Modified Polytropic Models for Charged Anisotropic Compact Objects

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This thesis is dedicated to **my beloved parents**.

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Abstract

In this thesis, we present new solutions for charged anisotropic matter distributions by modifying the generalized polytropic equation of state. Using the Einstein-Maxwell equations in static and spherically symmetric spacetimes, we obtain solutions with non-singular metric potentials, densities, and pressures. These exact solutions for different variations of the adjustable parameters satisfy all physically admissible conditions. The Tolman-Oppenheimer-Volkoff equation, the relativistic adiabatic index, and Abreu's criterion are used to verify stability. The anisotropy factor starts at zero at the center and increases outward. Additionally, we study the behaviour of redshifts, compaction factors, and mass, all of which are found to be in acceptable ranges.

Chapter 1

Introduction

1.1 Historical Overview of General Relativity

More than three centuries ago, Isaac Newton proposed his well-known theory of gravity, which provides a comprehensive understanding of the universe. This theory applies the same principles and laws to describe the motion of both celestial bodies and objects on Earth. According to this theory, the world has three-dimensional Euclidean geometry and is an unbounded flat space. Three spatial coordinates, x^1, x^2, and x^3 as well as time, may be used to describe any event in this geometry. According to this view, time (t), is unchangeable, meaning it flows uniformly throughout the universe. It doesn't matter where you are or how you're moving; time passes at the same rate for everyone. If you and a friend have perfect clocks, they would always show the same time, even if your friend travelled far away or moved very quickly. This idea of absolute, unchangeable time helped scientists predict how things would behave in space and on Earth, making the universe seem very orderly. In the late 18th century, Joseph-Louis Lagrange extended Newtonian mechanics by introducing Lagrangian mechanics, using the principle of least action. This approach provided generalized framework for analyzing physical systems. Following this, in the 19th century, William Rowan Hamilton further developed classical mechanics, establishing a deeper mathematical foundation that not only enhanced the understanding of mechanical systems but also laid the groundwork for future advancements in fields such as electrodynamics [1].

1.1.1 Role of Electrodynamics

Our perspective of the universe was fundamentally altered by Maxwell's revolutionary discovery in 1864, which combined electricity, magnetism, and light into a single framework known as electromagnetism [2]. This great accomplishment was captured in a series of formulas that are now acknowledged as Maxwell's equations. These are set of coupled partial differential equations given as

$$\begin{aligned} \nabla \cdot \mathbf{E} &= \rho, \\ \nabla \cdot \mathbf{B} &= 0, \\ \nabla \times \mathbf{E} + \partial_t \mathbf{B} &= 0, \\ \nabla \times \mathbf{B} - \partial_t \mathbf{E} &= \mathbf{j}, \end{aligned}$$

where ρ is the electric charge density, $\mathbf{j}$ is the current density, $\mathbf{E}$ is the electric field, $\mathbf{B}$ is the magnetic field, and ∂_t represents the derivative with respect to t. These equations completely changed our understanding of the basic forces in nature. The presence of electromagnetic waves was predicted by Maxwell's equations, which also offered a mathematical framework for understanding their transmission. The speed of electromagnetic waves is represented by the formula $c = \frac{1}{\sqrt{(\mu_0 \epsilon_0)}}$, where μ_0 is the vacuum's permeability and ϵ_0 is its permittivity. Maxwell's groundbreaking theory challenged the Newtonian mechanics and raised profound questions about the nature of space and time [3]. The relationship between the constants μ_0, ϵ_0, and the speed of light emphasised how different aspects of physics are connected. This made scientists rethink basic ideas, such as the nature of the vacuum and the existence of an absolute frame of reference. Maxwell's contributions provided the foundation for modern electromagnetism and played a crucial role in the development of later theories, such as the Special Theory of Relativity (SR).

1.1.2 Special Theory of Relativity

In his well-known study "On the Electrodynamics of Moving Bodies," published in 1905, Albert Einstein proposed his SR [4]. The following two postulates served as the foundation for Einstein's hypothesis.

- Whenever two inertial frames are travelling in the same direction at a constant speed, all of the laws of physics apply.
- The speed of light, c, remains constant across all inertial frames.

For all coordinate systems in inertial frames, the first postulate suggests that all physical laws are the same. Thus, it may be concluded that there is no preferred set of coordinates for time and space. For instance, everything in the world appears to be moving in the opposite direction to an observer sitting still in a train travelling at an constant speed and looking out the window. On the other hand, an additional observer sitting outside the train will see that the first observer is travelling at the train's speed while everything else is still. The question is, how can the coordinates of the observer inside the train and the observer outside the train be defined? The time and distance are measured by both observers using their own frames of reference. The SR only applies to frames of reference that are travelling relative to one another at a constant velocity, v, meaning that there is no acceleration [5]. According to the second postulate, c, the speed of light, remains constant throughout all internal frames. We may argue that each observer has their own clock since various observers see time differently, meaning that time is not absolute for all of them.

1.1.3 General Theory of Relativity

Albert Einstein generalized his SR and gave a new theory called, General theory of Relativity (GR), which describes gravity as a geometric property of spacetime. In this theory, massive objects like planets and stars warp the fabric of spacetime, causing other objects to follow curved paths. This means that gravity is not simply a force pulling objects together; rather, it is the result of the curvature of spacetime created by mass and energy. The mathematical formulation of GR was significantly influenced by the work of mathematicians such as Marcel Grossmann, who introduced Einstein to Riemannian geometry, essential for understanding the curvature of spacetime [1]. The Einstein field equations are central to GR, as they mathematically relate the curvature of spacetime to the energy and momentum of the matter present. One of the key concepts in GR is the equivalence principle, which states that the effects of gravity are locally similar to acceleration. This principle leads to the understanding that the laws of physics are the same for all observers, regardless of their state of motion. The theory predicts several phenomena that have been confirmed by experiments and observations, such as the bending of light around massive objects and the existence of black holes [6]. The gravitational deflection of light was famously confirmed during a solar eclipse in 1919 by Arthur Eddington, providing one of the first experimental validations of GR. This observation demonstrated how light follows the curvature of spacetime around massive objects, confirming Einstein's predictions [7].

1.2 Tensors

A tensor is a mathematical object that generalizes the concepts of scalars, vectors, and matrices to higher dimensions. Tensors are defined as multilinear maps from vectors and dual vectors to real numbers, and they remain invariant under coordinate transformations. This invariance makes tensors fundamental in describing physical laws in a way that is independent of the choice of coordinate system. The number of indices that a tensor has, which can be classified as contravariant (higher indices) or covariant (lower indices), determines its rank.

$$T = T^{i_1 \cdots i_r}_{j_1 \cdots j_s}. \tag{1.1}$$

In this case, $T^{i_1\cdots i_r}_{j_1\cdots j_s}$ indicates a tensor with 'r' contravariant and 's' covariant indices.

To span the tensor space, we use a combination of basis vectors $\hat{e}^i$ and dual vectors $\hat{v}_j$ as

$$T = T^{i_1\cdots i_r}_{j_1\cdots j_s} e^{i_1} \otimes \cdots \otimes e^{i_r} \otimes v_{j_1} \otimes \cdots \otimes v_{j_s}, \tag{1.2}$$

where the symbol $\otimes$ stands for the tensor product, which is a basic tensor algebra operation. The elements of a tensor transform in accordance with specific guidelines while shifting between coordinate systems, making sure that the physical descriptions they offer stay constant.

$$T^{i'_1\ldots i'_r}_{j'_1\ldots j'_s} = T^{i_1\ldots i_r}_{j_1\ldots j_s} \prod_{k=1}^{r} \frac{\partial x^{i'_k}}{\partial x^{i_k}} \prod_{l=1}^{s} \frac{\partial x^{j_l}}{\partial x^{j'_l}}. \tag{1.3}$$

According to the eq (1.3), the fundamental descriptions of physical phenomena stay the same even when the observer's viewpoint or coordinate system changes. Tensors can be manipulated through various operations such as addition, subtraction, and multiplication, which are defined for tensors of compatible types and ranks. Furthermore, tensor contraction, which is an operation that sums over matched pairs of upper and lower indices to reduce a tensor rank, can be used to simplify tensor expressions

$$S^{\alpha\beta}_{\gamma} = T^{\alpha\eta\beta}_{\gamma\eta}. \tag{1.4}$$

Contracting indices η simplifies the tensor, reducing its complexity and focusing on essential features. The symmetric and skew-symmetric components of a tensor are defined as follows

$$T_{(i_1\ldots i_n)} = \frac{1}{n!} \sum_{\sigma\in S_n} T_{i_{\sigma(1)}\cdots i_{\sigma(n)}}, \tag{1.5}$$

here, σ represents a permutation of the set $\{1, \ldots, n\}$, and S_n is the symmetric group, which includes all possible permutations of n elements.

$$T_{[i_1\ldots i_n]} = \frac{1}{n!} \sum_{\sigma\in S_n} (\mathrm{sgn}(\sigma)) T_{i_{\sigma(1)}\cdots i_{\sigma(n)}}. \tag{1.6}$$

The function $\mathrm{sgn}(\sigma)$ denotes the sign of the permutation σ, which is $+1$ for an even permutation and -1 for an odd permutation.

1.2.1 Metric Tensor

The metric tensor emerges as the fundamental tensor for analysing the geometry of spacetime. This tensor is useful for measuring angles and distances as well as for converting other tensors and vectors between their covariant and contravariant forms. This symmetric rank−2 tensor, sometimes referred to as the metric, is essential for characterising gravitational fields. The usual matrix representation of the components of the metric tensor is $g_{\alpha\beta}$, where α and β are span from 0 to 3. This matrix's components are as follows

$$g_{\alpha\beta} = \begin{pmatrix} g_{00} & g_{01} & g_{02} & g_{03} \\ g_{10} & g_{11} & g_{12} & g_{13} \\ g_{20} & g_{21} & g_{22} & g_{23} \\ g_{30} & g_{31} & g_{32} & g_{33} \end{pmatrix}. \tag{1.7}$$

The metric tensor transforms between different coordinate systems according to

$$g_{\alpha\beta} = \frac{\partial x^\rho}{\partial x^\alpha} \frac{\partial x^\sigma}{\partial x^\beta} g_{\rho\sigma}. \tag{1.8}$$

This equation shows how the components of the metric tensor change under coordinate transformations. The metric tensor is used to define the line element, which represents the infinitesimal distance between two events in spacetime

$$ds^2 = g_{\alpha\beta}, dx^\alpha, dx^\beta. \tag{1.9}$$

This line element is crucial for computing intervals and distances in spacetime.
The contravariant metric tensor, $g^{\alpha\beta}$, is the inverse of the matrix (1.7). The link between the metric tensor and its inverse in tensor algebra guarantees that the identity matrix, represented by the Kronecker Delta, is produced when the metric and its inverse are multiplied.

$$g^{\alpha\gamma} g_{\gamma\beta} = \delta^\alpha_\beta = g_{\beta\gamma} g^{\gamma\alpha}, \tag{1.10}$$

where δ^α_β is the Kronecker Delta, which is defined as

$$\delta^\alpha_\beta = \begin{cases} 1 & \text{if } \alpha = \beta, \\ 0 & \text{if } \alpha \neq \beta. \end{cases}$$

The formula of Einstein's field equations is dependent on the determinant of the metric tensor, or g, which is essential in determining volume components in spacetime

$$g = \det(g_{\alpha\beta}). \tag{1.11}$$

1.2.2 Curvature Tensor

The curvature tensor provides a mathematical description of the curvature of spacetime caused by the presence of matter and energy. Christoffel symbols are used in the derivation of this tensor, which is essential in understanding the warping and bending of spacetime due to gravitational forces. The Christoffel symbols, denoted by $\Gamma^\alpha_{\beta\gamma}$, are essential for defining geometry and measuring curvature on curved spacetime surfaces. They are named after Elwin Bruno Christoffel. The metric tensor components are used to express them mathematically

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\eta}(g_{\eta\beta,\gamma} + g_{\eta\gamma,\beta} - g_{\beta\gamma,\eta}). \tag{1.12}$$

The Riemann curvature tensor is then defined as

$$R^{\alpha}{}_{\beta\gamma\delta} = \Gamma^{\alpha}_{\beta\delta,\gamma} - \Gamma^{\alpha}_{\beta\gamma,\delta} + \Gamma^{\alpha}_{\gamma\xi}\Gamma^{\xi}_{\beta\delta} - \Gamma^{\alpha}_{\delta\xi}\Gamma^{\xi}_{\beta\gamma}, \tag{1.13}$$

where the partial derivative is denoted by ",". To convert $R^{\alpha}{}_{\beta\gamma\delta}$ into covariant tensor form, use the transformation

$$R_{\alpha\beta\gamma\eta} = g_{\alpha\xi} R^{\xi}{}_{\beta\gamma\eta}, \tag{1.14}$$

in general, $R_{\alpha\beta\gamma\eta} \neq R^{\alpha}{}_{\beta\gamma\eta}$. Using some algebra, the covariant curvature tensor may be expressed explicitly as

$$R_{\alpha\beta\gamma\eta} = \frac{1}{2}\left(g_{\eta\alpha,\beta\gamma} + g_{\beta\gamma,\eta\alpha} - g_{\beta\eta,\alpha\gamma} - g_{\gamma\alpha,\beta\eta}\right) + g_{\xi\zeta}\left(\Gamma^{\zeta}_{\eta\alpha}\Gamma^{\xi}_{\gamma\beta} - \Gamma^{\zeta}_{\gamma\alpha}\Gamma^{\xi}_{\eta\beta}\right). \tag{1.15}$$

This tensor has a number of important properties and symmetries

- It is skew-symmetrical if the first two indices or the order of the last two are switched.
$$R_{\alpha\beta\gamma\eta} = -R_{\beta\alpha\gamma\eta}, \quad R_{\alpha\beta\gamma\eta} = -R_{\alpha\beta\eta\gamma}. \tag{1.16}$$
- If the first two pairs of indices swap with the last two, then it is symmetric
$$R_{\alpha\beta\gamma\delta} = R_{\gamma\delta\alpha\beta}. \tag{1.17}$$
- It satisfies the first and second kinds of Bianchi's identification, which are described as
$$R_{\alpha[\beta\gamma\eta]} = R_{\alpha\beta\gamma\eta} + R_{\alpha\eta\beta\gamma} + R_{\alpha\gamma\eta\beta} = 0, \tag{1.18}$$
$$R^{\alpha}_{p[\beta\gamma;\eta]} = R^{\alpha}_{p\beta\gamma;\eta} + R^{\alpha}_{p\eta\beta;\gamma} + R^{\alpha}_{p\gamma\eta;\beta} = 0. \tag{1.19}$$

Contracting the components of the Riemann tensor allows for the construction of the Ricci tensor

$$R_{\alpha\beta} = R^{\gamma}_{\alpha\gamma\beta}, \tag{1.20}$$

and the Ricci scalar, which is the trace of the Ricci tensor, provides a scalar measure of curvature

$$R = g^{\alpha\beta} R_{\alpha\beta}. \tag{1.21}$$

Being invariant under coordinate transformations, the Ricci scalar has an interesting property in that it determines the nature of a singularity together with other invariants. There are two types of singularities coordinate and essential, one arises with a bad choice of coordinates and is removable, the other occurs due to a problem in geometry that can not be removed. The nature of singularity can be determined by looking at the invariant quantities that are mentioned in eqs (1.22) to (1.25). The singularity is a coordinate singularity if the curvature invariants are finite and an essential singularity otherwise.

$$R_1 = R, \tag{1.22}$$
$$R_2 = R^{\alpha\beta}_{\ \ \gamma\eta} R^{\gamma\eta}_{\ \ \alpha\beta}, \tag{1.23}$$
$$R_3 = R^{\alpha\beta}_{\ \ \gamma\eta} R^{\gamma\eta}_{\ \ \xi\zeta} R^{\xi\zeta}_{\ \ \alpha\beta}, \tag{1.24}$$
$$R_4 = R^{\alpha\beta}_{\ \ \gamma\eta} R^{\gamma\eta}_{\ \ \xi\zeta} R^{\xi\zeta}_{\ \ \omega\rho} R^{\omega\rho}_{\ \ \alpha\beta}. \tag{1.25}$$

1.2.3 Einstein Tensor

The Einstein tensor is a fundamental geometric object in general relativity. It is an order-two tensor that plays a crucial role in describing the curvature of spacetime. The components of the Einstein tensor are written as [8]

$$G_{\alpha\beta} = R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta}. \tag{1.26}$$

One of the defining characteristics of the Einstein tensor is its symmetry

$$G_{\alpha\beta} = G_{\beta\alpha}. \tag{1.27}$$

The Einstein tensor is also divergence-free, in component form

$$G^{\alpha\beta}_{\ \ ;\beta} = 0. \tag{1.28}$$

1.2.4 Maxwell Tensor

The electromagnetic field is described by the Faraday tensor, also known as the electromagnetic field tensor or Maxwell tensor [9]. It is constructed from the four-vector potential A_α as follows

$$A_\alpha = (\phi, -\mathbf{A}), \tag{1.29}$$

where the vector potential is indicated by $\mathbf{A}$ and the scalar potential is represented by ϕ. The component of electromagnetic field tensor $F_{\alpha\beta}$ is expressed using these potentials as follows

$$F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha. \tag{1.30}$$

From the scalar and vector potentials, the electric ($\mathbf{E}$) and magnetic ($\mathbf{B}$) fields are found as

$$\mathbf{E} = -\nabla\phi - \partial_t \mathbf{A}, \tag{1.31}$$
$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{1.32}$$

From the eqs (1.30) to (1.32), we get $F_{0j} = E_j$ and $F_{ij} = \epsilon_{ijk} B^k$ where $i, j, k = 1, 2, 3$ and ϵ_{ijk} is the Levi-Civita tensor, defined as follows

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of 123,} \\ -1 & \text{if } ijk \text{ is an odd permutation of 123,} \\ 0 & \text{otherwise.} \end{cases} \tag{1.33}$$

Consequently, the components of the electromagnetic field tensor are given as

$$F_{\alpha\beta} = \begin{pmatrix} 0 & E_1 & E_2 & E_3 \\ -E_1 & 0 & B_3 & -B_2 \\ -E_2 & -B_3 & 0 & B_1 \\ -E_3 & B_2 & -B_1 & 0 \end{pmatrix}. \tag{1.34}$$

The representation of the metric tensor component in its contravariant form is as follows

$$F^{\alpha\beta} = g^{\alpha\gamma} g^{\beta\eta} F_{\gamma\eta}. \tag{1.35}$$

1.2.5 Stress-Energy Tensor

The fundamental second-rank symmetric tensor, in its component form $T_{\alpha\beta} = T_{\beta\alpha}$ is the stress-energy tensor, commonly referred to as the energy-momentum tensor. This tensor measures the distribution of energy, momentum, and stress within a specific region of spacetime. The elements of the stress-energy tensor are organised in a 4×4 matrix in a four-dimensional spacetime.

$$T^{\alpha\beta} = \begin{pmatrix} T^{00} & T^{01} & T^{02} & T^{03} \\ T^{10} & T^{11} & T^{12} & T^{13} \\ T^{20} & T^{21} & T^{22} & T^{23} \\ T^{30} & T^{31} & T^{32} & T^{33} \end{pmatrix}. \tag{1.36}$$

Each component in this matrix has a distinct physical meaning.

- The energy density of the matter is represented by T^{00}.
- In the i^{th} direction, energy flux $\times c^{-1}$ is represented by T^{0i}.

- In the i^{th} direction, momentum density $\times c$ is represented by T^{i0}.
- The i^{th} component of momentum per unit area flows in the j^{th} direction is described by T^{ij}

The spatial coordinates are represented by the indices i and j in this case. The covariant form of the stress-energy tensor is presented by

$$T_{\alpha\beta} = T^{\gamma\eta} g_{\gamma\alpha} g_{\eta\beta}, \tag{1.37}$$

and in the mixed form as

$$T^{\alpha}_{\beta} = T^{\alpha\gamma} g_{\gamma\beta}. \tag{1.38}$$

1.3 Derivation of the Einstein Field Equations

First we define the general form of the Einstein-Hilbert action, from which the Einstein field equations (EFEs) can be derived

$$S = \int \mathcal{L}\sqrt{-g}\, d^4x. \tag{1.39}$$

The Lagrangian $\mathcal{L}$ for the gravitational field and matter is expressed as $\mathcal{L} = \mathcal{L}_G + \mathcal{L}_m$.

$$S = \int \frac{1}{2\kappa} R\sqrt{-g}\, d^4x + \int \mathcal{L}_m \sqrt{-g}\, d^4x. \tag{1.40}$$

The gravitational Lagrangian $\mathcal{L}_G$ is given by $\mathcal{L}_G = \frac{1}{2\kappa}R$ and $\kappa = \frac{8\pi G}{c^4}$. Throughout this discussion, we will assume that the gravitational constant G and the speed of light c are both equal to 1.
We take $\delta S = 0$ to apply the principle of least action, using this and eq (1.21) in (1.40) gives

$$\begin{aligned}\delta S = \frac{1}{2\kappa}\int &\left(R_{\alpha\beta} g^{\alpha\beta}\delta\sqrt{-g} + R_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta} + \sqrt{-g} g^{\alpha\beta}(\delta R_{\alpha\beta})\right) d^4x \\ &+ \int \left(\mathcal{L}_m \delta\sqrt{-g} + \sqrt{-g}\delta(\mathcal{L}_m)\right) d^4x = 0.\end{aligned} \tag{1.41}$$

At an arbitrary point P where $\Gamma^{\alpha}_{\beta\gamma} = 0$, the Riemann tensor simplifies as

$$R^{\gamma}{}_{\alpha\eta\beta} = \Gamma^{\gamma}_{\alpha\beta,\eta} - \Gamma^{\gamma}_{\alpha\eta,\beta}, \tag{1.42}$$

or

$$\delta R^{\gamma}{}_{\alpha\eta\beta} = \delta\Gamma^{\gamma}_{\alpha\beta,\eta} - \delta\Gamma^{\gamma}_{\alpha\eta,\beta}. \tag{1.43}$$

At a point where the partial derivative is equivalent to the covariant derivative and commutes with variation, the well-known Palatini equation is obtained, given as

$$\delta R^{\gamma}{}_{\alpha\eta\beta} = \delta\Gamma^{\gamma}_{\alpha\beta;\eta} - \delta\Gamma^{\gamma}_{\alpha\eta;\beta}. \tag{1.44}$$

Contraction of the γ and η gives

$$\delta R_{\alpha\beta} = \delta\Gamma^{\gamma}_{\alpha\beta;\gamma} - \delta\Gamma^{\gamma}_{\alpha\gamma;\beta}, \tag{1.45}$$

or

$$g^{\alpha\beta}\delta R_{\alpha\beta} = g^{\alpha\beta}\delta\Gamma^{\gamma}_{\alpha\beta;\gamma} - g^{\alpha\beta}\delta\Gamma^{\gamma}_{\alpha\gamma;\beta}, \tag{1.46}$$

$$= g^{\alpha\beta}\delta\Gamma^{\gamma}_{\alpha\beta;\gamma} - g^{\alpha\gamma}\delta\Gamma^{\beta}_{\alpha\beta;\gamma}, \tag{1.47}$$

$$= (g^{\alpha\beta}\delta\Gamma^{\gamma}_{\alpha\beta} - g^{\alpha\gamma}\delta\Gamma^{\beta}_{\alpha\beta})_{;\gamma}. \tag{1.48}$$

Writing

$$A^{\gamma} = g^{\alpha\beta}\delta\Gamma^{\gamma}_{\alpha\beta} - g^{\alpha\gamma}\delta\Gamma^{\beta}_{\alpha\beta}, \tag{1.49}$$

further simplification and integration over the volume v on both sides of eq (1.48) yields

$$\int_v g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g}\, d^4x = \int_v A^{\gamma}_{;\gamma}\sqrt{-g}\, d^4x, \tag{1.50}$$

Applying the divergence theorem, eq (1.50) gives

$$\int_v g^{\alpha\beta}\delta R_{\alpha\beta}\sqrt{-g}\, d^4x = 0. \tag{1.51}$$

Following the substitution of the identity $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}$ and eq (1.51), eq (1.41) takes on the following form

$$\begin{aligned} &\frac{1}{2\kappa}\int_v \left(R_{\alpha\beta}g^{\alpha\beta}\left(-\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}\right) + R_{\alpha\beta}\sqrt{-g}\delta g^{\alpha\beta}\right) d^4x \\ &+ \int_v \left(\mathcal{L}_m\left(-\frac{1}{2}\sqrt{-g}g_{\alpha\beta}\delta g^{\alpha\beta}\right) + \sqrt{-g}\delta(\mathcal{L}_m)\right) d^4x = 0. \end{aligned} \tag{1.52}$$

As $\mathcal{L}_m = \mathcal{L}_m(g_{\alpha\beta})$, this implies $\delta\mathcal{L}_m = \dfrac{\partial\mathcal{L}_m}{\partial g^{\alpha\beta}}\delta g^{\alpha\beta}$, thus substituting it in the eq (1.41)

$$\begin{aligned} &\frac{1}{2\kappa}\int_v \left(-\frac{1}{2}Rg_{\alpha\beta}\delta g^{\alpha\beta} + R_{\alpha\beta}\delta g^{\alpha\beta}\right)\sqrt{-g}\, d^4x \\ &-\frac{1}{2}\int_v \left(-2\frac{\partial\mathcal{L}_m}{\partial g^{\alpha\beta}} + \mathcal{L}_m g_{\alpha\beta}\right)\delta g^{\alpha\beta}\sqrt{-g}\, d^4x = 0. \end{aligned} \tag{1.53}$$

In terms of Lagrangian, the energy momentum tensor is defined as

$$T_{\alpha\beta} = -2\frac{\partial \mathcal{L}_m}{\partial g^{\alpha\beta}} + \mathcal{L}_m g_{\alpha\beta}. \tag{1.54}$$

Therefore, eq (1.53) has this form

$$\frac{1}{2\kappa}\int_v \left(R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R - \kappa T_{\alpha\beta}\right)\delta g^{\alpha\beta}\sqrt{-g}\,d^4x = 0. \tag{1.55}$$

We use the fundamental lemma of the calculus of variations, which states that if an integral of a product of a function and an arbitrary variation equals zero, then the integrand must be zero. Thus, setting the integrand of our integral to zero, we obtain

$$R_{\alpha\beta} - \frac{1}{2}g_{\alpha\beta}R = \kappa T_{\alpha\beta}. \tag{1.56}$$

Eq (1.56) represents the EFEs. On the left side, the curvature determines the presence of a gravitational source, while on the right side, the energy-momentum tensor describes the matter content.

1.4 Solutions of Einstein Field Equations

This section discusses well-known black hole solutions to the EFEs.

1.4.1 The Schwarzschild Solution

German physicist Karl Schwarzschild solved EFEs for the first time in 1916. This static vacuum solution, which assumes a zero stress tensor ($T_{\alpha\beta} = 0$), represents the geometry around a spherically symmetric empty region surrounding a huge spherical body [10]. For this solution, the metric in curvature coordinates is given by

$$ds^2 = \left(1 - \frac{2m}{r}\right)c^2dt^2 - \left(1 - \frac{2m}{r}\right)^{-1}dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right),\ m = \frac{GM}{c^2}. \tag{1.57}$$

The mass of the source field is denoted by M in this case. The metric is not dependent on the time component that is why it is called static. The Schwarzschild radius, $r = 2m$, is the location of the horizon determined by setting $g_{rr} = 0$. Coordinate and essential singularities are indicated at $r = 2m$ and $r = 0$, respectively, by $g_{rr} \to \infty$ and $g_{tt} \to \infty$. Although the coordinate singularity can be removed by selecting suitable coordinates, the essential singularity cannot be removed. One of the characteristics that distinguishes a Schwarzschild black hole is the Schwarzschild radius $r = 2m$, which denotes the event horizon beyond which nothing can escape, not even light [10]. The solution is also asymptotically flat, which is far away from the source; the metric reduces to the flat Minkowski spacetime. Mathematically, this is expressed as

$$r \to \infty \Rightarrow ds^2 = c^2dt^2 - dr^2 - r^2\left(d\theta^2 + \sin^2\theta\,d\phi^2\right). \tag{1.58}$$

1.4.2 The Reissner-Nordström Solution

This Schwarzschild metric is extended by the Reissner-Nordström solution by adding charge. The region surrounding a charged spherical body is described by the geometry around it in this static solution. The Maxwell field equations must also be taken into consideration in order to account for the charge. Two parameters, total mass M and total charge Q, define this solution. The metric for this solution is as follows [11]

$$\begin{aligned} ds^2 = & \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) c^2 dt^2 - \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 \\ & - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right), \; m = \frac{GM}{c^2}. \end{aligned} \tag{1.59}$$

For $r = 0$, $g_{tt} \to \infty$ is the essential singularity, and for $\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) = 0$, $g_{rr} \to \infty$ is the coordinate singularity. The location of the horizons is determined by solving

$$\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) = 0, \tag{1.60}$$

or

$$r^2 - 2mr + Q^2 = 0. \tag{1.61}$$

The above equation has the following solution

$$r_\pm = m \pm \sqrt{m^2 - Q^2}. \tag{1.62}$$

The radii of the black hole's inner (r_-) and outer (r_+) horizons are represented by these solutions. When $m < Q$, a naked singularity occurs, as no coordinate singularities exist, and this situation is often considered physically unrealistic. For $m > Q$, the standard Reissner-Nordström black hole forms with two coordinate singularities at the surfaces $r = r_\pm$. In the case where $m = Q$, an extremal Reissner-Nordström black hole forms with a single horizon and the region between r_- and r_+ is removed. We will consider only the non-extremal case for our purpose. The solution is also asymptotically flat, which is far away from the source; the metric reduces to the flat Minkowski spacetime [10]. Mathematically, this is expressed as

$$r \to \infty \Rightarrow ds^2 = c^2 dt^2 - dr^2 - r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

The metric reduces to the Schwarzschild solution if the charge is set to zero ($Q = 0$).

1.5 Compact Objects

Laplace presented the theoretical idea of a large object whose gravitational field blocks any particle, "even light," from escaping earlier in 1798. However, because of its strange properties, the idea did not attract much attention. But because of its unique properties, gravity has gained attention. Astronomers believe that gaseous mass collapses under the force of gravity to generate stars. It was referred to as Laplace's nebular hypothesis in the 18th century, but the solar nebular disk model overruled it in the 1970s, mostly because of its lack of information about the dissemination of angular momentum between planets and the sun. Nebulae, which are massive disk-like structures of interstellar dust and gas in the form of molecular clouds mostly made up of hydrogen (75%) and helium (23%) [12], are what lead to star formation. The conservation of momentum from the movement of the particles causes this cloud to begin spinning, and the rapid rotation flattens the cloud into a protoplanetary disk. As areas of increased gravity let gas and dust condense, they collapsed under the increased gravitational field, raising the temperature and eventually leading to the creation of a protostar [13]. The star would eventually stay in that state for thousands of years until the compressed hydrogen atoms were ignited by fusion to make helium, then carbon to form iron. There is no energy generated during the fusion process that makes iron. Because the collapse of the star's core, a process caused by gravity, occurs during nuclear fusion when the gravitational force of inward pressure is greater than the pressure pushed outward [14]. The formation of stars, sometimes known as "compact objects," starts at this stage. The different kinds of compact objects are white dwarfs, neutron stars, and black holes.

- **White Dwarfs:**
  A star turns into a white dwarf after a long period of stellar evolution, typically spanning millions to billions of years. Smaller stars, up to 1.4 solar masses, become white dwarfs due to a lack of their nuclear fuel, and they shrink because of their own gravitational pull. With masses equal to the sun's, white dwarfs are very dense objects. These objects have a volume comparable to that of Earth. Sirius B is one of the closest known white dwarfs to Earth [15]. It is approximately the size of Earth but has a mass nearly equal to that of the Sun. This white dwarf is an excellent example of a dense stellar remnant left after a star has exhausted its nuclear fuel.

- **Neutron Stars:**
  Neutron stars form when the core of a massive star undergoes gravitational collapse after a supernova explosion. For a core with a mass greater than approximately 1.4 solar masses (the Chandrasekhar limit), the force of gravity will cause the star to collapse into a neutron star instead of becoming a white dwarf [16].

Neutron stars have radii of about 10 kilometers and masses ranging from 1.4 to around 3 solar masses. They are among the densest objects in the universe. PSR J1614-2230 is a heavy neutron star, with a mass of about 1.97 solar masses [17].

- **Black Holes:**
  Among the most important findings of general relativity are black holes. In 1916, Albert Einstein made his first prediction about them. The term "black hole" was first used in 1967 by American theorist John Wheeler. Among the weirdest and most interesting phenomena in the universe are black holes. A black hole is a region of space with so much gravity inside of it that light cannot escape. Gravity is incredibly powerful because much of the matter has been crammed into a very small region. When a star dies, this can happen. According to Einstein's theory of relativity, "nothing can travel faster than light." Because of this, everything is dragged back by the gravitational field; if light cannot escape, then neither can anything else. So, one has a series of occurrences in a region of space from which one cannot leave to reach a distant observer. We now refer to this region as a black hole [10, 18]. A black hole, sometimes referred to as a star's last stage, is the result of a star's death. Since black holes do not emit light, they are invisible to the naked eye. By catching the light of the accretion disk from various angles, space telescopes such as the Event Horizon Telescope, working in collaboration with several other telescopes at various places, can create a combined image of a black hole. Black holes range in size from the size of an atom to the size of a galaxy. Stellar black holes, which are found in large numbers in our galaxy Milky Way, have masses between $10^{\frac{1}{2}}$ and 10^2 solar masses. Massive black holes, known as supermassive black holes, are found at the centres of the largest galaxies. A recent discovery reveals that Sagittarius A*, a supermassive black hole with a mass of approximately 4 million times that of the Sun [19], is located at the centre of our galaxy, the Milky Way. Intermediate black holes, ranging from 10^3 to 10^5 solar masses, are believed to form through star cluster collisions, though their existence is still debated. Supermassive black holes, located at galaxy centers, form from the merger of numerous stellar and intermediate black holes, typically ranging from 10^6 to 10^9 solar masses.

Chapter 2

Charged Compact Objects with Generalized Polytropic Equation of State

Compact objects like white dwarfs and neutron stars have long been modelled using linear and quadratic polytropic equations of state, dating back to the seminal work of Lane [20], and further developed by Chandrasekhar [21] and Tooper [22]. More recently, generalised polytropic equations of state (GPEoS) have been researched as a GR model for charged anisotropic compact objects. Noureen *et al.* [23] used GPEoS for different polytropic indices n to develop new solutions for charged compact objects. After examining models with $n = 1, 2, \frac{2}{3}$, and $\frac{1}{2}$, they were able to obtain exact solutions for the Einstein-Maxwell field equations. By varying n, the authors were able to model different types of compact objects and regain masses matching observed strange stars. The authors were able to create mathematical models that represented various types of dense stellar objects by varying the polytropic index n. These models successfully reproduced the measured masses of known strange stars, which are a hypothetical type of ultra-dense star. Their work builds on earlier studies of polytropes in general relativity by authors like Tooper [24, 25] and more recent explorations of charged anisotropic models by researchers such as Takisa and Maharaj [26].
We will examine Noureen *et al.* [23] application of GPEoS to charged compact objects. The spacetime is considered to be static and spherically symmetric, described by the following metric

$$ds^2 = e^{2\nu(r)}\, dt^2 - e^{2\lambda(r)} \left[dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]. \tag{2.1}$$

Metric tensor of line element (2.1) can be written as

$$g_{\alpha\beta} = diag\left(e^{2\nu(r)}, -e^{2\lambda(r)}, -r^2 e^{2\lambda(r)}, -r^2 e^{2\lambda(r)} \sin^2\theta\right), \tag{2.2}$$

and inverse metric tensor for line element (2.1) can be written as

$$g^{\alpha\beta} = diag\left(e^{-2\nu(r)}, -e^{-2\lambda(r)}, -\frac{e^{-2\lambda(r)}}{r^2}, -\frac{e^{-2\lambda(r)}}{r^2\sin^2\theta}\right). \tag{2.3}$$

The gravitational potential is represented in the static radial coordinate r by the functions $\nu(r)$ and $\lambda(r)$.
The line element (2.1) has the following Christofell symbols

$$\Gamma^0_{01} = \nu', \tag{2.4}$$
$$\Gamma^1_{00} = \nu' e^{2(\nu-\lambda)}, \tag{2.5}$$
$$\Gamma^1_{11} = \lambda', \tag{2.6}$$
$$\Gamma^1_{22} = -r - \lambda' r^2, \tag{2.7}$$
$$\Gamma^1_{33} = -r\sin^2\theta(\lambda' r + 1), \tag{2.8}$$
$$\Gamma^2_{12} = \Gamma^3_{13} = \lambda' + \frac{1}{r}, \tag{2.9}$$
$$\Gamma^2_{33} = -\sin\theta\cos\theta, \tag{2.10}$$
$$\Gamma^3_{32} = \cot\theta. \tag{2.11}$$

So the non-vanishing components of the Ricci curvature tensor are

$$R_{00} = \frac{-e^{2(\nu-\lambda)}}{r}\left(\nu'\lambda' r + \nu'^2 r + \nu'' r + 2\nu'\right), \tag{2.12}$$
$$R_{11} = -\frac{1}{r}\left(\lambda'\nu' r - \nu'^2 r - \nu'' r - 2\lambda'' r - 2\lambda'\right), \tag{2.13}$$
$$R_{22} = \lambda'^2 r^2 + \lambda'\nu' r^2 + \lambda'' r^2 + 3\lambda' r + \nu' r, \tag{2.14}$$
$$R_{33} = R_{22}\sin^2\theta, \tag{2.15}$$

and the Ricci scalar is

$$R = \frac{-2}{e^{2\lambda} r}\left(\lambda'^2 r + \lambda'\nu' r + \nu'^2 r + \nu'' r + 2\lambda'' r + 4\lambda' + 2\nu'\right). \tag{2.16}$$

In the equations, the primes denote differentiation with respect to r.
The distribution of matter inside the core of this star is described by the energy-momentum tensor. The energy momentum tensor with charge for anisotropic fluid is as follows

$$T_{\alpha\beta} = \mathrm{diag}\left(-\rho - \frac{1}{2}E^2,\ p_r - \frac{1}{2}E^2,\ p_t + \frac{1}{2}E^2,\ p_t + \frac{1}{2}E^2\right). \tag{2.17}$$

E represents the intensity of the electric field, while ρ denotes the energy density. p_t is the tangential pressure, while p_r is the radial pressure.

Thus, the Einstein-Maxwell equations become

$$8\pi\rho + \frac{1}{2}E^2 = -\frac{1}{e^{2\lambda}}\left(\lambda'^2 + 2\lambda'' + \frac{4\lambda'}{r}\right), \tag{2.18}$$

$$8\pi p_r - \frac{1}{2}E^2 = \frac{1}{e^{2\lambda}}\left(\lambda'^2 + 2\lambda'\nu' + \frac{2\lambda'}{r} + \frac{2\nu'}{r}\right), \tag{2.19}$$

$$8\pi p_t + \frac{1}{2}E^2 = \frac{1}{e^{2\lambda}}\left(\nu'^2 + \lambda'' + \nu'' + \frac{\lambda'}{r} + \frac{\nu'}{r}\right), \tag{2.20}$$

$$\sigma = \frac{1}{4\pi r^2 e^{\lambda}}\left(r^2 E\right)'. \tag{2.21}$$

Assuming the transformations are

$$x = r^2,\ L = e^{-\lambda(r)},\ G = Le^{\nu(r)}. \tag{2.22}$$

Applying the set of transformed equations (2.22) in eqs (2.18)-(2.21)

$$8\pi\rho + \frac{1}{2}E^2 = 4[2xLL_{xx} - 3(xL_x - L)L_x], \tag{2.23}$$

$$8\pi p_r + \frac{1}{2}E^2 = 4L(L - 2xL_x)\frac{G_x}{G} - 4(2L - 3xL_x)L_x, \tag{2.24}$$

$$8\pi p_t + \frac{1}{2}E^2 = 4xL^2\frac{G_{xx}}{G} + 4L(L - 2xL_x)\frac{G_x}{G} - 4(2L - 3xL_x)L_x - 8xLL_{xx}, \tag{2.25}$$

$$\sigma^2 = \frac{1}{4\pi x^2}L^2(E + xE_x)^2. \tag{2.26}$$

Considering the polytropic equation of state

$$p_r = \beta\rho + \alpha\rho^{1+\frac{1}{n}}. \tag{2.27}$$

The system of the eqs (2.23)-(2.26) with the polytropic equation becomes

$$8\pi\rho = 4(2xLL_{xx} - 3(xL_x - L)L_x) - \frac{1}{2}E^2, \tag{2.28}$$

$$p_r = \beta\left(\frac{8(2xLL_{xx} - 3(xL_x - L)L_x) - E^2}{16\pi}\right) + \alpha\left(\frac{8(2xLL_{xx} - 3(xL_x - L)L_x) - E^2}{16\pi}\right)^{1+\frac{1}{n}}, \tag{2.29}$$

$$p_t = \frac{\Delta}{8\pi} + p_r, \tag{2.30}$$

$$\Delta = 4xL^2\frac{G_{xx}}{G} + 4L(L-2xL_x)\frac{G_x}{G} - 4(2L-3xL_x)L_x - 8xLL_{xx} - \frac{1}{2}E^2 - \frac{\beta}{2}$$
$$\left(8(2xLL_{xx} - 3(xL_x - L)L_x) - E^2\right) - 8\pi\alpha\left(\frac{8(2xLL_{xx} - 3(xL_x - L)L_x) - E^2}{16\pi}\right)^{1+\frac{1}{n}}, \tag{2.31}$$

$$\frac{G_x}{G} = \frac{\beta}{8L(L-2xL_x)}\left(8(2xL_{xx} - 3(xL_x - L)L_x) - E^2\right) + \frac{2\alpha\pi}{L(L-2xL_x)}$$
$$\left(\frac{8(2xL_{xx} - 3(xL_x - L)L_x) - E^2}{16\pi}\right)^{1+\frac{1}{n}} + \frac{1}{4L(L-2xL_x)}\left(4(2L-3xL_x)L_x - \frac{1}{2}E^2\right), \tag{2.32}$$

$$\sigma^2 = \frac{1}{4\pi^2 x}L^2\left(E + xE_x\right)^2, \tag{2.33}$$

where Δ is the ansiotropic factor and σ is the charge density. There are some typing errors in [23]. The corrected version is given by eq (2.32).
A physically reasonable form of gravitational potential L and electric field intensity E are chosen i.e.

$$L = a + bx, \tag{2.34}$$
$$E^2 = c + dx, \tag{2.35}$$

where a, b, c, and d are real constants. We choose L and E^2 to be a linear function in the variable x. This ensures that the potential and the charge are finite at the centre and are regular in the interior.
Therefore, the eqs (2.28), (2.29), (2.32) and (2.33) by substituting L and E become

$$8\pi\rho = \left(\frac{24ab - c}{2}\right) - \frac{xd}{2}, \tag{2.36}$$

$$p_r = \beta\left(\frac{24ab - c - xd}{16\pi}\right) + \alpha\left(\frac{24ab - c - xd}{16\pi}\right)^{1+\frac{1}{n}}, \tag{2.37}$$

$$\frac{G_x}{G} = \frac{\left(16\pi\alpha\left(\frac{24ab-c-dx}{16\pi}\right)^{1+\frac{1}{n}} + \beta(24ab - c - dx) + 16ab - 8b^2x - c - dx\right)}{8(a+bx)(a-bx)}, \tag{2.38}$$

$$\sigma^2 = \frac{(a+bx)^2(2c+3xd)^2}{16\pi^2 x(c+xd)}. \tag{2.39}$$

2.1 Generalized Polytropic Models

The models presented in this article are derived by varying the values of n as $1, 2, \frac{2}{3}$, and $\frac{1}{2}$.

Model 1

The radial and tangential pressures for $n = 1$ are as follows

$$p_r = \beta\left(\frac{24ab - c - xd}{16\pi}\right) + \alpha\left(\frac{24ab - c - xd}{16\pi}\right)^2, \tag{2.40}$$

$$\begin{aligned} p_t = {} & \frac{1}{32768\pi^3(a - bx)^2}\Bigg[(36864a^4b^2\alpha(2\pi + 9b^2x\alpha) - 6144a^3b\left(c\alpha(\pi + 9b^2x\alpha)\right. \\ & \left. + 2\pi(dx\alpha - 4\pi\beta) + 3b^2x\alpha\left(3dx\alpha - 8\pi(1 + 3\beta)\right)\right) - 32abx\left(3c^3\alpha^2 + c^2\alpha\left(9dx\alpha\right.\right. \\ & \left. -8\pi(7 + 9\beta)\right) + c\left(192b^2\pi x\alpha + 9d^2x^2\alpha^2 + 128\pi^2\beta(4 + 3\beta) - 16d\pi x\alpha(7 + 9\beta)\right) \\ & \left. +x\left(3d^3x^2\alpha^2 - 1536b^2\pi^2\beta + 128d\pi^2\beta(4 + 3\beta) - 8d^2\pi x\alpha(7 + 9\beta)\right)\right) + x\left(c^4\alpha^2\right. \\ & +4c^3\alpha\left(dx\alpha - 8\pi(1 + \beta)\right) + 2c^2\Big(3d^2x^2\alpha^2 + 128\pi^2(1 + \beta)^2 + 16\pi x\alpha\left(4b^2 - 3d(1\right. \\ & \left. +\beta)\right)\Big) + dx^2\left(-128b^2\pi\left(16\pi + dx\alpha\right) + d\left(dx\alpha - 16\pi(1 + \beta)\right)^2\right) + 4cx\left(-512b^2\pi^2(2\right. \\ & \left.\left. + \beta) + d\left(d^2x^2\alpha^2 - 24d\pi x\alpha(1 + \beta) + 128\pi^2(1 + \beta)^2\right)\right)\Big) + 128a^2\left(c^2\alpha(\pi + 27b^2x\alpha)\right. \\ & -2c\left(-27b^2dx^2\alpha^2 + 8\pi^2(2 + \beta) + 2\pi x\alpha\left(-d + 12b^2(5 + 9\beta)\right)\right) + x\left(576b^4\pi x\alpha\right. \\ & +d\pi\left(3dx\alpha - 16\pi(3 + 2\beta)\right) + 3b^2\left(9d^2x^2\alpha^2 - 16d\pi x\alpha(5 + 9\beta) + 128\pi^2(1\right. \\ & \left.\left.\left. +2\beta + 3\beta^2)\right)\right)\right)\Bigg]. \end{aligned} \tag{2.41}$$

Integrating eq (2.38) for $n = 1$ yields the gravitational potential G

$$G = D_1(a + bx)^H(a - bx)^P exp(N(x)), \tag{2.42}$$

where D_1 is integration constant. The constants H and P and the variable $N(x)$ are given as

$$\begin{aligned}
H &= \frac{1}{256\pi ab^3}\Big(\alpha\left(24ab^2+ad-bc\right)^2+16\pi b\beta\left(24ab^2+ad-bc\right) \\
&\quad +16\pi b\left(24ab^2+ad-bc\right)\Big), \\
P &= \frac{1}{256\pi ab^3}\Big(-\alpha\left(24ab^2-ad-bc\right)^2-16\pi b\beta\left(24ab^2-ad-bc\right) \\
&\quad -16\pi b\left(8ab^2-ad-bc\right)\Big), \\
N(x) &= -\frac{\alpha d^2 x}{128\pi b^2}.
\end{aligned}$$

For $D_1^2 = k_1$ the eq (2.1) becomes

$$ds^2 = k_1(a+br^2)^{2(H-1)}(a-br^2)^{2P}exp(2N(r^2))\,dt^2-(a+br^2)^{-2}\left[dr^2+r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)\right]. \tag{2.43}$$

Model 2

The radial and tangential pressures for $n = 2$ are as follows

$$p_r = \beta\frac{f(x)}{16\pi}+\alpha\left(\frac{f(x)}{16\pi}\right)^{\frac{3}{2}}, \tag{2.44}$$

$$\begin{aligned}
P_t &= \frac{1}{2048\pi^2(a-bx)^2}\Bigg[768a^3b\left(\sqrt{\pi f(x)}\alpha+18b^2x\alpha^2+4\pi\beta\right)+x\left(-c^3\alpha^2\right. \\
&\quad +c^2\left(-3dx\alpha^2+8\sqrt{\pi f(x)}\alpha(1+\beta)+16\pi(1+\beta)^2\right)+dx^2\left(-16b^2\left(8\pi\right.\right. \\
&\quad \left.\left.-\sqrt{\pi f(x)}\alpha\right)+d\left(-dx\alpha^2+8\sqrt{\pi f(x)}\alpha(1+\beta)+16\pi(1+\beta)^2\right)\right)+cx \\
&\quad \times\left(d\left(-3dx\alpha^2+16\sqrt{\pi f(x)}\alpha(1+\beta)+32\pi(1+\beta)^2\right)-32b^2\left(\sqrt{\pi f(x)}\alpha\right.\right. \\
&\quad \left.\left.\left.+4\pi(2+\beta)\right)\right)\right)+8abx\left(9c^2\alpha^2-2c\left(-9dx\alpha^2+8\sqrt{\pi}\sqrt{f(x)}\alpha(2+3\beta)\right.\right. \\
&\quad \left.+16\pi\beta(4+3\beta)\right)+x\left(96b^2\left(\sqrt{\pi f(x)}\alpha+4\pi\beta\right)+d\left(9dx\alpha^2-16\sqrt{\pi f(x)}\alpha\right.\right. \\
&\quad \left.\left.\left.\times(2+3\beta)-32\pi\beta(4+3\beta)\right)\right)\right)-16a^2\left(2c\left(\sqrt{\pi f(x)}\alpha+54b^2x\alpha^2+4\pi(2\right.\right.
\end{aligned}$$

$$+\beta)) + x\left(5d\sqrt{\pi f(x)}\alpha + 8d\pi(3+2\beta) - 12b^2\left(-9dx\alpha^2 \;+ 8\sqrt{\pi f(x)}\alpha(1\right.\right.$$
$$\left.\left. +3\beta) + 16\pi(1+\beta(2+3\beta)))\right)\right)\Bigg]. \tag{2.45}$$

Integrating eq (2.38) for $n = 2$ yields the gravitational potential G

$$G = D_2(a+bx)^H(a-bx)^P\left(U(x)\right)^\Omega\left(V(x)\right)^\gamma exp(N(x)), \tag{2.46}$$

where D_2 is integration constant. The constants H, P, Ω, and γ and the variables $f(x)$ $U(x)$, $V(x)$, and $N(x)$ are given as

$$\begin{aligned}
H &= \frac{1}{16ab^2}(\beta+1)\left(24ab^2 + ad - bc\right),\\
P &= \frac{1}{16ab^2}\left(\beta\left(ad - 24ab^2 + bc\right) + ad - 8ab^2 + bc\right),\\
\Omega &= \frac{\alpha}{64a}\left[\frac{\left(a(24b^2-d)-bc\right)^3}{\pi b^5}\right]^{\frac{1}{2}},\\
\gamma &= \frac{\alpha}{64a}\left[\frac{\left(a(24b^2+d)-bc\right)^3}{\pi b^5}\right]^{\frac{1}{2}},\\
f(x) &= 24ab - c - dx,\\
U(x) &= \frac{\left[a(24b^2-d)-bc\right]^{\frac{1}{2}} + \left(bf(x)\right)^{\frac{1}{2}}}{\left[a(24b^2-d)-bc\right]^{\frac{1}{2}} - \left(bf(x)\right)^{\frac{1}{2}}},\\
V(x) &= \frac{\left[a(24b^2+d)-bc\right]^{\frac{1}{2}} - \left(bf(x)\right)^{\frac{1}{2}}}{\left[a(24b^2+d)-bc\right]^{\frac{1}{2}} + \left(bf(x)\right)^{\frac{1}{2}}},\\
N(x) &= \frac{d\alpha}{16b^2}\left[\frac{f(x)}{\pi}\right]^{\frac{1}{2}}.
\end{aligned}$$

For $D_2^2 = k_2$ the eq (2.1) becomes

$$\begin{aligned}
ds^2 &= k_2(a+br^2)^{2(H-1)}(a-br^2)^{2P}\left(U(r^2)\right)^{2\Omega}\left(V(r^2)\right)^{2\gamma} exp(2N(r^2))\,dt^2\\
&\quad - (a+br^2)^{-2}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right].
\end{aligned} \tag{2.47}$$

Model 3

The radial and tangential pressures for $n = \frac{2}{3}$ are as follows

$$p_r = \frac{\beta f(x)}{16\pi} + \alpha \left(\frac{f(x)}{16\pi}\right)^{\frac{5}{2}}, \tag{2.48}$$

$$\begin{aligned}
p_t = {} & \frac{1}{524288\pi^4(a-bx)^2}\Bigg[7962624a^5b^5x\alpha^2 + 36864a^4b^2\alpha\left(8\pi^{3/2}\sqrt{f(x)} - 45b^2x\right.\\
& \left.\times(c+dx)\alpha\right) + 3072a^3b\left(-18d\pi^{3/2}x\sqrt{f(x)}\alpha + 45b^2c^2x\alpha^2 + c\left(-8\pi^{3/2}\sqrt{f(x)}\alpha\right.\right.\\
& \left.\left.+90b^2dx^2\alpha^2\right) + 256\pi^3\beta + 3b^2x\alpha\left(15d^2x^2\alpha + 64\pi^{3/2} \times \sqrt{f(x)}(1+3\beta)\right)\right)\\
& - x\left(c^5\alpha^2 + 5c^4dx\alpha^2 + 2c^3\alpha\left(5d^2x^2\alpha + 64\pi^{3/2}\sqrt{f(x)}(1+\beta)\right) - 2c^2\left(-5d^3x^3\alpha^2\right.\right.\\
& \left.+2048\pi^3(1+\beta)^2 + 64\pi^{3/2}x\sqrt{f(x)}\alpha\left(4b^2 - 3d(1+\beta)\right)\right) + dx^2\left(256b^2\left(128\pi^3\right.\right.\\
& \left.\left.+3d\pi^{3/2}x\sqrt{f(x)}\alpha\right) + d\left(d^3x^3\alpha^2 + 128d\pi^{3/2}x\sqrt{f(x)}\alpha(1+\beta) - 4096\pi^3(1+\beta)^2\right)\right)\\
& +cx\left(d\left(5d^3x^3\alpha^2 + 384d\pi^{3/2}x\sqrt{f(x)}\alpha(1+\beta) - 8192\pi^3(1+\beta)^2\right) + 256b^2\left(d\pi^{3/2}\right.\right.\\
& \left.\left.\left.\times x\sqrt{f(x)}\alpha + 128\pi^3(2+\beta)\right)\right)\right) + 8abx\left(15c^4\alpha^2 + 60c^3dx\alpha^2 + 2c^2\alpha\left(45d^2x^2\alpha\right.\right.\\
& \left.+64\pi^{3/2}\sqrt{f(x)}(7+9\beta)\right) - 4c\left(768b^2\pi^{3/2}x\sqrt{f(x)}\alpha - 15d^3x^3\alpha^2 + 2048\pi^3\beta(4+3\beta)\right.\\
& \left.-64d\pi^{3/2}x\sqrt{f(x)}\alpha(7+9\beta)\right) + x\left(768b^2\left(d\pi^{3/2}x\sqrt{f(x)}\alpha + 128\pi^3\beta\right) + d\left(15d^3x^3\alpha^2\right.\right.\\
& \left.\left.\left.-8192\pi^3\beta(4+3\beta) + 128d\pi^{3/2}x\sqrt{f(x)}\alpha(7+9\beta)\right)\right)\right) - 128a^2\left(45b^2c^3x\alpha^2\right.\\
& +c^2\alpha\left(-4\pi^{3/2}\sqrt{f(x)} + 135b^2dx^2\alpha\right) + c\left(135b^2d^2x^3\alpha^2 + 256\pi^3(2+\beta) + 6\pi^{3/2}\right.\\
& \left.\times x\sqrt{f(x)}\alpha\left(-3d + 32b^2(5+9\beta)\right)\right) + x\left(-2304b^4\pi^{3/2}x\sqrt{f(x)}\alpha - 14d^2\pi^{3/2}\right.\\
& \times x\sqrt{f(x)}\alpha + 256d\pi^3(3+2\beta) - 3b^2\left(-15d^3x^3\alpha^2 - 64d\pi^{3/2}x\sqrt{f(x)}\alpha(5+9\beta)\right.\\
& \left.\left.\left.+2048\pi^3(1+2\beta+3\beta^2)\right)\right)\right)\Bigg].
\end{aligned} \tag{2.49}$$

Integrating eq (2.38) for $n = \frac{2}{3}$ yields the gravitational potential G

$$G = D_3(a+bx)^H(a-bx)^P \left(U(x)\right)^{\Omega} \left(V(x)\right)^{\gamma} exp(N(x)), \tag{2.50}$$

where D_3 is integration constant. The constants H, P, Ω, and γ and the variables $f(x)$, $U(x)$, $V(x)$, and $N(x)$ are given as

$$\begin{aligned}
H &= \frac{1}{16ab^2}(\beta+1)\left(24ab^2 + ad - bc\right), \\
P &= \frac{1}{16ab^2}\Big(\beta\left(ad - 24ab^2 + bc\right) + ad - 8ab^2 + bc\Big), \\
\Omega &= \frac{\alpha}{1024a}\left[\frac{\left(a(24b^2 - d) - bc\right)^5}{\pi^3 b^7}\right]^{\frac{1}{2}}, \\
\gamma &= \frac{\alpha}{1024a}\left[\frac{\left(a(24b^2 + d) - bc\right)^5}{\pi^3 b^7}\right]^{\frac{1}{2}}, \\
f(x) &= 24ab - c - dx, \\
U(x) &= \frac{\left[a(24b^2 - d) - bc\right]^{\frac{1}{2}} + \left(bf(x)\right)^{\frac{1}{2}}}{\left[a(24b^2 - d) - bc\right]^{\frac{1}{2}} - \left(bf(x)\right)^{\frac{1}{2}}}, \\
V(x) &= \frac{\left[a(24b^2 + d) - bc\right]^{\frac{1}{2}} - \left(bf(x)\right)^{\frac{1}{2}}}{\left[a(24b^2 + d) - bc\right]^{\frac{1}{2}} + \left(bf(x)\right)^{\frac{1}{2}}}, \\
N(x) &= \frac{d\alpha}{768b^2}\left[\frac{f(x)}{\pi^3}\right]^{\frac{1}{2}}[168ab - 7c - dx].
\end{aligned}$$

For $D_3^2 = k_3$ the eq (2.1) becomes

$$\begin{aligned}
ds^2 &= k_3(a+br^2)^{2(H-1)}(a-br^2)^{2P}\left(U(r^2)\right)^{2\Omega}\left(V(r^2)\right)^{2\gamma} exp(2N(r^2))\, dt^2 \\
&\quad - (a+br^2)^{-2}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right].
\end{aligned} \tag{2.51}$$

Model 4

The radial and tangential pressures for $n = \frac{1}{2}$ are as follows

$$p_r = \beta\left(\frac{24ab - c - xd}{16\pi}\right) + \alpha\left(\frac{24ab - c - xd}{16\pi}\right)^3, \tag{2.52}$$

$$p_t = \frac{1}{8388608\pi^5(a-bx)^2}\Big[191102976a^6b^6x\alpha^2 - 1769472a^5b^3\alpha(-16\pi^2 + 27b^2x(c+dx)\alpha)$$
$$+ 110592a^4b^2\alpha(-64d\pi^2x + 45b^2c^2x\alpha + c(-32\pi^2 + 90b^2dx^2\alpha) + b^2x(45d^2x^2\alpha$$
$$+ 512\pi^2(1+3\beta))) - 16abx(9c^5\alpha^2 + 45c^4dx\alpha^2 + 2c^3\alpha(45d^2x^2\alpha + 256\pi^2(5+6\beta))$$
$$+ 6c^2x\alpha(-1536b^2\pi^2 + 15d^3x^2\alpha + 256d\pi^2(5+6\beta)) + c(45d^4x^4\alpha^2 + 65536\pi^4\beta(4$$
$$+ 48b^2x + 3\beta) + 1536d^2\pi^2x^2\alpha(5+6\beta)) + x(9d^5x^4\alpha^2 + 65536d\pi^4\beta(4+3\beta)$$
$$+ 512d^3\pi^2x^2\alpha(5+6\beta) + 3072b^2(3d^2\pi^2x^2\alpha + 256\pi^4\beta + 512d\pi^4x\beta)))$$
$$+ 6144a^3b(4608b^4\pi^2x^2\alpha - 45b^2c^3x\alpha^2 + 3c^2\alpha(8\pi^2 - 45b^2dx^2\alpha) + 8(9d^2\pi^2x^2\alpha$$
$$+ 256\pi^4\beta - 512d\pi^4x\beta) - 9b^2dx^2\alpha(5d^2x^2\alpha + 256\pi^2(1+2\beta)) - 3cx\alpha(-32d\pi^2$$
$$+ 3b^2(15d^2x^2\alpha + 256\pi^2(1+2\beta)))) + x(c^6\alpha^2 + 6c^5dx\alpha^2 + c^4\alpha(15d^2x^2\alpha + 512\pi^2(1+\beta))$$
$$+ 4c^3x\alpha(-512b^2\pi^2 + 5d^3x^2\alpha + 512d\pi^2(1+\beta)) + c^2(15d^4x^4\alpha^2 + 3072d^2\pi^2x^2\alpha(1+\beta)$$
$$+ 65536\pi^4(1 + 2(1+8b^2x)\beta + \beta^2)) + dx^2(4096b^2(d^2\pi^2x^2\alpha + 128\pi^4(-1+\beta)) + d(d^2x^2\alpha$$
$$+ 256\pi^2(1+\beta))^2) + 2cx(3d^5x^4\alpha^2 + 1024d^3\pi^2x^2\alpha(1+\beta) + 65536d\pi^4(1+\beta)^2$$
$$+ 1024b^2(3d^2\pi^2x^2\alpha + 256\pi^4(-2+\beta+2dx\beta)))) + 64a^2(135b^2c^4x\alpha^2 + 4c^3\alpha(-8\pi^2$$
$$+ 135b^2dx^2\alpha) + 4c(135b^2d^3x^4\alpha^2 + 2048\pi^4(-2 + (-1+2dx)\beta) - 72\pi^2x^2\alpha(192b^4 + d^2$$
$$- 64b^2d(2+3\beta))) + 6c^2x\alpha(-32d\pi^2 + 3b^2(45d^2x^2\alpha + 512\pi^2(2+3\beta)))$$
$$+ x(9437184b^4\pi^4x\beta - 128d\pi^2(d^2x^2\alpha - 64\pi^2(-3-\beta+2dx\beta)) + 3b^2(45d^4x^4\alpha^2$$
$$+ 3072d^2\pi^2x^2\alpha(2+3\beta) + 65536\pi^4(1+2\beta+3\beta^2))))\Big]. \tag{2.53}$$

Integrating eq (2.38) for $n = \frac{1}{2}$ yields the gravitational potential G

$$G = D_4(a+bx)^H(a-bx)^P exp(N(x)), \tag{2.54}$$

where D_4 is integration constant. The constants H and P and the variable $N(x)$ are given as

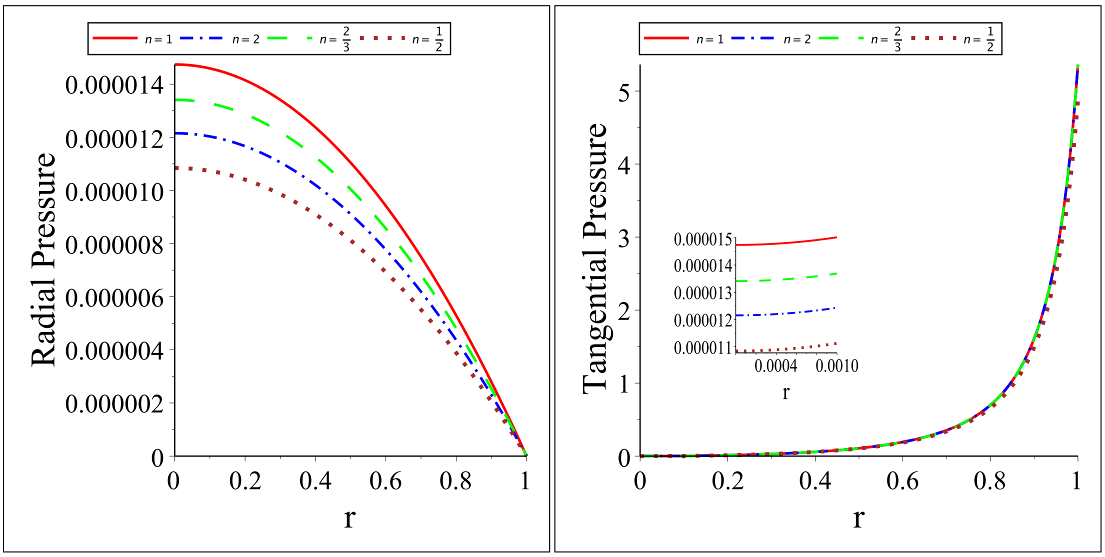
$$H = \frac{1}{4096\pi^2ab^4}\left(\alpha\left(a\left(24b^2+d\right)-bc\right)^3 + 256\pi^2b^2\beta\left(a\left(24b^2+d\right)-bc\right)\right.$$
$$\left.+256\pi^2b^2\left(a\left(24b^2+d\right)-bc\right)\right),$$
$$P = \frac{1}{4096\pi^2ab^4}\left(\alpha\left(a\left(d-24b^2\right)+bc\right)^3 + 256\pi^2b^2\beta\left(a\left(d-24b^2\right)+bc\right)\right.$$
$$\left.+256\pi^2b^2\left(a\left(d-8b^2\right)+bc\right)\right),$$

$$N(x) = \frac{\alpha d^2 x[6(c - 24ab) + xd]}{4096\pi^2 b^2}.$$

For $D_4^2 = k_4$ the eq (2.1) becomes

$$ds^2 = k_4(a+br^2)^{2(H-1)}(a-br^2)^{2P} exp(2N(r^2))\, dt^2 - (a+br^2)^{-2}\left[dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right]. \tag{2.55}$$

In order to understand the internal structure and stability of charged compact objects, we present several key graphical representations from the study. Through visual analysis of these distributions, we may gain a better understanding of how charge and anisotropy affect these objects' gravitational behaviour and what this means for their physical viability. The variation of radial pressure, p_r, and tangential pressure, p_t, are shown in Fig 2.1, both p_r and p_t decrease radially outward. Fig 2.2 shows the graph of density and anisotropy, density decreases with increase in radius and anisotropy is zero at the centre and positive otherwise.



(a) p_r (b) p_t

Figure 2.1: The radial and tangential pressures both are well defined and non-negative and plotted by taking $a = 1$, $b = 0.772$, $c = 0$, $d = 0.0804$, and $\alpha = 0.025$.

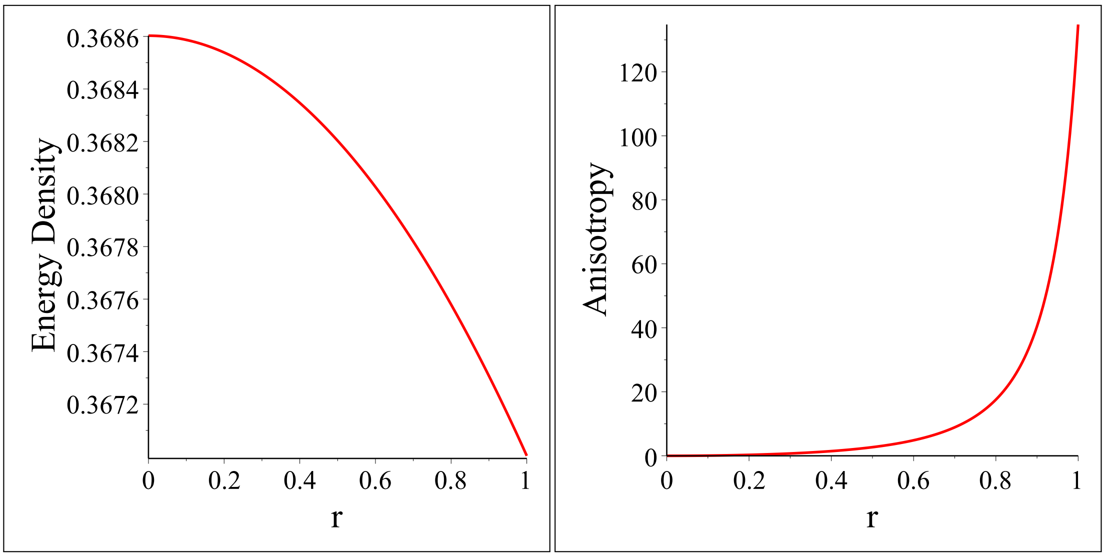


Figure 2.2: Variation of energy density and anisotropy are plotted by taking the same values of the constants mentioned in Fig 2.1.

Discussion

In [23], the authors used a GPEoS to present exact solutions to the EMFEs. Their main area of focus is how charge and anisotropy affect the interior structure of extremely dense stars. According to the study, tangential pressure and anisotropy increase towards the star's boundary, while energy density and radial pressure decrease monotonically from the star's centre to its boundary. Furthermore, the model satisfies all the energy conditions, which are necessary. The stability factor and adiabatic index fall within the limits.

Chapter 3

Charged Compact Objects with $p_r = \beta\rho + \alpha\rho^\Gamma + \gamma\rho^2$

3.1 Exact Solutions with Modified Generalized Polytropic Equation of State

In this chapter, we present a solution of the EMFEs for a charged anisotropic matter distribution. Our aim is to extend the work of Noureen *et al.* [23] by incorporating an additional quadratic term in the generalized polytropic equation of state. The equation of state of our model is chosen as

$$p_r = \beta\rho + \alpha\rho^\Gamma + \gamma\rho^2, \tag{3.1}$$

where α, β and γ are arbitrary constants. The system of the eqs (2.23)-(2.26) with the polytropic equation becomes

$$8\pi\rho = 4(2xLL_{xx} - 3(xL_x - L)L_x) - \frac{1}{2}E^2, \tag{3.2}$$

$$\begin{aligned} p_r = {} & \beta\left(\frac{8(2xLL_{xx} - 3(xL_x - L)L_x) - E^2}{16\pi}\right) \\ & + \alpha\left(\frac{8(2xLL_{xx} - 3(xL_x - L)L_x) - E^2}{16\pi}\right)^{1+\frac{1}{n}} \\ & + \gamma\left(\frac{8(2xLL_{xx} - 3(xL_x - L)L_x) - E^2}{16\pi}\right)^2, \end{aligned} \tag{3.3}$$

$$p_t = \frac{\Delta}{8\pi} + p_r, \tag{3.4}$$

$$\Delta = 4xL^2\frac{G_{xx}}{G} + 4L(L-2xL_x)\frac{G_x}{G} - 4(2L-3xL_x)L_x - 8xLL_{xx} - \frac{1}{2}E^2 - \frac{\beta}{2}$$
$$\left(8(2xLL_{xx} - 3(xL_x - L)L_x) - E^2\right) - \frac{\gamma}{32\pi}\left(8(2xLL_{xx} - 3(xL_x - L)L_x) - E^2\right)^2$$
$$- 8\pi\alpha\left(\frac{8(2xLL_{xx} - 3(xL_x - L)L_x) - E^2}{16\pi}\right)^{1+\frac{1}{n}}, \tag{3.5}$$

$$\frac{G_x}{G} = \frac{\beta}{8L(L-2xL_x)}\left(8(2xL_{xx} - 3(xL_x - L)L_x) - E^2\right) + \frac{2\alpha\pi}{L(L-2xL_x)}$$
$$\left(\frac{8(2xL_{xx} - 3(xL_x - L)L_x) - E^2}{16\pi}\right)^{1+\frac{1}{n}} + \frac{1}{4L(L-2xL_x)}\left(4(2L-3xL_x)L_x - \frac{1}{2}E^2\right)$$
$$+ \frac{\gamma}{128\pi L(L-2xL_x)}\left(8(2xL_{xx} - 3(xL_x - L)L_x) - E^2\right)^2, \tag{3.6}$$

$$\sigma^2 = \frac{1}{4\pi^2 x}L^2\left(E + xE_x\right)^2. \tag{3.7}$$

Substituting the expressions of L and E given by eq (2.34) and eq (2.35) respectively in eq (3.6), we get

$$\begin{aligned}\frac{G_x}{G} = \frac{1}{8(a+bx)(a-bx)}\Bigg[&16\pi\alpha\left(\frac{24ab-c-dx}{16\pi}\right)^{1+\frac{1}{n}} + \beta(24ab-c-dx) \\ &+ \frac{\gamma}{16\pi}(24ab-c-dx)^2 + 16ab - 8b^2x - c - dx\Bigg].\end{aligned} \tag{3.8}$$

The expressions of energy density, p_r and charge density take the form

$$8\pi\rho = \left(\frac{24ab-c}{2}\right) - \frac{xd}{2}, \tag{3.9}$$

$$p_r = \beta\left(\frac{24ab-c-xd}{16\pi}\right) + \alpha\left(\frac{24ab-c-xd}{16\pi}\right)^{1+\frac{1}{n}} + \gamma\left(\frac{24ab-c-xd}{16\pi}\right)^2, \tag{3.10}$$

$$\sigma^2 = \frac{(a+bx)^2(2c+3xd)^2}{16\pi^2 x(c+xd)}. \tag{3.11}$$

The total mass within the radius 'r' of the sphere is given by

$$M(x) = 2\pi\int_0^x \frac{1}{\sqrt{\tau}}\left[\tau\rho(\tau) + \frac{E^2}{8\pi}\right]d\tau. \tag{3.12}$$

By using eqs (3.9) and (2.35) within eq (3.12), the expression for the mass function is

$$M(x) = \frac{1}{60}\sqrt{x}(120abx - 5c(x-6) + dx(10-3x)). \tag{3.13}$$

3.1.1 Modified Generalized Polytropic Models

In this section, exact solutions to field equations using polytropic indices $n = 1, 2, \frac{2}{3}$, and $\frac{1}{2}$ are presented.

MODEL I

When $n = 1$, Model 1 of the [23] is recovered, since in this case

$$p_r = \beta\left(\frac{24ab - c - xd}{16\pi}\right) + (\alpha+\gamma)\left(\frac{24ab - c - xd}{16\pi}\right)^2. \tag{3.14}$$

Model II

When $n = 2$, radial and tangential pressures are given as

$$p_r = \beta\left(\frac{24ab - c - xd}{16\pi}\right) + \alpha\left(\frac{24ab - c - xd}{16\pi}\right)^{\frac{3}{2}} + \gamma\left(\frac{24ab - c - xd}{16\pi}\right)^2, \tag{3.15}$$

$$\begin{aligned}
p_t = {} & \frac{1}{32768\pi^3(a-bx)^2}\Bigg[36864a^4b^2\gamma(2\pi + 9b^2x\gamma) + 6144a^3b\Big(2\pi^{3/2}\sqrt{f(x)}\alpha + 8\pi^2\beta \\
& -\pi(c+2dx)\gamma + 18b^2x\sqrt{\pi f(x)}\alpha\gamma - 9b^2x(c+dx)\gamma^2 + 12b^2\pi x(3\alpha^2 + 2\gamma + 6\beta\gamma)\Big) \\
& + x\Bigg(c^4\gamma^2 - 4c^3\Big(2\sqrt{\pi f(x)}\alpha\gamma - dx\gamma^2 + 4\pi(\alpha^2 + 2(1+\beta)\gamma)\Big) + 2c^2\Big(64\pi^{3/2}\sqrt{f(x)} \\
& \times\alpha(1+\beta) + 128\pi^2(1+\beta)^2 - 12dx\sqrt{\pi f(x)}\alpha\gamma + 3d^2x^2\gamma^2 - 8\pi x(-8b^2\gamma + 3d(\alpha^2 \\
& +2(1+\beta)\gamma))\Big) + dx^2\Bigg(-128b^2\pi(16\pi - 2\sqrt{\pi f(x)}\alpha + dx\gamma) + d\Big(128\pi^{3/2}\sqrt{f(x)}\alpha(1 \\
& +\beta) + 256\pi^2(1+\beta)^2 - 8dx\sqrt{\pi f(x)}\alpha\gamma + d^2x^2\gamma^2 - 16d\pi x(\alpha^2 + 2(1+\beta)\gamma)\Big)\Bigg) \\
& +4cx\Bigg(-128b^2(\pi^{3/2}\sqrt{f(x)}\alpha + 4\pi^2(2+\beta)) + d\Big(64\pi^{3/2}\sqrt{f(x)}\alpha(1+\beta) + 128\pi^2(1+\beta)^2 \\
& -6dx\sqrt{\pi f(x)}\alpha\gamma + d^2x^2\gamma^2 - 12d\pi x(\alpha^2 + 2(1+\beta)\gamma)\Big)\Bigg)\Bigg) - 128a^2\Big(-c^2\gamma(\pi + 27b^2x\gamma) \\
& +2c\Big(2\pi^{3/2}\sqrt{f(x)}\alpha + 8\pi^2(2+\beta) + 54b^2x\sqrt{\pi f(x)}\alpha\gamma - 27b^2dx^2\gamma^2 + 2\pi x(-d\gamma + 6b^2 \\
& \times(9\alpha^2 + 2(5+9\beta)\gamma))\Big) + x\Big(-576b^4\pi x\gamma + d\pi(10\sqrt{\pi f(x)}\alpha + 16\pi(3+2\beta) - 3dx\gamma)
\end{aligned}$$

$$-3b^2\Big(64\pi^{3/2}\sqrt{f(x)}\alpha(1+3\beta)+128\pi^2(1+2\beta+3\beta^2)-36dx\sqrt{\pi f(x)}\alpha\gamma+9d^2x^2\gamma^2$$
$$-8d\pi x(9\alpha^2+2(5+9\beta)\gamma))\Big)\Big)+32abx\Big(-3c^3\gamma^2+c^2\Big(36\pi\alpha^2+18\sqrt{\pi f(x)}\alpha\gamma$$
$$+8\pi(7+9\beta)\gamma-9dx\gamma^2\Big)+c\Big(-64\pi^{3/2}\sqrt{f(x)}\alpha(2+3\beta)-128\pi^2\beta(4+3\beta)+36dx$$
$$\times\sqrt{\pi f(x)}\alpha\gamma-9d^2x^2\gamma^2+8\pi x(9d\alpha^2-24b^2\gamma+2d(7+9\beta)\gamma)\Big)+x\Big(384b^2(\pi^{3/2}\sqrt{f(x)}\alpha$$
$$+4\pi^2\beta)+d\Big(-64\pi^{3/2}\sqrt{f(x)}\alpha(2+3\beta)-128\pi^2\beta(4+3\beta)+18dx\sqrt{\pi f(x)}\alpha\gamma$$
$$-3d^2x^2\gamma^2+4d\pi x(9\alpha^2+2(7+9\beta)\gamma))\Big)\Big)\Big]. \qquad (3.16)$$

Integrating eq (3.8) by substituting $n=\frac{2}{3}$, we get the gravitational potential G as

$$G=K_2(a+bx)^I(a-bx)^W\big(U(x)\big)^\Omega\big(V(x)\big)^\zeta exp(Z(x)), \qquad (3.17)$$

where K_2 is the integration constant. The constants I, W, Ω, and ζ and the variables $f(x)$, $U(x)$, $V(x)$, and $Z(x)$ are given as

$$I=\frac{1}{16ab^2}(\beta+1)\left(24ab^2+ad-bc\right)+\frac{\gamma}{256ab^3\pi}(-24ab^2-ad+bc)^2,$$
$$W=\frac{1}{16ab^2}\Big(\beta\left(ad-24ab^2+bc\right)+ad-8ab^2+bc\Big)-\frac{\gamma}{256ab^3\pi}(-24ab^2+ad+bc)^2,$$
$$\Omega=\frac{\alpha}{64a}\left[\frac{\left(24ab^2-da-bc\right)^3}{\pi b^5}\right]^{\frac{1}{2}},$$
$$\zeta=\frac{\alpha}{64a}\left[\frac{\left(24ab^2+da-bc\right)^3}{\pi b^5}\right]^{\frac{1}{2}},$$
$$f(x)=24ab-c-dx,$$
$$U(x)=\frac{\left[24ab^2-da-bc\right]^{\frac{1}{2}}+\left(bf(x)\right)^{\frac{1}{2}}}{\left[24ab^2-da-bc\right]^{\frac{1}{2}}-\left(bf(x)\right)^{\frac{1}{2}}},$$
$$V(x)=\frac{\left[24ab^2+da-bc\right]^{\frac{1}{2}}-\left(bf(x)\right)^{\frac{1}{2}}}{\left[24ab^2+da-bc\right]^{\frac{1}{2}}+\left(bf(x)\right)^{\frac{1}{2}}},$$
$$Z(x)=\frac{d\alpha}{16b^2}\left[\frac{f(x)}{\pi}\right]^{\frac{1}{2}}-\frac{d^2r^2\gamma}{128b^2\pi}.$$

The line element (2.1) becomes

$$\begin{aligned} ds^2 = K_2^2(a+br^2)^{2(I-1)}(a-br^2)^{2W}\left(U(r^2)\right)^{2\Omega}\left(V(r^2)\right)^{2\zeta} exp(2Z(r^2))\,dt^2 \\ -(a+br^2)^{-2}\left[dr^2+r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)\right]. \end{aligned} \tag{3.18}$$

Model III

When $n=\frac{2}{3}$, radial and tangential pressures are given as

$$p_r=\beta\left(\frac{24ab-c-xd}{16\pi}\right)+\alpha\left(\frac{24ab-c-xd}{16\pi}\right)^{\frac{5}{2}}+\gamma\left(\frac{24ab-c-xd}{16\pi}\right)^{2}, \tag{3.19}$$

$$\begin{aligned} p_t=&\frac{1}{524288\pi^4(a-bx)^2}\Bigg[7962624a^5b^5x\alpha^2+36864a^4b^2\Big(8\pi^{3/2}\sqrt{f(x)}\alpha-45b^2x(c+dx)\alpha^2 \\ &+32\pi^2\gamma+72b^2\sqrt{\pi}x\sqrt{f(x)}\alpha\gamma+144b^2\pi x\gamma^2\Big)-3072a^3b\Big(18d\pi^{3/2}x\sqrt{f(x)}\alpha-45b^2c^2x\alpha^2 \\ &-256\pi^3\beta+64d\pi^2x\gamma+2c\Big(4\pi^{3/2}\sqrt{f(x)}\alpha-45b^2dx^2\alpha^2+16\pi^2\gamma+72b^2\sqrt{\pi}x\sqrt{f(x)}\alpha\gamma \\ &+144b^2\pi x\gamma^2\Big)-3b^2x\Big(15d^2x^2\alpha^2+64\pi^{3/2}\sqrt{f(x)}\alpha(1+3\beta)-48d\sqrt{\pi}x\sqrt{f(x)}\alpha\gamma \\ &-96d\pi x\gamma^2+256\pi^2(\gamma+3\beta\gamma)\Big)\Big)-x\Bigg(c^5\alpha^2+2c^3\Big(5d^2x^2\alpha^2+64\pi^{3/2}\sqrt{f(x)}\alpha(1+\beta) \\ &-16d\sqrt{\pi}x\sqrt{f(x)}\alpha\gamma+256\pi^2(1+\beta)\gamma-32d\pi x\gamma^2\Big)-2c^2\big(-5d^3x^3\alpha^2+2048\pi^3(1+\beta)^2 \\ &+64\pi^{3/2}x\sqrt{f(x)}\alpha(4b^2-3d(1+\beta))+24d^2\sqrt{\pi}x^2\sqrt{f(x)}\alpha\gamma+256\pi^2x(4b^2-3d(1+\beta))\gamma \\ &+48d^2\pi x^2\gamma^2\big)+c^4\left(5dx\alpha^2-8(\sqrt{\pi}\sqrt{f(x)}\alpha\gamma+2\pi\gamma^2)\right)+cx\Bigg(256b^2\Big(d\pi^{3/2}x\sqrt{f(x)}\alpha \\ &+128\pi^3(2+\beta)\Big)+d\Big(5d^3x^3\alpha^2+384d\pi^{3/2}x\sqrt{f(x)}\alpha(1+\beta)-8192\pi^3(1+\beta)^2-32d^2 \\ &\times x^2\sqrt{\pi f(x)}\alpha\gamma+1536d\pi^2x(1+\beta)\gamma-64d^2\pi x^2\gamma^2\Big)\Bigg)+dx^2\Bigg(256b^2\Big(128\pi^3+3d\pi^{3/2} \\ &\times x\sqrt{f(x)}\alpha+8d\pi^2x\gamma\Big)+d\Big(d^3x^3\alpha^2+128d\pi^{3/2}x\sqrt{f(x)}\alpha(1+\beta)-4096\pi^3(1+\beta)^2 \\ &-8d^2\sqrt{\pi}x^2\sqrt{f(x)}\alpha\gamma+512d\pi^2x(1+\beta)\gamma-16d^2\pi x^2\gamma^2\Big)\Bigg)\Bigg)-128a^2\big(45b^2c^3x\alpha^2-c^2 \\ &\times\Big(4\pi^{3/2}\sqrt{f(x)}\alpha-135b^2dx^2\alpha^2+16\pi^2\gamma+216b^2\sqrt{\pi}x\sqrt{f(x)}\alpha\gamma+432b^2\pi x\gamma^2\Big) \\ &+c\Big(135b^2d^2x^3\alpha^2+256\pi^3(2+\beta)+6\pi^{3/2}x\sqrt{f(x)}\alpha(-3d+32b^2(5+9\beta))-432b^2d \\ &\times x^2\sqrt{\pi f(x)}\alpha\gamma+64\pi^2x(-d+12b^2(5+9\beta))\gamma-864b^2d\pi x^2\gamma^2\Big)+x\big(256d\pi^3(3+2\beta) \end{aligned}$$

$$-2304b^4(\pi^{3/2}x\sqrt{f(x)}\alpha+4\pi^2x\gamma)-2d^2(7\pi^{3/2}x\sqrt{f(x)}\alpha+24\pi^2x\gamma)-3b^2\left(-15d^3x^3\alpha^2\right.$$
$$-64d\pi^{3/2}x\sqrt{f(x)}\alpha(5+9\beta)+2048\pi^3(1+2\beta+3\beta^2)+72d^2\sqrt{\pi}x^2\sqrt{f(x)}\alpha\gamma-256d\pi^2$$
$$\left.\left.\times x(5+9\beta)\gamma+144d^2\pi x^2\gamma^2)\right)\right)+8abx\left(15c^4\alpha^2+c^2\left(90d^2x^2\alpha^2+128\pi^{3/2}\sqrt{f(x)}\alpha(7\right.\right.$$
$$\left.+9\beta)-288d\sqrt{\pi}x\sqrt{f(x)}\alpha\gamma+512\pi^2(7+9\beta)\gamma-576d\pi x\gamma^2\right)-4c\left(-15d^3x^3\alpha^2+2048\pi^3\right.$$
$$\times\beta(4+3\beta)+72d^2\sqrt{\pi}x^2\gamma(\sqrt{f(x)}\alpha+2\sqrt{\pi}\gamma)-64d\pi^{3/2}x(7+9\beta)(\sqrt{f(x)}\alpha+4\sqrt{\pi}\gamma)$$
$$\left.+768b^2(\pi^{3/2}x\sqrt{f(x)}\alpha+4\pi^2x\gamma)\right)+12c^3\left(5dx\alpha^2-8(\sqrt{\pi}\sqrt{f(x)}\alpha\gamma\right.$$
$$\left.+2\pi\gamma^2)\right)+x\left(768b^2\left(d\pi^{3/2}x\sqrt{f(x)}\alpha+128\pi^3\beta\right)+d\left(15d^3x^3\alpha^2-8192\pi^3\beta(4+3\beta)\right.\right.$$
$$\left.\left.\left.\left.-96d^2\sqrt{\pi}x^2\gamma(\sqrt{f(x)}\alpha+2\sqrt{\pi}\gamma)+128d\pi^{3/2}x(7+9\beta)(\sqrt{f(x)}\alpha+4\sqrt{\pi}\gamma)\right)\right)\right)\right]. \tag{3.20}$$

Integrating eq (3.8) by substituting $n=\frac{2}{3}$, we get the gravitational potential G as

$$G=K_3(a+bx)^S(a-bx)^J\left(F(x)\right)^\xi\left(T(x)\right)^\chi exp(D(x)), \tag{3.21}$$

where K_3 is the integration constant. The constants H, P, Ω, and ζ and the variables $f(x)$, $U(x)$, $V(x)$, and $N(x)$ are given as

$$\begin{aligned}
S &= \frac{1}{16ab^2}(\beta+1)\left(24ab^2+ad-bc\right)+\frac{\gamma}{256ab^3\pi}(-24ab^2-ad+bc)^2,\\
J &= \frac{1}{16ab^2}\left(\beta\left(ad-24ab^2+bc\right)+ad-8ab^2+bc\right)-\frac{\gamma}{256ab^3\pi}(-24ab^2+ad+bc)^2,\\
\xi &= \frac{\alpha}{1024a}\left[\frac{\left(24ab^2-ad-bc\right)^5}{\pi^3b^7}\right]^{\frac{1}{2}},\\
\chi &= \frac{\alpha}{1024a}\left[\frac{\left(24ab^2+ad-bc\right)^5}{\pi^3b^7}\right]^{\frac{1}{2}},\\
f(x) &= 24ab-c-dx,\\
F(x) &= \frac{\left[24ab^2-ad-bc\right]^{\frac{1}{2}}+\left(bf(x)\right)^{\frac{1}{2}}}{\left[24ab^2-ad-bc\right]^{\frac{1}{2}}-\left(bf(x)\right)^{\frac{1}{2}}},\\
T(x) &= \frac{\left[24ab^2+ad-bc\right]^{\frac{1}{2}}-\left(bf(x)\right)^{\frac{1}{2}}}{\left[24ab^2+ad-bc\right]^{\frac{1}{2}}+\left(bf(x)\right)^{\frac{1}{2}}},
\end{aligned}$$

$$D(x)=\frac{d}{768b^2\pi^{\frac{3}{2}}}\left[\left(f(x)\right)^{\frac{1}{2}}(168ab-7c-dx)\alpha-6dx\sqrt{\pi}\gamma\right].$$

The line element (2.1) becomes

$$\begin{aligned}ds^2&=K_3^2(a+br^2)^{2(S-1)}(a-br^2)^{2J}\left(F(r^2)\right)^{2\xi}\left(T(r^2)\right)^{2\chi}exp(2D(r^2))\,dt^2\\&-(a+br^2)^{-2}\left[dr^2+r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)\right].\end{aligned}\tag{3.22}$$

Model IV

When $n=\frac{1}{2}$, radial and tangential pressures are given as

$$p_r=\beta\left(\frac{24ab-c-xd}{16\pi}\right)+\alpha\left(\frac{24ab-c-xd}{16\pi}\right)^3+\gamma\left(\frac{24ab-c-xd}{16\pi}\right)^2,\tag{3.23}$$

$$\begin{aligned}p_t&=\frac{1}{8388608\pi^5(a-bx)^2}[191102976a^6b^6x\alpha^2+1769472a^5b^3\alpha(16\pi^2-27b^2x(c+dx)\alpha\\&+144b^2\pi x\gamma)+36864a^4b^2(135b^2c^2x\alpha^2+64\pi^2(-3dx\alpha+8\pi\gamma)+6c\alpha(-16\pi^2+45b^2dx^2\alpha\\&.-240b^2\pi x\gamma)+3b^2x(45d^2x^2\alpha^2-480d\pi x\alpha\gamma+256\pi^2(2\alpha+6\alpha\beta+3\gamma^2)))+6144a^3b\\&\times(4608b^4\pi^2x^2\alpha-45b^2c^3x\alpha^2+3c^2\alpha(8\pi^2-45b^2dx^2\alpha+240b^2\pi x\gamma)+8\pi^2(9d^2x^2\alpha\\&+256\pi^2\beta-64d\pi x\gamma)-3b^2x(15d^3x^3\alpha^2-240d^2\pi x^2\alpha\gamma-2048\pi^3(1+3\beta)\gamma+768d\pi^2x\\&\times(\alpha+2\alpha\beta+\gamma^2))-c(32\pi^2(-3dx\alpha+8\pi\gamma)+9b^2x(15d^2x^2\alpha^2-160d\pi x\alpha\gamma+256\pi^2\\&\times(\alpha+2\alpha\beta+\gamma^2))))+x(c^6\alpha^2+2c^5\alpha(3dx\alpha-16\pi\gamma)+dx^2(d(d^2x^2\alpha+256\pi^2(1+\beta)\\&-16d\pi x\gamma)^2-4096b^2\pi^2(128\pi^2-d^2x^2\alpha+8d\pi x\gamma))+c^4(15d^2x^2\alpha^2-160d\pi x\alpha\gamma+256\pi^2\\&\times(2\alpha(1+\beta)+\gamma^2))-4c^3(512b^2\pi^2x\alpha-5d^3x^3\alpha^2+80d^2\pi x^2\alpha\gamma+2048\pi^3(1+\beta)\gamma\\&-256d\pi^2x(2\alpha(1+\beta)+\gamma^2))+c^2(15d^4x^4\alpha^2+65536\pi^4(1+\beta)^2-320d^3\pi x^3\alpha\gamma+8192\pi^3x\\&\times(4b^2-3d(1+\beta))\gamma+1536d^2\pi^2x^2(2\alpha(1+\beta)+\gamma^2))+2cx(-1024b^2\times(-3d^2\pi^2x^2\alpha\\&+256\pi^4(2+\beta))+d(3d^4x^4\alpha^2+65536\pi^4(1+\beta)^2-80d^3\pi x^3\alpha\gamma-12288d\pi^3x(1+\beta)\gamma\\&\times\pi x\gamma)-4c(-135b^2d^3x^4\alpha^2+2048\pi^4(2+\beta)+2160b^2d^2\pi x^3\alpha\gamma+512\pi^3x(-d+12b^2(5\\&+9\beta))\gamma+72\pi^2x^2(192b^4\alpha+d^2\alpha-32b^2d(4\alpha+6\alpha\beta+3\gamma^2)))+2c^2(32\pi^2(-3dx\alpha+8\pi\gamma)\\&+9b^2x(45d^2x^2\alpha^2-480d\pi x\alpha\gamma+256\pi^2(4\alpha+6\alpha\beta+3\gamma^2)))+x(294912b^4\pi^3x\gamma-128d\pi^2\\&\times(d^2x^2\alpha+64\pi^2(3+2\beta)-12d\pi x\gamma)+3b^2(45d^4x^4\alpha^2+65536\pi^4(1+2\beta+3\beta)^2-960d^3\pi\\&\times x^3\alpha\gamma-8192d\pi^3x(5+9\beta)\gamma+1536d^2\pi^2x^2(4\alpha+6\alpha\beta+3\gamma^2))))-16abx(9c^5\alpha^2+15c^4\alpha\\&\times(3dx\alpha-16\pi\gamma)+c^2(-9216b^2\pi^2x\alpha+90d^3x^3\alpha^2-1440d^2\pi x^2\alpha\gamma-4096\pi^3(7+9\beta)\gamma\\&+1536d\pi^2x(5\alpha+6\alpha\beta+3\gamma^2))+2c^3(45d^2x^2\alpha^2-480d\pi x\alpha\gamma+256\pi^2(\alpha(5+6\beta)+3\gamma^2))\\&+c(45d^4x^4\alpha^2-960d^3\pi x^3\alpha\gamma-8192d\pi^3x(7+9\beta)\gamma+32768\pi^3(2\pi\beta(4+3\beta)+3b^2x\gamma)\\&+1536d^2\pi^2x^2(\alpha(5+6\beta)+3\gamma^2))+x(3072b^2(3d^2\pi^2x^2\alpha-256\pi^4\beta)+d(9d^4x^4\alpha^2+65536\end{aligned}$$

$$\times \pi^4\beta(4+3\beta) - 240d^3\pi x^3\alpha\gamma - 4096d\pi^3 x(7+9\beta)\gamma + 512d^2\pi^2x^2(5\alpha+6\alpha\beta+3\gamma^2))))]. \quad (3.24)$$

Integrating eq (3.8) by substituting $n = \frac{1}{2}$, we get the gravitational potential G as

$$G = K_4(a+bx)^Y(a-bx)^O exp(\kappa(x)), \quad (3.25)$$

where K_4 is the integration constant. The constants O and Y and the variable $\kappa(x)$ are given as

$$Y = \frac{1}{4096\pi^2ab^4}\Big(\alpha\left(24ab^2+da-bc\right)^3 + 256\pi^2b^2\beta\left(24ab^2+ad-bc\right)$$
$$+256\pi^2b^2\left(24ab^2+ad-bc\right) + 16b\pi\gamma(-24ab^2-ad+bc)^2\Big),$$

$$O = \frac{1}{4096\pi^2ab^4}\Big(\alpha\left(ad-24ab^2+bc\right)^3 + 256\pi^2b^2\beta\left(ad-24ab^2+bc\right)$$
$$+256\pi^2b^2\left(ad-8ab^2)+bc\right) - 16b\pi\gamma(-24ab^2 - +ad+bc)^2\Big),$$

$$\kappa(x) = \frac{d^2x[\alpha(6(c-24ab)+xd) - 32\pi\gamma]}{4096\pi^2b^2}.$$

The line element (2.1) becomes

$$ds^2 = K_4^2(a+br^2)^{2(Y-1)}(a-br^2)^{2O}exp(2\kappa(r^2))\,dt^2 - (a+br^2)^{-2}\left[dr^2 + r^2\left(d\theta^2+\sin^2\theta\,d\phi^2\right)\right]. \quad (3.26)$$

3.2 Boundary Conditions

We ensure boundary conditions by requiring that the metric coefficients and their first derivatives are continuous in both the interior and exterior solutions up to and on the boundary. The continuity of metric coefficients defines the first fundamental form, while the continuity of their derivatives defines the second fundamental form [27, 28]. By matching the first and second fundamental forms for the interior spacetime metric (2.1) and the exterior spacetime metric (1.59), we obtain the conditions at the boundary. These conditions are given by

$$R_s = r_s e^{2\lambda_s}, \quad (3.27)$$

$$e^{2\nu_s} = \left(1 - \frac{2M}{R_s} + E^2R_s^2\right), \quad (3.28)$$

$$(1+r\lambda')_s = \left(1 - \frac{2M}{R_s} + E^2R_s^2\right)^{\frac{1}{2}}, \quad (3.29)$$

$$r_s(\nu')_s = \left(\frac{M}{R_s} - E^2R_s^2\right)\left(1 - \frac{2M}{R_s} + E^2R_s^2\right)^{-1}, \quad (3.30)$$

where the subscript "s" denotes the boundary of the star. The total mass in our case by using the eq (3.29) which is

$$M = \frac{R_s}{2}\left(-\left(1-\frac{2bR_s^2\left(a+bR_s^2\right)^2}{a\left(a+bR_s^2\right)^2+bR_s^2}\right)^2 + R_s^2\left(c+dR_s^2\right)+1\right). \tag{3.31}$$

Applying the boundary condition yields the integration constant for all models.

$$K_2^2 = \frac{\left(\left(a+bR_s^2\right)^2\left(a-2bR_s^2\right)+bR_s^2\right)^2}{\left(a\left(a+bR_s^2\right)^2+bR_s^2\right)^2}(a+bR_s^2)^{-2(I-1)}(a-bR_s^2)^{-2W}\left(U(R_s^2)\right)^{-2\Omega}\left(V(R_s^2)\right)^{-2\zeta}exp(-2Z(R_s^2)), \tag{3.32}$$

$$K_3^2 = \frac{\left(\left(a+bR_s^2\right)^2\left(a-2bR_s^2\right)+bR_s^2\right)^2}{\left(a\left(a+bR_s^2\right)^2+bR_s^2\right)^2}(a+bR_s^2)^{-2(S-1)}(a-bR_s^2)^{-2J}\left(F(R_s^2)\right)^{-2\xi}\left(T(R_s^2)\right)^{-2\chi}exp(-2D(R_s^2)), \tag{3.33}$$

$$K_4^2 = \frac{\left(\left(a+bR_s^2\right)^2\left(a-2bR_s^2\right)+bR_s^2\right)^2}{\left(a\left(a+bR_s^2\right)^2+bR_s^2\right)^2}(a+bR_s^2)^{-2(Y-1)}(a-bR_s^2)^{-2O}exp(-2\kappa(R_s^2)). \tag{3.34}$$

By using the boundary condition $p_r(r_s = R_s) = 0$, we get

$$\beta = -\alpha\left(\frac{24ab-c-dR_s^2}{16\pi}\right)^{\frac{1}{n}} - \gamma\left(\frac{24ab-c-dR_s^2}{16\pi}\right). \tag{3.35}$$

To ensure the physical viability of the models, the parameter c is set to zero, eliminating the singularity in the charge density equation at $r = 0$. This choice transforms the eq (3.11) to the well-behaved form

$$\sigma^2 = \frac{d[3(a+br^2)]^2}{(4\pi)^2}. \tag{3.36}$$

For our analysis, we adopt the parameter values as $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$. We examine the models over $0 \leq r \leq 1$. The parameters β and the constant of integration for the four models are determined using eqs (3.32)-(3.34).

Models	β	K
II.($n = 2$)	-0.1018170771	$0.7101238658 - 0.1049644613i$
III.($n = \frac{2}{3}$)	-0.0507397379	$0.6403356786 - 0.0045403350i$
IV.($n = \frac{1}{2}$)	-0.0479803487	0.6315680368

Table 3.1. Values of the parameters β and K.

3.3 Physical Conditions

Metric potential

There should be no singularity in metric potential within the radius of the object. The metric potential $e^{2\nu}$ is monotonically increasing. The behaviour of metric potentials for different values of n is shown in Figure 3.1.

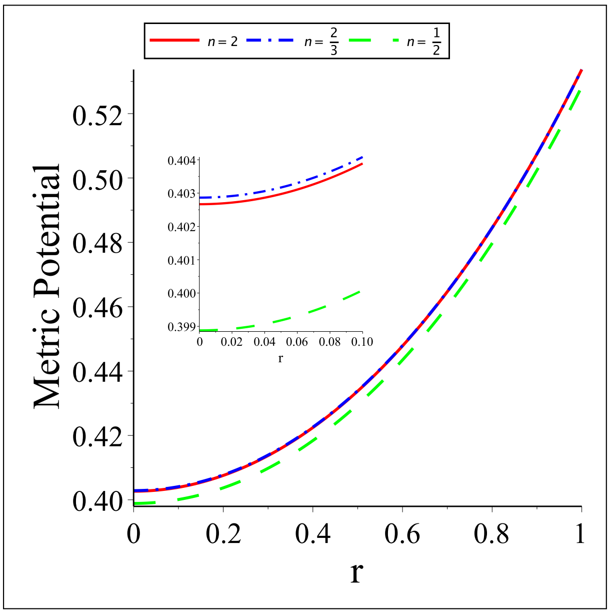


Figure 3.1: Metric potential $e^{2\nu}$ as a function of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

Electric Field Intensity

The electric field intensity, as shown in Figure 3.2, starts at zero at the star's center and increases as one moves towards the boundary.

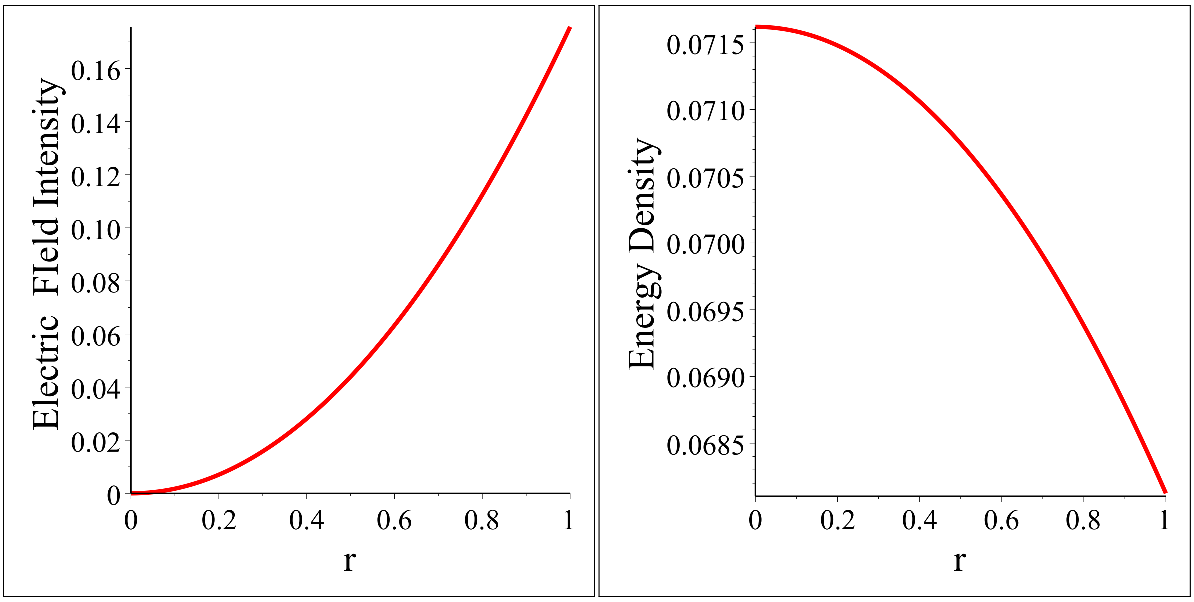


Figure 3.2: Electric field intensity E^2 and energy density ρ as functions of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

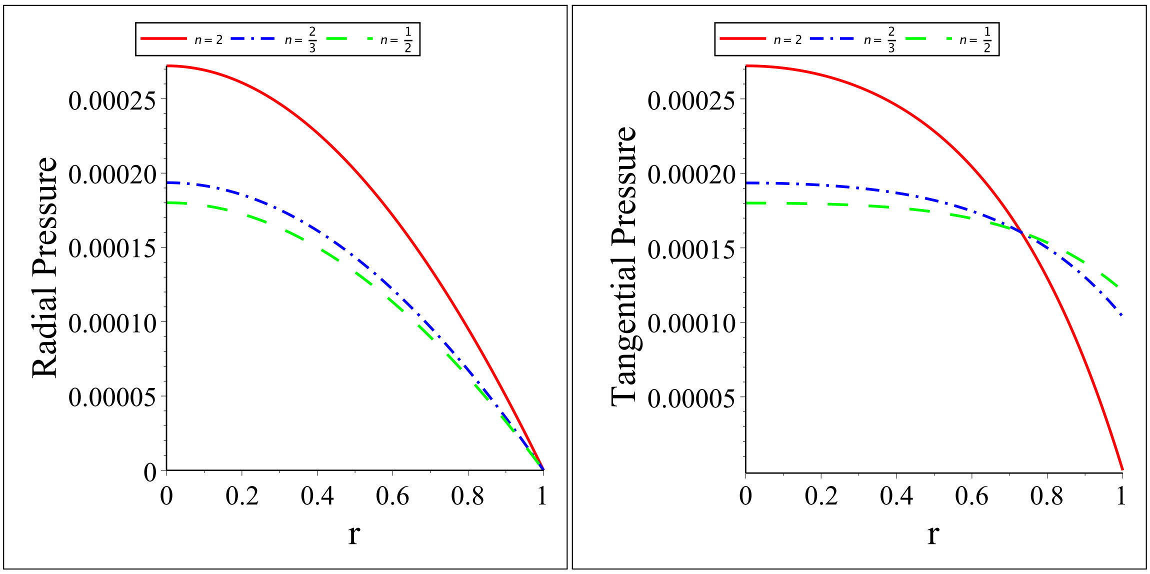
Density and Pressures

The density and pressures in the model do not exhibit any singularities. At $r = 0$, the density and pressures are given by

$$\rho(r = 0) = \frac{24ab - c}{16\pi} > 0,$$

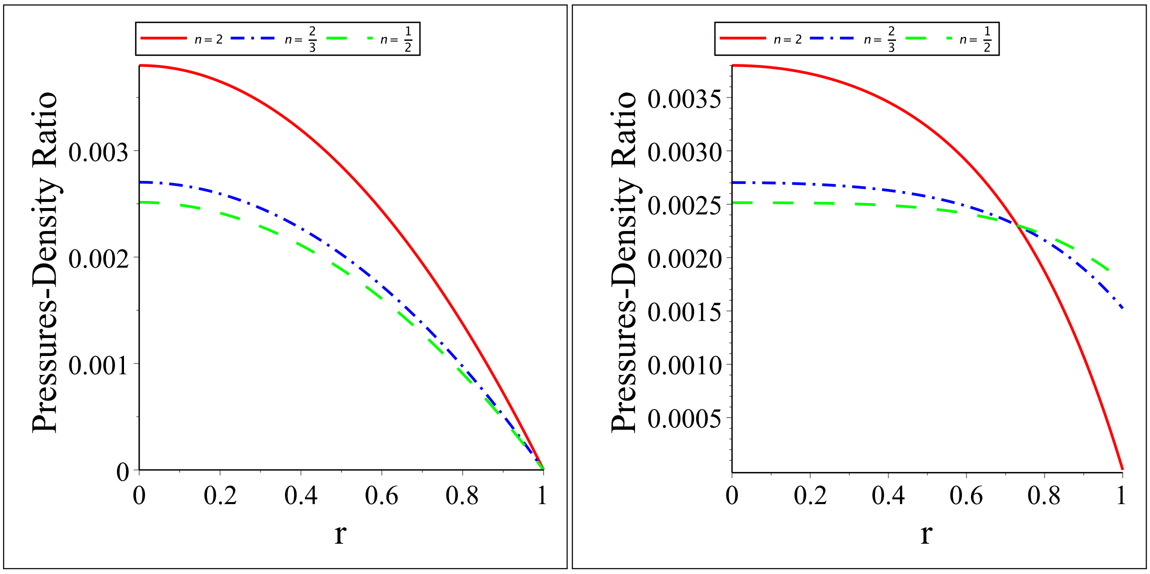
$$p_r(r = 0) = p_t(r = 0) = \beta \left(\frac{24ab - c}{16\pi}\right) + \alpha \left(\frac{24ab - c}{16\pi}\right)^{\Gamma} + \gamma \left(\frac{24ab - c}{16\pi}\right)^{2} > 0.$$

Here, the radial and tangential pressures are equal and positive at the center. Figures 3.2 and 3.3 illustrate the energy density and pressures, which decrease monotonically. Additionally, the radial pressure at the surface, $p_r(r = R)$, is zero. The model satisfies Zeldovich's [29] criteria, which state that the pressure-density ratio must be less than 1 throughout the compact object, as shown in Figure 3.4.



(a) p_r (b) p_t

Figure 3.3: Radial and tangential pressures as functions of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.



(a) p_r/ρ (b) p_t/ρ

Figure 3.4: Pressure-density ratios as functions of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

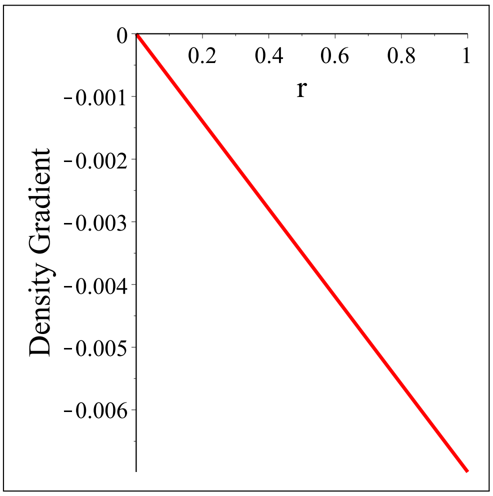
Gradients

The gradients decrease when we use appropriate parameter values. The generalized expressions for density gradient and radial pressure gradient are as follows

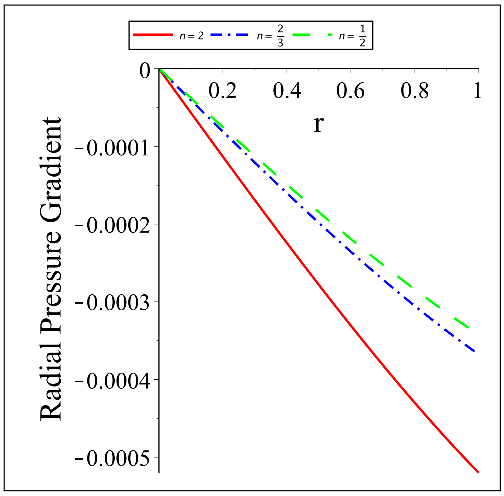
$$\frac{d\rho}{dr} = -\frac{dr}{8\pi}, \tag{3.37}$$

$$\frac{dp_r}{dr} = -\frac{dr\beta}{8\pi} - 2^{1-4\Gamma} d\pi^{-\Gamma} r(24ab - c - dr^2)^{-1+\Gamma}(\alpha+\gamma)\Gamma, \tag{3.38}$$

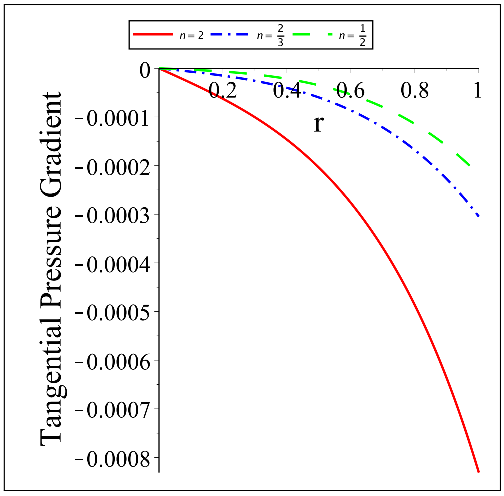
and the expression for tangential pressure gradient is shown in Appendix A1. Figure 3.5 displays graphs showing how the gradients change for different values of n.



(a) $d\rho/dr$



(b) dp_r/dr



(c) dp_t/dr

Figure 3.5: Density and pressure gradients as functions of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

Trace of the Energy-Momentum Tensor

According to Bondi's [30] condition for an anisotropic fluid sphere, compact objects are only suitable if the energy-momentum tensor trace is positive. The condition $\rho - p_r - 2p_t > 0$ is satisfied for our model, and a graph is shown in Figure 3.6.

Anisotropy

The anisotropic factor, defined as $\Delta = 8\pi(p_t - p_r)$, quantifies the difference between tangential and radial pressures in a compact object. For a physically viable model, Δ must be zero at the center and increase outward. Our calculations show that this condition is met, as shown in Figure 3.6. The graph shows Δ starting at zero at the object's center and increasing towards its boundary, validating our model's physical consistency.

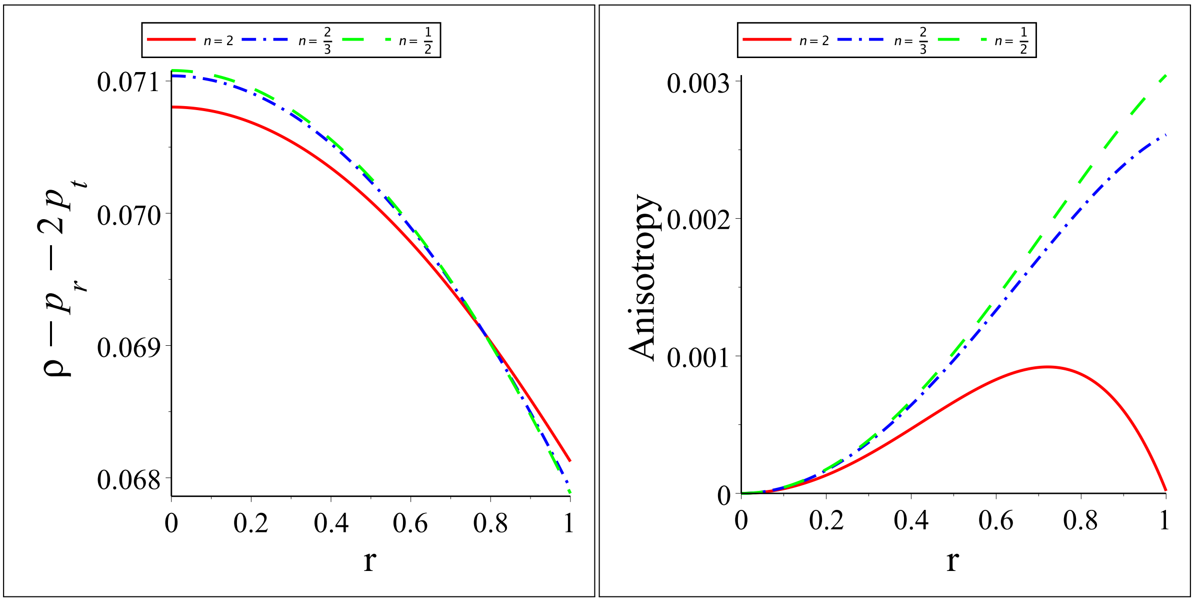


Figure 3.6: Trace of energy-momentum tensor and anisotropy as functions of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

Mass-Radius Relation

We previously determined the mass function using eq (3.31). According to Buchdal [31], the ratio of mass to radius for a compact star should not exceed 4/9 (i.e., $\frac{M}{R} < \frac{4}{9}$). In our current models, the mass-to-radius ratio is $\frac{M}{R} = 0.32$, which is within this limit, as shown in Figure 3.7. This figure illustrates the mass function, which is positive and smoothly varying throughout the interior of the star. The compactness factor for our model is given by

$$\mu(r) = \frac{M(r)}{r}.$$

Figure 3.7 displays the profile of this factor, which increases monotonically and remains below the 4/9 limit. Our models are therefore consistent with the theoretical constraints for stable stars. The results suggest that the star's structure remains physically plausible and stable under these conditions.
The gravitational redshift is expressed as

$$z = e^{-\nu} - 1$$

The gravitational redshift (z) shown in Figure 3.8 has a decreasing nature for our model.

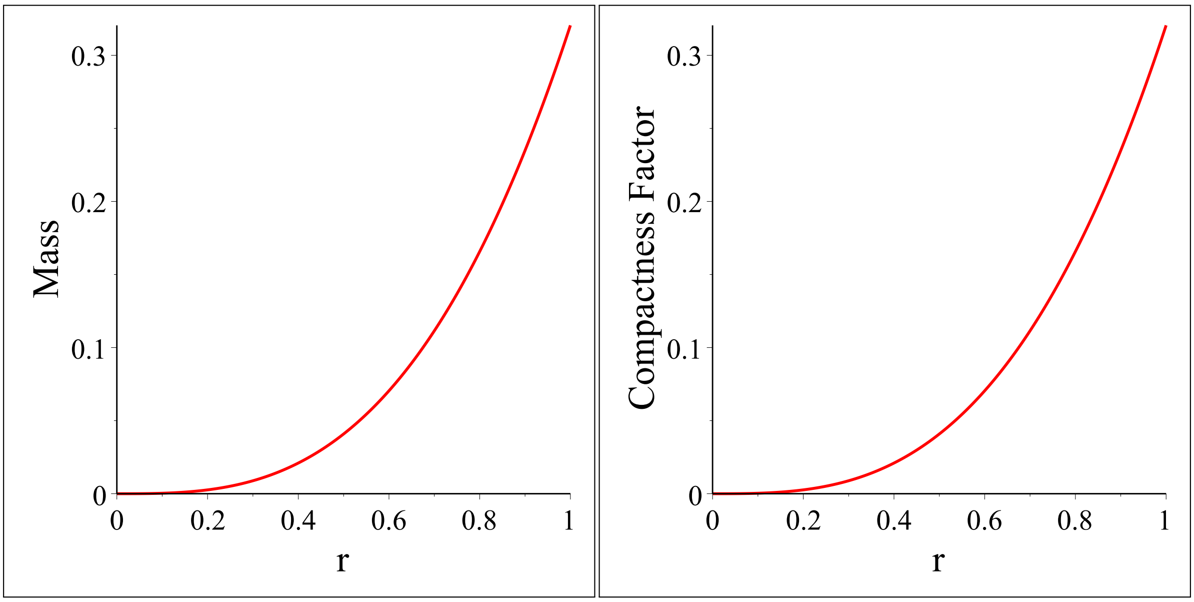


Figure 3.7: Mass $M(r)$ and compactness factor μ as functions of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

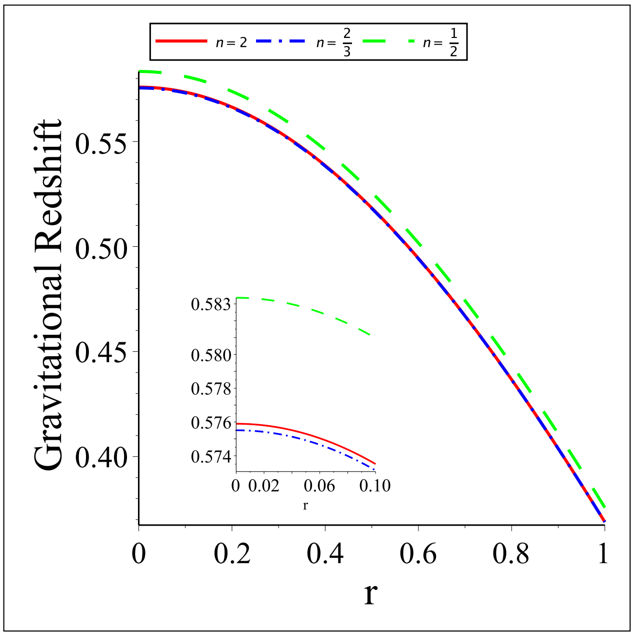


Figure 3.8: Gravitational redshift as a function of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

3.4 Stability Analysis

Energy Conditions

To ensure that the system is physically viable, it must satisfy the following energy conditions throughout the stellar interior.

- Null Energy Condition: $\rho(r) \geq 0$,
- Weak Energy Condition: $\rho + p_r \geq 0$ and $\rho + p_t \geq 0$,
- Strong Energy Condition: $\rho + p_r + 2p_t \geq 0$,
- Dominant Energy Condition: $\rho \geq |p_r + 2p_t|$.

Figure 3.9 illustrate that these energy conditions are well-satisfied in our model, indicating a stable configuration.

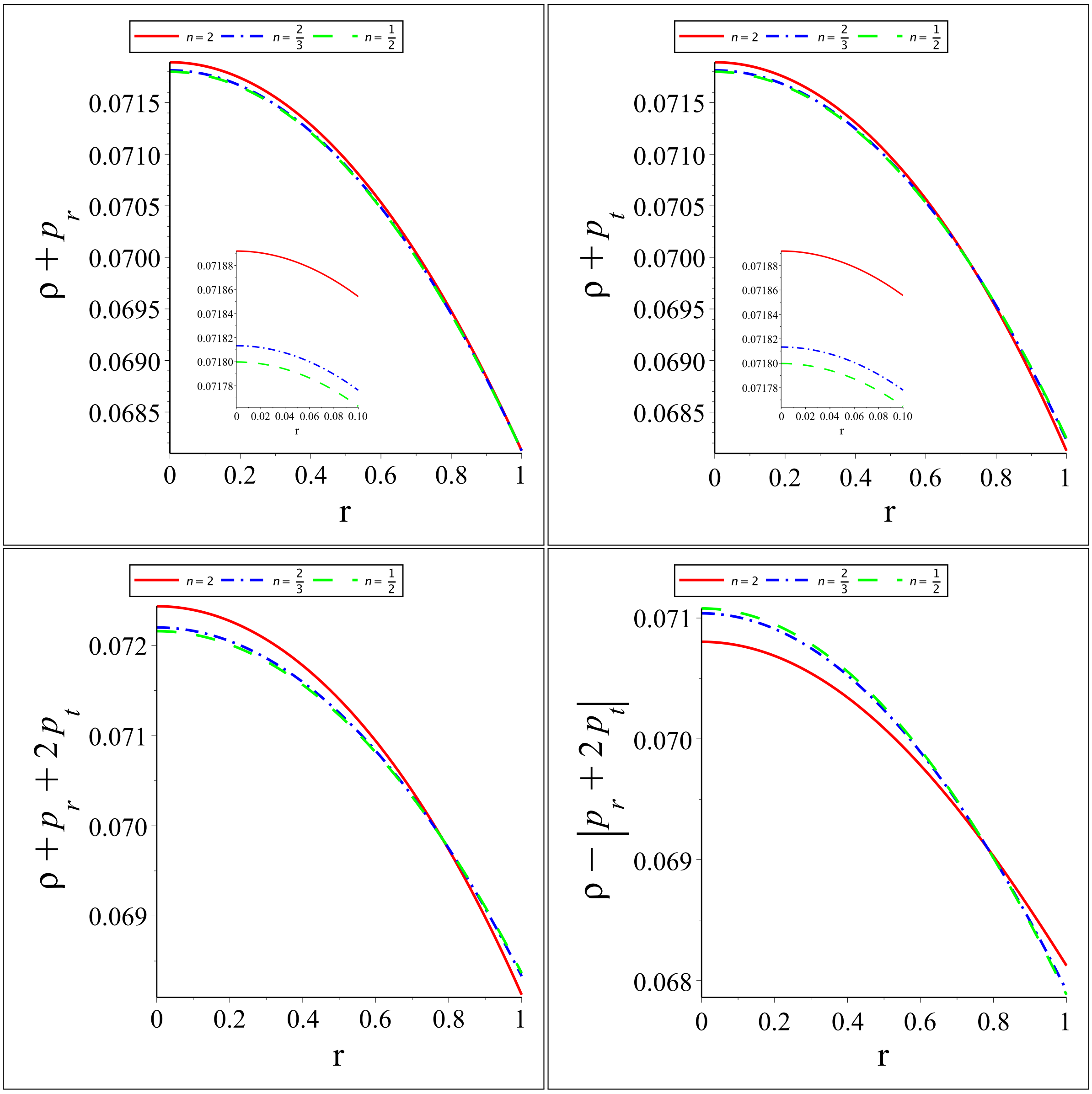


Figure 3.9: weak, strong, and dominant energy conditions as functions of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

Causality Condition

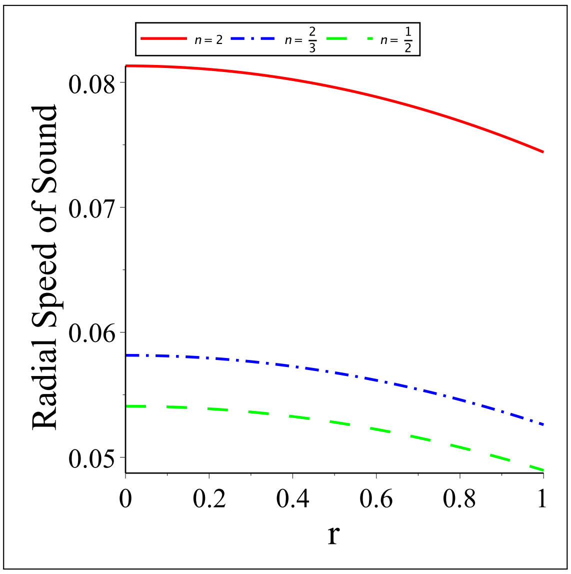
The speed of sound, v_α^2, in an anisotropic fluid distribution is defined as

$$v_\alpha^2 = \frac{dp_\alpha}{d\rho} = \left(\frac{dp_\alpha/dr}{d\rho/dr}\right).$$

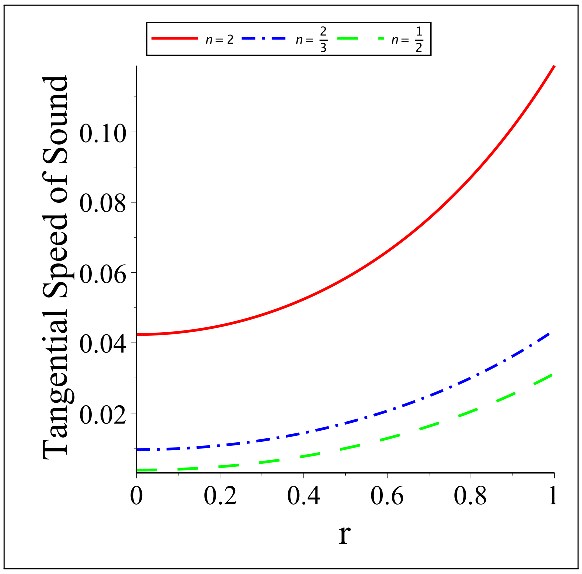
The generalized expression for radial velocity is

$$v_r^2 = \Gamma(16\pi)^{1-\Gamma}(\alpha+\gamma)\left(24ab - c - dr^2\right)^{\Gamma-1} + \beta, \tag{3.39}$$

and the expression for tangential velocity is shown in Appendix A1. The speed of sound in the radial and tangential directions inside the compact object must be less than 1, i.e., $0 < v_r^2, v_t^2 < 1$. Figure 3.10 illustrates the graphs of the radial and tangential speeds of sound, indicating that these speeds fall within the range of 0 to 1, thereby satisfying the causality condition. Additionally, our model satisfies the condition $0 \le |v_t^2 - v_r^2| \le 1$, as demonstrated in Figure 3.11.



(a) v_r^2

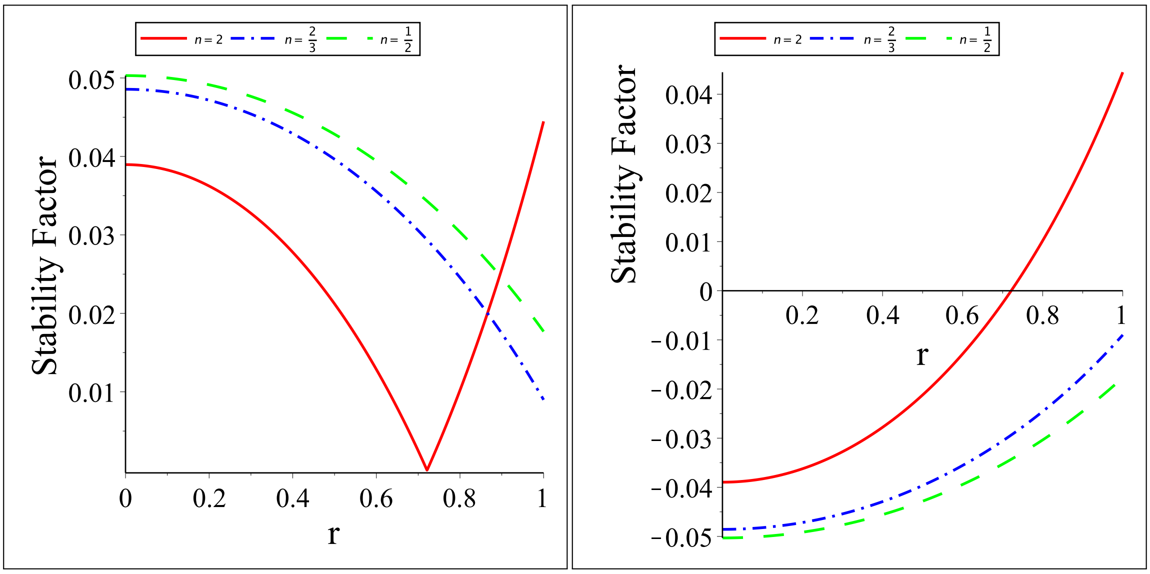


(b) v_t^2

Figure 3.10: Radial and tangential speed of sound as functions of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

Herrera[32] and Abreu [33] introduced the concept of "Cracking," which identifies potentially stable and unstable regions in anisotropic matter distributions. The compact object is stable if $0 \le |v_t^2 - v_r^2| \le 1$. Specifically, it is potentially stable when $-1 \le v_t^2 - v_r^2 \le 0$ and potentially unstable when $0 \le v_t^2 - v_r^2 \le +1$. Figure 3.11

illustrates the stability parameter, indicating large stable regions and some unstable areas. By choosing suitable values for parameter n, nearly complete stability can be achieved.



(a) $|v_t^2 - v_r^2|$ (b) $v_t^2 - v_r^2$

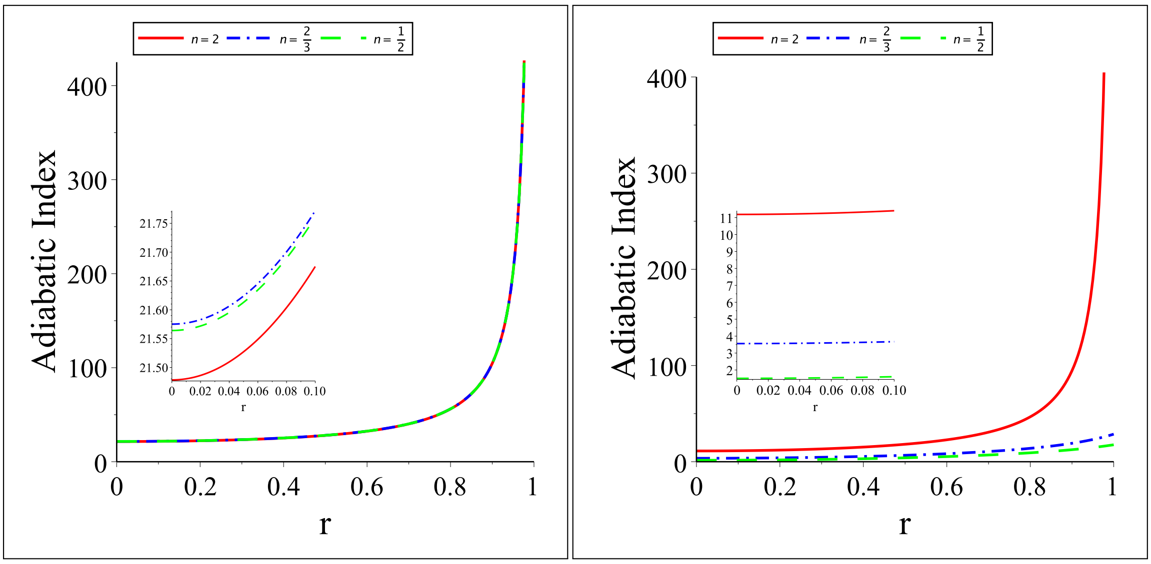
Figure 3.11: Stability factor as a function of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

Adiabatic Index

The adiabatic index is denoted by Γ_α. Chen *et al.* [34] explained the two specific heat ratios for stable systems, which are

$$\Gamma_\alpha = \frac{\rho + p_\alpha}{p_\alpha} \frac{dp_\alpha}{d\rho}. \tag{3.40}$$

For our model to be physically acceptable, it must maintain a Γ_α value exceeding 4/3. This condition is necessary to ensure the model's stability against gravitational collapse, as indicated by Chandrasekhar [34] and further supported by Tooper [22]. Our analysis demonstrates that the proposed model satisfies these crucial criteria. To illustrate this, we have provided graphical representations of the adiabatic index in Figure 3.12, which visually confirm the model's stability across various conditions.



(a) Γ_r (b) Γ_t

Figure 3.12: Adiabatic index as a function of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

Forces in Stellar Equilibrium

A stellar configuration is in equilibrium if the Tolman-Oppenheimer-Volkoff (TOV) equation, given by [35, 36], is satisfied.

$$\frac{2}{r}(p_t - p_r) - \frac{dp_r}{dr} + \sigma E e^{\lambda/2} - \frac{(\rho + p_r)\nu'}{2} = 0. \tag{3.41}$$

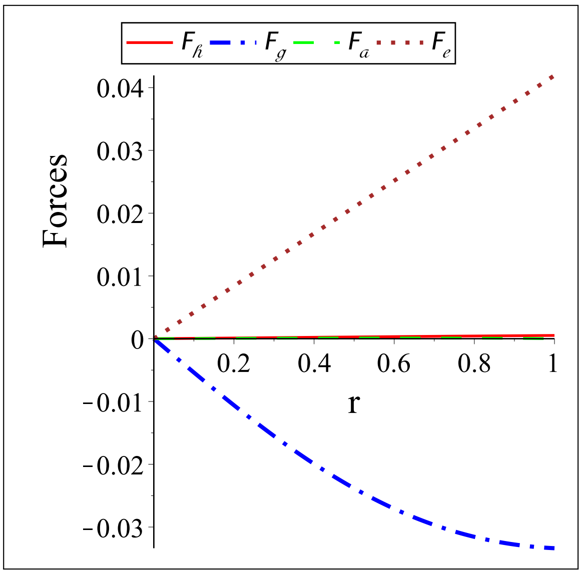
This equation can be expressed as

$$F_a + F_h + F_g + F_e = 0, \tag{3.42}$$

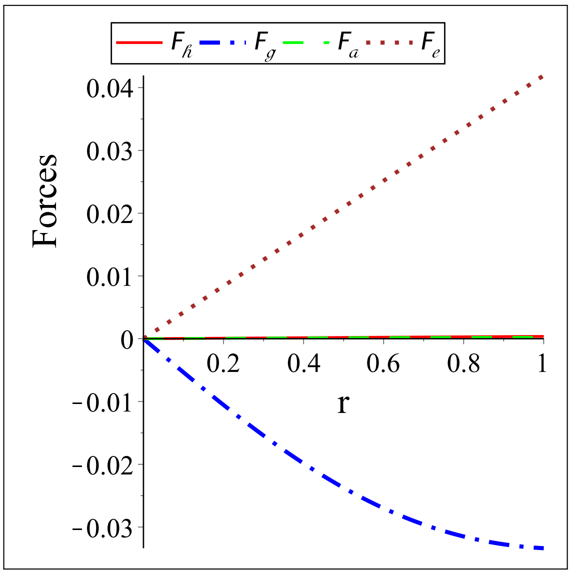
where F_a, F_h, F_e, and F_g represent anisotropic, hydrostatic, electric, and gravitational forces respectively, defined as

$$F_a = \frac{2}{r}(p_t - p_r), \quad F_h = -\frac{dp_r}{dr}, \quad F_e = \sigma E e^{\lambda/2}, \quad F_g = -\frac{(\rho + p_r)\nu'}{2}.$$

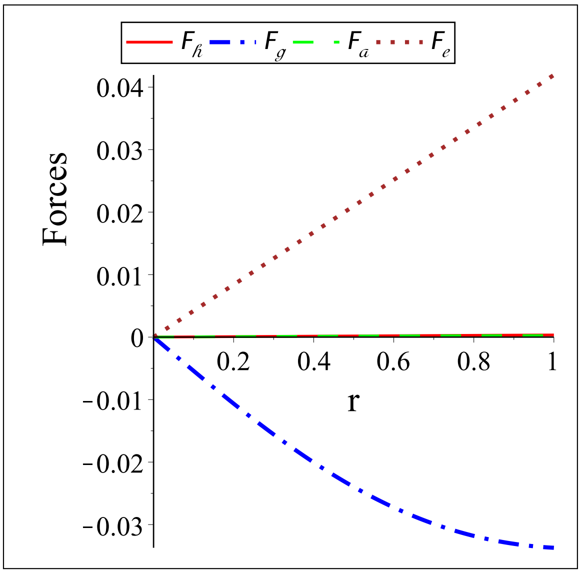
We may conclude that the configuration of our compact object is in static equilibrium because the above four forces counterbalance each other.



(a) $n = 2$



(b) $n = 2/3$



(c) $n = 1/2$

Figure 3.13: Graph of different forces as functions of r when $a = 1$, $b = 0.15$, $c = 0$, $d = 0.1757$, $\alpha = 0.21$, and $\gamma = 0.69$.

Chapter 4

Conclusion

In this thesis, we began by exploring the historical development of the concept of space-time and tracing its evolution to the formulation of general relativity by Einstein. The Einstein-Maxwell field equations and exact solutions of field equations are discussed in detail. Afterward, tensors are also discussed in Chapter 1. In Chapter 2, models of charged compact objects having a generalized polytropic equation of state are examined.

In Chapter 3, the properties of charged compact objects using a modified generalized polytropic equation of state given by $p_r = \beta\rho + \alpha\rho^{\Gamma} + \gamma\rho^2$ are explored. To ensure a physically viable model, the analysis is conducted under the framework of general relativity with specific gravitational potentials and electric field intensities. It is shown through the calculations that the density ρ, radial pressure p_r, and tangential pressure p_t are positive at the center and monotonically decrease towards the boundary. The negative gradients of these quantities confirm this behaviour. The parameters such as anisotropy, mass, and compactness factors exhibit an increasing trend towards the boundary, which is consistent with the requirements for a physically acceptable configuration. Moreover, all energy conditions are satisfied by the proposed model, indicating its physical viability. The configuration's stability is confirmed by the stability factor, showing that the stability criteria are within acceptable limits. The forces acting on the charged matter distribution are in equilibrium, resulting in a static and stable structure. The addition of the $\gamma\rho^2$ term in the radial pressure equation enhances the central pressure, leading to a steeper pressure gradient. This modification leads to a more compact, stable configuration, with all energy conditions satisfied and forces in equilibrium.

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Appendix A

1

$$\frac{dp_t}{dr} = \frac{4^{-4\Gamma-7}\pi^{-2\Gamma-3}r}{\left(a-br^2\right)^3\left(-24ab+c+dr^2\right)^2}\Bigg[729\cdot 4^{9+4\Gamma}a^7b^6\pi^{2\Gamma}\Gamma^2+27\cdot 256^{2+\Gamma}a^6b^3\pi^{2\Gamma}\Gamma$$
$$\Big(-4d\pi+108b^4r^2\Gamma+3b^2\left(16\pi(2+3\beta)-3(3c+5dr^2)\Gamma\right)\Big)+9\cdot 16^{3+\Gamma}a^5b^2\pi^\Gamma$$
$$\Bigg(7d^2r^2\Gamma\left(4^{1+2\Gamma}\pi^{1+\Gamma}+45b^2(16\pi)^\Gamma r^2\Gamma\right)-2^{1+4\Gamma}d\pi^\Gamma\left(16\pi^2(3+2\beta)-12\pi(c\right.$$
$$\left.-32b^2r^2(2+3\beta))\Gamma+9b^2r^2(-25c+24b^2r^2)\Gamma^2\right)\Bigg)+3b^2\Big(4^{5+2\Gamma}\pi^{2+\Gamma}\beta+1536$$
$$\times\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma-2^{5+4\Gamma}\pi^{1+\Gamma}\left(-24b^2r^2(2+3\beta)+c(11+15\beta)\right)\Gamma$$
$$-27\cdot 16^{1+\Gamma}b^2c\pi^\Gamma r^2\Gamma^2+(16\pi)^\Gamma\left(256\pi^2(1+3\beta^2)+45c^2\Gamma^2\right)\Big)+3\cdot 2^{11+4\Gamma}a^4b\pi^\Gamma\times$$
$$\Bigg(18b^4r^2\Big(4^{5+2\Gamma}\pi^{2+\Gamma}\beta+1536\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma-2^{5+4\Gamma}\pi^{1+\Gamma}\big(2dr^2(2+3\beta)$$
$$+c(11+15\beta)\big)\Gamma+15\cdot 2^{1+4\Gamma}cd\pi^\Gamma r^2\Gamma^2+(16\pi)^\Gamma\big(256\pi^2(1+3\beta^2)+15(3c^2-d^2$$
$$\times r^4)\Gamma^2\big)\Big)+2d\pi\Bigg(-3c^2(16\pi)^\Gamma\Gamma+c\Big(3\cdot 2^{3+4\Gamma}\pi^{1+\Gamma}+(16\pi)^\Gamma\left(16\pi\beta-7dr^2\Gamma\right)\Big)$$
$$+4\left(2^{1+4\Gamma}d\pi^{1+\Gamma}r^2(3+2\beta)-d^2(16\pi)^\Gamma r^4\Gamma-32\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma\right)\Bigg)-3b^2$$
$$\times\Big(-2048\pi^3(24ab-c-dr^2)^\Gamma\alpha(2+3\beta)-2^{3+4\Gamma}d^2\pi^{1+\Gamma}r^4(47+66\beta)\Gamma+15c^3$$
$$(16\pi)^\Gamma\Gamma^2+45d^3(16\pi)^\Gamma r^6\Gamma^2-3c^2\Gamma\left(4^{3+2\Gamma}\pi^{1+\Gamma}+5(16\pi)^\Gamma\left(16\pi\beta-5dr^2\Gamma\right)\right)$$
$$+3c\Big(2^{7+4\Gamma}\pi^{2+\Gamma}(1+3\beta)+512\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma-4^{3+2\Gamma}d\pi^{1+\Gamma}r^2(3+4\beta)\Gamma$$
$$+(16\pi)^\Gamma\left(256\pi^2\beta^2+35d^2r^4\Gamma^2\right)\Big)+64d\pi^2r^2\Big(5(16\pi)^\Gamma+2^{1+4\Gamma}\pi^\Gamma\beta(13+9\beta)+6$$
$$\times(24ab-c-dr^2)^\Gamma\alpha\Gamma(6+\Gamma)\Big)\Bigg)+br^2\Big(256^\Gamma c^6\pi^{2\Gamma}\Gamma^2+2^{1+8\Gamma}c^5\pi^{2\Gamma}\Gamma\big(-16\pi(1$$

$$+\beta)+dr^2\Gamma\Big)+c^4(16\pi)^\Gamma\Big(2^{9+4\Gamma}\pi^{2+\Gamma}\beta+512\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma-4^{3+2\Gamma}d\pi^{1+\Gamma}r^2$$

$$\times(1+\beta)\Gamma+(16\pi)^\Gamma\big(256\pi^2(1+\beta^2)-5d^2r^4\Gamma^2\big)\Big)+c^2\Big(65536\pi^4(24ab-c-dr^2)^{2\Gamma}\alpha^2$$

$$+256^{1+\Gamma}d^3\pi^{1+2\Gamma}r^6(1+\beta)\Gamma-25\cdot 256^\Gamma d^4\pi^{2\Gamma}r^8\Gamma^2+256^{1+\Gamma}b^2d\pi^{1+2\Gamma}r^4\big(8\pi+dr^2\Gamma\big)$$

$$+2^{13+4\Gamma}d\pi^{3+\Gamma}r^2(24ab-c-dr^2)^\Gamma\alpha(1+\beta)(-2+\Gamma)-3\cdot 2^{9+4\Gamma}d^2\pi^{2+\Gamma}r^4(24ab-c$$

$$-dr^2)^\Gamma\alpha\Gamma\Gamma\Big)+4^{1+2\Gamma}c^3\pi^\Gamma\Big(-2048\pi^3(24ab-c-dr^2)^\Gamma\alpha(1+\beta)+d^2(16\pi)^{1+\Gamma}r^4(1$$

$$+\beta)\Gamma-5d^3(16\pi)^\Gamma r^6\Gamma^2+128d\pi^2r^2\Big((16\pi)^\Gamma+2^{1+4\Gamma}\pi^\Gamma\beta+(16\pi)^\Gamma\beta^2+2(24ab-c$$

$$-dr^2)^\Gamma\alpha\Gamma-(24ab-c-dr^2)^\Gamma\alpha\Gamma\Gamma\Big)\Big)-2cdr^2\Big(-7d^3(16\pi)^{1+2\Gamma}r^6(1+\beta)\Gamma+7\cdot 256^\Gamma$$

$$\times d^4\pi^{2\Gamma}r^8\Gamma^2-256^{1+\Gamma}b^2d\pi^{1+2\Gamma}r^4\big(8\pi+dr^2\Gamma\big)+65536\pi^4(24ab-c-dr^2)^{2\Gamma}\alpha^2(-1$$

$$+\Gamma)+4096d\pi^3r^2(24ab-c-dr^2)^\Gamma\alpha(1+\beta)\Big((16\pi)^\Gamma-2^{1+4\Gamma}\pi^\Gamma\Gamma\Big)+256d^2\pi^2r^4\Big(256^\Gamma$$

$$\pi^{2\Gamma}(1+\beta)^2+2^{1+4\Gamma}\pi^\Gamma(24ab-c-dr^2)^\Gamma\alpha\Gamma+3(16\pi)^\Gamma(24ab-c-dr^2)^\Gamma\alpha\Gamma^2\Big)\Big)+d^2$$

$$\times r^4\Big(16^{2+\Gamma}b^2\pi^{1+\Gamma}r^2\Big(2^{3+4\Gamma}d\pi^{1+\Gamma}r^2+d^2(16\pi)^\Gamma r^4\Gamma+128\pi^2(24ab-c-dr^2)^\Gamma\alpha(-1$$

$$+\Gamma)\Gamma\Big)-\Big(-256\pi^2(24ab-c-dr^2)^\Gamma\alpha+d(16\pi)^{1+\Gamma}r^2(1+\beta)-d^2(16\pi)^\Gamma r^4\Gamma\Big)(d$$

$$\times(16\pi)^{1+\Gamma}r^2(1+\beta)-3d^2(16\pi)^\Gamma r^4\Gamma-256\pi^2(24ab-c-dr^2)^\Gamma\alpha(-1+2\Gamma)\Big)\Big)-64$$

$$\times a^3\Big(9\cdot 2^{5+4\Gamma}b^4\pi^\Gamma r^2\Big(-2048\pi^3(24ab-c-dr^2)^\Gamma\alpha(2+3\beta)+2^{3+4\Gamma}d^2\pi^{1+\Gamma}r^4(7+6$$

$$\times\beta)\Gamma+15c^3(16\pi)^\Gamma\Gamma^2-15d^3(16\pi)^\Gamma r^6\Gamma^2-3c^2\Gamma\Big(4^{3+2\Gamma}\pi^{1+\Gamma}+5(16\pi)^\Gamma(16\pi\beta-dr^2\Gamma)\Big)$$

$$+3c\Big(2^{7+4\Gamma}\pi^{2+\Gamma}(1+3\beta)+512\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma-4^{3+2\Gamma}d\pi^{1+\Gamma}r^2\beta\Gamma+(16\pi)^\Gamma$$

$$\big(256\pi^2\beta^2-48d\pi r^2\Gamma-5d^2r^4\Gamma^2\big)\Big)+64d\pi^2r^2\Big(5(16\pi)^\Gamma+2^{1+4\Gamma}\pi^\Gamma\beta(4+3\beta)-6(24ab$$

$$-c-dr^2)^\Gamma\alpha\Gamma(-2+\Gamma)\Big)\Big)+4^{1+2\Gamma}d\pi^{1+\Gamma}\Big(-2^{1+4\Gamma}c^3\pi^\Gamma\Gamma+c^2\Big(3\cdot 2^{3+4\Gamma}\pi^{1+\Gamma}+(16\pi)^\Gamma$$

$$(16\pi\beta-7dr^2\Gamma)\big)+8c\Big(2^{1+4\Gamma}d\pi^{1+\Gamma}r^2(3+2\beta)-d^2(16\pi)^\Gamma r^4\Gamma-32\pi^2(24ab-c-dr^2)^\Gamma$$

$$\alpha\Gamma\big)+dr^2\Big(2^{3+4\Gamma}d\pi^{1+\Gamma}r^2(3+2\beta)-3d^2(16\pi)^\Gamma r^4\Gamma-128\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma(1$$

$$+\Gamma)\big)\Big)-3b^2\Big(196608\pi^4(24ab-c-dr^2)^{2\Gamma}\alpha^2-3\cdot 4^{3+4\Gamma}d^3\pi^{1+2\Gamma}r^6(11+14\beta)\Gamma+45$$

$$\times 256^\Gamma c^4\pi^{2\Gamma}\Gamma^2+165\cdot 256^\Gamma d^4\pi^{2\Gamma}r^8\Gamma^2+4^{1+4\Gamma}c^3\pi^{2\Gamma}\Gamma\big(-16\pi(13+15\beta)+75dr^2\Gamma\big)+3$$

$$\times 2^{1+4\Gamma}c^2\pi^\Gamma\Big(2^{9+4\Gamma}\pi^{2+\Gamma}+1536\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma-2^{7+4\Gamma}d\pi^{1+\Gamma}r^2(5+6\beta)\Gamma$$

$$+\times(16\pi)^{\Gamma}\left(256\pi^2\beta(5+3\beta)+105d^2r^4\Gamma^2\right)\Big)-2^{13+4\Gamma}d\pi^{3+\Gamma}r^2(24ab-c-dr^2)^{\Gamma}\alpha\,(10$$
$$+\Gamma+3\beta(4+\Gamma)\big)+2^{9+4\Gamma}d^2\pi^{2+\Gamma}r^4\left(29(16\pi)^{\Gamma}\beta+2^{1+4\Gamma}\pi^{\Gamma}(4+9\beta^2)+9(24ab-c\right.$$
$$\left.-dr^2)^{\Gamma}\alpha\Gamma(4+\Gamma)\right)+4^{1+2\Gamma}c\pi^{\Gamma}\left(-2048\pi^3(24ab-c-dr^2)^{\Gamma}\alpha(7+9\beta)-16d^2\pi r^4\,(5\right.$$
$$\times16^{1+\Gamma}\pi^{\Gamma}+99(16\pi)^{\Gamma}\beta\Big)\,\Gamma+135d^3(16\pi)^{\Gamma}r^6\Gamma^2+128d\pi^2r^2\left(27(16\pi)^{\Gamma}\beta^2+2^{1+4\Gamma}\pi^{\Gamma}(7\right.$$
$$\left.\left.\left.+22\beta)+9(24ab-c-dr^2)^{\Gamma}\alpha\Gamma(6+\Gamma)\right)\right)\right)+a\left(256^{\Gamma}c^6\pi^{2\Gamma}\Gamma^2-2^{1+8\Gamma}c^5\pi^{2\Gamma}\Gamma\left(16\pi(1\right.\right.$$
$$+\beta)+(72b^2-5d)r^2\Gamma\big)+c^4\,(16\pi)^{\Gamma}\left(2^{9+4\Gamma}\pi^{2+\Gamma}\beta+512\pi^2\left(24ab-c-dr^2\right)^{\Gamma}\alpha\Gamma+16^{2+\Gamma}\right.$$
$$\pi^{1+\Gamma}r^2\left(-d(1+\beta)+b^2(14+15\beta)\right)\Gamma-1516^{1+\Gamma}b^2d\pi^{\Gamma}r^4\Gamma^2+(16\pi)^{\Gamma}\left(256\pi^2(1+\beta^2)\right.$$
$$\left.+35d^2r^4\Gamma^2)\right)-4^{1+2\Gamma}c^3\pi^{\Gamma}\left(2048\pi^3\left(24ab-c-dr^2\right)^{\Gamma}\alpha(1+\beta)+11d^2\,(16\pi)^{1+\Gamma}\,r^4(1\right.$$
$$+\beta)\Gamma-15d^3\,(16\pi)^{\Gamma}\,r^6\Gamma^2+8b^2r^2\left(2^{7+4\Gamma}\pi^{2+\Gamma}(5+11\beta)+1536\pi^2\left(24ab-c-dr^2\right)^{\Gamma}\alpha\Gamma\right.$$
$$\left.-34^{3+2\Gamma}d\pi^{1+\Gamma}r^2\beta\Gamma+(16\pi)^{\Gamma}\left(768\pi^2\beta^2-176d\pi r^2\Gamma-15d^2r^4\Gamma^2\right)\right)-128d\pi^2r^2\left(3\,(16\pi)^{\Gamma}\right.$$
$$\left.+32^{1+4\Gamma}\pi^{\Gamma}\beta+3\,(16\pi)^{\Gamma}\,\beta^2+6\left(24ab-c-dr^2\right)^{\Gamma}\alpha\Gamma+\left(24ab-c-dr^2\right)^{\Gamma}\alpha\Gamma\Gamma\right)-c^2$$
$$\left(-65536\pi^4\left(24ab-c-dr^2\right)^{2\Gamma}\alpha^2+72^{7+8\Gamma}d^3\pi^{1+2\Gamma}r^6(1+\beta)\Gamma-55256^{\Gamma}d^4\pi^{2\Gamma}r^8\Gamma^2\right.$$
$$+2^{5+4\Gamma}b^2\pi^{\Gamma}r^2\left(-2048\pi^3\left(24ab-c-dr^2\right)^{\Gamma}\alpha(8+9\beta)+2^{3+4\Gamma}d^2\pi^{1+\Gamma}r^4(19+18\beta)\Gamma\right.$$
$$-45d^3\,(16\pi)^{\Gamma}\,r^6\Gamma^2+64d\pi^2r^2\left(17\,(16\pi)^{\Gamma}+2^{1+4\Gamma}\pi^{\Gamma}\beta(16+9\beta)-18\left(24ab-c-dr^2\right)^{\Gamma}\right.$$
$$\left.\left.\alpha\Gamma(-2+\Gamma)\right)\right)+2^{13+4\Gamma}d\pi^{3+\Gamma}r^2\left(24ab-c-dr^2\right)^{\Gamma}\alpha(1+\beta)(4+\Gamma)-32^{9+4\Gamma}d^2\pi^{2+\Gamma}r^4$$
$$\times\left(2^{1+4\Gamma}\pi^{\Gamma}(1+\beta)^2+\left(24ab-c-dr^2\right)^{\Gamma}\alpha\Gamma(4+\Gamma)\right)\Big)-2cr^2\left(32^{11+8\Gamma}b^4d\pi^{1+2\Gamma}r^4\,(8\pi\right.$$
$$+dr^2\Gamma)-8b^2\left(-196608\pi^4\left(24ab-c-dr^2\right)^{2\Gamma}\alpha^2-3256^{1+\Gamma}d^3\pi^{1+2\Gamma}r^6(1+\beta)\Gamma+75\right.$$
$$\times256^{\Gamma}d^4\pi^{2\Gamma}r^8\Gamma^2-4096d\pi^3r^2\left(24ab-c-dr^2\right)^{\Gamma}\alpha\left(2^{1+4\Gamma}\pi^{\Gamma}\left(-5+3\beta(-2+\Gamma)\right)+5\right.$$
$$\left.\left.\times(16\pi)^{\Gamma}\,\Gamma\right)+d^2\,(16\pi)^{2+\Gamma}\,r^4\left(-2^{1+4\Gamma}\pi^{\Gamma}+(16\pi)^{\Gamma}\,\beta+18\left(24ab-c-dr^2\right)^{\Gamma}\alpha\Gamma\right)\right)$$
$$+dr^4\left(-316^{3+2\Gamma}b^4d\pi^{1+2\Gamma}r^4\left(8\pi+dr^2\Gamma\right)+d\left(256\pi^2\left(24ab-c-dr^2\right)^{\Gamma}\alpha-d\,(16\pi)^{1+\Gamma}\right.\right.$$
$$\left.r^2(1+\beta)+d^2\,(16\pi)^{\Gamma}\,r^4\Gamma\right)\left(-3d\,(16\pi)^{1+\Gamma}\,r^2(1+\beta)+5d^2\,(16\pi)^{\Gamma}\,r^4\Gamma+256\pi^2\,(24ab\right.$$
$$\left.-c-dr^2\right)^{\Gamma}\alpha(1+2\Gamma)\Big)+16b^2\left(-21d^3\,(16\pi)^{1+2\Gamma}\,r^6(1+\beta)\Gamma+21256^{\Gamma}d^4\pi^{2\Gamma}r^8\Gamma^2\right.$$
$$+196608\pi^4\left(24ab-c-dr^2\right)^{2\Gamma}\alpha^2\,(-1+\Gamma)-2^{11+4\Gamma}d\pi^{3+\Gamma}r^2\left(24ab-c-dr^2\right)^{\Gamma}\alpha$$

$$
\Big(-4 + 9\Gamma + \Gamma^2 + 6\beta\,(-1+2\Gamma)\,) + 32^{7+4\Gamma} d^2 \pi^{2+\Gamma} r^4 \left((16\pi)^{\Gamma} + 2^{1+4\Gamma}\pi^{\Gamma}\beta\,(2+\beta)\right.
$$

$$
\left.\left. +2\left(24ab - c - dr^2\right)^{\Gamma}\alpha\Gamma\,(2+3\Gamma)\right)\right) 16a^2 b \left(-9\cdot 256^{\Gamma} c^5 \pi^{2\Gamma}\Gamma^2 + c^4\Gamma\left(7\cdot 2^{5+8\Gamma}\pi^{1+2\Gamma}\right.\right.
$$

$$
\left. +15(16\pi)^{1+2\Gamma}\beta + 15\cdot 256^{\Gamma}(36b^2 - 5d)\pi^{2\Gamma} r^2 \Gamma\right) - 2^{1+4\Gamma} c^3 \pi^{\Gamma}\left(2^{7+4\Gamma}\pi^{2+\Gamma}(5+11\beta)\right.
$$

$$
+1536\pi^2(24ab - c - dr^2)^{\Gamma}\alpha\Gamma + 4^{3+2\Gamma}\pi^{1+\Gamma} r^2 \left(-d(11+12\beta) + b^2(78+90\beta)\right)\Gamma - 45
$$

$$
\left.\times 2^{3+4\Gamma} b^2 d\pi^{\Gamma} r^4 \Gamma^2 + 3(16\pi)^{\Gamma}\left(256\pi^2\beta^2 + 35d^2 r^4 \Gamma^2\right)\right) + c\left(-196608\pi^4(24ab - c - dr^2)^{2\Gamma}\right.
$$

$$
\alpha^2 + 3\cdot 2^{5+8\Gamma} d^3 \pi^{1+2\Gamma} r^6 (25+28\beta)\Gamma - 165\cdot 256^{\Gamma} d^4 \pi^{2\Gamma} r^8 \Gamma^2 + 3\cdot 16^{1+\Gamma} b^2 \pi^{\Gamma} r^2 \left(-2048\pi^3\right.
$$

$$
\times(24ab - c - dr^2)^{\Gamma}\alpha(7+9\beta) + 16d^2\pi r^4\left(5\cdot 2^{1+4\Gamma}\pi^{\Gamma} + 9(16\pi)^{\Gamma}\beta\right)\Gamma - 45d^3(16\pi)^{\Gamma} r^6
$$

$$
\left.\left.\Gamma^2 + 128d\pi^2 r^2\left(9(16\pi)^{\Gamma}\beta^2 + 2^{1+4\Gamma}\pi^{\Gamma}(4+7\beta) - 9(24ab - c - dr^2)^{\Gamma}\alpha\Gamma(-2+\Gamma)\right)\right)\right)
$$

$$
+ 2^{13+4\Gamma} d\pi^{3+\Gamma} r^2 (24ab - c - dr^2)^{\Gamma}\alpha\left(11 + 2\Gamma + 3\beta(4+\Gamma)\right) - d^2\,(16\pi)^{2+\Gamma}\, r^4 \left(65(16\pi)^{\Gamma}\right.
$$

$$
\left.\times\beta + 2^{1+4\Gamma}\pi^{\Gamma}(13+18\beta^2) + 18(24ab - c - dr^2)^{\Gamma}\alpha\Gamma(4+\Gamma)\right) + c^2\left(4096\pi^3(24ab - c\right.
$$

$$
-dr^2)^{\Gamma}\alpha\left(2^{3+4\Gamma}\pi^{\Gamma} + 9(16\pi)^{\Gamma}\beta\right) + d^2\,(16\pi)^{1+2\Gamma}\, r^4\,(179+198\beta)\,\Gamma - 135\cdot 2^{1+8\Gamma} d^3 \pi^{2\Gamma}
$$

$$
\times r^6 \Gamma^2 + 92^{3+4\Gamma} b^2 \pi^{\Gamma} r^2 \left(2^{9+4\Gamma}\pi^{2+\Gamma} + 1536\pi^2(24ab - c - dr^2)^{\Gamma}\alpha\Gamma - 2^{5+4\Gamma} d\pi^{1+\Gamma} r^2 (5\right.
$$

$$
\left.+6\beta)\Gamma + (16\pi)^{\Gamma}\left(256\pi^2\beta(5+3\beta) - 15d^2 r^4 \Gamma^2\right)\right) - 2^{7+4\Gamma} d\pi^{2+\Gamma} r^2 \left(41(16\pi)^{\Gamma} + 2^{1+4\Gamma}\right.
$$

$$
\left.\pi^{\Gamma}\beta(49+27\beta) + 18(24ab - c - dr^2)^{\Gamma}\alpha\Gamma(6+\Gamma)\right) + r^2\left(9\cdot 4^{5+4\Gamma} b^4 d\pi^{1+2\Gamma} r^4 (8\pi + dr^2\right.
$$

$$
\times\Gamma) - 12b^2\left(-196608\pi^4(24ab - c - dr^2)^{2\Gamma}\alpha^2 - 3\cdot 256^{1+\Gamma}\ d^3\pi^{1+2\Gamma} r^6 (1+\beta)\Gamma + 75\right.
$$

$$
\times 256^{\Gamma} d^4 \pi^{2\Gamma} r^8 \Gamma^2 - 2^{13+4\Gamma} d\pi^{3+\Gamma} r^2 (24ab - c - dr^2)^{\Gamma}\alpha\,(2+3\beta)\,(-2+\Gamma) + 2^{9+4\Gamma} d^2
$$

$$
\left.\pi^{2+\Gamma} r^4 \left(-2^{1+4\Gamma}\pi^{\Gamma} + (16\pi)^{\Gamma}\,\beta + 9(24ab - c - dr^2)^{\Gamma}\alpha\Gamma\right)\right) - d\left(-3d^3(16\pi)^{1+2\Gamma} r^6 (15\right.
$$

$$
+17\beta)\Gamma + 39\cdot 256^{\Gamma} d^4 \pi^{2\Gamma} r^8 \Gamma^2 + 196608\pi^4(24ab - c - dr^2)^{2\Gamma}\alpha^2(1+\Gamma) + 2^{11+4\Gamma} d\pi^{3+\Gamma}
$$

$$
r^2(24\times ab - c - dr^2)^{\Gamma}\alpha\left(-28 - 9\Gamma + \Gamma^2 - 6\beta(5+2\Gamma)\right) + 3\cdot 2^{7+4\Gamma} d^2 \pi^{2+\Gamma} r^4 \left(7(16\pi)^{\Gamma}\right.
$$

$$
\left.\left.\left. +2^{1+4\Gamma}\times\pi^{\Gamma}\beta(9+5\beta) + 2(24ab - c - dr^2)^{\Gamma}\alpha\Gamma(10+3\Gamma)\right)\right)\right] \tag{A.1}
$$

$$v_t^2 = -\frac{8\cdot 4^{-4\Gamma-7}\pi^{-2\Gamma-2}r}{rd\left(a-br^2\right)^3\left(-24ab+c+dr^2\right)^2}\Bigg[729\cdot 4^{9+4\Gamma}a^7b^6\pi^{2\Gamma}\Gamma^2+27\cdot 256^{2+\Gamma}a^6b^3\pi^{2\Gamma}$$

$$\Gamma\left(-4d\pi+108b^4r^2\Gamma+3b^2\left(16\pi(2+3\beta)-3(3c+5dr^2)\Gamma\right)\right)+9\cdot 16^{3+\Gamma}a^5b^2\pi^\Gamma$$

$$\Bigg(7d^2r^2\Gamma\left(4^{1+2\Gamma}\pi^{1+\Gamma}+45b^2(16\pi)^\Gamma r^2\Gamma\right)-2^{1+4\Gamma}d\pi^\Gamma\left(16\pi^2(3+2\beta)-12\pi(c\right.$$

$$\left.-32b^2r^2(2+3\beta))\Gamma+9b^2r^2(-25c+24b^2r^2)\Gamma^2\right)\Bigg)+3b^2\Bigg(4^{5+2\Gamma}\pi^{2+\Gamma}\beta+1536$$

$$\times\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma-2^{5+4\Gamma}\pi^{1+\Gamma}\left(-24b^2r^2(2+3\beta)+c(11+15\beta)\right)\Gamma$$

$$-27\cdot 16^{1+\Gamma}b^2c\pi^\Gamma r^2\Gamma^2+(16\pi)^\Gamma\left(256\pi^2(1+3\beta^2)+45c^2\Gamma^2\right)\Bigg)+3\cdot 2^{11+4\Gamma}a^4b\pi^\Gamma\times$$

$$\Bigg(18b^4r^2\Bigg(4^{5+2\Gamma}\pi^{2+\Gamma}\beta+1536\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma-2^{5+4\Gamma}\pi^{1+\Gamma}\left(2dr^2(2+3\beta)\right.$$

$$\left.+c(11+15\beta)\right)\Gamma+15\cdot 2^{1+4\Gamma}cd\pi^\Gamma r^2\Gamma^2+(16\pi)^\Gamma\left(256\pi^2(1+3\beta^2)+15(3c^2-d^2\right.$$

$$\left.\times r^4)\Gamma^2\right)\Bigg)+2d\pi\Bigg(-3c^2(16\pi)^\Gamma\Gamma+c\left(3\cdot 2^{3+4\Gamma}\pi^{1+\Gamma}+(16\pi)^\Gamma\left(16\pi\beta-7dr^2\Gamma\right)\right)$$

$$+4\left(2^{1+4\Gamma}d\pi^{1+\Gamma}r^2(3+2\beta)-d^2(16\pi)^\Gamma r^4\Gamma-32\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma\right)\Bigg)\Bigg)-3b^2$$

$$\times\Bigg(-2048\pi^3(24ab-c-dr^2)^\Gamma\alpha(2+3\beta)-2^{3+4\Gamma}d^2\pi^{1+\Gamma}r^4(47+66\beta)\Gamma+15c^3$$

$$(16\pi)^\Gamma\Gamma^2+45d^3(16\pi)^\Gamma r^6\Gamma^2-3c^2\Gamma\left(4^{3+2\Gamma}\pi^{1+\Gamma}+5(16\pi)^\Gamma\left(16\pi\beta-5dr^2\Gamma\right)\right)$$

$$+3c\Big(2^{7+4\Gamma}\pi^{2+\Gamma}(1+3\beta)+512\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma-4^{3+2\Gamma}d\pi^{1+\Gamma}r^2(3+4\beta)\Gamma$$

$$+(16\pi)^\Gamma\left(256\pi^2\beta^2+35d^2r^4\Gamma^2\right)\Big)+64d\pi^2r^2\Big(5(16\pi)^\Gamma+2^{1+4\Gamma}\pi^\Gamma\beta(13+9\beta)+6$$

$$\times(24ab-c-dr^2)^\Gamma\alpha\Gamma(6+\Gamma)\Big)\Bigg)+br^2\Bigg(256^\Gamma c^6\pi^{2\Gamma}\Gamma^2+2^{1+8\Gamma}c^5\pi^{2\Gamma}\Gamma\left(-16\pi(1\right.$$

$$\left.+\beta)+dr^2\Gamma\right)+c^4(16\pi)^\Gamma\Big(2^{9+4\Gamma}\pi^{2+\Gamma}\beta+512\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma-4^{3+2\Gamma}d\pi^{1+\Gamma}r^2$$

$$\times(1+\beta)\Gamma+(16\pi)^\Gamma\left(256\pi^2(1+\beta^2)-5d^2r^4\Gamma^2\right)\Big)+c^2\Big(65536\pi^4(24ab-c-dr^2)^{2\Gamma}\alpha^2$$

$$+256^{1+\Gamma}d^3\pi^{1+2\Gamma}r^6(1+\beta)\Gamma-25\cdot 256^\Gamma d^4\pi^{2\Gamma}r^8\Gamma^2+256^{1+\Gamma}b^2d\pi^{1+2\Gamma}r^4\left(8\pi+dr^2\Gamma\right)$$

$$+2^{13+4\Gamma}d\pi^{3+\Gamma}r^2(24ab-c-dr^2)^\Gamma\alpha(1+\beta)(-2+\Gamma)-3\cdot 2^{9+4\Gamma}d^2\pi^{2+\Gamma}r^4(24ab-c$$

$$-dr^2)^\Gamma\alpha\Gamma\Gamma\Big)+4^{1+2\Gamma}c^3\pi^\Gamma\Big(-2048\pi^3(24ab-c-dr^2)^\Gamma\alpha(1+\beta)+d^2(16\pi)^{1+\Gamma}r^4(1$$

$$+\beta)\Gamma-5d^3(16\pi)^\Gamma r^6\Gamma^2+128d\pi^2r^2\Big((16\pi)^\Gamma+2^{1+4\Gamma}\pi^\Gamma\beta+(16\pi)^\Gamma\beta^2+2(24ab-c$$

$$-dr^2)^\Gamma\alpha\Gamma-(24ab-c-dr^2)^\Gamma\alpha\Gamma\Gamma\Big)\Big)-2cdr^2\Big(-7d^3(16\pi)^{1+2\Gamma}r^6(1+\beta)\Gamma+7\cdot 256^\Gamma$$

$$\times d^4\pi^{2\Gamma}r^8\Gamma^2-256^{1+\Gamma}b^2d\pi^{1+2\Gamma}r^4\left(8\pi+dr^2\Gamma\right)+65536\pi^4(24ab-c-dr^2)^{2\Gamma}\alpha^2(-1$$

$+\Gamma) + 4096d\pi^3r^2(24ab-c-dr^2)^\Gamma\alpha(1+\beta)\left((16\pi)^\Gamma - 2^{1+4\Gamma}\pi^\Gamma\Gamma\right) + 256d^2\pi^2r^4\left(256^\Gamma\right.$

$\pi^{2\Gamma}(1+\beta)^2 + 2^{1+4\Gamma}\pi^\Gamma(24ab-c-dr^2)^\Gamma\alpha\Gamma + 3(16\pi)^\Gamma(24ab-c-dr^2)^\Gamma\alpha\Gamma^2\Big)\Big) + d^2$

$\times r^4\left(16^{2+\Gamma}b^2\pi^{1+\Gamma}r^2\left(2^{3+4\Gamma}d\pi^{1+\Gamma}r^2 + d^2(16\pi)^\Gamma r^4\Gamma + 128\pi^2(24ab-c-dr^2)^\Gamma\alpha(-1\right.\right.$

$+\Gamma)\Gamma\Big) - \left(-256\pi^2(24ab-c-dr^2)^\Gamma\alpha + d(16\pi)^{1+\Gamma}r^2(1+\beta) - d^2(16\pi)^\Gamma r^4\Gamma\right)(d$

$\times(16\pi)^{1+\Gamma}r^2(1+\beta) - 3d^2(16\pi)^\Gamma r^4\Gamma - 256\pi^2(24ab-c-dr^2)^\Gamma\alpha(-1+2\Gamma)\Big)\Big) - 64$

$\times a^3\left(9\cdot 2^{5+4\Gamma}b^4\pi^\Gamma r^2\left(-2048\pi^3(24ab-c-dr^2)^\Gamma\alpha(2+3\beta) + 2^{3+4\Gamma}d^2\pi^{1+\Gamma}r^4(7+6\right.\right.$

$\times\beta)\Gamma + 15c^3(16\pi)^\Gamma\Gamma^2 - 15d^3(16\pi)^\Gamma r^6\Gamma^2 - 3c^2\Gamma\left(4^{3+2\Gamma}\pi^{1+\Gamma} + 5(16\pi)^\Gamma(16\pi\beta - dr^2\Gamma)\right)$

$+3c\left(2^{7+4\Gamma}\pi^{2+\Gamma}(1+3\beta) + 512\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma - 4^{3+2\Gamma}d\pi^{1+\Gamma}r^2\beta\Gamma + (16\pi)^\Gamma\right.$

$\left(256\pi^2\beta^2 - 48d\pi r^2\Gamma - 5d^2r^4\Gamma^2\right)\Big) + 64d\pi^2r^2\left(5(16\pi)^\Gamma + 2^{1+4\Gamma}\pi^\Gamma\beta(4+3\beta) - 6(24ab\right.$

$-c-dr^2)^\Gamma\alpha\Gamma(-2+\Gamma)\Big)\Big) + 4^{1+2\Gamma}d\pi^{1+\Gamma}\left(-2^{1+4\Gamma}c^3\pi^\Gamma\Gamma + c^2\left(3\cdot 2^{3+4\Gamma}\pi^{1+\Gamma} + (16\pi)^\Gamma\right.\right.$

$(16\pi\beta - 7dr^2\Gamma)\Big) + 8c\left(2^{1+4\Gamma}d\pi^{1+\Gamma}r^2(3+2\beta) - d^2(16\pi)^\Gamma r^4\Gamma - 32\pi^2(24ab-c-dr^2)^\Gamma\right.$

$\alpha\Gamma) + dr^2\left(2^{3+4\Gamma}d\pi^{1+\Gamma}r^2(3+2\beta) - 3d^2(16\pi)^\Gamma r^4\Gamma - 128\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma(1\right.$

$+\Gamma))\Big) - 3b^2\left(196608\pi^4(24ab-c-dr^2)^{2\Gamma}\alpha^2 - 3\cdot 4^{3+4\Gamma}d^3\pi^{1+2\Gamma}r^6(11+14\beta)\Gamma + 45\right.$

$\times 256^\Gamma c^4\pi^{2\Gamma}\Gamma^2 + 165\cdot 256^\Gamma d^4\pi^{2\Gamma}r^8\Gamma^2 + 4^{1+4\Gamma}c^3\pi^{2\Gamma}\Gamma\left(-16\pi(13+15\beta) + 75dr^2\Gamma\right) + 3$

$\times 2^{1+4\Gamma}c^2\pi^\Gamma\left(2^{9+4\Gamma}\pi^{2+\Gamma} + 1536\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma - 2^{7+4\Gamma}d\pi^{1+\Gamma}r^2(5+6\beta)\Gamma\right.$

$+\times(16\pi)^\Gamma\left(256\pi^2\beta(5+3\beta) + 105d^2r^4\Gamma^2\right)\Big) - 2^{13+4\Gamma}d\pi^{3+\Gamma}r^2(24ab-c-dr^2)^\Gamma\alpha\,(10$

$+\Gamma + 3\beta(4+\Gamma)\big) + 2^{9+4\Gamma}d^2\pi^{2+\Gamma}r^4\left(29(16\pi)^\Gamma\beta + 2^{1+4\Gamma}\pi^\Gamma(4+9\beta^2) + 9(24ab-c\right.$

$-dr^2)^\Gamma\alpha\Gamma(4+\Gamma)\Big) + 4^{1+2\Gamma}c\pi^\Gamma\left(-2048\pi^3(24ab-c-dr^2)^\Gamma\alpha(7+9\beta) - 16d^2\pi r^4\,(5\right.$

$\times 16^{1+\Gamma}\pi^\Gamma + 99(16\pi)^\Gamma\beta\Big)\,\Gamma + 135d^3(16\pi)^\Gamma r^6\Gamma^2 + 128d\pi^2r^2\left(27(16\pi)^\Gamma\beta^2 + 2^{1+4\Gamma}\pi^\Gamma(7\right.$

$+22\beta) + 9(24ab-c-dr^2)^\Gamma\alpha\Gamma(6+\Gamma)\Big)\Big)\Big) + a\left(256^\Gamma c^6\pi^{2\Gamma}\Gamma^2 - 2^{1+8\Gamma}c^5\pi^{2\Gamma}\Gamma\left(16\pi(1\right.\right.$

$+\beta) + (72b^2-5d)r^2\Gamma\Big) + c^4\,(16\pi)^\Gamma\left(2^{9+4\Gamma}\pi^{2+\Gamma}\beta + 512\pi^2\left(24ab-c-dr^2\right)^\Gamma\alpha\Gamma + 16^{2+\Gamma}\right.$

$\pi^{1+\Gamma}r^2\left(-d(1+\beta) + b^2(14+15\beta)\right)\Gamma - 1516^{1+\Gamma}b^2d\pi^\Gamma r^4\Gamma^2 + (16\pi)^\Gamma\left(256\pi^2(1+\beta^2)\right.$

$$+35d^2r^4\Gamma^2)\Big) - 4^{1+2\Gamma}c^3\pi^\Gamma \Big(2048\pi^3 \left(24ab - c - dr^2\right)^\Gamma \alpha(1+\beta) + 11d^2 \left(16\pi\right)^{1+\Gamma} r^4(1$$

$$+\beta)\Gamma - 15d^3 \left(16\pi\right)^\Gamma r^6\Gamma^2 + 8b^2r^2 \Big(2^{7+4\Gamma}\pi^{2+\Gamma}(5+11\beta) + 1536\pi^2 \left(24ab - c - dr^2\right)^\Gamma \alpha\Gamma$$

$$-34^{3+2\Gamma}d\pi^{1+\Gamma}r^2\beta\Gamma + \left(16\pi\right)^\Gamma \left(768\pi^2\beta^2 - 176d\pi r^2\Gamma - 15d^2r^4\Gamma^2\right)\Big) - 128d\pi^2r^2 \Big(3\left(16\pi\right)^\Gamma$$

$$+32^{1+4\Gamma}\pi^\Gamma\beta + 3\left(16\pi\right)^\Gamma\beta^2 + 6\left(24ab - c - dr^2\right)^\Gamma \alpha\Gamma + \left(24ab - c - dr^2\right)^\Gamma \alpha\Gamma\Gamma\Big) - c^2$$

$$\Big(-65536\pi^4 \left(24ab - c - dr^2\right)^{2\Gamma} \alpha^2 + 72^{7+8\Gamma}d^3\pi^{1+2\Gamma}r^6(1+\beta)\Gamma - 55256^\Gamma d^4\pi^{2\Gamma}r^8\Gamma^2$$

$$+2^{5+4\Gamma}b^2\pi^\Gamma r^2 \Big(-2048\pi^3 \left(24ab - c - dr^2\right)^\Gamma \alpha(8+9\beta) + 2^{3+4\Gamma}d^2\pi^{1+\Gamma}r^4(19+18\beta)\Gamma$$

$$-45d^3\left(16\pi\right)^\Gamma r^6\Gamma^2 + 64d\pi^2r^2 \Big(17\left(16\pi\right)^\Gamma + 2^{1+4\Gamma}\pi^\Gamma\beta(16+9\beta) - 18\left(24ab - c - dr^2\right)^\Gamma$$

$$\alpha\Gamma(-2+\Gamma)\Big)\Big) + 2^{13+4\Gamma}d\pi^{3+\Gamma}r^2 \left(24ab - c - dr^2\right)^\Gamma \alpha(1+\beta)(4+\Gamma) - 32^{9+4\Gamma}d^2\pi^{2+\Gamma}r^4$$

$$\times\Big(2^{1+4\Gamma}\pi^\Gamma(1+\beta)^2 + \left(24ab - c - dr^2\right)^\Gamma \alpha\Gamma(4+\Gamma)\Big)\Big) - 2cr^2 \Big(32^{11+8\Gamma}b^4d\pi^{1+2\Gamma}r^4 (8\pi$$

$$+dr^2\Gamma) - 8b^2 \Big(-196608\pi^4 \left(24ab - c - dr^2\right)^{2\Gamma} \alpha^2 - 3256^{1+\Gamma}d^3\pi^{1+2\Gamma}r^6(1+\beta)\Gamma + 75$$

$$\times 256^\Gamma d^4\pi^{2\Gamma}r^8\Gamma^2 - 4096d\pi^3r^2 \left(24ab - c - dr^2\right)^\Gamma \alpha \Big(2^{1+4\Gamma}\pi^\Gamma \left(-5 + 3\beta(-2+\Gamma)\right) + 5$$

$$\times\left(16\pi\right)^\Gamma\Gamma\Big) + d^2\left(16\pi\right)^{2+\Gamma} r^4 \Big(-2^{1+4\Gamma}\pi^\Gamma + \left(16\pi\right)^\Gamma\beta + 18\left(24ab - c - dr^2\right)^\Gamma \alpha\Gamma\Big)\Big)$$

$$+dr^4 \Big(-316^{3+2\Gamma}b^4d\pi^{1+2\Gamma}r^4 \left(8\pi + dr^2\Gamma\right) + d\Big(256\pi^2 \left(24ab - c - dr^2\right)^\Gamma \alpha - d\left(16\pi\right)^{1+\Gamma}$$

$$r^2(1+\beta) + d^2\left(16\pi\right)^\Gamma r^4\Gamma\Big) \Big(-3d\left(16\pi\right)^{1+\Gamma} r^2(1+\beta) + 5d^2\left(16\pi\right)^\Gamma r^4\Gamma + 256\pi^2 (24ab$$

$$-c - dr^2)^\Gamma \alpha(1+2\Gamma)\Big) + 16b^2 \Big(-21d^3\left(16\pi\right)^{1+2\Gamma} r^6(1+\beta)\Gamma + 21256^\Gamma d^4\pi^{2\Gamma}r^8\Gamma^2$$

$$+196608\pi^4 \left(24ab - c - dr^2\right)^{2\Gamma} \alpha^2\left(-1+\Gamma\right) - 2^{11+4\Gamma}d\pi^{3+\Gamma}r^2 \left(24ab - c - dr^2\right)^\Gamma \alpha$$

$$\Big(-4 + 9\Gamma + \Gamma^2 + 6\beta\left(-1+2\Gamma\right)\Big) + 32^{7+4\Gamma}d^2\pi^{2+\Gamma}r^4 \Big(\left(16\pi\right)^\Gamma + 2^{1+4\Gamma}\pi^\Gamma\beta\left(2+\beta\right)$$

$$+2\left(24ab - c - dr^2\right)^\Gamma \alpha\Gamma\left(2+3\Gamma\right)\Big)\Big) 16a^2b \Big(-9\cdot 256^\Gamma c^5\pi^{2\Gamma}\Gamma^2 + c^4\Gamma \Big(7\cdot 2^{5+8\Gamma}\pi^{1+2\Gamma}$$

$$+15(16\pi)^{1+2\Gamma}\beta + 15\cdot 256^\Gamma(36b^2 - 5d)\pi^{2\Gamma}r^2\Gamma\Big) - 2^{1+4\Gamma}c^3\pi^\Gamma \Big(2^{7+4\Gamma}\pi^{2+\Gamma}(5+11\beta)$$

$$+1536\pi^2(24ab - c - dr^2)^\Gamma\alpha\Gamma + 4^{3+2\Gamma}\pi^{1+\Gamma}r^2 \left(-d(11+12\beta) + b^2(78+90\beta)\right)\Gamma - 45$$

$$\times 2^{3+4\Gamma}b^2d\pi^\Gamma r^4\Gamma^2 + 3(16\pi)^\Gamma \left(256\pi^2\beta^2 + 35d^2r^4\Gamma^2\right)\Big) + c\Big(-196608\pi^4(24ab - c - dr^2)^{2\Gamma}$$

$$\alpha^2 + 3\cdot 2^{5+8\Gamma}d^3\pi^{1+2\Gamma}r^6(25+28\beta)\Gamma - 165\cdot 256^\Gamma d^4\pi^{2\Gamma}r^8\Gamma^2 + 3\cdot 16^{1+\Gamma}b^2\pi^\Gamma r^2 \Big(-2048\pi^3$$

$$\times(24ab - c - dr^2)^\Gamma\alpha(7+9\beta) + 16d^2\pi r^4 \Big(5\cdot 2^{1+4\Gamma}\pi^\Gamma + 9(16\pi)^\Gamma\beta\Big)\Gamma - 45d^3(16\pi)^\Gamma r^6$$

$$
\begin{aligned}
&\Gamma^2 + 128d\pi^2 r^2 \left(9(16\pi)^\Gamma \beta^2 + 2^{1+4\Gamma}\pi^\Gamma(4+7\beta) - 9(24ab-c-dr^2)^\Gamma \alpha\Gamma(-2+\Gamma)\right)\Big) \\
&+ 2^{13+4\Gamma} d\pi^{3+\Gamma} r^2 (24ab-c-dr^2)^\Gamma \alpha \left(11+2\Gamma+3\beta(4+\Gamma)\right) - d^2 (16\pi)^{2+\Gamma} r^4 \Big(65(16\pi)^\Gamma \\
&\times\beta + 2^{1+4\Gamma}\pi^\Gamma(13+18\beta^2) + 18(24ab-c-dr^2)^\Gamma\alpha\Gamma(4+\Gamma)\Big) + c^2 \Big(4096\pi^3(24ab-c \\
&-dr^2)^\Gamma\alpha\left(2^{3+4\Gamma}\pi^\Gamma + 9(16\pi)^\Gamma\beta\right) + d^2(16\pi)^{1+2\Gamma} r^4 (179+198\beta)\Gamma - 135\cdot 2^{1+8\Gamma} d^3\pi^{2\Gamma} \\
&\times r^6\Gamma^2 + 92^{3+4\Gamma} b^2 \pi^\Gamma r^2 \Big(2^{9+4\Gamma}\pi^{2+\Gamma} + 1536\pi^2(24ab-c-dr^2)^\Gamma\alpha\Gamma - 2^{5+4\Gamma} d\pi^{1+\Gamma} r^2(5 \\
&+6\beta)\Gamma + (16\pi)^\Gamma \left(256\pi^2\beta(5+3\beta) - 15d^2r^4\Gamma^2\right)\Big) - 2^{7+4\Gamma} d\pi^{2+\Gamma} r^2 \Big(41(16\pi)^\Gamma + 2^{1+4\Gamma} \\
&\pi^\Gamma\beta(49+27\beta) + 18(24ab-c-dr^2)^\Gamma\alpha\Gamma(6+\Gamma)\Big) + r^2 \Big(9\cdot 4^{5+4\Gamma} b^4 d\pi^{1+2\Gamma} r^4 (8\pi + dr^2 \\
&\times\Gamma) - 12b^2 \Big(-196608\pi^4(24ab-c-dr^2)^{2\Gamma}\alpha^2 - 3\cdot 256^{1+\Gamma}\ d^3\pi^{1+2\Gamma} r^6(1+\beta)\Gamma + 75 \\
&\times 256^\Gamma d^4 \pi^{2\Gamma} r^8 \Gamma^2 - 2^{13+4\Gamma} d\pi^{3+\Gamma} r^2 (24ab-c-dr^2)^\Gamma \alpha (2+3\beta)(-2+\Gamma) + 2^{9+4\Gamma} d^2 \\
&\pi^{2+\Gamma} r^4 \left(-2^{1+4\Gamma}\pi^\Gamma + (16\pi)^\Gamma\beta + 9(24ab-c-dr^2)^\Gamma\alpha\Gamma\right) - d\Big(-3d^3(16\pi)^{1+2\Gamma} r^6 (15 \\
&+17\beta)\Gamma + 39\cdot 256^\Gamma d^4\pi^{2\Gamma} r^8\Gamma^2 + 196608\pi^4(24ab-c-dr^2)^{2\Gamma}\alpha^2(1+\Gamma) + 2^{11+4\Gamma} d\pi^{3+\Gamma} \\
&r^2(24\times ab-c-dr^2)^\Gamma\alpha\left(-28-9\Gamma+\Gamma^2-6\beta(5+2\Gamma)\right) + 3\cdot 2^{7+4\Gamma} d^2\pi^{2+\Gamma} r^4 \Big(7(16\pi)^\Gamma \\
&+2^{1+4\Gamma}\times\pi^\Gamma\beta(9+5\beta) + 2(24ab-c-dr^2)^\Gamma\alpha\Gamma(10+3\Gamma)\Big)\Big)\Bigg]
\end{aligned}
\tag{A.2}
$$