Soliton solutions for the two-mode Gardner equation through the Jacobi elliptic function and Kudryashov method

by

Mariyam Mukhtar

A thesis
submitted in partial fulfillment of the
requirements for the degree of
Master of Science
in
Mathematics

School of Natural Sciences,
National University of Sciences and Technology,
H-12, Islamabad, Pakistan

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July, 2024

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MS THESIS WORK

We hereby recommend that the dissertation prepared under our supervision by: **"Mariyam Mukhtar"** Regn No. **00000365328** Titled: **"Soliton Soultion for the Two-Mode Gardner Equation Through the Jacabi Elliptic Funtion and Kudryashov Method"** accepted in partial fulfillment of the requirements for the award of **MS** degree.

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Dedicated

to

My Beloved Parents

and

Honorable Teachers

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First and foremost, I am incredibly appreciative to **Allah** Ta'ala for giving me the chance, grit, and capacity to advance knowledge. I would like to express my sincere gratitude and affection to Prophet **Hazrat Muhammad (PBUH), Whose** teachings have helped me acknowledge our Creator and ingrained in me the noble ideals of Islam.

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Abstract

Dual-mode equations focus on nonlinear models that explain two-way wave motion concurrently under the impact of phase variations. First, Korsunsky introduced a dual-mode model that enhanced the Kottweg De Vries equation (KDVe) by transforming it into a second-order expression. The main goal of our research is to find an exact solution to the two-mode Gardner equation obtained from the ideal fluid model. We will find the analytical solutions by applying the Jacobi elliptic functions approach and the Kudryashov approach. By using the strategies, the issue may be viewed from several perspectives, offering a comprehensive comprehension of how the equation functions and the variety of answers it produces. Moreover, both 2D and 3D graphs will be used to highlight the findings. This graphic will provide a concise representation of how changes in the phase velocity parameter influence the solutions, en- hancing our comprehension of the flow of waves in the two-mode Gardner equation. The solutions that we get in this research have important significance for our comprehension of the propagation of solitons in the context of nonlinear optics. The outcomes of the research advance our knowledge of how light travels via optical fibers and how periodic solutions and solitons impact wave behavior in shallow water

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Chapter 1

Introduction

Partial differential equations (PDEs) are essential for understanding complex natural processes in engineering, physics, and applied mathematics. These equations facilitate the study of how various physical, biological, and chemical processes evolve over time and space. PDEs are utilized across numerous fields, including physics, engineering, ecology, chemistry, finance, biology, and telecommunications. They are essential to the simulation of physics concepts like wave dynamics and heat conduction. PDEs are frequently used in ecology to simulate populations. These equations are used in chemistry to describe the reaction and diffusion of compounds. PDEs simulate signal propagation. This includes wave equations for electromagnetic waves in transmission lines, fiber optics, and diffusion equations for modeling data traffic and network congestion. Furthermore, PDEs are crucial in the related fields of fluid dynamics, quantum mechanics, electrical systems, plasma physics, and shallow water wave research. It is becoming more and more crucial to comprehend both conventional and contemporary methods for solving PDEs due to their crucial role in simulating both natural and man-made systems. The capacity to use these techniques in real-world situations is equally essential.

Recently, there has been an increased interest in nonlinear PDEs, particularly in engineering, physics, shallow water waves, and earth sciences. Numerous solitary wave solutions for these equations have been discovered in physics-related areas such as fluid mechanics, plasma theory, and nonlinear optics. This growing focus highlights the importance of nonlinear PDEs in advancing our understanding of solitary wave behavior in various scientific domains [2–6].

Furthermore, nonlinear evolution equations (NLEEs) provide essential mathematical models for understanding physical processes in applied sciences, making them fundamental to mathematical physics and engineering. Exact solutions that describe wave behavior in these equations are crucial for comprehending these phenomena. Various systems, including structured materials, plasma, chemicals, elastic materials, optical cables, fluid movement, and quantum mechanics, display wave-like characteristics.

NLEEs are indispensable tools for addressing problems in mathematical physics and engineering, with numerous methods developed to solve them. These methods include the G'/G-expansion method, tanh expansion approach, Kudryshov technique, extended F-expansion procedure, rational sin/cos strategy, inverse scattering transform, Hirota bilinear method, Darboux transformation, Painlevé analysis, Backlund Darboux transformation, tanh method, Exp-function method, improved F-expansion method, modified extended tanh-function method, $\exp(-\phi(\xi))$-expansion method, sine-cosine method, modified simple equation method, the novel G'/G-expansion method, homogeneous balance method, and the homotopy perturbation technique, among others [7–13].

Additionally, the periodic nature of the Jacobi elliptic functions method is particularly effective for capturing dual-mode behavior, where waves propagate in two interacting modes simultaneously. While the Kudryashov approach is valuable for finding rational and singular solutions, the Jacobi elliptic functions method excels in generating periodic wave solutions and extending trigonometric functions to elliptic integrals.

Furthermore, when examining wave behavior, the KdV equation, which describes the propagation of small waves in shallow water, is particularly insightful. This equation is crucial for understanding the balance between linear dispersion and nonlinear steepening of waves. The wave motion is notably influenced when the wave size differs significantly from the water depth. Similar principles apply to higher-order KdV equations, which are related to the derivation of (2+1)-dimensional Gardner equations from ideal fluid models. In analyzing this system, three non-dimensional parameters are particularly important: the amplitude parameter $\alpha = a/H$, the wavelength parameter $\beta = H^2/L^2$, and the bond number $\tau = T/(\rho g H^2)$. These parameters are critical for comprehending the system's behavior. Here, g is the gravitational acceleration, H is the average depth upstream, a and L are the typical wave amplitude and wavelength values, respectively, ρ represents the water's density and T is the surface tension coefficient.

In conclusion, due to their unique characteristics and numerous applications in physics and engineering, surface waves in shallow water require thorough study. Understanding and investigating the exact solutions to fundamental equations that describe these physical processes is essential. Even when a clear physical explanation is lacking, precise solutions can serve as models to evaluate the accuracy of various computational and mathematical techniques. With advancements in computer technology, the field of nonlinear science has gained significant relevance in recent years. A crucial aspect of this field is finding accurate solutions for nonlinear PDEs, which are essential for interpreting physical phenomena. For instance, the Gardner equations have special solutions such as kink waves, periodic waves and solitary waves. These solitons represent nonlinear wave processes in various domains, including shallow water waves, fluid dynamics and plasmas. By applying methods like the Kudryashov approach and the Jacobi elliptic functions method, unique solutions can be obtained, providing deeper insights into wave interactions. Computational tools such as MATLAB, MAPLE and others are often used to generate and compute many of these solutions, further enhancing our understanding of nonlinear wave behavior.

1.0.1 Nonlinear Evolution Equations (NLEEs)

Dynamic partial differential equations (PDEs) using both temporal and spatial variables as independent variables are known as evolution equations. In this particular category, PDEs with nonlinear components and temporal derivatives are referred to as NLEEs. These equations are effective tools for comprehending how dynamical systems change over time. These equations accurately describe phenomena where the system's dynamics are nonlinearly influenced by its state, incorporating nonlinear variables and temporal derivatives. This nonlinearity manifests in various forms, including chaotic behavior, self-organization, wave interactions and pattern development. Ongoing research on nonlinear evolution equations (NLEEs) aims to develop new analytical techniques, uncover unexpected phenomena and clarify the fundamental principles underlying nonlinear dynamics.

Applications of the NLEEs

NLEEs are used in a variety of fields, comprising biology, materials science, fluid dynamics, and plasma physics [17, 18]. They provide insightful information on how biological populations behave, how waves travel via optical fibers, how fluid flows behave, and how complex systems self-organize. NLEEs exhibit a broad range of phenomena, including pattern generation, turbulence, wave breaking, and soliton propagation. These equations offer a framework for the investigation of emergent phenomena in biological, engineering, and physical systems that result from nonlinear interactions.
Example of NLEEs:

$$p_t - 6p^2 p_x + p_{xxx} = 0.$$

1.0.2 Gardner Equation

The Gardner equation provides a crucial milestone while examining nonlinear wave equations and is often considered a combination of both KdV and mKdV equations. Compared to the KdV or mKdV equations alone, it offers a more complete framework for simulating wave interactions as it captures the effects of both quadratic and cubic nonlinearity. The Gardner equation is used in fluid dynamics to simulate internal waves in stratified fluids and shallow water waves. The research of ion-acoustic waves in plasma is another application of the Gardner equation. Because of its all-encompassing framework, it can be used to analyze the dynamics and stability of these waves in a plasma environment. The Gardner equation is useful in various contexts, such as optical fiber networks, where it clarifies the interactions and transmission of optical

solitons.
The Gardner equation is written as [23]:

$$\Phi_t + \alpha\Phi\Phi_x + \beta\Phi^2\Phi_x + \gamma\Phi_{xxx} = 0.$$

1.1 A travelling wave

When the medium travels in the path of the wave's propagation, the wave is said to be traveling. Maintaining a constant velocity during propagation is similarly related to a traveling wave. These waves have been seen in many other domains, such as combustion, which is a chemical process. The visible impulses in nerve fibers are represented as moving waves in mathematical biology. Additionally, problems with fluid dynamics are linked to conservation laws. The features of shock are described in terms of traveling waves.

1.1.1 Traveling wave solutions

A key concept in the research of nonlinear partial differential equations (PDEs) is traveling wave solutions. The permanent form solution moving at a constant speed is known as the traveling wave solution. Usually, the equations for the nonlinear evolution are transformed into corresponding ordinary differential equations to find the traveling wave solutions. The traveling wave solutions are mathematically represented as follows:

$$\Phi(x,t) = V(\xi), \;\; \xi = x - \omega t,$$

where the wave speed can be denoted by ω, while t and x represent the time and space variables, respectively. ξ represents the traveling wave variable.
Solitary wave theory, which is rapidly expanding in a range of scientific disciplines from shallow water waves to plasma physics, is particularly interested in various types of traveling wave solutions. As previously said, there are many different forms of traveling waves, and only a few of them will be discussed.

1.1.2 Pulse waves

Instead of continuous waveforms, traveling wave solutions known as pulse waves are defined by isolated pulses. These waves consist of brief, transient disturbances that move through a medium and carry energy in a localized manner. Pulse waves are essential in techniques such as ultrasonic imaging, where short bursts of sound waves are sent into the body to provide finely detailed pictures of interior structures. Moreover, pulse waves can be described mathematically using different functions, depending on their specific characteristics and applications. One common representation is through a Gaussian pulse due to its smooth, bell-shaped curve.

1.1.3 Peakons

Single-wave solutions with peaks are called peakons. With the exception of a peak near the crest's corner, the traveling wave solutions are smooth. These are solitons that continue to move at the same speed and form even after colliding. The integrable Camassa-Holm equation gives the peakon solution in the form,

$$p(x,t) = we^{-\left|(x-wt)\right|}, -4 \leqslant x, t \leq 4.$$

Peakons have the following unique properties;

- Peakons exhibit discontinuities in their first derivatives due to their peaked structure, which consists of a strong peak at the crest and a zero amplitude trough.
- Peakons propagate non-smoothly, with the trough remaining stationary and the peak keeping its shape and speed constant.
- Peakons interact and show intriguing collision dynamics, such as the creation and destruction of peaks and the momentum and energy exchange.

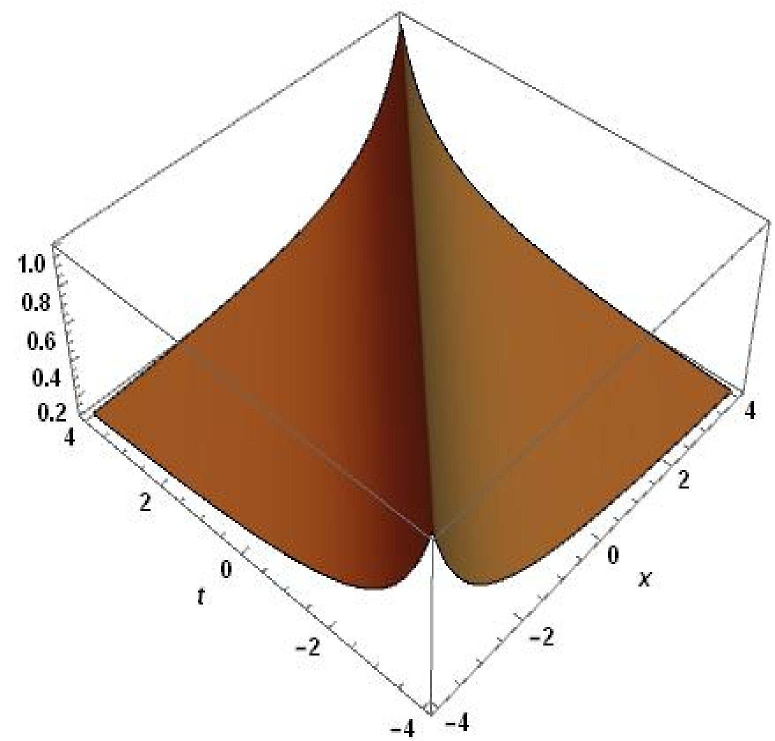


Figure 1.1: Graph of peakon solution for $p_1 = (1/(1 + Abs[x - t])),\ -4 \leq x,\ t \leq 4.$

1.1.4 Kink waves

Kink waves are characterized by smooth transitions between stable states, usually from one asymptotic value to another. In contrast to other wave forms, Kink solutions approach a constant value at infinity. A prominent example of an equation yielding solutions for kink waves is the dissipative Burger equation[38], denoted by $u_t + u_{xx} = vu_{xx}$, where v represents the viscosity coefficient. The characteristic feature of kink waves is their stable, smooth transition between states, which makes them an essential solution in various physical contexts.

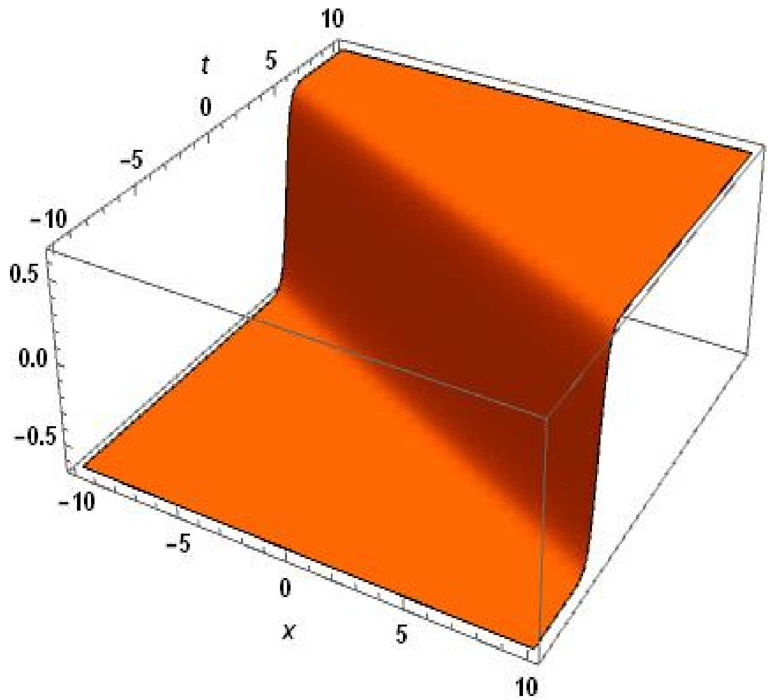


Figure 1.2: Graph of the kink wave, $p_1(x,t) = \tanh(x+8t),\ -10 \leq x,\ t \leq 10$.

1.1.5 Periodic waves

A basic class of wave solutions known as periodic waves are distinguished by their oscillatory and repeated behavior. In fluid mechanics, surface waves on water that exhibit regular patterns of wave crests and troughs are known as periodic waves. Crests and troughs highest and lowest points of the wave respectively. These waves are crucial to many different scientific phenomena and applications because they repeat themselves at predictable intervals in both space and time. Sine and cosine, two sinusoidal functions that capture periodic waves' regular and recurring nature, are frequently used to characterize them.

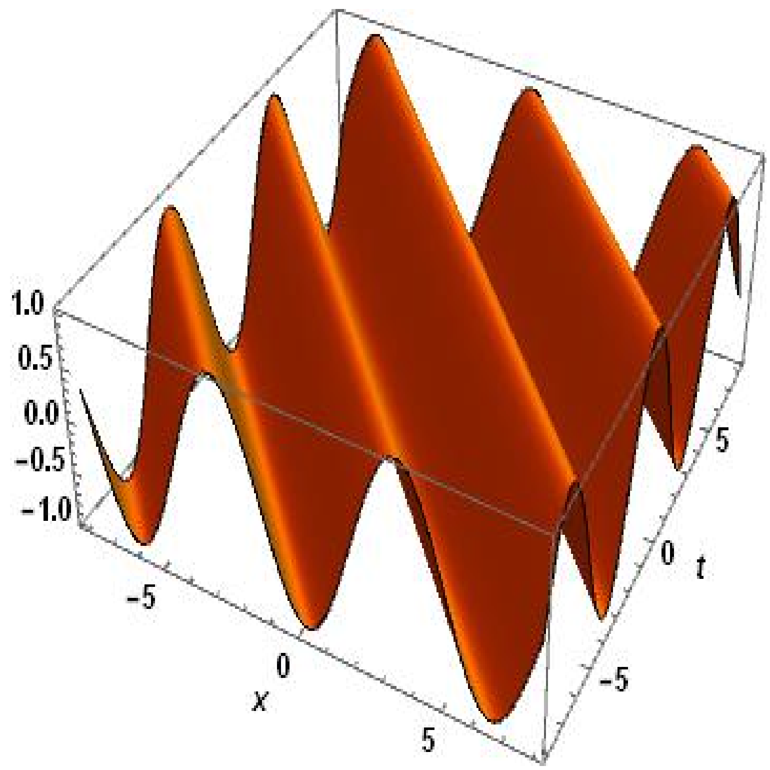


Figure 1.3: Graph of the periodic wave ,$p_1(x,t) = \sin(x-t)$, $-8 \leq x,\ t \leq 8$.

1.1.6 Importance of the traveling wave

Disturbances that travel across a medium and transfer energy are identified as traveling waves. Numerous disciplines, including fluid mechanics, chemistry, acoustics, biology, elasticity, and electromagnetic theory, depend on them. Understanding traveling waves helps in our comprehension of a number of processes, including electromagnetic wave transmission, water waves, combustion, sound propagation, seismic activity, and the spread of illness. We can better analyze and anticipate the flow of energy by examining these waves, which will result in important breakthroughs in these fields.

1.2 Soliton and Solitary waves

Martin Kruskal and Norman Zabusky made the revolutionary discovery of the solitons within the Korteweg-de Vries (KdV) equation [17]. A complete and accurate definition of a soliton is difficult to provide. However, we will use the word to refer to any solution of a nonlinear equation (or system) which (a) symbolizes a permanent shape wave; (b) is localized, declining, or approaching a constant as it approaches infinity; (c) may maintain its identity while having intense interactions with other solitons [18]. Moreover, solitons form because of the precise balance between nonlinear and dispersive effects in the medium through which they travel. **Solitons**, the solutions of nonlinear dispersive partial differential equations (PDEs), are essential for explaining a variety of physical phenomena.

A single wave crest that moves across a medium without changing form or depleting energy is known as a solitary wave. The light intensity in fiber optics and the surface height of the water in finite-amplitude water channels both result in solitary waves. There are many examples of solitary waves but the two distinct examples of solitary waves are:

- **Ocean solitary wave :** These waves are prime examples of solitary waves as they are well-studied in the field of fluid dynamics and are well-known for their capacity to go across great distances without significantly losing energy or form. Underwater disturbances such as landslides or tectonic activity might cause them to arise. Among the most well-known examples of solitary waves in the water are tsunamis, which retain their energy and form across enormous distances.

- **Heartbeat wave :** The ECG waveform is a recording of the electrical impulses in the heart that travel as waves. To examine how action potentials spread across heart tissue, these waves might occasionally be modeled as solitary waves.

Figure 1.4: Ocean solitary wave

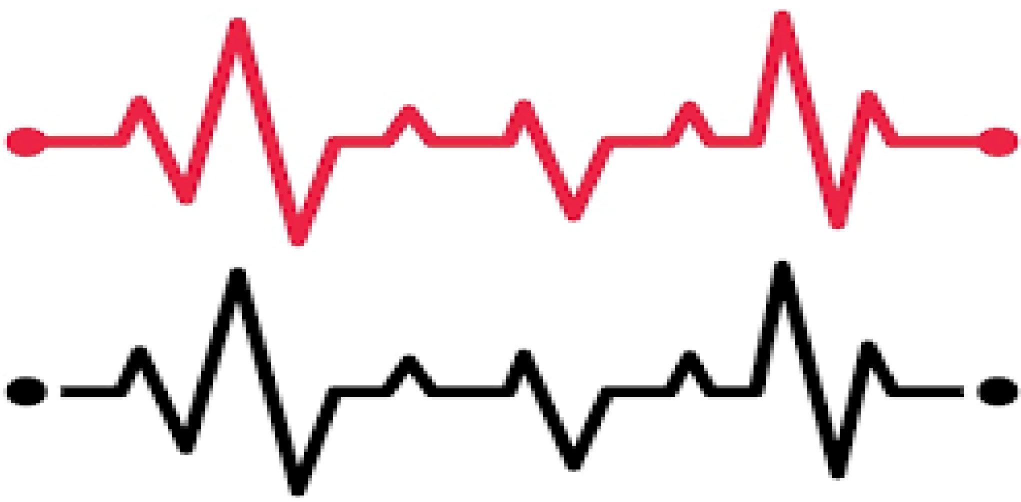


Figure 1.5: Heartbeat wave

1.2.1 Properties of soliton

Shape preservation:

Over extended distances and times, solitons retain their amplitude and shape. This occurs because the nonlinear effects in the medium exactly counteract the dispersive effects that would normally cause the wave to spread out and lose its shape.

Particle-like behavior:

The unique property of soliton is referred to as particle-like behavior. Solitons retain their integrity and form even after collisions, in contrast to typical waves that scatter and change as they move. Their extraordinary stability and resistance to perturbations contribute to their particle-like characteristics. Numerous nonlinear systems are explained by the special behavior of solitons. It is significant to remember that there isn't yet a well-established quantum theory that finds particles as solitons. On a macroscopic level, however, the idea of solitons may be used to elaborate the propagation of waves in domains like oceanography, communications, and optics. A clear framework for comprehending and analyzing the wave behavior in these systems is provided by the theory of solitons, opening up important new avenues for research and useful applications.

Stability:

Small perturbations cannot destabilize solitons. They do not change much in form or speed in response to minor perturbations or interactions with other waves. Solitons are resilient solutions to the governing nonlinear equations due to their inherent features. In contrast to other waveforms that may decay or scatter, the soliton often returns to its stable form when disturbed.

Elastic collisions:

When solitons interact with each other, they behave like elastic particles. After the interaction, their velocities and forms remain the same as they emerge, except for a possible phase shift. This amazing quality is generally due to the integrability of the governing nonlinear differential equations. For instance, soliton solutions can flow through one another and emerge unchanged in the KdV equation and the nonlinear Schrödinger equation, a property not observed in linear wave theory.

Localization:

Since soliton waves are localized, they can only exist in a particular region of space. They don't spread out or disperse over time. In order to prevent any dispersive spreading, the medium's nonlinearity offers a concentrating mechanism.

Versatility of Existence:

Solitons are found out in a vast range of physical systems, including water waves, optical fibers, plasmas, and biological systems. The widespread use of solitons in a variety of contexts emphasizes their essential characteristics and usefulness.

Integrability:

Integrability is a mathematical property that gives valuable predictive power and qualitative insights into the dynamics of a system. Solitons are frequently PDEs (integrable nonlinear equations) solutions. Solitons are stable and persistent because integrability entails the existence of an unlimited number of conserved quantities. One effective mathematical technique for examining and comprehending the integrability of soliton equations is the inverse scattering transform[18] .

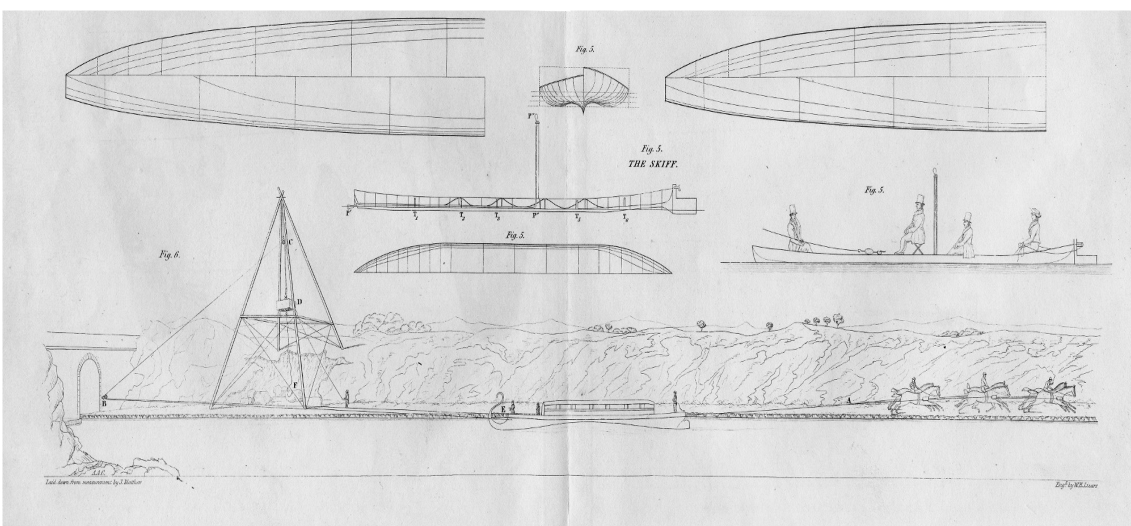
1.2.2 Discovery of soliton

In Edinburgh in 1834, Scottish engineer John Scott Russell (1808–1882) made the first known observation of solitary waves. He figured a massive swelling wave of water moving through the canal with a consistent shape. Russell expressed his observation in his own words as follows:

"I was observing the motion of a boat that was rapidly drawn along a narrow channel by a pair of horses, when the boat suddenly stopped not so the mass of water in the channel which it had put in motion; it accumulated round the prow of the vessel in a state of violent agitation, then suddenly leaving it behind, rolled forward with great velocity, assuming the form of a large solitary elevation, a rounded, smooth and well-defined heap of water, which continued its course along the channel apparently without

change of form or diminution of speed. I followed it on horseback and overtook it still rolling on at a rate of some eight or nine miles an hour, preserving its original figure some thirty feet long and a foot to a foot and a half in height. Its height gradually diminished, and after a chase of one or two miles, I lost it in the windings of the channel. The month of August 1834, was my first chance interview with that singular and beautiful phenomenon which I have called the wave of translation". He experimented with a water tank at home to better understand the concept of these waves. Scott Russell discovered some of the most significant traits of certain waves: Scott Russell identifies some of the most significant traits of certain waves:

- Greater speeds are correlated with larger waves.
- Lower speeds are characteristic of smaller waves.
- These waves are remarkably capable of traveling great distances while remaining stable.
- These waves don't merge like other waves do.



Scott Russell's work questioned a challenge to the established hydrodynamic theories of Isaac Newton and Daniel Bernoulli. Due to discrepancies with the existing water wave theories, George Gabriel Stokes and George Biddell Airy initially rejected Russell's experimental results. These contradictions stood unresolved until 1870s when Joseph Boussinesq and Lord Rayleigh published their solutions to the problem. The equation known as Korteweg-de Vries (KdV) was introduced by Diederik Korteweg and Gustav de Vries in 1895. It had solutions for solitary waves. It wasn't until 1965 that Martin Kruskal and Norman Zabusky discovered solitons through the implementation of a finite difference scheme applied to the KdV equation

1.2.3 Applications of soliton

Numerous fields of physics, including hydrodynamics, plasma physics, optics, and others, employ soliton waves, which are distinct wave phenomena with a multitude of real-world uses. They support the control of river flow, nuclear fusion plasma stability, optical communication signal integrity, and particle behavior in materials. Solitons play a vital role in the comprehension and advancement of several scientific and technical fields, including biology, low-temperature physics, astronomy, and industries as diverse as engineering and data transmission. Because of their ability to maintain their energy and form while traveling across many media, solitons are useful tools for understanding and affecting complex physical systems. Their dependability and consistency are essential for real-world innovations, spurring ongoing research and advancement throughout these several fields.

Biological system

It has been found that soliton motion is synchronous and sluggish in proteins and DNA. These solitons transmit energy in molecules during chemical processes. According to a contemporary neuroscience theory, neuronal transmissions resemble soliton signals. In the brain, they appear as information waves. According to this notion, soliton mobility and communication in the brain's networks depend on them. It provides new insights into how the brain works and processes information [19, 20].

Fluid dynamics

To explain wave behavior in fluid flows, especially in shallow water and open-channel flows, soliton theory is necessary [37]. They are essential for optimizing the management of water resources, forecasting floods, and comprehending river dynamics. Solitons are essential for designing effective irrigation systems and water distribution networks, as well as for enhancing agricultural and water conservation techniques. Furthermore, solitons are crucial to the development of more effective hydroelectric power plants and to the improvement of marine boat designs that lessen environmental impact and boost maritime safety.

Nonlinear Partial Differential Equations

Solitons are solutions to particular nonlinear partial differential equations (PDEs) that characterize different physical processes [34]. The sine-Gordon Equation is used in many areas, such as theoretical biology, magnetic spin waves, and crystal dislocation theory. Wave phenomena that are stable throughout time are described by soliton solutions to the sine-Gordon equation. The nonlinear Schrödinger Equation describes how complex wave fields evolve and is used in quantum mechanics and nonlinear optics. It illustrates how light pulses propagate via optical fibers, where soliton solutions aid in maintaining signal integrity over extended distances.

Nuclear physics[35]

It is conjectured that the whole nuclear wave function can have soliton-like properties at certain temperatures and energies. It is thought that these scenarios arise in the nucleus of certain stars. Under these conditions, nuclei are shown to pass one another unaffected, and even after a collision, their soliton waves do not alter. This fascinating phenomenon sheds light on the distinct dynamics and interactions occurring in these celestial objects by indicating the existence of strong and stable soliton structures within star nuclei.

Chemical Reaction Dynamics

Solitons are useful in the study of chemical reaction kinetics because they assist in clarifying how reaction fronts spread in nonlinear chemical systems. For instance, soliton-like waves may be used to illustrate the temporal and spatial development of concentration profiles in some autocatalytic processes. This can help explain the workings of intricate reaction networks.

Acoustics

Solitons are incredibly useful in studying acoustics [33], especially when examining nonlinear wave behavior. They function similarly to practical instruments for improving sound barriers and reducing noise. Underwater soliton research aids in our comprehension of sound propagation and ocean noise reduction techniques. This improves underwater communication efficiency and submarine navigation. Moreover, solitons are quite helpful in the field of medical ultrasound. They allow us to take pictures of the body without requiring any invasive procedures.

Meteorology

Solitons are crucial to studying and modeling many wave phenomena in meteorology that affect the weather and climate. They explain atmospheric gravity waves, which affect weather patterns and the jet stream by transferring energy and momentum inside the atmosphere. Additionally, Rossby waves—vast meanders in high-altitude winds—are modeled by solitons, and these waves are essential for long-term weather forecasting. At the interfaces between atmospheric layers, internal solitary waves are responsible for vertical mixing and temperature dispersion. Furthermore, solitons contribute to the understanding of atmospheric acoustic waves, which are relevant to events such as sonic booms, and bore waves linked to weather fronts, which influence local meteorological conditions. These soliton models have a major impact on climate research and weather prediction because they offer insightful information on the dynamics and propagation of atmospheric waves [36].

1.2.4 Difference between soliton and solitary waves

In the context of the KdV equation and other equations of a similar kind, the term **"solitary wave"** is typically used to describe a single soliton solution; however, when many solitary waves are present in a solution, they are referred to as **solitons**. To put it another way, when a soliton is infinitely separated from all other soliton, it transforms into a solitary wave. It is also important to note that the solitary-wave solution for equations other than the KdV equation could not be a sech function; for instance, we might encounter a sech function along with $\arctan(e^x)$. Furthermore, it should be noted that some nonlinear systems have solitary waves but not solitons, whereas others: like (the KdV equation) have solitary waves which are solitons [18].

1.3 Literature review

In 1994, two interacting waves that propagate concurrently and have nonlinear effects were described employing mathematical models known as the two-mode equations. Three important parameters—nonlinearity, linearity, and phase velocity—affect the way the waves interact. Korsunsky published one of the first examples of a two-mode equation, which is called the temporal-second-order Korteweg-de Vries (KdV) equation [21]. In the context of the Hirota-Satsuma model, Korsunsky noted the interaction of two long waves with different dispersion factors. The two-mode KdV (TMKdV) equation is an improved version of the KdV equation that Korsunsky established to account for the simultaneous motion and overlap of these waves.

$$w_{\xi\xi} + (c_1 + c_2)w_{\eta\xi} + c_1c_2w_{\eta\eta} + ((\alpha_1 + \alpha_2)\frac{\partial}{\partial\xi} + (c_2\alpha_1 + c_1\alpha_2)\frac{\partial}{\partial\eta})ww_\eta + (\beta_1 + \beta_2)\frac{\partial}{\partial\xi}+ (c_2\beta_1 + c_1\beta_2)\frac{\partial}{\partial\eta})w_{\eta\eta\eta} = 0. \tag{1.1}$$

Scaled space and time coordinates are represented in the analysis in this case by the variables ξ and η. The height of the free surface of the water above a level bottom is represented by the function $w(\eta, \xi)$. α_1 and α_2 stand for the nonlinearity parameters, while c_1 and c_2 stand for the phase velocities of the two interacting waves. Furthermore,

β_1 and β_2 represent the dispersion parameters[21]. Each mode's assessment is explained by its own KDV equation when there are no other waves present.

$$w_t + 6ww_x + w_{xxx} = 0, \tag{1.2}$$

which has solution

$$w = \frac{c}{2}\mathrm{sech}^2(\frac{\sqrt{c}}{2}(x - ct - R)). \tag{1.3}$$

Now we find the solution of equation (1.2). Let us assume a solution

$$\begin{aligned}
&w(x,t) = y(s), \quad \text{where} \;\; s = x - ct,\\
&w_t = -cy'(s),\\
&w_x = y'(s),\\
&w_{xxx} = y'''(s).
\end{aligned} \tag{1.4}$$

Putting these in equation (1.2) we get

$$-cy'(s) + y(s)y'(s) + y'''(s) = 0. \tag{1.5}$$

By integrating above

$$-cy(s) + \frac{6}{2}(y(s))^2 + y''(s) = d, \tag{1.6}$$

where d is an integration constant. Multiply above equation with $y'(s)$ we get

$$-cy(s)y'(s) + 3(y(s))^2(y'(s)) + y''(s)y'(s) = dy'(s). \tag{1.7}$$

Again integration, we get

$$\begin{aligned}
&-c\frac{1}{2}(y(s))^2 + \frac{3}{3}(y(s))^3 + \frac{1}{2}(y'(s))^2 = dy(s) + b,\\
\text{or} \quad &-c\frac{1}{2}(y(s))^2 + (y(s))^3 + \frac{1}{2}(y'(s))^2 = dy(s) + b,
\end{aligned} \tag{1.8}$$

where b is integration constant

$$y,\ y',\ y'' \to 0, \quad as \;\; s \to \pm\infty, \tag{1.9}$$

Note: $w(x,t) = y(x-ct) = y(s), \quad s = x-ct$, is called solitory wave solution as $s \to \pm 1$.

By (1.6) and (1.8) we get $d = 0$, and $b = 0$. Thus from (1.8)

$$-c\frac{1}{2}(y(s))^2 + (y(s))^3 + \frac{1}{2}(y'(s))^2 = 0, \tag{1.10}$$

$$\implies (y'(s))^2 = (y(s))^2(c - 2y(s). \tag{1.11}$$

By taking square root on both side we get

$$\frac{dy}{ds} = y(s)\sqrt{c - 2y(s)}, \tag{1.12}$$

here we use only the -ve sign for computational convenience.

$$\int ds = -\int \frac{dy}{y(s)\sqrt{c - 2y(s)}}. \tag{1.13}$$

By integrating we get

$$s = -\int \frac{dy}{y(s)\sqrt{c - 2(y(s)}} + A, \tag{1.14}$$

where A is an integration constant. Let us assume

$$y = \frac{c}{2}\text{sech}^2\theta, \tag{1.15}$$

$$dy = c\ \text{sech}^2\theta \tanh\theta d\theta. \tag{1.16}$$

By putting these values in (1.14) we get

$$s = \int \frac{c\ \text{sech}^2\theta \tanh\theta d\theta}{\frac{c}{2}\text{sech}^2\theta\sqrt{(c - c\ \text{sech}^2\theta)}} + A, \tag{1.17}$$

$$s = \int \frac{c\ 2\tanh\theta d\theta}{\sqrt{(c\ \ \tanh\theta)}} + A, \tag{1.18}$$

$$s = \frac{2}{\sqrt{c}}\int d\theta + A, \tag{1.19}$$

$$s = \frac{2}{\sqrt{c}}\theta + A, \tag{1.20}$$

$$\frac{\sqrt{c}}{2}(s - B) = \theta. \tag{1.21}$$

By putting in (1.15), we get

$$y(s) = \frac{c}{2}\mathrm{sech}^2(\frac{\sqrt{c}}{2}(s - A)), \tag{1.22}$$

or

$$y(x - ct) = \frac{c}{2}\mathrm{sech}^2(\frac{\sqrt{c}}{2}(x - ct - A)), \tag{1.23}$$

$$w(x,t) = \frac{c}{2}\mathrm{sech}^2(\frac{\sqrt{c}}{2}(x - ct - A)), \tag{1.24}$$

which is the required solution and is called solitons solution. By taking $+$ve sign in Eq. (1.12), we get soliton in cosine hyperbolic form.

By using transformation

$$\begin{aligned} &\eta = (\beta_1 + \beta_2)^{-\frac{1}{2}}(x - ct),\ \xi = (\beta_1 + \beta_2)^{-\frac{1}{2}}t, \\ &c_0 = \frac{1}{2}(c_1 + c_2),\quad \theta(\zeta, T) = (\alpha_1 + \alpha_2)y(x,t). \end{aligned} \tag{1.25}$$

The TMKdV equation (1.1) has been modified, resulting in a new form expressed as follows:

$$\theta_{tt} - s^2\theta_{xx} + (\frac{\partial}{\partial t} - \alpha s\frac{\partial}{\partial x})(\theta\theta_x) + (\frac{\partial}{\partial t} - \beta s\frac{\partial}{\partial x})(\theta_{xxx}) = 0, \tag{1.26}$$

where

$$s = \frac{1}{2}(c_1 - c_2),\quad \alpha = \frac{\alpha_1 - \alpha_2}{\alpha_1 + \alpha_2} \leq 1,\quad \beta = \frac{\beta_1 - \beta_2}{\beta_1 + \beta_2} \leq 1,\quad c_2 \leq c_1, \tag{1.27}$$

where $\theta = \theta(\zeta, T)$ represents the unknown field function. The parameters β and α correspond to the dispersion and nonlinearity parameters, respectively, both of which are bounded by 1. The parameter s represents the phase velocity, which indicates the overlapping of two moving waves. It is important to note that when $s = 0$, there is no overlapping, and integrating the equation once with respect to time t leads to the well-known KdV equation, which is used to describe the motion of a single wave. This modified form of the TMKdV equation has been validated and studied with the specific conditions mentioned above.

The above equation possesses at least two conserved conserved quantities,

$$\frac{\partial T}{\partial t} + \frac{\partial X}{\partial x} = 0, \tag{1.28}$$

where X and T are conserved flux and conserved density respectively. Neither one involves derivative w.r.t time (t), is called a consumption law. This means X and T may depend on $x,\ t,\ \theta,\ \theta_x$ but not on θ_t. For most equations density flux pairs are polynomials in θ and derivative of θ w.r.t ζ . For polynomial type T and ζ, integration of (1.28) is :

$$P = \int_{-\infty}^{\infty} T dx = \text{constant}, \tag{1.29}$$

provided that ζ vanishes at infinity.

$$w_t - 6ww_x + w_{xxx} = 0, \tag{1.30}$$

integrating

$$T = w, \quad \zeta = w_{xx} - 3w^2,$$
$$\int w dx = \text{constant}.$$

Multiply (1.30) by w we get

$$\frac{\partial}{\partial t}(\frac{1}{2}w^2) + \frac{\partial}{\partial x}(ww_{xx} - \frac{1}{2}(w_x)^2 - 2w^3) = 0. \tag{1.31}$$

Which gives the second law of conservation

$$\int_{-\infty}^{\infty} w^2 dx = \text{constant}. \tag{1.32}$$

Multiplying (1.32) by $3w^2$

$$3w^2(w_t - 6ww_x + w_{xxx}) = 0. \tag{1.33}$$

Multiplying partial derivative of (1.30) w.r.t x by w_x

$$w_x(w_{xt} - 6(w_x)^2 + -6ww_{xx} + w_{xxxx}) = 0. \tag{1.34}$$

Adding the last two equations, we get

$$\frac{\partial}{\partial t}(w^3 - \frac{1}{2}(w_x)^2) + \frac{\partial}{\partial x}(\frac{-9}{2}w^4 + 3w^2 w_{xx} - 6w(w_x)^2 + w_x w_{xxx} - \frac{1}{2}(w_{xx})^2 = 0. \tag{1.35}$$

This gives the third conservation law of KDV

$$\int_{-\infty}^{\infty}(w^3 - \frac{1}{2}(w_x)^2 dx = \text{constant}, \tag{1.36}$$

the existence of conservation laws has been considerad as an indication of the integrability of KDV. By substituting

$$\theta(\zeta, T) = \theta(z), \quad z = \zeta - \alpha T, \tag{1.37}$$

in (1.26) we get

$$(-\alpha s - \alpha)(\theta')^2 + (\alpha^2 - s^2)\theta'' + (-\alpha s - \alpha)\theta\theta'' + (-\beta s - \alpha)u^4 = 0, \tag{1.38}$$

$$\frac{1}{2}(\alpha s - \alpha)\theta^2 + (\beta s - \alpha)\theta'' + (\alpha^2 - s^2)\theta + D = 0, \tag{1.39}$$

$$\implies (\beta s - \alpha)\theta'' + \frac{1}{2}(\alpha s - \alpha)\theta^2 + (\alpha^2 - s^2)\theta + D = 0, \tag{1.40}$$

$$s(\alpha - \beta)\theta'' + \frac{1}{2}(\alpha s^2 - \alpha s)\theta^2 + (\alpha^2 s^2 - s^2)\theta + D = 0, \tag{1.41}$$

$$\frac{1}{s}(s(\alpha - \beta)\theta'' + \frac{1}{2}(\alpha s^2 - \alpha s)\theta^2 + (\alpha^2 s^2 - s^2)\theta + D) = 0, \tag{1.42}$$

$$(\beta - \alpha)\theta'' - \frac{\alpha}{2}(s - \alpha)\theta^2 + s(1 - \alpha^2)\theta + D^* = 0. \tag{1.43}$$

By solving this we get

$$\theta = e^{\frac{\sqrt{s(\sqrt{\alpha^2 - 1})z}}{(-\beta + \alpha}}) + e^{\frac{\sqrt{s}(\sqrt{\alpha^2 - 1})z}{\sqrt{(-\beta + \alpha)}}} + \frac{D(-\beta + \alpha)}{s(\alpha^2 - 1)}. \tag{1.44}$$

The 2017 work by Abdul-Majid Wazwaz [24] on the two-mode Korteweg-de Vries (TKdV) equation made a substantial contribution to our knowledge of nonlinear wave interactions, in particular the two different wave modes propagating simultaneously. When defining situations in stratified fluids and plasmas with many ion species, the TKdV equation is essential. The significance of dispersion parameters as well as nonlinearity in the soliton solution creation process was emphasized by Wazwaz's work. The TKdV equation admits several soliton solutions for certain parameters' values, providing insights into the presence and interaction of various wave modes in a nonlinear medium. Wazwaz effectively generated multi-soliton solutions using the simplified Hirota's approach, revealing the conditions required for these phenomena. Furthermore, additional precise solutions with different physical structures were derived by using the tanh/coth and tan/cot approaches, expanding the range of wave behaviors that the TKdV equation models.

In the same year M. Syam et al. study [22] focused on examining a set of coupled dual-mode modified Korteweg-de Vries (mKdV) equations. They examined this problem using a streamlined bilinear approach to determine the prerequisites for solving it. The study emphasized the essential conditions for getting N soliton solutions while examining the characteristics of numerous solitary soliton and multiple soliton solutions throughout the system. The researchers also employed a trigonometric function-based technique to test their methodology, and this produced 27 unique solutions for the system. Crucially, these answers matched those found with the simpler bilinear technique, demonstrating the efficacy of their suggested methodology. This study presented fresh research and offered the possibility of using the same methodology to examine additional connected systems.

The differential-difference Jacobi elliptic functions sub-equation approach was employed in 2021 by Fendzi-Donfack et al.[25] to get accurate solutions for a nonlinear electrical circuit that had inherent fractional-order components, which were identified via Riemann-Liouville derivatives. They achieved a variety of soliton solutions, such as singular wave trains, singular kink-type solitons, doubly periodic solitons, and grey and anti-grey soliton-like structures, by employing the method of unified symbolic computing. Their results, built using the three Jacobi elliptic functions, provide accurate solutions for hyperbolic, trigonometric, exotic, and doubly periodic fractions, all of which had not before been published in the model under study. In particular, the work revealed unique propagative modes through the cn, dn, and sn Jacobi elliptic functions for the fractional nonlinear electrical passband circuit, revealing new exotic soliton-like solutions. Additionally, their study demonstrated the possibility of the Jacobi elliptic functions sub-equation approach for differential-difference to find more soliton solutions in other nonlinear discrete systems. By broadening the range of precise solutions for intricate nonlinear models, this novel method makes a substantial contribution to the field; especially when it comes to electrical circuits with fractional-order features.

A unique dual-mode generalization of the Cahn–Allen equation was presented in 2022, and its mathematical features were investigated by M. Alquran and R. Alhami [31]. Three mathematical approaches were applied to this novel model: Kudryashov expansion, rational sin/cos method, and rational sinh/cosh method. These analyses yielded many accurate traveling solutions. Significantly, it was discovered that these solutions represented symmetric bidirectional waves with distinct features such as peakon-soliton, kink-periodic, convex-periodic, and kink-soliton patterns. This work shows how flexible two-mode equations may be, indicating that they could be used to convey information between multiple places while maintaining their basic physical characteristics. There are several possible applications for the two-mode waves in these equations. It may make the connection between mathematical concepts and real-world requirements, particularly in terms of information transmission.

In the same year, Lu Tang [32] studied the dual-mode nonlinear Schrödinger equation with Kerr law nonlinearity, focusing on its dynamical behavior and optical solitons. He analytically generated dark soliton solutions and singular soliton solutions using the dynamical system's bifurcation approach. Using the whole discriminant technique and symbolic computation, the study also obtained various more soliton solutions, such as rational function solutions and several families of Jacobian elliptic solutions. These techniques have yielded answers that not only supplement and maybe enhance earlier findings but also hold promise for stimulating more investigation into novel complicated physical phenomena within this system.

In 2023, Badar e Alam [26] carried out a thorough investigation to get explicit solutions for the dual-mode nonlinear Schrödinger equation (NLSE) utilizing the Kudryashov technique, the tanh-coth approach, and the extended exponential function expansion scheme. These sophisticated mathematical methods played a key role in revealing the complex dynamics controlling the propagation of waves in both directions when phase velocity is considered. The research yielded significant insights into the behavior of solitons in nonlinear optics through rigorous analytical inspection combined with in-depth 2D and 3D graphical analysis. Furthermore, the consequences encompassed a wide range of disciplines, including plasma physics, Bose-Einstein condensates, and other areas where dual-mode equations are essential for simulating complex wave events and developing theoretical models. In addition to improving our basic understanding of wave dynamics, this work creates new opportunities for investigation and application in a variety of scientific fields.

In the current year, Ahmed et al. [27] reformulated the nonlinear Schrödinger equation (NLSE) into a dual-mode framework using the improved modified extended tanh-function (IMETF) technique. Weierstrass elliptic doubly periodic solution, singular periodic solution, singular brilliant, rational, exponential, and Jacobi elliptic function (JEF) were among the kinds of explicit, accurate solutions that could be obtained using this approach. Their investigation centered on the geometric assessment and characterization of this new model, highlighting its significance in comprehending soliton propagation in nonlinear optics. The study illustrated the originality and significant contribution of their method to the state of nonlinear dynamics research by contrasting the traveling wave solutions they got with already available literature. Their method's effectiveness points to possible uses in various nonlinear problems in several domains, especially in soliton theory, where the model they investigated has broad relevance. Moreover, extensive 2D and 3D graphics were used to explain the behaviors of these solutions, giving a thorough visual comprehension of their dynamics. This graphic aid improves understanding of the derived solutions' unique features, which enriches the study's conclusions and consequences.

This year, a study by Sadiq et al. [28] introduced an analysis of the two-mode Gardner equation which is derived from the ideal fluid model. Their goal was to arrive at precise answers by applying two different approaches: the tan / cot method and the tanh / coth method.. Using these techniques, the researchers offered a thorough grasp of the problem from several angles. They found periodic, kink, and single solutions. 3D and 2D graphics were used to show their findings. These graphics illustrated how the phase velocity parameter affected the solutions, which improved our comprehension of the wave dynamics in the dual-mode Gardner equation. In contrast to conventional models that depict the movement of a solitary wave, this dual-mode equation takes into consideration the concurrent movement of two waves.

For a deeper understanding, using distinct methodologies, we have expanded Sadiq et al.'s [28] analysis. In particular, we used the Kudryashov approach and the Jacobi elliptic function method. We were able to acquire a greater range of solutions, such as periodic, kink, and singular kink solutions, by using these sophisticated analytical approaches. To ensure the accuracy and correctness of the governing equation, all obtained solutions for the two-mode Gardner equation were validated through direct substitution. This process served as a means to verify the solutions and provide confidence in the reliability of the proposed model.

The two-mode Gardner equation's wave dynamics and phase velocity are better understood because of the 2D and 3D visualizations. My research's findings highlight the solutions' diversity and range of reliance on different factors. Also, it advances knowledge about soliton propagation in optical fibers and shallow water by investigating the impact of phase velocity.

The structure of the thesis is as follows:

- In chapter 2, the description of the Jacobi elliptic functions technique and Kudryashov method are illustrated.
- Chapters 3 consists of the applications of the Jacobi elliptic functions method and Kudryashov method, to address the Gardner equation which is derived from

the ideal fluid model.

- In chapter 4, a conclusion is drawn.

Chapter 2

Methodologies

The approaches used to expand the study of the two-mode Gardner equation obtained from the ideal fluid model are described in this chapter. To get precise solutions, two main techniques were applied: the Kudryashov approach and the Jacobi elliptic functions technique. The chapter describes each method's procedures and mathematical framework.

2.1 Description of the Jacobi elliptic functions technique

The Jacobi elliptic functions approach is one efficient way to create accurate solutions to nonlinear differential equations. Let's consider a generic nonlinear partial differential equation.

$$R(p,\ p_x,\ p_t,\ p_{tt},\ p_{xt}, p_{xx}, ...) = 0. \tag{2.1}$$

In the above equation, R consists of polynomials that involve the function $p(x,t)$, its partial derivatives, and nonlinear terms. The subscripts denote the partial derivatives of p(x, t) with respect to x and t.

The following is an outline of the major steps;

- **Initial step:** Examine the transformation;

$$p(x,t) = g(\xi), \quad \xi = x - ct. \tag{2.2}$$

Where c represents the speed of a wave. By using the substitution, We will turn equation (2.1) into an ordinary differential equation (ODE) as follows:

$$R(g,\ g',\ g'', g''', ...) = 0. \tag{2.3}$$

R is a polynomial that includes the function $g(\xi)$ and its ordinary derivatives (denoted as $' = d/d\xi$).

Step 2:

We suppose the Jacobi elliptic function method solutions of (2.3) in the following form [29]:

$$g(\xi) = \sum_{i=0}^{n} a_i \text{JacobiSN}^i(\xi). \tag{2.4}$$

Where $\xi = x - ct$, $\text{sn}\xi = \text{sn}(\xi|k)$, and k is called a modulus. The parameters $a_0,\ a_1$ and c are to be determined.

Step 3: The number "n" is positive. We can use the balancing principle to find the value of n: by equating the non-linear terms with the highest-order derivative terms in Eq. (2.3).

Step 4: Upon substituting Eqs. (2.4), and its derivatives when required into Eq. (2.3), we derive polynomials that encompass terms involving powers of $\text{sn}^i(\xi)$. We obtain a set of algebraic equations by collecting the coefficients of these polynomials and setting them to zero. Solving this system using Maple with different methods provides unique exact solutions for each method.

2.2 Description of Kudryashov method

A strong technique to figure out precise solutions to nonlinear differential equations is the Kudryashov method. It entails breaking down a problem into algebraic equations and expressing the solution in a certain way.

Consider a generic nonlinear partial differential equation:

$$R(p,\ p_x,\ p_t,\ p_{tt},\ p_{xt}, p_{xx}, ...) = 0. \tag{2.5}$$

Where, R consists of polynomials that involve the function $p(x,t)$, its partial derivatives, and nonlinear terms. The subscripts denote the partial derivatives of $p(x,t)$ to x and t

The following is an outline of the major steps;

- **Initial step:** Examine the transformation wave transformation;

$$p(x,t) = g(\xi), \quad \xi = x - ct, \tag{2.6}$$

where c represents the speed of a wave. By using the substitution, We will turn equation (2.1) into an ordinary differential equation (ODE) as follows:

$$R(g,\ g',\ g'', g''', ...) = 0. \tag{2.7}$$

R is a polynomial that includes the function $g(\xi)$ and its ordinary derivatives (denoted as $' = d/d\xi$).

Step 2:

The following form represents our assumption for the Kudryashov method solutions of Eq. (2.3)[30]:

$$g(\xi) = \sum_{i=0}^{n} a_i Y^i(\xi) + \sum_{i=1}^{n} b_i Y^i(\xi)^{-1}. \tag{2.8}$$

Assuming that the constants a_i (where $i = 0, 1, ..., n$) are determined algebraically, with the condition $a_n \neq 0$,

whereas,

$$Y' = \delta Y(\xi)(Y(\xi) - 1). \tag{2.9}$$

The solution of (2.9) is;

$$Y(\xi) = a_0 + \frac{a_1}{1 + de^{\delta(-ct+x)}} + b_1(1 + de^{\delta(-ct+x)}), \quad d \neq 0. \tag{2.10}$$

d is free constants. $a_0,\ \delta,\ a_1$ and b_1 are to be identified.

Step 3: The number "n" is positive. We can use the homogeneous balancing method to find the value of n: by equating the non-linear terms with the highest-order derivative terms in Eq. (2.3).

Step 4: Upon substituting Eqs. (2.8),(2.9), (2.10) and their derivatives into Eq. (2.3), we derive polynomials that encompass terms involving powers of $Y(\xi)$. We obtain a set of algebraic equations by collecting the coefficients of these polynomials and setting them to zero. Solving this system using Maple with different methods provides unique exact solutions for each method.

Chapter 3

Soliton solutions for the two-mode Gardner equation through the Jacobi elliptic function and Kudryashov method

This chapter uses the Jacobi elliptic functions approach and the Kudryashov technique. The topic of our study is the Gardner equation, which is well-known in mathematical physics and is derived from an ideal fluid model. Our objective is to provide a sophisticated and useful solution. The primary goal of this work is to explore the two-wave dynamics in the context of GE and analyze how they interact with different parameter values. This model's theoretical foundation was presented and described in a paper by [39].

$$p_t + p_x + \frac{3}{2}\,\alpha\, pp_x - \frac{3}{8}\,\alpha^2 p^2 p_x + \beta\,\left(\frac{1-3\tau}{6}\right) p_{xxx} = 0. \tag{3.1}$$

The above equation, essential to the study, is used to examine how some non-dimensional parameters behave: the aspect ratio $\alpha = a/H$, the wavelength parameter $\beta = H^2/L^2$ and the bond number $\tau = T/\rho g H^2$. These parameters are crucial for characterizing the physical properties and stability of the system under various conditions. Additionally, understanding these non-dimensional quantities aids in comparing theoretical predictions with experimental observations, thereby validating the model's accuracy and applicability. Here, g is the gravitational acceleration, H is the average upstream depth, and a and L are typical wave amplitude and wavelength values, respectively. In addition, ρ is the density of water, and T is the coefficient for surface tension.

3.1 Two-mode Gardner equation

In this study, we initially reformulate Eq. (3.1) within a two-mode framework. This expansion is accomplished by utilizing the reformulated versions of Eqs. (1.26) and (1.27), as described below:

$$p_{tt} - s^2 p_{xx} + \left(\frac{\partial}{\partial t} - \eta s \frac{\partial}{\partial x}\right)\left(\frac{3}{2}\alpha p p_x - \frac{3}{8}\alpha^2 p^2 p_x\right) + \left(\frac{\partial}{\partial t} - \mu s \frac{\partial}{\partial x}\right)\left(\frac{\beta(1-3\tau)}{6} p_{xxx} + p_x\right) = 0. \tag{3.2}$$

The term p_t has been modified to $p_{tt} - s^2 p_{xx}$, where s represents the phase velocity. In the second and third mappings, the manipulation involves the operator $(\frac{\partial}{\partial t} - \eta s \frac{\partial}{\partial x})$ acting on non-linear terms and the operator $(\frac{\partial}{\partial t} - \mu s \frac{\partial}{\partial x})$ acting on linear terms. Here, μ acts as the dispersion parameter, while η represents the non-linearity.

In this section, we will apply the previously discussed approaches to derive new exact solutions for the two-mode Gardner equation, outlined in Eq. (3.2). A brief summary of these techniques was provided in the earlier section. Assuming that $p(x,t)$ in Eq. (3.2) is a real-valued function, we make the subsequent presumption:

$$p(x,t) = g(\xi), \qquad \xi = k(x - ct). \tag{3.3}$$

This transformation is known as a traveling wave transformation. By substituting this expression into Eq. (3.2), the subsequent ordinary differential equation (ODE) is generated:

$$\begin{aligned}
&(-\frac{3}{2}\alpha\eta s - \frac{3}{2}\alpha c)g'^2 + (\frac{3}{4}\alpha^2 c + \frac{3}{4}\eta s\alpha^2)gg'^2 \\
&+ (c^2 - c - s^2 - \mu s)g'' \\
&+ (\frac{3}{8}\eta\alpha^2 s + \frac{3}{8}\alpha^2 c)g^2 g'' + (-\frac{3}{2}\eta\alpha s - \frac{3}{2}\alpha c)gg'' \\
&+ (-\frac{1}{6}\beta\mu s - \frac{1}{6}\beta\ c + \frac{1}{2}\beta c\tau + \frac{1}{2}\beta\mu\tau)g'''' = 0.
\end{aligned} \tag{3.4}$$

The simplified form that results from integrating this differential equation and supposing that the integration constants are zero is as follows:

$$\frac{1}{8}(\alpha^2\eta s+\alpha^2 c)g^3+\frac{3}{4}(-\alpha\eta s-\alpha c)g^2+(c^2-\mu s-s^2-c)g \qquad (3.5)$$
$$+\frac{1}{6}\beta\left(3\tau\mu s+3c\tau-\mu s-c\right)g''=0.$$

We now use the Jacobi elliptic functions approach and the Kudryashov technique, as detailed in Section 2, to explore possible solutions.

3.1.1 Applications of the Jacobi elliptic functions approach

The value of n is identified by equating the dispersive term in Eq. (3.5) with the leading nonlinear term. Upon entering the value of n =1 into Eq. (2.4), the following outcome is obtained:

$$p(x,t)=a_0+a_1 JacobiSN(x-ct,k). \qquad (3.6)$$

By Eq. (3.6) is substituted into Eq. (3.2), and the coefficients of $\text{JacobiSN}^i(x-ct,k)$ for $i=0$ are compared to 5 to get the values for unidentified parameters:

Initial case: $\eta\neq\mu$

$$c=\pm s,\ a_0=\frac{2}{\alpha},\quad \mu=\frac{3\tau\beta k^2-k^2\beta+3\beta\tau-\beta-9\eta+15}{3\tau\beta k^2-k^2\beta+3\beta\tau-\beta+6}, \qquad (3.7)$$
$$a_1=\frac{2}{\alpha}\left(\sqrt{-\frac{18\beta\tau+6\beta}{3\beta k^2\tau-\beta k^2+3\tau\beta-\beta+6}}ck\right),\qquad (18\beta\tau+6\beta)\cdot(3\beta k^2\tau-\beta k^2+3\tau\beta-\beta+6)<0.$$

Thus, a kink solution is produced by this equation.

$$p_1(x,t)=\frac{2}{\alpha}\left(1+\sqrt{-\frac{18\beta\tau+6\beta}{3\beta k^2\tau-\beta k^2+3\tau\beta-\beta+6}}kJacobiSN(-ct+x,k)\right). \qquad (3.8)$$

Here is a graph of $p_1(x,t)$, that shows the separate left and right waves, and their combined plot showing both modes together.

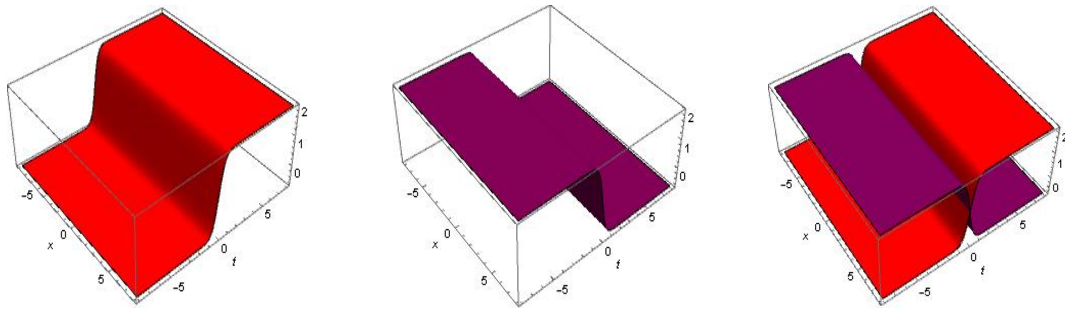


Fig.1.(a) Graphs of the left wave, right wave, and twofold waves for $p_1(x,t)$ while $s=2,\ \tau=-2,\ k=1,\ \alpha=3$ and $\beta=2$.

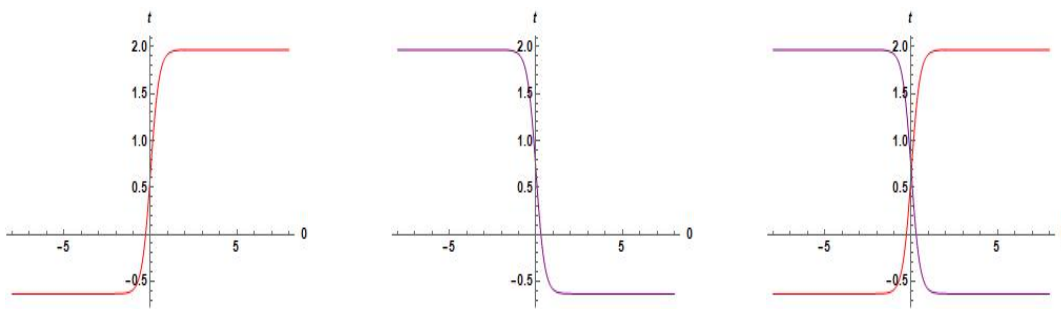


Fig.1.(b) Two-dimensional Plot of the left, right, and both waves for $p_1(x,t)$ while $s=2,\ \tau=-2,\ k=1,\ \alpha=3$ and $\beta=2$

Additionally, the variance in two-mode behavior when the phase velocity is varied is shown in both 2D and 3D graphs.

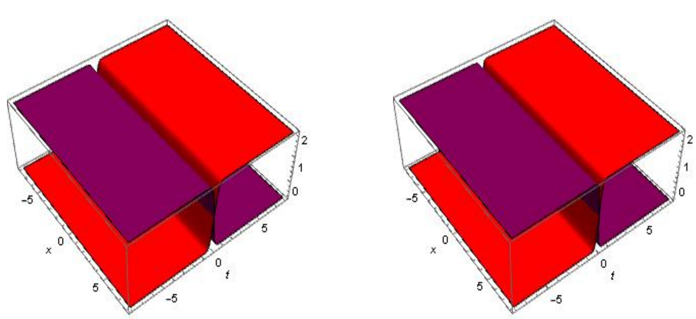


Fig.2.(a) 3D graph is plotted to visualize the two-mode waves for $p_1(x,t)$ while $\beta=2,\ \tau=-2,\ k=1,\ \alpha=3$ and $s=4$ *and* 6.

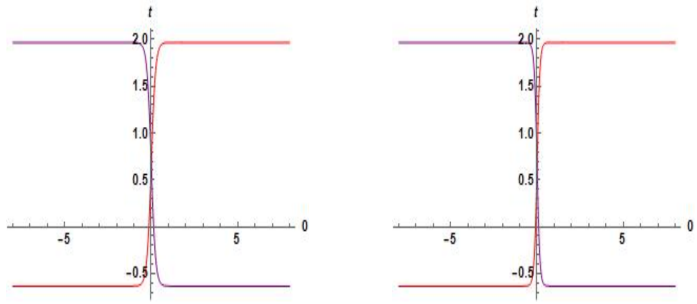


Fig.2.(b) Two-dimensional plot of the pair of waves for $p_1(x,t)$ while $\beta = 2,\ \tau = -2,\ k = 1,\ \alpha = 3$ and $s = 4$ and 6.

Second case: $\mu = \eta$

After replacing this supposition with $\mu = \eta$ in the PDE, the coefficients involving $\text{JacobiSN}^i(x - ct, k)$ for $i = 0$ to 5, are collected. Afterward, by resolving the generated system, the collection of parameters' values that are unknown is as follows:

$$a_0 = \frac{2}{\alpha},\ a_1 = \frac{2\sqrt{-2\beta\tau + \frac{2}{3}\beta}k}{\alpha}, \quad \alpha \neq 0 \quad and \quad -2\beta\tau + \frac{2}{3}\beta > 0, \tag{3.9}$$
$$c = \frac{1}{4}\tau\beta k^2 - \frac{1}{12}\beta k^2 + \frac{1}{4}\tau\beta - \frac{1}{12}\beta + \frac{5}{4} + \frac{1}{12}\sqrt{H},$$

where,

$$H = 9\tau^2\beta^2k^4 - 6\tau\beta^2k^4 + 18\tau^2\beta^2k^2 + 72\tau\beta\eta k^2 s + \beta^2k^4 - 12\tau\beta^2k^2 - 24\beta\eta k^2 s$$
$$+9\tau^2\beta^2 + 72\tau\beta\eta s + 90\tau\beta k^2 + 2\beta^2k^2 - 6\tau\beta^2 - 24\beta\eta s - 30\beta k^2 + 90\tau\beta + \beta^2$$
$$+360\eta s + 144s^2 - 30\beta + 225.$$

And

$$9\tau^2\beta^2k^4 - 6\tau\beta^2k^4 + 18\tau^2\beta^2k^2 + 72\tau\beta\eta k^2 s + \beta^2k^4 - 12\tau\beta^2k^2 - 24\beta\eta k^2 s$$
$$+9\tau^2\beta^2 + 72\tau\beta\eta s + 90\tau\beta k^2 + 2\beta^2k^2 - 6\tau\beta^2 - 24\beta\eta s - 30\beta k^2 + 90\tau\beta + \beta^2$$
$$+360\eta s + 144s^2 - 30\beta + 225 > 0.$$

Consequently, the periodic solution has been obtained.

$$p_2(x,t) = \frac{2}{\alpha} + \frac{2}{\alpha}\sqrt{-2\beta\tau + \frac{2}{3}\beta k JacobiSN(-ct + x, k)}. \tag{3.10}$$

Now, displaying the two-dimensional and three-dimensional plots of a solution, $p_2(x,t)$.

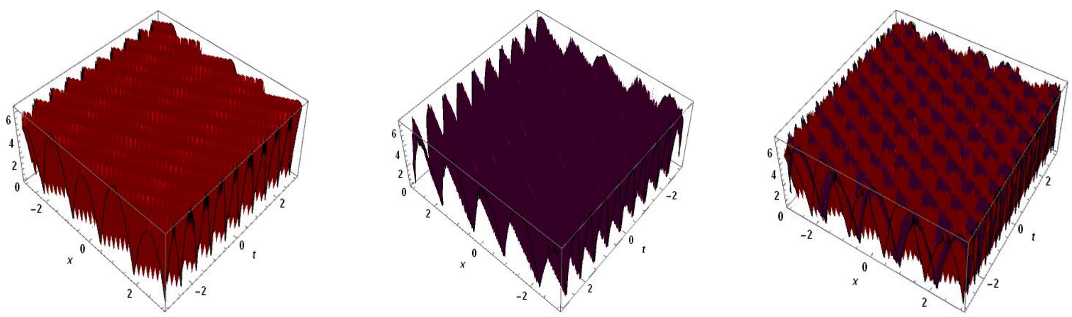


Fig.3.(a) Graphs of the left wave, right wave, and both waves for $p_2(x,t)$ while $s = 1,\ \tau = -1,\ k = 1,\ \alpha = 1,\ \eta = 0.1$ and $\beta = 0.5$.

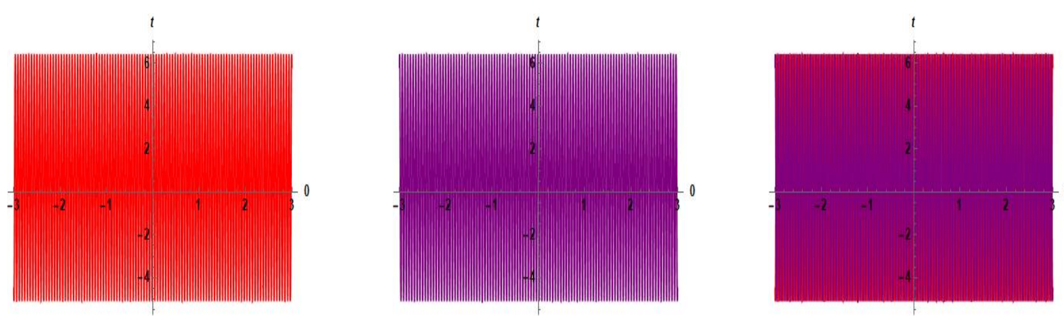


Fig.3.(b) Two-dimensional plot of the left, right, and both waves for $p_2(x,t)$ while $s = 1,\ \tau = -1,\ k = 1,\ \alpha = 1,\ \eta = 0.1$ and $\beta = 0.5$.

Following this, 3D and 2D graphs depict the variation in two-mode behavior in $p_2(x,t)$ when the phase velocity is changed.

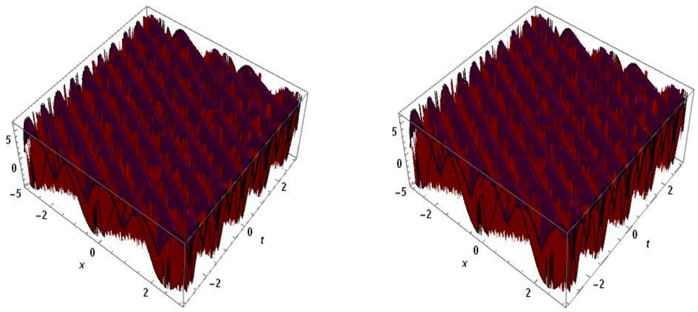


Fig.4.(a) 3D graph is plotted to visualize the combined waves for $p_2(x,t)$ while $\beta = 0.5,\ \tau = -1,\ k = 1,\ \alpha = 1,\ \eta = 0.1$ and $s = 3\ and\ 5$.

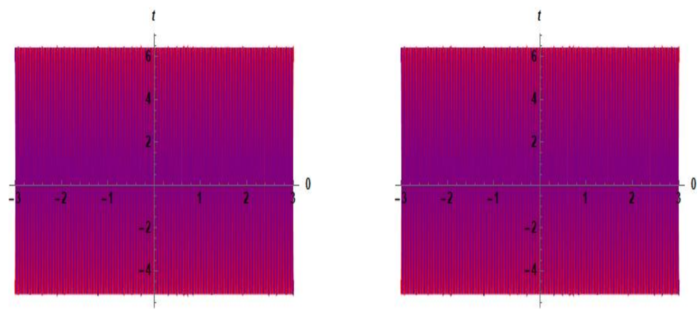


Figure 4.(b) shows the dual waves' 2D plot for $p_2(x,t)$ while $\beta = 0.5,\ \tau = -1,\ k = 1,\ \alpha = 1,\ \eta = 0.1$ and $s = 3$ and 5.

3.1.2 Applications of the Kudryashov method

Here, the analysis addresses Eq. (3.2) using the Kudryashov approach for more possible solutions. The assumed solution to Eq. (3.5) is the following expression:

$$g(\xi) = \sum_{i=0}^{n} a_i Y^i(\xi) + \sum_{i=1}^{n} b_i Y^i(\xi)^{-1}. \tag{3.11}$$

To find the appropriate index 'n' in Eq. (3.11), we put the highest nonlinear term, g^2g'' equals to the term with the highest order derivative, g'''' from (3.4). So, based on the comparison; n =1. Consequently, the following solution is obtained:

$$p(x,t) = g(\xi) = a_0 + a_1 Y(\xi) + b_1 Y(\xi)^{-1}. \tag{3.12}$$

Here the auxiliary differential equation's solution is denoted by $Y(\xi)$, where,

$$Y' = \delta Y(\xi)(Y(\xi) - 1), \tag{3.13}$$

$$Y'' = \delta^2 Y(\xi)\left(Y(\xi) - 1\right)\left(2Y(\xi) - 1\right), \tag{3.14}$$

$$Y''' = \delta^3 Y(\xi)(Y(\xi) - 1)(6Y^2(\xi) - 6Y(\xi) + 1). \tag{3.15}$$

The solution for Eq. (3.13) is;

$$Y(\xi) = a_0 + \frac{a_1}{1 + de^{\delta(-ct+x)}} + b_1(1 + de^{\delta(-ct+x)}). \tag{3.16}$$

d is free constants. a_0, δ, a_1 and b_1 are to be identified.

Eq. (3.12) and Eq. (3.13) are substituted into Eq. (3.5) and the coefficients of the same power of Y are collected. The resultant system is then figured out, yielding the values for the unidentified parameters. Now, by substituting the determined parameter values into Eq. (3.16) and then substituting the resulting equation into Eq. (3.2), we can derive the subsequent solutions. The following solutions can be derived.

First case: $\eta \neq \mu$

$$\begin{aligned} &c = \pm\mu s,\ a_0 = \frac{2}{\alpha},\ \eta = -1, \\ &a_1 = 0,\quad b_1 = \frac{\beta}{2}\ \textit{and}\ \delta = \sqrt{6}\sqrt{\frac{1}{3\tau\beta - \beta}}, \end{aligned} \tag{3.17}$$

where,

$$(1) \cdot (3\tau\beta - \beta) > 0. \tag{3.18}$$

Consequently, the acquired parameter values, when entered into Eq. (3.16), give a singular kink wave solution.

$$p_3(x,t) = \frac{2}{\alpha} + \frac{1}{2}\beta\left(1 + d\exp\left(\sqrt{6}\sqrt{\frac{1}{3\tau\beta - \beta}}(-st + x)\right)\right). \tag{3.19}$$

Here is a graph of p_3(x,t) that shows the separate left and right waves and their combined graph showing both modes together.

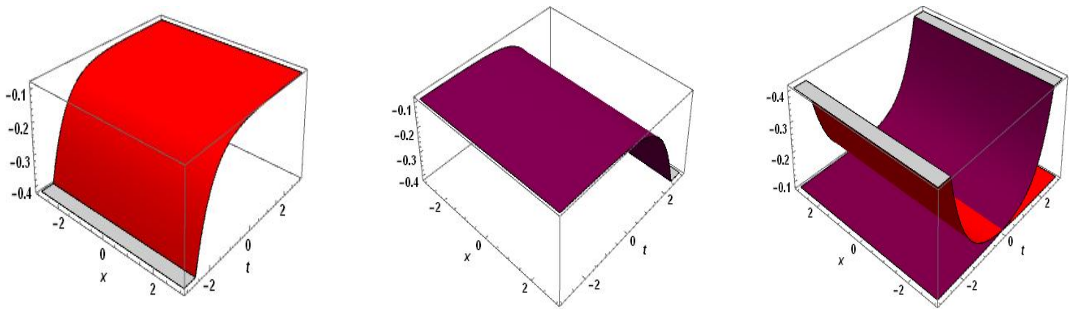


Fig.5.(a) Plot of the left wave, right wave, and combined waves for $p_3(x,t)$ while $s = 2,\ \tau = -2,\ d = 0.01,\ \alpha = 3$, and $\beta = -1.5$.

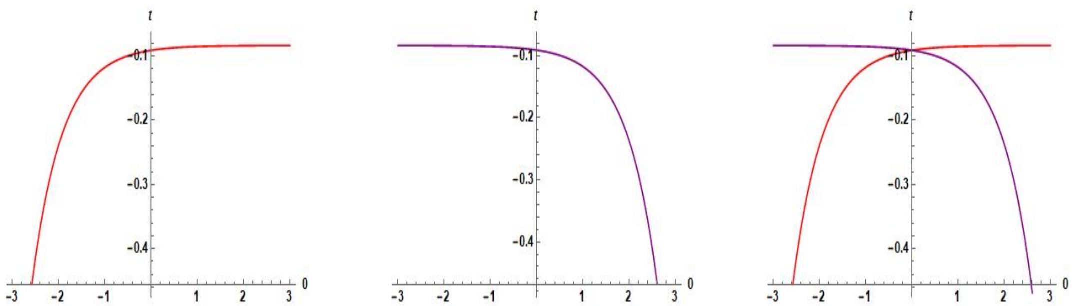


Fig.5.(b) 2D Plot of the left, right, and combined waves for $p_3(x,t)$ while $s = 2,\ \tau = -2,\ d = 0.01,\ \alpha = 3$ and $\beta = -1.5$.

Additionally, The variance in two-mode behavior when the phase velocity is varied is shown in both 2D and 3D graphs.

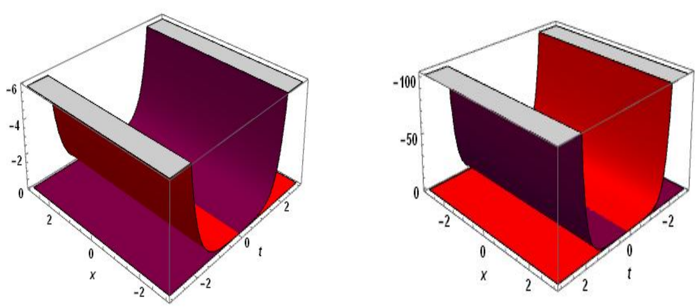


Fig.6.(a) 3D graph is plotted to visualize the combined waves for $p_3(x,t)$ while $\beta = -1.5,\ \tau = -2,\ k = 2,\ d = 0.01,\ \alpha = 3$ and $s = 4$ *and* 6.

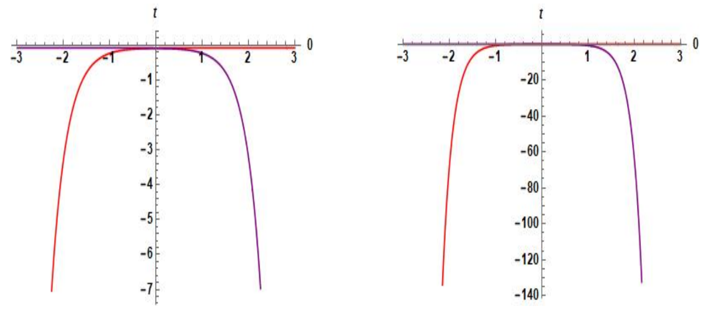


Fig.6.(b) Two dimensional plot of the combined waves for $p_3(x,t)$ while $\beta = -1.5,\ \tau = -2,\ k = 2,\ d = 0.01,\ \alpha = 3$ and $s = 4$ *and* 6.

Second case: $\mu = \eta$:

When this assumption is puted into PDE with $\mu = \eta$ the following set of solutions for the unidentified parameters results from gathering the coefficients of the terms that include $\exp(-\delta i(ct-x))$ for $i = 0$ to 7:

$$a_0 = \frac{2(\eta s + \sqrt{-2c^2\eta s + 5\eta^2 s^2 + 2\eta s^3 - 2c^3 + 10c\eta s + 2cs^2 + 5c^2} + c)}{(\eta s + c)\alpha}, \tag{3.20}$$

$$\delta = \sqrt{-\frac{-12c^2 + 30\eta s + 12s^2 + 30c}{3\tau\beta\eta s + 3\tau\beta c - \beta\eta s - \beta c}},\ b_1 = 0, \tag{3.21}$$

$$a_1 = \frac{8(2c^2 - 5\eta s - 2s^2 - 5c)}{\alpha(\eta s + c)\left(\frac{2(\eta s + \sqrt{-2c^2\eta s + 5\eta^2 s^2 + 2\eta s^3 - 2c^3 + 10c\eta s + 2cs^2 + 5c^2} + c)}{\eta s + c} - 2\right)},$$

where,

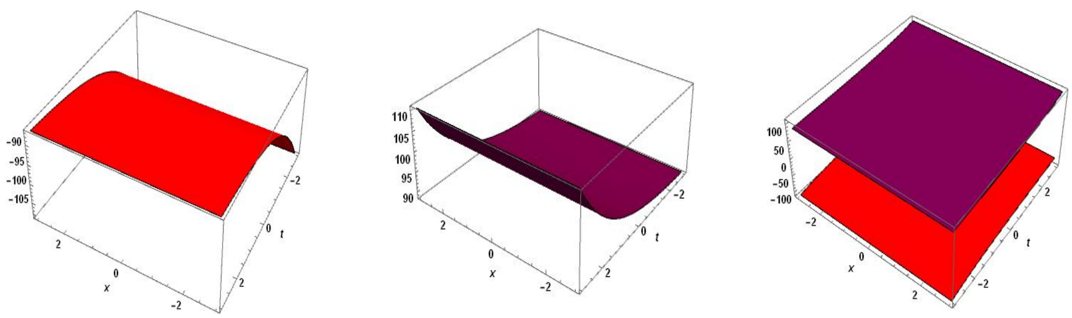
$$(-2c^2\eta s + 5\eta^2 s^2 + 2\eta s^3 - 2c^3 + 10c\eta s + 2cs^2 + 5c^2) > 0$$

$$and\ \ (-12c^2 + 30\eta s + 12s^2 + 30c)\cdot(3\tau\beta\eta s + 3\tau\beta c - \beta\eta s - \beta c) < 0,\ \ \alpha \neq 0,$$

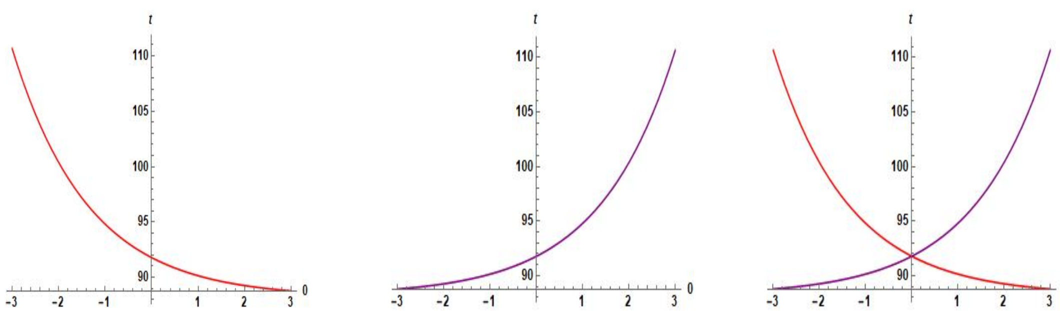
This ultimately results in the creation of a kink wave solution.

$$
\begin{aligned}
p_4(x,t) &= \frac{2(\eta s + \sqrt{-2c^2\eta s + 5\eta^2 s^2 + 2\eta s^3 - 2c^3 + 10c\eta s + 2cs^2 + 5c^2} + c)}{(\eta s + c)\alpha} \\
&+ \frac{8(2c^2 - 5\eta s - 2s^2 - 5c)}{\alpha(\eta s + c)\left(\frac{2(\eta s + \sqrt{-2c^2\eta s + 5\eta^2 s^2 + 2\eta s^3 - 2c^3 + 10c\eta s + 2cs^2 + 5c^2} + c)}{\eta s + c - 2}\right)} \\
&\times \left(1 + d\exp\left(\sqrt{-\frac{-12c^2 + 30\eta s + 12s^2 + 30c}{3\tau\beta\eta s + 3\tau\beta c - \beta\eta s - \beta c}}(-ct + x)\right)\right).
\end{aligned} \tag{3.22}
$$

Using two-dimensional and three-dimensional plots, we show the left, right, and two-mode depiction of $p_4(x,t)$.



Left, right, and two-mode wave plots for $p_4(x,t)$ while $s = 1$, $\tau = -5$, $\alpha = 2$, $\eta = 1$, $d = 0.04$, $b_1 = 0$, and $\beta = 5$ are shown in Fig. 7.(a).



The left, right, and combined wave 2D plots for $p_4(x,t)$ while $s = 1$, $\tau = -5$, $\alpha = 2$, $\eta = 1$, $d = 0.04$, $b_1 = 0$, and $\beta = 5$ are shown in Fig. 7.(b).

The subsequent graphical representations, both in 2D and 3D, illustrate how the two-mode behavioral acts as the phase velocity varies.

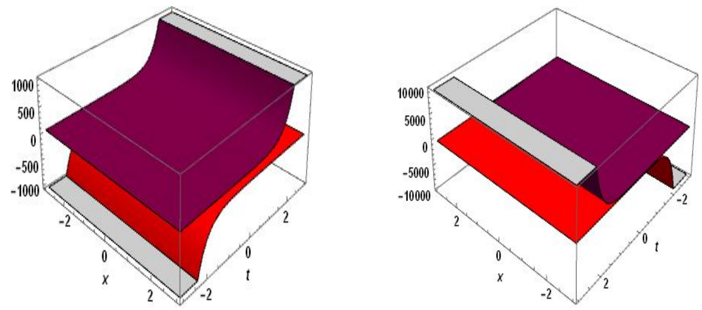


Fig.8.(a) 3D graph is plotted to visualize the combined wave for $p_4(x,t)$ while $\tau=-5$, $\alpha=2$, $\eta=1$, $d=0.04$, $b_1=0$, $\beta=5$ and $s=4$ and 6.

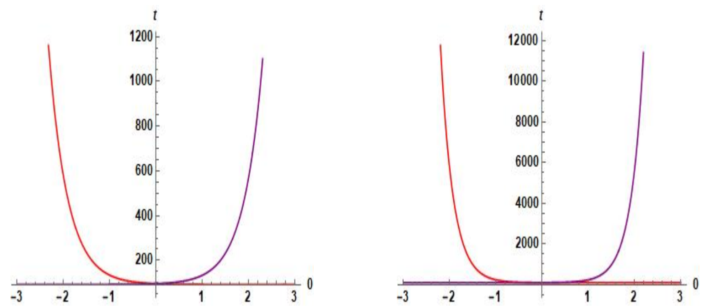


Fig.8.(b) two dimensional plots of the two-mode wave for $p_4(x,t)$ while $\tau=-5$, $\alpha=2$, $\eta=1$, d $=0.04$ $b_1=0$, $\beta=5$ and $s=4$ and 6.

3.2 Results and graphical analysis

In this study, we delve into the dual significance of the c sign, representing two unique values. The following dual representation reflects the unique two-mode motion inherent in the nonlinear equation under examination, where dual waves propagate simultaneously, specifically left-wave and right-wave. We focus on two approaches to the ideal fluid model yielded by the two-mode Gardner equation: the Jacobi elliptic function approach and the Kudryashov approach.

$p_1(x,t)$ has left, right, and two-mode graphs, which are shown in Figures 1(a) and (b). Furthermore, Figures 2(a) and (b) show the ways, phase velocity parameter s, impacts the results. The waves on the left and right merge and become much more near as the phase velocity parameter rises.

Next, Using the technique of Jacobi elliptic function, we examine the scenario where the solution $p_2(x,t)$ is acquired by equating the dispersive parameter to the nonlinearity parameter. We present two-dimensional and three-dimensional visualizations of $p_2(x,t)$ in Figures 3(a) and (b) to highlight the phase velocity's impact. The waves once again become closer to one another as s increases.

Next, we examine a case in which the dispersive and nonlinearity parameters are not identical using the Kudryashov approach. In Figures 5(a) and (b), we display two- and three-dimensional graphs of $p_3(x,t)$, which reveal the singular kink solution. We observe its effects in Figures 6(a) and (b) by varying the phase velocity.

Finally, we uncover two solutions in the last case: kink and singular waves. So we visualize these solutions through graphs of $p_4(x,t)$ in Figures 8(a) and (b). In all of our examinations, we find that the phase velocity has an effect when the values of the corresponding constants are changed.

Numerous wave characteristics, such as kink waves, singular solutions, and periodic patterns, have been identified by our research. Every single finding is significant: Periodic patterns find uses in signal processing and laser technologies, singular solutions help study black hole dynamics, and kink waves are relevant to fluid dynamics. This comprehensive analysis underscores the importance of understanding parameter effects in nonlinear wave propagation, offering practical insights across diverse scientific domains.

3.3 Advantages and limitations of the methods

Here, we go over the benefits and drawbacks of each of the two approaches. **Advantages:** A wide range of periodic solutions are offered by Jacobi elliptic functions, which help to solve several technical and physical problems. It provides precise solutions, which are essential for comprehending how nonlinear systems behave. As the modulus parameter varies, the Jacobi elliptic functions technique makes the transition from periodic to solitary wave solutions easier.

The Kudryashov technique can provide many kinds of solutions like solitons, periodic, and rational solutions. This approach is incredibly effective in locating exact solutions to nonlinear differential equations. Furthermore, the technique can be effortlessly used in Maple and Mathematica by employing polynomial forms with certain substitutions. Different kinds of nonlinearities and equations with varying orders can be handled using this technique. Moreover, in comparison to other approaches, it frequently yields results in an easy-to-understand way with fewer computing steps.

Demerits: The Jacobi elliptic functions approach can get complicated when used in higher-dimensional or highly nonlinear systems. The physical interpretation of the solutions involving Jacobi elliptic functions may not always be straightforward, especially in applied contexts.

It is not appropriate to use the Kudryashov technique to find two periodic solutions. Due to the growing algebraic complexity, the approach may not scale well to higher-dimensional systems or equations with many variables.

Chapter 4

Conclusion

This study has effectively handled the dual-mode Gardner equation (which is derived from an ideal fluid model) using the Kudryashov and Jacobi elliptic function techniques. Both approaches have enabled us to explore a range of solutions, comprising kink, kink singular, and periodic waves, thereby providing a thorough understanding of the equation's behavior. The 2D and 3D visualizations have elucidated the relationship between variations in the phase velocity parameter and the resulting resolution, enhancing our comprehension of wave dynamics. Furthermore, these findings have significant implications for optical fiber technology and shallow water wave behavior, contributing valuable insights to the scientific community. This study enriches scientific knowledge by unraveling the complexities of the two-mode Gardner equation. It paves the way for further research in related areas, ultimately advancing our understanding of wave propagation in various mediums.

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