-order calculus by extending the concepts of differintegration to fractional order. While integer-order calculus deals with derivatives and integrals of integer orders FC generalizes these operations to include arbitrary fractional orders. This extension allows FC to capture complex phenomena that exhibit non-local behaviour, which is not adequately described by classical calculus. In integer-order calculus, derivatives represent rates of change or slopes of functions, while integrals correspond to accumulation or area under the curves. In contrast, fractional differintegrals introduce non-local effects, where the value of a function at a certain point depends not only on its local behavior but also on its history over a specific interval. This distinction enables FC to provide more accurate models for systems with long-range interactions, anomalous diffusion, and fractional dynamics, making it a valuable tool in various scientific and engineering applications.

2.2 Special functions

Certain mathematical functions are assigned unique notations and names because of their significant roles in mathematical analysis, physics, and other fields. In this context, we will discuss some special functions relevant to our work, such as the gamma, beta function, error, and M–L Function.

2.2.1 Gamma function

Swiss mathematician Leonhard Euler developed the gamma function in the 18th century to extend the concept of factorials to include non-integers. This mathematical extension allows for the calculation of factorials for any positive real number, rather than just whole numbers.

The factorial function extended to complex numbers is denoted by the Gamma function, denoted by $\Gamma(\zeta)$ and defined by [9]

$$\Gamma(\zeta) = \int_0^\infty s^{\zeta-1} e^{-s}\, ds \qquad \zeta > 0, \tag{2.1}$$

where Re(ζ) denotes the real part of ζ, which is called Euler's integral of the second kind and Euler's Gamma function. The Gamma function has several important properties and applications in various fields of mathematics, including complex analysis, number theory, and probability theory.

2.2.1.1 Properties of gamma function

The following are some of the properties of gamma function [9]

(i) If $\zeta > 0$ then, $\Gamma(\zeta + 1) = \zeta\Gamma(\zeta)$, known as functional equation for gamma function.

Proof. Putting $\zeta = \zeta + 1$ in the definition of Gamma function (2.1), we have

$$\Gamma(\zeta + 1) = \int_0^\infty s^{\zeta} e^{-s}\, ds. \tag{2.2}$$

Integrating by parts, we get

$$\Gamma(\zeta + 1) = -s^{\zeta} e^{-s}\Big|_{s=0}^{s=\infty} + \int_0^\infty \zeta s^{\zeta-1} e^{-x}\, dx. \tag{2.3}$$

The first term on the right side of (2.3) now goes to 0 in the case where $\zeta > 0$. The required result was obtained by applying the definition of gamma function. However, the first term on the right side of (2.3) is undefined in the case where $\zeta < 0$. Therefore, the integral formula doesn't work for a negative ζ. To evaluate the gamma function for $\zeta < 0$ aside from the non-positive integers, we thus use the mathematical analytical approach known as analytic continuation. □

(ii) The function $\Gamma(\zeta)$ can be extended analytically to be meromorphic across the entire complex plane, exhibiting simple poles at the non-positive integers. The residues at these poles are given by $(\Gamma, -n) = \frac{(-1)^n}{n!}$.

(iii) For positive integers ζ, the gamma function satisfies $\Gamma(\zeta) = (\zeta - 1)!$.

Proof. The proof uses the mathematical induction technique. First, in the base case when $\zeta = 1$, we get that $\Gamma(1) = 0! = 1$, as it satisfied by property (i). Before moving on to the induction stage, we make use of the property (i) as well as the assumption that comes from the induction process.

$$\Gamma(\zeta + 1) = \zeta\Gamma(\zeta) = \zeta(\zeta - 1)! = \zeta!.$$

The desired outcome is reached by following this process.

□

(iv) $\Gamma(1) = 1$.

Proof. This property is straightforward. By substituting $\zeta = 1$ into the property (iii), we directly obtain this result. □

(v) $\Gamma(\zeta) = \lim\limits_{n\to\infty} \frac{n!\,n^\zeta}{\zeta(\zeta+1)(\zeta+2)\cdots(\zeta+n)}$, which is called Guass's product formula.

2.2.1.2 Extension domain of gamma function

The Gamma function can be extended to a larger domain using the following property:

$$\begin{aligned} \zeta\Gamma(\zeta) &= \Gamma(\zeta + 1) \\ \Gamma(\zeta) &= \frac{\Gamma(\zeta + 1)}{\zeta}. \end{aligned} \tag{2.4}$$

This allows us to define the Gamma function for $\zeta \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}$, where it has simple poles. Furthermore, by recursively applying Equation (2.4), we obtain

$$\Gamma(\zeta) = \frac{\Gamma(\zeta + n)}{\zeta(\zeta + 1)(\zeta + 2)\cdots(\zeta + n - 1)} \quad \text{for} \quad \zeta \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}, \tag{2.5}$$

where n is a positive integer. Thus the gamma function is extended for all real numbers except $\zeta = 0, -1, -2, \ldots$

2.2.2 Beta function

The beta function, denoted as $B(\zeta, \eta)$, is a special function in mathematics that is closely associated with the gamma function. It was first studied by Leonhard Euler, a Swiss mathematician, and Adrien–Marie Legendre, a French mathematician. The beta function as defined [9], involves the integrating the product of two power functions over the interval $[0, 1]$ as

$$\int_0^1 x^{\zeta-1}(1-x)^{\eta-1}dx = \frac{\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\zeta+\eta)},$$

where $\zeta, \eta \in \mathbb{R}^+$. The relation between beta and gamma function can be written as

$$B(\zeta, \eta) = \frac{\Gamma(\zeta)\Gamma(\eta)}{\Gamma(\zeta+\eta)}.$$

The integral in the first equation is known as Euler's integral of the first kind or Euler's Beta function $B(\zeta, \eta)$.

2.2.3 Mittag–Leffler function

Extensive research has focused on the M-L function, which is named after the Swedish mathematician Gosta Magnus Mittag-Leffler, and has been generalized in various ways. Wiman [50] was the first to introduce a generalization of this function, and it was later rediscovered and thoroughly analyzed by Humbert and Agarwal in 1953 [51].

Dzherbashian [52] introduced integral representations for the two-parametric M–L function and established formulas for its asymptotic behaviour at infinity. These formulas were then employed to develop Fourier-type integrals and establish theorems regarding the pointwise convergence of these integrals. The details of the M–L function can be found in [52] and the references included there.

One parameter Mittag–Leffler function. It is a special function, known as the one–parameter M–L function, which is defined by the series

$$E_\zeta(s) = \sum_{\jmath=0}^{\infty} \frac{s^\jmath}{\Gamma(\zeta\jmath+1)}, \tag{2.6}$$

where $\zeta > 0$, $z \in \mathbb{C}$

Gosta Mittag–Leffler's work on this function has led to various generalizations and applications, particularly in the fields of mathematical analysis, integral equations, and FC.

The M–L function satisfies specific relations for special cases [52], as outlined below:

$$E_1(\pm s) = \sum_{\jmath=0}^{\infty} (\pm 1)^{\jmath} \frac{s^{\jmath}}{\Gamma(\jmath+1)} = e^{\pm s},$$

$$E_2(-s^2) = \sum_{\jmath=0}^{\infty} (-1)^{\jmath} \frac{s^{2\jmath}}{\Gamma(2\jmath+1)} = \cos(s),$$

$$E_2(s^2) = \sum_{\jmath=0}^{\infty} \frac{s^{2j}}{\Gamma(2\jmath+1)} = \cosh(s),$$

$$E_{\frac{1}{2}}(\pm s^{1/2}) = \sum_{\jmath=0}^{\infty} (\pm 1)^{\jmath} \frac{s^{\jmath/2}}{\Gamma(\frac{1}{2}\jmath+1)} = e^{s}\sqrt{\pi}(1 \pm \mathrm{erf}(\pm s^{1/2})) = e^{s}\mathrm{erfc}(\mp s^{1/2}).$$

In this context, the symbols erf(s) reffered as error function and erfc(s) stand for the complementary error function.

Two parameter Mittag–Leffler function The two-parameter M–L function is a generalization of the classical M–L function 2.6, defined as

$$E_{\zeta,\eta}(s) = \sum_{\jmath=0}^{\infty} \frac{s^{\jmath}}{\Gamma(\zeta\jmath+\eta)} \quad (\zeta > 0,\ \eta \in \mathbb{C}). \tag{2.7}$$

And, 2.6 is a special case of 2.7, where $\eta = 1$, denoted as $E_{\zeta,1}(s) = E_{\zeta}(s)$.

2.3 Function spaces

Certain function spaces are introduced here, which will be utilized in the sequal.

Definition 2.3.1. *[8] Let* $\xi = [c, d]$*, and* $r \geq 1$*. The Lebesgue space denoted by* L^r*, is the set of all measurable function* z*, such that,* $\{\mathrm{z} : \|\mathrm{z}\|_r < \infty\}$*, where*

$$\|\mathrm{z}\|_{L^r(\xi)} = \left(\int_c^d |\mathrm{z}(\tau)|^r \, d\tau \right)^{\frac{1}{r}},$$

If $r = \infty$ *then, we have*

$$\|\mathrm{z}\|_{L^\infty(\xi)} = \underset{\tau\in\xi}{esssup}|\mathrm{z}(\tau)|$$

where esssup is the essential supremum of a function z.

Definition 2.3.2. *The set of functions represented by* $AC^n[c,d]$ *contains functions which posses derivatives up to order* $(n-1)^{th}$ *on the interval* $[a,b]$ *and are absolutely continuous. This means that for a function* u*, there is a corresponding function* z *in* $L^1[c,d]$ *such that the following holds*

$$\mathrm{u}^{(n-1)}(x) = \mathrm{u}^{(n-1)}(c) + \int_c^x \mathrm{z}(\tau)\, d\tau,$$

where, $\mathrm{z} = \mathrm{u}^{(n)}$.

Russian mathematician Sergei Lvovich Sobolev defined the Sobolev spaces, denoted by $W^{m,r}(c,d)$, in the 1930s. These spaces consist of functions whose generalized derivatives up to the m^{th} order belong to the $L^r(c,d)$ space and whose partial derivatives satisfy specific integrability conditions.

Definition 2.3.3. *[65–67] Let* $1 \leq r \leq \infty$*,* m *be an arbitrary non-negative integer and* (c,d) *be an open set in* $\mathbb{R}^n$ *. Then, we have the following definition of Sobolev space*

$$W^{m,r}(c,d) = \left\{\mathrm{z} \in L^r(c,d) : \mathcal{D}^\sigma \mathrm{z} \in L^r(c,d), 0 \leq |\sigma| \leq m\right\}.$$

2.4 Fundamentals of fractional calculus

This section will discuss the fundamentals of FC, including the definition of fractional operators and their properties.

2.4.1 RL fractional integrals and differential operators

The RL fractional integral operators extend the classical Riemann integrals to the case of non-integer order. It is defined on the space of Lebesgue integrable functions $L^1[c,d]$.

Definition 2.4.1. *[8, 9] If* $\sigma > 0$*,* $c \leq x \leq d$*, and* $\mathrm{u} \in L^1[c,d]$*. Then, then left and right RL fractional integral operators are defined as*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^\sigma_x \mathrm{u}(x) &= \frac{1}{\Gamma(\sigma)} \int_c^x \mathrm{u}(s)(x-s)^{\sigma-1}\, ds, \quad x > c, \\
{}^{RL}_{x}\mathcal{I}^\sigma_{d^-} \mathrm{u}(x) &= \frac{1}{\Gamma(\sigma)} \int_x^d \mathrm{u}(s)(s-x)^{\sigma-1}\, ds, \quad x < d,
\end{aligned}
$$

respectively. When the order σ *of the integral operator is set to zero, then it corresponds to the identity operator as* ${}^{RL}_{c^+}\mathcal{I}^0_x = {}^{RL}_{x}\mathcal{I}^0_{d^-} = I$.

These operators extend the concept of integration to fractional orders, providing a powerful tool for various applications in mathematical analysis and applied fields. Next, we define RL fractional differential operators, which extend integer-order differentiation to noninteger (fractional) orders in FC.

Definition 2.4.2. *[8, 9] The left and right RL fractional differential operators of order* $\sigma > 0$ *of a function* $\mathrm{u} \in AC^n[c, d]$*, can be expressed as*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}\mathrm{u}(x) &= \frac{1}{\Gamma(n-\sigma)} (\mathcal{D})^n \int_c^x \frac{\mathrm{u}(s)}{(x-s)^{\sigma-n+1}}\, ds, \\
{}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) &= \frac{1}{\Gamma(n-\sigma)} (-\mathcal{D})^n \int_x^d \frac{\mathrm{u}(s)}{(s-x)^{\sigma-n+1}}\, ds,
\end{aligned}
$$

respectively, where $\mathcal{D}^n = \frac{d}{dx}$ *and* $n = \lfloor \sigma \rfloor + 1$.

Notice that the RL fractional derivatives defined in Definition 2.4.2 can be seen as an analytic continuation of the RL fractional integrals outlined in Definition 2.4.1. This means that the RL fractional derivatives in Definition 2.4.2 can be considered an extension of the RL fractional integrals in Definition 2.4.1 through analytic continuation as follows

$$
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}\mathrm{u}(x) = {}^{RL}_{c^+}\mathcal{I}^{-\sigma}_{x}\mathrm{u}(x), \quad \sigma > 0, \quad x > c, \tag{2.8}
$$

$$
{}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) = {}^{RL}_{x}\mathcal{I}^{-\sigma}_{d^-}\mathrm{u}(x), \quad \sigma > 0, \quad x < d. \tag{2.9}
$$

Using analytic continuation principles, we can extend the findings from fractional integrals to include fractional derivatives.

Example 2.4.1. [8, 9] If κ^{μ} and γ^{μ} be power functions, where $\kappa = (x - c)$, $\gamma = (d - x)$, $\alpha > -1$ and $\sigma > 0$. Then, we have the following power rule formulae

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}\kappa^{\mu} &= \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\sigma)}\kappa^{\sigma+\mu}, \\
{}^{RL}_{x}\mathcal{I}^{\sigma}_{d^-}\gamma^{\mu} &= \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\sigma)}\gamma^{\sigma+\mu}.
\end{aligned} \tag{2.10}
$$

Proof. Using the definition of RL integrals, we obtain

$$
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}\kappa^{\mu} = \frac{1}{\Gamma(\sigma)} \int_c^x \frac{\kappa^{\mu}}{(x-s)^{1-\sigma}}\, ds,
$$

substitute $s = c + t(\kappa) \implies ds = dt(\kappa)$, where $t \to 1$ as $s \to x$ and $t \to 0$ as $s \to c$, then we have

$$\begin{aligned} {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}\kappa^{\mu} &= \frac{1}{\Gamma(\sigma)}\int_0^1 \frac{(t\kappa)^{\mu}(\kappa)}{((\kappa)(1-t))^{1-\sigma}}\,dt \\ &= \frac{1}{\Gamma(\sigma)}\int_0^1 \frac{\kappa^{\sigma+\mu}t^{\mu}}{(1-t)^{1-\sigma}}\,dt \\ &= \frac{\kappa^{\sigma+\mu}}{\Gamma(\sigma)}\int_0^1 \frac{t^{\mu}}{(1-t)^{1-\sigma}}\,dt. \end{aligned}$$

By the definition of the beta function, we obtain

$$= \frac{\kappa^{\sigma+\mu}}{\Gamma(\sigma)}\mathrm{B}(\mu+1,\sigma) = \frac{\Gamma(\mu+1)\kappa^{\sigma+\mu}}{\Gamma(\mu+\sigma+1)}.$$

Similarly, we can do this for the second identity and by using the fact analytic continuation, we can obtain the result for the derivatives of RL as follows

$${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}\kappa^{\mu} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\sigma+1)}\kappa^{\mu-\sigma}, \quad \sigma > 0,\ \mu > -1.$$

□

Using the power rule, we can easily evaluate the integral and differentials of constant functions and other simple power functions.

Example 2.4.2. Consider σ=$\frac{3}{2}$ and $\mu = 0$ in Example. (2.4.1), we get

$${}^{RL}_{c^+}\mathcal{I}^{\frac{3}{2}}_{x}x^0 = \frac{1}{\Gamma(\frac{3}{2}+1)}x^{\frac{3}{2}} = \frac{x^{\frac{3}{2}}}{\Gamma(\frac{5}{2})} = \frac{4}{3\sqrt{\pi}}x^{\frac{3}{2}}.$$

Theorem 2.4.3. *[8] If* $\mathrm{u} \in L^r(c,d)$, $\mathrm{v} \in L^q(c,d)$, $\frac{1}{r} + \frac{1}{q} \leq 1 + \sigma$ *with* $r \neq 1$, $q \neq 1$ *in case of* $\frac{1}{r} + \frac{1}{q} = \sigma + 1$. *Then the following formulae holds*

$$\begin{aligned} \int_c^d \mathrm{u}(x){}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}(\mathrm{v})\,dx &= \int_c^d \mathrm{v}(x){}^{RL}_{x}\mathcal{I}^{\sigma}_{d^-}(\mathrm{u})\,dx, \\ \int_c^d \mathrm{u}(x){}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}(\mathrm{v})\,dx &= \int_c^d \mathrm{v}(x){}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-}(\mathrm{u})\,dx. \end{aligned} \tag{2.11}$$

To ensure that functions u, v *satisfies (2.11), a simple sufficient condition is that both functions are the elements of space* $C[c,d]$. *Furthermore it required that the fractional derivatives* ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}$ *and* ${}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-}$ *exists for every* $x \in [c,d]$.

2.4.2 Properties of fractional integrals and derivatives

This subsection discusses some fundamental properties of fractional differintegrals, including semigroup properties and composition relations.

Lemma 2.4.4. *[8, 9, 12] If* σ_1, $\sigma_2 > 0$ *and* $\mathrm{u} \in L^r[c,d]$. *Then, the following properties hold almost everywhere on* $x \in [c,d]$

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma_1}_{x}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma_2}_{x}\mathrm{u}(x) &= {}^{RL}_{c^+}\mathcal{I}^{\sigma_1+\sigma_2}_{x}\mathrm{u}(x),\\
{}^{RL}_{x}\mathcal{I}^{\sigma_1}_{d^-}\,{}^{RL}_{x}\mathcal{I}^{\sigma_2}_{d^-}\mathrm{u}(x) &= {}^{RL}_{x}\mathcal{I}^{\sigma_1+\sigma_2}_{d^-}\mathrm{u}(x).
\end{aligned}
$$

Additionally, if $\mathrm{u} \in C[c,d]$, *then the properties hold to the entire interval* $[c,d]$.

Theorem 2.4.5. *[8, 9, 12] Assume that two functions* u_1 *and* u_2 *on interval* $[c,d]$, *with RL fractional integrals and derivatives existing almost everywhere, and constant* ζ_1, $\zeta_2 \in \mathbb{R}$. *Then* ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}(\zeta_1\mathrm{u}_1 + \zeta_2\mathrm{u}_2)$, ${}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-}(\zeta_1\mathrm{u}_1 + \zeta_2\mathrm{u}_2)$ *exists and*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}(\zeta_1\mathrm{u}_1 + \zeta_2\mathrm{u}_2) &= \zeta_1{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}\mathrm{u}_1 + \zeta_2{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}\mathrm{u}_2,\\
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}(\zeta_1\mathrm{u}_1 + \zeta_2\mathrm{u}_2) &= \zeta_1{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}\mathrm{u}_1 + \zeta_2{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}\mathrm{u}_2,
\end{aligned}
$$

which shows the linearity of fractional operators.

The next results describe the compositions of RL fractional integral operators with the RL fractional differential operators. The RL fractional derivatives are the left inverse of RL fractional integrals, but the converse is not true.

Lemma 2.4.6. *[8, 9, 12] If* $\sigma > 0$, *then for the function* $\mathrm{u} \in L^r[c,d]$, *the following relation holds*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}\mathrm{u}(x) &= \mathrm{u}(x),\\
{}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-}\,{}^{RL}_{x}\mathcal{I}^{\sigma}_{d^-}\mathrm{u}(x) &= \mathrm{u}(x),
\end{aligned}
$$

almost everywhere on $[c,d]$.

Lemma 2.4.7. *[12] If* $\sigma_1 > \sigma_2 > 0$, *then for* $1 \le r < \infty$ *the function* $\mathrm{u}(x) \in L^r[c,d]$, *the following relations hold*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{D}^{\sigma_1}_{x}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma_2}_{x}\mathrm{u}(x) &= {}^{RL}_{c^+}\mathcal{I}^{\sigma_2-\sigma_1}_{x}\mathrm{u}(x),\\
{}^{RL}_{x}\mathcal{D}^{\sigma_1}_{d^-}\,{}^{RL}_{x}\mathcal{I}^{\sigma_2}_{d^-}\mathrm{u}(x) &= {}^{RL}_{x}\mathcal{I}^{\sigma_2-\sigma_1}_{d^-}\mathrm{u}(x).
\end{aligned}
$$

almost everywhere on $[c,d]$. *Additionally, if* $\sigma_1 = n \in \mathbb{N}$, *where* $\mathcal{D} = \frac{d}{dx}$, $\sigma_2 > n$, *then the following relations holds*

$$\mathcal{D}^n \, {}^{RL}_{c^+}\mathcal{I}^{\sigma_2}_{x}\mathrm{u}(x) = {}^{RL}_{c^+}\mathcal{I}^{\sigma_2-n}_{x}\mathrm{u}(x),$$
$$\mathcal{D}^n \, {}^{RL}_{x}\mathcal{I}^{\sigma_2}_{d^-}\mathrm{u}(x) = (-1)^n {}^{RL}_{x}\mathcal{I}^{\sigma_2-n}_{d^-}\mathrm{u}(x).$$

Lemma 2.4.8. *[12] Assume that* $\sigma > 0$, $n \in \mathbb{N}$, *and the fractional derivatives* ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}$, ${}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-}$, ${}^{RL}_{c^+}\mathcal{D}^{n+\sigma}_{x}$ *and* ${}^{RL}_{x}\mathcal{D}^{n+\sigma}_{d^-}$ *exists. Then*

$$\mathcal{D}^n \, {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}\mathrm{u}(x) = {}^{RL}_{c^+}\mathcal{D}^{\sigma-n}_{x}\mathrm{u}(x),$$
$$\mathcal{D}^n \, {}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) = (-1)^n {}^{RL}_{x}\mathcal{D}^{\sigma-n}_{d^-}\mathrm{u}(x).$$

Proposition 2.4.9. *[8, 9, 12] If* $\mathrm{u} \in AC^n[c,d]$, *where* $n = \lfloor \sigma \rfloor + 1$. *The following composition relation holds true for order* $\sigma > 0$

$${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x} \, {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}\mathrm{u}(x) = \mathrm{u}(x) - \sum_{\jmath=0}^{n-1} \frac{(x-c)^{\sigma-\jmath}}{\Gamma(\sigma-\jmath+1)} \cdot \lim_{x\to c^+} \mathcal{D}^{\sigma-\jmath}\mathrm{u}(x),$$
$${}^{RL}_{x}\mathcal{I}^{\sigma}_{d^-} \, {}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) = \mathrm{u}(x) - \sum_{\jmath=0}^{n-1} \frac{(d^- - x)^{\sigma-\jmath}}{\Gamma(\sigma-\jmath+1)} \cdot \lim_{x\to d^-} \mathcal{D}^{\sigma-\jmath}\mathrm{u}(x),$$

for $0 < \sigma < 1$,

$${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x} \, {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}\mathrm{u}(x) = \mathrm{u}(x) - \frac{(x-c)^{\sigma-1}}{\Gamma(\sigma+1)} \lim_{x\to c^+} \mathcal{D}^{\sigma-1}\mathrm{u}(x),$$
$${}^{RL}_{x}\mathcal{I}^{\sigma}_{d^-} \, {}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) = \mathrm{u}(x) - \frac{(d^- - x)^{\sigma-1}}{\Gamma(\sigma+1)} \lim_{x\to d^-} \mathcal{D}^{\sigma-1}\mathrm{u}(x).$$

2.4.3 Caputo fractional derivatives and their properties

The CFD was studied by Michele Caputo in 1967, while investigatingn a solution to boundary value problems in viscoelasticity theory [74]. It has a primary advantage in that its initial and boundary conditions are similar to those of integer-order differential equations, making for easier interpretation. It is widely used for solving fractional differential equations in practical applications due to its ability to simplify the modelling process.

Definition 2.4.10. *[9] The left and right CFD of order $\sigma > 0$ of a function $\mathrm{u} \in AC^n$, can be defined as*

$$
\begin{aligned}
{}_{c^+}^{C}\mathcal{D}_x^{\sigma}\mathrm{u}(x) &= \frac{1}{\Gamma(n-\sigma)}\int_c^x \frac{\mathcal{D}^n\mathrm{u}(s)}{(x-s)^{\sigma-n+1}}\,ds, \quad x > c,\\
{}_{x}^{C}\mathcal{D}_{d^-}^{\sigma}\mathrm{u}(x) &= \frac{(-1)^n}{\Gamma(n-\sigma)}\int_x^d \frac{\mathcal{D}^n\mathrm{u}(s)}{(s-x)^{\sigma-n+1}}\,ds, \quad x < d,
\end{aligned}
$$

where $n = \lfloor\sigma\rfloor + 1$, and $\mathcal{D} = \frac{d^n}{dx^n}$.

Example 2.4.3. [9] If κ^μ is the power function, where $\kappa = x - c^+$. Then, for $\mu \geq 0$, $n = \lfloor\sigma\rfloor + 1$, we have the following power rule

$$
{}_{c^+}^{C}\mathcal{D}_x^{\sigma}(\kappa)^{\mu} = \begin{cases} 0, & \text{if } \mu \in \{0,1,2,\dots,n-1\}, \\ \frac{\Gamma(\mu+1)}{\Gamma(\mu-\sigma+1)}(\kappa)^{\mu-\sigma}, & \text{if } \mu \in \mathbb{N} \text{ and } \mu \geq n \text{ or } \mu \notin \mathbb{N} \text{ and } \mu > n-1. \end{cases}
$$

By the Definition 2.4.10, it is clear that $\mathcal{D}^n(\kappa(x))^\mu = 0$, if $\alpha = 0,1,2,\dots,n-1$. Thus ${}_{c^+}^{C}\mathcal{D}_x^{\sigma} = 0$. Similarly, when $\mu \in \mathbb{N}$ and $\mu \geq n$ or $\mu \notin \mathbb{N}$ and $\mu > n-1$, we have

$$
\mathcal{D}^n(\kappa(x))^\mu = \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}(\kappa)^{\mu-n}. \tag{2.12}
$$

Substituiting (2.12) in Definition 2.4.10, we get

$$
\begin{aligned}
{}_{c^+}^{C}\mathcal{D}_x^{\sigma}(\kappa)^{\mu} &= \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\left(\frac{1}{\Gamma(n-\sigma)}\int_c^x \frac{(\kappa)^{\alpha-n}}{(x-s)^{\mu-n+1}}\,ds\right)\\
&= \frac{\Gamma(\mu+1)}{\Gamma(\mu-n+1)}\,{}_{c^+}^{RL}\mathcal{I}_x^{n-\sigma}(\kappa)^{\mu-n} = \frac{\Gamma(\mu+1)}{\Gamma(\mu-\sigma+1)}(\kappa)^{\mu-\sigma}, \quad \text{by (2.10)}.
\end{aligned}
$$

Next, we will present some basic properties of CFD and the composition relation of RL integrals and Caputo derivatives.

The CFD is also a left inverse of the RL integral but not the right inverse of the RL integral.

Lemma 2.4.11. *[9, 12] If $\sigma > 0$ and function u is continuous. Then, we have*

$$
\begin{aligned}
{}_{c^+}^{C}\mathcal{D}_x^{\sigma}\,{}_{c^+}^{RL}\mathcal{I}_x^{\sigma}\mathrm{u}(x) &= \mathrm{u}(x),\\
{}_{x}^{C}\mathcal{D}_{d^-}^{\sigma}\,{}_{x}^{RL}\mathcal{I}_{d^-}^{\sigma}\mathrm{u}(x) &= \mathrm{u}(x).
\end{aligned}
$$

Proposition 2.4.12. *[9, 12] Assume that* $\sigma \geq 0$ *and function* $\mathrm{u} \in AC^n[c,d]$*, where* $n = \lfloor \sigma \rfloor + 1$*. Then*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}\,{}^{C}_{c^+}\mathcal{D}^{\sigma}_{x}\mathrm{u}(x) &= \mathrm{u}(x) - \sum_{\jmath=0}^{n-1} \frac{\mathcal{D}^{\jmath}\mathrm{u}(c)}{\jmath!}(x-c)^{\jmath}, \\
{}^{RL}_{x}\mathcal{I}^{\sigma}_{d^-}\,{}^{C}_{x}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) &= \mathrm{u}(x) - \sum_{\jmath=0}^{n-1} \frac{(-1)^n\mathcal{D}^{\jmath}\mathrm{u}(d^-)}{\jmath!}(d^- - x)^{\jmath}.
\end{aligned}
$$

Additionally, for $0 < \alpha < 1$, $\mathrm{u} \in AC[c,d]$ *or* $\mathrm{u} \in C[c,d]$*, then*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}\,{}^{C}_{c^+}\mathcal{D}^{\sigma}_{x}\mathrm{u}(x) &= \mathrm{u}(x) - \mathrm{u}(c), \\
{}^{RL}_{x}\mathcal{I}^{\sigma}_{d}\,{}^{C}_{x}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) &= \mathrm{u}(x) - \mathrm{u}(d).
\end{aligned}
$$

Chapter 3

Fractional calculus with respect to functions

The concept of FC extends the ideas of differintegration to non-integer orders. Several types of FC operators have been proposed and can be categorized into general classes. FC with respect to functions, is a significant and important area within the field of FC. The concept was first introduced by Liouville in 1835, who suggested fractional integration with respect to functions [22]. Later, in 1865, Holmgren formally proposed the notion of fractional integrals in this context [23]. Gradually, the idea of operators of FC with respect to functions gained attention in several subsequent papers. Notably, Erdélyi made significant contributions in 1964 [24] and 1970 [25], followed by Talenti in 1965 [26]. In 1971, Chrysovergis examined some fundamental properties of integral fractional operators with respect to functions [27]. Subsequently, many renowned researchers, such as Osler in 1970 [28, 29] and Samko et al. [8], further advanced the study of this topic.

In 2017 [39], Almeida defined CFD with respect to functions, explored their properties, and presented various miscellaneous results. Later on, Sousa et al. presented the ψ–Hilfer fractional derivatives, which combines the ψ–RL and Caputo derivatives. Almeida et al. [55] in 2018, considered equations with ψ–CFD and their applications. In 2019, Almeida [58] described additional properties of fractional integrals and derivatives and conducted a numerical study with ψ–CFD, describing fractional relaxation oscillation. Almeida and Malinowska in 2021 [59] studied systems of equations with ψ-CFD.

In recent studies, Fahad et al. [30] have contributed to the field by studying this class of fractional operators. They focus on algebraic expressions of conjugation relations (which were first mentioned by Samko et al. [8] and in the book by Kilbas et al. [12]) between the original classical operators and their corresponding generalized versions, offering valuable perspectives for both theoretical and practical developments in FC.

This chapter focuses on a generalized class of FC known as φ–FC. We aim to establish a base by introducing spaces and key definitions that will be utilized and interconnected throughout the chapter. We recall the generalizations of L space, denoted by $L^r_\varphi(c,d)$, which have been discussed in the literature. Here, we simply start this chapter with the definition of spaces without an explanation of its properties or application. Throughout this chapter, the following notations will be adopted: ${}^{RL}_{c^+}\mathcal{I}^\sigma_\varphi$ and ${}^{RL}_{\varphi}\mathcal{I}^\sigma_{d^-}$ denote the left and right fractional integrals with respect to functions, respectively. Similarly, ${}^{RL}_{c^+}\mathcal{D}^\sigma_\varphi$, ${}^{RL}_{\varphi}\mathcal{D}^\sigma_{d^-}$, ${}^{C}_{c^+}\mathcal{D}^\sigma_\varphi$, and ${}^{C}_{\varphi}\mathcal{D}^\sigma_{d^-}$ denote the left and right RL and CFD with respect to functions, respectively.

3.1 Function spaces

First, we mention the definitions of $L^r_\varphi(c,d)$ space (r-integrable functions with respect to a function φ) and AC^n_φ.

Definition 3.1.1. *[62] The $L^r_\varphi(c,d)$ space is a collection of functions defined on $[c,d]$ that are integrable with respect to the function φ that is $L^r_\varphi(c,d) = \Big\{\mathrm{z} : \|\mathrm{z}\|_{L^r_\varphi} < \infty\Big\}$, where $1 \le r < \infty$, where*

$$\|\mathrm{z}\|_{L^r_\varphi} = \left(\int_c^d |\mathrm{z}(s)|^r \varphi'(s)\, ds\right)^{\frac{1}{r}},$$

and

$$\|\mathrm{z}\|^\infty_\varphi = \underset{c\le x\le d}{esssup}|\mathrm{u}(s)|.$$

Definition 3.1.2. *The space $AC^n_\varphi[c,d]$ is defined as*

$$AC^n_\varphi[c,d] = \left\{\mathrm{z} : [c,d] \to \mathbb{R} : \left(\frac{1}{\varphi'(s)}\frac{d}{ds}\right)^n \mathrm{z}(s) \in AC[c,d]\right\}.$$

Note that, $L^r_\varphi(c,d) = L^r(c,d)$, if φ' is bounded on $[c,d]$. Moreover, by making the change of variable in the definition of φ–RL integral as $\varphi(s) = \xi$, we establish a relationship between these spaces. Specifically, a function u belongs to $L^r_\varphi(c,d)$ if and only if the composition $\mathrm{u} \circ \varphi^{-1}$ belongs to $L^r \varphi(c), \varphi(d))$.

Lemma 3.1.3. *[62] For any* $\mathrm{u} \in L^r_{\varphi}(c,d)$*,* $r \in [1,\infty)$*, we have*

$$ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathrm{u}(x) = ({}^{RL}\mathcal{I}^{\sigma}_{\varphi(c^+)}\mathrm{u} \circ \varphi^{-1}) \circ \varphi(x). $$

Proof. We start with the definition of ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathrm{u}(x)$

$$ \mathcal{I}^{\sigma}_{\varphi}\mathrm{u}(x) = \frac{1}{\Gamma(\sigma)} \int_c^x \varphi'(s)(\varphi(x) - \varphi(s))^{\sigma-1}\mathrm{u}(s)\, ds. $$

By making the change of variable $\varphi(s) = \xi$, the integral transforms. Specifically, we substitute s with $\varphi^{-1}(\xi)$, and the differential ds changes accordingly. Let $\xi = \varphi(s)$, then $d\xi = \varphi'(s)\, ds$. As, $s = c \implies \xi = \varphi(c^+)$, $s = x \implies \xi = \varphi(x)$. Thus, the integral becomes

$$ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathrm{u}(x) = \frac{1}{\Gamma(\sigma)} \int_{\varphi(c^+)}^{\varphi(x)} (\varphi(x) - \xi)^{\sigma-1}\mathrm{u}(\varphi^{-1}(\xi))\, d\xi. $$

This integral now looks like the definition of the original RL fractional integral $\mathcal{I}^{\sigma}_{\varphi(c^+)}$ but applied to the function $\mathrm{u} \circ \varphi^{-1}$:

$$ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathrm{u}(x) = \left({}^{RL}_{\varphi(c^+)}\mathcal{I}^{\sigma}(\mathrm{u} \circ \varphi^{-1})\right)(\varphi(x)). $$

This means that the fractional integral ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}$, can be expressed as the standard fractional integral ${}^{RL}_{\varphi(c^+)}\mathcal{I}^{\sigma}_{\varphi}$ of the function $\mathrm{u} \circ \varphi^{-1}$, evaluated at $\varphi(x)$. Therefore

$$ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathrm{u}(x) = \left({}^{RL}_{\varphi(c^+)}\mathcal{I}^{\sigma}(\mathrm{u} \circ \varphi^{-1})\right)(\varphi(x)). $$

This completes the proof by showing that the φ–fractional integral is equivalent to the original integral of a transformed function. □

Theorem 3.1.4. *[62] If* $0 < \sigma < 1$ *and* $1 \leq r < \infty$*, then the following statements are valid*

(i) The operators ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}$*,* ${}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}$ *are continuous from* $L^r_{\varphi}[c,d]$ *to* $L^q_{\varphi}[c,d]$ *for every* $1 \leq q < \frac{r}{1-\sigma r}$*, if* $1 \leq r < \frac{1}{\sigma}$*.*

(ii) The fractional operators ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}$*,* ${}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}$ *are continuous from* $L^r_{\varphi}[c,d]$ *to* $L^q_{\varphi}[c,d]$ *for every* $1 \leq q < \infty$*, if* $\sigma = \frac{1}{r}$*.*

(ii) The fractional operators ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}$*,* ${}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}$ *are continuous from* $L^r_{\varphi}[c,d]$ *to* $C[c,d]$*, if* $\frac{1}{r} < \sigma < 1$*. Additionally, the operators* ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}$*,* ${}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}$ *are continuous from* $L^r_{\varphi}[c,d]$ *to* $L^q_{\varphi}[c,d]$ *for every* $r \leq q \leq \infty$*.*

3.2 Fractional operators with respect to functions

In this section, we mention the key definitions with suitable function spaces.

Definition 3.2.1. *[30] Let* u *be an integrable function and* $\varphi \in C[c,d]$, *with* φ *be an increasing positive montone function on the interval* $[c,d]$, *such that, for all* $x \in [c,d]$ $\varphi \neq 0$. *Then the operators* ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathrm{u}(x)$ *and* ${}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}\mathrm{u}(x)$, *of order* $\sigma > 0$, *are defined as*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathrm{u}(x) &= \frac{1}{\Gamma(\sigma)}\int_c^x \frac{\mathrm{u}(s)\varphi'(s)}{(\varphi(x)-\varphi(s))^{1-\sigma}}\,ds, \quad x > c,\\
{}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}\mathrm{u}(x) &= \frac{1}{\Gamma(\sigma)}\int_x^d \frac{\mathrm{u}(s)\varphi'(s)}{(\varphi(s)-\varphi(x))^{1-\sigma}}\,ds, \quad x < d,
\end{aligned}
$$

respectively.

Definition 3.2.2. *[30] If* $\sigma > 0$ *and* $\varphi \in C^n[c,d]$. *Then the operators* ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\mathrm{u}(x)$ *and* ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\mathrm{u}(x)$ *can be define as*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\mathrm{u}(x) &= \mathcal{D}^n_{\varphi}\left({}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi}\mathrm{u}(x)\right), \quad x > c,\\
{}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) &= (-1)^n\mathcal{D}^n_{\varphi}\left({}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-}\mathrm{u}(x)\right), \quad x < d,
\end{aligned}
$$

where $\mathcal{D}_{\varphi} = \frac{1}{\varphi'(x)}\frac{d}{dx}$.

Also, the φ–RL derivatives in Definition 3.2.2 is the analytic continuation of the φ–RL integrals 3.2.1. Therefore, we can efine both ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}$ and ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}$ for all values of $\sigma > 0$, as previously menioned.

Definition 3.2.3. *[30, 39] Let* $\sigma > 0$, $-\infty \leq c < d \leq \infty$, u,φ *be two functions in space* $C^n[c,d]$, *where* φ *be an increasing positive monotone function in interval* $[c,d]$ *and* $\varphi'(x) \neq 0$, $n = \lfloor\sigma\rfloor + 1$. *Then the operators* ${}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi}$ *and* ${}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi}$ *of order* σ *can be defined as*

$$
\begin{aligned}
{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\mathrm{u}(x) &= {}^{C}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi}\left(\mathcal{D}^n_{\varphi}\mathrm{u}(x)\right), \quad x > c,\\
{}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) &= (-1)^n {}^{C}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-}\left(\mathcal{D}^n_{\varphi}\mathrm{u}(x)\right), \quad x < d.
\end{aligned}
$$

where $\mathcal{D}_{\varphi} = \frac{1}{\varphi'(x)}\frac{d}{dx}$.

When $\varphi(x) = x$ in the above definitions, they coincides with the original RL operators as defined in Definition 2.4.1, 2.4.2 and 2.4.10. These generalized fractional operators can be expressed as the conjugation of the original fractional operators with the operation of composition with the function φ. This is obtained using the functional operator $\mathcal{Q}_\varphi$, defined as

$$\mathcal{Q}_\varphi \mathrm{u}(x) = \mathrm{u}(\varphi(x)).$$

Proposition 3.2.4. *[8, 12, 30] The operators in 3.2.1, 3.2.2, and 3.2.3 can be expressed as the conjugation of the original RL fractional operators*

$$ {}^{RL}_{c^+}\mathcal{I}^\sigma_\varphi = \mathcal{Q}_\varphi \circ {}^{RL}_{\varphi(c^+)}\mathcal{I}^\sigma_x \circ \mathcal{Q}^{-1}_\varphi, \quad {}^{RL}_{\varphi}\mathcal{I}^\sigma_{d^-} = \mathcal{Q}_\varphi \circ {}^{RL}_{x}\mathcal{I}^\sigma_{\varphi(d^-)} \circ \mathcal{Q}^{-1}_\varphi, \tag{3.1}$$

$$ {}^{RL}_{c^+}\mathcal{D}^\sigma_\varphi = \mathcal{Q}_\varphi \circ {}^{RL}_{\varphi(c^+)}\mathcal{D}^\sigma_x \circ \mathcal{Q}^{-1}_\varphi \quad {}^{RL}_{\varphi}\mathcal{D}^\sigma_{d^-} = \mathcal{Q}_\varphi \circ {}^{RL}_{x}\mathcal{D}^\sigma_{\varphi(d^-)} \circ \mathcal{Q}^{-1}_\varphi, \tag{3.2}$$

$$ {}^{C}_{c^+}\mathcal{D}^\sigma_\varphi = \mathcal{Q}_\varphi \circ {}^{C}_{\varphi(c^+)}\mathcal{D}^\sigma_x \circ \mathcal{Q}^{-1}_\varphi \quad {}^{C}_{\varphi}\mathcal{D}^\sigma_{d^-} = \mathcal{Q}_\varphi \circ {}^{C}_{x}\mathcal{D}^\sigma_{\varphi(d^-)} \circ \mathcal{Q}^{-1}_\varphi. \tag{3.3}$$

Many properties of generalized fractional operators can be difficult to establish directly. However, the concept of conjugation relations, introduced in works like Kilbas et al. [12] and samko et al.[8], offers a powerful tool. These relations connect a generalized operator to well studied classical operators, such as the RL and Caputo operators. This connection allows us to efficiently prove properties in the generalized setting by utilizing the existing knowledge from the classical framework.

Example 3.2.1. [8, 12] Assume that the power functions are given by κ^α and ρ^α, where $\kappa = \varphi(x) - \varphi(c)$, $\rho = \varphi(d) - \varphi(x)$. Then for $\sigma > 0$ and $\alpha > -1$, the power rules for φ–RL fractional operators can be expressed as

$$\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^\sigma_\varphi \kappa^\alpha &= \frac{\Gamma(1+\alpha)}{\Gamma(\alpha+\sigma+1)}\kappa^{\sigma+\alpha}, \\
{}^{RL}_{\varphi}\mathcal{I}^\sigma_{d^-} \rho^\alpha &= \frac{\Gamma(1+\alpha)}{\Gamma(\alpha+\sigma+1)}\rho^{\sigma+\alpha}, \\
{}^{RL}_{c^+}\mathcal{D}^\sigma_\varphi \kappa^\alpha &= \frac{\Gamma(1+\alpha)}{\Gamma(\alpha-\sigma+1)}\kappa^{\alpha-\sigma}, \\
{}^{RL}_{\varphi}\mathcal{D}^\sigma_{d^-} \rho^\alpha &= \frac{\Gamma(1+\alpha)}{\Gamma(\alpha-\sigma+1)}\rho^{\alpha-\sigma}.
\end{aligned}$$

We can easily verify this power rule by using conjugation relations as

$$\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\kappa^{\alpha} &= \mathcal{Q}_{\varphi} \circ {}^{RL}_{\varphi(c^+)}\mathcal{I}^{\sigma}_{x} \circ \mathcal{Q}^{-1}_{\varphi}\kappa^{\alpha} \\
&= \mathcal{Q}_{\varphi} \circ \left[{}^{RL}_{\varphi(c^+)}\mathcal{I}^{\sigma}_{x}(x-\varphi(c))^{\alpha}\right] \\
&= \mathcal{Q}_{\varphi} \circ \left[\frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma+1)}(x-\varphi(c))^{\alpha+\sigma}\right] \\
&= \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\sigma+1)}\kappa^{\alpha+\sigma}.
\end{aligned}$$

Similarly, the other three relations, the right φ–RL integrals power rule, and the left and right φ–RL derivatives power rule can be derived using analogous methods. By applying the original results and the conjugation relations, we can find or show the remaining relations.

Example 3.2.2. Consider the power functions κ^{α} and ρ^{α}, where $\kappa = \varphi(x) - \varphi(c)$, $\rho(x) = \varphi(d) - \varphi(x)$ and $\alpha \geq 0$. Then for $\sigma > 0$, we have

$$\begin{aligned}
{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\kappa^{\alpha} &= \begin{cases} 0, & \text{if } \alpha \in \{0,1,2,\ldots,n-1\}, \\ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\sigma+1)}\kappa^{\alpha-\sigma}, & \text{if } \alpha \in \mathbb{N} \text{ and } \alpha \notin \mathbb{N} \text{ or } \alpha \geq n \text{ and } \alpha > n-1, \end{cases} \\
{}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-}\rho^{\alpha} &= \begin{cases} 0, & \text{if } \alpha \in \{0,1,2,\ldots,n-1\}, \\ \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\sigma+1)}\rho^{\alpha-\sigma}, & \text{if } \alpha \in \mathbb{N} \text{ and } \alpha \notin \mathbb{N} \text{ or } \alpha \geq n \text{ and } \alpha > n-1. \end{cases}
\end{aligned}$$

We have already proved the left φ–RL integral power rule using the conjugation relation. Similarly, we have conjugation relations for the left and right φ–CFD, as

$$\begin{aligned}
{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\kappa^{\alpha} &= \mathcal{Q}_{\varphi} \circ {}^{C}_{\varphi(c^+)}\mathcal{D}^{\sigma}_{x} \circ \mathcal{Q}^{-1}_{\varphi}\kappa^{\alpha}, \\
{}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-}\rho^{\alpha} &= \mathcal{Q}_{\varphi} \circ {}^{C}_{x}\mathcal{D}^{\sigma}_{\varphi(d^-)} \circ \mathcal{Q}^{-1}_{\varphi}\rho^{\alpha},
\end{aligned}$$

for $n-1 < \alpha \notin \mathbb{N}$ or $n \leq \alpha \in \mathbb{N}$. We can verify this by using a similar method as we have done in the above example. And $\mathcal{D}^{n}_{\varphi}\kappa^{\alpha} = \mathcal{D}^{n}_{\varphi}\rho^{\alpha} = 0$, if $\alpha = 0,1,2,\ldots,n-1$. Thus, ${}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\kappa^{\alpha} = {}^{C}_{d^-}\mathcal{D}^{\sigma}_{x}\rho^{\alpha} = 0$.

Example 3.2.3. For the function $\varphi(x) = x$ in Definitions 3.2.1, 3.2.2, and 3.2.3, the resulting operators are the RL fractional integral, derivatives, and Caputo derivatives of order σ, as mentioned in Definitions 2.4.1, 2.4.2, and 2.4.10.

Example 3.2.4. Consider the function $\varphi(x) = \log(x)$ in Definitions 3.2.1, 3.2.2, and 3.2.3, we have the following Hadamard fractional operators

$$
\begin{aligned}
{}^{H}_{c^+}\mathcal{I}^{\sigma}_{\log}\mathrm{u}(x) &= \frac{1}{\Gamma(\sigma)}\int_c^x \frac{\mathrm{u}(s)}{(\log(\frac{x}{s}))^{1-\sigma}}\frac{ds}{s}, \quad x > c,\\
{}^{H}_{\log}\mathcal{I}^{\sigma}_{d^-}\mathrm{u}(x) &= \frac{1}{\Gamma(\sigma)}\int_x^d \frac{\mathrm{u}(s)}{(\log(\frac{s}{x}))^{1-\sigma}}\frac{ds}{s}, \quad x < d,
\end{aligned}
$$

called left and right Hadamard fractional integrals, respectively. The corresponding left and right Hdamard fractional derivatives are

$$
\begin{aligned}
{}^{H}_{c^+}\mathcal{D}^{\sigma}_{\log}\mathrm{u}(x) &= \left(x\frac{d}{dx}\right)^n {}^{H}_{c^+}\mathcal{I}^{n-\sigma}_{\log}\mathrm{u}(x), \quad x > c,\\
{}^{H}_{\log}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) &= \left(-x\frac{d}{dx}\right)^n {}^{RL}_{\log}\mathcal{I}^{n-\sigma}_{d^-}\mathrm{u}(x), \quad x < d,
\end{aligned}
$$

respectively.

$$
\begin{aligned}
{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\log}\mathrm{u}(x) &= {}^{C}_{c^+}\mathcal{I}^{n-\sigma}_{\log}\left(x\frac{d}{dx}\right)^n \mathrm{u}(x), \quad x > c,\\
{}^{C}_{\log}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) &= {}^{C}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-}\left(-x\frac{d}{dx}\right)^n \mathrm{u}(x), \quad x < d,
\end{aligned}
$$

called left and right Caputo–Hadamard fractional derivatives respectively.

3.3 Properties

This section provides some fundamental properties of generalized fractional operators by using conjugation relations.

Lemma 3.3.1. *Let σ_1, $\sigma_2 > 0$, $\mathrm{u} \in L^1[c,d]$, $\varphi \in C^1[c,d]$ and an increasing monotonic function. Then the following semigroup property hold*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma_1}_{\varphi}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma_2}_{\varphi}\mathrm{u}(x) &= {}^{RL}_{c^+}\mathcal{I}^{\sigma_1+\sigma_2}_{\varphi}\mathrm{u}(x),\\
{}^{RL}_{\varphi}\mathcal{I}^{\sigma_1}_{d^-}\,{}^{RL}_{\varphi}\mathcal{I}^{\sigma_2}_{d^-}\mathrm{u}(x) &= {}^{RL}_{\varphi}\mathcal{I}^{\sigma_1+\sigma_2}_{d^-}\mathrm{u}(x).
\end{aligned}
$$

Proof. We will prove semigroup properties by using conjugation relations as follows

$$\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma_1+\sigma_2}_{\varphi}\mathrm{u}(x) &= \mathcal{Q}_{\varphi} \circ {}^{RL}_{\varphi(c^+)}\mathcal{I}^{\sigma_1+\sigma_2}_{x} \circ \mathcal{Q}^{-1}_{\varphi}\mathrm{u}(x);\\
\mathrm{u} &: x \to \mathrm{u}(x);\\
\mathcal{Q}^{-1}_{\varphi}(\mathrm{u}) &: x \to \mathrm{u}(\varphi^{-1}(x));\\
{}_{\varphi(c^+)}\mathcal{I}^{\sigma_1+\sigma_2}_{x} \circ \mathcal{Q}^{-1}_{\psi}(\mathrm{u}) &: x \to {}_{\varphi(c^+)}\mathcal{I}^{\sigma_1}_{x}\,{}_{\varphi(c^+)}\mathcal{I}^{\sigma_2}_{x}\mathrm{u}(\varphi^{-1}(x));\\
\mathcal{Q}_{\varphi} \circ {}_{\varphi(c^+)}\mathcal{I}^{\sigma_1+\sigma_2}_{x} \circ \mathcal{Q}^{-1}_{\varphi}(\mathrm{u}) &: x \to \mathcal{Q}_{\varphi}\left({}_{\varphi(c^+)}\mathcal{I}^{\sigma_1}_{x}\,{}_{\varphi(c^+)}\mathcal{I}^{\sigma_2}_{x}\,\mathcal{Q}^{-1}_{\varphi}\mathrm{u}(x)\right);\\
&: x \to \mathcal{Q}_{\varphi}\left({}_{\varphi(c^+)}\mathcal{I}^{\sigma_1}_{x}\left(\mathcal{Q}^{-1}_{\varphi}\mathcal{Q}_{\varphi}\right){}_{\varphi(c^+)}\mathcal{I}^{\sigma_2}_{x}\,\mathcal{Q}^{-1}_{\varphi}\mathrm{u}(x)\right);\\
\mathcal{Q}_{\varphi} \circ {}_{\varphi(c^+)}\mathcal{I}^{\sigma_1+\sigma_2}_{x} \circ \mathcal{Q}^{-1}_{\varphi}(\mathrm{u}) &: x \to \mathcal{Q}_{\psi}\left({}_{\varphi(c^+)}\mathcal{I}^{\sigma_1}_{x}\left(\mathcal{Q}^{-1}_{\varphi}\mathcal{Q}_{\varphi}\right){}_{\varphi(c^+)}\mathcal{I}^{\sigma_2}_{x}\,\mathcal{Q}^{-1}_{\varphi}\mathrm{u}(x)\right);\\
&: x \to \left(\mathcal{Q}_{\varphi\varphi(c^+)}\mathcal{I}^{\sigma_1}_{x}\mathcal{Q}^{-1}_{\varphi}\right)\left(\mathcal{Q}_{\varphi\varphi(c^+)}\mathcal{I}^{\sigma_2}_{x}\mathcal{Q}^{-1}_{\varphi}\right)\mathrm{u}(x).
\end{aligned}$$

By using relation (3.1) we get,

$${}^{RL}_{c^+}\mathcal{I}^{\sigma_1+\sigma_2}_{\varphi}\mathrm{u}(x) = {}_{c^+}\mathcal{I}^{\sigma_1}_{\varphi(x)}\,{}_{c^+}\mathcal{I}^{\sigma_2}_{\varphi(x)}\mathrm{u}(x).$$

We can obtain the same result by using the same method in a slightly different way, as follows

$$\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma_1}_{\varphi}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma_2}_{\varphi}\mathrm{u}(x) &= \left(\mathcal{Q}_{\varphi} \circ {}^{RL}_{\varphi(c^+)}\mathcal{I}^{\sigma_1}_{x} \circ \mathcal{Q}^{-1}_{\varphi}\right)\left(\mathcal{Q}_{\varphi} \circ {}^{RL}_{\varphi(c^+)}\mathcal{I}^{\sigma_2}_{x}\mathrm{u}(x) \circ \mathcal{Q}^{-1}_{\varphi}\right)\mathrm{u}(x)\\
&= \mathcal{Q}_{\varphi} \circ {}^{RL}_{\varphi(c^+)}\mathcal{I}^{\sigma_1}_{x}\,{}^{RL}_{\varphi(c^+)}\mathcal{I}^{\sigma_2}_{x} \circ \mathcal{Q}^{-1}_{\varphi}\mathrm{u}(x).
\end{aligned}$$

Since we have a semigroup property for original RL integrals as mentioned in Lemma 2.4.4

$${}^{RL}_{c^+}\mathcal{I}^{\sigma_1}_{\varphi}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma_2}_{\varphi}\mathrm{u}(x) = \left(\mathcal{Q}_{\varphi} \circ {}^{RL}_{\varphi(c^+)}\mathcal{I}^{\sigma_1+\sigma_2}_{x} \circ \mathcal{Q}^{-1}_{\varphi}\right)\mathrm{u}(x) = {}^{RL}_{c^+}\mathcal{I}^{\sigma_1+\sigma_2}_{\varphi}\mathrm{u}(x).$$

□

The properties of fractional operators were summarized in the previous chapter. The properties of φ–fractional operators can be immediately derived using conjugation relations. Specifically, the semigroup property for φ–fractional integrals has been demonstrated through this relation. This conjugation relation serves as the main tool for deriving the properties of generalized operators. To avoid repetition, the remaining properties, which are straightforward to prove, have been omitted.

Lemma 3.3.2. *For $n \in \mathbb{N}$, $\sigma > 0$. Then the integer order φ–derivatives and φ–RL intgerals have a semigroup property as follow*

$${}_{c^+}\mathcal{D}^{n}_{\varphi}\,{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\mathrm{u}(x) = {}^{RL}_{c^+}\mathcal{D}^{n+\sigma}_{\varphi}\mathrm{u}(x).$$

Once again, we find that the φ–RL and φ–CFD of a function act as a left inverse of the φ–RL fractional integrals. However, the φ–RL and φ–CFD are not the right inverse of φ–RL fractional integrals.

Lemma 3.3.3. *Let $\sigma > 0$, function $\mathrm{u}, \varphi \in C[c,d]$. Then, for an increasing positive function φ, we have*

$$ {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\, {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathrm{u}(x) = \mathrm{u}(x) $$
$$ {}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-}\, {}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}\mathrm{u}(x) = \mathrm{u}(x). $$

Lemma 3.3.4. *Let $\sigma > 0$, φ be an increasing positive function and $\mathrm{u} \in C^n[c,d]$. Then*

$$ {}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\, {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathrm{u}(x) = \mathrm{u}(x), $$
$$ {}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-}\, {}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}\mathrm{u}(x) = \mathrm{u}(x). $$

Theorem 3.3.5. *If $\sigma > 0$, φ be an increasing positive function and $\mathrm{u}, \varphi \in C^n[c,d]$ such that*

$$ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\, {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\mathrm{u}(x) = \mathrm{u}(x) - \sum_{\jmath=0}^{n-1} \lim_{x \to c^+} \mathcal{I}^{\jmath-\sigma}\mathrm{u}(x) \cdot \frac{(\varphi(x) - \varphi(c))^{\sigma-\jmath}}{\Gamma(\sigma - \jmath + 1)}, $$
$$ {}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}\, {}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) = \mathrm{u}(x) - \sum_{\jmath=0}^{n-1} \lim_{x \to d^-} \mathcal{I}^{\jmath-\sigma}\mathrm{u}(x) \cdot \frac{(\varphi(d) - \varphi(x))^{\sigma-\jmath}}{\Gamma(\sigma - \jmath + 1)}. $$

Additionally, for $0 < \sigma < 1$, we have

$$ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\, {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\mathrm{u}(x) = \mathrm{u}(x) - \frac{(\varphi(x) - \varphi(c))^{\sigma-1}}{\Gamma(\sigma)} \lim_{x \to c^+} \mathcal{I}^{1-\sigma}\mathrm{u}(x), $$
$$ {}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}\, {}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) = \mathrm{u}(x) - \frac{(\varphi(d) - \varphi(x))^{\sigma-1}}{\Gamma(\sigma)} \lim_{x \to d^-} \mathcal{I}^{1-\sigma}\mathrm{u}(x). $$

Proposition 3.3.6. *Let functions $\mathrm{u}, \varphi \in C^n[c,d]$. Then for $\sigma > 0$, we have*

$$ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\, {}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\mathrm{u}(x) = \mathrm{u}(x) - \sum_{\jmath=0}^{n-1} \lim_{x \to c^+} \mathcal{D}^{\jmath}_{\varphi}\mathrm{u}(x) \cdot \frac{(\varphi(x) - \varphi(c))^{\jmath}}{\jmath!}, $$
$$ {}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}\, {}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) = \mathrm{u}(x) - \sum_{\jmath=0}^{n-1} \lim_{x \to d^-} \mathcal{D}^{\jmath}_{\varphi}\mathrm{u}(x) \cdot \frac{(-1)^{\jmath}(\varphi(d) - \varphi(x))^{\jmath}}{\jmath!}. $$

Additionally, for $0 < \sigma < 1 \implies n = 1$, we have the following composition relations

$$ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\, {}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\mathrm{u}(x) = \mathrm{u}(x) - \mathrm{u}(c), $$
$$ {}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-}\, {}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-}\mathrm{u}(x) = \mathrm{u}(d) - \mathrm{u}(x). $$

Proof. Proof of these composition relations is straightforward via conjugation of the original RL operators in Theorem 2.4.9. □

Chapter 4

Weighted fractional calculus

Weighted fractional calculus (WFC) and WFC with respect to functions is another generalized class of FC that extends the original FC. In 2012, Agrawal introduced the concept of weighted/scaled fractional differintegrals with respect to functions, marking the inception of scaled FC. This work, detailed in his studies [31, 32], explored the applications of these operators in variational calculus and probabilistic modelling [32]. This approach can be viewed as a combination of FC of a function by functions and WFC.

In 2020, significant progress was made in this field. Abdeljawad et al. [33, 35] investigated WFOs with respect to functions, studied their properties, proposing a modified Laplace transform suitable for these operators, and investigating the existence of positive solutions for WF order differential equations. Al-Refai [34] examined weighted Atangana–Baleanu fractional operators in the same year. Then, in 2021, Liu et al. [36] provided some general results related to the weighted CFD, further advancing the mathematical framework for these operators.

In 2022, Fahad et al. [16] presented a comprehensive study on WFC and its extension to WFC with respect to functions. Their work emphasized the significance of conjugation relations with classical fractional operators and studied various fundamental properties. They explored some special cases of these operators named as Erdélyi-Kober, Hadamard-type, and tempered operators and suggested modifications to the Laplace transform and convolution operations. Additionally, they effectively dealt with ODEs within the framework of these general operator classes.

In 2023, a new weighted fractional operator was introduced by Sabri T.M. Thabet et al. [72]. This operator was based on a modified generalized M–L law. Their research, centred on the study of generalized WF derivatives and integrals in the Caputo and RL sense. The research also identified various special cases and key properties such as series versions and weighted Laplace transforms.

This chapter provides the definitions, properties, and some new insights related to these operators. Throughout this chapter, the following notations will be adopted: ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}$ and ${}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-;\omega}$ denote the left and right WF integrals with respect to functions, respectively. Similarly, ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$, ${}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega}$, ${}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$, and ${}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega}$ denote the left and right weighted RL and CFD with respect to functions, respectively.

4.1 Weighted fractional calculus

In this section we mentioned the definitions and properties of weighted fractional calculus.

Definition 4.1.1. *[16, 32] If function* $\mathrm{u} \in L^1(c,d)$ *and* $\sigma > 0$, *then the operators* ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x;\omega}$ *and* ${}^{RL}_{x}\mathcal{I}^{\sigma}_{d^-;\omega}$ *can be defined as*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x;\omega}\mathrm{u}(x) &= \frac{1}{\omega(x)\Gamma(\sigma)}\int_c^x \omega(s)\mathrm{u}(s)\,(x-s)^{\sigma-1}\,ds, \quad x > c,\\
{}^{RL}_{x}\mathcal{I}^{\sigma}_{d^-;\omega}\mathrm{u}(x) &= \frac{\omega(x)}{\Gamma(\sigma)}\int_x^d \frac{\mathrm{u}(s)\,(s-x)^{\sigma-1}}{\omega(s)}\,ds, \quad x < d,
\end{aligned}
$$

respectively, where $\omega \in L^{\infty}(c,d)$.

Definition 4.1.2. *[16, 31, 32] If* $\omega, \mathrm{u} \in AC^n[c,d]$. *Then the operators* ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x;\omega}$ *and* ${}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-;\omega}$ *of order* $\sigma > 0$ *can be defined as*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x;\omega}\mathrm{u}(x) &= \left(\omega(x)^{-1}\frac{d}{dx}\omega(x)\right)^n {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{x;\omega}\mathrm{u}(x),\\
{}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-;\omega}\mathrm{u}(x) &= \left(-\omega(x)\frac{d}{dx}\omega(x)^{-1}\right)^n {}^{RL}_{x}\mathcal{I}^{n-\sigma}_{d^-;\omega}\mathrm{u}(x),
\end{aligned}
$$

respectively, where $\sigma > 0$ *and* $n = \lfloor \sigma \rfloor + 1$.

Definition 4.1.3. [*16*, *31*, *32*] *Let* $\mathrm{u} \in C^n[c,d]$, *weight function* ω *with* $\omega \neq 0$. *Then the operators* ${}^{C}_{c^+}\mathcal{D}^{\sigma}_{x;\omega}$ *and* ${}^{C}_{x}\mathcal{D}^{\sigma}_{d^-;\omega}$ *are defined as*

$$
\begin{aligned}
{}^{C}_{c^+}\mathcal{D}^{\sigma}_{x;\omega}\mathrm{u}(x) &= {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{x;\omega}\left(\omega(x)^{-1}\frac{d}{dx}\omega(x)\right)^n \mathrm{u}(x),\\
{}^{C}_{x}\mathcal{D}^{\sigma}_{d^-;\omega}\mathrm{u}(x) &= {}^{RL}_{x}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(-\omega(x)\frac{d}{dx}\omega(x)^{-1}\right)^n \mathrm{u}(x),
\end{aligned}
$$

respectively, where $\sigma > 0$ *and* $n = \lfloor \sigma \rfloor + 1$.

The above definitions can be wriiten as conjugations with the original RL and Caputo fractional operators as following.

Proposition 4.1.4. *The operators defined in Definition 4.1.1, 4.1.2, and 4.1.3 can be represented as the result of conjugating the original left and right RL fractional operators with the operator $\mathcal{M}_\omega$, where $\mathcal{M}_\omega \mathrm{u}(x) = \omega(x)\mathrm{u}(x)$.*

$$\begin{aligned}
{}_{c^+}^{RL}\mathcal{I}_{x;\omega}^{\sigma} &= \mathcal{M}_\omega^{-1} \circ {}_{c^+}^{RL}\mathcal{I}_{x}^{\sigma} \circ \mathcal{M}_\omega, \\
{}_{c^+}^{RL}\mathcal{D}_{x;\omega}^{\sigma} &= \mathcal{M}_\omega^{-1} \circ {}_{c^+}^{RL}\mathcal{D}_{x}^{\sigma} \circ \mathcal{M}_\omega, \\
{}_{c^+}^{C}\mathcal{D}_{x;\omega}^{\sigma} &= \mathcal{M}_\omega^{-1} \circ {}_{c^+}^{C}\mathcal{D}_{x}^{\sigma} \circ \mathcal{M}_\omega.
\end{aligned}$$

In the case of right WFO with respect to functions, conjugation relations are given by

$$\begin{aligned}
{}_{x}^{RL}\mathcal{I}_{d^-;\omega}^{\sigma} &= \mathcal{M}_\omega \circ {}_{x}^{RL}\mathcal{I}_{d^-}^{\sigma} \circ \mathcal{M}_\omega^{-1}, \\
{}_{x}^{RL}\mathcal{D}_{d^-;\omega}^{\sigma} &= \mathcal{M}_\omega \circ {}_{x}^{RL}\mathcal{D}_{d^-}^{\sigma} \circ \mathcal{M}_\omega^{-1}, \\
{}_{x}^{C}\mathcal{D}_{d^-;\omega}^{\sigma} &= \mathcal{M}_\omega \circ {}_{x}^{C}\mathcal{D}_{d^-}^{\sigma} \circ \mathcal{M}_\omega^{-1}.
\end{aligned}$$

The weighted RL derivative extends the concept of the weighted RL integral of order $\sigma > 0$, where integrals of negative order are interpreted as derivatives of positive order as ${}_{c^+}^{RL}\mathcal{D}_{x;\omega}^{\sigma} = {}_{c^+}^{RL}\mathcal{I}_{x;\omega}^{-\sigma}$. This property allows both of these operators to define all values of $\sigma > 0$.

Proof. The conjugation relations for the left WFO have been comprehensively proven in Fahad's work [16]. Now, we extend this by proving the conjugation relations for the right WFO as follows The initial result ${}_{x}^{RL}\mathcal{I}_{d^-;\omega}^{\sigma} = \mathcal{M}_\omega \circ {}_{x}^{RL}\mathcal{I}_{d^-}^{\sigma} \circ \mathcal{M}_\omega^{-1}$ is straightforward. Both types of WF derivatives involve the composition of the WF integral with the operator $\frac{d}{dx} - \frac{\omega'}{\omega}$, repeated n times. It is enough to demonstrate that this integer-order operator also satisfies the conjugation relation $\frac{d}{dx} - \frac{\omega'}{\omega} = \mathcal{M}_\omega \circ \frac{d}{dx} \circ \mathcal{M}_\omega^{-1}$.

$$\begin{aligned}
\mathcal{M}_\omega \circ \frac{d}{dx} \circ \mathcal{M}_\omega^{-1}\mathrm{u}(x) &= \mathcal{M}_\omega \circ \left[\frac{d}{dx}\left(\frac{\mathrm{u}(x)}{\omega(x)}\right)\right] \\
&= \mathcal{M}_\omega \circ \frac{1}{(\omega(x))^2}\left[\omega(x)\cdot \mathrm{u}'(x) - \mathrm{u}(x)\omega'(x)\right] \\
&= \mathrm{u}'(x) - \frac{\mathrm{u}(x)}{\omega(x)}\cdot \omega'(x) = \left(\frac{d}{dx} - \frac{\omega'(x)}{\omega(x)}\right)\mathrm{u}(x).
\end{aligned}$$

□

Example 4.1.1. If $\kappa(x)=\frac{(x-c)^{\mu}}{\omega(x)}$, $\sigma>0$ and $\mu>-1$, then

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x;\omega}\kappa(x) &= \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\sigma)}\frac{(x-c)^{\mu+\sigma}}{\omega(x)},\\
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\kappa(x) &= \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\sigma)}\frac{(x-c)^{\mu-\sigma}}{\omega(x)}.
\end{aligned}
$$

And for some $\mu\geq\lfloor\sigma\rfloor$, we have

$$
{}^{C}_{c^+}\mathcal{D}^{\sigma}_{x;\omega}\kappa(x)=\begin{cases}0, & \text{if } \mu\in\{0,1,2,\ldots,n-1\},\\ \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\sigma)}\frac{(x-c)^{\mu-\sigma}}{\omega(x)}, & \text{otherwise.}\end{cases} \tag{4.1}
$$

Proposition 4.1.5. *If $\sigma>0$ and $\mathrm{u}(x)=\kappa^{\mu}$, where $\kappa=x-c$ for some $\mu>-1$, then we have the following power rules for weighted RL operators*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x;\omega}\mathrm{u}(x) &= \sum_{\ell=0}^{\infty}\frac{\varsigma_\ell\Gamma(1+\mu+\ell)}{\ell!\Gamma(1+\sigma+\mu+\ell)}\frac{(\kappa)^{\sigma+\mu+\ell}}{\omega(x)},\\
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x;\omega}\mathrm{u}(x) &= \sum_{\ell=0}^{\infty}\frac{\varsigma_\ell\Gamma(1+\mu+\ell)}{\ell!\Gamma(1+\mu-\sigma+\ell)}\frac{(\kappa)^{\mu-\sigma+\ell}}{\omega(x)},
\end{aligned}
$$

where for $\ell=0,1,2,\ldots$, $\varsigma_\ell=\lim_{x\to c^+}\mathcal{D}^{\ell}_{x}\omega(x)$ satisfies the following infinite series

$$
\omega(x)=\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell}{\ell!}(\kappa)^{\ell}. \tag{4.2}
$$

Proof. By using conjugation relations together with (4.4), we can write

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x;\omega}(\kappa)^{\mu} &= \mathcal{M}^{-1}_{\omega}\circ{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}\circ\mathcal{M}_{\omega}\kappa^{\mu}\\
&= \mathcal{M}^{-1}_{\omega}\circ\left[{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}\omega(x)(\kappa)^{\mu}\right]=\mathcal{M}^{-1}_{\omega}\circ\left[{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell}{\ell!}(\kappa)^{\ell}(\kappa)^{\mu}\right]\\
&= \mathcal{M}^{-1}_{\omega}\circ\left[\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell}{\ell!}{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x}\kappa^{\mu+\ell}\right]\\
&= \mathcal{M}^{-1}_{\omega}\circ\left[\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell\Gamma(1+\mu+\ell)}{\ell!\Gamma(1+\sigma+\mu+\ell)}\kappa^{\sigma+\mu+\ell}\right]\\
&= \sum_{\ell=0}^{\infty}\frac{\varsigma_\ell\Gamma(1+\mu+\ell)}{\ell!\Gamma(1+\sigma+\mu+\ell)}\frac{\kappa^{\sigma+\mu+\ell}}{\omega(x)}.
\end{aligned}
$$

We have successfully proven the first identity with the detailed steps provided above. The second identity can be derived directly from the first by using analytic continuation [16]. □

We obtain the following corollary by putting $\mu = 0$ in the above proposition.

Corollary 4.1.6. *If $\sigma > 0$, then for $\kappa = x - c$, we have*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x;\omega}(1) &= \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell}{\Gamma(1+\sigma+\ell)} \frac{\kappa^{\sigma+\ell}}{\omega(x)},\\
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x;\omega}(1) &= \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell}{\Gamma(1+\ell-\sigma)} \frac{\kappa^{\ell-\sigma}}{\omega(x)}.
\end{aligned}
$$

Proof. By setting $\mu = 0$ in Proposition 4.2.7, we get

$$
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x;\omega}(1) = \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell}{\Gamma(1+\sigma+\ell)} \frac{\kappa^{\sigma+\ell}}{\omega(x)}.
$$

Similarly, we get the result for the second part by using analytic continuation. □

4.1.1 Properties

This subsection outlines the essential properties of WFC, including the semigroup property and composition relations.

Lemma 4.1.7. *[16] For $\sigma_1 > 0,\ \sigma_2 > 0$, and $\mathrm{u} \in L^1(c,d)$ the WFO have the following semigroup properties*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma_1}_{x;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma_2}_{x;\omega}\mathrm{u}(x) &= {}^{RL}_{c^+}\mathcal{I}^{\sigma_1+\sigma_2}_{x;\omega}\mathrm{u}(x),\\
{}^{RL}_{x}\mathcal{I}^{\sigma_1}_{d^-;\omega}\,{}^{RL}_{x}\mathcal{I}^{\sigma_2}_{d^-;\omega}\mathrm{u}(x) &= {}^{RL}_{x}\mathcal{I}^{\sigma_1+\sigma_2}_{d^-;\omega}\mathrm{u}(x)
\end{aligned}
$$

Lemma 4.1.8. *[16] For $\sigma_1 > 0,\ \sigma_2 > 0$, and $\mathrm{u} \in AC^{n+\lfloor\sigma_1\rfloor+1}[c,d]$, we have*

$$
\mathcal{D}^{n}\,{}^{RL}_{c^+}\mathcal{D}^{\sigma_1}_{x;\omega}\mathrm{u}(x) = {}^{RL}_{c^+}\mathcal{D}^{n+\sigma_1}_{x;\omega}\mathrm{u}(x), \quad \sigma_1 > 0,\ n \in \mathbb{N},
$$

Proposition 4.1.9. *[16] For the weighted RL fractional integrals, the following composition property is valid when the semigroup property is not*

$$
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x;\omega}\,{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x;\omega}\mathrm{u}(x) = \mathrm{u}(x) - \sum_{\ell=1}^{n} \frac{(x-c)^{\sigma-\ell}}{\Gamma(\sigma-\ell+1)} \cdot \frac{\omega(c)}{\omega(x)} \cdot \lim_{x\to c^+} {}^{RL}_{c^+}\mathcal{D}^{\sigma-\ell}_{x;\omega}\mathrm{u}(x),
$$

$$
{}^{RL}_{x}\mathcal{I}^{\sigma}_{d^-;\omega}\,{}^{RL}_{x}\mathcal{D}^{\sigma}_{d^-;\omega}\mathrm{u}(x) = \mathrm{u}(x) - \sum_{\ell=1}^{n} \frac{(d-x)^{\sigma-\ell}}{\Gamma(\sigma-\ell+1)} \cdot \frac{\omega(x)}{\omega(d)} \cdot \lim_{x\to d^-} {}^{RL}_{x}\mathcal{D}^{\sigma-\ell}_{d^-;\omega(x)}\mathrm{u}(x)
$$

where $\sigma > 0$ and $n = \lfloor\sigma\rfloor + 1$. The function u *can be any function for which the relevant expressions are well-defined. In particular, it is sufficient for* u *to belong to* $C^n[c,d]$.

Proposition 4.1.10. *[16] For the Caputo weighted derivatives, the following composition relations are valid when the semigroup properties are not*

$$
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{x;\omega}\,{}^{C}_{c^+}\mathcal{D}^{\sigma}_{x;\omega}\mathrm{u}(x) = \mathrm{u}(x) - \sum_{\ell=0}^{n-1} \frac{(x-c)^{\ell}}{\ell!} \cdot \frac{\omega(c)}{\omega(x)} \cdot \lim_{x\to c^+}\left(\frac{d}{dx} + \frac{\omega'}{\omega}\right)^{\ell}\mathrm{u}(x),
$$

$$
{}^{RL}_{x}\mathcal{I}^{\sigma}_{d^-;\omega}\,{}^{C}_{x}\mathcal{D}^{\sigma}_{d^-;\omega}\mathrm{u}(x) = \mathrm{u}(x) - \sum_{\ell=0}^{n-1} \frac{(d-x)^{\ell}}{\ell!} \cdot \frac{\omega(x)}{\omega(d^-)} \cdot \lim_{x\to d^-}\left(\frac{d}{dx} - \frac{\omega'}{\omega}\right)^{\ell}\mathrm{u}(x)
$$

where $\sigma > 0$, $n = \lfloor\sigma\rfloor + 1$ and $\mathrm{u} \in C^n[c,d]$ is sufficient.

4.2 Weighted fractional Calculus with respect to functions

This sections explore the definition, properperties and some new insights related to this generalized class.

Definition 4.2.1. *[33] The generalized weighted Lebesgue space $L^r_{\varphi;\omega}(c,d)$ as the class of measurable function* z*, is defined as $L^r_{\varphi;\omega}(c,d) = \left\{\mathrm{z} : \|\mathrm{z}\|_{L^r_{\varphi;\omega}} < \infty\right\}$, where*

$$
\|\mathrm{z}\|_{L^r_{\varphi;\omega}} = \left(\int_c^d |\omega(s)\mathrm{z}(s)|^r \varphi'(s)\,ds\right)^{\frac{1}{r}},
$$

and when $r = \infty$, then

$$
\|\mathrm{z}\|^{\infty}_{\varphi;\omega} = \underset{c\le x\le d}{esssup}|\mathrm{z}(s)|,
$$

where weight function $\omega \neq 0$ and φ be a strictly increasing function on interval $[c,d]$.

Considering $\varphi(s) = s$, $\omega(s) = 1$ in above the definition, the space $L^r_{\varphi;\omega}(c,d)$ concides with the Lebesgue space $L^r(c,d)$ [8].

Next, we define the WFO with respect to functions.

Definition 4.2.2. *[16, 31, 32] If $\mathrm{u} \in L^1_{\varphi}(c,d)$ and $\sigma > 0$. Then the operators ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}$ and ${}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-;\omega}$ are defined as*

$$
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\mathrm{u}(x) = \frac{1}{\Gamma(\sigma)\omega(x)} \int_c^x \omega(s)\mathrm{u}(s)\varphi'(s)\big(\varphi(x)-\varphi(s)\big)^{\sigma-1}\,ds, \quad x > c,
$$

$$
{}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-;\omega}\mathrm{u}(x) = \frac{\omega(x)}{\Gamma(\sigma)} \int_x^d \mathrm{u}(s)\varphi'(s)\omega(s)^{-1}\left(\varphi(s)-\varphi(x)\right)^{\sigma-1}\,ds, \quad x < d,
$$

respectively, where $\varphi \in C^1[c,d]$ and is strictly increasing function and $\omega \in L^\infty(c,d)$ is a weight function.

Definition 4.2.3. *[16, 31, 32] If $\mathrm{u}, \omega \in AC^n_\varphi[c,d]$ where φ is an increasing positive function with $\varphi' > 0$. Then the operators ${}^{RL}_{c^+}\mathcal{D}^\sigma_{\varphi;\omega}$ and ${}^{RL}_{\varphi}\mathcal{D}^\sigma_{d^-;\omega}$ of order $\sigma > 0$ are defined as*

$$ {}^{RL}_{c^+}\mathcal{D}^\sigma_{\varphi;\omega}\mathrm{u}(x) = \left(\frac{\omega(x)^{-1}}{\varphi'(x)} \cdot \frac{d}{dx}\omega(x)\right)^n {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega}\mathrm{u}(x), $$

$$ {}^{RL}_{\varphi}\mathcal{D}^\sigma_{d^-;\omega}\mathrm{u}(x) = \left(-\frac{\omega(x)}{\varphi'(x)} \cdot \frac{d}{dx}\omega(x)^{-1}\right)^n {}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\mathrm{u}(x), $$

respectively, where $\sigma > 0$ and $n = \lfloor \sigma \rfloor + 1$. Additionally, for $0 < \sigma < 1$, we have

$$ {}^{RL}_{c^+}\mathcal{D}^\sigma_{\varphi;\omega}\mathrm{u}(x) = \left(\frac{1}{\omega(x)\varphi'(x)} \cdot \frac{d}{dx}\omega(x)\right) {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega}\mathrm{u}(x), $$

$$ {}^{RL}_{\varphi}\mathcal{D}^\sigma_{d^-;\omega}\mathrm{u}(x) = \left(-\frac{\omega(x)}{\varphi'(x)} \cdot \frac{d}{dx}\frac{1}{\omega(x)}\right) {}^{RL}_{\varphi}\mathcal{I}^{1-\sigma}_{d^-;\omega}\mathrm{u}(x), $$

Definition 4.2.4. $[16,\ 31,\ 32]$ *Let $\mathrm{u} \in C^n_\varphi[c,d]$, weight function ω and increasing function φ are in $C^n_\varphi[c,d]$ with $\varphi' > 0$ and $\omega \neq 0$. Then the operators ${}^{C}_{c^+}\mathcal{D}^\sigma_{\varphi;\omega}$ and ${}^{C}_{\varphi}\mathcal{D}^\sigma_{d^-;\omega}$ of order $\sigma >$ are defined as*

$$ {}^{C}_{c^+}\mathcal{D}^\sigma_{\varphi;\omega}\mathrm{u}(x) = {}^{RL}_{c^+} \cdot \mathcal{I}^{n-\sigma}_{\varphi;\omega}\left(\frac{\omega(x)^{-1}}{\varphi'(x)} \cdot \frac{d}{dx}\omega(x)\right)^n \mathrm{u}(x), $$

$$ {}^{C}_{\varphi}\mathcal{D}^\sigma_{d^-;\omega}\mathrm{u}(x) = {}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(-\frac{\omega(x)}{\varphi'(x)} \cdot \frac{d}{dx}\omega(x)^{-1}\right)^n \mathrm{u}(x), $$

respectively, where $\sigma > 0$ and $n = \lfloor \sigma \rfloor + 1$. Additionally, for $0 < \sigma < 1$, we have

$$ {}^{C}_{c^+}\mathcal{D}^\sigma_{\varphi;\omega}\mathrm{u}(x) = {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega}\left(\frac{\omega(x)^{-1}}{\varphi'(x)} \cdot \frac{d}{dx}\omega(x)\right) \mathrm{u}(x), $$

$$ {}^{C}_{\varphi}\mathcal{D}^\sigma_{d^-;\omega}\mathrm{u}(x) = {}^{RL}_{\varphi}\mathcal{I}^{1-\sigma}_{d^-;\omega}\left(-\frac{\omega(x)}{\varphi'(x)} \cdot \frac{d}{dx}\omega(x)^{-1}\right) \mathrm{u}(x), $$

Remark 1. [16] By defining operators of WFC through conjugation relations, as described in Section §4.2.1, we can recover the existing fractional operators by appropriately choosing the weight functions ω and the functions φ. This framework provides a unified approach to studying and generalizing various special cases, including

- when $\omega(x) = 1$ or any constant k and $\varphi(x) = x$, the Definitions 4.2.2, 4.2.3, 4.2.4 reduce to the original RL and Caputo fractional operators [8, 9].

- the tempered fractional operators obtained as a special case using $\omega(x) = e^{\eta}$ and $\varphi(x) = x$. This definition can be found [37] and the references listed therein.
- when the functions $\omega(x) = e^{\eta}$ and $\varphi(x) = \log(x)$ are used, they gives the Hadamard-type FC, as mentioned in the literature [14, 15].
- when the function $\omega(x) = 1$, it can be simplified to the φ–fractional operators [8, 30]
- $\omega(x) = e^{\eta\varphi(x)}$, permits the φ–tempered fractional operators [15, 37].
- when $\omega(x) = e^{\eta\zeta}$ and $\varphi(x) = x^{\eta}$, we obtain the operators Erdelyi–Kober FC [8, 12].
- when $\varphi(x) = log(\varphi(x))$ and $\omega(x)$=$\varphi(x)^{\lambda}$, it gives the Hadamard–type fractional operators with respect to functions [15].

4.2.1 Conjugation relations and their importance

Conjugation relations play a significant role in FC, facilitating the understanding of new generalized operators and their relationships. In the context of WFC with respect to functions, the operators involve conjugation relations. Fahad and Fernandez [16] provide a detailed analysis of left WFOs in the context of generalized WFC. In the case of right WFOs with respect to functions, a conjugation relation is established by dividing by the weight function ω, applying the φ–fractional operators with of the same order, and then multiplying by the weight function ω again. One of the advantages of conjugation relations is their ability to serve as a powerful tool for establishing properties and proving fundamental results for numerous generalized fractional operators in terms of certain fundamental fractional operators.

Proposition 4.2.5. *[16, 30] The operators in Definition 4.2.2, 4.2.3 and 4.2.4 can be expressed by conjugation of the original left and right RL fractional operators or φ–fractional operators with the operator $\mathcal{M}_\omega$ defined as $\mathcal{M}_\omega \mathrm{u}(x) = \omega(x)\mathrm{u}(x)$*

$$\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} &= \mathcal{M}_\omega^{-1} \circ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi} \circ \mathcal{M}_\omega, \\
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} &= \mathcal{M}_\omega^{-1} \circ {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi} \circ \mathcal{M}_\omega, \\
{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} &= \mathcal{M}_\omega^{-1} \circ {}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi} \circ \mathcal{M}_\omega.
\end{aligned}$$

In the case of right WFO with respect to functions, conjugation relations are given by

$$
\begin{aligned}
{}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-;\omega} &= \mathcal{M}_{\omega} \circ {}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-} \circ \mathcal{M}_{\omega}^{-1}, \\
{}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega} &= \mathcal{M}_{\omega} \circ {}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-} \circ \mathcal{M}_{\omega}^{-1}, \\
{}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega} &= \mathcal{M}_{\omega} \circ {}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-} \circ \mathcal{M}_{\omega}^{-1}.
\end{aligned}
$$

4.2.2 Examples and properties

This subsection examines the properties of WFOs with respect to functions. Many properties have been extensively studied in previous works [16, 31, 33]. We will discuss some of these established properties as well as new results specific to these operators. Previous studies typically focused on power rules that include weight in the denominator and composition relations for $\sigma > 0$. However, this work introduces the power rule of these operators without including any weight in the denominator and composition relations for $0 < \sigma < 1$. These new results are used to derived the main results.

Proposition 4.2.6. *[16] Let* $\mathrm{u}(x) = \frac{\kappa^{\mu}}{\omega(x)}$ *be a power function, where* $\kappa = \varphi(x) - \varphi(c)$ *and* $\sigma > 0$ *and* $\mu > -1$*, then*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\mathrm{u}(x) &= \frac{\Gamma(1+\mu)}{\Gamma(1+\mu+\sigma)}\frac{\kappa^{\mu+\sigma}}{\omega(x)}, \\
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{u}(x) &= \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\sigma)}\frac{\kappa^{\mu-\sigma}}{\omega(x)}.
\end{aligned}
$$

And for some $\mu \geq \lfloor \sigma \rfloor$*, we have*

$$
{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{u}(x) = \begin{cases} 0, & \textit{if } \mu \in \{0, 1, 2, \ldots, n-1\}, \\ \frac{\Gamma(1+\mu)}{\Gamma(1+\mu-\sigma)}\frac{\kappa^{\mu-\sigma}}{\omega(x)}, & \textit{otherwise}. \end{cases} \tag{4.3}
$$

Proposition 4.2.7. *Consider a power function* κ^{μ}*, where* $\kappa = \varphi(x) - \varphi(c)$ *and* $\sigma > 0$ *for some* $\mu > -1$*, then*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\kappa^{\mu} &= \sum_{\ell=0}^{\infty} \frac{\varsigma_{\ell}\Gamma(\mu+\ell+1)}{\ell!\Gamma(\sigma+\mu+\ell+1)}\frac{\kappa^{\sigma+\mu+\ell}}{\omega(x)}, \\
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\kappa^{\mu} &= \sum_{\ell=0}^{\infty} \frac{\varsigma_{\ell}\Gamma(\mu+\ell+1)}{\ell!\Gamma(\mu-\sigma+\ell+1)}\frac{\kappa^{\mu-\sigma+\ell}}{\omega(x)},
\end{aligned}
$$

where for $\ell = 0, 1, 2, \ldots$, $\varsigma_\ell = \lim_{x \to c^+} \mathcal{D}_\varphi^\ell \omega(x)$ *satisfies the following infinite series*

$$\omega(x) = \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell}{\ell!} \kappa^\ell. \tag{4.4}$$

Proof. By using conjugation relations together with (4.4), we can write

$$\begin{aligned}
{}_{c^+}^{RL}\mathcal{I}_{\varphi;\omega}^{\sigma} \kappa^\mu &= \mathcal{M}_\omega^{-1} \circ {}_{c^+}^{RL}\mathcal{I}_{\varphi}^{\sigma} \circ \mathcal{M}_\omega \kappa^\mu \\
&= \mathcal{M}_\omega^{-1} \circ \left[{}_{c^+}^{RL}\mathcal{I}_{\varphi}^{\sigma} \omega(x) \kappa^\mu \right] \\
&= \mathcal{M}_\omega^{-1} \circ \left[{}_{c^+}^{RL}\mathcal{I}_{\varphi}^{\sigma} \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell}{\ell!} \kappa^\ell \kappa^\mu \right] \\
&= \mathcal{M}_\omega^{-1} \circ \left[\sum_{\ell=0}^{\infty} \frac{\varsigma_\ell}{\ell!} {}_{c^+}^{RL}\mathcal{I}_{\varphi}^{\sigma} \kappa^{\mu+\ell} \right] \\
&= \mathcal{M}_\omega^{-1} \circ \left[\sum_{\ell=0}^{\infty} \frac{\varsigma_\ell \Gamma(1+\mu+\ell)}{\ell! \Gamma(1+\sigma+\mu+\ell)} \kappa^{\sigma+\mu+\ell} \right] \\
&= \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell \Gamma(1+\mu+\ell)}{\ell! \Gamma(1+\sigma+\mu+\ell)} \frac{\kappa^{\sigma+\mu+\ell}}{\omega(x)}.
\end{aligned}$$

We have successfully proven the first identity with the detailed steps provided above. The second identity can be derived directly from the first by using analytic continuation [16]. $\square$

By putting $\mu = 0$ in the above proposition, we get the following corollary.

Corollary 4.2.8. *If* $\sigma > 0$*, then for* $\kappa(x) = \varphi(x) - \varphi(c)$

$$\begin{aligned}
{}_{c^+}^{RL}\mathcal{I}_{\varphi;\omega}^{\sigma}(1) &= \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell}{\Gamma(1+\sigma+\ell)} \frac{\kappa^{\sigma+\ell}}{\omega(x)}, \\
{}_{c^+}^{RL}\mathcal{D}_{\varphi;\omega}^{\sigma}(1) &= \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell}{\Gamma(1+\ell-\sigma)} \frac{\kappa^{\ell-\sigma}}{\omega(x)}.
\end{aligned}$$

Proof. By setting $\mu = 0$ in Proposition 4.2.7, we get

$${}_{c^+}^{RL}\mathcal{I}_{\varphi;\omega}^{\sigma}(1) = \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell}{\Gamma(1+\sigma+\ell)} \frac{\kappa^{\sigma+\ell}}{\omega(x)}.$$

Similarly, we get the result for the second part by using analytic continuation. $\square$

Lemma 4.2.9. *For $\sigma_1 > 0$, $\sigma_2 > 0$, $\mathrm{u} \in L^1_\varphi(c,d)$ the following semi-group properties hold for WFO with respect to functions*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma_1}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma_2}_{\varphi;\omega}\mathrm{u}(x) &= {}^{RL}_{c^+}\mathcal{I}^{\sigma_1+\sigma_2}_{\varphi;\omega}\mathrm{u}(x),\\
{}^{RL}_{\varphi}\mathcal{I}^{\sigma_1}_{d^-;\omega}\,{}^{RL}_{\varphi}\mathcal{I}^{\sigma_2}_{d^-;\omega}\mathrm{u}(x) &= {}^{RL}_{\varphi}\mathcal{I}^{\sigma_1+\sigma_2}_{d^-;\omega}\mathrm{u}(x).
\end{aligned}
$$

The composition relations of WFO with respect to functions also referred to as the right inverse property, which have been already studied [16, 33].

Proposition 4.2.10. *[16] If $\mathrm{u},\ \omega \in C^n_\varphi[c,d]$, and $\kappa = \varphi(x) - \varphi(c)$, where φ is an increasing positive function, then the following composition properties for WFO holds*

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{u}(x) &= \mathrm{u}(x) - \sum_{\ell=1}^{n} \frac{\kappa^{\sigma-\ell}}{\Gamma(\sigma-\ell+1)} \cdot \frac{\omega(c)}{\omega(x)} \cdot \lim_{x\to c^+} {}^{RL}_{c^+}\mathcal{D}^{\sigma-\ell}_{\varphi;\omega}\mathrm{u}(x),\\
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\,{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{u}(x) &= \mathrm{u}(x) - \sum_{\ell=0}^{n-1} \frac{\kappa^{\ell}}{\ell!} \cdot \frac{\omega(c)}{\omega(x)} \cdot \lim_{x\to c^+} \mathcal{D}^{\ell}_{\varphi;\omega}\mathrm{u}(x).
\end{aligned}
$$

If $0 < \sigma < 1$, then

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{u}(x) &= \mathrm{u}(x) - \lim_{x\to c^+} {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega}\mathrm{u}(x) \cdot \frac{(\kappa)^{\sigma-1}}{\Gamma(\sigma)} \cdot \frac{\omega(c)}{\omega(x)} \cdot,\\
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\,{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{u}(x) &= \mathrm{u}(x) - \frac{\omega(c)}{\omega(x)}\mathrm{u}(c).
\end{aligned}
$$

Proof. As this result has already been addressed in [16], we follow similar steps of proof presented in that paper, but here we follow for $0 < \sigma < 1$. Since $n = \lfloor \sigma \rfloor + 1$, this implies that $n = 1$.

□

4.3 Mean value theorem

The MVT for fractional operators provides a generalization of the classical MVT to the field of FC, enabling the analysis of functions with non-integer order. The classical and fractional forms of this theorem are further detailed in [38]. This section aims to prove the MVT for WFO with respect to functions. Before presenting our main results, we will first review some preliminary findings that will be helpful in proving our main theorems.

Theorem 4.3.1. *[[57], Theorem 4.2d] If y, z are integrable, then convolution product $y * z$ exists and is integrable*

$$\int |y * z| = \int |y| \int |z| \, .$$

Remark 2. Since y and z are integrable with a strictly monotonic increasing function φ and continuous on a closed interval then by [[56], Theorem 7.2.7–8], $y\circ\varphi$ is also integrable. Since the inverse of a strictly increasing function is also an increasing function, and the inverse of a continuous function is also continuous then by [[56], Theorem 7.3.14], $y \circ \varphi^{-1}$ is also integrable. Then by Theorem 4.3.1 φ-convolution $y *_\varphi z$ is integrable, where $y *_\varphi z$ is defined as

$$y *_\varphi z = \mathcal{Q}_\varphi \left(* \left(y \circ \varphi^{-1}, z \circ \varphi^{-1} \right) \right).$$

Generalized MVT in terms of classical integral calculus [38] can be expressed as

$$\int_c^d y(x) z(x) \, dx = y(\zeta) \int_c^d z(x) \, dx, \tag{4.5}$$

where some constant $\zeta \in (c, d)$, $y \in C[c, d]$, z is Lebesgue integrable on closed interval $[c, d]$ and is sign consistent in $[c, d]$.

Theorem 4.3.2. *Let $\sigma > 0$, $\varphi \in C^1[c, d]$ be an increasing function, z be Lebesgue integrable on $[c, d]$ and sign consistent in the interval $(c, d]$. Assume that $y \in C[c, d]$, then $\forall\, x \in (c, d]$ $\exists$ some constant $\zeta \in (c, x)$, sucn that the MVT for ${}^{RL}_{c^+}\mathcal{I}^\sigma_\varphi$ can be written as*

$${}^{RL}_{c^+}\mathcal{I}^\sigma_\varphi z(x) y(x) = y(\zeta) \mathcal{I}^\sigma_\varphi z(x). \tag{4.6}$$

If $\sigma \geq 1$ and $z \in C[c, d]$, then this result holds for every $x \in (c, d]$.

Proof. By using the definition of ${}^{RL}_{c^+}\mathcal{I}^\sigma_\varphi$, we get

$${}^{RL}_{c^+}\mathcal{I}^\sigma_\varphi y(x) z(x) = \frac{1}{\Gamma(\sigma)} \int_c^x \frac{y(s) z(s) \varphi'(s)}{\big(\varphi(x) - \varphi(s)\big)^{1-\sigma}} \, ds.$$

Assume that $\sigma \geq 1$ and $x \in (c, d]$, then $\big(\varphi(x) - \varphi(s)\big)^{1-\sigma}$ is continuous. Thus, the function $h(s) = \frac{\left(\varphi(x) - \varphi(s)\right)^{\sigma-1} z(s) \varphi'(s)}{\Gamma(\sigma)}$ is integrable and has a consistent value in the interval $(c, d]$, then we get

$${}^{RL}_{c^+}\mathcal{I}^\sigma_\varphi z(x) y(x) = \int_c^x y(s) h(s) \, ds.$$

By the classical generalized MVT, we obtain

$$
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi} z(x)y(x) = y(\zeta)\int_c^x h(s)\,ds = y(\zeta)\mathcal{I}^{\sigma}_{\varphi} z(x). \tag{4.7}
$$

Here, further, we have two cases.

(i) If $0 < \sigma < 1$ and function z and φ are continuous then the same line of proof works.

(ii) If $0 < \sigma < 1$ and function z is only integrable and φ is a continuous strictly monotone increasing function φ then by Remark 2 integrability of function h holds for almost all x.

□

If we consider $\varphi(x) = x$, then (4.6) reduces to the RL fractional MVT mentioned in [38]. If we put $z(x) = 1$ in Theorem 4.3.2, we get the following relation

$$
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi} y(x) = y(\zeta)\frac{\big(\varphi(x) - \varphi(c)\big)^{\sigma}}{\Gamma(\sigma+1)}. \tag{4.8}
$$

Theorem 4.3.3. *Assume the hypothesis of Theorem 4.3.2 and the weight function* $\omega \in C[c,d]$*. If* $x \in (c,d]$*, then there exists some constant* $c \in (c,x)$*, such that the MVT for* ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}$ *can be written as*

$$
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} z(x)y(x) = y(\zeta)\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} z(x). \tag{4.9}
$$

Proof. We prove the MVT for ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}$ by using conjugation relations

$$
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} = \mathcal{M}_{\omega}^{-1} \circ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi} \circ \mathcal{M}_{\omega}.
$$

We begin by applying the operator $\mathcal{M}_{\omega}$ to the product $z(x)y(x)$

$$
\mathcal{M}_{\omega} z(x)y(x) = \omega(x)z(x)y(x).
$$

Now, applying the integral operator ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}$, we obtain

$$
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathcal{M}_{\omega} z(x)y(x) = {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\omega(x)z(x)y(x).
$$

Assume that $h(x) = \omega(x)z(x)$ then, we get

$$
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathcal{M}_{\omega} z(x)y(x) = {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi} h(x)y(x).
$$

Then, by the MVT (4.6), we have

$$ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathcal{M}_{\omega}z(x)y(x) = y(\zeta)\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}h(x) = y(\zeta)\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}z(x)\omega(x). $$

Finally, applying the operator $\mathcal{M}_{\omega}^{-1}$ yields the MVT for ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}$ as follow

$$ \begin{aligned} \mathcal{M}_{\omega}^{-1}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathcal{M}_{\omega}z(x)y(x) &= \mathcal{M}_{\omega}^{-1}y(\zeta)\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}z(x)\omega(x) \\ &= y(\zeta)\mathcal{M}_{\omega}^{-1}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}\mathcal{M}_{\omega}z(x) \\ &= y(\zeta)\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}z(x). \end{aligned} $$

□

As a special case, if we set $\omega(x) = 1$ in the above theorem, we recover the MVT for the operator ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi}$. Furthermore, if we take $z(x) = 1$ in Theorem 4.3.3 and apply Corollary 4.2.8, we obtain the following simplified relation

$$ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y(x) = y(\zeta)\sum_{\ell=0}^{\infty}\frac{\varsigma_{\ell}}{\Gamma(\sigma+\ell+1)}\frac{(\varphi(x)-\varphi(c))^{\sigma+\ell}}{\omega(x)}. \tag{4.10} $$

By taking $\omega(x) = e^{\lambda\varphi(x)}$ in (4.10), we recover the MVT for tempered fractional integrals with respect to functions, as established in [37]. This shows that our result includes the tempered fractional integrals with respect to functions as a special case.

Proposition 4.3.4. *If* $\sigma > 0$, $p \in \mathbb{N}$, *and* $\mathrm{u} \in AC^{n+p}_{\varphi}[c, d]$, *then* $\forall\ q \in \mathbb{N}$, *and for some* $\zeta \in (c, x) \subset (c, d), \delta \in (x, d) \subset (c, d)$, *we get the following expressions*

$$ \left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\right)^{q}\left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{r}\mathrm{u}(x) = \left({}^{RL}_{c^+}\mathcal{D}^{q\sigma}_{\varphi;\omega}\right)^{r}\mathrm{u}(\zeta)\sum_{\ell=0}^{\infty}\frac{\varsigma_{\ell}}{\Gamma(\ell+q\sigma+1)}\frac{(\varphi(x)-\varphi(c))^{\ell+q\sigma}}{\omega(x)}, $$

$$ \left({}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-;\omega}\right)^{q}\left({}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega}\right)^{r}\mathrm{u}(x) = \left({}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega}\right)^{r}\mathrm{u}(\delta)\sum_{\ell=0}^{\infty}\frac{\varsigma_{\ell}}{\Gamma(\ell+q\sigma+1)}\omega(x)(\varphi(d^-)-\varphi(x))^{\ell+q\sigma}. $$

Proof. By using the semigroup property for fractional integrals, we can write

$$ \underbrace{{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\circ\cdots\circ{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}}_{q-time} = {}^{RL}_{c^+}\mathcal{I}^{q\sigma}_{\varphi;\omega}. $$

Utilizing the Definition 4.2.2 together with (4.10), we have

$$\begin{aligned}\left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\right)^q \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^r \mathrm{u}(x) &= {}^{RL}_{c^+}\mathcal{I}^{q\sigma}_{\varphi;\omega} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^r \mathrm{u}(x) \\ &= \frac{1}{\omega(x)\Gamma(q\sigma)} \int_c^x \frac{\omega(s)\left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^r \mathrm{u}(s)\varphi'(s)}{\big(\varphi(x)-\varphi(s)\big)^{1-q\sigma}}\, ds \\ &= \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^r \mathrm{u}(\zeta)\, {}^{RL}_{c^+}\mathcal{I}^{q\sigma}_{\varphi;\omega}(1). \\ &= \left({}^{RL}_{c^+}\mathcal{D}^{q\sigma}_{\varphi;\omega}\right)^r \mathrm{u}(\zeta) \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell}{\Gamma(\ell+q\sigma+1)} \frac{(\varphi(x)-\varphi(c))^{\ell+q\sigma}}{\omega(x)},\end{aligned}$$

with this final step, our proof is complete. Similarly, the second part can be proved by using a similar strategy used to prove the first part. □

Theorem 4.3.5. *Assume the hypothesis of Theorem 4.3.3 with* $0 < \sigma < 1$. *Then,* $\forall$ $x \in (c, d]$, $\exists$ *some constant* $\zeta \in (c, x)$*, such that the MVT for* ${}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$ *can be written as*

$$ {}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta) = \frac{\omega(x)y(x)-\omega(c)y(c)}{\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell}{\Gamma(\ell+\sigma+1)}(\varphi(x)-\varphi(c))^{\ell+\sigma}}. \tag{4.11}$$

Proof. Using the Definition 4.2.2 together with MVT (4.10), we get

$$\begin{aligned}{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\,{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} &= {}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta)\frac{1}{\Gamma(\sigma)\omega(x)}\int_c^x \frac{\omega(s)\varphi'(s)}{\big(\varphi(x)-\varphi(s)\big)^{1-\sigma}}\, ds \\ &= {}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta)\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell}{\Gamma(\ell+\sigma+1)}\frac{(\varphi(x)-\varphi(c))^{\sigma+\ell}}{\omega(x)}.\end{aligned}$$

By using Proposition 4.2.10, we obtain

$$\begin{aligned}y(x)-\frac{\omega(c)}{\omega(x)}y(c) &= {}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta)\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell}{\Gamma(\ell+\sigma+1)}\frac{(\varphi(x)-\varphi(c))^{\ell+\sigma}}{\omega(x)} \\ \omega(x)y(x)-\omega(c)y(c) &= {}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta)\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell}{\Gamma(\ell+\sigma+1)}(\varphi(x)-\varphi(c))^{\ell+\sigma} \\ &= \frac{\omega(x)y(x)-\omega(c^+)y(c^+)}{\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell}{\Gamma(\ell+\sigma+1)}(\varphi(x)-\varphi(c))^{\sigma+\ell}}.\end{aligned}$$

This concludes the proof. □

Theorem 4.3.6. *Assume the hypothesis of Theorem 4.3.3 with* $0 < \sigma < 1$*. Then,* $\forall\, x \in (c, d]$*,* $\exists\, c \in (c, x)$*, such that the MVT for* ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$ *can be written as*

$$
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta) = \frac{\left(\omega(x)y(x) - \frac{(\varphi(x)-\varphi(c))^{\sigma-1}}{\Gamma(\sigma)}\omega(c^+) \lim\limits_{x\to c^+} {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega} y(x)\right)}{\sum_{\ell=0}^{\infty} \frac{\varsigma_\ell(\varphi(x)-\varphi(c))^{\ell+\sigma}}{\Gamma(\ell+\sigma+1)}}. \tag{4.12}
$$

Proof. By using the Definition 4.2.2 and (4.10), we obtain

$$
\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\, {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(x) &= {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta) \frac{1}{\Gamma(\sigma)\omega(x)} \int_c^x \frac{\omega(s)\varphi'(s)}{\big(\varphi(x)-\varphi(s)\big)^{1-\sigma}}\, ds \\
&= {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta)\, {}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}(1) = {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta) \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell(\varphi(x)-\varphi(c))^{\ell+\sigma}}{\omega(x)\Gamma(\ell+\sigma+1)}.
\end{aligned}
$$

By using Proposition 4.2.10, we have

$$
\begin{aligned}
&y(x) - \frac{\big(\varphi(x)-\varphi(c)\big)^{\sigma-1}}{\Gamma(\sigma)} \cdot \frac{\omega(c)}{\omega(x)} \cdot \lim_{x\to c^+} {}^{RL}_{c^+}\mathcal{I}^{\sigma-1}_{\varphi;\omega} y(x) \\
&\qquad\qquad = {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta) \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell(\varphi(x)-\varphi(c))^{\ell+\sigma}}{\omega(x)\Gamma(\ell+\sigma+1)} \\
&\omega(x)y(x) - \frac{(\varphi(x)-\varphi(c))^{\sigma-1}}{\Gamma(\sigma)} \cdot \omega(c^+) \lim_{x\to c^+} {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega} y(x) \\
&\qquad\qquad = {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta) \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell(\varphi(x)-\varphi(c))^{\ell+\sigma}}{\Gamma(\ell+\sigma+1)}
\end{aligned}
$$

$$
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(\zeta) = \frac{\left(\omega(x)y(x) - \frac{(\varphi(x)-\varphi(c))^{\sigma-1}}{\Gamma(\sigma)}\omega(c^+) \lim\limits_{x\to c^+} {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega} y(x)\right)}{\sum_{\ell=0}^{\infty} \frac{\varsigma_\ell(\varphi(x)-\varphi(c))^{\ell+\sigma}}{\Gamma(\ell+\sigma+1)}}.
$$

This completes the proof. □

Remark 3. The results presented in this section extend the classical mean value theorems to the context of fractional operators. Specifically, we have demonstrated that these generalized theorems hold for a broader class of functions and operators. Specifically, if we consider $\sigma = 1$, $\omega = 1$, $\varphi = x$, the results reduce to the classical mean value theorems. Chossing $\varphi = x$ and $\omega = 1$, the results reduces to the fractional operators [38, 41]. Setting the weight function $\omega = 1$, the results simply reduces to the φ– fractional operators [39]. Additionally, when $\omega(x) = e^{\lambda\varphi(x)}$ our results corresponds to the φ–tempered fractional operators [37], incorporating an exponential tempering effect. These special cases demonstrate the flexibility and consistency of our generalized approach, providing a coherent link to classical and modern results in fractional calculus.

4.4 Taylor's theorem

This section presents Taylor's theorem for weighted CFD and RL derivatives with respect to functions, which represents a more generalized form of Taylor's theorem. Many different types of generalizations of Taylor's theorem have been explored in the literature [37, 39, 40, 63], including RL and Caputo fractional operators.

Theorem 4.4.1. *Assume that* $0 < \sigma < 1$, $m \in \mathbb{N}$*, and* y *be any function such that* ${}^{C}_{c^+}\mathcal{D}^{\ell\sigma}_{\varphi;\omega}y$ *exists and is continuous* $\forall\ \ell = 0, 1, 2, \ldots, m+1$. *Then,* $\forall\ x \in [c, d]$ *and for some constant* $\zeta \in (c, x)$*, we have*

$$y(x) = \frac{1}{\omega(x)}\Bigg\{\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{m+1} y(\zeta)\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell(\varphi(x)-\varphi(c))^{\ell+\sigma(m+1)}}{\Gamma(1+\ell+\sigma(m+1))} + \omega(c)\sum_{\ell=0}^{m}\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(c)\frac{(\varphi(x)-\varphi(c))^{\ell\sigma}}{\Gamma(1+\ell\sigma)}\Bigg\}.$$

Proof. For different values of ℓ, we find the difference between the given function, by using the semigroup property, we get

$$\begin{aligned}&\left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\right)^{\ell}\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(x) - \left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\right)^{\ell+1}\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell+1} y(x)\\ =&{}^{RL}_{c^+}\mathcal{I}^{\ell\sigma}_{\varphi;\omega}\left[\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(x) - {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\left(\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(x)\right)\right].\end{aligned}$$

By using Proposition 4.2.10, we get

$$\begin{aligned}
&\left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\right)^{\ell}\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(x)-\left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\right)^{\ell+1}\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell+1} y(x)\\
&={}^{RL}_{c^+}\mathcal{I}^{\ell\sigma}_{\varphi;\omega}\left[\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(x)-\left\{\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(x)-\frac{\omega(c)}{\omega(x)}\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(c)\right\}\right]\\
&=\omega(c)\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(c)\cdot{}^{RL}_{c^+}\mathcal{I}^{\ell\sigma}_{\varphi;\omega}\left(\frac{1}{\omega(x)}\right)\\
&=\omega(c)\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(c)\cdot\frac{1}{\omega(x)}{}^{RL}_{c^+}\mathcal{I}^{\ell\sigma}_{\varphi(x)}(1)\\
&=\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(c)\cdot\frac{\omega(c)}{\omega(x)}\left(\frac{\varphi(x)-\varphi(c)^{\ell\sigma}}{\Gamma(\ell\sigma+1)}\right).
\end{aligned}$$

Summing this result over ℓ from 0 to m, we obtain

$$y(x)-\left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\right)^{m+1}\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{m+1} y(x)=\frac{\omega(c)}{\omega(x)}\sum_{\ell=0}^{m}\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(c)\cdot\frac{(\varphi(x)-\varphi(c))^{\ell\sigma}}{\Gamma(1+\ell\sigma)}. \tag{4.13}$$

By using the MVT (4.10), we get

$$\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma(m+1)}_{\varphi;\omega}\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{m+1} y(x)&=\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{m+1} y(\zeta)\\
&\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell}{\Gamma(\ell+\sigma(m+1)+1)}\frac{(\varphi(x)-\varphi(c))^{\ell+\sigma(m+1)}}{\omega(x)}.
\end{aligned} \tag{4.14}$$

Substituting (4.14) in (4.13), we obtain

$$\begin{aligned}
y(x)=&\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{m+1} y(\zeta)\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell}{\Gamma(\ell+\sigma(m+1)+1)}\frac{(\varphi(x)-\varphi(c))^{\ell+\sigma(m+1)}}{\omega(x)}\\
&+\frac{\omega(c)}{\omega(x)}\sum_{\ell=0}^{m}\left({}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell} y(c)\frac{(\varphi(x)-\varphi(c))^{\ell\sigma}}{\Gamma(\ell\sigma+1)}.
\end{aligned}$$

Thus, we obtain the desired result. □

Theorem 4.4.2. *Assume that $m\in\mathbb{N}$, $0<\sigma<1$ and y be any function such that ${}^{RL}_{c^+}\mathcal{D}^{\ell\sigma}_{\varphi;\omega}y$ exists and is continuous $\forall\ \ell=0,1,2,\dots,m+1$. Then, $\forall\ x\in[c,d]$ and for some $\zeta\in(c,d)$,* *we have*

$$y(x) = \frac{1}{\omega(x)} \left\{ \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{m+1} y(\zeta) \sum_{\ell=0}^{\infty} \frac{\varsigma_\ell(\varphi(x)-\varphi(c))^{\ell+\sigma(m+1)}}{\Gamma(\ell+\sigma(m+1)+1)} \right.$$
$$\left. + \sum_{\ell=0}^{m} \frac{\left(\varphi(x)-\varphi(c)\right)^{\sigma(\ell+1)-1}}{\Gamma(\sigma(\ell+1))} \omega(c) \lim_{x\to c^+} {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x) \right\}.$$

Proof. Initially, we find the difference between the function for different values of ℓ, by using semi-group property, we get

$$\left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} \right)^{\ell} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x) - \left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} \right)^{\ell+1} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell+1} y(x)$$
$$= {}^{RL}_{c^+}\mathcal{I}^{\ell\sigma}_{\varphi;\omega} \left[\left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x) - {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \left(\left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x) \right) \right].$$

By using Proposition 4.2.10, we get

$$\left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} \right)^{\ell} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x) - \left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} \right)^{\ell+1} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell+1} y(x)$$
$$= {}^{RL}_{c^+}\mathcal{I}^{\ell\sigma}_{\varphi;\omega} \left[\left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x) \right.$$
$$\left. - \left\{ \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x) - \frac{\left(\varphi(x)-\varphi(c)\right)^{\sigma-1}}{\Gamma(\sigma)} \cdot \frac{\omega(c)}{\omega(x)} \cdot \lim_{x\to c^+} {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x) \right\} \right]$$
$$= {}^{RL}_{c^+}\mathcal{I}^{\ell\sigma}_{\varphi;\omega} \left[\frac{\left(\varphi(x)-\varphi(c)\right)^{\sigma-1}}{\Gamma(\sigma)} . \frac{\omega(c)}{\omega(x)} . \lim_{x\to c^+} {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x) \right]$$
$$= {}^{RL}_{c^+}\mathcal{I}^{\ell\sigma}_{\varphi;\omega} \left(\frac{\left(\varphi(x)-\varphi(c)\right)^{\sigma-1}}{\omega(x)} \right) \frac{\omega(c)}{\Gamma(\sigma)} \cdot \lim_{x\to c^+} {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x)$$
$$= \left(\frac{\left(\varphi(x)-\varphi(c)\right)^{\sigma(\ell+1)-1}}{\omega(x)\Gamma(\sigma(\ell+1))} \right) \omega(c) \cdot \lim_{x\to c^+} {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x).$$

Summing this result over ℓ from 0 to m, we get

$$y(x) - \left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} \right)^{m+1} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{m+1} y(x)$$
$$= \sum_{\ell=0}^{m} \frac{\left(\varphi(x)-\varphi(c)^{\sigma(\ell+1)-1}\right.}{\omega(x)\Gamma(\sigma(\ell+1))} \omega(c) \lim_{x\to c^+} {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega} \left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} \right)^{\ell} y(x). \tag{4.15}$$

By using (4.10), we get

$$\left({}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\right)^{m+1}\left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{m+1}y(x)=\left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{m+1}y(\zeta)\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell(\varphi(x)-\varphi(c))^{\sigma(m+1)+\ell}}{\omega(x)\Gamma(\ell+\sigma(m+1)+1)}. \tag{4.16}$$

Using (4.16) in (4.15), we get

$$\begin{aligned}y(x)=&\left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{m+1}y(\zeta)\sum_{\ell=0}^{\infty}\frac{\varsigma_\ell(\varphi(x)-\varphi(c))^{\ell+\sigma(m+1)}}{\omega(x)\Gamma(\ell+\sigma(m+1)+1)}\\&+\sum_{\ell=0}^{m}\frac{\left(\varphi(x)-\varphi(c)\right)^{\sigma(\ell+1)-1}}{\omega(x)\Gamma(\sigma(\ell+1))}\omega(c)\lim_{x\to c^+}{}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega}\left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\right)^{\ell}y(x),\end{aligned}$$

which is the required result.

Remark 4. In this section, we have proved Taylor's theorem for WFOs with respect to functions, which represents the generalization of both classical Taylor's theorem and the fractional Taylor's theorem. When we consider $\sigma=1$ with $\varphi(x)=x$ and the weight function $\omega(x)=1$, the theorems reduces to the classical Taylor's theorem. This reduction confirms the correctness of our generalization, as it aligns with the well-established classical result. By setting $\omega(x)=1$ and $\varphi(x)=x$, the theorem reduces to the Taylor's theorem for fractional operators. When $\omega(x)=1$, the theorem simplifies to the φ–fractional operators. Moreover, considering $\omega(x)=e^{\lambda\varphi(x)}$, the theorem corresponds to the φ-tempered fractional operators, demonstrating the inclusion of tempered fractional calculus as yet another special cases. These reductions illustrate the correctness of our generalization, as it seamlessly integrates and extends existing mathematical frameworks.

□

4.5 Integration by parts

Fractional integration by parts generalizes the classical integration by parts formula to handle fractional derivatives and integrals. In classical calculus, integration by parts simplifies the integral of a product of functions by integrating one function and differentiating the other. Fractional integration by parts extends this concept to fractional orders, allowing for a more flexible approach in modeling complex systems. Love and Young [64] obtained the fractional generalization of integration by parts for Lebesgue integrals, Riemann–Stieltjes integrals, and generalized Stieltjis integrals. Fractional and generalized fractional versions of integration by parts formulae are commonly used in literature. They have a significant impact on the generalized fractional variational formulation [8, 39, 60, 61]. While Agrawal [32] has already covered the result for weighted RL fractional operators with respect to functions and their applications. In this section we revisit the theorem and provide a comprehensive proof for both weighted RL and CFD with respect to functions. In this section, we derive the results of integration by parts for ${}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$ and ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$ in Theorem 4.5.2 and 4.5.3, respectively.

Lemma 4.5.1. *Let $\sigma > 0$ and $\omega \neq 0$ be a continuous function with an increasing positive monotone function φ, on a closed interval $[c,d] \subset \mathbb{R}$ and $r \geq 1, q \geq 1$ with $\frac{1}{r} + \frac{1}{q} \leq 1+\sigma$ ($r \neq 1, q \neq 1$ in case of $\frac{1}{r} + \frac{1}{q} = 1+\sigma$). If $\mathrm{u} \in L^r_{\varphi;\omega}(c,d)$ and $\mathrm{v} \in L^q_{\varphi;\omega}(c,d)$, then the following relation holds*

$$\int_c^d \varphi'(x)\mathrm{u}(x)\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx = \int_c^d \varphi'(x)\mathrm{v}(x)\,{}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-;\omega}\mathrm{u}(x)\,dx. \tag{4.17}$$

Proof. Using the Definition 4.2.2 in (4.17), we obtain

$$\int_c^d \mathrm{u}(x)\varphi'(x)\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx = \int_c^d \frac{\varphi'(x)\mathrm{u}(x)}{\Gamma(\sigma)\omega(x)} \int_c^x \frac{\omega(s)\mathrm{v}(s)\varphi'(s)}{(\varphi(x)-\varphi(s))^{1-\sigma}}\,ds\,dx.$$

We change the order of integration by using Fubini's theorem

$$\begin{aligned}\int_c^d \varphi'(x)\mathrm{u}(x)\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx &= \int_c^d \varphi'(s)\mathrm{v}(s)\frac{\omega(s)}{\Gamma(\sigma)} \int_s^d \frac{\mathrm{u}(x)\varphi'(x)}{\omega(x)(\varphi(x)-\varphi(s))^{1-\sigma}}\,dx\,ds \\ &= \int_c^d \varphi'(s)\mathrm{v}(s)\,{}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-;\omega}\mathrm{u}(s)ds = \int_c^d \varphi'(x)\mathrm{v}(x)\,{}^{RL}_{\varphi}\mathcal{I}^{\sigma}_{d^-;\omega}\mathrm{u}(x)\,dx.\end{aligned}$$

This completes the proof. □

When we consider $\omega(x) = 1$, then the above relation reduces to the ψ–fractional integrals [60].

Remark 5. [33] The space $L^r_{\varphi;\omega}(c,d)$, $1 \leq r \leq \infty$ is the set of all weighted Lebesgue measurable function u defined on a closed interval $[c,d]$ for which $\|\mathrm{u}\|_{L^r_{\varphi,w}(c,d)} < \infty$. One should observe that $\mathrm{u} \in L^r_{\varphi;\omega}(c,d)$, if and only if $\omega(x)\mathrm{u}(x)(\varphi'(x))^{\frac{1}{r}} \in L_r(c,d)$ for $1 \leq r < \infty$ and $\mathrm{u} \in L^{\infty}_{\varphi;\omega}(c,d)$, if and only if $\omega(x)\mathrm{u}(x)(\varphi'(x))^{\frac{1}{r}} \in L_{\infty}(c,d)$.

Theorem 4.5.2. *Assume that $\sigma > 0$, $n = \lfloor \sigma \rfloor + 1$ with an increasing positive monotone function φ on closed interval $[c, d]$, and $1 \leq r \leq \infty$. If $\mathrm{u} \in L^r_{\varphi;\omega}(c, d)$ and $\mathrm{v} \in AC^n_{\varphi;\omega}[c, d]$, then the following integration by parts formula for ${}^{C}_{c^+}\mathcal{D}^{\mu}_{\varphi;\omega}$ holds*

$$\int_c^d \mathrm{u}(x){}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx = \int_c^d \varphi'(x)\mathrm{v}(x){}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)dx + \left[\sum_{j=0}^{n-1} {}_{\varphi}\mathcal{D}^{j}_{d^-;\omega}{}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right){}_{c^+}\mathcal{D}^{n-j-1}_{\varphi;\omega}\mathrm{v}(x)\right]_c^d,$$

$$\int_c^d \mathrm{u}(x){}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega}\mathrm{v}(x)\,dx = \int_c^d \varphi'(x)\mathrm{v}(x){}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)dx + \left[\sum_{j=0}^{n-1}(-1)^{n-j}{}_{c^+}\mathcal{D}^{j}_{\varphi;\omega}{}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right){}_{\varphi}\mathcal{D}^{n-j-1}_{d^-;\omega}\mathrm{v}(x)\right]_c^d,$$

where ${}_{c^+}\mathcal{D}_{\varphi;\omega}(\cdot) = \left(\frac{1}{\varphi'\omega} \cdot \frac{d}{dx} \cdot \omega\right)(\cdot)$ *and* ${}_{\varphi}\mathcal{D}_{d^-;\omega}(\cdot) = \left(\frac{\omega}{\varphi'} \cdot \frac{d}{dx}\frac{1}{\omega}\right)(\cdot)$.

Proof. By using Definition 4.2.4, we get

$$\begin{aligned}\int_c^d \mathrm{u}(x){}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx &= \int_c^d \mathrm{u}(x){}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega}{}_{c^+}\mathcal{D}^{n}_{\varphi;\omega}\mathrm{v}(x)\,dx \\ &= \int_c^d \varphi'(x)\frac{\mathrm{u}(x)}{\varphi'(x)} \cdot {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega}\left({}_{c^+}\mathcal{D}^{n}_{\varphi;\omega}\mathrm{v}(x)\right)dx.\end{aligned}$$

By using Lemma 4.5.1, we have

$$\begin{aligned}\int_c^d \mathrm{u}(x){}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx &= \int_c^d \varphi'(x){}_{c^+}\mathcal{D}^{n}_{\varphi;\omega}\mathrm{v}(x) \cdot {}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)dx \\ &= \int_c^d \varphi'(x)\left(\frac{1}{\varphi'(x)\omega(x)}\frac{d}{dx}\omega(x)\right){}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x)\,{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)dx \\ &= \int_c^d \frac{d}{dx}\left(\omega(x){}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x)\right) \cdot \frac{1}{\omega(x)}{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)dx.\end{aligned}$$

Integrating by parts, we get

$$\begin{aligned}\int_c^d \mathrm{u}(x)_{c^+}^{C}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx &= \left[\frac{1}{\omega(x)}\,{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)\cdot\omega(x)_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x)\right]_c^d \\ &\quad -\int_c^d \omega(x)_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x)\cdot\frac{d}{dx}\frac{1}{\omega(x)}\,{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)dx \\ &= \left[{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)\cdot{}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x)\right]_c^d \\ &\quad +\int_c^d \varphi'(x)_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x)\cdot\left(-\frac{\omega(x)}{\varphi'(x)}\frac{d}{dx}\frac{1}{\omega(x)}\right){}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)dx \\ &= \left[{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)\cdot{}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x)\right]_c^d \\ &\quad +\int_c^d \varphi'(x)_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x)\cdot{}_{\varphi}\mathcal{D}^{1}_{d^-;\omega}{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)dx \\ &= \left[{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)\cdot{}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x)\right]_c^d \\ &\quad +\int_c^d \frac{d}{dx}\left(\omega(x)_{c^+}\mathcal{D}^{n-2}_{\varphi;\omega}\mathrm{v}(x)\right)\cdot\frac{1}{\omega(x)}{}_{\varphi}\mathcal{D}^{1}_{d^-;\omega}\,{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)dx.\end{aligned}$$

Again integrating by parts, we get

$$\begin{aligned}\int_c^d \mathrm{u}(x)_{c^+}^{C}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx &= \left[{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)\cdot{}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x)\right]_c^d \\ &\quad +\left[\frac{1}{\omega(x)}{}_{\varphi}\mathcal{D}^{1}_{d^-;\omega}{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)\cdot\omega(x)_{c^+}\mathcal{D}^{n-2}_{\varphi;\omega}\mathrm{v}(x)\right]_c^d \\ &\quad -\int_c^d \omega(x)_{c^+}\mathcal{D}^{n-2}_{\varphi;\omega}\mathrm{v}(x)\cdot\frac{d}{dx}\left(\frac{1}{\omega(x)}{}_{\varphi}\mathcal{D}^{1}_{d^-;\omega}{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)\right)dx\end{aligned}$$

$$= \left[{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) \cdot {}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x) \right]_c^d + \left[{}_{\varphi}\mathcal{D}^{1}_{d^-;\omega}\, {}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) \cdot {}_{c^+}\mathcal{D}^{n-2}_{\varphi;\omega}\mathrm{v}(x) \right]_c^d$$
$$+ \int_c^d \varphi'(x)_{c^+}\mathcal{D}^{n-2}_{\varphi;\omega}\mathrm{v}(x) \cdot \left(-\frac{\omega(x)}{\varphi'(x)} \frac{d}{dx} \frac{1}{\omega(x)} \right) {}_{\varphi}\mathcal{D}^{1}_{d^-;\omega}{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) dx$$
$$= \left[{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) \cdot {}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega}\mathrm{v}(x) \right]_c^d + \left[{}_{\varphi}\mathcal{D}^{1}_{d^-;\omega}{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) \cdot {}_{c^+}\mathcal{D}^{n-2}_{\varphi;\omega}\mathrm{v}(x) \right]_c^d$$
$$+ \int_c^d \varphi'(x)_{c^+}\mathcal{D}^{n-2}_{\varphi;\omega}\mathrm{v}(x) \cdot {}_{\varphi}\mathcal{D}^{2}_{d^-;\omega}\, {}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) dx.$$

After applying integration by parts n times, we obtain the following result

$$\int_c^d \mathrm{u}(x){}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{v}(x)\, dx = \left[\sum_{j=0}^{n-1} {}_{\varphi}\mathcal{D}^{j}_{d^-;\omega}{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) \cdot {}_{c^+}\mathcal{D}^{n-j-1}_{\varphi;\omega}\mathrm{v}(x) \right]_c^d$$
$$+ \int_c^d \varphi'(x)\mathrm{v}(x)_{\varphi}\mathcal{D}^{n}_{d^-;\omega}{}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) dx$$
$$= \left[\sum_{j=0}^{n-1} {}_{\varphi}\mathcal{D}^{j}_{d^-;\omega}{}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) \cdot {}_{c^+}\mathcal{D}^{n-j-1}_{\varphi;\omega}\mathrm{v}(x) \right]_c^d$$
$$+ \int_c^d \varphi'(x)\mathrm{v}(x)\, {}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) dx,$$

which is required. Similarly, the proof of the second part of the theorem follows steps similar to those of the first part; therefore, it is not explicitly stated here. □

If we consider $\omega(x) = 1$, the result reduces to the integration by parts formula for the ${}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi}$ [39].

Theorem 4.5.3. *Let* $\sigma > 0$, $n = \lfloor \sigma \rfloor + 1$ *with an increasing positive monotone function* φ *on a closed interval* $[c, d]$ *and* $1 \leq p \leq \infty$. *If* $\mathrm{u} \in AC^{n}_{\varphi;\omega}[c, d]$ *and* $\mathrm{v} \in L^{r}_{\varphi;w}(c, d)$, *then the* *following integration by parts for* ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;w}$ *holds*

$$\int_c^d \mathrm{u}(x)^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx = \int_c^d \varphi'(x)\mathrm{v}(x)\,{}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)\,dx$$

$$+\sum_{j=0}^{n-1}\left[{}_{\varphi}\mathcal{D}^{j}_{d^-;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right){}^{RL}_{c^+}\mathcal{I}^{j-\sigma+1}_{\varphi;\omega}\mathrm{v}(x)\right]_c^d,$$

$$=\sum_{j=0}^{n-1}\left[(-1)^{n-j}{}_{c^+}\mathcal{D}^{j}_{\varphi;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right){}^{RL}_{\varphi}\mathcal{I}^{j-\sigma+1}_{d^-;\omega}\mathrm{v}(x)\right]_c^d$$

$$+\int_c^d \varphi'(x)\mathrm{v}(x){}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)\,dx.$$

Proof. Starting with the definition of ${}^{RL}\mathcal{D}^{\sigma}_{\varphi;\omega}$

$$\int_c^d \mathrm{u}(x)^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx = \int_c^d \mathrm{u}(x)_{c^+}\mathcal{D}^{n}_{\varphi;\omega}{}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx$$

$$= \int_c^d\left(\frac{\mathrm{u}(x)}{\varphi'(x)\omega(x)}\right)\cdot\frac{d}{dx}\omega(x)\left(\frac{1}{\varphi'(x)\omega(x)}\frac{d}{dx}\omega(x)\right)^{n-1}{}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega}\mathrm{v}(x)\,dx.$$

Integrating by parts, we get

$$\int_c^d \mathrm{u}(x)^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega}\mathrm{v}(x)\,dx = \left[\left(\frac{\mathrm{u}(x)}{\varphi'(x)\omega(x)}\right)\omega(x)\left(\frac{1}{\varphi'(x)\omega(x)}\frac{d}{dx}\omega(x)\right)^{n-1}{}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega}\mathrm{v}(x)\right]_c^d$$

$$-\int_c^d\omega(x)\left(\frac{1}{\varphi'(x)\omega(x)}\frac{d}{dx}\omega(x)\right)^{n-1}{}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega}\mathrm{v}(x)\cdot\left(\frac{d}{dx}\left(\frac{\mathrm{u}(x)}{\varphi'(x)\omega(x)}\right)\right)\,dx$$

$$=\left[\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)\left(\frac{1}{\varphi'(x)\omega(x)}\frac{d}{dx}\omega(x)\right)^{n-1}{}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega}\mathrm{v}(x)\right]_c^d$$

$$+\int_c^d\varphi'(x)\left(\frac{1}{\varphi'(x)\omega(x)}\frac{d}{dx}\omega(x)\right)^{n-1}{}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega}\mathrm{v}(x)\cdot\left(-\frac{\omega(x)}{\varphi'(x)}\frac{d}{dx}\frac{1}{\omega(x)}\right)\left(\frac{\mathrm{u}(x)}{\varphi'(x)}\right)\,dx$$

$$= \left[\left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) {}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega} \, {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) \right]_c^d + \int_c^d \varphi'(x) {}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega} \, {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) {}_{\varphi}\mathcal{D}^{1}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) dx$$

$$= \left[\left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) {}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega} \, {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) \right]_c^d$$

$$+ \int_c^d \varphi'(x) \left(\frac{1}{\varphi'(x)\omega(x)} \frac{d}{dx} \omega(x) \right) {}_{c^+}\mathcal{D}^{n-2}_{\varphi;\omega} \, {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) {}_{\varphi}\mathcal{D}^{1}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) dx$$

$$= \left[\left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) {}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega} \, {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) \right]_c^d$$

$$+ \int_c^d \left(\frac{d}{dx} \omega(x) {}_{c^+}\mathcal{D}^{n-2}_{\varphi;\omega} \, {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) \right) \cdot \frac{1}{\omega(x)} {}_{\varphi}\mathcal{D}^{1}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) dx.$$

Again applying integrating by parts, we get

$$= \left[\left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) {}_{c^+}\mathcal{D}^{n-1}_{\varphi;\omega} \, {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) \right]_c^d + \left[{}_{d}\mathcal{D}^{1}_{\varphi;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) {}_{c^+}\mathcal{D}^{n-2}_{\varphi;\omega} \, {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) \right]_c^d$$

$$+ \int_c^d \varphi'(x) {}_{c^+}\mathcal{D}^{n-2}_{\varphi;\omega} \, {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) {}_{\varphi}\mathcal{D}^{2}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) dx.$$

Applying integration by parts n times, we get

$$\int_c^d \mathrm{u}(x) {}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega} \mathrm{v}(x) \, dx = \sum_{j=0}^{n-1} \left[{}_{d}\mathcal{D}^{j}_{\varphi;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) {}_{c^+}\mathcal{D}^{n-j-1}_{\varphi;\omega} \, {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) \right]_c^d$$

$$+ \int_c^d \varphi'(x) {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) {}_{\varphi}\mathcal{D}^{n}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) dx.$$

Using Lemma 4.5.1 on the integral from the right–hand side

$$\int_c^d \mathrm{u}(x) {}^{RL}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega} \mathrm{v}(x) \, dx = \sum_{j=0}^{n-1} \left[{}_{\varphi}\mathcal{D}^{j}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) \mathcal{D}^{n-j-1}_{\varphi;\omega} \, {}^{RL}_{c^+}\mathcal{I}^{n-\sigma}_{\varphi;\omega} \mathrm{v}(x) \right]_c^d$$

$$+ \int_c^d \varphi'(x) \mathrm{v}(x) \, {}^{RL}_{\varphi}\mathcal{I}^{n-\sigma}_{d^-;\omega} \, {}_{\varphi}\mathcal{D}^{n}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) dx$$

$$= \sum_{j=0}^{n-1} \left[{}_{\varphi}\mathcal{D}^{j}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) {}^{RL}_{c^+}\mathcal{I}^{j-\sigma+1}_{\varphi;\omega} \mathrm{v}(x) \, dx \right]_c^d$$

$$+ \int_c^d \mathrm{v}(x) {}^{C}_{\varphi}\mathcal{D}^{\sigma}_{d^-;\omega} \left(\frac{\mathrm{u}(x)}{\varphi'(x)} \right) \varphi'(x) \, dx.$$

which completes the proof. Similarly, the proof of the second part of this theorem follows a similar approach as the proof of the first part. □

Remark 6. When the fractional order $\sigma = 1$, $\omega = 1$, and $\varphi(x) = x$, the fractional integration by parts formula simplifies to the classical result. This demonstrates how fractional calculus builds on and extends traditional methods. Additionally, setting $\omega = 1$ and $\varphi(x) = x$ reduces the formulae to standard fractional operators [8], while considering $\omega = 1$ aligns the formula with φ–fractional operators [39]. This unified approach ensures that fractional integration by parts encompasses classical principles and provides a broader framework for handling non-integer orders.

4.6 Leibniz' rule

Leibniz' rule, also known as the product rule in classical calculus, provides a method for differentiating the product of two functions. In the setting of fractional operators, a rich literature on Leibniz' rule can be found in [9]. In this section, we focus on the generalized Leibniz rule already discussed in [28] but present a different approach utilizing conjugation relations here. After that, we set up the proof for weighted Leibniz' rule for integer orders, subsequently extending it to the generalized case that is ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;w}$. We will prove Leibniz' rule for ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}$ in Proposition 4.6.1, weighted derivatives of integer order in Proposition 4.6.2, and ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$ in Proposition 4.6.3.

Leibniz rule for RL fractional derivatives

Let $\sigma > 0$ and assume that u and v are analytic functions on $(a-h, a+h)$with some $h > 0$. Then,

$$
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{x}(\mathrm{u}(x)\mathrm{v}(x)) = \sum_{\ell=0}^{\lfloor\sigma\rfloor} \binom{\sigma}{\ell} {}_{c^+}\mathcal{D}^{\ell}\mathrm{u}(x) {}^{RL}_{c^+}\mathcal{D}^{\sigma-\ell}_{x}\mathrm{v}(x) + \sum_{\ell=\lfloor\sigma\rfloor+1}^{\infty} \binom{\sigma}{\ell} \mathcal{D}^{\ell}\mathrm{u}(x) {}^{RL}_{c^+}\mathcal{I}^{\ell-\sigma}_{x}\mathrm{v}(x), \tag{4.18}
$$

where ℓ is non-negative integer. Since $\mathcal{I}^{\sigma} = \mathcal{D}^{-\sigma}$ we can write relation (4.18) as,

$$\begin{aligned}
{}_{c^+}^{RL}\mathcal{D}_x^{\sigma}(\mathrm{u}(x)\mathrm{v}(x)) &= \sum_{\ell=0}^{\lfloor\sigma\rfloor}\binom{\sigma}{\ell}_{c^+}\mathcal{D}^{\ell}\mathrm{u}(x){}_{c^+}^{RL}\mathcal{D}_x^{\sigma-\ell}\mathrm{v}(x) + \sum_{\ell=\lfloor\sigma\rfloor+1}^{\infty}\binom{\sigma}{\ell}_{c^+}\mathcal{D}^{\ell}\mathrm{u}(x){}_{c^+}^{RL}\mathcal{D}_x^{\sigma-\ell}\mathrm{v}(x) \\
&= \left(\sum_{\ell=0}^{\lfloor\sigma\rfloor} + \sum_{\ell=\lfloor\sigma\rfloor+1}^{\infty}\right)\binom{\sigma}{\ell}_{c^+}\mathcal{D}^{\ell}\mathrm{u}(x){}_{c^+}^{RL}\mathcal{D}_x^{\sigma-\ell}\mathrm{v}(x) \\
&= \sum_{\ell=0}^{\infty}\binom{\sigma}{\ell}\mathcal{D}^{\ell}\mathrm{u}(x){}_{c^+}^{RL}\mathcal{D}_x^{\sigma-\ell}\mathrm{v}(x). \qquad (4.19)
\end{aligned}$$

Proposition 4.6.1. *Assume that* u *and* v *are analytic functions on* $[c,d]$ *with an increasing monotonic positive function* φ*, which is also an analytic function on* $[c,d]$*, then for* $\sigma > 0$*, we have*

$$ {}_{c^+}^{RL}\mathcal{D}_{\varphi}^{\sigma}(\mathrm{u}(x)\mathrm{v}(x)) = \sum_{\ell=0}^{\infty}\binom{\sigma}{\ell}{}_{c^+}\mathcal{D}_{\varphi}^{\ell}(\mathrm{u}(x)){}_{c^+}^{RL}\mathcal{D}_{\varphi}^{\sigma-\ell}(\mathrm{v}(x)).$$

Proof. We use the conjugation relations to prove Leibniz' rule for ${}_{c^+}^{RL}\mathcal{D}_{\varphi}^{\sigma}$. To begin, we apply $\mathcal{Q}_{\varphi}^{-1}$ to the product of functions u and v, as follows

$$\mathcal{Q}_{\varphi}^{-1}(\mathrm{u}(x)\mathrm{v}(x)) = \mathrm{u}(\varphi^{-1}(x))\mathrm{v}(\varphi^{-1}(x)).$$

Applying ${}_{\varphi(c^+)}^{RL}\mathcal{D}_x^{\sigma}$ on both sides, and by using the Leibniz' rule for RL fractional derivatives [9], we obtain

$$\begin{aligned}
{}_{\varphi(c^+)}^{RL}\mathcal{D}_x^{\sigma}\circ\mathcal{Q}_{\varphi}^{-1}(\mathrm{u}(x)\mathrm{v}(x)) &= {}_{\varphi(c^+)}^{RL}\mathcal{D}_x^{\sigma}(\mathrm{u}(\varphi^{-1}(x))\mathrm{v}(\varphi^{-1}(x))) \\
&= \sum_{\ell=0}^{\infty}\binom{\sigma}{\ell}{}_{\varphi(c^+)}\mathcal{D}_x^{\ell}\mathrm{u}(\varphi^{-1}(x)){}_{\varphi(c^+)}^{RL}\mathcal{D}_x^{\sigma-\ell}\mathrm{v}(\varphi^{-1}(x)).
\end{aligned}$$

Now applying $\mathcal{Q}_{\varphi}$ on both sides

$$\begin{aligned}
\mathcal{Q}_{\varphi}\circ{}_{\varphi(c^+)}^{RL}\mathcal{D}_x^{\sigma}\circ\mathcal{Q}_{\varphi}^{-1}(\mathrm{u}(x)\mathrm{v}(x)) &= \mathcal{Q}_{\varphi}\left(\sum_{\ell=0}^{\infty}\binom{\sigma}{\ell}{}_{\varphi(c^+)}\mathcal{D}_x^{\ell}\mathrm{u}(\varphi^{-1}(x)){}_{\varphi(c^+)}^{RL}\mathcal{D}_x^{\sigma-\ell}\mathrm{v}(\varphi^{-1}(x))\right) \\
&= \sum_{\ell=0}^{\infty}\binom{\sigma}{\ell}\left(\mathcal{Q}_{\varphi}{}_{\varphi(c^+)}\mathcal{D}_x^{\ell}\mathrm{u}(\varphi^{-1}(x))\right)\left(\mathcal{Q}_{\varphi}{}_{\varphi(c^+)}^{RL}\mathcal{D}_x^{\sigma-\ell}\mathrm{v}(\varphi^{-1}(x))\right) \\
&= \sum_{\ell=0}^{\infty}\binom{\sigma}{\ell}\left(\mathcal{Q}_{\varphi}{}_{\varphi(c^+)}\mathcal{D}_x^{\ell}\mathcal{Q}_{\varphi}^{-1}\right)\mathrm{u}(x)\left(\mathcal{Q}_{\varphi}{}_{\varphi(c^+)}^{RL}\mathcal{D}_x^{\sigma-\ell}\mathcal{Q}_{\varphi}^{-1}\right)\mathrm{v}(x) \\
{}_{c^+}^{RL}\mathcal{D}_{\varphi(x)}^{\sigma}(\mathrm{u}(x)\mathrm{v}(x)) &= \sum_{\ell=0}^{\infty}\binom{\sigma}{\ell}{}_{c^+}\mathcal{D}_{\varphi}^{\ell}(\mathrm{u}(x)){}_{c^+}^{RL}\mathcal{D}_{\varphi}^{\sigma-\ell}(\mathrm{v}(x)),
\end{aligned}$$

and this completes the proof. $\square$

Proposition 4.6.2. *Let* u *and* v *be analytic functions with weight function* ω *which is also analytic on the interval* $[c,d]$*, then for* $n \in \mathbb{N}$ *and* $x \in (c,d)$*, we have the classical Leibniz' rule for weighted derivatives, as follows*

$$\mathcal{D}_{\omega}^{n}\left(\mathrm{u}(x)\mathrm{v}(x)\right) = \sum_{\ell=0}^{n}\binom{n}{\ell}\mathcal{D}_{\omega}^{n-\ell}\mathrm{v}(x)\mathrm{u}^{(\ell)}(x), \tag{4.20}$$

where $\mathcal{D}_{\omega} = \left(\frac{d}{dx} + \frac{\omega'}{\omega}\right)$ *and* $\mathrm{u}^{(\ell)} = \frac{d^{\ell}}{dx^{\ell}}\mathrm{u}$.

Proof. We will prove this result by induction.
Basic step: When $n = 1$, (4.20) becomes

$$\begin{aligned}
\mathcal{D}_{\omega}^{1}\left(\mathrm{u}(x)\mathrm{v}(x)\right) &= \left(\frac{d}{dx} + \frac{\omega'(x)}{\omega(x)}\right)\left(\mathrm{u}(x)\mathrm{v}(x)\right) = \frac{d}{dx}\left(\mathrm{u}(x)\mathrm{v}(x)\right) + \frac{\omega'(x)}{\omega(x)}\mathrm{u}(x)\mathrm{v}(x) \\
&= \mathrm{v}(x)(\mathrm{u}'(x)) + \mathrm{u}(x)\mathrm{v}'(x) + \frac{\omega'(x)}{\omega(x)}\mathrm{u}(x)\mathrm{v}(x) \\
&= \mathrm{v}(x)(\mathrm{u}'(x)) + \mathrm{u}(x)\left(\frac{d}{dx} + \frac{\omega'(x)}{\omega(x)}\right)\mathrm{v}(x) \\
&= \mathrm{u}(x)\mathcal{D}_{\omega}\left(\mathrm{v}(x)\right) + \mathrm{v}(x)\mathrm{u}'(x) = \sum_{\ell=0}^{1}\binom{n}{\ell}\mathcal{D}_{\omega}^{1-\ell}\mathrm{v}(x)(\mathrm{u}^{(\ell)}(x)).
\end{aligned}$$

Thus (4.20) is true for $n = 1$. Now assume that the (4.20) is true for $n = m$

$$\mathcal{D}_{\omega}^{m}\left(\mathrm{u}(x)\mathrm{v}(x)\right) = \sum_{\ell=0}^{m}\binom{m}{\ell}\mathcal{D}_{\omega}^{m-\ell}\mathrm{v}(x)\mathrm{u}^{(\ell)}(x).$$

For $n = m + 1$, we have

$$\begin{aligned}
\mathcal{D}_{\omega}^{m+1}\left(\mathrm{u}(x)\mathrm{v}(x)\right) &= \mathcal{D}_{\omega}^{1}\left(\mathcal{D}_{\omega}^{m}\left(\mathrm{u}(x)\mathrm{v}(x)\right)\right) = \mathcal{D}_{\omega}^{1}\left(\sum_{\ell=0}^{m}\binom{m}{\ell}\mathcal{D}_{\omega}^{m-\ell}\mathrm{v}(x)\mathrm{u}^{(\ell)}(x)\right) \\
&= \sum_{\ell=0}^{m}\binom{m}{\ell}\mathcal{D}_{\omega}^{1}\left(\mathcal{D}_{\omega}^{m-\ell}\mathrm{v}(x)\mathrm{u}^{(\ell)}(x)\right).
\end{aligned}$$

Since the formula holds for $n = 1$, thus, we obtain

$$\mathcal{D}_{\omega}^{m+1}\left(\mathrm{u}(x)\mathrm{v}(x)\right) = \sum_{\ell=0}^{m}\binom{m}{\ell}\left(\mathrm{u}^{(\ell)}(x)\mathcal{D}_{\omega}^{m-\ell+1}\mathrm{v}(x) + \mathcal{D}_{\omega}^{m-\ell}\mathrm{v}(x)\mathrm{u}^{(\ell+1)}(x)\right)$$

$$= \sum_{\ell=0}^{m} \binom{m}{\ell} \mathrm{u}^{(\ell)}(x) \mathcal{D}_{\omega}^{m-\ell+1} \mathrm{v}(x) + \sum_{\ell=0}^{m} \binom{m}{\ell} \mathcal{D}_{\omega}^{m-\ell} \mathrm{v}(x) \mathrm{u}^{(\ell+1)}(x)$$

$$= \sum_{\ell=0}^{m} \binom{m}{\ell} \mathrm{u}^{(\ell)}(x) \mathcal{D}_{\omega}^{m-\ell+1} \mathrm{v}(x) + \sum_{\ell=1}^{m+1} \binom{m}{\ell-1} \mathcal{D}_{\omega}^{m-\ell+1} \mathrm{v}(x) \mathrm{u}^{(\ell)}(x)$$

$$= \mathrm{u}(x) \mathcal{D}_{\omega}^{m+1} \mathrm{v}(x) + \sum_{\ell=1}^{m} \binom{m}{\ell} \mathrm{u}^{(\ell)}(x) \mathcal{D}_{\omega}^{m-\ell+1} \mathrm{v}(x) + \sum_{\ell=1}^{m+1} \binom{m}{\ell-1} \mathcal{D}_{\omega}^{m-\ell+1} \mathrm{v}(x) \mathrm{u}^{(\ell)}(x)$$

$$= \mathrm{u}(x) \mathcal{D}_{\omega}^{m+1} \mathrm{v}(x) + \sum_{\ell=1}^{m} \binom{m}{\ell} \mathrm{u}^{(\ell)}(x) \mathcal{D}_{\omega}^{m-\ell+1} \mathrm{v}(x)$$

$$+ \sum_{\ell=1}^{m} \binom{m}{\ell-1} \mathcal{D}_{\omega}^{m-\ell+1} \mathrm{v}(x) \mathrm{u}^{(\ell)}(x) + \binom{m}{m} \mathrm{v}(x) \mathrm{u}^{(m+1)}(x)$$

$$= \mathrm{u}(x) \mathcal{D}_{\omega}^{m+1} \mathrm{v}(x) + \sum_{\ell=1}^{m} \left(\binom{m}{\ell} + \binom{m}{\ell-1} \right) \mathcal{D}_{\omega}^{m-\ell+1} \mathrm{v}(x) \mathrm{u}^{(\ell)}(x) + \mathrm{v}(x) \mathrm{u}^{(m+1)}(x).$$

Using relation

$$\binom{m+1}{\ell} + \binom{m}{\ell-1} = \binom{m+1}{\ell},$$

we get

$$\mathcal{D}_{\omega}^{m+1} \big(\mathrm{u}(x)\mathrm{v}(x)\big) = \mathrm{u}(x) \mathcal{D}_{\omega}^{m+1} \mathrm{v}(x) + \mathrm{v}(x) \mathrm{u}^{(m+1)}(x) + \sum_{\ell=1}^{m} \binom{m+1}{\ell} \mathcal{D}_{\omega}^{m-\ell+1} \mathrm{v}(x) \mathrm{u}^{(\ell)}(x)$$

$$= \sum_{\ell=0}^{m+1} \binom{m+1}{\ell} \mathcal{D}_{\omega}^{m-\ell+1} \mathrm{v}(x) \mathrm{u}^{(\ell)}(x).$$

Hence, the result holds for all $n \in \mathbb{N}$. □

The next result is Leibniz' rule for the ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$, which will be proved by using conjugation relations.

Proposition 4.6.3. *Assume the hypothesis of Proposition 4.6.1. Let* ω *be an analytic function on* $[c, d]$*, then the following formulae hold true*

$$ {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}(\mathrm{u}(x)\mathrm{v}(x)) = \sum_{\ell=0}^{\infty} \binom{\sigma}{\ell} {}_{c^+}\mathcal{D}^{\ell}_{\varphi;\omega}(\mathrm{u}(x)) {}^{RL}_{c^+}\mathcal{D}^{\sigma-\ell}_{\varphi}(\mathrm{v}(x)), \tag{4.21}$$

$$ {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}(\mathrm{u}(x)\mathrm{v}(x)) = \sum_{\ell=0}^{\infty} \binom{\sigma}{\ell} {}_{c^+}\mathcal{D}^{\ell}_{\varphi}(\mathrm{u}(x)) {}^{RL}_{c^+}\mathcal{D}^{\sigma-\ell}_{\varphi;\omega}(\mathrm{v}(x)). \tag{4.22}$$

Proof. We will prove Leibniz' rule for WFOs with respect to functions by using conjugation relations

$$ {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}(\mathrm{u}(x)\mathrm{v}(x)) = \mathcal{M}_{\omega}^{-1} \circ {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi} \circ \mathcal{M}_{\omega}(\mathrm{u}(x)\mathrm{v}(x)). $$

Applying $\mathcal{M}_{\omega}$ to the product of u and v, as follows

$$ \mathcal{M}_{\omega}(\mathrm{u}(x)\mathrm{v}(x)) = \omega(x)\mathrm{u}(x)\mathrm{v}(x). $$

Now applying ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}$ on both sides, then by Proposition 4.6.1, we get

$$ \begin{aligned} {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\mathcal{M}_{\omega}(\mathrm{u}(x)\mathrm{v}(x)) &= {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}(\omega(x)\mathrm{u}(x)\mathrm{v}(x)) \\ &= \sum_{\ell=0}^{\infty} \binom{\sigma}{\ell} {}_{c^+}\mathcal{D}^{\ell}_{\varphi}(\omega(x)\mathrm{u}(x)) {}^{RL}_{c^+}\mathcal{D}^{\sigma-\ell}_{\varphi}(\mathrm{v}(x)). \end{aligned} $$

Applying $\mathcal{M}_{\omega}^{-1}$ on both sides

$$ \begin{aligned} \mathcal{M}_{\omega}^{-1}\left({}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi}\mathcal{M}_{\omega}\left(\mathrm{u}(x)\mathrm{v}(x)\right)\right) &= \mathcal{M}_{\omega}^{-1}\left(\sum_{\ell=0}^{\infty} \binom{\sigma}{\ell} {}_{c^+}\mathcal{D}^{\ell}_{\varphi}(\omega(x)\mathrm{u}(x)) {}^{RL}_{c^+}\mathcal{D}^{\sigma-\ell}_{\varphi}(\mathrm{v}(x))\right) \\ &= \sum_{\ell=0}^{\infty} \binom{\sigma}{\ell} \left(\mathcal{M}_{\omega(x)}^{-1} {}_{c^+}\mathcal{D}^{\ell}_{\varphi}\mathcal{M}_{\omega}(x)\right) \mathrm{u}(x) {}^{RL}_{c^+}\mathcal{D}^{\sigma-\ell}_{\varphi}(\mathrm{v}(x)) \\ &= \sum_{\ell=0}^{\infty} \binom{\sigma}{\ell} {}_{c^+}\mathcal{D}^{\ell}_{\varphi;\omega}(\mathrm{u}(x)) {}^{RL}_{c^+}\mathcal{D}^{\sigma-\ell}_{\varphi}(\mathrm{v}(x)). \end{aligned} $$

Two formulae are presented in this proposition, both of which are nearly identical, differing only in the way of multiplication of weight function. Using a similar approach of (4.21), we can prove the (4.22) by multiplying the weight function ω with v. □

Remark 7. If we consider $\omega = 1$, $\varphi(x) = x$, the formula reduces to the RL fractional operators. In the context of the Leibniz formula for RL operators, the binomial coefficient $\binom{\sigma}{\ell}$ is zero for $\ell > \sigma$ due to the definition of binomial coefficients, which is defined as

$$ \binom{\sigma}{\ell} = \frac{\Gamma(\sigma+1)}{\Gamma(\sigma-\ell+1)\Gamma(\ell+1)} $$

For σ is an integer and $\ell > \sigma$, the Gamma function $\Gamma(\sigma-\ell+1)$ becomes undefined (since it involves the Gamma of a negative number), which by convention makes $\binom{\sigma}{\ell} = 0$. This property ensures that when $\ell > \sigma$, the terms in the series with $\binom{\sigma}{\ell}$ automatically vanish, simplifying the expression.

This property is significant because it allows the weighted RL fractional Leibniz formula with respect to functio φ to the classical product rule when σ is an integer, $\omega = 1$ and $\varphi(x) = x$ eliminating the additional terms introduced by fractional differentiation. This illustrates that while the Leibniz rule for weighted RL operators with respect to functions includes these extra terms when using fractional orders, they naturally disappear when dealing with integer orders, maintaining consistency with classical calculus. This seamless transition is essential in fractional calculus, as it provides a unified approach to understanding and applying differentiation across different contexts.

4.7 Existence and uniqueness

In this section, we develop the existence and uniqueness solution for IVP of weighted Caputo fractional differential equation with respect to functions within the framework of Sobolev space [65–67]. Our approach, motivated by the work of [45], studied the solution of the IVP of ${}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$ without requiring the continuity for a function with respect to an independent variable x. By building upon the equivalence with an integral equation under appropriate conditions, we prove an existence theorem in the space $W^{m,r}_{\varphi;\omega}(c,d)$. Particularly, our result allows the weak singularities and discontinuities. Before presenting the main results, we will define a function space and examine some additional results, which will be required to prove our main results.

Definition 4.7.1. ***Fixed point theorem****[9]. Let $(\mathcal{X}, \mathrm{d})$ be a complete metric space. Assume that the mapping $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ satisfies the inequality*

$$d(\mathcal{T}\mathrm{u}, \mathcal{T}\mathrm{v}) \leq \sigma d(\mathrm{u}, \mathrm{v}), \quad \textit{for every} \quad \mathrm{u}, \mathrm{v} \in \mathcal{X}, \quad 0 \leq \sigma < 1,$$

Then, mapping $\mathcal{T}$ has a unique fixed point.

Definition 4.7.2. ***Schauder's fixed point theorem*** *[9]. Assume that $(\mathcal{X}, \mathrm{d})$ be a complete metric space and $\mathcal{U}$ be a closed convex subset of $\mathcal{X}$. Assume that, mapping $\mathcal{T}; \mathcal{U} \to \mathcal{U}$ such that the set having fixed point $\mathcal{T}\mathrm{u} = \mathrm{u}$ is aa relatively compact in $\mathcal{X}$. Then $\mathcal{T}$ has at least one fixed point.*

Definition 4.7.3. ***Relatively Compact*** *[9]. Assume that $(\mathcal{X}, \mathrm{d})$ be a metric space and $\mathcal{Y} \subset \mathcal{X}$. Then, the set $\mathcal{Y}$ is relatively compact in $\mathcal{X}$, if the closure of $\mathcal{Y}$ is a compact subset ($\mathcal{Y}$ is said to be compact if and only if every class of open sets which covers $\mathcal{Y}$ has finite subclass which also covers $\mathcal{Y}$) of $\mathcal{X}$.*

Definition 4.7.4. *Let y, $\omega \in L^r_{\varphi;\omega}(c,d)$ with $\omega \neq 0$, ϕ be a test function (smooth function with compact support that is used to define weak derivatives) [54] with an increasing positive monotone function φ then, $\mathcal{D}_{\varphi;\omega}(\frac{y}{w})$ is called weighted weak derivative of y, if*

$$\int_c^d \mathcal{D}_{\varphi;\omega}\left(\frac{\phi(x)}{\omega(x)}\right) y(x)\omega(x)\varphi'(x)\,dx = -\int_c^d \mathcal{D}_{\varphi;\omega}\left(\frac{y(x)}{\omega(x)}\right) \phi(x)\omega(x)\varphi'(x)\,dx, \quad (4.23)$$

where $\mathcal{D}_{\varphi;\omega} f(x) = \frac{1}{\varphi'(x)\omega(x)} \cdot \frac{d}{dx}(\omega(x)f(x))$.

Definition 4.7.4 introduces the concept of weak derivatives in the context of weighted derivatives with respect to functions. They have a significant role in the study of PDEs. Since not all functions have classical derivatives, weak derivatives provide a necessary extension, enabling us to define a notion of derivative for functions that may not be differentiable in the classical sense. In the above definition, we consider ϕ to be a test function, which means that ϕ is a smooth function with compact support (infinitely differentiable and vanishes outside of a compact set). If we consider $\varphi(x) = x$, $\omega(x) = 1$, then the definition coincides with the usual definition of weak derivatives in [68, 69].

Definition 4.7.5. *Let m be a non-negative integer, $1 \leq r < \infty$, and φ be a strictly increasing function in the interval $[c,d]$. Then the generalized weighted Sobolev space is defined by*

$$W^{m,r}_{\varphi;\omega}(c,d) = \left\{ y \in L^r_{\varphi;\omega}(c,d) : \mathcal{D}^m_{\varphi;\omega} y \in L^r_{\varphi;\omega}(c,d), m = 1,2,3,\ldots \right\}.$$

The norm of this space is defined by

$$\|y\|_{W^{m,r}_{\varphi;\omega}(c,d)} = \left(\sum_{i=1}^m \|\mathcal{D}^i_{\varphi;\omega} y\|_{L^r_{\varphi;\omega}(c,d)} \right)^{\frac{1}{r}}, \quad (4.24)$$

for $1 \leq r < \infty$.

This space is a generalization of the classical Sobolev space [65, 66], where the derivatives are understood in weighted ordinary derivatives. The weighted Sobolev space is essential for subsequent proofs of existence and uniqueness results for weighted Caputo fractional differential equations with respect to functions.

Lemma 4.7.6. *Let function $y \in L^r_{\varphi;\omega}(c,d)$, $r \geq 1$. Then ${}_{c^+}\mathcal{D}_{\varphi;\omega}\, {}_{c^+}\mathcal{I}_{\varphi;\omega} y = y$ and ${}_{c^+}\mathcal{I}_{\varphi;\omega} y \in W^{1,r}_{\varphi;\omega}(c,d)$.*

Proof. Using the definition of ${}_{c^+}\mathcal{D}_{\varphi;\omega}$, ${}_{c^+}\mathcal{I}_{\varphi;\omega}$ and Dirichlet formula, we get

$$\begin{aligned}
&\int_c^d \omega(x)^2\varphi'(x)\,{}_{c^+}\mathcal{D}_{\varphi;\omega}\left(\frac{\phi(x)}{\omega(x)}\right){}_{c^+}\mathcal{I}_{\varphi;\omega}y(x)\,dx = \int_c^d \frac{d}{dt}\phi(x)\int_c^x \omega(s)y(s)\varphi'(s)\,ds\,dx\\
&\quad= \int_c^d\int_s^d \frac{d}{dx}\phi(x)\omega(s)y(s)\varphi'(s)\,dx\,ds = \int_c^d\int_x^d \frac{d}{ds}\phi(s)\omega(x)y(x)\varphi'(x)\,ds\,dx\\
&\quad= \int_c^d \omega(x)y(x)\varphi'(x)(\phi(d)-\phi(x))\,dx\\
&\quad= -\int_c^d \omega(x)y(x)\phi(x)\varphi'(x)\,dx, \quad \phi\in C_c^\infty(c,d), \quad \text{so} \quad \phi(d)=0
\end{aligned}$$

thus, we have ${}_{c^+}\mathcal{D}_{\varphi;\omega}\,{}_{c^+}\mathcal{I}_{\varphi;\omega}y = y$ in a weak sense. Therefore, ${}_{c^+}\mathcal{I}_{\varphi;\omega}y \in W^{1;r}_{\varphi;\omega}(c,d)$.

□

Lemma 4.7.7. *Let* $0<\sigma<1$, $y\in L^r_{\varphi;\omega}(c,d)$*, where* φ *is an increasing positive monotonic function and* $r\geq 1$*. Then, we have* ${}^{RL}_{\ a}\mathcal{D}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y = y$*.*

Proof. By the definition of ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$ together with semigroup property and Lemma 4.7.6, we get

$${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y(x) = {}_{c^+}\mathcal{D}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y(x) = {}_{c^+}\mathcal{D}_{\varphi;\omega}\,{}_{c^+}\mathcal{I}_{\varphi;\omega}y(x) = y(x).$$

□

Lemma 4.7.8. *Let* $0<\sigma<1$, $y\in L^r_{\varphi;\omega}(c,d)$*, where* φ *be an increasing positive monotonic function and* $r\geq 1$*. Then* ${}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y = y$*.*

Proof. By the definition of ${}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$ and semigroup property, we get

$$\begin{aligned}
{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\,{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y(x) &= {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega}\,{}_{c^+}\mathcal{D}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y(x)\\
&= {}^{RL}_{c^+}\mathcal{I}_{\varphi;\omega}\,{}_{c^+}\mathcal{D}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y(x) = {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y(x).
\end{aligned}$$

Applying ${}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}$, and by Lemma 4.7.7, we obtain

$$\begin{aligned}
{}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\,{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y(x) &= {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y(x),\\
{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}y(x) &= y(x),
\end{aligned}$$

which is required. □

Lemma 4.7.9. *Let* $0 < \sigma < 1$, $y \in L^r_{\varphi;\omega}(c,d)$ *with* $\omega(x) \neq 0$, $r > \frac{1}{\sigma}$. *Then* ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} y(c^+) = 0$ *and*

$$|{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} y(x)| \leq \frac{1}{\omega(x)\Gamma(\sigma)} \frac{\big(\varphi(x) - \varphi(c)\big)^{\sigma - \frac{1}{r}}}{\big(q(\sigma - \frac{1}{r})\big)^{\frac{1}{q}}} \|y\|^r_{\varphi;\omega}. \tag{4.25}$$

Proof. By applying limit $x \to c^+$ in the definition of ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}$, we get

$$\lim_{x \to c^+} {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} y(x) = 0.$$

By using Holder's inequality with $\frac{1}{r} + \frac{1}{q} = 1, \ x \in [c,d]$, we have

$$\begin{aligned}
|{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} y(x)| &= \frac{1}{\Gamma(\sigma)\omega(x)} \int_c^x \frac{|\omega(s) y(s)| \varphi'(s)}{(\varphi(x) - \varphi(s))^{1-\sigma}} \, ds \\
&\leq \frac{1}{\Gamma(\sigma)\omega(x)} \left(\int_c^x (\varphi(x) - \varphi(s))^{q(\sigma-1)} \varphi'(s) \, ds \right)^{\frac{1}{q}} \left(\int_c^x |y(s)\omega(s)|^r \varphi'(s) \, ds \right)^{\frac{1}{r}} \\
&= \frac{1}{\Gamma(\sigma)\omega(x)} \frac{(\varphi(x) - \varphi(c))^{\sigma - \frac{1}{r}}}{\Big(q(\sigma - \frac{1}{r})^{\frac{1}{q}}\Big)} \|y\|^r_{\varphi;\omega} \leq \frac{(\varphi(b) - \varphi(c^+))^{\sigma - \frac{1}{r}}}{\omega(x)\Gamma(\sigma)\Big(q(\sigma - \frac{1}{r})^{\frac{1}{q}}\Big)} \|y\|^r_{\varphi;\omega},
\end{aligned}$$

which is required. □

Lemma 4.7.10. *[35] Let* $0 < \sigma < 1, y \in L^r_{\varphi;\omega}(c,d)$, $r > \frac{1}{\sigma}$. *Then,* ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega} y(x)$ *is continuous function for* $x \in (c,d)$.

Consider the IVP

$$\begin{aligned}
{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega} y(x) &= g(x,y), \ x \in (c,d], \qquad (4.26) \\
y(c) &= y_c, \quad 0 < \sigma < 1. \qquad (4.27)
\end{aligned}$$

First, we prove that the IVP (4.26, 4.27) is equivalent to the integral equation

$$y(x) = \frac{\omega(c^+)}{\omega(x)} y(c^+) + \frac{1}{\Gamma(\sigma)\omega(x)} \int_c^x \frac{\varphi'(s)\omega(s) g(s, y(s))}{(\varphi(x) - \varphi(s))^{1-\sigma}} \, ds. \tag{4.28}$$

Theorem 4.7.11. *Let* $g(x, y(x))$ *be a function in* $L^r_{\varphi;\omega}(c,d)$ *and function* $y \in W^{1,r'}_{\varphi;\omega}(c,d)$, *where* $r > \frac{1}{\sigma}$ *and* $r' > 1$. *Then,* y *is a solution of the IVP (4.26, 4.27), if and only if* y *is a solution of the integral equation (4.28).*

Proof. Let us consider that $y \in W^{1,r'}_{\varphi;\omega}(c,d)$ is a solution of the IVP (4.26, 4.27). Now, apply the integral operator ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}$ to both sides of (4.26). This yields

$$ {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}y(x) = {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}g(x,y) $$

Afterward, by using the semigroup property together with Proposition 4.2.10, we get

$$ \begin{aligned} {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}{}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega}{}_{c^+}\mathcal{D}_{\varphi;\omega}y(x) &= {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}g(x,y), \\ {}_{c^+}\mathcal{I}_{\varphi;\omega}{}_{c^+}\mathcal{D}_{\varphi;\omega}y(x) &= {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}g(x,y), \\ y(x) - \frac{\omega(c^+)}{\omega(x)}y(c^+) &= {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}g(x,y). \end{aligned} $$

Conversely, assume that $y \in W^{1,r'}_{\varphi;\omega}(c,d)$ is solution of integral equation (4.28). First, it follows from the Proposition 4.2.10 that ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}y(x) = {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}f(x,y(x))$. Then by using the semigroup property, we obtain

$$ \begin{aligned} {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega}\,{}_{c^+}\mathcal{D}_{\varphi;\omega}y(x) &= {}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}g(x,y), \\ {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}\,{}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega}\,{}_{c^+}\mathcal{D}_{\varphi;\omega}y(x) &= {}^{RL}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}{}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}g(x,y), \\ {}^{RL}_{c^+}\mathcal{I}^{1-\sigma}_{\varphi;\omega}\,{}_{c^+}\mathcal{D}_{\varphi;\omega}y(x) &= g(x,y), \\ {}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}y(x) &= g(x,y). \end{aligned} $$

Furthermore, from Lemma 4.7.9 and integral equation (4.28), we see that $y(c) =_c$. □

Theorem 4.7.12. *Let* $d' \geq \omega(c)y(c) + \frac{1}{\Gamma(\sigma)}$*. Assume that when* $|y||\frac{1}{w}| \leq d'$*, the following conditions hold*

(i) there exist some $r > \frac{1}{\sigma}$ *such that* $g(x,y(x)), w(x,y) \in L^r_{\varphi;\omega}(c,d)$*;*

(ii) for $x \in [c,d]$*,* $(\varphi(x)-\varphi(c))^{\sigma}g(x,y)$ *is* y*–dependently continuous;*

(iii) there exist some $r' > 1$ *such that* ${}^{RL}_{c^+}\mathcal{I}^{\sigma}_{\varphi;\omega}g(x,y) \in W^{1,r'}_{\varphi;\omega}(c,d)$*.*

Then, the IVP (4.26, 4.27) has a solution y *in a space* $W^{1,r'}_{\varphi;\omega}(c,\xi)$*, where* $\xi \in (c,x] \subset [c,d]$ *such that* $\frac{z^{\sigma}}{\left(q(\sigma-\frac{1}{r})\right)^{\frac{1}{q}}}\|y\|_{L^r_{\varphi;\omega}} \leq 1$ *where* $z = (\varphi(\xi)-\varphi(c)$*.*

Proof. Let $X = \left\{y \in C[c,d] : |y| \leq d'\right\}$ be a closed convex subset of Banach space $C[c,\xi]$. Define the map $\mathcal{T} \in X$

$$ \mathcal{T}y(x) = \frac{1}{\omega(x)}\left\{\omega(c)y(c) + \frac{1}{\Gamma(\sigma)}\int_c^x \frac{\omega(s)\varphi'(x)g(s,y(s))}{(\varphi(x)-\varphi(s))^{1-\sigma}}\,ds\right\}. $$

From condition (i), Lemma 4.7.9 and Lemma 4.7.10, we observe that $\mathcal{T}y \in C[c,d]$ and

$$|\mathcal{T}y(x)| \leq |\frac{1}{w}| \left\{ |\omega(c)y(c)| + \frac{1}{\Gamma(\sigma)} \frac{z^{(\sigma-\frac{1}{r})}}{(q(1-\frac{1}{r}))^{\frac{1}{q}}} \|y\|_{L^r_{\varphi;\omega}} \right\} \leq d',\ \xi \in (c,x]. \tag{4.29}$$

Now, for $\epsilon > 0$ and using condition (ii), when $|y-y'| \leq \delta$, we have $(\varphi(x)-\varphi(c))^\sigma |g(x,y)-g(x,y')| \leq \epsilon$, then for all $\mathcal{T}y, \mathcal{T}y' \in X$

$$\begin{aligned}
|\mathcal{T}y - \mathcal{T}y'| &\leq \frac{1}{\omega(x)\Gamma(\sigma)} \int_c^x (\varphi(x)-\varphi(s))^{\sigma-1}\omega(s)\varphi'(s) \frac{(\varphi(s)-\varphi(c^+))^\sigma}{(\varphi(s)-\varphi(c^+))^\sigma} |g(s,y)-g(s,y')|\, ds \\
&\leq \frac{1}{\omega(x)\Gamma(\sigma)} \int_c^x (\varphi(x)-\varphi(s))^{\sigma-1}\omega(s)\varphi'(s)(\varphi(s)-\varphi(c^+))^{-\sigma}\epsilon\, ds \\
&\leq {}^{RL}_{c^+}\mathcal{I}^\sigma_{\varphi;\omega}(\varphi(x)-\varphi(c))^{-\sigma}\epsilon = \Gamma(1-\sigma)\epsilon
\end{aligned}$$

which shows that the map $\mathcal{T}$ is equicontinuous, as the continuity of mapping $\mathcal{T}$ is uniform with respect to x into set X. According to condition (iii), we have $\mathcal{T}y = \frac{\omega(c)}{\omega(x)}y(c) + {}^{RL}_{c^+}\mathcal{I}^\sigma_{\varphi;\omega}g \in W^{1,r'}_{\varphi;\omega}(c,d)$. This yields that

$$\|\mathcal{T}y\|_{W^{1,r'}_{\varphi;\omega}(c,d)} \leq \|\frac{\omega(c)}{\omega(x)}y(c)\|_{W^{1,r'}_{\varphi;\omega}(c,d)} + \|\mathcal{I}^\sigma_{\varphi;\omega}g\|_{W^{1,r'}_{\varphi;\omega}(c,d)} \leq M$$

which shows that image of $y \in X$, that is, $\mathcal{T}y$ is a bounded in $W^{1,r'}_{\varphi;\omega}(c,d)$ and the space $W^{1,r'}_{\varphi;\omega}(c,d)$ compactly embeds into $C[c,d]$. Thus, by the Arzela–Ascoli theorem, the operator $\mathcal{T}$ is relatively compact and maps $\mathcal{T}$ into itself. Then by the Schauder fixed point theorem, $\mathcal{T}$ has a fixed point, that is, $y = \mathcal{T}y$ in set X, which implies that y is a local solution of integral equation (4.28). Then by the equivalence theorem, since $y \in W^{1,r'}_{\varphi;\omega}(c,d)$, y is also solution of IVP (4.26, 4.27). □

Theorem 4.7.12 guarantees the existence of a local solution, while the subsequent result guarantees the global existence of the solution.

Theorem 4.7.13. *Assume the hypothesis of Theorem 4.7.12, and replace the condition* $|y| \leq d'$ *by* $|y| < \infty$*. Then the IVP (4.26, 4.27) has a solution* $y \in W^{1,r'}_{\varphi;\omega}(c,d)$.

Proof. Consider the mapping T defined in Theorem 4.7.12

$$\mathcal{T}y(x) = \frac{\omega(c^+)}{\omega(x)}y(c^+) + \frac{1}{\omega(x)\Gamma(\sigma)} \int_c^x \frac{\omega(s)g(s,y(s))\varphi'(s)}{(\varphi(x)-\varphi(s))^{1-\sigma}}\, ds.$$

From the proof of Theorem 4.7.12, we know that $\mathcal{T}$ is a compact map from $C[c,d]$ into itself, $\mathcal{T}$ takes bounded sets in $C[c,d]$ to relatively compact sets. Now, for any parameter $0 \leq \rho \leq 1$, we introduce the map $\mathcal{T}_\rho$ defined as $\mathcal{T}_\rho y = \rho T y$. We can observe that $\mathcal{T}_\rho$ is a compact map from $C[c,d]$ into itself. $\mathcal{T}_\rho$ takes bounded sets in $C[c,d]$ to relatively compact sets, due to the compactness of $\mathcal{T}$. The motivation for introducing the map $\mathcal{T}_\rho$ is that it will allow us to apply a suitable fixed point theorem to establish the existence of a solution. However, we need to find a bound M for all $y \in C[c,d]$ satisfying $\mathcal{T}_\rho y = y$, by inequality (4.29), we have

$$|\mathcal{T}_\rho y(x)| = |\rho \mathcal{T} y(x)| \leq |\mathcal{T} y(x)| = \frac{\omega(c^+)}{\omega(x)} y(c^+) + \frac{1}{\omega(x)\Gamma(\sigma)} \int_c^x \frac{\omega(s) g(s, y(s)) \varphi'(s)}{(\varphi(x) - \varphi(s))^{1-\sigma}} \, ds < M.$$

Therefore, for any $y \in [c,d]$ satisfying $y = \mathcal{T}_\rho y$, we have $|\mathcal{T}_\rho y| < M$. Since $\mathcal{T}_\rho$ is compact by Theorem 4.7.12 and we have found a bound M for all $y \in [c,d]$ satisfying $y = \mathcal{T}_\rho y$. Thus, the Leray-Schauder fixed point theorem can be applied. This theorem ensures the existence of a fixed point $y \in C[c,d]$ such that $y = \mathcal{T}_\rho y$ for some $\rho \in [0,1]$. Finally, from condition (iii) in Theorem 4.7.12, we have that $y(x) = \mathcal{T} y(x) = \frac{\omega(c^+)}{\omega(x)} y(c^+) + \frac{1}{\omega(x)\Gamma(\sigma)} \int_c^x (\varphi(x) - \varphi(s))^{\sigma-1} \omega(s) \varphi'(s) g(s, y(s)) \, ds \in W^{1,r'}_{\varphi;\omega}(c,d)$, which means y is a global solution of the fractional problem 4.26. This signifies the conclusion of the proof. □

In the following theorem, we establish the conditions necessary for the uniqueness of the solution to the IVP (4.26, 4.27).

Theorem 4.7.14. *Assume that the conditions in Theorem 4.7.12 are satisfied, except that condition (ii) is replaced by a Lipchitz-like condition.*

$$(\varphi(x) - \varphi(c))^\sigma |g(x,y) - g(x,y')| \leq L|y - y'|, \; x \in [c,d], \tag{4.30}$$

where $L \leq \frac{1}{\Gamma(1-\sigma)} < 1$ *is a constant and* $y,\ y'$ *are bounded within the closed interval* $[-d', d']$. *Then, the IVP (4.26, 4.27) has a unique solution in* $W^{1,r'}_{\varphi,w}(c,d)$.

Proof. Assume that y and y' are the two solutions of the IVP (4.26, 4.27). Assume that the condition (ii) in Theorem 4.7.12 satisfied and condition (4.30) holds, then by the integral equation (4.28), we obtain

$$\begin{aligned} \|y - y'\|_\infty &\leq \frac{1}{\omega(x)\Gamma(\sigma)} \int_c^x (\varphi(x) - \varphi(s))^{\sigma-1} \varphi'(s) \omega(s) \frac{(\varphi(x) - \varphi(c))^\sigma}{(\varphi(x) - \varphi(c))^\sigma} \|g(s,y) - g(s,y')\|_\infty \, dx \\ &\leq {}^{RL}_{c^+}\mathcal{I}^\sigma_{\varphi;\omega} (\varphi(s) - \varphi(c^+))^{-\sigma} L \|y - y'\|_\infty = \Gamma(1-\sigma) L \|y - y'\|_\infty < \|y - y'\|_\infty. \end{aligned}$$

there can only be one solution because the maximum absolute difference between y and y' is strictly less than itself, which is a contradiction. □

Example 4.7.1. Consider an IVP

$$
\begin{aligned}
{}^{C}_{c^+}\mathcal{D}^{\sigma}_{\varphi;\omega}y(x) &= \frac{(\varphi(x)-\varphi(c))^{-\mu}}{\omega(x)}, \quad x \in (c,d], \quad 0<\mu<\sigma<1,\\
y(c^+) &= y_{c^+},
\end{aligned}
$$

where $g(x,y) = \frac{(\varphi(x)-\varphi(c))^{-\mu}}{\omega(x)}$ is singular at $x = c^+$.
Clearly, function $g(x,y)$ satisfies the conditions in Theorem 4.7.12 and 4.7.13 with $\frac{1}{\sigma} < r < \frac{1}{\mu}$ and $1 < r' < \frac{1}{\mu-\sigma+1}$. Then the unique solution of this problem is $y(x) = \frac{\omega(c^+)}{\omega(x)}y(c^+) + \frac{\Gamma(1-\mu)}{\Gamma(\sigma-\mu+1)} \cdot \frac{(\varphi(x)-\varphi(c))^{\sigma-\mu}}{\omega(x)}$.

Chapter 5

Conclusion

The thesis thoroughly investigated FC, specifically focusing on WFC with respect to functions. It explored historical developments, special functions, and function spaces to built a foundation for understanding fractional operators and their properties. The core of the research focused on the extension theory of WFC with respect to functions, which present a generalized class of FC. We derived several fundamental theorems, including the MVT and Taylor's theorem, for the operators of WFC. Furthermore, we have derived the result of integration by parts formulae as initially addressed by Agrawal [32]. However, our contribution includes detailed proof of these results, thoroughly examining the function spaces. Considering suitable function spaces ensures that under which conditions the integration by parts formulae hold, providing a thorough understanding of the formulae. After that, we established Leibniz' rule for weighted RL fractional derivatives, building on the work of Osler [28] but utilizing a different approach to derive this result. Moreover, we have studied the Leibniz rule for classical weighted integer order derivatives and generalized weighted RL fractional derivatives with respect to functions.

Finally, we addressed the existence and uniqueness solutions for the IVP of the weighted Caputo fractional differential equation with respect to functions within a framework of Sobolev space. By utilizing the concept of weak derivatives, we demonstrated that this approach has the significant advantage of not requiring any continuity assumptions on the function with respect to the independent variable x. This allowed us to ensure the existence of weak solutions for discontinuous functions in the Sobolev space, expanding the applicability of the theory.

Overall, the developed theory and derived results have implications for further research and applications, particularly in areas such as the calculus of variations, where WFC finds significant utility. Future studies have the potential to explore more intricate systems and diverse fractional operators, thereby advancing both the theoretical framework and practical applications of WFC.

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