Four Step $\mathfrak{F}$-Stable Iterative Technique for Garcia-Falset Mapping with Improved Convergence and Various Applications

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THESIS ACCEPTANCE CERTIFICATE

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DEDICATION

This effort is dedicated to my parents, sister, and late grandparents, for their infinite support and encouragement. May ALLAH Rub ul Izzat bless them with the best of health, life and akhira.

Ameen

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I lower my head to **ALLAH Almighty**, Who is most merciful and beneficial, Who utterly has all the power and knowledge, and Who blessed me with such an opportunity to enhance my knowledge.
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Abstract

In this thesis, First we present a new form of iterative technique. This new iterative strategy for contraction mapping outperforms previous methods such as Picard, Thakur et al., and Asghar Rahimi and many more. We compared these iterative methods against a new iterative strategy and presented the findings graphically. The research investigates the convergence of this new iteration to the fixed point in uniformly convex Banach spaces using Garcia Falset operator and worked on its stability. We also present the working of this iterative technique on to a boundary value problem to support our findings. Moreover, exhibit the applicability of four-step iteration process in the delay differential equations. Finally, we design a training problem for an implicit neural network that can be considered as an extension of traditional Feed forward network.

Chapter 1

Introduction

Fixed point theorems are derived for abstract metric space mappings with single or set values. The fixed-point theorems for set-valued mappings, in particular, are very useful in optimal control theory and have been widely employed to solve a variety of economic and game theory problems. However, if F is not self-mapping, the aforementioned equation may not have a fixed point. In this scenario, it is worthwhile to find an approximation answer $\mathfrak{x}$ such that the error $d(\mathfrak{x}, \mathfrak{T}\mathfrak{x})$ is minimized. This is the concept underlying the best approximation theory.

The appropriate fixed/best-proximity point theorem, together with the unique features of the underlying function space, can have a significant impact on the solvability of nonlinear operator equations. We want to create a venue for scholars to promote, communicate, and discuss new issues and discoveries in this field.

In the early 1900s, scholars expanded the Banach contraction principle to encompass multiple abstract spaces and developed additional types of contraction mapping. Initially, the notion of fixed points focuses solely on locating them. However, in other cases, it is impossible to locate fixed locations. Using direct approaches, particularly for nonlinear mapping. As a result, iterative approximations are offered as an alternative method for estimating the fixed point. Many issues in applied physics and mathematical engineering are too complex to solve with traditional analytical approaches [1, 2, 3, 4, 5]. In such instances, fixed-point theory proposes several alternate strategies for getting the desired results. To begin, we write the problem as a fixed point equation, ensuring that the fixed point set and solution set are equal. Establishing the existence of a fixed point for an equation implies the existence of a solution for it.

In 1922, Banach introduced the Banach contraction principle (BCP), which states that each contraction has a unique fixed point.

In 1931, Caccioppoli described the Banach fixed point theorem for the first time. Numerous studies have now been published on the generalization and extension of Banach's Principle to both single-valued and multi-valued mappings. Essentially, the generalization and extension of Banach's result is dependent on two factors: changing the underlying structure of space to a more generalized form or changing the nature or conditions of contractive mapping, e.g. [6] see and all the relevant references.

In 1965, Kirk [7], Browder [8], and Gohde [9] independently developed a Fixed-point theorem for nonexpansive mappings in a uniformly convex Banach space (UCBS). Soon after, Goebel [10] established a basic proof for the Kirk-Browder-Gohde Theorem. Several writers have demonstrated that nonexpansive mappings occur spontaneously while studying nonlinear problems in various distance space topologies . Thus, it is entirely reasonable to investigate extensions of these mappings.

Iterative approaches are extremely effective at solving nonlinear equations and systems of equations. Some notable approaches are the Picard [17] , Mann [18], Ishikawa [19], Noor [20], Abbas and Nazir [22], Thakur et al.[23] iterations. The Picard technique [17], one of the earliest, uses fixed-point iteration, making it crucial in numerical analysis. The Mann iteration [18] provides a relaxation approach, which improves convergence rates. Noor and Thakur's solutions apply these concepts to a broader range of situations, including multi-step processes to improve efficiency. Nazir and Abbass iterations [22] improve these techniques by solving specific convergence and stability issues, making them useful in applied mathematics and computational science. Researchers are constantly improving these iterative strategies to obtain faster convergence and higher accuracy.

In Chapter 2, basic terminology and notations such as fixed point, contraction, metric space with its completeness and convergence, and Garcia-Falset mapping will be covered. For clarification, examples of the Banach's fixed point theorem, Iterative Methods, $\mathfrak{T}$-Stable fixed point, and implicit neural network are provided.

In Chapter 3, we will see iterative approach in Banach space by considering the reviewed papers. Additionally, proofs of the theorems pertaining to the existence, uniqueness, and robustness of the fixed point of these kinds of mappings are provided, along with application of these kind of iterative technique.

In Chapter 4, we will cover our primary findings about the iterative approach in Banach space. Additionally, proofs of the theorems pertaining to the existence, uniqueness, and robustness of the fixed point of these kinds of mappings will be provided, along with certain corollaries that stem from our primary findings.

In Chapter 5, we investigate the applications of our proven conclusions including the well-posedness of the Fixed point problem.
We wrap up our thesis here and outline some ideas for more research.

Chapter 2

Preliminaries

This chapter compiles, from literature, a number of definitions and outcomes of the well-known theorems related to contraction, contractive, non-expansive and Garcia-Falset mappings of Banach space into itself from literature along with some examples.

2.1 Fundamental Concepts

We denote the set of positive integers by $\mathbb{N}$ and the set of real numbers by $\mathbb{R}$. Let $\mathfrak{S}$ be the a uniformly convex Banach space and $\mathfrak{C}$ be a nonempty closed convex subset of $\mathfrak{S}$.

Fixed point theory is an area of mathematics that studies the existence and uniqueness of specific mappings in abstract spaces. This theory is based on the Banach contraction principle, which demonstrates that the fixed point of contraction mapping is unique in entire metric spaces.

Many problems in science and engineering defined by nonlinear functional equations can be solved by converting them to an analogous fixed-point problem. In fact, an operator equation $\mathfrak{G}\mathfrak{x} = 0$ can be represented as a fixed-point equation is a self-mapping over a suitable domain. Fixed point theory provides essential tools for solving problems in various branches of mathematical analysis, including split feasibility problems, variational inequality problems, nonlinear optimization problems, equilibrium problems, complementarity problems, selection and matching problems, and proving the existence of integral and differential equation solutions.

2.2 Banach Space

Definition 2.2.1. *A Banach space is a normed linear space that is a complete metric space.*

Definition 2.2.2. *Let* $(\mathfrak{X}, d)$ *be a metric space and* $\{\mathfrak{a}_\mathfrak{n}\}$ *be a sequence in* $\mathfrak{X}$. *Then,* $\{\mathfrak{a}_\mathfrak{n}\}$ *is said to be convergent sequence if there exists* $\mathfrak{p} \in \mathfrak{X}$ *such that*

$$lim_{\mathfrak{n}\to\infty} \|\mathfrak{a}_\mathfrak{n} - \mathfrak{p}\| = 0$$

where $\mathfrak{p}$ *represents the limit of the given sequence* $\{\mathfrak{a}_\mathfrak{n}\}$ *written as* $lim_{\mathfrak{n}\to\infty}\mathfrak{a}_\mathfrak{n} = \mathfrak{p}$, *we say that the sequence converges to* $\mathfrak{p}$.

Definition 2.2.3. *A sequence of numbers* $\{\mathfrak{a}_\mathfrak{n}\} \in \mathfrak{X}$ *is a cauchy sequence if for each* $\epsilon > 0$, *there exists* $\mathbb{N} = \mathbb{N}(\epsilon)$ *such that* $\|\mathfrak{a}_\mathfrak{m} - \mathfrak{a}_\mathfrak{n}\| < \epsilon$ *for each* $\mathfrak{m}, \mathfrak{n} > \mathbb{N}$.

Definition 2.2.4. *A metric space* $(\mathfrak{X}, d)$ *is said to be complete if every Cauchy sequence in* $\mathfrak{X}$ *converges to a point in* $\mathfrak{X}$. *Note that the limit of the convergent sequence belongs to* $\mathfrak{X}$ *and is always unique.*

Definition 2.2.5. *A Banach space* $\mathfrak{S}$ *is called uniformly convex if for each* $\epsilon > 0$, *there is a* $\delta > 0$ *such that, for all* $\|\mathfrak{x}\| \leq 1$, $\|\mathfrak{y}\| \leq 1$ *and* $\| \mathfrak{x} - \mathfrak{y}\| \geq \epsilon$, $\| \mathfrak{x} + \mathfrak{y}\| \leq 2(1 - \delta)$ *holds.*

Lemma 2.2.6. *[32]. Let* $\mathfrak{C}$ *be a uniformly convex Banach space and* $0 < \mathfrak{a} \leq \mathfrak{l}_\mathfrak{n} \leq \mathfrak{b}_\mathfrak{n} < 1$ *for all* $\mathfrak{n} \in \mathbb{N}$. *Let* $\mathfrak{x}_\mathfrak{n}$ *and* $\mathfrak{y}_\mathfrak{n}$ *be two sequences such that*

$$\lim_{\mathfrak{n}\to\infty} \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| \leq \mathfrak{r}$$

$$\lim_{\mathfrak{n}\to\infty} \|\mathfrak{y}_\mathfrak{n} - \mathfrak{p}\| \leq \mathfrak{r}$$

and

$$\lim_{\mathfrak{n}\to\infty} \left\|\mathfrak{l}_\mathfrak{n}\mathfrak{x}_\mathfrak{n} + (1 - \mathfrak{l}_\mathfrak{n})\mathfrak{y}_\mathfrak{n}\right\| = \mathfrak{r}$$

hold for some $\mathfrak{r} \geq 0$. *Then*

$$\lim_{\mathfrak{n}\to\infty} \|\mathfrak{x}_\mathfrak{n} - \mathfrak{y}_\mathfrak{n}\| = \mathfrak{r}$$

Definition 2.2.7. *[33] Let* $\{\mathfrak{a}_n\}$ *and* $\{\mathfrak{b}_\mathfrak{n}\}$ *be two sequences of real numbers converging to* $\mathfrak{a}$ *and* $\mathfrak{b}$, *respectively. If*

$$\lim_{\mathfrak{n}\to\infty} \frac{|\mathfrak{a}_\mathfrak{n} - \mathfrak{a}|}{|\mathfrak{b}_\mathfrak{n} - \mathfrak{b}|} = 0,$$

then $\{\mathfrak{a}_\mathfrak{n}\}$ *converges faster than* $\{\mathfrak{b}_\mathfrak{n}\}$.

Definition 2.2.8. *[33] Suppose that for two fixed point iteration processes* $\{\mathfrak{x}_\mathfrak{n}\}$ *and* $\{\mathfrak{u}_\mathfrak{n}\}$, *both converging to the same fixed point* $\mathfrak{p}$, *the error estimates*

$$\begin{cases} ||\mathfrak{x}_\mathfrak{n} - \mathfrak{p}|| & \leq \mathfrak{a}_\mathfrak{n}, \quad \forall \mathfrak{n} \in \mathbb{N} \\ ||\mathfrak{u}_\mathfrak{n} - \mathfrak{p}|| & \leq \mathfrak{b}_\mathfrak{n}, \quad \forall \mathfrak{n} \in \mathbb{N} \end{cases}$$

are available where $\{\mathfrak{a}_\mathfrak{n}\}$ *and* $\{\mathfrak{b}_\mathfrak{n}\}$ *are two sequences of positive numbers converging to zero. If* $\{\mathfrak{a}_\mathfrak{n}\}$ *converges faster than* $\{\mathfrak{b}_\mathfrak{n}\}$*, then* $\{\mathfrak{x}_\mathfrak{n}\}$ *converges faster than* $\{\mathfrak{u}_\mathfrak{n}\}$ *to* $\mathfrak{p}$.

Lemma 2.2.9. *[34] Let* $\mathfrak{a},\ \mathfrak{b} \in \mathfrak{C}$*. If* $\|\mathfrak{a}\| \leq 1$ *,* $\|\mathfrak{b}\| \leq 1$ *and* $\|\mathfrak{a} - \mathfrak{b}\| \geq \epsilon > 0$ *, then*

$$\left\|\lambda\mathfrak{a} + (1-\lambda)\mathfrak{b}\right\| \leq 1 - 2\lambda(1-\lambda)\ \delta(\epsilon) \qquad for \quad 0 \leq \lambda < 1$$

.

2.3 Fixed Point Theorems

A **fixed point** of any mapping is such a point which remains unchanged or invariant under the given mapping. If $\mathfrak{p}$ is a fixed point of a function $\mathfrak{f}$ i.e. $\mathfrak{f}(\mathfrak{p}) = \mathfrak{p}$, then $\mathfrak{p}$ belongs to both the domain and the codomain of $\mathfrak{f}$.

Definition 2.3.1. ***(Fixed Point)*** *Given a non-empty set* $\mathfrak{X}$ *and a mapping* $\mathfrak{f} : \mathfrak{X} \to \mathfrak{Y}$*, the problem of finding a point* $\mathfrak{p} \in \mathfrak{X}$ *such that* $\mathfrak{f}(\mathfrak{p}) = \mathfrak{p}$ *is called fixed point problem and the point* $\mathfrak{p}$ *is called fixed point of the mapping.*

Remark 1. Root finding problem in terms of $\mathfrak{f}$, $\mathfrak{f}(\mathfrak{p}) = 0$, is equivalent to fixed point problem $\mathfrak{g}(\mathfrak{p}) = \mathfrak{p}$ in terms of $\mathfrak{g}$ where $\mathfrak{f}(\mathfrak{p}) = \mathfrak{p} - \mathfrak{g}(\mathfrak{p})$.

Definition 2.3.2. *A mapping* $\mathfrak{T}$ *of a metric space* $\mathfrak{C}$ *into itself satisfies* ***Lipschitz condition*** *if there exists a real number* $\mathfrak{H}$ *such that*

$$\|\ \mathfrak{T}\mathfrak{x} -\ \mathfrak{T}\mathfrak{y}\| \leq \mathfrak{H}\ \|\ \mathfrak{x} - \mathfrak{y}\| \quad for\ \ all\ \ \mathfrak{x}, \mathfrak{y} \in \mathfrak{C}$$

.

Definition 2.3.3. *It is well known that a mapping* $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ *is called* **contraction mapping** *if there is a constant* $\mathfrak{H} \in\ [0,1)$ *that*

$$\|\ \mathfrak{T}\mathfrak{x} -\ \mathfrak{T}\mathfrak{y}\| \leq \mathfrak{H}\ \|\ \mathfrak{x} -\ \mathfrak{y}\|$$

for all $\mathfrak{x}, \mathfrak{y} \in \mathfrak{C}$*. It is called* **nonexpansive** *if* $\mathfrak{H} = 1$*. An element* $\mathfrak{p} \in \mathfrak{C}$ *is said to be a fixed point of* $\mathfrak{T}$ *if* $\mathfrak{T}\mathfrak{p} = \mathfrak{p}$ *and the set of fixed points of* $\mathfrak{T}$ *is denoted by* $\mathfrak{F}(\mathfrak{T})$*.*

Lemma 2.3.4. *[25] Suppose $\mathfrak{T}$ is non-expansive self-map whose domain of definition is possibly a subset $\mathfrak{C}$ of Banach space with a fixed point, namely, $\mathfrak{p}$. In such a case, the estimate $\|\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{T}\mathfrak{p}\| \leq \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\|$ holds for all $\mathfrak{p} \in \mathfrak{C}$ and $\mathfrak{p} \in \mathfrak{F}(\mathfrak{T})$.*

Definition 2.3.5. *A mapping is called* **contractive – like mapping** *if there is a constant $\mathfrak{H} \in [0,1)$*

$$\| \mathfrak{T}\mathfrak{x} - \mathfrak{p}\| \leq \mathfrak{H} \| \mathfrak{x} - \mathfrak{p}\| \quad \forall\ \mathfrak{x} \in \mathfrak{C}, \ and \quad \mathfrak{p} \in \mathfrak{F}(\mathfrak{T}). \tag{2.1}$$

Now, we will define **Garcia – Falset mapping**.

Definition 2.3.6. *[11] Let $\mathfrak{C} \neq \phi$ is a subset of a Banach space, and $\mathfrak{T}$ is a self-map on $\mathfrak{C}$. Then $\mathfrak{T}$ is a Garcia-Falset mapping with $\mathfrak{F}(\mathfrak{T}) \neq \phi$, then $\mathfrak{T}$ satisfies*

$$\|\mathfrak{x} - \mathfrak{T}\mathfrak{y}\| \leq \mu \|\mathfrak{x} - \mathfrak{T}\mathfrak{x}\| + \|\mathfrak{x} - \mathfrak{y}\| \tag{2.2}$$

for some $\mu \geq 1$ and $\forall\ \mathfrak{x}, \mathfrak{y} \in \mathfrak{C}$.

Proposition 1. [11] Assume that $\mathfrak{C} \neq \phi$ is a subset of a Banach space, and $\mathfrak{T}$ is a self-map on $\mathfrak{C}$. If $\mathfrak{T}$ is a Garcia-Falset mapping with $\mathfrak{F}(\mathfrak{T}) \neq \phi$, then for every $\mathfrak{x} \in \mathfrak{C}$ and $\mathfrak{p} \in \mathfrak{F}(\mathfrak{T})$, we have $\|\mathfrak{T}\mathfrak{x} - \mathfrak{T}\mathfrak{p}\| \leq \|\mathfrak{x} - \mathfrak{p}\|$

Now, we will see an example for a self-map $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$, endowed with condition (2.2), but it is not a nonexpansive mapping.

Example 2.3.1. Consider $\mathfrak{C} = [2, 7]$ a self-map $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ defined by

$$\mathfrak{T}\mathfrak{x} = \begin{cases} \frac{2\ \mathfrak{x}+5}{3}, & if \quad \mathfrak{x} \in \mathfrak{C}_1 = [2,7) \\ \\ 5, & if \quad \mathfrak{x} \in \mathfrak{C}_2 = \{7\}. \end{cases}$$

To observe that $\mathfrak{T}$ satisfies condition (2.2), we will check, we have $\|\mathfrak{x} - \mathfrak{T}\mathfrak{y}\| \leq \mu \|\mathfrak{x} - \mathfrak{T}\mathfrak{x}\| + \|\mathfrak{x} - \mathfrak{y}\|$, *for some* $\mu \geq 1$ and $\forall\ \mathfrak{x}, \mathfrak{y} \in \mathfrak{C}$.

For this, we will fix the value of $\mu = 4$ and discuss the following cases.

(M_1) : Choose any $\mathfrak{x}, \mathfrak{y} \in \mathfrak{C}_2$, then $\mathfrak{T}\mathfrak{x} = 5 = \mathfrak{T}\mathfrak{y}$. We have,

$$\begin{aligned} \|\mathfrak{x} - \mathfrak{T}\mathfrak{y}\| &= |\mathfrak{x} - \mathfrak{T}\mathfrak{y}| \\ &= |\mathfrak{x} - 5| \\ &= |\mathfrak{x} - \mathfrak{T}\mathfrak{x}| \\ &\leq 4\ |\mathfrak{x} - \mathfrak{T}\mathfrak{x}| + |\mathfrak{x} - \mathfrak{y}| \\ &= 4\ ||\mathfrak{x} - \mathfrak{T}\mathfrak{x}|| + ||\mathfrak{x} - \mathfrak{y}||. \end{aligned} \tag{2.3}$$

(M_2) : Choose any $\mathfrak{x}, \mathfrak{y} \in \mathfrak{C}_1$, then $\mathfrak{T}\mathfrak{x} = \frac{2x+5}{3}$ and $\mathfrak{T}\mathfrak{y} = \frac{2y+5}{3}$. We have,

$$
\begin{aligned}
\|\mathfrak{x} - \mathfrak{T}\mathfrak{y}\| &= |\mathfrak{x} - \mathfrak{T}\mathfrak{y}| \\
&\leq |\mathfrak{x} - \mathfrak{T}\mathfrak{x}| + |\mathfrak{T}\mathfrak{x} - \mathfrak{T}\mathfrak{y}| \\
&= |\mathfrak{x} - \mathfrak{T}\mathfrak{x}| + |\frac{2\mathfrak{x}}{3} - \frac{2\mathfrak{y}}{3}| \\
&= |\mathfrak{x} - \mathfrak{T}\mathfrak{x}| + \frac{2}{3}|\ \mathfrak{x} - \mathfrak{y}| \\
&\leq |\mathfrak{x} - \mathfrak{T}\mathfrak{x}| + |\mathfrak{x} - \mathfrak{y}| \\
&\leq 4\ |\mathfrak{x} - \mathfrak{T}\mathfrak{x}| + |\mathfrak{x} - \mathfrak{y}| \\
&= 4\ ||\mathfrak{x} - \mathfrak{T}\mathfrak{x}||\ +\ ||\mathfrak{x} - \mathfrak{y}||\ .
\end{aligned}
$$

(M_3) : Choose any $\mathfrak{x} \in \mathfrak{C}_1$ and $\mathfrak{y} \in \mathfrak{C}_2$, then $\mathfrak{T}\mathfrak{x} = \frac{2\mathfrak{x}+5}{3}$ and $\mathfrak{T}\mathfrak{y} = 5$. We have,

$$
\begin{aligned}
\|\mathfrak{x} - \mathfrak{T}\mathfrak{y}\| &= |\mathfrak{x} - \mathfrak{T}\mathfrak{y}| \\
&= |\mathfrak{x} - 5| \\
&= 3|\frac{\mathfrak{x} - 5}{3}| \\
&= 3|\mathfrak{x} - \frac{2\mathfrak{x} + 5}{3}| \\
&= 3|\mathfrak{x} - \mathfrak{T}\mathfrak{x}| \\
&\leq 4|\mathfrak{x} - \mathfrak{T}\mathfrak{x}| \\
&\leq 4|\mathfrak{x} - \mathfrak{T}\mathfrak{x}| + |\mathfrak{x} - \mathfrak{y}| \\
&= 4\ ||\mathfrak{x} - \mathfrak{T}\mathfrak{x}||\ +\ ||\mathfrak{x} - \mathfrak{y}||\ .
\end{aligned}
$$

(M_4) : Choose any $\mathfrak{x} \in \mathfrak{C}_2$ and $\mathfrak{y} \in \mathfrak{C}_1$, then $\mathfrak{T}\mathfrak{x} = 5$ and $\mathfrak{T}\mathfrak{y} = \frac{2\mathfrak{x}+5}{3}$. We have,

$$\begin{aligned}
||\mathfrak{x}-\mathfrak{T}\mathfrak{y}|| &= |\mathfrak{x}-\mathfrak{T}\mathfrak{y}| \\
&= |\mathfrak{x}-\frac{2\mathfrak{y}+5}{3}| \\
&= |\frac{3\mathfrak{x}-2\mathfrak{y}-5}{3}| \\
&= |\frac{2\mathfrak{x}+\mathfrak{x}-2\mathfrak{y}-5}{3}| \\
&\leq |\frac{2(\mathfrak{x}-\mathfrak{y})}{3}|+|\frac{\mathfrak{x}-5}{3}| \\
&= \frac{2}{3}|\mathfrak{x}-\mathfrak{y}|+\frac{1}{3}|\mathfrak{x}-5| \\
&\leq |\mathfrak{x}-\mathfrak{T}\mathfrak{x}|+|\mathfrak{x}-\mathfrak{y}| \\
&\leq 4|\mathfrak{x}-\mathfrak{T}\mathfrak{x}|+|\mathfrak{x}-\mathfrak{y}| \\
&= 4\ ||\mathfrak{x}-\mathfrak{T}\mathfrak{x}||\ +\ ||\mathfrak{x}-\mathfrak{y}||\ .
\end{aligned}$$

From above, we proved that in each case $\mathfrak{T}$ satisfies condition (2.2). For choosing $\mathfrak{x} = 6.5$ and $\mathfrak{y} = 7$ *and fixed point of this is* $\mathfrak{p} = 5$, it is easily can be seen that $||\mathfrak{T}\mathfrak{x} - \mathfrak{T}\mathfrak{y}|| = 1 > ||\mathfrak{x} - \mathfrak{y}|| = 0.5$ but $||\mathfrak{T}\mathfrak{x} - \mathfrak{T}\mathfrak{p}|| = 1 < ||\mathfrak{x} - \mathfrak{p}|| = 1.5$. Hence $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ is Garcia-Falset mapping.

Remark 2. Every Garcia Falset mapping is not a nonexpansive mapping.

A key concept in the theory of metric spaces is the well-known Banach contraction principle, which both ensures the existence and uniqueness of fixed points of specific self-maps of metric spaces and offers a useful technique for obtaining them.

Theorem 2.3.7. *[17] Let $\mathfrak{C}$ be a complete metric space and $\mathfrak{f} : \mathfrak{C} \to \mathfrak{C}$ be a function such that there exists $\mathfrak{H} \in [0,1)$*

$$||\mathfrak{f}\mathfrak{x} - \mathfrak{f}\mathfrak{y}|| \leq \mathfrak{H}||\mathfrak{x} - \mathfrak{y}||$$

then, there exists a unique $\mathfrak{p} \in \mathfrak{C}$ such that $\mathfrak{f}(\mathfrak{p}) = \mathfrak{p}$.

Generally, we write $\mathfrak{f}\mathfrak{p}$ instead of $\mathfrak{f}(\mathfrak{p})$.

Example 2.3.2. Let $\mathfrak{C} = \mathbb{R}$ be the usual metric space. Define a function $\mathfrak{f} : \mathbb{R} \to \mathbb{R}$ by $\mathfrak{f}(\mathfrak{p}) = \imath + \frac{\mathfrak{p}}{2}$. Then, for all $\mathfrak{x}, \mathfrak{y} \in \mathbb{R}$

$$||\mathfrak{f}\mathfrak{x} - \mathfrak{f}\mathfrak{y}|| \leq \frac{1}{2}||\mathfrak{x} - \mathfrak{y}||.$$

Thus, $\mathfrak{f}$ is a contraction on $\mathbb{R}$ and by Banach theorem, it possess a unique fixed point $\mathfrak{p} = 2$.

2.4 $\mathfrak{T}$-Stable fixed point

Definition 2.4.1. *Harder and Hicks [35] Let $\mathfrak{T}:\mathfrak{C}\to\mathfrak{C}$ be a map and $(\mathfrak{C},\mathfrak{d})$ be a complete metric space. Assume that the set of fixed points of $\mathfrak{T}$ is $\mathfrak{F}(\mathfrak{T})$. Let $\{\mathfrak{b}_\mathfrak{n}\}_{\mathfrak{n}=0}^{\infty}\subset\mathfrak{C}$ denote the sequence generated by an iterative operation involving $\mathfrak{S}$ defined by*

$$\mathfrak{b}_{\mathfrak{n}+1}=\mathfrak{f}(\mathfrak{T},\mathfrak{b}_\mathfrak{n}),\quad \mathfrak{b}_\mathfrak{o}\in\mathfrak{C},\qquad \mathfrak{n}=0,1,2,...$$

The first approximation is $\mathfrak{b}_\mathfrak{o}\in\mathfrak{C}$ and $\mathfrak{f}$ is some function. Assume that the sequence $\{\mathfrak{b}_\mathfrak{n}\}_{\mathfrak{n}=0}^{\infty}$ converges to a fixed point $\mathfrak{p}$ of $\mathfrak{T}$. Let $\{\mathfrak{c}_\mathfrak{n}\}_{\mathfrak{n}=0}^{\infty}\in\mathfrak{C}$ and set $\epsilon_n=\mathfrak{d}(\mathfrak{c}_{\mathfrak{n}+1},\mathfrak{f}(\mathfrak{T},\mathfrak{c}_\mathfrak{n}))$, $\mathfrak{n}=0,1,2,...$ and the iterative technique above is $\mathfrak{T}$-stable if and only if $\lim_{\mathfrak{n}\to\infty}\epsilon_n=0$ implies $\lim_{\mathfrak{n}\to\infty}\mathfrak{c}_\mathfrak{n}=\mathfrak{p}$.

Lemma 2.4.2. *[17] Let $\mathfrak{x}_\mathfrak{n}$, $\mathfrak{n}$ be nonnegative sequences satisfying $\mathfrak{x}_{\mathfrak{n}+1}\leq\mathfrak{H}\mathfrak{x}_\mathfrak{n}+\epsilon_n$, for all $\mathfrak{n}\in\mathbb{N}$, $0\leq\mathfrak{H}<1$, $\lim_{\mathfrak{n}\to\infty}\epsilon_\mathfrak{n}=0$. Then, $\lim_{\mathfrak{n}\to\infty}\mathfrak{x}_\mathfrak{n}=0$.*

2.5 Iterative techniques

A mathematical process known as an iterative technique creates a series of better approximations for a class of problems starting with an initial value. The $\mathfrak{n}$-th approximation is then derived from the preceding ones.
Iterative techniques can solve a variety of issues, including minimization, equilibrium, and viscosity approximation in various domains [13, 14, 15, 16].
There exists some iteration processes which are often used to approximate fixed points. In 1890, Picard [17] presented an iterative scheme for approximating the fixed point which is defined by the sequence $\mathfrak{x}_\mathfrak{n}$ as

$$\begin{cases}\mathfrak{x}_1 &= \mathfrak{x}\in\mathfrak{C}\\ \mathfrak{x}_{\mathfrak{n}+1} &= \mathfrak{T}\mathfrak{x}_\mathfrak{n},\qquad \mathfrak{n}\in\mathbb{N}\end{cases}\tag{2.4}$$

Mann iterative process [18] is

$$\begin{cases}\mathfrak{x}_1 &= \mathfrak{x}\in\mathfrak{C}\\ \mathfrak{z}_\mathfrak{n} &= \mathfrak{T}((1-\mathfrak{M}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n}),\quad \mathfrak{n}\in\mathbb{N}\end{cases}\tag{2.5}$$

In 2000, Noor introduced the following three-step iteration process [20]

$$\begin{cases} \mathfrak{x}_{n+1} &= (1-\mathfrak{M}_n)\mathfrak{x}_n + \mathfrak{M}_n\mathfrak{T}\mathfrak{y}_n \\ \mathfrak{y}_n &= \mathfrak{T}((1-\mathfrak{N}_n)\mathfrak{x}_n + \mathfrak{N}_n\mathfrak{T}\mathfrak{z}_n) \\ \mathfrak{z}_n &= \mathfrak{T}((1-\mathfrak{O}_n)\mathfrak{x}_n + \mathfrak{O}_n\mathfrak{T}\mathfrak{x}_n), \quad \mathfrak{n} \in \mathbb{N} \end{cases} \tag{2.6}$$

where introduced the following $\{\mathfrak{M}_n\}$, $\{\mathfrak{N}_n\}$ and $\{\mathfrak{O}_n\}$ are in (0,1).

In 2007, Agarwal et al. introduced the following iteration process [21]

$$\begin{cases} \mathfrak{x}_{n+1} &= (1-\mathfrak{M}_n)\mathfrak{T}\mathfrak{x}_n + \mathfrak{M}_n\mathfrak{T}\mathfrak{y}_n \\ \mathfrak{y}_n &= \mathfrak{T}((1-\mathfrak{N}_n)\mathfrak{x}_n + \mathfrak{N}_n\mathfrak{T}\mathfrak{x}_n) \quad \mathfrak{n} \in \mathbb{N} \end{cases} \tag{2.7}$$

where introduced the following $\{\mathfrak{M}_n\}$, $\{\mathfrak{N}_n\}$ are in (0,1).

Recently, Abbas and Nazir introduced the following three-step iteration process [22]

$$\begin{cases} \mathfrak{x}_{n+1} &= (1-\mathfrak{M}_n)\mathfrak{T}\mathfrak{y}_n + \mathfrak{M}_n\mathfrak{T}\mathfrak{z}_n \\ \mathfrak{y}_n &= \mathfrak{T}((1-\mathfrak{N}_n)\mathfrak{T}\mathfrak{x}_n + \mathfrak{N}_n\mathfrak{T}\mathfrak{z}_n) \\ \mathfrak{z}_n &= \mathfrak{T}((1-\mathfrak{O}_n)\mathfrak{x}_n + \mathfrak{O}_n\mathfrak{T}\mathfrak{x}_n), \quad \mathfrak{n} \in \mathbb{N} \end{cases} \tag{2.8}$$

where the following $\{\mathfrak{M}_n\}$, $\{\mathfrak{N}_n\}$ and $\{\mathfrak{O}_n\}$ are in (0,1).They proved that this process converges faster than iteration process (2.7) for contractive mappings.

Thakur iteration [23] is

$$\begin{cases} \mathfrak{u}_{n+1} &= (1-\mathfrak{M}_n)\mathfrak{T}\mathfrak{z}_n + \mathfrak{M}_n\mathfrak{T}\mathfrak{y}_n \\ \mathfrak{y}_n &= \mathfrak{T}((1-\mathfrak{N}_n)\mathfrak{z}_n + \mathfrak{N}_n\mathfrak{T}\mathfrak{z}_n) \\ \mathfrak{z}_n &= \mathfrak{T}((1-\mathfrak{O}_n)\mathfrak{u}_n + \mathfrak{O}_n\mathfrak{T}\mathfrak{u}_n), \quad \mathfrak{n} \in \mathbb{N} \end{cases} \tag{2.9}$$

where introduced the following $\{\mathfrak{M}_n\}$, $\{\mathfrak{N}_n\}$ and $\{\mathfrak{O}_n\}$ are in (0,1). They proved that this process converges faster than iteration process (2.8) for contractive mappings.

JF iteration [24] is

$$\begin{cases} \mathfrak{a}_{n+1} &= \mathfrak{T}((1-\mathfrak{M}_n)\mathfrak{y}_n + \mathfrak{M}_n\mathfrak{T}\mathfrak{y}_n) \\ \mathfrak{y}_n &= \mathfrak{T}\mathfrak{z}_n \\ \mathfrak{z}_n &= \mathfrak{T}((1-\mathfrak{O}_n)\mathfrak{a}_n + \mathfrak{O}_n\mathfrak{T}\mathfrak{a}_n), \quad \mathfrak{n} \in \mathbb{N} \end{cases} \tag{2.10}$$

where introduced the following $\{\mathfrak{M}_n\}$ and $\{\mathfrak{O}_n\}$ are in (0,1).
Asghar Rahim iteration [25] is

$$\begin{cases} \mathfrak{x}_{n+1} &= (1-\mathfrak{M}_n)\mathfrak{T}\mathfrak{y}_n + \mathfrak{M}_n\mathfrak{T}\mathfrak{z}_n \\ \mathfrak{y}_n &= \mathfrak{T}((1-\mathfrak{N}_n)\mathfrak{x}_n + \mathfrak{N}_n\mathfrak{z}_n) \\ \mathfrak{z}_n &= \mathfrak{T}((1-\mathfrak{O}_n)\mathfrak{x}_n + \mathfrak{O}_n\mathfrak{T}\mathfrak{x}_n), \quad \mathfrak{n} \in \mathbb{N} \end{cases} \tag{2.11}$$

where the following $\{\mathfrak{M}_n\}$, $\{\mathfrak{N}_n\}$ and $\{\mathfrak{O}_n\}$ are in (0,1). They proved that this process converges faster than iteration process (2.9) for contractive mappings.

Picard-Thakur hybrid iteration [26] is

$$\begin{cases} \mathfrak{j}_{n+1} &= \mathfrak{V}\mathfrak{k}_n \\ \mathfrak{k}_n &= (1-\mathfrak{M}_n)\mathfrak{V}\mathfrak{m}_n + \mathfrak{M}_n\mathfrak{V}\mathfrak{l}_n \\ \mathfrak{l}_n &= (1-\mathfrak{N}_n)\mathfrak{m}_n + \mathfrak{N}_n\mathfrak{V}\mathfrak{m}_n \\ \mathfrak{m}_n &= (1-\mathfrak{O}_n)\mathfrak{j}_n + \mathfrak{O}_n\mathfrak{V}\mathfrak{j}_n, \quad \mathfrak{n} \in \mathbb{N} \end{cases} \tag{2.12}$$

where introduced the following $\{\mathfrak{M}_n\}$, $\{\mathfrak{N}_n\}$ and $\{\mathfrak{O}_n\}$ are in (0,1). They proved that this process converges faster than iteration processes (2.5, 2.6, 2.7, 2.8, 2.9) for contraction mappings.

Akanimo iteration [27] is

$$\begin{cases} \mathfrak{x}_{n+1} &= (1-\mathfrak{M}_n)\mathfrak{T}\mathfrak{z}_n + \mathfrak{M}_n\mathfrak{T}\mathfrak{y}_n \\ \mathfrak{y}_n &= \mathfrak{T}(\mathfrak{T}\mathfrak{z}_n) \\ \mathfrak{z}_n &= \mathfrak{T}((1-\mathfrak{N}_n)\mathfrak{x}_n + \mathfrak{N}_n\mathfrak{T}\mathfrak{x}_n), \quad \mathfrak{n} \in \mathbb{N} \end{cases} \tag{2.13}$$

where introduced the following $\{\mathfrak{M}_{\mathfrak{n}}\}$, $\{\mathfrak{N}_{\mathfrak{n}}\}$ are in (0,1). They proved that this process converges faster than iteration process (2.5, 2.6, 2.7, 2.8, 2.9) for contractive mappings.

Iterative methods based on fixed-point theory are essential in order to solve the heat equation. Time delays add complexity to delay differential equations, which is why fixed-point iterative techniques (2.5, 2.9, 2.11, 2.12) are employed. By approximating the system's state at each time step, these methods iteratively approach a solution, aiding in the stabilization and precise solution of equations when delays impact future behavior.

Chapter 3

Literature Review

In this Chatpter, we shall describe a comprehensive summary of papers which proved to be thought provoking and led our attention to convert the results in more generalized way. The papers and work of **Asghar Rahimi** [25], **Mohd Jubair** [28] and **Jie Jia** [26] were major landmark of our research.

3.1 Convergence Theorems For Rahimi iteration

In this subsection, we will discuss how Asghar Rahimi iteration process (2.11) converges faster than the process (2.9) in contraction mapping. For this, we presenting you convergence theorems in uniformly convex Banach spaces which theoretically proving this.

Theorem 3.1.1. *Let* $\mathfrak{C}$ *be a nonempty closed convex subset of a uniformly convex Banach space. Let* $\mathfrak{T}$ *be a contraction mapping with some constant* $\mathfrak{H} \in [0,1)$ *and fixed point* $\mathfrak{p}$*. Let* $\{\mathfrak{u}_\mathfrak{n}\}$ *be defined by the iteration process (2.9) and* $\{\mathfrak{x}_\mathfrak{n}\}$ *by (2.11) , where* $\{\mathfrak{M}_\mathfrak{n}\}$*,* $\{\mathfrak{N}_\mathfrak{n}\}$*,* $\{\mathfrak{O}_\mathfrak{n}\}$ *are in [*ϵ *, 1-* ϵ*] for any* $\mathfrak{n} \in \mathbb{N}$ *and some* ϵ *in* $(0,1)$*. Then* $\{\mathfrak{x}_\mathfrak{n}\}$ *converges faster than* $\{\mathfrak{u}_\mathfrak{n}\}$*.*

Proof. As proved in Theorem 3.1 of [23] , we have

$$\|\mathfrak{u}_{\mathfrak{n}+1} - \mathfrak{p}\| \leq \mathfrak{H}^\mathfrak{n}[1 - (1 - \mathfrak{H})\ \mathfrak{O})]^\mathfrak{n}\, \|\mathfrak{u}_1 - \mathfrak{p}\| \tag{3.1}$$

Let

$$\mathfrak{a}_{\mathfrak{n}} \;=\; \mathfrak{H}^{\mathfrak{n}}[1-(1-\mathfrak{H})\mathfrak{O}]^{\mathfrak{n}}\left\|\mathfrak{u}_1-\mathfrak{p}\right\|. \tag{3.2}$$

Now with notion to process (2.11), we have

$$\begin{aligned}
\|\mathfrak{z}_{\mathfrak{n}}-\mathfrak{p}\| &= ||\mathfrak{T}[(1-\mathfrak{O}_{\mathfrak{n}})\mathfrak{x}_{\mathfrak{n}}+\mathfrak{O}_{\mathfrak{n}}\mathfrak{T}\mathfrak{x}_{\mathfrak{n}}]-\mathfrak{p}|| \\
&\leq \mathfrak{H}\;||(1-\mathfrak{O}_{\mathfrak{n}})(\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p})+\mathfrak{O}_{\mathfrak{n}}(\mathfrak{T}\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p})|| \\
&\leq \mathfrak{H}[(1-\mathfrak{O}_{\mathfrak{n}})\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|+\mathfrak{O}_{\mathfrak{n}}\left\|\mathfrak{T}\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|] \\
&\leq \mathfrak{H}[(1-\mathfrak{O}_{\mathfrak{n}})\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|+\mathfrak{O}_{\mathfrak{n}}\mathfrak{H}\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|] \\
&= \mathfrak{H}[1-(1-\mathfrak{H})\mathfrak{O}_{\mathfrak{n}}]\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|.
\end{aligned}$$

so that

$$\begin{aligned}
\|\mathfrak{y}_{\mathfrak{n}}-\mathfrak{p}\| &= ||\mathfrak{T}[(1-\mathfrak{N}_{\mathfrak{n}})\mathfrak{x}_{\mathfrak{n}}+\mathfrak{N}_{\mathfrak{n}}\mathfrak{z}_{\mathfrak{n}}]-\mathfrak{p}|| \\
&\leq \mathfrak{H}\;||(1-\mathfrak{N}_{\mathfrak{n}})(\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p})+\mathfrak{N}_{\mathfrak{n}}(\mathfrak{z}_{\mathfrak{n}}-\mathfrak{p})|| \\
&\leq \mathfrak{H}[(1-\mathfrak{N}_{\mathfrak{n}})\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|+\mathfrak{N}_{\mathfrak{n}}\left\|\mathfrak{z}_{\mathfrak{n}}-\mathfrak{p}\right\|] \\
&\leq \mathfrak{H}[(1-\mathfrak{N}_{\mathfrak{n}})\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|+\mathfrak{N}_{\mathfrak{n}}\mathfrak{H}^2(1-(1-\mathfrak{H})\mathfrak{O}_{\mathfrak{n}})\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\| \\
&< \mathfrak{H}[(1-\mathfrak{N}_{\mathfrak{n}})\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|+\mathfrak{N}_{\mathfrak{n}}\mathfrak{H}(1-(1-\mathfrak{H})\mathfrak{O}_{\mathfrak{n}})\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\| \\
&= \mathfrak{H}[1-(1-\mathfrak{H})\mathfrak{O}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}}]\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|.
\end{aligned} \tag{3.3}$$

Thus

$$\begin{aligned}
\|\mathfrak{x}_{\mathfrak{n}+1}-\mathfrak{p}\;\| &= ||[(1-\mathfrak{M}_{\mathfrak{n}})\mathfrak{T}\mathfrak{y}_{\mathfrak{n}}+\mathfrak{M}_{\mathfrak{n}}\mathfrak{T}\mathfrak{z}_{\mathfrak{n}}]-\mathfrak{p}|| \\
&\leq \mathfrak{H}[(1-\mathfrak{M}_{\mathfrak{n}})\left\|\mathfrak{y}_{\mathfrak{n}}-\mathfrak{p}\right\|+\mathfrak{M}_{\mathfrak{n}}\left\|\mathfrak{z}_{\mathfrak{n}}-\mathfrak{p}\right\|] \\
&< \mathfrak{H}[1-(1-\mathfrak{H})(\mathfrak{O}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}})]\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|.
\end{aligned}$$

Let

$$\mathfrak{b}_{\mathfrak{n}} \;=\; \mathfrak{H}^{n}[1-(1-\mathfrak{H})\mathfrak{O}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}}]^{\mathfrak{n}}\left\|\mathfrak{x}_1-\mathfrak{p}\right\|.$$

Then

$$
\begin{aligned}
\frac{\mathfrak{b}_{\mathfrak{n}}}{\mathfrak{a}_{\mathfrak{n}}} &= \frac{\mathfrak{H}^{n}[1-(1-\mathfrak{H})\mathfrak{O}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}}]^{\mathfrak{n}}\left\|\mathfrak{x}_{1}-\mathfrak{p}\right\|}{\mathfrak{H}^{\mathfrak{n}}[1-(1-\mathfrak{H})\mathfrak{O})]^{\mathfrak{n}}\left\|\mathfrak{u}_{1}-\mathfrak{p}\right\|} \\
&= \frac{[1-(1-\mathfrak{H})\mathfrak{O}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}}]^{\mathfrak{n}}\left\|\mathfrak{x}_{1}-\mathfrak{p}\right\|}{[1-(1-\mathfrak{H})\mathfrak{O})]^{\mathfrak{n}}\left\|\mathfrak{u}_{1}-\mathfrak{p}\right\|} \to 0 \quad \mathfrak{n}\to\infty.
\end{aligned} \tag{3.4}
$$

Consequently $\mathfrak{x}_{\mathfrak{n}}$ converges faster than $\mathfrak{u}_{\mathfrak{n}}$.

□

Lemma 3.1.2. *Let $\mathfrak{C}$ be a nonempty closed convex subset of a Banach space $\mathfrak{S}$ and $\mathfrak{T}: \mathfrak{C}\to\mathfrak{C}$ be a generalized α-nonexpansive mapping. If the sequence $\mathfrak{x}_{\mathfrak{n}}$ be define by (2.11) and $\mathfrak{F}(\mathfrak{T})\neq\varnothing$, then $lim_{\mathfrak{n}\to\infty}\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|$ exists for all $\mathfrak{p}\in\mathfrak{F}(\mathfrak{T})$.*

Proof. $\mathfrak{F}(\mathfrak{T})\neq\varnothing$, let $\mathfrak{p}\in\mathfrak{F}(\mathfrak{T})$. By Proposition 1 , then we have

$$
\begin{aligned}
\left\|\mathfrak{z}_{\mathfrak{n}}-\mathfrak{p}\right\| &= ||\mathfrak{T}[(1-\mathfrak{O}_{\mathfrak{n}})\mathfrak{x}_{\mathfrak{n}}+\mathfrak{O}_{\mathfrak{n}}\mathfrak{T}\mathfrak{x}_{\mathfrak{n}}]-\mathfrak{p}|| \\
&\leq [(1-\mathfrak{O}_{\mathfrak{n}})\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|+\mathfrak{O}_{\mathfrak{n}}\left\|\mathfrak{T}\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|] \\
&\leq [(1-\mathfrak{O}_{\mathfrak{n}})\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|+\mathfrak{O}_{\mathfrak{n}}\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|] \\
&\leq \left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|.
\end{aligned} \tag{3.5}
$$

so that

$$
\begin{aligned}
\left\|\mathfrak{y}_{\mathfrak{n}}-\mathfrak{p}\right\| &= ||\mathfrak{T}[(1-\mathfrak{N}_{\mathfrak{n}})\mathfrak{x}_{\mathfrak{n}}+\mathfrak{N}_{\mathfrak{n}}\mathfrak{z}_{\mathfrak{n}}]-\mathfrak{p}|| \\
&\leq [(1-\mathfrak{N}_{\mathfrak{n}})\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|+\mathfrak{N}_{\mathfrak{n}}\left\|\mathfrak{z}_{\mathfrak{n}}-\mathfrak{p}\right\|] \\
&\leq \left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|.
\end{aligned} \tag{3.6}
$$

Thus

$$
\begin{aligned}
\left\|\mathfrak{x}_{\mathfrak{n}+1}-\mathfrak{p}\right\| &= ||\mathfrak{T}[(1-\mathfrak{M}_{\mathfrak{n}})\mathfrak{T}\mathfrak{y}_{\mathfrak{n}}+\mathfrak{M}_{\mathfrak{n}}\mathfrak{T}\mathfrak{z}_{\mathfrak{n}}]-\mathfrak{p}|| \\
&\leq [(1-\mathfrak{M}_{\mathfrak{n}})\left\|\mathfrak{y}_{\mathfrak{n}}-\mathfrak{p}\right\|+\mathfrak{M}_{\mathfrak{n}}\left\|\mathfrak{z}_{\mathfrak{n}}-\mathfrak{p}\right\|] \\
&= \left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|.
\end{aligned} \tag{3.7}
$$

This implies that $\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|$ is bounded and nonincreasing for all $\mathfrak{p}\in\mathfrak{F}(\mathfrak{T})$. Hence $lim_{\mathfrak{n}\to\infty}\left\|\mathfrak{x}_{\mathfrak{n}}-\mathfrak{p}\right\|$ exists.

□

Lemma 3.1.3. *Let $\mathfrak{C}$ be a nonempty closed convex subset of a Banach space E and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be a generalized α-nonexpansive mapping. Suppose the sequence $\mathfrak{x}_\mathfrak{n}$ be define by (2.11) and $\mathfrak{F}(\mathfrak{T}) \neq \varnothing$, then $lim_{\mathfrak{n}\to\infty} \|\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{x}_\mathfrak{n}\| = 0$.*

Proof. As from above Lemma 3.1.2, $lim_{\mathfrak{n}\to\infty} \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\|$ exists for each $p \in \mathfrak{F}(\mathfrak{T})$. Suppose that for some $\mathfrak{T} \geq 0$, we have

$$\lim_{\mathfrak{n}\to\infty} \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| \;=\; \mathfrak{t} \tag{3.8}$$

It is proved in the proof of Lemma 3.1.2 that $\|\mathfrak{z}_\mathfrak{n} - \mathfrak{p}\| \leq \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\|$. Accordingly, one has

$$\lim_{\mathfrak{n}\to\infty} sup \|\mathfrak{z}_\mathfrak{n} - \mathfrak{p}\| \leq \lim_{\mathfrak{n}\to\infty} sup \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| \;=\; \mathfrak{t} \tag{3.9}$$

Now $\mathfrak{p}$ is the point, by Proposition 1 .It follows that $\|\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| \leq \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\|$

$$\lim_{\mathfrak{n}\to\infty} sup \|\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| \leq \lim_{\mathfrak{n}\to\infty} sup \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| \;=\; \mathfrak{t}. \tag{3.10}$$

We know again by the proof Lemma 3.1.2 that $\|\mathfrak{y}_\mathfrak{n} - \mathfrak{p}\| \leq \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\|$. So, we have

$$\|\mathfrak{x}_{\mathfrak{n}+1} - \mathfrak{p}\| \leq [(1 - \mathfrak{M}_\mathfrak{n}) \|\mathfrak{y}_\mathfrak{n} - \mathfrak{p}\| + \mathfrak{M}_\mathfrak{n} \|\mathfrak{z}_\mathfrak{n} - \mathfrak{p}\|] \tag{3.11}$$

$$\leq [(1 - \mathfrak{M}_\mathfrak{n}) \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| + \mathfrak{M}_\mathfrak{n} \|\mathfrak{z}_\mathfrak{n} - \mathfrak{p}\|]. \tag{3.12}$$

It follows that

$$\|\mathfrak{x}_{\mathfrak{n}+1} - \mathfrak{p}\| - \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| \leq \frac{\|\mathfrak{x}_{\mathfrak{n}+1} - \mathfrak{p}\| - \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\|}{\mathfrak{M}_\mathfrak{n}} \leq \|\mathfrak{z}_\mathfrak{n} - \mathfrak{p}\| - \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\|$$

.

So, we have

$$\|\mathfrak{x}_{\mathfrak{n}+1} - \mathfrak{p}\| \leq \|\mathfrak{z}_\mathfrak{n} - \mathfrak{p}\| .$$

From (3.9) , we have

$$\mathfrak{t} \leq \lim_{\mathfrak{n}\to\infty} inf \left\| \mathfrak{z}_\mathfrak{n} - \mathfrak{p} \right\|. \tag{3.13}$$

Hence, from (3.9) and (3.13), we have

$$\mathfrak{t} = \lim_{\mathfrak{n}\to\infty} \left\| \mathfrak{z}_\mathfrak{n} - \mathfrak{p} \right\|$$

From (2.11) and from (3.13), we have

$$\begin{aligned}
\mathfrak{t} &= \lim_{\mathfrak{n}\to\infty} \left\| \mathfrak{z}_\mathfrak{n} - \mathfrak{p} \right\| \\
&= \lim_{\mathfrak{n}\to\infty} ||\mathfrak{T}[(1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n}] - \mathfrak{p}|| \\
&\leq \lim_{\mathfrak{n}\to\infty} ||(1-\mathfrak{O}_\mathfrak{n})(x_n - \mathfrak{p}) + \mathfrak{O}_\mathfrak{n}(\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{p})|| \\
&\leq \lim_{\mathfrak{n}\to\infty} [(1-\mathfrak{O}_\mathfrak{n}) \left\| \mathfrak{x}_\mathfrak{n} - \mathfrak{p} \right\| + \lim_{n\to\infty} \mathfrak{O}_\mathfrak{n} \left\| \mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{p} \right\|] \\
&\leq \mathfrak{t}. \qquad (3.14)
\end{aligned}$$

Hence

$$\mathfrak{t} = \lim_{\mathfrak{n}\to\infty} ||(1-\mathfrak{O}_\mathfrak{n})(\mathfrak{x}_\mathfrak{n} - \mathfrak{p}) + \mathfrak{O}_\mathfrak{n}(\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{p})|| \tag{3.15}$$

Now from (3.9), (3.10), (3.15) and Lemma 2.2.6, we conclude that $lim_{\mathfrak{n}\to\infty} \left\| \mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{x}_\mathfrak{n} \right\| = 0$.

□

3.2 $\mathfrak{T}$-Stable Fixed Point

The purpose of this subsection is to see how **Mohd Jubair** [28] proved the stability stability for contractive-like mapping via iteration process (2.10).

Throughout this section, we presume that $\mathcal{U}$ is a non-empty, closed and convex subset of a Banach space $\mathcal{W}$ and $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ is a contractive-like mapping. we prove the stability of iteration process to approximate the fixed point.

Theorem 3.2.1. *[28] Let* $\{\mathfrak{a}_\mathfrak{n}\}$ *be an iterative process be defined by iteration (2.10). Then iteration process becomes* $\mathcal{F}$*-stable.*

Proof. Suppose $\mathfrak{a}_\mathfrak{n}$ is an arbitrary sequence in $\mathcal{U}$ and $\mathfrak{a}_{\mathfrak{n}+1} = \mathfrak{f}(\mathcal{F}, \mathfrak{a}_\mathfrak{n})$ is sequence generated by (2.10) converging to a fixed point $\mathfrak{p}$ (by Theorem 3.1.1) and $\epsilon_n = \left\|\mathfrak{a}_{\mathfrak{n}+1} - \mathfrak{f}(\mathcal{F}, \mathfrak{a}_\mathfrak{n})\right\|$ for all $\mathfrak{n} \in \mathbb{Z}+$. We have to prove that $\lim_{\mathfrak{n}\to\infty} \epsilon_n = 0$ which implies that $\lim_{\mathfrak{n}\to\infty} \mathfrak{a}_\mathfrak{n} = \mathfrak{p}$. Suppose $lim_{\mathfrak{n}\to\infty}\epsilon_n = 0$, then by iteration process (2.10), we have

$$\begin{aligned} \|\mathfrak{a}_{\mathfrak{n}+1} - \mathfrak{p}\| &\leq \left\|\mathfrak{a}_{\mathfrak{n}+1} - \mathfrak{f}(\mathcal{F}, \mathfrak{a}_\mathfrak{n})\right\| + \left\|\mathfrak{f}(\mathcal{F}, \mathfrak{a}_\mathfrak{n}) - \mathfrak{p}\right\| \\ &\leq \epsilon_n + \left\|\mathfrak{f}(\mathcal{F}, \mathfrak{a}_\mathfrak{n}) - \mathfrak{p}\right\| \\ &\leq \epsilon_n + \mathfrak{H}^2[(1-(1-\mathfrak{H})\mathfrak{M}_\mathfrak{n})(1-(1-\mathfrak{H})\mathfrak{O}_\mathfrak{n})] \|\mathfrak{a}_\mathfrak{n} - \mathfrak{p}\| \end{aligned} \tag{3.16}$$

since

$$0 < (1-(1-\mathfrak{H})\mathfrak{M}_\mathfrak{n}) \leq 1,$$

and

$$0 < (1-(1-\mathfrak{H})\mathfrak{O}_\mathfrak{n}) \leq 1,$$

we get

$$\|\mathfrak{a}_{\mathfrak{n}+1} - \mathfrak{p}\| \leq \epsilon_n + \mathfrak{H}^2 \|\mathfrak{a}_\mathfrak{n} - \mathfrak{p}\| .$$

Define

$$\mathfrak{q}_\mathfrak{n} = \|\mathfrak{a}_\mathfrak{n} - \mathfrak{p}\| ,$$

then

$$\mathfrak{a}_{\mathfrak{n}+1} \leq \mathfrak{H}^2\mathfrak{q}_\mathfrak{n} + \epsilon_n ,$$

since $\lim_{\mathfrak{n}\to\infty} \epsilon_n = 0$, so , we have $\lim_{\mathfrak{n}\to\infty} \mathfrak{q}_\mathfrak{n} = 0$ i.e., $\lim_{\mathfrak{n}\to\infty} \mathfrak{a}_\mathfrak{n} = \mathfrak{p}$. Conversely, suppose $\lim_{\mathfrak{n}\to\infty} \mathfrak{a}_\mathfrak{n} = \mathfrak{p}$, we have

$$\begin{aligned} \epsilon_n &= \left\|\mathfrak{a}_{\mathfrak{n}+1} - \mathfrak{f}(\mathcal{F}, \mathfrak{a}_\mathfrak{n})\right\| \\ &\leq \|\mathfrak{a}_{\mathfrak{n}+1} - \mathfrak{p}\| + \left\|\mathfrak{f}(\mathcal{F}, \mathfrak{a}_\mathfrak{n}) - \mathfrak{p}\right\| \\ &\leq \|\mathfrak{a}_{\mathfrak{n}+1} - \mathfrak{p}\| + \mathfrak{H}^2[(1-(1-\mathfrak{H})\mathfrak{M}_\mathfrak{n})(1-(1-\mathfrak{H})\mathfrak{O}_\mathfrak{n})] \|\mathfrak{a}_\mathfrak{n} - \mathfrak{p}\| \\ &\leq \|\mathfrak{a}_{\mathfrak{n}+1} - p\| + \mathfrak{H}^2 \|\mathfrak{a}_\mathfrak{n} - \mathfrak{p}\| . \end{aligned}$$

This implies that $\lim_{\mathfrak{n}\to\infty} \epsilon_n = 0$. Hence iteration process (2.10) $\mathcal{F}$-stable.

□

3.3 Application: Delay Differential Equations

In this subsection, we see how **Jie Jia** [26] using Picard-Thakur hybrid iterative scheme find the solution of delay differential equations. Let the space of all continuous real-valued functions be denoted by $\mathfrak{C}([\mathfrak{u}, \mathfrak{v}])$ on closed interval $[u, v]$ endowed with th chebyshev norm $||\mathfrak{j} - \mathfrak{m}||_\infty$ and defines as $||\mathfrak{j} - \mathfrak{m}||_\infty = sup_{\mathfrak{r}\in[\mathfrak{u},\mathfrak{v}]}|\mathfrak{j}(\mathfrak{r}) - \mathfrak{m}(\mathfrak{r})|$, and it is clear that in [30] that $(\mathfrak{C}([\mathfrak{u}, \mathfrak{v}], ||.||_\infty))$ is a Banach space. Now, consider the following delay differential equation

$$\mathfrak{j}'(\mathfrak{r}) = \psi(\mathfrak{r}, \mathfrak{j}(\mathfrak{r}), \mathfrak{j}(\mathfrak{r} - \gamma)), \qquad \mathfrak{r} \in [\mathfrak{r}_o, \mathfrak{v}] \tag{3.17}$$

with initial condition

$$\mathfrak{j}(\mathfrak{r}) = \zeta(\mathfrak{r}), \qquad \mathfrak{r} \in [\mathfrak{r}_o - \gamma, \mathfrak{r}_o] \tag{3.18}$$

By the solution of above delay differential equation, we mean a function $\mathfrak{j} \in \mathfrak{C}([\mathfrak{r}_o - \gamma, \mathfrak{v}], \mathbb{R}) \cap \mathfrak{C}^1([\mathfrak{r}_o, \mathfrak{v}], \mathbb{R})$ satisfying (3.17) and (3.18).

Assume that the following conditions are satisfied.

(1) $\mathfrak{r}_o, \mathfrak{v} \in \mathbb{R}, \gamma > 0$

(2) $\psi \in \mathfrak{C}([\mathfrak{r}_o, \mathfrak{v}] \times \mathbb{R}^2, \mathbb{R})$

(3) $\zeta \in \mathfrak{C}([\mathfrak{r}_o - \gamma, \mathfrak{v}], \mathbb{R})$

(4) There exists $\mathfrak{L}_\psi > \mathfrak{o}$ such that

$$|\psi(\mathfrak{r}, \mathfrak{s}_1, \mathfrak{s}_2) - \psi(\mathfrak{r}, \mathfrak{t}_1, \mathfrak{t}_2)| \le \mathfrak{L}_\psi \sum_{\mathfrak{i}=1}^{2} |\mathfrak{s}_\mathfrak{i} - \mathfrak{t}_\mathfrak{i}|, \quad \mathfrak{s}_\mathfrak{i}, \mathfrak{t}_\mathfrak{i} \in \mathbb{R}, \mathfrak{r} \in [\mathfrak{r}_o, \mathfrak{v}] \tag{3.19}$$

(5) $2\mathfrak{L}_\psi(\mathfrak{v} - \mathfrak{r}_o) < 1$

Now, we construct (3.17) and (3.18) by the integral equation as

$$\mathfrak{j}(\mathfrak{r}) = \begin{cases} \zeta(\mathfrak{r}), & \mathfrak{r} \in [\mathfrak{r}_o - \gamma,\ \mathfrak{v}] \\ \zeta(\mathfrak{r}_o) + \int_{\mathfrak{r}_o}^{\mathfrak{r}} \psi(\mathfrak{t},\ \mathfrak{j}(\mathfrak{t}), \mathfrak{j}(\mathfrak{t} - \gamma))\partial\mathfrak{t}, & \mathfrak{r} \in [\mathfrak{r}_o,\ \mathfrak{v}]. \end{cases} \tag{3.20}$$

The following result is the generalization of the result of Coman et al. [31].

Theorem 3.3.1. *Let the conditions $*_1$) to $*_5$) be satisfied. Then (3.17) and (3.18) have unique solution* $\mathfrak{j}^* \in \mathfrak{C}([\mathfrak{r}_o - \gamma, \mathfrak{v}], \mathbb{R}) \cap \mathfrak{C}^1([\mathfrak{r}_o, \mathfrak{v}], \mathbb{R})$ *and*

$$\mathfrak{j}^* = lim_{\mathfrak{n}\to\infty}\mathfrak{V}^\mathfrak{n}(\mathfrak{j}) \tag{3.21}$$

Now, by using the Picard-Thakur hybrid iterative scheme (2.12), we prove the following result.

Theorem 3.3.2. *Let the conditions $*_1)$ to $*_5)$ be satisfied. Then (3.17) and (3.18) have unique solution $\mathfrak{j}^* \in \mathfrak{C}([\mathfrak{r}_o - \gamma, \mathfrak{v}], \mathbb{R}) \cap \mathfrak{C}^1([\mathfrak{r}_o, \mathfrak{v}], \mathbb{R})$ and the Picard-Thakur hybrid iterative scheme (2.12) converges to $\mathfrak{j}^*$*

Proof. Let $\mathfrak{j}_\mathfrak{n}$ be a sequence generated by the Picard-Thakur hybrid iterative scheme (2.12) for an operator $\mathfrak{V}$ defined by

$$\mathfrak{V}\mathfrak{j}(\mathfrak{r}) = \begin{cases} \zeta(\mathfrak{r}), & \mathfrak{r} \in [\mathfrak{r}_o - \gamma, \mathfrak{v}] \\ \zeta(\mathfrak{r}_o) + \int_{\mathfrak{r}_o}^{\mathfrak{r}} \psi(\mathfrak{p},\ \mathfrak{j}(\mathfrak{p}), \mathfrak{j}(\mathfrak{p}-\gamma))\partial\mathfrak{p}, & \mathfrak{r} \in [\mathfrak{r}_o,\ \mathfrak{v}]. \end{cases} \tag{3.22}$$

Let $\mathfrak{j}^*$ be a fixed point of $\mathfrak{V}$. Now, we prove that $\mathfrak{j}_\mathfrak{n} \to \mathfrak{j}^*$ as $\mathfrak{n} \to \infty$ for each $\mathfrak{r} \in [\mathfrak{r}_o - \gamma, \mathfrak{r}_o]$. Now, for each $\mathfrak{r} \in [\mathfrak{r}_o, \mathfrak{v}]$, we have

$$\begin{aligned}
||\mathfrak{j}_{\mathfrak{n}+1} - \mathfrak{j}^*||_\infty &\leq ||\mathfrak{V}\mathfrak{k}_\mathfrak{n} - \mathfrak{j}^*|| \\
&\leq \sup_{\mathfrak{r}\in[r_0,v]} |\mathfrak{V}\mathfrak{k}_\mathfrak{n} - \mathfrak{V}\mathfrak{j}^*| \\
&\leq sup_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]} \\
&\quad \left|\zeta(\mathfrak{r}_o) + \int_{\mathfrak{r}_o}^{\mathfrak{r}} \psi(\mathfrak{p}, \mathfrak{k}_\mathfrak{n}(\mathfrak{p}), \mathfrak{k}_\mathfrak{n}(\mathfrak{p}-\gamma))\partial\mathfrak{p} - (\zeta(\mathfrak{r}_o) + \int_{\mathfrak{r}_o}^{\mathfrak{r}} \psi(\mathfrak{p}, \mathfrak{j}^*(\mathfrak{p}), \mathfrak{j}^*(\mathfrak{p}-\gamma))\partial\mathfrak{p})\right| \\
&\leq \int_{\mathfrak{r}_o{}^r} \left|\psi(\mathfrak{p}, \mathfrak{k}_\mathfrak{n}(\mathfrak{p}), \mathfrak{k}_\mathfrak{n}(\mathfrak{p}-\ \gamma))\partial\mathfrak{p} - \psi(\mathfrak{p}, \mathfrak{j}^*(\mathfrak{p}), \mathfrak{j}^*(\mathfrak{p}-\gamma))\partial\mathfrak{p})\right| \\
&\leq sup_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]} \int_{\mathfrak{r}_o{}^r} \mathfrak{L}_\psi(\left|\mathfrak{k}_\mathfrak{n}(\mathfrak{p}) -\ \mathfrak{j}^*(\mathfrak{p})\right| + \left|\mathfrak{k}_\mathfrak{n}(\mathfrak{p}-\gamma) - \mathfrak{j}^*(\mathfrak{p}-\gamma)\right|)\partial\mathfrak{p} \\
&\leq \int_{\mathfrak{r}_o{}^r} \mathfrak{L}_\psi sup_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]}(\left|\mathfrak{k}_\mathfrak{n}(\mathfrak{p}) -\ \mathfrak{j}^*(\mathfrak{p})\right| + \left|\mathfrak{k}_\mathfrak{n}(\mathfrak{p}-\gamma) - \mathfrak{j}^*(\mathfrak{p}-\gamma)\right|)\partial\mathfrak{p} \\
&\leq \int_{\mathfrak{r}_o{}^r} \mathfrak{L}_\psi(||\mathfrak{k}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty + ||\mathfrak{k}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty)\partial\mathfrak{p} \\
&\leq 2\mathfrak{L}_\psi(v - r_0)||\mathfrak{k}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty.
\end{aligned} \tag{3.23}$$

Now

$$\begin{aligned}
||\mathfrak{k}_\mathfrak{n} - \mathfrak{j}^*||_\infty &= ||(1-\mathfrak{M}_\mathfrak{n})\mathfrak{V}\mathfrak{m}_\mathfrak{n} + \mathfrak{M}_\mathfrak{n}\mathfrak{V}\mathfrak{l}_\mathfrak{n} - \mathfrak{j}^*||_\infty \\
&\leq (1-\mathfrak{M}_\mathfrak{n})||\mathfrak{V}\mathfrak{m}_\mathfrak{n} - \mathfrak{j}^*||_\infty + \mathfrak{M}_\mathfrak{n}||\mathfrak{V}\mathfrak{l}_\mathfrak{n} - \mathfrak{j}^*||_\infty
\end{aligned} \tag{3.24}$$

As

$$
\begin{aligned}
||\mathfrak{V}\mathfrak{l}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty &\leq sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \\
&\left|\zeta(\mathfrak{r}_\mathfrak{o}) + \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}} \psi(\mathfrak{p}, \mathfrak{l}_\mathfrak{n}(\mathfrak{p}), \mathfrak{l}_\mathfrak{n}(\mathfrak{p}-\gamma))\mathfrak{d}\mathfrak{p} - (\zeta(\mathfrak{r}_\mathfrak{o}) + \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}} \psi(\mathfrak{p}, \mathfrak{j}^*(\mathfrak{p}), \mathfrak{j}^*(\mathfrak{p}-\gamma))\mathfrak{d}\mathfrak{p})\right| \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}{}^r} \left|\psi(\mathfrak{p}, \mathfrak{l}_\mathfrak{n}(\mathfrak{p}), \mathfrak{l}_\mathfrak{n}(\mathfrak{p}-\gamma))\mathfrak{d}\mathfrak{p} - \psi(\mathfrak{p}, \mathfrak{j}^*(\mathfrak{p}), \mathfrak{j}^*(\mathfrak{p}-\gamma))\mathfrak{d}\mathfrak{p})\right| \\
&\leq sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \int_{\mathfrak{r}_\mathfrak{o}{}^r} \mathfrak{L}_\psi(\left|\mathfrak{l}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})\right| + \left|\mathfrak{l}_\mathfrak{n}(\mathfrak{p}-\gamma) - \mathfrak{j}^*(\mathfrak{p}-\gamma)\right|)\mathfrak{d}\mathfrak{p} \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}{}^r} \mathfrak{L}_\psi sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]}(\left|\mathfrak{l}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})\right| + \left|\mathfrak{l}_\mathfrak{n}(\mathfrak{p}-\gamma) - \mathfrak{j}^*(\mathfrak{p}-\gamma)\right|)\mathfrak{d}\mathfrak{p} \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}{}^r} \mathfrak{L}_\psi(||\mathfrak{l}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty + ||\mathfrak{l}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty)\mathfrak{d}\mathfrak{p} \\
&\leq 2\mathfrak{L}_\psi(r - r_0)||\mathfrak{l}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty \leq 2\mathfrak{L}_\psi(v - r_0)||\mathfrak{l}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty. \qquad (3.25)
\end{aligned}
$$

$$
\begin{aligned}
||\mathfrak{l}_\mathfrak{n} - \mathfrak{j}^*||_\infty &= ||(1-\mathfrak{N}_\mathfrak{n})\mathfrak{V}\mathfrak{m}_\mathfrak{n} + \mathfrak{N}_\mathfrak{n}\mathfrak{V}\mathfrak{m}_\mathfrak{n} - \mathfrak{j}^*||_\infty \\
&\leq (1-\mathfrak{N}_\mathfrak{n})||\mathfrak{V}\mathfrak{m}_\mathfrak{n} - \mathfrak{j}^*||_\infty + \mathfrak{N}_\mathfrak{n}||\mathfrak{V}\mathfrak{m}_\mathfrak{n} - \mathfrak{j}^*||_\infty \qquad (3.26)
\end{aligned}
$$

For

$$
\begin{aligned}
||\mathfrak{V}\mathfrak{m}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty &\leq sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \\
&\left|\zeta(\mathfrak{r}_\mathfrak{o}) + \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}} \psi(\mathfrak{p}, \mathfrak{m}_\mathfrak{n}(\mathfrak{p}), \mathfrak{m}_\mathfrak{n}(\mathfrak{p}-\gamma))\mathfrak{d}\mathfrak{p} - (\zeta(\mathfrak{r}_\mathfrak{o}) + \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}} \psi(\mathfrak{p}, \mathfrak{j}^*(\mathfrak{p}), \mathfrak{j}^*(\mathfrak{p}-\gamma))\mathfrak{d}\mathfrak{p})\right| \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}{}^r} \left|\psi(\mathfrak{p}, \mathfrak{m}_\mathfrak{n}(\mathfrak{p}), \mathfrak{m}_\mathfrak{n}(\mathfrak{p}-\gamma))\mathfrak{d}\mathfrak{p} - \psi(\mathfrak{p}, \mathfrak{j}^*(\mathfrak{p}), \mathfrak{j}^*(\mathfrak{p}-\gamma))\mathfrak{d}\mathfrak{p})\right| \\
&\leq sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \int_{\mathfrak{r}_\mathfrak{o}{}^r} \mathfrak{L}_\psi(\left|\mathfrak{m}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})\right| + \left|\mathfrak{m}_\mathfrak{n}(\mathfrak{p}-\gamma) - \mathfrak{j}^*(\mathfrak{p}-\gamma)\right|)\mathfrak{d}\mathfrak{p} \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}{}^r} \mathfrak{L}_\psi sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]}(\left|\mathfrak{m}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})\right| + \left|\mathfrak{m}_\mathfrak{n}(\mathfrak{p}-\gamma) - \mathfrak{j}^*(\mathfrak{p}-\gamma)\right|)\mathfrak{d}\mathfrak{p} \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}{}^r} \mathfrak{L}_\psi(||\mathfrak{m}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty + ||\mathfrak{m}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty)\mathfrak{d}\mathfrak{p} \\
&\leq 2\mathfrak{L}_\psi(r - r_0)||\mathfrak{m}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty \leq 2\mathfrak{L}_\psi(v - r_0)||\mathfrak{m}_\mathfrak{n}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty. \qquad (3.27)
\end{aligned}
$$

$$
\begin{aligned}
||\mathfrak{m}_\mathfrak{n} - \mathfrak{j}^*||_\infty &= ||(1-\mathfrak{O}_\mathfrak{n})\mathfrak{V}\mathfrak{j}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{V}\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty \\
&\leq (1-\mathfrak{O}_\mathfrak{n})||\mathfrak{V}\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty + \mathfrak{O}_\mathfrak{n}||\mathfrak{V}\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty \qquad (3.28)
\end{aligned}
$$

As

$$\begin{aligned}
||\mathfrak{V}\mathfrak{j}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty &\leq sup_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]} \\
&\Bigg|\zeta(\mathfrak{r}_o) + \int_{\mathfrak{r}_o}^{\mathfrak{r}} \psi(\mathfrak{p},\ \mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}),\ \mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}-\gamma))\partial\mathfrak{p} - (\zeta(\mathfrak{r}_o) \\
&+ \int_{\mathfrak{r}_o}^{\mathfrak{r}} \psi(\mathfrak{p},\ \mathfrak{j}^*(\mathfrak{p}),\ \mathfrak{j}^*(\mathfrak{p}-\gamma))\partial\mathfrak{p})\Bigg| \\
&\leq \int_{\mathfrak{r}_o}{}^{r} \big|\psi(\mathfrak{p},\mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}),\mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}-\gamma))\partial\mathfrak{p} - \psi(\mathfrak{p},\mathfrak{j}^*(\mathfrak{p}),\mathfrak{j}^*(\mathfrak{p}-\gamma))\partial\mathfrak{p})\big| \\
&\leq sup_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]} \int_{\mathfrak{r}_o}{}^{r} \mathfrak{L}_\psi(\big|\mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})\big| + \big|\mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}-\gamma) - \mathfrak{j}^*(\mathfrak{p}-\gamma)\big|)\partial\mathfrak{p} \\
&\leq \int_{\mathfrak{r}_o}{}^{r} \mathfrak{L}_\psi sup_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]}(\big|\mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})\big| + \big|\mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}-\gamma) - \mathfrak{j}^*(\mathfrak{p}-\gamma)\big|)\partial\mathfrak{p} \\
&\leq \int_{\mathfrak{r}_o}{}^{r} \mathfrak{L}_\psi(||\mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty + ||\mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty)\partial\mathfrak{p} \\
&\leq 2\mathfrak{L}_\psi(\mathfrak{r}-\mathfrak{r}_o)||\mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty \leq 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o) \\
&\quad ||\mathfrak{j}_{\mathfrak{n}}(\mathfrak{p}) - \mathfrak{j}^*(\mathfrak{p})||_\infty. \qquad (3.29)
\end{aligned}$$

Putting (3.28) in (3.29), we get

$$\begin{aligned}
||\mathfrak{m}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty &= (1-\mathfrak{O}_{\mathfrak{n}})||\mathfrak{j}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty + \mathfrak{O}_{\mathfrak{n}} 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)||\mathfrak{j}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty \\
&\leq [1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o))]||\mathfrak{j}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty \qquad (3.30)
\end{aligned}$$

Putting (3.30) in (3.27), we get

$$||\mathfrak{m}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty \leq 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)[1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)]||\mathfrak{j}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty. \qquad (3.31)$$

Putting (3.31) and (3.30) in (3.26), we get

$$\begin{aligned}
||\mathfrak{l}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty &\leq (1-\mathfrak{O}_{\mathfrak{n}})[1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)]||\mathfrak{j}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty \\
&\quad + \mathfrak{O}_{\mathfrak{n}} 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)[1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)]||\mathfrak{j}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty \\
&\leq (1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)\mathfrak{O}_{\mathfrak{n}})(1-(1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)\mathfrak{N}_{\mathfrak{n}})))||\mathfrak{j}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty \\
&\leq (1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)\mathfrak{O}_{\mathfrak{n}}) - (1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)\mathfrak{O}_{\mathfrak{n}}) \\
&\quad (1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)\mathfrak{N}_{\mathfrak{n}})||\mathfrak{j}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty \\
&\leq [1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o))(\mathfrak{O}_{\mathfrak{n}} - \mathfrak{N}_{\mathfrak{n}} + \mathfrak{N}_{\mathfrak{n}}\mathfrak{O}_{\mathfrak{n}})]||\mathfrak{j}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty \qquad (3.32)
\end{aligned}$$

Putting (3.32) in (3.25), we get

$$\begin{aligned}
||\mathfrak{V}\mathfrak{l}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty &\leq 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)[1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o))(\mathfrak{O}_{\mathfrak{n}} - \mathfrak{N}_{\mathfrak{n}} + \mathfrak{N}_{\mathfrak{n}}\mathfrak{O}_{\mathfrak{n}})] \\
&\quad ||\mathfrak{j}_{\mathfrak{n}} - \mathfrak{j}^*||_\infty. \qquad (3.33)
\end{aligned}$$

Putting (3.33) and (3.27) in (3.24), we get

$$\begin{aligned} ||\mathfrak{k}_\mathfrak{n} - \mathfrak{j}^*||_\infty &\leq (1-\mathfrak{M}_\mathfrak{n})2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})||\mathfrak{m}_\mathfrak{n} - \mathfrak{j}^*||_\infty + \mathfrak{M}_\mathfrak{n}2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})||\mathfrak{l}_\mathfrak{n} - \mathfrak{j}^*||_\infty \\ &\leq 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})[(1-\mathfrak{M}_\mathfrak{n})||\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty + \mathfrak{M}_\mathfrak{n}||\mathfrak{l}_\mathfrak{n} - \mathfrak{j}^*||_\infty] \\ &\leq 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})[(1-\mathfrak{M}_\mathfrak{n})[(1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})\mathfrak{O}_\mathfrak{n})]||\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty \\ &\quad + \mathfrak{M}_\mathfrak{n}[1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o}))(\mathfrak{O}_\mathfrak{n} - \mathfrak{N}_\mathfrak{n} + \mathfrak{N}_\mathfrak{n}\mathfrak{O}_\mathfrak{n})]||\mathfrak{l}_\mathfrak{n} - \mathfrak{j}^*||_\infty] \\ &\leq 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})[(1-\mathfrak{M}_\mathfrak{n})[(1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})(\mathfrak{O}_\mathfrak{n} - \mathfrak{N}_\mathfrak{n} + \mathfrak{N}_\mathfrak{n}\mathfrak{O}_\mathfrak{n}))]||\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty \\ &\leq 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})[(1-\mathfrak{M}_\mathfrak{n})[(1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})(\mathfrak{O}_\mathfrak{n} + \mathfrak{N}_\mathfrak{n}\mathfrak{M}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}\mathfrak{M}_\mathfrak{n}))] \\ &\quad ||\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty \end{aligned}$$

Let $\mathfrak{O}_\mathfrak{n} + \mathfrak{N}_\mathfrak{n}\mathfrak{M}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}\mathfrak{M}_\mathfrak{n} = \mathfrak{p}_\mathfrak{n}$, and by using condition $*_5$), we have

$$||\mathfrak{k}_\mathfrak{n} - \mathfrak{j}^*||_\infty \leq [1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})\mathfrak{p}_\mathfrak{n}]||\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty. \tag{3.34}$$

Now, putting (3.34) in (3.23), we have

$$||\mathfrak{j}_{\mathfrak{n}+1} - \mathfrak{j}^*||_\infty \leq 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})[1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})\mathfrak{p}_\mathfrak{n}]||\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty. \tag{3.35}$$

Again using condition $*_5$), we have

$$||\mathfrak{j}_{\mathfrak{n}+1} - \mathfrak{j}^*||_\infty \leq [1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})\mathfrak{p}_\mathfrak{n}]||\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty. \tag{3.36}$$

Let $(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})\mathfrak{p}_\mathfrak{n}) = \tau_\mathfrak{n} < 1$ and $||\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty = \mathfrak{r}_\mathfrak{n}$. So, the conditions of Lemma 3 of [29] are satisfied. Hence, $lim_{\mathfrak{n}\to\infty}||\mathfrak{j}_\mathfrak{n} - \mathfrak{j}^*||_\infty = 0$ □

By considering above sections (3.1), (3.2), (3.3) , we are willingly proposing new iterative scheme (2.11) which has faster convergence rate and also with improved stability.

Chapter 4

Rate of Convergence For Proposed Iterative Technique

In this Chapter, we shall discuss our man theorem and results regarding convergence. In this paper, We prove that iterative process (4.1) converges faster than iteration processes (2.5), (2.9), (2.11) and we present numerical example in support of the proof. Our new proposed iteration is given by

$$\begin{cases} \mathfrak{x}_{\mathfrak{n}+1} & = \mathfrak{T}(\mathfrak{T}(\mathfrak{p}_{\mathfrak{n}})) \\ \mathfrak{p}_{\mathfrak{n}} & = \mathfrak{T}((1-\mathfrak{M}_{\mathfrak{n}})\mathfrak{T}\mathfrak{z}_{\mathfrak{n}} + \mathfrak{M}_{\mathfrak{n}}\mathfrak{T}\mathfrak{y}_{\mathfrak{n}}) \\ \mathfrak{y}_{\mathfrak{n}} & = \mathfrak{T}((1-\mathfrak{N}_{\mathfrak{n}})\mathfrak{x}_{\mathfrak{n}} + \mathfrak{N}_{\mathfrak{n}}\mathfrak{T}\mathfrak{z}_{\mathfrak{n}}) \\ \mathfrak{z}_{\mathfrak{n}} & = \mathfrak{T}((1-\mathfrak{O}_{\mathfrak{n}})\mathfrak{x}_{\mathfrak{n}} + \mathfrak{O}_{\mathfrak{n}}\mathfrak{T}\mathfrak{x}_{\mathfrak{n}}), \quad \mathfrak{n} \in \mathbb{N} \end{cases} \tag{4.1}$$

where the following $\{\mathfrak{M}_{\mathfrak{n}}\}$, $\{\mathfrak{N}_{\mathfrak{n}}\}$ and $\{\mathfrak{O}_{\mathfrak{n}}\}$ are in (0,1).

Theorem 4.0.1. *Let $\mathfrak{C}$ be a nonempty closed convex subset of a uniformly convex Banach space. Let $\mathfrak{T}$ be a contraction mapping with some constant $\mathfrak{H} \in [0,1)$ and fixed point $\mathfrak{p}$. Let $\{\mathfrak{u}_{\mathfrak{n}}\}$ be defined by the iteration process (2.11) and $\{\mathfrak{x}_{\mathfrak{n}}\}$ by (4.1) , where $\{\mathfrak{M}_{\mathfrak{n}}\}$, $\{\mathfrak{N}_{\mathfrak{n}}\}$, $\{\mathfrak{O}_{\mathfrak{n}}\}$ are in [ϵ , 1- ϵ] for any $\mathfrak{n} \in \mathbb{N}$ and some ϵ in $(0,1)$. Then $\{\mathfrak{x}_{\mathfrak{n}}\}$ converges faster than $\{\mathfrak{u}_{\mathfrak{n}}\}$.*

Proof. As proved in Theorem 2 of [25] , we have

$$\|\mathfrak{u}_{\mathfrak{n}+1} - \mathfrak{p}\| \leq \mathfrak{H}^{\mathfrak{n}}[1-(1-\mathfrak{H})\mathfrak{N}\mathfrak{O})]^{\mathfrak{n}} \tag{4.2}$$

Let

$$\mathfrak{a}_n = \mathfrak{H}^n[1-(1-\mathfrak{H})\mathfrak{N}\mathfrak{O}]^n \|\mathfrak{u}_1 - \mathfrak{p}\| \tag{4.3}$$

Now with notion to process (4.1), we have

$$\begin{aligned}
\|\mathfrak{z}_n - \mathfrak{p}\| &= ||\mathfrak{T}[(1-\mathfrak{O}_n)\mathfrak{x}_n + \mathfrak{O}_n\mathfrak{T}\mathfrak{x}_n] - \mathfrak{p}|| \\
&\leq \mathfrak{H}\ ||(1-\mathfrak{O}_n)(\mathfrak{x}_n - \mathfrak{p}) + \mathfrak{O}_n(\mathfrak{T}\mathfrak{x}_n - \mathfrak{p})|| \\
&\leq \mathfrak{H}[(1-\mathfrak{O}_n)\|\mathfrak{x}_n - \mathfrak{p}\| + \mathfrak{O}_n\|\mathfrak{T}\mathfrak{x}_n - \mathfrak{p}\|] \\
&\leq \mathfrak{H}[(1-\mathfrak{O}_n)\|\mathfrak{x}_n - \mathfrak{p}\| + \mathfrak{O}_n\mathfrak{H}\|\mathfrak{x}_n - \mathfrak{p}\|] \\
&= \mathfrak{H}[1-(1-\mathfrak{H})\mathfrak{O}_n]\|\mathfrak{x}_n - \mathfrak{p}\|
\end{aligned}$$

so that

$$\begin{aligned}
\|\mathfrak{y}_n - \mathfrak{p}\| &= ||\mathfrak{T}[(1-\mathfrak{N}_n)\mathfrak{z}_n + \mathfrak{N}_n\mathfrak{T}\mathfrak{z}_n] - \mathfrak{p}|| \\
&\leq \mathfrak{H}\ ||(1-\mathfrak{N}_n)(\mathfrak{z}_n - \mathfrak{p}) + \mathfrak{N}_n(\mathfrak{T}\mathfrak{z}_n - \mathfrak{p})|| \\
&\leq \mathfrak{H}[(1-\mathfrak{N}_n)\|\mathfrak{z}_n - \mathfrak{p}\| + \mathfrak{N}_n\|\mathfrak{T}\mathfrak{z}_n - \mathfrak{p}\|] \\
&\leq \mathfrak{H}[(1-\mathfrak{N}_n)\|\mathfrak{z}_n - \mathfrak{p}\| + \mathfrak{N}_n\mathfrak{H}\|\mathfrak{z}_n - \mathfrak{p}\|] \\
&= \mathfrak{H}[1-(1-\mathfrak{H})\mathfrak{N}_n]\|\mathfrak{z}_n - \mathfrak{p}\| \\
&< \mathfrak{H}(1-(1-\mathfrak{H})\mathfrak{N}_n)(1-(1-\mathfrak{H})\mathfrak{O}_n)\|\mathfrak{x}_n - \mathfrak{p}\|
\end{aligned}$$

and

$$
\begin{aligned}
\|\mathfrak{p}_\mathfrak{n}-\mathfrak{p}\| &= ||\mathfrak{T}[(1-\mathfrak{M}_\mathfrak{n})\mathfrak{T}\mathfrak{z}_\mathfrak{n}+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}]-\mathfrak{p}|| \\
&\leq \mathfrak{H}\ ||(1-\mathfrak{M}_\mathfrak{n})(\mathfrak{T}\mathfrak{z}_\mathfrak{n}-\mathfrak{p})+\mathfrak{M}_\mathfrak{n}(\mathfrak{T}\mathfrak{y}_\mathfrak{n}-\mathfrak{p})|| \\
&\leq \mathfrak{H}[(1-\mathfrak{M}_\mathfrak{n})\|\mathfrak{T}\mathfrak{z}_\mathfrak{n}-\mathfrak{p}\|+\mathfrak{M}_\mathfrak{n}\|\mathfrak{T}\mathfrak{y}_\mathfrak{n}-\mathfrak{p}\|] \\
&\leq \mathfrak{H}[(1-\mathfrak{M}_\mathfrak{n})\mathfrak{H}\|\mathfrak{z}_\mathfrak{n}-\mathfrak{p}\|+\mathfrak{M}_\mathfrak{n}\mathfrak{H}\|\mathfrak{y}_\mathfrak{n}-\mathfrak{p}\|] \\
&\leq \mathfrak{H}[(1-\mathfrak{M}_\mathfrak{n})\mathfrak{H}^2(1-(1-\mathfrak{H})\mathfrak{O}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n}-\mathfrak{p}\|+\mathfrak{M}_\mathfrak{n}\mathfrak{H}^2(1-(1-\mathfrak{H})\mathfrak{O}_\mathfrak{n}) \\
&\quad (1-(1-\mathfrak{H})\mathfrak{N}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n}-\mathfrak{p}\|] \\
&\leq \mathfrak{H}^3(1-(1-\mathfrak{H})\mathfrak{O}_\mathfrak{n})[(1-\mathfrak{M}_\mathfrak{n})+\mathfrak{M}_\mathfrak{n}(1-(1-\mathfrak{H}))\mathfrak{N}_\mathfrak{n}]\|\mathfrak{x}_\mathfrak{n}-\mathfrak{p}\| \\
&= \mathfrak{H}^3(1-(1-\mathfrak{H})\mathfrak{O}_\mathfrak{n})[(1-(1-\mathfrak{H}))\mathfrak{N}_\mathfrak{n}]\|\mathfrak{x}_\mathfrak{n}-\mathfrak{p}\| \\
&= \mathfrak{H}^3[1-(1-\mathfrak{H})\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}-(1-\mathfrak{H})\mathfrak{O}_\mathfrak{n}+(1-\mathfrak{H})^2\mathfrak{O}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}\mathfrak{M}_\mathfrak{n}]\|\mathfrak{x}_\mathfrak{n}-\mathfrak{p}\| \\
&\leq \mathfrak{H}^3[1-(1-\mathfrak{H})\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}-(1-\mathfrak{H})\mathfrak{O}_\mathfrak{n}+(1-\mathfrak{H})\mathfrak{O}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}\mathfrak{M}_\mathfrak{n}]\|\mathfrak{x}_\mathfrak{n}-\mathfrak{p}\| \\
&\leq \mathfrak{H}^3[1-(1-\mathfrak{H})(\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}+\mathfrak{O}_\mathfrak{n}-\mathfrak{O}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}\mathfrak{M}_\mathfrak{n}]\|\mathfrak{x}_\mathfrak{n}-\mathfrak{p}\| \\
&< \mathfrak{H}[1-(1-\mathfrak{H})(\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}+\mathfrak{O}_\mathfrak{n}-\mathfrak{O}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}\mathfrak{M}_\mathfrak{n}]\|\mathfrak{x}_\mathfrak{n}-\mathfrak{p}\|
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|\mathfrak{x}_{\mathfrak{n}+1}-\mathfrak{p}\| &= \left\|\mathfrak{T}(\mathfrak{T}\mathfrak{P}_\mathfrak{n})-\mathfrak{p}\right\| \\
&\leq \mathfrak{H}\|\mathfrak{T}\mathfrak{p}_\mathfrak{n}-\mathfrak{p}\| \\
&\leq \mathfrak{H}^2\|\mathfrak{p}_\mathfrak{n}-\mathfrak{p}\| \\
&\leq \mathfrak{H}^3[1-(1-\mathfrak{H})(\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}+\mathfrak{O}_\mathfrak{n}-\mathfrak{O}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}\mathfrak{M}_\mathfrak{n}]\|\mathfrak{x}_\mathfrak{n}-\mathfrak{p}\| \qquad (4.4)
\end{aligned}
$$

Let

$$
\mathfrak{b}_\mathfrak{n} = \mathfrak{H}^{3n}[1-(1-\mathfrak{H})(\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}+\mathfrak{O}_\mathfrak{n}-\mathfrak{O}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}\mathfrak{M}_\mathfrak{n}]^\mathfrak{n}\|\mathfrak{x}_1-\mathfrak{p}\|
$$

Then

$$
\begin{aligned}
\frac{\mathfrak{b}_\mathfrak{n}}{\mathfrak{a}_\mathfrak{n}} &= \frac{\mathfrak{H}^{3\mathfrak{n}}[1-(1-\mathfrak{H})(\mathfrak{M}\mathfrak{N}+\mathfrak{O}-\mathfrak{O}\mathfrak{N}\mathfrak{M}]^\mathfrak{n}\|\mathfrak{x}_1-\mathfrak{p}\|}{\mathfrak{H}^n[1-(1-\mathfrak{H})\mathfrak{N}\mathfrak{O}]^\mathfrak{n}\|\mathfrak{u}_1-\mathfrak{p}\|} \\
&= \frac{\mathfrak{H}^3[1-(1-\mathfrak{H})(\mathfrak{M}\mathfrak{N}+\mathfrak{O}-\mathfrak{O}\mathfrak{N}\mathfrak{M}]^\mathfrak{n}\|\mathfrak{x}_1-\mathfrak{p}\|}{[1-(1-\mathfrak{H})\mathfrak{N}\mathfrak{O}]^\mathfrak{n}\|\mathfrak{u}_1-\mathfrak{p}\|} \to 0 \quad \mathfrak{n}\to\infty
\end{aligned}
\qquad (4.5)
$$

Thus $\mathfrak{x}_\mathfrak{n}$ converges faster than $\mathfrak{u}_\mathfrak{n}$.

□

Remark 3. As proved in [36], Thakur iteration (2.9) is faster than Ishikawa [19], Noor [20], Abbas and Nazir [22]. So, we will compare iteration (4.1) with Picard (2.5), Thakur (2.9), Asghar Rahimi (2.11).

We, now present an example which shows that our iteration process (4.1) converges at a faster rate than Picard iteration process (2.5), Thakur iteration process (2.9) and Asghar Rahimi iteration process (2.11).

Example 4.0.1. Let $\mathfrak{S} = \mathbb{R}$ and $\mathfrak{C} = [1, 21]$. Let $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be a mapping defined by $\mathfrak{T}(\mathfrak{x}) = \sqrt{\mathfrak{x}^2 - 6\mathfrak{x} + 30}$ for any $\mathfrak{x} \in \mathfrak{C}$. Choose $\mathfrak{M}_\mathfrak{n} = \mathfrak{N}_\mathfrak{n} = \mathfrak{O}_\mathfrak{n} = \frac{1}{2}$, with the initial value $\mathfrak{x}_1 = 30$. It is obvious that $\mathfrak{x} = 5$ is fixed point of $\mathfrak{T}$. The table below show behaviour all iteration processes mentioned to fixed point of $\mathfrak{T}$ in 21 iteration.

No. of iteration	Picard	Thakur	Asghar Rahimi	New iteration
1	30	30	30	30
2	27.3861278753	25.4605027258	23.2381722116	14.6168424629
3	24.8129650132	21.0691659384	16.8793944946	5.5974190240
4	22.2891328380	16.8924163440	11.2697340648	5.0032637416
5	19.8260093221	13.0431036573	7.1650814763	5.0000149868
6	17.4388815498	9.7163929865	5.3849641099	5.0000000688
7	15.1486402165	7.2125863864	5.0454052746	5.0000000003
8	12.9842003647	5.7772680541	5.0049230360	5.0000000000
9	10.9856386670	5.2154250367	5.0005284002	5.0000000000
10	9.2070855823	5.0535678237	5.0000566520	5.0000000000
11	7.7154333272	5.0128902394	5.0000060732	5.0000000000
12	6.5753563754	5.0030760332	5.0000006510	5.0000000000
13	5.8123294135	5.0007325602	5.0000000698	5.0000000000
14	5.3767273252	5.0001743757	5.0000000075	5.0000000000
15	5.1622507474	5.0000415029	5.0000000008	5.0000000000
16	5.0670828190	5.0000098778	5.0000000001	5.0000000000
17	5.0272091045	5.0000023509	5.0000000000	5.0000000000
18	5.0109456945	5.0000005595	5.0000000000	5.0000000000
19	5.0043883329	5.0000001332	5.0000000000	5.0000000000
20	5.0017569502	5.0000000317	5.0000000000	5.0000000000
21	5.0007030393	5.0000000075	5.0000000000	5.0000000000

Table 4.1. Compairing iterations convergence

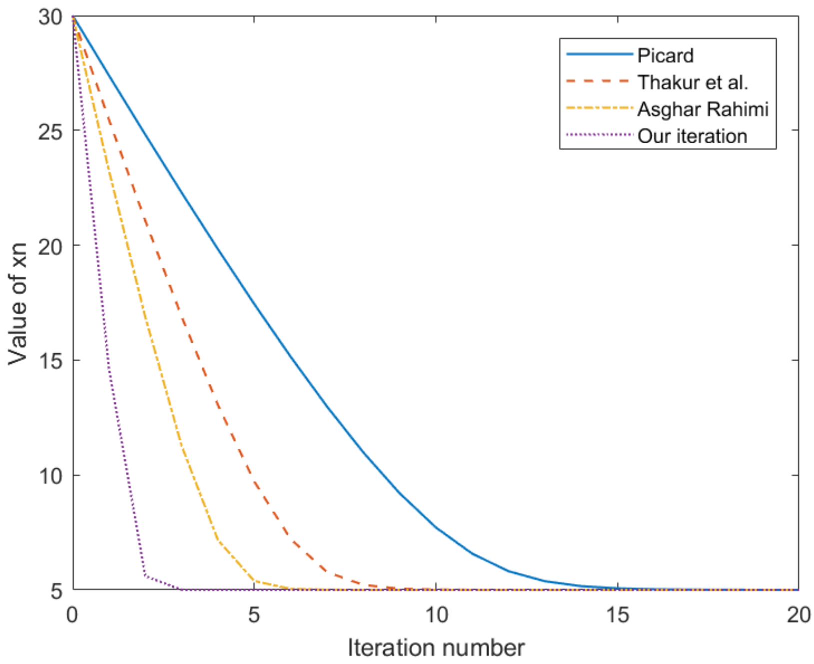

Figure 4.1: Comparison of iteration processes convergence

4.1 Convergence Results for Garcia-Falset Mapping

Lemma 4.1.1. *Let* $\mathfrak{C}$ *be a nonempty closed convex subset of a Banach space* $\mathfrak{S}$ *and* $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ *is a mapping satisfying condition (E). If the sequence* $\mathfrak{x}_\mathfrak{n}$ *be by (4.1) and* $\mathfrak{F}(\mathfrak{T}) \neq \varnothing$, *then* $lim_{\mathfrak{n}\to\infty} \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\|$ *exists for all* $\mathfrak{p} \in \mathfrak{F}(\mathfrak{T})$.

Proof. $\mathfrak{F}(\mathfrak{T}) \neq \varnothing$, Let $\mathfrak{p} \in \mathfrak{F}(\mathfrak{T})$. By Proposition 1 , then we have

$$\begin{aligned}
\|\mathfrak{z}_\mathfrak{n} - \mathfrak{p}\| &= ||\mathfrak{T}[(1-\mathfrak{D}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{D}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n}] - \mathfrak{p}|| \\
&\leq ||(1-\mathfrak{D}_\mathfrak{n})(\mathfrak{x}_\mathfrak{n} - \mathfrak{p}) + \mathfrak{D}_\mathfrak{n}(\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{p})|| \\
&\leq [(1-\mathfrak{D}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| + \mathfrak{D}_\mathfrak{n}\|\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\|] \\
&\leq [(1-\mathfrak{D}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| + \mathfrak{D}_\mathfrak{n}\|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\|] \\
&= \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| \qquad (4.6)
\end{aligned}$$

so that

$$\begin{aligned}
\|\mathfrak{y}_{\mathfrak{n}} - \mathfrak{p}\| &= ||\mathfrak{T}[(1-\mathfrak{N}_{\mathfrak{n}})\mathfrak{z}_{\mathfrak{n}} + \mathfrak{N}_{\mathfrak{n}}\mathfrak{T}\mathfrak{z}_{\mathfrak{n}}] - \mathfrak{p}|| \\
&\leq ||(1-\mathfrak{N}_{\mathfrak{n}})(\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}) + \mathfrak{N}_{\mathfrak{n}}(\mathfrak{T}\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p})|| \\
&\leq [(1-\mathfrak{N}_{\mathfrak{n}})\|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\| + \mathfrak{N}_{\mathfrak{n}}\|\mathfrak{T}\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\|] \\
&\leq [(1-\mathfrak{N}_{\mathfrak{n}})\|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\| + \mathfrak{N}_{\mathfrak{n}}\|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\|] \\
&= \|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\| \\
&< \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| \qquad (4.7)
\end{aligned}$$

and

$$\begin{aligned}
\|\mathfrak{p}_{\mathfrak{n}} - \mathfrak{p}\| &= ||\mathfrak{T}[(1-\mathfrak{M}_{\mathfrak{n}})\mathfrak{T}\mathfrak{z}_{\mathfrak{n}} + \mathfrak{M}_{\mathfrak{n}}\mathfrak{T}\mathfrak{y}_{\mathfrak{n}}] - \mathfrak{p}|| \\
&\leq ||(1-\mathfrak{M}_{\mathfrak{n}})(\mathfrak{T}\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}) + \mathfrak{M}_{\mathfrak{n}}(\mathfrak{T}\mathfrak{y}_{\mathfrak{n}} - \mathfrak{p})|| \\
&\leq [(1-\mathfrak{M}_{\mathfrak{n}})\|\mathfrak{T}\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\| + \mathfrak{M}_{\mathfrak{n}}\|\mathfrak{T}\mathfrak{y}_{\mathfrak{n}} - \mathfrak{p}\|] \\
&\leq [(1-\mathfrak{M}_{\mathfrak{n}})\|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\| + \mathfrak{M}_{\mathfrak{n}}\|\mathfrak{y}_{\mathfrak{n}} - \mathfrak{p}\|] \\
&\leq [(1-\mathfrak{M}_{\mathfrak{n}})\|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| + \mathfrak{M}_{\mathfrak{n}}\|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\|] \\
&= \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| \qquad (4.8)
\end{aligned}$$

Thus

$$\begin{aligned}
\|\mathfrak{x}_{\mathfrak{n}+1} - \mathfrak{p}\| &= \left\|\mathfrak{T}(\mathfrak{p}_{\mathfrak{n}}) - \mathfrak{p}\right\| \\
&\leq \|\mathfrak{p}_{\mathfrak{n}} - \mathfrak{p}\| \\
&\leq \|\mathfrak{T}\mathfrak{p}_{\mathfrak{n}} - \mathfrak{p}\| \\
&\leq \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| \qquad (4.9)
\end{aligned}$$

(4.10)

This implies that $\|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\|$ is bounded and nonincreasing for all $\mathfrak{p} \in \mathfrak{F}(\mathfrak{T})$. Hence $lim_{\mathfrak{n}\to\infty}\|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\|$ exists.

□

Lemma 4.1.2. *Let $\mathfrak{C}$ be a nonempty closed convex subset of a Banach space E and $\mathfrak{T} : \mathfrak{C} \to \mathfrak{C}$ be a Garcia-Falset mapping. Suppose the sequence $\mathfrak{x}_{\mathfrak{n}}$ be by (4.1) and $\mathfrak{F}(\mathfrak{T}) \neq \varnothing$, then $lim_{\mathfrak{n}\to\infty}\|\mathfrak{T}\mathfrak{x}_{\mathfrak{n}} - \mathfrak{x}_{\mathfrak{n}}\| = 0$.*

Proof. As from above Lemma 4.1.1, $lim_{\mathfrak{n}\to\infty}\|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\|$ exists for each $p \in \mathfrak{F}(\mathfrak{T})$. Suppose that for some $\mathfrak{T} \geq 0$, we have

$$\lim_{\mathfrak{n}\to\infty} \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| = \mathfrak{t} \tag{4.11}$$

It is proved in the proof of Lemma 4.1.1 that $\|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\| \leq \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\|$. Accordingly, one has

$$\lim_{\mathfrak{n}\to\infty} sup \|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\| \leq \lim_{\mathfrak{n}\to\infty} sup \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| = \mathfrak{t} \tag{4.12}$$

Now $\mathfrak{p}$ is the point, by Proposition 1. It follows that $\|\mathfrak{T}\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| \leq \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\|$

$$\lim_{\mathfrak{n}\to\infty} sup \|\mathfrak{T}\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| \leq \lim_{\mathfrak{n}\to\infty} sup \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| = \mathfrak{t} \tag{4.13}$$

We know again by the proof Lemma 4.1.1 that $\|\mathfrak{y}_{\mathfrak{n}} - \mathfrak{p}\| \leq \|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\|$. So, we have

$$\|\mathfrak{x}_{\mathfrak{n}+1} - \mathfrak{p}\| \leq [(1 - \mathfrak{M}_{\mathfrak{n}}) \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| + \mathfrak{M}_{\mathfrak{n}} \|\mathfrak{y}_{\mathfrak{n}} - \mathfrak{p}\|]$$

$$\leq [(1 - \mathfrak{M}_{\mathfrak{n}}) \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| + \mathfrak{M}_{\mathfrak{n}} \|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\|]$$

It follows that

$$\|\mathfrak{x}_{\mathfrak{n}+1} - \mathfrak{p}\| - \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\| \leq \frac{\|\mathfrak{x}_{\mathfrak{n}+1} - \mathfrak{p}\| - \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\|}{\mathfrak{M}_{\mathfrak{n}}} \leq \|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\| - \|\mathfrak{x}_{\mathfrak{n}} - \mathfrak{p}\|$$

So, we have

$$\|\mathfrak{x}_{\mathfrak{n}+1} - \mathfrak{p}\| \leq \|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\|$$

From (4.11) , we have

$$\mathfrak{t} \leq \lim_{\mathfrak{n}\to\infty} inf \|\mathfrak{z}_{\mathfrak{n}} - \mathfrak{p}\| \tag{4.14}$$

Hence, from (4.12) and (4.14), we have

$$\mathfrak{t} = \lim_{\mathfrak{n}\to\infty} \|\mathfrak{z}_\mathfrak{n} - \mathfrak{p}\|$$

From (4.1) and from (4.11), we have

$$\begin{aligned}
\mathfrak{t} &= \lim_{\mathfrak{n}\to\infty} \|\mathfrak{z}_\mathfrak{n} - \mathfrak{p}\| \\
&= \lim_{\mathfrak{n}\to\infty} ||\mathfrak{T}[(1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n}] - \mathfrak{p}|| \\
&\leq \lim_{\mathfrak{n}\to\infty} ||(1-\mathfrak{O}_\mathfrak{n})(x_n - \mathfrak{p}) + \mathfrak{O}_\mathfrak{n}(\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{p})|| \\
&\leq \lim_{\mathfrak{n}\to\infty} [(1-\mathfrak{O}_\mathfrak{n}) \|\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\| + \lim_{n\to\infty} \mathfrak{O}_\mathfrak{n} \|\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{p}\|] \\
&\leq \mathfrak{t} \qquad (4.15)
\end{aligned}$$

Hence

$$\mathfrak{t} = \lim_{\mathfrak{n}\to\infty} ||(1-\mathfrak{O}_\mathfrak{n})(\mathfrak{x}_\mathfrak{n} - \mathfrak{p}) + \mathfrak{O}_\mathfrak{n}(\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{p})|| \qquad (4.16)$$

Now from (4.12, 4.13, 4.16) and Lemma 2.2.6, we conclude that $lim_{\mathfrak{n}\to\infty} \|\mathfrak{T}\mathfrak{x}_\mathfrak{n} - \mathfrak{x}_\mathfrak{n}\| = 0$.

□

4.2 Stability

Throughout this section, we presume that $\mathcal{U}$ is a non-empty, closed and convex subset of a Banach space $\mathcal{W}$ and $\mathcal{F} : \mathcal{U} \to \mathcal{U}$ a Garcia Falset mapping. The purpose of this section is to prove stability for Garcia Falset mappings via iteration process (4.1).

Theorem 4.2.1. *Let $\{\mathfrak{x}_\mathfrak{n}\}$ be an iterative process be defined by iteration (4.1).Then iteration process becomes $\mathcal{F}$-stable.*

Proof. Suppose $\mathfrak{a}_\mathfrak{n}$ is an arbitrary sequence in $\mathcal{U}$ and $\mathfrak{a}_{\mathfrak{n}+1} = \mathfrak{f}(\mathcal{F}, \mathfrak{a}_\mathfrak{n})$ is the sequence generated by (4.1) converging to a fixed point $\mathfrak{p}$ (by Theorem 4.0.1) and $\epsilon_n = \left\|\mathfrak{a}_{\mathfrak{n}+1} - \mathfrak{f}(\mathcal{F}, \mathfrak{a}_\mathfrak{n})\right\|$ for all $\mathfrak{n} \in \mathbb{Z}+$. We have to prove that $\lim_{\mathfrak{n}\to\infty} \epsilon_n = 0$ which implies that $\lim_{\mathfrak{n}\to\infty} \mathfrak{a}_\mathfrak{n} = \mathfrak{p}$. Suppose $lim_{\mathfrak{n}\to\infty}\epsilon_n = 0$, then by iteration process (4.1), we have

$$
\begin{aligned}
\|\mathfrak{a}_{\mathfrak{n}+1}-\mathfrak{p}\| &\leq \left\|\mathfrak{a}_{\mathfrak{n}+1}-\mathfrak{f}(\mathcal{F},\mathfrak{a}_{\mathfrak{n}})\right\| + \left\|\mathfrak{f}(\mathcal{F},\mathfrak{a}_{\mathfrak{n}})-\mathfrak{p}\right\| \\
&\leq \epsilon_n + \left\|\mathfrak{f}(\mathcal{F},\mathfrak{a}_{\mathfrak{n}})-\mathfrak{p}\right\| \\
&\leq \epsilon_n + \mathfrak{H}^3[1-(1-\mathfrak{H})(\mathfrak{M}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}}+\mathfrak{O}_{\mathfrak{n}}-\mathfrak{O}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}}\mathfrak{M}_{\mathfrak{n}}]\,\|\mathfrak{a}_{\mathfrak{n}}-\mathfrak{p}\|
\end{aligned} \tag{4.17}
$$

since

$$0 < (1-(1-\mathfrak{H})(\mathfrak{M}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}})) \leq 1,$$

and

$$0 < (1-(1-\mathfrak{H})(\mathfrak{M}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}}+\mathfrak{O}_{\mathfrak{n}}-\mathfrak{O}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}}\mathfrak{M}_{\mathfrak{n}}) \leq 1,$$

we get

$$\|\mathfrak{a}_{\mathfrak{n}+1}-\mathfrak{p}\| \leq \epsilon_n + \mathfrak{H}^3\,\|\mathfrak{a}_{\mathfrak{n}}-\mathfrak{p}\|$$

Define

$$\mathfrak{q}_{\mathfrak{n}} = \|\mathfrak{a}_{\mathfrak{n}}-\mathfrak{p}\|\ ,$$

then

$$\mathfrak{a}_{\mathfrak{n}+1} \leq \mathfrak{H}^3\mathfrak{q}_{\mathfrak{n}} + \epsilon_n\ ,$$

since $\lim_{\mathfrak{n}\to\infty}\epsilon_n = 0$, so , we have $\lim_{\mathfrak{n}\to\infty}\mathfrak{q}_{\mathfrak{n}} = 0$ i.e., $\lim_{\mathfrak{n}\to\infty}\mathfrak{a}_{\mathfrak{n}} = \mathfrak{p}$. Conversely, suppose $\lim_{\mathfrak{n}\to\infty}\mathfrak{a}_{\mathfrak{n}} = \mathfrak{p}$, we have

$$
\begin{aligned}
\epsilon_n &= \left\|\mathfrak{a}_{\mathfrak{n}+1}-\mathfrak{f}(\mathcal{F},\mathfrak{a}_{\mathfrak{n}})\right\| \\
&\leq \|\mathfrak{a}_{\mathfrak{n}+1}-\mathfrak{p}\| + \left\|\mathfrak{f}(\mathcal{F},\mathfrak{a}_{\mathfrak{n}})-\mathfrak{p}\right\| \\
&\leq \|\mathfrak{a}_{\mathfrak{n}+1}-\mathfrak{p}\| + \mathfrak{H}^3[1-(1-\mathfrak{H})(\mathfrak{M}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}}+\mathfrak{O}_{\mathfrak{n}}-\mathfrak{O}_{\mathfrak{n}}\mathfrak{N}_{\mathfrak{n}}\mathfrak{M}_{\mathfrak{n}}]\,\|\mathfrak{a}_{\mathfrak{n}}-\mathfrak{p}\| \\
&\leq \|\mathfrak{a}_{\mathfrak{n}+1}-p\| + \mathfrak{H}^3\,\|\mathfrak{a}_{\mathfrak{n}}-\mathfrak{p}\|
\end{aligned}
$$

This implies that $\lim_{\mathfrak{n}\to\infty}\epsilon_n = 0$. Hence iteration process (4.1) $\mathcal{F}$-stable.

□

Chapter 5

APPLICATION

Thermal analysis and engineering both make extensive use of the heat equation, which represents the temperature distribution over time in a specific location. By adding delays, delay differential equations expand on conventional differential equations and can be used in systems where past states affect future behavior, such as population dynamics. Implicit functions are used by designed implicit neural networks to define outputs, resulting in reliable and effective solutions to challenging issues.

5.1 Heat Equation

In this section, we approximate the solution of one dimensional heat equation

$$\frac{\partial \mathfrak{u}}{\partial \mathfrak{t}} = \alpha \frac{\partial^2 \mathfrak{u}}{\partial \mathfrak{x}^2} \quad , \; 0 < \mathfrak{x} < \mathfrak{L}, \quad 0 < \mathfrak{t} < \infty, \tag{5.1}$$

subject to the initial condition:

$$\mathfrak{u}(\mathfrak{x}, 0) = \mathfrak{f}(\mathfrak{x}), \tag{5.2}$$

and homogeneous boundary conditions:

$$\mathfrak{u}(0, \mathfrak{t}) = 0 \quad \text{and} \quad \mathfrak{u}(\mathfrak{L}, \mathfrak{t}) = 0, \tag{5.3}$$

where $\mathfrak{f}(\mathfrak{x})$ is continuous function. Now we approximate the solution of problem (5.1) by utilizing iterative scheme (4.1) with the following assumptions

$$|\mathfrak{f}(\mathfrak{t},\mathfrak{a}) - \mathfrak{f}(\mathfrak{t},\mathfrak{b})| \leq max|\mathfrak{a}-\mathfrak{b}|, \quad for\ \ all \quad 0 < \mathfrak{t} < \infty, \tag{5.4}$$

Theorem 5.1.1. *Let $\mathfrak{S}$ be a Banach space. Let $\mathfrak{x}_\mathfrak{n}$ be a sequence defined by (4.1) for operator $\mathfrak{T}: \ \mathfrak{C} \to \mathfrak{C}$ defined by*

$$\mathfrak{T}(\mathfrak{u}(\mathfrak{t},\ \mathfrak{x})) = \ \mathfrak{B}_\mathfrak{n}\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t},\mathfrak{x})), \ \ \mathfrak{x} \in \mathfrak{C} \tag{5.5}$$

assume that condition (5.4) is satisfied. Then the sequence defined by iterative scheme converges to a solution, say $\mathfrak{u}(\mathfrak{t},\mathfrak{x})$ of problem (5.1).

Proof. Observe $\mathfrak{u}(\mathfrak{t},\mathfrak{x})$ is a solution of (5.1) if and only if $\mathfrak{u}(\mathfrak{t},\mathfrak{x})$ is a solution of

$$\mathfrak{u}(\mathfrak{t},\mathfrak{x}) = \ \mathfrak{B}_\mathfrak{n}\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t},\mathfrak{x})), \ \ \mathfrak{x} \in \mathfrak{C} \tag{5.6}$$

Let $\mathfrak{u},\mathfrak{v} \in \mathfrak{S}$ and using (5.4), we get

$$\begin{aligned} |\mathfrak{T}(\mathfrak{u}(\mathfrak{t},\mathfrak{x})) - \mathfrak{T}(\mathfrak{v}(\mathfrak{t},\mathfrak{x}))| &= |\mathfrak{B}_\mathfrak{n}\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t},\mathfrak{x})), -\mathfrak{B}_\mathfrak{n}\mathfrak{f}(\mathfrak{t},\mathfrak{v}(\mathfrak{t},\mathfrak{x}))| \\ &\leq \mathfrak{B}_\mathfrak{n}|\mathfrak{f}(\mathfrak{t},\mathfrak{u}(\mathfrak{t},\mathfrak{x})) - \mathfrak{f}(\mathfrak{t},\mathfrak{v}(\mathfrak{t},\mathfrak{x}))| \\ &\leq \mathfrak{B}_\mathfrak{n}max|(\mathfrak{u}(\mathfrak{t},\mathfrak{x})) - (\mathfrak{v}(\mathfrak{t},\mathfrak{x}))| \\ &\leq \mathfrak{B}_\mathfrak{n}||(\mathfrak{u}(\mathfrak{t},\mathfrak{x})) - (\mathfrak{v}(\mathfrak{t},\mathfrak{x}))|| \end{aligned} \tag{5.7}$$

As

$$\mathfrak{B}_\mathfrak{n} = \begin{cases} 0 & for \quad even \quad \mathfrak{n} \\ \frac{4}{\mathfrak{n}\pi} & for \quad odd \quad \mathfrak{n} \end{cases} \tag{5.8}$$

To apply our theorem, we must have nonexpansive mapping. To make (5.7) nonexpansive, we will make $\mathfrak{B}_\mathfrak{n} = 1$. As we see in (5.8), we will take only odd case. when we take $\mathfrak{n} = \frac{4}{\pi}$. We obtain

$$\left\|\mathfrak{T}(\mathfrak{u}(\mathfrak{t},\mathfrak{x})) - \mathfrak{T}(\mathfrak{v}(\mathfrak{t},\mathfrak{x}))\right\| = \ \left\|(\mathfrak{u}(\mathfrak{t},\mathfrak{x})) - (\mathfrak{v}(\mathfrak{t},\mathfrak{x}))\right\| \tag{5.9}$$

Thus $\mathfrak{T}$ is nonexpansive mapping. Hence our iterative scheme converges to the solution of (5.1).

□

Example 5.1.1. Consider the following equation

$$\frac{\partial \mathfrak{u}}{\partial \mathfrak{t}} = \alpha \frac{\partial^2 \mathfrak{u}}{\partial \mathfrak{x}^2}, \qquad 0 < \mathfrak{x} < 2, \quad \mathfrak{t} = 0.5, \quad \alpha = 0.5, \tag{5.10}$$

subject to the initial condition:

$$\mathfrak{u}(\mathfrak{x}, 0) = 1,$$

and homogeneous boundary conditions:

$$\mathfrak{u}(0, \mathfrak{t}) = 0 \quad \text{and} \quad \mathfrak{u}(2, \mathfrak{t}) = 0.$$

The exact solution of problem (5.10) is given by

$$\mathfrak{u}(\mathfrak{t}, \mathfrak{x}) = \sum_{\mathfrak{n}=1,\text{odd}}^{\infty} \frac{4}{\mathfrak{n}\pi} e^{-0.25\left(\frac{\mathfrak{n}\pi}{2}\right)^2} \sin\left(\frac{\mathfrak{n}\pi x}{2}\right). \tag{5.11}$$

The operator $\mathfrak{T}: \mathfrak{C} \to \mathfrak{C}$ is defined by

$$\mathfrak{T}\mathfrak{u}(\mathfrak{t}, \mathfrak{x}) = \sum_{\mathfrak{n}=1,\text{odd}}^{\infty} \frac{4}{\mathfrak{n}\pi} e^{-0.25\left(\frac{\mathfrak{n}\pi}{2}\right)^2} \sin\left(\frac{\mathfrak{n}\pi\mathfrak{x}}{2}\right). \tag{5.12}$$

We observe that iterative scheme 4.1 converges to the exact solution of problem(5.10) for the operator defined in equation(5.12) which is shown in Table (5.1) and Figure (5.1).

Remark 4. It is visible that after 7 iterations, new iteration (4.1) converges to solution of problem (5.10) upto 2-decimal faster than the iterative techniques [23], [25], [27] and after 50 iterations, convergence rate is faster than iterative techniques [23], [25], [27] and shown in Table 5.1.

No. of iteration	Akanimo	Thakur	Asghar Rahimi	New iteration
1	0.1	0.1	0.1	0.1
2	-0.0644720187	-0.0869402962	-0.1630096033	-0.3803139327
3	-0.1922289988	-0.2264595551	-0.3322517610	-0.5494920044
4	-0.2915201418	-0.3306827011	-0.4414827416	-0.6100326607
5	-0.3687681179	-0.4086549542	-0.5122262545	-0.6318737760
6	-0.4289375510	-0.4670790018	-0.5581749961	-0.6397788231
7	-0.4758578291	-0.5109175395	-0.5880826744	-0.6426433848
8	-0.5124838838	-0.5438511695	-0.6075783795	-0.6436818788
9	-0.5410994344	-0.5686166193	-0.6202998349	-0.6440584262
10	-0.5634729424	-0.5872541993	-0.6286065855	-0.6441949665
⋮	⋮	⋮	⋮	⋮
48	-0.6442665057	-0.6442716845	-0.6442726463	-0.6442726473
49	-0.6442678310	-0.6442719208	-0.6442726467	-0.6442726473
50	-0.6442688702	-0.6442720992	-0.6442726469	-0.6442726473

Table 5.1. Compairing iterations convergence

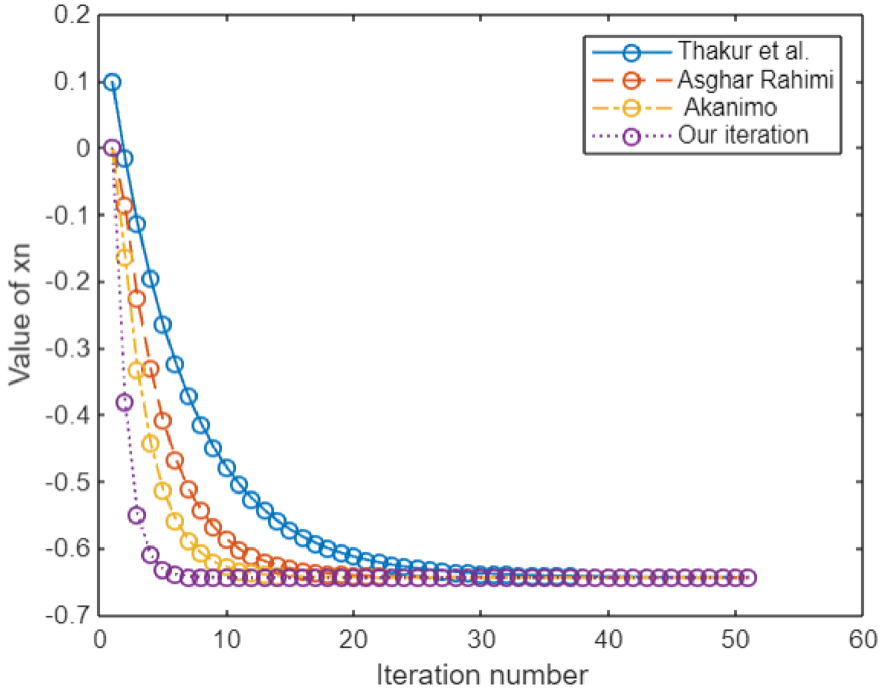

Figure 5.1: Comparison of iteration processes convergence

5.2 Delay Differential Equations

A delay differential equation (DDE) is a type of differential equation that includes terms dependent on past values of the solution. It takes the form:

$$\frac{\partial \mathfrak{y}(\mathfrak{t})}{\partial \mathfrak{t}} = \mathfrak{f}(\mathfrak{t}, \mathfrak{y}(\mathfrak{t}), \mathfrak{y}(\mathfrak{t} - \tau))$$

where $\mathfrak{y}(\mathfrak{t})$ is the unknown function of time $\mathfrak{t}$, τ is the delay parameter, and $\mathfrak{f}$ is a function that describes how the rate of change of $\mathfrak{y}$ depends on both the current state $\mathfrak{y}(\mathfrak{t})$ and the state at an earlier time $\mathfrak{y}(\mathfrak{t} - \tau)$. This inclusion of past states allows DDEs to model systems where historical information impacts future dynamics, such as in biological, engineering, or economic processes In this subsection, we using proposed iterative scheme (4.1) to find the solution of delay differential equations.
Let the space of all continuous real-valued functions be denoted by $\mathfrak{C}([\mathfrak{u}, \mathfrak{v}])$ on closed interval $[u, v]$ endowed with th chebyshev norm $||\mathfrak{x} - \mathfrak{z}||_\infty$ and defines as $||\mathfrak{x} - \mathfrak{z}||_\infty = sup_{\mathfrak{r} \in [\mathfrak{u}, \mathfrak{v}]} |\mathfrak{x}(\mathfrak{r}) - \mathfrak{z}(\mathfrak{r})|$, and it is clear that in [30] that $(\mathfrak{C}([\mathfrak{u}, \mathfrak{v}], ||.||_\infty))$ is a Banach space.
Now, consider the following delay differential equation

$$\mathfrak{x}'(\mathfrak{r}) = \psi(\mathfrak{r}, \mathfrak{x}(\mathfrak{r}), \mathfrak{x}(\mathfrak{r} - \gamma)), \qquad \mathfrak{r} \in [\mathfrak{r}_o, \mathfrak{v}] \tag{5.13}$$

with initial condition

$$\mathfrak{x}(\mathfrak{r}) = \zeta(\mathfrak{r}), \qquad \mathfrak{r} \in [\mathfrak{r}_o - \gamma, \mathfrak{r}_o] \tag{5.14}$$

By the solution of above delay differential equation, we mean a function $\mathfrak{x} \in \mathfrak{C}([\mathfrak{r}_o - \gamma, \mathfrak{v}], \mathbb{R}) \cap \mathfrak{C}^1([\mathfrak{r}_o, \mathfrak{v}], \mathbb{R})$ satisfying (5.13) and (5.14).
Assume that the following conditions are satisfied.
(1) $\mathfrak{r}_o, \mathfrak{v} \in \mathbb{R}, \gamma > 0$
(2) $\psi \in \mathfrak{C}([\mathfrak{r}_o, \mathfrak{v}] \times \mathbb{R}^2, \mathbb{R})$
(3) $\zeta \in \mathfrak{C}([\mathfrak{r}_o - \gamma, \mathfrak{v}], \mathbb{R})$
(4) There exists $\mathfrak{Q}_\psi > \mathfrak{o}$ such that

$$|\psi(\mathfrak{r}, \mathfrak{s}_1, \mathfrak{s}_2) - \psi(\mathfrak{r}, \mathfrak{t}_1, \mathfrak{t}_2)| \leq \mathfrak{Q}_\psi \sum_{\mathfrak{i}=1}^{2} |\mathfrak{s}_\mathfrak{i} - \mathfrak{t}_\mathfrak{i}|, \quad \mathfrak{s}_\mathfrak{i}, \mathfrak{t}_\mathfrak{i} \in \mathbb{R}, \mathfrak{r} \in [\mathfrak{r}_o, \mathfrak{v}] \tag{5.15}$$

(5) $2\mathfrak{Q}_\psi(\mathfrak{v} - \mathfrak{r}_o) < 1$
Now, we construct (5.13) and (5.14) by the integral equation as

$$\mathfrak{x}(\mathfrak{r}) = \begin{cases} \zeta(\mathfrak{r}), & \mathfrak{r} \in [\mathfrak{r}_o - \gamma, \mathfrak{v}] \\ \zeta(\mathfrak{r}_o) + \int_{\mathfrak{r}_o}^{\mathfrak{r}} \psi(\mathfrak{t}, \mathfrak{x}(\mathfrak{t}), \mathfrak{x}(\mathfrak{t} - \gamma)) \partial \mathfrak{t}, & \mathfrak{r} \in [\mathfrak{r}_o, \mathfrak{v}]. \end{cases} \tag{5.16}$$

The following result is the generalization of the result of Coman et al. [31].

Theorem 5.2.1. *Let the conditions $*_1)$ to $*_5)$ be satisfied. Then (5.13) and (5.14) have unique solution $\mathfrak{x}^* \in \mathfrak{C}([\mathfrak{r}_o - \gamma, \mathfrak{v}], \mathbb{R}) \cap \mathfrak{C}^1([\mathfrak{r}_o, \mathfrak{v}], \mathbb{R})$ and*

$$\mathfrak{x}^* = lim_{\mathfrak{n}\to\infty}\mathfrak{T}^{\mathfrak{n}}(\mathfrak{x}) \tag{5.17}$$

Now, by using the iterative scheme (4.1), we prove the following result.

Theorem 5.2.2. *Let the conditions $*_1)$ to $*_5)$ be satisfied. Then (5.13) and (5.14) have unique solution $\mathfrak{x}^* \in \mathfrak{C}([\mathfrak{r}_o - \gamma, \mathfrak{v}], \mathbb{R}) \cap \mathfrak{C}^1([\mathfrak{r}_o, \mathfrak{v}], \mathbb{R})$ and the iterative scheme (4.1) converges to $\mathfrak{x}^*$*

Proof. Let $\mathfrak{x}_{\mathfrak{n}}$ be a sequence generated by the iterative scheme (4.1) for an operator $\mathfrak{T}$ defined by

$$\mathfrak{T}\mathfrak{x}(\mathfrak{r}) = \begin{cases} \zeta(\mathfrak{r}), & \mathfrak{r} \in [\mathfrak{r}_o - \gamma, \mathfrak{v}] \\ \zeta(\mathfrak{r}_o) + \int_{\mathfrak{r}_o}^{\mathfrak{r}} \psi(\mathfrak{p}, \mathfrak{x}(\mathfrak{p}), \mathfrak{x}(\mathfrak{p} - \gamma))\partial\mathfrak{p}, & \mathfrak{r} \in [\mathfrak{r}_o, \mathfrak{v}]. \end{cases} \tag{5.18}$$

Let $\mathfrak{x}^*$ be a fixed point of $\mathfrak{T}$. Now, we prove that $\mathfrak{x}_{\mathfrak{n}} \to \mathfrak{x}^*$ as $\mathfrak{n} \to \infty$ for each $\mathfrak{r} \in [\mathfrak{r}_o - \gamma, \mathfrak{r}_o]$. Now, for each $\mathfrak{r} \in [\mathfrak{r}_o, \mathfrak{v}]$, we have

$$
\begin{aligned}
\|\mathfrak{z}_\mathfrak{n} - \mathfrak{x}^*\|_\infty &= \|\mathfrak{T}((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n}) - \mathfrak{x}^*\|_\infty \\
&\leq \sup_{\mathfrak{r}\in[r_0,v]} \left|\mathfrak{T}(((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n}))(\mathfrak{t}) - \mathfrak{T}\mathfrak{x}^*(\mathfrak{t})\right| \\
&\leq \sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \\
&\quad \Bigg|\zeta(\mathfrak{r}_\mathfrak{o}) + \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}} \psi(\mathfrak{s},((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})(\mathfrak{s}), \\
&\quad ((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s} \\
&\quad -(\zeta(\mathfrak{r}_\mathfrak{o}) + \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}} \psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s})\Bigg| \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}} \Big|\psi(\mathfrak{s},((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})(\mathfrak{s}), \\
&\quad ((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s} - \psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s})\Big| \\
&\leq \sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}} \mathfrak{L}_\psi \Big(\left|((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})(\mathfrak{s}) - \mathfrak{x}^*(\mathfrak{s})\right| \\
&\quad + \left|((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})(\mathfrak{s}-\gamma) - \mathfrak{x}^*(\mathfrak{s}-\gamma)\right|\Big)\,\mathfrak{d}\mathfrak{s} \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}} \mathfrak{L}_\psi \Bigg(\sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \left|((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})(\mathfrak{s}) - \mathfrak{x}^*(\mathfrak{s})\right| \\
&\quad + \sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \left|((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})(\mathfrak{s}-\gamma) - \mathfrak{x}^*(\mathfrak{s}-\gamma)\right|\Bigg)\,\mathfrak{d}\mathfrak{s} \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}} \mathfrak{L}_\psi \left(\|((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})(\mathfrak{s}) - \mathfrak{x}^*(\mathfrak{s})\|_\infty\right. \\
&\quad \left.+\|((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})(\mathfrak{s}) - \mathfrak{x}^*(\mathfrak{s})\|_\infty\right)\mathfrak{d}\mathfrak{s} \\
&\leq 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})\|((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n} + \mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})(\mathfrak{s}) - \mathfrak{x}^*(\mathfrak{s})\|_\infty. \qquad (5.19)
\end{aligned}
$$

$$
\begin{aligned}
\|((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})-\mathfrak{x}^*\|_\infty &= \|((1-\mathfrak{O}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{O}_\mathfrak{n}\mathfrak{T}\mathfrak{x}_\mathfrak{n})-\mathfrak{x}^*\|_\infty \\
&\leq (1-\mathfrak{O}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*\|_\infty+\mathfrak{O}_\mathfrak{n}\|\mathfrak{T}\mathfrak{x}_\mathfrak{n}-\mathfrak{T}\mathfrak{x}^*\|_\infty \\
&\leq (1-\mathfrak{O}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*\|_\infty+\mathfrak{O}_\mathfrak{n} sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \\
&\quad (\left|\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\psi(\mathfrak{s},(\mathfrak{x}_\mathfrak{n}(\mathfrak{s}),\mathfrak{x}_\mathfrak{n}(\mathfrak{s}-\gamma)\mathfrak{d}\mathfrak{s}-\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s}\right|) \\
&\leq (1-\mathfrak{O}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*\|_\infty \\
&+ \mathfrak{O}_\mathfrak{n}\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\mathfrak{L}_\psi||\mathfrak{x}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty+||\mathfrak{x}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty\mathfrak{d}\mathfrak{s} \\
&\leq (1-\mathfrak{O}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*\|_\infty+\mathfrak{O}_\mathfrak{n}2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})||\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*||_\infty \\
&\leq [1-\mathfrak{O}_\mathfrak{n}(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o}))]||\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*||_\infty \qquad (5.20)
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\|\mathfrak{y}_\mathfrak{n}-\mathfrak{x}^*\|_\infty &= \|\mathfrak{T}((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})-\mathfrak{x}^*\|_\infty \\
&\leq \sup_{\mathfrak{r}\in[r_0,v]}\left|\mathfrak{T}(((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n}))(\mathfrak{t})-\mathfrak{T}\mathfrak{x}^*(\mathfrak{t})\right| \\
&\leq \sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \\
&\quad \left|\zeta(\mathfrak{r}_\mathfrak{o})+\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\psi(\mathfrak{s},((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})(\mathfrak{s}),\right. \\
&\quad \left.((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s}-(\zeta(\mathfrak{r}_\mathfrak{o})+\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s})\right| \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\left|\psi(\mathfrak{s},((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})(\mathfrak{s}),\right. \\
&\quad \left.((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s}-\psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s})\right| \\
&\leq \sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]}\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\mathfrak{L}_\psi\left(\left|((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})\right|\right. \\
&\quad \left.+\left|((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})(\mathfrak{s}-\gamma)-\mathfrak{x}^*(\mathfrak{s}-\gamma)\right|\right)\mathfrak{d}\mathfrak{s} \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\mathfrak{L}_\psi\left(\sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]}\left|((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})\right|\right. \\
&\quad \left.+\sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]}\left|((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})(\mathfrak{s}-\gamma)-\mathfrak{x}^*(\mathfrak{s}-\gamma)\right|\right)\mathfrak{d}\mathfrak{s} \\
&\leq \int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\mathfrak{L}_\psi\left(\|((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})\|_\infty\right. \\
&\qquad (5.21)
\end{aligned}
$$

$$
\begin{aligned}
&+\|((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})\|_\infty\big)\,\mathfrak{d}\mathfrak{s}\\
\leq\ & 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})\|((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})\|_\infty. \qquad (5.22)
\end{aligned}
$$

$$
\begin{aligned}
\|((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})-\mathfrak{x}^*\|_\infty &= \|((1-\mathfrak{N}_\mathfrak{n})\mathfrak{x}_\mathfrak{n}+\mathfrak{N}_\mathfrak{n}\mathfrak{T}\mathfrak{z}_\mathfrak{n})-\mathfrak{x}^*\|_\infty\\
&\leq (1-\mathfrak{N}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*\|_\infty+\mathfrak{N}_\mathfrak{n}\|\mathfrak{T}\mathfrak{z}_\mathfrak{n}-\mathfrak{T}\mathfrak{x}^*\|_\infty\\
&\leq (1-\mathfrak{N}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*\|_\infty+\mathfrak{N}_\mathfrak{n} sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]}\\
&\quad (\left|\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\psi(\mathfrak{s},(\mathfrak{z}_\mathfrak{n}(\mathfrak{s}),\mathfrak{z}_\mathfrak{n}(\mathfrak{s}-\gamma)\mathfrak{d}\mathfrak{s}-\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s}\right|)\\
&\leq (1-\mathfrak{N}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*\|_\infty\\
&+ \mathfrak{N}_\mathfrak{n}\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\mathfrak{L}_\psi||\mathfrak{z}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty+||\mathfrak{z}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty\mathfrak{d}\mathfrak{s}\\
&\leq (1-\mathfrak{N}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*\|_\infty+\mathfrak{N}_\mathfrak{n}2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})||\mathfrak{z}_\mathfrak{n}-\mathfrak{x}^*||_\infty\\
&\leq (1-\mathfrak{N}_\mathfrak{n})\|\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*\|_\infty+\mathfrak{N}_\mathfrak{n}2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})\\
&\quad [1-\mathfrak{O}_\mathfrak{n}(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o}))]||\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*||_\infty\\
&\leq [1-\mathfrak{N}_\mathfrak{n}(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o}))][1-\mathfrak{O}_\mathfrak{n}(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o}))]\\
&\quad ||\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*||_\infty \qquad (5.23)
\end{aligned}
$$

and

$$
\begin{aligned}
\|\mathfrak{p}_\mathfrak{n}-\mathfrak{x}^*\|_\infty &= \|\mathfrak{T}\left((1-\mathfrak{M}_\mathfrak{n})\mathfrak{T}\mathfrak{z}_\mathfrak{n}+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}\right)-\mathfrak{x}^*\|_\infty\\
&\leq \sup_{\mathfrak{r}\in[r_0,v]}\left|\mathfrak{T}\left((1-\mathfrak{M}_\mathfrak{n})\,\mathfrak{T}\mathfrak{z}_\mathfrak{n}+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}\right)(\mathfrak{t})-\mathfrak{T}\mathfrak{x}^*(\mathfrak{t})\right|\\
&\leq \sup_{\mathfrak{r}\in[r_0,v]}\left|\zeta(\mathfrak{r}_\mathfrak{o})+\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\psi\left(\mathfrak{s},(1-\mathfrak{M}_\mathfrak{n})\,\mathfrak{T}\mathfrak{z}_\mathfrak{n}(\mathfrak{s})+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}(\mathfrak{s}),\right.\right.\\
&\quad (1-\mathfrak{M}_\mathfrak{n})\,\mathfrak{T}\mathfrak{z}_\mathfrak{n}(\mathfrak{s}-\gamma)+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}(\mathfrak{s}-\gamma)\big)\,d\mathfrak{s}\\
&\quad \left.-\left(\zeta(\mathfrak{r}_\mathfrak{o})+\int_{r_0}^{r}\psi\left(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma)\right)d\mathfrak{s}\right)\right|\\
&\leq \int_{r_0}^{r}\Big|\psi\left(s,(1-\mathfrak{M}_\mathfrak{n})\,\mathfrak{T}\mathfrak{z}_\mathfrak{n}(\mathfrak{s})+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}(\mathfrak{s}),\right.\\
&\quad \left.(1-\mathfrak{M}_\mathfrak{n})\,\mathfrak{T}\mathfrak{z}_\mathfrak{n}(\mathfrak{s}-\gamma)+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}(\mathfrak{s}-\gamma)\right)-\psi\left(s,\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma)\right)\Big|\,d\mathfrak{s}\\
&\leq \int_{r_0}^{r}\mathfrak{L}_\psi\left(\sup_{\mathfrak{r}\in[r_0,v]}\left|(1-\mathfrak{M}_\mathfrak{n})\,\mathfrak{T}\mathfrak{z}_\mathfrak{n}(\mathfrak{s})+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})\right|\right.
\end{aligned}
$$

$$
\begin{aligned}
&+ \sup_{\mathfrak{r}\in[r_0,v]} \left|(1-\mathfrak{M}_\mathfrak{n})\,\mathfrak{T}\mathfrak{z}_\mathfrak{n}(\mathfrak{s}-\gamma)+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}(\mathfrak{s}-\gamma)-\mathfrak{x}^*(\mathfrak{s}-\gamma)\right|\Big)\, d\mathfrak{s} \\
\leq\ & \int_{r_0}^{r} \mathfrak{L}_\psi \left(\|\,(1-\mathfrak{M}_\mathfrak{n})\,\mathfrak{T}\mathfrak{z}_\mathfrak{n}(\mathfrak{s})+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})\|_\infty\right. \\
&\left.+\|\,(1-\mathfrak{M}_\mathfrak{n})\,\mathfrak{T}\mathfrak{z}_\mathfrak{n}(\mathfrak{s}-\gamma)+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}(\mathfrak{s}-\gamma)-\mathfrak{x}^*(s-\gamma)\|_\infty\right) d\mathfrak{s} \\
\leq\ & 2\mathfrak{L}_\psi(v-r_0)\|(1-\mathfrak{M}_\mathfrak{n})\mathfrak{T}\mathfrak{z}_\mathfrak{n}+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n}-\mathfrak{x}^*(\mathfrak{s})\|_\infty. \qquad (5.24)
\end{aligned}
$$

$$
\begin{aligned}
\|((1-\mathfrak{M}_\mathfrak{n})\mathfrak{T}\mathfrak{z}_\mathfrak{n}+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n})-\mathfrak{x}^*\|_\infty &= \|((1-\mathfrak{M}_\mathfrak{n})\mathfrak{T}\mathfrak{z}_\mathfrak{n}+\mathfrak{M}_\mathfrak{n}\mathfrak{T}\mathfrak{y}_\mathfrak{n})-\mathfrak{T}\mathfrak{x}^*\|_\infty \\
&\leq (1-\mathfrak{M}_\mathfrak{n})\|\mathfrak{T}\mathfrak{z}_\mathfrak{n}-\mathfrak{T}\mathfrak{x}^*\|_\infty+\mathfrak{M}_\mathfrak{n}\|\mathfrak{T}\mathfrak{y}_\mathfrak{n}-\mathfrak{T}\mathfrak{x}^*\|_\infty \\
&\leq (1-\mathfrak{M}_\mathfrak{n})sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \\
&\quad (\left|\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\psi(\mathfrak{s},(\mathfrak{z}_\mathfrak{n}(\mathfrak{s}),\mathfrak{z}_\mathfrak{n}(\mathfrak{s}-\gamma)\mathfrak{d}\mathfrak{s}-\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s}\right|) \\
&+ \mathfrak{M}_\mathfrak{n}sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]} \\
&\quad (\left|\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\psi(\mathfrak{s},(\mathfrak{y}_\mathfrak{n}(\mathfrak{s}),\mathfrak{y}_\mathfrak{n}(\mathfrak{s}-\gamma)\mathfrak{d}\mathfrak{s}-\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s}\right|) \\
&\leq (1-\mathfrak{M}_\mathfrak{n})\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\mathfrak{L}_\psi||\mathfrak{z}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty+||\mathfrak{z}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty\mathfrak{d}\mathfrak{s} \\
&+ \mathfrak{M}_\mathfrak{n}\int_{\mathfrak{r}_\mathfrak{o}}^{\mathfrak{r}}\mathfrak{L}_\psi||\mathfrak{y}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty+||\mathfrak{y}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty\mathfrak{d}\mathfrak{s} \\
&\leq (1-\mathfrak{M}_\mathfrak{n})2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})||\mathfrak{z}_\mathfrak{n}-\mathfrak{x}^*||_\infty \\
&+ \mathfrak{M}_\mathfrak{n}2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})||\mathfrak{y}_\mathfrak{n}-\mathfrak{x}^*||_\infty \\
&\leq (1-\mathfrak{M}_\mathfrak{n})2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})[1-\mathfrak{O}_\mathfrak{n}(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o}))]+\mathfrak{M}_\mathfrak{n}2 \\
&\quad \mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o})[1-\mathfrak{N}_\mathfrak{n}(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o}))] \\
&\quad [1-\mathfrak{O}_\mathfrak{n}(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o}))]||\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*||_\infty \\
&= 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o}) \\
&\quad [1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_\mathfrak{o}))(\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}+\mathfrak{O}_\mathfrak{n}-\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}\mathfrak{O}_\mathfrak{n})] \\
&\quad ||\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*||_\infty \qquad (5.25)
\end{aligned}
$$

Finally,

$$
\begin{aligned}
\|\mathfrak{x}_{\mathfrak{n}+1}-\mathfrak{x}^*\|_\infty &= \|\mathfrak{T}(\mathfrak{T}\mathfrak{p}_\mathfrak{n})-\mathfrak{x}^*\|_\infty \\
&\leq \sup_{\mathfrak{r}\in[r_0,v]}\left|\mathfrak{T}(\mathfrak{T}\mathfrak{p}_\mathfrak{n})(\mathfrak{t})-\mathfrak{T}\mathfrak{x}^*(\mathfrak{t})\right| \\
&\leq sup_{\mathfrak{r}\in[\mathfrak{r}_\mathfrak{o},\mathfrak{v}]}
\end{aligned}
$$

$$
\begin{aligned}
&\left|\zeta(\mathfrak{r}_o)+\int_{\mathfrak{r}_o}^{\mathfrak{r}}\psi(\mathfrak{s},\mathfrak{T}\mathfrak{p}_\mathfrak{n}(\mathfrak{s}),\mathfrak{T}\mathfrak{p}_\mathfrak{n}(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s}-(\zeta(\mathfrak{r}_o)+\int_{\mathfrak{r}_o}^{\mathfrak{r}}\psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s})\right|\\
\leq\ & \int_{\mathfrak{r}_o{}^r}\left|\psi(\mathfrak{s},\mathfrak{T}\mathfrak{p}_\mathfrak{n}(\mathfrak{s}),\mathfrak{T}\mathfrak{p}_\mathfrak{n}(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s}-\psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s})\right|\\
\leq\ & sup_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]}\int_{\mathfrak{r}_o}^{\mathfrak{r}}\mathfrak{L}_\psi(\left|\mathfrak{T}\mathfrak{p}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})\right|+\left|\mathfrak{T}\mathfrak{p}_\mathfrak{n}(\mathfrak{s}-\gamma)-\mathfrak{x}^*(\mathfrak{s}-\gamma)\right|)\mathfrak{d}\mathfrak{s}\\
\leq\ & \int_{\mathfrak{r}_o}^{\mathfrak{r}}\mathfrak{L}_\psi(\mathfrak{sup}_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]}(\left|\mathfrak{T}\mathfrak{p}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})\right|+\mathfrak{sup}_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]}\left|\mathfrak{T}\mathfrak{p}_\mathfrak{n}(\mathfrak{s}-\gamma)-\mathfrak{x}^*(\mathfrak{s}-\gamma)\right|))\mathfrak{d}\mathfrak{s}\\
\leq\ & \int_{\mathfrak{r}_o}^{\mathfrak{r}}\mathfrak{L}_\psi(||\mathfrak{T}\mathfrak{p}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty+||\mathfrak{T}\mathfrak{p}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty)\mathfrak{d}\mathfrak{s}\\
\leq\ & 2\mathfrak{L}_\psi(v-r_0)||\mathfrak{T}\mathfrak{p}_\mathfrak{n}-\mathfrak{x}^*||_\infty. \qquad (5.26)
\end{aligned}
$$

$$
\begin{aligned}
\|\mathfrak{T}\mathfrak{p}_\mathfrak{n}-\mathfrak{x}^*\|_\infty &= \|\mathfrak{T}(\mathfrak{p}_\mathfrak{n})-\mathfrak{x}^*\|_\infty\\
&\leq \sup_{\mathfrak{r}\in[r_0,v]}\left|\mathfrak{T}(\mathfrak{p}_\mathfrak{n})(\mathfrak{t})-\mathfrak{T}\mathfrak{x}^*(\mathfrak{t})\right|\\
&\leq sup_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]}\\
&\quad\left|\zeta(\mathfrak{r}_o)+\int_{\mathfrak{r}_o}^{\mathfrak{r}}\psi(\mathfrak{s},\mathfrak{p}_\mathfrak{n}(\mathfrak{s}),\mathfrak{p}_\mathfrak{n}(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s}-(\zeta(\mathfrak{r}_o)+\int_{\mathfrak{r}_o}^{\mathfrak{r}}\psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s})\right|\\
&\leq \int_{\mathfrak{r}_o}^{\mathfrak{r}}\left|\psi(\mathfrak{s},\mathfrak{p}_\mathfrak{n}(\mathfrak{s}),\mathfrak{p}_\mathfrak{n}(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s}-\psi(\mathfrak{s},\mathfrak{x}^*(\mathfrak{s}),\mathfrak{x}^*(\mathfrak{s}-\gamma))\mathfrak{d}\mathfrak{s})\right|\\
&\leq sup_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]}\int_{\mathfrak{r}_o}^{\mathfrak{r}}\mathfrak{L}_\psi(\left|\mathfrak{p}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})\right|+\left|\mathfrak{p}_\mathfrak{n}(\mathfrak{s}-\gamma)-\mathfrak{x}^*(\mathfrak{s}-\gamma)\right|)\mathfrak{d}\mathfrak{s}\\
&\leq \int_{\mathfrak{r}_o}^{\mathfrak{r}}\mathfrak{L}_\psi(\mathfrak{sup}_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]}(\left|\mathfrak{p}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})\right|+\mathfrak{sup}_{\mathfrak{r}\in[\mathfrak{r}_o,\mathfrak{v}]}\left|\mathfrak{p}_\mathfrak{n}(\mathfrak{s}-\gamma)-\mathfrak{x}^*(\mathfrak{s}-\gamma)\right|))\mathfrak{d}\mathfrak{s}\\
&\leq \int_{\mathfrak{r}_o}^{\mathfrak{r}}\mathfrak{L}_\psi(||\mathfrak{p}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty+||\mathfrak{p}_\mathfrak{n}(\mathfrak{s})-\mathfrak{x}^*(\mathfrak{s})||_\infty)\mathfrak{d}\mathfrak{s}\\
&\leq 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)||\mathfrak{p}_\mathfrak{n}-\mathfrak{x}^*||_\infty.\\
&= 2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o)^2\\
&\quad[1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o))(\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}+\mathfrak{O}_\mathfrak{n}-\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}\mathfrak{O}_\mathfrak{n})]\\
&\quad||\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*||_\infty \qquad (5.27)
\end{aligned}
$$

Remark 5. By condition $*_5$) and as $[1-(1-2\mathfrak{L}_\psi(\mathfrak{v}-\mathfrak{r}_o))(\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}+\mathfrak{O}_\mathfrak{n}-\mathfrak{M}_\mathfrak{n}\mathfrak{N}_\mathfrak{n}\mathfrak{O}_\mathfrak{n})]=\tau_\mathfrak{n}<1$ and $||\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*||_\infty=\mathfrak{r}_\mathfrak{n}$. So, the conditions of Lemma 3 of [29] are satisfied. Hence, $lim_{\mathfrak{n}\to\infty}||\mathfrak{x}_\mathfrak{n}-\mathfrak{x}^*||_\infty=0$

□

5.3 Implicit Neural Network

In this subsection, we present a modified implicit neural network which can be regard as a multi-layer extension of the traditional feed-forward neural network.

Deep Equilibrium (DEQ) Models, an emerging class of implicit models that map inputs to fixed points in neural networks, are gaining popularity in the deep learning community. A deep equilibrium (DEQ) model deviates from classical depth by solving for the fixed point of a single nonlinear layer $\mathfrak{R}$. This structure allows for the decoupling of the layer's internal structure (which regulates representational capacity) from how the fixed point is determined (which affects inference-time efficiency), which is typically done using classic techniques.

We aim to build a neural network (5.2) that translates from a data space $\mathfrak{x}$ to an inference space $\mathfrak{y}$. The implicit component of the network utilizes a latent space $\mathfrak{X}$, and data is $\mathfrak{L}$ maps this latent space from $\mathfrak{x}$ to $\mathfrak{X}$. We defined the $\mathfrak{T}$ is a network operator that transforms $\mathfrak{X} \times \mathfrak{x} \rightarrow \mathfrak{X}$ by

$$\mathfrak{T}(\mathfrak{X},\ \mathfrak{x}) \triangleq \mathfrak{R}(\mathfrak{X}, \mathfrak{L}(\mathfrak{x}))$$

The objective is to find the unique fixed point $\mathfrak{X}^*_{\mathfrak{x}}$ of $\mathfrak{T}(;\mathfrak{x})$ given input data $\mathfrak{x}$. We will then use a final mapping $\mathfrak{J} : \mathfrak{X} \rightarrow \mathfrak{y}$ to transfer $\mathfrak{X}^*_{\mathfrak{x}}$ to the inference space $\mathfrak{y}$. Because of this, we can create an implicit network $\mathfrak{N}$ by

$$\mathfrak{N}(\mathfrak{x}) \triangleq \mathfrak{J}(\mathfrak{X}^*_{\mathfrak{x}}) \text{ where } \mathfrak{T}(\mathfrak{X}^*_{\mathfrak{x}} = \mathfrak{X}^*_{\mathfrak{x}}, \mathfrak{x})$$

Implicit models specify their outputs as solutions to nonlinear dynamical systems, as opposed to stacking a number of operators hierarchically. For instance, the outputs of DEQ models, which are the subject of this work, are defined as fixed points (a.k.a. equilibria) of a layer $\mathfrak{R}$ and input $\mathfrak{x}$; that is, output

$$\mathfrak{x}^* = \mathfrak{R}(\mathfrak{x}^*, \mathfrak{x})$$

Theorem 5.3.1. *Let $\mathfrak{C}$ be a nonempty closed convex subset of a uniformly convex Banach space and $\sigma : \mathfrak{C} \rightarrow \mathfrak{C}$ be a contraction mapping (activation function). Then equation (5.28) models a single input, well-posed and robust neural network provided that the weights $\{\mathfrak{M}_{\mathfrak{t}}\}$, $\{\mathfrak{N}_{\mathfrak{t}}\}$, $\{\mathfrak{O}_{\mathfrak{t}}\}$,$\{\mathfrak{W}_{(.)}\}$ and biases $\{\mathfrak{b}_{(.)}\}$ are in $[\epsilon$, 1- $\epsilon]$ for any $\mathfrak{n} \in \mathbb{N}$ and some ϵ in $(0, 1)$.*

$$\begin{aligned}
\mathfrak{q}_{\mathfrak{t}} &= \sigma((1-\mathfrak{O}_{\mathfrak{t}})\ \mathfrak{X} + \mathfrak{O}_{\mathfrak{t}}\ \sigma(\mathfrak{W}_1\ \mathfrak{X} + \mathfrak{W}_2\ \mathfrak{P}_{\mathfrak{t}-1} + \mathfrak{b}_{\mathfrak{q}})) \\
\mathfrak{r}_{\mathfrak{t}} &= \sigma((1-\mathfrak{N}_{\mathfrak{t}})\ \mathfrak{X} + \mathfrak{N}_{\mathfrak{t}}\ \sigma(\mathfrak{W}_3\ \mathfrak{q}_{\mathfrak{t}} + \mathfrak{b}_{\mathfrak{r}})) \\
\mathfrak{z}_{\mathfrak{t}} &= \sigma((1-\mathfrak{M}_{\mathfrak{t}})\ \sigma(\mathfrak{W}_3\ \mathfrak{q}_{\mathfrak{t}} + \mathfrak{b}_{\mathfrak{r}}) + \mathfrak{M}_{\mathfrak{t}}\ \sigma(\mathfrak{W}_4\ \mathfrak{r}_{\mathfrak{t}} + \mathfrak{b}_{\mathfrak{z}})) \\
\mathfrak{P}_{\mathfrak{t}} &= \sigma(\mathfrak{W}_5\ \mathfrak{z}_{\mathfrak{t}} + \mathfrak{b}_{\mathfrak{P}}) \\
\mathfrak{X}_{\mathfrak{t}} &= \sigma(\mathfrak{W}_6\ \mathfrak{P}_{\mathfrak{t}} + \mathfrak{b}_{\mathfrak{y}})
\end{aligned} \tag{5.28}$$

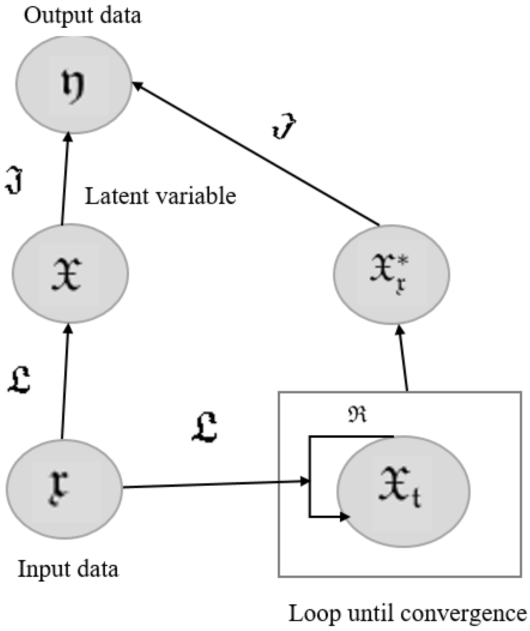

Figure 5.2: Feedforward networks act by computing $\mathfrak{J} \circ \mathfrak{L}$. Implicit networks add a fixed point condition using $\mathfrak{R}$. When $\mathfrak{R}$ is Garcia-Falset repeatedly applying $\mathfrak{R}$ to update a latent variable $\mathfrak{x}^{\mathfrak{k}}$ converges to a fixed point $\mathfrak{x}^* = \mathfrak{R}(\mathfrak{x}^*; \mathfrak{L} \circ (\mathfrak{x}))$.

Proof. Under the given conditions, equation (5.28) models a well-posed system as the existence of unique fixed point for the contraction mapping σ is guaranteed [17]. The robustness of system can be verified by Theorem 4.2.1 where the iterative scheme is proved to be $\mathcal{F}$-stable which shows that the smaller perturbation to the system does not effect the output.

□

Example 5.3.1. Training implicit neural network
Suppose we want to build a single input neural network that can predict the exam score based on the number of hours studied by assuming that whoever studies for $\mathfrak{x} \in [0, 3]$ hours tends to achieve an exam score of $\mathfrak{y} \in [0, 10]$ which can be define by function $\mathfrak{f}(\mathfrak{x}) = \frac{10}{3}\mathfrak{x}$. For this purpose, we train a network that takes an input of 3 (hours) and gives an output of 10 (score). Given:

- Maximum input (max) = 3
- Minimum input (min) = 0
- Input value = 3

Normalizing the input and output:

$$\text{normalized\_value} = \frac{\text{value} - \min}{\max - \min}$$

we get:

$$\text{normalized\_value} = \frac{3-0}{3-0} = \frac{3}{3} = 1$$

By normalization, the input 3 becomes 1 and similarly, the output 10 also becomes 1. We start Example by taking :

- Input $\mathfrak{X} = 1$
- Weights: $\mathfrak{W}_1 = 0.1,\ \mathfrak{W}_2 = 0.2,\ \mathfrak{W}_3 = 0.3,\ \mathfrak{W}_4 = 0.4,\ \mathfrak{W}_5 = 0.5,\ \mathfrak{W}_6 = 0.3$
- Bias: $\mathfrak{b}_{\mathfrak{q}} = 0.5, \mathfrak{b}_{\mathfrak{r}} = 0.3, \mathfrak{b}_{\mathfrak{z}} = 0.1, \mathfrak{b}_{\mathfrak{P}} = 0.3, \mathfrak{b}_{\mathfrak{y}} = 0.4$
- $\mathfrak{O} = 0.3, \quad \mathfrak{N} = 0.4, \quad \mathfrak{M} = 0.5$
- Learning rate $\alpha = 0.02$

New Proposed Method
We will compute $\mathfrak{y}_{\mathfrak{t}}$ using (5.28). For this, let's start by taking activation function as $\sigma(\mathfrak{x}) = \frac{\mathfrak{e}^{\mathfrak{x}} - \mathfrak{e}^{-\mathfrak{x}}}{\mathfrak{e}^{\mathfrak{x}} + \mathfrak{e}^{-\mathfrak{x}}}$ which is a contraction on $(-1, 1)$.

At Time Step $\mathfrak{t} = 1$:
Now applying our proposed iterative method, We get :

$$\mathfrak{q}_1 = 0.69$$
$$\mathfrak{r}_1 = 0.65$$
$$\mathfrak{z}_1 = 0.34$$
$$\mathfrak{P}_1 = 0.44$$
$$\mathfrak{X}_1 = 0.48$$

This will be output $\mathfrak{y}_{\mathfrak{t}}$at time step $\mathfrak{t} = 1$
Compute Loss
The mean square error cost function is defined as follows:
$\mathfrak{C}(\mathfrak{W}, \mathfrak{b}) = \sum_{\mathfrak{x}} ||\frac{1}{2}(\mathfrak{y}(\mathfrak{x}) - \mathfrak{y}_{\mathfrak{t}})^2||$
where,

- $\mathfrak{x}$: input
- $\mathfrak{y}_{\mathfrak{t}}$: output at time step $\mathfrak{t}$
- $\mathfrak{W}$: weights collected in the network
- $\mathfrak{b}$: biases
- $\|\mathfrak{v}\|$: usual length of vector $\mathfrak{v}$

Let's use Mean Squared Error (MSE) loss. The true output is $\mathfrak{y} = 1.00$

$$Loss \;=\; \frac{1}{2}(\mathfrak{y} - \mathfrak{y}_{1})^2$$

$$= \frac{1}{2}(1.00 - 0.48)^2 \approx 0.52$$

Backpropagation
Now, by using backpropagation, we will update weights and biases.

$$\frac{\partial loss}{\partial \mathfrak{y}_{1}} = -(1.00 - 0.48) = -0.52$$

Gradient Learning Algorithm
We will update weights and biases by gradient learning algorithm. This technique can be written as:

$$\mathfrak{P}_{\mathfrak{n}+1} = \mathfrak{P}_{\mathfrak{n}} - \alpha \nabla \mathfrak{f}(\mathfrak{P}_{\mathfrak{n}})$$

Updating $\mathfrak{W}_6$:

$$\frac{\partial loss}{\partial \mathfrak{W}_6} = \frac{\partial loss}{\partial \mathfrak{y}_{1}} \frac{\partial \mathfrak{y}_{1}}{\partial \sigma} \frac{\partial \sigma}{\partial \mathfrak{W}_6} = -0.52 \times (1 - tan^2(0.48)) \times \mathfrak{P}_{1} = -0.52 \times 0.76 \times 0.44 = -0.18$$

$$Updated\ \mathfrak{W}_6 = \mathfrak{W}_6(previous) - 0.2 \times \frac{\partial loss}{\partial \mathfrak{W}_6} = 0.33$$

Updating $\mathfrak{b}_{\mathfrak{y}}$:

$$\frac{\partial loss}{\partial \mathfrak{b}_{\mathfrak{y}}} = \frac{\partial loss}{\partial \mathfrak{y}_{1}} \frac{\partial \mathfrak{y}_{1}}{\partial \sigma} \frac{\partial \sigma}{\partial \mathfrak{b}_{\mathfrak{y}}} = -0.52 \times (1 - tan^2(0.48)) = -0.41$$

$$Updated\ \mathfrak{b}_{\mathfrak{y}} = \mathfrak{b}_{\mathfrak{y}}(previous) - 0.2 \times \frac{\partial loss}{\partial \mathfrak{b}_{\mathfrak{y}}} = 0.48$$

In similar way, we will calculate other weights and biases as well.
Updated Weights and Biases

Weights	Biases
$\mathfrak{W}_1 = 0.10$	$\mathfrak{b}_{\mathfrak{q}} = 0.50$
$\mathfrak{W}_2 = 0.20$	$\mathfrak{b}_{\mathfrak{r}} = 0.30$
$\mathfrak{W}_3 = 0.30$	$\mathfrak{b}_{\mathfrak{z}} = 0.10$
$\mathfrak{W}_4 = 0.40$	$\mathfrak{b}_{\mathfrak{P}} = 0.31$
$\mathfrak{W}_5 = 0.50$	$\mathfrak{b}_{\mathfrak{y}} = 0.48$
$\mathfrak{W}_6 = 0.33$	

At Time Step $\mathfrak{t} = 2$ **:**
We will calculate output $\mathfrak{y}_2$ by using updated weights and biases.
By doing same procedure we calculated

$$\mathfrak{y}_2 = 0.55$$

Compute Loss
Let's use Mean Squared Error (MSE) loss.

$$Loss\ \approx 0.45.$$

By continuing the same procedure :
At Time Step $\mathfrak{t} = 6$ **:**
After 6 iterations, our model is trained and we get loss:

$$Loss\ \approx 0.27.$$

Now as our model is trained, we will check at $\mathfrak{x} = 0.5$ and see what will be the output for this. Output $\mathfrak{y}$ at normalized input is :

$$\mathfrak{y} = 0.50$$

We get the estimated output $\mathfrak{y} = 5.0$ (score) for input $\mathfrak{x} = (1.5)$ hours .

It is proved in Subsection 4 that the rate of convergence of the iterative scheme (4.1) is higher than several others, therefore the convergence rate of the trained network (5.28) is higher than the other traditional networks like FNN, RNN, etc.

Since the proposed iterative method is faster in its convergence rate. Therefore, the implicit neural networks operate on the basis of Banach's theorem has also an improved convergence rate.

Chapter 6

SUMMARY

We have developed a novel iterative contraction mapping method that outperforms existing approaches by Asghar Rahimi [25], Thakur et al. [23], and Picard [17]. Our research compares various approaches to our new strategy through graphical comparisons, showing better performance. We use the Garcia Falset operator to study the convergence of our iteration to fixed points in uniformly convex Banach spaces, with a focus on approximation stability. As an application, we worked to find fixed point of heat equation and we exhibit the applicability of our four-step iteration process in delay differential equations. We designed implicit neural network models using our suggested iteration, verifying convergence and stability to fixed points. This work provides opportunities for future fixed-point estimate research in addition to validating our iteration's convergence and stability. According to our research, our iteration converges more quickly than current schemes, improving the computational efficiency of iterative procedures.

Chapter 7

CONCLUSIONS AND FUTURE RECOMMENDATION

In this thesis, we present a new iteration that converge faster than the iterative scheme of Picard [17], Thakur et al. [23], Asghar Rahimi [25], Sintunavarat iteration [37], Jubair et al. [24], Okeke and Abbas [38], Agarwal et al. [21], Akanimo [27]. The comparison of iterative scheme (4.1) with other iterative schemes through an Example 4.0.1 is also presented. Finally, an applications that raised from the various fields of sciences is given to show the applicability of our scheme. In future work, a single input neural network (5.28) can be extended into a multi-input network and apply on data set. Implicit framework creates an enormous number of new opportunities for innovative architectures, algorithms, robustness analysis, and design.

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