Energy Flux of Parallel Propagating Electromagnetic Waves in Cairns Distributed Plasmas

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THESIS ACCEPTANCE CERTIFICATE

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Abstract

The energy transport of the circularly polarized electromagnetic (EM) waves is studied in a collision less Cairns distributed plasmas, whose constituents are the mobile electrons and static ions. For this purpose, the set of Vlasov-Maxwell equations are employed to derive a modified dispersion relation in terms of plasma dispersion function for the parallel propagating EM waves. The energy flux of the EM waves is studied by taking into account the wave-particle interactions caused by non-resonant $\left[\xi = \frac{\omega-\Omega}{k_{\parallel} v_{T_{\parallel}}} \gg 1\right]$ and resonant $[\xi \ll 1]$ conditions. It is found that variation of Cairns parameter, thermal speed, wave frequency, and temperature anisotropy (assuming the perpendicular temperature $T_{\perp}$ to be larger than the parallel temperature $T_{\parallel}$ with respect to ambient magnetic field) significantly influence the curves of the energy flux. Moreover, one has also been able to find the possibility of long distance energy transport as more efficient in resonant case as compared to non-resonant case. The outcome of this investigation can be helpful in understanding the wave-particle interaction in space exploration, in heating of fusion plasmas, and energy deposition in laser-produced plasmas.

Contents

Chapter 1

Introduction

1.1 Plasma

A widely held opinion is that 99% of the matter in universe is in the state of plasma; which is made up of electrified gas with the atom segregated into negative electrons and positive ions. Solids go through a phase change and become liquid as they get heated. The liquids are turned into gasses by heating them. The addition of more energy resulting in ionization of some of the atoms. The majority of matter is in the ionized state (fourth state of matter), at temperatures exceeding $100,000^o$k. If there is a way to ionize the gas and the density is low enough to prevent rapid recombination, a plasma state can also exist below the specified temperature.

Plasma can be defined as a quasineutral gas (comprises neutral and charged particles) which shows collective behaviour. We can't specify any ionized gas a plasma unless it meets the two fundamental properties: quasineutrality and collective behavior. Quasineutrality describes plasma as neutral sufficiently to consider $n_i \simeq n_e = n$. Whereas collective behavior indicates that each individual particle affects the behavior of the plasma [1,2].

1.1.1 Space Plasma

Plasmas can be found in the solar wind, solar corona, comet tails, magnetosphere, interstellar and intergalactic media, and accretion disks surrounding black holes, among other places in the Solar System and beyond. Such type of a plasma also known as astrophysical plasma [3]. On

Earth, plasmas can also be found in candle flames extended to nuclear fusion reactor interiors. High temperature and density are seen in several locations of space plasma; the Sun is the finest example of this type of plasma. The Sun is Earth's primary energy source, and all energy that is sent to Earth from the Sun comes in the form of plasma.

1.2 Debye Shielding

In the stationary state, about equal amounts of positive and negative charges per volume element are required for the plasma to behave quasineutral. A volume element of this nature needs to be sufficiently large to accommodate a good number of particles and relatively small in relation to the characteristic lengths for changes in macroscopic parameters like density n and temperature T.

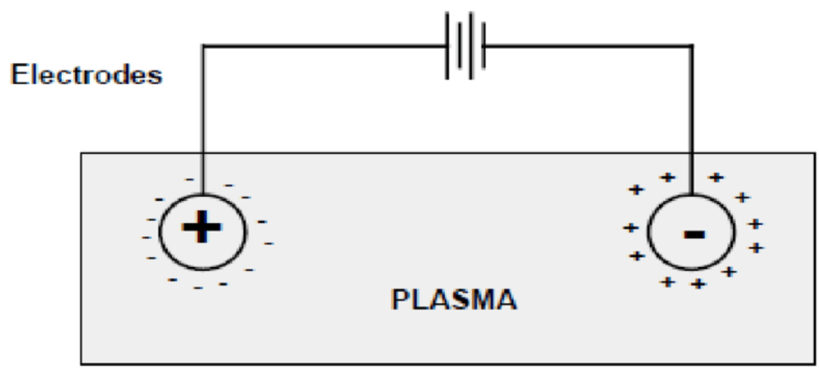

Figure 1.1. Debye Shielding

In order for the plasma to seem neutral, the electric Coulomb potential field Φ_C of every individual point charge q is shielded by other charges in the plasma and assumes the Debye potential form:

$$\Phi_D = \frac{q}{4\pi\varepsilon_o r}\exp(-\frac{r}{\lambda_D}) \tag{1.1}$$

where λ_D is the debye length; a measure of thickness of the sheath or shielding distance. Debye shielding can also happen in systems with single species as the electron streams in klystrons and magnetrons.

1.3 Criteria for Plasma

There are three necessary requirements which an ionized gas must be fulfilled to behave as plasma:

a. Quasineutrality $\lambda_D << L$

b. No. of particles in debye sphere $N_D >> 1$

c. Collision Time $\omega\tau >> 1$

The first condition shows that the λ_D must be less than the dimension of plasma. When a system's dimensions L are significantly greater than λ_D, any external potentials or localized concentrations of charge that may appear are shielded out within a shorter distance than L, preventing massive electric potentials or fields from entering the majority of the plasma. The second condition $N_D >> 1$ explains that debye sphere must have large number of particles. Debye shielding would not have any sense if there were only one or two particles in the sheath region. The third condition requires $\omega\tau >> 1$ for a gas to act like a plasma instead of a neutral gas, where ω is the typical plasma oscillation frequency and τ is the mean time between collisions with neutral atoms [1,2].

1.4 Models to Study Plasma

Numerous mathematical models are considered to study properties and associated phenomenon of plasma, e.g. fluid and kinetic model.

1.4.1 Fluid Model

Fluid model is the easiest model to understand the macroscopic behavior of plasmas ranging from laboratory experiments up to astrophysical phenomena. This model provide useful information of collective plasma behavior which helps greatly in theoretical and numerical studies. Fluid model works well in describing complex systems taking place on broader time and spatial scales compared to average particle mean free path and collision time, such as plasma confinement, plasma instabilities, etc. The commonly used equations in fluid model are Maxwell equations, equation of continuity and a momentum transport equation.

Fluid model is extended to Single Fluid Model [2]; considers plasma treated as one fluid and assumes no charge accumulation, Two Fluid Model [4,7]; distinguish between electrons and ions based on their unique dynamics, Magnetohydrodynamic (MHD) [5,7]; includes magnetic field dynamics and fit for the description of large scale plasmas. Fluid model is applicable in space physics, astrophysics and fusion research.

1.4.2 Kinetic Model

On small spatial and temporal scales, kinetic models are necessary with particle distributions that do not follow the Maxwellian distribution [6]. Despite being much more accurate than other types of models, the processing time required to simulate a macroscopic event with a kinetic model is enormous. The kinetic model precisely represents the landau damping and temperature anisotropy phenomenon. Kinetic theory uses microscopic models to understand how individual particles behave in a plasma, collisional and non-equilibrium phenomenon. Kinetic model explains the statistical behavior of particles through distribution function which depends on position and velocity and time.

$$f_{(\mathbf{r},\mathbf{v},t)} = f(x, y, z, v_x, v_y, v_z, t) \tag{1.2}$$

Kinetic model mainly encompasses Vlasov equation [7]

$$\frac{\partial f}{\partial t} + \mathbf{v}.\frac{\partial f}{\partial \mathbf{x}} + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}).\frac{\partial f}{\partial \mathbf{v}} = 0$$

By establishing the complexities of particle dynamics and collisions, kinetic model enhance our comprehension of plasma physics.

1.5 Plasma Waves

A plasma may consist of various phenomena in the form of waves. Waves in plasma physics are the basic events created by the association between charged particles and electromagnetic fields [8]. They give basic understanding of plasma behavior both on Earth and in space. According to various parameters like frequency, propagation mode, and the presence of magnetic fields, plasma waves can be classified into different types. The relation between the wave vector **k**, uniform magnetic field $\mathbf{B}_o$ and perturbed electric field $\mathbf{E}_1$ determines the direction of wave propagation [9].

1.5.1 Transverse Wave

A uniform, cold plasma lacking a magnetic field is the simplest case. Perturbations in the magnetic and electric fields can be studied in relation to the equilibrium state. The perturbation of the fields is described by the Maxwell's curl equations. In the presence of oscillating magnetic field $\mathbf{B}_1$ or ($\mathbf{B}_1 \neq 0$), the wave vector $\mathbf{k}$ would be perpendicular to the perturbed electric field $\mathbf{E}_1$ such that $\mathbf{k} \perp E_1$.

For a pure transverse case, we can write

$$\mathbf{k} \perp \mathbf{E}_1 \text{ and } \mathbf{k} \perp \mathbf{B}_1$$

since

$$\boldsymbol{\nabla} \times \mathbf{E}_1 = -\mathbf{B}_1$$

thus the wave would be electromagnetic if $\mathbf{B}_1$ is finite.

1.5.2 Longitudinal Wave

In the absence of oscillating magnetic field ($\mathbf{B}_1 = 0$), the wave vector $\mathbf{k}$ would be parallel to the perturbed electric field $\mathbf{E}_1$such that $\mathbf{k} \parallel E_1$ represents that wave is longitudinal and electrostatic as well. Langmuir waves and ion acoustic waves are the examples of electrostatic waves [10].

Ion Acoustic Waves

In an ion acoustic wave (low frequency range) the wave vector $\mathbf{k}$ is perpendicular to unperturbed magnetic field B_0 such that $\mathbf{k} \perp \mathbf{B}_0$. The motion of ions relative to electrons causes these waves in the plasma. The restoring force of moving ions generates them when they interact via electrostatic field with background electrons [8]. It describes ion fluid equations that especially dwell upon moving ions and their resultant electrostatic potential, being in the ion acoustic wave dynamics characterized sense. There are many instances where ion-acoustic waves have had notable impacts such as thermal agitation due to waves, acceleration of particles.

1.5.3 Parallel Propagating Wave

The wave is said to be parallel propagating wave when wave vector **k** is parallel to unperturbed magnetic field $\mathbf{B}_0$. [9]

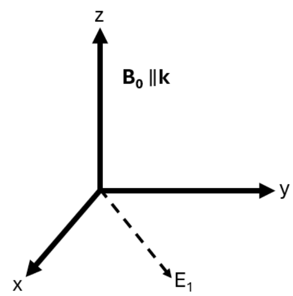

Figure 1.2. Wave geometry of parallel propagating wave

Electromagnetic waves such as radio waves, Alfven waves and R-L waves are few examples of parallel propagating waves.

1.5.4 Perpendicular Propagating Wave

If the wave vector **k** is perpendicular to the ambient magnetic field $\mathbf{B}_0$, then the wave would be known as perpendicular propagating wave. [9]

$$\mathbf{k} \perp \mathbf{B}_0$$

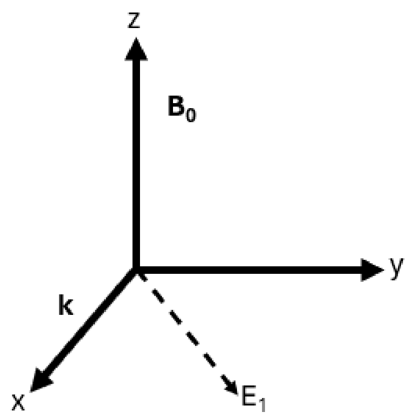

Figure 1.3. Wave geometry of perpendicular propagating wave

Extraordinary waves and magnetosonic waves are examples of perpendicular propagating waves.

1.6 Applications of Transportation of Wave Energy in Radiation Belt

Transport of wave energy takes into consideration large number of uses associated with interaction between charged particles and electromagnetic waves in space plasmas. These collaborations are very important to different sectors of astrophysics, and exploration. I have chosen a "Radiation Belt Region" to understand the energy flux of parallel propagating electromagnetic (PPEM) waves and some of the important applications of such specific region, are as follows:

1.6.1 Comprehending Wave Particle Interaction

PPEM waves have the capability to engage with charged particles of radiation belt by means of diverse mechanisms like cyclotron resonance. By employing the Cairns distribution function, we c an easily explain the behavior and distribution of such charged particles (especially electrons) inside radiation belts. Through the analysis of PPEM waves' energy flux, we can understand the ways in which electromagnetic waves affect particle interactions, particularly processes related to scattering, loss and acceleration [11].

1.6.2 Loss and Heating of Particles

PPEM waves plays a vital role in particles' loss and heating inside radiation belts. The energy flow of such waves controls the rate whereupon particles acquire or release energy from wave particle dynamics. Understanding how this loss and heating affect the overall behavior and stability of radiation belt, is also important for forecasting space weather developments [12,13].

1.6.3 Dynamics and Variance in Radiation Belts

It is clear from examining radiation belts that a variety of factors i.e, solar storm and geomagnetic disruptions, have an impact on these regions in various ways. This suggests that the degree of diversity inside radiation belts is largely determined by the way electromagnetic waves propagate along with their energy flow. Examining wave properties such as circular polarization and how they interact with Cairns dispersed plasmas can help us to comprehend the way radiation belts vary on a daily or long-term basis.[11]

1.6.4 Prediction of Space Weather

Van Allen Radiation belts are hazardous to satellites and spaceships because of the existence of severe cosmic particle bombardment within them. Expanding our understanding of wave particle interactions would be beneficial for predicting space weather, building spacecraft, and modeling radiation belts. It would also help humans to create the best designs possible (of spaceships) to lessen the impact of radiation belt' vulnerability on missions to space.[14]

1.7 Layout of Thesis

The initial section of my thesis encompasses brief introduction of plasma and its requirements. Moreover, a description of fluid and kinetic models, and an overview of plasma waves has also been discussed in later part of first section. The second section covers literature survey, and it includes the synopsis of maxwellian and non-maxwellian distribution functions, the vlasov kinetic model and the calculations of generalized dielectric tensor. The third section of my thesis explains how can we calculate the general dispersion relation of electromagnetic waves.

Furthermore, a derivation of general dispersion relation of right handed parallel propagating electromagnetic waves through cairns distribution function, calculations of the imaginary wave number by using pade approximation and plasma dispersion function for the limiting cases; non-resonant and resonant cases, are also a part of said section. The fourth section comprises results and discussion, which explains the effect of variation of cairns parameter, thermal anisotropy, thermal speed and wave frequency with respect to normalized poynting flux and distance, for both the limiting cases. At the end, conclusion of the thesis is included in section five.

Chapter 2

Literature Survey

2.1 Distribution Functions

Since it is rare for a plasma to be perfect at equilibrium, some regular techniques used in statistical mechanics cannot be used. Plasma treatment usually involves many species with different temperatures because each type of particle is nearly close to a state of local equilibrium. The conventional meaning of a thermal dynamic system is based on particle distribution function $f(\mathbf{r}, \mathbf{v}, t)$, where r depicts the coordinate in the configuration space, v represents the velocity space coordinate, and both collectively act as a phase space which is about six dimensions and also includes time. Averaged quantity "distribution function" implies that it is a kind of probability density for having particle at some volume in the phase space [8].

The distribution function comprises seven variables $f(x, y, z, v_x, v_y, v_z, t)$ and if f is normalized, it can be written as

$$\int_{-\infty}^{+\infty} f(\mathbf{r}, \mathbf{v}, t) dv = 1 \tag{2.1}$$

where dv represents three dimensional volume element in velocity space [1]. A fluid description is less complex when compared to following the movement of every single particle, but it is not as detailed as the change of f with time. Using the system's evolution to describe it does not involve looking at individual particle paths but seeking out groups of particles with the same configuration and velocity space coordinate [7]. Various types of distribution functions e.g. maxwellian and non-maxwellian distribution functions, are used to investigate the plasma environments.

2.1.1 Maxwellian Velocity Distribution Function

The Maxwellian distribution function $f(\mathbf{v})$ explains how likely it is to find a particle in a gas or plasma at temperature T moving with a velocity v. One dimensional and three dimensional [7] mathematical forms of such a distribution function are as, respectively:

$$f(\mathbf{v}) = \left[\frac{m}{2\pi k_B T}\right]^{\frac{1}{2}} \exp\left(-\frac{m|\mathbf{v}|^2}{2k_B T}\right) \tag{2.2}$$

$$f(\mathbf{v}) = \left[\frac{m}{2\pi k_B T}\right]^{\frac{3}{2}} \exp\left(-\frac{m|\mathbf{v}|^2}{2k_B T}\right) \tag{2.3}$$

The distribution of velocities attained by particles that have reached thermal equilibrium is described by the Maxwellian distribution function. This enables one to comprehend the movement of charged particles in various environments found within plasma structures. The Maxwellian distribution function is essential to compute the thermal and electrical conductivity, describing wave-particle interactions and equilibrium states. The Maxwellian distribution function can be graphically represented as:

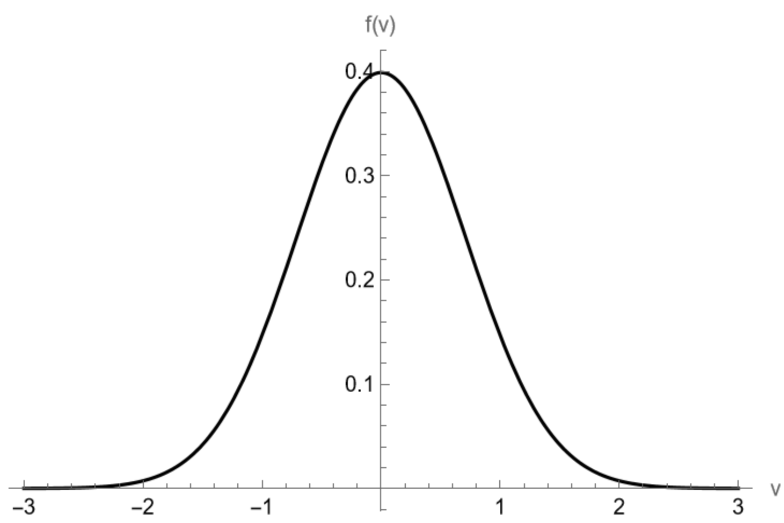

Figure 2.1. Plot of Maxwellian velocity distribution function

Bi-Maxwellian Velocity Distribution Function

The Bi-Maxwellian distribution function characterizes velocities of particles in thermal equilibrium while this distribution is expansion of Maxwellian distribution [1]. This function is a probabilistic density function which oftenly employed in plasma physics to illustrate distribution of particles' velocity, especially when considering two temperature plasmas is also not left out. Whereby particles' populations have divergent temperatures because of their heating mechanism hence it can occur under various conditions like astrophysical plasmas, laboratory plasmas or regions of space. Whether the temperature ratio ($\frac{T_\perp}{T_\parallel}$) of hot particles exceeds or falls below one, arbitrary anisotropy can be dealt with using the model bi-Maxwellian distribution function. The mathematical representation of bi-Maxwellian distribution function is as follows:

$$f(v) = \frac{1}{\pi^{\frac{3}{2}}} \frac{1}{v_{T_\parallel} v_{T_\perp}^2} \exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2} \right) \tag{2.4}$$

2.1.2 Non-Maxwellian Velocity Distribution Function

For a distribution function to be a Maxwellian, required enough collisions or various randomization processes. Many significant events have far shorter span than the period needed for a plasma to cool down to be a Maxwellian, because collisions in a hot plasmas happen sporadically. It is therefore necessary to employ a collisionless paradigm of plasma to explain such rapid processes. Due to the absence of randomization in a collisionless plasmas, entropy remains constant thus an obtained distribution function is usually known as non-Maxwellian [7]. Under these circumstances, thermodynamic equilibrium doesn't exist making most thermodynamic notion useless.

Kappa Distribution

A Kappa distribution refers to the statistical distribution followed by velocities of particle. Notably, Kappa distributions majorly represent situations of non-equilibrium wherein particles display long-tail characteristics which differs than Maxwellian distributions [15]. The distribution is determined by the Kappa parameter κ that affects the distribution's tails and shape. Mathematically, the kappa distribution function [16] can be written as:

$$f(v) = \frac{N}{(\pi\kappa)^{\frac{3}{2}}}\frac{1}{v_t^3}\frac{\Gamma(\kappa+1)}{\Gamma(\kappa-\frac{1}{2})}\left[1+\frac{1}{\kappa}\frac{v^2}{v_t^2}\right]^{-\kappa-1} \tag{2.5}$$

where κ represents kappa function / index and it is a real number. The index establishes the long tail of the distribution function's slope for the suprathermal particle energy spectrum. The kappa index must have values larger than $\frac{3}{2}$ because the distribution function is folded and thermal velocity would be undefined at $\kappa \leq \frac{3}{2}$. Moreover, kappa distribution function retrieves the Maxwellian distributions when κ approaches infinity [17,18]. Here Γ depicts gamma function and v_t shows thermal velocity wherein $v_t^2 = \frac{2k_BT}{m}$.

Tsallis Distribution

Tsallis distribution (also known as q non-extensive distribution) models systems that conform to non-extensive statistics, i.e. their behavior can not be described using the formalism of standard extensive statistics. Tsallis distribution can simulate heavy-tailed distributions. This distribution is appropriate for systems with existing correlations and clustering. q non-extensive distribution function can be defined as [19]:

$$f_o = A_q\left[1-(q-1)\frac{v^2}{v_t^2}\right]^{\frac{1}{q-1}} \tag{2.6}$$

where A_q represents normalization constant and q parameter shows extent of non-extensivity. Notably, q non-extensive distribution function retrieves the Maxwellian distributions when q approaches unity ($q = 1$) [20,21]. Hence, the value of normalization constant for $q \lessgtr 1$ can be defined as:

$$A_q = \frac{(1-q)^{\frac{3}{2}}\Gamma\left[\frac{1}{1-q}\right]}{\pi^{\frac{3}{2}}\nu_t^3\Gamma\left[-\frac{1}{2}-\frac{q}{q-1}\right]}; q < 1 \tag{2.7}$$

and

$$A_q = \frac{q\sqrt{q-1}\Gamma\left[\frac{5}{2}+\frac{1}{q-1}\right]}{\pi^{\frac{3}{2}}\nu_t^3\Gamma\left[2+\frac{1}{q-1}\right]}; q > 1 \tag{2.8}$$

Cairns Velocity Distribution

Long-tailed distributions are caused by the presence of energetic particles in space and in laboratory plasmas [22]. These distributions diverge from the Maxwellian equilibrium and could be found in the Universe's low-density plasma, where charged particle collisions are quite rare [23-24]. Cairns Distribution Function was developed to represent the tails through an abundance of particles that have high energy. A low energy component that resembles a Maxwellian is overlaid on augmented energetic tail in the Cairns distribution.

Consequently, it offers a valuable theoretical foundation for the non-Maxwellian, non-thermal class of space plasmas and has been extensively utilized to understand waves and instabilities in space plasmas. Mathematically, 3D anisotropic Cairns distribution function for plasma particles can be written as [25]:

$$f_{o\alpha} = \frac{1}{(1+\frac{15}{4}\Lambda)\pi^{\frac{3}{2}} v_{T\perp\alpha}^2 v_{T\parallel\alpha}} \left\{1+\Lambda(\frac{v_\perp^2}{v_{T\perp\alpha}^2}+\frac{v_\parallel^2}{v_{T\parallel\alpha}^2})^2\right\} \times \exp\left[-\frac{v_\perp^2}{v_{T\perp\alpha}^2}-\frac{v_\parallel^2}{v_{T\parallel\alpha}^2}\right] \tag{2.9}$$

Similarly one dimensional form of said function is as under:

$$f(v_x) = \frac{1}{v_T\sqrt{2\pi}}\frac{1}{1+3\Lambda}\left\{1+\Lambda\frac{v_x^4}{v_T^4}\right\} \times \exp\left[-\frac{v_x^2}{2v_T^2}\right] \tag{2.10}$$

where Λ is a spectral index that determines the number of particles with high energy in a system. $v_{T(\perp,\parallel)}^2 = \frac{2k_B T_{(\perp,\parallel)}}{m_\alpha}$ depicts perpendicular and parallel thermal velocities. The spectral index Λ (non-thermal parameter; $\Lambda \geq 0$) which demonstrates deflection from Maxwellian Distribution and distinguishes the overall amount of energetic non-thermal electrons [26]. At $\Lambda = 0$, Cairns distribution function transitioned to Maxwellian distribution and contort its shape at higher values; $\Lambda > 0$ [27]. 1D and 3D Cairns distribution functions can be plotted as:

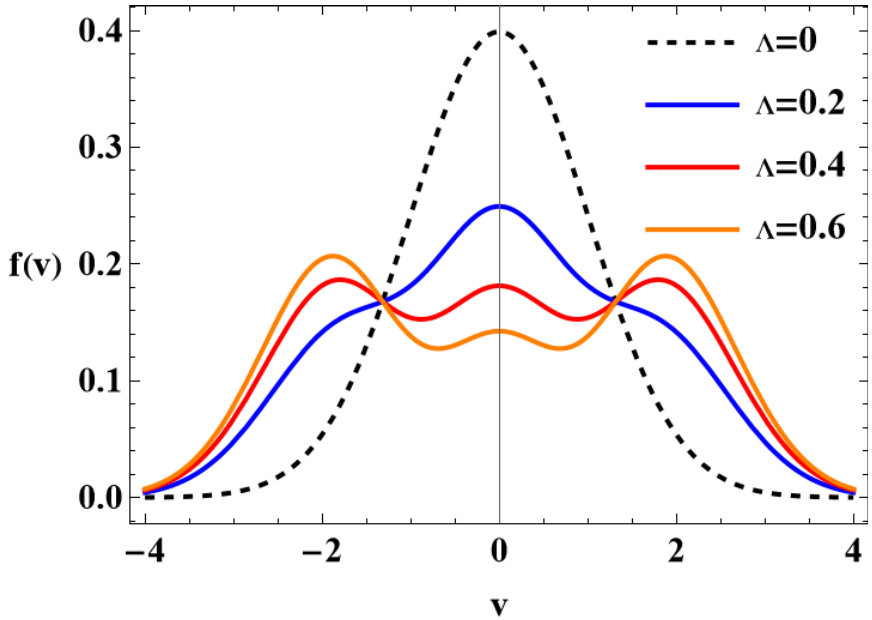

Figure 2.2. Plot of 1D Cairns velocity distribution function

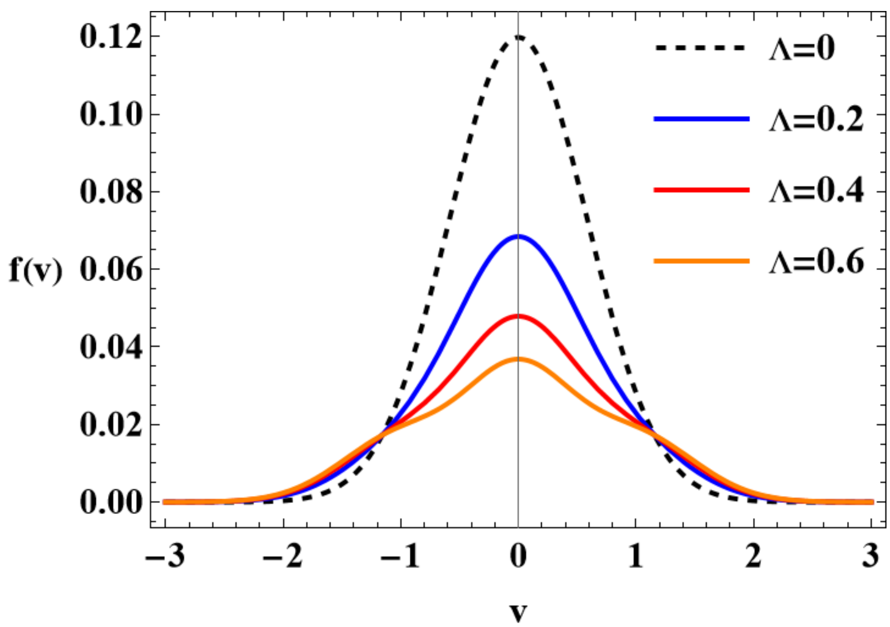

Figure 2.3. Plot of 3D Cairns velocity distribution function

Where humps in plots represents the presence of energetic particles.

2.2 The Kinetic Approach

Fluid model is the easiest model to understand the macroscopic behavior of plasmas, and is enough to explain most observed phenomena. The wave equations or magnetohydrodynamics are few examples of fluid equations that are sufficient to explain a large portion of astrophysics. However, there is certain phenomena that fluid theory is unable to explain. The distribution function $f(v)$ must be taken into consideration for such phenomena. Kinetic theory describes the development of the distribution function.

It is quite simple to understand the difference between fluid and kinetic theories [1]. The four scalar parameters are used in a function that defines the density $n = n(\mathbf{r}, t)$. In the velocity distribution function $f = f(\mathbf{r}, \mathbf{v}, t)$, there are seven distinct variables to take into consideration. The function $f(x, y, z, v_x, v_y, v_z, t)\ dv_x dv_y dv_z$ describes the number of particles in every point in space per unit volume and time. This function depends on the position of the particles as well as their speed, which is measured by three components along the three axes of coordinates. We are working in phase space because the position parameter $\mathbf{r}$, and the velocity parameter $\mathbf{v}$, are independent of one another in this situation [10]. The density of particles is represented as:

$$n(\mathbf{r}, t) = \int_{-\infty}^{\infty} f(\mathbf{r}, \mathbf{v}, t) d^3v \tag{2.11}$$

The normalized state of distribution function is:

$$\int_{-\infty}^{\infty} f(\mathbf{r}, \mathbf{v}, t) d^3v = 1 \tag{2.12}$$

therefore we can write the function as

$$f(\mathbf{r}, \mathbf{v}, t) = n(\mathbf{r}, t) \times f(\mathbf{r}, \mathbf{v}, t) \tag{2.13}$$

2.2.1 Vlasov Equation

The Boltzmann equation is satisfied by the distribution function $f(\mathbf{r}, \mathbf{v}, t)$.

$$\frac{\partial f}{\partial t} + \mathbf{v}.\boldsymbol{\nabla}\ f + \frac{\mathbf{F}}{m}.\frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t}\right)_c \tag{2.14}$$

where $\mathbf{F}$ depicts the force exerted on a particle whereas $\left(\frac{\partial f}{\partial t}\right)_c$ indicates the time rate of variation of f as a result of collisions. As the function $f(\mathbf{r}, \mathbf{v}, t)$ comprises seven distinct parameters, thus a complete derivative of function can be written as:

$$\begin{aligned} \frac{df}{dt} &= \frac{\partial f}{\partial t} + \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} + \frac{\partial f}{\partial v_x}\frac{dv_x}{dt} \\ &+ \frac{\partial f}{\partial v_y}\frac{dv_y}{dt} + \frac{\partial f}{\partial v_z}\frac{dv_z}{dt} \end{aligned} \tag{2.15}$$

by simplifications, we can write

$$\begin{aligned} \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} &= \left[\begin{array}{c} \left(\frac{dx}{dt}\mathbf{x} + \frac{dy}{dt}\mathbf{y} + \frac{dz}{dt}\mathbf{z}\right) . \\ \left(\frac{\partial f}{\partial x}\mathbf{x} + \frac{\partial f}{\partial y}\mathbf{y} + \frac{\partial f}{\partial z}\mathbf{z}\right) \end{array} \right] \\ &= \mathbf{v}.\boldsymbol{\nabla}\ f \end{aligned} \tag{2.16}$$

also

$$\begin{aligned} \frac{\partial f}{\partial v_x}\frac{dv_x}{dt} + \frac{\partial f}{\partial v_y}\frac{dv_y}{dt} + \frac{\partial f}{\partial v_z}\frac{dv_z}{dt} &= \left[\begin{array}{c} \left(\frac{\partial f}{\partial v_x}\mathbf{x} + \frac{\partial f}{\partial v_y}\mathbf{y} + \frac{\partial f}{\partial v_z}\mathbf{z}\right) . \\ \left(\frac{dv_x}{dt}\mathbf{x} + \frac{dv_y}{dt}\mathbf{y} + \frac{dv_z}{dt}\mathbf{z}\right) \end{array} \right] \\ &= \frac{\partial f}{\partial \mathbf{v}}.\frac{d\mathbf{v}}{dt} \\ &= \frac{\mathbf{F}}{m}.\frac{\partial f}{\partial \mathbf{v}} \end{aligned} \tag{2.17}$$

by using equation (2.16) and (2.17), we can write equation (2.15) as

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \mathbf{v}.\boldsymbol{\nabla}\ f + \frac{\mathbf{F}}{m}.\frac{\partial f}{\partial \mathbf{v}} \tag{2.18}$$

by comparing equation (2.14) and (2.18)

$$\frac{df}{dt} = \left(\frac{\partial f}{\partial t}\right)_c \tag{2.19}$$

here the time derivative of function f can be illustrated as convective derivative and it would be zero when there are no collisions.

thus

$$\frac{\partial f}{\partial t} + \mathbf{v}.\boldsymbol{\nabla}\ f + \frac{\mathbf{F}}{m}.\frac{\partial f}{\partial \mathbf{v}} = 0 \tag{2.20}$$

since lorentz force is

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \tag{2.21}$$

thus

$$\frac{\partial f}{\partial t} + \mathbf{v}.\boldsymbol{\nabla}\ f + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}).\frac{\partial f}{\partial \mathbf{v}} = 0 \tag{2.22}$$

The above equation is named as Vlasov equation and is generally used in kinetic model.

2.2.2 Generalized Dielectric Tensor

We will start with the relativistic form of Vlasov equation

$$\frac{\partial f_\alpha}{\partial t} + \mathbf{v} \cdot \frac{\partial f_\alpha}{\partial \mathbf{x}} + q_\alpha(\mathbf{E} + \frac{\mathbf{v} \times \mathbf{B}}{c}) \cdot \frac{\partial f_\alpha}{\partial \mathbf{p}} = 0 \tag{2.23}$$

where $\mathbf{v}$ represents the velocity and the $\mathbf{p}$ shows relativistic momentum, both terms can be co-related as

$$\mathbf{p} = \gamma m \mathbf{v}$$

and

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} = \sqrt{1 + \frac{p^2}{m^2c^2}}$$

thus

$$\mathbf{v} = \frac{c\mathbf{p}}{\sqrt{m^2c^2 + p^2}}$$

The plasma system is fully described by the relativistic Vlasov equation alongwith the curl Maxwell equations.

from Faraday's law

$$\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t} \tag{2.24}$$

from Ampere's law

$$\nabla \times \mathbf{B} = \frac{1}{c}\frac{\partial \mathbf{E}}{\partial t} + \frac{4\pi}{c}\mathbf{J} \tag{2.25}$$

by linearizing the equations (2.23 - 2.25) and performing Fourier Laplace transformation

$$(\mathbf{E}, \mathbf{B},\ f_\alpha) = \int_0^\infty dt\ e^{-st} \int_{-\infty}^{\infty} \frac{dx}{(2\pi)^3} e^{-i\ \mathbf{k}\ .\ \mathbf{x}}\ (\mathbf{E}_1, \mathbf{B}_1, \mathbf{f}_1)\ ,\ \mathcal{L}\left(\frac{\partial\left(\mathbf{E}_1, \mathbf{B}_1, \mathbf{f}_1\right)}{\partial t}\right) = s\mathcal{L}\left(\mathbf{E}_1, \mathbf{B}_1,\ f_{1\alpha}\right)$$

$$\text{and } \mathcal{F}\left(\frac{\partial\left(\mathbf{E}_1, \mathbf{B}_1,\ f_{1\alpha}\right)}{\partial t}\right) = i\ \mathbf{k}\ \mathcal{F}\left(\mathbf{E}_1, \mathbf{B}_1,\ f_{1\alpha}\right)\ ;\ s = -i\omega$$

by using the Maxwell curl equations (2.24 - 2.25), the Vlasov equation (2.23) would be

$$(s + i\mathbf{k}.\mathbf{v})\ \ f_{1\alpha} + \frac{q_\alpha}{c}(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_{1\alpha}}{\partial \mathbf{p}} + q_\alpha \left[\mathbf{E}_1 - \frac{i}{s}\mathbf{v} \times (\mathbf{k} \times \mathbf{E}_1)\right] \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{p}} = 0 \tag{2.26}$$

similarly the wave equation become

$$\left[\left(s^2 + c^2k^2\right)\delta_{ij}\ -\ c^2k_ik_j + 4\pi s\sigma_{ij}\right]E_j = \mathbf{0}$$

$$\left[\delta_{ij} + \frac{c^2k^2}{s^2}\delta_{ij} - \ \frac{c^2k_ik_j}{s^2} + \frac{4\pi s}{s^2}\sigma_{ij}\right]s^2E_j = 0$$

$$\left[\left(\delta_{ij} + \frac{4\pi}{s}\sigma_{ij}\ \right) + \frac{c^2k^2}{s^2}\delta_{ij} - \ \frac{ck_i}{s}\frac{ck_j}{s}\right]E_j = \mathbf{0}$$

Since $s = -i\omega$, so

$$\left[\left(\delta_{ij} + \frac{4\pi}{s}\sigma_{ij}\right) - N^2\delta_{ij} + N_iN_j\right] E_j = \mathbf{0}$$

therefore

$$\left[\epsilon_{ij} - N^2\left(\delta_{ij} - \frac{N_iN_j}{N^2}\right)\right] E_j = \mathbf{0} \tag{2.27}$$

here $\epsilon_{ij} \equiv \delta_{ij} + \frac{4\pi}{s}\sigma_{ij}$ represents the tensor of dielectric permittivity, whereas $N = \dfrac{ck}{\omega}$ depicts the refractive index and $N_{i,\ j} = \dfrac{ck_{i,\ j}}{\omega}$ are the i and j components of refractive index.

by considering wave vector $\mathbf{k}$ in x-z plane, we can write the equation (2.27) as

$$\frac{\partial f_{1\alpha}}{\partial\varphi} - \frac{\left(s + ik_\parallel v_\parallel + ik_\perp v_\perp \cos\varphi\right)}{\Omega} f_{1\alpha} = \frac{\Phi(\varphi)}{\Omega} \tag{2.28}$$

assuming cylindrical coordinates, we can write

$$\frac{q_\alpha}{c}(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_{1\alpha}}{\partial \mathbf{p}} = -\,\Omega\frac{\partial f_{1\alpha}}{\partial\varphi}$$

so

$$\Phi(\varphi) = q_\alpha(\mathbf{E}_1 - \frac{i}{s}\mathbf{v}\times(\mathbf{k} \times \mathbf{E}_1) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{p}}$$

and

$$\Omega_\alpha = \frac{q_\alpha B_0}{\gamma m_\alpha\, c} = \frac{\Omega_{0\alpha}}{\gamma}$$

at equilibrium point, the Vlasov equation would be time independent, thus

$$\frac{q_\alpha}{c}(\mathbf{v} \times \mathbf{B}_0) \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{p}} = 0$$

$$\Longrightarrow \frac{\partial f_{0\alpha}}{\partial\varphi} = 0$$

where f_0 depends on $p_\perp$ and $p_\parallel$ but it is independent of φ

On the other hand, we can write $\Phi(\varphi)$ as

$$\Phi(\varphi') = q_\alpha \left\{ \mathbf{E}_1 - \frac{i}{s}\, \mathbf{v} \times (\mathbf{k} \times \mathbf{E}_1) \right\} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{p}'}$$

and $\mathbf{p}'(p_\perp, \varphi', p_\parallel)$

$$\Phi(\varphi') = q_\alpha \left[\mathbf{E}_1 \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{p}'} - \frac{i}{s} \left\{ \left(\mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{p}'} \right) (\mathbf{v} \cdot \mathbf{E}_1) - \left(\mathbf{E}_1 \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{p}'} \right) (\mathbf{v} \cdot \mathbf{k}) \right\} \right]$$

$$\Phi(\varphi') = q_\alpha \left[\mathbf{E}_1 \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{p}'} \left(1 + \frac{i}{s} (\mathbf{k} \cdot \mathbf{v}) \right) - \frac{i}{s} \left(\mathbf{k} \cdot \frac{\partial f_{0\alpha}}{\partial \mathbf{p}'} \right) (\mathbf{v} \cdot \mathbf{E}_1) \right]$$

$$\Phi(\varphi') = q_\alpha \left[\begin{array}{c} \left\{ 1 + \frac{i}{s} \left(k_\perp v_\perp \cos\varphi' + k_\parallel v_\parallel \right) \right\} \left\{ (E_{1x} \cos\varphi' + E_{1y} \sin\varphi') \frac{\partial f_{0\alpha}}{\partial p_\perp} + E_{1z} \frac{\partial f_{0\alpha}}{\partial p_\parallel} \right\} \\ -\frac{i}{s} \left(k_\perp \cos\varphi' \frac{\partial f_{0\alpha}}{\partial p_\perp} + k_\parallel \frac{\partial f_{0\alpha}}{\partial p \parallel} \right) \left\{ (E_{1x} \cos\varphi' + E_{1y} \sin\varphi')\, v_\perp + E_{1z}\, v_\parallel \right\} \end{array} \right]$$

so

$$\begin{aligned} \Phi(\varphi') \;\; = \;\; & q_\alpha \left[\frac{\partial f_{0\alpha}}{\partial p_\perp} + \frac{i}{s} k_\parallel \left\{ v_\parallel \frac{\partial f_{0\alpha}}{\partial p_\perp} - v_\perp \frac{\partial f_{0\alpha}}{\partial p \parallel} \right\} \right] E_{1x} \cos\varphi' \\ & + q_\alpha \left[\frac{\partial f_{0\alpha}}{\partial p_\perp} + \frac{i}{s} k_\parallel \left\{ v_\parallel \frac{\partial f_{0\alpha}}{\partial p_\perp} - v_\perp \frac{\partial f_{0\alpha}}{\partial p \parallel} \right\} \right] E_{1y} \sin\varphi' \\ & + q_\alpha \left[\frac{\partial f_{0\alpha}}{\partial p_\parallel} - \frac{i}{s} k_\perp \cos\varphi' \left\{ v_\parallel \frac{\partial f_{0\alpha}}{\partial p_\perp} - v_\perp \frac{\partial f_{0\alpha}}{\partial p \parallel} \right\} \right] E_{1z} \end{aligned}$$

and

$$\Phi(\varphi') = q_\alpha (\chi_1 \cos\varphi', \chi_1 \sin\varphi', \chi_2 + \cos\varphi' \chi_3).E_1$$

here

$$\chi_1 = \left\{ \frac{\partial\, f_0}{\partial\, p_\perp} + \frac{i\, k_\parallel}{s} \left(v_\parallel \frac{\partial\, f_0}{\partial\, p_\perp} - v_\perp \frac{\partial\, f_0}{\partial\, p_\parallel} \right) \right\}$$

$$\chi_2 = \frac{\partial\, f_0}{\partial\, p_\parallel}$$

and

$$\chi_3 = -\frac{i\,k_\perp}{s}\left(v_\parallel \frac{\partial\, f_0}{\partial\, p_\perp} - v_\perp \frac{\partial\, f_0}{\partial\, p_\parallel}\right)$$

by solving the homogeneous portion of equation (2.28)

$$\frac{\partial f_{1\alpha}}{\partial \varphi} - \frac{\left(s + ik_\parallel v_\parallel + ik_\perp v_\perp \cos\varphi\right)}{\Omega} f_{1\alpha} = 0$$

thus the solution would be

$$G(\varphi, \varphi') = \exp\left[\frac{1}{\Omega_\alpha}\int_{\varphi'}^{\varphi} \left(s + ik_\parallel v_\parallel + ik_\perp v_\perp \cos\varphi\right) d\varphi''\right]$$

$$G(\varphi, \varphi') == \exp\left[\frac{1}{\Omega_\alpha}\left\{(s + ik_\parallel v_\parallel)(\varphi - \varphi^{'}) + ik_\perp v_\perp (\sin\varphi - \sin\varphi^{'})\right\}\right]$$

The solution to equation (2.28) can be written with the help of integrating factor approach

$$f_{1\alpha} = \frac{1}{\Omega_\alpha}\int_{\pm\infty}^{\varphi} G(\varphi, \varphi')\Phi(\varphi')d\varphi' \tag{2.29}$$

also

$$f_{1\alpha} = \frac{q_\alpha}{\Omega_\alpha}\int_{\pm\infty}^{\varphi} \exp\left[\begin{array}{c}\left(\frac{s+ik_\parallel v_\parallel}{\Omega_\alpha}\right)(\varphi - \varphi^{'}) \\ +i\frac{k_\perp v_\perp}{\Omega_\alpha}(\sin\varphi - \sin\varphi^{'})\end{array}\right] \times \mathbf{E}_1 \left[\begin{array}{c}\chi_1 \cos\varphi' \\ \chi_1 \sin\varphi' \\ \chi_2 + \cos\varphi' \chi_3\end{array}\right] d\varphi^{'} \tag{2.30}$$

Since

$$\mathbf{J} = \overleftrightarrow{\boldsymbol{\sigma}}.\mathbf{E}_1 = \sum_\alpha q_\alpha n_{0\alpha}\int \mathbf{v} f_{1\alpha} d^3p$$

thus we can write the perturbed current density as

$$
\begin{aligned}
4\pi s \overleftrightarrow{\boldsymbol{\sigma}}.\mathbf{E}_1 \;=\; & 4\pi s \sum_\alpha q_\alpha^2 n_{0\alpha} \int_{-\infty}^{\infty} dp_\parallel \int_0^\infty dp_\perp p_\perp \int_0^{2\pi} d\varphi \int_\infty^{\varphi} d\varphi' \\
& \times \frac{1}{\Omega_\alpha} \exp\left[\frac{1}{\Omega_\alpha} \left\{ \begin{array}{c} \left(s + ik_\parallel v_\parallel\right)\left(\varphi - \varphi'\right) \\ +ik_\perp v_\perp \left(\sin\,\varphi - \sin\,\varphi'\right) \end{array} \right\} \right] \\
& \times \mathbf{E}_1 . \begin{bmatrix} \chi_1 \cos\varphi' \\ \chi_1 \sin\varphi' \\ \chi_2 + \cos\varphi' \chi_3 \end{bmatrix} \times \begin{bmatrix} v_\perp \cos\,\varphi \\ v_\perp \sin\,\varphi \\ v_\parallel \end{bmatrix}
\end{aligned}
\tag{2.31}
$$

let $\beta = \varphi - \varphi'$ and by performing φ-integration, we get

$$
\begin{aligned}
& \int_0^{2\pi} d\varphi \exp[iz(\sin\varphi - \sin(\varphi - \beta))] \\
& \times \begin{pmatrix} v_\perp \cos\varphi \cos(\varphi-\beta)\chi_1 & v_\perp \cos\varphi \sin(\varphi-\beta)\chi_1 & v_\perp \cos\varphi\,(\chi_2 + \cos(\varphi-\beta)\chi_3) \\ v_\perp \sin\varphi \cos(\varphi-\beta)\chi_1 & v_\perp \sin\varphi \sin(\varphi-\beta)\chi_1 & v_\perp \sin\varphi\,(\chi_2 + \cos(\varphi-\beta)\chi_3) \\ v_\parallel \cos(\varphi-\beta)\chi_1 & v_\parallel \sin(\varphi-\beta)\chi_1 & v_\parallel(\chi_2 + \cos(\varphi-\beta)\chi_3) \end{pmatrix}
\end{aligned}
$$

also

$$
\begin{aligned}
= \;& 2\pi \sum_{n=-\infty}^{\infty} \exp[in\beta] \\
& \times \begin{pmatrix} \frac{n^2}{z^2} v_\perp \left[J_n(z)\right]^2 \chi_1 & \frac{in}{z} v_\perp J_n(z) J'_n(z)\chi_1 & v_\perp \frac{n}{z}[J_n(z)]^2 \left(\chi_2 + \frac{n}{z}\chi_3\right) \\ -\frac{in}{z} v_\perp J_n(z) J'_n(z)\chi_1 & v_\perp [J'_n(z)]^2 \chi_1 & -iv_\perp J_n(z) J'_n(z) \left(\chi_2 + \frac{n}{z}\chi_3\right) \\ \frac{n}{z} v_\parallel [J_n(z)]^2 \chi_1 & iv_\parallel J_n(z) J'_n(z)\chi_1 & v_\parallel [J_n(z)]^2 \left(\chi_2 + \frac{n}{z}\chi_3\right) \end{pmatrix}
\end{aligned}
\tag{2.32}
$$

here, we used Bessel function identities;

$$
\exp[iz\sin\varphi] = \sum_{n=-\infty}^{\infty} \exp[in\varphi] J_n(z)
$$

and

$$\exp[-\,iz\sin(\varphi-\beta)] = \sum_{m=-\infty}^{\infty} \exp[-im(\varphi-\beta)]J_m(z)$$

where $z = \frac{k_\perp v_\perp}{\Omega_\alpha}$

by performing the β-integration,

$$\int_0^{-\infty} d\beta \exp\left[\frac{1}{\Omega_\alpha}(s+ik_\parallel v_\parallel + in\Omega_\alpha)\beta\right] = \frac{-\Omega_\alpha}{s+ik_\parallel v_\parallel + in\Omega_\alpha} \tag{2.33}$$

by substituting the equations (2.32 - 2.33) in equation (2.31), we can rewrite the equation (2.27) as;

$$\left[\varepsilon_{ij} - N^2\left(\delta_{ij} - \frac{N_i N_j}{N^2}\right)\right] E_j = 0$$

thus

$$\begin{aligned}\epsilon_{ij} \;=\; & \delta_{ij} - \frac{2\pi}{s}\sum_\alpha m_\alpha\,\omega_{p\alpha}^2 \int_{-\infty}^{\infty} dp_\parallel \int_0^{\infty} p_\perp dp_\perp \sum_{n=-\infty}^{\infty} \frac{1}{s+ik_\parallel v_\parallel + in\Omega_\alpha} \\ & \times \begin{pmatrix} \frac{n^2}{z^2} v_\perp [J_n(z)]^2 \chi_1 & \frac{in}{z} v_\perp J_n(z) J'_n(z)\chi_1 & v_\perp \frac{n}{z}[J_n(z)]^2\left(\chi_2 + \frac{n}{z}\chi_3\right) \\ -\frac{in}{z} v_\perp J_n(z)J'_n(z)\chi_1 & v_\perp [J'_n(z)]^2\chi_1 & -iv_\perp J_n(z)J'_n(z)\left(\chi_2+\frac{n}{z}\chi_3\right) \\ \frac{n}{z}[J_n(z)]^2 v_\parallel \chi_1 & iJ_n(z)J'_n(z)v_\parallel\chi_1 & v_\parallel [J_n(z)]^2\left(\chi_2+\frac{n}{z}\chi_3\right)\end{pmatrix}\end{aligned} \tag{2.34}$$

Equation (2.34) can be simplified with the help of under-mentioned relation and Bessel function properties

$$\frac{v_\perp\left(\chi_2+\frac{n}{z}\chi_3\right)}{s+ik_\parallel v_\parallel + in\Omega_\alpha} = \frac{v_\parallel \chi_1}{s+ik_\parallel v_\parallel + in\Omega_\alpha} - \frac{1}{s}\left(v_\parallel \frac{\partial f_{0\alpha}}{\partial p_\perp} - v_\perp \frac{\partial f_{0\alpha}}{\partial p_\parallel}\right)$$

and

$$\sum_{n=-\infty}^{\infty}[J_n(z)]^2 = 1,\;\; \sum_{n=-\infty}^{\infty} n[J_n(z)]^2 = 0,\;\; \sum_{n=-\infty}^{\infty} J_n(z)J'_n(z) = 0$$

Therefore equation (2.34) can be simplified as

$$\epsilon_{ij} = \delta_{ij} - \frac{2\pi}{s} \sum_{\alpha} m_{\alpha} \omega_{p\alpha}^2 \int\limits_{-\infty}^{\infty} dp_{\parallel} \int\limits_{0}^{\infty} p_{\perp} dp_{\perp} \chi_1 \sum_{n=-\infty}^{\infty} \frac{M_{ij}}{s + ik_{\parallel} v_{\parallel} + in\Omega_{\alpha}} + L_{ij} \tag{2.35}$$

where

$$M_{ij} = \begin{pmatrix} \frac{n^2}{z^2} v_{\perp} [J_n(z)]^2 & \frac{in}{z} v_{\perp} J_n(z) J'_n(z) & \frac{n}{z} v_{\parallel} [J_n(z)]^2 \\ -\frac{in}{z} v_{\perp} J_n(z) J'_n(z) & v_{\perp} [J'_n(z)]^2 & -iv_{\parallel} J_n(z) J'_n(z) \\ \frac{n}{z} v_{\parallel} [J_n(z)]^2 & iv_{\parallel} J_n(z) J'_n(z) & \frac{v_{\parallel}^2}{v_{\perp}} [J_n(z)]^2 \end{pmatrix}$$

and

$$L_{zz} = \frac{2\pi}{s^2} \sum_{\alpha} m_{\alpha} \omega_{p\alpha}^2 \int\limits_{-\infty}^{\infty} dp_{\parallel} \int\limits_{0}^{\infty} p_{\perp} dp_{\perp} \left[\frac{v_{\parallel}}{v_{\perp}} \left(v_{\parallel} \frac{\partial f_{0\alpha}}{\partial p_{\perp}} - v_{\perp} \frac{\partial f_{0\alpha}}{\partial p_{\parallel}} \right) \right]$$

also

$$\chi_1 = \left\{ \frac{\partial f_{0\alpha}}{\partial p_{\perp}} + \frac{i k_{\parallel}}{s} \left(v_{\parallel} \frac{\partial f_{0\alpha}}{\partial p_{\perp}} - v_{\perp} \frac{\partial f_{0\alpha}}{\partial p_{\parallel}} \right) \right\}$$

The equation (2.35) is the representation of generalized dielectric tensor and it is applicable to any distribution function at equilibrium. The Onsager symmetric relations of generalized dielectric tensor's component are as follows

$$\epsilon_{xz} = \epsilon_{zx} \text{ , } \epsilon_{xy} = -\epsilon_{xy} \text{ and } \epsilon_{yz} = -\epsilon_{zy} \tag{2.36}$$

In a magnetized plasma, we can write the generalized dispersion relation as

$$\begin{vmatrix} \epsilon_{xx} - N_{\parallel}^2 & \epsilon_{xy} & \epsilon_{xz} + N_{\parallel} N_{\perp} \\ -\epsilon_{xy} & \epsilon_{yy} - N_{\parallel}^2 & \epsilon_{yz} \\ \epsilon_{xz} + N_{\parallel} N_{\perp} & -\epsilon_{yz} & \epsilon_{zz} - N_{\perp}^2 \end{vmatrix} = 0 \tag{2.37}$$

Chapter 3

Mathematical Model

Energy flux of parallel propagating electromagnetic waves has already been computed in Kappa distributed plasmas [28]. Based on a review of the literature, there haven't been any documented studies of the energy transportation of parallel propagating electromagnetic waves in Cairns distributed plasmas by using kinetic model, till date. Therefore, we are calculating the same in our thesis for the both scenarios; non-resonant and resonant limit.

3.1 Generalized Dispersion Relation for Electromagnetic Waves by Using Kinetic Model

As we calculated the general form of dielectric permittivity tensor in section 2.2, with specified values of M_{ij} and L_{ij}

$$\epsilon_{ij} = \delta_{ij} - \frac{2\pi}{s} \sum_{\alpha} m_{\alpha} \omega_{p\alpha}^2 \int_{-\infty}^{\infty} dp_{\parallel} \int_{0}^{\infty} p_{\perp} dp_{\perp} \chi_1 \sum_{n=-\infty}^{\infty} \frac{M_{ij}}{s + ik_{\parallel} v_{\parallel} + in\Omega_{\alpha}} + L_{ij} \tag{3.1}$$

and by considering the parallel propagation we assume $k_{\perp} = 0$, therefore the general dispersion relation in magnetized plasma mentioned at equation (2.39) reduces into following form

$$\left[\left(\epsilon_{xx} - N_{\parallel}^2\right)^2 + \epsilon_{xy}^2\right] \epsilon_{zz} = 0$$

where the z, z components of dielectric permittivity belongs to electrostatic case and it would be zero, so equation (3.1) becomes

$$\left(\epsilon_{xx} - N_{\parallel}^2\right)^2 + \epsilon_{xy}^2 = 0$$

$$N_{\parallel}^2 = \epsilon_{xx} \pm i\epsilon_{xy} \tag{3.2}$$

where (+) sign belongs to Right handed waves and (-) sign corresponds to Left handed waves.

By substituting the values of ϵ_{xx} and ϵ_{xy}, we obtain the generalized dispersion relation of right handed parallel propagating electromagnetic waves;

$$\frac{c^2k_{\parallel}^2}{\omega^2} = 1 + \frac{\omega_{p_e}^2}{\omega}\int d^3V \frac{v_{\perp}}{2}\frac{\left[\frac{\partial f_0}{\partial v_{\perp}} - \frac{v_{\parallel}k_{\parallel}}{\omega}\left\{\frac{\partial f_0}{\partial v_{\perp}} - \frac{v_{\perp}}{v_{\parallel}}\frac{\partial f_0}{\partial v_{\parallel}}\right\}\right]}{\left(\omega - \Omega - v_{\parallel}k_{\parallel}\right)}$$

3.2 Dispersion Relation of Right Handed Parallel Propagating Electromagnetic Waves by Using Cairns Velocity Distribution Function

We consider cylindrical geometry to solve our problem, where ambient magnetic field $\mathbf{B}_o$ is parallel to wave vector $\mathbf{k}$ along z-axis, and electric field lies in x-y plane.

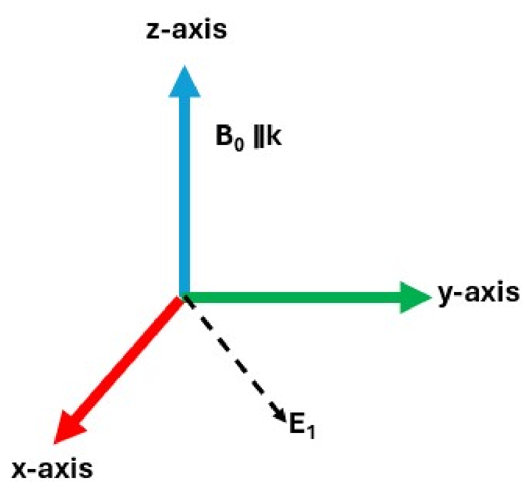

Figure 3.1. Geometry of right handed parallel propagating electromagnetic wave

In order to derive the dispersion relation for right handed circularly polarized wave, we assume collisionless magnetized plasma and consider the dynamics of electrons whereas we treat the ions as merely a neutral background that at some point secures charge neutrality. The generalized dispersion relation of parallel propagating electromagnetic waves can be written as

$$\frac{c^2k_{\parallel}^2}{\omega^2} = 1 + \frac{\omega_{p_e}^2}{\omega}\int d^3V \frac{v_\perp}{2}\frac{\left[\frac{\partial f_0}{\partial v_\perp} - \frac{v_\parallel k_\parallel}{\omega}\left\{\frac{\partial f_0}{\partial v_\perp} - \frac{v_\perp}{v_\parallel}\frac{\partial f_0}{\partial v_\parallel}\right\}\right]}{\left(\omega - \Omega - v_\parallel k_\parallel\right)} \tag{3.3}$$

where $k_{\parallel}$, ω & $\Omega = \frac{qB_o}{m_e c}$ are the wave number, wave frequency and gyro-frequency of electrons respectively. $\omega_{p_e} = \sqrt{\frac{4\pi n_o e^2}{m_e}}$ represents plasma frequency of electrons.

We employ 3D anisotropic Cairns distribution function for plasma particles [29]

$$f_o = \frac{1}{\left(1+\frac{15}{4}\Lambda\right)\pi^{\frac{3}{2}} v_{T_\perp}^2 v_{T_\parallel}} \left\{1 + \Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2\right\} \times \exp\left[-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right] \tag{3.4}$$

where Λ is a spectral index that determines the number of particles with high energy in a system.

let the normalization constant

$$A = \frac{1}{\left(1+\frac{15}{4}\Lambda\right)\pi^{\frac{3}{2}} v_{T_\perp}^2 v_{T_\parallel}}$$

also we have the perpendicular and parallel thermal velocities of electrons

$$v_{T(\perp,\parallel)}^2 = \left(\frac{2k_B T_{(\perp,\parallel)}}{m_e}\right)$$

thus

$$f_o = A\left\{1 + \Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2\right\} \times \exp\left[-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right] \tag{3.5}$$

$$\frac{\partial f_o}{\partial v_\perp} = \frac{\partial}{\partial v_\perp}\left[A\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right) + A\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2 \exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\right]$$

$$\frac{\partial f_o}{\partial v_\perp} = A\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2}-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2v_\perp}{v_{T_\perp}^2}\right)+\left[\begin{array}{c} 2A\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2}+\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(\frac{2v_\perp}{v_{T_\perp}^2}\right)\times\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2}-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)+ \\ A\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2}+\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2}-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2V_\perp}{v_{T_\perp}^2}\right) \end{array}\right]$$

$$\frac{\partial f_o}{\partial v_\perp} = A\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2}-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2v_\perp}{v_{T_\perp}^2}\right)\left[1-2\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2}+\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)+\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2}+\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2\right] \quad (3.6)$$

similarly

$$\frac{\partial f_o}{\partial v_\parallel} = \frac{\partial}{\partial v_\parallel}\left[A\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2}-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)+A\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2}+\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2}-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\right]$$

$$\frac{\partial f_o}{\partial v_\parallel} = A\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2}-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2v_\parallel}{v_{T_\parallel}^2}\right)+\left[\begin{array}{c} 2A\Lambda\left(\frac{V_\perp^2}{v_{T\perp\alpha}^2}+\frac{V_\parallel^2}{v_{T\parallel\alpha}^2}\right)\left(\frac{2v_\parallel}{v_{T_\parallel}^2}\right)\times\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2}-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)+ \\ A\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2}+\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2}-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2v_\parallel}{v_{T_\parallel}^2}\right) \end{array}\right]$$

$$\frac{\partial f_o}{\partial v_\parallel} = A\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2}-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2v_\parallel}{v_{T_\parallel}^2}\right)\left[1-2\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2}+\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)+\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2}+\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2\right] \quad (3.7)$$

by substituting equation (3.6 - 3.7) in equation (3.3)

$$\frac{c^2k_\parallel^2}{\omega^2} = 1 + \frac{\omega_{p_e}^2}{\omega} A \int d^3V \left(\frac{v_\perp}{2}\right) \frac{\left[\begin{array}{c} \exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2v_\perp}{v_{T_\perp}^2}\right)\left[\begin{array}{c} 1 - 2\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right) \\ +\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2 \end{array}\right] \\ -\frac{v_\parallel k_\parallel}{\omega}\left\{\begin{array}{c} \exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2v_\perp}{v_{T_\perp}^2}\right)\left[\begin{array}{c} 1 - 2\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right) \\ +\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2 \end{array}\right] - \\ \frac{v_\perp}{v_\parallel}\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2v_\parallel}{v_{T_\parallel}^2}\right)\left[\begin{array}{c} 1 - 2\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right) \\ +\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2 \end{array}\right] \end{array}\right\} \end{array}\right]}{\left(\omega - \Omega - v_\parallel k_\parallel\right)}$$

The volume integeral is

$$\int d^3V = \int_0^{2\pi} d\phi \int_0^{\infty} v_\perp dv_\perp \int_{-\infty}^{\infty} dv_\parallel$$

since

$$\int_0^{2\pi} d\phi = 2\pi$$

$$\int d^3V = 2\pi \int_0^{\infty} v_\perp dv_\perp \int_{-\infty}^{\infty} dv_\parallel$$

thus

$$\frac{c^2 k_\parallel^2}{\omega^2} = 1 + \frac{\omega_{p_e}^2}{\omega} A\,(2\pi) \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty v_\perp dv_\perp \left(\frac{v_\perp}{2}\right)$$

$$\times \frac{\left[\begin{array}{c} \exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2v_\perp}{v_{T_\perp}^2}\right)\left[1 - 2\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right) + \Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2\right] \\ -\frac{v_\parallel k_\parallel}{\omega}\left\{\begin{array}{c} \exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2v_\perp}{v_{T_\perp}^2}\right)\left[\begin{array}{c} 1 - 2\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right) \\ +\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2 \end{array}\right] - \\ \frac{v_\perp}{v_\parallel}\exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left(-\frac{2v_\parallel}{v_{T_\parallel}^2}\right)\left[\begin{array}{c} 1 - 2\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right) \\ +\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2 \end{array}\right] \end{array}\right\} \end{array}\right]}{\left(\omega - \Omega - v_\parallel k_\parallel\right)}$$

$$\frac{c^2 k^2}{\omega^2} = 1 + \frac{\omega_{p_e}^2}{\omega} A(2\pi) \int_{-\infty}^{\infty} dv_\parallel \int_0^\infty v_\perp dv_\perp \left(\frac{v_\perp}{2}\right)\left(-\frac{2v_\perp}{v_{T_\perp}^2}\right)$$

$$\times \frac{\left[\begin{array}{c} \exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left[1 - 2\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right) + \Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2\right] \\ \times\left\{1 - \frac{v_\parallel k_\parallel}{\omega} + \frac{v_\parallel k_\parallel}{\omega}\frac{v_{T_\perp}^2}{v_{T_\parallel}^2}\right\} \end{array}\right]}{\left(\omega - \Omega - v_\parallel k_\parallel\right)}$$

$$\frac{c^2 k_\parallel^2}{\omega^2} = 1 - \frac{\omega_{p_e}^2}{\omega}\frac{A(2\pi)}{v_{T_\perp}^2} \int_{-\infty}^{\infty} dv_\parallel \times \tag{3.8}$$

$$\int_0^\infty v_\perp^3 dv_\perp \frac{\left[\begin{array}{c} \exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left[1 - 2\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right) + \Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2\right] \\ \times\left\{1 - \frac{v_\parallel k_\parallel}{\omega} + \frac{v_\parallel k_\parallel}{\omega}\frac{T_\perp}{T_\parallel}\right\} \end{array}\right]}{\left(\omega - \Omega - v_\parallel k_\parallel\right)}$$

let

$$\int_0^\infty v_\perp^3 \exp\left(-\frac{v_\perp^2}{v_{T_\perp}^2} - \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left[1 - 2\Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right) + \Lambda\left(\frac{v_\perp^2}{v_{T_\perp}^2} + \frac{v_\parallel^2}{v_{T_\parallel}^2}\right)^2\right] dv_\perp = I \tag{3.9}$$

thus

$$\frac{c^2 k_\parallel^2}{\omega^2} = 1 - \frac{\omega_{p_e}^2}{\omega}\frac{A(2\pi)}{v_{T_\perp}^2}\int\limits_{-\infty}^{\infty} dv_\parallel (I)\frac{\left\{1 - \frac{v_\parallel k_\parallel}{\omega} + \frac{v_\parallel k_\parallel}{\omega}\frac{T_\perp}{T_\parallel}\right\}}{\left(\omega - \Omega - v_\parallel k_\parallel\right)} \tag{3.10}$$

3.2.1 Perpendicular Integration

Now by solving the integral I first, let change the parameters

$$v_{T_\parallel}^2 = \alpha^2 \;\; ; \;\; v_{T_\perp}^2 = \beta^2 \;\; ; v_\perp = x \; ; dv_\perp = dx$$

thus equation (3.9) becomes

$$I = \int_0^\infty x^3 \exp\left(-\frac{v_\parallel^2}{\alpha^2} - \frac{x^2}{\beta^2}\right)\left[1 - 2\Lambda\left(\frac{v_\parallel^2}{\alpha^2} + \frac{x^2}{\beta^2}\right) + \Lambda\left(\frac{v_\parallel^2}{\alpha^2} + \frac{x^2}{\beta^2}\right)^2\right] dx$$

by simplifying

$$I = \frac{\exp\left(-\frac{v_\parallel^2}{\alpha^2}\right) v_{T_\perp}^4 \left\{v_{T_\parallel}^4 + \Lambda\left(v_\parallel^4 + 2v_{T_\parallel}^2 v_\parallel^2 + 2v_{T_\parallel}^4\right)\right\}}{2v_{T_\parallel}^4} \tag{3.11}$$

by putting equation (3.11) in equation (3.10)

$$\frac{c^2 k_\parallel^2}{\omega^2} = 1 - \frac{\omega_{p_e}^2}{\omega}\frac{A(2\pi)}{v_{T_\perp}^2}\int\limits_{-\infty}^{\infty} dv_\parallel \frac{\left\{I - I\frac{v_\parallel k_\parallel}{\omega} + I\frac{v_\parallel k_\parallel}{\omega}\frac{T_\perp}{T_\parallel}\right\}}{\left(\omega - \Omega - v_\parallel k_\parallel\right)}$$

$$
\begin{aligned}
\frac{c^2k^2}{\omega^2} &= 1-\frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\left(1+\frac{15}{4}\Lambda\right)\pi^{\frac{3}{2}}v_{T_\perp}^2\,v_{T_\parallel}}\right)\left(\frac{2\pi}{v_{T_\perp}^2}\right) \\
&\quad \int_{-\infty}^{\infty}\frac{\exp\left(-\frac{v_\parallel^2}{\alpha^2}\right)v_{T_\perp}^4\left\{v_{T_\parallel}^4+\Lambda\left(v_\parallel^4+2v_{T_\parallel}^2v_\parallel^2+2v_{T_\parallel}^4\right)\right\}}{2v_{T_\parallel}^4}dv_\parallel \\
&\quad -\frac{k_\parallel}{\omega}\int_{-\infty}^{\infty}v_\parallel\frac{\exp\left(-\frac{v_\parallel^2}{\alpha^2}\right)v_{T_\perp}^4\left\{v_{T_\parallel}^4+\Lambda\left(v_\parallel^4+2v_{T_\parallel}^2v_\parallel^2+2v_{T_\parallel}^4\right)\right\}}{2v_{T_\parallel}^4}dv_\parallel \\
&\quad \times\frac{+\frac{k_\parallel}{\omega}\frac{T_\perp}{T_\parallel}{}_\parallel\int_{-\infty}^{\infty}v_\parallel\frac{\exp\left(-\frac{v_\parallel^2}{\alpha^2}\right)v_{T_\perp}^4\left\{v_{T_\parallel}^4+\Lambda\left(v_\parallel^4+2v_{T_\parallel}^2v_\parallel^2+2v_{T_\parallel}^4\right)\right\}}{2v_{T_\parallel}^4}dv_\parallel}{\left(\omega-\Omega-v_\parallel k_\parallel\right)}
\end{aligned}
$$

since $v_{T_\parallel}^2=\alpha^2$

let

$$
I_1=\left(\frac{2}{\left(1+\frac{15}{4}\Lambda\right)\pi^{\frac{1}{2}}v_{T_\perp}^4\,v_{T_\parallel}}\right)\int_{-\infty}^{\infty}\frac{\exp\left(-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)v_{T_\perp}^4\left\{v_{T_\parallel}^4+\Lambda\left(v_\parallel^4+2v_{T_\parallel}^2v_\parallel^2+2v_{T_\parallel}^4\right)\right\}}{2v_{T_\parallel}^4\left(\omega-\Omega-v_\parallel k_\parallel\right)}dv_\parallel
$$

$$
I_2=\left(\frac{2}{\left(1+\frac{15}{4}\Lambda\right)\pi^{\frac{1}{2}}v_{T_\perp}^4\,v_{T_\parallel}}\right)\int_{-\infty}^{\infty}v_\parallel\frac{\exp\left(-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)v_{T_\perp}^4\left\{v_{T_\parallel}^4+\Lambda\left(v_\parallel^4+2v_{T_\parallel}^2v_\parallel^2+2v_{T_\parallel}^4\right)\right\}}{2v_{T_\parallel}^4\left(\omega-\Omega-v_\parallel k_\parallel\right)}dv_\parallel
$$

thus we can write as

$$
\frac{c^2k_\parallel^2}{\omega^2}=1-\frac{\omega_{p_e}^2}{\omega}\left[I_1-I_2\frac{k_\parallel}{\omega}+I_2\frac{k_\parallel}{\omega}\frac{T_\perp}{T_\parallel}\right]
$$

or

$$
\frac{c^2k_\parallel^2}{\omega^2}=1-\frac{\omega_{p_e}^2}{\omega}\left[I_1+I_2\frac{k_\parallel}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right)\right] \tag{3.12}
$$

3.2.2 Parallel Integration

Now we will solve the integrals I_1 and I_2 separately

Since

$$I_1 = \left(\frac{2}{\left(1+\frac{15}{4}\Lambda\right)\pi^{\frac{1}{2}} v_{T_\parallel}} \right) \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)\left\{v_{T_\parallel}^4 + \Lambda\left(v_\parallel^4 + 2v_{T_\parallel}^2 v_\parallel^2 + 2v_{T_\parallel}^4\right)\right\}}{2v_{T_\parallel}^4\left(\omega - \Omega - v_\parallel k_\parallel\right)} dv_\parallel$$

by simplifying

$$I_1 = \left(\frac{1}{\left(1+\frac{15}{4}\Lambda\right)\sqrt{\pi} v_{T_\parallel}} \right) \int_{-\infty}^{\infty} \frac{\exp(-\frac{v_\parallel^2}{v_{T_\parallel}^2})\left\{\frac{v_{T_\parallel}^4}{v_{T_\parallel}^4} + \frac{v_\parallel^4\Lambda}{v_{T_\parallel}^4} + \frac{2v_\parallel^2 v_{T_\parallel}^2\Lambda}{v_{T_\parallel}^4} + \frac{2v_{T_\parallel}^4\Lambda}{v_{T_\parallel}^4}\right\}}{\left(\omega - \Omega - v_\parallel k_\parallel\right)} dv_\parallel$$

$$I_1 = \left(\frac{1}{\left(1+\frac{15}{4}\Lambda\right) v_{T_\parallel}} \right) \left[\begin{array}{c} \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)}{\left(\omega-\Omega-v_\parallel k_\parallel\right)} dv_\parallel + \frac{\Lambda}{v_{T_\parallel}^4}\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} v_\parallel^4 \frac{\exp\left(-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)}{\left(\omega-\Omega-v_\parallel k_\parallel\right)} dv_\parallel \\ + \frac{2\Lambda}{v_{T_\parallel}^2}\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} v_\parallel^2 \frac{\exp\left(-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)}{\left(\omega-\Omega-v_\parallel k_\parallel\right)} dv_\parallel + 2\Lambda\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp\left(-\frac{v_\parallel^2}{v_{T_\parallel}^2}\right)}{\left(\omega-\Omega-v_\parallel k_\parallel\right)} dv_\parallel \end{array} \right]$$

by changing variables

$$v_{T_\parallel}^2 = \alpha^2 \quad ; \quad \frac{v_\parallel^2}{v_{T_\parallel}^2} = s^2 \text{ and } v_\parallel^2 = \alpha^2 s^2 \quad \text{also } v_\parallel = s v_{T_\parallel} \quad ; \quad dv_\parallel = v_{T_\parallel} ds$$

let

$$\frac{1}{\left(\omega - \Omega - v_\parallel k_\parallel\right)} = \frac{1}{k_\parallel v_{T_\parallel}\left(\frac{\omega-\Omega-v_\parallel k_\parallel}{k_\parallel v_{T_\parallel}}\right)} = \frac{1}{k_\parallel v_{T_\parallel}\left(\frac{\omega-\Omega}{k_\parallel v_{T\parallel\alpha}} - \frac{v_\parallel}{v_{T_\parallel}}\right)}$$

also

$$\xi = \frac{\omega - \Omega}{k_\parallel v_{T\parallel\alpha}} \text{ and } s = \frac{v_\parallel}{v_{T_\parallel}}$$

so

$$\implies \frac{1}{\left(\omega - \Omega - v_\parallel k_\parallel\right)} = -\frac{1}{k_\parallel v_{T_\parallel}(s-\xi)}$$

therefore

$$I_1 = \left(\frac{1}{\left(1+\frac{15}{4}\Lambda\right) v_{T_\parallel}}\right) \left[\begin{array}{c} \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{\exp\left(-s^2\right)}{-k_\parallel v_{T_\parallel}(s-\xi)} v_{T_\parallel} ds + \frac{\Lambda}{v_{T_\parallel}^4} \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} s^4 v_{T_\parallel}^4 \frac{\exp\left(-s^2\right)}{-k_\parallel v_{T_\parallel}(s-\xi)} v_{T_\parallel} ds \\ + \frac{2\Lambda}{v_{T_\parallel}^2} \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} s^2 v_{T_\parallel}^2 \frac{\exp\left(-s^2\right)}{-k_\parallel v_{T_\parallel}(s-\xi)} v_{T_\parallel} ds + 2\Lambda \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{\exp\left(-s^2\right)}{-k_\parallel v_{T_\parallel}(s-\xi)} v_{T_\parallel} ds \end{array} \right]$$

$$I_1 = \left(\frac{1}{\left(1+\frac{15}{4}\Lambda\right) v_{T_\parallel}}\right) \left[\begin{array}{c} \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{\exp\left(-s^2\right)}{-k_\parallel (s-\xi)} ds + \frac{\Lambda}{v_{T_\parallel}^4} \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} s^4 v_{T_\parallel}^4 \frac{\exp\left(-s^2\right)}{-k_\parallel (s-\xi)} ds \\ + \frac{2\Lambda}{v_{T_\parallel}^2} \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} s^2 v_{T_\parallel}^2 \frac{\exp\left(-s^2\right)}{-k_\parallel (s-\xi)} ds + 2\Lambda \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{\exp\left(-s^2\right)}{-k_\parallel (s-\xi)} ds \end{array} \right]$$

$$I_1 = -\left(\frac{1}{\left(1+\frac{15}{4}\Lambda\right) v_{T_\parallel}}\right) \frac{1}{k_\parallel} \left[\begin{array}{c} \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{\exp\left(-s^2\right)}{(s-\xi)} ds + \frac{\Lambda}{v_{T_\parallel}^4} \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} s^4 v_{T_\parallel}^4 \frac{\exp\left(-s^2\right)}{(s-\xi)} ds \\ + \frac{2\Lambda}{v_{T_\parallel}^2} \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} s^2 v_{T_\parallel}^2 \frac{\exp\left(-s^2\right)}{(s-\xi)} ds + 2\Lambda \frac{1}{\sqrt{\pi}} \int\limits_{-\infty}^{\infty} \frac{\exp\left(-s^2\right)}{(s-\xi)} ds \end{array} \right] \tag{3.13}$$

Pade Approximation

A complex plasma dispersion function $Z(\xi)$ (responsible for well known Landau damping) develops when we examine the wave-particle interaction of parallel propagating waves. In order to handle such a complicated function, we use the Pade approximation which is a distinctive and conventional kind of rational fraction approximation. The method of this approximation involves expanding a function as a ratio of two power series and using the coefficients of the Taylor series expansion of a function $f(x)$ to figure out the coefficients of the denominator and numerator [30,31]. The most precise approximation of a function by a rational function of a specific order is commonly referred as the Pade approximation.

When functions have poles, the Pade approximation is typically better since rational functions can be used to demonstrate them accurately. When the Taylor series fails to converge [32], the Pade approximation frequently provides a more accurate approximation of the function than truncating it. These factors make the Pade approximation a popular tool in computer calculations.

Plasma dispersion function can be defined as [32]

$$Z(\xi) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{\exp(-s^2)}{(s-\xi)} ds \tag{3.14}$$

where $\xi = \frac{\omega - \Omega}{k_\parallel v_{T_\parallel}}$, and $s = \frac{v_\parallel}{v_{T_\parallel}}$

The Landau integral cannot be computed analytically, unless the plasma dispersion function is simplified. Therefore, we use the Pade approximation [32] to manage the complexity involved in plasma dispersion function

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} s \frac{\exp(-s^2)}{(s-\xi)} ds = 1 + \xi Z(\xi) \tag{3.15}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} s^2 \frac{\exp(-s^2)}{(s-\xi)} ds = \xi + \xi^2 Z(\xi) \tag{3.16}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} s^3 \frac{\exp(-s^2)}{(s-\xi)} ds = \frac{1}{2} + \xi^2 + \xi^3 Z(\xi) \tag{3.17}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} s^4 \frac{\exp(-s^2)}{(s-\xi)} ds = \frac{1}{2}\xi + \xi^3 + \xi^4 Z(\xi) \tag{3.18}$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} s^5 \frac{\exp(-s^2)}{(s-\xi)} ds = \frac{3}{4} + \frac{\xi^2}{2} + \xi^4 + \xi^5 Z(\xi) \tag{3.19}$$

thus by using pade approximation results, we substitute equations (3.14, 3.16 & 3.18) in equation (3.13)

$$I_1 = -\left(\frac{1}{\left(1+\frac{15}{4}\Lambda\right) v_{T_\parallel}}\right) \frac{1}{k_\parallel} \left[Z(\xi) + \frac{1}{2}\Lambda\xi + \Lambda\xi^3 + \Lambda\xi^4 Z(\xi) + 2\Lambda\xi + 2\Lambda\xi^2 Z(\xi) + 2\Lambda Z(\xi)\right]$$

$$I_1 = -\left(\frac{1}{1+\frac{15}{4}\Lambda}\right)\frac{1}{k_\parallel v_{T_\parallel}}\left[Z(\xi)\left\{1+\Lambda\xi^4+2\Lambda\xi^2+2\Lambda\right\}+\frac{5}{2}\Lambda\xi+\Lambda\xi^3\right] \tag{3.20}$$

Similarly, we will now solve the integral I_2;

$$I_2 = \left(\frac{2}{\left(1+\frac{15}{4}\Lambda\right)\pi^{\frac{1}{2}}v_{T_\perp}^4 v_{T_\parallel}}\right)\int_{-\infty}^{\infty} v_\parallel \frac{\exp(-\frac{v_\parallel^2}{v_{T_\parallel}^2})v_{T_\perp}^4\left\{v_{T_\parallel}^4+\Lambda\left(v_\parallel^4+2v_{T_\parallel}^2 v_\parallel^2+2v_{T_\parallel}^4\right)\right\}}{2v_{T_\parallel}^4\left(\omega-\Omega-v_\parallel k_\parallel\right)}dv_\parallel$$

by changing variables

$$\frac{v_\parallel^2}{v_{T_\parallel}^2} = s^2 \ ; \ v_\parallel^2 = v_{T_\parallel}^2 s^2 \ \text{ so } v_\parallel = s v_{T_\parallel} \ \text{ and } dv_\parallel = v_{T_\parallel} ds$$

$$I_2 = \left(\frac{1}{\left(1+\frac{15}{4}\Lambda\right)v_{T_\parallel}}\right)\left[\begin{array}{c}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} s v_{T_\parallel}\frac{\exp(-s^2)}{-k_\parallel v_{T_\parallel}(s-\xi)}v_{T_\parallel}ds+\frac{\Lambda}{v_{T_\parallel}^4}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(s v_{T_\parallel})s^4 v_{T_\parallel}^4\frac{\exp(-s^2)}{-k_\parallel v_{T_\parallel}(s-\xi)}v_{T_\parallel}ds \\ +\frac{2\Lambda}{v_{T\parallel\alpha}^2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}(s v_{T_\parallel})s^2 v_{T_\parallel}^2\frac{\exp(-s^2)}{-k_\parallel v_{T_\parallel}(s-\xi)}v_{T_\parallel}ds+2\Lambda\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}s v_{T_\parallel}\frac{\exp(-s^2)}{-k_\parallel v_{T_\parallel}(s-\xi)}v_{T_\parallel}ds\end{array}\right]$$

$$I_2 = \left(\frac{1}{\left(1+\frac{15}{4}\Lambda\right)v_{T_\parallel}}\right)\left[\begin{array}{c}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} s v_{T_\parallel}\frac{\exp(-s^2)}{-k_\parallel(s-\xi)}ds+\frac{\Lambda}{v_{T_\parallel}^4}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}s^5 v_{T_\parallel}^5\frac{\exp(-s^2)}{-k_\parallel(s-\xi)}ds \\ +\frac{2\Lambda}{v_{T_\parallel}^2}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}s^3 v_{T_\parallel}^3\frac{\exp(-s^2)}{-k_\parallel(s-\xi)}ds+2\Lambda\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}s v_{T_\parallel}\frac{\exp(-s^2)}{-k_\parallel(s-\xi)}ds\end{array}\right]$$

$$I_2 = -\left(\frac{1}{\left(1+\frac{15}{4}\Lambda\right)v_{T_\parallel}}\right)\frac{v_{T_\parallel}}{k_\parallel}\left[\begin{array}{c}\left\{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}s\frac{\exp(-s^2)}{(s-\xi)}ds\right\}+\Lambda\left\{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}s^5\frac{\exp(-s^2)}{(s-\xi)}ds\right\} \\ +2\Lambda\left\{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}s^3\frac{\exp(-s^2)}{(s-\xi)}ds\right\}+2\Lambda\left\{\frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty}s\frac{\exp(-s^2)}{(s-\xi)}ds\right\}\end{array}\right]$$

$$I_2 = -\left(\frac{1}{1+\frac{15}{4}\Lambda}\right)\frac{1}{k_\parallel}\left[\begin{array}{c} \left\{\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} s\frac{\exp(-s^2)}{(s-\xi)}ds\right\} + \Lambda\left\{\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} s^5\frac{\exp(-s^2)}{(s-\xi)}ds\right\} \\ +2\Lambda\left\{\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} s^3\frac{\exp(-s^2)}{(s-\xi)}ds\right\} + 2\Lambda\left\{\frac{1}{\sqrt{\pi}}\int\limits_{-\infty}^{\infty} s\frac{\exp(-s^2)}{(s-\xi)}ds\right\} \end{array}\right] \tag{3.21}$$

Now with reference to pade approximation results, we substitute equations (3.15, 3.17 & 3.19) in equation (3.21);

$$I_2 = -\left(\frac{1}{1+\frac{15}{4}\Lambda}\right)\frac{1}{k_\parallel}\left[\begin{array}{c} \{1+\xi Z(\xi)\} + \Lambda\left\{\frac{3}{4}+\frac{\xi^2}{2}+\xi^4+\xi^5 Z(\xi)\right\} \\ +2\Lambda\left\{\frac{1}{2}+\xi^2+\xi^3 Z(\xi)\right\} + 2\Lambda\{1+\xi Z(\xi)\} \end{array}\right]$$

$$I_2 = -\left(\frac{1}{1+\frac{15}{4}\Lambda}\right)\frac{1}{k_\parallel}\left[\begin{array}{c} 1+\xi Z(\xi)+\frac{3\Lambda}{4}+\frac{\Lambda\xi^2}{2}+\Lambda\xi^4+\Lambda\xi^5 Z(\xi) \\ +\Lambda+2\Lambda\xi^2+2\Lambda\xi^3 Z(\xi)+2\Lambda+2\Lambda\xi Z(\xi) \end{array}\right]$$

$$I_2 = -\left(\frac{1}{1+\frac{15}{4}\Lambda}\right)\frac{1}{k_\parallel}\left[Z(\xi)\left\{\xi+\Lambda\xi^5+2\Lambda\xi^3+2\Lambda\xi\right\}+1+\frac{15\Lambda}{4}+\frac{5\Lambda\xi^2}{2}+\Lambda\xi^4\right] \tag{3.22}$$

by substituting both integrals equations (3.20 & 3.22) in equation (3.12), as;

$$\frac{c^2k_\parallel^2}{\omega^2} = 1 - \frac{\omega_{p_e}^2}{\omega}\left[I_1 + I_2\frac{k_\parallel}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right)\right]$$

$$\frac{c^2k_\parallel^2}{\omega^2} = 1 - \frac{\omega_{p_e}^2}{\omega}\left[\begin{array}{c} -\left(\frac{1}{1+\frac{15}{4}\Lambda}\right)\frac{1}{k_\parallel v_{T_\parallel}}\left[Z(\xi)\left\{1+\Lambda\xi^4+2\Lambda\xi^2+2\Lambda\right\}+\frac{5}{2}\Lambda\xi+\Lambda\xi^3\right] \\ -\left(\frac{1}{1+\frac{15}{4}\Lambda}\right)\frac{1}{k_\parallel}\left[\begin{array}{c} Z(\xi)\left\{\xi+\Lambda\xi^5+2\Lambda\xi^3+2\Lambda\xi\right\} \\ +1+\frac{15\Lambda}{4}+\frac{5\Lambda\xi^2}{2}+\Lambda\xi^4 \end{array}\right]\frac{k_\parallel}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right) \end{array}\right]$$

$$\frac{c^2k_\parallel^2}{\omega^2} = 1 + \frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{(1+\frac{15}{4}\Lambda)}\right)\frac{1}{k_\parallel v_{T_\parallel}}\left[\begin{array}{c} \left[Z(\xi)\left\{1+\Lambda\xi^4+2\Lambda\xi^2+2\Lambda\right\}+\frac{5}{2}\Lambda\xi+\Lambda\xi^3\right] \\ +\left[\begin{array}{c} Z(\xi)\left\{\xi+\Lambda\xi^5+2\Lambda\xi^3+2\Lambda\xi\right\} \\ +\left(1+\frac{15\Lambda}{4}\right)+\frac{5\Lambda\xi^2}{2}+\Lambda\xi^4 \end{array}\right]\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right) \end{array}\right]$$

let

$$\Lambda_1 = 1 + \frac{15}{4}\Lambda$$

$$\frac{c^2 k_\parallel^2}{\omega^2} = 1 + \frac{\omega_{p_e}^2}{\omega}\frac{1}{\Lambda_1 k_\parallel v_{T_\parallel}}\left[\begin{array}{c} \left[Z(\xi)\left\{1 + \Lambda\xi^4 + 2\Lambda\xi^2 + 2\Lambda\right\} + \frac{5}{2}\Lambda\xi + \Lambda\xi^3\right] \\ + \left[\begin{array}{c} Z(\xi)\left\{\xi + \Lambda\xi^5 + 2\Lambda\xi^3 + 2\Lambda\xi\right\} \\ +\Lambda_1 + \frac{5\Lambda\xi^2}{2} + \Lambda\xi^4 \end{array}\right]\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel} - 1\right) \end{array}\right] \tag{3.23}$$

3.3 Expansion of Plasma Dispersion Function

Expansion of $Z(\xi)$ presents as a useful mathematical depiction of how plasma reacts to electromagnetic waves. Therefore we use the expansions for a large argument ($\xi >> 1$)(non-resonant case), and for small argument ($\xi << 1$) (resonant case). Moreover, the rationale behind selecting a large argument would result in a relatively weaker wave particle interaction, which would negate the imaginary component of the wave number. In contrast, a small argument for the plasma dispersion function exhibits a strong wave particle contact.

3.3.1 Non-Resonant Case ($\xi >> 1$)

We will use the following expansion of plasma dispersion function for large argument [32]

$$Z(\xi) = i\sqrt{\pi}\exp(-\xi^2) - \frac{1}{\xi} - \frac{1}{2\xi^3} - \frac{3}{4\xi^5} - ... \tag{3.24}$$

by neglecting the higher order terms

$$Z(\xi) = -\frac{1}{\xi} - \frac{1}{2\xi^3} \tag{3.25}$$

by substituting equation (3.25) in equation (3.23), we get

$$\frac{c^2k_\parallel^2}{\omega^2} = 1 + \frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\frac{1}{k_\parallel v_{T_\parallel}}\left[\begin{array}{c} \left[\left(-\frac{1}{\xi}-\frac{1}{2\xi^3}\right)\left\{1+\Lambda\xi^4+2\Lambda\xi^2+2\Lambda\right\}+\frac{5}{2}\Lambda\xi+\Lambda\xi^3\right] \\ +\left[\begin{array}{c}\left(-\frac{1}{\xi}-\frac{1}{2\xi^3}\right)\left\{\xi+\Lambda\xi^5+2\Lambda\xi^3+2\Lambda\xi\right\} \\ +\Lambda_1+\frac{5\Lambda\xi^2}{2}+\Lambda\xi^4\end{array}\right]\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right)\end{array}\right]$$

$$\frac{c^2k_\parallel^2}{\omega^2} = 1 + \frac{\omega_{p}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\frac{1}{k_\parallel v_{T\parallel\alpha}}\left[\begin{array}{c} \left[\begin{array}{c}-\frac{1}{\xi}-\Lambda\xi^3-2\Lambda\xi-\frac{2\Lambda}{\xi}-\frac{1}{2\xi^3} \\ -\frac{\Lambda\xi}{2}-\frac{\Lambda}{\xi}-\frac{\Lambda}{\xi^3}+\frac{5}{2}\Lambda\xi+\Lambda\xi^3\end{array}\right] \\ +\left[\begin{array}{c}-1-\Lambda\xi^4-2\Lambda\xi^2-2\Lambda-\frac{1}{2\xi^2} \\ -\frac{\Lambda\xi^2}{2}-\Lambda-\frac{\Lambda}{\xi^2}+\Lambda_1+\frac{5\Lambda\xi^2}{2}+\Lambda\xi^4\end{array}\right]\frac{k_\parallel v_{T\parallel\alpha}}{\omega}(\frac{T_\perp}{T_\parallel}-1)\end{array}\right]$$

$$\frac{c^2k_\parallel^2}{\omega^2} \;=\; 1 + \frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\frac{1}{k_\parallel v_{T\parallel\alpha}}\left[\begin{array}{c}\left[-\frac{1}{\xi}\left\{1+\Lambda+2\Lambda\right\}-\Lambda\xi\left\{2+\frac{1}{2}-\frac{5}{2}\right\}-\frac{1}{\xi^3}\left\{\frac{1}{2}+\Lambda\right\}\right] \\ +\left[-1-3\Lambda+\Lambda_1-\Lambda\xi^2\left\{2+\frac{1}{2}-\frac{5}{2}\right\}-\frac{1}{\xi^2}\left\{\frac{1}{2}+\Lambda\right\}\right]\frac{k_\parallel v_{T\parallel\alpha}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right)\end{array}\right]$$

$$\frac{c^2k_\parallel^2}{\omega^2} = 1 + \frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\frac{1}{k_\parallel v_{T_\parallel}}\left[\begin{array}{c}\left[-\frac{1}{\xi}\left\{1+3\Lambda\right\}-\frac{1}{\xi^3}\left\{\frac{1}{2}+\Lambda\right\}\right] \\ +\left[-1-3\Lambda+\Lambda_1-\frac{1}{\xi^2}\left\{\frac{1}{2}+\Lambda\right\}\right]\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right)\end{array}\right]$$

by ignoring higher order term ξ^3

$$\frac{c^2k_\parallel^2}{\omega^2} = 1 + \frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\frac{1}{k_\parallel v_{T_\parallel}}\left[\begin{array}{c}-\frac{1}{\xi}\left\{1+3\Lambda\right\} \\ +\left[-1-3\Lambda+\Lambda_1-\frac{1}{\xi^2}\left\{\frac{1}{2}+\Lambda\right\}\right]\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right)\end{array}\right]$$

as; $\xi = \frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}$

$$\frac{c^2k_{\parallel}^2}{\omega^2}=1+\frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\frac{1}{k_{\parallel}v_{T_{\parallel}}}\left[\begin{array}{c} -\frac{k_{\parallel}v_{T_{\parallel}}}{\omega-\Omega}\left\{1+3\Lambda\right\} \\ +\left[\begin{array}{c} -1-3\Lambda+\Lambda_1 \\ -\left\{\frac{k_{\parallel}^2v_{T_{\parallel}}^2}{(\omega-\Omega)^2}\right\}\left\{\frac{1}{2}+\Lambda\right\}\end{array}\right]\frac{k_{\parallel}v_{T_{\parallel}}}{\omega}\left(\frac{T_{\perp}}{T_{\parallel}}-1\right)\end{array}\right]$$

$$\frac{c^2k_{\parallel}^2}{\omega^2}=1+\frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\left[\begin{array}{c} -\frac{k_{\parallel}v_{T_{\parallel}}}{\omega-\Omega}\frac{1}{k_{\parallel}v_{T_{\parallel}}}\left\{1+3\Lambda\right\} \\ -\left[\begin{array}{c} 1+3\Lambda-\Lambda_1 \\ +\left\{\frac{k_{\parallel}^2v_{T_{\parallel}}^2}{(\omega-\Omega)^2}\right\}\left\{\frac{1}{2}+\Lambda\right\}\end{array}\right]\frac{1}{k_{\parallel}v_{T_{\parallel}}}\frac{k_{\parallel}v_{T_{\parallel}}}{\omega}\left(\frac{T_{\perp}}{T_{\parallel}}-1\right)\end{array}\right]$$

$$\frac{c^2k_{\parallel}^2}{\omega^2}=1-\frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\left[\begin{array}{c} \frac{1}{\omega-\Omega}\left\{1+3\Lambda\right\} \\ +\left[1+3\Lambda-\Lambda_1+\left\{\frac{k_{\parallel}^2v_{T_{\parallel}}^2}{(\omega-\Omega)^2}\right\}\left\{\frac{1}{2}+\Lambda\right\}\right]\frac{1}{\omega}\left(\frac{T_{\perp}}{T_{\parallel}}-1\right)\end{array}\right]$$

$$\frac{c^2k_{\parallel}^2}{\omega^2}=1-\frac{\omega_{p_e}^2}{\omega^2}\left(\frac{1}{\Lambda_1}\right)\left[\frac{\omega}{\omega-\Omega}\left\{1+3\Lambda\right\}+\left[\begin{array}{c} 1+3\Lambda-\Lambda_1 \\ +\left\{\frac{k_{\parallel}^2v_{T_{\parallel}}^2}{(\omega-\Omega)^2}\right\}\left\{\frac{1}{2}+\Lambda\right\}\end{array}\right]\left(\frac{T_{\perp}}{T_{\parallel}}-1\right)\right] \quad (3.26)$$

we can retrieve the bi-maxwellian distribution by substituting $\Lambda=0$;

$$\frac{c^2k_{\parallel}^2}{\omega^2}=1-\frac{\omega_p^2}{\omega^2}\left[\frac{\omega}{\omega-\Omega}\left\{1\right\}+\left[1+0-1+\left\{\frac{k_{\parallel}^2v_{T_{\parallel}}^2}{(\omega-\Omega)^2}\right\}\left\{\frac{1}{2}+0\right\}\right]\left(\frac{T_{\perp}}{T_{\parallel}}-1\right)\right]$$

$$\frac{c^2k_{\parallel}^2}{\omega^2}=1-\frac{\omega_{p_e}^2}{\omega^2}\left[\frac{\omega}{\omega-\Omega}+\frac{1}{2}\left\{\frac{k_{\parallel}^2v_{T_{\parallel}}^2}{(\omega-\Omega)^2}\right\}\left(\frac{T_{\perp}}{T_{\parallel}}-1\right)\right]$$

The above equation is a comparable validation of generalized dispersion relation of R-waves [33].

thus from equation (3.26)

$$\frac{c^2k_{\parallel}^2}{\omega^2}=1-\frac{\omega_{p_e}^2}{\omega^2}\left[\frac{\omega}{\omega-\Omega}\frac{\left\{1+3\Lambda\right\}}{\Lambda_1}+\left\{\begin{array}{c} \frac{(1+3\Lambda-\Lambda_1)}{\Lambda_1} \\ +\left(\frac{k_{\parallel}^2v_{T_{\parallel}}^2}{(\omega-\Omega)^2}\right)\left(\frac{\frac{1}{2}+\Lambda}{\Lambda_1}\right)\end{array}\right\}\left(\frac{T_{\perp}}{T_{\parallel}}-1\right)\right]$$

$$c^2k_{\parallel}^2 = \omega^2 - \omega_{p_e}^2 \left[\frac{\omega}{\omega - \Omega} \frac{\{1+3\Lambda\}}{\Lambda_1} + \left\{ \begin{array}{c} \frac{(1+3\Lambda-\Lambda_1)}{\Lambda_1} \\ + \left(\frac{k_{\parallel}^2 v_{T_{\parallel}}^2}{(\omega-\Omega)^2} \right) \left(\frac{\frac{1}{2}+\Lambda}{\Lambda_1} \right) \end{array} \right\} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) \right]$$

$$\frac{c^2k_{\parallel}^2}{\omega_{p_e}^2} = \left[\begin{array}{c} \frac{\omega^2}{\omega_{p_e}^2} - \frac{\omega}{\omega-\Omega} \frac{\{1+3\Lambda\}}{\Lambda_1} - \frac{(1+3\Lambda-\Lambda_1)}{\Lambda_1} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) \\ - \left\{ \frac{k_{\parallel}^2 v_{T_{\parallel}}^2}{(\omega-\Omega)^2} \right\} \left\{ \frac{\frac{1}{2}+\Lambda}{\Lambda_1} \right\} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) \end{array} \right]$$

$$\frac{c^2k_{\parallel}^2}{\omega_{p_e}^2} \left[1 + \frac{\omega_{p_e}^2}{c^2} \left\{ \frac{v_{T_{\parallel}}^2}{(\omega-\Omega)^2} \right\} \left\{ \frac{\frac{1}{2}+\Lambda}{\Lambda_1} \right\} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) \right] = \left[\begin{array}{c} \frac{\omega^2}{\omega_p^2} - \frac{\omega}{\omega-\Omega} \frac{\{1+3\Lambda\}}{\Lambda_1} \\ - \frac{(1+3\Lambda-\Lambda_1)}{\Lambda_1} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) \end{array} \right]$$

$$\frac{c^2k_{\parallel}^2}{\omega_{p_e}^2} = - \left(\frac{\frac{\omega}{\omega-\Omega} \frac{\{1+3\Lambda\}}{\Lambda_1} + \frac{(1+3\Lambda-\Lambda_1)}{\Lambda_1} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) - \frac{\omega^2}{\omega_p^2}}{\left[1 + \frac{\omega_p^2}{c^2} \left\{ \frac{v_{T_{\parallel}}^2}{(\omega-\Omega)^2} \right\} \left\{ \frac{\frac{1}{2}+\Lambda}{\Lambda_1} \right\} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) \right]} \right)$$

$$\frac{ck_{\parallel i}}{\omega_{p_e}} = \sqrt{\frac{\frac{\omega}{\omega-\Omega} \frac{\{1+3\Lambda\}}{\Lambda_1} + \frac{(1+3\Lambda-\Lambda_1)}{\Lambda_1} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) - \frac{\omega^2}{\omega_p^2}}{\left[1 + \frac{\omega_p^2}{c^2} \left\{ \frac{v_{T\parallel\alpha}^2}{(\omega-\Omega)^2} \right\} \left\{ \frac{\frac{1}{2}+\Lambda}{\Lambda_1} \right\} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) \right]}}$$

$$k_{\parallel i} = \frac{\omega_{p_e}}{c} \sqrt{\frac{\frac{\omega}{\omega-\Omega} \frac{\{1+3\Lambda\}}{\Lambda_1} + \frac{(1+3\Lambda-\Lambda_1)}{\Lambda_1} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) - \frac{\omega^2}{\omega_p^2}}{\left[1 + \frac{\omega_p^2}{c^2} \left\{ \frac{v_{T\parallel\alpha}^2}{(\omega-\Omega)^2} \right\} \left\{ \frac{\frac{1}{2}+\Lambda}{\Lambda_1} \right\} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right) \right]}} \tag{3.27}$$

It is clear that k is not real for $\omega >> \omega_{pe}$ so it would be imaginary under such condition. Thus

$$\frac{\omega}{\omega_{pe}} > \frac{\frac{\omega}{\omega_{pe}}}{\frac{\omega}{\omega_{pe}} - \frac{\Omega}{\omega_{pe}}} \frac{1+3\Lambda}{\Lambda_1} + \frac{1+3\Lambda-\Lambda_1}{\Lambda_1} \left(\frac{T_{\perp}}{T_{\parallel}} - 1 \right)$$

3.3.2 Resonant Case ($\xi << 1$)

Similarly, expansion of Plasma Dispersion Function for the smaller argument would be

$$Z(\xi) = i\sqrt{\pi}\exp(-\xi^2) - 2\xi + \frac{4}{3}\xi^3 - \frac{8}{15}\xi^5 + ... \tag{3.28}$$

by ignoring higher order terms

thus equation (3.23) becomes

$$\frac{c^2k_\parallel^2}{\omega^2} = 1+\frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\frac{1}{k_\parallel v_{T_\parallel}}\left[\begin{array}{c} \left\{\begin{array}{c} i\sqrt{\pi}\exp(-\xi^2)\left\{1+\Lambda\xi^4+2\Lambda\xi^2+2\Lambda\right\} \\ +\frac{5}{2}\Lambda\xi+\Lambda\xi^3 \end{array}\right\} \\ +\left\{\begin{array}{c} i\sqrt{\pi}\exp(-\xi^2)\left\{\xi+\Lambda\xi^5+2\Lambda\xi^3+2\Lambda\xi\right\} \\ +\Lambda_1+\frac{5\Lambda\xi^2}{2}+\Lambda\xi^4 \end{array}\right\}\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right) \end{array}\right] \tag{3.29}$$

by ignoring higher order terms of ξ

$$\frac{c^2k_\parallel^2}{\omega^2} = 1+\frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\frac{1}{k_\parallel v_{T_\parallel}}\left[\begin{array}{c} \left\{i\sqrt{\pi}\exp(-\xi^2)\left\{1+2\Lambda\right\}+\frac{5}{2}\Lambda\xi\right\} \\ +\left\{i\sqrt{\pi}\xi\exp(-\xi^2)\left\{1+2\Lambda\right\}+\Lambda_1\right\}\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right) \end{array}\right]$$

$$\frac{c^2k_\parallel^2}{\omega^2} = 1+\frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\frac{1}{k_\parallel v_{T_\parallel}}\left[\begin{array}{c} i\sqrt{\pi}\exp(-\xi^2)\left\{1+2\Lambda\right\}+\frac{5}{2}\Lambda\xi \\ +i\sqrt{\pi}\exp(-\xi^2)\xi\left\{1+2\Lambda\right\}\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right) \\ +\Lambda_1\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right) \end{array}\right]$$

$$\frac{c^2k_\parallel^2}{\omega^2} = 1+\frac{\omega_{p_e}^2}{\omega}\left(\frac{1}{\Lambda_1}\right)\frac{1}{k_\parallel v_{T_\parallel}}\left[\begin{array}{c} i\sqrt{\pi}\exp(-\xi^2)\left\{1+2\Lambda\right\}\left\{1+\xi\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right)\right\} \\ +\frac{5}{2}\Lambda\xi+\Lambda_1\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right) \end{array}\right]$$

as $\xi = \frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}$

$$c^2k_\parallel^2 = \omega^2+\omega_{p_e}^2\left(\frac{1}{\Lambda_1}\right)\frac{\omega}{k_\parallel v_{T_\parallel}}\left[\begin{array}{c} i\sqrt{\pi}\exp\left[-\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)^2\right]\left\{1+2\Lambda\right\} \\ \times\left\{1+\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right)\right\} \\ +\frac{5}{2}\Lambda\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)+\Lambda_1\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right) \end{array}\right]$$

$$k_\parallel^2 = \frac{\omega^2}{c^2} + \frac{\omega_{p_e}^2}{c^2}\left(\frac{1}{\Lambda_1}\right)\frac{\omega}{k_\parallel v_{T_\parallel}}\left[\begin{array}{c} i\sqrt{\pi}\exp\left[-\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)^2\right]\{1+2\Lambda\} \\ \times\left\{1+\left(\frac{\omega-\Omega}{\omega}\right)\left(\frac{T_\perp}{T_\parallel}-1\right)\right\} \\ +\frac{5}{2}\Lambda\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)+\Lambda_1\frac{k_\parallel v_{T_\parallel}}{\omega}\left(\frac{T_\perp}{T_\parallel}-1\right)\end{array}\right]$$

$$k_\parallel^2 = \frac{\omega^2}{c^2} + \frac{\omega_{p_e}^2}{c^2}\left[\begin{array}{c} i\sqrt{\pi}\exp\left[-\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)^2\right]\frac{\{1+2\Lambda\}}{\Lambda_1}\frac{\omega}{k_\parallel v_{T_\parallel}}\left\{1+\left(\frac{\omega-\Omega}{\omega}\right)\left(\frac{T_\perp}{T_\parallel}-1\right)\right\} \\ +\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)\frac{\omega}{k_\parallel v_{T_\parallel}}+\frac{k_\parallel v_{T\parallel\alpha}}{\omega}\frac{\omega}{k_\parallel v_{T_\parallel}}\left(\frac{T_\perp}{T_\parallel}-1\right)\end{array}\right]$$

$$k_\parallel^2 = \frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2}\left[\begin{array}{c} i\sqrt{\pi}\exp\left[-\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)^2\right]\frac{\{1+2\Lambda\}}{\Lambda_1}\frac{1}{k_\parallel}\left\{\frac{\omega}{v_{T_\parallel}}+\frac{\omega}{v_{T_\parallel}}\left(\frac{\omega-\Omega}{\omega}\right)\left(\frac{T_\perp}{T_\parallel}-1\right)\right\} \\ +\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)\frac{\omega}{k_\parallel v_{T_\parallel}}+\left(\frac{T_\perp}{T_\parallel}-1\right)\end{array}\right]$$

$$k_\parallel^2 - \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}\left[i\sqrt{\pi}\exp\left[-\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)^2\right]\frac{\{1+2\Lambda\}}{\Lambda_1}\frac{1}{k_\parallel}\left\{\frac{\omega}{v_{T_\parallel}}+\left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)\left(\frac{T_\perp}{T_\parallel}-1\right)\right\}\right]$$
$$-\frac{\omega_p^2}{c^2}\left\{\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)\frac{\omega}{k_\parallel v_{T_\parallel}}+\left(\frac{T_\perp}{T_\parallel}-1\right)\right\} = 0 \quad (3.30)$$

let the wave number as complex and wave frequency as real

$$k_\parallel = k_r + ik_i$$
$$k_\parallel^2 = k_r^2 - k_i^2 + 2ik_rk_i$$

by considering the spatial profile of the wave $k_i << k_r$, thus

$$k_\parallel^2 = k_r^2 + 2ik_rk_i$$

$$k_r^2 + 2ik_rk_i - \frac{\omega^2}{c^2} - \frac{\omega_{p_e}^2}{c^2}\left[\begin{array}{c} i\sqrt{\pi}\exp\left[-\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)^2\right]\frac{\{1+2\Lambda\}}{\Lambda_1} \\ \times\frac{1}{k_\parallel}\left\{\frac{\omega}{v_{T_\parallel}} + \left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)\left(\frac{T_\perp}{T_\parallel}-1\right)\right\} \end{array}\right]$$

$$-\frac{\omega_p^2}{c^2}\left\{\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)\frac{\omega}{k_\parallel v_{T_\parallel}} + \left(\frac{T_\perp}{T_\parallel}-1\right)\right\} = 0 \qquad (3.31)$$

by splitting the real part and imaginary parts;

Real Part

$$k_r^2 - \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}\left(\frac{T_\perp}{T_\parallel}-1\right) - \frac{\omega_p^2}{c^2}\left\{\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{k_\parallel v_{T_\parallel}}\right)\frac{\omega}{k_\parallel v_{T_\parallel}}\right\} = 0$$

$$k_r^2 - \frac{\omega^2}{c^2} - \frac{\omega_p^2}{c^2}\left(\frac{T_\perp}{T_\parallel}-1\right) - \frac{\omega_p^2}{c^2k_\parallel^2}\left\{\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)\frac{\omega}{v_{T_\parallel}}\right\} = 0$$

since $c^2k_\parallel^2 = \omega^2$

$$k_r = \sqrt{\frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2}\left(\frac{T_\perp}{T_\parallel}-1\right) + \frac{\omega_p^2}{\omega^2}\left\{\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)\frac{\omega}{v_{T_\parallel}}\right\}} \qquad (3.32)$$

Imaginary Part

$$2k_rk_i = \frac{\omega_{p_e}^2}{c^2}\left[\begin{array}{c} \sqrt{\pi}\exp\left[-\left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)^2\frac{1}{k_r^2}\right]\frac{\{1+2\Lambda\}}{\Lambda_1} \\ \times\frac{1}{k_r}\left\{\frac{\omega}{v_{T_\parallel}} + \left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)\left(\frac{T_\perp}{T_\parallel}-1\right)\right\} \end{array}\right]$$

$$k_i = \frac{\sqrt{\pi}}{2}\frac{\omega_{p_e}^2}{c^2}\exp\left[-\left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)^2\frac{1}{\frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2}\left(\frac{T_\perp}{T_\parallel}-1\right) + \frac{\omega_p^2}{\omega^2}\left\{\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)\frac{\omega}{v_{T_\parallel}}\right\}}\right]\frac{\{1+2\Lambda\}}{\Lambda_1}$$

$$\times\frac{1}{\frac{\omega^2}{c^2} + \frac{\omega_p^2}{c^2}\left(\frac{T_\perp}{T_\parallel}-1\right) + \frac{\omega_p^2}{\omega^2}\left\{\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)\frac{\omega}{v_{T_\parallel}}\right\}}\left\{\frac{\omega}{v_{T_\parallel}} + \left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)\left(\frac{T_\perp}{T_\parallel}-1\right)\right\}$$

$$k_i = \frac{\sqrt{\pi}}{2}\frac{\omega_p^2}{c^2}\exp\left[-\left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)^2 \frac{\frac{c^2}{\omega_p^2}}{\frac{\omega^2}{\omega_p^2}+\left(\frac{T_\perp}{T_\parallel}-1\right)+\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{\omega}\right)\frac{c^2}{v_{T_\parallel}^2}}\right]\frac{\{1+2\Lambda\}}{\Lambda_1}$$
$$\times\frac{\frac{c^2}{\omega_p^2}}{\frac{\omega^2}{\omega_p^2}+\left(\frac{T_\perp}{T_\parallel}-1\right)+\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{\omega}\right)\frac{c^2}{v_{T_\parallel}^2}}\left\{\frac{\omega}{v_{T_\parallel}}+\left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)\left(\frac{T_\perp}{T_\parallel}-1\right)\right\}$$

$$k_i = \frac{\sqrt{\pi}}{2}\exp\left[-\left(\frac{\omega-\Omega}{\omega_p}\right)^2 \frac{\frac{c^2}{v_{T\parallel\alpha}^2}}{\frac{\omega^2}{\omega_p^2}+\left(\frac{T_\perp}{T_\parallel}-1\right)+\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{\omega}\right)\frac{c^2}{v_{T_\parallel}^2}}\right]\frac{\{1+2\Lambda\}}{\Lambda_1}$$
$$\times\frac{\frac{\omega_p^2}{c^2}\times\frac{c^2}{\omega_p^2}}{\frac{\omega^2}{\omega_p^2}+\left(\frac{T_\perp}{T_\parallel}-1\right)+\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\omega-\Omega}{\omega}\right)\frac{c^2}{v_{T_\parallel}^2}}\left\{\frac{\omega}{v_{T\parallel\alpha}}+\left(\frac{\omega-\Omega}{v_{T\parallel\alpha}}\right)(\frac{T_\perp}{T_\parallel}-1)\right\}$$

$$k_i = \frac{\sqrt{\pi}}{2}\exp\left[-\left(\frac{\omega}{\omega_p}-\frac{\Omega}{\omega_p}\right)^2 \frac{\frac{c^2}{v_{T_\parallel}^2}}{\frac{\omega^2}{\omega_p^2}+\left(\frac{T_\perp}{T_\parallel}-1\right)+\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\frac{\omega}{\omega_p}-\frac{\Omega}{\omega_p}}{\frac{\omega}{\omega_p}}\right)\frac{c^2}{v_{T_\parallel}^2}}\right]\frac{\{1+2\Lambda\}}{\Lambda_1}$$
$$\times\frac{1}{\frac{\omega^2}{\omega_p^2}+\left(\frac{T_\perp}{T_\parallel}-1\right)+\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\frac{\omega}{\omega_p}-\frac{\Omega}{\omega_p}}{\frac{\omega}{\omega_p}}\right)\frac{c^2}{v_{T_\parallel}^2}}\left\{\frac{\omega}{v_{T_\parallel}}+\left(\frac{\omega-\Omega}{v_{T_\parallel}}\right)\left(\frac{T_\perp}{T_\parallel}-1\right)\right\}$$

so the imaginary wave number would be

$$k_i = \frac{\omega_p}{c}\times\frac{\sqrt{\pi}}{2}\exp\left[-\left(\frac{\omega}{\omega_p}-\frac{\Omega}{\omega_p}\right)^2 \frac{\frac{c^2}{v_{T_\parallel}^2}}{\frac{\omega^2}{\omega_p^2}+\left(\frac{T_\perp}{T_\parallel}-1\right)+\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\frac{\omega}{\omega_p}-\frac{\Omega}{\omega_p}}{\frac{\omega}{\omega_p}}\right)\frac{c^2}{v_{T_\parallel}^2}}\right]\frac{\{1+2\Lambda\}}{\Lambda_1}$$
$$\times\frac{1}{\frac{\omega^2}{\omega_p^2}+\left(\frac{T_\perp}{T_\parallel}-1\right)+\frac{5}{2}\frac{\Lambda}{\Lambda_1}\left(\frac{\frac{\omega}{\omega_p}-\frac{\Omega}{\omega_p}}{\frac{\omega}{\omega_p}}\right)\frac{c^2}{v_{T_\parallel}^2}}\left\{\begin{array}{c}\frac{c}{v_{T_\parallel}}\frac{\omega}{\omega_p}\\ +\frac{c}{v_{T_\parallel}}\left(\frac{\omega}{\omega_p}-\frac{\Omega}{\omega_p}\right)\left(\frac{T_\perp}{T_\parallel}-1\right)\end{array}\right\} \quad (3.33)$$

3.4 Energy Flux

The focus of my thesis is to calculate the energy flux of parallel propagating electromagnetic waves. Therefore, we can comprehend the energy flux density of electromagnetic waves by using the Poynting vector [34]. Positive and negative divergence ($\nabla.\mathbf{S} \gtrless \mathbf{0}$) represents the increase and decrease of energy density in the environment. Understanding of the divergence in the poynting vector is vital in exploring how the electromagnetic energy behaves in various materials henceforth providing insights into energy propagation, as well as absorption and scattering phenomena. Such a concept is very helpful to understand electromagnetic wave dynamics in fields like astrophysics, optics and telecommunication.

The poynting vector provides the way to calculate the energy flux density during the propagation of electromagnetic waves, and is important in determining where waves come from. The steady state of poynting flux theorem; $\nabla.\mathbf{S} = -P$ (where $\mathbf{S}$ represents poynting vector and P depicts power dissipation), is used to find out the energy transportation from electromagnetic waves to plasma particles [35-38]. The poynting vector and power dissipation are written as:

$$\mathbf{S} = \frac{R_e}{2\mu_o}(\mathbf{E}^* \times \mathbf{B}) \tag{3.34}$$

and

$$P = \frac{1}{2}R_e(\mathbf{J}^*.\mathbf{E}) \tag{3.35}$$

where $\mathbf{J}$ is the current density and μ_o is the permeability of free space. By using the current density which is calculated from Ampere's law ($\mathbf{J} = \frac{1}{\mu_o}(\nabla \times \mathbf{B})$), we can write equation (3.35) as:

$$P = R_e[\frac{ik(B_x E_y - B_y E_x)}{2\mu_o}]$$

$$P = -R_e[ikS] \tag{3.36}$$

thus steady state of poynting flux theorem becomes

$$\frac{dS}{dz} = -P = -k_i S_z$$

$$S(z) = S(0)\exp[-k_i z] \tag{3.37}$$

where $S(0)$ shows the energy per unit time per unit area; from where wave begins to travel, whereas value of k_i has already been calculated in equations (3.27) and (3.33) for non-resonant and resonant cases, respectively.

Chapter 4

Results and Discussions

We have derived the imaginary wave numbers for both the non-resonant and resonant cases of right handed parallel propagating electromagnetic waves with the help of kinetic theory. The focus of our study was to calculate the energy flux of right circularly polarized waves in a collision-less cairns distributed plasmas. In this regard, plots of equations (27 & 33) in accordance with equation (37) have been prepared by choosing the parameters of outer radiation belt [39]; $n_e = 1\ cm^{-3}$, $\omega_{ce} = 10^5\ s^{-1}$ and $\omega_p = 6 \times 10^4\ s^{-1}$.

4.1 Variation of Cairns Parameter with Normalized Poynting Flux Versus Distance In Non-Resonant and Resonant Cases

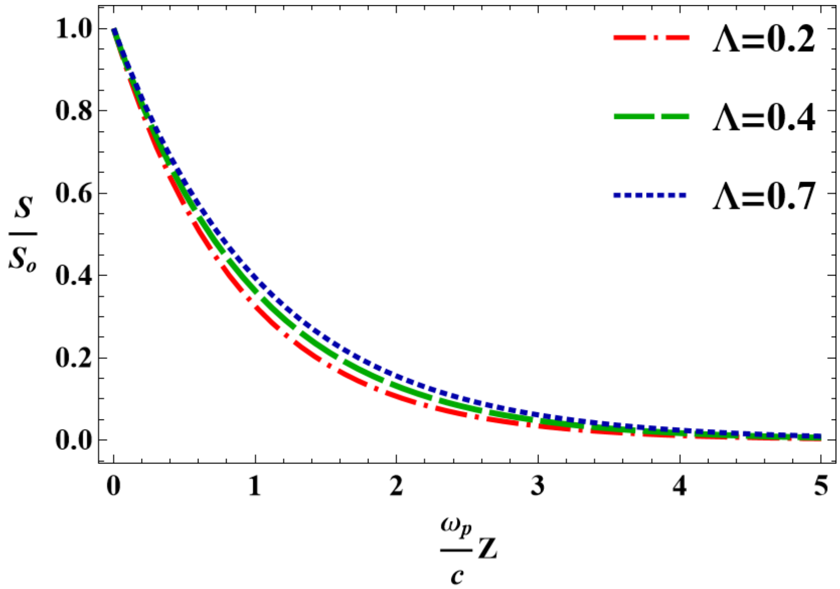

Figure 4.1. Normalized Poynting Flux ($\frac{S}{S_0}$) versus distance ($\frac{\omega_p}{c}Z$) for different values of Cairns parameter in a non-resonant case, with a fixed values; (a) $\frac{\omega}{\omega_p} = 2$. (b) $\frac{\omega_{ce}}{\omega_p} = 1.66$. (c) $\frac{v_{th}}{c} = 0.09$. (d) $\frac{T_\perp}{T_\parallel} = 2$.

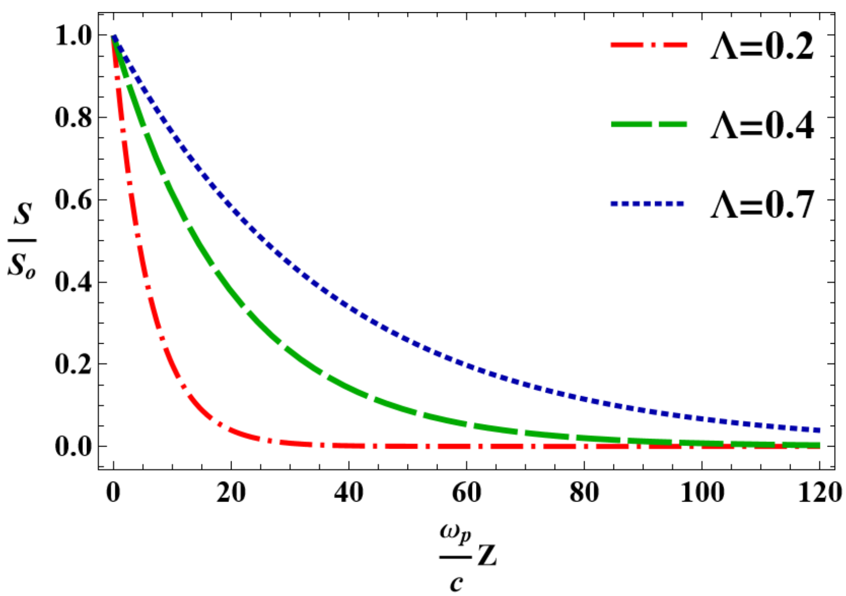

Figure 4.2. Normalized Poynting Flux ($\frac{S}{S_0}$) versus distance ($\frac{\omega_p}{c}Z$) for different values of Cairns parameter in a resonant case, with a fixed values; (a) $\frac{\omega}{\omega_p} = 0.8$. (b) $\frac{\omega_{ce}}{\omega_p} = 1.66$. (c)

$\frac{v_{th}}{c} = 0.09$. (d) $\frac{T_\perp}{T_\parallel} = 2$.

Figures 1 and 2 depicts how the energy is transported over a specific range of distance for non-resonant and resonant cases, respectively. By varying the non-thermality/Cairns parameter Λ, the wave transports its energy slowly over the shorter distance in a non-resonant case because of efficient resonance effects at higher values of non-thermality parameter (see figure 1). However, in the resonant case, the wave dissipates energy quickly over the shorter distance for smaller values of non-thermal parameter, whereas it may deliver its energy over longer distance by increasing Λ, probably due to the significant thermal effects as can be seen from figure 2.

4.2 Change in Temperature Anisotropy with Normalized Poynting Flux Versus Distance in Non-Resonant and Resonant Cases

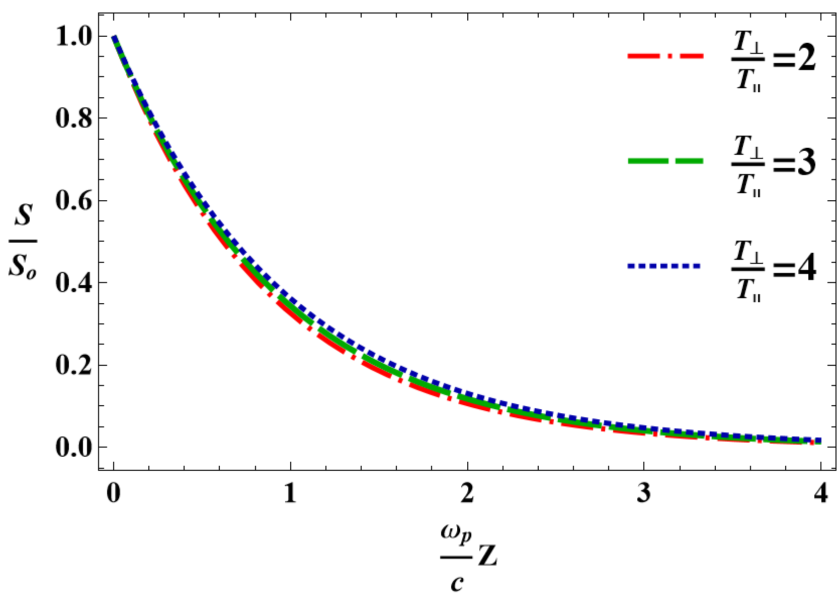

Figure 4.3. Normalized Poynting Flux ($\frac{S}{S_0}$) versus distance ($\frac{\omega_p}{c}Z$) for different values of temperature anisotropy in a non-resonant case, with a fixed values; (a) $\frac{\omega}{\omega_p} = 2$. (b) $\frac{\omega_{ce}}{\omega_p} = 1.66$. (c) $\frac{v_{th}}{c} = 0.09$. (d) $\Lambda = 0.2$.

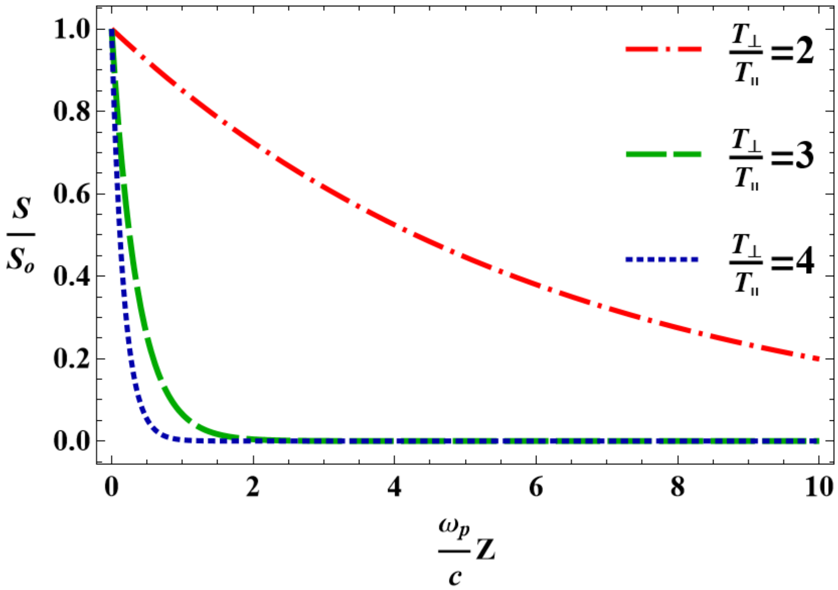

Figure 4.4. Normalized Poynting Flux ($\frac{S}{S_0}$) versus distance ($\frac{\omega_p}{c}Z$) for different values of temperature anisotropy in a resonant case, with a fixed values; (a) $\frac{\omega}{\omega_p} = 0.8$. (b) $\frac{\omega_{ce}}{\omega_p} = 1.66$. (c) $\frac{v_{th}}{c} = 0.09$. (d) $\Lambda = 0.2$.

Figures 3 and 4 represent the energy transportation under the influence of temperature anisotropy (considering $T_\perp > T_\parallel$). Temperature anisotropy causes the transportation of energy by affecting the spatial apportioning. The wave delivers its energy gradually over the distance with the increase in temperature anisotropy in a non-resonant case (figure 3) probably due to weaker thermal effects. Contrariwise, the wave transports energy over a larger distance at smaller value of temperature anisotropy. Whereas, the wave looses its energy abruptly at higher values of temperature anisotropy in a resonant case (see figure 4), because of larger number of resonant particles.

4.3 Variation of Thermal Speed with Normalized Poynting Flux Versus Distance In Non-Resonant and Resonant Cases

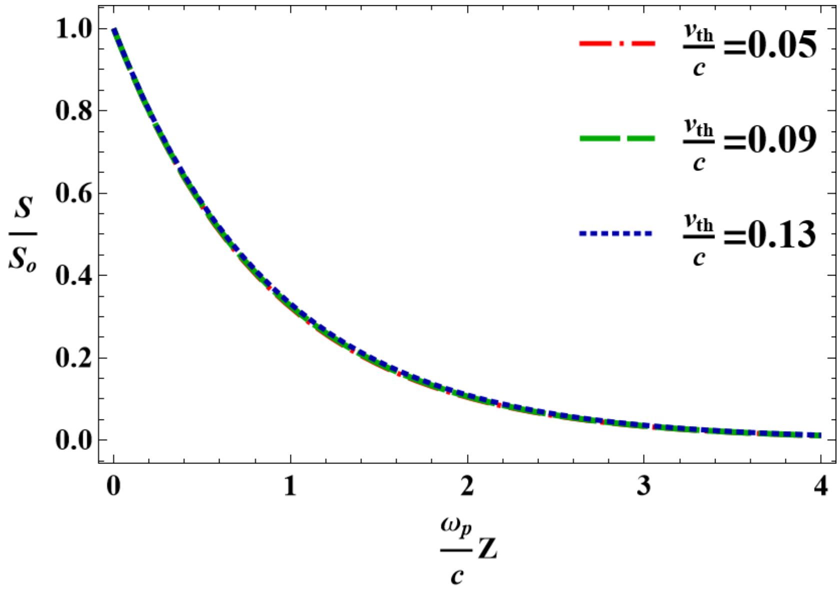

Figure 4.5. Normalized Poynting Flux ($\frac{S}{S_0}$) versus distance ($\frac{\omega_p}{c}Z$) for different values of thermal speed $\frac{v_{th}}{c}$ in a non-resonant case, with a fixed values; (a) $\frac{\omega}{\omega_p} = 2$. (b) $\frac{\omega_{ce}}{\omega_p} = 1.66$. (c) $\frac{T_\perp}{T_\parallel} = 2$. (d) $\Lambda = 0.2$.

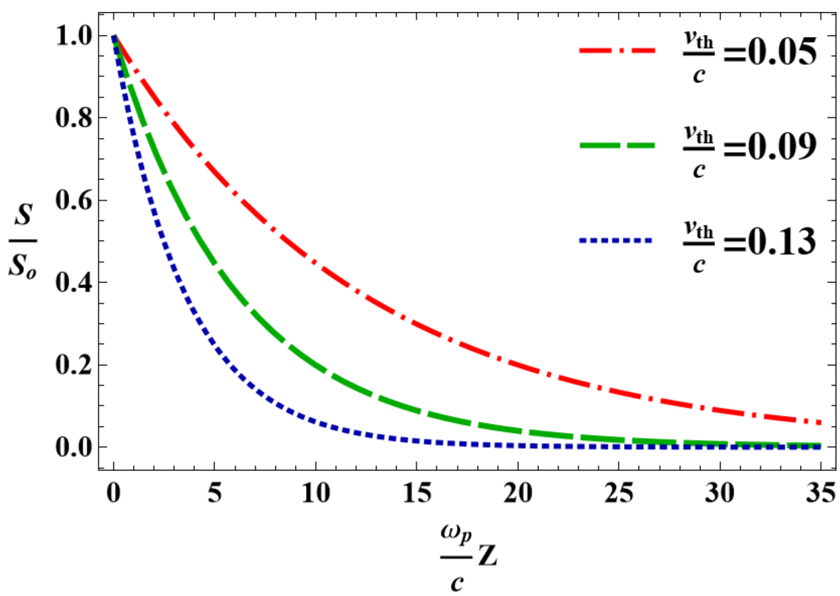

Figure 4.6. Normalized Poynting Flux ($\frac{S}{S_0}$) versus distance ($\frac{\omega_p}{c}Z$) for different values of thermal speed $\frac{v_{th}}{c}$ in a resonant case, with a fixed values; (a) $\frac{\omega}{\omega_p} = 0.8$. (b) $\frac{\omega_{ce}}{\omega_p} = 1.66$. (c)

$\frac{T_\perp}{T_\parallel} = 2$. (d) $\Lambda = 0.2$.

Figures 5 and 6 display the variation of thermal speed on the energy transportation during wave-particle interaction. The wave carries energy over a certain distance in a non-resonant case (figure 5), which clearly indicates that variation of thermal effects is not mainly affecting the energy transportation in a limiting case $\xi >> 1$. On the other hand, the wave transports energy over a longer distance at smaller value of thermal speed, however wave surrenders energy instantaneously over shorter distances by increasing the contribution of thermal effects (figure 6) in the limiting case $\xi << 1$ because for higher thermal speed, maximum number of particles would gain velocity near the resonance requirement, resulting in the loss of energy of the wave.

4.4 Variation of Wave Frequency with Normalized Poynting Flux Versus Distance In Non-Resonant and Resonant Cases

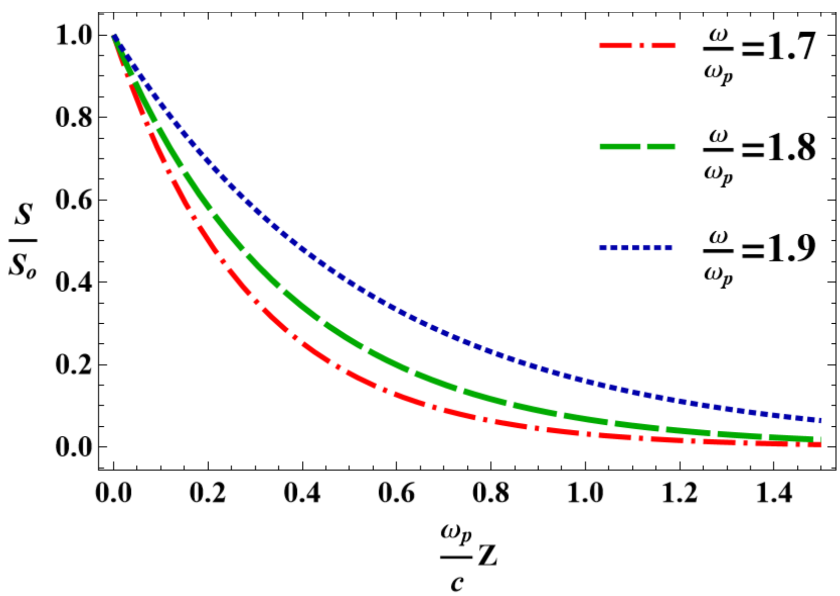

Figure 4.7. Normalized Poynting Flux ($\frac{S}{S_0}$) versus distance ($\frac{\omega_p}{c}Z$) for different values of wave frequency $\frac{\omega}{\omega_p}$ in a non-resonant case, with a fixed values; (a) $\frac{\omega_{ce}}{\omega_p} = 1.66$. (b) $\frac{v_{th}}{c} = 0.09$. (c) $\frac{T_\perp}{T_\parallel} = 2$. (d) $\Lambda = 0.2$.

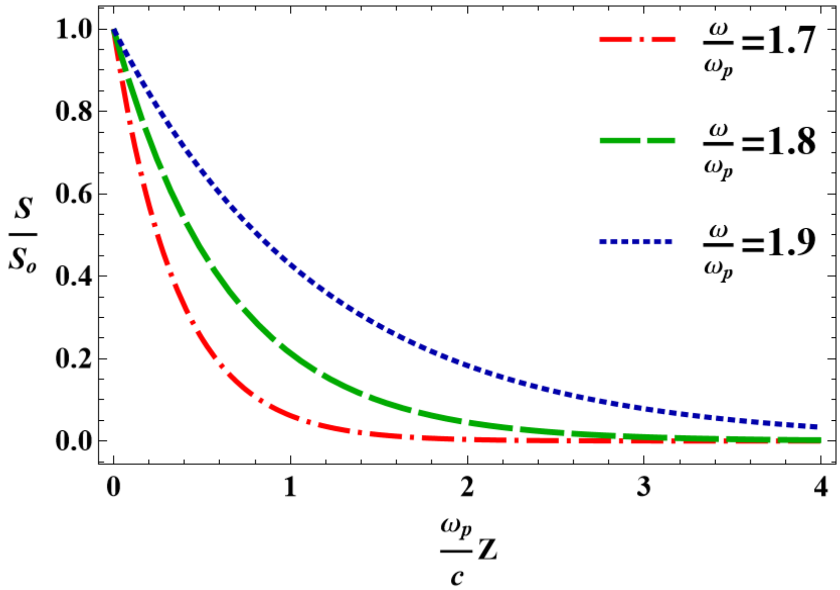

Figure 4.8. Normalized Poynting Flux ($\frac{S}{S_0}$) versus distance ($\frac{\omega_p}{c}Z$) for different values of wave frequency $\frac{\omega}{\omega_p}$ in a resonant case, with a fixed values; (a) $\frac{\omega_{ce}}{\omega_p} = 1.66$. (b) $\frac{v_{th}}{c} = 0.09$. (c) $\frac{T_\perp}{T_\parallel} = 2$. (d) $\Lambda = 0.2$.

In figures 7 and 8, we have shown the effect of the normalized wave frequency ($\frac{\omega}{\omega_{pe}}$) on the energy transportation caused by non-resonant and resonant interactions, respectively. In both cases, the wave delivers its energy quickly at smaller frequencies because resonant interactions permit waves to deliver energy more effectively to the particles. However, wave is carrying energy over longer distances at higher frequencies because of the less absorption and wave frequencies do not match with the resonance condition.

Chapter 5

Conclusion

To summarize, we have studied the energy transport of the circularly polarized electromagnetic waves in a Cairns distributed plasma. In this regard, the dynamical behavior of electrons is described by the coupled set of Vlasov-Maxwell equations. A significant change in the energy flux has been found by varying the nonthermal parameter, electron thermal speed, wave frequency and electron temperature anisotropy effects. It may be observed that wave transports its energy over a longer distance for resonant interaction caused by more resonant particles while at shorter distance in non-resonant case possibly due to weaker thermal effects in the Cairns distributed plasmas. One also confirms that a significant deflection from the Maxwellian results occurs in the presence of nonthermal distributed plasmas. The results are relevant for heating plasmas in fusion research, understanding astrophysical phenomena, diagnosing plasmas and developing advanced laser and tele-communications equipment.

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