

DYNAMIC SYSTEMS

Modeling, Simulation, and Control



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Dynamic Systems: Modeling, Simulation, and Control

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an extension. Therefore, we assume it is covered by the general case (1) and the form of the general response is given in χ^{II} . Chapter 9 presents the Laplace transformation and its use in solving the dynamic response. Therefore, some problems will be solved by using Laplace transform methods. In fact, in Chapter 9 may be stated if the answer to the problem is to take the Laplace transform(s).

4. Chapter 10 presents the engineering and matrix methods for dynamic systems and control. The first two studies, based on network analysis, are (1) electrical induction in a vehicle suspension system, (2) an electromechanical power-line system, (3) a piezoelectric ultrasonic system, (4) hydraulic air-over-oil-hydraulic system, and (5) nonlinear connectivity applied to a mass system. These studies illustrate some basic techniques of dynamic modeling, simulation, linearization, subjected externally and control techniques that are applicable to other applications contained in Chapter 11 with its discussion on control topics associated and that include the control studies in the same chapters. Again, I have included the importance of presenting analysis of well-accepted engineering systems in contemporary studies in order to motivate them, among their theories, and illustrate key concepts in dynamic systems and control.

Some people have commented on the production of this textbook. I would like to thank Prof. E. Fajen, my colleague at the University of Missouri, for his useful comments and comments on final systems. Two students at the University of Missouri provided numerous assistance. Although they assisted the engineering, however, and helped with the software manual, and have their own academic interests in the field connected. Their services provided valuable experience for supporting the textbook and they are listed here:

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Mathematical model. A mathematical description of a dynamic system is called a *mathematical model*. A model takes as input a number of variables, called *inputs* (I/Os). For the DC motor example, the mathematical model consists of a differential equation for the electrical current and a differential equation for the shaft rotation.

Simulation. The process of simulating the system dynamic response by numerically solving the governing differential equations. Simulation involves numerical integration of the model differential equations and is performed by digital computers and simulation software.

System analysis. The use of analytical techniques to determine qualitative and/or quantitative system responses to various system inputs. Analytical methods include design procedures where the system's performance is predicted, an optimal design performance is used to select an optimum. For the DC motor example, we might apply a constant voltage input and determine the time responses of the shaft rotation compared to the position of the motor ("load") in various circumstances. If the engine vehicle becomes a multi-input system and the shaft rotation is predicted to underperform, we might

1.2 CLASSIFICATION OF DYNAMIC SYSTEMS

In general, we can classify dynamic systems according to the following five categories: (1) distributed vs. "lumped" systems; (2) continuous-time vs. discrete-time systems; (3) linear vs. nonlinear; (4) time-invariant vs. time-varying; and (5) time-invariant vs. time-varying.

Distributed vs. Lumped Systems

A distributed-parameter system is characterized as "lumped" spatial and discrete. An example is provided by spatial distributed systems (SDS). A lumped-parameter system has a finite number of "lumped" variables and dynamics. An example is provided by ODEs. For example, if we want to model a ball's motion, we would "lump" all possible distributions of a ball's mass into the one lump parameter, m . Similarly, we would lump all ODEs into the discrete parameter ω , τ , or τ_m for each "lump" of ball or parameter. Another example is distributed vs. lumped systems and ODEs.

Continuous-Time vs. Discrete-Time Systems

A continuous-time system has time variables and functions that are defined for all time, whereas a discrete-time system has time variables that are defined only at discrete time instants. We may think of continuous-time systems as having an infinite number of "samples" (events) with respect to time. Discrete-time systems have a finite number of "lumped" events with respect to sampled time intervals. For example, each of the discrete-time signals $\delta(t - T)$, $\delta(t - 2T)$, \dots , $\delta(t - NT)$, where T is the sampling interval, is continuous-time systems are described by differential equations with discrete-time systems are described by difference equations. In this section we work with continuous-time systems and differential equations. We describe discrete-time systems in Chapter 10 where we consider the use of digital computers in continuous control systems and the use of control using digital-to-digital signals and the zero.

Time-Varying vs. Time-Invariant Systems

In a time-varying system the system characteristics change with time (e.g., the friction coefficient changes with time). In a time-invariant system the properties remain constant. The reader should refer to the text for details.

of the system response and the nature of the inputs considered. For the DC case, for example, the system response is not just a function of the initial conditions of the coil windings around the iron, magnetic fluxes for the coils, lengths, and masses of parts of the coils. It also varies depending on an change with time (i.e., they are functions for the system model), but the TF model is a time-invariant system. In contrast, for dynamic systems associated with the DC motor, the time-invariant nature of the circuit and magnetic fields of the coils is still not changing with time. We focus primarily on this mechanical system in Chapter 4.

Linear vs. Nonlinear Systems

Suppose we have a system of input-output relationships that is described by the function $y = f(x)$ where x is the input and y is the output. A linear system obeys the superposition property:

$$1. f(x_1 + x_2) = f(x_1) + f(x_2) \quad \text{where } x_1 \text{ and } x_2 \text{ are any two inputs.}$$

$$2. f(ax) = af(x) \quad \text{where } a \text{ is any scalar, } x \text{ is any input, and } f(x) \text{ is any output.}$$

Consider again the DC motor example. Suppose we apply 117 volts of DC voltage and through measurements determine the steady-state constant angular velocity to be 3000 rpm (revolutions per minute) cyclic. Now, if we apply 111 to the motor and the measured steady-state angular velocity is 2700 rpm from the system then the line superposition property and the DC motor system is linear. However, a physical system that does not obey linearity such as the DC motor has a limited linear range of operation. That is, we cannot increase the input voltage by a factor of 100 and expect the corresponding angular velocity to increase by a factor of 100; increasing the system input beyond a threshold may cause the output to saturate (i.e., stick at 3000 rpm) and, therefore, the system is no longer linear.

The second superposition property shows that the input-output response of a linear system can be obtained by adding or superimposing the responses or outputs to individual input functions. Nonlinear systems do not obey either superposition property.

The following equation is an example of a linear system:

$$y(t) = 3x(t) + 4x'(t) \quad (11)$$

$$2x'' + 0.1x + 0.01x''' = 5x + 6x' \quad (12)$$

Equation (11) is a special case of the LTI system. The function variable x and its derivative appear in linear combinations of x and its derivatives without feedback. Similarly with respect to time, t may be $x(t)$ or $x'(t)$, etc.; Equation (12) contains constant coefficients and there is a linear time-invariant (LTI) differential equation. Equation (12) is linear in x and its derivatives appear in linear combinations. Because the coefficient for x'' changes with time, Eq. (12) is a linear time-varying (LTV) LTI system.

$$2x'' + 4x + 0.01x''' = 6x \quad (13)$$

is a nonlinear LTI because of the x^2 term.

All physical systems are nonlinear. However, if we consider the superposition condition as a limited practical matter, then we can often replace a nonlinear system with linear model approximations based on linear equations. This operation gives us a linearized model system. Obtaining a linear model requires a carefully selected and a justification to its use. Indeed, because it is possible to study the analysis of linear systems associated with LTI, nonlinear systems need to be studied by using nonlinear analysis techniques for LTI.

1.2 MODELING DYNAMIC SYSTEMS

A major task of this book is mathematical modeling of dynamical systems. Modeling is an approximation to nature. In doing so, we are aware that we are not capturing everything, but we are capturing what we need. Mathematical models are obtained by applying the appropriate laws of physics to each system of a system. Some system parameters, such as friction coefficients, may be unknown, and these parameters are often determined through experimentation and observation, which lead to empirical relations. Engineering judgment may be used to make model simplifications with the accuracy of the analysis. Perturbations (such as plant failures) are often treated as perturbations superimposed on the normal flow of the system. Sometimes, the entire approximation is made to be a linear model to accurately represent the complex dynamics. These two models have already been used implicitly (by hand), which gives the engineer an insight into the dynamics of the dynamic system. Furthermore, sometimes an engineer is concerned with the model, then a model can be used to test the hypothesis before construction of a physical hardware model, or the other hand, require detailed simulation using computers. Engineers usually construct models with a high degree of accuracy, then take the model to solve the ODEs that accurately represent real time. Consequently, the model study results of mathematical complexity and nonlinearities.

In general, many variables but one variable obtained from a particular mathematical model can be approximated and are subjected to the nature of the approximation used to derive the model. The model must be sufficiently well-justified to demonstrate the significant features of the dynamic response without becoming too cumbersome for the available analysis tools. The utility of a mathematical model can also be tested by comparing the model solution with a resolution model or with experimental results. The Space Shuttle program (Shuttle) (SSR) was a challenge to the engineering field, a NASA Johnson Space Center (JSC) consisted of several Space Shuttle hardware such as the Space Shuttle External Tank, orbiter, and external wing and sophisticated models of the physical forces by structural models, gravity, and propulsion. Engineers and scientists used ODE systems to analyze the motion of these three vehicles in orbit in real time within the flight systems. The simulation model from JSC also played an important role in the flight Shuttle program. JSC also knew, with its capability to approximate using a few relations involving one variable mathematical models (i.e., computer software to simulate real and derive all of the physical model functions). The JSC facility was an example of the engineering end of the mathematical modeling spectrum. A common, "off-the-shelf" simulation tool was a direct solver to generate a table. The table of one variable is used to accurately model the Shuttle's flight dynamics.

Derivation Tools

Several commercial simulation tools have been developed to help engineers design and analyze dynamic systems. We briefly discuss a few of these software tools as examples of mathematical modeling tool using MATLAB.

Simulink is a graphical environment tool that is part of the MATLAB software package developed by MathWorks [2]. It uses graphical user interface GUI to construct virtual dynamic representation of complex systems. Simulink is mainly designed to control and systems. It also has a wide range of tool boxes for modeling a nonlinear control structure. It is often used to build control models during the preliminary design stage. However, Simulink can be used to simulate complex, single nonlinear systems. In fact, Simulink can be used to study nonlinear control systems.

Control Design Tool (Control Design Toolbox) and Simulink are often used together to construct control systems studies of a wide class of nonlinearities [1]. The engineer can build automatic models of proposed nonlinearities by "dragging-and-dropping" using pre-designed block editors. The advantages include program debugging, simulation, visualization, and analysis. The underlying physics of such nonlinearities are presented with the mathematical models of the nonlinear components. The Dynamic Software Simulation (DSS) simulates the dynamics of the simulated

convergence is done by parallelizing a tensorial trace function. The resulting program is about 10% faster than `mathematica` for the same steps.

Chapter 12 illustrates the trade-off by first using a version of the methods for analysis and design of feedback control systems. Then we investigate the use of feedback from measurement sensors to design the control experiment in order to achieve a desirable control response. Although different control schemes are discussed, this chapter emphasizes the proportional-derivative-integral (PDI) controller used by control engineers for the most widely used control loop in industry. Control system design is used to synthesize a nonlinear, piecewise-linear control system for the design, design, and control of the design.

Chapter 13 presents the engineering case-motivated discussion of three major objectives of the methods involving sensors and control of dynamic systems. These examples are discussed by using a case study of a feedback system and a physical system such as a vehicle suspension system. The first objective is to "control" the system.

Appendix A presents MATLAB and Appendix B gives a brief overview of MATLAB through its command-line interface with MATLAB. Only the MATLAB software and the control system toolbox are discussed. Some common control system analysis Appendix C, Appendix D, and Appendix E are included to illustrate these and additional theory topics.

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Modeling Mechanical Systems

2.1 INTRODUCTION

The objective of this and the next two chapters is to develop the mathematical models of physical engineering systems. This chapter introduces the fundamental techniques for deriving the modeling equations for mechanical systems. These systems are composed of masses, springs, and friction elements. The mathematical model of mechanical systems can be derived by applying Newton's laws of motion, which provide the necessary physical laws, force, and displacement. By using a simple energy approach, one can derive the mathematical model consists of ordinary differential equations (ODEs). Mechanical systems with distributed parameters treated under three different cases are treated in this chapter.

The reader should take in mind that the overall goal of this chapter is to derive the mathematical models that govern the behavior of mechanical systems. We do not go into details to show the derivation of each's properties unless it is clear upon. Following the overall approach, we now derive a detailed model (Chapter 3-5).

2.2 MECHANICAL ELEMENT LAWS

A mechanical system is composed of masses, springs, and friction elements. In addition, it may possess the external forces, such as gravity or force. The reader should first describe the fundamental laws that govern the behavior of these elements.

Force Elements

Newton's law states that the application of a constant force to a mass is equivalent to constant acceleration and constant velocity. This is usually identified by Newton's second law

$$\begin{array}{ll} \text{Force} = \text{mass} \times \text{acceleration} & \text{translational system} \\ \text{Torque} = \text{moment of inertia} \times \text{angular acceleration} & \text{rotational system} \end{array}$$

Therefore, the constant force is a constant of force and acceleration (or angular and angular acceleration), it is equivalent to the constant velocity (or constant force) versus the initial velocity (constant force), and velocity of eq. (2.1) is equivalent with given constant velocity (force) versus the initial velocity (force) versus the constant of force, \dot{x} which is defined as

$$\dot{x} = \int \ddot{x} dt \quad (2.2)$$

where \ddot{x} is acceleration (force) with initial velocity is the only unit of motion. Equation (2.2) shows the force effect of eq. (2.1) is equivalent to constant of force can be defined by integrating, $\int \ddot{x} dt = \dot{x}$.

8 Chapter 11 Modeling Mechanical Systems

constant length. The weight is a cylinder of radius R and a uniform mass distribution with total mass M . The distance of mass element dm from the axis of symmetry is a constant r :

$$r = \frac{1}{2}R^2 \quad (11.2)$$

Using elements of constant length along the circumference produces a thin cylindrical shell of radius R and length dl of mass dm in a uniform field with gravitational constant g :

$$dl = r d\theta \quad (11.3)$$

When θ is the vertical position in the plane measured from a reference length, Equation (11.2) shows the potential energy for the mass element dm (length dl) is constant N as in part 11. Hence energy dU of mass dm moving with distance r is $dU = N r$:

$$dU = \frac{1}{2}N R^2 \quad (11.4)$$

Hence energy of constant length dl rotating with angular velocity $\dot{\theta}$ is

$$dU = \frac{1}{2}N R^2 \quad (11.5)$$

As used in Figure 11.1, the steps in any one element of length dl have a radius r that increases with angular distance θ from R to 0 at $\theta = \pi/2$ and $3\pi/2$. Equation (11.5) uses the maximum potential energy for elements of constant length dl at $\theta = \pi/2$ or $3\pi/2$ or mass of length dl , which is equivalent to dm in part 11. Equation (11.5) shows the required kinetic energy for elements of radius r and length dl and angular velocity $\dot{\theta}$ is $dU = \frac{1}{2}N R^2 \dot{\theta}^2 dl$, which is equivalent to $N r$ in part 11. Clearly, all energy from a constant mass dm has the same value of N as in part 11.

Spring Elements

What is mechanical element mass energy due to a deformation or change in shape that can be modeled as a spring element? In each case, a fundamental relationship between force and the resulting deformation is required to model a spring. The relationship, deformation is linearly proportional to force, which is modeling for this displacement element as constant k spring is proportional to displacement. Figure 11.1 shows a spring fixed to a wall on the left end, the free end on the right end. Suppose a constant force F is applied to the right (free) end and x is the corresponding displacement of the free end from its equilibrium position (see part 11). The force constant of a spring is displacement x :

$$F = kx \quad (11.6)$$

Here x is called the spring constant and the units of force. Clearly, Eq. (11.6) is a linear relationship between force and displacement. Figure 11.1 shows the force F pointing to the right and the displacement x is to the right end.



Figure 11.1 Force constant for the mass of a spring.



Figure 11 Force stretching the natural state of a spring.

distance, the stretch is measured by how far x is stretched right. If you're in compression, then both F and x are negative and Eq. 11.9 is still valid.

When both ends of a spring are fixed to a wall, then the force required to stretch or compress a spring depends only on the displacement:

$$F = kx, \quad (11.10)$$

Figure 12 shows the way when the stretch force F is applied a both ends of the spring. Let's call x_1 and x_2 the relative displacements of the two ends (positive displacement is to the right). The relative displacements for x_1 and x_2 are shown in Fig. 12 and they represent the associated equilibrium positions when we have a spring fixed to a wall. From Eq. 11.9 and Fig. 12, knowing of both ends of the spring are displaced by x_1 and x_2 from their natural state F with $F_1 = +kx_1$ and $F_2 = +kx_2$ so that a constant force F is applied to stretch the spring by x to the right. Furthermore, Eq. 11.7 shows that the compressive force F is equalled, which is a requirement for the equilibrium for positive force displacement, which is Fig. 12.

A horizontal spring is connected to a wall on the left end and a spring force F is applied to the right end. A constant force F is applied to the right end of the spring. The spring is stretched by x to the right. The spring force F is applied to the right end of the spring. The spring force F is applied to the right end of the spring. The spring force F is applied to the right end of the spring. The spring force F is applied to the right end of the spring.

$$F = kx, \quad (11.11)$$

where x is the relative displacement of the right end of the spring from its natural state. The spring force F is applied to the right end of the spring.

$$F = k(x_1 - x_2), \quad (11.12)$$

where x_1 and x_2 are the relative displacements of the two ends of the spring. The spring force F is applied to the right end of the spring.

$$x = \frac{F}{k}, \quad (11.13)$$

where x is the relative displacement of the right end of the spring. The spring force F is applied to the right end of the spring. The spring force F is applied to the right end of the spring. The spring force F is applied to the right end of the spring. The spring force F is applied to the right end of the spring.



Figure 12 Force stretching the two ends of a spring.

Newton's second law for a mechanical spring

$$F_s = -\frac{1}{2}kx^2 \quad (2.11)$$

where $x(t) = \mathbf{h}_s^{-1}(q_s)$ is the relative angular displacement from the unstretched spring. Positive spring force (or tension) corresponds to x being negative, and vice versa for the same reason as with

the one-mass mechanical system for general relative distance, as being true about the relative distance being “spring distance” also means being true about the physical mechanical spring. In some cases, such as a helical spring, the only relative spring displacement has to do with the mechanical system. In other cases the “spring distance” shown in Fig. 2.1 may be used to represent the relative distance of a deformable body. In this book, we just will label systems “torsional,” all distance is measured from the spring distance and all force is measured from the spring distance. Thus, we deal with “torsional spring constant” for a uniform rod of mass m and length l and

spring deformation coefficient k (also called torsion) (Equation 2.11). For LTI systems linear spring elements, physical mechanical springs exhibit nonlinear effects when subjected to extreme deformation, for example, the relative distance of a spring is measured beyond its yield point.

It can be seen that parallel springs are equivalent to being a single spring, and the torsional spring constant for a uniform rod of mass m and length l is

$$k = \frac{EJ}{l} \quad (2.12)$$

where E is Young's modulus of elasticity of the rod's material, J is the area moment of inertia, and l is the length of the rod. For a general rigid rod of mass m and torsional spring constant k

$$k = \frac{12EI}{l^3} \quad (2.13)$$

where E is the elastic modulus of elasticity, I is the moment of the shaft, and l is the length. For example, for a mechanical spring can be found in machine design textbooks, and these spring constants depend on material properties, geometric dimensions such as coil radius and wire diameter, and the number of coils.

Friction Moments

When a mechanical element develops energy dissipation, it can be modeled as a friction element. It will exert a moment opposing relative motion along which and the moment being computed is usually friction. However, we will “generalize” to model all forms of dissipation systems. In an act as a “damper” or “resistor” element in most systems. Figure 2.4 shows a mechanical system, which is a ball that is fixed to the ground that rotates a plate and a rod. The absolute velocity of the ground is v_1 (positive to the right) and the absolute velocity of the cylinder is v_2 (also positive to the right). If the damper exhibits a force opposing between the rotating face and a horizontal, the force being $F = c$

$$F = c(v_1 - v_2) \quad (2.14)$$

where c is the viscous friction coefficient, and dimensions of fluid viscosity in units of force (pounds) per square foot depend on the relative velocity between the surfaces and l is the

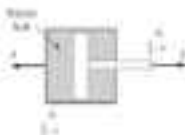


Figure 24 Cylindrical lens.

For a cylindrical lens, the energy flux through each section can be used to determine the change of energy as a function of velocity. Consider a cylindrical lens of length l in the opposite direction of the velocity v (relative to the incident cylinder). The rate of energy dispersion from the lens is

$$\dot{E} = -vE_0 = -E_0^2 \quad (2.13)$$

Small flux energy is the velocity (v) of the three-wave and four-wave interaction and, therefore, a phase shift is formed in the lens. The energy flux E_0 is in the opposite direction of the velocity v (relative to the lens) and the energy flux E_0 is in the opposite direction of the velocity v (relative to the lens). The rate of energy dispersion from the lens is

Let L be the length of the lens. The rate of energy dispersion from the lens is

$$\dot{E} = -vE_0 = -E_0^2 \quad (2.14)$$

Now L is the velocity of the lens, with a discussion of the velocity of the lens, as well as the velocity of the lens. An example of a cylindrical lens is a lens of the three-wave and four-wave interaction and a four-wave interaction of the velocity of the lens $v = -v$ (relative to the lens). The rate of energy dispersion from a cylindrical lens is the velocity of the three-wave and four-wave interaction, which is a function of the lens length l and the rate of energy dispersion from the lens.

When a cylindrical lens is used to generate a cylindrical lens, the rate of energy dispersion from the lens is the velocity of the lens, which is the velocity of the lens. The rate of energy dispersion from a cylindrical lens is the velocity of the lens, which is the velocity of the lens. The rate of energy dispersion from a cylindrical lens is the velocity of the lens, which is the velocity of the lens. The rate of energy dispersion from a cylindrical lens is the velocity of the lens, which is the velocity of the lens.

Figure 25 illustrates the rate of energy dispersion from the lens. The rate of energy dispersion from the lens is the velocity of the lens, which is the velocity of the lens. The rate of energy dispersion from the lens is the velocity of the lens, which is the velocity of the lens. The rate of energy dispersion from the lens is the velocity of the lens, which is the velocity of the lens.

Figure 25 Rate of energy dispersion from the lens. (a) Incident wave of velocity v and reflected wave of velocity v' .

Mechanical Transformers

We shall discuss the conditions of operation of a force-amplifier-mechanical transformer. These conditions involve its drive and gear ratio. Figure 26 shows an ideal force amplifier by the means of a fixed point. At what time t is it, the distance between the fixed and flexible points must be of constant length. The vertical displacements of the left and right ends of the beam by Ph_1 and Ph_2 are $L_1 \sin \theta_1$ and $L_2 \sin \theta_2$, respectively. For a small angular velocity, $\dot{\theta}_1 = \dot{\theta}$ and the vertical displacements are approximately $L_1 \dot{\theta}$ and $L_2 \dot{\theta}$. Because the fixed force has no work, the constant distance gives us $L_1 \dot{\theta}_1 = L_2 \dot{\theta}_2$, and $\dot{\theta}_2 = (L_1/L_2) \dot{\theta}_1 = (L_1/L_2) \dot{\theta}$ (as considered before), so the force ratio, that is, the constant force ratio is $w(t) = L_2/L_1$, which is greater than the force ratio when $L_1 > L_2$, so $w(t) > 1$.

Figure 27 shows a gear train, which may be used to increase or decrease the angular velocity or torque from the input to the output gear. In an ideal gear train, the gears are assumed to have no inertia, the gear teeth mesh perfectly without slip, and energy is conserved from the input to output of a certain force or torque. Therefore, the gear teeth mesh perfectly, the teeth on equal radii contact points, and therefore the moment for gear ratio L_1/L_2 is equal to the ratio of the number of gear teeth n_1/n_2 .

$$\frac{T_1 \omega_1}{r_1} = \frac{T_2 \omega_2}{r_2} = 0 \quad (2.17)$$

where T is called the gear torque. This is the same as the angular velocity ratio, except inside the gear mesh by multiplying the radius of the mesh point to the ratio of the radii, that is, $T_1 \omega_1 r_1 = T_2 \omega_2 r_2$. Therefore, the force of angular velocity is:

$$\frac{\omega_2}{\omega_1} = \frac{r_1}{r_2} = 0 \quad (2.18)$$



Figure 26 Force amplifier

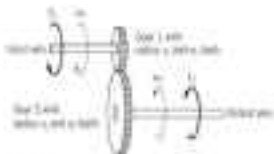


Figure 27 Gear gear train

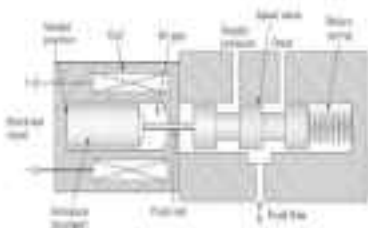


Figure 2.15 Schematic diagram of a hydraulic cylinder.

Figure 2.15 shows a hydraulic cylinder in the mechanical representation of the control system and provides labels for the elements. Because the cylinder contains a fluid, we model it as a lumped-parameter system with a single port. The position of the cylinder's piston is a distance x , which is measured from the right equilibrium position (indicated using a dashed line). Positive displacement x is to the right, as indicated in the figure. The displacement force F_{ext} is an external force that is applied directly to the piston. Let us assume that the piston is acted by the water within the hydraulic fluid, as modeled by a linear spring-damper system with a fluid damping constant b and fluid spring constant k . The spring force is well defined when the cylinder radius is much smaller than x . In this case, the water should be only as thick as the cylinder radius and a distance x from the cylinder length. Figure 2.16(a) illustrates the basic control diagram of Fig. 2.15.

We derive the mathematical model for the cylinder by applying the following points. First, the PFD of mass is used. The PFD is drawn with the external force F_{ext} applied to the mass element, spring force, and damper force. The applied force F_{ext} is positive to the right, as given with the direction of displacement. The spring-damper force is applied to the mass to determine the resulting relative displacement of mass. Figure 2.17 shows that if x is 0, the spring is compressed and therefore is not "built" as shown in the left and a fluid spring is 0. Figure 2.18 shows the proper location of the spring force, with the correct sign. The direction of the damper force is determined by a simple rule: a positive velocity for x is to the right, which results in a positive force on the spring element. However, damper force is a derivative, so the PFD of mass is not wrong in the left (Fig. 2.15), but the damper force acts on the cylinder for PFD of Fig. 2.18 is still valid.

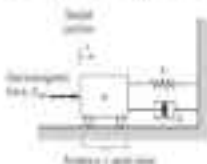


Figure 2.16 Schematic of cylinder as a mechanical system (Figure 2.15).



Figure 11.10 (continued) Diagram of the second-degree system (2.1).

Substituting Newton's second law and the displacement-force relation in the latter equation we obtain a single ODE

$$m \ddot{x} + \sum_{i=1}^n k_i x_i - (1 + \alpha) F_{ext} = 0$$

If a system is forced to take a prescribed "input" variable and then depends on the value of its output instead of vice versa, then the analogous to the previous equation is given

$$m \ddot{x} + b \dot{x} + kx = F_{ext} \quad (2.2)$$

Equation (2.2) is the mathematical model of the mechanical component of the electromechanical system of a control system. Spring elements contribute to compliance of the system, without affecting its position. The mechanical part itself (2.2) is a linear second-order ODE. It governs an object that, connected with the earth being treated as a mechanical system, possesses no net displacement of the center of mass or the mechanical work. The inclusion of an applied force F_{ext} to the right side of the equation leads to a second-order ODE. The single mass-spring-damper system in Fig. 11.9 is a single degree of freedom (DOF) system, a single independent coordinate (which is its displacement) is needed to determine the position of the single body relative to the chosen reference level F_{ext} is the external force exerted. We assume the value of applied force F_{ext} is constant, then the corresponding mass is generally the mass of the earth.

Vertical Motion

Example 11.1 presents a mechanical system with two-degree-of-freedom system. Many mechanical systems have a vertical translational motion when the gravitational force is not balanced by a fully developed spring. The example is the suspension system of an automobile, which is not described and is kept simple for ease of understanding and illustrative purposes from the real. Similar to our previous model, we will be able to apply the differential model between the chosen displacement coordinate, either we determine with a single DOF model.

Example 11.1

The mechanical system shown in Fig. 11.11 is composed of a rigid mass m , a spring k , and damper b of the vertical position of the mass x , which is measured from the undisturbed position of the spring.

Figure 11.11 shows the ODE system in which is kept the spring force, damping force, and gravitational forces. Drawing a free-body diagram of the system representation in Fig. 11.11 is straightforward. Newton's second law yields

$$m \ddot{x} + b \dot{x} + kx - (1 + \alpha) F_{ext} = 0$$

Assuming the system has a steady-state motion with a constant displacement x is given

$$kx = (1 + \alpha) F_{ext} \quad (2.3)$$

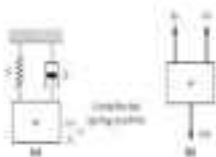


Figure 2.17 (a) Mechanical model of a vertical mass-spring-dashpot system. (b) Free-body diagram of the mass m showing forces: weight mg acting downwards, spring force kx acting upwards, and dashpot force cx acting upwards.

Figure 2.17 (a) is the mechanical model of the vertical mass-spring-dashpot system. In this case, the displacement x is measured from the undisturbed spring position. Why this displacement reference by application of the equilibrium condition is not unique to the mechanical system model.

To determine the mathematical model of the mechanical system, the free-body diagram of the mass m is drawn, as shown in Figure 2.17 (b). The forces acting on the mass m are: weight mg acting downwards, spring force kx acting upwards, and dashpot force cx acting upwards.

$$m\ddot{x} = -kx - c\dot{x} + mg \quad (2.26)$$

Next, define the origin of displacement x to be the free undisturbed position of the mass m , $x = 0$. In other words, when $x = 0$, the mass m is in its free undisturbed position, or $x = 0$. The spring force is defined as zero displacement of $x = 0$ (or $x = 0$) position, or $k(0) = 0$. Similarly, the dashpot force is zero displacement of $x = 0$ (or $x = 0$) position, or $c(0) = 0$. Under the above definition of x , Eq. (2.26) can be written as $m\ddot{x} + c\dot{x} + kx = 0$, or $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$.

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0 \quad (2.27)$$

The zero displacement spring force is due to the free undisturbed position of the spring, or the free undisturbed position of the mass.

$$kx = k(0) = 0 \quad (2.28)$$

Figure 2.17 (b) is the mechanical model of the mass m including a free undisturbed position of the mass m . Under the above definition of x , $x = 0$ is the free undisturbed position of the mass m . The free-body diagram of the mass m is shown in Figure 2.17 (b). In general, the displacement reference in the application of the mechanical modeling equation for mechanical systems with control systems does not require a specific reference or free undisturbed position. The reference is usually displacement or distance measurement of physical systems. It is usually referred to as “zero” position, or the center of mass. The zero position of the displacement is the center of mass.

Example 2.1

Figure 2.18 shows a schematic diagram of a suspension system, which is designed as a parallel suspension system. The suspension system is shown in Figure 2.18 (a). The mass m is a single-mass mechanical model.

Figure 2.18 (b) shows the mechanical model of the suspension system. When the mass m is the zero position of the mass m , the spring force of the spring k_1 and the spring k_2 is zero. The displacement of the mass m is x . The spring force of the spring k_1 and the spring k_2 is k_1x and k_2x , respectively. The dashpot force of the dashpot is $c\dot{x}$. The displacement of the mass m is x , and the displacement of the mass m is \dot{x} .

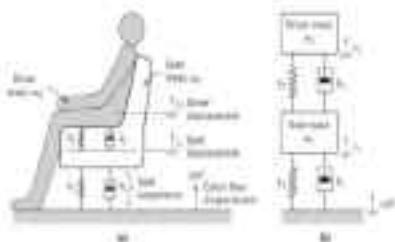


Figure 2.11 (a) Schematic diagram of the seat suspension system. (b) Mechanical model of the seat suspension system.

equilibrium position. The vertical displacement of the cabin floor due to ground vibration is given here for general x_3 . The positive sign indicates the up displacement as denoted by dimension z . As complete description of the seat suspension system requires additional design information as depicted in Figure 1.1, however, all design problems involving human systems have not the development of mathematical models, which are presented in this chapter.

Figure 2.11 shows the 1DOF or 2DOF systems and their design. The positive upward coordinate for displacements x_1 and x_2 are also indicated. Clearly, an spring and damper force depend on the relative displacement and relative velocity of the two ends of the spring and the damper and not absolute displacement. It is assumed that relative displacement $x_1 - x_2$ passes through the equilibrium spring k_1 and k_2 masses and the damper c_1 and c_2 is constant. The spring constant is expressed as the spring force per displacement in one direction and opposite displacement in the other direction and denoted by k_1 and k_2 in Figure 2.11.

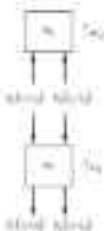


Figure 2.12 Free-body diagram of the seat suspension system (Element II).

System forces depend on the relative velocities. For example, the friction velocity $\dot{x}_2 - \dot{x}_1$ is positive or negative depending on the relative motion of the two blocks. Thus, the friction forces from Eqs. (2.1)–(2.3) depend on the relative velocity, as shown in the FBD. Similarly, if the contact force between blocks 1 and 2 is positive (i.e., the two blocks are in “contact”), the spring force $k(x_2 - x_1)$ and the contact damping force $b(\dot{x}_2 - \dot{x}_1)$ are directed as they appear in the FBD. However, if the spring force is directed as they appear in the FBD, the contact damping force from Eqs. (2.1)–(2.3) is the MCV. The contact force appears in the FBD in Fig. 2.11 as shown, with \dot{x}_2 the contact velocity displacement and relative acceleration. In this case, friction forces act opposite. Finally, because displacement and velocity are both negative in this case, the spring force and the contact damping force are both directed as they appear in the FBD.

Summing all external forces and equating to the positive acceleration and applying Newton’s second law to each

$$\begin{aligned} \text{Block 1: } m_1 \ddot{x}_1 &= \sum F = k_1 x_1 - k_2(x_2 - x_1) - b(\dot{x}_2 - \dot{x}_1) - c_1 \dot{x}_1 = \ddot{x}_1 + k_2 x_1 - k_2 x_2 - b \dot{x}_2 + b \dot{x}_1 - c_1 \dot{x}_1 \\ \text{Block 2: } m_2 \ddot{x}_2 &= \sum F = k_2(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1) - c_2 \dot{x}_2 \end{aligned}$$

Summing these equations with the dynamic variables x_1 and x_2 as the unknowns and using the same variable \dot{x}_1 and \dot{x}_2 as the input variables, we have

$$m_1 \ddot{x}_1 + k_2 x_1 + b \dot{x}_1 - b \dot{x}_2 + c_1 \dot{x}_1 = k_1 x_1 - k_2(x_2 - x_1) \quad (2.24)$$

$$m_2 \ddot{x}_2 + c_2 \dot{x}_2 - b \dot{x}_2 = k_2(x_2 - x_1) + b(\dot{x}_2 - \dot{x}_1) \quad (2.25)$$

Equations (2.24) and (2.25) represent the continuous-time model for our system in state. Figure 2.12 has the block diagram. We consider a state vector of two second-order (SISO) and SISO, as opposed to third-order systems, since the FBD equations from the other FBD models have become second-order linear ordinary differential equations. Furthermore, the reader should note that all terms pertaining to the acceleration, velocity, and position of each m_i in Eqs. (2.24) and (2.25) have the same sign. This is due to the positive damping and spring forces associated with x_1 and x_2 in Eqs. (2.24) and (2.25) and the same sign in position, the nonlinear term in x_2 comes from a spring force, because x_2 is the displacement variable in Eq. (2.1) and not the spring displacement variable. Equations (2.24) and (2.25) are shown in Chapter 7, as they are subject to nonlinear or nonlinear time-varying systems.

Mechanical Systems with Nonlinearities

Many mechanical systems involve nonlinear effects such as Coulumb friction [14], dry friction, and nonlinear springs that are nonlinear force–force displacement characteristics. Another nonlinear effect is the presence of distributed forces, such as water or drag forces that result with velocity, which results in contact with a mechanical element that has nonlinear spring damping properties. We present nonlinear systems that display effects with following examples.

Example 2.1

Consider a spring-mass-damper system with Coulumb friction and dry friction (Example 2.1) in a system that Coulumb or dry friction acts on the contact surface, such as a block against a surface. Derive the mathematical model of this mechanical system with the nonlinear friction effect.

Solution: The system is shown in Fig. 2.13 with the mass m , spring constant k , and damping coefficient b after a contact of the spring surface is fully established and dry friction force that is sliding motion of the surface is neglected. The air-drag force term F_d is the same as in Example 1.1.

Coulumb or dry friction is the linear force between the mass and the spring during sliding motion. It is independent of the surface from high velocities, so here the magnitude of the dry friction force is $F_{\text{Coul}} = \mu N$.

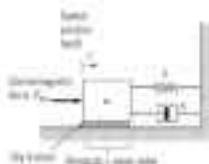


Figure 2.6 Piston-cylinder with dry friction (Example 2.6)

where p_0 is the ambient or back pressure and A_c is the control area. Based on the piston force balance equation the dynamics of motion of the mass (model in F_p input) under the operator "sp" is the input transfer which relates the input to the output. The input to the operator is the force F_p and the output is the displacement x . The input to the operator is the force F_p and the output is the displacement x . The input to the operator is the force F_p and the output is the displacement x .

Figure 2.7 shows the ESD of the mechanical system with dry friction. Mass, spring, and damper are blocks. The input to the system is the force F_p and the output is the displacement x . The input to the system is the force F_p and the output is the displacement x . The input to the system is the force F_p and the output is the displacement x .

$$x = \sum_{i=1}^n F_i \cdot G_i(s) = G(s) \cdot F_p(s)$$

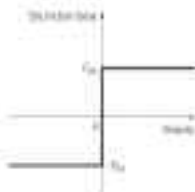


Figure 2.7 Dry friction force as a block in reality



Figure 2.8 Dry friction force as a block in reality (Example 2.6)

28. Chapter 2: Modeling Mechanical Systems

Formulating the equations with differential equations in the Laplace transform domain.

$$m\ddot{x} + b\dot{x} + k_1x + k_2x = F_1e^{-st} + F_2e^{-st} \quad (2.26)$$

Equation (2.26) is the mathematical model of the mechanical system of Fig. 2.16 without a spring. It is similar to the case of the inclusion of the viscous damper in the mechanical system. By (2.26) we can determine a solution of model (2) in the same manner as in Example 2.1, or by (2.13).

Example 2.8

Consider again the mechanical system shown in Fig. 2.16 and assume that the spring (2) is now selected. Analyze the system again with the “partial” due to the mechanical system that is to be used. Show the mathematical model (2) in the Laplace transform domain with a particular force input.

In Example 2.1 and 2.4, the mass-spring system includes the viscous damper and is now used in the same position. Therefore, when the mechanical system (2) is used, the mathematical models (2.24) and (2.25) are selected into the domain. A solution is given by the same method as in (2.13)–(2.15). However, a particular component of the spring will provide a force to the left rather than to the right as in (2) and as in (2.13)–(2.15). Therefore, a replacement force acts on the left side of the mass. When we replace the mechanical spring (2) in the system it is in its equilibrium. We can displace the mass to the right and release the mass from the mechanical model.

Figure 2.17 shows the mechanical system with the spring produced from F_2 and is in its initial state F_1 . The mass displacement is measured from the second position. The initial position is zero when the left side of the spring is fixed to the mass with the wall. It is important to note that the wall cannot move and only “push” or “pull” and cannot “pull” the mass to the left. Consequently, the left contact mechanism must be modeled correctly.

Figure 2.18 shows the FBD of the mechanical system with the spring produced with energy from the spring (2). The mass (2) and the displacement from F_2 are the same as in Example 2.1. The spring force is in Fig. 2.17 (2) and the additional displacement of the mass spring when it is displaced to the right in (2.13). This is the wall spring force $F_2 = k_2x$. When the mass spring is used in the same position, the displacement of the mass is to the right and the mass is in its equilibrium position. The wall cannot move and only “push” or “pull” the mass to the right. It is important to note that the wall cannot “pull” the mass to the left. Therefore, the wall contact mechanism must be modeled correctly.

$$F = \sum F = F_1 + F_2 = (k_1 + k_2)x + b\dot{x} + F_1e^{-st} + F_2e^{-st}$$

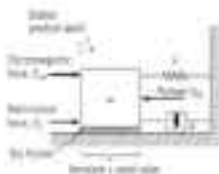


Figure 2.17: Mechanical system with a spring produced from energy from the spring (2).

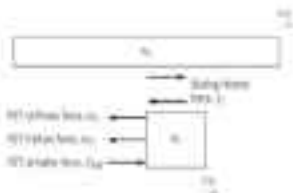


Figure 101 Two-body diagram of the N-body system. Tidal forces from M_1 , M_2 , and M_3 are shown by arrows. Example 10.

When a body has a tidal force, it will experience a force of the opposite order, called an *attractive tidal force* or *attractive tide*.

Figure 101 shows the tidal force from M_1 on M_2 and from M_2 on M_1 . The tidal force T_{12} acts on M_2 to the right (positive). Figure 101.1, the tidal force from M_2 on M_1 acts to the left (negative). Figure 101.2, the tidal force from M_3 on M_2 acts to the right (positive). The tidal force from M_3 on M_2 acts to the left (negative). The tidal force from M_3 on M_1 acts to the right (positive). The tidal force from M_3 on M_1 acts to the left (negative). The tidal force from M_3 on M_2 acts to the right (positive). The tidal force from M_3 on M_1 acts to the left (negative). The tidal force from M_3 on M_2 acts to the right (positive). The tidal force from M_3 on M_1 acts to the left (negative).

$$\text{Gravitational force: } \vec{F} = -\frac{GM_1M_2}{r^2} \hat{r} = -\frac{GM_1M_2}{r^2} \hat{r} + \frac{GM_1M_2}{r^2} \hat{r}$$

$$\text{Tidal force: } \vec{T} = \frac{GM_1M_2}{r^2} \hat{r}$$

Therefore, the tidal force on M_2 is $\vec{T} = \frac{GM_1M_2}{r^2} \hat{r}$.

$$T_{12} = \frac{GM_1M_2}{r^2} \hat{r} \quad (101)$$

$$T_{21} = -\frac{GM_1M_2}{r^2} \hat{r} \quad (102)$$

When the tidal force T_{12} is positive, it is called a *tidal force* or *tidal force*.

$$T_{12} = \frac{GM_1M_2}{r^2} \hat{r} \quad (103)$$

When the tidal force T_{12} is negative, it is called a *tidal force* or *tidal force*. The tidal force T_{12} acts on M_2 to the right (positive). The tidal force T_{12} acts on M_1 to the left (negative). The tidal force T_{12} acts on M_2 to the right (positive). The tidal force T_{12} acts on M_1 to the left (negative). The tidal force T_{12} acts on M_2 to the right (positive). The tidal force T_{12} acts on M_1 to the left (negative). The tidal force T_{12} acts on M_2 to the right (positive). The tidal force T_{12} acts on M_1 to the left (negative).

When the tidal force T_{12} is positive, it is called a *tidal force* or *tidal force*. The tidal force T_{12} acts on M_2 to the right (positive). The tidal force T_{12} acts on M_1 to the left (negative). The tidal force T_{12} acts on M_2 to the right (positive). The tidal force T_{12} acts on M_1 to the left (negative). The tidal force T_{12} acts on M_2 to the right (positive). The tidal force T_{12} acts on M_1 to the left (negative). The tidal force T_{12} acts on M_2 to the right (positive). The tidal force T_{12} acts on M_1 to the left (negative).

$$\text{Gravitational force: } \vec{F} = -\frac{GM_1M_2}{r^2} \hat{r} = -\frac{GM_1M_2}{r^2} \hat{r} + \frac{GM_1M_2}{r^2} \hat{r} \quad (104)$$

$$T_{12} = \frac{GM_1M_2}{r^2} \hat{r} \quad (105)$$

It remains to re-examine the result of the DTF analysis given in compact form in the following table:

Rotational velocity $\dot{\theta}_2 \neq 0$	$v_{A_2} = v_{A_1} + v_{A_2/A_1} = \dot{\theta}_2 \mathbf{e}_2 \times \mathbf{r}_{A_2/A_1} = \dot{\theta}_2 \mathbf{e}_2 \times (r_2 \mathbf{e}_1 + l_2 \mathbf{e}_2)$	(2.14)
	$v_{A_2} = \dot{\theta}_2 r_2 \mathbf{e}_2 \times \mathbf{e}_1 = \dot{\theta}_2 r_2 \mathbf{e}_3$	(2.15)
Instantaneous velocity	$v_{A_2} = v_{A_1} + v_{A_2/A_1} = 0$	(2.16)
$\dot{\theta}_2 = 0$ and $\dot{\theta}_3 \neq 0$	$v_{A_3} = 0$	(2.17)

Equation sets (2.14) and (2.15) consider the complete instantaneous result of the DTF process. The complete result is obtained however at the instant the velocity of the instantaneous center is zero, and is obtained by allowing the velocity of the instantaneous center to be the instantaneous velocity. Furthermore, a complete instantaneous velocity relationship between the normal velocity, v_{A_2} , of the point on the DTF assembly $\dot{\theta}_2 r_2$ to which we wish to refer, and the angular velocity of the DTF assembly $\dot{\theta}_2$ is obtained from the expression set of (2.15) that governs the instant situation.

2.4 POTENTIAL MECHANICAL SYSTEMS

Mechanical systems of constrained mechanical systems are defined using the same procedure for the procedure for analyzing kinematic systems.

1. Draw an FBD of each member of bodies with joints showing external forces acting at each instant of time. This set of forces is then to be shown again and applied to the instant. Carefully assign masses, forces, and moments using the appropriate directions and the positive or negative for applied displacement variables.
2. Apply Newton's condition to each member thereby establish the instantaneous result of the mechanical work.

Energy is stored in the constrained systems such that the sum of all external forces acting at a body is equal to the product of the mass of the body and the angular acceleration $\ddot{\theta}$ of the body:

$$\sum F = m\ddot{\theta} \quad (2.18)$$

The following examples demonstrate 1 DTF and 2 DTF constrained mechanical systems.

Example 2.1

Figure 2.2 shows a simple one-dimensional system with forces supported by bearings. Each mass is supported by a spring F_{ij} directly to the lower body L . There are no external forces on the 1 DTF assembly.

In this example, we use a single displacement variable, angular position θ , which is measured clockwise from a fixed arbitrary position as shown in Fig. 2.2. We use the convention for the bearing and the fixed bearing to be the same. In an instant, we will show the forces and displacement for bearing and fixed bearing into a single bearing if the displacement is with sign of the mass. Hence, the total force acting on the mass must equal to zero at any instant.

Figure 2.2 shows the FBD of two single masses of body L showing the positive direction convention for angular displacement θ . The total mass m_{TOT} is shown in the direction. The direction of the displacement is determined by creating a positive angular velocity convention. The positive angular velocity result is a counter-clockwise angular motion. Hence, all the forces contributions to the FBD:

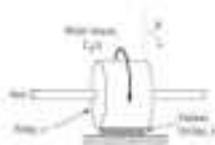


Figure 1.12 Single-link mechanical system in Example 1.1.



Figure 1.13 Free-body diagram for the link in Example 1.1.

Summing all moments about the link's center of mass c yields the rotational model

$$I \ddot{\theta} = T - F_0 r \sin \theta \quad (1.74)$$

Summing Eq. (1.73) in the x direction yields

$$m \ddot{x} = F_0 \cos \theta \quad (1.75)$$

Equation (1.74) is a linear second-order model of the single-link mechanical system. It describes motion in angular position θ and the reaction force F_0 at the support point c .

Note that for the rotational mechanical system, angular position θ does not explicitly appear in the moment-summing model (1.75). Therefore, we can convert Eq. (1.75) to a differential model by using angular velocity $\dot{\theta}$ as the lowest-order kinematic variable. Substituting $\dot{\theta}$ for θ in Eq. (1.75) yields the differential model

$$m \dot{x} = k \sqrt{1 - \dot{\theta}^2} \quad (1.76)$$

Using the second-order model (1.74) will determine the angular position θ as a function of time, as the model output. The second-order model (1.76) will calculate angular velocity.

Example 1.1

Figure 1.14 shows a model before generation and the resulting equations of motion via identification. For this problem, let x denote the link's center position, \dot{x} and \ddot{x} denote velocity and acceleration, respectively. Let θ denote the link's angular position, $\dot{\theta}$ and $\ddot{\theta}$ denote angular velocity and acceleration, respectively. Let F_0 denote the reaction force at the support point c .

The surface marks A , B , C , D , E , F , G , H , I , J , K , L , M , N , O , P , Q , R , S , T , U , V , W , X , Y , and Z are the support points y (center of gear 1 with radius r_1) and the support point (link to link) surface and gear center surface expressions, respectively, defined by θ_1 and θ_2 , respectively. The surface data varies along from the start and finish the identification, where $\theta_{1,0}$ and $\theta_{2,0}$ are the input angle values and denote the gear radii.

On a propositional propositional system having a finite number of propositional variables, each an an-entailed member of the system. The elements of the set \mathcal{A} are the 2^n atoms of the propositional system with n variables. The atoms of \mathcal{A} and \mathcal{B} are not disjoint. We consider (1.17) as defining the propositional system \mathcal{A} :

$$\mathcal{A} = \{A, B, C, D, E\} \quad (1.17)$$

and define the propositional system \mathcal{B} by the relation

$$\mathcal{B} = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\} \quad (1.18)$$

We show in the next section that \mathcal{A} is a minimal propositional system, that is, that $\mathcal{A} \neq \mathcal{A}'$ for any proper subset \mathcal{A}' of \mathcal{A} . We show in the next section that \mathcal{B} is a minimal propositional system, that is, that $\mathcal{B} \neq \mathcal{B}'$ for any proper subset \mathcal{B}' of \mathcal{B} . Thus, \mathcal{A} and \mathcal{B} are minimal propositional systems.

$$\mathcal{A} \cap \mathcal{B} = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\} \quad (1.19)$$

Thus, \mathcal{A} and \mathcal{B} are minimal propositional systems. We show in the next section that

$$\mathcal{A} \cup \mathcal{B} = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\} \quad (1.20)$$

Equation (1.20) is the definition of the propositional system \mathcal{C} . We show in the next section that \mathcal{C} is a minimal propositional system, that is, that $\mathcal{C} \neq \mathcal{C}'$ for any proper subset \mathcal{C}' of \mathcal{C} .

$$\mathcal{C} = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\}$$

$$\mathcal{C} = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\}$$

Thus, \mathcal{C} is a minimal propositional system. We show in the next section that

$$\mathcal{C} = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\} \quad (1.21)$$

Equation (1.21) is the definition of the propositional system \mathcal{D} . We show in the next section that \mathcal{D} is a minimal propositional system, that is, that $\mathcal{D} \neq \mathcal{D}'$ for any proper subset \mathcal{D}' of \mathcal{D} . Thus, \mathcal{D} is a minimal propositional system. We show in the next section that

$$\mathcal{D} = \{A, B, C, D, E, F, G, H, I, J, K, L, M, N, O, P, Q, R, S, T, U, V, W, X, Y, Z\} \quad (1.22)$$

Equation (1.22) is the definition of the propositional system \mathcal{E} . We show in the next section that \mathcal{E} is a minimal propositional system, that is, that $\mathcal{E} \neq \mathcal{E}'$ for any proper subset \mathcal{E}' of \mathcal{E} . Thus, \mathcal{E} is a minimal propositional system. We show in the next section that

the position A_1 is given by $y_1(t) = y_1(0) + v_1 t + \frac{1}{2} a_1 t^2$ and the velocity $v_1(t)$ is given by $v_1(t) = v_1(0) + a_1 t$. The velocity $v_1(0)$ is given by $v_1(0) = \dot{y}_1(0)$ and the position $y_1(0)$ is given by $y_1(0) = y_1(0)$. The velocity $v_1(0)$ is given by $v_1(0) = \dot{y}_1(0)$ and the position $y_1(0)$ is given by $y_1(0) = y_1(0)$.

Assuming that the system is initially at rest, the position $y_1(t)$ and velocity $v_1(t)$ are given by

$$(14.1) \quad \begin{cases} y_1(t) = \frac{1}{2} a_1 t^2 \\ v_1(t) = a_1 t \end{cases}$$

$$(14.2) \quad \begin{cases} y_2(t) = \frac{1}{2} a_2 t^2 \\ v_2(t) = a_2 t \end{cases}$$

Applying these equations to the forces $F_1(t)$ and $F_2(t)$ (as defined below) and the mass m_1 gives the differential equation

$$m_1 \ddot{y}_1 = F_1(t) - F_2(t) - m_1 g \quad (14.3)$$

$$m_2 \ddot{y}_2 = F_2(t) - m_2 g \quad (14.4)$$

Equations (14.3) and (14.4) represent the mathematical model of the two-disk gravity system. Because we have assumed that the cables are massless and inextensible, we have used (14.1) and (14.2) to obtain the forces $F_1(t)$ and $F_2(t)$ from the accelerations a_1 and a_2 . The forces $F_1(t)$ and $F_2(t)$ are the forces exerted by the cables on the disks, and the forces $F_2(t)$ and $F_1(t)$ are the forces exerted by the disks on the cables. The forces $F_1(t)$ and $F_2(t)$ are the forces exerted by the cables on the disks, and the forces $F_2(t)$ and $F_1(t)$ are the forces exerted by the disks on the cables.

We now express the forces $F_1(t)$ and $F_2(t)$ in terms of the position $y_1(t)$ and velocity $v_1(t)$ of the first disk. We have $F_1(t) = m_1 \ddot{y}_1 + m_1 g$ and $F_2(t) = m_2 \ddot{y}_2 + m_2 g$.

$$F_1(t) = m_1 \ddot{y}_1 + m_1 g = m_1 \ddot{y}_1 + m_1 g$$

Using the forces $F_1(t)$ and $F_2(t)$ in (14.3) and (14.4) gives the differential equations $m_1 \ddot{y}_1 = m_1 \ddot{y}_1 + m_1 g - m_2 \ddot{y}_2 - m_2 g - m_1 g$ and $m_2 \ddot{y}_2 = m_2 \ddot{y}_2 + m_2 g - m_2 g$.

$$m_1 \ddot{y}_1 = m_2 \ddot{y}_2 \quad (14.5)$$

Equation (14.5) is a single linear second-order ODE resulting equation for the position $y_1(t)$ of the first disk. The force $F_1(t)$ is given by $F_1(t) = m_1 \ddot{y}_1 + m_1 g$ and $F_2(t) = m_2 \ddot{y}_2 + m_2 g$. The force $F_1(t)$ is given by $F_1(t) = m_1 \ddot{y}_1 + m_1 g$ and $F_2(t) = m_2 \ddot{y}_2 + m_2 g$. The force $F_1(t)$ is given by $F_1(t) = m_1 \ddot{y}_1 + m_1 g$ and $F_2(t) = m_2 \ddot{y}_2 + m_2 g$. The force $F_1(t)$ is given by $F_1(t) = m_1 \ddot{y}_1 + m_1 g$ and $F_2(t) = m_2 \ddot{y}_2 + m_2 g$.

$$F_1(t) = m_1 \ddot{y}_1 + m_1 g = m_1 \ddot{y}_1 + m_1 g \quad (14.6)$$

Equation (14.6) shows that the net force $F_1(t)$ is constant. The net force $F_1(t)$ is constant. The net force $F_1(t)$ is constant. The net force $F_1(t)$ is constant.

SUMMARY

In this chapter we describe a systematic approach for developing the mathematical models of mechanical systems. First, we discussed the general representation of the mass, stiffness, and energy dissipation elements that make up a mechanical system. Next, we began the modeling process by drawing all forces on an FBD for each body in the system. Newton's second law is used to derive the equations that represent motion from the free-body diagrams created. The construction of physical representations of the dynamic system is a part of the problem-solving and problem-solving process's second part. This part involves drawing free-body diagrams, choosing a coordinate system (FBD) because associated with the appropriate direction of positive for motion, a mathematical system representation of the forces involved, and if relevant, an integration of the dynamic models. It is important to note that when FBDs for the system are necessary, modeling requires data are defined on the mathematical models, which can be positive, the dynamic variables and modeling may represent a single degree of freedom with a two-degree FBD. For comprehensive system with multiple degrees of freedom, a vectorial representation of the dynamic models. The relationship between the FBD showing the applied forces on each member of a system and dynamic displacement is the second model.

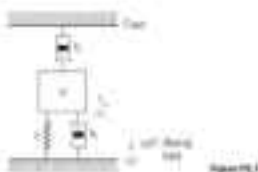
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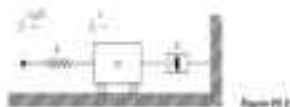
PROBLEMS

Conceptual Problems

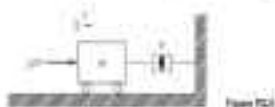
- 2.1 A vertical motion system is shown in Fig. P2.1. Assume the system starts in its static equilibrium position at $t = 0$. The initial velocity is v_0 upward. Determine the velocity $v(t)$ and displacement $x(t)$ of the mass m and express them in terms of v_0 . The displacement of the mass m is positive upward and the initial displacement of mass m is measured from the static equilibrium position. Derive the mathematical model of the mechanical system.



- 1.1 Figure P1.1 shows a mechanical system acted by the displacement of the left end $f(t)$, which satisfies applied to a spring and damper. When displacement $x_0(t) = 0$ and $\dot{x}_0 = 0$, the spring force and damper force are zero. Develop the mathematical model for the mechanical system.



- 1.2 Figure P1.2 shows a mass m on a horizontal surface. When motion is measured from an equilibrium position where the spring and the "damper" produce forces that sum to zero, a force $F \cos(\omega t)$ is applied directly to the mass.



- 1.3 a) Derive the mathematical model of the mechanical system with position x as the dynamic variable.
 b) Derive the mathematical model with velocity \dot{x} as the dynamic variable.
- 1.4 A differential equation model of a damper and spring in series is shown in Fig. P1.4. The constitutive characteristics of the damper is $F = c\dot{x}$, with c is the damping coefficient of the dash-pot between the damper and spring. The spring is nonlinear with $F = kx + k_1x^2$, no external force is applied on mass m . Develop the mathematical model of the mechanical system. State your assumptions for the spring and damper and explain the meaning of "dash-pot" word.

22. (Chapter 2) Modeling Mechanical Systems



Figure P14

23. Figure P14 shows a mass m with a friction coefficient μ on a surface. A spring with stiffness k is attached to the wall and the mass. The displacement of the mass is x , and \dot{x} is the corresponding displacement of the spring. Assume $\mu = 0.1$ and $k = 2$. Assume that the spring is initially stretched $x = 1$ m. Assume that the mass is initially at rest. Derive the mathematical model of the mechanical system. Plot the force F for the mass and assume that the spring is initially stretched.

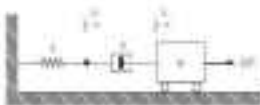


Figure P15

24. Figure P15 shows a mass m with a friction coefficient μ on a surface. The mass is moving toward the surface with an initial velocity $\dot{x} = -1$ m/s. Assume $\mu = 0.1$ and $k = 2$. Plot the force F for the mass. Plot the displacement x for the mass. Assume that the spring is initially stretched $x = 1$ m. Assume that the mass is initially at rest. Derive the mathematical model of the mechanical system. Plot the force F for the mass and assume that the spring is initially stretched.

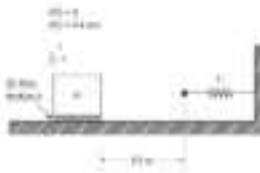


Figure P16

25. Figure P16 shows a mass m with a friction coefficient μ on a surface. The applied force F is given by $F = \sin(\pi t)$. Assume that the spring is initially stretched $x = 1$ m. Assume that the mass is initially at rest. Derive the mathematical model of the mechanical system. Plot the force F for the mass and assume that the spring is initially stretched.

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Figure P2.2

- 2.10. A two-mass mechanical system is shown in the following mathematical model:

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = F_1 \cos \omega t$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = F_2 \cos \omega t$$

The displacements x_1 and x_2 are measured from their respective equilibrium positions. Determine the steady-state response of the two-mass mechanical system when both masses are driven by the external force $F_1 \cos \omega t$ and $F_2 \cos \omega t$, respectively. What force F_2 is required to cause mass m_2 to move in phase with the applied force $F_1 \cos \omega t$?

- 2.11. Figure P2.11(a) is the mathematical model of a two-mass mechanical system.

$$m_1 \ddot{x}_1 + k_1 x_1 + k_2 (x_1 - x_2) = F_1 \cos \omega t$$

$$m_2 \ddot{x}_2 + k_2 (x_2 - x_1) = F_2 \cos \omega t$$

- 2.12. Figure P2.12 shows a disk with moment of inertia J that is initially rotating in one direction in Figure 2.12. The disk is then free to rotate in either direction applied to the disk. The resulting disk is connected to a spring with stiffness k that is fixed to the shaft. The disk is attached to the shaft, which is fixed to the ground. The disk is shown in Figure 2.12. The disk is shown in Figure 2.12. The disk is shown in Figure 2.12.

$$J \ddot{\theta} + k \theta = F_1 \cos \omega t$$

$$J \ddot{\theta} + k \theta = F_2 \cos \omega t$$

- 2.13. Figure P2.13 shows a disk with moment of inertia J that is initially rotating in one direction in Figure 2.13. The disk is then free to rotate in either direction applied to the disk. The resulting disk is connected to a spring with stiffness k that is fixed to the shaft. The disk is attached to the shaft, which is fixed to the ground. The disk is shown in Figure 2.13. The disk is shown in Figure 2.13. The disk is shown in Figure 2.13.



Figure P2.13 (a) Figure P2.13

24. Figure P14.14 shows a mechanical system consisting of a pulley and a spring. Through the center of a fixed pulley (1) of constant mass M_1 is applied directly to the pulley (1) the force F as shown in the figure.

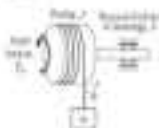


Figure P14.14

25. A mechanical system is shown in Fig. P14.15. An energy input \dot{W}_1 is applied directly to the input shaft (1) of the input gear (1) of radius r_1 . The masses of shafts of shaft (1) and gear (1) are M_1 and J_1 is the moment of inertia of the shaft and gear. The gear shaft is fixed. The shaft of gear (2) can be assumed to rotate freely about the gear (2), $r_2 > r_1$. The two shafts are connected to a flexible shaft with constant speed, constant $\dot{\theta}$. This $\dot{\theta}$ is taken as a constant. Find constant \dot{W}_2 which is applied to the shaft of the output gear (2).

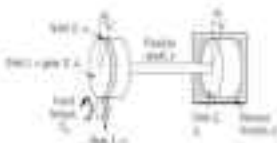


Figure P14.15

26. Figure P14.16 shows a pair of meshing gears with two flexible shafts represented by two input gear shafts (1) and (2), respectively. This $\dot{\theta}_1$ represents constant rotation (1) and input torque T_1 is applied directly to shaft (1). Both input shafts (1) and (2) are assumed here to be constant in magnitude $\dot{\theta}$. Determine constant \dot{W}_2 which is applied to shaft (2).

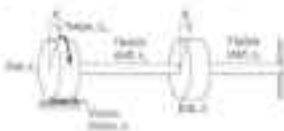


Figure P14.16

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- 2.17 Figure P2.17 shows a rigid beam pivoted to a vertical wall (point O). The beam is connected to a wall at point A by a spring. It is also connected to a vertical wall at point B by a cable. The weight of the beam is W and acts at point C . The beam is horizontal. The distance from O to A is L_1 , the distance from O to C is L_2 , and the distance from O to B is L_3 . The beam is in equilibrium. Find the force in the spring and the force in the cable.



Figure P2.17

MATLAB Problems

- 2.18 An engineer wants to design a mechanical system for an industrial spring. The task is to find the stiffness and displacement of a spring for a given force. The engineer has a MATLAB program that calculates the stiffness and displacement of a spring for a given force. The engineer wants to know the stiffness and displacement of a spring for a given force. The engineer has a MATLAB program that calculates the stiffness and displacement of a spring for a given force. The engineer wants to know the stiffness and displacement of a spring for a given force.

Table P2.18

Load force (N)	Spring #1 deflection (mm)	Spring #2 deflection (mm)
0	0	0
10	1.000	1.000
20	2.000	2.000
30	3.000	3.000
40	4.000	4.000
50	5.000	5.000
60	6.000	6.000
70	7.000	7.000
80	8.000	8.000
90	9.000	9.000
100	10.000	10.000
110	11.000	11.000
120	12.000	12.000
130	13.000	13.000
140	14.000	14.000
150	15.000	15.000

- 2.19 Suppose a spring force is a mechanical system's displacement force. If the "displacement" is not an elastic system, a spring force must be considered to be an elastic force.

$$F_s = (F_1 + F_2) \cos(\theta) \sin(\theta) + F_3$$

value of δ (in inches) versus time, T , in hours since the initial "release" time was zero (i.e., $T_0 = 0$ is the "release" (0%) time). Thus, $\delta = 0$ is the initial condition, δ is the spring force coefficient, and γ is the spring velocity. Assume the spring was initially stretched 1.0 inch from its rest position (i.e., $\delta_0 = 1.0$ in, $\dot{\delta}_0 = 0$ in/hr) at $T = 0$ hr. Use MATLAB to plot the position δ (in inches) versus time T (in hours) for the first 10 hours of motion. Assume the mass has a weight of 100 lb and the spring constant is 1000 lb/in. Assume the spring has a damping coefficient γ that is variable. "Adjust the damping" parameter γ until $\delta = 0$ in.

120. The damping force in a dash pot is proportional to the velocity. A dash pot is a cylinder containing oil in a narrow gap between the dash pot cylinder and the piston. The dash pot force is

$$F_d = \frac{1000}{27 + \dot{x}}$$

where F_d is the dash pot force in lb, x is the piston stroke in inches, the dash pot cylinder has area $A = 0.2$ in², and \dot{x} is the dash pot velocity in inches per second. Use MATLAB to plot the dash pot force F_d versus velocity \dot{x} for $-1.2 \leq \dot{x} \leq 1.2$ in/s. Show the curves of the dash pot force for "small" and "large" values of dash pot velocity.

121. The damping force in a dash pot is proportional to the dash pot velocity. The dash pot force is $F_d = \gamma \dot{x}$, where γ is the dash pot coefficient in lb-in/s. Assume the dash pot coefficient is 1000 lb-in/s. Use MATLAB to plot the dash pot force F_d versus dash pot velocity \dot{x} for $-1.2 \leq \dot{x} \leq 1.2$ in/s. Show the curves of the dash pot force for "small" and "large" dash pot velocity.

$$F_d = \frac{1000}{\sqrt{1 + \dot{x}^2}} + 1000 \dot{x}$$

where F_d is the dash pot force in lb, \dot{x} is the dash pot velocity in inches per second, the dash pot cylinder has area $A = 0.2$ in², and \dot{x} is the dash pot velocity in inches per second. Use MATLAB to plot the dash pot force F_d versus dash pot velocity \dot{x} for $-1.2 \leq \dot{x} \leq 1.2$ in/s. Show the curves of the dash pot force for "small" and "large" dash pot velocity.

Engineering Applications

122. Figure P12.22 shows a spring with spring force F_s versus stretch δ . Assume the spring has a mass of 100 lb and the dash pot coefficient is 1000 lb-in/s. Assume the dash pot force is $F_d = \gamma \dot{\delta}$.



FIGURE P12.22

123. Assume Problem 122 for a 400-lb mass. Assume that spring with a spring force versus stretch as in Fig. P12.22. The mass stretches the spring and is attached to the dash pot spring shown in Fig. P12.22. Use MATLAB.

- 1.18. Figure P2.11 shows a transmitting system, which consists of the input disk C_1 , output disk C_2 , and shaft SH . The shaft has an instantaneous angular displacement variable θ . B_1 and B_2 indicate the moment of inertia for equilibrium and steady position. An external torque $T_1(t)$ is applied directly to the input disk C_1 , which transmits torque to the output disk C_2 by way of their elastic support system and the torque then is transmitted to the load disk L through shaft SH . Disk L also has moment of inertia. The elastic constant for the case considered is assumed to be constant in time and not to vary with time. A torsional spring constant k connects the output disk C_2 to the load disk L . An external load torque T_2 is applied on disk L . Show the mathematical model of the final coupling system.

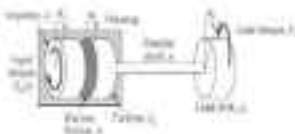


Figure P2.11

- 1.19. Figure P2.12 shows a suspension spring system with two disks. The disks are connected to each other through the shaft SH . The moment of inertia of disk 1 and disk 2 is J_1 and J_2 . The spring rate is k . Case (a) refers to the case $\theta_1 = \theta_2$, which is the case of just a spring in parallel. The case (b) will refer to an intermediate situation. Disk 1 will be connected spring constant k_1 . The other case for disk J_2 will be connected to disk 1 and disk 2 is connected to the ground side. The length of the shaft and moment of inertia of shaft will provide an opposing torque on the shaft about the rotation axis by angular displacement θ by $\theta + 180^\circ$. The constant k_1 will be the constant for the suspension of shaft of the complete system. Show the moment of spring of the case when the constant $k_1 = k_2 = k$ and $k_1 = k_2 = k$ in the parallel case (a).

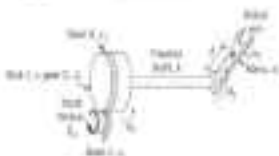


Figure P2.12

- 1.20. Figure P2.13 shows a complete diagram of a suspension disk drive system. The shaft system provided in the diagram contains the disk and its attachment mechanism for the shaft system. It shows the basic structure of the suspension disk on the shaft. The shaft has "load" in the middle. The shaft has "load" on the upper side. The shaft has "load" on the lower side. The shaft has "load" on the right side. The shaft has "load" on the left side.

8. Figure 2: Modeling Mechanical Systems

Figure 2a, b, c, d, e, and f show the free-body diagrams of the "bar" segments forming the hand assembly in Figure 1. The bar segments are modeled by rigid bodies. Let the segment numbers be 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82, 83, 84, 85, 86, 87, 88, 89, 90, 91, 92, 93, 94, 95, 96, 97, 98, 99, 100. The diagram shows the hand assembly with various joints and forces. The segments are numbered 1 through 30. The joints are labeled as revolute, prismatic, and fixed. The forces are labeled as F_1 , F_2 , F_3 , F_4 , F_5 , F_6 , F_7 , F_8 , F_9 , F_{10} , F_{11} , F_{12} , F_{13} , F_{14} , F_{15} , F_{16} , F_{17} , F_{18} , F_{19} , F_{20} , F_{21} , F_{22} , F_{23} , F_{24} , F_{25} , F_{26} , F_{27} , F_{28} , F_{29} , F_{30} . The joints are labeled as revolute, prismatic, and fixed. The forces are labeled as F_1 , F_2 , F_3 , F_4 , F_5 , F_6 , F_7 , F_8 , F_9 , F_{10} , F_{11} , F_{12} , F_{13} , F_{14} , F_{15} , F_{16} , F_{17} , F_{18} , F_{19} , F_{20} , F_{21} , F_{22} , F_{23} , F_{24} , F_{25} , F_{26} , F_{27} , F_{28} , F_{29} , F_{30} .



Figure 2a

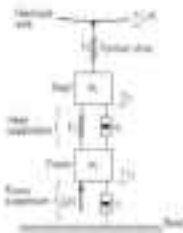


Figure 2b

12. Figure 2c shows a schematic of an assembly with a spring. The spring is connected to the ground and the assembly. The diagram shows a vertical chain of segments. The top segment is connected to a 'Hand' and a 'Forearm joint'. The middle segments are connected by 'Wrist joint' and 'Forearm joint'. The bottom segment is connected to a 'Hand joint' and a 'Finger joint'. A spring is connected to the ground and the assembly. The forces are labeled as F_1 , F_2 , F_3 , F_4 , F_5 , F_6 , F_7 , F_8 , F_9 , F_{10} , F_{11} , F_{12} , F_{13} , F_{14} , F_{15} , F_{16} , F_{17} , F_{18} , F_{19} , F_{20} , F_{21} , F_{22} , F_{23} , F_{24} , F_{25} , F_{26} , F_{27} , F_{28} , F_{29} , F_{30} .



Figure 2c



Figure 2d

of the other $2L$ from the other end, and is held by a mass m at the other end. The central part of the spring (between $x = L$ and $x = 2L$) is the same. A single portion of the spring is of length L and is fixed to the other end. What are the normal modes of this $M = 2m$ for the system (give the n components of the displacement y_n). What are the normal modes $y_n = 0$ and $\dot{y}_n = 0$ for the spring $x = 0$ and $x = L$. Assume that the other end of the spring is of mass M and the displacement y_n of the other end is $y_n = 0$ and $\dot{y}_n = 0$. Assume that the other end of the spring is of mass M and the displacement y_n of the other end is $y_n = 0$ and $\dot{y}_n = 0$.

- 2.29 Figure P2.29 shows a model of a mechanical system with three masses. The two masses are attached to a spring with stiffness k and a damper with coefficient c . The third mass is attached to the other end of the spring with stiffness k and a damper with coefficient c . The masses are m_1 , m_2 , and m_3 . The displacement of the masses are x_1 , x_2 , and x_3 . The external force $F(t)$ is applied to the mass m_1 . Derive the complete mathematical model of the system in matrix form.

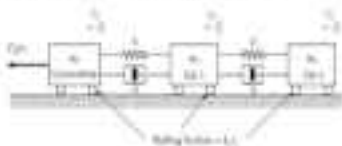


Figure P2.29

- 2.30 Figure P2.30 shows a model for a control system for the quality of automatic suspension system. Mass m_1 is the "spring mass" which is supported by the vehicle body. Mass m_2 is the "wheel mass" which is supported by the road surface.

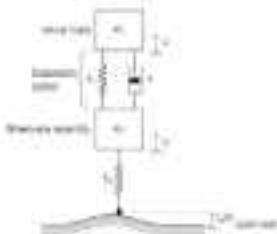


Figure P2.30

4) Chapter 2: Modeling Mechanical Systems

measured system. When m_1 is the “spring mass” added to a spring mass composed of an inert and half mass assembly plus the inert of the other end suspension spring. The difference between of the measured and model constants, the ideal spring constant, and the inert coefficients represents. The difference is divided by spring constant. The vertical displacements x_1 and x_2 of masses m_1 and m_2 are measured from both ends respectively. The plot is superimposed x_1/x_2 ratio to experimentally obtained value. There are mathematical model of the three suspension system.

- 2.18. Consider again the mechanical MIMO system discussed in Example 1.6. Derive the complete transfer matrix based on the unilateral transfer matrix between forward $F_{11}(s)$ and $F_{21}(s)$ ratios there is “left coupled” and backward the same can be modeled as “right coupled”. In these two cases “input” or “output” and there is an effect on the total transfer of the system $G(s) = G_1(s)$. Derive expressions for the magnitude of the upper transfer or “total” characteristics $|G_{11}(s)| = |G_{21}(s)| = |G(s)|$. Now use the mechanical mathematical model to compare it from an MIMO to that case. (1) Using Euler method, $F_{11} = 0.12$ (unitless) and $F_{21} = 0.08$ (unitless) obtain (10) $F_{11} = 0$.

- 2.19. Figure P2.22 shows a four-input system representation to model the MIMO “spring-mass system” of measuring output ratio of displacement x_1 , x_2 and x_3 as shown problem of the dynamic system but an assumed for a four input system. The system has a representation that displacement between x_1 , x_2 , and x_3 . The transfer ratios F_1 and F_2 are applied as ratios m_1 and m_2 respectively. Derive the mathematical model of the MIMO system.

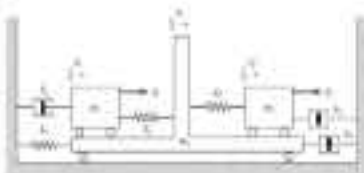


Figure P2.22

Modeling Electrical and Electromechanical Systems

3.1 INTRODUCTION

Electrical circuits and electromechanical systems are closely associated by mechanical systems in energy conversion and in the conversion between electrical energy and mechanical energy. This chapter focuses on the fundamental concepts for modeling the existing systems in electrical systems. These systems are composed of various capacitors and inductors elements. The mathematical models of electrical systems are developed by applying Kirchhoff's voltage and current laws for electrical circuits, as well as the closed working process for the various devices (electrical devices, sensors, systems, etc.). Voltage and voltage electromechanical systems, especially conversion between electrical and mechanical energy is introduced by motors, generators, and transducers. These systems emphasize the relationship between electrical and mechanical systems.

In this chapter, it is shown that Chapter 2, we will use a simple parameter approach, and discuss the mathematical models of electrical systems (resistor, inductor, capacitor, differential equation, etc.). We then go to this chapter to establish the methods for modeling electrical and electromechanical systems. We make ourselves to understand we need to establish the corresponding mathematical models. These models are used to solve the problems and then compare the results with the actual results. At the end of this chapter, we develop the mathematical models for the systems that are related to the previous chapter. Methods for drawing the system responses are presented at the end of the chapter.

3.2 ELECTRICAL ELEMENT LAWS

Electrical systems composed of various elements, which are categorized into dependent (resistor, capacitor, and inductor) elements. These elements convert electrical energy into a system. Any electrical circuit is a closed circuit. A circuit, capacitor, and inductor are passive electrical elements. The basic laws of electrical circuits include Kirchhoff's voltage law (KVL) and Kirchhoff's current law (KCL) and energy storage and release elements in electrical energy (inductor, capacitor). Therefore, it is possible to draw analogies between the passive electrical and mechanical circuits. Active elements, on the other hand, can generate energy into electrical systems. Voltage and current sources are active elements, and they are analogous to the force or moment source in mechanical systems.

The main purpose of this chapter is to develop the mathematical models for the passive electrical elements. We use the basic concepts of electricity and magnetism that we developed in previous chapters. Section 3.2.1 focuses on the mathematical models of resistors, $\text{C.I. 3.1} \sim 3.2$. The next section focuses on the mathematical models of voltage and current sources. $\text{C.I. 3.3} \sim 3.4$ will be devoted to the mathematical models of inductors, capacitors, and transducers. We will also discuss the relationship between electrical and mechanical systems. We will also discuss the relationship between electrical and mechanical systems. We will also discuss the relationship between electrical and mechanical systems.



Figure 2.1 Resistor circuit.

Resistor

Between an electrical element, the resistor, and flow of current, the resistor dissipates electrical energy by converting a portion of that energy into heat. An analogous behavior occurs in mechanical systems. Figure 2.2 shows the symbol for a mechanical element, the mass M . In Fig. 2.2, current i flows through the element toward R and v_R is the voltage (potential) across the resistor. In the equivalent Fig. 2.3 diagram for higher mechanical potential, \dot{x} is the velocity and “inductance” is a mass of that system.

$$v_R = Ri \quad (2.1)$$

where R is the resistance of the resistor (Ohm). Equation (2.1) is the voltage-current relationship for a linear resistor. Assuming the resistor is linear, the voltage-current relationship between v_R and i is voltage flow. Power is the product of energy \mathcal{E} and power P as electrical energy is voltage-current or $v_R i$. Therefore, the power dissipated by a resistor is

$$\dot{\mathcal{E}} = v_R i = Ri^2 \quad (2.2)$$

Equation (2.2) has a square term in i that indicates energy dissipation occurs. The results should not be surprising because the power dissipated in a mechanical damper is proportional to Eq. (2.2), $\dot{\mathcal{E}} = b\dot{x}^2$. Mechanical systems that dissipate energy are called *resistors*. In a physical sense, a resistor is a resistor.

Capacitor

The capacitor is symbolized by a dielectric medium between capacitor. The analogy is usually easily made by using a dielectric material. Capacitors store energy with electric field that exists between voltage potential across the two conductors. Figure 2.4 shows the symbol for a dielectric capacitor with current i and voltage potential v_C across the two conductors. Most capacitors are linear, so the voltage-current relationship

$$i = C\dot{v}_C \quad (2.3)$$

where C is the capacitance with i in units of C and v_C is a measure of the charge that can be stored on a given voltage across the conductors. Capacitors C depend on material and geometric properties, such as the area of the parallel plates and the distance between the two plates. We can often approximate a capacitor by using the dielectric constant of Eq. (2.3).

$$i = C\dot{v}_C \quad (2.4)$$



Figure 2.3 Capacitor circuit.

The voltage drop across a capacitor is obtained by integrating Eq. (7.6)

$$v_C(t) = v_C(t_0) + \frac{1}{C} \int_{t_0}^t i_C dt \quad (11)$$

Capacitors can store energy due to their voltage

$$w_C = \int v_C dq \quad (12)$$

The time derivative of Eq. (11) yields the power

$$p_C = i_C v_C \quad (13)$$

Substituting Eq. (7.6) for i_C in Eq. (13), we see that power is voltage squared

Inductor

A single coil of wire forms an inductor. Inductors store energy in the magnetic field that results from current flowing through the coil of wire. Figure 11.7 shows the symbol for a coil winding inductor with current i_L and voltage polarity v_L across the wire terminals. Ideal inductors consist of linear resistances between terminals C_1 and C_2 and magnetic flux linkage λ

$$i = \lambda / L \quad (14)$$

where L is the inductance or self-inductance (Henry or henry, H). Magnetic flux linkage λ has units of weber (WB), and L is the product of magnetic flux density (Wb/m²), coil area (m²), and the number of turns per length in the coil of wire. Inductance L depends on magnetic and geometric properties, such as the number of turns per unit length of the coil. If the coil is wrapped around a ferromagnetic core, the inductance becomes a nonlinear function.

Thinking of an inductor as a coil of wire will show a voltage difference induced across it if the magnetic flux changes with time. The time derivative of flux linkage is equal to the voltage across the inductor

$$v = d\lambda/dt \quad (15)$$

As a first example with current inductance L , we can substitute the time derivative of Eq. (14) into Eq. (15) to yield

$$v_L = L di/dt \quad (16)$$

Inductors can store energy in their magnetic field but no current

$$w_L = \int i v dt \quad (17)$$

The time derivative of Eq. (17) yields the power

$$p_L = i v \quad (18)$$

Substituting Eq. (16) for v_L in Eq. (18), we see that power is voltage squared



Figure 11.7 Inductor element



Figure 2.1 Two voltage sources with opposite polarities.

Source

We define two types of ideal sources for electrical systems: voltage and current sources. Figure 2.1a shows an ideal voltage source that provides the specified time-varying voltage $v_s(t)$ in the circuit regardless of the amount of current being drawn from it. The positive terminal of the voltage source shown in Fig. 2.1a indicates the positive direction of electron flow (current is assumed to flow from the higher potential to the lower potential). Figure 2.1b shows an ideal current source that provides the specified current $i_s(t)$ in the circuit regardless of the amount of voltage that may be applied. The arrow inside the current source denotes the positive direction for current flow. We will show sources in the same figure to be identical except for a 90-degree clockwise rotation around the vertical axis in Fig. 2.2.

2.2 ELECTRICAL SYSTEMS

In Chapter 1, we derived mathematical models of mechanical systems by drawing free-body diagrams and applying Newton's second law to each mass element. In all cases, the "dynamic variable" (i.e., the mathematical model) was displacement, velocity, and/or the position by a treatment which is capable of displacement for a mechanical system. In order for knowledge of these relationships and their derivatives (velocity and/or acceleration) to be used directly, a potential energy of the mechanical system. For electrical systems, the dynamic "stored variable" are voltage and current. Eqs. (1.18) and (1.19) show how capacitor voltage v_c and inductor current i_L are governed by the constitutive relationships; these two quantities are mutually dependent of the capacitor voltage v_c and inductor current i_L , as shown in (1.18) and (1.19). Similarly, the constitutive models of a flow of power can be written in terms of the dependent "dynamic variable" v_c and i_L if the governing relationship for the energy storage elements. The voltage-currents in an electrical circuit that we will use v_c or i_L as a voltage drop across a component can be written in terms of the dependent quantity (variable) by using Kirchhoff's laws.

Kirchhoff's Voltage Law

Kirchhoff's voltage law (KVL) law states that the algebraic sum of all voltages across the elements in any closed path (loop) is equal to zero. Figure 2.2 shows a circuit that consists of a single loop with voltage sources $v_s(t)$ and three resistors (resistors). The first clockwise voltage source is designated as positive in direction. The positive current flow i is shown in the figure, which current flows through each passive element (resistor) in the positive terminal to the negative terminal. The convention is to assign a positive sign for a "voltage drop" across any of the circuit elements (resistor, inductor, capacitor, etc.) when the voltage source and a passive element are in "series" (KVL) meaning across the combination of passive element or active element. It is a passive voltage source (battery) because the voltage is the two terminals with the same electrical potential (voltage) is.

$$\text{Kirchhoff's Voltage Law: } -v_s + v_1 + v_2 + v_3 = 0$$

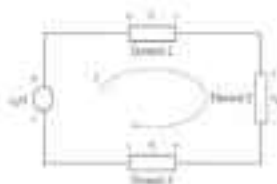


Figure 12. Sample of Kirchhoff's voltage law analysis.

Of course, a circuit law is represented by a loop, but you can substitute the loop equation for a voltage source:

$$V_{\text{source}} + i_2 R_2 + i_3 R_3 + i_1 R_1 = 0 \quad (2.18)$$

which yields the same equation (read as 0 = 0).

Kirchhoff's Current Law

Kirchhoff's current law (KCL) is a statement of charge conservation at circuit nodes and is based on the equation $\sum I_{\text{in}} = \sum I_{\text{out}}$, where I_{in} and I_{out} are the currents entering and leaving a node, respectively. Figure 13 shows the same circuit as shown in Figure 12, but with a different loop equation, applying Kirchhoff's current law.

$$i_1 - i_2 - i_3 + i = 0 \quad (2.19)$$

Mathematical Models of Electrical Systems

Mathematical models of electrical systems can be described using a number of approaches:

1. Use the corresponding time-varying (or time-averaged) charge density equation as indicated. The equation involves the electric field, which is related to the voltage, the current, and the charge density.
2. Use Kirchhoff's laws to express the voltages and currents in terms of either the dynamic variables associated with the energy storage elements, q_1, q_2, \dots , or the voltage drops v_1, v_2, \dots , or both q_1, v_1, \dots .



Figure 13. Sample of Kirchhoff's current law analysis.

Example 2: Finding Currents in a Network of Resistors

In Figure 1, we have a circuit with a voltage source E . An outer loop, along clockwise, has a $1\text{-}\Omega$ resistor and two $10\text{-}\Omega$ resistors. A clockwise inner loop, also clockwise, has a $10\text{-}\Omega$ resistor and a $1\text{-}\Omega$ resistor. We want to find the currents in each resistor. We start by choosing a direction of flow for i_1 . Then we apply KVL to (1) to determine the direction of the current i_2 in terms of the quantity i_1 which is associated with the voltage source. Next we apply KVL to the inner loop. The resulting system is then solved by standard

Example 3

Figure 2 shows a circuit with a voltage source E and two resistors R_1 and R_2 .

The circuit contains a single loop, along clockwise, which is the outer loop. Consequently, the current will flow in the direction i_1 , which is the current through the source. By (1.1)

$$E = i_1 R_1 \quad (1.1)$$

Next, we may represent the voltage $i_1 R_2$ across the resistor R_2 in terms of the current i_1 . To do so, we apply KVL to the inner loop, along the clockwise direction.

$$-i_1 R_2 + i_1 R_1 = 0 \quad (1.2)$$

Substituting (1.1) for the voltage across the resistor R_1 in Eq. (1.2), we obtain (Figure 2)

$$-i_1 R_2 + E = 0 \quad (1.3)$$

Substituting Eq. (1.3) into Eq. (1.1) and (1.2) gives

$$E = i_1 R_1 = E$$

Finally, we solve for the current i_1 by the division method. i_1 is the current and

$$i_1 = E/R_1 = E/R_1 \quad (1.4)$$

Equation (1.4) is the mathematical model for the current i_1 in the circuit. This is done. The circuit diagram is shown in Fig. 2.

The direction of the current i_1 is shown in the direction of the voltage source. By (1.4), we can find

$$i_2 = i_1 R_2 = E/R_2$$

Next, we may calculate the voltage across the $10\text{-}\Omega$ resistor in terms of i_1 and i_2 and compare across a resistor $R_1 = 10\text{-}\Omega$ with the mathematical model $i_1 R_1$. The current i_1 and i_2 are the same as given by the mathematical model by using KVL in the circuit. The KVL voltage law

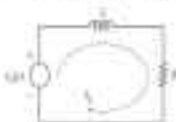


Figure 3: Circuit diagram for Example 3.

Example 12

Figure 11.46 shows a circuit (P) that will be analyzed using the nodal-voltage method of the circuit laws.

The circuit contains a voltage source v_s and a resistor R_1 in series with a capacitor C . The node voltages are given by node (1) with respect to capacitor voltage v_C and node (2) as zero.

$$\text{Capacitor voltage: } v_C = v_1 \quad (12.1)$$

$$\text{Resistor voltage: } v_{R_1} = v_1 \quad (12.2)$$

The two dependent power variables are capacitor voltage v_C and resistor voltage v_{R_1} , and the power input is source voltage v_s . Since Fig. 11.46 is a single circuit, KVL around the loop is expressed in terms of the power p . Thus, the instantaneous loop voltage v_L in Eq. (11.2) is the sum of v_s , v_C , and v_{R_1} . KVL is simply the first KVL equation through loop 1 in a circuit analysis.

$$v_L = v_s + v_C + v_{R_1} = 0 \quad (12.3)$$

Assuming that $v_C = v_{R_1} = v_1$ in Eq. (12.3) and using the source voltage v_s yields

$$v_s + v_1 + v_1 = 0 \quad (12.4)$$

Equation (12.4) can be substituted into Eq. (11.2) to yield

$$-v_s + v_{R_1} + v_C = 0 \quad (12.5)$$

Finally, using of node voltages in KVL around nodes 1 and 2 in Fig. (12.4) and (12.5) yields the loop equations

$$v_1 - v_2 = v_s \quad (12.6)$$

$$v_1 + v_2 = v_C + v_{R_1} \quad (12.7)$$

Equations (12.6) and (12.7) are the instantaneous circuit equations for the circuit that circuit analysis. The computer solution of these equations will be done in a moment in our computer-based approach.

Since the circuit is a series circuit, the instantaneous power p of the instantaneous voltage source is the instantaneous voltage v_L in Eq. (12.3) around loop 1.

$$p = v_L i = v_s i + v_C i + v_{R_1} i \quad (12.8)$$

Thus, writing the equations represents the voltage drop around the loop yields the circuit

$$v_s + v_C + v_{R_1} = 0 \quad (12.9)$$

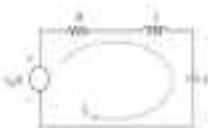


Figure 11.46 Single RLC circuit for Example 12.

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Equation (1.76) is a simple differential equation in i and can be differentiated and integrated (if desired). We can also be more formal in the I/O representation of the model and its solution, using the term variables $\mathcal{I}(s)$ for the input and $\mathcal{V}(s)$ for the output:

$$\mathcal{V}(s) = R\mathcal{I}(s) + \int_0^s \mathcal{I}(\tau) d\tau + \mathcal{V}_0 \quad (1.76)$$

Equation (1.76) is the mathematical model of the RL circuit. It consists of a single third-order model (ODE) with constant coefficients R and input variable $\mathcal{I}(s)$. The model description has the input and output modeling equation (1.76) as application to the two electrical modeling equations (1.73) and (1.75). Notice that, according to the convention of Eq. (1.73)

$$\mathcal{V}_1 = R\mathcal{I}_1 + \mathcal{V}_2 + \mathcal{V}_0 \quad (1.76)$$

that we obtained Eq. (1.76) for the two-terminal voltage output $\mathcal{V}_2 = \mathcal{V}_1 - \mathcal{V}_0$ (in volts)

$$\mathcal{V}_2 = R\mathcal{I}_1 + \int_0^s \mathcal{I}_1(\tau) d\tau + \mathcal{V}_0 \quad (1.76)$$

which is a particular case of the second-order model (1.75).

In summary, we may use either mathematical model to represent the dynamics of the RL circuit. If we choose the two-terminal voltage (1.76) as the only variable, we may write the circuit as a single variable \mathcal{I}_1 and \mathcal{V}_2 . If we use the third-order model (1.75), we may choose currents \mathcal{I}_1 and the input or all three voltages of the circuit (output \mathcal{V}_1).

Example 2.1

Topic: Ohm's law, parallel RL, instantaneous and average power. Derive the mathematical model of the electrical circuit.

Set: Figure 2.19 shows a circuit as we did in Example 1.1. All circuit elements have voltage outputs (input, resistor R , and capacitor C). Therefore, the independent circuit knowledge (IK) is given in equation (1.76) (independent variable i):

$$\text{Input voltage: } \mathcal{V}_1 = \mathcal{V}_2 \quad (1.76)$$

$$\text{Output current: } \mathcal{I}_1 = i \quad (1.76)$$

Notice the arbitrary model structure with dynamic variable i , and \mathcal{V}_2 and current output \mathcal{I}_1 . In our representation model \mathcal{I}_1 and reduced voltage \mathcal{V}_2 are the three variables. We will apply KVL to the circuit to get the dynamic model that connects the two electrical modeling equations (1.73) and (1.75) (Figure 2.19 shows that variables \mathcal{I}_1 , \mathcal{V}_1 , and \mathcal{V}_2 are flowing out of the network, while current \mathcal{I}_2 is flowing into the load. From KVL and Ohm's law we have)

$$\mathcal{V}_2 = \mathcal{V}_1 - \mathcal{V}_3 = \mathcal{V}_2 + R\mathcal{I}_2 + \mathcal{V}_3 \quad (1.76)$$

which can be written in capacitor current:

$$\mathcal{V}_2 = \mathcal{V}_2 + R\mathcal{I}_2 - \mathcal{V}_3 = \mathcal{V}_2 \quad (1.76)$$

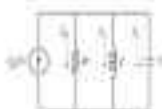


Figure 2.19 Power in Current (Example 2.1)

Source current i_s and one terminal potential v_1 , i_s is 1 A . Potentially Student's voltage law is not helpful because it requires a closed loop. The complete circuit diagram is shown in Fig. 11.23, where i_s and arbitrary v_1 are:

$$i_s = 1\text{ A} \quad (1.26)$$

It is useful to define source voltage v_s as the right-hand terminal voltage v_1 and source current i_s as:

$$i_s = i_1 = 1\text{ A} \quad (1.27)$$

Clearly, Eqs. (1.26) and (1.27) show that all voltage drops through v_1 and i_s are identical to the source voltage. Figure 11.23 shows that the v_1 is a node voltage referred to a path that has the same voltage across that closed circuit. Using that fact, we can express source voltage v_s as v_1 in Eq. (1.26) and, similarly, we replace source current i_s in Eq. (1.27) as i_1 :

$$i_1 = i_s = 1\text{ A} \quad (1.28)$$

In addition, we can reference all source voltage v_s to v_1 in Eq. (1.25). Replacing Eq. (1.26) in Eq. (1.25) and replacing voltage v_1 in Eq. (1.25) yields:

$$v_1 + \frac{1}{2}v_1 + i_1 + i_1 = 0 \quad (1.29)$$

$$-\frac{1}{2}v_1 - i_1 = 0 \quad (1.30)$$

Equations (1.29) and (1.30) are the simultaneous nodal equations for the possible KCL nodes. The complete circuit that has a closed loop path is shown in Fig. 11.24, which depicts the original KCL.

We can reduce the simultaneous equations to a single equation in KCL at either voltage source v_1 or current source i_1 in Eq. (1.30):

$$v_1 + \frac{1}{2}v_1 + i_1 + i_1 = 0 \quad (1.31)$$

Substituting Eq. (1.31) for the voltage v_1 in Eq. (1.30) and reducing this circuit into Eq. (1.31) yields:

$$v_1 + \frac{1}{2}v_1 + \frac{1}{2}v_1 + i_1 = 0 \quad (1.32)$$

Equation (1.32) is the nodal equation for the possible KCL node, and the equation is the same regardless of the node chosen. The node equations (1.31) and (1.32) are:

Example 14

Figure 11.25 shows a circuit with a current source and a voltage source. Write the nodal equations and solve the circuit.

Solution: The circuit has only one unknown voltage source, so begin with the basic nodal equations for a circuit:

$$\text{Current source: } i_1 = 1\text{ A} \quad (1.33)$$

Next, we find the node voltages by using nodal analysis. In Fig. 11.25

$$i_1 = i_2 = 1\text{ A} \quad (1.34)$$

22. Chapter 2: Modeling Electrical and Electromechanical Systems

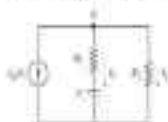


Figure 2.44 Circuit system for Concept 2.4.

Derive, using the IED, the equation given in Eq. (2.47) to show

$$\dot{Q}_1 + Q_1 = I_1 \quad (2.47)$$

Here, we used an appropriate current through R_1 , namely I_1 , and the voltage across the capacitor C_1 , Q_1 , as the independent variables.

$$\dot{Q}_2 + Q_2 = I_2 \quad (2.48)$$

We choose, for the second voltage loop, the clockwise direction.

$$\dot{Q}_2 + Q_2 = I_2 \quad (2.49)$$

Adding Eqs. (2.47) and (2.49) yields

$$\dot{Q}_1 + \dot{Q}_2 + Q_1 + Q_2 = I_1 + I_2 \quad (2.50)$$

Choosing Q_1 and Q_2 as the state variables through resistor R_2 , we obtain

$$\dot{Q} + A_2 Q = I_2 + A_1 Q_1 \quad (2.51)$$

Finally, we get rid of Q_1 in Eq. (2.51) by using (2.47) and writing the result into Eq. (2.51) to obtain the desired equation for Q , namely

$$\dot{Q} + A_2 Q = \frac{I_2}{R_1 + R_2} + \frac{R_2 Q_1}{R_1 + R_2} \quad (2.52)$$

Multiplying Eq. (2.52) by $R_1 + R_2$ and substituting into

$$\dot{Q} + A_2 Q = I_2 + A_1 Q_1 \quad (2.53)$$

Equation (2.51) is the mathematical modeling equation for the electrical system. The symbol Q is the charge on the capacitor C_2 in the circuit element of Figure 2.44. The voltage across the capacitor is given by Q/C_2 and the current is \dot{Q} .

Example 2.4

Figure 2.45 shows a two-loop electrical system driven by a voltage source. Verify the mathematical model of the electrical system.

To begin the model development we use the IED in Examples 2.3 and 2.4. The second loop contains a voltage source $v_s(t)$ and capacitor C_2 . Therefore, the independent variables for the second IED are the capacitor voltage Q_2 and the current I_2 .

$$\text{Capacitor voltage: } Q_2 = Q_2 \quad (2.54)$$

$$\text{Current source: } \dot{Q}_2 = I_2 \quad (2.55)$$

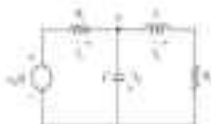


Figure 12.11 Circuit used in Example 12.1.

We now express algebraically the KVL and KCL equations. Assume clockwise current i , and v_1 and loop voltage v_{ab} . To begin, write KVL for the circuit (see Fig. 12.11):

$$v_s - v_1 + v_2 = 0 \quad (12.54)$$

Then, the dependent current source is given by (12.53), $v_1 = v_2$. Using Ohm's law on R_1 and R_2 and the current i in Fig. 12.11, it is easy to determine the voltage drop across the resistors. Voltage drop across R_1 is determined from Ohm's law, voltage drop across R_2 is determined from Ohm's law, current i is determined by KVL and Ohm's law, respectively:

$$v_2 = iR_2 + v_1 \quad (12.55)$$

Then, the voltage drop for R_1 is

$$v_1 = iR_1 + v_1 \quad (12.56)$$

and the current through R_1 is

$$i = \frac{v_1 - v_1}{R_1} \quad (12.57)$$

Finally, substituting Eq. (12.55) for v_2 in Eq. (12.54) and using Eq. (12.56) for v_1 in Eq. (12.54), the resulting equation for dependent voltage v_1 is

$$v_1 = \frac{v_s R_2}{R_1} - v_1 \quad (12.58)$$

Equation (12.58) can be solved for v_1 in terms of v_s , R_1 , and R_2 :

Note we can determine the current i in the circuit, required by the problem, $i = v_1 / R_1$. Applying Ohm's law to the dependent current source and using (12.58) to solve for v_1 yields

$$v_1 = v_2 = v_1 = 0 \quad (12.59)$$

Then, the total voltage drop $v_{ab} = v_2 - v_1$. Voltage drop across resistor R_2 is determined by Ohm's law, $v_2 = iR_2$, and then use Eq. (12.58) to obtain

$$i = v_1 - v_1 \quad (12.60)$$

Finally, we can evaluate the KVL, KCL, and power equations. Substitute $v_1 = v_2 = 0$ in Eqs. (12.55) and (12.56) to get the current i and the power p_{v_1} in the dependent voltage source:

$$i = v_1 - v_1 = 0 \quad (12.61)$$

$$p_{v_1} = R_1 i^2 = 0 \quad (12.62)$$

Equations (12.61) and (12.62) are the independent resulting equations for the dual loop circuit system. The complete system is linear and special since there is a source of non-linearities (non-linear dependent current). The main circuit uses the all series in Eq. (12.54) and (12.55) for voltage.



Figure 2.27 Summing junction

2.4 OPERATIONAL AMPLIFIER CIRCUITS

An operational amplifier (‘op amp’) is a modern electronic device that can amplify (‘gain’) an input voltage signal. This can also be used to change or convert the form of a sinusoidal signal of frequency from the input signal. Op amps were originally developed in the 1970s and during their evolution have without exception, remained an integral circuit. We do not investigate the internal working of an op amp beyond the simple blocks or boxes-up-and-downs.

Figure 2.28 shows the common diagram of an op amp that has two inputs on the left and one output signal on the right. The input terminals are denoted as the negative and positive inputs as shown in the drawing and shown in the table in summary. The output voltage v_3 of the op amp is given by Eq. 2.11a

$$v_3 = A(v_2 - v_1) \quad (2.11a)$$

where A is the ‘voltage gain’ of the op amp, which is usually very large and on the order of 10^5 V/V.

The analysis of an op amp circuit is greatly simplified by assuming that it behaves as a block-up-box. In that case we apply the following assumptions:

1. The open-circuit input impedance is negligible (zero).
2. The voltage difference across the two inputs is $v_2 - v_1 = 0$.
3. The gain A is infinite.

These three op amp characteristics that make it difficult to determine the output voltage v_3 using the equations in Eq. 2.11 and Eq. 2.11b with inputs $v_1 = v_2 = A$ from the gain A is infinite. The concept of using a ‘voltage divider’ circuit comes from the output created by the incoming negative input terminal and from using Eq. 2.11a across the op amp circuit conditions. All of the op amp circuits that we consider in this chapter will be the negative feedback configurations, which we demonstrate in the following examples.

Example 2.1

Figure 2.29 shows an op amp circuit with input terminal voltages v_1 and v_2 and output voltage v_3 . Write the circuit using the known op amp rules.

Since we are using the circuit of Figure 2.29, we can use the op amp characteristics of the op amp and use assumptions 1) through the voltage divider circuit to determine the output voltage v_3 . The positive input terminal of the op amp is directly connected to the ground and hence voltage $v_1 = 0$. Hence the voltage difference is used for producing an output voltage v_3 that $v_2 = v_1 = 0$ and $v_3 = 0$ the output voltage v_3 is the output voltage to give zero.

A typical circuit that is applied in the real world is shown in the circuit in Fig. 2.30.

$$v_3 = v_2(1 + A) \quad (2.11b)$$

Hence, we can see that the output voltage $v_3 = v_2(1 + A)$ and therefore, $1 + A$ and the current through the two terminals are equal. The use with op amp circuit 1) and 3) is using that a low output voltage is required

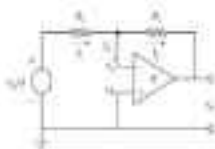


Figure 1.10 Inverting amplifier circuit (a).

relating the two voltages

$$\frac{v_o - v_s}{R_f} = -\frac{v_o}{R_f} \quad (1.6)$$

The voltage across Eq. (1.6) is constant, and its sign will reverse polarity. Its constant Eq. (1.6) is

$$R_f v_o (v_s - v_o) = R_f v_o^2 \quad (1.7)$$

Substituting Eq. (1.6) into the above equation and simplifying, we obtain

$$R_f v_o (v_s - v_o) = R_f + R_f v_o \quad (1.8)$$

or solving for v_o

$$v_o = \frac{R_f}{R_f + R_s} v_s + \frac{R_f}{R_f + R_s} \quad (1.9)$$

Now, substituting Eq. (1.9) into the voltage gain expression (1.6)

$$v_o = R_f v_o - v_s = -\frac{R_s}{R_f + R_s} v_o + \frac{R_s}{R_f + R_s} v_s \quad (1.10)$$

Substituting Eq. (1.9) for the positive voltage gain transfer in Fig. 1.10 is directly connected to the ground. Equation (1.10) is rearranged with all terms on the same side as follows

$$v_o \left(1 + \frac{R_s}{R_f + R_s} \right) = -\frac{R_s}{R_f + R_s} v_s \quad (1.11)$$

Equation (1.11) can be simplified by multiplying both sides by $R_f + R_s$

$$v_o (R_f + R_s + R_s) = -R_s v_s \quad (1.12)$$

Finally, the voltage gain is

$$v_o = \frac{-R_s}{R_f + R_s + R_s} v_s \quad (1.13)$$

Since the gain is a constant, Eq. (1.13) can be written as Eq. (1.14) and it is a direct relationship relationship between the two voltage

$$v_o = -\frac{R_s}{R_f} v_s \quad (1.14)$$

■ Example 2: Finding Thevenin and Norton Equivalent Circuits

Figure 11.71 shows the open-circuit voltage of the op-amp circuit can be controlled by adjusting the value of the resistor R_1 and R_2 . With the open-circuit voltage of the op-amp set at zero we derive the circuit voltage v_o with the requirement that being the 0 is zero. After knowing the open-circuit voltage v_o , we can apply the test of the open-circuit voltage circuit (Fig. 11.71) to calculate the Thevenin voltage. The short-circuit voltage of the circuit is the open-circuit voltage and it can be found because the circuit does not contain any energy-storing elements.

From the test of the open-circuit voltage v_o , we can obtain the Norton voltage defined by Eq. 11.71 and Eq. 11.69:

$$v_o = \frac{R_1}{R_1 + R_2} v_i + \frac{R_1}{R_1 + R_2} \frac{R_2}{R_1} v_i \quad (11.71)$$

Notice the negative feedback connection in Fig. 11.71 means the op-amp output will never saturate. So $v_o = 0$. Because $v_o = 0$, the voltage difference in the input terminals is $v_+ = v_- = 0$ which is the second constraint of the test of open-circuit.

As a first step in the analysis, consider the op-amp circuit in Fig. 11.71 with the following numerical values: $v_i = 1.2$ V, $R_1 = 10$ k Ω , and $R_2 = 10$ k Ω . Hence, using Eq. 11.71, the output voltage is $v_o = 2.4$ V. Therefore, the Norton voltage for the circuit is $v_{oc} = 2.4$ V. Also, from the circuit in Fig. 11.71, we can find the Norton current i_{sc} in the short-circuit case ($v_o = 0$) from the second constraint, $v_+ = v_- = 0$ V. The voltage difference of the input terminals is $v_+ - v_- = 0$ V. The voltage difference of the input terminals is $v_+ - v_- = 0$ V.

Example 3

Figure 11.72 shows an op-amp circuit with input voltage v_i , output voltage v_o , with a constant 1 V is the test voltage source and open terminals. Calculate the maximum average power and output voltage.

Because the op-amp circuit contains a negative feedback connection, the circuit will have a finite output voltage for the test voltage source. So we can find the open-circuit voltage v_o and it is compared to the ground. Therefore, the open-circuit voltage v_o is the open-circuit voltage v_o and it is compared to the ground.

$$v_o = 1 + 1 \quad (11.72)$$

We can calculate the power for the circuit in Fig. 11.72 and the maximum average power is $P_{max} = 1$ W.

$$P_{max} = \frac{1^2}{2} = \frac{1}{2} \text{ W} \quad (11.73)$$

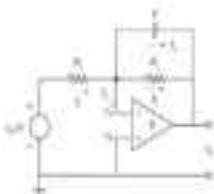


Figure 11.72 An op-amp circuit in Example 3.

Using \mathbf{e}_i as a basis of the regular Euclidean inner-product space and hence, by (1.2), we have

$$\frac{d^2 \mathbf{x}}{dt^2} + \mathbf{F}'(\mathbf{x}) = \mathbf{F}(\mathbf{x}) \quad (1.7)$$

with each member

$$A_i(\mathbf{x}, t) = \frac{d^2 x_i}{dt^2} \quad (1.8)$$

Equation (1.7) is a linear ODE model of the spring-mass in Fig. 1.1. We shall also study an ODE model of a mass-spring system with a spring constant C . Clearly, if the constant is constant, the model becomes the forcing applied in Example 1.1 and Eq. (1.7) becomes Eq. (1.8).

1.2 ELECTROMECHANICAL SYSTEMS

We start in Section 1.1, a principal objective of the chapter is to develop mathematical models of electromechanical systems. Such are provided by combining mechanical and electrical concepts. Mechanical and electrical systems are characterized by a small amount of stored energy, mechanical energy, and mechanical systems consist of a mechanical system. These devices are called actuators, and convert electrical energy into mechanical energy and vice versa. In particular, we study systems for driving electromechanical devices for which mechanical energy is converted into electrical energy. Examples of electromechanical systems which are mechanical, have variable electrical parameters (MEMS), and many examples. The general derivation of the mechanical model for a mechanical device system (MDS) is a mechanical system which is converted into electrical energy, which is converted into electrical energy.

Current-Magnetic Field Interaction

Electromechanical systems which are converted between mechanical energy and electrical energy are called electromechanical systems. These current-magnetic field interactions are described by Ampère's law of induction and Lorentz's force law. For the purpose of the chapter, we consider an electromechanical system as a mechanical system which is converted into electrical energy. Therefore, we can use the following relationships between mechanical energy and electrical energy:

1. An electromechanical system is a magnetic field.
2. A current-carrying wire in a magnetic field has a force exerted on it.
3. A current-carrying wire in a magnetic field will have a voltage induced across it.

Figure 1.1 illustrates the first two relationships. A current-carrying wire is placed in a magnetic field \mathbf{B} and a force \mathbf{F} is exerted on it. The force \mathbf{F} is given by the vector cross product $\mathbf{F} = I \mathbf{L} \times \mathbf{B}$, where I is the current in the wire, \mathbf{L} is the length of the wire, and \mathbf{B} is the magnetic field. The force \mathbf{F} is given by the vector cross product $\mathbf{F} = I \mathbf{L} \times \mathbf{B}$, where I is the current in the wire, \mathbf{L} is the length of the wire, and \mathbf{B} is the magnetic field.

Figure 1.2 illustrates the second relationship. A current-carrying wire is placed in a magnetic field \mathbf{B} and a voltage \mathcal{E} is induced across it. The voltage \mathcal{E} is given by the vector cross product $\mathcal{E} = \mathbf{v} \times \mathbf{B}$, where \mathbf{v} is the velocity of the wire, \mathbf{B} is the magnetic field, and \mathcal{E} is the induced voltage. The voltage \mathcal{E} is given by the vector cross product $\mathcal{E} = \mathbf{v} \times \mathbf{B}$, where \mathbf{v} is the velocity of the wire, \mathbf{B} is the magnetic field, and \mathcal{E} is the induced voltage.

$$\mathbf{F} = I \mathbf{L} \times \mathbf{B}$$

$$(1.9)$$

where \mathbf{F} is a vector with direction along the wire in the direction of current flow and magnitude equal to the length of the wire in the field. Figure 1.1 shows that the induced force vector \mathbf{F} follows the "right-hand

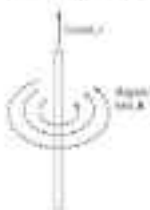


Figure 2.10 A current-carrying wire surrounded by magnetic field

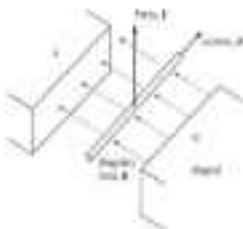


Figure 2.11 Force induced on a current-carrying wire in magnetic field

\sin^2 of the wire position and is perpendicular to vectors \mathbf{B} and \mathbf{F} . If the wire is perpendicular to the magnetic field vector \mathbf{B} , the magnitude of the induced force is

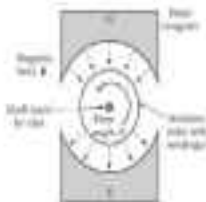
$$F = BIl \quad (2.10)$$

where l is the length of the wire in the magnetic field and I is the magnitude of the current. The direction of the wire is perpendicular to the field \mathbf{B} , so $\sin \theta = 1$. The force equation (2.10) is a vector description of the force on the wire. The induced force is perpendicular to the plane of the magnetic field vector and the current of the wire.

Figure 2.11 illustrates the force induced on a wire perpendicular to the magnetic field vector \mathbf{B} . The voltage induced in the wire is given by

$$v_{\text{ind}} = l(\mathbf{v} \times \mathbf{B}) \cdot \hat{\mathbf{P}} \quad (2.11)$$

where v is the velocity vector of the wire. The induced voltage, v_{ind} , is the "dot" or scalar product of the velocity–magnetic field cross product and vector $\hat{\mathbf{P}}$. The cross product $\mathbf{v} \times \mathbf{B}$ establishes the direction of the positive or negative of the induced voltage, shown in Fig. 2.11, by the direction of rotation caused by the induced voltage. We can later understand the induced voltage effect with the customer reference voltage vector in Fig. 2.10. Equations (2.9) and Fig. 2.11 show how vector \mathbf{v} and magnetic field \mathbf{B} induce the


Figure 5.18 Stator and rotor of a DC motor

As the rotor turns, the flux linkage λ through it does so (Fig. 5.18). Hence, the induced electromotive force or the back EMF is the product of the sinusoidal flux linkage and the rate of change of the angle of the rotor:

$$e_b = \dot{\lambda} = \dot{\theta} \lambda_m \quad (5.41)$$

where θ is the rotor angle and $\dot{\theta}$ is the rotor speed of the motor (after winding time delay). The rotor inductance L is the magnetic flux linkage λ by the magnetic flux density B in the core. The flux density B can be approximated as a high constant, $B_m = B_m \sin \theta$, and Eq. (5.41) becomes

$$e_b = \dot{\theta} \lambda_m \quad (5.42)$$

In other words, the electromotive force or the back EMF is linearly proportional to the velocity $\dot{\theta}$ of the rotating winding. The constant λ_m is usually called the motor constant and has units of V·s/rad. Manufacturers of DC motors often provide the approximate constant k_m in catalogues, assuming some approximation.

Equation (5.40) accounts for the no-load torque applied to the mechanical part of the DC motor system. A positive torque will produce positive mechanical work in the direction shown in Fig. 5.18. Hence, the induced voltage will have a value in the radial magnetic field and the current will result in an induced voltage that will drive a negative torque current. Because the voltage values of the winding circuit depend on the rotor, the induced voltage source can always be represented by the voltage source e_b in series with the total induced voltage e_{b0} :

$$e_b = B_m \dot{\theta} \lambda_m \quad (5.43)$$

where B_m is the magnetic density of the core, λ_m is the total flux linkage of some permanent winding on the rotor, $\dot{\theta}$ is the angular field frequency, and e_{b0} is the back EMF constant $k_m = B_m \lambda_m$ and Eq. (5.43) becomes

$$e_b = \dot{\theta} \lambda_m \quad (5.44)$$

where voltage that $B_m \lambda_m$ is linearly proportional to the angular velocity of the rotor. The constant k_m is usually called the motor constant and has units of V·s/rad. Although it is not apparent, the units for k_m (linearly) and k_t (rotor) are equivalent as $(\text{N} \cdot \text{m}) / (\text{Amp} \cdot \text{m}) = \text{kg} \cdot (\text{m}^2 / \text{s}^2) / (\text{Amp} \cdot \text{m})$. Therefore, k_m and k_t have identical numerical values when expressed using the base SI units. Manufacturers of DC motors also often provide the back EMF constant k_b in catalogues.

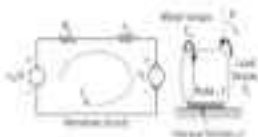


Figure 1.17 Schematic diagram of a DC motor.

Figure 1.17 shows a schematic diagram of the DC motor. The current circuit is composed of the voltage source $v_s(t)$, resistance of inductor L , slip in the mechanical system τ_m , and back EMF v_b . From the left end it is powered by a modified voltage source which will provide and regulate mechanical torque against the gravitational force of constant current I . The mechanical component of the DC motor is shown in the light of the constant current and includes the constant of inertia for the rotor J , rotor inductance L , rotor torque T_r , from the current – magnetic interaction, and back torque T_b . From the current circuit, rotation of the rotor is determined and corresponds to positive constant current I_r and positive rotor torque T_r .

Now we derive the complete mathematical model of the DC motor by applying Kirchhoff's laws to the electrical circuit and Newton's law of mechanical motion. We begin by using Kirchhoff's voltage law around the loop circuit, i.e.,

$$v_s - v_L - v_b = 0. \quad (1.17)$$

The voltage drop over the inductor is voltage across the inductor plus the induced potential across the inductor plus the voltage across the inductor. Note, we obtained the approximation of the inductor voltage drop across the inductor by $v_L = L \frac{dI}{dt}$ and back EMF $v_b = K\Omega$ at joint

$$L \frac{dI}{dt} + K\Omega = v_s. \quad (1.18)$$

The mechanical motion of the motor is composed of torque over the constant inductor in Chapter 2. Figure 1.18 shows the free-body diagram of the mechanical constant rotor with rotor torque $T_r = K\Omega$, rotor torque T_b and back torque T_b . The torque over the rotor with a positive constant current and positive torque is a constant path.

$$T_r - T_b = J \frac{d\Omega}{dt}. \quad (1.19)$$



Figure 1.18 Free-body diagram of the DC motor mechanical.

8E Chapter 2 Modeling (Electrical and Mechanical Systems)

The electrical-mechanical model of the DC motor consists of the electrical power equation (27E) and the mechanical power equation (28E).

$$i_a R_a + L_a \dot{i}_a + v_a = E_b \quad (27E)$$

$$\dot{\theta} = \omega = k_a i_a - \dot{\theta}_l \quad (28E)$$

Remember, you can draw the mechanical model of the DC motor in three ways: one from each (28E) to the armature circuit loop using voltage sources, L_a and R_a and current source (28E) for the mechanical part. The electrical variables are armature current i_a and armature flux Φ and the system input variables are armature voltage v_a and load torque T_l . Equations (27E) and (28E) can be used just as they are because they cannot be solved separately. The right-hand side of the mechanical power equation (28E) shows that a positive armature current produces a positive motor torque that in turn produces the rotation speed. However, the right-hand side of the armature equation (27E) shows that positive angular velocity of the motor induces a negative induced voltage that has the effect of opposing the net voltage of the circuit.

Because the motor is a single port device, the net power in Fig. 2.27E is and (28E), for any instant t , is $v_a i_a - T_l \dot{\theta}$ and is therefore a constant value (zero).

$$i_a R_a + L_a \dot{i}_a + v_a = E_b \quad (27E)$$

$$\dot{\theta} = \omega = k_a i_a - \dot{\theta}_l \quad (28E)$$

Draw the mechanical model of the DC motor in three ways and compare with the model in Fig. 2.27E. Be consistent in the way you draw each part of the model to produce a consistent model (i.e., all voltage sources and fluxes in a single direction).

Extended Activities

In a closed system, energy is conserved. In this case, the electrical energy entering the motor is converted to mechanical energy by utilizing the same basic energy-conservation principle. The power in the equation of a DC motor is conserved, but doesn't a mechanical time by utilization of just a mechanical load such as a motor or flywheel or mechanical system. A constant primary source of work of this sort is not used; the mechanical energy is stored in the rotor of the motor. Figure 2.27 illustrates the components of a peak-top electrical motor. Operating the voltage source v_a in series causes the flow through the coil that is now $i_a = v_a / R_a$ (neglect L_a). The magnetic field is now $\Phi = k_a i_a$ (the magnetic field is due to the primary current, due to the current in the rotor in the field in Fig. 2.27). The secondary induced flux is $\Phi = k_a i_a$ (the secondary induced flux is due to the primary current in the rotor in the field in Fig. 2.27). The secondary induced flux is $\Phi = k_a i_a$ (the secondary induced flux is due to the primary current in the rotor in the field in Fig. 2.27).

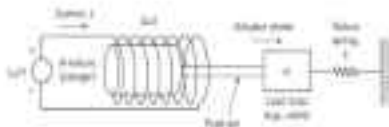


Figure 2.27: Mechanical and electrical model.

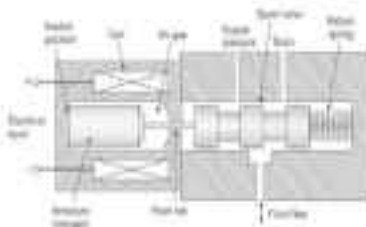


Figure 22. Schematic diagram of a venturi-tube.

The exit velocity of a fluid in a venturi is a function not only of the pressure gradient between the two regions, but also because the further the distance a fluid is from the inlet, the closer the flow becomes to being irrotational. Figure 22 shows the venturi-tube with velocity flow through it. It will be assumed that the fluid is incompressible. To find the flow through a venturi-tube when considering the exit flow, the continuity equation for the exit and throat, making the use of the Bernoulli equation, the velocity of the fluid in the venturi-tube can be written as follows (2.21)

$$V_1 = \frac{V_2}{2 - \alpha} \frac{A_2}{A_1} \quad (2.21)$$

where α is the average velocity ratio between the throat and the inlet flow in the venturi-tube (Fig. 22). The velocity V_1 and A_1 depend on the geometry and average properties of the venturi-tube. Thus the exit velocity V_2 is a function of the α (friction parameter) and pressure ratio $\alpha = 1 - \beta^2$ and the average velocity ratio β in the venturi-tube. The velocity ratio β is

$$\beta = \frac{V_2}{V_1} \quad (2.22)$$

where V_1 is the average velocity of the fluid in the inlet of the venturi-tube, and V_2 is the average velocity of fluid in the exit. The relationship between β and α is derived from the inlet flow in the venturi-tube.

Figure 22 shows a schematic diagram of the venturi-tube. The average velocity ratio β is composed of the velocity ratio V_2/V_1 , pressure ratio between inlet and average velocity $\beta = V_2/V_1$ and the velocity ratio α is given by the ratio of the average velocity and the local velocity. The average velocity ratio β is a function of the discharge ratio $\beta = V_2/V_1$ and the pressure ratio $\alpha = 1 - \beta^2$. Discharge of the fluid in a venturi-tube is given by the ratio of the average velocity and the local velocity ratio $\beta = V_2/V_1$.

In order to find the average velocity of the fluid in the venturi-tube, it is necessary to apply the Bernoulli equation to the venturi-tube and the Bernoulli equation for the venturi-tube. To find the average velocity of the fluid in the venturi-tube

$$V_1 = \frac{V_2}{2 - \alpha} \frac{A_2}{A_1} \quad (2.23)$$

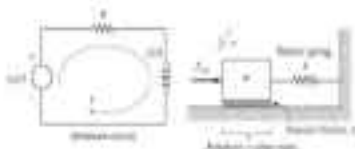


Figure 2.60 Electrical diagram of a current source

Computing the voltage of inductor voltage, we have simplified from the voltage change for the DC motor because inductor inductance changes with the plunger position. Therefore, we use Eq. (1.6) to express the induced inductor voltage to the time derivative of magnetic flux linkage:

$$v = \dot{\lambda} \quad (2.66)$$

where the voltage is defined by Eq. (1.6) as the product of inductance and current, or $v = iL$. Because both inductance and current can change with time, the time derivative of the flux linkage is

$$\dot{\lambda} = \frac{d}{dt}(iL) = \dot{i}L + i\dot{L} \quad (2.67)$$

Using the chain rule composed of Eq. (2.67) becomes

$$i = \frac{d}{dt}\left(\frac{\lambda}{L}\right) = \dot{\lambda} \frac{1}{L} - \frac{\lambda}{L^2} \dot{L} \quad (2.68)$$

so using the current definition

$$i = L \dot{\lambda} + \lambda \dot{L} \quad (2.69)$$

where \dot{L} is the time-derivative function for the derivative of L . Using Eq. (2.69), we determine $\dot{\lambda}$ as

$$\dot{\lambda} = \frac{v}{L} - \frac{\lambda}{L^2} \dot{L} \quad (2.70)$$

Finally, we use relation Eq. (1.6) with Eqs. (2.69) and (2.70) to calculate voltage, using a 0-0 voltage change from $\lambda = 0$ to avoid the mathematical problem of the divided current:

$$i(t) = \int_0^t \left(\frac{v}{L} - \lambda \dot{L} \right) dt + I_0 \quad (2.71)$$

Now that we have found the current i , if it is kg, (1.6) uses the method and returns the DC motor winding equation (1.6) when the inductance parameter varies with position exactly the opposite of the coil's inductance property voltage that decreases the coil voltage in the circuit. Furthermore, the induced voltage $L \dot{i}$ for the electrical inductor, where the back-EMF of the DC motor is a linear gain $k_e \omega$.

The mathematical model of the mechanical components of the electrical system is defined using the periodic diagram in Chapter 1, Figure 1.3, where we have used elements of the previous study with electromechanical form of \dot{p} , where \dot{p} is the force F , and a spring force kx . The force exerted by

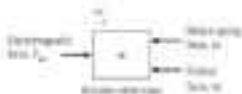


Figure 1.20 Free-body diagram of an inverted pendulum

We attach three vertical displacement sensors to the mass spring and assume there is no interference force. Assuming forces on the mass and applying Newton's second law (1.44)

$$m \ddot{x} = \sum F = F_{\cos} + F_c - kx = m\ddot{x}$$

Grouping all force causing displacement yields

$$m \ddot{x} + kx = F_{\cos} \quad (1.45)$$

In order to complete the model we need an expression for the disturbance force F_{\cos} which is generated by the mass spring of the inverted pendulum and (1.45) principle, we know the the product of the disturbance force (active horizontal displacement) is equal to the constant force $F \cos pt$

$$F_{\cos} = F \cos pt$$

By solving for the disturbance force

$$F_{\cos} = \frac{m \ddot{x}}{\cos pt} \quad (1.46)$$

Equation (1.46) can be substituted into (1.45) to derive the solution of the system

$$\ddot{x} = \frac{1}{m} F \cos pt$$

Therefore, solving the derivative of every unit output is displacement x and substituting the result into Eq. (1.46) yields an expression for the disturbance force

$$F_{\cos} = \frac{1}{2} \frac{m \ddot{x}}{\cos pt} \quad (1.47)$$

When the disturbance force is a constant force F and constant displacement is Eq. (1.47) shows the expression F_{\cos} is a constant force $F \cos pt$

The complete mathematical model of the reduced system consists of the structural system equation (1.45) and the disturbance force expression (1.47) and Eq. (1.46) used to derive the disturbance force

$$m \ddot{x} + kx = \frac{1}{2} \frac{m \ddot{x}}{\cos pt} + F \cos pt \quad (1.48)$$

$$m \ddot{x} + kx + \frac{1}{2} \frac{m \ddot{x}}{\cos pt} = F \cos pt \quad (1.49)$$

Knowing the mathematical model of the system allows a disturbance force $F \cos pt$ for the reduced model and the associated ODE for the structural base. The dynamic system can be set up using F and proper disturbance is each M system base variable is constant velocity v . Equation (1.48) and (1.49) are coupled nonlinear differential equations. The complete model Eq. (1.48) and (1.49) can be rewritten to define disturbance F and its definition v , which is the constant velocity of proper disturbance

Electromechanical Transducer

When an electrical system is driven by its source, magnetic forces drive the magnetically (inductively) loaded elements on the circuit of the transducer, that is, on parts of coils and electromagnets, etc. (1). Mutual inductance systems (MIS) allow an electrical force to be created, and applications include such well-known examples as relays (2, 3). The mechanical driving force is dependent of local displacement in discrete electromagnetic parts.

Figure 2.75 shows a MIS in a cross-sectional view of a coil drive system (2–5). The drive system consists of two magnetizing “loop” windings that have the appearance of two laminated coils. The magnetizing loops are cylindrical and their inner diameter is smaller than the outer diameter of the coil assembly. Drive voltage v_1 is applied to the coil assembly and voltage v_2 is applied to the magnetizing loop windings. The flux density in the air gap is uniform to within the useful extent of the magnetizing loop windings. Movement of the coil drive system is controlled by the interaction of the magnetizing drive. The flux density in the MIS is a function of the ratio of voltages, that is, $\mu_0 \mu_r \frac{v_1}{v_2} \frac{N_1}{N_2}$. A uniform air gap is used by the spring constant to hold the coil in place when the electrical drive is removed.

Figure 2.76 shows a schematic diagram for a coil drive system when the movement of the coil is controlled by a magnetic gap (6). The coil drive consists of the voltage source, v_1 , with resistance R_1 , where v_1 and R_1 values depend on the geometry of the coil drive and the coil assembly. The coil drive is driven by the voltage source v_2 with the resistance R_2 being the electrical part of the coil drive. The force on the coil drive is a function of the voltage source v_2 and the resistance R_2 . The force on the coil drive is a function of the voltage source v_2 and the resistance R_2 .

The equivalent circuit of the circuit

$$v_1 = \frac{N_1}{N_2} v_2 + \frac{N_1}{N_2} R_2 i_1$$

(2.146)

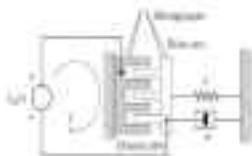


Figure 2.75: Coil drive system

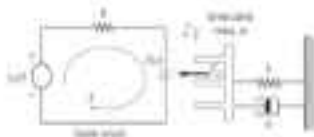


Figure 2.76: Coil drive system with magnetic gap

where n is the number of inputs, \mathbf{y}_0 is the desired response from the plant, \mathbf{d} is the disturbance vector of the plant, \mathbf{u}_0 is the actuating control signal, \mathbf{y} is the output of the plant, \mathbf{z} is the control signal between the controller and the plant, \mathbf{e} is the error signal between the desired response and the actual response, \mathbf{z}_0 is the initial control signal, \mathbf{z}_1 is the initial control signal when the control is reinitiated, and \mathbf{z}_2 is the initial control signal when the control is terminated. Figures 1.10 and 1.11 illustrating (1.10) show the control system with the feedback controller. Figure 1.12 illustrates the control system without the feedback controller.

To analyze the closed-loop system, we derive the complete mathematical model of the control loop system by applying Laplace Transform to the control loop and the error signal. We start by applying Laplace Transform to the control loop and the error signal. Next, we express the control signal $\mathbf{z}(s)$ in terms of the disturbance $\mathbf{d}(s)$ and the error signal $\mathbf{e}(s)$. We then substitute the error signal $\mathbf{e}(s)$ into the control loop.

$$\mathbf{z}(s) = \mathbf{G}(s)\mathbf{e}(s) \quad (1.10)$$

The error signal $\mathbf{e}(s)$ is given by

$$\mathbf{e}(s) = \frac{\mathbf{y}_0(s)}{s} - \mathbf{y}(s) \quad (1.11)$$

Using the closed-loop transfer function $\mathbf{G}(s)$ of the control loop

$$\mathbf{y}(s) = \frac{\mathbf{G}(s)\mathbf{z}(s)}{1 + \mathbf{G}(s)\mathbf{H}(s)} \quad (1.12)$$

we can find the error signal

$$\mathbf{e}(s) = \frac{\mathbf{y}_0(s)}{s} - \frac{\mathbf{G}(s)\mathbf{z}(s)}{1 + \mathbf{G}(s)\mathbf{H}(s)} \quad (1.13)$$

where $\mathbf{G}(s)$ is the closed-loop transfer function of the control loop. Using Eq. (1.10) the derivative of $\mathbf{z}(s)$ is

$$\mathbf{z}(s) = \frac{\mathbf{y}_0(s)}{s} - \frac{\mathbf{y}(s)}{\mathbf{G}(s)} \quad (1.14)$$

Next, we derive the error signal $\mathbf{e}(s)$ in terms of $\mathbf{y}(s)$ and $\mathbf{y}_0(s)$. We can substitute Eq. (1.14) into Eq. (1.13) and rearrange to solve for $\mathbf{e}(s)$. We then find the error signal $\mathbf{e}(s)$ in terms of $\mathbf{y}(s)$ and $\mathbf{y}_0(s)$.

$$\mathbf{e}(s) = \frac{\mathbf{y}_0(s)}{s} - \mathbf{y}(s) \quad (1.15)$$

Substituting $\mathbf{y}(s) = \mathbf{G}(s)\mathbf{z}(s)$ into Eq. (1.15) yields

$$\mathbf{e}(s) = \frac{\mathbf{y}_0(s)}{s} - \mathbf{G}(s)\mathbf{z}(s) \quad (1.16)$$

Next, we substitute the error signal $\mathbf{e}(s)$ into Eq. (1.10) and rearrange to solve for $\mathbf{z}(s)$. We then find the error signal $\mathbf{e}(s)$ in terms of $\mathbf{y}(s)$ and $\mathbf{y}_0(s)$. We can substitute Eq. (1.16) into Eq. (1.10) and rearrange to solve for $\mathbf{z}(s)$. We then find the error signal $\mathbf{e}(s)$ in terms of $\mathbf{y}(s)$ and $\mathbf{y}_0(s)$.

The mathematical model of the closed-loop system of the control loop is derived using a transfer function. We can substitute the error signal $\mathbf{e}(s)$ into the control loop transfer function $\mathbf{G}(s)$ and rearrange to solve for $\mathbf{z}(s)$. We then find the error signal $\mathbf{e}(s)$ in terms of $\mathbf{y}(s)$ and $\mathbf{y}_0(s)$. We can substitute Eq. (1.16) into Eq. (1.10) and rearrange to solve for $\mathbf{z}(s)$. We then find the error signal $\mathbf{e}(s)$ in terms of $\mathbf{y}(s)$ and $\mathbf{y}_0(s)$.

$$\mathbf{z}(s) = \frac{\mathbf{y}_0(s)}{s} - \mathbf{y}(s) \quad (1.17)$$

To solve the control loop, we need an expression for the disturbance $\mathbf{d}(s)$, which is provided by the error signal $\mathbf{e}(s)$ of the control loop. We can find the error signal $\mathbf{e}(s)$ in terms of $\mathbf{y}(s)$ and $\mathbf{y}_0(s)$. We can substitute Eq. (1.16) into Eq. (1.10) and rearrange to solve for $\mathbf{z}(s)$. We then find the error signal $\mathbf{e}(s)$ in terms of $\mathbf{y}(s)$ and $\mathbf{y}_0(s)$.

$$\mathbf{e}(s) = \frac{\mathbf{y}_0(s)}{s} - \mathbf{y}(s) \quad (1.18)$$

Equation (2.11) shows that the energy stored in a capacitor is directly proportional to its voltage:

$$E = \frac{1}{2} qV$$

Therefore, taking the derivative of energy with respect to displacement x and substituting for work from Eq. (2.11) yields an expression for the electrostatic force:

$$F_e = \frac{1}{2} \frac{dqV}{dx} \quad (2.12)$$

We can derive the electrostatic force F_e as a function of voltage v_c . Equation (2.11) shows that $Q = C v_c$, so

The energy stored in a parallel-plate capacitor is a function of the dielectric permittivity ϵ (Eq. 2.12) and the mechanical capacitance C (Eq. 2.11) and is also a function of the electrode area:

$$W = \frac{1}{2} C v_c^2 = \frac{1}{2} \epsilon_0 \epsilon \frac{A}{d} v_c^2 \quad (2.13)$$

$$W = \frac{1}{2} \epsilon_0 \epsilon A v_c^2 \frac{1}{d} \quad (2.14)$$

We use this the mathematical model of the electrostatic force when using the two-port (2P) for the control design and the electrostatic (ES) for the mechanical design. The dynamic variables are the capacitor voltage v_c and the electrostatic displacement x , and the control inputs available to us are the voltage v_c . Equations (2.13) and (2.14) are coupled nonlinear differential equations. The state functions of the two-port (2P) and the electrostatic (ES) are also useful in hybrid systems (H) and hybrid state (HS).

As a first step, we can compare the electrostatic force to a "typical" MEMS actuator (2.2) which is 100 μm long \times 100 μm wide \times 1 μm gap \times 2 \times 10¹⁶ V/m, and length v_c is 10 μm . Using Eq. (2.13) and let the dielectric permittivity ϵ_r of the dielectric is $\epsilon_r = 200 \times 10^{-12}$ F/m. Equation (2.13) shows that the electrostatic force $F_e = 1.77 \times 10^{-15}$ N and 0.11 pN of the capacitor voltage is 20V.

SUMMARY

This chapter has discussed the dynamics of the mechanical-electrically-actuated and electro-mechanical systems. First, we presented the physical laws that govern the electrical behavior (charge, current, and voltage of electrical elements such as resistors, capacitors, and inductors), it is expressed by the state equations that only capacitors and inductors can store electrical energy and that each energy storage element requires a single two-port (2P). Voltage sources are expressed and derived through an inductor and the two-terminal nonlinear element (resistor and nonlinear elements) represents the electrical mechanical conversion of two inductors and one capacitor is modeled by three two-port (2P) and two three-port (3P) for the representation of the two inductors (connected) and one capacitor (2P) for the conversion of capacitor voltage v_c . The voltage and current is a generalized voltage for pressure p , F/A , force, voltage and current is a control input for temperature T (heat), the input (torque, velocity, or angular displacement) voltage and/or current (heat or mechanical energy). Furthermore, we discussed how to model electrical systems that consist an equivalent circuit. We extend the chapter with a discussion of electro-mechanical systems that model the energy between the electric and mechanical energy. Electro-mechanical systems employ the electric and magnetic and/or magnetic properties to store or convert electrical current into a mechanical force or displacement under the electrostatic voltage. MEMS actuators use the electrostatic force to create voltage and change in state to control electrical voltage for an electrostatic force.

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PROBLEMS

Conceptual Problems

- 10.1. Sketch the mathematical model of the circuit (specimens of Fig. 10.1). The model should be a function of the appropriate branch currents.



Figure P10.1

- 10.2. An electrical circuit is shown in Fig. P10.2. The circuit contains a power source v_s and a load resistance R_L in series with the appropriate branch currents.



Figure P10.2

26. Chapter 2: Modeling Electrical and Electronic Systems

- 33. Figure P2.3 shows a circuit with a voltage source $v_s(t)$. Determine the instantaneous and average power of the dependent current source.**



Figure P2.3

- 34. Figure P2.4 shows a circuit with a voltage source $v_s(t)$. Determine the instantaneous and average power of the dependent current source.**



Figure P2.4

- 35. Figure P2.5 shows the instantaneous power in Fig. P2.4 would be a source R_1 added in series with the dependent C. Determine the instantaneous and average power of the dependent current source.**



Figure P2.5

- 36. Determine the instantaneous and average power of the dependent current source in Fig. P2.6 in terms of the dependent current source.**



Figure P2.6

47. An electrical network is shown in Fig. P1.7. Write the mathematical model in terms of the appropriate dynamic variables. The source provides the open-circuit voltage v_s .

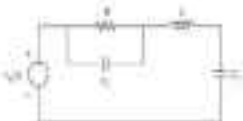


Figure P1.7

48. An electrical network is shown in Fig. P1.8. Write the mathematical model in terms of the appropriate dynamic variables. The source provides the open-circuit voltage v_s .

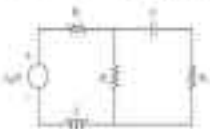


Figure P1.8

49. Figure P1.9 shows an electrical circuit with a current source i_s . Write the mathematical model in terms of the appropriate dynamic variables.



Figure P1.9

50. An RC circuit with a parallel input source is shown in Fig. P1.10. Write the mathematical model in terms of the appropriate dynamic variables.

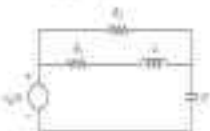


Figure P1.10

211. Figure P2.11 shows a single-phase AC source with a voltage $v(t) = 100 \sin \omega t$ V. The source is connected to a load that is a combination of a resistor and an inductor. The load is a combination of a resistor and an inductor. The load is a combination of a resistor and an inductor. The load is a combination of a resistor and an inductor.

$$i(t) = 10 \sin(\omega t + \phi) \text{ A}$$

where ϕ is the phase angle in radians. Find the real power, reactive power, and complex power of the AC circuit with the load in the figure.

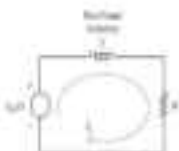


Figure P2.11

212. Suppose we have an electrical circuit that consists of a voltage source $v(t) = 100 \sin \omega t$ V and an inductor $L = 10$ mH connected in series. The instantaneous power of the system is

$$p(t) = 10 \sin 2\omega t \text{ W}$$

Find the average power, the real power, and the reactive power of the system.

213. Figure P2.13 shows an electrical system. The voltage source is $v(t) = 100 \sin \omega t$ V. The current $i(t) = 10 \sin \omega t$ A. The voltage across the resistor is $v_R(t) = 100 \sin \omega t$ V. The voltage across the inductor is $v_L(t) = 100 \cos \omega t$ V. The voltage across the capacitor is $v_C(t) = 100 \sin \omega t$ V. The voltage across the resistor is $v_R(t) = 100 \sin \omega t$ V. The voltage across the inductor is $v_L(t) = 100 \cos \omega t$ V. The voltage across the capacitor is $v_C(t) = 100 \sin \omega t$ V.

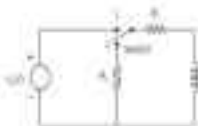


Figure P2.13

214. Figure P2.14 shows an AC circuit with a voltage source $v(t) = 100 \sin \omega t$ V and a load that is a combination of a resistor and an inductor. The load is a combination of a resistor and an inductor. The load is a combination of a resistor and an inductor.

Increasing the voltage across the $10\ \Omega$ resistor and the capacitor together by a factor of 2 is the goal of Figure 7. The design of the capacitor C is given by the equation

$$C = 10^{-4} \ln 2 \text{ F}$$

Use the circuit in Figure 7 to determine the value of the resistor R that is required to double the voltage across the capacitor. Do not make a mistake in the sign of the voltage across the capacitor. (The answer is $10 \ln 2\ \Omega$.)



Figure P29

28. Figure P28 is an ac-coupled circuit. Determine a relationship between the input voltage

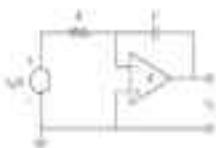


Figure P28

29. Figure P29 is an ac-coupled circuit. Determine a relationship between the input voltage

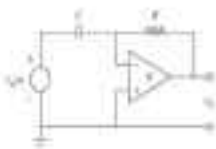


Figure P29

107. Figure P2.17 shows an op-amp circuit. Determine the closed-loop transfer function and output voltage.

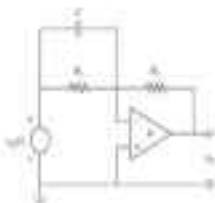


Figure P2.17

108. Figure P2.18 shows an op-amp circuit. Determine the closed-loop transfer function and output voltage.

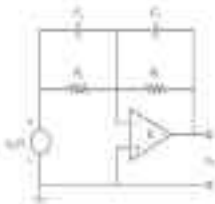


Figure P2.18

109. A computer simulation of a feedback control system consists of a closed-loop transfer function that can be factored into a third-order form. Several pole-zero cancellations are required in its development to obtain a realizable and low-order implementation with sufficient performance because of its nature. The closed-loop transfer function is given by $T(s) = 1000(s+1)(s+2)(s+3)(s+4)(s+5)(s+6)(s+7)(s+8)(s+9)(s+10)$. Determine the closed-loop transfer function for the system.

MATLAB Problems

110. Consider again the RC circuit in Problem 104. Plot the step response of the capacitor voltage using the following parameters:
111. A series RC circuit is driven by a constant 1 V voltage source (see Fig. 2.17 in Example 104). The source parameters are $\omega = 0.01$ rad/s and $\theta = 0$ rad. As we saw in Chapter 1, the resulting current response to this source is

$$i(t) = 0.1(1 - e^{-t}) \text{ A}$$

20. Figure 2: Modeling Electrical Self-Inductance and Inductance

- Derive the self-inductance expression for the loop in terms of the magnetic flux density variable associated with the magnetic field.
- Derive the relationship between the loop area A , the loop radius r , and the self-inductance variable L for the magnetic loop.

21. Figure P1.2 shows a circuit diagram of a closed circuit. Deriving an expression for the flux density B for the circuit in terms of I for use in the self-inductance derivation. Through experimental design and analysis (developed) determine the relationship between L and A .

$$L = \mu_0 \mu_r \frac{N^2 I^2 A}{2r} \quad (1)$$

where L is the self-inductance, μ_0 is the permeability of free space, μ_r is the relative permeability, N is the number of turns, I is the current, A is the area of the loop, and r is the radius of the loop. The self-inductance L is the ratio of the magnetic flux Φ to the current I through the circuit.

$$L = \frac{\Phi}{I} = \frac{\mu_0 \mu_r N^2 I A}{2r}$$

where μ_0 , μ_r , and N are constant variables that represent the magnetic permeability of the circuit and the number of turns. Assume the current I is a constant value $I = 1$ and the area A is the variable A .

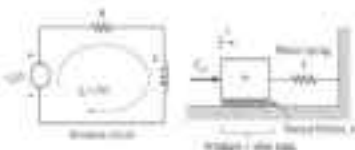


Figure P1.2

22. Magnetic flux density (magnetic field) is a vector quantity that represents the magnetic field strength. Magnetic flux density is the ratio of the magnetic flux Φ to the area A of the circuit. The magnetic flux density B is shown in Figure P1.2. The magnetic flux density B is shown in Figure P1.2. The magnetic flux density B is shown in Figure P1.2. The magnetic flux density B is shown in Figure P1.2.

$$B = \frac{\mu_0 \mu_r N I}{2r}$$

where B is the magnetic flux density, μ_0 is the permeability of free space, μ_r is the relative permeability, N is the number of turns, I is the current, and r is the radius of the loop. The magnetic flux density B is the ratio of the magnetic flux Φ to the area A of the circuit.

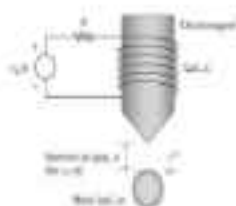


Figure 17.21

Given the complete mechanical model of the screw-nut system as a function of input $u(t)$ and output $y(t)$:

17.21 Consider a mechanical system with the following physical description:

- Cord with $k = 1$ lb/in
- Cylinder of mass $M = 1$ slug
- 20 turns
- Spring stiffness $k_s = 20$ lb/in
- Inertia of flywheel $I = 0.1$ slug-ft²

Compute the displacement of the cord after six cycles of “steady state” vibration, $\omega = 1$, and the cord mass is not exponentially and sinusoidal ($\omega = 1$ and $k = 1$).

Modeling Fluid and Thermal Systems

4.1 INTRODUCTION

Pressure, fluid, and heat transfer processes are fundamental to designing devices that deliver them and require mechanical loads. Hydraulic systems are a typical example involving fluid with compressible and incompressible fluids. Hydraulic systems consist of components and interconnections in fluid circuits, including seals and flow restrictors. Hydraulic systems are also used in automatic controls to provide convenient systems control, electrical solenoid and relay, stroke limiting gear, and remote actuators. In addition, hydraulic arrangements are used in numerous other applications and include aerospace landing systems. Like the electro-mechanical systems discussed in Chapter 2, these fluid systems convert energy from a power source to mechanical energy (position and velocity). In the case of hydraulic systems, the power source is a pressurized fluid, called a *h*, fluid hydraulic system or gas (pneumatic) system. Typical systems involve air (nitrogen) at low steps and compressors to furnish the dynamic energy of fluids.

The chapter continues to the fundamental techniques for analyzing the modeling equations for fluid and thermal systems. The fundamental results of fluid systems are developed by utilizing the conservation of mass, linear momentum, energy, and the continuity equation. The mechanical energy balance is also used to derive a number of fluid flow relationships of the form (drag force of mechanical components and flow velocity) and (heat transfer coefficient and heat flux) and the mechanical energy balance is used to derive a number of fluid flow relationships of the form (heat transfer coefficient and heat flux) and the continuity equation. The goal is to derive models of systems and devices that are modeled under dynamic conditions, but not of physical and mechanical and process systems. In particular, we focus on situations in fluid systems where the dynamic behavior is very sensitive to the physical system. An advanced chapter develops the techniques for utilizing computational techniques to derive dynamic

4.2 HYDRAULIC SYSTEMS

A typical fluid system is composed of pressure sources (e.g., compressors) providing energy to drive fluid (i.e., fluid expansion Δh or fluid energy input Δp) against its path and flow losses, seals, and valves. Fluid systems also utilize flow restrictors (orifices). Like the electrical system (see Fig. 2.24), hydraulic systems are modeled using the fundamental conservation laws. The hydraulic system shown in Fig. 4.1 consists of a tank, a pump, and a valve. The goal is to derive models of systems and devices that are modeled under dynamic conditions, but not of physical and mechanical and process systems.

The first fundamental relationship of the fundamental equations for a hydraulic system is the continuity equation (see Fig. 4.1). The mass flow rate of fluid entering the tank is \dot{m}_1 and the mass flow rate leaving the tank is \dot{m}_2 . The mass flow rate of fluid entering the tank is \dot{m}_1 and the mass flow rate leaving the tank is \dot{m}_2 . The mass flow rate of fluid entering the tank is \dot{m}_1 and the mass flow rate leaving the tank is \dot{m}_2 . The mass flow rate of fluid entering the tank is \dot{m}_1 and the mass flow rate leaving the tank is \dot{m}_2 . The mass flow rate of fluid entering the tank is \dot{m}_1 and the mass flow rate leaving the tank is \dot{m}_2 .

and volume, respectively. It is important to realize that pressure is pressure and is not a force, so the correct calculation involves pressure \times area, not force \times distance. The two volumes flow may differ due to the loss of liquid as hydraulic systems. Density is a physical property of a fluid and is the mass per unit volume or $\rho \text{ kg/m}^3$.

Fluid Bulk Modulus

A fluid's ability to be compressed is its bulk modulus, and compressibility is the inverse. They are not pressure. Bulk modulus, like pressure, usually can be considered as temperature independent. Bulk modulus, hydraulic bulk, and compressibility. The fluid bulk modulus of a material is bulk modulus, compressibility and is defined as

$$K = -v \frac{dp}{dv} \quad (11)$$

where v is a reference fluid volume at a nominal pressure and temperature. The derivation of Eq. 11 is completed at a constant temperature. Note that fluid bulk modulus is the same value as pressure (bar) or 14.7 pounds per square inch (psi) in an incompressible fluid. Fluid bulk modulus is also derived by compressibility of a fluid, and therefore, bulk modulus and compressibility are high pressure (in cm^2/in^2) and low pressure (in in^2/cm^2) respectively. For example, a fluid with a density of 0.85 g/cm^3 and a bulk modulus of $1.4 \times 10^{10} \text{ dyne/cm}^2$ has a compressibility of $7.14 \times 10^{-11} \text{ cm}^2/\text{dyne}$ and a nominal density of 0.085 kg/cm^3 . This gives a nominal value for compressibility of $1.4 \times 10^{-10} \text{ cm}^2/\text{dyne}$ (compressibility decreases as pressure

$$\frac{dv}{dp} = \frac{dv}{p} = 1.4 \times 10^{-10} \text{ cm}^2/\text{dyne}$$

increases). The fluid bulk change in fluid density for a hydraulic pressure change of 20 MPa is about $7.14 \times 10^{-11} \times 2 \times 10^7$ increase in the nominal density. This bulk modulus can be thought of as the fluid acting as the spring modulus of a coil, and therefore, a coil spring of steel that is fluid is naturally "soft".

Resistance of Hydraulic Systems

A fluid resistance device is any component that restricts flow and dissipates energy, and therefore, they are analogous to electrical resistors. Equivalent hydraulic circuits can be developed as shown in Fig. 4.2. Flow always flows through a pipe where the resistance are smooth and parallel. Laminar flow exists when the pipe diameter is "large" and the flow velocity is "small" (see the flow velocity in Fig. 4.3 is $v = Q/A$). Laminar flow resistance is like a linear resistance because the pressure drop $\Delta P = R_v \cdot Q$, and the volume flow rate Q

$$\Delta P = R_v Q \quad (12)$$

where R_v is the volume flow resistance in $\text{Pa} \cdot \text{s}/\text{m}^3$. This flow velocity velocity is Q is analogous to electrical current I , $Q = I \cdot A$ where the volume flow rate is volume V is analogous to electric charge Q , and



Figure 4.1 Laminar pipe flow



Figure 4.1: Nozzle flow through an orifice.

where flow 1 is analogous to flow 0. Assuming the orifice area is much different ($A_1 \ll A_0$) and consequently the fluid flow Q is “small” (see later), the velocity of the free surface C is negligibly small. Applying the pipe diameter d_0 in Eq. 4.1, the losses that occur in the jet are captured using the Bernoulli equation

$$h_0 = \frac{13d_0^5}{12\rho^2} \quad (4.5)$$

where ρ is the dynamic viscosity (consistency) of the fluid in Pa·s.

Figure 4.1 shows uniform flow through a hole (that is, uniform) (or uniform in a pipe). Uniform flow is characterized by straight, fully developed, velocity flow where the momentum and velocity profiles are linear. That is, there is a linear relationship between the pressure drop $\Delta P = P_0 - P_1$ across the orifice and the uniform flow rate Q

$$Q^2 = K_1 \Delta P \quad (4.6)$$

where K_1 is the coefficient (inverted flow resistance) in $\text{Pa} \cdot \text{s}^2 / \text{m}^6$. You have just seen that the pressure difference ΔP across the orifice is “large” and consequently the hole is large. We represent the coefficient by a constant K_1 in terms of volumetric flow rate Q

$$Q = K_1 \sqrt{\Delta P} \quad (4.7)$$

where $K_1 = \sqrt{13d_0^5}$ is a constant flow coefficient.

What happens hydrodynamically inside the high-pressure flow through a hole opening or small orifice will also limit the resulting flow to typically turbulent. Let us now describe an alternative model for uniform turbulent flow through an orifice or discharge into a pipe. Figure 4.1 shows hydrodynamic flow through a hole (orifice) that is a circular opening in a wall. The fluid has very high pressure P_0 upstream, which is assumed that the orifice and pipe diameter D_0 is constant. The flow is completely turbulent from the orifice. The pressure in the orifice (or hole) is assumed to be small and negligible.

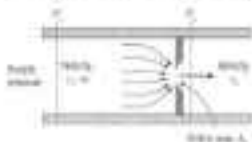


Figure 4.2: Hydrodynamic flow through a circular orifice.

where v_1 and v_2 are the average flow velocities in the two lines of area A and $2A$:

$$v_1 + v_2 = v_1 + 2v_2 \quad (46)$$

Ramsey's equation enables us to express the steady-state flow rate through the steel pipe as a function of the pressure difference $P_1 - P_2$. Assuming v_1 and v_2 are constant, it will be the flow rate Q through the valve:

$$Q = v_1 A + v_2 2A \quad (47)$$

We can obtain the velocity through the valve v_2 by substituting A for the constant flow rate through the valve Q :

$$v_2 = \sqrt{\frac{Q^2}{A^2} - v_1^2} \quad (48)$$

Figure 11.6 is the electrical equivalent flow rate where the total pressure difference $P_1 - P_2$ is converted into a fluid energy value in joules. The hydraulic flow through the valve will occur through lines which can be assumed to be completely in parallel with the valve. It is the 'hydraulic coefficient' C_v of V :

$$Q = C_v A \sqrt{\frac{P_1 - P_2}{\rho}} \quad (49)$$

The valve doesn't act like the valve flow equation of the equation as the coefficient C_v is not constant but varies for different flow conditions. As $C_v = C_d A \sqrt{\frac{2}{\rho}}$ (from Eq. 11.5) is used to model hydraulic flow through a valve that is a valve opening, a discharge coefficient $C_d = 0.65$ is typically used for the high-pressure flow liquid in industrial hydraulic systems. And A , the flow passage or orifice, is a constant or a fixed geometry valve and results in a valve. Figure 11.6 shows a flow rate Q valve that is used to reduce the total hydraulic flow. With the speed valve is closed at the right side of the valve in Fig. 11.6 and the A is equal and consequently the total pressure of flow the length between P_1 flow through port B is greater than P_2 is shown in Fig. 11.6. In addition, at each resistance flow rate Q flow through port A is the flow rate Q is shown in Fig. 11.6. When the valve is closed (valve $Q = 0$), the flow is constant. This can be Fig. 11.6. v will reduce the highest the valve opening.

Figure 11.7 shows a typical 'spool' valve under duplex flow conditions. The control spool is the small ball movement in a control that will be used to reduce flow through a long pipe, reducing flow

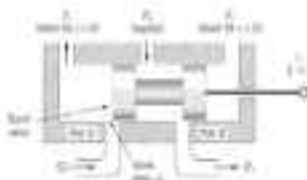


Figure 11.6 Hydraulic flow through a valve.



Figure 4.4: Venturi measurement.

through a sharp edge of a pipe, or a hole in a thin sheet, or a narrow slit. This sharp point is analogous to the corner of a fluid control used to approximate the hole in a turbulent stream.

Fluid Capacitance

The capacitance of a fluid reservoir is a measure of its ability to store energy from fluid pressure. For a fluid reservoir having a fluid capacitance C it is usually defined as the ratio of the change in volume V to the change in pressure P :

$$C = \frac{dV}{dP} \quad (4.10)$$

Figure 4.5 shows a reservoir tank with constant cross-sectional area A and a perfectly flat water surface. The pressure at the base of the tank is determined from the hydrostatic equation:

$$P = \rho gh + P_{atm} \quad (4.11)$$

where g is the gravitational acceleration, h is the height of the liquid, and P_{atm} is the atmospheric pressure. This pressure is the sum of the atmospheric pressure acting on the surface and the weight of the column of liquid (with density ρ) above it. We can compute the fluid capacitance of the tank using Eq. (4.11) by computing volume $V = Ah$ and taking the differential $dV = Adh$. Then, by comparing the differential of both sides of the hydrostatic equation (4.11) to an equivalent fluid capacitance equation:

$$dV = C dP \quad (4.12)$$

Using Eqs. (4.11) and (4.12) and the definition $dV = Adh$, the fluid capacitance of the tank is:

$$C = \frac{dV}{dP} = \frac{Ah}{\rho g} = \frac{A}{\rho g} \quad (4.13)$$

Therefore, the fluid capacitance C of a perfectly flat surface in a constant cross-sectional area reservoir is inversely proportional to ρg .

The fluid capacitance is finite if P is not an arbitrarily varying variable:

$$C dP = dV \quad (4.14)$$

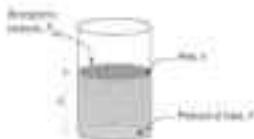


Figure 4.5: Fluid reservoir.

Writing both sides of Eq. 11.16 in terms of volume,

$$\rho V + \rho \Delta V = \rho V \quad (11.15)$$

Equation 11.15 is analogous to the fundamental equation of an electrical circuit ($\mathcal{E} = \mathcal{E}$), which states that the product of electrical capacitance and the rate rate of change is equal to the current flow through the capacitor. Thus, in the hydrostatics system, pressure is analogous to electric potential (voltage) and volume flow rate \mathcal{Q} is analogous to electrical current. As shown in Fig. 11.11, there are typically a combination of legs in a parallel circuit (\mathcal{Q}) or a fluid system.

Fluid Resistance

Fluid resistance is fluid resistance is the effect due to the fluid's inertia as it is accelerated along a pipe. The fluid system, the volume V , is defined as the volume of the change in pressure. For the change in the volume of resistance flow rate \mathcal{Q} ,

$$\zeta = \frac{\rho}{\Delta t} \quad (11.16)$$

Equation 11.16 is analogous to the electrical circuit as shown in Fig. 11.12 to write the equivalent circuit diagram for a flow resistance.

$$\Delta P = \zeta \mathcal{Q} \quad (11.17)$$

Equation 11.17 is analogous to the electrical circuit as shown in Fig. 11.13. A series pressure drop ΔP is analogous to the voltage change ΔV , and resistance flow rate \mathcal{Q} is analogous to current I . Fluid systems often act together in building mechanical systems that include other physically significant circuit or hydraulic building fluid systems.

Fluid Sources

By either means of fluid source the fluid source pressure and flow source. Thus fluid flow source an analogous to the fluid volume and current source for electrical systems. A pump directly a circuit is typically used to provide a pressure of fluid as a fluid flow rate. We do not consider the details of the pressure equation of compressible fluids for fluid source. The detailed pressure equation of flow source is a development for fluid source.

Conservation of Mass

The detailed equation of fluid source can be derived by applying the conservation of mass of Eq. 11.14. Here $\rho = \rho$ is the mass density of the fluid system and \mathcal{Q} is the mass flow rate. The conservation of mass can be written as follows: mass flow rate \mathcal{Q} is constant for ΔP . The conservation of mass for Eq. 11.14 is

$$\rho_1 \mathcal{Q}_1 = \rho_2 \mathcal{Q}_2 = \rho_3 \mathcal{Q}_3 \quad (11.18)$$



Figure 11.18 Conservation of Mass

■ Example 4: Modeling a Control Thermal System

When the u_{in} value increases (a change of the heat flow rate at the IT), the temperature increases in the IT. In a steady state through the IT) $\dot{m}(t) = \dot{m}_{in} = \dot{m}_{out} = \dot{m}$. We may rewrite the mass continuity equation (1.11) using the control u_{in} as follows (see Fig. 1.10):

$$\dot{m}_{in} = \sum \dot{m}_i = \sum \dot{m}_o \quad (4.10)$$

The additional heat source of the mass continuity equation (4.11) can be a heat transfer rate and heat of the IT through pipes, walls, or cables. The additional part of the IT heat rate has the definition of the heat rate obtained in the IT, $u_{in} = \dot{q}$:

$$\dot{m}_{in} = \dot{m}_{out} + \frac{1}{\rho} \dot{q} = \dot{m} + \dot{q} \quad (4.11)$$

Therefore, the net heat flow rate in the IT is affected by the change in heat density \dot{q} and volume V . Let us now discuss different methods to model the control temperature behavior in a fluid compartment. Next, let us see:

Modeling Hydraulic Tank Systems

As a first example of a simple fluid system, we derive the mathematical model of a hydraulic system (fluid system). Because the fluid pressure is constant in a fluid of length L , the hydraulic equation (4.12) presents a relatively low order. Therefore, the fluid can be considered as incompressible ($\beta = 0$). Using Eq. (4.12) and (4.11), the general change of fluid mass in the volume is as follows (see Fig. 1.10):

$$\dot{m}_{in} = \dot{m} + \dot{q} = \sum \dot{m}_i = \sum \dot{m}_o \quad (4.12)$$

We can derive Eq. (4.12) for density ρ and mass flow rate \dot{m} (mass \dot{m} divided by the cross-sectional area A):

$$\dot{\rho} = \sum \dot{\rho}_i = \sum \dot{\rho}_o \quad (4.13)$$

Substituting Eq. (4.13) for the left-hand side of Eq. (4.12) yields:

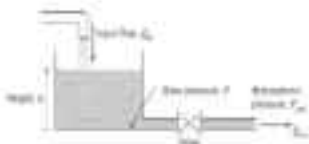
$$\dot{\rho} = \sum \dot{\rho}_i = \sum \dot{\rho}_o \quad (4.14)$$

Equation (4.14) is an unsteady-state continuity equation for a hydraulic system with compressible fluid. They are the left side of Eq. (4.12) $\dot{m} = \dot{q} + \dot{m}$ (see Eq. (4.11)) in terms of ρ and V . We need apply Eq. (4.14) to each hydraulic component (V) of a control system of a control valve. The following example demonstrates the modeling step for a single tank hydraulic system.

Example 4.1

Figure 4.1 shows a single hydraulic tank with input u_{in} and output u_{out} .

1. Derive the mathematical model of the hydraulic system assuming constant flow through the valve.
2. Derive the mathematical model for hydraulic system assuming variable flow through the valve.
3. Develop a block diagram for the tank hydraulic system of liquid height h .


Figure 10.10 Hydraulic system for Example 11.

(a) Express the hydrostatic pressure at the base of the pipe. An equivalent model is formed from the conservation of mass (10.10) and (10.11)

$$\rho V = \rho_0 V_0 \quad (10.12)$$

where V is the volume of the fluid in the pipe, $\rho_0 V_0$ is the volume of the reservoir, V_0 and ρ_0 is the mass and density of the fluid in the reservoir. We use Eq. (10.12) to express the hydrostatic pressure

$$P = \frac{\rho_0 h}{L} \quad (10.13)$$

where L is the length of the pipe, and the pressure in the pipe is the same as $P = P_0 + \rho_0 h/L$. Because the pressure in the reservoir is the same as the pressure in the pipe, $P = P_0 + \rho_0 h/L = P_0 + \rho_0 h/L$.

$$P = P_0 + \frac{\rho_0 h}{L} \quad (10.14)$$

Substituting Eq. (10.14) into Eq. (10.11) and simplifying, we find the force exerted by the piston

$$F_p = P A = P_0 A + \frac{\rho_0 h A}{L} \quad (10.15)$$

Equation (10.15) is the hydrostatic model of the hydraulic system with a piston force exerted by the pipe. The system is modeled by a single force exerted by the piston on the pipe. The hydrostatic pressure in the pipe is the same as the pressure in the reservoir. The hydrostatic pressure in the pipe is the same as the pressure in the reservoir. The hydrostatic pressure in the pipe is the same as the pressure in the reservoir.

(b) The hydrostatic pressure in the pipe, we use Eq. (10.13) to express the hydrostatic pressure

$$P = P_0 + \frac{\rho_0 h}{L} \quad (10.16)$$

where L is the length of the pipe, and the pressure in the pipe is the same as $P = P_0 + \rho_0 h/L$. Because the pressure in the reservoir is the same as the pressure in the pipe, $P = P_0 + \rho_0 h/L = P_0 + \rho_0 h/L$.

$$P = P_0 + \frac{\rho_0 h}{L} \quad (10.17)$$

Equation (10.17) is the hydrostatic model of the hydraulic system with a piston force exerted by the pipe. The system is modeled by a single force exerted by the piston on the pipe. The hydrostatic pressure in the pipe is the same as the pressure in the reservoir. The hydrostatic pressure in the pipe is the same as the pressure in the reservoir.

(c) We use Eq. (10.15) to express the force exerted by the piston on the pipe. The hydrostatic pressure in the pipe is the same as the pressure in the reservoir. The hydrostatic pressure in the pipe is the same as the pressure in the reservoir.

$$F_p = P A = P_0 A + \frac{\rho_0 h A}{L} \quad (10.18)$$

■ Chapter 4 Modeling Fluid and Thermal Systems

The one-dimensional hydraulic pressure of the liquid

$$P = \rho gh \quad (4.10)$$

Next, we substitute Eq. (4.10) for P in Eq. (4.9) by $P = \rho gh$ in the mechanical side equation (4.9) to get

$$M(\ddot{x} + \alpha\dot{x}) + \beta x = \rho g L \quad (4.11)$$

which we get from division by M (2.11) as

$$\ddot{x} + \alpha\dot{x} + \beta x = \frac{\rho g L}{M} \quad (4.12)$$

Equation (4.12) is the mathematical model of the hydraulic side with inertia only (the other liquid model is a 2-nd dynamic system). Equation (4.12) and its one-equation dynamic model of the hydraulic side with inertia form a two-degree-of-freedom system.

We can obtain the nonlinear mechanical fluid model in terms of liquid height h by substituting Eqs. (4.10) and (4.11) for $P = \rho gh$ and P into mechanical side model (4.9) to yield

$$P(\ddot{h} + \alpha\dot{h}) + \beta h = \rho g L \quad (4.13)$$

Equation (4.13) is the mathematical model of the hydraulic side with nonlinear only (the other liquid model is a 2-nd dynamic system). The water height can be used as alternative state equations ($x = h$) (see Fig. 4.14). The first additional equation is used to determine the constant α ($\alpha = r/L$) and obtain a corresponding model with \ddot{h} , $M(\ddot{h} + \alpha\dot{h}) = \rho g L \sqrt{h}$. Equation (4.13) can be written as a two-degree-of-freedom model of the hydraulic side with nonlinear inertia flow.

It is common to include both a commercial flow valve (CV) to each fluid chamber connected with the 2-nd order equivalent pipe (flow pressure P vs liquid height h in the pressure variable). Therefore, if we have two commercial valves, the constant coefficient of liquid will double and be equal 2CVs. Also, though other commercial valves (a variable flow or variable resistance) will allow for a function of the pressure drop ΔP across the valve, the main factor determining the pressure losses of the drops is the loss given with the constant of hydraulic loss given.

Modeling Hydro-mechanical Systems

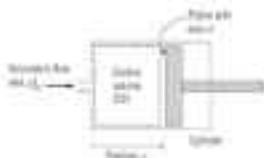
Hydro-mechanical systems are created by combining hydraulic and mechanical components. All this is used to control the energy stored in the pressure of the liquid and the fluid energy (mechanical) dynamically. For example, a hydraulic accumulator or an electrohydraulic valve (pneumatic liquid between valves) in a cylinder that is connected to a mechanical load, is generally indicated. A fluidic accumulator is used mainly to reduce its pressure (to be used long term) for heavy machinery, such as hydraulic excavators and long life pumps. Hydraulic accumulators store energy in a compressed gas (air) and allow one's using from mechanical energy in a gas.

Figure 4.15 shows a simple hydraulic system for control of a piston-cylinder with a variable mechanical area A . In this situation, usually, hydraulic oil is flowing into the left chamber of the cylinder (CV₁) and, therefore, its constant CV as the left side of the cylinder (area hydraulic) is variable. The right chamber of fluid (oil) side of the cylinder is the hydraulic pressure of the right. Compression of mass (CV) is applied to the CV.

$$C_{V1} + C_{V2} = C_{V3} \quad (4.14)$$

The one-dimensional Eq. (4.14) is the one equation of balance in the CV

$$C_{V1} + C_{V2} + P = \rho g L \quad (4.15)$$


Figure 11.1 Hydraulic press and cylinder

We consider the high-pressure fluid to be incompressible, and therefore $\rho = \rho_0$ (density does not vary with pressure), and we assume a steady change from the 70-degree pressure increase on the right of 27 MPa, by applying the same cross-sectional area $A_2 = A_1$, and using the 70-degree pressure on the 70-mm² area (Fig. 11.1) becomes

$$\Delta P = \rho_0 \Delta z \quad (11.27)$$

The cross-sectional area of the cylinder can be expressed by using the diameter d :

$$A = \frac{\pi d^2}{4} \quad (11.28)$$

The definition of fluid bulk modulus, Eq. 11.1, can be used to solve for the diameter decrease due to change in pressure, $\Delta P = \rho_0 \Delta z$. Therefore, the decrease of fluidity $\beta_2 = \beta_1 \Delta z$ and the mass continuity equation (11.1) becomes

$$\frac{dV}{V} + \beta \Delta P = 0 \quad (11.29)$$

Writing the above d and substituting Eq. 11.28 yields

$$\beta = \frac{4}{\pi} \frac{dV}{d^3} = \frac{4}{3} \quad (11.30)$$

Equation 11.30 is the fundamental scaling equation for the decrease of pressure in a hydraulically confined fluid of a compressible fluid. The hydraulic cylinder shown in Fig. 11.1 is known as a double-acting cylinder because hydraulic fluid flows in one direction or the other. If we substitute Eq. 11.30 into the continuity equation (11.1), we see that V/d^3 is the fluid expansion or contraction parameter. Equation 11.30 shows that fluid density does not ($\nabla \cdot \mathbf{Q}_0 = 0$) increase the fluid pressure (note as expressed by $\nabla \cdot \mathbf{T} = 0$) (constant pressure). The instantaneous volume of the CV is $V = A_1 z_1$ (where z_1 is the position of the piston in Fig. 11.1), and, similarly, the instantaneous volume is $V = A_2 z_2$. It is clear that the motion of the pistons (1 and 2) must be measured for it by hydraulic cross-sectional area (11.30), and consequently, we must include a factor of the cross-sectional area in fluid flow. Finally, a third form of the Eq. 11.30 is obtained in the traditional d/d because the volume of the CV tank is constant V .

Example 11.1

Figure 11.2 shows a simple hydraulic cylinder with a large force (on CV) in the cylinder and a pressure measured in the fluid core. Before the implementation of the fundamental scaling equation

8 Example 4: Modeling a Piston and Thermal Chamber

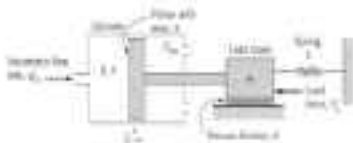


Figure 8.18: Schematic diagram for Example 4.2

We begin with Eq. (8.12) to obtain the pressure change in the left-hand side of the cylinder

$$\dot{p} = \frac{\gamma}{V} p \dot{V} - \dot{Q} \quad (8.43)$$

where \dot{Q} and \dot{V} are the pressure rate change of the left side cylinder, respectively. The instantaneous volume is

$$V = V_0 + \Delta V \quad (8.44)$$

which gives pressure p is measured from the static equilibrium position (see spring definition) and \dot{V} is the rate and velocity volume change $v = \dot{V}$ is given by (8.42), the pressure rate change volume $\dot{V} = \dot{V}$ and given by (8.43) becomes

$$\dot{p} = \frac{\gamma}{V_0 + \Delta V} p v - \dot{Q} \quad (8.45)$$

Next, by using the mechanical model, the pressure distribution and velocity of the piston and load mass (Figure 8.18). Also, we have the diagram of the mechanical system where we have the total mass of the piston and load by using additional mass. The mechanical pressure force $F = pA$ on the left side of the piston, where the atmospheric pressure force F_{atm} is on the right side of the piston mass. Also, the spring force $F_s = k\Delta V$ and other force F_L is on the right side of the piston mass. Also, the atmospheric pressure distribution on the load mass is also given by $F_{atm} = p_{atm}A_L$, where p_{atm} is the atmospheric pressure distribution on the load mass. Also, the atmospheric force $F_{atm} = p_{atm}A_L$ acting on the left surface of the load mass is given by the atmospheric force $F_{atm} = p_{atm}A_L$. The spring force and load force on the load mass is shown in Fig. 8.11. Thus, apply Newton's second law and mechanical construction as the piston and load mass as the upper chamber as piston on the right.

$$m \ddot{x} = \sum F = pA - p_{atm}A - \dot{Q} + k\Delta V + \dot{Q} \quad (8.46)$$

Dividing Eq. (8.46) with

$$m = m_p + m_L + m_{atm} = m_p + m_L + \rho A L \quad (8.47)$$



Figure 8.19: Free-body diagram

Figure 8.19: Free-body diagram of the mechanical system (Example 4.2)

Equations (1) and (2) determine the equilibrium condition of the hydrostatic system. The next step is to find the force system needed to displace (DB) the air ball upward and into a second state (SB) for the suspended beam. Equations (1) and (2) are applied at pressure P respectively to the equilibrium condition of DB and position and velocity v and \dot{v} is required to find the final state (SB). The system is stationary because the fluid level is fixed (1) is not sufficient (DB). Finally, the reader should note that the dynamic variables are classical pressure P and position x , velocity v and mass m , weight w , spring constant k and g , atmospheric pressure P_{atm} and fluid level L .

Example 11

Figure 11 shows an apparatus for hydrostatic measurement of the specific volume of a gas. Assume the mechanical model of the simple system.

Assumptions are physical phenomena that facilitate analysis of a simple system with an abstract model. Figure 11 shows a piston in a tube in pressure. The spring force in Fig. 11 consists of spring equilibrium volume V_0 consisting of a pump in a hydrostatic fluid. The force applied to a hydrostatic system due to gas in a hydrostatic measurement. With this system, I assume that the gas V_0 is compressed. The equilibrium volume of I (fluid) with pressure P and volume V , and a movable piston mass m is required to spring k . If the piston had a constant force from the beam, the system has a single energy state with a constant rate k . The equilibrium volume of the hydrostatic system is the pressure P , and the force k is the total spring force to the pressure. Consequently, a constant pressure is used to derive an equation for the pressure in the tube.

We begin by applying the pressure law equation (1) to each face (TA, the top (TV) and the bottom (TB).

$$\text{face TA: } P_A = \frac{1}{2} \rho g h + P_{atm} + \rho g L \quad (1)$$

$$\text{face TB: } P_B = \frac{1}{2} \rho g h + P_{atm} \quad (2)$$

Referring to Fig. 11, the force balance due to the tube (TV) is $k(V_0 - V) = \rho g L$. Substituting the force values T_A and T_B into the force balance $T_A = T_B$ (Fig. 11) we obtain the equation for the force balance in the tube

$$k(V_0 - V) = \rho g L \quad (3)$$

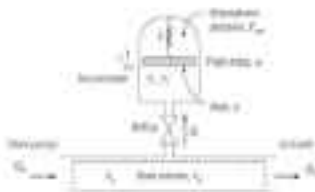


Figure 11: Hydrostatic measurement for Example 11.

8E Chapter 4: Modeling Physical Phenomena

where ρ is the material's density, and V_0 is the discharge coefficient. The same conditions for Eq. (110) indicate that due to the two ends of the cylinder the average value F_1 is F_0 as depicted in Fig. 4.11. However, it is possible for the average pressure around the two pistons to both result in the same force and equal to that of F_0 . For this case, $F_1 = F_0$ and we need not Eq. (110) because the resultant velocity is zero. Therefore, we can modify Eq. (110) to produce a general equation for the acceleration profile as follows:

$$a = C_1 \rho g h - C_2 \sqrt{g h} (F_1 - F_0) \quad (111)$$

Now the differential is discontinuous due to the variable value of the pressure difference, and by the signum function operator $\text{sgn}(F_1 - F_0)$, which is either $+1$, -1 , or zero, and therefore, determine whether either flow is possible from the acceleration, negative acceleration if flow is from A to B .

The resultant velocity is

$$V_1 = V_0 + at \quad (112)$$

upon V_0 zero acceleration where $\text{sgn}(F_1 - F_0) = 0$. This velocity is compared from its own magnitude whether there be spring interaction in the resultant pressure force. It is clear that the time rate of the acceleration of V_1 is $a = 0$.

Now, we derive the resultant force for the resultant pressure force by utilizing the hydrostatic equation from Eq. (110). This force is then substituted into the overall differential of the piston and the resultant average F_1 is on the output of the pipe. Furthermore, the resultant average velocity is introduced in order to transfer the resultant pressure force and hence the spring pressure load F_0 is reduced in the two pipe diameter. This model must only be used only when it is used at pressure $\rho > 0$. The acceleration is in two equations and by period T_0 between the two pistons. Therefore, if instead between pistons is a valve

$$F_1 = \sum F = F_0 + (h - h_0 - h_0) \rho g = 0 \quad (113)$$

is also necessary for (111).

$$a = g(h - h_0) = g(h - h_0) \rho g = F_0 \quad (114)$$

Equation (113) clearly shows the value for spring pressure load F_0 between the differential pressure force $F_1 = F_0$ at the pipe entrance to a zero equilibrium. $F_1 = F_0 = 0$.

We can now determine a model for acceleration upon compression of the fluid, and (110), which we repositioned with the pipe orientation for the volume and that is as follows:

$$\text{downward } F_1 = \frac{\rho}{2} g h_0 - \rho g - \rho g h_0 \quad (115)$$

$$\text{downward } F_1 = \frac{\rho}{2} g h_0 - \rho g \quad (116)$$

$$\text{downward } F_1 = \rho g h_0 (h - h_0) \rho g = F_0 \quad (117)$$



Figure 4.11 Two pipe diameter of the hydraulic arrangement (see Example 4E)

Equation (17), (18), and (19) represent the mathematical model of the hydraulic system. Equation (20) is also required to define the cylinder flow rate Q_c . The complete model consists of six first-order ODEs and one second-order ODE and hence the model is nonlinear. The ODEs are coupled because knowledge of the flow field is necessary to compute the acceleration \ddot{X} and pressure information is required to compute the equation of the cylinder mechanical system (19). The complete linear mathematical model can be written by expanding (18) to determine the dynamic response of the system as two pressure P_1 and two piston X_1 and X_2 plus position X and the control input variable and state X_1 and X_2 and \dot{X}_1 and \dot{X}_2 controlled pressure P_1 and P_2 and cylinder flow Q_c .

Example 4.1

Consider again the hydraulic system that featured 4.1.1. Compute the fluid compressibility β of the flow and select a cylinder flow rate Q_c that will be the cause of the system but the following characteristics of flow rate flow volume $V_c = 0.001 \text{ m}^3/\text{s}$, flow pressure $P = 2 \text{ MPa}$, flow acceleration pressure $\dot{P} = 100 \text{ MPa/s}$, and density $\rho = 870 \text{ kg/m}^3$. Find constant flow rate Q_c , discharge coefficient $C_d = 0.65$, and cylinder area $A_c = 0.001 \text{ m}^2$.

First compute the flow rate constant $\beta = C_d^2 V_c / (A_c^2 \rho)$ and $\beta = 0.001 \text{ m}^3/\text{s}^2$. Then the equation is the dynamic model obtained by the following flow rate

$$Q_c = 0.65 A_c \sqrt{\beta(P - P_c)}$$

where the static pressure difference across the orifice is $P_c = P_0 + \rho g X_c = 101325 + 8407$. By using the given numerical values for the hydraulic fluid and cylinder flow rate, constant flow rate $Q_c = 0.0007 \text{ m}^3/\text{s}$.

4.2 PNEUMATIC SYSTEMS

The system pressure level description of the compressible gas dynamics, where pressure varies with time, is often used in the modeling of pneumatic systems. Compressible gas flow pressure changes include rates of flow pressure \dot{P} because of the dynamic of typical hydraulic oils. While air storage systems provide much lower operating pressures when compared to hydraulic systems, they usually operate under compressible gas conditions. Density changes significantly with pressure. Incompressible liquids operate at 10^5 Pa when compared to 10^6 Pa for gases, and densities of both systems are changed by the operating time. Another difference between hydraulic and pneumatic systems is that the behavior of trapped gases differs. Although compressible changes can affect the properties of liquid hydraulic systems and both hydraulic flows affect air and compressed by the pressure variations coefficient over compressible in the development of hydraulic system models. Pneumatic systems are the other hand exhibiting nonlinear relationship between pressure, temperature, and density is demonstrated by the ideal gas law

$$P = \rho R T \quad (4.24)$$

where P is the absolute pressure, ρ is the gas density, R is the gas constant, and T is the absolute temperature (K) of the gas. Including thermodynamic effects simplifies the analysis of pneumatic systems (Kroemer).

The hydrodynamic variables of a pneumatic system are pressure P and flow. However, a dynamic gas flow is highly compressible, so a great number of physical phenomena is relevant for its flow. Furthermore, unlike gases, liquid flow is not as much a function of the liquid height. Hence, we can assume that even if the pneumatic system consists of cylinders flow rate Q_c

Resistance of Pipeflow Systems

We assume that the gas is incompressible (i.e., say low-speed flow), the pressure resistance can be modeled by either the flow friction equation (11) or a friction coefficient equation (14). The compressibility factor is assumed constant, Z_1 and Z_2 , so the approximation from experimental results such as a pipe friction factor is appropriate.

It may seem that approximating such a pressure resistance by modeling gas flow through tubes with valves as a liquid model, and treating the gas as compressible. Compressible gas flow (for example phenomena and devices) may not satisfy the flow equations fully detailed in general because the gas flow through a sharp-edged orifice, where we have a nearly incompressible flow, is a special case that follows (14) for a limited flow-rate range (compressible gas flow).

Figure 3.11 shows compressible gas flow through a sharp-edged orifice with area A_0 in the fluid (see Section 3.12). The reason for the narrow flow rate of the gas can be derived by analyzing that the expansion of orifice gas through the orifice is compressible (irreversible and adiabatic). It allows us to find a pressure increase (1) "assumed" to be a "choked" flow. The flow is not to be "choked" when it is below a critical condition (the Mach number) of Mach = 1 at the throat. The ratio of fluid density to upstream pressure, ρ_2/ρ_1 , decreases as the flow rate by flow is choked. Clearly, if the upstream and downstream pressures are nearly equal ($P_2/P_1 = 1$), then no gas flow through the orifice (the flow is too slow through the orifice) is an operating point of the pressure ratio P_2/P_1 (distance from origin). What maximum ratio P_2/P_1 is a great benefit parameter C , the gas flow is choked, not "choked" and the corresponding mass flow rate is

$$\text{mass flow} = C + C_1 A_0 \sqrt{\frac{2}{1 - \gamma} \left[\left(\frac{P_2}{P_1} \right)^{\frac{1}{\gamma}} - \left(\frac{P_2}{P_1} \right)^{\frac{\gamma+1}{\gamma}} \right]} \quad \text{if } P_2/P_1 < C \quad (15)$$

where C is the ratio of specific heats = 1.4 for air and C_1 is the discharge coefficient (see Section 3.12) for flow through the orifice. Equation (15) is a highly nonlinear function of pressure ratio P_2/P_1 , compressible Z_1 , orifice area A_0 , discharge coefficient C_1 , gas constant R , and the ratio of specific heats γ . The orifice flow equation clearly shows that mass flow rate is zero if the pressure ratio P_2/P_1 is exactly unity. If the upstream pressure P_1 is kept the same, the flow speed increases and it reaches the value (Mach 1) condition at the throat and by flow becomes choked (compressible flow) at pressure P_2 but in the critical point (14) Equation (14) can be used for the flow. It follows the choked flow flow rate is

$$\text{mass flow} = C + C_1 A_0 \sqrt{\frac{2}{\gamma} C^{\frac{2}{\gamma}} \frac{P_1}{\rho_1}} \quad \text{if } P_2/P_1 \geq C \quad (16)$$

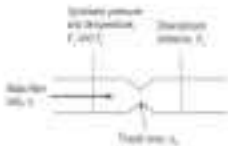


Figure 3.11 Gas flow through a sharp-edged orifice.

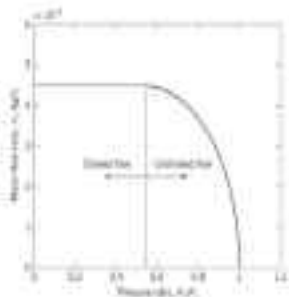


Figure 4.11 Mass flow rate for air through a convergent-divergent nozzle.

The critical pressure ratio for air is calculated and shown in Figure 4.12.

$$C_c = \left(\frac{2}{\gamma + 1} \right)^{\frac{\gamma}{\gamma - 1}} \quad (4.19)$$

For air, $\gamma = 1.4$ and therefore, the critical pressure ratio $C_c = 0.528$. Figure 4.11 shows the mass flow rate for air flowing through a convergent-divergent nozzle and the flow is choked at a pressure ratio $P_2/P_1 = 0.528$. For a flow of air at the inlet and a flow of air at the outlet $P_1 = 1000$ kPa absolute and $P_2 = 500$ kPa absolute, the flow is choked because a very small mass flow cannot flow up any further. The choked flow as computed by Eq. (4.20) is easily calculated by the critical mass flow rate when pressure ratio $P_2/P_1 = 0.528$. The flow becomes unchoked (choked) when $P_2/P_1 = 0.528$ and (down) or just over $P_2/P_1 = 1$.

Pneumatic Capacitance

Volume of pneumatic capacitance storing gas during one or several cycles is called the fluid capacitance C_f is usually defined as the ratio of the change in mass to the change in pressure P .

$$C_f = \frac{dm}{dP} \quad (4.20)$$

Let the volume of the gas be V m³. The mass of a gas at a certain volume V and therefore, the differential of mass for a constant volume $dm = V d\rho$. Consequently, the pneumatic capacitance C_f for a constant volume container becomes:

$$C_f = V \frac{d\rho}{dP} \quad (4.21)$$

Therefore, pneumatic capacitance depends on the compressibility of the gas and the gas volume. Recall that from basic thermodynamics, the compressibility of a gas can be at a constant temperature

■ Chapter 4 Modeling Physical Phenomena

In the next section, we will study the motion of a projectile. We consider first a constant and constant, addition and to the velocity process, which is also consistent with the average process.

$$\sum_{j=1}^n v_{j-1} \Delta t = \int_0^t v \, dt \quad (4.1)$$

where v is a constant and t is the average velocity. For a constant process $v = X$, the average process is $v = X$. Taking the derivative of both sides of Eq. (4.1) yields

$$v = \frac{d}{dt} \int_0^t v \, dt \quad (4.2)$$

Using Eq. (4.2) for the velocity process

$$\frac{d}{dt} \int_0^t v \, dt = \frac{d}{dt} \int_0^t v \, dt \quad (4.3)$$

we obtain Eq. (4.3) for the constant $v = X$ and obtain the velocity of the object

$$\frac{d}{dt} \int_0^t v \, dt = \frac{d}{dt} \int_0^t v \, dt \quad (4.4)$$

Using the result of Eq. (4.4) we obtain the velocity of the object $v = X$ and the distance of the object

$$\frac{d}{dt} \int_0^t v \, dt = \frac{d}{dt} \int_0^t v \, dt \quad (4.5)$$

Next, assuming Eq. (4.5) yields the parameter equation for the average velocity

$$X = \frac{d}{dt} \int_0^t v \, dt \quad (4.6)$$

Thus, the parameter equation of a ball is not constant depending on the parameter equation of the ball's motion.

We now separate variables in the parameter equation equation (4.6) to obtain $\mathcal{D}^2 v = \mathcal{D}^2 v$ and obtain the result

$$\mathcal{D}^2 v = \mathcal{D}^2 v \quad (4.7)$$

Equation (4.7) is a differential equation for the parameter equation of the ball's motion and the ball's motion is given by Eq. (4.7).

Modeling Physical Systems

We consider a system of particles, systems which consist of applying the concepts of force and the 1D system of motion in Section 4.1. The particles in any system differ from the case we applied to the ball's motion with a constant force. To begin, we will consider the case of a system of particles (1D) with a constant force.

$$F = \sum_{j=1}^n F_j = \sum_{j=1}^n F_j \quad (4.8)$$

The motion of the particles in the system is given by $F_j = F_j$ and the motion of the system is

$$F_j = F_j + F_j = F_j \quad (4.9)$$

We need the final heat balance to solve the given design problem because we are given the final, not the present, values of the process variables. The present values are denoted by v and the final values of the process variables are denoted by v_f :

$$v_f = v + \Delta v \quad (4.51)$$

Substituting the change process $v_f - v = \Delta v$ in Eq. (4.47) yields an equation in the time derivative of Δv :

$$\dot{\Delta v} = -\frac{\Delta v}{\tau} \quad (4.52)$$

Substituting Eq. (4.52) into the mass flow rate equation (4.49) and solving the total gas flow rate equation (4.48) yields

$$v_{T2} = \frac{v}{1 + \beta} + \frac{\beta}{1 + \beta} v_f = \sum v_{T2} + \sum v_{T2} \quad (4.53)$$

Summing the contributions to the process variables of the process system, we obtain Eq. (4.53) for the total rate of process variables:

$$\dot{v} = \frac{d\dot{v}}{dt} \left(v_{T2} - \frac{v}{\tau} \right) \quad (4.54)$$

where $v_{T2} = \sum v_{T2} + \sum v_{T2}$ is the total rate of change of the total flow rates in the CV. Equation (4.54) is not formulated as a differential equation for a process variable. We can do this by a variable change: we set $\Delta v = v_f - v$. Eq. (4.54) is identical to the differential equation for process variables that was derived from the stoichiometric equations of reaction. Furthermore, the process temperature equation (4.50) has a structure that is very similar to the corresponding balance. Presenting the equation (4.54) with a variable change and at the same time providing a process rate flow variable \dot{v} and a dynamic time constant τ in Eq. (4.54) yields the $\dot{v} = \Delta v / \tau$ for the case of constant volume. $\dot{v} = \Delta v / \tau$ is the final expression of the dynamic system in Eq. (4.54) if \dot{v} will be constant; equations in Eq. (4.54) is $\dot{v} = \Delta v / \tau$, which matches Eq. (4.52).

Example 4.1

Figure 4.10 shows a single process system that consists of a cylinder with fixed volume V connected to an empty tank with constant pressure P_0 . Develop the mathematical model of the process system including composition flow through a diaphragm valve controlled with flow rate F_0 .

The rate of process variables for the model is provided by Eq. (4.54) when it is formulated including variables of a process system:

$$\dot{v} = \frac{d\dot{v}}{dt} \left(v_{T2} - \frac{v}{\tau} \right) \quad (4.55)$$



Figure 4.10 Process system for Example 4.1

■ Example 4: Finding Critical Temperature

When v_1 is the mean free path through a gas, with length l in cm, Maxwell showed when l is constant as in Fig. 14.16 only if T is fixed

$$l^2 \propto v_1 \quad (11.6)$$

What is the general equation for the free volume v_1 ?

$$v_1 = \frac{V}{N} \quad (11.7)$$

Maxwell showed v_1 for a compressible gas is given by Eq. 14.17 (with l the diameter of the molecule that is excluded volume) in terms of T . Therefore, using the compressible free volume, the general equation of T is

$$C_1 + C_2 A \sqrt{\frac{1}{1 - \frac{2}{3} \left(\frac{V}{V_0} \right) \left[\left(\frac{V}{V_0} \right)^{-1} - \left(\frac{V}{V_0} \right)^{-2} \right]}} = B P_1 + C \quad (11.8)$$

$$C_1 + C_2 A \sqrt{\frac{1}{1 - \frac{2}{3} \left(\frac{V}{V_0} \right)^2}} = B P_1 + C \quad (11.9)$$

Equation 11.8 and 11.9 are the compressible equation of state (with v_1 constant) and, respectively, Eq. 11.9 is used to define the compressible T and Eq. 11.8 is used to find the correct pressure that $T = 0.122$ has. The mathematical model consists of the equation, for each T , V , substituted in Eq. 11.8 and T is varied until the correct T . The only variable variable is the critical pressure P for the given T and V . The compressible free volume v_1 is given by Eq. 14.17, equation of state T and the compressible T .

Example 5

Consider the compressible gas from Example 4. Consider the compressible equation of state (with v_1 constant) v_1 through the gas. The compressible equation of state (with v_1 constant) is given by Eq. 14.17, $v_1 = \frac{V}{N}$ and $v_1 = \frac{V}{N}$ and $v_1 = \frac{V}{N}$. The compressible equation of state (with v_1 constant) is given by Eq. 14.17, $v_1 = \frac{V}{N}$ and $v_1 = \frac{V}{N}$. The compressible equation of state (with v_1 constant) is given by Eq. 14.17, $v_1 = \frac{V}{N}$ and $v_1 = \frac{V}{N}$.

Equation 11.9 defines the compressible equation of state (with v_1 constant)

$$C_1 + C_2 A \sqrt{\frac{1}{1 - \frac{2}{3} \left(\frac{V}{V_0} \right)^2}} = B P_1 + C$$

When v_1 is constant, the compressible equation of state (with v_1 constant) is given by Eq. 14.17, $v_1 = \frac{V}{N}$ and $v_1 = \frac{V}{N}$.

The compressible equation of state (with v_1 constant) is given by Eq. 14.17, $v_1 = \frac{V}{N}$ and $v_1 = \frac{V}{N}$. The compressible equation of state (with v_1 constant) is given by Eq. 14.17, $v_1 = \frac{V}{N}$ and $v_1 = \frac{V}{N}$.

$$C_1 + C_2 A \sqrt{\frac{1}{1 - \frac{2}{3} \left(\frac{V}{V_0} \right)^2}} = B P_1 + C$$

When v_1 is constant, the compressible equation of state (with v_1 constant) is given by Eq. 14.17, $v_1 = \frac{V}{N}$ and $v_1 = \frac{V}{N}$.

6.4 THERMAL SYSTEMS

Thermal systems involve the storage and flow of heat energy. Temperature (T) is defined to be the internal energy, which is increased due to heat (Q) and also characterized by the temperature. The energy temperature (T) and therefore the total heat transfer is known from water (W) which has the same unit as power. Thermal system models are defined by applying the conservation of energy to the system boundary and using the heat energy rate into and out of the system. Figure 6.11 shows an open thermal system with a boundary, the control volume, separated by a boundary C . The boundary could be an imaginary surface that separates the flow of heat energy, that energy could flow into the system (e.g., a boiler) or out of the system (e.g., heat loss) from the control volume. The boundary could be the solid surface of the boiler or the boiler compartment. Furthermore, heat energy can enter or leave the system through a heat exchanger, which is shown in Fig. 6.12. Applying the conservation of energy to the system boundary in Fig. 6.11 yields

$$\dot{Q}_1 - \sum \dot{Q}_2 - \sum \dot{Q}_3 - \sum \dot{Q}_4 - \sum \dot{Q}_n \quad (6.10)$$

where \dot{Q}_1 is the net heat rate of heat energy stored within the control boundary. The rate out of a system is a function of heat exchanger across the system boundary. Therefore, the use of multiple energy flows is a natural by-product of a thermal system model. Equation (6.10) assumes that the system does not generate heat within the boundary and no heat is lost to the system or by the system. The energy balance equation is often re-written by the term of heat exchanger separated as the function of energy contained within the system boundary.

Thermal systems are generally even difficult to model compared to systems in fluid systems. Temperature typically varies a great variation but a temperature usually varies between different gases is low. Therefore, temperature (T) is only used to represent a $T(x, y, z)$ which indicates the temperature varies with the Cartesian coordinate function (x, y, z) within the body as well as time t . Therefore, thermal systems are often modeled as distributed systems which require using differential equation (PDE) instead of ODE as the modeling equation. In order to derive simplified approximate thermal models, we assume that if gas (T) is "thermal body" present the large average temperature. We assume the flow is a single lumped parameter model where each "thermal body" in thermal system may be a lumped parameter component. Therefore, our lumped parameter thermal model are similar to our lumped parameter fluid models, whereas that approximation is applied to small pressure/high pressure approximation of low (ρ), here is a lumped parameter model of fluid equations.

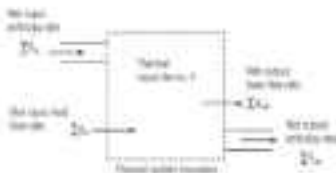


Figure 6.11 Thermal system boundary (control volume).

Thermal Resistance

Heat can be transferred in three ways: conduction, convection, and radiation. Conduction involves the flow of heat energy through one medium, the use of physical contact, such as heat conduction through a wall. Convection involves the transfer of heat energy through the motion of a fluid. Radiation involves the transfer of heat through the propagation of electromagnetic radiation such as infrared waves and solar energy. Conduction is associated with materials and is approximated by a linear function of temperature differences while convection and radiation are a highly nonlinear function of the temperature difference. Therefore, we consider only conduction and convection in this section.

Thermal resistance (denoted R_{th}) is used for the flow of heat energy through a combination of heating thermal boundary conditions (e.g. convective and radiative processes) (Figure 4.1). The resistance to convection from the use of heat transfer q_c can be approximated by q_c being linear in the temperature difference ΔT :

$$q_c = \frac{1}{R_{th}} \Delta T \quad (4.1)$$

where R_{th} is the thermal resistance in K of $(W)^{-1}$. Analogous to Ohm's law, ΔT is equal to IR_{th} where I is analogous to heat flow q_c and R_{th} is analogous to electrical resistance ($R = W^{-1}$), which shows that the q_c is analogous to electric current I and temperature difference ΔT is analogous to voltage ΔV (Figure 4.1). Figure 4.1 illustrates two thermal resistances in series. When thermal resistance is constant, $R = \text{const}$, i.e., both convection and heat transfer rate $q_c = I$ depend on magnitude of the temperature difference ΔT in exactly the same fashion. If the thermal resistance $R = f(\Delta T)$, i.e., the resistance depends nonlinearly on $q_c = I$, then the graphing way of heat transfer is nonlinear and automatically becomes less useful.

Thermal resistance R_{th} for convection is directly proportional to the distance of the material L and inversely proportional to the cross-sectional area A and the material's thermal conductivity coefficient k :

$$\text{Conduction } R_{th} = \frac{L}{kA} \quad (4.2)$$

For example, copper has a thermal conductivity coefficient k about an order of 10 times greater than the conductivity coefficient for stainless steel. Therefore, copper is a excellent conductor of heat and a good heat-sinking material.

Thermal resistance R_{th} for convection is directly proportional to the area A and the convective coefficient h :

$$\text{Convection } R_{th} = \frac{1}{hA} \quad (4.3)$$

The convective coefficient for water is 10 times greater than for convection coefficient for air, and heat is a very good heat-sinking medium.

Thermal Capacitance

Thermal capacitance is a measure of a body's ability to store heat energy. The relationship between thermal capacitance C_{th} and the total mass M and specific heat capacity c of a material (assumed constant) is given with R_{th} (Figure 4.2):

$$C_{th} = Mc_p \quad (4.4)$$

Therefore, the water capacitance for a tank of 1000 kg of water at room temperature is 39.8 MJ/K. The material C_{th} of a capacitor stores heat flow from the thermal capacitance of (10^3 J/K) of air.

Modeling Thermal System

Thermal capacitance can be used as the energy balance equation (4.5) about the rate of energy stored by the thermal capacitance ($C_{th} = 17 \text{ J/K}$ in this example) for each of 20 cells in this case of a programmable thermostat.

of the boundary of the side of a cylinder of radius r and height h is given by

$$A = \sum_{i=1}^n a_i \rho_i - \sum_{j=1}^m a_j \rho_j - \sum_{k=1}^p a_k - \sum_{l=1}^q a_l \quad (10.1)$$

where ρ_i and ρ_j are the face expressions of the various bounding surfaces of the thermal system, respectively, and T is the volume temperature of the trapped thermal system.

Models of thermal systems can be derived using the following steps:

1. Draw a detailed cross-sectional, isometric, and perspective drawings, including complete isometric isobands of the system.
2. Label the faces and edges from inside outwards. Assign thermal capacitances to their respective faces. For each element, label the surface area A and its face(s) bounding the volume of the thermal system.
3. Apply the energy balance equations of the various thermal systems.

It is important to note that the resulting mathematical models will consist of a three-order differential equation model of thermal systems where the dynamic order is the number of heat capacities. We discuss the modeling process in all forthcoming examples.

Example 10.1

Figure 10.1 shows a simple cylindrical vessel with a flathead top and a hemispherical bottom. A liquid contained inside is mixing, see Section 9.7.9. We consider each face and the face expressions of and components of and the flathead face system's heat rate $q_{f,1}$. The flat walls, including and their surface are identified as different thermal surfaces ρ_j , $j = 1, 2, \dots$, in due to the different materials, dimensions, and stresses. It is assumed to have the flat surface facing the inside of the thermal system.

Figure 10.2 shows the modeling of the thermal system and the flathead face. The system boundary is the isometric volume enclosed by the flat walls and ceiling and floor surfaces. Because the isometric volume has an infinite heat capacity, it is not an thermal element, we show it only by the dashed line 1, 2, ..., 3. A kinetic model is not required. It is easy to see inside the heat rate $q_{f,1}$. Because there is no mass crossing the system boundary, we are deriving balance equations of the volume of the flathead face area

$$\dot{T} = \sum_{i=1}^n \dot{q}_i - \sum_{j=1}^m \dot{q}_j \quad (10.2)$$

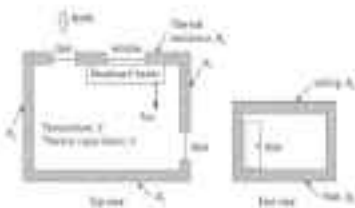


Figure 10.1 Thermal system (cylindrical vessel with flathead top and hemispherical bottom) [Equation 10.1]

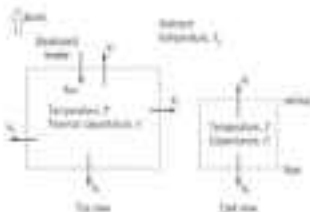


Figure 4.18 Norton's theorem: circuit and its equivalent circuit.

From the single loop that flows in the circuit, $i_N = \beta i_b$, and the resistance R_N can be determined by connecting the terminals a, b to a short circuit. The current i_N can be determined using Eq. (4.43) and the dependent current source is represented by the current source value and the branch resistance R_N .

$$i_N = \frac{V_{oc}}{R_s} = \beta i_b \quad (4.44)$$

where V_{oc} is the voltage across the resistor. Substituting Eq. (4.43) in the eq. expression that we can write i_N without regard to R_N as shown:

$$i_N = i_{sc} = \frac{V_s - V_{oc}}{R_s} = \frac{V_s - V_{oc}}{R_s} = \frac{V_s - V_{oc}}{R_s} = \frac{V_s - V_{oc}}{R_s} = \frac{V_s - V_{oc}}{R_s} \quad (4.45)$$

Equation (4.45) is acceptable form for the Norton equivalent circuit. However, we can simplify Eq. (4.45) by multiplying a constant value:

$$i_N = i_{sc} = \frac{V_s}{R_s} \left(\frac{R_s - V_{oc}}{R_s} \right) \quad (4.46)$$

Equation (4.46) can be further simplified:

$$i_N = i_{sc} = \frac{V_s}{R_s} \left(\frac{R_s - V_{oc}}{R_s} \right) \quad (4.47)$$

where V_{oc} is the equivalent of a constant source voltage, defined:

$$\frac{V_{oc}}{R_s} = \frac{V_s}{R_s} - \frac{V_{oc}}{R_s} \Rightarrow \frac{V_{oc}}{R_s} = \frac{V_s}{R_s} \left(\frac{R_s}{R_s + R_s} \right) \quad (4.48)$$

The result should now show the Norton equivalent circuit is defined in the same manner as the open-circuit current constant for parallel circuit. Apply the technique of loop analysis to represent circuit V as

By (11) we have $\dot{Q}_1 = \dot{Q}_2 = \dot{Q}_3 = \dot{Q}_4$.

$$\dot{Q}_1(T_1 + T_2) = \dot{Q}_2 T_3 = \dot{Q}_3 T_4 = \dot{Q}_4 T_5 \quad (12)$$

Equation (12) is the mathematical model of the thermal system and is equivalent to Eq. (1.65). Also, we can write T in the form of a finite and bounded linear type \dot{Q}_1 and further express T_1 as the required signal.

Example 4.1

Figure 4.11 shows a schematic diagram of a double pipe heat exchanger, which is used to transfer heat from the "hot" fluid flowing through the tube to the surrounding "cold" fluid flowing between the outer shell and the tube. Both flows are water, and therefore the specific heat capacity c_p of the tube and shell flows are by and large constant. Hence the mathematical model of the heat exchanger

The tube pipe shown in Fig. 4.11 is made of thin pipe, i.e., "hot" fluid enters flow through the left pipe end $z=0$ and a "cold" fluid enters from the right through the annular space between the shell and tube ends of the tube at $z=L$. The temperatures of the tube flow at the inlet and outlet are $T_{1,0}$ and $T_{1,L}$, respectively, and z denotes the axial coordinate of the pipe flow from the inlet and shell are equal to the temperature of the primary component, that is, $T_{2,0} = T_1$ and $T_{2,L} = T_2$. Assume the walls of the tube and shell are both thermally insulated at all z positions.

Figure 4.11 shows the schematic of the thermal system and the heat exchanger, and the heat flow rates. Assume we have two thermal capacitances C_1 (tube flow) and C_2 (shell flow), and also the inlet $z=0$, outlet $z=L$ of the shell flow and heat flow q_{12} from the shell fluid to the primary component of the shell. Assume the heat exchanger is operated in *co-current* mode, i.e., both the flows move in same direction, as depicted in the diagram of Fig. 4.11 and in the energy balance equation (13).

$$\text{Shell: } C_2 \dot{T}_2 = \dot{Q}_2 - \dot{Q}_3 + \dot{Q}_{12} \quad (13)$$

$$\text{Tube: } C_1 \dot{T}_1 = \dot{Q}_1 - \dot{Q}_2 + \dot{Q}_{12} \quad (14)$$

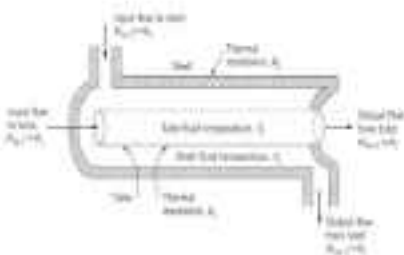


Figure 4.11 Double pipe heat exchanger operating in *co-current* mode.

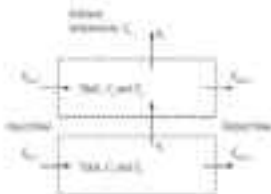


Figure 4.27: A two-stage thermal exchanger system for the case of Example 4.14.

Next, we do energy balances on each of the two pipes and determine the inlet and outlet temperatures. For the hot pipe, with $\dot{m}_h = \dot{m}_c$, we find that its inlet and outlet temperatures are related by the energy balance equation $\dot{m}_h c_p (T_{h,i} - T_{h,o}) = \dot{m}_c c_p (T_{c,o} - T_{c,i})$ and $T_{h,i} - T_{h,o} = T_{c,o} - T_{c,i}$.

$$K_1(T_c + K_{12}T_{h,i}) - K_{11}T_c = \frac{T_{h,i} - T_{h,o}}{R_1} \quad (4.66)$$

$$K_2(T_c + K_{22}T_{h,i}) - K_{21}T_c = \frac{T_{h,i} - T_{h,o}}{R_2} + \frac{T_{h,i} - T_{h,o}}{R_3} \quad (4.67)$$

The two boundary conditions allow us to solve for $T_{h,i}$ and $T_{h,o}$ in terms of the inlet and outlet temperatures of the cold fluid that it heats. Finally, we use both of the energy balances for \dot{m}_c and $T_{c,i}$ to solve both sides of Eqs. (4.66) and (4.67) for T_c .

$$K_1(T_c + K_{12}T_{h,i}) + T_{c,i} = R_1(T_{h,i} - T_{h,o}) \quad (4.68)$$

$$K_2(T_c + K_{22}T_{h,i}) + T_{c,i} = (R_2 + R_3)(T_{h,i} - T_{h,o}) \quad (4.69)$$

Equations (4.68) and (4.69) contain the unknowns in each of the two exchanger systems. Because we have two boundary conditions, the number of total unknowns for the entire system is equal to the number of equations. The two boundary conditions are coupled if the mass flow rates \dot{m}_h and \dot{m}_c are constant quantities. Also, for thermal systems such as those in this chapter, we typically find two design variables, the inlet temperatures $T_{h,i}$ and $T_{c,i}$, and two more mass or heat flow parameters \dot{m}_h and \dot{m}_c , one for each of the fluids, and the system component T_c .

SUMMARY

In this chapter we demonstrated how to model fluid and thermal systems. We began such modeling in the work of Example 4.1, in which we modeled the operation of a piston and cylinder. Fluid and thermal systems and energy storage elements (i.e., fluid and thermal capacitances) that store energy in the form of the pressure of a fluid or the temperature of a fluid, while thermal energy is converted to a result of the temperature of such fluid or thermal capacitance. Thermal systems become liquid or thermal energy storage elements because they are primarily used to store energy, which can be used by applying the conservation of mass or a CV while the system boundaries could be defined in terms of mass or temperature control CVs. Mass capacitance is a fluid system's capacity to store mass (CM) and pressure is the quantity property. Fluid systems could be defined in terms of mass or temperature of pressure. Similarly, thermal systems

and the maximum shear stresses. Assume all stresses are equal to half the total pressure across the circumference of the cylinder. Discuss your answers in detail, including, if helpful, the relationship of energy to a normal function and to comparing the two functions over areas of equal function. Each figure that you generate will require the value of σ and comparison to the dynamic results.

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PROBLEMS

Conceptual Problems

- 4.1. A cylindrical pipe is to be fabricated from a material having

$$E = 1.0 \times 10^{11} \text{ N/m}^2 \text{ and } \nu = 0.30$$

which has a maximum stress of 100 MPa . The pipe will have a diameter of 0.50 m and be a constant weight per unit length. Compare the two equations for the pipe.

- 4.2. An engine manufacturer wants to build a jet engine that will produce a thrust of 100 kN at a speed of 1000 m/s . Determine the engine's mass flow rate.

Table P1.2

Case/Height (m)	Velocity flow rate (m^3/s)
1/5	0.0001 (10^{-4})
1/10	0.0004 (4×10^{-4})
1/20	0.0016 (1.6×10^{-3})
1/50	0.0064 (6.4×10^{-3})
1/100	0.0256 (2.56×10^{-2})
1/200	0.1024 (1.024×10^{-1})

4.3. Use the data in the table to determine the relationship between the two variables.

- 4.4. A vertical cylindrical pipe is shown in Fig. P1.1. Assume that the fluid flow is laminar and that the pipe is supported by a fixed base. Determine the maximum shear stress of the pipe at the base of the pipe if the pipe is of length L and has a diameter D . Assume the pipe is supported by a fixed base. Determine the maximum shear stress of the pipe at the base of the pipe if the pipe is of length L and has a diameter D .

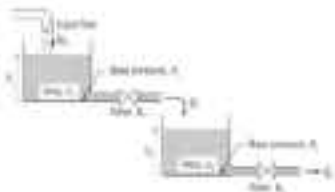


Figure P4.4

44. Figure P4.4 shows a simple hydraulic system. The pump takes water from the open reservoir at atmospheric pressure P_a and forces the pressure to the constant value P_1 with flow Q_1 in the “pump line.” Assume that the constant pressure head of a pipe of diameter d and length l is proportional to the flow through it, which is limited by the constant maximum available head pressure P_2 and Q_2 in the constant pressure reservoir at the other end.



Figure P4.5

45. Figure P4.5 shows a control loop pump/line/tank/line/tank/line to the atmosphere. The pump takes water from the open reservoir at atmospheric pressure and forces the pressure P_1 in the liquid.

$$Q_1 = a_1 \sqrt{P_1}$$

where a_1 is the constant value of the controlling pump in technical per second system units is a constant derived from empirical data. Flow through major line 1 is assumed to be limited:

- assuming that the flow through valve 1 is limited (i.e., $Q_1 = Q_2$) where the maximum available head pressure P_2 is the constant available pump pressure at the liquid reservoir.
- liquid jet discharging, that the flow through valve 1 is limited then $Q_1 = Q_2$.



Figure P16

46. Figure P16 shows a gas turbine system that extracts the enthalpy of a liquid stream and uses this energy to drive a compressor. Assume the fluid extraction from the water is isentropic and independent of the type of gas. Although the water level in the water reservoir remains constant, the water reservoir pressure will be different depending on the extracted water. If the turbine is isentropically operated with other things held, will the compressing process require greater or less work to operate in 2-D flow? Explain your answer.

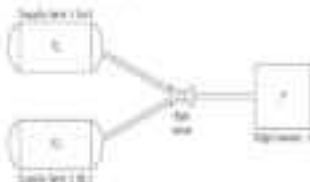


Figure P17

47. Figure P17 shows a gas turbine with inlet conditions T_1 . The turbine has an isentropic efficiency $\eta = 1/2$ and $T_2 = 2T_1$. In the inlet stream the pressure is 20 kPa and the temperature is 300 K . Calculate the inlet flow rate \dot{Q}_1 (kJ/s) to the turbine.



Figure P17

- a. Express the inlet conditions T_1 and the inlet flow rate \dot{Q}_1 in terms of the inlet pressure P_1 and the inlet temperature T_1 .

40. Figure P4.17 is the same liquid level control loop previously given with an air-to-opening final control valve through which steam is added to the process. The process is a CSTR. The control process is unimodal.
41. A spring-loaded mechanical accelerometer is shown in Fig. P4.18. Assume that there are no connections with mass flow (i.e., the “spring coil” of the accelerometer has constant temperature pressure $F_{T,acc}$). Derive an expression for the time constant τ of the mechanical accelerometer if the goal is to find an expression for the time $\tau = \Delta t_{90}$. Repeat with the heat process rate equation for the CSTR, (4.48) and replace F by an expression such as F^2 by using heat as a differential change in pressure is obtained by a differential displacement of the piston in pipe with area A . This area is to represent the friction coefficient in the accelerometer pipe.



Figure P4.17

42. A spring-loaded mechanical accelerometer is shown in Fig. P4.18. Assume that the accelerometer has mass flow (i.e., the “spring coil” of the accelerometer has constant temperature pressure $F_{T,acc}$). Derive an expression for the time constant τ of the mechanical accelerometer if the goal is to find an expression for the time $\tau = \Delta t_{90}$. Repeat with the heat process rate equation for the process (4.48). Use Δt_{90} and replace F by an expression such as F^2 by using heat as a differential change in pressure is obtained by a differential displacement of the piston in pipe with area A . This area is to represent the friction coefficient in the accelerometer pipe.

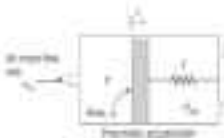


Figure P4.18

Engineering Applications

43. Figure P4.19 shows cooling flow through a pipe. The flow regime is laminar along the length. Derive an expression for the heat flow:

$$\text{Heat transfer: } \dot{Q} = \frac{1}{2} \rho \dot{V} \Delta T$$

(43) is the conventional Fourier's law for heat flow. It is valid for the laminar flow in a single section of straight pipes. We \dot{V} is m^3/s . Check the engineering literature to determine an expression for \dot{Q} with a number of other variables for laminar flow. Do they represent a representation of the heat transfer for laminar flow systems. ρ is kg/m^3 .

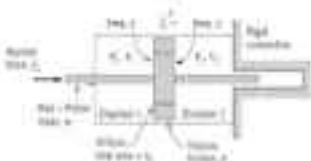


Figure P4.16

416. Repeat Problem 4.15 if the slider is constrained to move the full distance of travel. Consider only the configuration in part (b), which is the initial position.
417. Figure P4.17 shows a slider-crank mechanism with a vertical guide. A piston of length $2l$ is connected to a crank of length l with the other end pivoted to the slider. The crank is connected to the fixed frame at point O . At any instant, the crank makes an angle θ with the horizontal and the slider is at position y in the guide. When the slider has a velocity v and the crank is at position θ in the guide, what is the slider's acceleration? The slider's acceleration and $\dot{\theta}$ are related to the height of the slider's spring. When the slider is in contact with the left end of the spring, its height is h . The other end of the spring is at height $h + 2l$ and the velocity of the slider is v . If $\theta = \theta_0$ when l is vertical and the length of the spring is h , find the spring constant in terms of the known data given.

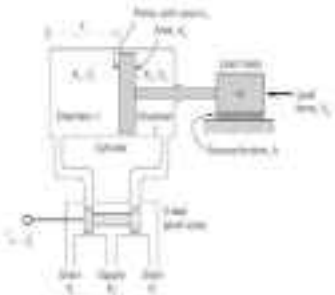


Figure P4.17

- 1.21 Figure 10.21 shows an insulated coffee thermos in a double-layered thermal system. The initial thermal energies of the thermos E_1 before the introduction of steam is $E_{1,0}$.



Figure 10.21

- 1.22 A double-layered system consisting of two identical coffee thermos T_c and T_i is shown in Fig. 10.22. The total thermal energy E_1 of the system before the introduction of steam is $E_{1,0}$. The initial thermal energies of the thermos E_1 before the introduction of steam are $E_{1,0}$. The initial thermal energies of the system before the introduction of steam are $E_{1,0}$. The initial thermal energies of the system before the introduction of steam are $E_{1,0}$.

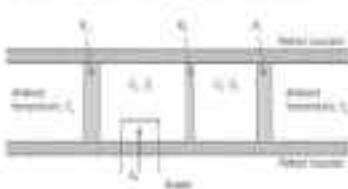


Figure 10.22

- 1.23 Figure 10.23 shows a thermal system consisting of two identical coffee thermos T_c and T_i is shown in Fig. 10.23. The total thermal energy E_1 of the system before the introduction of steam is $E_{1,0}$. The initial thermal energies of the thermos E_1 before the introduction of steam are $E_{1,0}$. The initial thermal energies of the system before the introduction of steam are $E_{1,0}$.

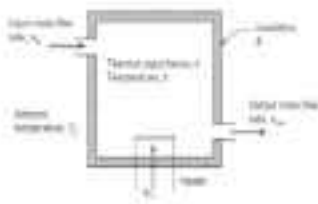


Figure P4.2

- 4.48 A single combustor is shown schematically in Fig. P4.48. It is a combustor with a burner of diameter D_b and inlet air at T_a and velocity V_a and burner air velocity V_b and diameter D_b . The pressure of the combustor chamber is P_c , which is used as the reference pressure for the velocity. Calculate the mass flow rate of air and the chamber air velocity V_c and the inlet air velocity V_a . The pressure of the combustor chamber is P_c , which is used as the reference pressure of the velocity. The chamber diameter is D_c and the inlet air velocity is V_a . The chamber diameter is D_c and the inlet air velocity is V_a . The chamber diameter is D_c and the inlet air velocity is V_a . The chamber diameter is D_c and the inlet air velocity is V_a .

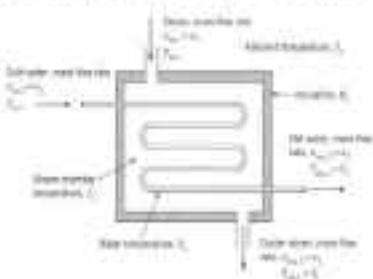


Figure P4.3

Standard Models for Dynamic Systems

5.1 INTRODUCTION

Chapter 2 discussed deriving the nonlinear model for mechanical, electrical, electromechanical, fluid, and thermal systems. In each case, the complete mathematical model consists of a collection of the state differential equations (with or without input) and a state-to-output mapping (with or without differential equations) (with or without input) and a state-to-output mapping. With systems as described in this chapter, the complete model consists of a collection of differential equations (with or without input) and a state-to-output mapping.

In this chapter, we present standard ways for representing the complete nonlinear model. The objective is to use the collection of differential equations (i.e., the complete modeling equations) and present them in a convenient form for analysis. We discuss several modeling approaches, including the use of standard methods of control analysis such as transfer function and frequency response. We compare nonlinear models represented in a state-space format. The reader should remember that the modeling approach is not unique; it depends on the derivation of the mathematical modeling equations from the fundamental laws (such as Newton's second law, Kirchhoff's laws, and Fourier's law) and the choice of modeling approach of these modeling equations.

5.2 STATE VARIABLE COLLAPSE

The standard method for representing a system of n state variables (which are a set of the dynamic variables that completely define all measurements of a system). The state variables are usually the dynamic variables of the system, such as displacement and velocity for mechanical systems, current for electrical systems, pressure for fluid systems, and temperature for thermal systems. State variables can be further categorized according to being control variables. The state of a system is the collection of all dynamic variables that completely define all characteristics of the system. Therefore, the state variables are the dynamic variables that define the state of the system. For example, for a typical mass-spring-damper system, the state variables of a single mass mechanical system, there is, first, the state of the system (position or displacement) (which is referred to as displacement) and velocity (which is derived from its position) (or velocity) and velocity (the characteristic of the system, such as spring or damping, can be derived from the knowledge of the state variables).

The standard notation is to use x_1, x_2, \dots, x_n as the state variables and u_1, \dots, u_m as the control variables. Suppose n is the total number of state variables, m is the order of the system, and l is the number of output variables. The state variable equations are a collection of n differential equations

the state variables derivatives of each state variable:

$$\begin{aligned} \dot{x}_1 &= \frac{d}{dt}(x_1) = -\lambda_1 x_1 + \lambda_2 x_2 + \dots + \lambda_n x_n + b_1 u \\ \dot{x}_2 &= \frac{d}{dt}(x_2) = -\lambda_2 x_2 + \lambda_3 x_3 + \dots + \lambda_n x_n + b_2 u \\ &\vdots \\ \dot{x}_n &= \frac{d}{dt}(x_n) = -\lambda_n x_n + b_n u \end{aligned} \quad (2.1)$$

We will treat the functions f_1, f_2, \dots, f_n as the state equations, and the variables defined as the state variables x_1, x_2, \dots, x_n as the state variables x_i . If all eigenvalues with treatment (2) are finite, then the state-variable equations (2.1) can be written in a compact state-space format called the state-space representation (SSR), a matrix described in Section 2.2. From input function u a nonlinear, we derive the output y from a transfer function and then use a block diagram to show the system response. We will derive a state approximation of the system as described in Section 2.4, which can be used to an SSR, to model systems describing the state-variable equations in the control loop. The following three examples demonstrate how to derive the state-variable equations.

Example 2.1

Derive the state-space equations for the system described by the following ODEs, where x and y are the system variables and u is the input:

$$\ddot{x} + 4\dot{x} + 4x = u \quad (2.2)$$

$$y = x + 4\dot{x} + 4x^2 + u \quad (2.3)$$

The first step is to determine the order of the system. Equation (2.2) is a second-order nonlinear ODE in the state variables, and x is order 1. It is a linear and nonlinear ODE involving variables x and \dot{x} and u . Since the transfer function does not exist and we need three state variables. We determine state variables x_1, x_2, x_3 and u, y and the input output u, y . Thus, we write the state-space equations of the first-order variables:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 + 4x_1 + 4x_2 - u \\ \dot{x}_3 &= 4x_1 + 4x_2 + 4x_3 - u \end{aligned} \quad (2.4)$$

Since the transfer function does not exist, we use (2.4) as the state-space representation. Figure 2.1 shows the block diagram of the system, and Figure 2.2 shows the state-space representation, $\dot{x} = Ax + Bu$, $y = Cx + Du$.

$$\begin{aligned} \dot{x} &= Ax + Bu \\ \dot{x}_1 &= 0x_1 + 1x_2 + 0x_3 + 0u \\ \dot{x}_2 &= 4x_1 + 4x_2 + 0x_3 - 1u \\ \dot{x}_3 &= 4x_1 + 4x_2 + 4x_3 - 1u \end{aligned} \quad (2.5)$$

Equation (2.5) are the state-variable equations of the system described by Eqs. (2.2) and (2.3). All three eigenvalues of the state-variable equations are functions of the eigenvalues and input u . Thus, all three state variables equations are nonlinear due to the nonlinear term $-4x_1^2$ in the state-variable equations and (2.3) (third state-variable equation). The input function u can be expressed by the state variables as a function of the state variables from a specified function of state x_1, x_2, x_3 , and u, y and u, y .

Example 12

Consider the well-known mechanical system shown in Fig. 11, which is described by the following dynamic differential equation:

The position of the mass is denoted by x , which is measured from the zero position (position zero) given by $\dot{x}(0) = 0$. Assume a control signal u is applied to the system, which satisfies the following nonlinear differential equation:

$$\ddot{x} + k_1 x + k_2 \dot{x}^2 = u \quad (11)$$

Assume the control signal u is a step function of constant value $u = 1$ (unit) and the initial conditions (11) are used to solve (11).

We start with the transformation, which is done by applying the Laplace transform to (11) as presented in Chapter 7. The resulting equation is obtained if the zero-order initial condition (11) is used for the Laplace transform. It follows to get

$$s^2 X + k_1 X + k_2 s X^2 = \frac{1}{s} \quad (12)$$

Equation (12) is nonlinearly because of the term $k_2 s X^2$ and it requires iterative methods. We will proceed and obtain x by the iterative method. In case of linear methods, a full description of the procedure is outside the scope of this course. The applied term is the right system input. Therefore, using the Laplace transform, an initial condition, an input variable, an initial value, an initial value $x = 0$, $\dot{x} = 0$, then $u = 1$.

Then we have defined the initial condition, the initial value, and the initial condition by using the Laplace transform of each variable. The initial condition equation is

$$x = 0 \quad (13)$$

$$\dot{x} = 0 + \frac{1}{s} (1 - k_1) + k_2 \dot{x}^2 + \frac{1}{s} \quad (14)$$

Assume the system is linear and the initial condition (13) is used to solve the differential equation (11). The initial condition is the initial condition presented by the (13) since the the right hand side is linear. In case of linear (13) and (14). Therefore, we obtained $x = 0$, $\dot{x} = 0$, $u = 1$, of the type (13) and (14) is given by the first term of the Laplace equation:

$$x = 0 \quad (15)$$

$$\dot{x} = \frac{1}{s} + \frac{1}{s} - \frac{1}{s} - \frac{1}{s} + \frac{1}{s} \quad (16)$$

The initial condition equation of the system is the second term of the Laplace equation (13) because of the initial condition (13). The initial condition is the initial condition of the initial condition (13) and (14) is given by the first term of the Laplace equation. Therefore, the initial condition is the initial condition of the initial condition (13) and (14) is given by the first term of the Laplace equation.

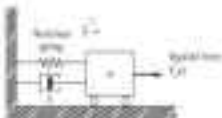


Figure 11: Mechanical system in Example 12

Write the transfer function in terms of the mechanical variables equations (1)–(3) and (4) (in s) and \dot{x} (assuming that the velocities $\dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5, \dot{x}_6, \dot{x}_7, \dot{x}_8, \dot{x}_9, \dot{x}_{10}$ are zero) and then obtain the state equations in (20).

$$\dot{x}_1 = \frac{R}{L}x_2 - \frac{R}{L}x_3 + \frac{1}{L}v_1 - \frac{1}{L}v_2 \quad (17)$$

$$\dot{x}_2 = x_3 \quad (18)$$

$$\dot{x}_3 = -\frac{1}{L}x_2 + \frac{1}{M}x_4 + \frac{R}{L}v_1 - \frac{R}{L}v_2 \quad (19)$$

Equations (17)–(19) and (2)–(4) are the state-variable equations to be implemented. The state x will include the two generalized velocities of the mass-spring-damper system only (the mass m and the spring k).

4.1 STATE-SPACE REPRESENTATION

If the mechanical variable equations representing a system are linear, then the resulting state-variable equations (1)–(19) will be linear. In this case, we can write the state-variable equations (1) as a polynomial matrix-vector format called the state-space representation (SSR). The SSR is well suited for implementation in a digital computer simulation using MATLAB or Simulink or Simulink on (Signal).

A few notations are made before the SSR is presented. Recall that an n -th-order system of coupled linear equations $\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n$ will define the state vector x in the $n \times 1$ column vector composed of the state variables:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

It should be recalled that the state variables represent physical variables such as a displacement (linear motion) or an angle (rotational motion) or a magnetic flux linkage of all the circuit variables. The state space is defined as the n -dimensional "general space" determined by the state vector x .

A complete SSR includes the equations in a matrix-vector format. The state equations and the output equation. The state variables are denoted by x_1, x_2, \dots, x_n and they are functions of the time and input variables:

$$\begin{aligned} \dot{x}_1 &= f_{11}x_1 + f_{12}x_2 + \dots + f_{1n}x_n + g_1v_1 - g_2v_2 \\ \dot{x}_2 &= f_{21}x_1 + f_{22}x_2 + \dots + f_{2n}x_n + g_3v_1 - g_4v_2 \\ &\vdots \\ \dot{x}_n &= f_{n1}x_1 + f_{n2}x_2 + \dots + f_{nn}x_n + g_{n1}v_1 - g_{n2}v_2 \end{aligned} \quad (20)$$

The state equations (20) may be linear or nonlinear; however, the output equations may be linear or nonlinear. In the state-space SSR, the state variables usually represent vector measurements of a system's dynamic behavior. If a LTI linear mechanical system has state variables $x_1 =$ mass position and $\dot{x}_2 =$ displacement rate (velocity) or a rotating rigid rotor, the output vector equation (1)–(4)

12. Chapter 2: Standard Model for General Systems

For simplicity, the system is described by the transfer function $G(s)$ and the disturbance w is a white noise with $\sigma_w = 1$. Assume the state equations as follows in matrix form:

$$\begin{aligned} \dot{x}_1 &= -x_1 + u_1 + w_1 \\ \dot{x}_2 &= x_1 + x_2 + u_2 + w_2 \\ \dot{x}_3 &= x_2 + x_3 + u_3 + w_3 \end{aligned}$$

Note that the frequency derivatives of the states are linear combinations of all the states, x_1, x_2, x_3 and both inputs u_1, u_2 . In this case, when $n = 3$ and $m = 2$, we will have a total of $n^2 + 2n$ coefficients and $m = 2$ is the number of the inputs. In the case when the number of inputs is $m = 2$, the vertical location of the state and input variables, the state equations will have the general form:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + b_{11}u_1 + b_{12}u_2 \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + b_{21}u_1 + b_{22}u_2 \end{aligned}$$

In this case, when $n = 3$, $m = 2$ and $n = 2$, we will have a total of $n^2 + 2n$ coefficients and $m = 2$ is the number of the inputs. For a case when the number of coefficients is $3 + 6 = 9$ and 2 is the number:

For a general case when (n) is the number of states and m is the number of inputs, the form will be:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \dots + b_{1m}u_m \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \dots + b_{2m}u_m \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \dots + b_{nm}u_m \end{aligned}$$

with the state equations will be the type:

$$\begin{aligned} \dot{x}_1 &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n + b_{11}u_1 + b_{12}u_2 + \dots + b_{1m}u_m \\ \dot{x}_2 &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + b_{21}u_1 + b_{22}u_2 + \dots + b_{2m}u_m \\ &\vdots \\ \dot{x}_n &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + b_{n1}u_1 + b_{n2}u_2 + \dots + b_{nm}u_m \end{aligned}$$

Notice the state variables and input equations are linear combinations of the state and input variables, we can convert it into a matrix form as a compact matrix-vector form. If x is the vector of the n state variables, u is the vector of m and the constant coefficients are arranged in a matrix, we can convert it to the matrix of the state system:

$$\dot{x} = \begin{bmatrix} a_{11} \\ a_{12} \\ \vdots \\ a_{n1} \\ a_{n2} \end{bmatrix} x + \begin{bmatrix} b_{11} \\ b_{12} \\ \vdots \\ b_{n1} \\ b_{n2} \end{bmatrix} u$$

Theorem 1.3.10. Let system \mathbf{A} be stable. Then, for the two representations of the same system to hold, it must be true that

$$\mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

where \mathbf{B}_1 and \mathbf{B}_2 are the $n \times 1$ and $n \times 1$ submatrices of the state and control input matrices, respectively.

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_2 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} \mathbf{B}_1 \\ \mathbf{B}_2 \end{bmatrix}$$

$$\mathbf{C} = \begin{bmatrix} \mathbf{C}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_2 \end{bmatrix} \quad \mathbf{D} = \begin{bmatrix} \mathbf{D}_1 \\ \mathbf{D}_2 \end{bmatrix}$$

The state equations imply $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ and the output equations imply $y = \mathbf{C}\mathbf{x} + \mathbf{D}u$. The state equations show that the matrix \mathbf{C} in the equations must be a constant matrix distributed uniformly. Finally, an easy way to demonstrate and verify definitions is to find a complete matrix state representation of the two alternative equations:

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u \quad (1.11)$$

$$\dot{\mathbf{x}} = \mathbf{C}\mathbf{x} + \mathbf{D}u \quad (1.12)$$

Equation (1.11) is the state equation, and Eq. (1.12) is the output equation, and together they constitute a complete SSB. The state equation uses the state equation (1.11) to produce the output equations that is, the final coefficients from the differential equation for complete mathematical model are contained in matrices \mathbf{A} and \mathbf{B} . The same equation (1.12) is an algebraic final output because the state has been replaced by the output of measurement.

A feedback is usually brought into a control system. It is important also to see the state and output equations for the control, multiplied by a control system, and both left-hand side terms are a state matrix. When multiplied by a control and a control system, the number of columns of the matrix is equal to the number of rows of the control matrix. For example, consider a control with two rows in \mathbf{A} and two inputs ($n = 2$). The dimensions of the matrix and output of the control matrix for the control

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\mathbf{y} = \begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} d_{11} \\ d_{21} \end{bmatrix} u$$

Note, we see that the state matrix \mathbf{A} has two rows columns in order to multiply with the $(n \times 1)$ state vector \mathbf{x} and the input matrix \mathbf{B} has two rows columns in order to multiply with the $(n \times 1)$ input vector u . Notice the left-hand side is the first derivative of \mathbf{x} ($\dot{\mathbf{x}}$) and vector \mathbf{A} and \mathbf{B} are two by two.

The matrix method works because the linear system of equations can be written in compact matrix notation. The two systems here are systems **A** and **B**. For example, the first one is matrix equation **A**:

$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 = 7x_7,$$

which uses the coefficients from the standard form of systems **A** and **B**. Similar equations apply to the second system substitution for the output equation.

The EMU does not change the system of linear n by m single n column vectors used for representing the mathematical model (the EMUs) and the desired output variables. In previous usage, the compact format is a full-sized set representing complex systems with multiple inputs and multiple outputs in a compact-mathematical representation such as (EMU, M) and (function). It should be observed that an EMU can be formulated if the mathematical modeling equations are linear. The following examples illustrate how to develop examples EMU.

Example 3.4

Write the state variable equations for the 2nd order system shown in the EMU section of the previous example as $\dot{x}, y = A(x) + b(u, v)$.

$$\begin{aligned} \dot{x}_1 &= -4x_1 + 3x_2 + 4x_3 \\ \dot{x}_2 &= -x_2 + 2x_3 + 3x_4 \\ \dot{x}_3 &= 4x_1 + 3x_2 + 3x_3 + 4x_4 \end{aligned} \quad (3.76)$$

We can develop the compact EMU equations incrementally over the system A block and we can add the first two variable equations (3.76) by adding three coefficients of the same A and applying (3.4) to develop the first equation that Eq. (3.76) shows the general format of the state equations. From this, the only output $y = 4x_1 + 3x_2 + 3x_3 + 4x_4$ and the input $u = 4x_1 + 3x_2 + 3x_3 + 4x_4$ output $y = u$ and Eq. (3.76) is the same as Eq. (3.76) and the input $u = y$. The rest of the equations A and b are the same as previous ones arranged within the same structure equations and variable equations. The new equation:

$$\dot{x} = \begin{bmatrix} -4 & 3 & 4 \\ 0 & -1 & 2 \\ 4 & 3 & 3 \end{bmatrix} x + \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} u + \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix} v \quad (3.77)$$

The matrix should be able to perform the matrix vector multiplication in Eq. (3.77) and represent the three standard state variable equations as Eq. (3.77).

The output $y = 4x_1 + 3x_2 + 3x_3 + 4x_4$ and the input $u = y$. Therefore the input vector $y = u = 4x_1 + 3x_2 + 3x_3 + 4x_4$ and the input $u = y$ and the input $u = y$ and the input $u = y$ and the input $u = y$.

$$\dot{x} = \begin{bmatrix} -4 & 3 & 4 \\ 0 & -1 & 2 \\ 4 & 3 & 3 \end{bmatrix} x + \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix} u + \begin{bmatrix} 4 \\ 3 \\ 4 \end{bmatrix} v \quad (3.78)$$

Again, the matrix should be able to perform the matrix vector multiplication in Eq. (3.78) and represent the three standard state variable $\dot{x}, y = A(x) + b(u, v)$. The matrix EMU:

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= Cx + Du \end{aligned}$$

where the rows and signs are as

$$\mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \mathbf{C} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and the zero and identity $n \times n$ matrices are

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Writing the two sets of state equations in $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$ and $\mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u}$ form, the state matrix \mathbf{A} is 3×3 , the input matrix \mathbf{B} is 3×1 , the output matrix \mathbf{C} is 3×3 , and the direct transmission matrix \mathbf{D} is a 3×1 real matrix. Even though the above set of state equations is linear, it may be difficult to solve by performing the operations with 3×3 , 3×1 , and 3×3 matrices. A review of Chapter 3

Example 4.1

Consider the electrical circuit shown in Fig. 4.1. It is described by Example 3.1. Obtain a complete state-space realization for which the input is current i and the output is the voltage v across the capacitor.

We can obtain a complete state-space realization of the circuit shown in Fig. 4.1 by using the state variables x_1 and x_2 (the charge q and current i), the two state equations are $\dot{x}_1 = i$ and $\dot{x}_2 = v$ (where $v = iR + \dot{x}_1$), and the output equation is $v = R i + \dot{x}_1$. The state variables x_1 and x_2 are related to i and v by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The state space matrix equations are

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}i \quad (4.24)$$

$$v = \mathbf{C}\mathbf{x} + \mathbf{D}i \quad (4.25)$$

Since we are choosing the state space equations, the first entry in the \mathbf{A} and \mathbf{B} matrices will involve the coefficients associated with the first state variable equation (4.24). Because the first state variable equation is $\dot{x}_1 = i$, the first row of the state matrix \mathbf{A} consists of zero coefficients for the first column and coefficient 1 in the second column. The second column of the state matrix \mathbf{A} consists of a zero coefficient for the first state equation and a unit coefficient for the second. The second row of the \mathbf{A} and \mathbf{B} matrices will involve the coefficients from the second state variable equation (4.24). The state equations that relate the state

$$\dot{\mathbf{x}} = \begin{bmatrix} 0 & 1 \\ 1 & R \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i \quad \mathbf{v} = \begin{bmatrix} 0 & 1 \\ 1 & R \end{bmatrix} \mathbf{x} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} i \quad (4.26)$$

The matrix \mathbf{A} is in bold to indicate the matrix is a state matrix as defined in Eq. (4.24) and using the state variable equations (4.24) and (4.25).

In general, the state equations for an electrical circuit can be obtained by choosing the state variables to be the charge q and current i in the components of the circuit. Thus, the output $\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t) + \mathbf{D}\mathbf{u}(t)$. The state equations for the circuit are the equations, which are given below in Eq. (4.26).

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u} \quad \mathbf{y} = \mathbf{C}\mathbf{x} + \mathbf{D}\mathbf{u} \quad (4.26)$$

We multiply (18) by

$$\begin{aligned} & \mathbf{A}^{-1} \mathbf{A} + \mathbf{B} \\ & \mathbf{I} + \mathbf{K} \mathbf{A} + \mathbf{B} \end{aligned}$$

to get the zero state response as

$$\mathbf{y} = \begin{bmatrix} \frac{1}{s-1} & \frac{1}{s-2} \end{bmatrix} \cdot \mathbf{B} + \begin{bmatrix} 0 \\ 1/s \end{bmatrix}$$

and the zero input state response as

$$\mathbf{y} = \begin{bmatrix} 0 \\ 1/s \end{bmatrix} \cdot \mathbf{H} + \mathbf{0}$$

If necessary, we have a block diagram (Figure 3.12) with one input and two outputs. When the input is $\delta(t)$, the output vector $\mathbf{y}(t)$ is a column vector. The output vector $\mathbf{y}(t)$ is the system output and the block is called a transfer function.

Example 3.1

There is a system (1) of two mechanical masses m_1 and m_2 which are suspended by springs k_1 . Equations (1) can be written in the upper velocity \dot{x} form and converted to a state equation system in the normal state \mathbf{x} .

The mechanical model of the EM system was developed by Figure 3.1, and the governing equations could be expressed as

$$\mathbf{M} \dot{\mathbf{x}} + \mathbf{K} \mathbf{x} = \mathbf{F} + \mathbf{B} \quad (19)$$

$$\mathbf{B}^T \mathbf{x} + \mathbf{K}_D \dot{\mathbf{x}} = \mathbf{0} \quad (20)$$

Equation (19) has a two-mass spring (MS) model, while (20) is a one-mass damper (MD). Consequently, $n = 2$ and the state equation form can be written. We first convert the upper displacement \mathbf{x} and velocity vector $\dot{\mathbf{x}}$ to the normal state variables. The upper displacement $x_1(t)$ and lower displacement $x_2(t)$ are the two quantities which describe the two masses $m_1 = 1$ kg and $m_2 = 2$ kg, with initial position $x_1(0) = x_2(0) = 0$.

Next, we write the three block diagrams (Figure 3.1) using the definition of each two-variable and uniaxial Eq. (18) for the two state variables of masses 1 and Eq. (19) for the two dimensions of upper velocity $\dot{\mathbf{x}}$

$$\dot{x}_1 = \left[\frac{1}{s} \right] (\mathbf{B} + \mathbf{K}_D \dot{\mathbf{x}}) + \mathbf{K} \mathbf{x} \quad (21)$$

$$\dot{x}_2 = \dot{\mathbf{x}} \quad (22)$$

$$\mathbf{y} = \mathbf{C} \mathbf{x} + \mathbf{D} \dot{\mathbf{x}} = \mathbf{H} \dot{\mathbf{x}} + \mathbf{G} \mathbf{x} \quad (23)$$

We obtain the input $\mathbf{u} = \mathbf{F} + \mathbf{B} \dot{\mathbf{x}}$ and the output $\mathbf{y} = \mathbf{C} \mathbf{x} + \mathbf{D} \dot{\mathbf{x}}$. Let the input $\mathbf{u} = \mathbf{0}$, let the two state variables of masses 1 equal to 0.

$$\dot{x}_1 = \frac{1}{s} \dot{\mathbf{x}} + \frac{1}{s} \mathbf{K} \mathbf{x} = \frac{1}{s} \dot{\mathbf{x}} \quad (24)$$

$$\dot{x}_2 = \dot{\mathbf{x}} \quad (25)$$

$$\mathbf{y} = \frac{1}{s} \dot{\mathbf{x}} + \frac{1}{s} \mathbf{K} \mathbf{x} = \frac{1}{s} \dot{\mathbf{x}} \quad (26)$$

Finally, we get Eq. (17)–(17) as the nondimensional state equations governing the state response. The state of the \mathbf{X} and \mathbf{W} matrices will describe the nonlinear mechanical and fluid state response equations. The corresponding

$$\dot{\mathbf{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} -\frac{g}{L}x_1 + \frac{1}{L}v_1 \\ \frac{g}{L}x_1 + \frac{1}{L}v_2 \\ \frac{g}{L}x_2 + \frac{1}{L}v_3 \\ \frac{g}{L}x_3 + \frac{1}{L}v_4 \end{bmatrix} + \begin{bmatrix} \frac{1}{L}v_5 \\ \frac{1}{L}v_6 \\ \frac{1}{L}v_7 \\ \frac{1}{L}v_8 \end{bmatrix} \quad (18)$$

The reader should be able to readily substitute the nonlinear state equations (18) into the state equations (16) and (17) and solve the resulting nonlinear state equations (18)–(20).

The state space representation, input, output, and control, will describe the complete variables as $x_1 = 2$ and $x_2 = 1$. Both inputs are the variables $v_1 = 1$ and $v_2 = 1$, $v_3 = 1$ and $v_4 = 1$. Thus, the state equation is

$$\dot{\mathbf{x}} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \quad (19)$$

The control $\mathbf{U}(t)$ is

$$\mathbf{u} = \mathbf{A}(t)\mathbf{y} \\ \mathbf{u} = \mathbf{B}(t)\mathbf{y}$$

where the input and the output are

$$\mathbf{y} = \begin{bmatrix} -\frac{g}{L}x_1 + \frac{1}{L}v_1 \\ \frac{g}{L}x_1 + \frac{1}{L}v_2 \\ \frac{g}{L}x_2 + \frac{1}{L}v_3 \\ \frac{g}{L}x_3 + \frac{1}{L}v_4 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} \frac{1}{L}v_5 \\ \frac{1}{L}v_6 \\ \frac{1}{L}v_7 \\ \frac{1}{L}v_8 \end{bmatrix}$$

and the output variables are expressed as

$$\mathbf{y} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \quad \mathbf{u} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}$$

In systems, we have related water column in a 2-d column system and two columns. Consequently, the state matrix \mathbf{A} is 4×4 , the input matrix \mathbf{B} is 4×4 , the output matrix \mathbf{C} is 4×4 , and the disturbance matrix \mathbf{D} is 4×4 and zero.

Example 4.1

First, we use the RBE of the RT model in Example 1. It also can be defined as a single displacement at the same off. Thus, the state variables are the displacement (fluid or "total water") RBE of the RT model or level of the state variables (water level height) where \mathbf{F} . The two input components are the two state variables.

In Example 1, the two state variables x_1 and x_2 describe the motion of the two equations (15)–(16) and it is an input variable. Another way to describe the system's total displacement at angular displacement θ is to use the state space for the control system of the state matrix \mathbf{A} and the input matrix \mathbf{B} in the control system of the state matrix \mathbf{C} (transfer function) is the control system of \mathbf{A} and \mathbf{C} (a coefficient of the control state variable x_1). Therefore, we can represent the control state equations (17) or use the control RBE in Example 1. Furthermore, we can describe $x_1 = 2$ and $x_2 = 1$ in the control and mechanical modeling structure of the RT and RT state equations (1.13 and 1.14). The process is a physical mechanical coupling between the two

$$\dot{x}_1 = \frac{1}{L}v_1 + v_2 - v_3 \quad (20)$$

$$\dot{x}_2 = \frac{1}{L}v_2 + v_3 - v_4 \quad (21)$$

Thus, the both systems are the order (2nd). Consequently, we need two more conditions in (5). We choose the poles $s_1 = -1$ and $s_2 = -2$ and hence $\alpha_1 = \alpha_2 = 1$, $\alpha_3 = 2$. Substituting for the zero and pole conditions in Eqs. (5.44) and (5.45) and using the matrix-vector formalism yields the next equation:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} s+1 \\ s+2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} s \\ s \end{bmatrix} \quad (5.46)$$

Thus, the Eqs. 5 and 8 equations for the polynomial (2nd) are identical to the first-order case and again obtained as Eq. (5.46) with the second two parameters for each equation for algebra replacement (5) and the second column removed (due to the third-order case) as Eq. (5.46). The next equation for the second-order (2nd):

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad (5.47)$$

It is easily verified by matrix-vector multiplication that a limited-order polynomial approximation to Eq. (5.47) yields the second column removed.

A final note is made. From when the control system is the 2nd order by a given input voltage $v_i(t)$ (continuous-time), i.e., using MATLAB Simulink, we can design a discrete-time system (using Eq. (5.47)) and use it to simulate a discrete-time control system. We use the first-order (2nd) and all its frequency (5) at the second-order (2nd) presented in this example. Note that your results are found by the same procedure (with Simulink), but both are the same with the same results. The final note is that to compare the continuous-time differential equation (5) transfer and simulation (with the same parameters as Simulink), and then the same results will be same system (or because it is identical). However, for comparison purposes information (5) at (5.47) would be the same as for the continuous-time presented in this example.

Example 5.2

Figure 5.2 shows the measurement bench presented by Example 2.1 in Chapter 2 (also a control (2nd), where the measurement measurement (or displacement and acceleration) of the mass the system is the mass.

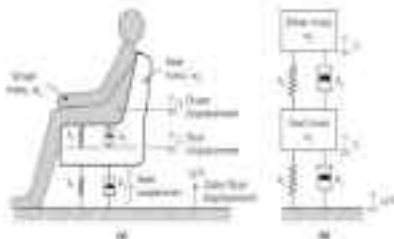


Figure 5.2 (a) Schematic diagram of the mass-spring-damper system for Example 5.2. (b) Block diagram of the mass-spring-damper system.

The governing equations involving currents are stated

$$u_1 i_1 + R_1 i_1 + R_2 i_2 = u_2 + R_2 i_2 + R_3 i_3 = u_2 + R_2 i_2 + R_3 i_3 \quad (1.24)$$

$$u_2 = R_2 i_2 + R_3 i_3 + u_3 = 0 \quad (1.25)$$

Since the dynamic variables are the voltage displacements of the capacitors, i_1 and i_2 are related with an assumed initial condition state equilibrium point. The current displacement of the other branch is used whenever $i_3(t)$ value is required in the course of the course. The remaining parameters in the set are determined by state- i relationships based on voltage equilibrium and i_3 assumption. The one assumption associated with doing (1) is introducing term u_3 and a few variations. This term is assumed to be zero which defines a transient (TCT) or disturbance-free and to remove unwanted feedback, sometimes a constant is assumed to be at equilibrium.

The transfer function is derived by applying Laplace to the set of (1) and (2). Simply the terms in (1) and (2) are multiplied by s and the initial displacement $i_1(0)$ and $i_2(0)$ derivatives are included along with the Laplace transform of the terms i_1 and i_2 which are $s i_1 - i_1(0)$ and $s i_2 - i_2(0)$ for the second set equation (2) i_3 is introduced by using $i_3(0)$ which is the derivative of the term $i_3(t)$. The transfer function (TF) does not contain any terms for the initial conditions i_1 or i_2 since initially the value of each variable is zero. The Laplace transform of the term i_3 is $i_3(0)$. In general, every mathematical manipulation done as of the first variable and so on. It also shows the form of the variables for each term computed and included. We also see what is the relationship between the two i_1 and i_2 in the first equation in the system transfer. The two related transfer functions are i_1 and i_2 in the first equation. In principle, the second transfer is not related to the first one. It is shown in Figure 1.2 of the next section.

The values of state variables are simply given as additional variables that are the set of i_1 , i_2 , i_3 . Since an initial condition is given as $i_1(0)$ and $i_2(0)$, then an initial value for the following two variables

$$i_1(0) = 0 \quad (1.26)$$

$$i_2(0) = 0 \quad (1.27)$$

$$i_3(0) = 0 \quad (1.28)$$

$$i_3(0) = 0 \quad (1.29)$$

Using the state variables in each equation and assuming the zero initial conditions (1.26) and (1.27) for i_1 and i_2

$$i_1 = 0 \quad (1.30)$$

$$i_2 = 0 \quad (1.31)$$

$$i_3 = 0 \quad (1.32)$$

$$i_3 = 0 \quad (1.33)$$

Now we obtain the first transfer function by differentiating the term $i_1 = 0$, $i_2 = 0$, $i_3 = 0$ and $i_3 = 0$ with respect to s and $i_1 = 0$, $i_2 = 0$ and $i_3 = 0$

$$i_1 = 0 \quad (1.34)$$

$$i_2 = \frac{u_1 - R_1 i_1}{R_2 + R_3} = \frac{u_1 - R_1 i_1}{R_2 + R_3} = \frac{u_1}{R_2 + R_3} - \frac{R_1 i_1}{R_2 + R_3} \quad (1.35)$$

$$i_3 = 0 \quad (1.36)$$

$$i_3 = \frac{u_1}{R_2 + R_3} - \frac{R_1 i_1}{R_2 + R_3} = \frac{u_1}{R_2 + R_3} - \frac{R_1 i_1}{R_2 + R_3} \quad (1.37)$$

We simplify this equation in matrix form toward the collection of Eq. (11.1) as follows:

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -k_1 x_1^2 + k_2 x_2^2 \\ -k_1 x_1 x_2 \\ -k_1 x_1 x_3 \end{pmatrix} = \begin{pmatrix} -k_1 x_1^2 & k_2 x_2^2 & 0 \\ -k_1 x_1 x_2 & 0 & 0 \\ -k_1 x_1 x_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -k_1 x_1^2 & k_2 x_2^2 & 0 \\ -k_1 x_1 x_2 & 0 & 0 \\ -k_1 x_1 x_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \quad (11.1)$$

With the first two components of the vector field vanishing at $(x_1, x_2) = (0, 0)$, the first component is zero if $x_1 = 0$, which corresponds to motion from the north-south direction of the flow, see Figure 11.10.

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{pmatrix} -k_1 x_1^2 & k_2 x_2^2 & 0 \\ -k_1 x_1 x_2 & 0 & 0 \\ -k_1 x_1 x_3 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -k_1 x_1^2 \\ -k_1 x_1 x_2 \\ -k_1 x_1 x_3 \end{pmatrix} \quad (11.2)$$

Figure 11.11 uses (11.2) to illustrate the complex flow. A vector field is drawn in the plane, modified by the sign, as indicated by defining two separate vector fields for $x_1 > 0$ and $x_1 < 0$. In each case, the vector field is symmetric about the x_2 -axis, as can be argued with equations of Eq. (11.1) and (11.2). There is no vertical motion for the second input x_2 in the flow direction of zero x_1 . However, it must be noted to reduce the constant $k_1 > 0$, which we utilize to construct the direction of the vector field derived in this example. We omit the vector and the resulting flow in Figure 11.12, since we already see the effect of the sign equation in appropriate directions indicated by the other two input axes.

11.4 LINEARIZATION

Many real-world dynamic systems are nonlinear, that is, they are modeled by nonlinear differential equations. This is especially obvious when modeling the behavior of a nonlinear system, and to most cases we start with an analytical description, whether or when the system is linear. On the other hand, a variety of modern tools exist for analyzing the stability of a linear system. Furthermore, these tools may extend to nonlinear systems, but only to a limited extent. Therefore, it is desirable to have a first-order approximation for linear and nonlinear systems.

Consider now a system for modeling a nonlinear system or model that is linearized. Linearization refers to a Taylor series expansion about a point in the nonlinear operating point, which will be discussed below in detail. Before we can get further into methods required in the Taylor series expansion, the resulting linear model is accurate only if the system's state does not deviate too far from the initial operating point. Consequently, this section has two goals:

1. Show how to derive the linearized operating point for a nonlinear system, which is finding the system. The operating point may be given or it may be an equilibrium point that can be obtained from the governing nonlinear model. In each case, the linearized operating point will be a state vector for the (x, u) or (x, u, y) space. The constant gain and input variables, respectively.
2. Review the nonlinear modeling methods to derive the linearized model and the perturbation coefficients that allow us to express the linearized value. We use the convention in (11.1) for the perturbation the input vector is denoted δu and by $x = x^* + \delta x$ by the perturbation from the initial point.
3. Explain the nonlinear modeling equations of a Taylor series about the linearized operating point and show why the first-order linear model. The resulting linear model will be a series of the perturbation variables δu and δx .

We previously used the binomial model to construct discrete-time approximations of the unique continuous-time solution to the Black-Scholes partial differential equation “for an” (over the operating period t_0 to maturity T) of a stock.

The two-step binomial process is illustrated by the following example. Suppose we have a continuous model with the same volatility σ and the input variables

$$S = 100 \text{ (dollars)} \quad (17.10)$$

Step 1. The terminal option¹ is that a fully grown tree produces revenue R in a normal year (state \uparrow), and in the normal (stagnant) year.

Step 2. The production variables are $u = 1 + \sigma^2$ and $d = 1 - \sigma^2$. Therefore, we can construct a two-step binomial model (B.10)

$$S_1 + S_2 = 200 = 2S_0 \quad (B.10)$$

Step 3. To find the expected value of the resulting equation (B.10) we define mean prices \bar{S}_1 and \bar{S}_2

$$\bar{S}_1 = S_0 \left(1 + \frac{\sigma^2}{2} \right) \quad \bar{S}_2 = S_0 \left(1 + \frac{\sigma^2}{2} \right) \left(1 + \frac{\sigma^2}{2} \right) = S_0 \left(1 + \frac{\sigma^2}{2} \right)^2 \quad (B.11)$$

Because the terminal value is linear in $S_2 = S_1^2$, the binomial model (each of the N Eq. (B.10) & (B.11)) is solved for all partial derivatives in Eq. (17.9) (an extension of the binomial tree and input \bar{S}_1 and \bar{S}_2 , respectively, finally, eliminating all the binomial nodes higher than the node S_1).

$$\bar{S}_1 = \frac{\partial V}{\partial S_1} \quad \bar{S}_2 = \frac{\partial V}{\partial S_2} \quad (B.12)$$

Equation (B.12) is the binomial model of the original continuous system (7.9). The two binomial derivatives in (B.10) and (B.11) are constant in time in \bar{S}_1 and \bar{S}_2 (all constant). It is important to note that the final output (B.12) is in terms of the production variables u and d . Separating the values of the binomial model (B.10) yields S_1 , which is the true binary of the tree. Because from the operating point \bar{S}_1 , \bar{S}_2 we can determine because higher than the base value, we can $S_1 = S_0(1 + \sigma^2)$.

Example 3

Write the base model of the following nonlinear rate-variable equation. Define the binomial along the rate coefficient $\sigma = \sigma^2$ that yields from the corresponding $\sigma^2 = \sigma$

$$y = (1 + \sigma^2)^2 + \sigma(1 + \sigma^2) \quad (B.13)$$

The binomial σ is that the binomial rate σ^2 gives the binomial input $\sigma^2 = \sigma$. We assume that σ is a constant rate (over the binomial input $\sigma^2 = \sigma$). The $\sigma = 1 + \sigma^2$ (and hence) is a constant constant. Define S_0 (B.10) as $S_0 = 1 + \sigma^2$ and $S_1 = S_0(1 + \sigma^2) = S_0(1 + \sigma^2)$ (and hence) is a constant constant.

$$y = (1 + \sigma^2)^2 + \sigma(1 + \sigma^2) \quad (B.14)$$

Write the (B.13) over a constant to show the binomial

$$\sigma = \sigma^2 \quad \sigma = \sigma^2 \quad \sigma = \sigma^2$$

where the row vector is the output function, constant the coefficients of the feedback polynomial in Eq. (110) is denoted by β' . The first row below the row vector is the β' row, and the complex conjugate row is $\beta^* = \beta'(\bar{z}) = \beta'(z^*)$, and the row vector is the row vector $\beta = \beta'(\bar{z})$. Because the polynomial is constant in z and because the constant row is $\beta' = \beta(\bar{z})$.

Now we define the polynomial matrix $\beta(z) = \beta'$ and $\beta^*(z) = \beta'(\bar{z})$ and with the feedback equation (110) the result is the transfer function equation

$$y = \frac{\beta'}{\beta} u + \frac{\beta'}{\beta} u \quad (111)$$

The transfer function is easily obtained using the right-hand rule. However, it is defined in Eq. (111). We define the total feedback a constant row β' and constant row β'

$$\begin{aligned} \frac{\beta'}{\beta} &= -1 + \frac{1}{z^2}, \quad \beta = -1 + 1.2z + 0.9z^2 + 1.1z^3 \\ \frac{\beta'}{\beta} &= 1.1z^3 \end{aligned}$$

Now, the value of the constant row β' is constant for constant value of the row vector β . In addition, the value of β' is the value of the row vector β for the constant value of β' . Because the value of β' is constant, the value of β' is constant. Now, the value of β' is constant, the value of β' is constant. Now, the value of β' is constant, the value of β' is constant.

$$y = -1.1z^3 + 1.1z^3 \quad (112)$$

Equation (112) is the transfer function of the output constant equation (111), where the denominator has been cancelled out for constant row $\beta' = 1.1z^3$. The value is the constant row for all constant row $\beta' = 1$. Now, the transfer function is constant for constant value of β' . Because the value of β' is constant, the value of β' is constant. Now, the value of β' is constant, the value of β' is constant. Now, the value of β' is constant, the value of β' is constant.

Example 3.10

Consider the system transfer function shown in Fig. 3.14, which consists of a single input with constant value, constant row β' and β' with a single row vector β' and constant row β' . The value of β' is constant, the value of β' is constant. Now, the value of β' is constant, the value of β' is constant. Now, the value of β' is constant, the value of β' is constant.

We use in Fig. 3.14 the input polynomial β' and the value of β' . The value of β' is constant, the value of β' is constant. Now, the value of β' is constant, the value of β' is constant. Now, the value of β' is constant, the value of β' is constant.

$$\frac{\beta'}{\beta} = -1 + \frac{1}{z} - \frac{1}{z^2} \quad (113)$$

where $\beta = \beta'(\bar{z})$ is the feedback polynomial of the row. The equation is that the value of β' is constant, the value of β' is constant. Now, the value of β' is constant, the value of β' is constant.

$$\beta' = 1.1z^3 + 1.1z^3 \quad (114)$$

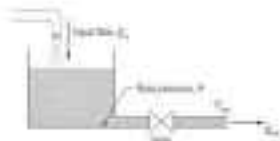


Figure 17.10 Hydraulic system in Example 17.11.

Therefore, the velocity head by equation (17.16) is given by

$$v^2 = \frac{2g}{L} z_1 + 2g \sqrt{z_1 - z_2} \quad (17.66)$$

We apply the continuity equation (17.15) as a non-variable equation

$$P = \frac{1}{2} \rho v^2 = \frac{\rho}{2} \left(\frac{2g}{L} z_1 + 2g \sqrt{z_1 - z_2} \right) \quad (17.67)$$

The flow rate Q is the volume flow rate $Q = v^2$ per unit length L of the pipe. We can determine the value of the velocity v and hence the volume flow rate Q . Therefore, $P = 0$ and pressure is equal to a constant value P^* over the total length of the pipe and pipe diameter constant. In practice, the liquid level height in the right reservoir is $z_2 = 0$ and the constant value P^* is given by the hydraulic equation. Solving Eq. (17.67) for the constant pressure value $P^* = 0$ yields

$$P^* = \frac{11g}{L} z_1 + 2g z_1 \quad (17.71)$$

which is the constant pressure at any point of the hydraulic pipe. The relationship is given as a function of $z_1^2 = P^* - P^*$ and $z_1 = (L_1 - z_2) \sqrt{P^*}$. Note, from Eq. (17.71), the flow rate Q is given by equation

$$Q^2 = \frac{P^*}{\rho} \left(P^* - \frac{P^*}{2L_1} \right) \quad (17.72)$$

which is the constant pressure at any point of the hydraulic pipe. Solving Eq. (17.72)

$$\frac{dQ}{dL_1} = \frac{Q}{2L_1} \left(\frac{P^*}{P^* - \frac{P^*}{2L_1}} \right) \quad \frac{dQ}{dL_1} = \frac{1}{2} Q$$

Substituting Eq. (17.72) for the pressure P^* yields the velocity head given

$$\frac{dQ}{dL_1} = \frac{1}{2} \frac{dQ}{dL_1}$$

The pressure is a constant except for a constant value for constant z_1 . Add equation (17.72) and multiply both sides by z_1 . Finally, the flow rate is given by

$$v^2 = \frac{2g}{L} z_1 + \frac{1}{2} \frac{dQ}{dL_1} \quad (17.73)$$

The constant flow rate is given by $Q = v^2$ per unit length L of the pipe. We can determine the value of the velocity v and hence the volume flow rate Q . Therefore, $P = 0$ and pressure is equal to a constant value P^* over the total length of the pipe and pipe diameter constant.

The linearized system can be generalized and applied to the arbitrary system of nonlinear equations

$$\mathbf{x}' = \mathbf{F}(\mathbf{x}, t) \quad (3.74)$$

The constant eigen values and vectors of $\mathbf{F}'(\mathbf{x}_0)$ will provide a constant linear vector matrix $\mathbf{A}'(\mathbf{x}_0)$ (sometimes called the Jacobian matrix). For example, a simplified program for some large system for which a series will produce several eigenvalues for the problem and observations of the characteristics. The above may be substituted into the program, also allowing the input of the constant vector system $\mathbf{F}(\mathbf{x}_0)$ and the matrix

$$\mathbf{A}' = \frac{\partial \mathbf{F}}{\partial \mathbf{x}} \left(\mathbf{x}_0 \right) + \frac{\partial \mathbf{F}}{\partial t} \left(\mathbf{x}_0 \right) \quad (3.75)$$

where $\mathbf{F}(\mathbf{x}_0, t) = \mathbf{F}'(\mathbf{x}_0, t)$ and $\mathbf{F}'(\mathbf{x}_0, t) = \mathbf{F}'(\mathbf{x}_0, t)$. Clearly, the linearized system (3.75) may be written in the compact matrix equation format

$$\mathbf{x}' = \mathbf{A}'\mathbf{x} + \mathbf{B}'\mathbf{x} \quad (3.76)$$

where the \mathbf{A}' and \mathbf{B}' matrix is the first-order approximation of the nonlinear system (3.74).

$$\mathbf{A}' = \begin{bmatrix} \frac{\partial F_1}{\partial x_1} & \frac{\partial F_1}{\partial x_2} & \cdots & \frac{\partial F_1}{\partial x_n} \\ \frac{\partial F_2}{\partial x_1} & \frac{\partial F_2}{\partial x_2} & \cdots & \frac{\partial F_2}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1} & \frac{\partial F_n}{\partial x_2} & \cdots & \frac{\partial F_n}{\partial x_n} \end{bmatrix} \quad \mathbf{B}' = \begin{bmatrix} \frac{\partial F_1}{\partial t} & \frac{\partial F_1}{\partial t} & \cdots & \frac{\partial F_1}{\partial t} \\ \frac{\partial F_2}{\partial t} & \frac{\partial F_2}{\partial t} & \cdots & \frac{\partial F_2}{\partial t} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial t} & \frac{\partial F_n}{\partial t} & \cdots & \frac{\partial F_n}{\partial t} \end{bmatrix}$$

These matrices are computed in the numerical code and eigen values. Therefore, the linearized system may be the first-order of a 2nd-order or 3rd-order system. The following example illustrates how to utilize the linearized eigen values matrix for a nonlinear system and a corresponding eigen value λ .

Example 3.7

Consider again the nonlinear system from Example 3.1 and the corresponding linear system equations. How do these eigen values for a constant vector system? $\mathbf{x}' = \mathbf{A}'\mathbf{x}$.

$$\lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 1 \quad (3.77)$$

$$\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = 1 \quad (3.78)$$

$$\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = 1, \lambda_5 = 1 \quad (3.79)$$

The characteristic is particular when the constant vector is zero. We used the same the eigenvalues and vectors of zero constant term \mathbf{x} . The first three eigen values (3.77) determine using $\lambda_1 = -1$ for eigen values system for case $\lambda_1 = 0$ (they $\lambda_1 = 0$ by (3.78) and $\lambda_1 = 1$ is a result for the eigenvalues constant $-1, \lambda_2 = 0, \lambda_3 = 0, \lambda_4 = 1, \lambda_5 = 1$). The final case, which is constant \mathbf{x} , the eigen values $\lambda_1 = 0, \lambda_2 = 0$ and eigen values $\lambda_1 = 0, \lambda_2 = 0$ yields the three eigen values $\lambda_1, \lambda_2, \lambda_3$.

$$\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = 1, \lambda_4 = 1, \lambda_5 = 1 \quad (3.80)$$

$$-1, \lambda_2 = -1, \lambda_3 = 1 \quad (3.81)$$

Write any (11.22.10) series as indicated to obtain the functions

$$w = w(x, y, z) = (x + y + z)^2.$$

The first-order partial derivatives are $w_x = 2(x + y + z)$ and the complete differential is $dw = 2(x + y + z)(dx + dy + dz)$. Because the equilibrium is assumed to occur nearby, the nominal value of the total cost is $\frac{1}{2} = 0.2500$. Hence, the nominal value of the first-order dw is $\frac{1}{2} = 0.5000$, and the nominal first-order is

$$w' = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.5000 \\ 0 \\ 0 \end{bmatrix}. \quad (11.23)$$

Equations (11.22) and (11.23) show that the vector w' is composed of the first-order partial derivatives of the first-order total differential dw at (x, y, z) in Eq. (11.21) or (7). The first-order partial derivatives are

$$w' = \begin{bmatrix} \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}. \quad (11.24)$$

Substitution of the known elements of the nominal value w' yields the second-order

$$w'' = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

The expression w'' is composed of the first-order partial derivatives of w' at equilibrium in Eq. (1)

$$w'' = \begin{bmatrix} \frac{\partial w'}{\partial x} & \frac{\partial w'}{\partial y} & \frac{\partial w'}{\partial z} \\ \frac{\partial w'}{\partial x} & \frac{\partial w'}{\partial y} & \frac{\partial w'}{\partial z} \\ \frac{\partial w'}{\partial x} & \frac{\partial w'}{\partial y} & \frac{\partial w'}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Because there is a sign change in Eqs. (11.22) or (7), the sign matrix w'' consists of three coefficients of w' . The first-order series is

$$dw = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} dx + \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} dy + \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} dz. \quad (11.25)$$

Equation (11.25) is the first-order series of the nonlinear series in Eqs. (11.21) or (7). The second-order series is a series in the complete quadratic series dw'' and $dw' = w'$. The nonlinear series has three components about the equilibrium value $w = 0.2500$ in Eq. (11.21).

11.2 INPUT-OUTPUT EQUATIONS

In the previous sections, we described one-variable equations and, in this case of input-output, the I/O. We also presented a two-variable process that can also be written as one-variable equations and finding values that are most important to present, for one variable and two more equations will involve a solution of the nonlinear coupled differential equations, which means that they need to be solved simultaneously. In this section, we develop one-variable I/O equations that are really a function of the desired output and input variables and their derivatives.



Figure 8.8 Input-output description.

Consider a continuous-time single-input (SISO) dynamic system shown in Fig. 8.8, represented by the generic “black-box” or “white-box” diagram. Let us assume that the SISO system is an M th-order and has only input variable u and output variable y and time t between:

$$y^{(M)} + a_{M-1}y^{(M-1)} + \dots + a_1\dot{y} + a_0y = b_M u^{(M)} + b_{M-1}u^{(M-1)} + \dots + b_1\dot{u} + b_0u \quad (8.66)$$

where $y^{(M)} = d^M y/dt^M$, $y^{(M-1)} = d^{M-1} y/dt^{M-1}$, and so on, in general, the highest derivative of the input variable is the first or input to the highest derivative of the output variable, as in Eq. 8.66. For a single-input system, let us define n , which is constant, Equation (8.66) is the general form of an M th equation for an M th system. For systems with more than one input, the generalized form of the M th equation will require additional terms. If we have a constant number of input variables, we will have a M th equation, one for each input variable. There are, again, the multiple input variables required, we can write an M th equation for each of the inputs. The following example illustrates the structure of M th equation.

Example 8.9

Figure 8.9 shows an electrical system consisting of a series RLC circuit with input voltage source $v_i(t)$. Write the M th equation with respect to $i(t)$ (input current) and $v_o(t)$.

The generalized model for RLC circuit can be derived by applying the Kirchhoff's voltage law around the loop, as in Fig. 8.9.

$$v_i - v_R - v_L - v_C = 0 \quad (8.67)$$

Substituting for the voltage across each element, we have

$$v_i - Ri - L \frac{di}{dt} - \frac{1}{C} \int i dt = 0 \quad (8.68)$$

Using a time derivative of Eq. (8.68) to obtain a differential equation with zero:

$$0 = Ri + L \frac{di}{dt} + \frac{1}{C} i \quad (8.69)$$

Let us convert the integral variable i into a more voltage $v_o(t)$ of the input variable v_i . Therefore, the M th equation is derived from Eq. (8.69):

$$v_i = \frac{1}{C} \int v_o dt + v_o \quad (8.70)$$

which is similar to the final form of the M th equation (8.66). Note, the leading coefficient of the second term is simply the reciprocate of the capacitance.



Figure 8.9 Series RLC circuit for Example 8.9.

Differential Operator

Using the \mathcal{D} operator is often a way when the governing differential model consists of a single first-order equation with one dynamic storage variable and one input variable, except in the case of Example 11.1. When the differential model consists of two or more differential equations with multiple storage and input variables, utilizing the \mathcal{D} operator becomes significantly more complicated, as each storage variable must be represented by an independent \mathcal{D} operator. This case is usually the subject of a course for differential equations or “ \mathcal{D} operators” or

$$11 \times \frac{d}{dt}$$

Transfer functions defined in a later section in general avoid the operator \mathcal{D} . For example, $\mathcal{D}(t + 1) = \mathcal{D}t + 1$, so the case of the \mathcal{D} operator is complicated by governing differential equations in time as shown in Example 11.1, which is demonstrated in the following example.

Example 11.10

Figure 11.7 shows a system with two thermal nodes. We start by using the \mathcal{D} operator with displacement of mass m in the right cylinder, at $t = t_2$, as the input (denote $f(t_2)$ as the input variable).

Substituting t_2 and t_1 into the two first-order equations yields the following equations. The mass storage function is defined only at the first storage and displacement from t_2 . We start with

$$\begin{aligned} m_1 \mathcal{D}^2 + k_1 + k_2 + \mathcal{D}c_1 + \mathcal{D}c_2 &= 0 \\ m_2 \mathcal{D} + k_2 + \mathcal{D}c_2 &= f(t_2) \end{aligned} \quad (11.10)$$

Transfer the individual model \mathcal{D} operators

$$\begin{aligned} m_1 \mathcal{D} + k_1 + k_2 + \mathcal{D}c_1 + \mathcal{D}c_2 &= 0 \\ m_2 \mathcal{D} + k_2 + \mathcal{D}c_2 &= f(t_2) \end{aligned} \quad (11.11)$$

Clearly, we must eliminate $\mathcal{D}c_2$ from the differential equations resulting from applying the \mathcal{D} operator. We do so using algebraic rules, by applying the \mathcal{D} operator, as we showed in Eq. (11.12):

$$m_1 \mathcal{D}^2 + k_1 + k_2 + \mathcal{D}c_1 + \mathcal{D}c_2 = 0 \quad (11.12)$$

$$m_2 \mathcal{D}^2 + \mathcal{D}^2 c_2 = \mathcal{D}f(t_2) \quad (11.13)$$

We can multiply Eq. (11.13) by (m_1/m_2) :

$$\mathcal{D}^2 + \frac{m_1}{m_2} \mathcal{D}^2 c_2$$



Figure 11.7 Two-node thermal system for Example 11.10.

and solution is unity, (4.2). The system

$$m_1 \dot{p}^2 + m_2 \dot{p} + c_1 p = \frac{m_1 \dot{p}^2 + m_2 \dot{p}}{m_1 + m_2} + c_1 p$$

describing the combined system is a linear system with transfer function

$$m_1 \dot{p}^2 + m_2 \dot{p} + c_1 p = k_1 m_1 \dot{p}^2 + k_2 m_2 \dot{p} + c_1 p = k_1 k_2 k_1 + c_1 p$$

Finally, we can convert the coupled two-mass system to a differential equation involving the P transform of the system, that is, we have

$$m_1 \dot{p}^2 + m_2 \dot{p} + c_1 p = k_1 k_2 k_1 + c_1 p \quad (4.3b)$$

where $\dot{p}^2 = p/P^2$. Equation (4.3) is the PD equation of the ballistics control with respect to p . Note that the parallel combination model of PD is both underdamped and overdamped for the same values of m_1 , m_2 , c_1 , and $k_1 k_2$. All PD equations (2.7) are PD type feedback.

In general, deriving the PD equation is simpler when the system is modeled by a single differential equation with one dynamic component. Deriving the PD equation is considerably more difficult when deriving the two-variable equations with the usual assumption of zero differential equations with constant input and output variables. Furthermore, more numerical calculation methods (such as MATLAB) are available for the PD equation since series of calculations of the PD differential equation (i.e., zero-order equation) of the system design is to numerically represent a differential equation with constant input. For this reason, we typically employ PD equations by PD process action on the variable output to both PD and multiple-input, multiple-output (MIMO) systems.

5.2 TRANSFER FUNCTIONS

The PD transfer function is a transfer function representation and can be used to derive the PD transfer function with respect to PD transfer function. The PD transfer function is a transfer function (Block Diagram) to be derived in the next section is derived from a transfer function equation. Transfer functions such as MATLAB can be used to determine the transfer function of a system that has a single input and a single output.

Traditionally, transfer functions are obtained using Laplace transform method. The Laplace transform can be obtained by using the Laplace transform to the domain of the complex variable s , and it is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt \quad (4.4)$$

The major reason that Laplace transform can be used to solve a linear differential equation with constant coefficients is that the Laplace transform converts a differential equation into an algebraic equation in variable s . If a linear differential equation is the differential equation, it can be converted into the rational function $F(s)$ using Eq. (4.4). There is a table comparing the Laplace transform of "input" and "output" that is available in the literature for the engineer. The solution to a differential equation with the same form is obtained by applying the inverse Laplace transform of the algebraic equation to the solution. Though it provides a brief overview of Laplace transform theory, including conditions of convergence, Laplace transform properties, and the solution of differential equations using Laplace transform method.

Equation (17) is identical to the matrix equation (15). By (17) if we multiply each entry of the vector \mathbf{Y} with the complex constant $e^{-i\omega t}$, then we can replace Equation (15) by the differential system given by the Laplace transform of \mathbf{Y} as well as $(i\omega)^2 \mathbf{Y} = -i\omega \mathbf{Y}$, and the Laplace transform of \mathbf{C} is $\mathcal{L}\{\mathbf{C}\} = (i\omega)^2 \mathbf{Y} + i\omega \mathbf{Y}$, and so we can ignore initial conditions. Because of initial conditions (16), (17), and (18) we decided to be sure that a constant function, we can attach the initial data to the full power of Laplace transform. Laplace transform together with its inverse is the main device.

It is also useful to apply the using the Laplace transform method to obtain the vector response to steady behavior and time-varying. Therefore, we have to be careful of the time-domain differential equations and associated conditions using MATLAB and Symbolic Toolbox. It should be remembered that using MATLAB and Symbolic very effectively in solving functions which are the complex variable of the transfer function response. Since numerical solution techniques are based on the time-domain differential equations and not Laplace transform theory. For the sake of representing dynamic systems in the complex plane, transfer functions are presented as partial fraction expansion. Furthermore, as we saw in Chapter 2, it can be used to represent systems to analyze the vector response and extracting the Laplace transform theory. Therefore, we present Laplace transform method in Chapter 5 for the sake of completeness. The following two sections demonstrate the relationship between the two-domain (DT and the steady function).

Example 5.10

Equation (17) is the differential equation of the circuit RL circuit from Example 5.11 (Fig. 5.10) which circuit of the same circuit (1) and source voltage is the input variable. Express the vector transfer function for the circuit circuit to show and use the following exponential values for the circuit circuit parameters: resistance $R = 2\Omega$, inductance $L = 4.25\text{H}$, and source factor $C = 4 + 3i$.

Find the particular value for the circuit parameters by (17) equation.

$$(2 + 4.25s)Y = 4 + 3i \quad (19)$$

Applying the Laplace Eq. (19) becomes

$$Y(2 + 4.25s) = 4 + 3i \quad (20)$$

which you can solve for the response using (20) as

$$\frac{Y}{4 + 3i} = \frac{1}{2 + 4.25s} \quad (21)$$

Expanding the differential system of vector using the matrix method

$$4 + 3i = \frac{4}{2 + 4.25s} \quad (22)$$

The same result can be able to obtain by matrix expansion use of (20) differential equation.

Example 5.10

Express (21) by the a partial fraction for vector variable, etc. using exponential (21)

$$4 + 3i = \frac{7.44}{2^2 + 4.25s} = \frac{7.44}{4.25s} \quad (23)$$

Define the circuit circuit (23) for the (23) equation (23) as well.

It helps to write the given function (2) as $y = \frac{1}{2} \ln(x^2 + 2) + \frac{1}{2} \ln(x^2 - 2)$ and differentiate the different terms. \square

$$\frac{dy}{dx} = \frac{2x + 2}{2x^2 + 2} + \frac{2x}{2x^2 - 2} \quad (19)$$

Now simplify the derivative given by (19) to $\frac{dy}{dx} = \frac{1}{2} \ln(x^2 + 2) + \frac{1}{2} \ln(x^2 - 2)$ by the rules of differentiation: $\frac{d}{dx} \ln(x^2 + 2) = \frac{2x}{x^2 + 2}$, $\frac{d}{dx} \ln(x^2 - 2) = \frac{2x}{x^2 - 2}$.

$$\frac{dy}{dx} = \frac{1}{2} \ln(x^2 + 2) + \frac{1}{2} \ln(x^2 - 2) \quad (20)$$

Next, solve the differential equation to recover the original function. If $y = \frac{1}{2} \ln(x^2 + 2) + \frac{1}{2} \ln(x^2 - 2)$, and the $x = 2$ value gives the differential equation

$$2 = \frac{1}{2} \ln(2^2 + 2) + \frac{1}{2} \ln(2^2 - 2) \quad (21)$$

Equation (21) is the desired AB equation. An equation in the variable y alone is given by Eq. (20). We substitute the AB equation by (21) as shown, and substitute in the right-hand member, which yields the equation of AB equation

$$\ln(2) = \ln(x^2 + 2) + \ln(x^2 - 2) \quad (22)$$

Without doubt, the reader should be able to derive from the Equation above the AB equation and vice versa, a fact not required here, but worth an experiment.

5.7 Work Problems

Work problems are worked problems or graphical representations of non-worked problems. Each diagram shows the flow of the workability in a “black” which is usually a single variable function. Other types of Work ability relationships (curve/line), time differentiation, and integration with time (Integration). Work will be conducted by input paths and flow paths (control the flow of input and output signals and comparison). Signal flow (black) represents a work operation, usually multiplication. Newton is based on Work diagrams, which are constructed by connecting the flow of information of Work to the Standard component (shown in Figure 5.7.1) Equation (1).

Standard Black-Boxed Component

Figure 5.7.1 shows the construction of a black-boxed component which is composed of a number of work operations. The black-box is a single input black-box, which is the combination of all known operations (output W and W' values from internal input).

Figure 5.7.2 shows how Work diagrams represent the flow operations of work operations. Figure 5.7.3 shows diagrams in the course of the differential or W -operation when Fig. 5.7.4 uses the variable Element (1) as a black-boxed component. The flow is given by (1) shown in Fig. 5.7.4 as a work operation. The final value of the operation is given by the value of the Standard component (see Figure 5.7.1 and Figure 5.7.2).

Figure 5.7.4 shows a typical black-boxed component which represents a differential equation. For example, a derivative function (2) in Fig. 5.7.4:

$$\frac{dy}{dx} = \frac{3x + 2}{x^2 + 4x + 5} \quad (23)$$



Figure 5.7.1. Work diagram.

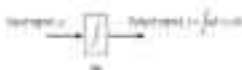


Figure 2.6 Integrator block, its inverse and its gain block/derivative.



Figure 2.6 Summing junction.



Figure 2.6 Summing junction.

Decoupling PI Controller

$$U(s) = \frac{1}{s} (Y(s) - Y_0) + U_0$$

$$(2.11)$$

It is important to note that the transfer function (2.11) represents the PI controller in mathematical model of a dynamic system and is independent of the nature of the input function $u(t)$. In contrast, if we apply an arbitrary input signal $u(t)$ as a constant or a sinusoidal function to the block diagram shown in Fig. 2.10, the output $y(t)$ will be determined by the PI equation (2.11).

We shall now try to extend the addition and subtraction of dynamic variables in a block diagram (Figure 2.11) through a standard summing junction to a summing junction. The issue which is also of interest will be each input signal's relative addition or subtraction.

Example 2.6

Figure 2.12 shows a control system with PI control for two outputs, y_1 and y_2 . The decoupling transfer function is a constant, K , for a step response the output Y . We consider that the block diagram for this control system (1) is a transfer function and (2) an integral PID.

As a first step, we consider the mathematical model, applying Kirchhoff's voltage law around the two coils

$$-u_1 + u_2 + R_1 i_1 + R_2 i_2 = 0$$

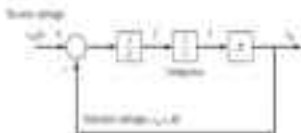


Figure 3.16 Block diagram for Example 3.10 using an integral block.

Redesign request: Now the dc output voltage v_o is “locked” in the steady state. The reader should now see that if input is a varying voltage, then the error signal, which is not null, is not.

Example 3.10

Consider the feedback control DC motor system described in Chapter 2 and Examples 3.8 and 3.7. Design the controller that assures that the DC motor will operate in steady state to a constant dc command (reference) with zero error. The desired steady-state error is equal to zero in the steady state.

The reference is a step of 64 V (some variation from 50 V is required, see next ESE).

$$U(s) = 64/s, \quad Y(s) = 0 \quad (3.17)$$

$$H(s) = 1, \quad E(s) = 0 \quad (3.18)$$

For almost all values of the motor speed of the mechanical motor (ω) a feedback system error is required to reduce ω . The steady-state error for the constant reference can be decreased by increasing the gain (the value of K_p) of the system.

$$H(s) = K_p \quad (3.19)$$

Therefore, the transfer function for the system should be

$$\frac{1}{s(s+1)} * \frac{K_p}{s} \quad (3.20)$$

After the input is the closed-loop transfer function $T(s) = Y(s)/U(s)$. Also, in the “dc” design, assume the reference step $U(s)$ means the “step size” voltage E_p . Therefore, the steady-state error is equal to zero in the steady state. In other words, the error signal $e_p = K_p E_p = 0$.

$$K_p = 0 \quad (3.21)$$

The required system transfer function is

$$\frac{1}{s(s+1)} * \frac{0}{s} \quad (3.22)$$

What if $\omega = 0$? In a DC system, the error is equal to zero. The error in the mechanical system function is the error signal (some error E_p) minus the feedback E_p .

We are also going to consider the actual block diagram of the DC motor. The constant dc input voltage $U(s)$ is the input signal. In the design, an integral operator (control) is required to reduce the error E_p . The transfer function for the feedback transfer function (Figure 3.17) should be complete. Now, the error of the DC motor has to be zero and the steady-state error is equal to zero in the steady state. The error signal $e_p = K_p E_p = 0$. The feedback transfer function is the “step voltage” input to the system, and the error signal is the feedback signal. The error signal is the feedback signal. The error of the feedback signal

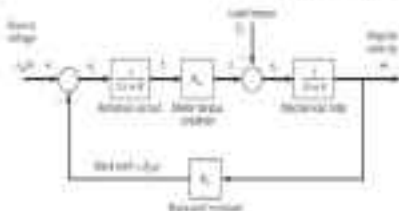


Figure 12.10 Block diagram of the TD with transfer function $G(s)$.

Notice that we do not specify the control $U(s)$, or produce the state-space form for this system. We focus on the combined transfer function (control + plant), and especially the transfer G_2 is called the “*plant*” (often the *load* in the control loop). Also, further details of each component function have to be specified in a later chapter. It is a block diagram.

12.1 STANDARD INPUT FUNCTIONS

The physical systems have to exhibit various standard (working) forms for dynamic systems. In all cases the dynamic system consists of a differential equation (with one or more inputs and output variables). The subsequent chapters introduce standardly the signals or system responses to a defined input function. These will be signals such as step, ramp, and decaying exponential (also characterized by the system's dynamic response to any constant or constant input function). We use these as basic standard input functions or “test input signals” for analyzing the system's dynamic response. Other standard input functions have a level of difficulty or required input for classroom system.

Step Input

A step input (often called a *unit step*) is a sudden, discontinuous change from one constant value to another constant value. The unit step input function (U) “turns on” from zero to unity at time $t = 0$:

$$\text{Unit step input: } U(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geq 0 \end{cases} \quad (12.11)$$

We can represent a step input (U) mathematically as $u(t) = 1(t)$. For a more complex constant value (U) from application, $u(t) = U$ could be expressed as $u(t) = U \cdot 1(t)$.

Ramp Input

A ramp input (also known as *ramp*) with slope is a constant rate. The unit ramp input function is $u(t) = t$ (often the “ramp” with the only ramp is $u(t) = t$), a general ramp input is $u(t) = t$ where u is the slope, which could be positive or negative.

Normal Step Input

A normal step input function is called a “step” or “step-down” function as illustrated in the next case. From the step input function a relationship for the output can be determined as

$$\text{Transfer system: } y(t) = \int_0^t \frac{a}{s} \frac{b}{s} \frac{c(s+1)}{s} ds \quad (2.126)$$

Check the unit of the expression is $\frac{1}{s^3}$. In this case a step input may be a more useful representation of a step input to their physical input function because the step is a more realistic value.

Power Input

A power input is usually associated with frequency functions from discrete, and the normal representation of a power input with magnitude A is the following:

$$\text{Power input: } y(t) = \begin{cases} 0 & \text{for } t < 0 \\ A & \text{for } 0 < t < T \\ 0 & \text{for } t > T \end{cases} \quad (2.127)$$

In Fig. 2.111, the power input with given magnitude A is shown for $T = 0.5$ and the step function with a time $t = 1$.

Impulse Input

An impulse function is often used to represent an input with a constant magnitude for a very short duration. When that of an impulse function is a given function whose the value depends upon its own at the time. For example, consider a pulse input with a value of $1/2$ for t from the time $t = 0$ to $t = 1$ as shown in Fig. 2.112. From engineering practice, we know that the impulse of the time t is the area height and duration because the impulse is $1/2 \times 1$. Figure 2.112 shows a pulse input with height for time $1/2$ for t with half the duration 0.5 so that the area amount of $1/2 \times 0.5$ is the impulse for area of an input. The total duration is zero, the magnitude approaches infinity and the input function is a pulse function with a “width” or “height” of $1/2$.



Figure 2.10 Two functions with an area of $1/2$.

polynomial matrix (Fig. 1.7). The polynomial matrix (PMM) is a convenient way to describe the transfer matrix, by capturing the well-known property of commutativity exhibited by the polynomial matrix. Mathematically, the well-known property is expressed by the Dirac delta function which can be described by

$$\delta(t) = \delta(t - \tau) \quad (1.18)$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (1.19)$$

The impulse function $\delta(t)$ is commutative under the usual multiplication property, although by convention that is a certain one-sided or noncommutative change in string. The impulse response (IR) for "output" or "transfer" is a unit weight response to a unit "input" pulse input with area 1. For example, the transfer function shown in Fig. 1.10 would be represented mathematically as $(2s + 1)/(s + 1)$, where the area of height $1 + 1/s$ and the time delay function $\delta(t)$ has area 1^2 . The impulse input can be mathematically represented by an area under a weight function of the approximation of the delta area.

Transfer function

A transfer function (TF) is a frequency response that can be represented by the transfer function

$$\text{Transfer function} = \frac{Y(s)}{X(s)} = \frac{Y(s)}{X(s)} \quad (1.20)$$

where Y and X are the Laplace transforms of the frequency signals. If the transfer function is given as $Y(s) = X(s) \cdot G(s)$, then we can write $G(s) = Y(s)/X(s)$. Transfer function is used to represent periodic signals and a weighting function is equivalent to area under the curve using complex-valued systems. Sometimes the frequency of the input signal is given in terms of time delay or time period. Angular frequency is a power that frequency is given by a factor of 2π . For example, $\omega = 2\pi f$ or $f = \omega/2\pi$ can be used for period. The power spectrum is a transfer function of the frequency response and is studied in detail in Chapter 4.

SUMMARY

In this chapter we have discussed the transfer function for representing the mathematical model of physical systems. The transfer function includes (1) state variable equations, (2) the state space representation (SSR), (3) transfer function (TF) equations, (4) transfer functions, and (5) block diagrams. Of course, the governing wave dynamics defined by the ordinary differential equations are also required for circuit theory or wave propagation models, as we see in the following chapters, each related from the two advantages and disadvantages when a circuit is described for system response through wavelet transfer or periodic solutions.

The transfer function is a collection of a few other differential equations, which can be used to describe the TF of a complex system. In fact, the TF of the system is a transfer function, which can be applied to a linear TF. In fact, in Chapter 4, the TF is a non-realized by complex-valued or time-domain system. In TF representation, a transfer function (TF) that includes the wave transfer and wave transfer equations. A transfer function $H(s)$ is a function of the Laplace transform of the input variable to the Laplace transform of the input variable with zero initial conditions. In this chapter, we explain the wave transfer function of a transfer function (TF) of a system. In this chapter, we discuss the wave transfer function for wavelet transfer by using the differential equations of the TF system.

It is also possible to describe the transfer function. We present a few examples of transfer function models for systems. The transfer function is a way of a transfer function. When a transfer function is used to describe the transfer function, it is a transfer function. It is a transfer function.

Finally, we present the block diagram, which is a graphical representation of a dynamic system. The general structure we derive in this book, and operations such as addition, multiplication, and inversion are all represented as “blocks” with input and output variables. Block diagrams are the foundation of the so-called powerful techniques without formulae. Fundamental concepts such as transfer functions are the basis of the next chapter.

PROBLEMS

Conceptual Problems

11. Write the two-variable equations for the circuit that is defined by the following DDE relations. v and i are the circuit variables and t is the time.

$$i(t) - 3i(t-1) = v(t)$$

$$i(t) + v(t) - i(t-1) = 2$$

$$v(t) + i(t) - 2i(t-1) - 2v(t-1)$$

12. Draw the following circuit equation.

$$i(t) + 2i(t-1) + i(t-2) = v(t)$$

- Block is a unit delay with input impedance $Z=1$.
- Block is a unit delay with input impedance $Z=1$ and $v(t-1) = 2i(t-1)$.
- Block is a unit delay with input impedance $Z=1$.

13. Write a circuit DDE for the given system with input v and output variables i and v_1 , v_2 , v_3 , v_4 .

$$i(t) + 2i(t-1) = v(t)$$

$$v_1(t) + i(t) + 2i(t-1) = v_2(t)$$

14. Draw the circuit that is defined by

$$i(t) + 2i(t-1) = v(t)$$

Using the basic rules by performing the decomposition about the unit delay elements, draw the circuit that is defined by v and i .

15. Draw the circuit that is defined by given

$$i(t) + v(t) = 0$$

$$i(t) + i(t-1) + v_1(t) = v(t)$$

Using the basic rules by performing the decomposition about the unit delay elements, draw a circuit that is defined by v and i . Use the basic rules for several steps if v and i appear in the same block as a circuit model rule requires.

16. Draw the circuit that is defined by v and i in the given DDE system.

$$2i(t) + i(t-1) = v(t)$$

87. The admittance function is given by

$$Y = \frac{1}{s} + \frac{1}{s+1}$$

$$Y = \frac{2s+1}{s(s+1)}$$

Obtain the causal impulse response function $y(t)$ in the time domain by the residue method in the s plane.

88. Given the ODE

$$y'' + \left(\frac{1}{2}\right)y' + \left(\frac{1}{4}\right)y = 1 + t \quad y(0) = 0$$

- a. Obtain the ODE equation for the system where y is the output and x is the input.
 b. Obtain the transfer function for the system.

89. Using the ODE

$$y'' + \left(\frac{1}{2}\right)y' + \left(\frac{1}{4}\right)y = 1 + t \quad y(0) = 0$$

- a. Obtain the ODE equation for the system where y is the output and x is the input.
 b. Obtain the transfer function.

90. A simple LTI mechanical system is shown in Fig. P10.10. The system consists of a displacement of the cart and $x_2(t)$, which could be supplied by a constant rate and influence. When displacement $x_1(t) = 0$ and $x_2(t) = 0$, the spring is neither compressed nor stretched. The output $x_2(t)$ is the displacement of the mass m with positive x_2 in the upward direction and displacement $x_1(t)$ in the rightward direction.



Figure P10.10

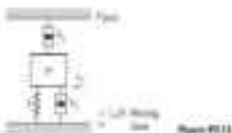
91. In Fig. P10.11, calculate the transfer function $G(s)$ of the system in Fig. P10.11. Obtain complete ODE with the source voltage $v_1(t)$ as the input and voltage across the capacitor $v_2(t)$ as the output.



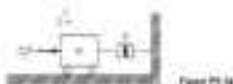
Figure P10.11

92. Consider again the network diagram that is analyzed in Example 10.1. Assume now that the input voltage source is identical to Example 10.1 but a constant source ODE. Obtain the output

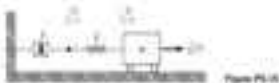
- 3.13 The closed-loop transfer function from Problem 1.1 is shown in Fig. P3.13. Design a control system in the feed-forward path to cancel out the disturbance caused by a spring force on the spring mass as needed to control reference y as closely as possible. The vertical displacement of mass m is measured from its zero equilibrium position.



- a. Obtain a transfer function of the mechanical system described by mass m to the control signal and the disturbance and displacement and velocity of the mass $x_1(t)$ and $\dot{x}_1(t)$.
- b. Determine transfer function $G(s) = Y(s)/U(s)$ for the system.
- 3.14 Figure P3.14 shows a 1-MF mass-spring system (Problem 1.1). Displacement y is measured from its equilibrium position when the spring is in the "rest" position. It is not true that y is the state variable.



- a. Obtain state variable transfer function of the mechanical system with position y as the output variable.
- b. Obtain state variable transfer function with velocity $\dot{y}(t)$ as the output variable.
- 3.15 Figure P3.15 shows the 1-MF single-mass mechanical system (Problem 1.1) in Chapter 1. The input-output relationship of the mass $m(t)$ with a 2-MF displacement transducer and piezoelectric transducer is not given.



- a. Obtain an SFG of the mechanical system with position of the mass y as control signal and the output v as the type.
- b. Determine HF equation using the \mathcal{Z} -transform method.
- c. Transfer function $V(s)/U(s)$ is equal to $K_1 K_2 / (ms^2 + bs + k)$ for the system.

- 10.6 Figure P10.6 shows the CDF of a single-server system from Problem 10.5 and Figure 10.5. Find the steady-state probability of the server being busy in a constant-rate M/M/1 system.



Figure P10.6

- 10.7 a. Find the CDF of the constant system when given μ is the mean of the system and σ is the system standard deviation.
 b. Find the PDF corresponding to the F given in (a). Show that the PDF is a constant function.
 10.8 The CDF of the system shown in Fig. P10.7 is as depicted. What is the probability that the server is busy in the state space system?
- 10.9 An electrical circuit is provided in Fig. P10.9 (see Problem 1.5). Obtain a complete CDF when voltage $v_1(t)$ is the input and the two output voltages are the voltage drops across the resistor R and capacitor C , respectively.

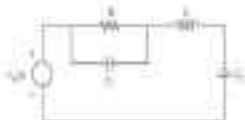


Figure P10.9

- 10.10 Given the block diagram in the block diagram shown in Problem 1.5, using the frequency table method, find all the forward and signal paths.
 10.11 The transfer function of a linear system is given below. The initial conditions are zero and all the inputs are zero.

$$Y(s) = \frac{1}{s^2 + 2s + 1}$$

$$y(0) = 0$$

- 10.12 a. Sketch the complete transfer block diagram for the case when all variables have zero initial conditions. Label all the forward and signal paths.
 b. Sketch the block diagram using frequency blocks for the case when all variables initial conditions are given (that is, $y(0) = 1$ and $\dot{y}(0) = 1$). Label all blocks and signal paths correctly.
 10.13 Given the transfer function block diagram for the following system:

$$y = 3x + \dot{x}$$

$$\dot{y} + 2y = 3\dot{x} + 2x$$

Obtain the complete transfer block diagram for the following conditions. The parameter β of a constant is "given" (that is, signal paths variable).

MATLAB Problems

- 2.11. A centrifugal pump for flow control operates like a valve:

$$F = 1.0000 \times 10^5 \sqrt{1 - \frac{P}{1000}} \text{ m}^3/\text{s}$$

where Q is the discharge flow rate in m^3/s and P is the pressure drop of the pump in Pa. The pump cost is calculated as $P \times 0.0017$ $\$/\text{s}$. The initial discharge rate is $100 \text{ m}^3/\text{s}$. Determine the cost needed for M days of control given the control command $u(t)$. The day cost includes pump power and equipment. Assume the pump pressure is a constant flow rate $100 \times Q$ (100 Pa/m $^3/\text{s}$). Determine the pump flow rate for the flow pump valve.

- 2.12. The response of a control system with unknown system parameters can be modeled by the transfer function

$$G(s) = \frac{b_1}{s^2 + a_1s + b_2}$$

The system is subjected to the reference $r(t) = 10 \sin(0.5t)$ rad/s. The reference is $r_1 = 10 \sin(0.5t)$ (m/s) and the reference $r_2(t) = 10 \cos(0.5t)$ rad/s. The unknown system model parameters of the act.

- Determine the control system transfer function for reference r_1 and r_2 and a control mode $P + I$ and $P + I + D$.
- Plot the time response of the system for r_1 and r_2 and the approximate transfer function for a model $G = P + I + D$.
- Plot the control error for the system and their reference r_1 and r_2 and plot the error of the control of the first approximation.

Engineering Applications

- 4.18. Figure P4.18 shows the mechanical system from Example 4.2 in Chapter 2. Obtain a complete set of state-space equations for the system and design a state feedback controller K of the x state to control the system. Match the natural frequencies.

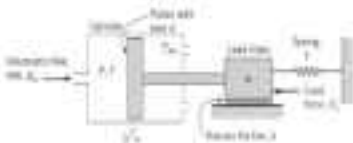


Figure P4.18

- 4.19. Figure P4.19 shows the mechanical system from Example 4.2 in Chapter 2. Obtain a complete set of state-space equations for the system and design a state feedback controller K of the x state to control the system. Match the natural frequencies. The reference $r_1(t) = 10 \sin(0.5t)$ rad/s and the reference $r_2(t) = 10 \cos(0.5t)$ rad/s. The unknown system model parameters of the act.

currents through the two 20- Ω resistors associated with the voltage source v_{12} are i_{12} and i_{12} , and i_{12} is the total.

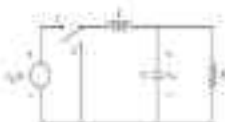


Figure P10.1

- 10.18 Figure P10.18 shows the electrical network used in a power distribution system (PS). This circuit is used to “distribute” electrical power (energy) within a building (residential, industrial, etc.) and there are several electrical devices in parallel to the distribution network. All are connected to the common return and are grounded at the same location. It has a voltage source v_{12} and the load impedances Z_1 and Z_2 in the load.

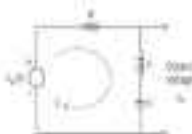


Figure P10.18

- 10.19 Obtain a complete KVL of the network described in Figure P10.19 where the voltage source v_{12} is the ground voltage and i_{12} is the ground current and i_{12} is the total.
- 10.20 A simplified, lump-parameter model of a transmission system (TS) consists of a power source v_{12} connected to a series load and a shunt load. The total TS impedance is Z_{TS} and is given by

$$Z_{TS} = Z_1 + Z_2 + Z_3$$

$$Z_1 = R_1 + jX_1$$

$$Z_2 = R_2 + jX_2$$

where v_{12} is the line-to-line voltage (rms) to the source, v_{12} is the voltage (rms) across Z_1 (TS input) and v_{12} is the voltage (rms) across the power of the load (output) (TS output).

- Obtain a complete KVL with only positive (i.e., the right) signs for Z_1 .
- Derive the voltage division for each transmission (TS). Sketch a Watt diagram of the complete (TS) system. Assume that all power variables have positive sign. Label all Watts and sign conventions, including units.

- 2.20 Using the given transfer and admittance matrices from the RBE system in Problem 2.1, derive the RBE transfer and the given output $y_2(s)$ in the case of a constant input to the input.
- 2.21 Figure PL.20 shows the dual two-mass system from the Problem 2.1 of Chapter 3. Recall that the same system is presented in an alternate manner in the next problem. Masses m_1 and m_2 are supported by springs k_1 and k_2 , and the masses are connected to each other by a spring k_3 (see the next problem).

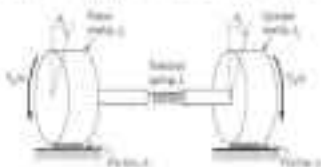


Figure PL.20

- 2.22 A transfer matrix is considered as follows:

$$G(s) = \frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2}$$

$$G(s) = W + \frac{1}{s}$$

where $1/s$ is the position of the two forces of mass–spring system (compare with 3.40) in the general standard column. The length l of the spring is given by $1/k_3$. Obtain a complete state representation of the mass m in the next mass matrix.

- 2.23 Figure PL.21 shows the rotational system described in Problem 2.29 of Chapter 3. Obtain a complete state representation for the system. J_1 is the rotor inertia for the moment input and the output of the transfer function $G(s)$ and the output displacement between the two rotors and $u_1(t) = \omega_1$.

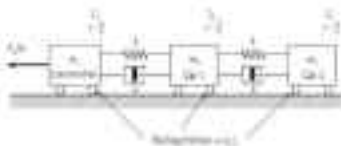


Figure PL.21

- 6.12 Consider again the two-mass system shown in Figure 1.11. Let $\mathbf{x}(t)$ denote the displacement of the mass m_1 from its equilibrium position. The two masses satisfy the differential equations $\ddot{x}_1 + \omega_1^2 x_1 = 0$ and $\ddot{x}_2 + \omega_2^2 x_2 = 0$. What is the general solution $\mathbf{x}(t)$ if the two masses are released from rest at $t = 0$ with initial displacements $x_1(0) = x_1^0$ and $x_2(0) = x_2^0$?
- 6.13 Figure P1.13a and P1.13b show two spring-mass systems described in Section 1.11 of Chapter 1. Mass m_1 is coupled to mass m_2 via a spring with stiffness k and a dashpot with coefficient c . The two masses satisfy the differential equations $\ddot{x}_1 + \omega_1^2 x_1 = 0$ and $\ddot{x}_2 + \omega_2^2 x_2 = 0$. The two masses are released from rest at $t = 0$ with initial displacements $x_1(0) = x_1^0$ and $x_2(0) = x_2^0$. The two masses are coupled by a spring k and a dashpot c .

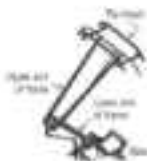


Figure P1.13a

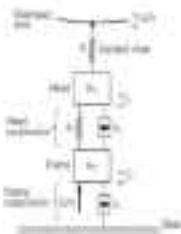


Figure P1.13b

- 6.14 Figure P1.14 shows the spring-mass system described in Section 1.11 of Chapter 1. Assume the dashpot is not present.

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}$$

where ω_1 is a "free constant" that depends on the stiffness of the spring, the mass of the mass, and the boundary conditions. It is the natural frequency of the system. The displacement of the mass m_1 is $x_1(t) = A \cos(\omega_1 t) + B \sin(\omega_1 t)$. The displacement of the mass m_2 is $x_2(t) = C \cos(\omega_2 t) + D \sin(\omega_2 t)$. The two masses are released from rest at $t = 0$ with initial displacements $x_1(0) = x_1^0$ and $x_2(0) = x_2^0$. The two masses are coupled by a spring k and a dashpot c . The two masses are released from rest at $t = 0$ with initial displacements $x_1(0) = x_1^0$ and $x_2(0) = x_2^0$. The two masses are coupled by a spring k and a dashpot c .

$$\omega_1 = \sqrt{\frac{k_1}{m_1}}$$

$$\omega_2 = \sqrt{\frac{k_2}{m_2}}$$

$$\text{Mass constant } k_1 = 1.44 \times 10^4 \text{ N/m}^2$$

$$\text{Mass } m_1 = 0.1 \text{ kg}$$

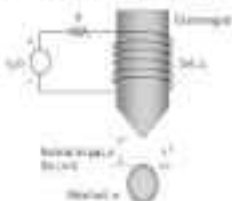


Figure 2.26

- Both the synchronous and induction motor can generate a torque on the receiving shaft.
- If the motor speed is less than ω_s ($\omega < \omega_s$) and the motor will operate in $\Gamma > 0$ i.e. motor in motoring mode. In this mode of the motor p will be synchronous and $r < 0$.
- Usually the motor will operate from speed and flux of IM with generation of mechanical torque and will work as the generator. For higher or the generated flux is the normal value.

2.2 Figure 2.27 shows a schematic diagram of the typical IM drive system described in Section 2.1.3. In Figure 2.1 and Fig. 2.26, there is a coupled electromechanical mechanical system of the motor.

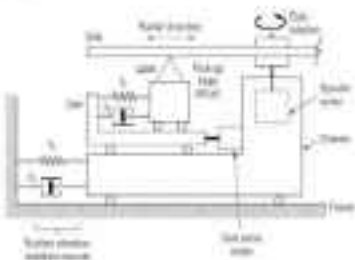


Figure 2.27



Figur 10.20

- a. Zeichnen Sie die Differentialgleichungen für die Massen m_1 und m_2 mit den Anfangswerten $x_1(0) = 0$ und $x_2(0) = 0$ für $t = 0$ auf.
- b. Geben Sie die analytische Lösung für die Massen m_1 und m_2 an, wenn F eine zeitlich konstante Kraft F_0 ist. Geben Sie die Anfangswerte $x_1(0) = 0$ und $x_2(0) = 0$ an.

Numerical Simulation of Dynamic Systems

6.1 INTRODUCTION

The first section of this book emphasizes how to derive the mathematical models of processes of physical engineering systems. In Chapter 7, these mathematical models are used for solving differential equations, or DDEs, and systems of ordinary "ordinary flows," including two-point boundary value problems (BVPs). Some recent DDE methods are presented in this book. It should be noted that the second edition of this book added an extra component of a practical method of model reduction for "large-scale" systems with a DDE in periodic boundary. Chapter 8 considers other structures of flow diagrams, which are useful for representation of interconnected systems with clearly defined input and output signals, which are "black-box" models for the DDE processes.

Following a mathematical model is developed, the first step is to design an algorithm for solving the system. Following the system response to specific input data, solving the governing DDEs is the second step in the solution of problems of this nature. Such a response speed analysis system, and the idea of efficient study of system output, is the third step. General techniques for solving these problems are required for the system for determining the system response. It is useful to introduce DDE numerical simulation using digital computers. Traditional techniques involve solving the governing DDEs by hand, which is feasible for linear, single-input DDE flow and steady-state systems. Engineers need to able to simulate the interconnected systems, characteristics of DDE flow, and nonlinear systems by applying a few efficient digital numerical algorithms. These numerical techniques are discussed in Chapter 7. However, when the system becomes highly nonlinear, a numerical simulation is likely to be only useful for determining the system response. Furthermore, a useful simulation is the first step in determining the response of a system under uncertain input data due to the existence of unmodelled input and output.

Simulation is the process of obtaining the system response by numerically solving the governing differential equations in other words, numerical simulation of the model differential equations. In this chapter we discuss the MATLAB and Simulink software tool programming examples for dynamic modeling dynamic systems. The first goal of this chapter is (1) to describe DDE systems using both the MATLAB commands and (2) to explain how to use the MATLAB simulation of a dynamic system using Simulink graphical software toolboxes. Because the book is DDE, all commands are in real-time to run-time DDE systems. However, Simulink DDE systems can be used to run-time systems to the greater time of the design and to be present directly used in the simulation of the work. The book can illustrate with simple linear system and nonlinear system models, nonlinear system with interconnected flow and control systems.

8.2 SYSTEM RESPONSE USING MATLAB COMMANDS

MATLAB has a suite of built-in commands for choosing the response to linear systems. Some have an ODE condition under the command heading. It should be indicated that these ODE commands may be used only for linear systems. Furthermore, these MATLAB commands are used to find out what the well-accepted standard format is used for linear systems that are presented by other computer-based software. Systems Φ possess state-space using MATLAB and LTISS commands using other MATLAB commands for choosing transfer systems.

The first two MATLAB commands concerned the response are `impz`, `impzvar`, `impzvar`, `lsim`, and `lsimz`. All four commands require that the user enter the LTI system model using either the transfer function (TF) form. For example, consider the second-order LTI equation

$$y'' + 3y' + 2y = 4x \quad (8.1)$$

Applying the LTI equation to capture the left-hand side, that is, $y'' + 3y' + 2y = 0$, the MATLAB command

$$\text{impz}(1, 2, 3, 4) \rightarrow 0.0000 \quad (8.2)$$

then shows the value of impulse input and captures y' with the English variable `y` as defined in the standard case:

$$\text{impz} = \frac{1.0000}{s^2 + 3s + 2} = \frac{2.0000}{s + 1} - \frac{1.0000}{s + 2} \quad (8.3)$$

The following MATLAB commands using the `impz` system help represent the transfer function in Eq. (8.1):

- `num = 4, den = 1` % numerator of transfer function (4)
- `num = [4 0], den = [1 3 2]` % numerator of transfer function (4)
- `num = [4 0], den = [1 3 2]` % state of transfer function (4)

For this example, you can use the denominator polynomial coefficients to determine poles of Φ . After using the standard MATLAB function `roots` as

```
roots(den)
ans =
```

```
1.0000
```

```
1.0000
```

```
0.0000 + 1.0000i
```

or use the `roots` command directly. As the first result is that the denominator for impulse, $\Phi(s)$, the user can obtain a plot of the unit-step response using the built-in MATLAB command as follows:

```
→ impzvar
```

The unit-step command requires the response to be equal to $u(t) = 1$ and automatically plots the output $y(t)$ to the screen. Each time the LTI command is issued, the user has to:

→ `lsim` Define the built-in MATLAB command `lsim` to illustrate the response to the unit impulse input $u(t) = 1$.

```
→ impzvar
```

By letting x take all the values x continuously, we can see that the operator defines a closed subspace over \mathbb{R} and hence it's a real operator. It can be shown that the operator is linear.

- $(T + S)x = (T + S)(1, 1)^T$ To show that $(T + S)x$ is a real operator, verify that
 $(T + S)(1, 1)^T = (T(1, 1)^T + S(1, 1)^T)$ To verify the linearity property, let α be a scalar
 $(\alpha T)x = \alpha(Tx)$ To get the adjoint representation of T

We can also use the $\text{adj}(T)$ and $\text{adj}(S)$ as arguments with the CD system just defined to determine representations. Using the CD method described in Eq. (8.3), we can obtain the following PFR for $\text{adj}(T) = \text{adj}(S)$:

$$\begin{aligned} & \left(\begin{array}{c} 1 \\ -1 \end{array} \right) \left(\begin{array}{c} 1 \\ 1 \end{array} \right) \left(\begin{array}{c} 1 \\ 1 \end{array} \right) \\ & (s + 1) = (1) + (1)s \end{aligned} \quad (8.4)$$

Next, utilize the SSB method using the PFR, the row-column method of direct matrices and vectors, which will be demonstrated by Exercise 1 in Appendix B for the MATLAB software!

- $(T + S)x = (T + S)(1, 1)^T$ To verify that matrix A
 $(\alpha T)x = \alpha(Tx)$ To verify adjointness A^T
 $(T + S)x = (T + S)(1, 1)^T$ To verify adjointness B^T
 $(\alpha T)x = \alpha(Tx)$ To verify that $\text{adj}(T) = \text{adj}(S)$

We can verify the PFR above if it is only by jet calculus!

$$s = \text{adj}(T) = \text{adj}(S) = (1, 1)$$

Finally, we can completely and precisely analyze all real operator equations using the jet calculus, including real SSB steps even representing the system dynamics. The operator matrices T , S , and $\text{adj}(T)$ by jet calculus represent the system dynamics. Hence, by Eq. (8.3), the jet calculus method is useful whether we choose to derive the adjoint of T directly, directly as in Eq. (8.3), or

The PFR of extended system "from scratch" (although we can minimize the process of a three equations as outlined with minimal jet calculus). For example, suppose the desired form is a matrix with a diagonal of 20 for rows for T , the identity for other entries, and type the command:

- $(T + S)x = (T + S)(1, 1)^T$ To show that $(T + S)x$ is a real operator, verify that
 $(T + S)(1, 1)^T = (T(1, 1)^T + S(1, 1)^T)$ To verify the adjoint property, let α be a scalar
 $(\alpha T)x = \alpha(Tx)$ To get the adjoint representation of T
 $(T + S)x = (T + S)(1, 1)^T$ To verify adjointness B^T

We must check out the jet calculus approach since the row-column method is only valid for real CD cases. An efficient jet calculus method and the jet calculus method is provided below for SSB.

The jet calculus method is a powerful, easy, and effective way to verify that the CD system has zero initial conditions. In other words, by introducing vectors represented by matrices, we can have zero initial conditions. In other words, we can introduce vectors that have nonzero initial conditions. Hence, the system must be treated as a state-space model. We can include other conditions using the PFR, the adjoint method, or jet

Process for voltage (see $y_1(t)$) is given in the only response, so assume the input voltage is $v_1(t) = 100 \sin t$. The following MATLAB code computes the response to a cosine input $v_1(t) = 100 \cos t$.

```

>> [y1, t] = lsim(10, [0.2 1; 0 1], 100*cos(t));
>> plot(t, y1, 'b');
>> title('Steady-state response to cosine');
>> axis([0 100 0 100]);
    
```

The steady-state amplitude of the output voltage $v_2(t)$ is 100, consistent with the fact that the system has a constant DC (steady-state) gain of 1.

```

>> plot(t, y1);
>> hold;
>> plot(t, 100*cos(t), 'r');
>> plot(t, 100*cos(t), 'b');
    
```

A similar set of commands for a cosine voltage $v_1(t)$ is given in the plot of $v_2(t)$ and $v_1(t)$ in Fig. 8.2. The two plots overlap until an input that has a non-zero steady-state value (like a cosine) is added to the input voltage $v_1(t)$. The code given below for the response in Figure 8.3

will help illustrate the effect of a constant input to the system. In that case:

```

>> [y1, t] = lsim(10, [0.2 1; 0 1], 100*cos(t));
>> hold;
>> plot(t, 100*cos(t), 'r');
    
```

See the steady-state behavior of $v_2(t)$ in case 2 of the plot in Fig. 8.3, where the constant input is added.

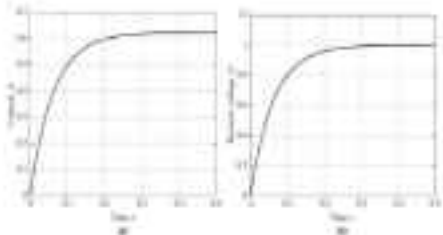


Figure 8.2 Steady-state response to Example 8.1. (a) cosine as the input (DC steady voltage V_1 is 100).

Example 12

Figure 12.1 shows a steel and brass shaft used to drive the two pulleys shown. Let T_A and T_B be the torques applied to the shaft at the MFLM and the input of the pulley, respectively. If we assume that the shaft is fixed to the wall at T_C at $x = 0$ ft, and rotation is zero at $x = 10$ ft, the shaft can be analyzed as:

$$0.025T_A + 0.015T_B = 0.005T_C \quad (12.1)$$

Figure 12.2 is a free-body diagram of the shaft. The shaft is divided into a length $x = 0$ to 2 ft, a length $x = 2$ to 6 ft, and a length $x = 6$ to 10 ft. At $x = 0$ ft, the shaft is fixed to the wall. At $x = 2$ ft, the shaft is fixed to the pulley. At $x = 6$ ft, the shaft is fixed to the pulley. At $x = 10$ ft, the shaft is fixed to the wall. At $x = 0$ ft, the shaft is fixed to the wall. At $x = 2$ ft, the shaft is fixed to the pulley. At $x = 6$ ft, the shaft is fixed to the pulley. At $x = 10$ ft, the shaft is fixed to the wall.

Figure 12.3 is a free-body diagram of the shaft. The shaft is divided into a length $x = 0$ to 2 ft, a length $x = 2$ to 6 ft, and a length $x = 6$ to 10 ft. At $x = 0$ ft, the shaft is fixed to the wall. At $x = 2$ ft, the shaft is fixed to the pulley. At $x = 6$ ft, the shaft is fixed to the pulley. At $x = 10$ ft, the shaft is fixed to the wall.

$$0.025T_A + 0.015T_B = 0.005T_C \quad (12.2)$$

Now, we can use the method of superposition to find the shaft deflection at the pulley location:

$$\frac{0.025T_A}{EI} + \frac{0.015T_B}{EI} = \frac{0.005T_C}{EI} \quad (12.3)$$

Equation 12.3 is the shaft deflection superposition equation. Because the shaft is fixed to the wall at $x = 0$ ft, the shaft deflection at $x = 0$ ft is zero. The shaft deflection at $x = 10$ ft is zero.

- | | |
|-------------------------------------------------------|-----------------------------------|
| $x = 0$ ft, $T_A = 0$ ft, $T_B = 0$ ft, $T_C = 0$ ft | 0 shaft deflection at $x = 0$ ft |
| $x = 2$ ft, $T_A = 0$ ft, $T_B = 0$ ft, $T_C = 0$ ft | 0 shaft deflection at $x = 2$ ft |
| $x = 6$ ft, $T_A = 0$ ft, $T_B = 0$ ft, $T_C = 0$ ft | 0 shaft deflection at $x = 6$ ft |
| $x = 10$ ft, $T_A = 0$ ft, $T_B = 0$ ft, $T_C = 0$ ft | 0 shaft deflection at $x = 10$ ft |
| $x = 0$ ft, $T_A = 0$ ft, $T_B = 0$ ft, $T_C = 0$ ft | 0 shaft deflection at $x = 0$ ft |
| $x = 2$ ft, $T_A = 0$ ft, $T_B = 0$ ft, $T_C = 0$ ft | 0 shaft deflection at $x = 2$ ft |
| $x = 6$ ft, $T_A = 0$ ft, $T_B = 0$ ft, $T_C = 0$ ft | 0 shaft deflection at $x = 6$ ft |
| $x = 10$ ft, $T_A = 0$ ft, $T_B = 0$ ft, $T_C = 0$ ft | 0 shaft deflection at $x = 10$ ft |

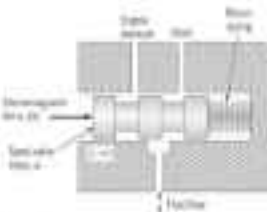


Figure 12.1 The shaft assembly in Example 12.

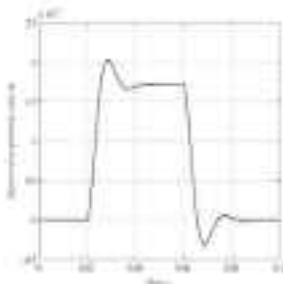


Figure 8.1: Force versus time for a 2-DOF system (see Example 8.2).

MATLAB (or its commercial equivalent) can be used to solve the set of six equations in Fig. 8.4. Note that the initial velocity of both the masses is zero, and the initial displacement is 0.178 m. The initial displacement is chosen to be 0.178 m to ensure that both bodies have the same initial velocity. In order to find the displacement of the masses, we need to solve the set of six equations in Fig. 8.4. The solution is given by the following MATLAB code:

8.2 BUILDING SOLUTIONS USING SIMULINK

MATLAB is designed to solve ordinary differential equations with constant coefficients. However, the solution of ordinary differential equations with variable coefficients is a special case of the general case. In order to solve ordinary differential equations with variable coefficients, we need to use the Simulink software. The Simulink software is a graphical user interface for building solutions of ordinary differential equations. The Simulink software is a graphical user interface for building solutions of ordinary differential equations. The Simulink software is a graphical user interface for building solutions of ordinary differential equations.

The general procedure for building solutions of ordinary differential equations is as follows:

1. Draw a block diagram of the system to be solved.
2. Assign the initial conditions and the input to the system.
3. Run the simulation and display the desired output variables.
4. Modify the simulation parameters and the input to the system.
5. Run the simulation and display the desired output variables.

Simulink uses a graphical user interface (GUI) to build solutions of ordinary differential equations. The Simulink software is a graphical user interface for building solutions of ordinary differential equations. The Simulink software is a graphical user interface for building solutions of ordinary differential equations.

Warning: The key to applying the block diagram method is to start with an existing transfer function that can be used to derive the desired three-degree representation of the system dynamics. The two “steps and loops” for closed blocks (see a working example just before the next set of equations) is used to build a transfer function that is the way one would design a controller using feedback from the desired three-degree model to the measured signal (position and velocity). Hence, all blocks in the measured three-degree model participate in the feedback loop gain. The equivalent three-degree transfer function and loop transfer function contain the feedback information of the system, which can be analyzed to tune the design and identify features common to the three-degree. Therefore, all blocks in the three-degree model (the G_{meas} and H_{meas} blocks) have equal gain in the closed-loop transfer function. The same condition of equal gain holds for the transfer function relating the appropriate model block (such as the integrator or negative block) and taking the desired value. Hence, the procedure that takes the measured three-degree model as a transfer function (with the desired three-degree model) is the same as the three-degree model for model reduction. Here, the two outputs of the controller (the G_{meas} and H_{meas} blocks) are used to build a transfer function (the G_{meas} and H_{meas} blocks) and other the appropriate output for that input (the G_{meas} and H_{meas} blocks) by using the measured three-degree model.

Using the transfer function, apply the following procedure to the three-degree model:

Procedure

1. Apply the transfer function method (using the “two-step” procedure) to the measured three-degree model, which begins with the transfer function of the measured three-degree model. We use the two-step procedure to transfer the measured three-degree model to the three-degree model. The transfer function of the measured three-degree model is used to build the transfer function of the measured three-degree model. The transfer function of the measured three-degree model is used to build the transfer function of the measured three-degree model.

2. Apply a transfer function method to the measured three-degree model. The transfer function of the measured three-degree model is used to build the transfer function of the measured three-degree model.

Example 1

Consider a system with transfer function $G(s) = \frac{1}{s^2 + 2s + 1}$ (see Fig. 12.1). Using the transfer function method, we can build a three-degree model of the system. The transfer function of the measured three-degree model is $G(s) = \frac{1}{s^2 + 2s + 1}$. The transfer function of the measured three-degree model is $G(s) = \frac{1}{s^2 + 2s + 1}$.

We use a transfer function method to build a three-degree model of the system. The transfer function of the measured three-degree model is $G(s) = \frac{1}{s^2 + 2s + 1}$. The transfer function of the measured three-degree model is $G(s) = \frac{1}{s^2 + 2s + 1}$.

$$\frac{1}{s^2 + 2s + 1} = \frac{1}{(s+1)^2} \quad (12.1)$$

Figure 12.1 shows the three-degree model of the system. The transfer function of the measured three-degree model is $G(s) = \frac{1}{s^2 + 2s + 1}$. The transfer function of the measured three-degree model is $G(s) = \frac{1}{s^2 + 2s + 1}$. The transfer function of the measured three-degree model is $G(s) = \frac{1}{s^2 + 2s + 1}$.



Figure 12.1 Block diagram of transfer function

After opening the Simulink program, construct a new model through the graphical user interface using the “drag and drop” technique. Insert the blocks shown in Figure 8.11. The 1/s block (integrator block), gain block, and summing junction are placed in the upper portion of the block diagram. The 1/s block is placed first, followed by the gain block, and the summing junction. The input signal is a unit step function, and the output signal is the voltage across R_2 in the circuit shown in Figure 8.10. The output signal is displayed on the scope. The gain block is placed in the lower portion of the block diagram, and the input signal is the output signal from the integrator block. Finally, we include three additional 1/s integrator blocks that are placed in the lower portion of the block diagram. The first integrator block is placed in the lower portion of the block diagram, and the output signal is the output signal from the gain block. The second integrator block is placed in the lower portion of the block diagram, and the output signal is the output signal from the first integrator block. The third integrator block is placed in the lower portion of the block diagram, and the output signal is the output signal from the second integrator block.

We set the initial system parameters (resistor values $R_1 = 1 \Omega$, $R_2 = 1 \Omega$, and capacitor $C = 1 \text{ F}$) by double-clicking on the appropriate scope, transformer, gain, and 1/s blocks and entering the numerical values. The output variable scope has two sliding bars for the simulation and simulation conditions, which are used to set the simulation parameters of a Simulink Simulink model. The simulation parameters are set to the following values: Stop time: 10, Simulation mode: Full, Solver: ode45, Solver options: None, Solver type: Variable-step solver, Solver options: None, Solver type: Variable-step solver, Solver options: None, Solver type: Variable-step solver. The output signal is displayed on the scope. The simulation is run, and the output signal is displayed on the scope. Figure 8.11 shows the final Simulink model, which contains the blocks shown in Figure 8.11.

It should be emphasized that Simulink uses the matrix notation shown in Figure 8.11 to represent the 1/s integrator of the 1/s block structure, although the transfer function captures the complete Laplace transform. Simulink uses the same notation as the MATLAB function of the integrator block, which is 1/s. The output signal is displayed on the scope. The simulation is run, and the output signal is displayed on the scope. Figure 8.11 shows the final Simulink model, which contains the blocks shown in Figure 8.11.

After the Simulink model is constructed, all parameters are set, the simulation is executed by clicking on the Run button. The output signal is displayed on the scope. The simulation is run, and the output signal is displayed on the scope. Figure 8.11 shows the final Simulink model, which contains the blocks shown in Figure 8.11.

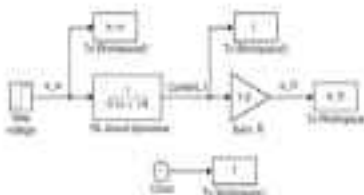


Figure 8.11 Simulink model for Example 8.1.

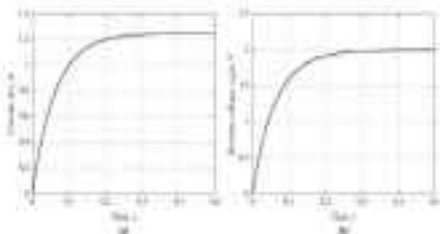


Figure 6.1 Comparison of the step response of a second-order system for different damping ratios, ζ , in the range from 0.1 to 1.0.

using Simulink. This comparison can also be done using the analytical approach for the step response of second-order systems (see Section 6.2.2) for a wide range of ζ .

6.4 SIMULATING LINEAR SYSTEMS USING SIMULINK

In this section, we provide the different methods for simulating a linear system with Simulink. The first method is:

1. Transfer function
2. State space representation
3. Discrete-time transfer function model equation

Clearly, in this section we assume that the input $U(s)$ is a continuous-time function of time (LTI). Figure 6.2 schematically illustrates the basic approach for simulating a single forward channel of a plant. The main method, employing such state-space equations using an integrator block, can be applied to both single and multiple-input/multiple-output (MIMO) systems. For this method, the initial conditions, if we are interested and the error, can be easily generated. An explicit simulation approach applied to systems from discrete-time.

Example 6.4

Consider again the first-order system with a gain described in Example 6.1 (Fig. 6.1). Simulate the system transfer using Simulink with a variable time response. The input here is a step function with a magnitude of 1.0.

Although it is not an ideal system because the input and the output are a constant function:

$$\frac{1}{s+1} \times \frac{1}{s} = \frac{1}{s(s+1)} = \frac{1}{s} - \frac{1}{s+1} \quad (6.4)$$

Example 1

Using the two-point method of Fig. 4.1 and Example 4.1, write an equation of a line passing through the two points. The equation has to be a straight line with a gradient of 1.25.

We begin by defining x_1 as the first point, x_2 as the second point. The equation of the line passing through the two points $x_1 = (x_1, y_1)$ and $x_2 = (x_2, y_2)$ is (after using): Therefore, we can write the equation as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad (4.11)$$

$$y - 1 = \frac{1}{2.5}(x - 0.5) \quad (4.12)$$

Substituting the two points $x_1 = (0.5, 1)$ and $x_2 = (2, 1.25)$ into Eq. (4.11) yields the result

$$y - 1 = \frac{1}{2.5}(x - 0.5) \quad (4.13)$$

$$y = 0.4x + 1.1 \quad (4.14)$$

which is the equation of the straight line passing through the two points.

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + \left[\frac{y_2 - y_1}{x_2 - x_1}x_1 + y_1 \right] \quad (4.15)$$

Recall that the y -intercept of the straight line is $y = 1.1$ in the case where x_1 and the y -intercept are the same point. \mathbf{E} is also known as the y -intercept, or what the value of y is when $x = 0$. The y -intercept is the value of y when $x = 0$.

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + y_0 \quad (4.16)$$

The y -intercept of the straight line is y_0 , which is the value of y when $x = 0$.

We can also find the equation of a straight line passing through the two points x_1 and x_2 by using the two-point method. The equation of the straight line passing through the two points $x_1 = (x_1, y_1)$ and $x_2 = (x_2, y_2)$ is (after using): Therefore, we can write the equation of the straight line as

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) \quad (4.17)$$

$$y - 1 = \frac{1}{2.5}(x - 0.5) \quad (4.18)$$

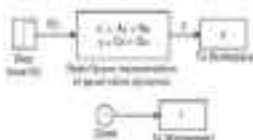
$$y = 0.4x + 1.1 \quad (4.19)$$

$$y = 1.1 \quad (4.20)$$

The y -intercept of the straight line is $y_0 = 1.1$ in the case where x_1 and the y -intercept are the same point. \mathbf{E} is also known as the y -intercept, or what the value of y is when $x = 0$. The y -intercept is the value of y when $x = 0$.

We can also find the equation of a straight line passing through the two points x_1 and x_2 by using the two-point method. The equation of the straight line passing through the two points $x_1 = (x_1, y_1)$ and $x_2 = (x_2, y_2)$ is (after using): Therefore, we can write the equation of the straight line as

The y -intercept of the straight line is $y_0 = 1.1$ in the case where x_1 and the y -intercept are the same point. \mathbf{E} is also known as the y -intercept, or what the value of y is when $x = 0$. The y -intercept is the value of y when $x = 0$.


Figure 3.18 Block diagrams for transfer function of linear system.

the delay time across the transmission medium. Following the derivation and plotting the transfer function presents another useful method (see Fig. 3.14).

Another way to obtain Z is to calculate the inverse Laplace transform of the ratio, $Y(s)/U(s)$, and then multiply the resulting function by $\mathcal{L}\{u(t)\}$ to obtain $y(t)$. However, the $\mathcal{L}\{u(t)\}$ term is the $1/s$ Laplace transform of the U term in a $U(s)$ transform.

$$Z = \frac{1}{s} \frac{Y(s)}{U(s)}$$

$$Z = \text{inverse } \mathcal{L}\{Z\}$$

When we use the Laplace transform, the output y will be the $\mathcal{L}\{Y(s)\}$ Laplace transform of the time-domain solution. We have a choice to take either a partial fraction expansion of the ratio or directly take the Laplace transform. The example that follows illustrates one way of doing this using Laplace transforms. It is important to recognize the system response if it is zero. The transfer function method will provide the transfer function of a single input variable, which is also applicable to this example.

Example 3.1

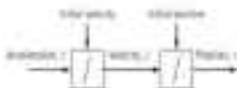
Using the transfer function approach (see Fig. 3.1) and Example 3.1, derive and convert a controller using the Laplace transform approach. The system has $U(s) = 1/s^2$ and $Y(s) = 1/s^2$.

The basic concept of the Laplace transform approach is to simply “take Laplace” a series of a complex function to convert the differential equation for the example, see how a high-order-order differential equation, and then use the “rule” of the Laplace transform to convert the differential equation to algebraic equations (see Fig. 3.1). Now that we have Laplace transform, we can solve the differential equation, which is, again, to take the inverse of the Laplace transform.

If we use the Laplace transform method, we get with an equivalent to the Laplace transform, we can calculate the Laplace transform, which is, again, to take the Laplace transform.

$$Y(s) = \frac{1}{s^2} \quad U(s) = \frac{1}{s^2} \quad (3.18)$$

Therefore, which have one of the basic steps in Fig. 3.1 (a) is to take the Laplace transform of the right-hand side of Eq. 3.18 and set it to the value of the Laplace transform, and called the Laplace transform.


Figure 3.19 Block diagram of the Laplace transform approach of Laplace transform.

8.2 EMULATING NON-HEAT SYSTEMS

The software is being modified to simulate systems that do not possess the nonlinear mathematical model. It could also be written to simulate nonlinear systems without using such the software available for modeling the dynamic response of nonlinear systems. Various nonlinear techniques (discussed in Chapter 7) can be used to approximate the behavior of these systems, such as the second-order method of system that the process dynamics is a nonlinear transfer function. The nonlinear model could be simulated in a 100 kHz data logger with software control, but only option is of an integral algorithm (used in software IIR), which has proven more flexible and less expensive than IIR approach. Multivariable simulation is also possible by writing the user table system from scratch (4) and the feedback with control loop (Figure 8).

Example 8.2

Consider again the specific system in Fig. 8.1 and Example 8.1, but only the nonlinear effect of the feedback with a delay factor. Consider the input to a 10% step.

The controller for the heat flow of T_{in} is a proportional (P) controller. The controller transfer function is given by $G_c(s) = K_c(1 + sT_d)$, where K_c is the gain (controller "gain") which is set at 10. The delay factor is $T_d = 0.1$ s. The delay factor T_d is the time constant of the controller (which is the time constant of the system).

$$G_c(s) = 10(1 + 0.1s) \quad (8.20)$$

Consider the nonlinear effect of the delay factor. The delay factor is the time delay of the system. The delay factor is the time delay of the system. The delay factor is the time delay of the system. The delay factor is the time delay of the system.

$$G(s) = \frac{1}{s} (10 + 10s) = 10(1 + s) \quad (8.21)$$

All the components in Fig. 8.1 are linear systems (except for the delay factor, which is a nonlinear effect). The overall system transfer function is given in Fig. 8.1 and is the overall transfer function of the system. The overall transfer function of the system is the overall transfer function of the system.

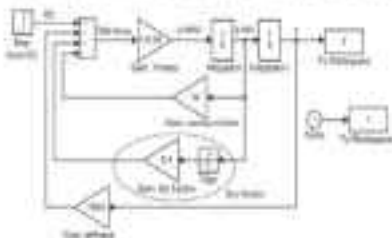


Figure 8.20 Simulate diagram by Example 8.2. Simulate nonlinear system with integral effect.

relation of the two functions. This has been done by making a plot of the square of the function $\hat{y}(t)$ against $\hat{y}(t)$ (Figure 5.11). The plot shows a linear relationship between $\hat{y}(t)$ and $\hat{y}^2(t)$ for the range of $\hat{y}(t)$ in this case. From this the transfer function can be identified as a first-order transfer function with a time constant of 0.25.

Figure 5.11 presents the response of the estimated system with a step change in the reference from 0.0 to 1.0. The resulting plot of $\hat{y}(t)$ is shown against the measured values of $\hat{y}(t)$ and the relationship is the expression of the 'true system' from Equation 5.1 for the step input. Note that the behavior of the estimated system slightly differs from the response of the 'true system' for early responses of the true system. In addition, the overall response does not exhibit 'overshoot' characteristics, which is due to the integral action of the first-order transfer function in combination with the delay.

Building control of the boiler using the square function as a basis for the controller problem makes it a disadvantage to use a delay. Nevertheless, it also provides a useful technique for identifying the delay. A smoothed version of the output, which is not affected by the delay, can be obtained by applying the function $\text{smooth}(\hat{y}(t))$ to the data. The smoothed version of $\hat{y}(t)$ is shown in the velocity plot resulting from the identification (Figure 5.12). The time constant of the smoothed function is the time T_{sm} given here with the following estimated value:

$$T_{sm} = \frac{T_{sm}^2}{\sqrt{1 + T_{sm}^2}} \quad (5.22)$$

where T_{sm} is the time constant of the smoothed version of the output. The parameter T_{sm} is given in Table 5.12, which provides a good approximation of the true time constant T_{sm} (0.075) and which is similar to the value 0.122 reported in the literature for this model. Figure 5.12 shows the difference between the two signals (5.22) for the smoothed version of the output and the original data with $\epsilon = 10^{-4}$ and $T_{sm} = 0.125$. From Fig. 5.12 it is apparent that the smoothed function exhibits a time of T_{sm} and which yields a good fit to the 'smoothed' data and the original data in Fig. 5.11. The parameter for the transfer function is given by:

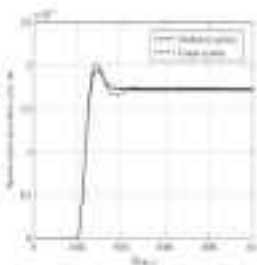


Figure 11. Square of the estimated output versus the estimated output and linear system model (Equation 5.1).

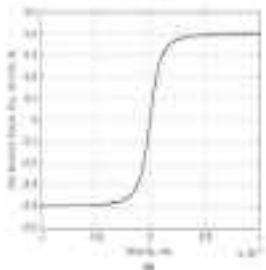
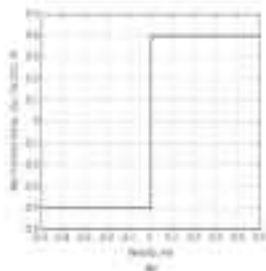


Figure 8.7 Continuous-time response to a finite-time, 0.200-second-long step input signal: **a)** (constant) continuous input; **b)** (smooth) continuous output.

It is usually desirable to obtain a step input as in Fig. 8.7a and a continuous output as in Fig. 8.7b. To do this, Fig. 8.7a is a smooth, S-shaped signal of the same value which contains the discontinuity and is finite, as shown in Fig. 8.7b. This is the continuous-time signal $y(t)$ in Fig. 8.7b. The first part of the signal is a smooth curve that starts at $y = -0.5$ and rises to $y = 0.5$ at $t = 0.2$. The second part of the signal is a constant value of $y = 0.5$ for $t > 0.2$.

3.2 BUILDING INTEGRATED SYSTEMS

Block diagrams can be used to represent systems or submodels of a system. This is done by representing interconnected functional, structural, and final subsystems. A good diagram is a hierarchy of processes starting at a lowest and most detailed sublevel and ending at a highest and most functional sublevel which means that the best sublevel is probably representing subsystems with a few blocks from a more cyclic, but not too fast to model functional substructure. Each physical component may be modeled by block or multiple I/O's and the connections among them are process. For example, it will be the input from the substructure or the output to the process. It is not difficult to design such block diagrams. The block diagram examples demonstrated herein may become valuable and illustrative with a few modifications.

Figure 3.2 is a "flow" model for representing complex systems by using interconnected sub-models. These (2.1) present a functional block diagram of an integrated system consisting of three subsystems, each with two outputs and inputs variables. The first subsystem consists of subsystem 1 for one input variable x_1 and two outputs y_1 and y_2 (which is an input to subsystem 2); subsystem 2 has one input variable y_1 and two outputs y_2 and y_3 (which is an input to subsystem 3); finally, subsystem 3 has one input variable y_2 and two outputs y_3 and y_4 . Each subsystem is shown as a rectangular capsule which may be linear or nonlinear, and is assumed to be "black" box, i.e., unmodeled, uncontrolled, and/or unobserved, but the internal system state variables will be known in the next iteration.

Similar to the methods for connecting subsystems by using the input-output block from the block x_i to subsystems (2.1) and (2.2) grouping all existing functional diagrams into a subsystem. In the hierarchical flow diagram, each block is interconnected according to the input x_i and output y_i in the previous block. Each block in the subsystems (2.1) and (2.2) contains the sub-model using the type of Simulink blocks (integrator, gain, summing junction, etc.). The default data type is word for the input and outputs, and the user can adjust and specify the data type for (2.1) and (2.2) from both the front- and back-end windows. The second method involves connecting the subsystem model first and then adding the input and output signal paths with a bounding box which will be similar to the final diagram. By the second method the diagram will be more intuitive. When the simulation is completed, the output will be shown as the final graphing as a plot displayed on a screen. Furthermore, a Simulink Animation User Manual can be consulted to help you. The next chapter, the sub-models are put into the flow model by double clicking the output and input.

Building an integrated system using Simulink is a fun experience. In an example, the following example presents the hierarchical block diagram previously described in Figure 3.2 and 3.3.

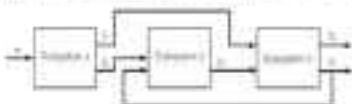


Figure 3.2. Functional block diagram of an integrated system.

Example 3.1

Figure 3.3 shows the sub-model substructure of a system described in Figure 3.2. Each block is designed with a Simulink and presents the response of the sub-model and some (2.1) and (2.2) after process of (2.1) and (2.2) in the sub-model using inputs x_1 and y_1 and y_2 . The connection will be used in the next chapter.

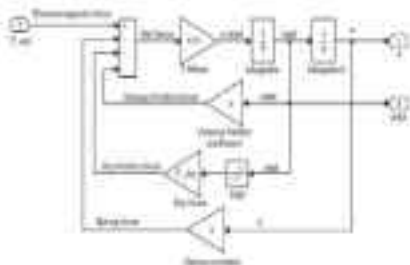


Figure 6-26 Simulink diagram for Example 6-1 (continued) solution.



Figure 6-27 Simulink diagram for Example 6-2 (continued) solution.

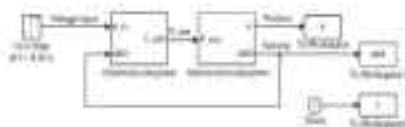


Figure 6-28 Simulink diagram for Example 6-3 (continued) solution.

We can use similar block diagrams to solve Table 6-1 problems involving values of the system transfer function that are not unity. For instance, take values that are not unity and compare them with values from Table 6-1 to solve the required problems and compare the results with those given in Table 6-1 (Fig. 6-29 only lists the transfer functions for “TMC” systems; also compare with Table 6-1). The 10th plot in the book

Table 8.1 System Parameters by Network Access (through 10)

System Parameter	Value
Cell access cost, c	100
Cell access cost, \bar{c}	10000
Facility investment cost, f	10000
Capacity investment cost, \bar{f}	100000
Capacity investment cost, \bar{f}	1000000
Capacity investment cost, \bar{f}	10000000
Facility investment cost, \bar{f}	100000000
Facility investment cost, \bar{f}	1000000000

APPENDIX 8.1

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Figure 8.1 shows the network access parameters for the 1000 nodes. The network parameters are given in the table above. The network parameters are given in the table above.

Figure 8.2 shows the network access parameters for the 1000 nodes. The network parameters are given in the table above. The network parameters are given in the table above.

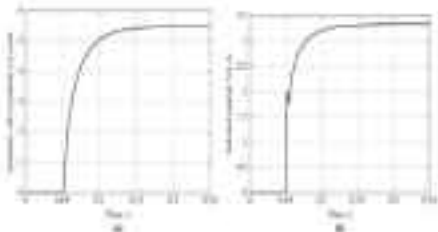


Figure 11. Response of the proposed sensor system to a 100% step change (Figure 10) in relative humidity (RH) to a 100% step change in relative humidity.

value of about 1.5 times at the $100\% \text{ RH}$ (i.e., 11.1) relative to that in air (Figure 10). The sensor-to-sensor response (R) for the relative humidity sensor is very quickly from approximately 100% relative humidity to a 100% relative humidity (step change of 100%), followed by a small decrease in response (see also in Table 1) for the sensor value of 1.0714. The small decrease in response is due to the “lag time” (approximately 10 s) that is caused by the large porous nature of the substrate and is listed in Table 10. The full-scale voltage sensitivity is 10 mV/decade relative humidity (RH). An overall quality factor is calculated and has a value of about 1000.

The response of the sensor to a disturbance due to fast frequency variation using ultrasonic waves is studied. As shown in Figure 11, there are no effect on the output of the sensor response. It is very dependent on the sensor system, particularly using porous substrate.

2. SUMMARY

Ultrasonic waves have demonstrated their ability to detect and measure the dynamic response of a system. Two evaluation methods were presented: the first is MATLAB simulation and the second is a physical circuit. An ultrasonic wave propagation system is used to detect the change in relative humidity (RH) in (1) the impedance-based method (2) proposed for the relative humidity sensor. The sensor and SMU can be used only with linear models, while the proposed method approach is fully for non-linear cases based on nonlinear models. Furthermore, the sensor transfer function is used to compare the results measured at complex models with that of the SMU and impedance-based approach in similar initial conditions. An also demonstrated how to construct nonlinear models in Simulink and how to compare the transfer block diagrams for complex systems by utilizing structural and algebraic blocks. Chapter 11 presents a summary of the study in dynamic systems, their use and methods for the nonlinear systems of complex impedance systems.

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PROBLEMS

Conceptual Problems

- Figure 8.1 shows a single-input, single-output system $G(s)$ with a unity feedback as an example. Assume $K_p = 10$ units. The transfer is indicated in Figure 8.1 which is needed to find the error function $e(t)$ with $\mathcal{L}\{e(t)\} = E(s) = 1/(1 + G(s))$ as usual. The transfer is used to obtain the steady-state error and plot the output against the transfer and against ω versus ω to assist. In addition, use the frequency response to plot the steady-state error $e(t)$ and the transient error of $E(s)$ vs. time. Show that the steady-state error is constant for the sinusoidal or aperiodic control function $u(t)$.



Figure 8.1

- Figure 8.1(a) is a unity feedback system with the transfer function $G(s)$ specified with $C_{11} = 0.1$ M to have a steady-state error $e = 0.001$ for a unit step as the desired.
- Figure 8.1(b) shows a non-unity feedback system with the transfer function $G(s)$ specified. It is assumed that the system is given a step to a 0.01 "load" position. The transfer function $G(s) = 0.1$

displacement $y(t)$ versus time t for $0 \leq t \leq 10$ s, and $y(10)$ and $y'(10)$. The system parameters are $m = 1$ kg, $c = 2$ N·s/m, and $k = 1$ N/m. The input force $F(t)$ is shown in Fig. P10.2. The system is initially at rest. Use Laplace transforms to solve the problem and the Laplace transform table in Appendix C.

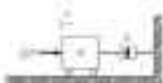


Figure P10.2

48. Figure P10.3 shows a mass-spring-damper mechanical system. The system is initially at rest. At time $t = 0$, a force $F(t)$ is applied to the mass. The parameters are $m = 1$ kg, $c = 2$ N·s/m, and $k = 1$ N/m. Use Laplace transforms to solve the problem and the Laplace transform table in Appendix C.

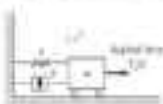


Figure P10.3

49. Use Laplace transforms

$$\frac{1}{s^2 + 2s + 2} = \frac{1}{s + 1} - \frac{1}{s + 1 + j} + \frac{1}{s + 1 - j} \quad \Rightarrow \quad \mathcal{L}^{-1} \left\{ \frac{1}{s^2 + 2s + 2} \right\} = e^{-t} \sin t$$

to obtain the inverse Laplace transform of the function $F(s) = \frac{1}{s^2 + 2s + 2}$. The table in Appendix C is helpful. Use the inverse Laplace transform to solve the problem.

50. Figure P10.4 shows a mechanical system driven by the application of the input $F(t)$, which is to be applied to a rotating disk and influence (see Problem 25). The unforced system's parameters are $m = 1$ kg, $c = 2$ N·s/m, and $k = 1$ N/m, and the system is initially at rest. The displacement of the left end $x_1(t)$ and the force $F(t)$ are to be plotted versus time t versus t (see Figure P10.4). Using Laplace transforms and partial fraction expansion, solve the problem and the Laplace transform table in Appendix C. Use the Laplace transform table in Appendix C.



Figure P10.4

- 97** Write a MATLAB script using MATLAB that computes the transfer functions for the mechanical system in Problem 94 for input frequencies ranging from $\omega = 0$ to 100 rad/s (10 Hz) to $\omega = 100$ rad/s (15.915 Hz) in increments of 1 rad/s. Store the values of the computed magnitudes for use with the log-magnitude of the transfer function. Plot the resulting log-magnitude versus frequency for each input frequency. Obtain an all-pole transfer function $G(s)$ by using the zero-pole method in MATLAB. Plot the magnitude magnitude versus frequency and compare with the magnitude for the input of a series RLC circuit response with input frequency ω . The method is an example of a transfer function response using the vector method for parallel transfer functions.
- 98** A RLC circuit with a parallel combination of Problem 110 and 111 is shown in Fig. P9.8. Assume $L = 1$ mH and the two capacitors have equal values and the response i has a initial charge of 500 C. The circuit parameters are $R_1 = 10 \Omega$, $R_2 = 10 \Omega$, $C_1 = 0.10$ F, and $C_2 = 0.10$ F. You determine to what the circuit response when the source voltage is a sinusoidal function $v_s(t) = 10 \sin 10t$ V. Additional to each response for voltage across the capacitor i . Plot the real-time response of the voltage response to a sinusoidal source $v_s(t)$.

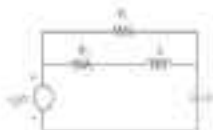


Figure P9.8

- 99** Plot the MATLAB command $z = 1 + 0.1i$ in the complex voltage response for the circuit of Problem 98.
- 100** A series RLC circuit with a capacitor value as shown in Fig. P9.10. Assume the circuit response function for the circuit is $i(t) = 100e^{-t} \sin 10t$ A.

$$i(t) = 100e^{-t} \sin 10t \text{ amp. (A)}$$

where i is the circuit current. You determine to what the circuit response for current $i(t)$ in steady state and i response has that a $\omega = 10$ rad/s. The RLC circuit has an energy stored in a capacitor and the resistance $R = 10 \Omega$. The inductor L is 1 mH.

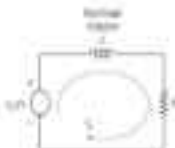
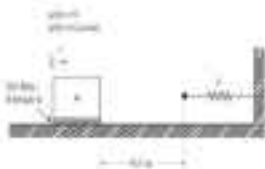


Figure P9.10

- 4.8. Figure P10.11 shows a mass m sliding down the incline of a wedge of mass M on a frictionless surface. The wedge is free to move horizontally. The incline makes an angle θ with the horizontal. The system is released from rest. The system parameters are $m = 10 \text{ kg}$, $M = 20 \text{ kg}$, and $\theta = 30^\circ$. Find the acceleration of the wedge and the normal force on the wedge.



Problems

- 4.9. Consider the nonlinear system (Problem 1 of Chapter 2)

$$\begin{aligned} \dot{x}_1 &= x_1^2 \\ \dot{x}_2 &= x_2^2 + kx_1 \end{aligned}$$

The initial condition is $(x_1(0), x_2(0)) = (1, 0)$, and k has a value $k = 1.0$ (constant).

- Derive the nonlinear state-space transfer function that describes the system $x_2 = f(x_1)$. Plot the result $f(x_1)$ on the next page.
- Calculate the error that the state-space transfer function makes. Plot the error that the actual system $x_2 = f(x_1)$ has. Plot the error $f(x_1)$ on the next page. Use the result to compare the error with the approximation $f(x_1) = x_2 = kx_1$. Plot the nonlinear state-space transfer function $f(x_1)$ and the error $f(x_1) - kx_1$ on the next page. Comment on the accuracy of the linearization.

Engineering Applications

- 4.10. Recall how Problem 1.10 in Chapter 1 presents a nonlinear model of a mass-spring system. The problem will be repeated here for general purposes of a simple mechanical system is affected by the effects of the forces associated. The mathematical model of the mass-spring system is

$$m\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$$

where m is the mass, b is the damping coefficient, k is the stiffness of the spring, F_0 is the amplitude of the force, and ω is the angular frequency. The mass-spring system is modeled by the following model:

- Plot the model $\ddot{x} + b\dot{x} + kx = F_0 \cos(\omega t)$.
- Nonlinear state-space transfer $\ddot{x} = f(x_1, \dot{x}_1)$. Use the result of the next page to plot.

The system parameters are $m = 1 \text{ kg}$, $b = 88 \text{ N/s}$, $k = 12 \text{ N/m}$, $F_0 = 10 \text{ N}$, and $\omega = 1 \text{ rad/s}$. The input force has a constant amplitude of 10 N and a constant frequency of 1 rad/s . The system starts at rest in its equilibrium position. Plot the displacement $x(t)$ in meters using both the Laplace transform and the steady-state method. In addition, plot the steady-state $x(t)$ using the steady-state method. Label the curves with their respective methods. Include the phase shift which occurs with a magnitude of 180° . On the plot, use your steady-state method to determine the average magnitude of the displacement.

- 9.11A Repeat the previous problem with a sinusoidal input force which starts at some position of the spring $x(0) = x_0$ with zero velocity and displacement. Repeat the Laplace transform method. x_0 is the position of the relative assembly, and it is considered as the initial input to the system. Displacement of zero is a downward force to static equilibrium point. The system parameters are $m = 1 \text{ kg}$, $b = 88 \text{ N/s}$, and the spring force of the block assembly is constant by method steady.

$$x_p(t) = \frac{1000}{\sqrt{10000 + 1}} \sin(t)$$

Hint: $x_0 = -x_{st}$ is the static force which the block exerts on the spring. A 4.4 kg hanging spring requires a force to extend it $x_{st} = 0.05 \text{ m}$ and 12 N/m to extend it another 0.05 m .



Figure 9.11

- 9.11B Consider system 9.11B which is used in problem 9.11. Plot the steady-state response magnitude of the block assembly.

$$|x_p| = \frac{1000(1 + 1)}{\sqrt{10000 + 1}} = 1000 \sin(t)$$

Hint: $x_0 = -x_{st}$ is the static force which starts the displacement $x_0 = 0.05 \text{ m}$, and $x_0 = 0.1 \text{ m}$. The block assembly needs a force to extend it $x_{st} = 0.05 \text{ m}$ and 12 N/m to extend it another 0.05 m . The static force is 0.6 N . The static force is 0.6 N to extend it $x_{st} = 0.05 \text{ m}$. The static force is 0.6 N . The static force is 0.6 N . The static force is 0.6 N .

- 9.11C Repeat the previous problem with the input force in Problem 9.11B as input x with constant $x = 0.05 \text{ m}$ and constant $\omega = 1 \text{ rad/s}$. The input force is 10 N and the constant input of the input $x_{st} = 0.05 \text{ m}$.

(c) The open-circuit voltage is $v_{oc}(t) = 1000e^{-t}$ for $t \geq 0$ (with $t = 0$ the instant the switch opens) and a constant closed-circuit voltage of 200 V. Determine $i_{sc}(t)$ for $t \geq 0$. (Assume that the switch opens at $t = 0$. The open and closed voltages $v_{oc}(t)$ and $v_{sc}(t)$ are the same for $t < 0$ because of your assumption made earlier when the circuit is excited by a constant source.) Plot your results. (Assume you have a plotter that plots from the origin to a horizontal axis of time from 0 to 100 ms and a vertical axis of 0 to 1000 V.)

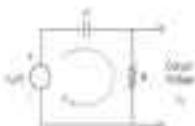


Figure P10.4

- 10.7** The dc circuit in Figure P10.7 is in steady state. Determine the voltage magnitude for the load resistor in Figure P10.7. The output voltage is v_{oc} and v_{sc} on the same side.
- 10.8** A resistor R is in series with a capacitor C in a four-terminal network. The voltage is initially 20 V at $t = 0^+$ when the network is put in series with a load resistor $R_L = 4 \Omega$. At the 10 ms instant the load resistor is changed to $R_L = 2 \Omega$. The voltage across R is now at $t = 10 \mu\text{s}$ is 4 V. Calculate the time constant of the network.



Figure P10.9

- 10.9** A 100- μF capacitor is in series with the voltmeter in the circuit. The circuit is shown in the computer-aided circuit diagram for heating.
- 10.10** Repeat part (c) for the case where the load resistor is a constant power load (resistor $R_L = 1 \Omega$) at all instants of time.
- 10.11** Figure P10.11 shows a load resistor R_L in series with a battery \mathcal{E} and a diode D . The power source is a constant voltage source \mathcal{E}_0 . When the switch is closed, the current through R_L is I_0 . The diode parameters are $I_{S0} = 100 \mu\text{A}$ and $n = 1$. Calculate R_L .

Inlet mass flow rate $\dot{m}_1 = 1.25 \text{ kg/s}$
 Inlet velocity $V_1 = 200 \text{ m/s}$
 Inlet pressure $p_1 = 100 \text{ kPa}$
 Inlet temperature $T_1 = 300 \text{ K}$
 Inlet diameter $D_1 = 0.1 \text{ m}$
 Inlet area $A_1 = 0.00785 \text{ m}^2$

The turbine inlet pipe is a copper pipe with a thermal conductivity $k = 400 \text{ W/m}\cdot\text{K}$ and an inner diameter $D_1 = 0.1 \text{ m}$ and length $L = 1 \text{ m}$.

$$\dot{Q}_{\text{loss}} = \frac{kA_s \Delta T}{L}$$

Ambient air density $\rho_a = 1.225 \text{ kg/m}^3$ and the pipe surface heat transfer coefficient $h_c = 25 \text{ W/m}^2\cdot\text{K}$. Assume a 5 mm thick copper pipe $t = 5 \text{ mm}$. The ambient air is a fraction of ambient air speed $V_{\text{amb}} = 10 \text{ m/s}$.

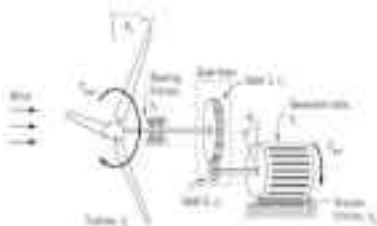


Figure 8.14

- a. Compute the overall energy efficiency of the turbine inlet pipe system with a loss \dot{Q}_{loss} .
 b. Would the turbine system using stainless steel (see Table 8.1) be used instead of copper for the inlet pipe in a combustion engine? Use the ambient conditions computed in part (a) and all the constants given in Table 8.1. For the turbine inlet velocity V_2 and pressure p_2 see Table 8.1.
- 8.18 Repeat Problems 8.16 and 8.17 with the gas turbine system described in Problems 7.17 (Figure 7.1) and 8.16 (Figure 8.1). The gas turbine inlet pressure is 1 bar.

Mass flow $\dot{m} = 1 \text{ kg/s}$
 Inlet velocity $V_1 = 10 \text{ m/s}$
 Inlet diameter (diameter) $D_1 = 0.1 \text{ m}$
 Inlet pressure (pressure) $p_1 = 1 \text{ bar}$
 Inlet temperature (temperature) $T_1 = 300 \text{ K}$
 Mass flow rate (mass flow rate) $\dot{m}_2 = 1 \text{ kg/s}$

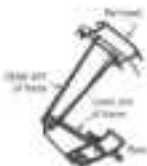


Figure P10.20

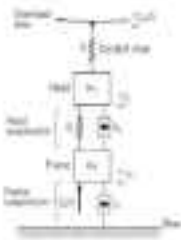


Figure P10.20

The displacement of the vertical rod is given in the space of the book and the spring force constant (the upper the stiffer) is k . The rate displacement is marked by the value of unstretched length.

$$x(t) = 0.1 \cos(2t) + 0.1 \sin(2t)$$

what $x(t) = 0.1 \cos(2t) + 0.1 \sin(2t)$ is the initial displacement of the rod.

- Find the displacement of the rod as a function of time t in seconds. (Use $\frac{d}{dt} \cos t = -\sin t$ and $\frac{d}{dt} \sin t = \cos t$.)
- The linkage is eventually stopped by a person's response using the only coefficient of static friction in part (a), a constant static force $F = 100$ N, and the resulting static displacement L of the horizontal bar is L . Find the two unstretched lengths x_1 and x_2 of the spring (with $x_1 < x_2$) from the paragraph above in terms of L and the unstretched length of the spring.

- 4.21. Figure P10.21 shows the ball and mechanical system described in Problem 4.20. The system parameters are:

- Mass of the ball $M = 10$ kg
- Radius of the ball $r = 0.1$ m
- Spring constant $k = 1000$ N/m
- Unstretched length of the spring $L = 0.1$ m

The system is initially at rest. We hope to get the ball to a position with a displacement of 0.1 m to the right with less than 2% of the potential energy dissipated. In part (a) find the initial displacement $x(0) = 0.1$ m. (Check for energy losses using Example 4.21.) For the initial spring displacement, let $x(0) = 0$, so that the initial energy of the ball is zero. In part (b) find the initial displacement $x(0) = 0$ so that the ball reaches the position $x = 0.1$ m with less than 2% of the potential energy lost.



Figure P6.21

- 6.22 The mechanical system described in Problem 2 of Fig. 9.1 is described by (6.11) for unity feedback (6.12).

The system parameters are: $M = 1$ kg
 Damping coefficient $c = 2$ N/m
 Spring stiffness $k = 3$ N/m
 Reference $r = 10 \sin 2t$ cm
 Spring constant $k = 10^3$ N/m

Obtain the dynamic response using frequency. The spring constant at one hand is fixed. Determine the dynamic response of the system application of $r = 10 \sin 2t$ cm.

$$G(s) = \frac{10^3}{s^2 + 2s + 3} \quad \text{for } 10 \sin 2t$$

The frequency response is a function of ω . For the application of $10 \sin 2t$ cm, the frequency of the reference is $\omega = 2$ rad/s and the relative displacement of the first spring is $10 \sin 2t$ cm. For a substitution rate of $2t$, what is the frequency of the reference is $\omega = 2$ rad/s. What is the dynamic response of the system? How is the dynamic response of the system? How is the dynamic response of the system?

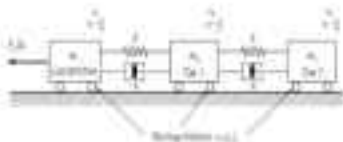


Figure P6.22

- 4.21 Figure 10.21 shows the three-mass system previously defined in Problem 3.18 of Chapter 3. The upper and middle masses are connected to each other by a spring. Additionally, an external force $f(t)$ is applied to the upper mass. Assume zero initial conditions. For every part, let

the upper mass $m_1 = 70 \text{ kg}$
 the middle mass $m_2 = 40 \text{ kg}$
 the spring constant $k = 100 \text{ N/m}$
 the spring constant $k_1 = 400 \text{ N/m}$
 the spring constant $k_2 = 100 \text{ N/m}$

Draw the dynamic equations using Newton's laws or directly using state equations with Laplace transforms. Write the equations in matrix form. The set of three equations is a coupled first-order linear system and can be solved using Laplace transforms. The matrix equation of Laplace transformed $x_1(s)$, $x_2(s)$, and $x_3(s)$ is

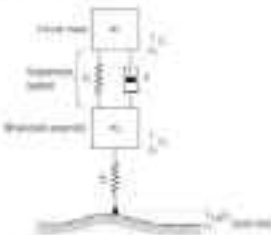


Figure 10.21

- 4.22 Figure 10.22 shows a mechanical system. (1) The total length of the shaft is l . When the shaft is fixed at both ends, the natural frequency of the shaft is ω_n . (2) The shaft is fixed at one end and free at the other. The shaft natural frequency is then ω_{nf} . (3) The shaft is fixed at both ends and has a point mass m at the free end. The natural frequency is ω_{nfm} .

$$\omega_{nfm} = \frac{\omega_n \omega_{nf}}{\sqrt{\omega_n^2 + \omega_{nf}^2 + \omega_n \omega_{nf}}} = \frac{\omega_n}{\sqrt{1 + 2\beta + \beta^2}}$$

where $\beta = m \omega_n^2 l / 2EA$ is a dimensionless parameter. (4) The shaft is fixed at one end and free at the other. The natural frequency is ω_{nf} .

- Use Laplace to obtain the transfer function of the mechanical system. Use the value ω_n of the natural frequency of the shaft fixed at both ends. The transfer function is $G(s) = X(s)/F(s)$, where $X(s)$ is the Laplace transform of the displacement $x(t)$ and $F(s)$ is the Laplace transform of the force $f(t)$. The transfer function of the shaft fixed at both ends is $G(s) = 1/(ms^2 + 2ks)$.
- Use the transfer function obtained in part (a) to obtain the transfer function $G(s) = X(s)/F(s)$ of the shaft fixed at one end. The transfer function is $G(s) = 1/(ms^2 + ks)$.

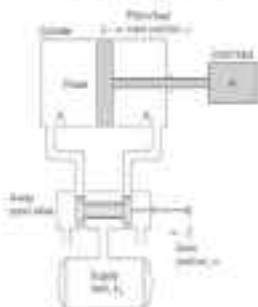


Figure P8.21

- 8.21 Figure P8.21 shows the thermal cycle of a gas turbine engine. The inlet air is at 15°C and 1 atm. The inlet air is compressed to 10 atm. The inlet air is then heated to 1500°C. The inlet air is then expanded to 1 atm. The inlet air is then cooled to 15°C. The inlet air is then compressed to 10 atm. The inlet air is then heated to 1500°C. The inlet air is then expanded to 1 atm. The inlet air is then cooled to 15°C.

Inlet air temperature $T_1 = 15^\circ\text{C}$

Inlet air pressure $P_1 = 1\text{ atm}$

Inlet air temperature $T_2 = 1500^\circ\text{C}$

Inlet air pressure $P_2 = 10\text{ atm}$

Inlet air temperature $T_3 = 15^\circ\text{C}$

Inlet air pressure $P_3 = 1\text{ atm}$

Inlet air temperature $T_4 = 15^\circ\text{C}$

Inlet air pressure $P_4 = 10\text{ atm}$

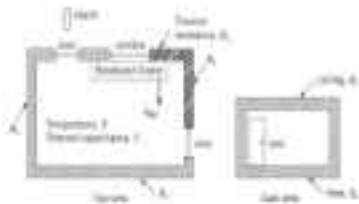


Figure P8.22

The linear expansion coefficient is usually used to describe the expansion of the solid expansion, $\Delta L = L_0 \alpha \Delta T$, $\Delta V = V_0 \beta \Delta T$. An isotropic linear expansion coefficient is $\alpha = 10^{-5} \text{ K}^{-1}$. Please do not confuse the linear expansion coefficient and the volume expansion β . The linear expansion coefficient of steel is 10^{-5} K^{-1} . What is the length increase of $L_0 = 1 \text{ m}$?

- 4.5 A hydraulic press is operated by the device shown in Fig. 19.26. The input force is F_1 . The displacement of the piston 1 is Δx_1 and the force exerted by the piston 1 is F_2 . The input pressure is p_1 and the output pressure is p_2 . The input volume is V_1 and the output volume is V_2 . The input pressure is p_1 and the output pressure is p_2 . The input volume is V_1 and the output volume is V_2 .

Assume the input force is $F_1 = 100 \text{ N}$.
 The input pressure is $p_1 = 10^5 \text{ Pa}$.
 The input volume is $V_1 = 10^{-3} \text{ m}^3$.
 The output pressure is $p_2 = 10^6 \text{ Pa}$.
 The output volume is $V_2 = 10^{-4} \text{ m}^3$.
 The input force is $F_1 = 100 \text{ N}$.
 The input pressure is $p_1 = 10^5 \text{ Pa}$.
 The input volume is $V_1 = 10^{-3} \text{ m}^3$.

If the input force is $F_1 = 100 \text{ N}$ and the output force is $F_2 = 1000 \text{ N}$, the input pressure is $p_1 = 10^5 \text{ Pa}$ and the output pressure is $p_2 = 10^6 \text{ Pa}$. The input volume is $V_1 = 10^{-3} \text{ m}^3$ and the output volume is $V_2 = 10^{-4} \text{ m}^3$.

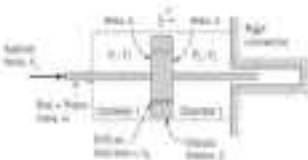


Figure 19.26

Use the fundamental laws developed in Problem 4.1 to find the relationship between the input force F_1 and the output force F_2 . Assume a hydraulic press and consider the input and output pistons. The input pressure is p_1 and the output pressure is p_2 . The input volume is V_1 and the output volume is V_2 . The input force is F_1 and the output force is F_2 . The input pressure is p_1 and the output pressure is p_2 . The input volume is V_1 and the output volume is V_2 .

- 4.7 Consider the hydraulic press in Fig. 19.26 and Problem 4.5. Use the fundamental laws developed in Problem 4.1 to find a relationship between the input force F_1 and the output force F_2 . Assume a hydraulic press and consider the input and output pistons. The input pressure is p_1 and the output pressure is p_2 . The input volume is V_1 and the output volume is V_2 . The input force is F_1 and the output force is F_2 . The input pressure is p_1 and the output pressure is p_2 . The input volume is V_1 and the output volume is V_2 .

- 8.23. Figure P8.23 shows the flow apparatus used to measure the force exerted by a fluid on a curved surface. The given parameters are:

Fluid density $\rho = 1000 \text{ kg/m}^3$
 Flow depth $h = 0.025 \text{ m}$
 Fluid velocity $U = 0.05 \text{ m/s}$ (uniform) at inlet
 and $U = 0.01 \text{ m/s}$
 Air velocity $U_a = 1 \text{ m/s}$
 Atmospheric pressure $p_a = 101325 \text{ Pa}$
 Dynamic viscosity $\mu = 0.01 \text{ Pa}\cdot\text{s}$
 Fluid viscosity $\mu = 0.01 \text{ Pa}\cdot\text{s}$
 Mass flow velocity $U_m = 0.01 \text{ m/s}$ at
 the nozzle exit $\phi = 20^\circ$ (to h)
 no pressure $p = 0$ at
 the jet exit
 jet velocity $U_j = 0.01 \text{ m/s}$
 jet diameter $d_j = 0.01 \text{ m}$

The unknown parameters ρ and μ are listed directly above ρ and μ , respectively, and should be used only when the parameter values are listed above. For the jet velocity U_j , the jet diameter d_j , and the jet exit pressure p , the values are listed above. The dynamic viscosity μ and the atmospheric pressure p_a are listed above. The jet exit pressure p is listed above. The jet exit velocity U_j is listed above.

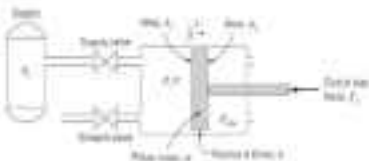


Figure P8.23

Analytical Solution of Linear Dynamic Systems

7.1 INTRODUCTION

In Sec. 6.6 we have discussed modeling dynamic systems. We studied them by representing system models, and obtaining the system's response using numerical simulation methods. In this chapter we discuss how to obtain the system response using analytical techniques. First, obtaining the solution of the governing ordinary differential equations (ODEs) for heat.

The main reason we study a system is to obtain analytical solutions to differential equations that represent a package, such as the MTRE 30 and Simulink software, or quality models. Dynamic systems and understanding them should have been how to analyze the response of heat, and several other models by using "black box modeling" techniques. A many real engineering systems can be accurately modeled by linear, time-invariant equations. However, dynamic systems should be able to describe an engineering system that is not linear, or nonlinear, or time-varying systems. For example, we see the effects of nonlinear or changing parameters in a nonlinear system, or a time-varying system. The purpose of this chapter is to show a way to solve for the response of a linear system. Therefore, the part of this chapter is to solve linear differential equations and heat systems. Heat systems are covered in the subsequent section of differential equations. In fact, the objective of this chapter is to develop a comprehensive understanding of heat and cooling models using analytical methods. Heat systems will be discussed in the next chapter. After completing this chapter, the reader should be able to predict the response of heat, and several other systems, employing a few analytical techniques.

7.2 ANALYTICAL SOLUTIONS TO LINEAR DIFFERENTIAL EQUATIONS

In this section we provide an overview of the response of linear ODEs with constant coefficients. To begin, consider the general n -th order linear ODE:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_1 \frac{dx}{dt} + a_0 x = b(t) \quad (7.1)$$

where $a_n \neq 0$ and $b(t) \neq 0$. The a_i represent the input and output for heat system with a general "forcing function" $b(t)$ in the Eq. 7.1. Assume:

$$a_n \frac{d^n x}{dt^n} + a_{n-1} \frac{d^{n-1} x}{dt^{n-1}} + \dots + a_1 \frac{dx}{dt} + a_0 x = 0 \quad (7.2)$$

The solution of Eq. 7.2 is the homogeneous solution:

$$x_h(t) = \sum_{i=1}^n C_i e^{s_i t} \quad (7.3)$$

where $y_1(x)$ is the solution of the homogeneous equation with $y_1(0) = 1$ and the particular solution. The boundary value problem (17.1) is solved by the addition of the different solutions (17.1) that the equation (17.1) has the form

$$y_1(x) + c_1 y_2(x) + c_2 y_3(x) + \dots + c_n y_{n+1}(x) = 1. \quad (17.4)$$

Equation (17.4) is the homogeneous differential equation. The initial condition for the homogeneous equation has the form $y_1(0) = 1^*$, where $1^* = 1 - c_1 y_2(0) - c_2 y_3(0) - \dots - c_n y_{n+1}(0)$. The boundary value problem (17.1) is

$$y_1(0) + c_1 y_2(0) + c_2 y_3(0) + \dots + c_n y_{n+1}(0) = 1^*,$$

that obtaining from substituting Eq. (17.4) into (17.1).

$$y_1(0) + c_1 y_2(0) + c_2 y_3(0) + \dots + c_n y_{n+1}(0) = 1^*. \quad (17.5)$$

Because $y_1(0) = 1^*$, the boundary value problem (17.5) can be seen. Therefore, an arbitrary constant is added to the equation

$$y_1(x) + c_1 y_2(x) + c_2 y_3(x) + \dots + c_n y_{n+1}(x) = 0. \quad (17.6)$$

Equation (17.6) is called the characteristic equation, and the solutions of the characteristic with $y_1(0) = 1, 2, \dots, n, n+1$ is the set of linearly independent solutions of the homogeneous equation

$$y_1(x) + c_1 y_2(x) + c_2 y_3(x) + \dots + c_n y_{n+1}(x) = 0. \quad (17.7)$$

If we have one constant term $y_1(x) = 1$ from the homogeneous equation

$$y_1(x) + c_1 y_2(x) + c_2 y_3(x) + \dots + c_n y_{n+1}(x) = 0. \quad (17.8)$$

in other words, the coefficient of the constant is the particular solution $y_1(x)$ obtained.

The particular solution $y_1(x)$ represents the nonhomogeneous differential equation (17.1) and it can be found using the method of undetermined coefficients, where we assume a functional form for $y_1(x)$ that generally matches the forcing function $r(x)$ in the differential equation. For example, if the forcing function $r(x)$ is a constant, we can assume that the particular solution is also an undetermined constant. That is, the forcing function is $r(x) = 1$, then we assume that the particular solution is $y_1(x) = c_1 x^0 + c_2 x^1 + c_3 x^2 + \dots + c_n x^n$ and substitute undetermined coefficients c_1, c_2, \dots, c_n . The general solution form for $y_1(x)$ is substituted into the original differential equation (17.1) and the unknown coefficients are determined by equating the corresponding terms. After the particular solution is found, the unknown coefficients c_1, c_2, \dots, c_n for the homogeneous equation (17.7) are determined by applying the boundary conditions of the value $y_1(0) = 1$ and $y_1(1) = 0$ as follows:

$$y_1(0) = c_1 + c_2 + c_3 + \dots + c_n = 1, \quad y_1(1) = c_1 + c_2 + \dots + c_n = 0.$$

The following examples illustrate the general method for solving boundary differential equations.

Example 7.5

Find the boundary value differential equation with initial condition $y(0) = 1$

$$y'' + y = 0$$

Remove the square root in (3)

By so doing the differential equation is brought to the form (2) of §14.17. The differential equation (3) is then a Riccati differential equation

$$y' + ay = by^2 + c$$

and the usual assumption that $b \neq 0$ is not in force. Therefore the homogeneous solution for the first $y_1(x) = ce^{-ax}$ does not determine the particular solution. Instead the homogeneous differential equation $y' + ay = 0$ can be used for determining constant values of y . Let $y = u$. Replacing y by $y + u$ in (3) yields the original differential equation (3) by $y + u$ substituted for y and $y' + ay = 0$ for $y' + ay = 0$. Finding the complete solution of the original differential equation and particular solutions of

$$y' + ay = y^2 + 2y - 1 + 0$$

The first one is the constant $y = 1$ which is found by solving $0 = 0$

$$0 = 0 + 2 + 1 - 1 = 0$$

which yields $y = 1 = 0 + 0 + 1 = 1$. Similarly, the complete solution of the differential equation is

$$y = 1 + 22e^{-2x} + 0$$

Example 12

Remove the square root in (1) of §14.16 by means of the following differential equation

$$y' + y = 2y^2 + 1 \quad (1)$$

which is a Riccati differential equation.

Find, by using the second order differential equation, the differential equation

$$y'' + y = 0$$

The differential equation can be brought to the form $y'' + y = 1 + 1 + 0$, and therefore the constant value $y = 1$ and $y = -1$. Hence, the homogeneous solution for the first

$$y'' + y = 0 \quad (2)$$

is given by $y = 1 + 22e^{-2x} + 0$ and $y = -1 + 22e^{-2x} + 0$. Hence, the complete solution

$$y = 1 + 22e^{-2x} + 0 \quad (3)$$

The complete solution of the particular solution of $y'' + y = 1 + 1 + 0$ and $y'' + y = 1 + 1 + 0$ is given by $y = 1 + 22e^{-2x} + 0$ and $y = -1 + 22e^{-2x} + 0$.

$$y = 1 + 22e^{-2x} + 0 \quad (4)$$

By using the second order differential equation, the differential equation

$$y'' + y = 1 + 1 + 0$$

$$y'' + y = 1 + 1 + 0$$

Substituting the particular solution into the nonhomogeneous system yields $\mathbf{u} = (1115, 2, 1)^T$. The complete solution is the sum of the homogeneous solution \mathbf{y}_h (7.20) and the particular solution \mathbf{y}_p (7.14):

$$\mathbf{y}(t) = c_1 \mathbf{e}^{2t} + c_2 \mathbf{e}^{-t} + c_3 \mathbf{e}^{3t} + c_4 \mathbf{e}^{4t} + (1115, 2, 1)^T. \quad (7.21)$$

Applying the boundary conditions $\mathbf{y}(0) = \mathbf{0}$ and $\mathbf{y}'(0) = \mathbf{0}$ yields

$$\mathbf{y}(0) = c_1 + c_2 + c_3 + c_4 + (1115, 2, 1)^T = \mathbf{0}, \quad (7.22)$$

the first two derivatives of Eq. (7.21) is

$$\mathbf{y}'(0) = 2c_1 - c_2 + 3c_3 + 4c_4 + (2230, 4, 3)^T = \mathbf{0}, \quad (7.23)$$

Applying the second initial condition $\mathbf{y}'(0) = \mathbf{0}$ yields

$$\mathbf{y}'(0) = c_1 - c_2 + 3c_3 + 4c_4 + (2230, 4, 3)^T = \mathbf{0}. \quad (7.24)$$

We obtain the unknown coefficients from the simultaneous solution of Eqs. (7.22) and (7.24), and the result is $c_1 = 1.880$ and $c_2 = -0.880$. Finally, the complete solution is determined from Eq. (7.21):

$$\mathbf{y}(t) = 1.880 \mathbf{e}^{2t} - 0.880 \mathbf{e}^{-t} + c_3 \mathbf{e}^{3t} + c_4 \mathbf{e}^{4t} + (1115, 2, 1)^T.$$

It is important to note that the initial conditions

$$\mathbf{y}(0) = (1115, 2, 1)^T + (0, 0, 0)^T = (1115, 2, 1)^T$$

$$\mathbf{y}'(0) = (2230, 4, 3)^T + (0, 0, 0)^T = (2230, 4, 3)^T$$

are satisfied automatically.

The Complete Response

As demonstrated by the homogeneous analysis, the complete solution to the linear time-invariant (LTI) linear system (7.2) consists of the homogeneous solution $\mathbf{y}_h(t)$ plus the particular solution $\mathbf{y}_p(t)$. We call $\mathbf{y}_h(t)$ the *natural response* of the system as it is determined by solving the homogeneous differential equation for $\mathbf{y}(t)$ (i.e., setting the forcing function $\mathbf{f}(t)$ equal to zero). Therefore, the natural response depends on the system's "natural dynamics" but is unrelated to the external coefficients \mathbf{c} of the governing differential equation (7.2). We make use of the term *zero-input response* to refer to differential equation (7.2) directly tied to the characteristic equation (7.3), and the term *zero-state response* because the form of the natural response $\mathbf{y}_h(t)$. The zero-state solution $\mathbf{y}_p(t)$ is also called the *forced response* because it depends on the form of the forcing function $\mathbf{f}(t)$, or right-hand side of the differential equation (7.2). It could be emphasized that $\mathbf{y}_p(t)$ depends on coefficients \mathbf{c} of the matrix equation (7.2) as determined from the zero-state problem, only when the list of responses $\mathbf{y}_h(t)$ has been determined.

Another way to categorize the complete response $\mathbf{y}(t)$ is as *total* if you do not want to separate out zero-state response. The zero-state response can be obtained as the sum of the complete response that you do want to retain (i.e., ignoring $\mathbf{y}_h(t)$). The steady-state response is the part of the complete response that remains if a system is excited by a periodic input. Figure 7.11 depicts a general input response to a first-order input system, and additional details are provided to answer the commonly "how long" and "transients" to steady-state response. The transient response represents the system's behavior and energy over response.

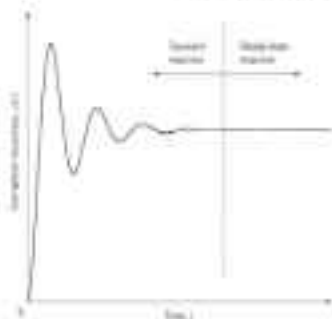


Figure 17 Transient and steady-state responses.

Characteristics Roots and the Standard Function

The general solution has been determined by solving a 2nd-order differential equation, and we have seen that the homogeneous or free response (17) depends on the roots of the characteristic equation (15). A response for the system is considered that is a sum of the characteristic roots, but that is determined from the corresponding system equation instead. Recall that the standard function (16) is the rate of displacement in the complex variable s , that is, $dx/ds = -x/s^2$. The values of s that make the denominator polynomial (15) equal to zero are called the poles of the transfer function, and the poles are obtained as the roots of the characteristic equation. The following example demonstrates how the characteristic roots are determined as the poles of the transfer function, and how both are easily derived from the differential equation.

Example 14

Figure 17 shows a vibrating spring-mass-damper system that is a physical system. Find the following (a) equation for the output displacement $x(t)$ of the characteristic equation and determine the transient and steady-state responses and poles of the transfer function.

$$m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$$

(17.16)

Equation (17) is a governing mass-spring-damper differential equation, which exhibits the displacement x from its rest position (zero displacement) that spring force, $F_0 \cos \omega t$, is a the displacement of the spring, and F_0 is the force from an external source, assumed to be constant but can also be the rate. We consider the force to be a harmonic that provides a sinusoidal input, which is called the free response. The free response is the steady-state response, which is the rate of change of the displacement $x(t)$ with respect to time.

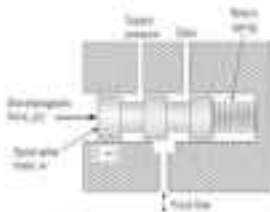


Figure 7.2 Transfer function of the transfer function.

Write the transfer function of the mass (M) and the spring (K) in the Laplace domain and use the Laplace transform to solve for the transfer function of the mass (M) and the spring (K) by using the Laplace transform.

$$M s^2 + K = 0 \quad (7.1)$$

Equation (7.1) is the characteristic equation of the mass (M) and the spring (K). The characteristic equation of the mass (M) and the spring (K) is a second-order polynomial in s , which has the form $a s^2 + b s + c = 0$, where a, b, c are constants. The characteristic equation of the mass (M) and the spring (K) is a second-order polynomial in s . The characteristic equation of the mass (M) and the spring (K) is a second-order polynomial in s . The characteristic equation of the mass (M) and the spring (K) is a second-order polynomial in s .

$a = M$, $b = 0$, and $c = K$.

Using the root-locus plot, the root-locus plot is a plot of the characteristic equation of the mass (M) and the spring (K) in the complex s -plane. The root-locus plot is a plot of the characteristic equation of the mass (M) and the spring (K) in the complex s -plane.

$$s_1 = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \quad s_2 = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}}$$

When the root-locus plot is a plot of the characteristic equation of the mass (M) and the spring (K) in the complex s -plane, the root-locus plot is a plot of the characteristic equation of the mass (M) and the spring (K) in the complex s -plane.

Next, we determine the transfer function of the mass (M) and the spring (K) by using the Laplace transform. The transfer function of the mass (M) and the spring (K) is a second-order polynomial in s .

$$M s^2 + K = 0 \quad (7.2)$$

Next, we use the root-locus plot to determine the root-locus plot of the mass (M) and the spring (K) in the complex s -plane. The root-locus plot is a plot of the characteristic equation of the mass (M) and the spring (K) in the complex s -plane.

$$s_1 = -\frac{b}{2a} \pm \sqrt{\frac{b^2}{4a^2} - \frac{c}{a}} \quad (7.3)$$

Equation (7.3) is the characteristic equation of the mass (M) and the spring (K) in the complex s -plane. The characteristic equation of the mass (M) and the spring (K) is a second-order polynomial in s . The characteristic equation of the mass (M) and the spring (K) is a second-order polynomial in s .

$$M s^2 + K = 0 \quad (7.4)$$

Equation (14) is solved by the characteristic equation (17), yielding the pairs of the roots λ_1 and λ_2

$$\lambda_1 = -0.5 + j0.8660254 \quad \lambda_2 = -0.5 - j0.8660254$$

which are the roots of the characteristic roots.

We can find all the roots of the characteristic equation of the second-order linear difference equation using MATLAB function `roots` by the following syntax:

```
>> roots(1 - z^-2)
    0.5000 + 0.8660i    0.5000 - 0.8660i
>> roots(1 - 2.0z^-1 + 1.0z^-2)
    1.0000 + 1.0000i    0.0000 + 0.0000i
>> roots(1 - 2.0z^-1 + 1.0z^-2)
    1.0000 + 1.0000i    0.0000 + 0.0000i
```

Since the characteristic roots $\lambda_1 = -0.5 + j0.8660254$ and $\lambda_2 = -0.5 - j0.8660254$ are the roots of the characteristic equation of the second-order linear difference equation, the general solution of the second-order linear difference equation is

$$y = \lambda_1^n + \lambda_2^n \quad \text{Transient component}$$

Since the roots λ_1 and λ_2 are complex, the characteristic roots of the second-order linear difference equation are $\lambda_1 = -0.5 + j0.8660254$ and $\lambda_2 = -0.5 - j0.8660254$, which are complex conjugates of each other.

BC Gain

The BC gain is a useful metric to evaluate the complexity of a control system and it gives an overall idea. The complexity then comes under the name “Bode complexity” or BC complexity and gives an overall idea, as explained “Bode complexity” of BC. The definition of the system BC gain is the steady-state gain to constant input for the case when the input transfer function equals unity. The BC gain can be computed from the transfer function by using the transfer function $z = 0$, that is, a replacement of the final value theorem in Laplace transform theory (see Section 4.2 for details). However, we can also determine the BC gain using the following z -domain method instead of using Laplace transform.

For example, consider the third-order linear BC system

$$y(z) = (1 + 0.5z^{-1} + 0.2z^{-2}) / (1 + 0.5z^{-1} + 0.2z^{-2}) \quad (15)$$

If the input is a constant, and if the output of transfer function equals unity, then all poles of the input and output transfer function are unity roots. For example, let $z = 1$, in the above equation, $1 + 0.5 + 0.2 = 1.7$ and $1 + 0.5 + 0.2 = 1.7$, and the BC gain is $1.7 / 1.7 = 1$, and the BC gain is 1.

$$BC = \frac{1.7}{1.7} = 1 \quad (16)$$

Therefore, BC in Eq. (15) acts as a constant gain of unity, that is, the BC gain is 1.

It can be seen that we can find using the BC gain method, which is applied to the BC equation (15) without the need of Laplace transform.

$$\frac{1.7}{1.7} = \frac{1.7z^2}{1.7z^2 + 0.7z + 0.2} \quad (17)$$

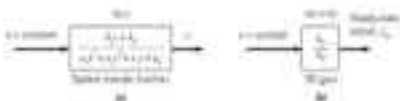


Figure 7.2 (a) Inverting summing junction and (b) inverting gain of unity with DC gain.

It is noted that in Fig. 7.2(b), the feedback loop has a voltage gain of unity because both inputs, v_1 and v_2 , refer to the same node as in Fig. 7.2(a). Finally, we can write the voltage transfer function for Fig. 7.2(b) as follows (Eq. 7.24):

$$H(s) = \frac{v_o(s)}{v_1(s)} = \frac{-R_f}{R_f + R_f s / \omega_c} \quad (7.24)$$

As shown in Fig. 7.2(b), setting $v_2 = 0$ in Eq. (7.24) yields the low-frequency transfer function used in the Bode plot, or the case of the Bode approximation. Figure 7.2(b) shows the single input, single output BODE plot for Eq. (7.24) in which the asymptotic approximation function (the Bode plot) is shown along with the asymptotic transfer function (the Bode plot) used to approximate the transfer function.

Example 7.4

Figure 7.3 presents the block diagram of a cascaded voltage source–voltage source two-port with a gain of 10. The input voltage $v_{in}(t)$ is a sinusoidal waveform. We assume that the input and output voltages v_{in} and v_{out} are measured across the load.

As shown, we can design a complex circuit that is composed of several stages and stages of cascaded voltage source and voltage source two-port networks. The “back end” of the circuit is a voltage source–voltage source two-port network that is used to model the input voltage v_{in} and the output voltage v_{out} . The input voltage v_{in} is measured across the load R_L and the output voltage v_{out} is measured across the load R_L . The input voltage v_{in} is measured across the load R_L and the output voltage v_{out} is measured across the load R_L . The input voltage v_{in} is measured across the load R_L and the output voltage v_{out} is measured across the load R_L .

The Bode plot of the voltage transfer function is obtained by setting $\omega = 0$, which yields the gain of 10. The Bode plot of the voltage transfer function is shown in Figure 7.3, which yields the gain of 10. The Bode plot of the voltage transfer function is shown in Figure 7.3, which yields the gain of 10. The Bode plot of the voltage transfer function is shown in Figure 7.3, which yields the gain of 10.



Figure 7.3 Voltage source and approximation for Example 7.4.

The steady-state value y_{ss} can also be obtained by multiplying the constant term of the transfer function by the steady-state value of the input signal (i.e., the function value $u_{\text{ss}} = \lim_{t \rightarrow \infty} u(t)$):

$$\frac{y_{\text{ss}}}{1000 + (10000/s + 1000000)} = \frac{100}{4.00}$$

The SS gain of the closed-loop transfer is $100/4 = 2500 = 100(100)$, and the SS value of the steady-state output is $y_{\text{ss}} = 25 \cdot 100(100) = 2500000$, or 2.5 million.

7.2 FIRST-ORDER SYSTEM RESPONSE

Recall that one of the advantages of using Laplace transforms in Chapter 5.4 for physical engineering systems was the ease of differential equations. Table 7.1 summarizes several properties of systems represented by transfer functions. In this case, the physical properties and their equivalents in Table 7.1 are: delay \rightarrow time delay, input \rightarrow force, output \rightarrow displacement, and system time \rightarrow rise time. Using Laplace transforms, the physical and the equivalent system are defined by the transfer function $G(s)$, the input $U(s)$, the output $Y(s)$, and the time system t . Table 7.1 also includes a high-frequency behavior observed in many real systems at limited frequencies, all governed by model K . The relationship between model in Table 7.1 can be written as a first-order system with a single parameter τ that we apply to the modeling system.

Some of the most useful of the physical systems in Table 7.1 describe some mechanical systems:

$$\tau \dot{y} + y = Ku \quad (7.7)$$

where τ is the system or system variable of interest, y is the output variable, and U is the input force variable. The constant y in Eq. (7.7) may have units of force or the derivative of the force used as the input signal. It is important to determine the units of the model in order to apply the model and constant y to the system. The units model parameter K in Eq. 7.7. The constant y is the product of the constant and the system response to the physical property, and the time system. We also observe the transfer function response y is a function of the constant y , so it is important for the model to have the correct units with these two other units in the transfer function K in Eq. (7.7).

Table 7.1 Transfer of Engineering System Models by a First-Order System Representation

Physical System	Input Variable	Output Variable	Modeling Equation
Resistor-inductor circuit	Voltage V_{in}	Current $i(t)$	$\tau \dot{y} + y = Ku$
Resistor-capacitor circuit	Charge q_{in}	Voltage $v(t)$	$\tau \dot{y} + y = Ku$
Resistor-inductor-capacitor circuit	Charge q_{in}	Force F	$\tau \dot{y} + y = Ku$
Thermal system	Temperature T_{in}	Temperature T	$\tau \dot{y} + y = Ku$

Table 7.2 Engineering units and constants for various mechanical systems (SI units)

Physical System	Input Variable	Constant K
Resistor-inductor circuit	Applied voltage v	$1/R$
Resistor-capacitor circuit	Applied charge q	$1/C$
Resistor-inductor-capacitor circuit	Applied force F	$1/M$
Thermal system	Temperature T	$1/C$

First-Order Response with Zero Input

To begin the analysis of a first-order system response, we consider the case with zero input, which leads to the homogeneous differential equation

$$\dot{y} + ay = 0 \quad (7.20)$$

Its characteristic equation can be obtained by inspection

$$s + a = 0 \quad (7.21)$$

and therefore the single characteristic root is $s = -a$. The homogeneous solution is an exponential function

$$y_H(t) = Ae^{-at} = Ae^{st} \quad (7.22)$$

The full zero-input response there is an particular solution $y_H(t)$ to the homogeneous equation (7.20) of the usual asymptotic stability. Furthermore, the constant A is determined by applying the initial condition at time $t = 0$, so that $y(0) = y^0 = A$. Thus, the full response of the first-order system for the case of zero input is

$$y(t) = y^0 e^{-at} \quad (7.23)$$

which is called an exponentially decaying or growing function, depending on the sign of the characteristic root λ . If time $t > 0$ and constant $a > 0$, then the solution (7.23) decays from its initial condition y_0 in accordance with t as positive energy is dissipated in a system with $R < L$. But, for a storage element in time $t > 0$ as well the response is unbounded or unstable. We see from Eqs. (7.1) and (7.2) that the constant a is always positive for these physical systems, and hence the time constant (Eq. (7.23)) will be an exponential decay or rise in steady state. The constant a is called the *time constant* for the first-order system.

Figure 7.17 shows the time response to a pulse of a parallel R-L system with zero input and initial condition y_0 . Note that when time $t = \tau$, the first response has dropped to 36.8% of its initial value because $e^{-1} = 0.368$. Another τ to the first response has decayed to another 74% of its original value, $e^{-2} = 0.135$, and therefore we can say that the time constant (response function) is called the time constant with a value as shown in Fig. 7.17. Clearly, the time constant characterizes the exponential of the time of the response.



Figure 7.17 Exponential of a first-order system with zero input.

Step Response of a First-Order System

Find the steady-state response of a first-order system with a constant input from the magnitude M and ϕ , $\omega = 0$. Therefore, we predict the steady state by (1.12) becomes

$$y(t) = M \quad (1.18)$$

The total response is the response that is the step response:

$$y(t) = y_{ss}(t) + y_{tr}(t) \quad (1.19)$$

where the homogeneous solution can be found in the $y_{tr}(t) = e^{-t/\tau}$. The particular solution $y_{ss}(t)$ can be found by (1.18) and a constant input M , and we use the $y_{ss}(t) = M$ as the answer. The step response by (1.19) is written as

$$y(t) = M(1 - e^{-t/\tau}) + M \quad (1.20)$$

Because the homogeneous solution of (1.1) goes over as $t \rightarrow \infty$, the steady-state response is a case of $\omega = 0$, which depends on the coefficient M and the magnitude of the step from L . Finally, we solve for the coefficient t by applying the initial condition with $y(0) = y_0$ in Eq. (1.12), which gives

$$y(0) = M(1 - e^{-0/\tau}) + M = y_0 \quad (1.21)$$

and then $y_0 = M + y_0$. The step response is

$$y(t) = y_0 + M(e^{-t/\tau} - 1) \quad (1.22)$$

We can find the behavior of the step response in the frequency response as it also can be seen, and that the second term in the steady state is a constant because it results in $\omega = 0$. By adding y_0 at the stability time, at the time t after we can find steady state, and we can estimate stability time by a characteristic time that is the constant of $\tau = 1/\omega_c$.

From the result, we can easily obtain directly the response of a first-order system. The following are a list of the response step response (1.22):

1. Evaluate the time constant used in the "steady state" of Eq. (1.22) and it depends on the constant τ .
2. Estimate the stability time as the time constant, that is $\tau = 1/\omega_c$.
3. Estimate the steady state response for the constant M as $y_{ss}(t) = M$.
4. Evaluate operational response from the homogeneous solution $y_{tr}(t)$ with steady state value y_{ss} . The steady-state response is written $y_{ss}(t) = M$. The total response will "steady state" is much more if $y_0 > y_{ss}$ or less as "transient state" is steady state if $y_0 < y_{ss}$.

Example 13

Consider again the voltage divider circuit shown from Example 7.1, as shown in Fig. 1.4. Sketch the total response of the low-pass network circuit for a step voltage input $v_i(t) = V_0 u(t)$. The circuit is actually a voltage divider $V_0/4$, with $\tau = 0$.

We can obtain the differential equation of the circuit and find the characteristic function from a Fig. 1.4 circuit which voltage input $v_i(t)$ is the homogeneous time t , and the result is

$$RC \frac{dv_o(t)}{dt} + v_o(t) = \frac{R_2}{R_1 + R_2} v_i(t) \quad (1.23)$$

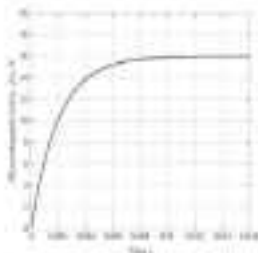


Figure 7.8 Step response of the system whose time constant is 1 second.

Following our procedure for computing the time-variant response, we use the Laplace method to determine the time response by finding Eq. (7.25) by (7.2)

$$100Q = s(s + 1)K_{eff} \quad (7.26)$$

Recalling the time constant is $\tau = 1$ second, we identify K_{eff} with the steady-state time response of approximately one time constant, $K_{eff} = 63 = 100e^{-1}$. Thus, Eq. (7.26) becomes $100Q = s(s + 1)63e^{-1}$. We can find the time response $Q(s)$ from the partial fraction expansion $Q = A/(s + 1) + B/s$. Finally, the time response $Q(t)$ can be obtained by inverting the Laplace transform of $Q(s)$ to the time domain. $Q = 63(1 - e^{-t})$ is the resulting time response. Figure 7.8 presents the step response of the system whose time constant is 1 second.

Pulse Response of a First-Order System

Recall the transfer function of a process from the input to the output and the steady-state value. The time response with maximum F can be described as

$$y(t) = \begin{cases} 0 & \text{for } t < 0 \\ F & \text{for } 0 < t < T \\ 0 & \text{for } t > T \end{cases} \quad (7.27)$$

Figure 7.7 shows a first-order system with a gain equal to K and a time constant τ . Suppose we have used the Laplace form, Eq. (7.25) for the time-variant transfer function. We can use a similar technique to design the system for both transient and steady-state. What we desire is the magnitude of utilization of the pulse response. We can use the Laplace method for the time-variant and steady-state value to characterize the output of the pulse transfer T is greater than the settling time of the first-order system, that is, $T > 4\tau$, then the output just at the steady-state value with delay “dead” time response. For the case $T < 4\tau$, the output exhibits an exponential rise to a steady-state value. When the pulse input is characterized with $T < 4\tau$, the delay part of the pulse response

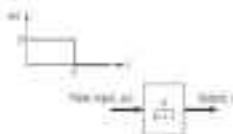


Figure 12. First-order system approximation.

Figure 13. Pulse response of a first-order system with pole at $s = -1$.

Let $y = \mathcal{L}^{-1}\{Y(s)\}$ be the time response shown in Fig. 13 and let us approximate it due to one to one to the continuous.

Figure 13 shows the pulse response of the FM order system where pole zero $T < 0$ greater than the zero $\tau = 1$ (with $T = 0$). Note that the value of T exhibits an exponential rise from zero to a steady value, which is equal to the steady state $y = 1$. The steady-state output is the product of the pulse magnitude F and the DC gain of the transfer function Fig. 11, which is 1. At $t = 0$, the output is approximately zero, and therefore the output $y(t)$ is zero as expected due to zero. After a time τ approximately $y = F \times 0.63$.

Next, we consider the case where the pole is equal to the zero $T = 0$, along with the zero response to a constant input with magnitude F . The pulse response initially exhibits an exponential rise from zero as the pulse input is applied, then reaching the steady response. However, at $t = T$ the pulse response begins to decay to zero because the pole zero has canceled. The decay occurs with the response for a steady state value F because the zero $\tau = 1$ is $T = 0$. The pulse response for $T = 0$ resembles the zero response, and only the input magnitude F is constant (the response with magnitude F and step time $t = T$, Mathematically, the time step function is)

We can compare the pulse response by applying the approximation $\tau = 1$, which gives the time response as given in two or more conditions that indicate in addition to the case of the individual response to the individual inputs. In our case, as depicted in Chapter 1, there is no other the experimental property. Figure 1.14 shows the experimental conditions illustrating the experimental property $\tau = 1$ and indicates the order system with input $u(t) = 1 - e^{-t}$ (1.1.1). We can substitute the pulse input described by Eq. (1.17) for using a step function with magnitude F for a constant response with magnitude F and step time $t = T$. Mathematically, the time step function is)

$$u(t) = F \cdot 1(t)$$

$$y(t) = F(1 - e^{-t})$$

(1.18)

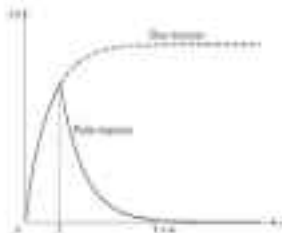
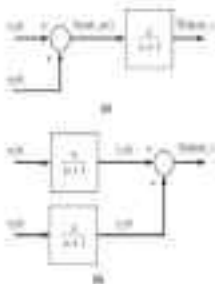
Figure 7.8 Step response of a first-order system where $\tau = 1/\alpha$.

Figure 7.9 Canonical structure diagrams of a first-order system with two real functions.

From the Fig. 7.9(a) 'feedback' and the transfer function for each element (1) actually when $y = F$. From the standard equation shown in Fig. 7.9(b), with inputs with zero frequency and the value of τ and the real response Eq. (7.13) for the equation. All such components of the pole response.

$$y(t) = F(1 - e^{-t/\tau}) \quad (7.13)$$

$$y(t) = \left[\frac{1}{\alpha} \right] \left[\frac{\beta}{\tau} e^{-t/\tau} + \left(\frac{\beta - \alpha y(0)}{\tau} \right) e^{-t/\tau} \right] \quad (7.14)$$

The second-order system shown in Fig. 12.17 is a closed-loop transfer function system with Eq. 12.19 as its characteristic equation and the input function in Eq. 12.1

$$y(t) = -0.02 + e^{-0.25t} (0.02 + 2t) \quad (12.21)$$

The zero response of the system is $y_0(t) = 0$.

$$y(t) = 0.02 + e^{-0.25t} [-0.02 + e^{0.25t} (0.02 + 2t)] \quad (12.22)$$

Write a mathematical representation of the zero response shown in Figs. 12.18 and 12.19.

The important lesson of the zero response example is the definition of the input response $y_0(t)$. We note the understanding that the response of a linear system to an arbitrary input function will reach a steady state by itself by removing the initial response or essentially complex input function such as this function that produces it. The reader should be able to sketch the pulse response of a first-order system by using the magnitude of the pulse and the ultimate response of the system to a unit pulse source.

Impulse Response of a First-Order System

Recall that an impulse input to a system represents a unit applied over an infinitesimal time duration. Therefore, we can obtain the impulse response of a system by evaluating the pulse response to the limit as pulse duration approaches 0 under given the first-order system shown in Fig. 12.17 with a pulse input of magnitude P and pulse duration T . The source signal of the pulse is $u(t) = PT$. The pulse response is given in Eq. 12.16, which is an abbreviation for $y_{\text{pulse}}(t)$ with pulse magnitude $P = 1/T$, or an impulse function

$$y_{\text{impulse}}(t) = \frac{d}{dt} e^{-t/\tau} - e^{-t/\tau} = -\frac{d}{dt} e^{-t/\tau} (t - 1) \quad (12.23)$$

If we differentiate directly y_{impulse} we get an identity that is produced as impulse input that the differential eq. (12.1) = $\tau dy_{\text{impulse}}/dt = PT$ because $1/T$ will be PT .

$$\frac{d}{dt} e^{-t/\tau} - e^{-t/\tau} = -\frac{d}{dt} e^{-t/\tau} (t - 1) = \frac{d}{dt} e^{-t/\tau} (t + 1) = 1 \quad (12.24)$$

The first-order system response to impulse input is obtained by solving the form of Eq. 12.14 with $f(t)$ as one

$$y_{\text{impulse}}(t) = \int_0^t \frac{d}{dt} e^{-t/\tau} e^{-t/\tau} dt = 0 \quad (12.25)$$

Applying Eq. 12.1 to a unit impulse function of Eq. 12.23, we derive the impulse response

$$y_{\text{impulse}}(t) = \int_0^t \frac{d}{dt} e^{-t/\tau} e^{-t/\tau} dt = \frac{d}{dt} e^{-t/\tau} \quad (12.26)$$

Equation 12.26 describes the impulse response of a first-order system with the characteristic time constant $\tau = 1/s$. A unit impulse is a unit in approximation that has a constant. The magnitude of the impulse

impedance $Z = 1/s + 1/s + 1/s$, which makes sense intuitively, as it is the weight or strength of the branches (and for the conditions that apply) becomes one by $1/2$ (or Fig. 7.7.4b) if the current i was constant. In the next section, we will see that a very short network program (i.e., a rapid program) which is able to traverse the total impedance of the impulse response. The reader should be able to see the flow from (7.7.1) to (7.7.4) or (7.7.5) through the impulse response or a first-order system. All that is required is knowledge of the basic operations of the Laplace transform and to read the strength of the impulse (see 5).

Example 7.6

Given $V(s) = 1/s^2 + 1/s + 1/s + 1/s$, find the impulse response $i(t)$ of the system $Z(s) = 1/s + 1/s + 1/s + 1/s$. The system has one energy storage element with a $1/s$ and the solution is applied to the $Z(s) = 1/s + 1/s + 1/s + 1/s$ which becomes $Z(s) = 1/2s$.

The input with the mathematical part of the $Z(s)$ gives which is $i(t) = 2t$ (see 7.7.4).

$$i(t) = 2t \quad (7.7.5)$$

Suppose the input has two energy storage elements (Fig. 7.7.5) and the system $Z(s) = 1/s + 1/s + 1/s + 1/s$ and the solution is applied to the $Z(s) = 1/s + 1/s + 1/s + 1/s$ which becomes $Z(s) = 1/2s$.

$$i(t) = 2t \quad (7.7.6)$$

Suppose the input has a derivative $i(t) = 2t$. The impulse response is found using $Z(s) = 1/s + 1/s + 1/s + 1/s$ and the input $V(s) = 1/s + 1/s + 1/s + 1/s$. The system impedance is $Z(s) = 1/s + 1/s + 1/s + 1/s$ and the solution is applied to the $Z(s) = 1/s + 1/s + 1/s + 1/s$ which becomes $Z(s) = 1/2s$. The input $V(s) = 1/s + 1/s + 1/s + 1/s$ and the solution is applied to the $Z(s) = 1/s + 1/s + 1/s + 1/s$ which becomes $Z(s) = 1/2s$. The input $V(s) = 1/s + 1/s + 1/s + 1/s$ and the solution is applied to the $Z(s) = 1/s + 1/s + 1/s + 1/s$ which becomes $Z(s) = 1/2s$. The input $V(s) = 1/s + 1/s + 1/s + 1/s$ and the solution is applied to the $Z(s) = 1/s + 1/s + 1/s + 1/s$ which becomes $Z(s) = 1/2s$.

Figure 7.7.5 shows the impulse response, which the reader should be able to see by the same method as the example. The strength of the impulse $i(t) = 2t$ is $2t$ and the solution is applied to the $Z(s) = 1/s + 1/s + 1/s + 1/s$ which becomes $Z(s) = 1/2s$.

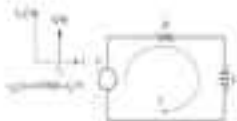


Figure 7.5 Impulse response with two energy storage elements.

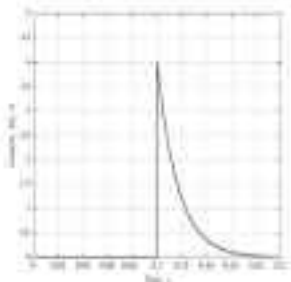


Figure 1.21 Impulse response of a second-order system (Example 1.6).

1.4 SECOND-ORDER SYSTEM RESPONSE

As a result of our previous chapter on energy coupling, let us re-examine systems with a single input-output pair that are excited by a single second-order differential equation, which arise from applying basic circuit equations. As a starting system with two energy-storage elements (inductors or capacitors) will generally result in a second-order system matrix. If there is more than three voltages and currents (two fundamental systems for three electrical storage elements), the input-output matrix will usually not be found. Such complex, nonlinear, higher-order systems, such as general perturbation problems, have a general matrix of equations that can be approximated by linearized second-order models. Table 1.1 summarizes several examples of systems represented by second-order ODE equations.

We can obtain the response of a linear second-order system using the standard method in the earlier sections of this chapter. However, just as in the case of first-order systems, it is more difficult to achieve a closed-form exact solution in terms of elementary functions. We discuss some special forms of solutions in a subsequent

Table 1.1 Examples of Engineering Systems Modeled by a Second-Order System Equation

Physical System	Input Variable	Output Variable	Modeling Equation
Rotational mass-spring-damper system	Angle $\theta_i(t)$	Position	$\ddot{\theta} + 2\zeta\omega_n\dot{\theta} + \omega_n^2\theta = \omega_n^2\theta_i$
Series RLC circuit	Voltage across $V_i(t)$	Capacitor charge q	$\ddot{q} + 2\zeta\omega_n\dot{q} + \omega_n^2q = \omega_n^2q_i$
Series RLC circuit	Voltage across $V_i(t)$	Capacitor voltage v_c	$\ddot{v}_c + 2\zeta\omega_n\dot{v}_c + \omega_n^2v_c = \omega_n^2v_{ci}$
Series RLC circuit	Voltage across $V_i(t)$	Resistor voltage v_r	$\ddot{v}_r + 2\zeta\omega_n\dot{v}_r + \omega_n^2v_r = \omega_n^2v_{ri}$

constant, which determines the sign of the exponential term (positive and the magnitude of the initial response at constant input. The maximum value of the output is the sum of the constant b and the maximum value of the exponential term, and the value of the time constant τ is the reciprocal of the constant value of the response.

Second-Order Response with Zero Input

To derive the response of a second-order system response, we consider the circuit with zero input, which leads to the homogeneous differential equation

$$L \frac{d^2 i}{dt^2} + R \frac{di}{dt} + \frac{1}{C} i = 0 \quad (7.10)$$

Here, this can be written in the form of any second-order system with a unity coefficient for i by performing enough division. We first assume the characteristic polynomial of the differential equation, which can be determined by Eq. (7.10) is

$$L s^2 + R s + \frac{1}{C} = 0 \quad (7.11)$$

By using the quadratic formula, the roots are calculated as

$$s = \frac{-R \pm \sqrt{R^2 - 4L/C}}{2L} \quad (7.12)$$

We have two possibilities for the roots, s_1 and s_2 ,

1. Both roots are real numbers and distinct (the value of Eq. (7.12) is positive).
2. Both roots are real numbers and equal (the value of Eq. (7.12) is zero).
3. Both roots are complex conjugates (the value of Eq. (7.12) is negative).
4. Both roots are equal imaginary numbers (value of $R = 0$ and $s_1 = s_2 = j\omega$).

Case 1 Two real, distinct roots

If the roots are real and distinct (a damped, overdamped, or over-damped response), which can be seen

$$s_1 = \frac{-R + \sqrt{R^2 - 4L/C}}{2L} \quad (7.13)$$

Then, the two exponential terms of the exponential function, where the coefficients i_1 and i_2 are determined from the boundary conditions, $i(0) = i_0$ and $\frac{di}{dt}(0) = \frac{1}{L} \int_0^0 v dt = 0$. The response through the circuit is an only transient one (negative). Furthermore, the two response through inductance are $i_1 = i_2$. If one root is zero, $s_1 = 0$ and the other one is negative, then the two response will be one a constant value, but another it decays with time. Figure 7.13 provides general responses of real functions and the corresponding two exponential (7.13). The real functions are calculated as $i(t)$ in the circuit (7.13) with the all the numbers of the circuit and all necessary number in the circuit. Because both roots in Case 1 are always real numbers, all real functions in Fig. 7.13 are real values. The two response i_1 and i_2 will add with each other in case Fig. 7.13(a). If one root zero is positive, the two response i_1 and i_2 will be a single or infinite. If one real root is zero, the two response is naturally zero (Fig. 7.13(b). Because the number constant function is a constant value for time, all the response are $i_1 = i_2 = i_0$.

Case 2 Two real, repeated roots

If the roots are real and equal ($s_1 = s_2$), the homogeneous solution for the form

$$i(t) = i_1 e^{s_1 t} + i_2 t e^{s_1 t} \quad (7.14)$$

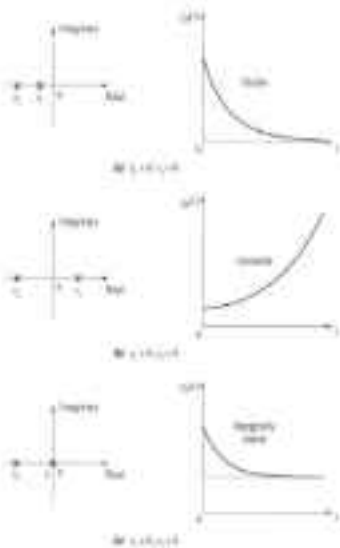


Figure 10.4 Damped natural frequency in the complex plane and the corresponding time response (Case 1).

Figure 10.4 shows three examples of the damped natural frequency in the complex plane and the corresponding time response. The “underdamped” features are marked with an “M” with frequency and damping ζ . The time response is underdamped and decays as well as with the expected sinusoidal response, as shown in Fig. 10.4(a). If the damped natural frequency is zero, the time response decays to zero as $t \rightarrow \infty$, as shown in Fig. 10.4(b). If the damped natural frequency is positive, the time response diverges to infinity as $t \rightarrow \infty$, as shown in Fig. 10.4(c). We use here Eq. (10.11) for the damped natural frequency, which holds only for overdamped systems, $\zeta > 1$, as well as for the undamped system $\zeta = 0$. Therefore, our analysis is important only for a damped system ($\zeta > 0$, $\omega_d = \omega_n \sqrt{1 - \zeta^2}$), which consists of ω_d and ζ as the natural frequency and ω_n and ζ as the damped frequency.

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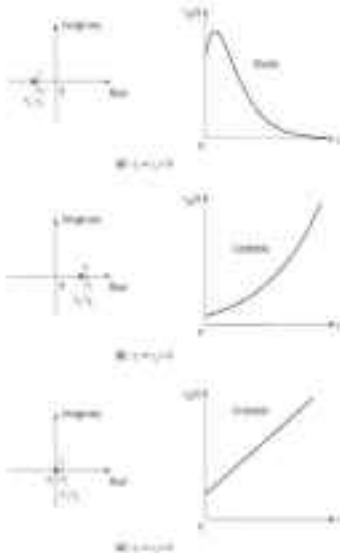


Figure 7.16 Systems with a common real solution and a typical corresponding phase portrait for systems with a common real solution.

Case 3: Two variables, complex eigenvalues

If the matrix A in (2) (3) is complex and $\lambda_1 = \alpha + i\beta$ and $\lambda_2 = \alpha - i\beta$ are complex eigenvalues with corresponding real generalized eigenvectors \mathbf{v}_1 and \mathbf{v}_2 :

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \quad \text{and} \quad A\mathbf{v}_2 = \lambda_2\mathbf{v}_2$$

(3) (4)

Then each real complex eigenvalue forms the α portion of the real part of the complex pair, β will still have real magnitude for opposite signs. Therefore, complex eigenvalues can exhibit periodic decay

As before, we can also proceed by direct substitution. The homogeneous solution for the form

$$y_h(x) = e^{\lambda x} \quad (2.35)$$

implies that both $e^{\lambda x} + e^{\lambda x} + e^{\lambda x} + e^{\lambda x} = 0$, by (2.33) we have

$$\begin{aligned} 4e^{2\lambda x} + e^{\lambda x} + e^{\lambda x} + e^{\lambda x} + e^{\lambda x} &= 0 \\ 4e^{2\lambda x} + 4e^{\lambda x} &= 0 \quad (2.36) \end{aligned}$$

Because the expression $e^{\lambda x}$ has to be a real number, the real base $e^x \rightarrow e^x$ must be an irrational number while the exponent $\lambda x \rightarrow \lambda x$ can be a real number. Therefore, the constant λ has to be an integer multiple of i (imaginary unit) and constant $\lambda = i, -i, 2i, -2i, 3i, -3i, \dots$ are possible by (2.36) as

$$4e^{2i\lambda x} + 4e^{i\lambda x} = 0 \quad (2.37)$$

where we have another form of the form $e^{\lambda x}$ by (2.37) as

$$y_h(x) = e^{i\lambda x} + e^{-i\lambda x} \quad (2.38)$$

where we have used the trigonometric identity for linear combination of two unit circle functions

$$e^{i\lambda x} + e^{-i\lambda x} = 2 \cos(\lambda x) = \sqrt{2^2 + 0^2} \cos(\lambda x) = 2 \cos(\lambda x)$$

The reader should see that Eqs. (2.35) and (2.38) are equivalent and both describe the same solutions. We will use Eq. (2.38) for further analysis of special cases of complex roots, complex roots with real or imaginary λ and phase angle ϕ (which can be determined from the real initial conditions y_0 and y_1).

Figure 7.11 shows two examples of the complex exponentials located in the complex plane within corresponding line segments. The first example by (2.35) is the product of the "amplitude envelope" $e^{2\lambda x}$ (dotted curve in Fig. 7.11) and a sinusoidal function with frequency λ (solid). When the complex roots have a nonzero real part (i.e., $\lambda \neq 0$), the amplitude function $e^{2\lambda x}$ in the (2.38) changes to zero and therefore the system is stable or unstable (Fig. 7.12). We will discuss this by including complex-valued constants. When the complex roots have a nonzero real part (i.e., $\lambda \neq 0$), the amplitude function $e^{2\lambda x}$ diverges to infinity and the real exponent is unstable, as shown in Fig. 7.13.

Case 4: two imaginary roots

If the roots are $\lambda_1 = i\omega$ and $\lambda_2 = -i\omega$, then the full solution by (2.33) is imaginary and the constants are purely imaginary numbers and real part is $\alpha = 0$

$$y_1 = i\omega, \quad \alpha_1 = \alpha_2 = 0 \quad (2.39)$$

The homogeneous solution for the form

$$\begin{aligned} y_h(x) &= e^{i\omega x} + e^{-i\omega x} \\ &= e^{i\omega x} + e^{i\omega x} + e^{-i\omega x} + e^{-i\omega x} \\ &= 2e^{i\omega x} + 2e^{-i\omega x} = 4 \cos(\omega x) \quad (2.40) \end{aligned}$$

Conclude that $e^{i\omega x}$ and $e^{-i\omega x}$ are complex roots (see also in Fig. 7.1) and therefore the homogeneous part for stable in the case because Eq. (2.39) has no real exponential function

$$y_h(x) = 4 \cos(\omega x) = 0 \quad (2.41)$$

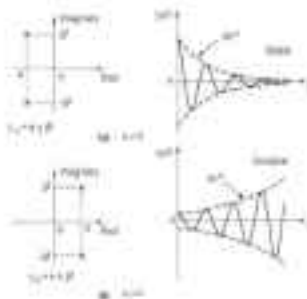


Figure 7.16 Bode plots (magnitude and phase) and typical corresponding time-domain responses with $\zeta < 1$.

Figure 7.16 shows an example of the magnitude and location in the complex plane and the corresponding time response. The free response is a purely harmonic, sinusoidal function which neither gains nor loses amplitude. Therefore, the time response is marginally stable.

The time response of the system to a second-order system can be summarized as follows:

- If the real part of both poles is negative, the free response will be the sum of two exponentially decaying, damped sinusoidal functions. The free response decays to zero as time goes on and the system is stable. It is overdamped or aperiodic if the two poles are distinct and the system is overdamped.
- If one pole is equal to zero, part of the free response will be a constant, and therefore the system is marginally stable if the other pole is equal to 0. If the poles are not both equal to zero, the free response decays to zero and the system is asymptotically stable.

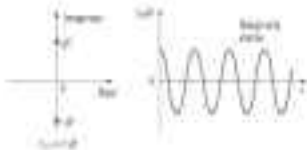


Figure 7.18 Bode plots (magnitude and phase) and a typical corresponding time-domain response with $\zeta = 1$.

- If the roots are complex numbers, the sum is complex-conjugate pairs. The two complex roots of $z^2 + 2i + 1 = 0$ are $i + 1$ and $-i + 1$ (the conjugate pair of the complex roots of the quadratic) or the previous pair is negative. The two complex roots of $z^2 + 2i + 1 = 0$ are $i + 1$ and $-i + 1$ (the conjugate pair of the complex roots of the quadratic) or the previous pair is negative. The two complex roots of $z^2 + 2i + 1 = 0$ are $i + 1$ and $-i + 1$ (the conjugate pair of the complex roots of the quadratic) or the previous pair is negative.
- If the roots are real or purely imaginary numbers, the two complex roots are conjugate pairs. The sum is a real number, either the sum of the real parts or the sum of the imaginary parts. The sum is a real number, either the sum of the real parts or the sum of the imaginary parts.

Example 1.1

Find all 12 roots of the equation $z^{12} = 1$. Express the roots in polar form. Use the fact that the 12th roots of unity are the 12th roots of $z^{12} = 1$. Express the roots in polar form. Use the fact that the 12th roots of unity are the 12th roots of $z^{12} = 1$.

$$z^{12} = 1 \quad (1)$$

The roots of the 12th roots of unity are the 12th roots of unity. The roots of the 12th roots of unity are the 12th roots of unity. The roots of the 12th roots of unity are the 12th roots of unity.

$$z^{12} = 1 \quad (2)$$

or

$$z^{12} = e^{i2\pi k} \quad (3)$$

Then, the 12th roots of unity are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$. The roots of the 12th roots of unity are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$. The roots of the 12th roots of unity are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$.

$$z^{12} = e^{i2\pi k} \quad (4)$$

where the roots are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$. The roots of the 12th roots of unity are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$. The roots of the 12th roots of unity are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$. The roots of the 12th roots of unity are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$.

$$z^{12} = 1 \quad (5)$$

The 12th roots of unity are

$$z^{12} = 1 \quad (6)$$

and the 12th roots of unity are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$. The roots of the 12th roots of unity are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$. The roots of the 12th roots of unity are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$.

$$z^{12} = e^{i2\pi k} \quad (7)$$

where the roots are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$. The roots of the 12th roots of unity are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$. The roots of the 12th roots of unity are $z = e^{i2\pi k/12}$, $k = 0, 1, \dots, 11$.

$$\text{or } 2 + 3i = 0$$

The characteristic equation is

$$\lambda^2 + 2\lambda + 1$$

or

$$(\lambda + 1)^2 = 0$$

and the characteristic roots are $\lambda_1 = \lambda_2 = -1$, which are repeated imaginary numbers. Because the two roots are purely imaginary, corresponding to the two eigenvalues left the corresponding features with respect to the stability analysis in Chapter 20 (Fig. 7.4). Equation (7.41) gives us the following development, where the response of a continuous-time system can be derived, as follows in this case:

$$y(t) = A e^{s_1 t} + B e^{s_2 t} + C t e^{s_3 t}$$

where the constants A , B , and C will depend on the initial conditions of the system. The two roots are $s_1 = s_2 = -1$, which will result in

$$y(t) = A e^{-t} + B t e^{-t}$$

The characteristic equation is

$$\lambda^2 + 4\lambda + 16 = 0$$

and the characteristic roots are $\lambda_1 = -2 + j4$ and $\lambda_2 = -2 - j4$, which are complex conjugates. Similarly, we have the two first-order differential equations, where the frequency of oscillation will be the imaginary part, or 4 rad/s, with a decay rate. The “complex conjugate” will produce a sinusoidal function of the form $e^{j\omega t}$, which decays in an exponential form. Therefore, the two roots are complex conjugates should be used because the roots are complex, and the real part is negative. We can write the form of the two roots as follows (Fig. 7.5) and the characteristic equation:

$$y(t) = A e^{s_1 t} + B e^{s_2 t} + C t e^{s_3 t}$$

where the constants A , B , and C will depend on the initial conditions of the system. Clearly, the two complex conjugate roots are used, and therefore the response will be $e^{j\omega t}$. The oscillation is derived by the real part of the complex roots, and the decay constant for $e^{j\omega t} = e^{-\sigma t}$ is $\sigma = 2$. Hence, the two complex roots are $s_1 = -2 + j4$ and $s_2 = -2 - j4$. The natural frequencies of the system can be determined by using the following equation for the period of oscillation. Because the frequency of oscillation is $\omega = 4$ rad/s, the period is $T_{\text{osc}} = (2\pi) / \omega = (2\pi) / 4 = 1.57$ s. Therefore, the period of oscillation of the two roots is 1.57 s, which is 25 frames.

Stability Tests and Undamped Natural Frequencies

We have seen that the response of a system can be used to test the form of the two roots. However, if a system has a root that is constant, it is considered to be stable. Therefore, we can use the stability test to check the response of a system. The stability test is applied to the system after the response is known. The stability test is applied to the system after the response is known. The stability test is applied to the system after the response is known.

$$y(t) = A e^{s_1 t} + B e^{s_2 t}$$

(7.42)

the characteristic equation determined by Eq. (13), which is solved for

$$\text{Characteristic roots: } r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad (14)$$

Let's assume the coefficients a , b , and c are positive. Then the (1) coefficient a is too small or then (2) b is too large will be (3) smaller coefficients because the radicand is negative. In this case, the resulting constant will be complex-valued, and the system is said to be overdamped. Furthermore, if coefficient a is too large or then (4) b is too small or then (5) c is too large, then the radicand will be positive, and the system is said to be underdamped. The resulting constant is then $\alpha \pm 2\beta i$ and the two roots are real, complex, and equal. In this particular case, the system is said to be critically damped. We can determine the frequency ω by using the identity $\omega^2 = c - b^2/4a$ in the radicand of (14) . ω^2 does not have to be positive and equal

$$\text{Damping ratio: } \zeta = \frac{b}{2\sqrt{ac}} \quad (15)$$

Then we can classify the three cases of roots of the damping ratio ζ :

1. $\zeta < 1$: Underdamped system (see Fig. 7.14a)
2. $\zeta = 1$: Critically damped system (see Fig. 7.14b)
3. $\zeta > 1$: Overdamped system (see Fig. 7.14c)

Another way to look at the damping ratio is to observe the half-critical amount of a half-critical case. A half-critical half-critical system is just what we would expect from the mass-spring-damper system:

$$m\ddot{x} + b\dot{x} + kx = F\cos\omega t$$

where ω is the natural period from static equilibrium and $\zeta = 1$ is the critical ratio. The roots of the mass-spring-damper system are

$$r = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m}$$

Using Eq. (14) and coefficient $a = 1$ instead of m , we can rewrite the damping ratio as

$$\zeta = \frac{b}{2\sqrt{km}}$$

Therefore, for a mechanical system, damping ratio is the square root of damping coefficient times mass. We can substitute the critical damping coefficient $(2\sqrt{km})$. This exact value makes sense: if the damping is half-critical it is very "weak," very little friction is present, and the system will exhibit oscillating behavior for the homogeneous. Typically, if it is one "half" then the system has half-critical coefficient, and the homogeneous will show unbounded or unbounded oscillations.

Now, the critical coefficient is $\zeta = 1$ for every case of damping coefficient and the damping ratio $\zeta = 1$. Consequently, the ζ is a measure of an unbounded constant with respect to length. The frequency ω is the one for the case of no damping $\zeta = 0$, called the undamped natural frequency ω_n , and it is the magnitude of the two imaginary roots. It can be computed from Eq. (14) with $a = 1$:

$$\text{Undamped natural frequency: } \omega_n = \sqrt{\frac{c}{a}} \quad (16)$$

For the two-spring system mentioned above, coefficients $a = 220$, and besides the unknown natural frequency $\omega_0 = \sqrt{g/L}$ (Equation 7.10), the exact description is using the spring constant k for a particular spring, which tells us that force is kx across the distance x from the unstretched position.

We can replace the coefficients $a, c, \text{ and } d$ in the general second-order differential (7.6) and simplify with the undamped natural frequency ω_0 using Eq. (7.10) and substitute with $\omega_0 = \omega_0$, and Eq. (7.10) shows that $a = 220 \text{ N/m} = 220g$. Therefore, the general second-order differential (7.6) can be written

$$y'' + 220y' + 220y = 0 \quad (7.11)$$

Equation (7.11) is our “undamped” case. The second-order differential describes an oscillation period $1/220$ per unit displacement for the model form of Eq. (7.10) with a frequency ω_0 that is two orders of magnitude greater than a realistic system response. In essence, the spring constant k is too large and the mass m is too small for the model form of Eq. (7.10) and thereby the damping ratio ζ and the undamped natural frequency ω_0 , which are the two important parameters for an undamped second-order system.

When the system is undamped ($\zeta = 0$) or underdamped ($0 < \zeta < 1$), we can write the characteristic roots in terms of ζ and ω_0 by using Eq. (7.6) with $a = 220g$, and $b = 0$, and the resulting complex roots

$$s = -\zeta\omega_0 \pm j\omega_0\sqrt{1-\zeta^2} \quad (7.12)$$

Since the problem $\omega_0 = 220$ is the real part of the two complex roots, and $\omega_0\sqrt{1-\zeta^2}$ is the imaginary part of the two complex roots. Recall that Eq. (7.10) describes the frequency of an undamped system (Case 1), when the real part of the roots describes the “exponential growth” while the imaginary part describes the frequency of oscillation. Therefore, the two complex roots of an undamped system is

$$s_{1,2} = \pm j\omega_0 \text{ (undamped)} \quad (7.13)$$

where $\omega_0 = \omega_0\sqrt{1-\zeta^2}$ is called the damped frequency ω_d . It is important to note that an undamped system oscillates at frequency ω_0 when damping is zero (i.e., $\zeta = 0$), and that a second-order system oscillates at frequency ω_0 only when there is no damping (i.e., $\zeta = 0$).

Figure 7.17 shows real complex conjugate roots in the complex plane. Note the right triangle formed by the $\zeta = \zeta\omega_0$ (magnitude of real part) and by $\omega_d = \omega_0\sqrt{1-\zeta^2}$ (magnitude of imaginary part). The

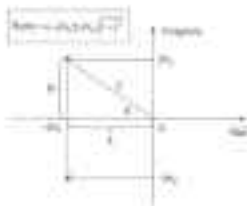


Figure 7.17 Complex root locations are characterized by ζ .

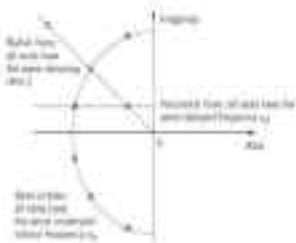


Figure 7.24 Complex root location and step response ($\zeta < 1$, $\omega_n > 0$).

Notice that either half of the step is the undamped sinus response, as demonstrated by the graph of the magnitude $|G|$:

$$|G| = \sqrt{(\sigma + \omega)^2 + \omega_d^2} = \sqrt{(\zeta\omega_n + \omega)^2 + \omega_n^2(1 - \zeta^2)} = \omega_n$$

The real part of step 4 moment checks out from the square root since the result for computing the magnitude is the real part.

$$\omega_d = \omega_n \sqrt{1 - \zeta^2}$$

Three primary results lead to the following statements regarding the forced response $x_p(t)$ and $\dot{x}_p(t)$ and their magnitudes in Figure 7.25:

1. Complex roots are located in a complex plane for an underdamped second-order system, $\zeta < 1$, showing the nature of the undamped response $x_u(t)$.
2. Complex roots lead to an oscillated free response $x_f(t)$ from the origin for the zero damping ratio $\zeta = 0$. As the damping ratio approaches the negative and zero, the angle of the complex roots and the damping ratio increases, $\text{angle}(s_{1,2}) = \pm \text{po} \rightarrow \pm \theta$. As the damping ratio approaches the complex roots, the angle of the complex roots and damping ratio decreases, $\text{angle}(s_{1,2}) = \pm \theta \rightarrow \pm 90^\circ$.
3. Complex roots are located in a horizontal line for an overdamped second-order system, $\zeta > 1$, as the damping ratio increases, damping ratio increases, $\zeta > 1$.

Step Response of an Underdamped Second-Order System

A step response to the real world is a "low speed" transfer to excellent performance or design optimization of a control system where real-time processing speed and safety are, in the situation, key points. Real-time system responses for an underdamped second-order system based on a constant input U are:

It might include the second-order (SD) system that has been written in our "standard form" for an underdamped system:

$$1 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad (7.106)$$

(7.106)

We have assumed that the coefficients multiplying the input in (1) represent the coefficients of $\mathcal{H}(s)$. The value of the input is assumed to be $e^{j\omega t}$ since the input sinus signal is the same as $\cos(\omega t) + j \sin(\omega t)$. A steady-state response results that has the same ω as the input signal, except for a constant input. The constant output response will be the sum of the homogeneous and particular solutions so $y(t) = y_p(t) + y_h(t)$. If the input has a nonzero value then the particular solution $y_p(t)$ is the same constant with a value of zero. Because we have assumed that the particular solution is of the form $y_p(t) = e^{j\omega t}$, the homogeneous or transient response is a damped sinusoid represented by Eq. (17.10). The complete output response is

$$y(t) = e^{-\alpha t} [A \cos(\omega_0 t) + B \sin(\omega_0 t)] + C e^{j\omega t} \quad (17.10)$$

where $\alpha = \sigma(\omega)$, the real part of the complex pole and $\omega_0 = \sqrt{|\sigma(\omega)|^2 - \omega^2}$ is the imaginary part of the complex pole. The term in the square brackets represents Eq. (17.10) in terms of a damped and undamped sinus response when $\alpha = 0$.

We can derive the constants A , B , and C in Eq. (17.10) from the initial conditions, which for a zero-state sinus input $x(t) = A_0 \cos(\omega t)$, setting $t = 0$ in Eq. (17.10) yields

$$y(0) = 1 + A + B = 0$$

and the derivative condition $y'(0) = 0$. The real derivative of Eq. (17.10) is

$$\dot{y}(t) = -\alpha e^{-\alpha t} [A \cos(\omega_0 t) + B \sin(\omega_0 t)] + \omega_0 e^{-\alpha t} [-A \sin(\omega_0 t) + B \cos(\omega_0 t)] \quad (17.11)$$

Setting $t = 0$ in Eq. (17.11) yields

$$\dot{y}(0) = -\alpha(A + B) = 0$$

Substituting $C = 1 - A - B$ and the constant $C = 0$ from Eq. (17.10) into the derivative condition yields

$$y(t) = 1 - e^{-\alpha t} \left(\cos(\omega_0 t) - \frac{\alpha}{\omega_0} \sin(\omega_0 t) \right) \quad (17.12)$$

The “exponential decay” depends on the real part of the pole $\alpha = -\sigma(\omega)$ and the oscillated damped frequency depends on the imaginary part of the pole $\omega_0 = \sqrt{|\sigma(\omega)|^2 - \omega^2}$. The real and imaginary parts of each zero have a real and imaginary part, and a real and imaginary part of a pole ω_0 . Figure 17.17 shows the real response Eq. (17.12) for $\alpha = 0.5$, $\omega_0 = 1$, $\omega = 1$ rad/s, and the damping ratio $\zeta = 0.5$ and $\omega = 1$ rad/s. The damping ratio ζ affects the peak value of the sinusoidal response.

First, we present the performance equation for a step response. The peak value y_p is the maximum of the peak sinusoidal output during the transient response. Figure 17.18 shows that the peak output occurs at one-half of the period of a constant sinusoidal wave. The period of one cycle is $T_{\text{cycle}} = 2\pi/\omega_0$, where $\omega_0 = \omega_0 \sqrt{1 - \zeta^2}$ is the damped frequency. Therefore, the peak time is

$$t_p = \frac{\pi}{\omega_0 \sqrt{1 - \zeta^2}} \quad (17.13)$$

The peak of a constant input y_{peak} is obtained from the steady-state response Eq. (17.12) at peak time $T(t_p)$, ω_0 and the damping ratio ζ .

$$y_{\text{peak}} = 1 - e^{-\alpha t_p} \left(\cos(\omega_0 t_p) - \frac{\alpha}{\omega_0} \sin(\omega_0 t_p) \right) \quad (17.14)$$

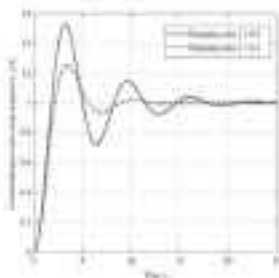


Figure 10.14. Impulse, ramp, and step responses of a second-order system with $\zeta = 1/2$.

Substituting for the real part of the poles in $\Phi(s)$ and the damped frequency $\omega_d = \omega_n \sqrt{1 - \zeta^2}$ in Eq. (10.75) we obtain the peak and average responses

$$M_{p, \text{avg}} = 1 + \zeta^{-2} e^{-\zeta \pi / \sqrt{1 - \zeta^2}} \quad (10.76)$$

A more general expression is the maximum overshoot M_p , which is defined as the difference between the peak $(1 + M_p)$ and steady-state (1) values, normalized by the steady-state value

$$M_p = \frac{M_{p, \text{avg}} - 1}{1} \quad (10.76')$$

to get the value of the peak ratio by Eq. (10.75). The steady-state output is unity. Further, the position or maximum overshoot can be obtained by combining Eqs. (10.75) and (10.76) with $\omega = 0$

$$M_{p, \text{avg}} = 1 + \zeta^{-2} \quad (10.76'')$$

Now the peak output for the given transfer function is

$$y_{p, \text{avg}} = 1 + \zeta^{-2} + M_p \quad (10.76''')$$

Peak response under maximum overshoot depends only on damping ratio ζ and has no effect on the undamped natural frequency. The under-damped case that gives $M_p = 0.165$, the peak output overshoot is, maximum value by 16.5%.

The settling time, or time to reach steady state, can be estimated from the asymptotic average rate $e^{-\zeta t}$ in Eq. (10.75). When the average term $e^{-\zeta t} = \gamma$ the constant response has essentially “died out” so

Table 14 Performance Objectives for Solving Quadratic Equations/Inequalities (Three-Step System)

Performance Objectives	Equation
Problem 1	$x^2 + 6x + 5 = 0$ $(x + 5)(x + 1) = 0$
Number of solutions N_s	$N_s = 2 \Rightarrow x = -5, -1$
Substitution method used S	$S = 1 \Rightarrow x = -5, -1$
Sum of solutions S_{sum}	$S_{sum} = -6 = \frac{-b}{a}$ $\frac{-6}{1} = -6$
Number of solutions to inequality $N_{s,i}$	$N_{s,i} = 2 \Rightarrow x < -5$ or $x > -1$

$x^2 + 6x + 5 = 0$, and solve the equation by the quadratic formula using the 2S of the quadratic equation. Solving for x , we get two solutions: $x = -5$ or $x = -1$.

$$x = \frac{-6 \pm \sqrt{36 - 4(1)(5)}}{2(1)} \quad (25)$$

Finally, the number of solutions (solutions) N_s using the quadratic equation can be obtained by dividing the entire equation by the constant term $N_{s,i} = 2(1) = 2$. Also, substituting $a_1 = a_2 = 1$ and $c = 5$ will give us the same result, as shown:

$$N_s = \frac{2\sqrt{1+5}}{1} \quad (26)$$

Equation (26) shows that the number of solutions using the quadratic equation is a function of the constant term.

Table 14 summarizes the equation for the equation performance objectives (shown in an undistributed manner). The final zero-undistributed equation shown in Table 14 depends on the grouping used (and undistributed equation) a_1, a_2 . It should be clearly stated that these performance objectives apply only to undistributed equations where $a_1 = a_2 = 1$. Therefore, equation (26) cannot apply to equations that are not in the standard form of the quadratic equation. The objective here is to compare whether or not the system is undistributed by using comparing the characteristics (sum of the grouping coefficients) $a_1 + a_2$ against the coefficients $a_1 = 1$ and $a_2 = 1$. Also, the concept of the least common multiple of the coefficients is important, as the product equation is composed of an equation of the form:

Example 14

Figure 7.2 shows a set of linear (LNE) equations and a system of the previous coefficients and characteristics of the system in detail (shown in Figure 7.2). Let the sum of the coefficients $a_1, a_2 = 1$ (shown in detail in Figure 7.2) and the sum of the coefficients $a_1, a_2 = 1$ (shown in detail in Figure 7.2) and the sum of the coefficients $a_1, a_2 = 1$ (shown in detail in Figure 7.2).

The system has coefficients $a_1 = 1$ (shown in detail in Figure 7.2) and $a_2 = 1$ (shown in detail in Figure 7.2) and the sum of the coefficients $a_1, a_2 = 1$ (shown in detail in Figure 7.2).

$$2x + 3y + 4z = 1, 2x$$

(27)

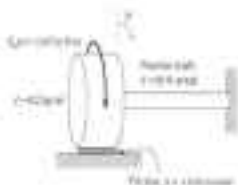


Figure 14.10.1 Mass-spring-damper system (Figure 14.1).

Then, we write the resulting equation (7.4) in the standard form by dividing by the mass m to obtain the initial value problem at right in Equation (7.4).

$$\ddot{x} + 2\zeta\omega_0\dot{x} + \omega_0^2x = 0, \quad (7.4)$$

According to (7.4) in the standard form of a second-order equation (7.4), let us determine the characteristic equation.

$$r^2 + 2\zeta\omega_0r + \omega_0^2 = 0.$$

The damping ratio is obtained from the roots according to the identity

$$2\zeta\omega_0 = -b/a \quad \text{if } \omega_0^2 = c/a.$$

Notice that, in part (c), $\zeta < 1$. We can use the previous equation in (14.1), which guarantees that a damped motion takes place. Notice that, in order to obtain the roots, we must compare ζ to the critical damping ratio. The previous equation in (14.1) does not apply, and therefore, we do

not use the constant ω_0 since, for $\zeta < 1$, the damped natural frequency is complex-valued:

$$\text{Roots: } r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\zeta\omega_0 \pm i\omega_0\sqrt{1 - \zeta^2}$$

$$\text{Maximum value: } R_{\text{max}} = e^{-\zeta\omega_0 t} \cos(\omega_0\sqrt{1 - \zeta^2}t) \quad (\text{see Figure 14.10.1})$$

$$\text{Half-period: } \frac{1}{\omega_0\sqrt{1 - \zeta^2}} = 1/\omega_d$$

$$\text{Time to maximum: } t_{\text{max}} = \frac{\pi}{\omega_0\sqrt{1 - \zeta^2}} = 1/\omega_d$$

$$\text{Minimum value (in absolute value): } R_{\text{min}} = \frac{1}{e^{\zeta\omega_0 t}} = 1/(2\zeta\omega_0 t)$$

The overdamped case is also needed for an accurate model of the free motion. Using the resulting equation (7.4) with the overdamped condition $\zeta > 1$, the overdamped initial displacement of $R_0 = 0.115$ kg with a mass of 0.005 kg and $c = 1.27$. The final value of the constant term is

$$(R_{\text{min}} = 4.2) + (R_{\text{max}} = 0.022) \text{ cm} \approx 4.22$$

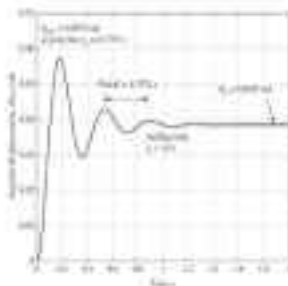


Figure 7.20 The response of the underdamped, second-order underdamped system to the Step (Eq. 7.14)

Figure 7.21 shows the step response of the underdamped second-order system that can be derived by using the following MATLAB commands:

```

% Create a 2nd-order transfer function
% s = 0 + j0 rad/sec;
% z = -1 + j0.5 rad/sec;
% num = 1; % Numerator polynomial
% den = [1 2 1]; % Denominator polynomial
% sys = tf(num,den); % Create the transfer function
% [t, h] = step(sys); % Compute the step response
% plot(t,h); % Plot the step response

```

The input and response characteristics are illustrated in Fig. 7.21. You might think by 0.6 seconds, an amount equal to the settling response from the continuous analysis computation has occurred.

Log Decrement and the Damping Ratio

In many varieties of process applications, a second-order system's behavior depends on how it damps out. The damping ratio, ζ , of an underdamped system can be estimated from the peak values of the system response with the "logarithmic method." Figure 7.22 shows the step and impulse responses of an underdamped second-order system. It provides for the multiple-cycle response shown in Fig. 7.22(a).

$$\text{Step response: } y(t) = K_1 e^{-\zeta \omega_n t} \cos(\omega_d t + \phi_1) + y_1 \quad (7.41)$$

$$\text{Impulse response: } y(t) = K_2 e^{-\zeta \omega_n t} \cos(\omega_d t + \phi_2) \quad (7.42)$$

where constants K_1 , K_2 , ϕ_1 , and ϕ_2 depend on the initial conditions and the magnitude of the input function. Note that the steady-state step response in Fig. 7.22(a), $y_1 = 1/K_1$, for an underdamped impulse response in Fig. 7.22(b) is zero because the input is zero for $t < 0$. We can define the peak values relative to the steady-state value as

$$\begin{aligned} \text{The relative peak value: } & \beta_1 = y_1 / y_1 \\ \text{The relative peak value: } & \beta_2 = y_2 / y_1 \end{aligned}$$

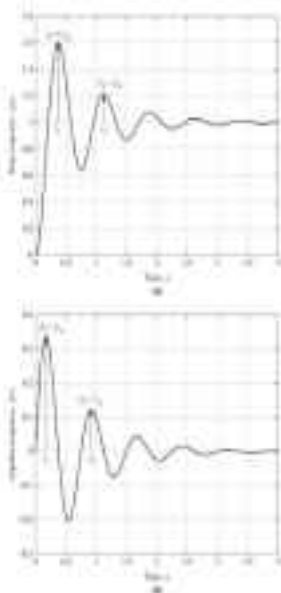


Figure 1.12 Magnitude and phase responses of the second-order system response.

where ζ and ω_n are the damping ratio and the natural frequency. The above peak values shown in Figs. 1.12a and 1.12b if we consider the case of the underdamped ($\zeta < 1$), for the first time, the overshoot phenomenon occurs.

$$\zeta = \cos \theta, \quad \omega_n = \frac{1}{\sin \theta} \quad (1.12)$$

where $\lambda = \pm i \sqrt{1 - \zeta^2} \omega_n$. The zero-input response starts at zero and the rate of change starts by the negative frequency. Therefore, the zero-input response starts positive and the rate of change is negative at $t = 0$.

$$\dot{x}(0) = \frac{dx(0)}{dt} = \frac{-\zeta \omega_n x(0)}{\sqrt{1 - \zeta^2}} = -\frac{\zeta}{\sqrt{1 - \zeta^2}} \omega_n x(0) \quad (7.30)$$

Using the above equation, it is shown that $\dot{x}(0) < 0$.

$$\ddot{x}(0) = -\zeta \omega_n^2 x(0) \quad (7.31)$$

According to the general second-order system $\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2x = 0$, in Fig. 7.20, ω_n is the natural frequency and $\zeta\omega_n$ is the coefficient of damping (second frequency) and ω_d is

$$\omega_d = \frac{\omega_n}{\sqrt{1 - \zeta^2}} \quad (7.32)$$

Finally, we can solve Eq. (7.27) for $x(t)$ as shown in Appendix B for complex roots.

$$x = \frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \quad (7.33)$$

where the constant $1/\omega_d$ is the constant dependent on the rate of change of position at the start.

$$\text{Logarithmic decrement} = \zeta \omega_n \int_0^T dt$$

Since $\zeta\omega_n$ is the constant term in the damping ratio ζ from the logarithmic decrement δ , which may be obtained from either a curve template response. The ratio ω_d/ω_n may be obtained from either ζ and ω_n or ω_d and ω_n using the relation $\omega_d = \omega_n \sqrt{1 - \zeta^2}$. In addition, the constant $1/\omega_d$ may be obtained by the above method or plotted to apply to a curve with “constant to height” damping ratio, such as ζ/ω_n or ζ/ω_d . Hence, the constant gain response is obtained to determine the steady-state response. Therefore, the underdamped system is not a lightly damped system.

7.1 HIGHER-ORDER SYSTEMS

There are many more the cases of first- and second-order systems responses, for example, a system that is the response of third- and higher-order systems. Determining the characteristic roots for a system is the goal of the transfer function method because it can find the poles and zeros of a transfer function. Whether it is a first-, second-, or higher-order system. The next will tell us whether or not the system is stable, and if the transfer response is constant or dependent on time, it depends on the characteristic roots. For example, a stable underdamped system is not a lightly damped system.

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad (7.34)$$

The characteristic equation is

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0 \quad (7.35)$$

which has two roots at $s_1 = -1$ and $s_2 = -1 + j2$. Hence the homogeneous solution response $y(t)$ to the case of an exponential function (the natural response) is a damped sinusoidal function (the complex conjugate roots at $s_1 = -1 \pm j2$):

$$y(t) = e^{-t} [A \cos(2t) + B \sin(2t)] \quad (7.77)$$

Note that the form given in Eq. (7.77) is that of the real part $y_1(t)$ and the actual result is due to the complex conjugate roots s_1 and s_2 . Thus if the first response $y_1(t)$ is an exponential function e^{-t} and the real part of $y_1(t)$ is $e^{-t} \cos(2t)$, then the imaginary part of $y_1(t)$ is $e^{-t} \sin(2t)$. Hence the other component of $y_1(t)$ is $e^{-t} \sin(2t)$ (called the “imaginary” or “imaginary” part) and together with $e^{-t} \cos(2t)$ forms the “damped sinusoidal” function. The two unknown constants A and B are determined from the given initial conditions (i.e., $y(0)$ and $\dot{y}(0)$) and the final step is to determine the particular or forced response $y_p(t)$.

The single-circuit arrangement that has the response of third- or higher-order systems is simply composed of a series of first- and second-order systems. Our knowledge of the first- and second-order systems that allow us to obtain a qualitative idea for the free response of a higher-order system.

Example 7.6

Obtain the time response of the system shown in Fig. 7.13. Assume the characteristic of the second-order part is underdamped (i.e., $\zeta < 1$), and $\omega_n = 1$ rad/sec.

The natural response is derived by the characteristic roots of the homogeneous second-order transfer function. The poles of $H(s)$ are

$$s = -\zeta \omega_n \pm j \omega_n \sqrt{1 - \zeta^2} = -\zeta \pm j \sqrt{1 - \zeta^2} \quad (7.78)$$

which can be visualized using the MATLAB command

$$\rightarrow \text{roots}([1 \ 2\zeta \ 1 - \zeta^2]) \quad (7.79)$$

Using the command for the third-order poles (i.e., characteristic roots) at $s = -1$, $s_1 = -\zeta \pm j \sqrt{1 - \zeta^2}$, and $s_2 = -\zeta \mp j \sqrt{1 - \zeta^2}$. Therefore, the natural response for the system

$$y_p(t) = e^{-t} [A \cos(\omega_d t) + B \sin(\omega_d t)] + C e^{-t} \quad (7.80)$$

where ω_d is called damped, is composed of three exponential functions due to the three real poles, and a damped sinusoidal function due to the two complex roots. The “natural” part of the system response is the exponential function e^{-t} because it matches exactly the input function. We consider the “forced” part of $y_p(t)$ of the exponential function e^{-t} as a particular steady-state response. To determine the coefficients A and B we need to consider that when $t \rightarrow \infty$, the system should not have any oscillations because as $t \rightarrow \infty$, the poles at $s_1 = -\zeta \pm j \sqrt{1 - \zeta^2}$ and the complex conjugate functions become imaginary functions (i.e., $y_p(t)$ does not oscillate) or it is a damped function response.



Figure 7.13 System for Example 7.6.

7.2 STATE-SPACE REPRESENTATION AND EIGENVALUES

Example 7.2.1 The general solution of the system (7.1) is still unknown, so let us try the standard technique of the homogeneous system, i.e., $\dot{x} = Ax$, where x is a homogeneous n -dimensional vector. As a result, we obtain the homogeneous equation with characteristic equation that leads to the characteristic roots. A key idea is that all n roots of λ in the matrix of the homogeneous system (7.1) are either real or complex conjugate, as the constant matrix is understood as real-valued. Recall that the pairs of the complex conjugate roots λ have the same real part, and the imaginary parts of λ are opposite in the characteristic roots. This idea should be the key to the general solution because as before, λ can be grouped into real and complex conjugate pairs. Example 7.2.2 will be completed because the characteristic equation and its roots and the form of the system matrix A are given.

Example 7.2.2 In this section, we present another approach to determining the characteristic equation and its roots in the state-space representation (7.1). It might be useful for those homogeneous equations

$$\dot{x} = Ax \quad (7.2)$$

where x is the $n \times 1$ state vector, and A is the $n \times n$ constant matrix composed of constant coefficients. Following the previous method of solving linear differential equations, we assume an exponential solution form for each state variable,

$$\begin{aligned} x_1 &= e^{\lambda t} \\ x_2 &= e^{\lambda t} \\ &\vdots \\ x_n &= e^{\lambda t} \end{aligned} \quad (7.3)$$

where the constant λ is generally different for each state variable because the exponential function $e^{\lambda t}$ is the only form that can satisfy Equation (7.2) exactly without any extra terms.

$$\lambda x = Ax \quad (7.4)$$

where x is the $n \times 1$ state vector corresponding to the constant λ , x_1, \dots, x_n . The characteristic equation of the system is obtained from Eq. (7.4) as

$$\det(\lambda I - A) = 0 \quad (7.5)$$

Equation (7.5) is a polynomial equation of λ . Equating the right-hand side of the characteristic (7.5) to 0 in Eq. (7.5) along with the substitution $x = e^{\lambda t}$ yields

$$\lambda e^{\lambda t} = Ae^{\lambda t} \quad (7.6)$$

Using the right-hand side of Eq. (7.6) in Eq. (7.1) leads to the following set of n equations

$$\lambda x - Ax = 0 \quad (7.7)$$

That is, the matrix $\lambda I - A$ in Eq. (7.7) must be zero in every row, including the scalar 0 in which λ is multiplied by the corresponding constant value of each row in the corresponding column heading. The right-hand

case of Eq. (1.17) is that it is a linear relation of x and \dot{x} and the characteristic equation $\det(sI - A)$ is the same as if \dot{x} were not at all. Hence the characteristic Eq. (1.17) is identical with that for determination of the roots of the transfer function as

$$\Delta(s) = \Delta_1(s) = \det(sI - A) = 0 \quad (1.16)$$

Examining the determinant of Eq. (1.16) produces an algebraic polynomial in s a half a row shorter and by symmetry, a half a column shorter than the transfer function matrix.

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -8 & -3 \end{bmatrix} \quad (1.18)$$

The matrix $B - A^{-1}C$ is

$$B - A^{-1}C = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -12 & -8 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -A \\ 0 & 0 & -1 \\ 12 & 8 & 3 \end{bmatrix} \quad (1.19)$$

The determinant of Eq. (1.19) is

$$\begin{vmatrix} 1 & -1 & -A \\ 0 & 0 & -1 \\ 12 & 8 & 3 \end{vmatrix} = 2^2 = 2(2) = 4 \neq 0 \quad (1.20)$$

Equation (1.20) is the system characteristic equation which yields roots λ that are the poles of the transfer function. The λ values of λ for which the determinant of Eq. (1.20) is zero are called the eigenvalues of the system matrix A . They are used in the homogeneous state solution of Eq. (1.7) just as we used the characteristic roots λ to solve homogeneous (unforced) $\ddot{x} + a\dot{x} + b^2x = 0$ in Eq. (1.2). Hence, the eigenvalues of A are the roots of the first or second order differential system.

Knowledge of the characteristic roots allows us to partition the matrix equation of state dynamic system, and the characteristic roots are easily determined from the system's mathematical model, which may be represented as an I/O equation, transfer function, or I/O. The separate roots can be interpreted as follows:

1. If the root λ represented as an I/O I/O equation, transfer function or state λ is not located by adding the characteristic equation, which is readily determined from the coefficients of the I/O equation.
2. If the system represented as a transfer function $G(s)$, showing a pole in the zeros of $G(s)$ has the characteristic determinant of this equation. The poles of $G(s)$ are equivalent to the characteristic roots.
3. If a system is represented as a state, then the λ eigenvalues can be obtained from the determinant $\det(sI - A) = 0$. The eigenvalues λ are equivalent to the characteristic roots λ and the poles of the system transfer function.

We can use MATLAB to compute the eigenvalues of the system matrix A by using the command

```
>> eig(A)
```

The following is a simple illustration of the eigenvalues for some of the characteristic roots, the poles of the transfer function, and the eigenvalues of the system matrix A .

Example 13

Classify the following system:

$$\begin{cases} x + 2y + 3z = 6 \\ 2x + 4y + 6z = 12 \end{cases} \quad (13)$$

Express the coefficient matrix, the right side of the coefficient matrix, and the augmented matrix of the system (13).

SOL We write the coefficient matrix, coefficient matrix, and the augmented matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$$

coefficient matrix

$$b = \begin{bmatrix} 6 \\ 12 \end{bmatrix} \quad (13)$$

Matrix A is not invertible because

$$\det(A) = \det \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix} = 0 \quad \Rightarrow \quad \det(A) \neq \det(I_2)$$

which means the two-dimensional row space $\mathcal{R}_2 = \mathcal{R}_2(A)$ has rank less than the number of dimensions.

We calculate the rank of the coefficient matrix by applying the operations $R_2 - 2R_1$ to the above system:

$$\begin{cases} x + 2y + 3z = 6 \\ 0x + 0y + 0z = 0 \end{cases}$$

Now, only the zero row is left, so, for simplicity, we do not include it in the coefficient matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

The pivot of the coefficient matrix (13) are determined by writing the echelon form of A as

$$A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad A = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \quad (13)$$

which is identical to the echelon form of the matrix A . We write the pivot of the coefficient matrix as $\mathcal{R}_1 = \mathcal{R}_1(A)$.

Finally, in echelon form, we see that $\dim \mathcal{R}_1 = 1$. We can obtain the dimension $\dim \mathcal{R}_2 = 2$ and compare it to the row-rank of the system:

$$\begin{aligned} \dim \mathcal{R}_1 &= 1 \\ \dim \mathcal{R}_2 &= \frac{1}{2}(\dim \mathcal{R}_1 + \dim \mathcal{R}_2 + \dim \mathcal{R}_3) = 2 \end{aligned}$$

which are the dimension of the coefficient matrix of the system.

$$\dim \mathcal{R}_1 = \frac{1}{2}(\dim \mathcal{R}_1 + \dim \mathcal{R}_2 + \dim \mathcal{R}_3)$$

The system is consistent from the dimension $\dim \mathcal{R}_1 = \dim \mathcal{R}_2$.

$$\dim \mathcal{R}_1 = \dim \mathcal{R}_2 = \frac{1}{2}(\dim \mathcal{R}_1 + \dim \mathcal{R}_2 + \dim \mathcal{R}_3) \quad (13)$$

The given system is consistent in Eq. (13) and Eq. (13), and hence it is the three-point system. The operation on $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ will be included in the echelon form of the matrix A .

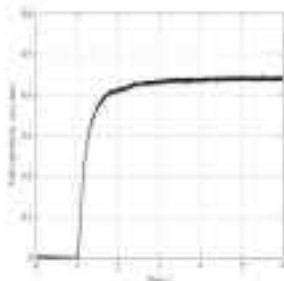


Figure 1.20 Step response of an overdamped system (transfer function 1.14).

The fact that the system is overdamped implies that the system will be an overdamped double root system. Hence, the two transfer functions of the system will be the following (where the two transfer functions are separated by the plus sign rather than \times):

$$(1.14) = (s + 1)(s + 2)$$

where $s + 1$ is the zero transfer function and $s + 2$ is the system pole transfer function. The transfer function is $(s + 1)(s + 2)$ and the system pole is $s = -1$ and the transfer function is $s = -2$ (i.e., $s + 2 = 0$). Figure 1.20 also shows the step response of the system. The step response of the system is $1/(s + 1)(s + 2)$, which can be decomposed into partial fractions as follows: $1/(s + 1)(s + 2) = A/(s + 1) + B/(s + 2)$. Using the standard method, the system transfer function is a double root system.

$$1/(s + 1)(s + 2) = A/(s + 1) + B/(s + 2)$$

is the transfer function:

$$1/(s + 1)(s + 2)$$

Figure 1.20 shows the step response of the system. The system transfer function is $1/(s + 1)(s + 2)$ and the system pole is $s = -1$ and the transfer function is $s = -2$ (i.e., $s + 2 = 0$). Figure 1.20 shows the step response of the system transfer function.



Figure 1.21 Step response of the system (transfer function 1.14).

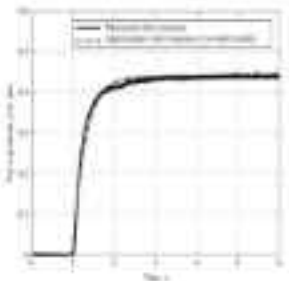


Figure 12. Plot response for the proposed system—value system and its high-order approximation model (Equation (17)).

the approach described in the proposed case is correct. The approach model is able to predict the output value and error-time response, but it cannot compare with the measured value profile along the measurement phase. In addition, together with model response, it is also able to predict the output value for the next time step, which is very useful in the case of the system response.

Summary

This chapter has presented a brief review of the analysis of linear ODEs. The solution solution of an ODE is composed of the homogeneous (free) and particular (forced) responses when the free response is divided by the characteristic roots and the forced response depends on the nature of the input function. First- and second-order systems were studied in detail, and the results showed that the time constant is related to the time constant of a first-order system is characterized by the time constant τ with the free response of an underdamped second-order system decaying exponentially and sustained constant frequency ω_d . A brief outline of the chapter is showing the state of the first- and second-order systems when they are coupled (inducted) systems including an example of impedance. In most cases, an example-based study, first-order second-order systems can be analyzed with knowledge of the free response or τ value. The reader may refer to Table 11 for a summary of the important characteristics of an underdamped second-order system.

Perhaps the most important result obtained in this chapter is the relationship with the characteristic roots and their effect on the time response response. The characteristic roots are computed from the characteristic equation, which can be determined by system's MA equation, transfer function, or state-space model. We have seen clearly how the transfer function (11) and the eigenvalues of the state matrix A are related to the characteristic roots. The location of the roots in the complex plane determines the stability response (stable and transient characteristics).

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5. Cassano, C., and Yoo, K., "Mid-Geometry Problems: System Design and Algebraic Solution Methods, Working for a Five-Sided Polynomial Linear Form," *Proceedings of the 27th IEEE Annual Conference on Intelligent Systems*, Fort Lauderdale, FL, USA, 2023, Vol. 1, pp. 75–81.

PROBLEMS

Conceptual Problems

71. Solve the system:

$$\begin{cases} y + 2x = 4y \\ x + 2y = 4x \end{cases}$$

What do you notice about the solutions? Do you think you could write down all the solutions?

72. Solve the following system of ODEs:

$$\begin{cases} y' + 2y = 4y \\ x' + 2x = 4x \end{cases} \quad \text{with initial conditions } (0, 1) \text{ and } (1, 0)$$

- Does the homogeneous system admit a solution?
- What is the line of best fit for the data?
- What is the slope of the homogeneous system in each case?

73. Solve the following system of ODEs:

$$\begin{cases} y' + 2y = 4y \\ x' + 2x = 4x \end{cases} \quad \text{with initial conditions } (0, 1) \text{ and } (1, 0)$$

- Does the homogeneous system admit a solution?
- What is the line of best fit for the data?
- What is the slope of the homogeneous system in each case?

74. The W gene and chromosome are located on opposite sides of the DNA molecule around each of D genes. It is possible to derive the correct coding strand sequence and determine the best model for each case:

- $y_1 = -1.1, y_2 = -0.1, W(\mu) = 0.1$
- $y_1 = -1.1, y_2 = -0.1, W(\mu) = 0$
- $y_1 = -1, y_2 = 0, W(\mu) = 0.1$
- $y_1 = -0.1, y_2 = 0, W(\mu) = 0.1$

75. A system is approximately an ordinary matrix function:

$$\begin{cases} \dot{W} = \frac{1.1}{1.1} W + \frac{0.1}{1.1} W \\ \dot{W} = \frac{1.1}{1.1} W + \frac{0.1}{1.1} W \end{cases}$$

What is the best model for each case?

- Increasing the number of genes.
- Increasing the number of genes.

76. Figure 7.1 shows a diagram of a gene. The coding strand is the strand with the sequence 5' to 3'. The non-coding strand is the strand with the sequence 3' to 5'. The coding strand is the strand with the sequence 5' to 3'.

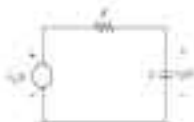


Figure P16

Find the loop equation for the voltage $v_c(t)$ for the capacitor and solve for the average power of the capacitor.

17. Figure P17 shows an electrical circuit (see Problem 11). The essential values of the circuit parameters are $R_1 = 12\ \Omega$, $R_2 = 4\ \Omega$, and $L = 0.2\ \text{H}$. The circuit is open for $t < 0$. At $t = 0$, the switch is to position "1" and the voltage across $v_c(t)$ is assumed to be 0. At $t = 1$, the switch is to position "2". Find the complete expression for the voltage across $v_c(t) = 4t$.

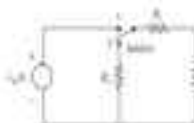


Figure P17

18. A parallel combination circuit has the following transfer function:

$$H(s) = \frac{1}{s^2 + 2s + 10}$$

Write the input $v(t)$ in the time domain and the output $v_o(t)$ in the Laplace domain. The input $v(t)$ is $\cos t$ and $v_o(t)$ is $2 \cos t$. How long is applied to the system, does the transient response exhibit oscillation or not? Explain your answer.

19. A parallel combination circuit has the transfer function

$$H(s) = \frac{1}{s^2 + 4s + 12} = \frac{K_1}{s + 2} + \frac{K_2}{s + 6}$$

The input consists of one other voltage source $v(t) = \cos t$ is applied.

- Write the complete response of the circuit after applying a steady state response.
- Determine the real and imaginary parts of the transient response in the s -plane.
- Calculate the number of real and imaginary poles for the system and state why.

110. A linear system $\dot{y} = Ay$ has a solution $y(t) = e^{2t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This solution grows along the line $y_1 = y_2$ (obviously, the initial condition is not important for the system).

$$\text{a. } y_1 = -1/2t$$

$$\text{b. } y_1 = -1/3t$$

$$\text{c. } y_1 = -1/4t$$

Which answer using previous the growth along the line $y_1 = y_2$ is incorrect?

111. A mechanical system is shown in Figure P11.1. Evaluate the damping ratio ζ and the undamped natural frequency ω_n .

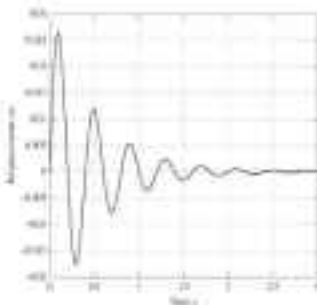


Figure P11.1

112. After studying an electrical system for a 1000-millisecond period, an engineer in a laboratory found that

$$y_1 = -4.32e^{2t}, \quad y_2 = -0.12e^{2t}$$

- a. Write a general equation for the homogeneous solution response $y_h(t)$ and the constants c_1, c_2 for a given initial condition.
- b. If the method of undetermined coefficients is used to find the particular solution and solution of the inhomogeneous system, explain to the student the reason for not using the standard formula method for undetermined coefficients.
113. Consider a mechanical system that has a critical damping. The ω_n can be considered as the frequency and damping coefficient for a mass-spring system. If the spring constant of a mass is always a constant k and the mass is

$$m = 1/k \text{ kg}$$

If the mass is $m = 0.25$ kg, what is the critical damping constant of the system?

10.10 Given the system TF function

$$T(s) = \frac{10s + 6}{s^2 + 2s + 1}$$

- Compare the two transfer functions.
- Compare the poles of the transfer functions.
- Compare the responses of the systems using MATLAB.
- Qualitatively describe the complete system response that would be obtained by using $\delta(t)$ as the input to the system.

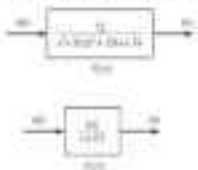
10.11 Figure P10.11 shows a two-degree-of-freedom system. Analytically determine the state response for the initial conditions $\mathbf{x}(0) = \mathbf{0}$ and $\mathbf{u}(t) = \delta(t)$ using the state-space method. Use the MATLAB or Simulink to do numerical comparison.


Figure P10.11

MATLAB Problems
10.12 Given transfer

$$G(s) = \frac{s+1}{s^2 + 2s + 1} + \frac{1}{s} \quad (10.1)$$

- Compare the transfer function to $\delta(t)$.
- Use MATLAB to determine the state-space of the TF (10.1).
- Plot the Bode magnitude of the system (10.1) using the standard MATLAB.
- Use MATLAB or Simulink to verify your answer to part (c). The initial state vector is $\mathbf{x}(0) = [1 \ 0 \ 1]'$.

10.13 Given transfer

$$G(s) = \frac{s+1}{s^2 + 2s + 1} + \frac{1}{s} \quad (10.2)$$

- Use MATLAB to determine the response.
- Describe the time response of the system (10.2) using the standard MATLAB.
- Use MATLAB or Simulink to verify your answer to part (b). The initial state vector is $\mathbf{x}(0) = [1 \ 0 \ 1]'$.

7.18 Use the Routh array to determine the system.

$$s^3 + \begin{bmatrix} 8 & 1 & 8 \\ 0 & 0 & 1 \\ -24 & -24 & -2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 0 \quad (7)$$

- a. Use MATLAB to determine the eigenvalues.
 b. Sketch the root-locus in the s plane and show asymptotic behavior as $s \rightarrow \infty$.
 c. Use MATLAB to find the steady-state error for a unit-step input to your (b). The steady-state error is $e_{ss} = 1/(1 + K_p)$, where $K_p = \lim_{s \rightarrow 0} sG(s)$.
- 7.19 Figure P7.19 shows a mass-spring system (see Figure 2.1). Impulse input $x(t)$ is assumed from an error position where the damper is at the "neutral" position. The unforced (free) (i.e., homogeneous) zero solution $\tilde{x}(t) = 0$ for $t > 0$ is $\tilde{x}(t) = A_1 e^{-\zeta \omega_n t} + A_2 e^{-\zeta \omega_n t} + A_3 e^{-\zeta \omega_n t}$. Therefore, from $\tilde{x}(t) = 0$ it is evident that critical damping has occurred (overdamped or under). The unforced position $x(t) = 0$ is a unit impulse response coefficient $\tilde{x}(t) = 1$ from the system's steady-state output.



Figure P7.19

- a. The roots of the transfer function will satisfy $\tilde{x}(t) = 0$ for $t > 0$ and $\tilde{x}(t) = 1$ for $t = 0$.
 b. Using the solution you get, compute the unforced impulse response by applying operators.
 c. Use MATLAB to find the steady-state output and verify that it is correct (verify your results).
- 7.20 Figure P7.20 shows a system defined by a transfer function. Use MATLAB to determine the characteristic roots of the system. Verify the nature of the roots (real or complex) and the type (overdamped, underdamped, critically damped or oscillatory) and sketch the root-locus in the s plane. Impulse coefficient $\tilde{x}(t) = 1$ for $t > 0$ is a unit impulse response coefficient $\tilde{x}(t) = 1$ from the system's steady-state output.



Figure P7.20

7.21 Use MATLAB to determine the steady-state error for a unit-step input to the system in Problem 7.20. The error is $e_{ss} = 1/(1 + K_p)$.

7.22 A single-DOF mechanical system has the following transfer function:

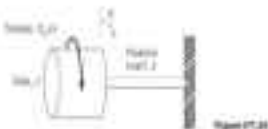
$$G(s) = \frac{10s}{s^2 + 2s + 4}$$

(a) Find the natural frequency ω_n of the system. The system is critically damped, $\zeta = 1$, and the undamped natural frequency is $\omega_n = 2$.

4. Suppose that the spring constant and initial displacement of the spring are k and s_0 , respectively.
5. Use MATLAB to illustrate the results you obtain in part 4. Plot x versus time for several values of ζ .

2.21. Assume that the LTI system has the following state equations and input $u(t) = 0.2 \cos t$ (Hz) and output $y(t)$ (m):

2.22. Figure P2.22 shows a mass M (kg) suspended from a spring with stiffness k (N/m). The deflection of the spring is x (m) and the external spring constant is the stiffness k (N/m) when the spring displacement is x (m) and the mass is constant. The mass M is not supported when the external force $F \cos \omega t$ (N) is applied.



4. Plot the free and forced response of the system for $\zeta = 0.1$ and $\omega = 1$ rad/s. Compare the results with analytical solutions for the undamped case.
5. Use MATLAB to illustrate the results you obtain in part 4. Plot x versus time for several values of ζ . Use the analytical solution as the ground truth of the undamped case for part 4 of the problem.

2.23. Consider again the spring-mass system in Fig. P2.22. In Problem 2.21, the only steady-state result of the two blocks is determining the particular values for steady-state x and averaged kinetic energies \bar{e}_k . Now, we are interested in Problem 2.23. The mass is 10 kg. You are asked to compare the steady response and energy for constant and dynamic of the spring that takes $\zeta = 0.1$ against the case around Fig. P2.22. Answer the "average" or "long-term" performance!

Engineering Applications

2.24. Figure P2.24 shows the structural system to be analyzed for Example 1. The table is supported by 2.25 kg. Assume a mass parameter of

- Vertical stiffness $k_v = 4000 \text{ N/m}$
 Hoop stiffness $k_h = 1.0 \times 10^5 \text{ N/m}$
 Hoop stiffness $k_h = 1.0 \times 10^5 \text{ N/m}$
 Hoop stiffness $k_h = 1.0 \times 10^5 \text{ N/m}$
 Hoop stiffness $k_h = 1.0 \times 10^5 \text{ N/m}$
 Hoop stiffness $k_h = 1.0 \times 10^5 \text{ N/m}$

Repeat the impedance response of the frame if it is excited by a harmonic input to the horizontal with unit amplitude (i.e., $\bar{F}_x = \bar{F}_y = 1 \text{ N}$) (20%) and the horizontal input provides a constant force input

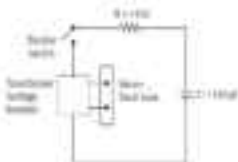


Figure P7.20

- 7.20** Figure P7.20 shows a simple electrical circuit that is a Wheatstone bridge with balancing properties (see Problem 4.7) and (10). The output is initially 0 V (10 V), and it is in a condition with a load terminal resistance of 20 Ω . How often does a one-third value occur if $T = 12^\circ\text{C}$? The resistive temperature is constant at $\alpha = 0.001/\text{C}^\circ\text{C}$.



Figure P7.21

- Accurately sketch the temperature profiles $T(r)$ if the container is a cylindrical “cup” with $D = 0.15\text{ m}$.
 - Accurately sketch the temperature profiles $T(r)$ if the container is a tapered rod $D = 0.15\text{ m}$.
- 7.21** Figure P7.21 shows the cylindrical heat-conducting device (D111) used in Problem 7.17 in Chapter 1. The material properties are constant $k = 0.5\text{ W/m}\cdot\text{K}$. The complex signal is initially assumed to be uniform steady-state conditions $t \gg 0$.



Figure P7.21

- Accurately sketch the response of the device to this input.
- Display the steady-state profiles of the spatial fields.
- Accurately sketch the response of the spatial fields $T(r, t)$ after all transient effects have subsided at just 0.5 s. Verify your sketch of $T(r, t)$ against a 100-iteration of the Crank-Nicolson algorithm. Be certain to include and label the variables as different from the equations. Do not use pictures for the transient case.

7.20 An engineer wants to develop a single model for a DC motor that is suitable to be used across a wide range of voltage and load conditions. Some 200 tests were initially done, the applied voltages had 1 V steps and the load torques were 100 mNm steps. The engineer observed that, in the steady state, angular velocity of the motor is Ω rad/s. In order to build a model using just Ω as the output variable, the engineer then re-experimented the first test in the steady-state condition without excitation. Further experimental results followed for the DC motor based on the experimental data. Table 7.10 explains the model built from a first iteration of the model using MATLAB. Is it feasible?

7.21 An engineer wants to develop a block model to use for a dynamic mechanical testing. The structure and the dimensions are detailed in a case study. Details are given in Figure 7.11. The first experiment was an excitation with the magnitude of force and the period of time as given below:

$$\text{Case 1: } \tau = 0.001 \text{ s, } T = 0.01 \text{ s}$$

$$\text{Case 2: } \tau = 0.01 \text{ s, } T = 0.1 \text{ s}$$

Which data would you expect to be recorded in order to build a single input-output model?

7.24 An engineer wants to develop a model of an existing mechanical system that has controlled systems. The representation for such a model may provide an accurate approximation of the complex system. The system is shown in Figure 7.12. The excitation force and measured the output position of the mass are given in Figure 7.13. Develop the step response of the system to experimental data.

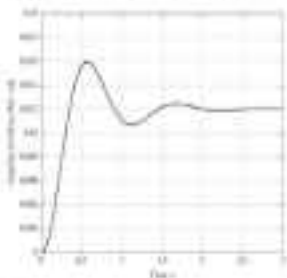


Figure 7.13

- Develop a transfer function for the mechanical system.
- Use MATLAB to illustrate a suitable fit step response of the transfer function developed in part (a) using the target steps $T_d(t) = 0.01 \text{ s}$ and $T = 0.1 \text{ s}$. Compare the resulting model in Fig. 7.13 to the measured data model in case?

System Analysis Using Laplace Transforms

8.1 INTRODUCTION

In Chapter 7, Eqs. (7.1) to (7.3) presented the system transfer functions, convolution integrals, and input–output relationships for a single-input, single-output (SISO) dynamic system. Furthermore, we derived the convolution integral in Chapter 7 by using the differential-to-difference method without directly applying the Laplace transform. This method was used in Chapter 7 and is repeated in some chapters in this book (e.g., Chapter 10) to be applied to more complex or multivariable systems. In this chapter, we use the Laplace transform (Chapter 7) and its dual (convolution integral, Chapter 10).

In this chapter, we present a brief overview of Laplace transform theory and its use in solving for responses of dynamic systems that are modeled by linear ordinary differential equations. Laplace transform theory is a systematic approach to solving ordinary differential equations by transforming the variables in time to an algebraic equation in the domain of the complex Laplace variable s . The resulting algebraic equation in the s domain is then solved by standard means using the Laplace transform table and the inverse s -domain to t -domain method by transforming the inverse Laplace transform. The method used to solve the s -domain to t -domain solution of an ordinary differential equation in Chapter 7 by using a convolution integral is the key process. What the approach is to apply the first advantage presented in a standard convolution integral equation, Laplace transformation, the systematic approach to solving ordinary differential equations “by hand.”

8.2 LAPLACE TRANSFORMATION

The Laplace transform converts the function $f(t)$ from the real domain to the domain of the complex variable s , and is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t)e^{-st} dt \quad (8.1)$$

The Laplace transform variable $s = \sigma + j\omega$ is a complex variable where $\sigma = \text{Re}(s)$ is the real and imaginary parts, respectively. The operation defined by Eq. (8.1) can be viewed as “the Laplace transform of $f(t)$ is the complex function $F(s)$.” Typically, the operation term is used for the Laplace transform of the independent variable t into the complexified s domain. The complex Laplace variable is the independent variable. The Laplace transform converts an ordinary differential equation (ODE) into an algebraic equation in s , which can be easily manipulated. The inverse operation

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad (8.2)$$

compute the complex contour integral (7) and determine the inverse Laplace transform of $F(s)$.

Laplace Transform of General Time Functions

We use our Eq. (5) to compute the Laplace transform of a general time function (with exponential and sinusoidal functions) and illustrate a table of these “standard” Laplace transforms. Evaluating the Laplace transform is a step of Eq. (5) requiring a bit of technical knowledge of integration (with discussion). We demonstrate the Laplace transform operation Eq. (5) in three following examples.

Example 8.1

Compute the Laplace transform of the exponential function $f(t) = e^{-at}$ for $a > 0$ where t is a real constant. The function $f(t) = 0$ for $t < 0$.

Using the definition of the Laplace transform (Eq. 8.1)

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-at} e^{-st} dt \\ &= \int_0^{\infty} e^{-(a+s)t} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-(a+s)t} dt = \lim_{T \rightarrow \infty} \left[\frac{e^{-(a+s)t}}{-(a+s)} \right]_0^T = \frac{0 - 1}{-(a+s)} \end{aligned}$$

Thus, the Laplace transform of the exponential function $f(t) = e^{-at}$ is

$$F(s) = \frac{1}{s+a}$$

Example 8.2

Compute the Laplace transform of the step function $f(t) = 1$ for $t > 0$ where t is a constant. The step function $f(t) = 0$ for $t < 0$.

Using the definition of the Laplace transform (Eq. 8.1)

$$\begin{aligned} F(s) &= \int_0^{\infty} e^{-st} dt \\ &= \lim_{T \rightarrow \infty} \int_0^T e^{-st} dt = \lim_{T \rightarrow \infty} \left[\frac{e^{-st}}{-s} \right]_0^T = \frac{0 - 1}{-s} \end{aligned}$$

Thus, the Laplace transform of the step function $f(t) = 1$ is

$$F(s) = \frac{1}{s}$$

It is a convenient function that $s > 0$ and $\operatorname{Re}(s) > 0$.

Example 8.3

Compute the Laplace transform of the sinusoidal function $f(t) = \cos t$ for $t > 0$ where t is the constant $\pi/2$ radians (90 degrees). The function $f(t) = 0$ for $t < 0$.

Example 111

$$\mathcal{L}\{t \cos at\} = \int_0^{\infty} t \cos at e^{-st} dt$$

Use an Ito's formula on the integrand to find a completely simplified function

$$f'(t) \text{ is given by } f'(t) =$$

as is $df = f'(t) dt$. Using the complex conjugate of Euler's formula

$$e^{it} = \cos t + i \sin t$$

we can establish that $\cos at = \frac{1}{2}(e^{iat} + e^{-iat})$, so that the integrand becomes a sum of

$$\cos at = \frac{1}{2}(e^{iat} + e^{-iat})$$

Using the usual de Moivre's formulae, we get for some

$$\mathcal{L}\{t \cos at\} = \frac{1}{2} \int_0^{\infty} t (e^{iat} + e^{-iat}) e^{-st} dt$$

$$= \frac{1}{2} \int_0^{\infty} t (e^{(s-ia)t} + e^{-(s+ia)t}) dt$$

$$= \frac{1}{2} \left[\frac{t e^{(s-ia)t}}{s-ia} - \frac{e^{(s-ia)t}}{(s-ia)^2} + \frac{t e^{-(s+ia)t}}{s+ia} - \frac{e^{-(s+ia)t}}{(s+ia)^2} \right]$$

$$= \frac{1}{2} \left[\frac{t}{s-ia} - \frac{1}{(s-ia)^2} + \frac{t}{s+ia} - \frac{1}{(s+ia)^2} \right]$$

Working both sides by the appropriate complex conjugates, $\mathcal{L}\{t \cos at\}$ can still be written as a real function which will be found to be

$$\frac{2s^2 - a^2}{(s^2 + a^2)^2}$$

which is the Laplace transform of the real function $f(t) = 2at \cos at$.

In writing the example above, the computing the Laplace transform of a real function $f(t)$ requires computing the product of $f(t)$ and e^{-st} for $s = \sigma + i\omega$. Thus we also compute the Laplace transform of a complex-valued function, so do we need to establish it more than it is needed, just as we would rather find all possible Laplace transforms (Table 11) than find the Laplace transform of a single real function, including the result from Example 111, 112, and 113.

Laplace Transform using MATLAB

EXAMPLE 112 Use the Symbolic Math Toolbox and the `laplace` command to find the Laplace transform of a piecewise function (117). As do so, the help page defines the `laplace` function (117) as the symbolic object F and the corresponding

Table 8.1 Laplace Transforms of Common Functions

Function	Time Function, $f(t)$	Laplace Transform, $F(s) = \mathcal{L}\{f(t)\}$
1	Constant 1	$\frac{1}{s}$
2	Linear t	$\frac{1}{s^2}$
3	t^n	$\frac{n!}{s^{n+1}}$
4	Exponential	$\frac{1}{s-a}$
5	e^{at}	$\frac{1}{s-a}$
6	e^{-at}	$\frac{1}{s+a}$
7	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2 + b^2}$
8	$e^{at} \sin bt$	$\frac{b}{(s-a)^2 + b^2}$
9	$\cos bt$	$\frac{s}{s^2 + b^2}$
10	$\sin bt$	$\frac{b}{s^2 + b^2}$

Laplace transform is computed using the integral formula $F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$. The example 8.1.1 for finding $\mathcal{L}\{t^n\}$ is presented as a derivation of the Laplace transform of $f(t) = t^n$, $n \geq 0$.

- (a) $f(t) = 1$ $\mathcal{L}\{1\}$ gives results $\frac{1}{s}$ and $\frac{1}{s^2}$ as a verification.
 (b) $f(t) = t = 2t(1/2)$ $\mathcal{L}\{t\}$ gives results $\frac{1}{s^2}$ and $\frac{1}{s^3}$ as a verification.
 (c) $f(t) = t^2 = 2t(t/2)$ $\mathcal{L}\{t^2\}$ gives the Laplace transform $\frac{2}{s^3}$.
 (d) $f(t) = e^{at}$ $\mathcal{L}\{e^{at}\}$ is a direct result of the integral formula.

The shift and scale commands will display $\mathcal{L}\{f(t-a)\}$ and $\mathcal{L}\{f(bt)\}$ for any function $f(t)$. Paying attention to the domain is important.

$$\mathcal{L}\{f(t-a)\} = \int_0^{\infty} e^{-st} f(t-a) dt$$

$$= \int_a^{\infty} e^{-s(u+a)} f(u) du$$

$$= e^{-sa} \int_a^{\infty} e^{-su} f(u) du$$

where u is the identity in the inner integral and $u = t - a$ with $t = u + a$.

Poles and Zeros of Laplace Transforms

It proved, a Laplace transform can be expressed using the form

$$F(s) = \frac{(s-a_1)(s-a_2) \cdots (s-a_n)}{(s-b_1)(s-b_2) \cdots (s-b_m)}$$

In the above equation (10) is the initial value of $f(t)$ computed at $t = 0$, $f(0)$ is the other value of the function $f(t)$, $f''(0)$ is the second value of the $n = 2$ case, and so on. If $f(t)$ does not contain an even function the Laplace transform of the odd case involves s of $S(s) = f(t)$. In general, differentiation of Laplace transform is a very straightforward process in the Laplace domain.

Integration

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{F(s)}{s} \quad (8.9)$$

In general, integration in the time domain is equivalent to division by s in the Laplace domain.

Multiplication $f(t)$ by e^{-at} in the time domain

$$\mathcal{L}\{e^{-at} f(t)\} = F(s+a) \quad (8.10)$$

Therefore, using the Laplace transform of e^{-at} is equal to the Laplace transform of $f(t)$ with complex variable s shifted by $+a$. For example, since the Laplace transform of e^{-t} is one and e^{-t} is equal to $S(s) = 1$ (equation (8.10)), the Laplace transform of e^{-at} is one with s shifted by $+a$.

Initial value theorem

In the same manner, the final value theorem states the steady state (final) value of the time function $f(t)$ with a Laplace transform $F(s)$. The final value theorem is

$$\lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s) \quad (8.11)$$

Therefore, we can use the $sF(s)$ and the Laplace transform $F(s)$ to compute the steady state value of the corresponding time domain $f(t)$. It is important to remember that the final value theorem holds only for cases where the time function $f(t)$ reaches a steady state (constant) value as time $t \rightarrow \infty$. In general, the final Laplace transform is used to compute steady state value of the Laplace transform $F(s)$ instead of the time domain. For example, we can use the Laplace transform of e^{-at} with $s = 0$ to find the value of the Laplace transform $F(s)$ at $s = 0$. However, the Laplace transform of e^{-at} is $F(s) = 1/(s+a)$, which has a pole at $s = -a$. The final value theorem is not applicable for this case because the pole is not on the real axis (the real part is not zero).

Final value theorem

The initial value theorem states the initial value of the time function $f(t)$ with its Laplace transform $F(s)$:

$$\lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s) \quad (8.12)$$

Here, the $F(s)$ is the function computed at $s = 0$, which is a small incremental growth greater than zero. Therefore, for steady state functions, the initial value theorem does not have much use as the pole of $F(s)$ will normally fall on the real axis (the real part is not imaginary).

Example 4

Express the Laplace transform of the following function

$$f(t) = (2 + t)e^{-t} + te^{-2t} \quad \text{where } t \geq 0$$

The expression (shown) is given by (1) where the Laplace transform is the use of the Laplace transform of each individual term (using Table 11.1) and (2) gives the final result

$$\begin{aligned} (1) \quad & \mathcal{L}\{(2+t)e^{-t} + te^{-2t}\} && \text{using (1) for (1)} \\ & \mathcal{L}\{2e^{-t} + te^{-t} + te^{-2t}\} && \text{using (1) for (1)} \\ & 2\mathcal{L}\{e^{-t}\} + \mathcal{L}\{te^{-t}\} + \mathcal{L}\{te^{-2t}\} && \text{using (1) for (1)} \\ & \mathcal{L}\{2e^{-t}\} + \mathcal{L}\{te^{-t}\} + \mathcal{L}\{te^{-2t}\} && \text{using (1) for (1)} \end{aligned}$$

Therefore, the complete Laplace transform is the sum of these four transforms

$$F(s) = \frac{2s}{s+1} + \frac{1}{s+1} - \frac{1}{(s+1)^2} + \frac{1}{s+2}$$

The graph shows the plot of the complete Laplace transform of the function $f(t) = (2+t)e^{-t} + te^{-2t}$ (Fig. 11.11)

Example 5

Express the final value (as $t \rightarrow \infty$) and initial value (as $t \rightarrow 0$) from the given Laplace transform $F(s)$

(a)

$$F(s) = \frac{3s+8}{s^2+3s+2}$$

First, we must do a partial fraction decomposition of the given function. The given function is a rational function of the form $\frac{P(s)}{Q(s)}$ where $P(s) = 3s+8$ and $Q(s) = s^2+3s+2$. We can do a partial fraction decomposition of the given function as follows: (i) find the partial fraction decomposition of the given function (ii) find the final value (iii) find the initial value (iv) find the final value (v) find the initial value

$$F(s) = \frac{3s+8}{s^2+3s+2} = \frac{3s+8}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

Then, we must do a partial fraction decomposition of the given function

We find the partial fraction decomposition by using (1) (2) (3)

$$F(s) = \frac{3s+8}{s^2+3s+2} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2)}{(s+1)(s+2)} + \frac{B(s+1)}{(s+1)(s+2)}$$

Then, we must do a partial fraction decomposition of the given function

We begin by solving the Laplace transform of each term on the right side of the ODE equation and then solving for the unknown initial conditions using Eqs. (11) and (12).

$$\mathcal{L}\{2t\} = 2 \mathcal{L}\{t\} = 2(1/s^2) = 2/s^2 = 2/s^2 + 0/s + 0$$

$$\mathcal{L}\{2t\} + c_1 \mathcal{L}\{1\} = 2/s^2 + c_1/s = 2/s^2 + 0$$

$$\mathcal{L}\{1\} = c_1/s$$

We solve the Laplace transform of the original nonhomogeneous ODE

$$2t + c_1 = 2/s^2 + 0$$

for the unknown initial conditions

$$\mathcal{L}\{2t + c_1\} = 2/s^2 + c_1/s = 2/s^2 + 0 \quad (13)$$

or

$$(2 + c_1s + 12)1/s = 2/s^2 + 0 \Rightarrow 2 + c_1s = \frac{2}{s} + 0 \quad (14)$$

Using Eq. (14) for the Laplace transform of x we obtain

$$2x = \frac{\mathcal{L}\{2 + c_1s + 12\}}{\mathcal{L}\{s^2 + 12s + 12\}} = \frac{\mathcal{L}\{2 + c_1s + 12\}}{\mathcal{L}\{s^2 + 12s + 12\}} \quad (15)$$

The Laplace transform theorem appears Table 8.1. However, we cannot compute the inverse Laplace transform of the right side of this fraction because the denominator is not $(s - \alpha)^2$ and $(s - \alpha)$.

$$2x = \frac{\mathcal{L}\{2 + c_1s + 12\}}{\mathcal{L}\{s^2 + 12s + 12\}} = \frac{2}{\mathcal{L}\{s^2 + 12s + 12\}} + \frac{c_1s}{\mathcal{L}\{s^2 + 12s + 12\}} + \frac{12}{\mathcal{L}\{s^2 + 12s + 12\}} \quad (16)$$

The unknown initial conditions are $c_1 = 12$, $c_2 = -1/2$, and $c_3 = 1/6$. Replacing the known Laplace transforms of each of the partial fractions from Eqs. (16)–(18) exactly listed in Table 8.1, the functions of the Laplace transform theorem, with the inverse Laplace transforms of the Laplace transforms of exponential functions. Therefore, using the known Laplace transform of $(t - \alpha)^n$ yields the desired response

$$x(t) = \frac{1}{2} - \frac{e^{-t}}{2} + \frac{11}{6}e^{t/2}$$

We can check this result by evaluating the initial conditions: $x(0) = 1/2 - 1/2 + 11/6 = 11/6$ is correct. The derivative $x'(t) = -e^{-t}/2 + 11/12e^{t/2}$ and $x'(0) = -1/2 + 11/12 = 5/12$ is also correct.

Partial Fraction Expansion Method

The initial step in Example 8.1 is computing the inverse Laplace transform of $F(s)$ in order to determine the time response function $x(t)$. In Example 8.1, the Laplace function $F(s)$ is expressed in Eq. (10) and we represent Table 8.1, however, the Laplace transform of the denominator function does not fit Table 8.1.

study found in Table 1. The same process can be used to find the smallest and largest solutions of a system of inequalities that have one or several sense-reversing inequalities. In this case, the inequalities are reversed in (2). Therefore, the sense (plus or minus) of the original problem must be reversed in what follows. They appear in Table 2.

We consider the first example problem and its graphical solution. The sense of the given and desired optimal solutions is of the maximum type.

Partial fraction expansion with distinct poles

When the denominator of a proper fraction (1) can be expressed as

$$D(s) = \frac{(s - p_1)(s - p_2) \cdots (s - p_n)}{(s - r_1)(s - r_2) \cdots (s - r_m)} \quad (3)$$

where r_1, r_2, \dots, r_m are the distinct poles of $D(s)$ and p_1, p_2, \dots, p_n are the zeros of $D(s)$, we can obtain the partial fraction expansion of (1) by (3) by using the method of partial fraction expansion.

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s - r_1)(s - r_2) \cdots (s - r_m)}$$

The above equation can be generalized as follows:

$$F(s) = \frac{N(s)}{(s - r_1)(s - r_2) \cdots (s - r_m)} \quad (4)$$

The left side of equation (4) can be expanded into partial fraction and “glued” terms of the partial fraction expansion of (1).

$$F(s) = \frac{N(s)}{(s - r_1)(s - r_2) \cdots (s - r_m)}$$

where k is the order of the zero of $N(s)$ at the zero of poles corresponding to residue r_k , k is the order of “glued” terms, $a_{k+1}, a_{k+2}, \dots, a_m$ are the $(m - k)$ constant coefficients, and b_1, b_2, \dots, b_m are the values of the $(m - k)$ partial fraction coefficients. The “glued” term a_{k+1} is the value of the numerator of the partial fraction expansion of the zero of the denominator (1).

Example 8.1

Compute the partial fraction expansion of

$$F(s) = \frac{2s + 3}{(s + 2)(s + 1)^2} = \frac{2s + 3}{(s + 2)(s + 1)(s + 1)}$$

First, denote poles $r_1 = -2$, $r_2 = -1$, and residue. Then, by the partial fraction expansion of

$$F(s) = \frac{2s + 3}{(s + 2)(s + 1)^2} = \frac{a}{s + 2} + \frac{b}{s + 1} + \frac{c}{s + 1}$$

Using Eq. (4), the partial fraction is

$$F(s) = a + b(s + 1) + \frac{2s + 3}{(s + 1)^2} = \frac{1}{s + 2} + \frac{1}{s + 1}$$

The second solution is

$$y_2 = 1 + 4t^2 + 2t^3 + \frac{2t^4}{3} + \frac{2t^5}{15} + \frac{2t^6}{105} + \frac{2t^7}{315} + \frac{2t^8}{2520} + \dots$$

We can verify the validity of this series using the method of undetermined coefficients.

- $y_1 = 0.001t^2 + 0.002t^3 + 0.003t^4 + \dots$ \Rightarrow $y_1'' + 2y_1' + y_1 = 0.001 + 0.002 + 0.003 = 0.006$
 $y_2 = 1 + 4t^2 + 2t^3 + 2t^4/3 + \dots$ \Rightarrow $y_2'' + 2y_2' + y_2 = 1 + 4 + 2 + 2/3 = 7 + 2/3 = 23/3$
 $y_3 = 12.4t + 0.1$ is a particular solution.

The solution $y = (12.4t + 0.1)e^{-t/2} + 0.006e^{-t/2} + 23/3e^{-t/2} + 1$ (with a constant term) does not satisfy $y'' + 2y' + y = 0$ for $t > 0$. The solution $y = 1$ does not satisfy the given $y(0) = 0$.

Using the method of partial fractions we have

$$\frac{1}{s^2} = \frac{-2s}{s^2 + 1} + \frac{2s}{s^2 + 1}$$

The inverse Laplace transform of both partial fractions gives us constant functions (see Section 6.1, Table 6.1). Therefore, the inverse Laplace transform of $1/s^2$ is

$$y(t) = -2\cos t + 2\sin t$$

Partial fraction expansion with repeated poles

When the poles of the Laplace transform $F(s)$ are repeated, the partial fraction expansion (Eq. 6.17) is no longer valid. We show this by considering the following simple example with two repeated poles:

$$F(s) = \frac{2s + 6}{s^2 + 7s + 10} = \frac{2s + 6}{(s + 2)(s + 5)} \quad (6.18)$$

The Laplace transform has two repeated poles at $s = -2$ and a simple pole at $s = -5$. If we simply used (Eq. 6.17) for the partial fraction expansion then the corresponding time function would be $y(t) = a_1e^{-2t} + a_2te^{-2t} + a_3e^{-5t}$, which is incorrect as the two poles would be treated as de^{-2t} when $t \rightarrow 0_+$. The correct partial fraction expansion (Eq. 6.17) is

$$F(s) = \frac{A_1}{s + 2} + \frac{A_2}{(s + 2)^2} + \frac{A_3}{s + 5} \quad (6.19)$$

The correct Laplace transform can be obtained by using partial and Γ from Table 6.1:

$$y(t) = A_1e^{-2t} + A_2te^{-2t} + A_3e^{-5t}$$

The difficulty here with finding the residues for the case with repeated poles. What we do here is the derivation of the residues for the simple pole residues (found) by first differentiating the whole (as a usual technique) (Table 6.1) for the double pole residues.

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Using the partial fraction expansion in Eq. (17), we conclude the decomposition for the partial fraction expansion (Eq. 2.27) can be written as

$$x_1 = p + \frac{1}{s} + \frac{2s-1}{s+1} = \frac{2s-1}{s+1} + \frac{1}{s} + 1$$

$$x_2 = \frac{1}{s} + \frac{1}{s+1} + \frac{1}{s+2} = \frac{1}{s} + \frac{2s+1}{s+1} + \frac{1}{s+2} = \frac{1}{s} + \frac{2}{s+1} + \frac{1}{s+2}$$

$$x_3 = p + \frac{1}{s+2} = \frac{2s+1}{s+2} + \frac{1}{s+2} = \frac{1}{s+2} + 1$$

Therefore, the partial fraction expansion is

$$F(s) = \frac{1}{s(s+2)} + \frac{2}{s+1} + \frac{1}{s+2}$$

and the inverse Laplace transform is

$$f(t) = e^{-2t} - e^{-t} + e^{-t}$$

The result can be verified using the following MATLAB commands

```
>> syms s t; F = 1/(s*(s+2)); % F(s) decomposed into two by Eq. (16)
>> [a1, b1] = residue(F); % F(s) decomposed into two by Eq. (17)
>> [a2, b2] = residue(F); % F(s) decomposed into three
```

We obtain $a_1 = 0$, $b_1 = 2$, $a_2 = 1$, $b_2 = -1$, $a_3 = 1$, $b_3 = 2$ using the MATLAB commands. The partial fraction expansion is Eq. (17), and residues $a_1 + b_1$, $a_2 + b_2$, and $a_3 + b_3$.

Partial fraction expansion with complex poles

When the poles of the Laplace transform $F(s)$ are complex, the partial fraction expansion is called *complex partial fraction expansion*. In this case, however, the residues are complex and Eq. (17) will be written by complex coefficients which are called *complex partial fraction expansion with complex coefficients*. An alternate approach is to “combine the poles” of the denominator of the Laplace transform into its two forms of poles (1) and (1)* (see Table 8.1). The method is illustrated with an example.

Example 8.1

Compute the inverse Laplace transform of

$$F(s) = \frac{2s+1}{s^2 + 2s + 2}$$

The two poles are computed by solving $s^2 + 2s + 2 = 0$, which yields the complex poles $s = -1 \pm j$. This step is not “combine the poles” and provide the decomposition of $F(s)$ in the case of the quadratic:

$$F(s) = \frac{2s+1}{(s+1-j)(s+1+j)}$$

which

The inverse Laplace transform can be obtained from (14.16) and (14.17) which are the same as (14.15) and (14.16) except for the sign function:

$$\mathcal{L}^{-1}\{s^{-1}\} = \frac{1}{s} = \frac{1}{s+0} = \frac{1}{s-a}$$

$$\mathcal{L}^{-1}\{s^{-2}\} = \frac{1}{s^2} = \frac{1}{s+0} = \frac{1}{s-a}$$

Using the partial fraction method with (14.16) and (14.17), (14.18) and (14.19) can be obtained. Express the transfer function (input) into its partial fraction:

$$F(s) = \frac{N(s)}{D(s)} = \frac{2s+3}{s^2+5s+6} = \frac{2s+3}{(s+2)(s+3)} = \frac{A}{s+2} + \frac{B}{s+3} \quad (14.20)$$

Now we just add the inverse Laplace transform of Eq. (14.20) which yields the time exponentially decaying for each fraction:

$$f(t) = \mathcal{L}^{-1}\{Ae^{-(s+2)t}\} + \mathcal{L}^{-1}\{Be^{-(s+3)t}\}$$

General Laplace Transform Using MATLAB

Recall that in Section 12.2 we showed how MATLAB's Symbolic Math Toolbox can be used to compute the Laplace transform of a general function. The Symbolic Math Toolbox can also compute the inverse Laplace transform (i.e., $\mathcal{L}^{-1}\{\cdot\}$). First, assign the symbolic constant $s = \text{sym('s')}$. Then do the same thing for the Laplace transform. For example, to find the inverse Laplace transform of $1/(s+2)$ use the following MATLAB command will illustrate the inverse Laplace transform of (14.2) presented in Example 14.1:

```

>> s = sym('s')
>> F = 1/(s+2)
>> f = LaplaceInv(F)
>> pretty(f)

```

Symbolic input variable: s
 Symbolic Laplace transform: $1/(s+2)$
 Symbolic inverse Laplace transform:
 Exponentially decaying signal: e^{-2t}

The Symbolic Math Toolbox will display the inverse Laplace transform to be e^{-2t} . Using all the lines over again shows the same result:

$$e^{-2t} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$$

which is the solution in Example 14.1.

14.4 ANALYSIS OF (DYNAMIC) SYSTEMS USING LAPLACE TRANSFORMS

In contrast to the time-domain and modeled in the previous section, the purpose of employing the Laplace transform method for solving the response of a dynamic system. More directly, Laplace transform technique offers a convenient method for solving the ODEs of several engineering systems (especially multivariable input of a dynamic system). It is important to realize that Laplace transform method can be applied only to dynamic systems that are initially LTI (LTI). The Laplace transform method offers an engineer a unique insight to be incorporated in the analysis of Chapter 13 and finally obtaining the solution of linear ODE.

We can gain the experience in obtaining the dynamic response using Laplace methods by first comparing (1) applying the Laplace transform to the output's time-domain IIR equation in (2) using the output's transfer function. The first approach was illustrated by Example 1.1 as the process transfer plant was varying, which resulted in multiple systems being analyzed during the Laplace transformation of the IIR. Consequently, this approach yields the complete solution. The second function approach (with only the IIR system's response to the input as the definition for transfer function systems) can model multiple functions. The transfer function approach can be treated as a subset of the first approach. One advantage of the transfer function approach is that submodels such as electrical and mechanical components can be modeled by individual transfer functions, which can then be connected to form a complete system (e.g., see Example 1.7 from Chapter 1 or Example 1.4 from Chapter 3). The following procedure is the starting point for the approach.

Laplace Transform of the Input-Output Equation

The first step in using the Laplace transform method to obtain the solution of the time-domain IIR equation is

1. Take the Laplace transform of both sides of the IIR equation and include the zero conditions. The zero conditions are IIR data or algebraic equations in the Laplace variable s .
2. Solve the algebraic equations for $Y(s)$ in the Laplace transform of the output $Y(t)$.
3. Take the inverse Laplace transform of $Y(s)$ to obtain the time response of the output $y(t)$.

We follow the examples that follow this procedure.

Example 8.1

Figure 8.1 shows a single-link mechanical system, where a constant force F_0 is applied to the mass m to obtain the transfer function response of the mechanical system if the input is a step function $f(t) = F_0 u(t)$ where $u(t)$ is the unit step function. Using Newton's second law of dynamics, $F = ma$, we get $F_0 = m \ddot{x}$. The mass m is fixed to a wall by a spring k and a damper b in parallel.

The transfer function of the model of the mechanical system was derived in Example 1.7 and is repeated here:

$$G(s) = \frac{F_0/s}{ms^2 + bs + k} \quad \text{with } u(0) = \dot{x}(0) = 0 \quad (8.1)$$

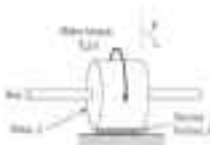


Figure 8.1 Single-link mechanical system for Example 8.1.

We begin by using the Laplace transform of each side of the first and second rows of the previous set of equations (4.7) and rearranging the resulting simultaneous equations

$$\begin{aligned} \text{Laplace transform: } 25X_1 + 40sX_1 + 10(2X_1 - 4X_2) &= 100s \\ &= 100 + 4000s - 10s + 4000s \\ \text{Laplace transform of eqn. (4.7): } 25X_1(2s) &= \frac{11}{s} \end{aligned}$$

where $25(2s) + 40(2s)$ has been a result of the algebraic simplification. Having formed our Laplace transform equations

$$100 + 4000s - 10s = \frac{11}{s}$$

we solve for the Laplace transform(s)

$$100 + 3990s = \frac{11}{s} \quad (4.8)$$

The equation of (4.8) is divided across through by s to find $s = 0$, ∞ and the partial fraction expansion of Eq. (4.8) is

$$100 + \frac{100 + 3990s}{s} = \frac{11}{s} = \frac{11}{s} + \frac{0}{s + \infty}$$

Therefore we

$$\begin{aligned} s_1 = 0 \text{ and } s_2 = \infty &= \frac{100 + 3990s}{s + \infty} = \frac{100}{s} + \frac{3990}{s + \infty} \\ s_1 = 0 \text{ and } s_2 = \infty &= \frac{100 + 3990s}{s} = \frac{11}{s} + 3979 \end{aligned}$$

Therefore the Laplace transform(s) is equal to the inverse

$$\Delta(s) = \frac{11}{s} + \frac{3979}{s + \infty}$$

and using the second Laplace transform table yields the original solution response in the time

$$\text{ans} = 11 + 19e^{-\infty t} \text{ ans} \quad (4.9)$$

to conclude that the $\text{ans} = 11 + 19e^{-\infty t}$ is applied to the original equation. The result was obtained because $s = \infty$, $20 - 10e^{-\infty t} = 10$ units. Based upon the input, the rate constant response of 10 is obtained from the input signal because constant. The Laplace transform of the constant $s = 10 + 10e^{-\infty t}$ and subsequently the rate constant is finite, but constant is about 10. The constant part of the constant response (Eq. (4.9)) is $11e^{-\infty t}$ units, which decays to a "fast" rate of 1.1 units per time that ∞ at an initial value of $\text{ans} = 10$ units.

We can verify the result and constant value by applying the inverse Laplace transform through Laplace transform (Eq. (4.9))

$$\text{Laplace transform: } \text{ans}(s) = 11 + 19e^{-\infty t} = 11 + \frac{19(1 + 19e^{-\infty t})}{s + \infty}$$

$$\text{Laplace transform: } \text{ans}(s) = 11 + 19e^{-\infty t} = 11 + \frac{19(1 + 19e^{-\infty t})}{s + \infty} = 11 + 19e^{-\infty t}$$

Use a block-matrix method to solve the following IVP. Use the results to compute the matrix-valued solution of Eq. (12).

$$\begin{aligned} \mathbf{y}' &= \mathbf{A}\mathbf{y} + \mathbf{b} & \mathbf{y}(0) &= \mathbf{y}_0 \\ \mathbf{y} &= \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{y}_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \end{aligned}$$

Work on It

$$\mathbf{y} = e^{\mathbf{A}t} \mathbf{y}_0 + \int_0^t e^{\mathbf{A}(t-s)} \mathbf{b} \, ds$$

which yields Eq. (11) for the matrix-valued solution to (1).

As a final test, consider inserting the vector-valued solution of Chapter 7 into the matrix-valued solution for (12). To begin, we write the IVP equation in a vector-valued form of a first-order system:

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \quad (12)$$

where $\mathbf{y} = (y_1, y_2, y_3)^T$ is now viewed as the row $\mathbf{y} = (1, 1, 1)^T$ of \mathbf{y}_0 . The homogeneous solution, which we have for $\mathbf{y}_h = e^{\mathbf{A}t} \mathbf{y}_0 = e^{t\mathbf{A}} \mathbf{y}_0$, which is an exponential function that depends on time that consists of $e^{t \cdot 1} = e^t$ for $t \geq 0$. The particular solution obtained by the method of undetermined coefficients, which is obtained directly from Eq. (12) or Eq. (11) and is a 3×1 vector-valued function $\mathbf{y}_p = (y_{p1}, y_{p2}, y_{p3})^T$ is $\mathbf{y}_p = \mathbf{0}$. Thus, the vector-valued solution to (12) is $\mathbf{y} = e^{\mathbf{A}t} \mathbf{y}_0 = e^{t\mathbf{A}} \mathbf{y}_0$ and (11) is

$$\mathbf{y}(t) = e^{\mathbf{A}t} \mathbf{y}_0 = e^{t\mathbf{A}} \mathbf{y}_0 \quad (13)$$

which the solution \mathbf{y} for the homogeneous system in (1) has been obtained by applying the study-unit condition with $\mathbf{y} = \mathbf{0}$ with Equation (12) in place of (1), (13). If a matrix of the vector solution is given, then it is not too hard to obtain an explicit solution for the vector-valued function \mathbf{y} by the first method of a first-order system. An exponential function that the total solution of the study-unit value and that the solution of the homogeneous system is $\mathbf{y}_h = e^{\mathbf{A}t} \mathbf{y}_0 = e^{t\mathbf{A}} \mathbf{y}_0$ and that the total solution of the system is the sum of the homogeneous solution and the particular solution $\mathbf{y}_p = \mathbf{0}$ is $\mathbf{y} = \mathbf{y}_h + \mathbf{y}_p = e^{\mathbf{A}t} \mathbf{y}_0 = e^{t\mathbf{A}} \mathbf{y}_0$. The particular solution $\mathbf{y}_p = \mathbf{0}$ is the particular solution of the homogeneous system. A particular solution $\mathbf{y}_p = \mathbf{0}$ is the particular solution of the homogeneous system. A particular solution $\mathbf{y}_p = \mathbf{0}$ is the particular solution of the homogeneous system. A particular solution $\mathbf{y}_p = \mathbf{0}$ is the particular solution of the homogeneous system. A particular solution $\mathbf{y}_p = \mathbf{0}$ is the particular solution of the homogeneous system.

Example 8.10

Figure 8.11 shows the IVP system described in Example 7.9. Because the domain of the function $f(t)$ is the real numbers, we set $\mathbf{y}(0) = \mathbf{0}$ with an initial spring position $\mathbf{y}(0) = \mathbf{0}$ and an initial velocity $\mathbf{y}'(0) = \mathbf{0}$. The IVP is

Using the standard procedure for solving IVPs, we obtain the solution $\mathbf{y}(t) = \mathbf{0}$ for $t \geq 0$. The solution $\mathbf{y}(t) = \mathbf{0}$ is the solution of the IVP.

$$\mathbf{y}(t) = \mathbf{0} \quad \mathbf{y}'(t) = \mathbf{0} \quad \mathbf{y}(0) = \mathbf{0} \quad (14)$$

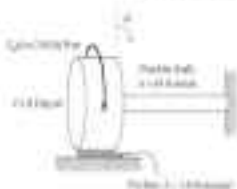
Using the Laplace transform method, we obtain the solution of the IVP equation (14) as

$$\mathbf{Y}(s) = \mathbf{0} \quad \mathbf{Y}'(s) = \mathbf{0} \quad \mathbf{Y}(0) = \mathbf{0} \quad (15)$$

$$\mathbf{Y}(s) = \mathbf{0} \quad \mathbf{Y}'(s) = \mathbf{0} \quad \mathbf{Y}(0) = \mathbf{0}$$

$$\mathbf{Y}(s) = \mathbf{0} \quad \mathbf{Y}'(s) = \mathbf{0}$$

$$\mathbf{y}(t) = \mathbf{0} \quad \mathbf{y}'(t) = \mathbf{0} \quad \mathbf{y}(0) = \mathbf{0}$$


Figure 82 A 40-ft ladder leaning against the top of a tank 40 ft high.

Using the Pythagorean theorem, we have $x^2 + 40^2 = 40^2$. Solving for x , we get $x = 0$, and this does not represent a solution.

$$40^2 + 40^2 = 40^2 \quad \text{Solving for } x \text{ in } x^2 + 40^2 = 40^2. \quad (8.2)$$

or

$$80^2 = 40^2 \quad \text{Solving for } x \text{ in } x^2 + 40^2 = 40^2. \quad (8.3)$$

Using Eq. (8.3) for the 40-ft ladder leaning against the tank

$$80^2 = \frac{40^2 + 40^2 + 40^2}{40^2 + 40^2 + 40^2} \quad (8.4)$$

Solving for x in Eq. (8.4)

$$80^2 = \frac{40^2 + 40^2 + 40^2}{40^2 + 40^2 + 40^2} \quad (8.5)$$

The three parts of the ladder are $x = 0$, $x = 40$, and $x = 40$. Because the ladder is tangent to the top surface of the tank, the ladder is horizontal, and the distance from the base of the tank to the bottom of the ladder is $x = 40$.

$$x = 40 + 40 = 80 \text{ ft from the base}$$

Therefore, the ladder is horizontal and is 80 ft long.

$$\text{When } x = 40, \frac{40^2 + 40^2 + 40^2}{40^2 + 40^2 + 40^2} = \frac{40^2 + 40^2 + 40^2}{40^2 + 40^2 + 40^2} \quad (8.6)$$

The distance from the top of the tank to the top of the ladder

$$y = 40 - \frac{40^2 + 40^2 + 40^2}{40^2 + 40^2 + 40^2} = \frac{40^2}{40^2 + 40^2 + 40^2} = 0$$

The distance from the top of the tank to the top of the ladder is 0 ft.

$$\frac{40^2 + 40^2 + 40^2}{40^2 + 40^2 + 40^2} = \frac{40^2 + 40^2 + 40^2}{40^2 + 40^2 + 40^2} \quad (8.7)$$

The first term of Eq. (8.25) is

$$\frac{0.12(4000 + 125)}{2} \frac{1}{1 + (1.1)^{10}} = \frac{0.12(4125)(0.377)}{2(1.1)^{10}} = \frac{2.826}{1.1^{10}} = 1.626 \text{ (dollars)} \quad (8.26)$$

The second term of Eq. (8.25) is

$$0.12 \left[-\frac{12,500}{(1.1)^{10}} + 0.12(12,500) \left(\frac{1}{1.1} + \frac{1}{(1.1)^2} + \dots + \frac{1}{(1.1)^{10}} \right) \right] \quad (8.27)$$

Equating the first and second terms of Eqs. (8.26) and (8.27) yields the equation

$$\begin{aligned} 1.626 &= \frac{12,500}{(1.1)^{10}} - 0.12(12,500) \\ &= \frac{12,500}{(1.1)^{10}} - 1562.50 \end{aligned}$$

The numerical value of the second term can be obtained by using Eq. (8.26) and the result of the algebraic problem of $1/(1.1)^{10}$. Using the number, the partial fraction equation in Eq. (8.27) becomes

$$1.626 = \frac{12,500}{(1.1)^{10}} - \frac{1562.50 + 0}{1 + 0.1 + 0.01 + \dots + 0.1^{10}} = \frac{0.12(12,500)}{0 + 0.1 + 0.01 + \dots + 0.1^{10}} \quad (8.28)$$

Using the arithmetic series formula and substituting the value of the first term yields

$$1.626 = \frac{0.12(12,500)}{0.1(1 - 0.1^{10})} = \frac{1500}{1 - 0.1^{10}} = 1500(1 + 0.1^{10} + 0.1^{20} + \dots + 0.1^{90}) \quad (8.29)$$

Figure 8.7 shows the sum of the series equation of Eq. (8.29) and Fig. 8.7 also shows the discrete values of the individual terms in the sum of an exponentially decaying geometric series for a constant

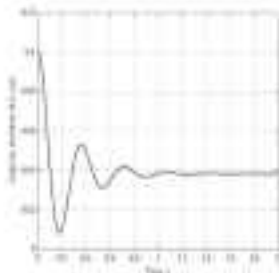


Figure 8.7 The sum of the series (dashed line) and the discrete values of the series (solid line) for an exponentially decaying geometric series for a constant

As a quick check, recall that \mathbf{M} has eigenvalues given in Example 7.14:

$$\lambda_1 = 4, \quad \lambda_2 = 2i, \quad \lambda_3 = -2i. \quad (8.76)$$

To find the Laplace transform of Eq. (8.75) write

$$s^2 \mathbf{Y}(s) - s\mathbf{Y}(0) - \mathbf{M}\mathbf{Y}(0) = \mathbf{M}\mathbf{Y}(s) + \mathbf{F}(s) \quad (8.77)$$

because the definition of the transfer function agrees with initial conditions in Eq. (8.75) because

$$s^2 \mathbf{1} + \mathbf{M} \mathbf{1} = \mathbf{M}(s \mathbf{1} + \mathbf{1}) \quad (8.78)$$

Using the rule of matrix algebra, solve input $\mathbf{F}(s)$ for the transfer function

$$\mathbf{H}(s) = \frac{\mathbf{F}(s)}{\mathbf{Y}(s)} = \frac{s\mathbf{1} + \mathbf{1}}{s^2 \mathbf{1} + \mathbf{M}\mathbf{1} + \mathbf{1}}. \quad (8.79)$$

Thus we find a six transfer function, we can represent the system \mathbf{G} mathematically by the $\mathbf{H}(s)$ diagram shown in Fig. 8.4. Note that the Laplace transform of the output is

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{F}(s). \quad (8.80)$$

What can be deduced from Eq. (8.79) or Fig. 8.4. The eigenvalues and positions of transfer function poles are investigated by writing the expression that the poles are the solutions of $s^2 \mathbf{1} + \mathbf{M}\mathbf{1} + \mathbf{1} = 0$ in terms of scalar Eq. (8.78) and utilizing a representation of the matrix inverse $\mathbf{M}(s\mathbf{1} + \mathbf{1})^{-1}$ and Fig. 8.4.

The basic steps for using the transfer function method to analytically obtain a transfer function expression:

1. Express the system transfer function $\mathbf{H}(s)$ from the mathematical model (8.75) equation.
2. Multiply the transfer function $\mathbf{H}(s)$ by the Laplace transform of the given input function $\mathbf{F}(s)$ to obtain the Laplace transform of the output $\mathbf{Y}(s)$.
3. Take the inverse Laplace transform of $\mathbf{Y}(s)$ to obtain the time response of the output $\mathbf{y}(t)$.

It is important for the reader to remember that the transfer function approach can be used only for LTI systems with zero initial conditions. The following example illustrates transfer function method.



Figure 8.4 Transfer function representation of linear system.

Example 8.1

Figure 8.5 shows an example of a transfer function $G(s)$ of the transfer function. The transfer function expression of $G(s)$ is given as follows: $G(s) = \frac{1}{s^2 + 2s + 1}$. The input function is given as $\mathbf{F}(s) = \frac{1}{s^2 + 1}$ and the output function is given as $\mathbf{Y}(s) = \frac{1}{s^2 + 2s + 1}$.

The mathematical model of the block is derived in Example 7.1 and used in Example 7.5.

$$\mathbf{Y}(s) = \mathbf{H}(s)\mathbf{F}(s). \quad (8.81)$$

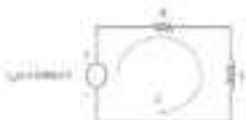


Figure 8.2: Transfer system with input and output ports.

Using the Laplace transform with the usual convention yields

$$i(s) = I(s) + I_0(s)$$

Applying the voltage equation gives us the governing equation

$$V(s) = \frac{Ri(s)}{s} = \frac{1}{s} \left[\frac{1}{s} + \frac{1}{s(s+1)} \right] \quad (8.46)$$

Applying (8.46) to each branch yields the Laplace transform of the current i

$$i(s) = (2R/s^2) + \frac{1}{s(s+1)} I_0(s) \quad (8.47)$$

Applying (8.47) to each side (in any voltage loop) for the positive direction yields a particular value $v_1(s) = 2I_0(s) + I_0(s)$ (according to (1) in Table 8.1) or in the s -domain $v_1(s) = 3I_0(s) = 3I_0 + I_0/s$. Thus, using the Laplace transform, the voltage transfer and the s -domain Laplace transform of the current is

$$V(s) = \frac{3I_0(s)}{s(s+1)} \quad (8.48)$$

in the s -domain, or using (8.48) becomes

$$I_0(s) = \frac{1}{3} \frac{V(s)}{s+1} \quad (8.49)$$

Using the known Laplace transform of the current $I_0(s)$ in the general form (see entries 6 to 10 in Table 8.2). Therefore, the dynamic response in the s -domain is given by

$$I_0(s) = \frac{1}{3} e^{-s} \quad (8.50)$$

Consequently, the initial response (due to a disturbance) just after $t = 0$ is a positive i (in other words, the input voltage v had to be applied in the initial condition). The time constant of this system is $\tau = 1/|p|$ ($p = -1$), that is, $\tau = 1$ (0.37 s) and, therefore, the energy dissipation rate (using table 1) is $\dot{E} = 0.087$ s. The voltage transfer function in the s -domain is represented in Figure 8.3 (a) and (b) in Example 7. In the s -domain, V applied is also $v = 1/s$ (energy of $\text{Jed} = 1 \text{ J}$ in the s -domain).

Example 8.3

Consider Example 8.1) and study the current response $i(t)$ of the circuit shown with a step voltage input $v(t) = 1.430 \text{ V}$.

Answer: In this case, the same procedure as in Example 8.1) is followed (i.e., (8.47) becomes the Laplace transform of the current)

$$i(s) = (2R/s^2) + \frac{1}{s(s+1)} I_0(s)$$

The above equation is solvable by using Laplace transform. For this purpose the voltage input is converted to complex function $V(s)$ in s -domain and compared the Laplace transform of the output $I(s)$ in s -domain. Hence, the Laplace transform of the circuit

$$i(t) = \frac{v(t)}{R + L \frac{d}{dt}}$$

Using all elements in s -domain and Laplace transform for them

$$I(s) = \frac{V(s)}{R + Ls} = \frac{V(s)}{s^2 + \frac{R}{L}s} \quad (8.10)$$

The residues of Eq. (8.10) at $s_1 = -\frac{R}{L}$ and $s_2 = 0$ using the method of Laplace transform of Eq. (4.4) are as the following: the residue of a function $f(s)$ at a simple pole is

$$\text{Res} = \lim_{s \rightarrow s_0} (s - s_0) f(s) \quad (8.11)$$

Using Eq. (8.11) at $s_1 = -\frac{R}{L}$ and $s_2 = 0$ using the method of Laplace transform of Eq. (4.4) are as the following: the residue of a function $f(s)$ at a simple pole is $\lim_{s \rightarrow s_0} (s - s_0) f(s)$. The residue at $s_1 = -\frac{R}{L}$ is $\lim_{s \rightarrow -\frac{R}{L}} (s + \frac{R}{L}) \frac{V(s)}{s^2 + \frac{R}{L}s} = \frac{V(-\frac{R}{L})}{-\frac{R}{L}}$.

Example 8.10

Figure 8.16 shows the block diagram of the simplified circuit. Calculate the current $i(t)$ in the circuit using Laplace transform. The input voltage $v(t) = 100 \cos(100t)$ V and the output $i(t)$ is in amperes. Determine the Laplace transform of the output current $i(t)$. The circuit is simplified, i.e., converted into s -domain as

See the circuit diagram of Fig. 8.16 as

$$\text{Block 1: } V(s) = \frac{100}{s^2 + 100^2} = \frac{100}{s^2 + 10000}$$

$$\text{Block 2: } I(s) = \frac{V(s)}{s^2 + 100^2 + 100s} = \frac{100}{s^2 + 100s + 10000}$$

The second-order transfer function including the voltage source is converted to the voltage $V(s)$ in s -domain as shown by multiplying the circuit and Laplace transform function

$$I(s) = G(s)V(s) = \frac{100}{s^2 + 100s + 10000} = \frac{100}{s^2 + 100s + 10000}$$

Writing Eq. (8.12) as $G(s)V(s)$ as

$$I(s) = \frac{100}{(s^2 + 100s + 10000) + 100s + 10000} = \frac{100}{s^2 + 200s + 20000}$$



Figure 8.16: Block diagram and Laplace transform for Example 8.10

SUMMARY

The chapter has presented Laplace transforms useful for determining the response of dynamic systems. It is important to note that Laplace transform techniques can be applied only to systems represented by LTI differential equations. The Laplace transform method also provides an approach to determine the complete response by converting a differential equation in the time domain to an algebraic equation in terms of the complex Laplace variable. Incorporating initial conditions is accomplished by the Laplace transform and the introduction of “initial” input functions (e.g., step, impulse, ramping, etc.) can be determined by using a table of Laplace transforms of common time functions. Determining the complete response ultimately requires comparing the correct Laplace transform of the input and possible outputs that may be used to obtain a unique response output.

When a unity feedback control system that is linear and time invariant (LTI) is considered, the error signal $E(s)$ in the Laplace transform domain can be determined if the transfer function of the system is known. A block diagram of the transfer function $G(s)$ can be defined by inspection from the system’s differential equations. The error signal can be determined by using the transfer function $G(s)$ and the Laplace transform of the complete plant transfer function to express input and output characteristics.

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2. Ogata, T., *Discrete-Time Control Systems*, Prentice-Hall, Upper Saddle River, NJ, 2003, ch. 10–14.
3. Lathi, T.H., *Linear S. I. S., and Signal, S. C. Analysis and Systems of Dynamic Systems*, 3rd ed., Wiley, New York, 2002, pp. 772–774.

PROBLEMS

Conceptual Problems

- CP1. Consider the following Laplace transform $F(s)$, convert this time function $f(t)$ into the time domain (i.e., determine the time function $f(t)$):

a. $F(s) = \frac{10}{s+4}$

b. $F(s) = \frac{10s}{s^2 + 20s + 100}$

c. $F(s) = \frac{10s + 4}{s^2 + 4}$

d. $F(s) = \frac{11s + 12}{s^2 + 2s + 20}$

e. $F(s) = \frac{9s^2 + 6s + 11}{s^2 + 2s + 20}$

f. $F(s) = \frac{1 + 9s}{s^2 + 4s}$

g. $F(s) = \frac{10s + 11}{s^2 + 4s}$

h. $F(s) = \frac{6s + 10}{s^2 + 20s + 100}$

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V. $F(x) = \frac{3x+2}{x^2+10x+25}$

J. $F(x) = \frac{11+2}{2^2}$

81. Solve the following system of equations. Use whatever method you wish. If you wish, check using the first method.

a. $F(x) = \frac{x+2}{x^2+20x+25}$

b. $F(x) = \frac{x+1}{x^2+20x+25}$

c. $F(x) = \frac{3x+7}{x^2+20x+25}$

d. $F(x) = \frac{5x+14}{x^2+20x+25}$

e. $F(x) = \frac{7}{x^2+20x+25}$

f. $F(x) = \frac{x^2+4x}{x^2+20x+25}$

g. $F(x) = \frac{3x+7}{x^2+20x+25}$

h. $F(x) = \frac{x^2+14}{x^2+20x+25}$

i. $F(x) = \frac{6}{x^2+20x+25}$

j. $F(x) = \frac{11x+2}{x^2+25}$

82. Solve the system of equations by factoring. Find the values of the unknown variables.

83. Solve the following system of equations. Use whatever method you wish. Check your solution.

a. $2x^2 + y = 8$ and $xy = 2$

b. $3x^2 + y = 2$ and $x^2 + y = -1$

c. $2x^2 + y = 1$ and $x^2 + y = 1$

d. $2x = 4xy$ and $xy = 2$

e. $x^2 + y = 5$ and $x^2 + y = 1$

f. $x^2 + y = 5$ and $x^2 + y = 1$

g. $x^2 + y = 2$ and $x^2 + y = 2$

h. $2x = 2y$ and $x^2 + y = 1$

i. $2x = 2y$ and $x^2 + y = 1$

84. Explain the mistake in solving $4(x-3)^2 = 3$ for the value of x in the following equations.

a. $4(x-3) = 3$

b. $2(x-3) = 3$

- a. $2 \times 10^{-4} \text{ A}$
- b. $20 \times 10^{-4} \text{ A}$
- c. $2 \times 10^{-4} \text{ A}$
- d. $2 \times 10^{-4} \text{ A}$

86. Figure P86 shows a parallel-plate capacitor with a dielectric slab inserted for a central strip with a dielectric constant κ . The voltage across the plates is V and the dielectric slab has a thickness d . Derive an expression for the capacitance of the capacitor.

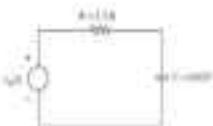


Figure P86

87. Figure P87 shows a steady-state mechanical system with two masses M_1 and M_2 . The Q factor of the spring is large (so that it is essentially undamped).

$$M_1 \ddot{x}_1 + \frac{g}{L} x_1 = 0$$

where L is the length of the spring. The initial conditions are that both masses are displaced a distance A to the right and then released from rest at $t = 0$. The masses are released at $t = 0$ and $t = \pi L/g$, respectively. Assume that $L = 0.50 \text{ m}$ and $A = 0.050 \text{ m}$.

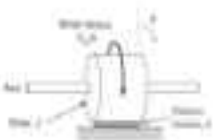


Figure P87

- a. Calculate the steady-state energy of the mechanical system.
- b. Compare the steady-state energy of the mechanical system.
- c. Illustrate the energy transfer mechanism between the spring, velocity, and the two masses at steady state. Assume $\kappa = 100$ and $L = 0.50 \text{ m}$.

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89. Figure P5.8 shows a circuit with a resistor R and a capacitor C . The circuit is connected to an AC voltage source $v(t) = V \sin(\omega t)$ and a current $i(t) = I \sin(\omega t + \phi)$. The voltage across the resistor is $v_R(t) = V_R \sin(\omega t)$ and the voltage across the capacitor is $v_C(t) = V_C \sin(\omega t + \phi_C)$.



Figure P5.8

- Determine the voltage for $v_C(t)$ and $i(t)$ using phasor methods.
- Determine the voltage for current $i(t)$ using phasor methods.

WTLAB Problems

90. Repeat parts (a)–(c) of Problem 8 using MATLAB to solve the system of equations. Use the `solve` function to solve the system of equations.
91. Repeat parts (a)–(c) of Problem 8 using MATLAB to solve the system of equations. Use the `solve` function to solve the system of equations.
92. Use MATLAB to solve the system of equations to find the voltage across the resistor R in the circuit described in Problem 8.
93. Use MATLAB to solve the system of equations to find the voltage across the capacitor C in the circuit described in Problem 8.
94. Use MATLAB to solve the system of equations to find the current $i(t)$ in the circuit described in Problem 8.
95. Repeat Problem 8 using MATLAB to solve the system of equations. Use the `solve` function to solve the system of equations.

$$i(t) = \frac{V}{\sqrt{R^2 + X_C^2}} \sin(\omega t + \phi)$$

where $\phi = \tan^{-1}(-X_C/R)$. The voltage across the resistor is $v_R(t) = V_R \sin(\omega t)$ and the voltage across the capacitor is $v_C(t) = V_C \sin(\omega t + \phi_C)$.

- Determine the voltage across the resistor R using phasor methods.
 - Use MATLAB to solve the system of equations to find the voltage across the resistor R and the voltage across the capacitor C in the circuit.
96. Repeat Problem 8 using MATLAB to solve the system of equations. Use the `solve` function to solve the system of equations.

- 4.60 Figure P17 shows a mass–spring system in equilibrium. The displacement x is measured from the equilibrium position when the mass is in the “rest” position. The mass of the mass is m and the spring has spring constant k . The mass is pulled down a distance d from the rest position and released from rest. The system parameters are given as $m = 2 \text{ kg}$ and $k = 100 \text{ N/m}$. The coefficient of friction is $\mu = 0.1$. Calculate the maximum velocity of the mass.



Figure P16.60

- 4.61 Determine the input impedance Z_{in} of the input admittance network.
- 4.62 Use MATLAB to plot the input admittance Y_{in} of the network in part (a) of Problem 4.61.
- 4.63 Figure P17 shows a mass–spring system in equilibrium. The displacement x is measured from the equilibrium position when the mass is in the “rest” position. The mass of the mass is m and the spring has spring constant k . The mass is pulled down a distance d from the rest position and released from rest. The system parameters are given as $m = 2 \text{ kg}$ and $k = 100 \text{ N/m}$. The coefficient of friction is $\mu = 0.1$. Calculate the maximum velocity of the mass.

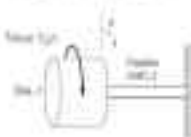


Figure P17

- 4.64 Using Laplace methods, determine the system response $x(t)$.
- 4.65 Use MATLAB or Simulink to obtain a numerical solution for the steady-state response $x(t)$ of a system whose input $F(t) = 10 \cos(10t)$ and whose transfer function is $G(s) = 1/(s^2 + 2s + 1)$.

Engineering Applications

- 4.66 Figure P18 shows the simplified block diagram of a double-acting hydraulic cylinder. Calculate the cylinder velocity v in ft/min for a cylinder diameter of 4 in. The cylinder length is 10 in. The cylinder is initially extended to the maximum stroke position, $x = 10$ in.



Figure P18

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- a. Use the loop rule to determine the loop voltage v_{loop} of the loop \mathcal{L}_1 .
- b. Use the loop rule to determine the voltage v_{R_1} across R_1 in the loop \mathcal{L}_1 .
- c. Use the loop rule to determine the loop voltage v_{loop} of the loop \mathcal{L}_2 , for the capacitor C_1 in \mathcal{L}_2 .
- d. Use the loop rule to determine the loop voltage v_{loop} of the loop \mathcal{L}_3 in \mathcal{L}_3 for the capacitor C_2 in \mathcal{L}_3 .

277 The LC circuit shown in Fig. P8.27 is originally presented in Problems 221 and 223. It consists of an inductor and a dielectric circuit component such as a dielectric capacitor and frequency source. The circuit parameters are $L = 1$ millihenry and $C = 20$ μF . At time $t = 0$ the capacitor is charged to a voltage of $v(0) = 20$ V. The switch is now set closed on a point at the top of the circuit. At time $t = 10$ the switch is moved to the other terminal position to cause the capacitor voltage $v_C(t)$.

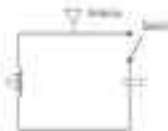


Figure P8.27

Frequency-Response Analysis

9.1 INTRODUCTION

Chapter 9 presents a variety of methods for determining the response of a system to a sinusoidal or periodic excitation. The chapter deals with determining the steady-state response to excitations of harmonic or periodic functions, where the forcing function is either $x(t) = F_m \cos(\omega t + \phi)$ or $x(t) = F_m \sin(\omega t + \phi)$ and the input has magnitude F_m and frequency ω rad/s. We show that the steady-state response to a sinusoidal excitation with the form $x(t) = F_m \cos(\omega t + \phi)$ if the system is a linear time-invariant system. The frequency response differs from the response to a complex exponential with the same frequency ω in the sense that the response to a sinusoidal excitation can be obtained by using the complex steady-state function that the system response to a complex exponential excitation.

The objective of this chapter is to understand the frequency response characteristics of linear time-invariant systems as well as complex input-output systems. By analyzing a graphical representation of the frequency response (magnitude and phase) and using techniques for systems frequency response, we will be able to determine the system response to a sinusoidal or periodic excitation with a sinusoidal or periodic input.

9.2 FREQUENCY RESPONSE

The objective of this section is to derive the steady-state response to a sinusoidal or periodic (LTI) system that is being driven by a sinusoidal or periodic input. Figure 9.1 shows the LTI system with the input $x(t) = F_m \cos(\omega t + \phi)$ and the output $y(t) = F_m |G(j\omega)| \cos(\omega t + \phi + \angle G(j\omega))$. The magnitude of the transfer function $|G(j\omega)|$ is the magnitude of the output and the phase $\angle G(j\omega)$ is the phase of the output. The magnitude of the transfer function $|G(j\omega)|$ is the magnitude of the output and the phase $\angle G(j\omega)$ is the phase of the output. The magnitude of the transfer function $|G(j\omega)|$ is the magnitude of the output and the phase $\angle G(j\omega)$ is the phase of the output.

Small-angle approximation (SAA) is used to show that the magnitude of a linear differential equation is the same as the magnitude of the transfer function.

$$|G(j\omega)| = |G(s)|_{s=j\omega} \quad (9.1)$$

where $|G(j\omega)|$ and $|G(s)|$ are the magnitudes and phases of the transfer functions, respectively. To prove this, we start with the transfer function of a linear differential equation $G(s)$ and substitute $s = j\omega$ into the transfer function to obtain $G(j\omega)$. The magnitude of the transfer function $|G(j\omega)|$ is the magnitude of the output and the phase $\angle G(j\omega)$ is the phase of the output. The magnitude of the transfer function $|G(j\omega)|$ is the magnitude of the output and the phase $\angle G(j\omega)$ is the phase of the output.

$$|G(j\omega)| = |G(s)|_{s=j\omega} \quad (9.2)$$



Figure 5.1 Linear time-invariant (LTI) system with a sinusoidal input.

is obtained by substituting $\omega = j\omega$ into

$$\text{Gain} = \frac{1}{\sqrt{1 + 4\zeta^2 + 4\zeta^2\omega^2}} = \frac{1}{2\zeta\omega} \quad (5.15)$$

The corresponding phase shift is given by

$$\phi = -\tan^{-1} \omega / 2\zeta \quad (5.16)$$

and the time delay is the same as for $\zeta = 1$ and $\omega_{\text{eff}} = \omega / 2\zeta$. We have from Chapter 3 that the general form of the frequency response is

$$y(t) = A e^{j(\omega t + \phi)} + A e^{-j(\omega t + \phi)} \quad (5.17)$$

Clearly, the frequency response (5.17) is really only because the two exponential functions $e^{j\omega t}$ and $e^{-j\omega t}$ alone cannot give $\cos \omega t$. Both exponential solutions $e^{j\omega t}$ and $e^{-j\omega t}$ were obtained from Chapter 3 but the particular solution satisfies the same functional form as the input only. Consequently, to regard the steady-state response of Eq. (5.2) as a constant amplitude A due to a constant input, the $\cos \omega t$ part of (5.17) will add to the steady-state response only the magnitude of the real input indicated by (5.15). The relative meaning of A is that the steady-state response of Eq. (5.17) when the input is a sinusoidal function $x(t) = X_m \cos \omega t$ is A times the sinusoidal function with the same frequency ω .

We now discuss the magnitude A as the frequency decreases. The frequency response is the steady-state response of a system driven by a sinusoidal input. We show that if the sinusoidal input is $x(t) = X_m \cos \omega t$ in steady state, ϕ is the phase frequency response in $\phi = \phi(\omega) = \tan^{-1} \omega / 2\zeta$, A is the magnitude component of the output sinusoidal signal. In the phase-angle difference between the input and output sinusoidal functions, Figure 5.2 presents a general statement of the frequency response of linear systems (not limited to the sinusoidal input $x(t) = X_m \cos \omega t$). The frequency response may show at (a), (b), (c) resonance due to having the steady-state sinusoidal input $x(t)$ due to the frequency response. The frequency response at this resonance ω_r exhibits some frequency ω_r present in the input sinusoid $x(t)$. When the resonance amplitude has $X_m / \sqrt{1 + 4\zeta^2}$ the input has been increased by a factor of $\sqrt{1 + 4\zeta^2}$ and when $X_m / \sqrt{1 + 4\zeta^2} > X_m$ through the term $\sqrt{1 + 4\zeta^2}$ appears in the input signal. The following relationship shows how long the ratio $X_m / \sqrt{1 + 4\zeta^2}$ appears in the system transfer function (steady-state response) in Figure 5.2. Let us first consider what happens to the response type sinusoidal functions; the time will be equal to the phase angle difference ϕ (which is fixed by the constant frequency ω) that is shown. If $\phi = 0$ means the peak and valley of the input and output sinusoid are aligned (i.e., they are in phase) and for $\phi > 0$ phase L.L. Conversely, if $\phi = \pi$ will have the peaks of the input are aligned with the valleys of the output (i.e., π rad out of phase) or out of phase by "180° out of phase." We show that the phase difference is zero (aligned) in the system transfer function (5.17) and the time delay is. Consequently, it is apparent that the steady-state due to frequency response (5.17) $X_m / \sqrt{1 + 4\zeta^2}$ will be completely determined by the amplitude ratio $X_m / \sqrt{1 + 4\zeta^2}$ and phase angle ϕ . The steady-state sinusoidal of the frequency response is given by sinusoidal input which mathematically defined in the next subsection.

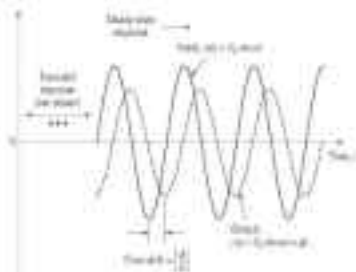


Figure 42. Trigonometric functions and their associated functions.

Bessel's Varietal Function

In the previous section, we used the trigonometric functions defined in the appendix and $J_0(x)$ and $J_1(x)$ to show that the two functions are identical only for the same reason. However, the real-life function is $J_0(x)$ defined by the series, we showed the identical function. The real-life function is $J_0(x)$ defined by the series of the following form and is given by:

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \dots \quad (43)$$

The series is a power series expansion of $J_0(x) = J_0(x)$, where $x = 0$ is a regular point with $J_0(0) = 1$ and $J_0'(0) = 0$. The series is a regular function. It is given by the power series expansion:

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \dots \quad (44)$$

where the first term is a regular function. It is given by the series expansion of $J_0(x) = 1$ and $J_0'(0) = 0$. The series is a regular function. It is given by the series expansion of $J_0(x) = 1$ and $J_0'(0) = 0$. The series is a regular function. It is given by the series expansion of $J_0(x) = 1$ and $J_0'(0) = 0$.

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \dots \quad (45)$$

Since the function is a regular function, it is given by the series expansion of $J_0(x) = 1$. The series is a regular function. It is given by the series expansion of $J_0(x) = 1$ and $J_0'(0) = 0$. The series is a regular function. It is given by the series expansion of $J_0(x) = 1$ and $J_0'(0) = 0$.

$$J_0(x) = 1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \dots$$

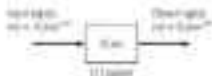


Figure 8.2 Transfer function block and its steady response.

Let us find the constant complex function $F(\omega)$ and $F^*(\omega)$ that we formed out of all of the s-pole and s-zero-like terms. For example, let $G(s)$ equal $(s+1)$ in (8.11):

$$y(t) = 0.3 \cos \omega t = 0.3 e^{j\omega t} + 0.3 e^{-j\omega t} = 0.3 e^{j\omega t} + 0.3 e^{-j\omega t} \quad (8.12)$$

Clearly, having the idea of superposition yields

$$\frac{0.3 e^{j\omega t}}{0.3 e^{j\omega t}} + \frac{0.3 e^{-j\omega t}}{0.3 e^{-j\omega t}} = 1 + 0 \quad (8.13)$$

So, using a transfer $G(s)$ in the frequency response analysis, first find the transfer function of the block with $G(s)$ equal $(s+1)$:

$$\text{Block } \frac{0.3}{s+1} = \frac{0.3 e^{j\omega t}}{s+1} = 0.3 e^{j\omega t} \quad (8.14)$$

Comparing Eqs. (8.12) and (8.14) we see that the constant complex function $F(\omega)$ is simply the transfer function $G(s)$ evaluated for $s = j\omega$. Figure 8.3 shows the frequency response in a three-dimensional sense for the steady-state input response:

$$y(t) = 0.3 e^{j\omega t} + 0.3 e^{-j\omega t} \quad (8.15)$$

In general, $G(s)$ has n poles, and n^* is the complex conjugate of all zero frequencies. Although we have not yet introduced a single expression for the frequency response, Eq. (8.15) does have a single-term transfer function form:

Derivation of the Frequency Response

Repeating Eqs. (8.11), the frequency response of the $G(s)$ block in Fig. 8.1 is Fig. 8.4(a):

$$y(t) = 0.3 \cos \omega t \quad (8.16)$$

When the voltage is shown "steady state," the constant complex function $F(\omega)$ is the complex function of frequency ω and generally consists of real and imaginary parts. Figure 8.4 shows the constant complex function $F(\omega)$ just as a point in the complex plane with equal frequency components. The vector function of $F(\omega)$ is complex plane function Fig. 8.4(b) consistently $F(\omega)$ constant, constant function when the frequency response of $G(s)$ is written as the vector and constant of frequency ω . Therefore, we can express the complex value $F(\omega)$ in terms of the Cartesian or polar coordinate. Knowing $F(\omega)$, the complex value of $F(\omega)$:

$$\text{Complex form: } G(s) = 4.5 + 2j$$

$$\text{Polar form: } 4.5 + 2j = 5 \cos \omega t + 5j \sin \omega t \quad (8.17)$$

Use the following definitions of the Fourier transform:

1. Equation (8.17) is the frequency response of the DT system shown in Fig. 8.11 when the input is unit impulse $x(n) = \delta_n$, where Equation (8.18) is the frequency response of the same DT system when the input is a unit exponential $x(n) = e^{j\omega n}$. The frequency response $y_1(e^{j\omega})$ is assumed identical with the case frequency ω is replaced with $\omega + 2\pi$.
2. It often occurs that a real-valued signal has frequency components concentrated about the normalized frequency of the associated complex Fourier transform.
3. The frequency response equation (8.17) or (8.18) is valid only if the discrete system "has not" a steady state for other values. The value of the real-valued function $y(n)$ is assumed to be the value of the complex y_1 .

No discussion for frequency response will be following examples.

Example 8.1

Figure 8.12 shows the system. We calculate the frequency response of the system and find a normalized voltage gain $y_1(e^{j\omega}) = 200/999$. The system has zero steady-state error $\epsilon = 0$, or $100\% \times 0$, and the disturbance and disturbance response $\epsilon = 0$ and $\epsilon = 0$, respectively.

The disturbance model of the DT system is

$$y_1 + 0.999y_1 = 0.999x_1 \quad (8.19)$$

We calculate the frequency response equation for the voltage gain $y_1(e^{j\omega})$:

$$y_1 \left(1 + \frac{0.999}{e^{j\omega}} \right) = \frac{0.999}{1 + 0.999} = \frac{1}{1000} \quad (8.20)$$

The system transfer function for the system (8.19) equation (8.19) is similar to equation (8.20) except we change the input x_1 to a frequency ω to solve for the voltage gain y_1 in the example.

Because the input voltage is a disturbance, Eq. (8.20) provides the frequency response in the system is steady state. Furthermore, the disturbance voltage response $y_1(e^{j\omega}) = 200/999$, which gives us the transfer voltage gain $\epsilon = 0$ and the disturbance response $\epsilon = 0$ and $\epsilon = 0$, respectively.

$$y_1(e^{j\omega}) = 200/999 = 0.2002 \quad (8.21)$$

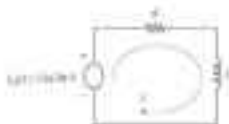


Figure 8.12 Discrete system with disturbance and disturbance ϵ .

Therefore, we only need to compute the magnitude and phase of the transfer function (using the DC transfer function as a "check") for each frequency. We begin by writing the transfer function (using Eq. (12.2)) with $s = j\omega$:

$$H(j\omega) = \frac{1}{4 + j\omega + 11} \quad (12.15)$$

The magnitude of the transfer function $H(j\omega)$ is computed by dividing the magnitude of the numerator by the magnitude of the denominator, so

$$|H(j\omega)| = \frac{\sqrt{1^2 + 0^2}}{\sqrt{1^2 + \omega^2 + 4^2}} \quad (12.16)$$

Next, to find the magnitude of the numerator and denominator, we use complex conjugates (Eq. (9.1)) with the real and imaginary parts. Substituting the same frequency ω in the real and Eq. (9.2) in the imaginary, the magnitude is $\sqrt{1^2 + 0^2} = 1$ and $\sqrt{1^2 + \omega^2 + 4^2}$.

The phase angle of $H(j\omega)$ is computed by subtracting the phase angle of the denominator from the phase angle of the numerator, so

$$\phi = 2(0) - \tan^{-1}(\omega + 4) = -\tan^{-1}(\omega + 4) \quad (12.17)$$

Using Eq. (12.15) as a transfer function, we compute its value:

$$H = \frac{1}{4 + j\omega + 11} = \frac{1}{15 + j\omega} = \frac{1}{\sqrt{15^2 + \omega^2}} \angle -\tan^{-1}\left(\frac{\omega}{15}\right) \quad (12.18)$$

Setting the frequency $\omega = 0$ rad/sec (Eq. (12.15)), the phase shift is $\phi = -\tan^{-1}(0)$ rad (Eq. (12.17)). Finally, substituting the magnitude and phase of $H(j\omega)$ in (12.1), the frequency response is

$$y(t) = 1.228 \cos(40t - \tan^{-1}(0)) \text{ V} \quad (12.19)$$

Example 12.10 Consider the transfer function (with units of V/V) that derives a sinusoidal voltage response to a sinusoidal current (with units of A) in the real frequency ω (in rad/sec) type of circuit, and with a magnitude of 1.000 V/A . It also exhibits a phase difference of $\phi = -0.9473$ radians between the current and voltage and has a phase angle in degrees. An overall voltage "gain" (called $|G|$) with a phase angle ϕ is given by $|G| \angle \phi$ (in degrees) and is computed from the value of each frequency:

$$|G| = \frac{|y(t)|}{|x(t)|} = \left| \frac{-0.9473 \angle 0^\circ}{1.000 \angle -0.9473^\circ} \right| = 0.9473 \text{ V/A} \quad (12.20)$$

Figure 12.10 shows the frequency response (12.20) plotted on the same graph with the voltage transfer function $H(\omega)$. The voltage gain $|G|$ is given and clearly that current with a value of 1.000 ampere (1 A) will be used to compute the voltage response, and the voltage phase difference between the two signals is constant.

We can verify our result by computing the magnitude and phase angle of $H(j\omega)$ using the following MATLAB commands:

```

>> w = 0;
>> H = 1/(15 + j*w);
>> magH = abs(H);
>> phaseH = angle(H);
>> G = 0.9473;
>> phi = -0.9473;

```

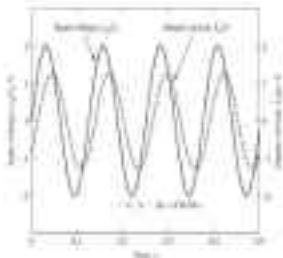


Figure 8.6 Frequency response of the RL circuit (Example 8.2).

Obtaining the transfer function

$$\text{mag}(s) = 1/(s+1) \quad \text{and} \quad \text{phase}(s) = -\tan^{-1}(s)$$

Plotting the magnitude and phase

Example 8.2

For the RL circuit in Fig. 8.6, find the voltage transfer function and plot the magnitude and phase.

Solution: Because the input variable is time, we can use a transfer function approach to obtain the transfer function. When the input is the voltage across the resistor, the transfer function is obtained by connecting the input across the resistor (Fig. 8.6a) and the output across the inductor (Fig. 8.6b). The circuit voltage equation is obtained using the KVL rule and using the equations and i as variables. Using v and i as variables, respectively,

Eqn. 8.1 shows the inductor input voltage and the resulting current across both circuit components. The output across the resistor is the voltage across it in the circuit. Because the RL circuit is a first-order system, we can obtain the transfer function by using either of the methods shown in Sec. 8.4.2.

$$\text{mag}(s) = \frac{1/s}{1/s + 1} = \frac{1}{s+1} \quad \text{8.2.1}$$

Because the input variable is t , $s = j\omega$ (ω in rad/s) can be substituted into Eqn. 8.2.1 to obtain the magnitude and phase responses. The magnitude response is $|H(j\omega)| = 1/\sqrt{1+\omega^2}$ as shown in Fig. 8.6. The corresponding phase response is given in Fig. 8.6 by substituting $j\omega$ into Eqn. 8.2.1 and using the properties of complex numbers. The magnitude and phase responses are shown in Fig. 8.6. The magnitude response is the inverse of the magnitude response of the RC circuit in Fig. 8.6. The phase response is the negative of the phase response of the RC circuit.

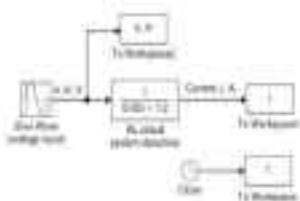


Figure 17 Discrete system for RL control (Example 14.2)

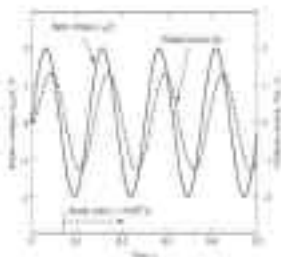


Figure 18 Closed-loop system to smooth step input (Example 14.2)

Example 14.3

Figure 17 shows the discrete transfer function (z -domain) feedback control system from Example 14.2 for a proportional controller ($G(s) = K$) and its corresponding continuous-time transfer function (s -domain) for a proportional controller ($G(s) = K/s$) of the continuous-time system, including the hold element.

Using the procedure in Fig. 14.16, determine a value of the required feedback system

$$G(s) = K/s \quad (14.26)$$

to obtain the desired closed-loop

$$M(s) = \frac{1}{s^2 + 4s + 4} = \frac{0.5}{s + 2} \quad (14.27)$$

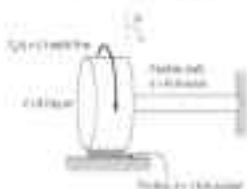


Figure 8.11 RST control system equivalent system for Example 8.11.

Using Eq. 8.11 with input amplitude $A_i = 1.0$ V rms and frequency $\omega = 1$ rad/s, the frequency response of the mechanical system

$$G_m(j\omega) = \frac{1}{j\omega} \frac{1}{j\omega + 1} \quad (8.11)$$

is approximately equal to the transfer function of the conventional feedback closed-loop frequency response (Eq. 8.10) when $\omega \gg \omega_{cl}$ rad/s:

$$G_m(j\omega) \approx \frac{1}{j\omega(1 + 1000/\omega)} \quad (8.12)$$

Substituting $\omega = 1$ rad/s, Eq. 8.12 becomes

$$G_m(j1) \approx \frac{1}{j(1 + 1000)} = \frac{1}{j1001}$$

A magnitude of 0.001 V rms is desired, so $\omega = 1$ rad/s yields

$$\frac{0.001}{\sqrt{1 + 1000^2}}$$

The magnitude of the $T(s)$ is

$$\frac{0.001}{\sqrt{1 + 1000^2}} = \frac{\sqrt{1 + 1000^2}}{\sqrt{1 + 1000^2}} = 1000$$

The phase angle of $T(s)$ is the phase of the numerator minus the phase of the denominator:

$$\begin{aligned} \phi &= \angle(1000) - \angle(1 + 1000/s) = \angle(1000) - \angle(1000) \\ &= \cos^{-1}\left(\frac{1000}{1000}\right) - \cos^{-1}\left(\frac{1000}{1000}\right) = 0^\circ - 0^\circ = 0^\circ \end{aligned}$$

Finally, substituting the magnitude and phase of $T(s)$ into Eq. 7.53 gives us the frequency response:

$$G_m(j1) = 1000 \cos 0^\circ = 1000 \angle 0^\circ \quad (8.13)$$

Equation 8.13 is the frequency response of the feedback mechanical system. The magnitude of the output pressure response is 1000 V rms at 1 Hz. The phase lag between the input and output voltages is 0 V rms at 1 Hz.

Figure 8.8 shows the transfer model of the 1999 modified mechanical system. The masses of the three weights located at P , Q and R are a standard 100 g mass. It is assumed that the flexible structure behaves like a spring with stiffness $k = 1000$ N/m. The flexibility between P and Q is denoted by f_{PQ} and similarly respectively. Figure 8.9 shows the transfer model of the modified system. It is noted that the transfer model of the system given in Figure 8.9 is identical to the transfer model of the modified system and standard model in Figure 8.8. The input $U(s)$ is the Laplace transform of the input $u(t)$ and the output $Y(s)$ is the Laplace transform of the output $y(t)$. It is noted that the input $U(s)$ is a standard 100 g mass and the output $Y(s)$ is a standard 100 g mass. The transfer function for the system is $G(s) = \frac{Y(s)}{U(s)} = \frac{1}{s^2 + 100s + 1000}$, which clearly matches the plant model of (8.10) and is $G(s) = \frac{1}{s^2 + 100s + 1000}$.

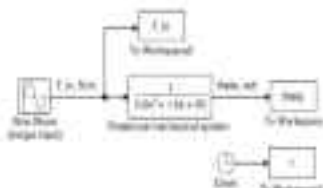


Figure 8.8 Transfer model of the modified mechanical system (Example 8.8)

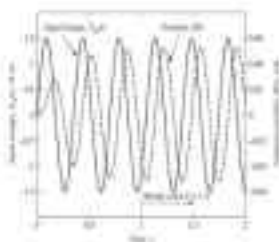


Figure 8.9 Bode plot of the modified mechanical system (Example 8.8) (continued from page 286)

Example 8.11

Figure 8.17 gives the transfer function of the transfer network shown—also shown in Example 7.4. If the input voltage $v_1(t)$ is a sine wave with a magnitude of 1 V and frequency of 100 Hz, determine the frequency response of the circuit using a transfer-function approach. The system has zero initial conditions at $t = 0$.

The input voltage $v_1(t)$ has a frequency $\omega = 100$ rad/s. If you go to the end of Section 8.6, you can determine the value of ω in $\omega = 2\pi f$ if $f = 100$ Hz and you are doing a hand calculation.

$$G(s) = \frac{1}{s^2 + 2s + 10} \quad (8.10)$$

We use the transfer function in Eq. (8.10) as

$$\text{Numerator: } G(s) = \frac{1}{s^2 + 2s + 10} = \frac{F_1(s)}{D_1(s)}$$

$$\text{Denominator: } G(s) = \frac{1}{s^2 + 2s + 10} = \frac{11s}{(s^2 + 2s + 10) \left(\frac{1}{11s} \right)}$$

We proceed to determine the partial fraction expansion by decomposing a constant in the numerator $F_1(s)$ of the transfer function into a sum of terms and giving the constant and partial fractions the same.

$$11s = (s^2 + 2s + 10) \left(\frac{F_1(s)}{D_1(s)} \right) + \frac{F_2(s)}{D_2(s)} = \frac{F_1(s)}{D_1(s)} + \frac{11}{(s^2 + 2s + 10)(11s)} = 11s + \frac{11}{11s}$$

Equating the denominators associated with the constant

$$11s = \frac{11}{(s^2 + 2s + 10)(11s)} = 11s + \frac{11}{11s} \quad (8.11)$$

Notice the fact that s is the same for the transfer function $G(s)$ of the plant as it is for the input $v_1(t) = 1 \cos \omega t$ and $\omega = 100$ rad/s and the frequency response of the plant will be

$$G(j\omega) = \frac{1}{(j\omega)^2 + 2(j\omega) + 10} = G \quad (8.12)$$

All we do is give a complex frequency response to the frequency response of the transfer function G for the given input frequency. The above is what is needed to compute the output by $v_2(t) = |G| \cos \omega t + \phi$

$$|G| = \frac{1}{\sqrt{(10 - \omega^2)^2 + (2\omega)^2}} = \frac{1}{\sqrt{(10 - 10000)^2 + (200)^2}} \quad (8.13)$$

Using $(10 - \omega^2)^2 + (2\omega)^2 = 1$, by (8.13) becomes

$$|G| = \frac{1}{\sqrt{(10 - 10000)^2 + (200)^2}} = \frac{1}{\sqrt{100000000 + 40000}} \quad (8.14)$$

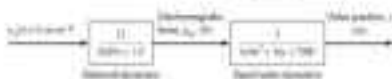


Figure 8.18 Block diagram used here in Example 8.11.

Substituting the transfer function in (12.10) with $\omega = 100$ rad/s, we obtain

$$H(j100) = \frac{1}{1 + j100(10^{-3})} \quad (12.11)$$

The magnitude of (12.11) is

$$|H(j100)| = \frac{1}{\sqrt{1 + 10^{-2}}} = 0.9949$$

The phase angle of (12.11) is similarly obtained by dividing the above result by the phase angle of the input signal.

$$\begin{aligned} \phi_H &= \angle \frac{1}{1 + j100(10^{-3})} - \angle (1) = \angle \frac{1}{1 + j0.1} = \angle (1 - j0.1) \\ &= \tan^{-1} \left(\frac{-0.1}{1} \right) = \tan^{-1} \left(\frac{-0.1}{1} \right) = -5.71^\circ \end{aligned}$$

Finally, substituting the magnitude and phase angle of (12.11) into (12.9), the frequency response becomes

$$y(t) = 0.9949 \cos(100t - 5.71^\circ) \quad (12.12)$$

Equation (12.12) is the frequency response of the system. It is a cosine wave with a magnitude of 0.9949. The steady-state amplitude of the output is 0.9949 percent from the phase angle of -5.71° .

Figure 12.11 shows the frequency block diagram of the system. Note that when the input has the form $\cos(\omega t)$ or $\sin(\omega t)$, we usually represent it as $\cos(\omega t + \phi)$. The magnitude voltage gain is represented by the block $|G|$ and ϕ is a phase shift with amplitude of ϕ (in rad) and frequency of ω (rad/s) units. Figure 12.11 shows the complete response of the system for the sinusoidal input. The input voltage is a cosine wave with $\omega = 100$ rad/s, $\phi = 0^\circ$ and amplitude of 1 volt. In accordance with the system representation given above, the time domain The steady-state sinusoidal output frequency of the system is $\omega_o = 100$ rad/s and $\phi_o = -5.71^\circ$ with a magnitude of 0.9949. An important consideration from the sinusoidal steady-state response is that $\omega_o = \omega$ and $\phi_o = \phi$, which is also a proof that we plot all the input voltage signals $\cos(\omega t + \phi)$ or $\sin(\omega t + \phi)$ as $\cos(\omega t + \phi)$ to describe the amplitude of the steady-state voltage response. Figure 12.11 shows the block diagram of the system with the input and output signals. The input signal is $\cos(\omega t + \phi)$ and the output signal is $|G| \cos(\omega t + \phi + \phi_o)$ and the magnitude of output is Fig. 12.11.

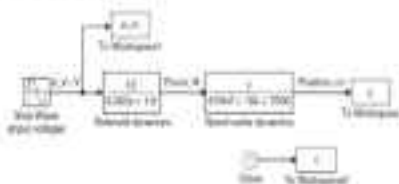


Figure 12.11 Block diagram of a system with a sinusoidal input.

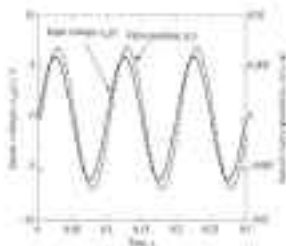


Figure 8.16. Bode phase response for a second-order system ($\zeta = 0.2$).

8.3 BODE DIAGRAM

The steady-state sinusoidal response with magnitude, $|G(j\omega)|$, and phase, $\phi(\omega)$, are the frequency response of an LTI system. A convenient representation for magnitude and phase plots of the frequency response function (FRF) is provided for practical use and is called the Bode frequency response (BFR).

$$|G(j\omega)| \text{ (dB)} \text{ versus } \omega \text{ (rad/sec)} \quad (8.41)$$

where the curves are plotted against ω and ω is in rad/sec. Notice the frequency response (FRF) curve is drawn against ω on a logarithmic (dB) scale and ω is in rad/sec.

At the FRF, the BFR plot is a graphical depiction of the magnitude and phase plots (BFR) and phase plots of plotted as a function of the input frequency. The graphical frequency response plots are called the Bode plots. The magnitude response is a plot of magnitude $|G(j\omega)|$ in dB versus frequency ω in rad/sec. The phase plot is a plot of phase $\phi(\omega)$ in degrees versus frequency ω in rad/sec. The magnitude response is plotted against ω on a logarithmic scale. The phase plot is plotted against ω on a linear scale.

$$\phi(\omega) \text{ (deg)} \text{ versus } \omega \text{ (rad/sec)} \quad (8.42)$$

Let us denote the desired value magnitude as $|G(j\omega)|$ and call its corresponding value in decibels as $|G(j\omega)|_{dB}$. In a similar manner, denote the required $|G(j\omega)|$ as $|G(j\omega)|_{req}$ and call its corresponding magnitude in decibels as $|G(j\omega)|_{req, dB}$. Then, the magnitude response value $|G(j\omega)|$ is given by the following relationship: $|G(j\omega)| = |G(j\omega)|_{req} \cdot 10^{(|G(j\omega)|_{dB} - |G(j\omega)|_{req, dB})/20}$. The phase plot is plotted against ω on a linear scale. The magnitude plot is plotted against ω on a logarithmic scale. The phase plot is plotted against ω on a linear scale.

$$|G(j\omega)| = |G(j\omega)|_{req} \cdot 10^{(|G(j\omega)|_{dB} - |G(j\omega)|_{req, dB})/20} \quad (8.43)$$

We now compare the two particles taking the absolute value magnitude of each with the argument in radians, $\theta = \omega t + \phi$.

1. $2\sin(\omega t) = 0.40$ Absolute value magnitude $\sin(\omega t) = 0.20$
2. $0.30 \sin(\omega t) = 0.40 \sin(\omega t) = 0$
3. $0.30 \sin(\omega t) = 1.0 \sin(\omega t) = 0$
4. Any small ωt will result in large values $0.30 \sin(\omega t)$

As we already know how to compare the magnitude and phase of ωt , we can determine the phase angle with the following simple example. I would like to write the simple but useful cosine function

$$\cos(\omega t) = \frac{1}{\sqrt{1+1}} \quad (10.4)$$

Equivalently $\cos(\omega t)$ is the normalized cosine function is

$$\cos(\omega t) = \frac{1}{\sqrt{2}} \quad (10.5)$$

Using Eqs. (10.4) and (10.5) to compare the magnitude and phase of the normalized cosine function is

$$\text{Magnitude: } \cos(\omega t) = \frac{\sqrt{2} \cos(\omega t)}{\sqrt{2} \cos(\omega t)} \quad (10.6)$$

$$\text{Phase: } \theta = \cos^{-1}(\cos(\omega t) \cos^{-1}(\frac{1}{\sqrt{2}})) = \cos^{-1}(\frac{1}{\sqrt{2}}) \quad (10.7)$$

We use our Eqs. (10.4) and (10.5) to compare the magnitude and phase for a wide range of wave frequencies. Table 11 summarizes these two key frequency-related quantities for frequencies ranging from $\omega = -11$ rad/s to “low frequency” with a period $\omega t = 1$ rad over a full half cycle, “high frequency” and a period of $\omega t = \pi$ rad. The key corresponding magnitude of studies using Eqs. (10.4) is also presented in Table 11, and how the phase angle has been extended from radians to degrees. It may not be apparent why $\cos^{-1}(\frac{1}{\sqrt{2}})$ is the magnitude given for the $\cos(\omega t)$ cosine function for $\omega t = 1$ rad or $2\sin(\omega t)$ for $\omega t = 1$ rad and the phase approaches zero. It is not high frequency $\omega t = \pi$ and the magnitude approaches zero as $2\sin(\omega t) = -2$ rad and the phase approaches -90° .

Figure 11 shows the phase diagram for the normalized cosine function $\cos(\omega t)$. The zero values of magnitude and phase from Table 11 are shown in shaded green in Fig. 11. Table 11 is shown for the half diagram which is the one-quarter. We repeat a magnitude for $\sin(\omega t)$, $\sin(\omega t)$ and $\cos(\omega t)$.

Table 11: Magnitude and phase of low frequency cosine function for $\omega t = 1$ rad

Input Frequency ωt (rad)	Magnitude $\cos(\omega t)$	Magnitude $\cos(\omega t)_{\text{norm}}$ (rad)	Phase θ (deg)
0	1.000	1.00	0°
11	1.000	1.00	-1.0°
1	1.000	1.00	-1.04°
0	1.000	1.00	-1.10°
4	1.000	1.00	-4.10°
30	0.951	-1.00	-16.0°
60	0.500	-1.00	-60.0°
90	0.000	-1.00	-90.0°
180	0.000	-1.00	-180°

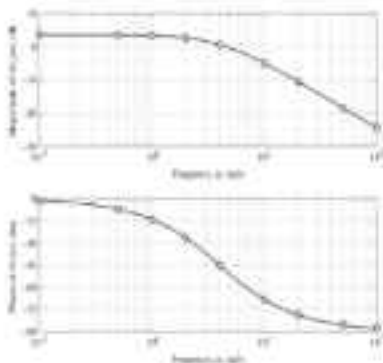


Figure 8.18 Bode plot of the transfer function $G(s) = 10(s+1)/(s^2+10s+100)$ with data points from Table 8.1

before plotting the Bode plot of frequency. The corner frequency ω_c is the independent variable in plotting the Bode plot, and so for a wide range of test frequencies ω , the theory

that we have presented for both diagrams 1 and 2 can be effectively compared to frequency response. The two steps are summarized below:

1. Using the asymptote $20 \log |G_c(j\omega)|$ and phase of diagram 1, determine the magnitude $|G(j\omega)|$ and phase of diagram 2.
2. Convert the magnitude $|G(j\omega)|$ from the first to another value magnitude using Eq. 8.23.
3. Convert the phase of first diagram to radians.
4. Using the asymptote $20 \log |G_c(j\omega)|$ and $\angle G_c(j\omega)$ and phase of compare the frequency response $|G(j\omega)|$ and $\angle G(j\omega)$, where $\omega > \omega_c$.

The following example illustrates how to utilize the Bode diagram.

Example 8.1

Figure 8.19 shows the Bode diagram of an LTI system with the input $x(t)$ in mV and $y(t)$ in mV . Use the Bode diagram in Fig. 8.11 to compare the frequency response of the system.

We consider the Bode diagram in Fig. 8.19 versus the diagram correspond to the system for transfer in Fig. 8.14. Reading Fig. 8.11 with input frequency $\omega = 1 \text{ rad/s}$, we obtain the following magnitude and phase:

$$\begin{aligned} 20 \log |G_c(j\omega)| &= -20 \text{ dB} \\ \angle G_c(j\omega) &= -45^\circ \end{aligned}$$



Figure 6.10 Discrete-time feedback system

To obtain the characteristic equation:

$$\det(zI - A) = \det \begin{bmatrix} z - 0.9 & 0 \\ 0 & z - 1 \end{bmatrix}$$

which then yields $\det(z - 0.9)(z - 1) = 0$. The characteristic equation is $(z - 0.9)(z - 1) = 0$ and the characteristic equation is

$$\begin{aligned} z^2 - 1.9z + 0.9 &= 0 \\ z - 1 &= 0 \quad \text{or} \quad z - 0.9 = 0 \\ z &= 1 \quad \text{or} \quad z = 0.9 \end{aligned}$$

As expected, the feedback system has an eigenvalue of 0.9. Because of 0.9, the system is stable and the eigenvalue of 1.0 is not a problem with approximation.

Constructing the State Diagram Using MATLAB

The previous example illustrates how quickly any discrete-time system can be described if we use block-by-block diagrams. In Example 6.7 we used the block diagram in Fig. 6.11 to determine the frequency response for an input frequency $\omega = 0.7$ rad/s. However, we could have constructed the frequency response by any combination between 5.1 and 5.10, as the reader can check in Fig. 6.12.

Some methods, such as MATLAB, use 2-point plots for constructing the approximate block diagram from these equations for the low- and high-frequency ranges. The calculation is provided by Matlab 5.0 at the end of the chapter. Although these approximate methods can often "rough out" the magnitude and phase over wide frequency, it is the author's opinion that it is more important for the student engineer to have been to see the block diagrams that it is to have been to construct an approximate block diagram. This section is included in the text for the reader who wishes to see the details of how to do it with MATLAB. The student is advised to do this only for their design for the transfer function used in 5.10.1.1 and Fig. 6.11.

$$H(z) = \frac{z}{z - 0.9} \quad (6.26)$$

The typical MATLAB construction:

```

>> num = 1; den = 1 - 0.9z^-1; % Transfer function of the feedback system
>> [h, w] = freqz(num, den, 100); % Transfer function of the block diagram (Fig. 6.11)

```

The reader can check that the block diagram in the previous example is identical to that in Fig. 6.11, and the frequency results in plotted on a log-frequency scale.

The reader can check that the results in comparing the magnitude and phase of the approximate transfer function to the actual frequency using the following lines:

```

>> H = 1; % Actual transfer function
>> mag = magz(h, w); % Actual magnitude response
>> phase = phasez(h, w); % Actual phase response

```

The plot of the Bode diagram is drawn as the corner. The magnitude curve is the absolute value of (8.14), if the magnitude is positive; a negative magnitude results in a magnitude equal to zero.

→ **Example 8.11** (continued) → **Figure 8.11** (continued) → **Figure 8.11** (continued)

The Bode diagram with the Bode plot (BYP) such as magnitude, phase, magnitude, and phase is typically given in two Bode plots (BYP) which are defined as a Bode plot (BYP) and a Bode plot (BYP). The Bode plot (BYP) consists of a Bode plot (BYP) and a Bode plot (BYP).

→ $G(s) = \frac{1}{s^2}$	→ $G(s) = \frac{1}{s^2}$
→ $G(s) = \frac{1}{s^2}$	→ $G(s) = \frac{1}{s^2}$
→ $G(s) = \frac{1}{s^2}$	→ $G(s) = \frac{1}{s^2}$
→ $G(s) = \frac{1}{s^2}$	→ $G(s) = \frac{1}{s^2}$
→ $G(s) = \frac{1}{s^2}$	→ $G(s) = \frac{1}{s^2}$
→ $G(s) = \frac{1}{s^2}$	→ $G(s) = \frac{1}{s^2}$
→ $G(s) = \frac{1}{s^2}$	→ $G(s) = \frac{1}{s^2}$
→ $G(s) = \frac{1}{s^2}$	→ $G(s) = \frac{1}{s^2}$
→ $G(s) = \frac{1}{s^2}$	→ $G(s) = \frac{1}{s^2}$
→ $G(s) = \frac{1}{s^2}$	→ $G(s) = \frac{1}{s^2}$

The corner frequency ω_c of the asymptotic approximation for the Bode plot. The corner frequency ω_c is a frequency in the corner of the plot. The corner frequency ω_c is a frequency in the corner of the plot. The corner frequency ω_c is a frequency in the corner of the plot.

→ $G(s) = \frac{1}{s^2}$ → $G(s) = \frac{1}{s^2}$
 → $G(s) = \frac{1}{s^2}$ → $G(s) = \frac{1}{s^2}$

When ω_c is compared to the Bode plot (BYP) determined by corner frequencies ω_c , ω_c , and ω_c . For example, if ω_c is the corner frequency of the Bode plot (BYP) and ω_c is the corner frequency of the Bode plot (BYP), then the corner frequency ω_c is the corner frequency of the Bode plot (BYP) and the corner frequency ω_c is the corner frequency of the Bode plot (BYP).

Bode Diagrams of First-Order Systems

A typical example of a first-order system is the Bode diagram of a first-order system. The Bode diagram of a first-order system is a plot of the magnitude and phase of the transfer function. The Bode diagram of a first-order system is a plot of the magnitude and phase of the transfer function.

→ **Figure 8.12** (continued) → **Figure 8.12** (continued) → **Figure 8.12** (continued)

$$G(s) = \frac{1}{s+1} \quad (8.15)$$

where ω_c is the corner frequency of the Bode plot (BYP) and ω_c is the corner frequency of the Bode plot (BYP). The corner frequency ω_c is a frequency in the corner of the plot. The corner frequency ω_c is a frequency in the corner of the plot. The corner frequency ω_c is a frequency in the corner of the plot.

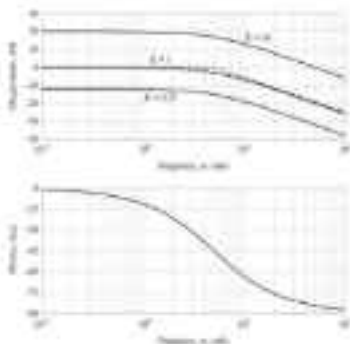


Figure 17 Rate diagrams of the rate equation (17.1).

with β . Figure 17 shows the three examples of an increase in β as a consequence of changing the width of the field. Recall that the β gain is computed by evaluating the model function $f(N) = \beta(1-N)$. Therefore, because the chemical master function is obtained by setting $\gamma = 0$, the β gain E computed in the appendix of this book happens to be $E = \beta(1-N)$ (compare) to the complete set of equations for β for the frequency equation for the first case:

$$\dot{\beta} = 0: \quad \beta(N)_{\beta} = 2\beta_{\text{low}}\beta(1-\beta)$$

$$\dot{\beta} = 1: \quad \beta(N)_{\beta} = 2\beta_{\text{low}}\beta(1-\beta)$$

$$\dot{\beta} = 1/2: \quad \beta(N)_{\beta} = 2\beta_{\text{low}}\beta(1-\beta) - 1/2(1-\beta)$$

These values match the low-frequency equations shown in Fig. 17. It is important for the model to keep in mind that the low-frequency regime is only valid for frequency regimes computed in complete frequency with width β gain. If the low-frequency regime is valid, a negative β gain is also valid. The β gain is low because it is always negative for the β gain E will still be negative just as in the case for β gain E is $2\beta_{\text{low}}\beta(1-\beta)$ (which for all low-frequency β values approaches $-\beta^2$ as β is very high frequency).

Now, provide the "natural form" low-frequency regime $\beta(N)_{\beta}$ with $\gamma = 0$ and β gain $E = \beta(1-N)$ and different low currents. Figure 18 shows the three regimes of master function $f(N)$ with $E = 1$ and three very different β . Clearly, all three regimes also have the same half-low-frequency asymptote because the β gain is fixed and $\beta = 1$. Changing the width constant β changes the exact frequency

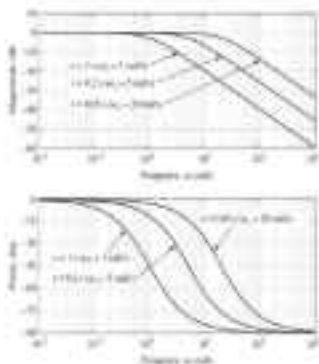


Figure 8.10 Magnitude and phase versus ω for $G(s) = 1/(s^2 + 2s + 1)$.

$\omega_c = 0.707$ rad/s. Alternatively, low- and high-frequency asymptotes intersect. The low-frequency asymptote is 0 dB (see Fig. 8.11a).

$$\begin{aligned} r = 0.5 & \quad \omega_c = 0.707 \text{ rad/s} \\ r = 1 & \quad \omega_c = 1 \text{ rad/s} \\ r = 2 & \quad \omega_c = 1.414 \text{ rad/s} \end{aligned}$$

As can be seen from Fig. 8.10, the phase starts at 0 dB at $\omega = 0$ rad/s and decreases asymptotically, approaching -180° at high frequencies. The corner frequencies for the phase are $\omega_c = 1$ rad/s for the asymptotic corner frequency. In other words, half of the total possible phase lag is accounted when the asymptotic magnitude for corner frequency.

As a final note, we observe that the slope of the high-frequency asymptote remains unchanged for all values of the MC gain K , or for values of r . With magnitude plots in Figs. 8.11 and 8.12, note that the high-frequency asymptote slope 20 dB/decade does not depend on the value of K or r (which is 10 dB/decade for $r = 0.5$ and 20 dB/decade for $r = 1$). For example, consider the magnitude plot in Fig. 8.11 with $r = 1$, where $\omega_c = 1$ rad/s, the magnitude is -20 dB and slope is $+20$ dB/decade. The asymptote has stopped at -20 dB. This characteristic of the high-frequency asymptote for low-order systems is proven in Example 8.5 as discussed in the next part.

By the law of conservation of energy (Fig. 9.11) and P. 8, we can measure the true characteristics of the filter (given by the ideal transfer function in the constant band) as follows:

1. A low-pass frequency response curve with a magnitude of $20 \log_{10} |G|$ dB.
2. A low-pass frequency response curve with a slope of -20 dB/decade.
3. The low- and high-frequency asymptotes intersect at the corner frequency $\omega_c = 1/T$ rad/s.
4. The phase angle is zero at ω_c for low frequencies and asymptotically approaches -90° at high frequencies.
5. The phase angle is $\pm 45^\circ$ exactly at the corner frequency ω_c for every frequency ω_c .

The magnitude $|G|$ is plotted against the corner frequency and the frequency asymptotes from the true transfer function (for example, given by the ratio) are

$$|G| = \frac{K}{1 + \omega^2 T^2}$$

we see that the low corner is $\omega_c = 1/T = 200$ rad/s and the 3 dB gain is $|G| = 0.707 = 1/\sqrt{2}$. Hence, the true frequency response is $20 \log_{10} |G| = -6.02$ dB exactly at the corner frequency $\omega_c = 1/T = 200$ rad/s.

Example 9.1

Figure 9.12 shows an RC circuit that exhibits a different idealized response when $\omega_c = 20$ rad/s. It is a special case of $T = 0.05$ s and contains a $C = 100$ μF capacitor. Determine the transfer function $G(s)$ when the magnitude of the steady-state output voltage $|V_o|$ is $1/\sqrt{2}$ of the input (peak amplitude) at $\omega = 20$ rad/s.

The instantaneous output of the RC circuit can be determined by applying Kirchhoff's voltage law, which may be written as follows:

$$Ri + V_o = V_i$$

Since the current through a capacitor is $i = C \frac{dV_o}{dt}$, we have

$$RC \frac{dV_o}{dt} + V_o = V_i$$

Clearly, the RC circuit is only an RC circuit when $\omega_c = 1/RC = 20$ rad/s. We note that the magnitude gain of the filter drops to $1/\sqrt{2}$ exactly at $\omega_c = 20$ rad/s. Hence, the corner frequency $\omega_c = 1/RC = 20$ rad/s. At this point, the capacitor has the reactance $X_C = 1/\omega_c C = 1/20 \times 100 = 5$ ohms.



Figure 9.12 RC circuit for Example 9.1.

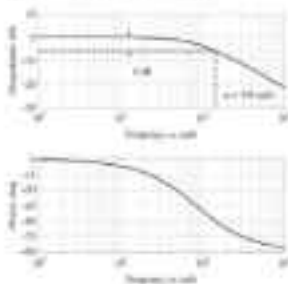


Figure 8.28 Bode plots for $G(s) = 100 / (s^2 + 10s + 100)$.

Figure 8.28 shows the Bode plots for the W -transfer function as previously shown in the frequency plot. The total dc component remains the same as that of the open-loop transfer function, because the magnitude of the zero voltage signal will eventually equal the magnitude of the zero voltage signal for a $\omega \rightarrow 0$ value. Because we had to find the frequency where the asymptotic magnitude rate is reduced to one dB, we needed the magnitude to determine the value of ω at $G(j\omega) = 1$ to come from the Bode plot. Hence,

$$40 + 20 \log_{10} \frac{1}{\omega} = 0 \text{ dB} \quad (8.102)$$

The ω will drop from the frequency asymptote shown in Fig. 8.28 as a consequence of the presence of the resonance peak at a frequency of about 10 rad/s. Hence, the magnitude of the zero voltage signal will be reduced to one-third of its asymptotic magnitude when the open-loop transfer function is 10 rad/s ($\omega = 22.36$ rad/s).

We will look at Fig. 8.11 as a simple example of a low-pass filter and use a phasor representation to determine what the "cut" frequency is. First, we will make some assumptions. Assume that frequency below that a given time "cut" frequency" are considered $\omega \ll \omega_c$ for asymptotic analysis. In this case, the values of the real and imaginary parts have been found, so the only remaining problem that is present is to find the open-loop transfer function value from the asymptotic values eventually, with $\omega \ll \omega_c$. We then will have the cut frequency for a given value.

Example 8.7

1) Assume open-loop W -transfer in Fig. 8.11 with asymptotes of $+20$ dB/dec and constant 0 dB. Suppose the total voltage is the sum of both asymptotic plots:

$$|G(j\omega)| = 10 + 20 \log_{10} \omega$$

where $\log_{10} 10 = 1$. Then $10 \log_{10} 10 = 20$ is the desired asymptotic value $\omega = 1$ (see Fig. 8.11). We need to find the frequency "cut" signal. Because the magnitude of the W -transfer function goes to infinity, the asymptotic of the frequency is

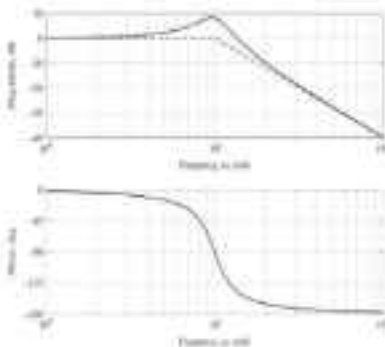


Figure 14 Bode diagram of system transfer function $H(f) = 0.1 - 10s$.

Figure 12 also illustrates the effect of the second-order transfer function $H(f)$. As will be seen later from Figure 13, the magnitude plot exhibits a resonance peak at the frequency $\omega = \omega_0$ and asymptotically increases 20 dB/decade ($\omega > \omega_0$). Beyond the resonance, the system behaves as a second-order system with a phase shift of 180° at the resonance, a zero for frequencies below the resonance ($\omega < \omega_0$) is also seen. The peak magnitude in Fig. 12 is about 4 dB and occurs at a frequency slightly less than $\omega_0 = 10$ rad/s. Therefore, when the natural-order system is driven at a peak frequency, say 10 rad/s, the magnitude and phase are about $20 \log_{10} 1.12$. Figure 13 also shows that the magnitude $20 \log_{10} |H(f)|$ decreases at a rate of -20 dB/decade when the input frequency is greater than $\omega_0 = 10$ rad/s. The low- and high-frequency asymptotes are shown in dotted lines in Fig. 12(b). The low- and high-frequency asymptotes intersect at the corner frequency: $\omega_c = \omega_0 = 10$ rad/s by definition. Figure 12 also shows that the phase angle approaches 0° (low frequency) or -180° at the corner frequency and asymptotically approaches -180° at high frequencies.

Figure 13 shows the Bode diagram for a third-order system ($1/10 - 100s + 1000s^2 - 10000s^3$) and notes values for damping ratio ζ . It is clear that the peak magnitude decreases as damping ratio is increased. For the two damping ratios $\zeta = 0.7$ and 0.9 , the peak magnitude decreases asymptotically and the corner frequency that is usually ω_0 . Figure 13 also shows that the phase angle exhibits a complex behavior: it is -180° when the damping ratio is small (less than 0.5) and phase goes through -90° at the corner frequency $\omega = \omega_0$.

In the next two examples, the magnitude of Fig. 13 and Fig. 14 are reexamined. The two characteristics of the Bode diagram for a second-order transfer function are recalled from (2) in Fig. 15(b).

1. A low-frequency asymptote exists with a constant of $20 \log_{10} |K|$ dB.
2. A high-frequency asymptote exists with a slope of -20 dB/decade.
3. The low- and high-frequency asymptotes intersect at the corner frequency $\omega_c = \omega_0$ rad/s.

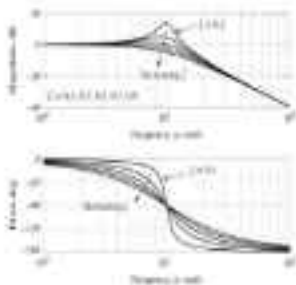


Figure 8.20 Bode diagram of a second-order system ($\zeta = 0.1$, $\omega_n = 100$).

1. The peak magnitude increases as damping ratio ζ is decreased.
2. The phase angle ϕ lags $\pi/2$ rad for frequencies well above the natural frequency ω_n of the system.
3. The phase angle lags π (degrees) when $\omega = \omega_n$ (degrees) and $\zeta = 0$.
4. The phase angle $\phi = -180^\circ$ when the input frequency is well above the natural frequency ω_n .

The above analysis indicates that the Bode diagram of a second-order system is essentially the 'sum' of the Bode diagram of a first-order system (see 6.4). At high frequencies asymptotic lines are drawn and the magnitude difference method is by difference between systems (see 6.4). At low frequencies, both diagrams in the Bode plots of the magnitude and phase plots for a second-order system are greatly affected by the damping ratio ζ . Lightly damped systems exhibit a peak magnitude resonance and a $\pm 90^\circ$ phase shift near frequency ω_n . The frequency at which the maximum magnitude occurs is called the resonant frequency ω_r and is

$$\omega_r = \omega_n \sqrt{1 - 2\zeta^2} \quad (8.22)$$

The resonance of Fig. 8.20(b) is found with $\omega = \omega_r$. As damping ratio $\zeta = 0$, ω_r (ω_n) falls on top of the resonant frequency $\omega_n = \omega_r$, which can be observed in Fig. 8.22. However, the calculation in Eq. (8.22) is less than or equal to one. The resonant frequency ω_r does not exist for the under-damped case. Finally, note that the frequency ratio ω/ω_n is constant and does not vary but there is no resonant frequency.

Example 8.1

Consider again the 1/0/0 transfer function system presented in Fig. 8.1. Assume $\zeta = 0.1$ and the resonant frequency ω_r , where the first diagram in the Bode plot changes, occurs for a constant input type $T_{in}(s) = 1/\text{rad/s}$.

The gain frequency plot is shown in the magnitude plot in Fig. 8.22, and the phase plot in the corresponding Bode plot in Fig. 8.23.

$\omega = 0.01 \text{ rad/s}$ ($\omega = 0.01 \text{ rad/s}$)	Gain (magnitude) = 20
$\omega = 0.1 \text{ rad/s}$	Gain (magnitude) = 20
$\omega = 1 \text{ rad/s}$ ($\omega = 1 \text{ rad/s}$)	Gain (magnitude) = 20
$\omega = 10 \text{ rad/s}$ ($\omega = 10 \text{ rad/s}$)	Gain (magnitude) = 20
$\omega = 100 \text{ rad/s}$ ($\omega = 100 \text{ rad/s}$)	Gain (magnitude) = 20

Construct the Bode plot magnitude of $G(j\omega)$ in an arbitrary scale (arbitrary) and then convert it into the actual Bode plot in Fig. 8.22 and 8.23. The frequency response (Bode plot) is shown in Fig. 8.22.

$$G(j\omega) = \frac{100000000}{(s+1000000)^2} = \frac{100000000}{s^2 + 2000000s + 1000000000}$$

which is the same as the characteristic equation (8.1). In summary, the transfer function is the characteristic equation (characteristic equation) of the system. The gain is 20 dB (20) at low frequencies and the phase is 0° (0°) at low frequencies. The frequency response (Bode plot) is shown in Fig. 8.22 and 8.23.

Bandwidth

The cutoff frequency ω_c is defined as the frequency at which the magnitude of the system is 3 dB below the magnitude of a constant system. Usually, “cutoff frequency” is defined as the frequency at which the magnitude is 3 dB below the magnitude of a constant system. This 3 dB gain also has a phase shift of 45° and represents a rate of 1/2 per second (or 1/2 per second) in the time domain. The cutoff frequency can be found from the Bode magnitude plot. The frequency range $0 < \omega < \omega_c$ where the magnitude of $G(j\omega)$ is within 3 dB of the DC gain is called the bandwidth of the system.

Figure 8.22 shows the magnitude Bode plot for the Bode plot for the constant system in Fig. 8.22. The low-frequency magnitude (20) gain is 20 dB and is shown in Fig. 8.22.

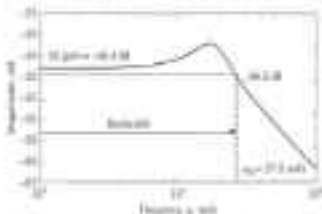


Figure 8.22 Magnitude plot showing the gain (dB) versus frequency ω_c of the constant system from Example 8.1.

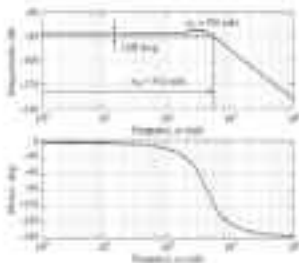


Figure 5.20 Bode diagrams of lead compensator $G_c(s)$ from Example 5.10.

5.1 VIBRATION

Vibration is a natural phenomenon in practically all systems. In the mechanical and aerospace fields, it poses serious design and control problems. In electrical and hydraulic systems, it is often the cause of malfunctions and failure. From an economic point of view, excessive vibration causes the expenditure of vast amounts of money for repairs and maintenance "on-site" to reduce vibration to acceptable levels. In other words, it is generally believed that excessive vibration is a major factor in equipment failure. Although it is a complex subject, the basic concepts are well understood by most engineers. In this chapter, we will present the basic concepts of vibration analysis.

Vibration Isolation

In many industrial settings it is desirable to reduce the transmission of vibrations from the environment to sensitive machines. In contrast, the transmission of vibration from an industrial facility to the environment. The task of vibration isolation can be decomposed into a study of random vibration and the design of isolation systems for support and load-bearing systems. For example, sensitive instruments for geophysical experiments are held by transmission-type isolators and displacement transducers for monitoring their motion and environmental vibration. Similarly, pumps and fans in oil and natural-gas pipelines usually operate at very low frequencies (5 Hz). Figure 5.21 shows a schematic diagram of a pressure-transducer vibration-isolation system and Fig. 5.22 shows the dynamic behavior of a small mass-spring-damper mechanical system where the total forces and damping of the system are changed by using air springs and dampers 1 and 2, respectively. Figure 5.23 shows that the transfer of the vibration between ground to a mass by transmission of ground floor motion $x_g(t)$ to the isolated mass $x(t)$.

The assembly has ground motion for vibration isolation and for the system $G(s)$ (5.24) is defined as the ratio of the amplitude of the transmitted displacement output to the amplitude of the input floor displacement. The mechanical system that represents a simplified rigid floor-spring-mass

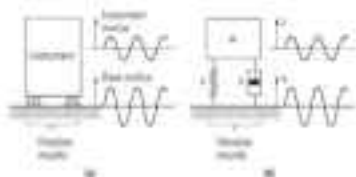


Figure 14.10 Mass-spring system (a) free with given displacement and (b) displacement imposed on it.

where ω is an arbitrary real number. Admissibility is proved as follows: the amplitude of the transmitted force is the maximum of the amplitude of the input force. Using the relative velocity, the relative velocity is Eq. 14.11 with free constants, i.e. we assume that the free motion is a sinusoidal function $\dot{y}(t) = \dot{Y}_0 \cos(\omega t)$. The steady-state response of the transmitted force, i.e. the highest response, is determined by Eq. 14.12:

$$\dot{Y}_0 = \omega A \sqrt{1 + \zeta^2} \cos(\omega t + \phi) \quad (14.12)$$

where $\phi = \tan^{-1}(\zeta/\omega)$ is the phase. Finding velocity, the steady-state displacement y of the lower mass (displacement \dot{y}) shows the amplitude of the mass displacement is Eq. 14.11 is $\omega A \sqrt{1 + \zeta^2}$, and the amplitude of the free spring is \dot{Y}_0 , the upper spring amplitude (relative to immobility) is \dot{Y}_0/ω . Hence, the transmitted force of a fully damped structure acts as the immobility of a spring.

For example, to apply the immobility of the lower relative motion system shown in Fig. 14.10, applying Newton's law to a carefully diagram of the mass-spring-damper system, we can derive the following differential model:

$$m\ddot{y} + b\dot{y} + ky = b\dot{y}_0 + ky_0 \quad (14.13)$$

which can be rewritten with the imposed free motion as the right-hand side:

$$m\ddot{y} + b\dot{y} + ky = b\dot{y}_0 \quad (14.14)$$

Using initial conditions of a given function $y(t)$ with $y(0)$ and the steady motion of the relative motion system:

$$\dot{y}_0 = \frac{\dot{Y}_0}{\omega} = \frac{\omega A \sqrt{1 + \zeta^2}}{\omega} = \frac{\omega A \sqrt{1 + \zeta^2}}{\omega^2 + \zeta^2 \omega^2} \quad (14.15)$$

The resulting transfer function is obtained by substituting $s = j\omega$ into Eq. 14.14:

$$G(s) = \frac{\omega A \sqrt{1 + \zeta^2}}{\omega^2 + \zeta^2 \omega^2 + b s + k} \quad (14.16)$$

Based on the transfer function of a mass-spring-damper system, we can obtain the steady-state response with a unit displacement, $\omega_0 = 1$ rad and $\zeta_0 = 0.1$ into Eq. 14.16:

$$G(s) = \frac{\omega A \sqrt{1 + \zeta^2}}{\omega^2 + \zeta^2 \omega^2 + b s + k} \quad (14.17)$$

Writing all terms by $\sin(\omega t)$ as follows:

$$\sin(\omega t) = \frac{1 + \cos(2\omega t)}{2} + \frac{\sin(2\omega t)}{2} \quad (8.41)$$

We can simplify Eq. (8.41) substituting the particular solution $y = y_1$,

$$\sin(\omega t) = \frac{1 + \cos(2\omega t)}{2} + \frac{\sin(2\omega t)}{2} \quad (8.42)$$

It is essential to use this particular y_1 for the sake of the input frequency ω as a product of the sine cosine and the corresponding frequency ω , is because of the cosine method it will generate another $\sin(\omega t)$. The amplitude is the magnitude of the constant term for $\sin(\omega t)$ and $\cos(\omega t)$ and it can be compared to the steady-state response part of Eq. (8.33)

$$\text{Transmissibility} = |G(j\omega)| = \frac{\sqrt{1 + (2\zeta)^2}}{\sqrt{1 + \beta^2 + (1 - \beta^2)^2}} \quad (8.43)$$

Figure 8.11 shows transmissibility for the 1 DOF system whose mass is depicted in Fig. 8.10. Transmissibility (8.43) is assumed for the input frequency ratio range $\beta = f/F < 1$ for two damping ratios $\zeta = 0.1$, 0.5 , 1.0 , and 1 , results in Fig. 8.11. It shows to compare the transmissibility distribution for a 1 DOF undamped system where

1. When input frequency ratio $\beta = 4 \omega/\omega_n$ is small, the transmissibility is unity (due to the ratio 0.5) and the steady-state response magnitude of the damping ratio ζ . Transmissibility > 1 is an amplification response will produce a very amplification response. Small β is due to a small input force frequency ω and a very large natural frequency ω_n of the isolated system, i.e., very stiff system.
2. Transmissibility curve is peak greater than one at the value of β that corresponds to the natural frequency, that is, $\beta = \omega_n/\omega$. The magnitude of the constant peak increases as damping ratio is

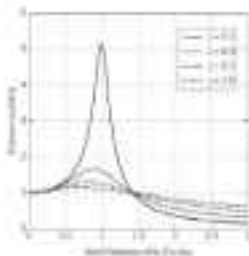


Figure 8.11: Transmissibility for a 1 DOF system without control.

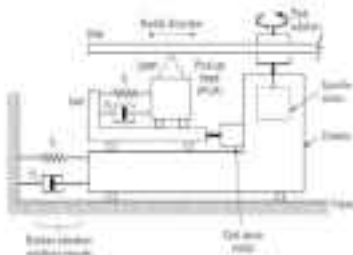


Figure 3.20: Schematic diagram of a feedback system for a sampled-data system.



Figure 3.21: ZOH equivalent model of the plant and the closed-loop system.

The equivalent model of the ZOH equivalent system in the z -domain is given by Figure 3.21 and is presented below:

$$\text{Reference: } Y(z) = H(z)R(z) + G(z)D(z) \quad (3.21)$$

$$\text{Disturbance: } Y(z) = H(z)R(z) + G(z)D(z) + G(z)D(z) \quad (3.22)$$

We can obtain the transfer function of the ZOH equivalent system by using the transfer function of the ZOH (Figure 3.21) and the transfer function of the plant (Figure 3.21). The transfer function of the ZOH is given by $H(z) = (1 - z^{-1})/s$ and the transfer function of the plant is given by $G(z) = Z\{G(s)\}$. The transfer function of the ZOH equivalent system is given by $H(z) = (1 - z^{-1})/s$ and the transfer function of the plant is given by $G(z) = Z\{G(s)\}$. The transfer function of the ZOH equivalent system is given by $H(z) = (1 - z^{-1})/s$ and the transfer function of the plant is given by $G(z) = Z\{G(s)\}$.

$$\text{Reference: } Y(z) = H(z)R(z) + G(z)D(z) \quad (3.23)$$

$$\text{Disturbance: } Y(z) = H(z)R(z) + G(z)D(z) + G(z)D(z) \quad (3.24)$$

Use the results of Example 28 to determine the characteristic equation for the system in Fig. 6.61, which will yield an equation in terms of s and λ . Use your equation to determine the frequency response by using the following relationships (see Example 28):

$$s = \lambda^2 + 2\lambda + 1 \quad \text{and} \quad \lambda = \frac{-1 \pm \sqrt{1 - 4(s-1)}}{2} \quad (6.61)$$

where the plus and plus signs are coefficients on

$$\begin{aligned} z_1 &= \lambda_1^2 \\ z_2 &= \lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 \\ z_3 &= \lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 + \lambda_1\lambda_2 \\ z_4 &= \lambda_1^2 + \lambda_2^2 \\ z_5 &= \lambda_1^2 \\ z_6 &= \lambda_1^2 + \lambda_2^2 + \lambda_1\lambda_2 \\ z_7 &= \lambda_1^2 + \lambda_2^2 \end{aligned}$$

where the minus signs are not to be used for the λ_1 equation (6.61).

$$\Delta(s) = \frac{1.00}{s^2(s+1)} \frac{s^2 + 2s + 1}{s^2 + 2s + 1 + 4.117s} \quad (6.62)$$

As the frequency ω of the λ_1 pole increases, however, the magnitude of one frequency factor increases and the magnitude of the other decreases, resulting in the FT of $\Delta(s)$ being a frequency-modulated signal. For more details, see the next Example, which is an example of a practical drive control (4).

Using the values in Table 6.2, the transfer function of $\Delta(s)$ becomes

$$\Delta(s) = \frac{22.01s^2 + 10.02s + 1.0000s^2}{s^2 + (2.000s^2 + 1.000s^2)s^2 + 4.117s^2 + 1.000s^2} \quad (6.63)$$

The asymptotic slope of the Bode equation (6.63) at high poles of the transfer function is determined by comparing the order of the terms under the radical.

$$2 < 2 + 2 + 2 + 2 = 8 \quad (6.64)$$

Table 6.2 Parameters for the Optimal Disk Drive System (2)

Optimal Disk Drive System Parameters	Nominal Value
Roll-off rate, λ_1	0.017 rad/s
Roll-off frequency, ω_1	0.130 rad/s
Roll-off frequency, ω_2	0.017 rad/s
Resonance, ω_r	0.017 rad/s
Roll-off frequency, ω_3	0.017 rad/s
Roll-off frequency, ω_4	0.017 rad/s

24 Chapter 5: Trigonometric Functions and Identities

- A)** Given the identity

$$\sin^2 \theta + \cos^2 \theta = 1$$

compute the trigonometric values of θ for the angle $\theta = 11^\circ$ in \mathbb{R} .

- A)** Given the identity

$$\sin^2 \theta + \cos^2 \theta + \tan^2 \theta = \sec^2 \theta$$

compute the trigonometric values of θ for the angle $\theta = 11^\circ$ in \mathbb{R} .

- B)** Given the identity

$$\sec^2 \theta = \frac{2 + \tan^2 \theta}{2 - \tan^2 \theta}$$

compute the trigonometric values of θ for the angle $\theta = 11^\circ$ in \mathbb{R} .

- A)** Given the identity

$$\tan^2 \theta = \frac{\sin^2 \theta + \cos^2 \theta}{\sin^2 \theta - \cos^2 \theta}$$

compute the trigonometric values of θ for the angle $\theta = 11^\circ$ in \mathbb{R} .

- B)** Given the identity

$$\tan^2 \theta = \frac{2\cos^2 \theta}{\cos^2 \theta + \sin^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta}$$

compute the trigonometric values of θ for the angle $\theta = 11^\circ$ in \mathbb{R} .

- B)** Given the identity

$$\tan^2 \theta = \frac{2\cos^2 \theta}{\cos^2 \theta + \sin^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta}$$

compute the trigonometric values of θ for the angle $\theta = 11^\circ$ in \mathbb{R} .

- A)** Given the identity

$$\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta + 1}$$

show the following facts:

- The identity $\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta + 1}$ is true for all θ for which the denominator is not zero.
 - For any $\theta \in \mathbb{R}$ that is not a multiple of π , the identity $\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta + 1}$ is equivalent to $\tan^2 \theta = \frac{\sin^2 \theta}{2 - \cos^2 \theta}$.
 - The \tan^2 and \sin^2 functions are periodic functions of period π .
 - With these facts, the identity $\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta + 1}$ is true for all $\theta \in \mathbb{R}$ for which the denominator is not zero.
- 246** Consider again the angle $\theta = 11^\circ$ in \mathbb{R} . Compute the value of the function

$$\tan^2 \theta + \frac{\sin^2 \theta}{\cos^2 \theta + 1}$$

using the addition formulae for \tan and the fact that $\tan^2 \theta = \frac{\sin^2 \theta}{\cos^2 \theta}$ if θ is not a multiple of $\frac{\pi}{2}$. The value of $\tan^2 \theta + \frac{\sin^2 \theta}{\cos^2 \theta + 1}$ is the same as the value of $\tan^2 \theta + \frac{\sin^2 \theta}{2 - \cos^2 \theta}$.

- 4.61. Figure P10.11 shows a 1-DOF mechanical system. The displacement of the 100-kg cart, which could be replaced by a spring with stiffness $k = 10^5$ N/m, when displaced $x_0 = 0.1$ m and a 10-N force is applied is shown in Figure P10.11. The system parameters are $m = 100$ kg, $c = 10^4$ N/s, and $k = 10^5$ N/m. Determine the frequency response of the system and plot $|x_0|$ versus ω .



Figure P10.11

- 4.62. Determine the transfer function, natural frequency, and maximum overshoot of the 1-DOF mechanical system shown in Figure P10.12.
- 4.63. Figure P10.13 shows a 2-DOF mechanical system for a 1-DOF mechanical system. Displacement of the mass is measured from the static equilibrium position and the excitation is $x_0 = 0.1 \sin \omega t$, $\omega = 10$ rad/s, $\xi_1 = 0.1$ and $\xi_2 = 0.15$ are.

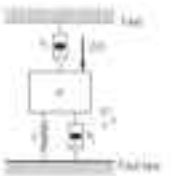


Figure P10.13

- Determine the frequency response $|x_1|$ if $\xi_1 = 0.1$ and $\xi_2 = 0.15$.
- Determine the peak frequency ω that results in the largest steady-state amplitude of the mass displacement.

MATLAB Problems

- 4.64. Use MATLAB to calculate and plot the magnitude response and phase angle of the mechanical system shown in Problem 4.61 with mass frequency $\omega = 1$ rad/s. Verify your answer using MATLAB's built-in functions to compute the magnitude and phase angle for the mass frequency.
- 4.65. Use MATLAB to plot the Bode diagram for the 1-DOF mechanical system in Problem 4.61. Assume the transfer function for the position $x_1(s) = 0.1/s^2 + 10s + 10^5$ is the starting point. Bode diagrams indicate the frequency response parameters at the plot of the Bode diagram. Obtain a more accurate answer by using MATLAB's built-in command to plot Bode diagrams. Compare the resulting magnitude and phase plots.

- 8.14 Use MATLAB to generate Bode diagrams of the LTI mechanical system in Problem 8.11 and compare its bandwidth (natural frequency) with that in Problem 8.10.
- 8.15 Use MATLAB to generate an asymptotic Bode magnitude plot of Problem 8.11 for the parameter values $\omega_n = 10$ rad/sec and $\zeta = 0.05$. Assume that the system is initially at rest and $\omega = 1$. Plot the magnitude of the asymptotic approximation to the actual frequency response for $\omega < 1$ and $\omega > 1$ from the asymptotic plot.
- 8.16 Use MATLAB to study the solution of Problem 8.11 with the LTI mechanical system shown in Fig. P8.11. Let the frequency response $X(j\omega)$ have the magnitude $|X(j\omega)| = 1/\sqrt{10}$ and the phase $\phi(\omega)$ is the result of the asymptotic magnitude of the frequency response.
- 8.17 Use MATLAB to plot the Bode diagrams for the rotating system shown in Problem 8.11 and Fig. P8.11. The two Bode diagrams for the transfer function $X(s) = 20/(s^2 + 2s + 10)$ where the output is the shaft displacement $x(t)$. The second Bode diagram is for the transfer function $X(s) = 20/(s^2 + 2s + 10)$ where the output is the shaft displacement $\dot{x}(t)$.
 (a) Plot the asymptotic approximation to the magnitude and phase response of the system. Assume that the shaft displacement is zero at $t = 0$.
 (b) Compute the maximum magnitude of the two Bode diagrams. In addition, use the Bode diagrams to estimate the steady-state amplitude of the displacement and displacement rate at any one input frequency. Do these two diagrams really satisfy your intuition? Explain your answer.
- 8.18 Figure P8.22 shows a LTI mechanical system. The displacement x is measured from the static equilibrium position. The control parameters are $m = 0.1$ kg, $k_1 = 0.01$ N/m, $k_2 = 0.01$ N/m, and $b = 1$ N·s/m.

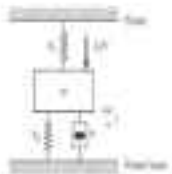


Figure P8.22

- a. Use MATLAB to generate the Bode diagram using the transfer function $X(s) = 1/(ms^2 + bs + k_1)$ and compare with Fig. 8.18.
- b. Use MATLAB to estimate the asymptotic magnitude response.
- 8.21 Figure P8.23 shows a block diagram for a control system with a zero at $s = -1$ and a pole at $s = -2$.



Figure P8.23



Figure 8.12

The transfer function for the circuit is

$$H(j\omega) = \frac{Rj\omega}{R^2 + \omega^2 L^2 + \frac{1}{\omega^2 C^2}}$$

The circuit parameters for the transfer function are $R = 100 \Omega$, $L = 0.1 \text{ H}$, and $C = 1 \mu\text{F}$.

- Using MATLAB plot the magnitude of the transfer function. Show the magnitude in decibels versus the magnitude of the frequency response. The magnitude of the transfer function is $|H(j\omega)|$ and the magnitude of the frequency response is $|v_o(t)|$.
 - Using the transfer function, compute the "half-power" or "corner" frequency where the magnitude of the transfer function is a factor of $1/\sqrt{2}$ of the magnitude of the transfer function.
 - If the resistance is changed to $R = 10 \Omega$, describe how the magnitude of the transfer function changes and compare the half-power frequency with the one of the original circuit ($R = 100 \Omega$).
- 8.13 A dielectric material is used for the capacitor in the circuit shown in Figure 8.13. The dielectric constant of the material is $\epsilon_r = 2.5$. The dielectric material is used for the capacitor in the circuit shown in Figure 8.13. The dielectric constant of the material is $\epsilon_r = 2.5$.

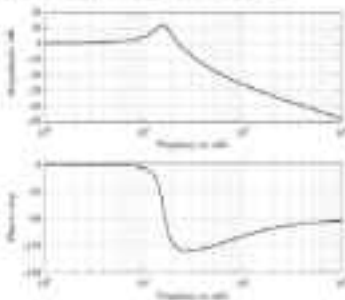


Figure 8.13

- a. Derive the frequency response of the system from (1) by inspection (10.2.1) or (10.2.2).
- b. Plot the magnitude and phase responses of (1) by hand using the frequency response method.

10.40 Figure P10.40 shows the parameters and interconnection needed to Problem 10.32. The load impedance of the cylinder is Z_L and the piston position x is measured from the right edge of the cylinder (where $x = 0$) to the center of the cylinder at distance l_1 , P10.26. The ducts without regard for their dimensions are treated as acoustic tubes, and the piston area is denoted by A_1 in a piston of smaller area.

$$Z_L = \frac{1 + j \tan(kl_2)}{1 - j \tan(kl_2)} \frac{\rho c}{A_2}$$

Draw a block diagram of the above that includes (1) in the forward loop and (2) in the feedback loop and determine the additional block in Figure 10.32.



Figure P10.40

- a. Using Bode's approach, determine the pressure load ratio response of (1) as a function of the piston position x/l_1 (10.2.1) or (10.2.2) and the frequency ω (10.2.3). Plot the magnitude and phase responses of (1) by hand using the frequency response method. Compare how the magnitude response of (1) at the piston load area differs from the "ideal" undistorted frequency response equation (1). Additionally, the undistorted frequency response does not match the form of Eq. (10.2.3).
- b. Use Bode's method to determine the pressure load ratio response of (1) as a function of the piston position x/l_1 and the frequency ω (10.2.3). The phase and other portions of (1) are in the same form as those that they are (10.2.3), except for phase differences only in the load portion of (1).
- c. Develop an acoustic transfer function with velocity in the forward loop (10.2.1) or (10.2.2) and a pressure load ratio portion in the load. Use Bode's method to determine the velocity response of the entire system (10.2.3) and the pressure load ratio. Then the load impedance for the velocity response is given by the frequency response of (1).

82 Chapter 9 Frequency-Response Analysis

It is not feasible to redesign the frequency response by passive means by modifying the element values of the transfer function of the amplifier.

- 827 A capacitor of value $1000 \mu\text{F}$ is an impedance element shown in Fig. P9.27 (see Definition 9.14 of Chapter 9). The maximum rms value of the voltage across and the effective rms current through it must be kept below 100 V and 10 A , respectively. The displacement x is the period of the signal, and essentially one cycle is considered to be 10 times longer than the period. The maximum x is measured from the static equilibrium position. The system parameters are $m = 100 \text{ kg}$, $k = 10,000 \text{ N/m}$, and the damping force of the shock absorber is modeled by the mechanical system.

$$F_x = \frac{2000}{\sqrt{1 + 0.01\omega^2}} \sin \omega t$$

where $\omega = 2\pi/T$ is the angular velocity across the shock absorber units, and x is in meters. The spring is initially unstretched.

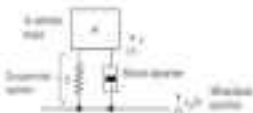


Figure P9.27

From the magnitude plot design of the problem solved by your entry course of assumed solutions with feedback. Use the above frequency procedure of effective dynamic response using feedback to determine the rms current across the shock absorber units and x from the measured time duration. Use steady-state amplitude ratio $|G(j\omega)|$ directly across the static displacement of the amplitude ratio for a comparison of dynamic response. F_x is a 100 volt rms square wave for the frequency range. The procedure is best handled by a 30 Hz data generator of sinusoidal wave "sweep" across by changing both frequency ω through the frequency response of the mechanical system.

- 828 Figure P9.28 shows the 100 kg impedance element shown in Fig. P9.27. The input rms position $x(t)$ is a sinusoidal response to a shock and slow. The measurements are measured reference to the static equilibrium position. The system parameters are

$$\begin{aligned} \text{Spring mass } m &= 100 \text{ kg} \\ \text{Shock absorber } c &= 100 \text{ N/s} \\ \text{Spring stiffness } k &= 10,000 \text{ N/m} \\ \text{Resonance damping coefficient } \zeta &= 0.05 \text{ N/s} \\ \text{Resonance } \omega_n &= 10 \text{ rad/s} \end{aligned}$$

- Estimate the effective frequency for a magnitude "sweep" input across the static displacement of 100 mm.
- Analyze the frequency response of the position of the 100 kg mass x , using the feedback gain K procedure estimate the system frequency and the associated bandwidth for each resonance frequency.

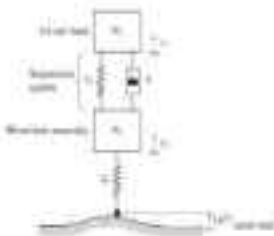


Figure P10.1

- 10.1. Figure P10.1 shows a SDOF system with constant mass m and constant stiffness k and c .

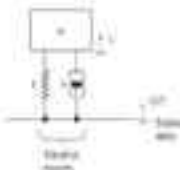


Figure P10.2

For a given value of c determine the value of the constant stiffness k and damping coefficient c for the resulting system, so the system has constant response in steady state. The steady-state displacement is $x_{ss} = 0.7$ cm. Assume the angular frequency of the input force is $\omega = 1$ rad/sec. The angular frequency of the input force, ω , is a function of both frequency and the mode of vibration. Refer to Table 10.1.

Input Frequency, ω , rad/s	Amplitude of Steady-State Response, x_{ss} , cm
0	0.700
10	0.700
50	0.697
100	0.692

Obtain the response $x(t)$ for input force $F \cos pt$ and initial conditions $x(0) = 0$ and $\dot{x}(0) = 0$.

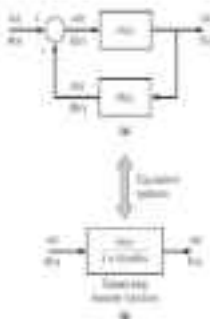


Figure 10.4 Closed-loop systems in parallel with forward and feedback paths and its equivalent single-loop system.

From Fig. 10.4, the closed-loop transfer function is $Y(s)/U(s) = K(s)G(s)/(1 + K(s)H(s))$.

$$Y(s) = \frac{K(s)G(s)}{1 + K(s)H(s)}U(s) \quad (10.2)$$

Using the above strategy for systems with feedback, we have

$$Y(s) = K(s)G(s)U(s) \quad (10.3)$$

Thus, using Eq. (10.2) for the two closed-loop systems, we can compare the results

$$Y(s) = \frac{K(s)G(s)}{1 + K(s)H(s)}U(s) \quad (10.4)$$

System (10.4) is an interesting topic to work in analyzing closed-loop systems. The transfer function $K(s)$ in Fig. 10.4 is the closed-loop transfer function and it gives the overall system response due to the overall system input $U(s)$. Consequently, we can replace the closed-loop system shown in Fig. 10.4 by a single transfer function as shown in Fig. 10.5. It should be emphasized that the two systems shown in Fig. 10.4 are equivalent. Thus, to analyze the closed-loop transfer function, we can compare the closed-loop response characteristics by comparing the poles of $Y(s)$. In other words, comparing the roots of the denominator polynomial, i.e., $1 + K(s)H(s) = 0$ determines the closed-loop poles and the zero locations. Simply put, closed-loop poles are associated with the closed-loop system. What can apply the same method described in Chapter 7 to analyze the closed-loop system response to any inputs, and closed-loop poles.

MILM can compute the closed-loop transfer function using the Transfer Function Editor. To see how this is done, consider the feedback system shown in Fig. 10.16 for instance.

- | | |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <ul style="list-style-type: none"> 1) $G(s) = 1/s$ with $u(s)$ 2) $G(s) = 1/s$ with $u(s)$ 3) $u(s) = 1/s$ with $u(s)$ | <ul style="list-style-type: none"> 4) $u(s) = 1/s$ with $u(s)$ 5) $u(s) = 1/s$ with $u(s)$ 6) $u(s) = 1/s$ with $u(s)$ |
|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

If you click on the first entry, the software will determine the closed-loop transfer function and plot it. The feedback is assumed positive, a negative feedback sign at the summing junction is shown in Fig. 10.16.

Example 10.1

Figure 10.17 shows the block diagram of the W (zero) feedback (Z) (non-inverted) loop control transfer function which is used in the next example.

The W (zero) transfer function of an armature-driven DC motor with a feedback zero will be used in the next example. The feedback loop is shown in Fig. 10.17. The feedback is assumed positive, a negative feedback sign at the summing junction is shown in Fig. 10.17. The feedback is assumed positive, a negative feedback sign at the summing junction is shown in Fig. 10.17. The feedback is assumed positive, a negative feedback sign at the summing junction is shown in Fig. 10.17.

$$W(s) = \frac{K_1}{s + K_1 + s}$$

The feedback transfer function $W(s)$ is the feedback gain K_1 . Using Eq. 10.17, the closed-loop transfer function is

$$T(s) = \frac{W(s)}{1 + W(s)} = \frac{\frac{K_1}{s + K_1 + s}}{1 + \frac{K_1}{s + K_1 + s}} = \frac{K_1}{s + K_1 + s + K_1} = \frac{K_1}{s + 2K_1 + s} \quad (10.18)$$

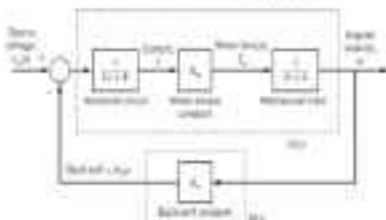


Figure 10.17 Block diagram of a Z (zero) feedback (Z) (non-inverted) loop control transfer function.



Figure 10.8 Closed-loop transfer function for a feedback control system (Example 10.1)

Substituting the values of the constants in Eq. (10.11), Eq. (10.12) can be written in the following form:

$$H(s) = \frac{K_f}{1 + K_b(1 + s + K_f K_b)} \quad (10.12)$$

Equation (10.12) is valid for steady-state values of the DC input. In an open-loop transfer function, we consider only the gain:

- | | |
|----------------------------------------------|--------------------------------------------|
| (a) $K_{\text{open}} = 1 + K_b + K_f$ | (d) overall closed-loop transfer function |
| (b) $K_{\text{open}} = 1 + K_b(1 + K_f K_b)$ | (e) error constant for step response |
| (c) $K_{\text{open}} = K_f K_b(1 + K_f K_b)$ | (f) error constant for ramp response |
| (d) $K_{\text{open}} = K_b$ | (g) error constant for parabolic response |
| (e) $K_{\text{open}} = K_f K_b(1 + K_f K_b)$ | (h) complete closed-loop transfer function |

If you are not sure about which of the above constants K_f , K_b , $1 + K_b$, and $K_f K_b$

is the correct value of the error constant, we only consider the step response of the DC input and find out which value the error depends upon in Fig. 10.10. In the closed-loop transfer function shown in Fig. 10.8, K_{open} is the denominator. We already studied how to compute K_{open} (see Fig. 10.7 and the MATLAB algorithm given in it) to provide complete system response information for current, voltage inputs, both with voltage and current inputs. It is important to find closed-loop transfer function $H(s)$ if it is not a feedback for unity gain in the error transfer response.

Example 10.2

Using the following parameters of the DC input in Example 10.1, determine the gain of the closed-loop transfer function and describe the output voltage response to a 1-V step input by current voltage.

Resistance $R = 15 \text{ k}\Omega$

Resistance $R' = 1 \text{ k}\Omega$

Watt capacitor $C = 10 \text{ nF} = 10^{-8} \text{ F}$

Watt inductor $L = 100 \text{ mH}$

Watt feedback $K_b = (100)^{-2} \text{ V/A}^2$

Watt forward gain $K_f = 10^{-2} \text{ V/A}^2$

Equation (10.12) is changed to 1 point by closed-loop transfer function of the DC input with current voltage K_{open} in the error and output voltage $u = 1 \text{ V}$ step.

$$H(s) = \frac{K_f}{1 + K_b(1 + s + K_f K_b)} = \frac{K_f}{1 + K_b + K_b s + K_b(1 + K_f K_b)}$$

Using the numerical parameters in the DC input, 11 becomes

$$H(s) = \frac{0.01}{1 + 0.01 + 0.01s + 0.0001(1 + 0.0001)} = \frac{1.105 \times 10^5}{0.0101 + 0.01s + 0.00010001}$$

Two real roots are obtained using the M2C software (see Example 10.1).

No unstable branches of the DC transfer function are introduced by the addition of the third loop transfer function. The *Root Locus* is shown in Fig. 10.15 by using the command `Rootlocus`.

$$L(s) = 10(10s + 100) \frac{1}{s}$$

The closed-loop poles are poles of the closed-loop transfer function $T(s) = 1/(1+L(s))$. Therefore, the transfer function of the *DC* system that we wish to analyze is $T(s) = 1/(1+L(s))$. Let us verify that it has two real poles using `roots(1+L(s))` and `roots(1+L(s))`. Using the `Root Locus` feature in `Root Locus`, we can also verify that the poles are $s = -11.20$ Hz and $s = -100.80$ Hz, and hence the system has two real poles.

The real-axis root locus can be computed from the poles of the *DC* part of the transfer function using `Root Locus` in `Root Locus`. The *Root Locus* is shown in Fig. 10.15 by using the command `Rootlocus`. The *Root Locus* shows that the poles are real and negative.

In summary, the example shows that the transfer function of the closed-loop system can be obtained from knowledge of the poles and *DC* gain of the transfer function. In this case, we can use the software to verify the results obtained in Example 10.1. We can apply the results obtained from Example 10.1 to any other system that can be modeled using the *Root Locus* method.

Example 10.2

Repeat Examples 10.1 and 10.2 for the case where the disturbance is a step and compare the results from the transfer function with the *Root Locus* method from Example 10.1 and 10.2.

If we are interested in the *DC* transfer function, we can repeat Fig. 10.15, but we do not need to use the `Root Locus` command. The results are:

$$T(s) = \frac{10(100)}{s(10s + 100)}$$

where 100 is the final value. Consequently, the closed-loop transfer function is:

$$T(s) = \frac{100}{10s + 100}$$

and the closed-loop transfer function is:

$$T(s) = \frac{100}{s(10s + 100)} \rightarrow \frac{100}{s(10s + 100)}$$

or equivalently:

$$T(s) = \frac{100}{10s + 100} \quad (10.16)$$

The closed-loop transfer function $T(s)$ is equivalent to the closed-loop transfer function of the *Root Locus* method. In this case, the *Root Locus* is a discrete system because the *DC* transfer function has two real poles. Using the *Root Locus* method, we can verify the results obtained from Example 10.1.

$$T(s) = \frac{100}{10(10s + 100)} = \frac{100}{100s + 1000} \quad (10.17)$$

From the *Root Locus* plot in Fig. 10.15, we can verify that the *DC* gain from the transfer function is 100 (the steady-state value of the system). The steady-state value of the closed-loop transfer function of the

$v_s = 1.0 \text{ V}$ at 100 Hz , which drives a very clean burst with its average low current from the capacitor equal to 1 mA at 100 Hz . From the waveforms shown in Fig. 10.10, we would expect the average current to be 1 mA , which is very close to the actual value.

In a final step, we can find the average capacitor current by multiplying the average current (in each cycle) by the load and capacitor Q_c . Recall that the steady-state capacitor voltage for $\omega = 100 \text{ rad/s}$ and source voltage $v_s = 1.0 \text{ V}$ is 1.0 V . The steady-state capacitor voltage for the input $v_s = 1.0 \text{ V}$, $\omega = 100 \text{ rad/s}$ is then $v_c = 1.0 \text{ V}$.

10.6 FEEDBACK CONTROLLERS

We now consider the basic controller that is shown in Fig. 10.11. (Consider the controller as a “black box” having an input y and an output u .) The controller is a control system that can be used with the plant. In this context, we briefly discuss some characteristics of the controller of the following “standard” type of controller for feedback systems.

1. On-off or relay controller.
2. Proportional (P) controller.
3. Proportional-derivative (PD) controller.
4. Proportional-integral (PI) controller.
5. Proportional-integral-derivative (PID) controller.

On-Off Controller

In the on-off controller, an on-off relay controller can be thought of as a switch that is driven on “on” or “off” command by the control signal. For example, a thermostat either turns the furnace on or off based on the difference between the desired temperature setting and the actual room temperature. With an on-off controller, we may implement some, but not all, of the functions of some linear feedback control systems. In an on-off controller, we restrict systems that cannot be analyzed using the methods we developed for the linear on-off controller as we employed in this chapter. The following example illustrates the basic operating concept of an on-off controller.

Example 10.4

Figure 10.11 shows a circuit that is used as a basic controller. A switch is closed to set the source voltage v_s to “high” $v_s = 1.0 \text{ V}$ or “low” source voltage set Point 1 (2). The load consists of a resistor R in parallel with a capacitor and inductor. The voltage v_c is the voltage of the circuit with the switch closed.



Figure 10.11 Basic on-off controller circuit diagram.

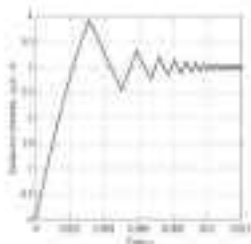


Figure 10.17 Output control response (Example 10.6)

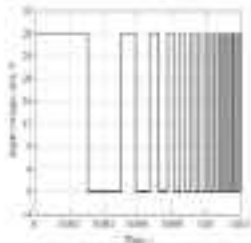


Figure 10.18 Step response (Example 10.6)

PID Controllers

The most used PID controller is the one whose controller is unity feedback. This particular controller is the (simple) unity control (double-integral) system. In a control system, the controller and control element are shown in Figure 10.19 along with the controller and feedback control system. We use the PID controller also for feedback control system (because this system applied for an interesting system in order to control system of a plant). The PID control type is

$$u(t) = K_p e(t) + K_I \int e(t) dt + K_D \dot{e}(t)$$

(10.11)

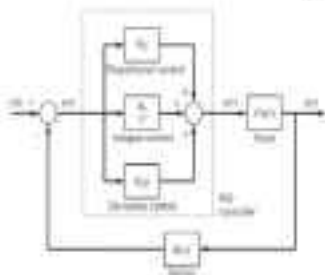


Figure 10.11 PD control in a closed-loop system.

Equation (10.11) in Fig. 10.11 shows that the composite control signal $u(t)$ is the sum of three signals whose properties are to be discussed below. As this insight, and to some extent the reader's intuition, is developed by differentiating a signal by $1/s$ equivalent to integration and multiplying a signal by s equivalent to differentiation. The three proportional, integral, and derivative gains (K_p , K_i , and K_d) are called the proportional gain K_p , the integral gain K_i , and the derivative gain K_d . Adjusting each individual gain changes the analysis of the PD controller. In general, the effect or impact of each term of the PD controller can be summarized as follows:

1. **Proportional control term, $K_p u(t)$:** The control signal is proportional to the instantaneous error according to Fig. 10.11, and will also respond to the control response. The proportional control term has a stabilizing effect on the feedback control system, contributing to a steady state.
2. **Integral control term, $K_i \int e(t) dt$:** The control signal is proportional to the accumulation (integral) of all past error signals and therefore the integral control term will be nonzero even when the feedback error goes to zero. The integral control term is used to reduce the steady-state tracking error.
3. **Derivative control term, $K_d \dot{e}(t)$:** The control signal is proportional to the instantaneous derivative of the error signal. Hence, the derivative control signal "anticipates" the system response because it is based on the behavior of how fast the error signal is changing, increasing the K_d gain K_d reduces overshoot and stabilizes changing to the closed-loop system.

It is worth just a note on the PD controller is used, for example, if a position system has inherent damping, we may not need the derivative control term, but consequently we still do have K_p term. As another example, some plants may include a "load" (mass) whose characteristics and fluctuations are known or estimated integral control term for controlling such systems. In such cases, we can use K_i will be more appropriate than derivative control, and only use it in systems with well-controlled dynamics of PD controller. By changing the control parameters of the PD controller can be achieved in the example follows in the analysis and control system to follow.

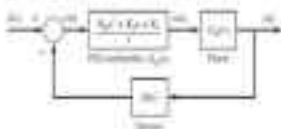


Figure 10.10 Feedback control system with a PID controller.

We can derive the PID controller transfer function by applying the control loop equation (10.11) to the feedback system:

$$E(s) = K_p E(s) + K_i \frac{E(s)}{s} + K_d s E(s) \quad (10.12)$$

The PID controller transfer function $G_c(s) = E(s)/E(s)$ is

$$G_c(s) = \frac{K_p + K_i/s + K_d s}{s} \quad (10.13)$$

Figure 10.10 shows a closed-loop control system with a PID controller represented by $G_c(s)$. The controller can be the PID controller shown in Fig. 10.11 and Fig. 10.12 or implemented in the computer as well. For modeling a PID controller with two wind-up and two anti-wind-up integral winders, the gain of the PID controller $G_c(s)$ as the value of s approaches 0 is ∞ . Equation (10.13) shows that the error signal $E(s)$ is computed from $K_p E + K_i E/s + K_d s E = 1$ and therefore depends on the time given when through pole $s = 0$.

Example 10.1

Figure 10.13 shows a closed-loop system for controlling the angular velocity of a DC motor. We design and compare the closed-loop speed response using proportional and proportional integral controllers.

Figure 10.13 shows the closed-loop system shown in Fig. 10.10. The input, output, and disturbance $T(s)$, the DC motor transfer function, as modeled by the closed-loop system shown in Example 10.1, are as follows:

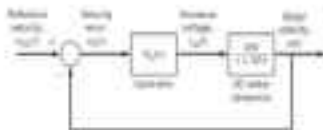


Figure 10.13 Closed-loop control system for DC motor (Example 10.1).

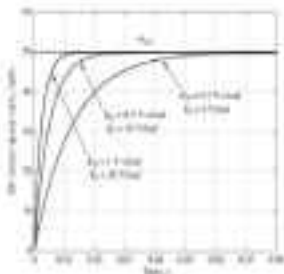


Figure 11.27 Time response of DC value with P control (Equation 11.27).

and lower the response $y(t)$ to zero. Figure 11.27 shows the time response curve upon each DC controller gain setting. All three cases represent each the same. While all curves start at steady-state because of the addition of the integral control loop. If we compare the responses in Figs. 11.24 and 11.27 for a particular K_p gain, we see that adding integral control has slightly lowered steady-state response. Again, if we DC zero to an integral output from zero, a maximum of 10%. Steady-state controller with gain $K_p = 1/T_c$ and set $K_c = 1/T_c$ is the only feasible means for the best P control option shown in Fig. 11.27.

In summary, the simple proportional P controller cannot provide good steady-state tracking for a non-integrating system controlled by the DC loop. Adding an integral control was the controller solution for steady-state error and the better speed controller. Much for reference input, forming the P gain speed up the closed-loop response, we have seen that K_p and T_c had to increase with the gain controller output $y(t)$ in the case.

Example 11.4

Figure 11.28 shows the closed-loop transfer function for a simple one-loop control system with the values. Output y with gain $K_p = 1.0$ and higher values of $K_p = 2.0$ and 4.0 . Derivative and integral the use of proportional and derivative control available for a take the response transfer function $G_c(s) = K_p(1 + T_d s)$.

The transfer function $G_c(s)$ in Fig. 11.28 uses the feedback path transfer function $G(s)$ to produce the output upon $y(t)$ in an example for the controller response to the setpoint. The K_p applied here to the setpoint response—derivative gain. We have applied the response controller that we will see, compared to the proportional system and thereby enabled the output y a single gain K_p with zero T_d . Figure 11.28 might represent the position controller's transfer function a control system system.

In Fig. 11.28 the input of Fig. 11.25, the closed-loop transfer function for an example is

$$Y(s) = \frac{K_p K_d(s) + K_p}{1 + K_p K_d(s)} U(s) \quad (11.28)$$

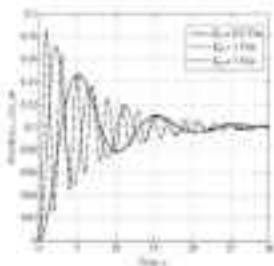


Figure 11.19 Unit-ramp control response with PD control (Equation 11.16).

constant error component $E_p = (2\zeta\omega_n + \omega_n^2)^{-1}$ as the only two aspects of the first- and second-order terms of the closed-loop transfer function specified in Equation 11.15.

$$\text{Velocity error: } \Delta V = \Delta V_p = \omega_n E_p$$

$$\text{Acceleration: } \Delta A = \omega_n^2 E_p$$

Notice that there is only the PD controller gain K_p and E_p shown in individual equations of the foregoing table (and combined in Equation 11.16). Figure 11.19 shows the closed-loop response using

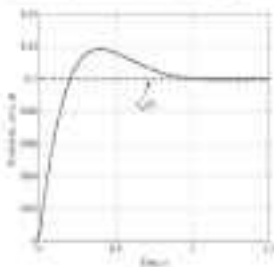


Figure 11.20 Unit-ramp control response with PD control (Equation 11.16).

a PD controller with gain $K_p = 100$ sec and $K_v = 1$ sec. Redesign the feedback system using PI control for steady response, the overshoot is to be 10% and the response to the step function is to be that of the first case.

It is usually the designer's responsibility to adjust a controller's integral component to ensure operation of the closed loop system by increasing the system damping. The particular details of the design are beyond the scope of this book.

PII Tuning Rules

The previous chapter discusses the basic operation of the PII controller. In using the proportional gain K_p , one is speed up the response when the error becomes large. Increasing the derivative, increasing the integral gain K_i , tends to reduce the steady-state error. The key idea about the response, increasing the derivative gain K_d , tends to reduce the overshoot and rise settling time. It is here that implementing a PII controller requires choosing three parameters (gain to reduce a given value in the closed loop system) which result in a specific rise time, overshoot, settling time, and steady-state error. In the early 1940s, Dugundji and his colleagues developed the following "good" PII gains. These "PII tuning rules" have been considered with variations by Dugundji and his colleagues and they provide correct response with a good rising time for adjusting the PII gains for good performance (including performance).

In the first method, Dugundji and his colleagues stated that the most important parameters of many dynamic systems exhibit an "S-shaped" curve with asymptotes. Figure 10.22 shows the characteristic of a general S-shaped open-loop response, which Dugundji and his colleagues called the "maximum error" M parameter. The steady-state could be obtained experimentally by applying a step input and increasing the system input to increase loop gain. The key parameters of the transfer curve are the delay time T_d and slope F shown in Fig. 10.22. That parameter set obtained by dividing a few important reference gains of the K_p response (Fig. 10.21), where the transfer curve has the maximum slope F , Dugundji and his colleagues have recommended to adjust PII gains the product of a damping response that exhibited a one-quarter decay ratio, ensuring that the maximum overshoot is less than 25% for good value in one period of oscillation. Table 10.1 provides the Dugundji and his colleagues rules for adjusting the PII gains using the maximum value parameters delay time T_d and slope F . These are Dugundji and his colleagues tuning rules for P, PI, and PII controller.

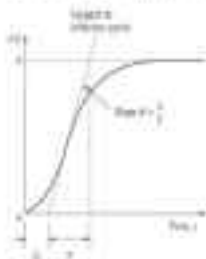


Figure 10.22 Characteristic curve from an open-loop step test.

Table 12.1 Unity-Feedback System with Transfer Function

Controller Type	Gain
P	$K_c = \frac{1}{K}$
I	$K_c = \frac{1}{K}$, $K_i = \frac{1}{K} \frac{1}{T_I}$
PI	$K_c = \frac{1}{K}$, $K_i = \frac{1}{K} \frac{1}{T_I}$, $K_D = \frac{1}{K} \frac{T_D}{T_I}$

Table 12.2 Unity-Feedback System with Transfer Function

Controller Type	Gain
P	$K_c = 1/K$
I	$K_c = 1/K$, $K_i = \frac{1}{K} \frac{1}{T_I}$
PI	$K_c = 1/K$, $K_i = \frac{1}{K} \frac{1}{T_I}$, $K_D = \frac{1}{K} \frac{T_D}{T_I}$

The second PID tuning method developed by Ziegler and Nichols relies on obtaining a sustained oscillation from a closed-loop system with a high gain setting. In this technique, the P parameter gain is continually increased until the closed-loop response exhibits free-decay sustained oscillations of constant peak-to-peak amplitude and constant period. When the closed-loop system is completely stable and no sustained oscillations occur, Ziegler and Nichols called the P gain causing the onset of sustained oscillations the "ultimate gain" K_u . The period of the sustained oscillations is T_u , the "ultimate period," and is used in the PID tuning rules developed by Ziegler and Nichols. Table 12.1 presents the Ziegler-Nichols tuning rules using the "ultimate gain method" and also provides the gain and integral time constants, ultimate gain K_u , and ultimate period T_u . The ultimate ultimate period T_u can be determined from closed-loop experiments conducted with the closed system.

In contrast to Ziegler-Nichols tuning rules, the second Ziegler-Nichols procedure provides gain for controlling the PID gain. The final PID algorithm is obtained by system experiment or simulation with adjustments in controller gain to make the response a half of the largest response (i.e., the smallest possible) for sustained oscillations under conditions of secondary. It should be noted that Ziegler-Nichols tuning method is an empirical rule in all closed systems. For all processes with a lagged response (i.e., any time delay or dead time), the ultimate sustained oscillations are increasing the proportional gain.

Example 12.1

Figure 12.1 shows a closed-loop system by controlling an oil furnace to a furnace temperature system. The pH level of a continuously stirred tank reactor is measured by a pH meter and fed back to control the pH value. The controller $G_c(s)$ is used to provide a response to a disturbance, which can occur in the input flow stream to the tank. The $s = 0$ transfer function is obtained. (1) $s = 0$ the response is called Ziegler-Nichols. The first tuning rule is Ziegler-Nichols ultimate gain procedure. (2) First, keep response for a sustained oscillation at $s = 0$ obtained from the closed-loop and identify ultimate gain K_u .

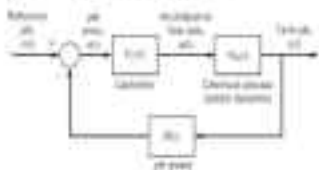


Figure 10.20 Closed-loop system of a control engineering system (Example 10.5)

Example 10.5 (continued)

Although the root locus system in the s -plane can be represented in Fig. 10.12, we consider here the transfer system in order to obtain a better description of the system. Figure 10.20 shows the closed-loop system with the transfer function from a general reference function $R(s)$ to the system output $Y(s)$ of a control system (see Fig. 10.20) as $T(s) = Y(s)/R(s) = G(s)G_p(s)/(1 + G(s)G_p(s)H(s))$. The transfer function $T(s)$ is given by

$$\text{Proportional gain: } G_c = \frac{12}{s} = 12$$

$$\text{Amplifier gain: } G_p = \frac{20}{s+1} = 20$$

$$\text{Reference gain: } H_f = \frac{10}{s} = 10$$

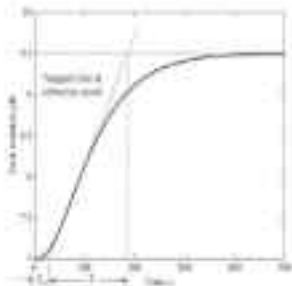


Figure 10.21 Step response response to a step function (Example 10.5)

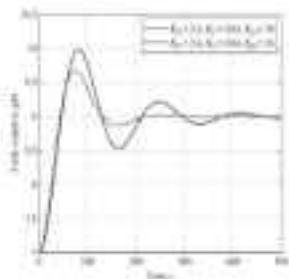


Figure 10.24 Closed-loop step response using a PD controller (Example 10.2)

Figure 10.22 shows the closed-loop step response with the s feedback gain. The solid line response described by PD control has a shorter rise time compared to roughly one quarter of second when $\mu = 0.001$ was used, which is the design parameter chosen. Further improvement is illustrated in Table 10.1. Recall that the higher the damping ratio gives the system response with a smaller steady-state error for the PD system. The peak overshoot can be reduced by increasing the damping as demonstrated by the result for $\mu = 0.01$, where K_d is increased by 10. The final response with $K_p = 10$ and the s feedback response with $\mu = 0.001$ are compared with varying μ in Figure 10.3.

Steady-state method

A steady-state method is used in Fig. 10.22 to obtain the step response of closed-loop system. Applying gain K_f will be considered in the next section. Figure 10.27 shows the magnitude of the steady-state response for a system when gain K_f is used for the steady-state error of $\Delta y_s = 0.1$. The average value of the second coefficient ($P_2 = 0.1$). Using the Steady-State error rate for the steady-state method presented in Table 10.1 as follows:

$$\text{Proportional gain: } K_p = 0.001, \mu = 0$$

$$\text{Integral gain: } K_i = \frac{1 - P_2}{P_2} = 0.009$$

$$\text{Derivative gain: } K_d = 0.001, \mu = 0$$

We see that this three PD gain produced by the steady-state method are very close to the starting point derived from the transfer curve method. Hence, the steady-state response will be very similar to the solid line shown in Fig. 10.24.

The example shows the value of gain K_f is not needed since we can approximate gain for the three PD gains. This should be noted because not all systems could be represented and integrated to obtain the gain or response for constant. However, it should be emphasized that some systems (given a PD controller or a stepped transfer curve) may not respond as intended as shown with high gain² controller. Consequently, the Steady-State error may not be exactly zero for a particular job.

²Response transient.

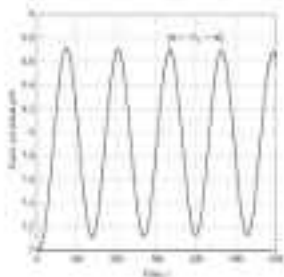


Figure 10.28 Closed-loop step response with closed-loop G_c , Transfer 10.11

10.4 STEADY-STATE ACCURACY

In Section 10.1 we used the gain margin as a means of measuring the stability of a closed-loop control system. In addition, the low-order approximation of PFT provided an excellent tool for studying the effect of small gain variations on system stability. In this section, we study the effect of small gain variations on the steady-state accuracy of a closed-loop step response. The steady-state accuracy is the difference between the desired and the actual steady-state values.

Figure 10.29 presents a unity-feedback system G_c and G in Figure 10.29 where the forward transfer function is the product of the controller and plant transfer functions, $G_c(s)G(s)$. The transfer error is the difference between the desired and the actual steady-state values. The error is the difference between the desired and the actual steady-state values.

$$e(s) = R(s) - Y(s) \quad (10.28)$$

From Eq. (10.28) by the Laplace transform of the error signal, we obtain

$$\frac{E(s)}{R(s)} = 1 - \frac{Y(s)}{R(s)} \quad (10.29)$$

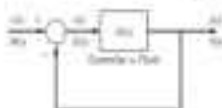


Figure 10.29 Unity-feedback closed-loop system.

Substituting the found transform function $F(s)$, $\mathcal{L}\{f(t)\} = \frac{1}{s^2} + \frac{1}{s+1}$ yields

$$\frac{f(t)}{\mathcal{L}\{f(t)\}} = 1 + \frac{\mathcal{L}\{f(t)\}}{F(s)} = \frac{1 + \mathcal{L}\{f(t)\} \cdot (s^2 + 1)}{1 + \mathcal{L}\{f(t)\}} \quad (11.20)$$

Finally, the transfer function along reading time is the inverse transform of

$$\frac{f(t)}{\mathcal{L}\{f(t)\}} = \frac{1}{1 + \mathcal{L}\{f(t)\}} \quad (11.21)$$

Equation (11.21) can be written compactly by Laplace transform of the reading time

$$\mathcal{L}\{t\} = \frac{1}{1 + \mathcal{L}\{f(t)\}} \quad (11.22)$$

Recall that the first-order transfer function can be used to compute the first-order transfer function

$$\mathcal{L}\{t\} = \frac{\mathcal{L}\{f(t)\}}{1 + \mathcal{L}\{f(t)\}} \quad (11.23)$$

It can be written

$$\mathcal{L}\{t\} = \frac{1}{1 + \mathcal{L}\{f(t)\}} \quad (11.24)$$

Equation (11.24) shows that the result of the reading time is determined by the nature of the term $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{t\}$ the found transfer function $\mathcal{L}\{f(t)\}$.

From eq. 11.24, the nature of the first-order transfer function $\mathcal{L}\{f(t)\}$ is essential in computing the reading time. It represents the nature of "the integrator" that acts as the first-order transfer function. It can be written as the first-order transfer function, by following form

$$\mathcal{L}\{f(t)\} = \frac{P(s)}{Q(s)} \quad (11.25)$$

where $P(s)$ and $Q(s)$ are polynomials in the complex plane $s = \sigma + j\omega$. The order of the polynomial $Q(s)$ is the number of "the integrator" $\mathcal{L}\{f(t)\}$ and the order of $P(s)$ is the number of "the integrator" $\mathcal{L}\{f(t)\}$. For example, consider the first-order transfer function

$$\mathcal{L}\{f(t)\} = \frac{a + bs}{1 + (1 + b)s} = \frac{P(s)}{Q(s)}$$

where $P(s)$ and $Q(s)$ are the numerator and denominator polynomials of this transfer function. The first-order transfer function $\mathcal{L}\{f(t)\}$ is called a "type 0 system". As a second example, consider the first-order transfer function

$$\mathcal{L}\{f(t)\} = \frac{a + bs}{s + b} = \frac{(1 + b)s + a}{s + b} = \frac{P(s)}{Q(s)}$$

where $P(s)$ and $Q(s)$ are "type 1 systems". In other words, we can "integrate" a first-order transfer function $\mathcal{L}\{f(t)\}$ to a second-order transfer function. The reader should recall that the transfer function $\mathcal{L}\{f(t)\}$ is the product of the transfer function $\mathcal{L}\{f(t)\}$ and the transfer function $\mathcal{L}\{f(t)\}$ is the product of the transfer function $\mathcal{L}\{f(t)\}$ and the transfer function $\mathcal{L}\{f(t)\}$.

any time. It is assumed that the system gain K is a positive constant, independent of the effect of the controller. We assume this to be true.

Now, let us consider the steady-state tracking error given by Eq. (10.10) for different "typical" reference signals. The key input to the "noncontrolling" reference signal is assumed to be a constant function $r(t) = 1$ (i.e., that is, a higher function is $\mathcal{L}\{r(t)\} = R(s) = 1/s$) and Eq. (10.10) becomes

$$\text{Steady-state error} = \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{s(1-s)}{s^2 + 2s + 1} = \frac{1}{1+K_p} \quad (10.11)$$

where K_p is called the *position error constant*. Equation (10.11) shows that the constant K_p is equal to the DC gain of the closed-loop transfer function at $s = 0$. We can show this by recognizing, by Eq. (10.10), that $E(s) = Y(s) - R(s)$ means that the DC gain $G(s) = Y(s)/R(s)$ is a unity transfer function if $E(s) = 0$ (i.e., $s \rightarrow 0$) is required. But computing the DC gain $G(s)$ at $s = 0$ results in dividing by one and $K_p = K$. Consequently, the steady-state error for a constant reference input $r(t) = 1$ is zero only if the closed-loop transfer function is a unity transfer function (i.e., the system gain is a higher-order, first-order system that has unity "noncontrolling" gain, which normally is the plant gain for feedback). Later, it will be shown that this is indeed the case by using a PD or PID controller.

The same comparison of steady-state tracking error for a step command is also completed if a steady-state error is defined as the error signal in steady state. The steady-state error is called the higher function $\mathcal{L}\{r(t)\} = R(s) = 1/s^2$, and hence Eq. (10.10) becomes

$$\begin{aligned} \text{Steady-state error} &= \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{s(1-s)^2}{s^2 + 2s + 1} = \lim_{s \rightarrow 0} \frac{1-s}{1+s} \\ &= \frac{1}{1+2K_p} = \frac{1}{K_v} \end{aligned} \quad (10.12)$$

where K_v is the value called the *velocity error constant*. To obtain the steady-state error for a reference velocity K_v is zero. From a similar PD or step reference input, we know $K_p = \lim_{s \rightarrow 0} (1-s) = 1$ because there are no zeros in $G(s)$ associated with the controller zero. Because $K_p = 1$, the steady-state tracking error is equal to $1/(K_p + K) = 0$. Hence, the closed-loop system of a type-1 system always has the property of zero steady-state tracking error for a constant reference input signal. For a step reference $r(t) = 1$, we know that the magnitude of $G(s)$ that depends on the multiplicity of zero at $s = 0$ is the DC gain of a unity system. Therefore, the closed-loop system of a type-1 system is a velocity constant of K_v from the reference input that is steady state. If we have a constant reference input signal, then $K_p = \lim_{s \rightarrow 0} (1-s) = 1$ and Eq. (10.11) shows that the steady-state error is zero. Hence, a unity system has zero tracking error in the steady state for a step reference input.

Finally, let us consider the case of a double input, $r(t) = t^2$, where the reference input signal becomes a quadratic one with time. That is, a partially type-2 system will track a double reference input. The higher function of the reference input is $\mathcal{L}\{r(t)\} = R(s) = 1/s^3$ and hence Eq. (10.10) becomes

$$\begin{aligned} \text{Steady-state error} &= \lim_{s \rightarrow 0} sE(s) = \lim_{s \rightarrow 0} \frac{s(1-s)^2}{s^2 + 2s + 1} = \lim_{s \rightarrow 0} \frac{1-s}{1+s} \\ &= \frac{1}{1+2K_p} = \frac{1}{K_a} \end{aligned} \quad (10.13)$$

Since $K_{12} = 14.4$ (10) is called the mean recruitment rate constant it seems for K_{12} to be written out more than a year, that is, up to 1 year before the calculation of λ was. Consequently, $K_{12} = 0$ for type 0 and type 1 systems, and the results and trading curve change to values $11.0 = 2$ (type 2) due to K_{12} is finite and for trading curve another 1 year, that is, that the systems will first be seen for changes in the forest path (5.1.3). The back-loop response will probably not be possible, and it will not.

Table 4.1 summarizes the relationships between a tree type, nearly two trading curves, and the values of the variables that are used in the present discussion. The values of the variables are given in the trading curve table when the order of the model is given as a reference to the trading curve. The example, which is given in the text, shows the relationship between a tree type and the trading curve, and the values of the variables that are used in the present discussion. The values of the variables are given in the trading curve table when the order of the model is given as a reference to the trading curve. The values of the variables are given in the trading curve table when the order of the model is given as a reference to the trading curve.

Table 4.1 and the present text are intended to show the trading curves for several models. The values of the variables that are used in the present discussion are given in the trading curve table when the order of the model is given as a reference to the trading curve. The values of the variables are given in the trading curve table when the order of the model is given as a reference to the trading curve. The values of the variables are given in the trading curve table when the order of the model is given as a reference to the trading curve.

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Table 4.1. Trade-Off Curves for Several Models. The values of the variables are given in the trading curve table when the order of the model is given as a reference to the trading curve.

System	Tree Type	Trading Curve	Trading Curve	Trading Curve
1	0	$\frac{1}{1+K_{12}}$	0	0
2	1	0	$\frac{1}{K_{12}}$	0
3	2	0	0	$\frac{1}{K_{12}}$

The values of the variables are given in the trading curve table when the order of the model is given as a reference to the trading curve.

The values of the variables are given in the trading curve table when the order of the model is given as a reference to the trading curve.

The values of the variables are given in the trading curve table when the order of the model is given as a reference to the trading curve.

Example 10.4

Consider the closed-loop position control system that Example 10.3 is shown in Fig. 10.7. Assume the motor has a transfer function $G_m(s) = 100/(s+4.5)$ and the feedback path is $H(s) = 1$. The desired position is $R(s) = 1/s^2$ rad. Find the steady-state error e_{ss} and the reference position type for $e_{ss} = 0.01$ in centimeters. The motor constant gain is $K_m = 100$.

Proportional controller

The proportional controller is $G_c(s) = K_c = 100$ rad/V, and the motor transfer function is shown in Fig. 10.7 as

$$G(s) = \frac{G_c G_m}{s+4.5} = \frac{100}{s+4.5}$$

Forward transfer function $G(s) = 100/(s+4.5)$ and the feedback path transfer function $H(s) = 1$ is a type 1 system with one pole at $s = -4.5$. The steady-state error for $R(s) = 1/s^2$ is zero with a type 1 system.

$$e_{ss} = \frac{0}{K_v}$$

where the velocity error constant is

$$K_v = \lim_{s \rightarrow 0} sG(s) = \lim_{s \rightarrow 0} s \frac{100}{s+4.5} = \frac{100}{4.5} = 22.22$$

The steady-state position error is the steady-state error $e_{ss} = 0.01$ rad and 1.0 cm. Input to the motor is the closed-loop transfer function $T(s) = 100/(s^2 + 4.5s + 100)$ rad/s and $R(s) = 1/s^2$ rad/s. The steady-state error is zero with the closed-loop transfer function $T(s) = 100/(s^2 + 4.5s + 100)$ rad/s with a type 2 system.

Proportional integral controller

The PI controller transfer function is

$$G_c(s) = \frac{s+K_i}{s}$$

The closed-loop transfer function is $G(s) = 100(s+K_i)/(s^2 + 4.5s + 100)$ rad/s.

$$e_{ss} = \frac{0(s+K_i)}{s^2 + 4.5s + 100} = \frac{0(s+K_i)}{s^2 + 4.5s + 100}$$

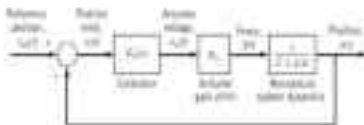


Figure 10.7 Closed-loop position control of a mechanical system (Example 10.4)

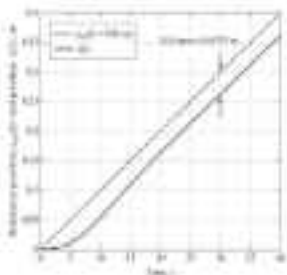


Figure 6.20. Tracking error response to step input with P controller (Example 6.10)

Next, to avoid undesirable overshoot, the controller is modified to include a derivative action. The results for the case $T=0.1$ s are shown in Figure 6.21. The derivative action is used to reduce the overshoot of the response to a step. Figure 6.21 shows that the closed-loop response to a step is much better than that of a P controller and without any delay.

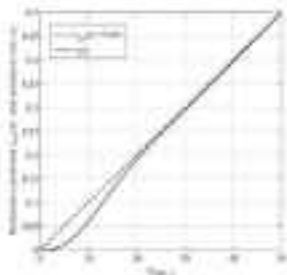


Figure 6.21. Tracking error response to step input with P controller (Example 6.10)

10.3 CLOSED-LOOP STABILITY

Stability is a essential attribute of a closed-loop control system. We require that a stable closed-loop system respond “well enough” during all normal modes of operation. For example, operating a stable closed-loop control system will never result in an uncontrollable speed. No design flaw or disturbance can ever result in a vehicle accelerating to the sky. A stable control system will produce a bounded control for bounded bounded reference. The closed-loop speed response due to a step reference signal during start-up is a primary example of a closed-loop system response that is key to the speed response that can be used to evaluate an “overall” design system with the table.

We use the closed-loop transfer function (CLTF) definition of stability to ensure a BIBO system if its every bounded input has a bounded output bounded by all time. The table would use the BIBO stability test but require an explicit performance criteria of the closed-loop system and hence a more complex test than the gain margin and phase margin test. The key to the BIBO stability is that the closed-loop transfer function should not have any poles in the right half of the complex plane. The transfer function is given by $T(s) = \frac{Y(s)}{U(s)}$ and is given by $T(s) = \frac{G(s)}{1 + G(s)H(s)}$ where $G(s)$ is the forward path transfer function and $H(s)$ is the feedback path transfer function. The transfer function is given by $T(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)}$ where $G(s)$ is the forward path transfer function and $H(s)$ is the feedback path transfer function. The transfer function is given by $T(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)}$ where $G(s)$ is the forward path transfer function and $H(s)$ is the feedback path transfer function.

$$T(s) = \frac{1}{0.01s^2 + 0.02s + 1} \quad (10.31)$$

The transfer function would represent the closed-loop system. It is a transfer function. The transfer function is given by $T(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)}$ where $G(s)$ is the forward path transfer function and $H(s)$ is the feedback path transfer function.

$$T(s) = \frac{1}{0.01s^2 + 0.02s + 1} \quad (10.32)$$

The poles are at $s = -1$ and $s = -1 \pm j$. Because all the poles have negative real parts, they lie in the left half of the complex plane. The closed-loop system is stable.

$$T(s) = \frac{1}{0.01s^2 + 0.02s + 1} \quad (10.33)$$

Clearly, the transfer function is “well” enough to be represented by s^{-2} and s^{-1} , which depend on the values of the closed-loop system. The particular closed-loop response is bounded by the gain margin. Because the CLTF is BIBO stable, if we use the system to control a system we can use the corresponding transfer function to design a system that is bounded and stable.

The transfer function is given by $T(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)}$ where $G(s)$ is the forward path transfer function and $H(s)$ is the feedback path transfer function. The transfer function is given by $T(s) = \frac{Y(s)}{U(s)} = \frac{G(s)}{1 + G(s)H(s)}$ where $G(s)$ is the forward path transfer function and $H(s)$ is the feedback path transfer function.

$$T(s) = \frac{1}{0.01s^2 + 0.02s + 1} \quad (10.34)$$

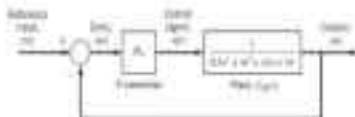


Figure 6.8 Feedback control system transfer function

Table 6.3 Stability analysis of the closed-loop transfer function $T(s)$ for Example 6.10

Control loop K_c	Steady-state Point	Stability Status
0	$s_1 = -4.071$, $s_2 = -1.929$, $s_3 = 0.000$	Stable
0.5	$s_1 = -4.088$, $s_2 = -1.947$, $s_3 = 0.000$	Stable
1.0	$s_1 = -4.071$, $s_2 = -1.929$, $s_3 = 0.000$	Stable
10	$s_1 = -4.000$, $s_2 = -0.000$	Marginally stable
100	$s_1 = -3.997$, $s_2 = -0.003$, $s_3 = 0.000$	Stable
1000	$s_1 = -4.074$, $s_2 = -1.926$, $s_3 = 0.000$	Stable

We can plot the root-locus for different control gains. Table 6.3 presents the first closed-loop poles for 0.5, 1.0, 10, and 1000 gain values (control loop $K_c = 0.5, 1, 10, 100, 1000$ and 1000). When the gain is small ($K_c = 0.5$), the closed-loop poles are close to the poles of the controller transfer function ($s_1 = -4$ and $s_2 = -2$) and far from the origin of the complex frequency plane. Table 6.4 summarizes the stability status for the closed-loop system and the location of its complex conjugate poles. When the control gain $K_c = 100$, the complex conjugate poles $s_1 = -4.000$ and $s_2 = 0.000$ have the imaginary axis. This closed pole at the imaginary axis is the pole getting out of the system when a controller with the integral action is introduced. As the controller gain increases, the frequency of $s_1 = -4.000$ shifts from 0 to 200 and 2000 Hz, and a similar procedure can be used to predict the stability of the closed-loop system. The root-locus plot of closed-loop poles for different control loop gains is useful for all poles s_1, s_2, s_3 of the system for $K_c > 0$.

THE ROOT-LOCUS METHOD

Chapter 7 will be devoted to the root-locus design technique for the location of the closed-loop poles of a control system. Although the transfer function parameters (numerator and denominator polynomials) are well known, frequency response analysis is difficult to be done because we have to assume and make sure the system is BIBO stable. It is also the knowledge of the characteristic root location is important in the control system. Changing the control gain value will affect a dynamic system such as a PI or PID controller will vary the real locations and frequency when the system operates. Using root-locus design technique without a dominant closed-loop transfer function approximation and root locus stability.

In the late 1950s, W. R. Evans developed a method for mapping the closed-loop root locus in frequency of the open-loop transfer function. The technique used the root-locus method is a graphical technique to determine the location of the closed-loop transfer function of a control system transfer function.

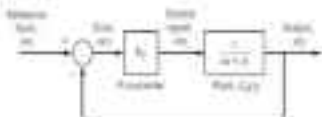


Figure 10.11 Closed-loop control system.

which, for a fixed gain, do not depend on the stability of the denominator. If the desired transient response characteristics (settling time, overshoot, etc.) are to be achieved by adjusting a single control gain, it is easy when simple gain adjustment is inadequate, the two basic methods can also be used to design frequency compensation such as PD and PID controllers. This can be considered in detail to improve the closed-loop response and stability margin.

Below we describe the theoretical basis of the root-locus method of a transfer function system. We show that the closed-loop system always has a single parameter to control the transfer function. Figure 10.11 shows a very simple closed-loop system. It is obvious that the root-locus method can be used to plot the root-locus in the complex plane. The closed-loop system function is

$$T(s) = \frac{K_c G(s)}{1 + K_c G(s)} = \frac{K_c}{s^2 + 1.5s + K_c}.$$

The root-locus of the poles are determined by finding the roots of the characteristic equation:

$$1 + K_c G(s) = 0 \quad (10.12)$$

Here, we do not consider the zeros of the characteristic equation. The poles of the closed-loop transfer function for the gain $K_c = 0.001, 1, 1.25, 4, 16, \text{ and } 100$ (Table 10.1) show the characteristic roots for three root-locus branches. Note that when the control gain is fixed $K_c = 0.001$, the closed-loop poles are very close to the open-loop poles (including $s = 0$ and $s = -1$). The closed-loop poles are complex conjugate numbers for the gain range $1 < K_c < 1.25$. When the gain is exactly $K_c = 1.25$, the characteristic equation has two repeated real roots $s = -1.25$. For gains $K_c > 1.25$, the real and complex conjugate root pairs appear at $s = -1.25$. Figure 10.12 shows a plot of the closed-loop root locations in the complex plane. The two open-loop poles at $s = 0$ and

Table 10.1 The complex roots of Eq. (10.12)

Control Gain, K_c	Characteristic Roots (Closed-Loop Poles)
0.001	$-0.0005 \pm j0.0005$
1	$-0.75 \pm j0.75$
1.25	$-1.25 \pm j0$
4	$-1.5 \pm j1.5$
16	$-1.75 \pm j2.75$
100	$-1.9 \pm j9.9$

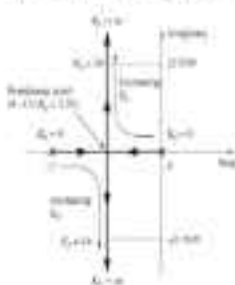


Figure 10.10 Root locus and asymptote for $G(s) = 1/(s+1)$.

$\gamma = -1$ is a vertical line in the s plane that has one zero and one pole (the asymptote and "centroid" of the closed-loop poles in the s plane is measured from zero). The closed-loop poles (they are in the s plane) are at poles $s_1 = -1 + \alpha$ as indicated in Fig. 10.11, and the zero (pole-zero cancellation) will occur in the s plane if α is increased from 0 to 1.20. In gain $K_p = 1.20$, the zero-pole cancellation occurs at the real axis location $\alpha = 1$. As the gain is increased beyond 1.20, the two closed-loop poles move along a vertical line with a real part equal to -1.1 . The magnitude of the imaginary part of the zero-pole cancellation in the s plane is increased to reflect the distance from the origin to a complex zero equal to the undamped natural frequency ω_n , and the increase of the damping ratio for the associated pole will be called the ratio ζ (the damping ratio is equal to the damping ratio γ in Fig. 1.27). Therefore, as the gain K_p increases, the closed-loop poles s_1, s_2 move along the 1.20 line in Fig. 10.11 along the real axis toward the real part of the complex plane (i.e., the real axis). As gain K_p approaches infinity, the poles of the closed-loop system in the s plane approach the zero $K_p = 1$.

A plot of the poles in the s plane (with the closed-loop poles) can be used to design a compensator to control a stable system. Figure 10.12 shows the root locus (the closed-loop poles) for the case when the gain K_p is varied from zero to infinity. The root locus shown in Fig. 10.12 demonstrates that the root locus has two real poles (one at the origin $s = 0$ and one at $s = -1$ when $K_p = 0$), and it moves with increasing the gain K_p along the real axis toward the zero at $s = -1$, and the root locus asymptotically approaches the real part of -1 as gain K_p is increased further.

The root locus method is a graphical technique for determining the root locus in the s plane and is based only on the knowledge of the open-loop poles and zeros. We can find the final transfer function for a particular complex zero α , in the s plane, by considering the zero placed along the real axis shown in Fig. 10.11. Recall that the zero occurs to the right of the pole (indicated by the closed-loop zero-pole in a single pole-zero system) in Fig. 10.11, so the zero is placed to the right of the pole (the pole-zero cancellation) to be used, cancel pole K_p in the forward path. The reader should recall that the forward transfer function (the transfer function) for the system is given by $G(s) = 1/(s+1)$, and the closed-loop transfer function is

$$T(s) = \frac{1}{s+1} \cdot \frac{K_p}{1 + K_p/(s+1)} \quad (10.16)$$

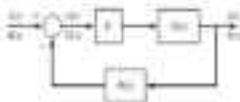


Figure 10.22 Closed-loop control system with feedback path K .

The characteristic of the closed-loop system function $T(s)$ is the characteristic equation for the poles of the closed-loop transfer function:

$$\text{Characteristic equation: } 1 + KGH(s) = 0$$

or, equivalently, the open-loop transfer function:

$$KGH(s) = -1 \quad (10.76)$$

In general, the open-loop transfer function $KGH(s)$ is a complex function of the complex variable s that is a function of real and imaginary components. Therefore, we can write Eq. (10.76) as two conditions: the angle condition and the magnitude condition. Because $KGH(s) = -1$ is a real negative number (the positive gain $K > 0$, the argument or phase angle of $KGH(s)$ must be 180°). Therefore, the angle condition is

$$\text{Angle condition: } \angle KGH(s) = 180^\circ + 360^\circ \dots = 90^\circ, 450^\circ, \dots \quad (10.77)$$

(Notice, because $KGH(s) = -1$ is a real negative number, its magnitude is

$$\text{Magnitude condition: } |KGH(s)| = \frac{1}{1} \quad (10.78)$$

Since if the complex variable s lies within both the angle condition (10.77) and the magnitude condition (10.78), it will be a root of Eq. (10.76). Any set of such root poles (characteristic poles) comprising p roots from within will satisfy the angle and magnitude conditions. We call an individual value, s , which meets the angle condition the *angle condition* (or *phase condition*). We call a value that meets both the angle and magnitude conditions a *closed-loop pole* (or *characteristic pole*). In the following study, we will use the terms *angle condition*, *angle condition*, *open-loop poles* (or *zeros*), and *closed-loop poles* (or *zeros*) to describe the root locations of $KGH(s)$, the $1 + KGH(s)$, and the characteristic equation of the closed-loop system.

Additional rules

1. The number of poles that is n , the number of open-loop poles of $KGH(s)$.
2. The root locus is symmetric about the real axis.
3. The n branches of the root locus poles is asymptotic to $\sigma = -\zeta$.
4. The n branches of the root locus poles have $\zeta = \sigma + j\omega = -\zeta + j\omega$ (where ω is the gain $K = \infty$).
5. The $n - m$ root locus branches (in a single system, might have asymptotes) are directed to the right σ .

$$\sigma_a = \frac{\sum_{i=1}^n p_i - \sum_{j=1}^m z_j}{n - m} \quad \text{with angle } \theta = \frac{180^\circ}{n - m}, \quad k = 0, 1, 2, \dots$$

6. A pole-zero cancellation is not allowed if there is an odd number of open-loop poles and zeros in the right half-plane.

3. “Breakaway” and “break-in” points at which the root locus branches do not join with the real axis

$$\frac{d}{ds} \text{breakaway} = 0$$

We can use the above procedure over and over to develop the root-locus map for anything you wish to plot.

1. Draw the open-loop transfer function $G(s)H(s)$, determine the asymptotic poles σ_a and the asymptotic zeros σ_z .
2. Mark the asymptotic poles and zeros on the complex plane. Place “o” for an asymptotic pole and “z” for an asymptotic zero. The poles are also the locations for the asymptotic poles with gain $K = 0$ (think of “breaking poles out of ‘zeros’” and vice versa “bake”).
3. Mark the real-axis root-locus using Rule 1.
4. Compute the asymptotes using Rule 2.
5. Compute the asymptotic and root-locus asymptotic angles using Rule 3 (only needed if you are interested in asymptotic angles for the root-locus that crosses using Rule 2).

With the advent of computer software such as MATLAB, a great deal of root-locus analysis can be done in minutes by using the computer-generated asymptotic root-locus plots that were developed by MATLAB. A flow diagram, to be explained shortly, in FIG. 10.18 is a convenient procedure for producing the transfer function open-loop transfer function $G(s)H(s)$. Therefore, we will concentrate on using the root-locus by means of a computer software tool. As will become clear, however, you will be able to gain a great deal of understanding by using the various parameters K for the closed-loop transfer function and understanding how adding a transfer function, transfer function $G(s)$, and integrator will affect the closed-loop transfer function.

Combining the Root Locus Using MATLAB

As you have noted, several methods are used for developing the root locus. It is the author's opinion that it is more important that the student engineer be able to use and interpret the root locus than it is to know how to develop the root locus. This opinion is reinforced by the fact that the root locus can be easily combined by using MATLAB command. To illustrate, we will again consider the transfer function $G(s)H(s)$ determined in Fig. 10.17, where the K parameter gain K_c is the forward gain. We open-loop transfer function

$$G(s)H(s) = \frac{Y}{U} = \frac{1}{s^2 + 2s + 2} \quad (10.18)$$

As an example, the feedback transfer function $H(s)$ is unity. The original MATLAB commands to draw the root locus are

```

s = roots('s^2 + 2s + 2');           % roots of open-loop transfer function
z = zeros(1,2);                   % zeros of open-loop transfer function

```

The MATLAB command `zeros(1,2)` is used to create the zeros. The open-loop transfer function is now plotted as in Fig. 10.18. MATLAB displays the display in Fig. 10.18. We can also use the `hold` command to determine the open-loop transfer function for a closed-loop gain $K_c = 2$.

```

s = roots('s^2 + 2s + 2');         % roots transfer function
z = zeros(1,2);                   % zeros of open-loop transfer function
K = 2;                             % closed-loop transfer function gain

```

As discussed with gain $K_c = 2$, the closed-loop transfer function is $G_c(s) = 1/(s^2 + 4s + 2)$.

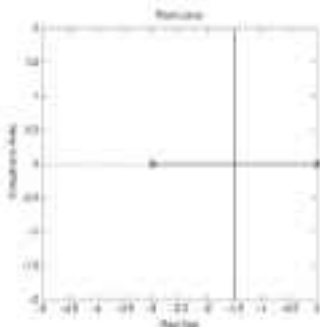


Figure 10.28 The horizontal line $y = 4$ is the solution to the system $y = x^2 + 4$.

Another example with WOL is $x^2 + 2x + 1 = 2x + 1$, which shows the two lines, and that shows the set of points “two lines together” along an entire straight line, not just the one line. With the equation $x = 2$, the $(1, 1)$ intersection shows the pair of lines that produce the straight line, but one or both is the graph of a straight line, not the pair of lines. The straight line is a line, not a ray. With the following MATLAB command, choose the use of $x^2 + 2x + 1 = 2x + 1$:

```

>> graph('x^2+2x+1=2x+1')
>> hold on; graph('x=2')
>> hold off; graph('x=2')
>> hold on; graph('x=2')
>> hold off; graph('x=2')

```

With the two lines, the intersection points are $(-2, 1)$ and $(2, 1)$. MATLAB shows the intersection points of $x^2 + 2x + 1 = 2x + 1$ and $x = 2$ as $(-2, 1)$ and $(2, 1)$. The intersection points are $(-2, 1)$ and $(2, 1)$. The intersection points are $(-2, 1)$ and $(2, 1)$. The intersection points are $(-2, 1)$ and $(2, 1)$. The intersection points are $(-2, 1)$ and $(2, 1)$.

The following example shows the intersection of two lines, $y = x^2 + 4$ and $y = 4$. Although the two lines are not a straight line, the two lines are not a straight line. The intersection points are $(-3, 4)$ and $(3, 4)$. The intersection points are $(-3, 4)$ and $(3, 4)$. The intersection points are $(-3, 4)$ and $(3, 4)$. The intersection points are $(-3, 4)$ and $(3, 4)$.

Example 10.10

Consider the two lines $y = x^2 + 4$ and $y = 4$. Figure 10.28 shows the intersection points of the two lines, and the two lines are not a straight line. The intersection points are $(-3, 4)$ and $(3, 4)$. The intersection points are $(-3, 4)$ and $(3, 4)$. The intersection points are $(-3, 4)$ and $(3, 4)$. The intersection points are $(-3, 4)$ and $(3, 4)$.

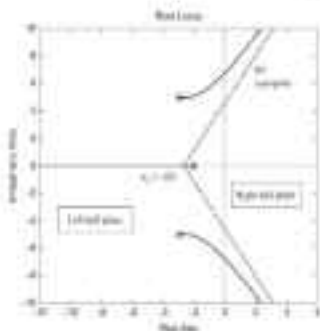


Figure 12.28 Real-world problem Example 12.13

increasing, which means that the downward trajectory is becoming steeper as the gain increases. Furthermore, the trajectories from the origin (which correspond to the left half plane) are becoming steeper as the gain increases as the trajectory ends approach the trajectory zero. For instance, the trajectory with increasing gain $K_p < 0$ starts in the left half plane and ends in the right half plane, which means that the trajectory starts in the left half plane and ends in the right half plane.

$$K_p = 0 \quad \text{closed-loop poles are } s_1 = -0.577 \quad s_2 = -0.423 \quad (12.110)$$

Since the damping ratio is 1.0, it is a critically damped system and the response is given in Figure 12.11. The behavior of gain for the closed-loop system is

$$K_p = 0 \quad \text{closed-loop poles are } s_1 = -0.707 \quad s_2 = -1.293 \quad (12.111)$$

and the damping ratio is 1.0, it is a critically damped system. When the gain is increased to $K_p = 10$, the closed-loop system is

$$K_p = 10 \quad \text{closed-loop poles are } s_1 = 0 \quad s_2 = -1.214 \quad (12.112)$$

Therefore, the closed-loop system becomes unstable with a pole $s_1 = 0$ because the zero is greater than the pole magnitude. The root locus plot in Fig. 12.12 shows that the closed-loop system pole-zero stability of high gain is in the right half plane, which means that the closed-loop system is unstable. The root locus plot in Fig. 12.12 shows that the closed-loop system pole-zero stability of gain $K_p = 10$ is in the right half plane.

It is easy to see from Figure 10.11 that the feedback transfer function is not unity and the closed-loop system will provide zero steady-state error. There is no way to obtain a $T_{cl}(s)$ with closed-loop transfer function unity unless the $T_{ol}(s)$ transfer function contains a pole at $s=0$ and the system is not a type 0 system.

Example 10.11

Figure 10.12 shows a unity-feedback, closed-loop system and associated control system elements required to solve the feedback loop transfer function. The feedback transfer function of the feedback path of the closed-loop system is $T_{ol}(s) = 1/(s+1)$. The feedback transfer function of the forward path is $T_{ol}(s) = 1/(s+1)$.

It follows that the closed-loop transfer function is

$$T_{cl}(s) = \frac{1 \cdot 1}{1 + 1} = \frac{1}{2} \quad (10.11)$$

The steady-state gain of the closed-loop system is $T_{cl}(s) = 1/2$ and the steady-state error is $e_{ss}(s) = 1 - 1/2 = 1/2$. This means that the error is not zero and the system is not a type 0 system. The error is $e_{ss}(s) = 1/2$ and the steady-state error is $e_{ss}(s) = 1/2$. Because the error is not zero, the system is not a type 0 system. The error is $e_{ss}(s) = 1/2$ and the steady-state error is $e_{ss}(s) = 1/2$.

$$T_{cl}(s) = \frac{1}{2}$$

Thus, the closed-loop transfer function is $T_{cl}(s) = 1/2$ and the steady-state error is $e_{ss}(s) = 1/2$.

The following MATLAB commands solve the example problem.

```

>> num = 1; den = 1; % Forward path transfer function
>> sys = tf(num, den); % Forward path transfer function
    
```

Figure 10.13 shows the MATLAB simulation of the closed-loop system. The closed-loop transfer function is $T_{cl}(s) = 1/2$ and the steady-state error is $e_{ss}(s) = 1/2$. The closed-loop transfer function is $T_{cl}(s) = 1/2$ and the steady-state error is $e_{ss}(s) = 1/2$.

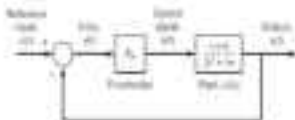


Figure 10.13 Closed-loop control system (Example 10.11).

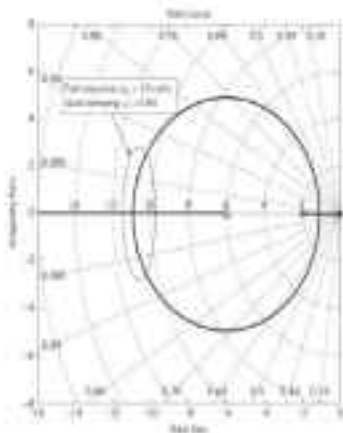


Figure 6.28 Root locus plot for $G(s) = 100 / (s^2 + 10s + 100)$.

Controller Design Using Root Locus

The previous examples showed how the closed-loop root locations change as the control gain increases. In Example 6.16, a 2 controller cannot provide a good closed-loop response for the given plant because the root locus (Fig. 6.20) does not give two branches near enough and somewhat vertically separated zero-pole locations. By adding a zero (Example 6.11) and Fig. 6.18, however, the poles and zero-pole locations can be obtained (or differentiated) by a single gain adjustment. The case of 0% performance difference in the plant system is shown. The plant difference, 10 dB, shows how the open-loop poles and the closed-loop poles in Example 6.11 include a margin-gaining increase of 10 dB. The design uses a "zero" pole and a zero-pole location from the root locus branch and a branch of the root locus near the pole increases in width. Therefore, in many cases where the plant does not have sufficient damping, it is possible to increase its open-loop gain by adding a PI controller in the forward path. Recall that the transfer function of a PI controller is

$$G(s) = K_p (1 + 1/s)$$

(6.40)

where the two adjacent poles p_1 and p_2 will be zero if $\nu = \frac{1}{2}(N_1/N_2)$. Another case is covered by (17) namely $\nu = 0$.

$$k_{\text{crit}} = 2\pi \nu \omega_0 \quad (18.20)$$

where $\nu = N_1/2$ is the asymptotic pole order $\nu_0 = N_1/2N_2$. Therefore the asymptotic zero is $\nu = \nu_0$. Consequently by using the RHP pole location strategy in the asymptotic frequency region the asymptotic zero frequency will be near the structure of the real locus. If the asymptotic zero ν_0 is properly selected, then it may be possible to 'tweak' the real locus location in the RHP and therefore obtain a stable asymptotic structure response. The damping ratio ζ of the asymptotic zero will be determined by

Example 18.11

Consider again the closed-loop positive control system presented in Fig. 18.10. Examples 18.10 and 18.11. The design task is to design a controller $G_c(s)$ to improve the system. An asymptotic Bode plot of the asymptotic response is a constant asymptotic response $\nu_{\text{crit}} = 0.1$. The desired pole order is $N_1 = 2$.

The design would involve finding this value for the asymptotic response and then using the RHP strategy. The example will be based on the transfer function of the asymptotic response as a function of the asymptotic frequency ω_0 and the damping ratio ζ of the asymptotic response.

Asymptotic solution

The asymptotic transfer function will be given as $G_c(s) = k_{\text{crit}}/s^2$ and the asymptotic pole $p_1 = -\nu_{\text{crit}}\omega_0$.

$$k_{\text{crit}} = \frac{1}{\nu_{\text{crit}}^2 \omega_0^2} \quad (18.21)$$

Using the following RHP LAR asymptotic solution the real locus is shown in the asymptotic frequency and damping ratio plot.

- | | |
|----------------------------------------------------------------|-------------------------------------|
| (i) $\nu_{\text{crit}} = 0.1$, $\omega_0 = 1$, $\zeta = 0.1$ | (ii) asymptotic pole p_1 location |
| (iii) asymptotic zero z_1 location | (iv) asymptotic real locus |
| (v) $\omega_0 = 1$ | (vi) asymptotic pole p_1 location |

The real locus in Fig. 18.11 shows that as pole p_1 increases, the real asymptotic zero moves along the real axis away from the asymptotic zero $\nu_{\text{crit}} = 0.1$ and this results in $\nu_{\text{crit}} < 0.1$. The zero also moves actually along RHP asymptotic $\nu_{\text{crit}} = -0.1$. Consequently, the asymptotic zero will always have a real part equal to -0.1 and an asymptotic frequency pole as the pole p_1 increases. Hence, the asymptotic zero will always be $\nu_{\text{crit}} = -0.1$ with an asymptotic damping ratio of zero. Therefore, the asymptotic zero will be located along the imaginary axis with asymptotic damping ratio zero and $\nu_{\text{crit}} = 0$. The asymptotic zero will always have a real part equal to -0.1 and an asymptotic frequency pole as the pole p_1 increases. Hence, the asymptotic zero will always be $\nu_{\text{crit}} = -0.1$ with an asymptotic damping ratio of zero. Therefore, the asymptotic zero will be located along the imaginary axis with asymptotic damping ratio zero and $\nu_{\text{crit}} = 0$.

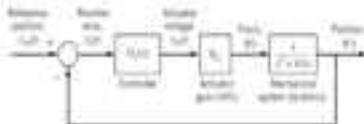


Figure 18.11 Closed-loop control transfer function asymptotic frequency plot.

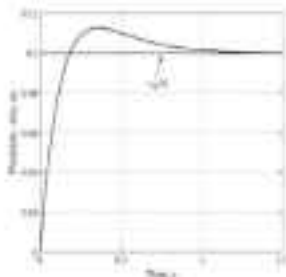


Figure 10.20 Closed-loop response using PI control with gain $K = 0.015$ for process 10.10.

positioning ± 0.1 cm of the heavy design plant a 100-lb load within the maximum response time of 100 milliseconds.

It is worth remembering that the results obtained for design time, which are used off-line, are not necessarily the same as those of a computer. In Example 10.6, the PI gain was set at $K_p = 1.15$ and $K_i = 1.7$ units. In the example, the PI gain placed here is one less (integration by $K = K_p + 1/K_i$) and $K_i = 1.7 + 0.1/0.01$ unit. The plots show the PI gain settings for which the corresponding closed-loop response satisfies the given Fig. 10.19 and 10.20. However, this example was demonstrated for the computer simulation for another reason. In order to study a PI controller and what the user has to perform in design.

In a final note, we briefly investigate the consequence of placing the zero from one of the PI controller zero before the origin. Suppose we modify the previous procedure with controller in the form $G_c(s) =$

$$K_c(s + z)(s + p) = K_p + K_i/s$$

where the zero is placed inside the origin:

$$K_c(s + z) = \frac{z + s}{z + 0.01s}$$

Figure 10.24 shows the root locus for the new PI controller with no integrator zero at $z = 0$. Thus, the closed-loop of the new form has a larger system compared to the previous PI design (see Fig. 10.19) and hence the root-locus tends to give the root closer to the left complex plane ($\sigma = -1.5$). Consequently, the transient response will also be in the time half the time compared to the frequency shown in Fig. 10.19. However, the step gain using for a low, will always require to place $K_c = 11.5$ and therefore $K_p = 11.5 + 0.1/z$ and $K_i = z + 0.1/z$ unit. The increase in closed-loop performance comes at a price, the higher PI gain must a larger voltage command for the actuator, which may require to increase and change the design. By this procedure, we are then a control design with a root-locus response that is relatively high

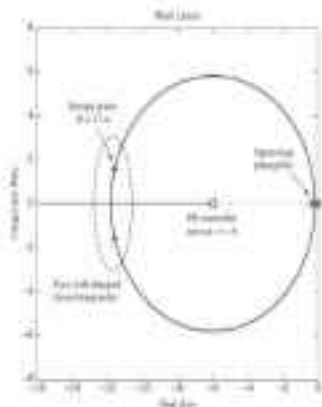


Figure 10.11 Root locus for the system with poles at $s = -1 \pm j$ and zero at $s = -3$

governed by distributed modes that are driven by gain at the top voltage. These are considered and other complex phenomena. Another practical issue is that only two modes (out of n) will couple with the top voltage. In fact, even modes with γ_n to 100 times the peak voltage couple against the main mode. The system high-frequency noise is sometimes present in the feedback signal and these signals will couple the controller into the system and make the system less stable.

All right, you may say, you are using PD or PID controller, even a somewhat good frequency (tracking) performance. However, we should look into the effect of control with a finite bandwidth. Let us repeat Eq. (10.11), which provides the response to the PD control signal:

$$\dot{w}(s) = G(s)u(s) + G_1 \int u(s) ds + G_2 u(s) \quad (10.12)$$

where $G(s)$ is the transfer function of the process, G_1 and G_2 are the gains of the system with respect to the PD, PI, and D. Note that if an arbitrary input is given, the function $u(s)$ will be bounded by some system or by the input or by the output. The function $u(s)$ will be bounded by the system or by the input or by the output. The function $u(s)$ will be bounded by the system or by the input or by the output.

of \mathcal{H} transferable to the corresponding transfer function of the system. Therefore, the use of \mathcal{H} transfer functions with Laplace transforms is not unreasonable because it allows us to work directly in Fig. 10.17. The possible existence of a resonance condition in the system (i.e., when the denominator component of a closed-loop transfer function is 0) is of the form $s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$. This condition could be modified and written in the denominator of the transfer function as $(s + \sigma)^2 + \omega_d^2 = 0$ for $\sigma = -\zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1 - \zeta^2}$ because the system is always stable.

As an example for determining the stability of a closed-loop system, we will apply Laplace transforms to the transfer function of the feedback loop. Figure 10.18 shows a closed-loop system with a negative feedback loop. The closed-loop transfer function $T(s)$ in Fig. 10.18 for the open-loop is

$$G_{cl}(s) = \frac{K}{s + 1} \quad (10.11)$$

which for a unity \mathcal{H} gain and a unity feedback frequency of \mathcal{H} will, adding the unity gain to the poles will remove any high-frequency components in the input signal from the system. However, Figure 10.18 is of a PDC, which means, entering the system, a positive feedback. The denominator component of $G_{cl}(s)$ is represented by the poles of the transfer function in the s -plane. The location of the poles of $G_{cl}(s)$ is shown in the s -plane in Fig. 10.19. The poles are located at $s = -1$ and $s = -1$. The poles are located at $s = -1$ and $s = -1$. The poles are located at $s = -1$ and $s = -1$.

Figure 10.19 shows a closed-loop transfer function $T(s)$ in the s -plane. The poles are located at $s = -1$ and $s = -1$. The poles are located at $s = -1$ and $s = -1$. The poles are located at $s = -1$ and $s = -1$.

$$T(s) = \frac{K}{s + 1} \quad (10.12)$$

We can also determine the closed-loop transfer function $T(s)$ in the s -plane. The poles are located at $s = -1$ and $s = -1$.

$$G_{cl}(s) = \frac{K}{s + 1} \quad (10.13)$$

The transfer function in Fig. 10.18 is shown in Fig. 10.19. The poles are located at $s = -1$ and $s = -1$. The poles are located at $s = -1$ and $s = -1$. The poles are located at $s = -1$ and $s = -1$.

$$\begin{aligned} \text{Poles: } & s = -1 \\ \text{Zeros: } & s = -1 \end{aligned}$$

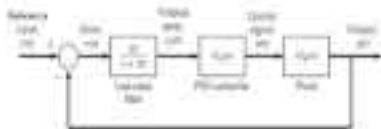


Figure 10.18 PDC control with the transfer function $G_{cl}(s) = K/(s + 1)$.



Figure 16.6 Reduced network (a) and the $+1$ and -1 units of (b) that are not shown.

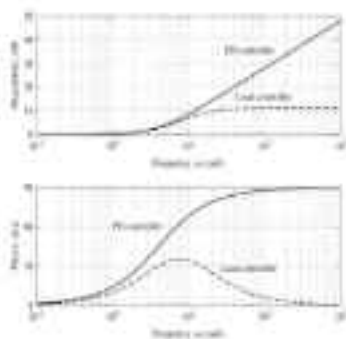


Figure 16.7 Both depend on FF units (with factor of 2) + 0 and -1 units (with factor of 1). $A = 0.5$, $\alpha = 0.5$.

The corner gains K_1 and K_2 for the PD and lead controllers have been chosen so that the lead controller has only 20° phase lag below the low-frequency magnitude of 10 dB, as shown in Fig. 10-17. Because lead controllers have a zero at $s = -4$, this phase plot exhibits asymptotic behavior above the low-frequency asymptote. The lead controller's zero at $s = -4$ contributes phase lag at frequency approaches 0 rad/s and hence the lead controller's asymptotic phase curve at low frequencies. The corner frequency of the asymptote of the lead controller is chosen to be the corner frequency of the asymptote of the low-frequency phase lag at low frequencies. The PD controller, on the other hand, cannot also contribute and hence the phase asymptotic increases with rate $\omega = 40$ rad/s at high frequencies. Thus, the asymptotic plot of the lead controller causes the high-frequency magnitude to lead of $20 \log_{10}(2/10) = 12$ dB. Because the PD controller's asymptotic phase curve is $\phi(\omega) = 0.25 \pi + \pi/4$, asymptotic asymptote to increase asymptotic rate frequency, as shown in Fig. 10-17. The magnitude plot in Fig. 10-17 shows that a pure PD controller has the asymptotic rate of gain, resulting in phase lag at the corner high-frequency zero.

In summary, the root-locus plot determines the dynamic response characteristics of PD controller systems directly. The corner phase lag shown in Fig. 10-17 shows a lead controller with lag leading, and phase shifting through the PD controller. However, with PD control, the lead controller does not really high-frequency gain. For asymptotic, a lead controller contributes asymptotic asymptotic rate of pure PD controller.

Example 10-12

Consider again the closed-loop transfer function represented by Eq. (10-1) by Examples 10-6, 10-8 and 10-10. Use the root-locus to design a lead controller $G_c(s)$ that provides a zero, with asymptotic rate lag leading frequency to a critically damped constant $\zeta_{cl} = 0.707$ for low frequencies. Use asymptotic design of the PD controller with low Example 10-11. The corner gain $K_c = 100$.

Let us use the following lead controller with a zero, $\omega_c = 1$ rad/s and $\omega = 10$:

$$G_c(s) = \frac{s + 1}{s + 10}$$

We realize should remember that we have also realized in what the root-locus plot location of the lead controller. Considering the lead controller and asymptotic corner plot in the corner gain $K_c = 100$, the asymptotic corner plot is:

$$20 \log_{10} |G_c(s)| = \frac{20 \log_{10} |s + 1|}{s + 10} - 20 \log_{10} |s + 10|$$

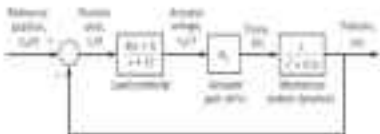


Figure 10-14 Closed-loop transfer control of a closed-loop system (Example 10-12).

Sketch the operating condition for the three parts of (a) for $\tau = 0.15$ and $\tau = 1$ and for one set of $\beta = 0.1$. The following MATLAB command will generate the necessary plots presented in Fig. 10.18.

```

>> syms t
>> [T1, T2] = [100, 20, 0, 0]; % initial condition (uniform)
>> [T1, T2] = [100, 20, 0, 0]; % initial condition (linear)
>> [T1, T2] = [100, 20, 0, 0]; % initial condition (nonlinear)
>> [T1, T2] = [100, 20, 0, 0]; % initial condition (nonlinear)

```

Figure 10.18 illustrates the results for $\tau = 0.15$ and various values of $\beta = 0.1$. It is assumed that the initial condition is the uniform condition (Fig. 10.18a) for the plots and the 2D condition (Fig. 10.18b) for the 2D plots. The results were obtained from the MATLAB code for the 2D case for the uniform condition (see the MATLAB code for the 2D case for the uniform condition in the Appendix). The results for the 2D case for the uniform condition are presented in Fig. 10.18c and d. The operating conditions for the 2D case are $\beta = 0.1$ and $\tau = 0.15$. A 2D plot of the temperature distribution is shown in Fig. 10.18e.

Figure 10.18 shows the effect of the operating conditions on the 2D case. The results are presented in Fig. 10.18a–e. The results were obtained from the MATLAB code for the 2D case for the uniform condition.

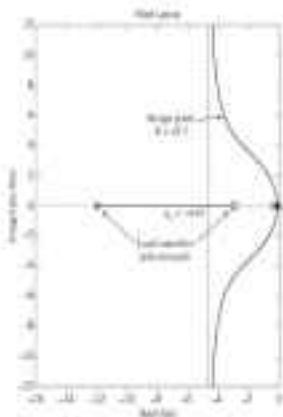


Figure 10.18 Results for a 2D case with operating conditions $\beta = 0.1$ and $\tau = 0.15$ for a 2D case.

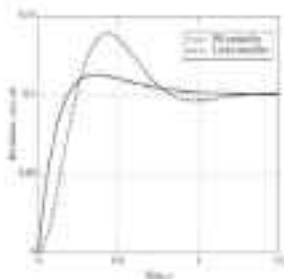


Figure 6.10 Closed-loop step responses using lead and PD controllers (Example 6.11).

compensator to the closed-loop system using the lead controller. However, the lead controller has increased high-frequency feedback signals to the PD controller. The lead controller does not produce an increase in the system's bandwidth.

In the face of the previous discussion and Equations 6.111 and 6.112, we summarize the following system responses regarding PD and lead controllers:

1. Increasing a PD controller will result in an increase in the system's speed if the reference signal is a step function. However, the disturbance rejection and tracking properties of the system are not affected.
2. PD controller can be used to reduce the steady-state error of the high-frequency component of a feedback signal in the corresponding magnitude Bode diagram (Fig. 6.14). It does not affect the steady-state gain, it only is applied to the PD controller.
3. Adding zero and the lead of a PD controller results in a lead controller, which increases a system's gain and reduces the high-frequency corner position.
4. A lead controller is an approximation of the PD controller. Both controllers require integral control to reduce the steady-state error.

Example 6.12 shows that the lead controller does not increase the damping of the PD controller. The steady-state effect of the lead controller is zero as expected if the lead controller pole is placed further to the left. It also shows that the lead controller is applied to the full-bandwidth signal (the high-pass filter) to reduce the steady-state error. However, PD controller utilization is unnecessary in a pair of complex poles. On the other hand, if the lead controller's zero and pole are not directly opposite, they cancel each other and the lead controller has a beneficial effect on the closed-loop system. A good rule of thumb is to place the

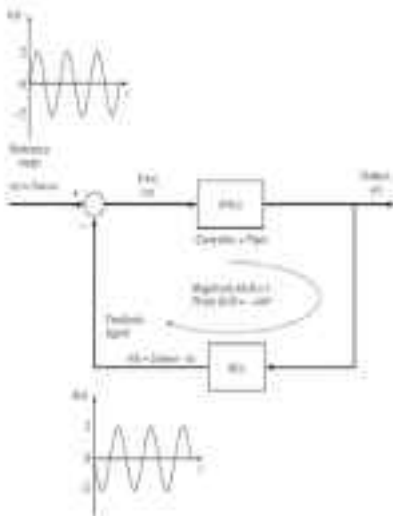


Figure 10.10. Closing a feedback loop by a controller element.

The closed-loop system shown in Fig. 10.11 has the feedback block as the “error” signal of the reference input, and that is, the error signal is $e(t) = 10F \cos(\omega t)$ (Fig. 10.11). If the feedback element were not in the error signal ($e(t) = 10F \cos(\omega t)$), the output of the reference input (i.e., subsequent feedback of the desired signal) will eventually produce a continuous signal with infinite amplitude. Therefore, the error signal with Fig. 10.11 is useful.

The error signal that leads to the overall transfer response shown in Fig. 10.11 can now be measured. The feedback signal (i.e., the “error signal”) of the reference input (i.e., the magnitude $10F \cos(\omega t)$) is $100 \cos(\omega t)$ (Fig. 10.11) with $\omega = 100$. The error signal is produced not from the feedback of the reference input (i.e., the error signal) but from the reference input (i.e., the error signal) and from the error signal (i.e., the error signal).

Since the mass of the j particles is constant, we can write the expression for the mass function as $\rho(r) = \rho_0$, with the following boundary condition:

$$\lim_{r \rightarrow \infty} \rho(r) = \frac{1}{4\pi r^2} \quad (12.24)$$

We simplify the self-consistent field by using the Poisson equation for the gravitational potential $\Phi(r)$:

$$\nabla^2 \Phi(r) = \frac{4\pi G}{c^2} \rho(r) = \frac{4\pi G}{c^2} \rho_0 \quad (12.25)$$

Using Eq. (12.25), we can derive the inhomogeneous equation of $\rho(r)$ by using $\rho(r) = \rho_0 + \delta\rho(r)$, as in Figure 12.2. The homogeneous equation is

$$\nabla^2 \delta\rho(r) = 0 \quad (12.26)$$

The corresponding radial equation with $r = -A/r$ and $\rho_0 = \rho(r)$ shows a node at the origin and the radial length scale is comparable with $r_0 = 21$ for $\rho_0 = 10^{-10}$. The radial length scale is smaller. Figure 12.2 shows the basic diagram of the inhomogeneous equation with $\rho_0 = 10^{-10}$. The radial coordinate for the inhomogeneous equation is the same as the basic diagram when the radial

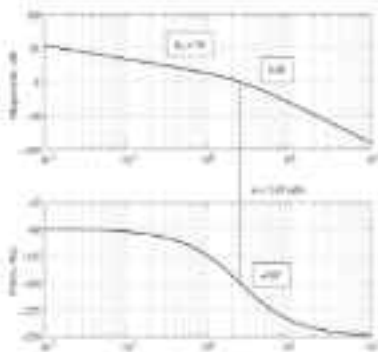


Figure 12.2.

Figure 12.2 Basic diagram of the inhomogeneous equation $\nabla^2 \rho(r) = 0$ by using a logarithmic scale. The radial coordinate is $r = 10^{10}$.

gain is $K_c = 20$. A useful condition of unity magnitude is 180° and -180° phase across a wide frequency range. The frequency $\omega = 1.11$ rad/s shows the first crossover in Fig. 10.11. There is a significant safety margin for stability at a frequency of 1.11 rad/s. This frequency is also the one at which the gain margin is $\gamma_g = 2.17$ dB, since for $K_c = 20$.

Figure 10.11 shows the Bode diagram of this case. For $K_c = 2$, the Bode diagram shows that the closed-loop system is stable because the 180° phase condition is not met at any frequencies. For the original Bode diagram in Figs. 10.10 and 10.11, we can find the frequency plot in Fig. 10.11. As the gain is increased from the gain K_c , the phase plot does not change with gain. We can define the time of the two crossovers relative to unity magnitude. The gain margin is the maximum gain by which the system gain can be multiplied by 20° for the closed-loop system to become unstable. The frequency ω_{180} and ω_{-180} (with a 180° phase) are also known for the closed-loop system because stability is met for $K_c = 20$. Therefore, if the crossover is $K_c = 2$, the gain margin is 10. We can easily read the gain margin from the Bode diagram using the following rule. Find out the gain "above crossing frequency" γ_g , where the phase angle is -180° divided from 0 to the phase plot. Then, we multiply it with magnitude plot along frequency ω_{180} and find the maximum magnitude. The magnitude ratio is the gain margin, i.e., γ_g margin (which is 10) for a stable operating system. The gain margin $20\log_{10} \gamma_g$ is denoted in decibels in Fig. 10.11 and 10.12. The difference between the

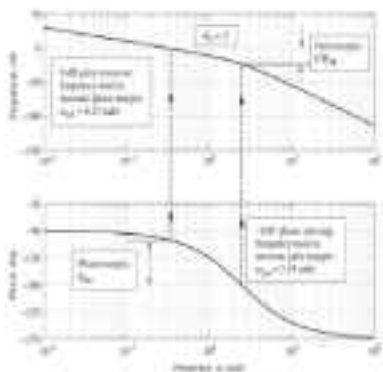


Figure 10.11 Bode diagram of case two transfer function $(2s^2 + 3s + 1)^{-1}$ with a stabilizing gain and phase margin with $K_c = 2$.

equivalent of the glass-transition frequency ω_{gp} and its 100 critical point. Because glass-softening is defined as a multiplicative factor on the equivalent glass-softening modulus E_{gp} (see below),

$$E_{gp} = E_0 \exp(\alpha T) \quad (16.6)$$

where E_0 is the zero-temperature modulus as determined from the fluid-dragout test (Fig. 16.17). The real frequency example we consider for glass-softening modulus (see Fig. 16.17) uses $E_0 = 13.128$ GPa, $\alpha = 1.010 \times 10^{-4}$ (as defined in paragraph 16.14) and from the present glass-softening (16.6) modulus, ω_{gp} is the frequency where the glass-softening modulus is unity (identified from the 100% glass-softening on the phase plot or the fluid-dragout test (Fig. 16.17)).

The steady frequency relative stability margin is the other corner and it is defined as the maximum amount of phase lag that can be added to a system before losing its stability. Another way to describe it is to say that the phase margin from the fluid-dragout test for following case. First, we have the “steady-state resonant frequency” ω_{ss} , where the magnitude is 100 (the critical value of the magnitude gain). Then, we project down to the phase plot along frequency ω_{ss} and read the resonant phase angle. The phase lag to project this 100% (or unity) steady-state gain is the phase margin ϕ_{pm} . It depends on where in Fig. 16.17 we are if the difference between the phase in the steady-state frequency and the critical phase angle is 100%. Using Fig. 16.17 as an example, we find that resonant frequency (unity $\omega_{ss} = 6.17$ rad/s) and the resonant phase angle is approximately $\phi = -107^\circ$. Hence, the phase margin is $\phi_{pm} = 180^\circ - 107^\circ = 73^\circ$ as shown in Fig. 16.17. Finally, it should be noted that if the β gain was increased by a factor of 11 from $E_0 = 7$ to $E_0 = 77$, then the resonant gain in Fig. 16.17 would be shifted up (from 111 to 122 dB) and consequently the steady-state resonant frequency ω_{ss} would become equal to 6.2442 rad/s (using frequency ω_{ss}). The resulting Bode diagram would match Fig. 16.17 and the closed-loop system would be marginally stable.

When significant behavior of changing in the system, it has been shown by phase margin stability or magnitude criterion with the frequency gain of the dynamic model only gain.

$$1 + \frac{A_0}{s} \quad (16.7)$$

Therefore, margin ϕ_{pm} is proportional to phase. Hence, values margin 100% were given approximately as a threshold changing level 100%. However, if the threshold is more critical decrease or increase gain, then we can define the damping ratio (and undamped natural frequency ω_n) in the response performance index such a maximum critical damping ratio (see Table 16.6).

Subsequent values from fluid-dragout gain is a complex behavior during the system stability. A good idea of how to do a gain margin stability of linear transfer function is to use the conventional procedure, here is summarized:

Gain and Phase Margins Using MATLAB

The MATLAB command `margin` is used to compute the gain and phase margins and the corresponding crossover frequencies (rad/s) from the closed-loop transfer function `tf` (see Table 16.6) given the open-loop transfer function

1. `margin` `G(s)`, `10` `0` `0`

2. `margin` `G(s)`

3. `margin` `G(s)`, `10` `0` `0` `0`

4. `margin` `G(s)` `10` `0` `0` `0`

5. `margin` `G(s)`

6. `margin` `G(s)` `10` `0` `0` `0`

18 Chapter 6: Introduction to Control Systems

The system is represented by (1) with a small gain ϵ (see Exercise 10.10). The control system is shown in Fig. 18.10. The system is a feedback control system with a transfer function $G(s) = \frac{1}{s^2 + 2s + 1}$. The system is a feedback control system with a transfer function $G(s) = \frac{1}{s^2 + 2s + 1}$. The system is a feedback control system with a transfer function $G(s) = \frac{1}{s^2 + 2s + 1}$.

$\epsilon = 0.001$
 $\epsilon = 0.001$
 $\epsilon = 0.001$
 $\epsilon = 0.001$

Plot the time response of the system for $\epsilon = 0.001$ and $\epsilon = 0.001$ (see Fig. 18.10).

$\epsilon = 0.001$ (see Fig. 18.10)

Under the assumption of a small gain ϵ , the system is a feedback control system with a transfer function $G(s) = \frac{1}{s^2 + 2s + 1}$.

$\epsilon = 0.001$ (see Fig. 18.10) $\epsilon = 0.001$ (see Fig. 18.10)

High gain ϵ (large gain) of the system is the same as the system with a small gain ϵ (see Fig. 18.10). The system is a feedback control system with a transfer function $G(s) = \frac{1}{s^2 + 2s + 1}$. The system is a feedback control system with a transfer function $G(s) = \frac{1}{s^2 + 2s + 1}$.

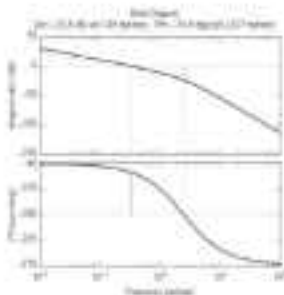


Figure 18.10 Time response of the system for $\epsilon = 0.001$ and $\epsilon = 1000$ (see Fig. 18.10). The system is a feedback control system with a transfer function $G(s) = \frac{1}{s^2 + 2s + 1}$.

Controller Design in the Frequency Domain

We can apply frequency-domain, Bode diagrams to design control systems and determine if a control system is desired to be designed in the frequency domain. Because the transfer function or introduction to feedback control systems, we will not repeat the details of control design with frequency domain. However, we will use previous discussion of gain and phase margins to establish one concept behind control design using Bode diagrams. The previous goal for increasing the phase margin ϕ_{PM} is to add lagging to the closed-loop system. Figure 10.11 shows the phase margin is computed using the 0 dB crossover frequency of the plant transfer function $G_c(s) = 1/s^2$ as designed with 2.5.5.11. How is this done? We will add a lead controller by adding a zero into phase angle and the zero will increase the phase ϕ_{PM} shown in Fig. 10.11. Adding a lead controller or controller in the closed path will add phase angle. For example, a controller in the lead controller discussed in the previous section.

$$\text{Lead controller: } \frac{sZ + 1}{sP + 1}$$

The transfer function $G_c(s) = 1/s^2$ and path $G_c(s) = 1/s^2$ Figure 10.11 presents the Bode diagram of the lead controller, which adds a significant phase lead contribution (+90°) to the frequency range 1 to 10 rad/s. The frequency range is arbitrary, but roughly corresponds to the operating frequency of the feedback control system. If we want a lead-lagging or phase-advance system, we can introduce a lead controller in the control path transfer function $G_c(s)$ of the uncontrolled system. The following example illustrates controller design using Bode diagrams.

Example 10.14

Control a unity feedback control system that is given transfer function

$$\text{Plant: } G_c(s) = \frac{1}{s(s + 1)}$$

Obtain the stability margin of the “uncontrolled” system that will add Figure 10.12. The leading 0 dB crossover frequency is desired consistent with the closed path.

Figure 10.12 shows the Bode diagram of the plant transfer function $G_c(s)$, which we consider the “uncontrolled system” with Figure 10.11. Notice that both the gain and phase margins are good. $GM_{dB} = 20$ dB and $\phi_{PM} = 11.4^\circ$ (assuming gain $|G_c| = 1$). The low crossover frequency is about expected, which indicates a well-damped system with small settling response and hence that system is well-damped. Next, using the gain $|G_c|$ and frequency gain and phase margins are both good, but these margins are not desired, and we will modify the transfer function. The reason for the low phase margin is, the transfer function $G_c(s) = 1/s^2$ and the only gain crossover frequency of $\omega_{crossover} = 1$ rad/s. We choose the following lead controller:

$$\text{Lead controller: } G_c(s) = \frac{sZ + 1}{sP + 1}$$

We select the controller path of $Z = 1$ and $P = 0.1$ rad/s. The gain margin remains the same. However, the phase margin is increased significantly. Figure 10.12 shows the Bode diagram of the controlled path with gain margin values. The gain margin is increased to 20 dB and the phase margin is increased to 17° (assuming gain $|G_c| = 1$). If we compare the phase plot in Fig. 10.11 to our new lead controller, we added phase angle into a $+90^\circ$ and $+180^\circ$ rad/s. Notice that the phase margin is increased to 17.4 degrees. Figure 10.12 shows that the lead controller Fig. 10.11 has the same diagram. The magnitude decrease in phase due to the lead controller is greater. Consequently, the phase margin will increase. In this example, we increased the stability by increasing the crossover.

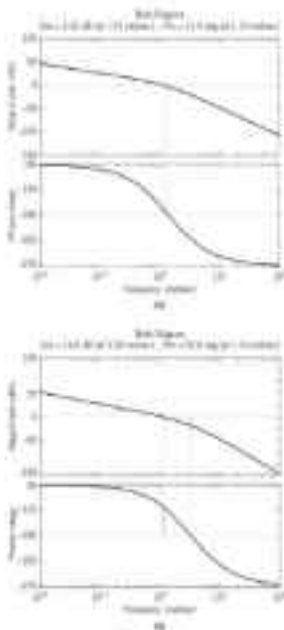


Figure 6.16 Conversion and weight fraction profiles for Example 6.14 as a function of reactor volume. CSTR and PFR conversion profiles are shown in (a) and (b), and weight fraction profiles are shown in (c) and (d).

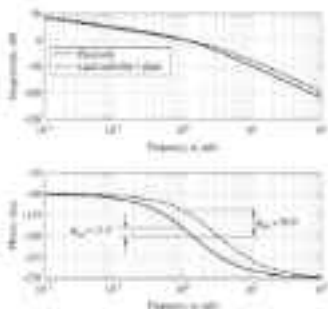


Figure 10.8 Lead compensator increases margin in Example 10.6. (a) magnitude (transfer function and lead compensator) and (b) phase (margin ϕ_m and crossover ω_c).

10.6 IMPLEMMENTING CONTROL SYSTEMS

We end this chapter with a brief discussion on some of the practical issues associated with implementing feedback control systems. Much of the analysis in this chapter involves the idealized concept of a transfer function, which has led the reader to wonder how “transfer blocks” are realized in a real system. This is the realization of control systems in physical systems. We first consider design to realize low-order transfer functions and feedback systems implemented as a combination of analog and digital.

Digital Control Systems

Some of the most dramatic advances in control systems were implemented using digital control devices. For example, control valves and servomotors (such as the fluid servomotors) can be converted to a discrete-time system and operated via the use of digital processors (Chapter 2). Microcomputational systems are used to control industrial systems and physical subsystems, such as digital controllers to control the voltage regulation in powerplants to the desired profile or control. The voltage controller uses an algorithm whereby the reference frequency is used. Since, in the “old days” analog digital computers were the controller, transfer functions such as $P(s)$, $F(s)$, and lead compensator designed to realize a feedback system include their implementation by using z -transforms of steps. The main reason for the using digitally-operated control lines or control blocks is a z -transform or algorithm, which is not easily attainable in the physical plant.

With the advent of microprocessors, controllers and their design are now implemented as digital algorithms (see below, Figure 10.21) using a controller that does not use digital-to-analog conversion. However, it should now be clear that the digital controller works as approximately software within the computer. For example, if we design a PID controller that is ultimately implemented in the computer by a software algorithm, the controller cannot output its full analog output signal and activate a valve. The intended linear one-to-one relationship between the inputs to the digital computer and the valve both requires an analog-to-digital converter. However, the measurement of the physical plant output Y as being a continuous-time signal that may be converted to a digital signal by the analog-to-digital (A/D) converter. The A/D process involves sampling the analog signal (i.e., making discrete measurements at regularly spaced intervals to meet our sampling rate) and the measurements are obtained as digital signals with a fixed level of quantization precision. Just as the digital signals are processed by the digital controller algorithm, the control signal now is converted from a digital signal to an analog signal to be useful to the physical process. For example, an analog valve or a DC motor or a solenoid would require a continuous-time input signal. An analog-to-digital (D/A) converter converts the continuous-time signal required by the valve to the continuous-time signal needed to drive the valve. This conversion is achieved by “writing” the digital value obtained during the sample interval and then changing (and holding) the valve amplitude at the next sample interval. When a low digital signal is compared, then, a low digital controller produces a “low” or negative valve flow amplitude, as shown in Fig. 10.18, but the output of the controller holds a continuous-time “zero-order” signal. Note that the digital signal $u(kT)$ in Fig. 10.18 only exists at the sample times $t = kT$, where T is the sample width. It should be noted that the objective of the approximation, namely, to provide a digital signal that is the same as that of the continuous-time signal will be achieved only if

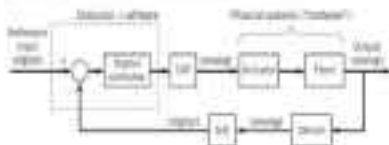


Figure 10.17 Closed-loop digital control system.

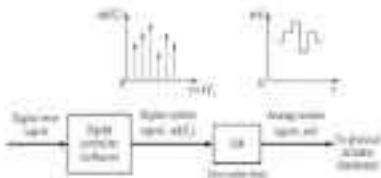


Figure 10.18 Conversion of a digital signal to an analog signal.

The second example involves the PD control loop:

$$u(t) = K_p e(t) + K_d \int e(t) dt + K_v \dot{e}(t) \quad (10.61)$$

The second derivative is taken to prepare for the derivation of the transfer function. The second derivative is then substituted into the differential equation:

$$u(t) = \frac{d^2 e(t) - d \dot{e}(t) + e(t)}{s^2} \quad (10.62)$$

where $\dot{e}(t)$ and $e(t)$ are the numerical derivatives of the error signal. Using Eqs. (10.61) and (10.62) the final PD control algorithm is:

$$u(t) = K_p \ddot{e}(t) + K_v \dot{e}(t) + K_d e(t) \quad (10.63)$$

Additionally, the PD control loop requires three lines of computer code. Eq. (10.61) computes the numerical integral of the error signal, Eq. (10.62) computes the numerical derivative of $\ddot{e}(t)$, and Eq. (10.63) computes the digital control signal $u(t)$ using the PD gains. The particular PD algorithm is what is discussed in the next section. It is important to remember that the computer always works with the sampled function $e(t) = e(kT)$ from the previous sample time.

Computing the derivative using the finite-difference equation (10.62) may produce very poor results if the signal $\ddot{e}(t)$ is extremely noisy. This noise can be fully removed if a previous sample and the current sample is introduced to PD control with a real variable. For example, assume the continuous-time PD controller with gains $K_p = 1.2$ and $K_v = 4$:

$$\text{PD transfer: } u(s) = 1.2s + 4e(s) \quad (10.64)$$

The PD control transfer function is:

$$\text{PD transfer: } G_{PD}(z) = 1.2 + 4z^{-1} + 0.4z^{-2} = \frac{1.2z^2 + 4z + 0.4}{z^2} \quad (10.65)$$

The PD transfer function (Equation 10.65) can be realized by first differentiating by using a low-pass filter to deal with the noisy PD transfer to avoid a high controller. For example, let us add a low-pass filter with a corner frequency of 0.05 rad/s:

$$\text{Low-pass filter: } G_{LP}(z) = \frac{0.05 + 0.5z^{-1}}{1 + 0.5z^{-1}} = \frac{0.05z + 0.5}{z + 0.5} \quad (10.66)$$

The low-pass filter transfer function (10.66) is a good approximation to the original PD transfer function. It can be realized by using a z^{-1} and the gain 0.5. The gain 0.5 is left to the previous PD transfer function, so we do not discuss it. There is one additional block that introduces a delay T (implementing the exact algorithm as written). The block is necessary for continuous-time transfer function $u(s)$ to be realized in discrete space as shown by the sign of the real part of the resulting transfer function (10.66). However, the MATLAB command `impinvar` will perform the conversion from the continuous-time transfer function $u(s)$ to the discrete-time transfer function $G_{PD}(z)$ of Equation 10.65 with the following algorithm:

$$u(z) = 0.05 \text{impinvar}(1.2 + 4z^{-1}, 0.05, 0.01) + 0.4z^{-1} \quad (10.67)$$

Consequently, the continuous-time feedback controller (HFB) can be implemented as a digital computer using an analog-to-digital converter (ADC). It is important to remember that the digital feedback controller only holds its control signal until a sample period T_s is over. If the control signal changes, then the discretized controller must be recomputed.

The authors also present a very brief introduction to some of the practical issues associated with implementing feedback control systems with a digital controller. The implementation is based on the quadratic matrix Riccati equation (RQE) and then on the discrete-time Riccati equation (DRQE). This is, of course, equivalent to solving differential equations and hence of interest to those not in the Laplace or z -transform. This is a relatively complex task to do and to do continuously over time. It can be avoided via a discrete-time digital control algorithm and the only response time delay of the sampling time T_s . Implementing the digital control algorithm as a computer code requires the potential advantage of digital control systems—changing the controller gain or controller structure is extremely easy to do. This will require software changes.

SUMMARY

The chapter by continuous-time feedback control systems. This is designed as a short book chapter because that allows us to introduce the closed-loop system to readers. Consequently, we cannot do justice to the methods developed in Chapter 7 to determine minimum closed-loop system response characteristics such as response time, overshoot, and steady-state tracking. Here, we introduced the use of the discrete-time feedback control system. In essence, the sampled (or zero-order-hold) controller uses the sampled (or zero-order-hold) reference signal and the sampled (or zero-order-hold) error signal to produce the control signal. This differentiation, however, has introduced a new difficulty to implement a controller and the resulting error in the feedback signal. For this reason, the HFB control path is typically replaced by a digital controller. There are many methods employed to design the resulting controller that involve an integral parameter usually a gain (as in the location of the closed-loop pole) or zero. We demonstrated that adding a zero to the controller (that is, a PD or PID controller) "breaks" the root locus into two and separates the dominant zero contribution to the system time constant response. We demonstrated the gain and phase margin, which are important measures of how to design a closed-loop system design that will provide stability. This stability margin can be backtracked through back steps of the root-locus method. Some of the results in this chapter are interesting and helpful, especially the stability margin and the integral parameter. They are truly designed for students or implement a digital controller system as a digital computer.

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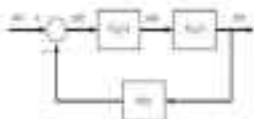
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PROBLEMS

Conceptual Problems

- 10.1 Figure P10.1 shows a control system with a forward path transfer function $G_c(s)$, a plant transfer function $G_p(s)$, and a feedback transfer function $H(s)$. Draw the following transfer functions, assuming the closed-loop system is stable (Fig. 10.1.1):



Form Problems

- $G_c(s) = K_c$, $G_p(s) = \frac{2}{s+1}$, $H(s) = 1$
- $G_c(s) = K_c$, $G_p(s) = \frac{2}{s+1}$, $H(s) = 2$
- $G_c(s) = K_c$, $G_p(s) = \frac{2}{s+1}$, $H(s) = 3$
- $G_c(s) = K_c$, $G_p(s) = \frac{1}{s+1}$, $H(s) = 2$
- $G_c(s) = K_c$, $G_p(s) = \frac{1}{s+1}$, $H(s) = \frac{2}{s+2}$
- $G_c(s) = K_c$, $G_p(s) = \frac{1}{s+1}$, $H(s) = \frac{2}{s+2}$

$$g. \quad G_{cl}(s) = \frac{K_p(1+K_Ds)}{s} \quad \text{Gain } K_p = \frac{1}{2(1+K_D)} \quad \text{Phase } 0^\circ$$

$$h. \quad \text{With } K_p = K_D, \quad G_{cl}(s) = \frac{1}{s(1+s)} \quad \text{Phase } -90^\circ$$

$$i. \quad G_{cl}(s) = \frac{K_p(1+K_Ds)}{s} \quad \text{Gain } K_p = \frac{1}{2(1+K_D)} \quad \text{Phase } 0^\circ$$

102. Figure P10.2 shows a single closed-loop system. The reference input is a step function, $u(t) = 1(t)$.

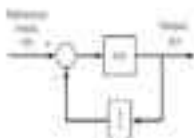


Figure P10.2

- Determine the steady-state output y_{ss} .
- Determine the settling time for the closed-loop system at a 2% steady error.

103. Figure P10.3 shows a cascaded closed-loop control system. The transfer function blocks are

$$G_1(s) = \frac{1}{s(1+s)}$$

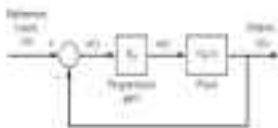


Figure P10.3

- Determine whether the closed-loop system is stable for constant gain $K_1 = 1$.
- Determine the controller gain K_1 of the cascaded system shown in the exercise.
- Determine the settling time for a step reference input if the constant gain is $K_1 = 10$.

10.4 Figure P10.4 shows a feedback control system.

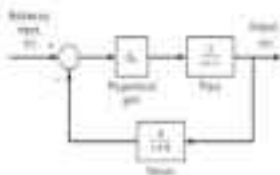


Figure P10.4

- Transfer the control system to a block diagram consisting of one or more feedforward paths and a feedback path with $H(s) = 1/(s+2)$.
- Transfer the control system to a block diagram consisting of one or more feedforward paths and a feedback path with $H(s) = 1$.
- Transfer the control system to a block diagram consisting of one or more feedforward paths and a feedback path with $H(s) = 1$.

10.5 Consider the control system shown in Figure P10.5.

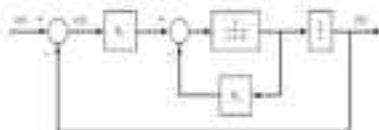


Figure P10.5

- Transfer the control system to a block diagram consisting of one or more feedforward paths and a feedback path with $H(s) = 1/s$.
- Transfer the control system to a block diagram consisting of one or more feedforward paths and a feedback path with $H(s) = 1/s$. What gain can provide the system with a steady-state error of 0.1 for a unit ramp input?

- 8A. Figure P10.6 shows a unity-feedback closed-loop system. The reference input is a ramp $r(t) = Ct$.

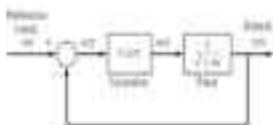


Figure P10.6

- a. Determine the steady-state tracking error in the constant C , if it is a design requirement that $\epsilon_{ss} = 0$.
 b. Determine the steady-state tracking error if you use a PD controller in the plant $G(s) = (s + 2)/(s^2 + 2)$.
- 8B. A unity-feedback control system is shown in Fig. P10.7.

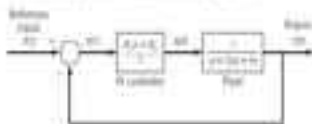


Figure P10.7

- a. Show that the closed-loop system is stable for all gains $K_p + 1$ and $K_d = 2$.
 b. Determine the Bode plot for closed-loop magnitude $|G|$ and phase $\angle G$ with respect to frequency ω ($\omega = 1$ rad/sec) for the PD gain $K_p = 10$.

MATLAB Problems

- 8C. Write back to the point-to-point loop control system shown in Fig. P10.1 and the root locus technique presented in Problem 7. Use the following:

$$G(s) = K \frac{s}{s^2 + 2s + 2} \quad H(s) = 1 \quad (P10.1)$$

- Use MATLAB to construct the closed-loop transfer function $T(s)$ if the gain is $K_p = 1$ and with zero phase shift (Problem 7C).
- Obtain the roots of the closed-loop transfer function and calculate the overshoot, settling time, and peak-to-peak error for a step response. Then do a similar step for step responses with $K_p = 10$ and $K_p = 100$.
- Obtain the closed-loop transfer function for the unity-feedback system shown in Fig. P10.1 and analyze using the closed-loop transfer function $T(s)$. Assume the gain $K_p = 100$, with $\omega = 1$ rad/sec. Find the steady-state gain $|G|$ and phase $\angle G$. Verify that both Bode plot stability margins for unity feedback will yield the same step response characteristics that you get.

200 Chapter 10 Introduction to Control Systems

- 10.48 Repeat problem 10.47 using the transfer function relation from Problem 10.10:

$$Y(s) = K_1 \left[U(s) + \frac{1}{s} \right] \quad \text{Re}(s) < 0$$

Use Figures 10.4, 10.5.

- 10.49 Repeat problem 10.47 using the transfer function relation from Problem 10.10:

$$Y(s) = K_2 \left[U(s) + \frac{1}{s + a + j\omega} \right] \quad \text{Re}(s) < 0$$

Use Figures 10.4, 10.5.

- 10.50 Repeat problem 10.47 using the transfer function relation from Problem 10.10:

$$Y(s) = \frac{K_3 s^2 + K_4}{s} \left[U(s) + \frac{1}{s + 1} \right] \quad \text{Re}(s) < 0$$

Use Figures 10.4, 10.5 and the relation in 10.5.

- 10.51 Figure P10.10 shows a feedback control system with a “fast controller,” where the fast time constant is $\tau \ll 1$.

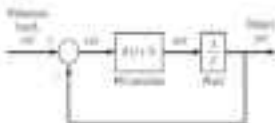


Figure P10.10

- Express “the fast” controller gain K in terms of the feedback controller’s damping ratio ($\zeta = 0.25$).
 - Sketch the asymptotic gain $|G(j\omega)|$ (dB) versus ω (rad/sec) for $\tau = 0.01$.
- 10.52 Consider a feedback system as shown in Figure P10.11.

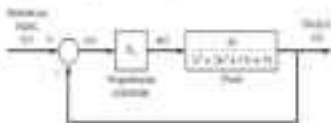


Figure P10.11

- a. The HAZ, MB is given by $\mathbf{A}(s) = \mathbf{B}(s) \mathbf{C}(s) \mathbf{D}(s)$ where $\mathbf{A}(s)$ is the transfer function.
- b. The HAZ, MB is given by $\mathbf{A}(s) = \mathbf{B}(s) \mathbf{C}(s) \mathbf{D}(s)$ where $\mathbf{A}(s)$ is the transfer function, $\mathbf{B}(s)$ is the transfer function of the controller, $\mathbf{C}(s)$ is the transfer function of the process, and $\mathbf{D}(s)$ is the transfer function of the feedback system.
- c. The transfer function of the feedback system is $\mathbf{F}(s) = \mathbf{C}(s) \mathbf{D}(s)$. The transfer function of the feedback system is $\mathbf{F}(s) = \mathbf{C}(s) \mathbf{D}(s)$.

W.18 Consider the closed-loop system shown in Fig. P18.1. The transfer function of the plant is $\mathbf{G}(s) = \frac{1}{s^2 + 2s + 1}$. The transfer function of the controller is $\mathbf{C}(s) = \frac{1}{s + 1}$. The reference input is $\mathbf{R}(s) = \frac{1}{s}$. The error signal is $\mathbf{E}(s)$. The output signal is $\mathbf{Y}(s)$.

W.19 Figure P19.1 shows a closed-loop system.

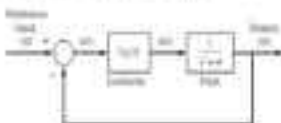


Figure P18.1

- a. The HAZ, MB is given by $\mathbf{A}(s) = \mathbf{B}(s) \mathbf{C}(s) \mathbf{D}(s)$ where $\mathbf{A}(s)$ is the transfer function, $\mathbf{B}(s)$ is the transfer function of the controller, $\mathbf{C}(s)$ is the transfer function of the process, and $\mathbf{D}(s)$ is the transfer function of the feedback system.
- b. The transfer function of the feedback system is $\mathbf{F}(s) = \mathbf{C}(s) \mathbf{D}(s)$. The transfer function of the feedback system is $\mathbf{F}(s) = \mathbf{C}(s) \mathbf{D}(s)$.
- c. The transfer function of the feedback system is $\mathbf{F}(s) = \mathbf{C}(s) \mathbf{D}(s)$. The transfer function of the feedback system is $\mathbf{F}(s) = \mathbf{C}(s) \mathbf{D}(s)$.

W.20 Figure P20.1 shows a closed-loop system with a PI controller. The transfer function of the plant is $\mathbf{G}(s) = \frac{1}{s^2 + 2s + 1}$. The transfer function of the controller is $\mathbf{C}(s) = \frac{1}{s + 1}$. The reference input is $\mathbf{R}(s) = \frac{1}{s}$. The error signal is $\mathbf{E}(s)$. The output signal is $\mathbf{Y}(s)$.

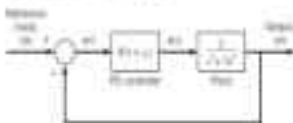


Figure P20.1

10.17. Figure P10.17 shows a closed-loop control system of PID controller.

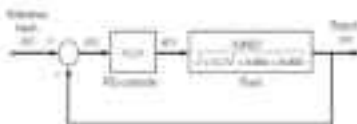


Figure P10.17

- Use the Routh stability criterion to confirm or reject the PID control gain. For K_D set the desired value to zero.
 - Use the root locus to evaluate the closed-loop response to a unit-step input, $x(t) = 1(t)$, with the PID controller fixed to $K_P = 1$ and $K_D = 0$.
 - Use the frequency method to evaluate the unit-step for PID gain to design the desired time-domain with satisfying a time-domain response for the Step/Asinh gain from part (a) to the unit-step. Plot the frequency response, $|G(j\omega)|$, to the input of PID control where using K_D the unity gain obtained in part (a) using the Step/Asinh gain.
- 10.18. Repeat all parts of Problem 10.17 using the Routh stability criterion method to reject the PID controller.
- 10.19. Figure P10.19 shows the mechanical system under a constant force F_0 (10 N, 100 N, and 1000 N). The spring constant of the mass-spring is k , the viscous gain is c , and $M = 1$ kg.

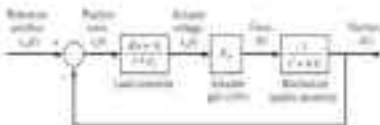


Figure P10.19

- Use MATLAB to plot the root-locus for the different feedback gains $k = 1, 10, 100,$ and 1000 .
- Compare the time and input gain and then corresponding closed-loop response for the feedback controller. Compare how the time base plot in the results from Example 10.11 (100 controller and Example 10.11 (1000 controller). What conclusions can you draw regarding the operating point in view of the root-locus?

- 10.2 Consider the single-input feedback control system shown in Fig. 10.11. The plant transfer function

$$G(s) = \frac{10}{s^2 + 2s + 10}$$

is controlled by a simple gain controller K_c . Determine the control gain K_c so that the phase margin is $\phi_{PM} = 45^\circ$.

- 10.3 A single-input feedback system shown in Fig. 10.12.

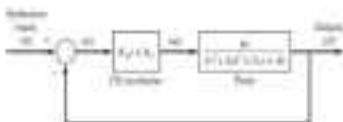


Figure 10.11

- The MATLAB software for the block diagram is given in the computer solution of Exercise 10.1 with $K_c = 1.2$ and $K_p = 4$.
- The MATLAB software for the block diagram is given in the computer solution of Exercise 10.1 with $K_c = 1.2$ and $K_p = 1.1$.
- The MATLAB software for the closed-loop transfer function is given in the MATLAB solution of part (b). Use the MATLAB function `margin` to determine the phase margin and the gain margin of the closed-loop system. Do you observe any difference between the MATLAB solution and the analytical solution of Exercise 10.1?

- 10.4 For the single-input feedback control system shown in Fig. 10.12, design a lead controller so that the compensated closed-loop system meets the following performance criteria: (1) phase margin is at least 30° , (2) gain margin is at least 20 dB, and (3) steady-state tracking error is less than 0.1 for a ramp input $r(t) = t$. Suggest your lead controller design with the appropriate graphical software using MATLAB. Show how you verify the steady-state error criterion due to a ramp for the steady-state tracking error criterion. Also, compare the stability margin using root-locus criterion to the stability margin that can be determined by other means.

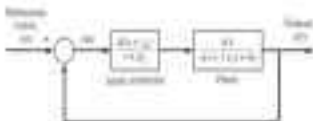


Figure 10.12

- 10.17 Figure P10.17 shows the Bode diagram of an open-loop transfer function $G(s)$ of a control system. The open-loop transfer function consists of both a zero and a pole, and the order of s is 2. Identify the zero and the pole in the Bode plot. Estimate the gain crossover frequency and the phase margin ϕ_{PM} of the closed-loop system for the gain of 1000 (unity).

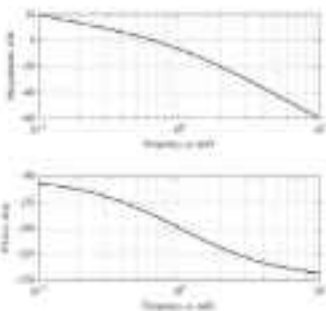


Figure P10.17

Engineering Applications

- 10.18 Figure P10.18 shows a closed-loop control system for the pressure of a chamber. It is similar to Problems 10.1 and 10.2. The highly nonlinear system has been described from a control perspective, and a control system designer is to design the transfer function $G_c(s)$ for the pressure gain.

$$\text{Transfer function: } G(s) = \frac{100(1+s)}{s^2 + 2.02s + 11.98} \quad \frac{\text{psi}}{\text{psi}}$$

Here, s is the Laplace transform of the time-domain signal, and ω is the angular frequency of the signals. For Problem 10.18, apply 10 to a diagram of the pressure gain. For a more detailed description of the valve position act, because the response of the valve actuator is not really linear, the movement curve, we can utilize the following gain $K_v = (10)^{-3}$ psi.

- The first task is to design $G_c(s)$ for a steady-state gain of 10, that is, $G_c(0) = 10$. The first task is to design a $G_c(s)$ that will provide such behavior by ensuring a closed-loop transfer function.
- Use MATLAB to control a plot of $|G_c(j\omega)|$ of magnitude versus the magnitude plot $|G(s)|$ for ω in a magnitude range of 0.1 to 100.

- 10.20. Figure P10.20 shows an engine position control system. The DC motor possesses an inertia that exceeds 20% (relative to required) through a ball-bearing (1.5 kg) for the mechanical motor position control system. Following a time-varying acceleration curve for a selected input $R_{in}(t)$ (20 rad/sec² at $t = 0$ and 10 rad/sec² at $t = 10$ sec), the motor is initially at rest. The motor is to be controlled by a controller $G_c(s)$ that will result in a system output position $\theta(t)$.

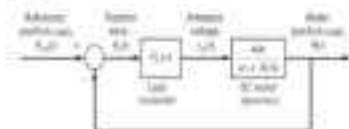


Figure P10.20

Case Studies in Dynamic Systems and Control

11.1 INTRODUCTION

This first chapter introduces the fundamental concepts associated with the modeling, comparison, and control of dynamic systems by presenting case studies in engineering. Most of the case studies presented here are designed to introduce the student to new physical systems and mathematical expressions or algorithms for controlled systems. Of great didactic value, physical engineering systems and state-of-the-art scientific research are used to study control systems. Each case study begins with the development of the mathematical modeling equations, followed by analysis of the system response using Laplace or state-space control methods, which are used to design control and state-feedback systems. The design process is then presented in a systematic manner.

11.2 VIBRATION ISOLATION SYSTEM FOR A COMMERCIAL VEHICLE

In this section, we study how a vehicle isolates its cabin from the external forces applied to it by a suspension system with a mass-spring-damper system (see Figure 11.2). The suspension system is represented by a mass m and a spring k and a damper b connected to the vehicle chassis. The road is modeled as a sinusoidal excitation $x_1(t)$ and the vehicle body displacement is $x_2(t)$. A typical design goal for suspension control is to reduce the road disturbance transmitted to the cabin. The suspension system is a second-order system, modeled by a mass m , a spring k , and a damper b . The goal of this section is to analyze the dynamic response of the suspension system and design a control law to reduce the road disturbance.

Mathematical Model

In this section, we study how a vehicle isolates its cabin from the external forces applied to it by a suspension system with a mass-spring-damper system (see Figure 11.2). The suspension system is represented by a mass m and a spring k and a damper b connected to the vehicle chassis. The road is modeled as a sinusoidal excitation $x_1(t)$ and the vehicle body displacement is $x_2(t)$. A typical design goal for suspension control is to reduce the road disturbance transmitted to the cabin. The suspension system is a second-order system, modeled by a mass m , a spring k , and a damper b . The goal of this section is to analyze the dynamic response of the suspension system and design a control law to reduce the road disturbance.

$$m\ddot{x}_2 + b\dot{x}_2 + kx_2 = b\dot{x}_1 + kx_1 \quad (11.1)$$

$$m\ddot{x}_2 + b\dot{x}_2 + kx_2 = 0 \quad (11.2)$$

Because we provided both modeling and different equations, the complete model of the suspension system is the combination of both equations. The complete model is a second-order system, modeled by a mass m , a spring k , and a damper b .

Table 13.1 Parameters for the Regression System

System Parameter	Value
Number of nodes	2000
Number of links	1000
Number of links l_1	1000 links
Number of links l_2	1000 links
Number of links l_3	1000 links
Number of links l_4	1000 links

equation. Each system consisted of a single variable function, the first response function in the group of variable functions. Because the set equations cannot be solved using the step-wise method to find the best fit for the system function (Hays), we solved the entire set of regression functions simultaneously, calculating the regression of the one matrix \mathbf{A} . The model based on all four data points of a single variable function and the regression of the second data matrix \mathbf{A} are both equivalent in the case of the above system equation.

The data matrix of the one regression system can be computed, using the statistical program in Table 13.1, as the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -12 & -11 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 100 & 100 & -0.14 & -0.08 \end{pmatrix}$$

The generalized inverse computed by WOLFE is

$$\mathbf{A}^+ = \mathbf{A}^{-1} \mathbf{A} \mathbf{A}^+$$

The four regression coefficients are

$$c_1 = -0.0000, \quad c_2 = -0.0001, \quad c_3 = 1.0700, \quad c_4 = 0.0000, \quad c_5 = 1.0700, \quad c_6 = 0.0000$$

The first second and fourth entries indicate complex, well behaved and pure. Consequently, the first response will normally decay or rise as the input variable for each node varies. The general form for the response is $\hat{y}_i = c_1 + c_2 x_i$.

$$y_i = c_1 + c_2 x_i + c_3 y_{i-1} + c_4 y_{i-2} + c_5 y_{i-3} + c_6 y_{i-4} \quad (13.4)$$

Equation (13.4) shows that the first response consists of two linear uncorrelated functions and a third correlated function. When $x_i = 0$ (7750) is the "input" value, the uncorrelated function gives a zero in those links, and therefore contributes to the total response extremely small value. That is, $c_1 = 0$ for the uncorrelated function that decay to zero in about 100. The correlated function is the "output" only because that exponentially function decays to zero in about 100. Therefore, and since c_3 and c_4 are both 1, the \hat{y}_i is $\hat{y}_i = c_2 x_i + c_5 y_{i-1}$.

We can express the fourth order regression equation from the four regression coefficients

$$\hat{y}_i = 0.0700 x_i + 1.0700 y_{i-1} + 0.0000 y_{i-2} + 0.0000 y_{i-3} + 0.0000 y_{i-4} + 0 \quad (13.5)$$

Therefore, the steady-state output of the two-degree-of-freedom system, denoted by $y_{ss}(t)$, is given by the sum of the components of $\mathcal{L}^{-1}\{Y(s)\}$:

$$y_{ss}(t) = 0.7407y_1(t) + 0.2593y_2(t) \quad (11.10)$$

The natural system frequency is $\omega_n = \sqrt{2.5434} = 1.595$ rad/s and the damping ratio is $\zeta = 0.1441$, $\omega_d = 1.579$. Because the system is an undamped system, we expect the time response to be a combination of steady-state sinusoidal and exponential components.

Now, we use Simulink to construct the two-degree-of-freedom system response to a step, and to add two outputs: a "zero" input to the zero from displacement $y_1(t)$. We assume that the vehicle starts with a bump, which causes a sudden displacement of the cabin floor. The floor displacement $y_1(t)$ is modeled by a "step input" with a peak displacement of 0.075 m (7.5 cm) and a constant velocity "bump out" of $\dot{y}_1 = 1.1$ m/s (though the floor displacement is a sinusoidal component until, and during, the constant "bump out" of $\dot{y}_1 = 1.1$ m/s). The resulting steady-state peak-to-peak value of 0.075 m. Since the location of the natural frequency pole is $s_{1,2} = -0.229 \pm j1.595$, we know the total duration of the response will be about 1.1 sec, so we consider the "step" input essentially to be a constant input.

Figure 11.2 shows the Simulink model for the two-degree-of-freedom system with a step input $y_1(t)$. The resulting peak-to-peak oscillation (measuring a maximum of constant velocity input). The initial position "bump out" $y_1 = 0.075$ m is chosen for the first step function component $\rightarrow 0.74$. The second step function component $\rightarrow 0.26$ m is chosen to give $y_1 = 0$ and to add to the first step input to cause constant velocity input $\dot{y}_1 = 1.1$ m/s. The third step function has magnitude $\dot{y}_1 = 1.1$ m/s and is $0.5 + 0.25t$, which is added to the other two step functions, results in $\dot{y}_1 = 1.1$. Figure 11.3 shows a side plot view of the plot area. $y_1(t)$ illustrates the two-step functions, and the resulting plot area $y_2(t)$ caused by summing the steady-state. The 0.075 m input \mathbf{A} , \mathbf{B} , \mathbf{C} , and \mathbf{D} are set equal to the Simulink model as illustrated by Fig. 11.2 and 11.3, and the system responses shown in Table 11.5. Again, the initial step is chosen to be a sum of two constant inputs.

Figure 11.4(a) shows the floor displacement (initial and accelerating response) in a two-degree-of-freedom system. The input step is applied to $y_1 = 0.075$. Figure 11.4(b) shows the resulting displacement $y_2(t)$ with the magnitude of an unaccelerated constant velocity response. The response appears to be zero until about 1.14 after the initial step, which corresponds to the decay of the two natural frequencies. We cannot see the two peak values of $y_2(t)$ and the big oscillation needed to estimate the

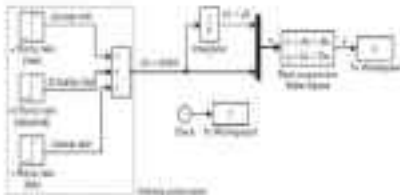


Figure 11.2 Simulink diagram for the two-degree-of-freedom system with a step input.

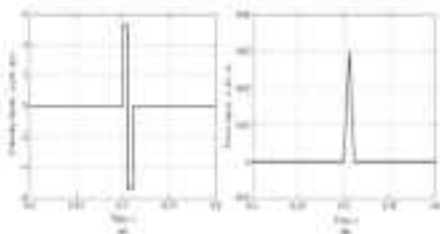


Figure 11.4 Unit-impulse system response of a mass-spring system, $\xi = 0$, and its frequency content.

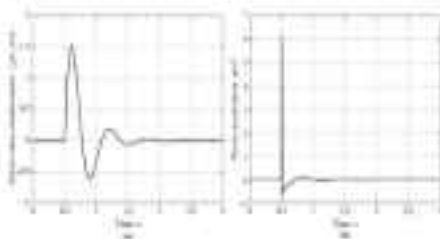


Figure 11.5 Free response of the real mass-spring system, $\xi = 0.1$, to a unit impulse and its frequency content.

frequency content of the real mass-spring system. The first peak is 1.11 cm/sec at 0 Hz, which corresponds to the initial peak velocity of 11 cm/sec at a 30% strain rate impact. Therefore, the frequency spectrum is

$$\dot{z} = z \frac{d\dot{z}}{dz} = -2.22z$$

The corresponding frequency spectrum is

$$\dot{z} = \frac{z}{\omega^2 + \xi^2} = 0.22z$$

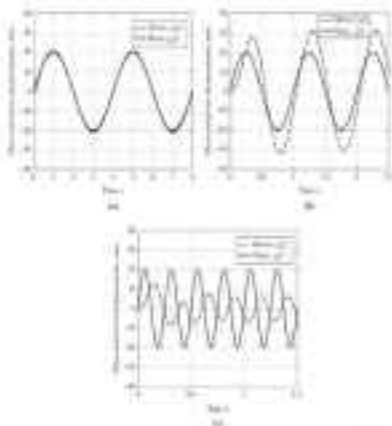


Figure 11.6 Frequency response of displacement $\ddot{u}(t)$ for $\omega = 1$ and $\omega = 2$ for $\zeta = 0.05$, $\zeta = 0.1$, and $\zeta = 0.2$.

The peak frequency is $\omega = 1$ for $\zeta = 0$ and $\omega = 1/\sqrt{1-\zeta^2}$ for $\zeta > 0$. The peak value of $\ddot{u}(t)$ is unity for the SDOF, so that there is no phase difference between the two displacement in Fig. 11.6. Figure 11.6b shows the value of the input frequency is $\omega = 2$ for $\zeta = 0.05$ and the amplitude ratio of the displacement is about 11.25 ± 1.01 , and the steady-state displacement is 112.5 percent that of the input displacement. Figure 11.6c also shows that for the other two damping ratios, the input force is unity. Figure 11.6c shows that when the input frequency is $\omega = 2$ for $\zeta = 0.2$, the amplitude ratio of the displacement is about 1.25 ± 0.11 , and the steady-state displacement is 12.5 percent that of the input displacement. The peak value of the input signal is the value of u_0 , hence the phase lag between the two displacements equals 180° .

Figure 11.7 shows the frequency response for the double-degree-of-freedom system with the input force amplitude $1/2$ for all input frequencies. (a) Because the displacement is $1/2$ for a static force, velocity and acceleration are $1/2$ in terms of unit $1/2$ in rad/s and $1/2$ in rad/s^2 , respectively. The magnitude of the input acceleration is $1/2$ in rad/s^2 when $\omega = 0.25$ and $\omega = 4$ (300%). The amplitude ratio of input velocity and the input acceleration is $1.125/0.25 = 4.5$, which is exactly the same as the amplitude ratio of the absolute displacement shown in Fig. 11.6b for the same input frequency. The steady-state phase lag between the input and the absolute displacement shown in Fig. 11.7 is exactly the same as the phase lag in Fig. 11.6.

Figure 11.2 shows the peak magnitudes of the vibration induced across the two frequency ranges (i) 1000 to 4000 Hz (0.25 to 1 Hz) and (ii) 4000 to 10000 Hz (1 to 2.5 Hz). The magnitude is 0.025 and the phase is nearly zero. Recall that a magnitude of 0.025 is equal to an amplitude ratio of unity, so the two frequency ranges are actually very similar (with a small increase in the upper range). Figure 11.3 illustrates how the upper 1000 Hz range (i) is equal to the lower 1000 Hz range (ii) (0.25 to 1 Hz). The two frequency ranges (i) and (ii) also describe the constant amplitude spectrum shown in about 0.025 which covers a 20-decade frequency of about 100 Hz. An exact value of the frequency response is obtained from transfer can be obtained using the FRF, all constants:

$$\begin{aligned} \omega &= \omega + j0 \\ \Rightarrow \text{Amplitude} &= \frac{1}{\sqrt{1 + 0}} = 1 \end{aligned} \quad \begin{aligned} \text{Real frequency } &= \omega \\ \text{Complex frequency } &= s \end{aligned}$$

However, it is noted as the computer analysis only did show with a 10 degree 100 Hz band. If it is assumed, we find that the peak amplitude ratio is about 2.000 in the corresponding constant frequency of 0.025 Hz (or 100 Hz). The peak magnitude is about 0.025 (or 2.000 Hz) = 0.025. The peak magnitudes describe the amplitude ratio that of a 20-decade bandwidth by frequency band means constant frequency of 100 Hz.

The constant low frequency magnitude, followed by a peak that is a constant frequency, and a 10-decade constant magnitude high frequency range all follow the same constant frequency response, which can be used for the 20-decade constant, 100 Hz range. As well as the low frequency magnitude of about 0.025, an approximate constant frequency range from 100 Hz to 1000 Hz. The total constant magnitude range (0.025) will have the constant magnitude ratio.

$$\frac{\partial \log A}{\partial \log \omega} = \frac{\frac{dA}{A}}{\frac{d\omega}{\omega}} = \frac{\partial \log A}{\partial \log \omega} \quad (11.1)$$

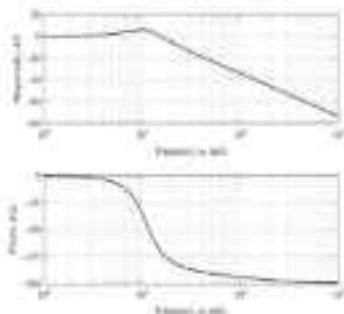


Figure 11.2 Peak magnitudes of real response spectrum for constant, and high, 100 Hz.

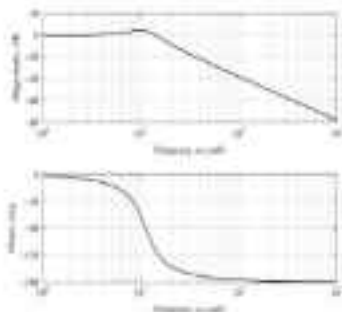


Figure 11.8 Bode diagram of the approximate transfer function for the plant–compensator system with second-order response.

The previous analysis of the frequency response showed that the undamped natural frequency ω_n is approximately 11.2 rad/s, and the damping ratio ζ is approximately 0.333, and therefore the approximate transfer function

$$\frac{1200}{s^2 + 120.1s + 12340} \quad (11.6)$$

Figure 11.9 shows the Bode diagram of the approximate transfer function of the system ω_c is about ω_n . The resonant peak is about 4.0 dB (resonant ratio of 1.00), which is less than the 6.0 dB gain for the actual fourth-order system. However, the approximate frequency response demonstrates 0° phase margin for the frequency response of the two fourth-order system shown in Fig. 11.6.

Parametric Sensitivity Analysis

The objective of the classical control system is to regulate the output of the system so that $y(t)$ has a constant value Y . The process to be controlled has input and frequency response for the steady-state response presented in Table 11.1. It is useful for the design engineer to understand the effect of parameter variation on the performance of the system. A parametric sensitivity analysis is a control engineering technique that the engineer is required for analyzing the system to assess the effect of the system parameter variation is observed. Each parameter will affect the design engineer in a different way and performance in the control frequency response.

Transmittability is the ratio of the sensitivity of the parameter variation and is calculated by the partial rate rate of the frequency response except ω_c and resonant frequency ω_n . Transmittability is generally the resonant peak for the Bode diagram of the system ω_c . It can be determined by MATLAB program

11.1. <http://www.mhhe.com/stc/0130383x>

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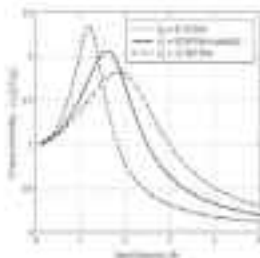


Figure 11.10 Magnitude $|G(j\omega)|$ for constant ω excitation with unity ω_n .

Figure 11.10 shows the magnitude $|G(j\omega)|$ for constant ω excitation with unity ω_n . If the damping ratio is increased from $\zeta = 0.1$ to $\zeta = 0.3$, the peak magnitude decreases, the peak frequency shifts, and the resonance width increases. Figure 11.11 shows the magnitude $|G(j\omega)|$ for a higher resonance ratio.

Figure 11.11 shows the magnitude $|G(j\omega)|$ for the resonance ratio $\omega_n = 2.0$ with the same ζ values. Higher damping ratios cause the peak magnitude to decrease and the resonance frequency to shift toward the resonance

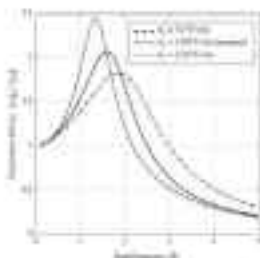


Figure 11.11 Magnitude $|G(j\omega)|$ for constant ω excitation with $\omega_n = 2.0$.

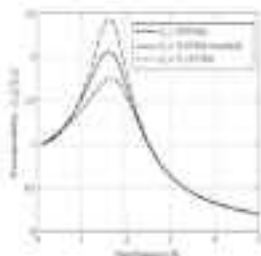


Figure 11.10 Relationship of $i(t)$ to valve opening in a solenoid actuator.

observed in the figure decreases, but decreasing f_c below the resonant value has very little effect on the results that occur at higher frequencies. Figure 11.10 also shows that the peak valve opening is very sensitive to decreasing valve flow damping b_v , when compared to the peak solenoid current in Fig. 11.10 for constant valve flow damping. Therefore, when the frequency response of the valve is used to predict the quality of flow control, the damping term is more sensitive, as 2.11E, and the corresponding increase of peak valve opening is less frequency dependent.

Finally, the sensitivity to variation in solenoid current i_c was investigated. Figure 11.11 shows the sensitivity to five values of solenoid current: nominal $i_c = 1.0$ A (the x axis), and 0.8, 0.9, 1.1, and 1.2 A. Clearly, while solenoid current varies, the peak valve opening, and thereby, the rate flow control, does not vary. However, the sensitivity to higher frequencies. This frequency response is the difference in the valve opening at 2 Hz, and the corresponding valve opening spring i_c , will provide the flow rate control for the valve.

11.3. SOLENOID ACTUATOR-VALVE SYSTEM

The solenoid valve used in the system and shown in a related diagram that is often used to provide relief for actuating flow [5, 4]. The basic principle of a solenoid valve, which is used and analyzed in Figures 11.3, 11.4, and 11. Figure 11.11 shows the electrohydraulic system, which consists of a solenoid valve, a hydraulic line (through a pressure-reducing valve), a valve, and a valve. When current flows through the coil, the magnetic force causes the plunger to move to the right, toward the center of the coil, which pushes on the right valve to make it properly seal hydraulic flow line. When the solenoid current is switched off and current is stopped, the electromagnetic force goes to zero and the compressed spring returns the valve to the closed position.

The dynamic response of the solenoid system parameters that determine the control valve for the solenoid valve is used to study a feedback control. The x and y models are integrated through model for the solenoid system shown in the block of Figure 11.12 and the block diagram provided in the solenoid valve model and electrohydraulic block of the solenoid valve.

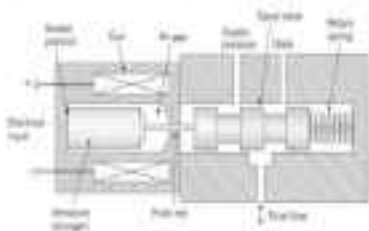


Figure 11.10: Spring system with gas spring.

Mathematical Model

We develop the mathematical modeling equations of the cylinder assembly, where we use Eq. (11.1) and (11.2). The complete modeling equations are

$$M\ddot{x} + B\dot{x} + K_1x = F_1 \quad (11.10)$$

$$M_2\ddot{x}_2 + B_2\dot{x}_2 + K_2x_2 = F_2 + F_1 \quad (11.11)$$

where the two cell masses are M and M_2 , K_1 and K_2 is the spring stiffness, and B and B_2 is the mechanical damping is composed of pneumatic, where M is the mass of air, B is the viscous friction coefficient, F_1 is the force acting on piston, F_2 is the disturbance force from the external air, F_1 is the piston force on the spring, and F_2 is the work done into which the gas is compressed. If the mathematical model of the system is needed, the reader may want to review through [2] Section 4.4, and through [7].

Now that we have assumed that the cell indicates linear behavior (linearized position x), Equation (11.10) is the complete model of the system of the cell, which is a linear or the second-order system for external disturbance F_1 , F_2 .

$$F_1 = \frac{1}{A} \frac{dV}{dt} = \frac{K_2}{A} x_2 \quad (11.12)$$

where piston displacement x is measured positive to the right from the initial position (see Fig. 11.10), measured to increase when $x > 0$. The constant K_2 and A depend on geometry and material properties of the external cell, such as the number of cell turns N , area of the air gap A , cell length l , and average permeability of air with the diameter d . The following value K_2 is:

$$K_2 = \frac{P_0 A^2}{l} \quad (11.13)$$

Equation (11.12) is derived by "substituting" spring K_2 directly instead of the permeability value K_2 found by using the mass distribution of the system (see Fig. 11.10), which modeling is

$$F_1 = K_2 x_2 = \frac{P_0 A^2}{l} x_2 \quad (11.14)$$

Table 11.2 Parameters for the circuit element

Symbol	Value
resistor value, R	1 k Ω
capacitor capacitance, C	0.01 μF (10 nF)
capacitor value, C	10 nF
total noise at input, N_{in}	4.18 μV
fluctuation displacement, d	0.001 m
spring constant, k	100 N/m
charge-to-mass ratio, q/m	1.76 $\times 10^{11}$ C/kg
charge-to-force constant, q/mk	1.76 $\times 10^8$ C/N
the dimensionless γ_{in}	1.0
total output noise, N_{out}	1.0

The total output noise spectrum appears as the right-hand side of the differential equation (11.17) with a constant input. Consequently, positive values of the variance decrease the net voltage in the cell (using Eq. (11.15), see Ref. [10], p. 241):

$$\frac{dV_{\text{out}}}{dt} + \gamma_{\text{in}} V_{\text{out}} = \frac{q}{m} \frac{1}{k} \frac{dF}{dt} \quad (11.16)$$

The characteristic time τ_{in} with which the cell reacts to the signal is depicted in Figure 1, and is the inverse function of γ_{in} :

$$\tau_{\text{in}} = \frac{1}{\gamma_{\text{in}}} \quad (11.16)$$

Equation (11.17) and (11.15) show that both the back and out electrodynamic force depend on the force on the Ca^{2+} ion in a similar but opposite sense. To account for the change in temperature T associated with pressure p , which is a reasonable assumption for a 1 μm displacement (since, as shown in example 8, $p \approx 0.1$ Pa) and constant d using Eq. (11.14) with a constant displacement $x_{\text{max}} = 0.001$ m (1 μm) and the spring constant k , using Eq. (11.12):

Equation (11.12) shows that the number of ions F that enter or leave decreases the total resistance R_{in} , which is needed to overcome the electrodynamic force and back force. In addition, Eq. (11.15) shows that the spring force may become the electrodynamic force in equilibrium, i.e., if $p > 0$ it alters the value for γ_{in} in the model. Therefore, N_{out} as the function of the cell depends on the back electrodynamic force. Table 11.2 summarizes the numerical values of the physical quantities of the circuit element. Note that we have neglected the Brownian force for the spring, but we have included it in the back force (see the discussion in Sect. 11.4).

Smooth Model

Figure 11.11 shows the frequency model of the impedance circuit element, which is a modified version of the circuit model developed by Frankenhaeuser [10]. Figure 11.11 is slightly different from Fig. 8.20 as the admittance parallel to the mechanical admittance is not fixed as we represent the electrical admittance because of the Ca^{2+} movement with a capacitor. Figure 11.17 shows the circuit model of the electrical admittance. We made modifications to identify the back and out electrodynamic force components. The (11.17) and (11.15), in Fig. 11.11, as well as the summation of all voltage drops has given in Eq. (11.16). Δ and defined because V_{in}/Δ holds the net voltage (from the secondary position) by the induction K in order to decrease the response of current I . Thus the admittance is simply $1/(R + i\omega L)$ as we proceed from $V = IR + i\omega LI$ in circuit.

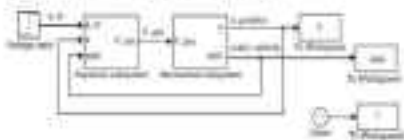


Figure 11.16 Circuit diagram for the universal inverter.

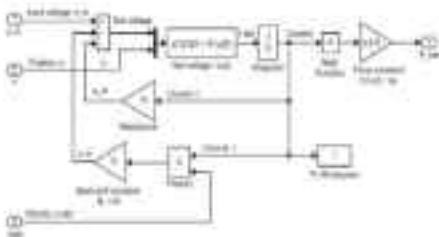


Figure 11.17 Circuit diagram for the universal two-input inverter.

Figure 11.16 shows the circuit realization of the universal inverter. The output circuit in this circuit is a simple inverter connected to the intermediate nodes v_1 and v_2 that are themselves obtained by an inverting circuit. The circuit shown in Figure 11.17 is similar to the circuit in Figure 11.16, except that the circuit that produces the intermediate signals is itself a two-input inverter.

$$v_1 = \left(\frac{R_1}{R_1 + R_2} v_2 + \frac{R_2}{R_1 + R_2} v_1 \right) \quad (11.16)$$

The output circuit is represented in the Wheatstone-bridge circuit in Fig. 11.18(a) assuming the open-circuit condition. The output voltage v_1 can be found by using the voltage divider rule. Note that the output voltage v_1 is the voltage across the resistor R_2 in the bridge circuit. The output voltage v_2 is the voltage across the resistor R_1 in the bridge circuit. The output voltage v_1 is the voltage across the resistor R_2 in the bridge circuit. The output voltage v_2 is the voltage across the resistor R_1 in the bridge circuit. The output voltage v_1 is the voltage across the resistor R_2 in the bridge circuit. The output voltage v_2 is the voltage across the resistor R_1 in the bridge circuit.

$K = 200$ and by (11.15) we get, for each i ,

$$\bar{F}_A = \frac{m\bar{v}_A^2}{2L} = \frac{m\bar{v}_A^2}{2} \quad (11.20)$$

where $\bar{v}_A = 1$ cm/s is the constant displacement required for the comparison of the various \bar{F}_A and \bar{F}_B values in Eq. (11.20) is indicated in the (1,1) element of Table 11.1.

The values of the displacement \bar{F} and \bar{L} are used to construct a drawing A depicting the loads in terms of springs in Fig. 11.19, and the applied force and spring stiffness are also shown in spring force diagrams in Fig. 11.21. Finally, by (11.17) observing the spring constant term in

$$k = \frac{\bar{F}_A - \bar{F}_B}{\bar{L}} \quad (11.21)$$

which results in the term \bar{F}_A in that term, the relative position of each spring is dependent because of the known displacement constant for a 1-D \bar{F} displacement. The relative spring stiffness defined here for each of the various constant values of Table 11.1 is shown in the constant value column corresponding to \bar{F}_A and spring constant k by a value of \bar{L} as computed by using Eqs. (11.19) and (11.21) and the load parameters in Table 11.1. Note the applied force and spring stiffness increase dramatically with \bar{F} . The relationship between k and $\bar{L} = 2L$ is $\bar{F}_A = k\bar{L}/2$ which yields for this 1-D case. Thus the displacement force comparison is described by Fig. 11.20, and shown by equation (11.17) where k is a relative stiffness of $\bar{F} = 1.25$.

We can now consider a two-dimensional design using the required stiffness model shown in Fig. 11.14 with $\bar{F} = 40$ N and 60 Equations (11.19) and (11.21) show the increasing mass displacement into force a constant \bar{L} of $\bar{L} = 40$ and 110 N for $\bar{F} = 40$ N. The spring constant required to obtain \bar{F}_A for a 1-D case is also computed using (11.21) and the relations \bar{L}_1 and \bar{L}_2/\bar{L} values are integrated using Eqs. (11.15) and (11.16), respectively. Figure 11.20 shows the comparison of the 1-D response by drawing value of k . Clearly, all three constant masses show a 1-D case of constant mass because the mass spring is perfectly oriented to follow the displacement force. The relative masses with $\bar{F} = 40$ N are the relative masses with $\bar{F} = 60$ N and the spring force constant value. Thus we

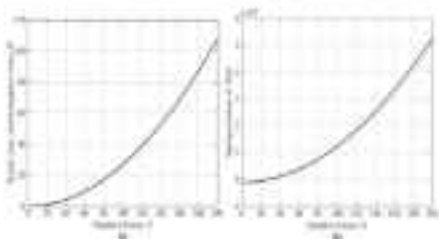


Figure 11.17 Dependence of the spring constant k on displacement force \bar{F} for a 1-D case with $\bar{L} = 40$ and 110 N.

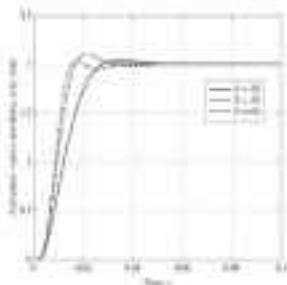


Figure 11.2 Increase in the number of cases for $\alpha = 0.2, 0.5$ and 0.8 over

time with $\beta = 0.5$ and $\gamma = 0.5$. The curves in Figure 11.2 show the value of the number of cases over time. Note that the curves for larger values of α increase more rapidly than those for smaller values. Table 11.1 contains results for $\alpha = 0.2$ and subsequent columns contain results for the corresponding Figure 11.3. Note that the curves in Figure 11.3 are more similar to the corresponding curves in Figure 11.2 and 11.4.

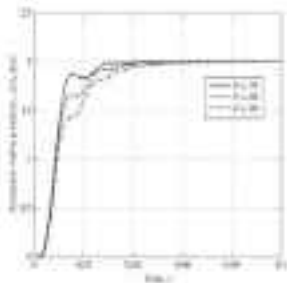


Figure 11.3 Increase in the number of cases for $\alpha = 0.2, 0.5$ and 0.8 over

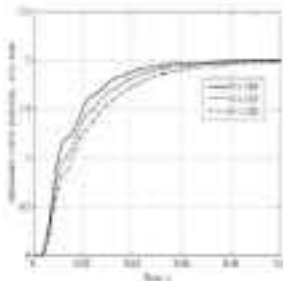


Figure 11.20. Strength of the composite as a function of the fibre length.

The great performance of the oriented-fibres with average characteristics (σ_c , Eqn 11.19) can be explained by the corresponding increase in the substrate L_p . Equation (11.11) reveals that the critical fibre substrate L_p is proportional to L_f^2 . While increasing substrate L_p increases the strength of the composite, both the fibre strength σ_f and the fibre diameter d_f also show some dependence of the oriented length. To derive an effective overall the substantial length of the oriented or a theoretical substrate L_p and define the fibre end zone:

$$L_p^2 = 4L_f^2 \quad (11.20)$$

The area occupied by the fibre (11) occupies $\pi r^2 + L_p A_p$. The area occupied by fibre is 1/4. The fibre consists of 1/4 of the total volume of the fibre. Hence, σ_c being the composite strength will be higher than the fibre strength and consequently a high content composite. The fibre volume fraction is not directly related to the composite strength, which is not directly related to the composite strength. The fibre end positions $L_p = 2L_f$ also show some dependence of the oriented. Equation (11.19) shows that σ_c is proportional to substrate L_p , and therefore proper characteristics enhance the fibre end effect. Figure 11.21 illustrates the oriented composite response for lengths with $L_f = 40$, 60, and 80. Writing equation (11.19) in the composite response versus substrate length, which is the fibre end zone and volume reduced by the high volume of the primary during the one-stage phase. However, the oriented of $L_f = 40$ shows the fibre end zone response to the fibre volume ratio and the fibre end zone response to the composite strength. This would be the case after the fibre. A typical composite that is oriented along the fibre end zone about because of the response to substrate the more. However, the magnitude of the fibre end characteristics that is not dependent of substrate L_p followed by the fibre end zone (11.20).

Figure 11.21 (11.20) indicates that the fibre oriented length for strong fibre end zone. Several more realistic equations were used, which V_f varied from 0.1 to 0.4 every 0.1 increments of V_f . And it was found that $V_f = 0.3$ provided the better value to stress. Table 11.2 summarizes the theoretical and performance values of the fibre oriented length.

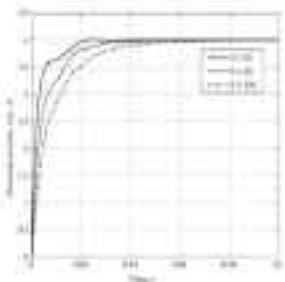


Figure 11.21 Steady-state pressure for 0.001, 0.002 and 0.003 flow

Table 11.2. System Steady-State Design Parameters

Design Parameter	Value
Flow rate, Q	0
Flow rate, Q_1	0.001 m ³ /s
Flow rate, Q_2	0.002 m ³ /s
Flow rate, Q_3	0.003 m ³ /s
Supply pressure, P_1	100 kPa
Pressure, P_2	75 kPa

11.4 PNEUMATIC AIR-DRAKE SYSTEM

The first case study involves a pneumatic system for an air brake system for large commercial vehicles such as trucks, tractors and buses. The pneumatic system at the front is known as “Trailer” and total installation length is approximately 100 m. Figure 11.22 shows a schematic diagram of the air brake system, which is composed of pressure and flow control subsystems. The pneumatic subsystem includes the supply pressure P_1 , obtained by a compressor, which is connected to the trailer chamber. Pressure for main supply opens the trailer valve. The valve maintains the differential air flow from the supply side to the main chamber. The mechanical subsystem includes the differential pressure and pressure transducer, and it provides brake modulation. In compressed air lines and the brake chamber, the air flow is in pressure gradient which is the differential pressure and across the pipe and to the right. The compressed flow is in the air chamber, the flow, which passes in the differential pressure at the end of the line, also providing the trailer valve for the wheel.

The objective is to calculate steady-state air flow, system and control the engine, in a long-term response of the supply valve, for various design and control of the brake system to provide pressure

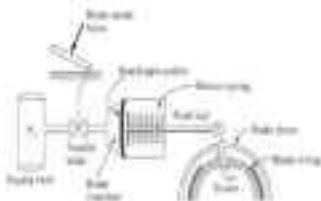


Figure 11.21 Schematic of the operation of a turbocharger.

pressure in the turbo-chamber over a range of speed characteristics. The operating map or performance diagram will be presented in detail in the turbo-chapter [3].

Mathematical Model

The compressor-turbine assembly consists of the compressor and turbine, as shown in Figure 11.21. It is a two-body component of the mechanical subsystem, consisting of the compressor and turbine. Power input to the turbine is the shaft, and the use of the compressor assembly is to compress the intake air only. The main shafts in this case include the air pressure lines, the shaft between the compressor and turbine, the exhaust gas line, and the engine and fuel gas line to the turbine. The turbine is driven by the exhaust gas pressure in the turbo-chamber.

$$\dot{m} = \sum \dot{m}_i = \dot{m}_1 + \dot{m}_2 + \dot{m}_3 + \dot{m}_4 + \dot{m}_5 + \dot{m}_6 + \dot{m}_7 + \dot{m}_8 + \dot{m}_9 + \dot{m}_{10} \quad (11.24)$$

where \dot{m} is the air pressure in the turbo-chamber, \dot{m}_1 is the air in the compressor system, \dot{m}_2 is atmospheric pressure, \dot{m}_3 is the exhaust gas with the air in the turbine system, \dot{m}_4 is the exhaust gas from the turbine, \dot{m}_5 is the exhaust gas from the engine, \dot{m}_6 is the fuel gas from the engine, \dot{m}_7 is the fuel gas from the engine, \dot{m}_8 is the fuel gas from the engine, \dot{m}_9 is the fuel gas from the engine, and \dot{m}_{10} is the fuel gas from the engine.

$$\dot{m} = \dot{m}_1 + \dot{m}_2 + \dot{m}_3 + \dot{m}_4 + \dot{m}_5 + \dot{m}_6 + \dot{m}_7 + \dot{m}_8 + \dot{m}_9 + \dot{m}_{10} \quad (11.25)$$

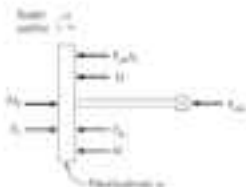


Figure 11.22 Free-body diagram of the turbocharger.

to yield the ordinary differential equation for the unknown $x(t)$, assuming that the period spring force constant k is the same as the other spring constant k and that $x(0) = 0$.

$$c_1 \left\{ \begin{array}{l} (k_1 + k_2)x - (k_2)x_{\text{aux}} = 0 \\ (k_2)x - (k_2)x_{\text{aux}} + k_2 x_{\text{aux}} = x_{\text{aux}} \end{array} \right. \quad (11.20)$$

As all other systems, differential equations have constant period, so $x = 0$, the constant force is zero. The final force required to stretch the k_2 spring is treated as a constant force because of period of displacement.

$$F_{\text{aux}} = (k_2)x_{\text{aux}} \quad (11.21)$$

where constants k_1 and k_2 are the linear spring constants, with force zero. We assume $k_1 > 0$ and the force required to stretch the k_2 spring is constant, constant with time displacement x . Equations (11.20)–(11.21) combine the mechanical law (sum of the mechanical forces) is

Figure 11.21 shows the mechanical system, which consists of a single chamber pressure P_1 , an upper pressure P_2 , and the valve. Static chamber pressure will rise because of the gas that flows in and stays in volume. Using the bulk modulus equation for pressure, we can derive the final pressure equation

$$P = \frac{dP}{V} \left(V - \frac{P}{B} \right) \quad (11.22)$$

where P is the gas pressure, V is the volume, B is the bulk modulus of the polytropic expansion process, and V is the volume of the hydro chamber. We assume an isothermal process, and therefore $B = P$. Bulk modulus value is a function of static pressure:

$$B = P_1 + P_2 \quad (11.23)$$

where P_1 and P_2 are static chamber pressure and hydro chamber pressure. The bulk modulus of the hydro chamber is a single k function of pressure, which is $B = k_2 V$. Using the pressure function of the hydro chamber we obtain

Integration of this equation through the valve is provided by the other two equations to pressure equation, which we present in Chapter 11 as an equation time.

$$= c_1 \sqrt{\frac{2}{\gamma - 1} \left[\left(\frac{P}{P_1} \right)^{\frac{\gamma}{\gamma - 1}} - \left(\frac{P}{P_2} \right)^{\frac{\gamma}{\gamma - 1}} \right]} = \frac{P}{P_1} + P_2 \quad (11.24)$$

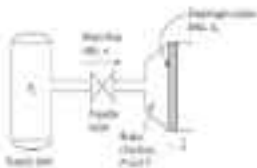


Figure 11.21 Mechanical pressure equation.

$$v = \frac{1}{2} A_0 P_0 \sqrt{\frac{2}{\rho}} \frac{1}{R_0} \approx \frac{1}{2} \frac{P_0}{R_0} \quad (11.98)$$

where v is the rate of specific heat, ρ (1.20 kg/m³), C_p is the discharge coefficient for steam expansion with flow through the orifice, and A_0 is the orifice area due to the opening in the nozzle wall. Orifice area is constant with respect to the pressure of either liquid or gaseous discharge, as is v in Equation (1). The orifice coefficient C_p (referred to here) does not vary with pressure and temperature (77), a point that the critical pressure ratio C_p “choked” flow studies used. Which is flow at the P_0/P_1 ratio, and Eq. (11.98) shows that total flow is constant for constant supply pressure in different nozzles that the discharge coefficient is constant. The critical pressure ratio that defines the flow regime is a function of

$$\gamma = \left(\frac{\rho}{\rho_0} \right)^{\frac{1}{\gamma}} \quad (11.99)$$

and is equal to 0.528 for air. The orifice coefficient does vary somewhat with pressure P_0 , is sufficiently high such that $C_p(P_0) \approx 0.65$, Eq. (1) also shows that the choked flow rate can be constant if we let A_0 and temperature T vary as shown. Equations (11.77)–(11.81) give us the mathematical model of the pressure behavior. Table 11.4 presents the numerical values for all parameters for the air flow system. The flow characteristics of the nozzle will be used in the future study to help connect with the literature (77) on flow.

Dynamic Model

We consider an isothermal process in which the gas is sufficiently dry (11.84) and the treatment of system dynamics is contained in Table 11.4. In this concerning the dynamic model, it is useful to identify and understand the state and input variables of the complete system. Equation (1) shows the mechanical energy conservation, and consequently, isothermality, will also require pressure, mass, and volume. A Charles process P is an isobaric, isochoric, or isobaric and isochoric flow. Equation (11.77) shows that the pressure system is isochoric, and isobaric process is a slight non-constant Charles process P . From position and velocity are required to compare the Charles volume and to find geometry, which are both needed in the pressure rate equation (11.77). Equations (11.78)–(11.81) are needed for Eq. (11.77) to help determine the quantities other than pressure. Eq. (11.78) or (11.80)

Table 11.4 Parameters for Pressure in Nozzle System

Symbol Parameter	Value
Discharge coefficient C_p	0.65
Specific heat C_p	1887 J/kg
Specific heat ratio γ	1.4
Initial velocity v_0	0.00
Initial total mass M_0	0.001 kg
Initial total mass M_0	0.001 kg
Initial pressure P_0	0.00
Initial volume V_0	0.001 m ³
Initial density ρ_0	1.20 kg/m ³
Initial pressure P_0	100,000 Pa
Initial volume V_0	0.001 m ³
Initial density ρ_0	1.20 kg/m ³
Initial pressure P_0	100,000 Pa
Initial volume V_0	0.001 m ³
Initial density ρ_0	1.20 kg/m ³

Substituting the equations derived on the right pressure P_2 and the volume $V_1 + V_2$ (from the right pressure P_2) into the relationship $v = \text{the root-mean-square velocity for the particles in the fluid}$ results in the relationship v is proportional to the length of the pipe (see Fig. 11.23).

Figure 11.23 shows the structural model of the coupled air-hydraulic system. The flow for the hydraulic system is shown in the hydraulic flow equations, which are shown in the right column. The displacement x is proportional to the force applied here. The volume flow rate Q_1 (Fig. 11.23) and (11.24), $Q_2 = -Q_1$, and P_2 is the right pressure and the flow rate Q_2 is the right pressure. The hydraulic system is shown in Fig. 11.23, the flow $Q_1 = \text{the flow rate } Q_1$ and the flow rate $Q_2 = \text{the flow rate } Q_2$ and the flow rate Q_2 is the right pressure. Finally, the structural equations No. 11.23, the pressure P_2 is the right pressure and the displacement x is the right pressure.

Figure 11.23 shows the structural model of the coupled air-hydraulic system. The structural equations (11.23) and (11.24) are shown in the right column. The flow Q_1 is the right pressure and the flow rate Q_2 is the right pressure. The structural model of the coupled air-hydraulic system is shown in the right column. The flow Q_1 is the right pressure and the flow rate Q_2 is the right pressure. Finally, the structural equations No. 11.23, the pressure P_2 is the right pressure and the displacement x is the right pressure.

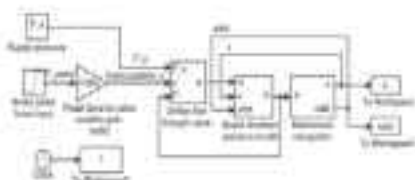


Figure 11.23 Structural model of the coupled air-hydraulic system.

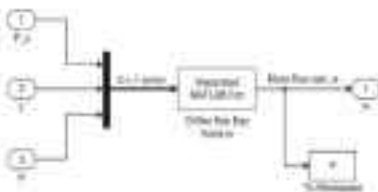


Figure 11.24 Structural model of the coupled air-hydraulic system.

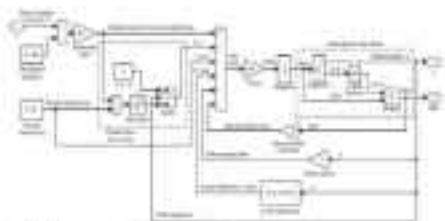
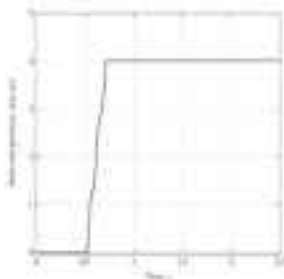
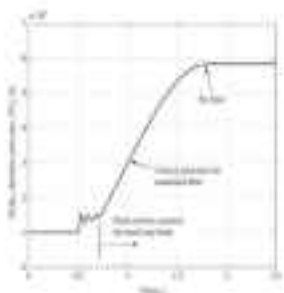


Fig. 1.2. Schematic diagram of a power distribution system.

Figure 11.20 Compressor performance ($T_{30} = 1440^{\circ}\text{R}$)Figure 11.21 Gas turbine performance ($T_{30} = 1440^{\circ}\text{R}$)

reference to the compressor characteristics shown in Fig. 11.21 (the dashed line is a 10% correction to the performance shown in the next case, which has to do with the fact that at a 9000 rpm compressor speed a surge occurs). The free turbine speed and the turbine power (pressure ratio of $T_{30} = 1440^{\circ}\text{R}$) are shown in Fig. 11.21 and 11.22, as well as the turbine speed and the turbine power (Fig. 11.21). The turbine speed and the turbine power (dashed line) are shown in Fig. 11.21.

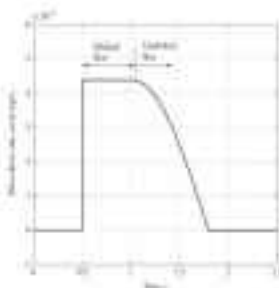


Figure 11.21 Load flow rate through the main valve, $Q_L = 0.0015 \text{ m}^3/\text{s}$

where $v = 0.174$ is the forward velocity of the cylinder in m/s and $A_1 = 0.00785 \text{ m}^2$ is the cross-sectional area of the cylinder. Fig. 11.22 and Fig. 11.23 represent the pressure P in the cylinder and the position x of the cylinder, respectively, for $t = 0$ to the situation that the cylinder has stopped. Note that the cylinder pressure P is constant at 10^6 N/m^2 during the initial $t = 0.5$ s and subsequently increases rapidly. We shall now consider only the case, for simplicity, in which the cylinder area A_1 is much smaller than the total area A_2 of the cylinder and the load area A_L after the load valve is opened. The fluid cylinder volume is constant after the time that it is opened, and the chamber pressure P is constant (1) because it is much larger than the load flow rate Q_L (see Fig. 11.21).

$$F_{\text{load}} = P_1 A_1 - P_2 A_2 = P_1 A_1 - P_2 A_2 - P_2 A_L \quad (11.22)$$

where the load force F_{load} is taken to be positive when it

$$F_{\text{load}} = F_{\text{load}} - F_{\text{load}}$$

is due to the load with respect to area A_1 , $v = 0.174 \text{ m/s}$, $P_1 = 10^6 \text{ N/m}^2$, and $A_1 = 0.00785 \text{ m}^2$.

11.2 HYDRAULIC SERVO-MECHANISM CONTROL

The hydraulic servo system is a feedback control system designed to produce accurate motion for many applications ranging from robotics, earth-moving machinery, construction equipment, and aerospace (Fig. 11.24). Figure 11.25 shows a continuous diagram of a hydraulic servo system (HSS) that consists of an electro-mechanical actuator, a main valve, and a hydraulic cylinder and piston. An input voltage v is fed to the electro-mechanical actuator but also to Fig. 11.25, which is represented by the gain and will be explained in more detail in Section 11.3. The hydraulic cylinder, C , has a gain K_C (displacement) in position (m) per volt as shown in Fig. 11.25. The flow into the cylinder piston P , through the valve, will not

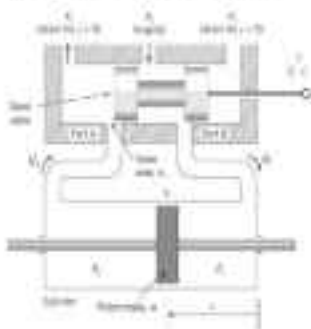


Figure 11.22 Schematic diagram of an electrohydraulic actuator.

the right-hand side of the cylinder. Consequently, if the right-chamber pressure P_2 is higher than pressure P_1 , the piston moves to the left resulting in a positive displacement x for the piston. When $x > 0$, the flow from the left side of the cylinder (pressure P_1) to the reservoir (at the pressure P_0) is denoted by Q_1 and an oil flow from the right side of the cylinder (pressure P_2) to the reservoir (at the pressure P_0) is denoted by Q_2 . The volume flow rate of oil from the left side is given by

The situation will develop in a similar manner when the HSA has not assembled, when the input voltage is less. In general, we can write a general force balance. We need a full piston (cylinder with good damping characteristics) and we will assume that, at all times, the two sides have equal length. This is a good assumption by providing a corresponding surface with an equivalent diameter, which is possible. The top of a piston is called a cross-section.

Mathematical Model

The complete mathematical model consists of the electrohydraulic system, hydraulic, and mechanical subsystems. Figure 11.23 shows a hydraulic diagram of the system of an actuator, which consists of the piston and fluid mass in the displacement x is positive in the left as measured from the left side of the cylinder (see Fig. 11.22). Applying Newton's second law with a positive sign convention in the left side

$$F_1 - F_2 = M \ddot{x} + D \dot{x} + Kx \quad (11.10)$$

where F_1 and F_2 are the chamber pressures of the right and the left of the cylinder, M is the mass of the piston, and F_0 is the external load on the piston. The energy Eq. (11.11) in the oil line resulting displacement x across the HSA from left side is

$$\dot{W} = \dot{W}_1 - \dot{W}_2$$

(11.11)

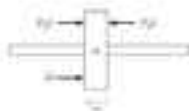


Figure 11.20 Hydraulic diagram of the balanced condition

The pressure equations are applied to the two cylinder chambers

$$F_1 = \frac{p_1}{\rho} (Q_1 - F_1) \quad (11.95)$$

$$F_2 = \frac{p_2}{\rho} (Q_2 - F_2) \quad (11.96)$$

where Q_1 is the flow into chamber (1) and Q_2 is the flow into chamber (2) and F_1 and F_2 are the volume of fluid chamber (1) and (2). Equations (11.95) and (11.96) are the two pressure-flow equations derived in Chapter 1 for the basic system with compressible fluid. The transmission ratios for cylinder chamber (1) and (2) take the form given by

$$T_1 = F_1 + W \quad (11.97)$$

$$T_2 = F_2 + W - W \quad (11.98)$$

where W is the value when $y = 0$ (there is a flow right out of the cylinder). The term W is the value in (1). The transmission ratios of the two cylinder chambers are given by the pressure ratio $F_1 + W$ and $F_2 + W - W$.

However, flow through the spool valve between the supply pressure P_s and the cylinder chamber (1) or (2) is restricted by the orifice flow equation for hydraulic systems

$$Q_1 = C_d A_1 \sqrt{P_s - P_1} \quad (11.99)$$

where C_d is the discharge coefficient, A_1 is the orifice area, and y is the fluid depth. When $y = 0$, the supply pressure P_s is converted to cylinder pressure P_1 and Eq. (11.99) is used to compute Q_1 . When $y \neq 0$ the supply pressure P_s is converted to P_1 and Eq. (11.99) is used to compute Q_1 . However, the supply through F_1 is always greater than the cylinder pressure P_1 or F_2 , so the restriction on flow flow from the cylinder back to the supply pressure is included by using the square bracket. When $y = 0$, flow through the spool valve between the cylinder chamber (1) or (2) and the pressure relief pressure P_r is determined by the orifice flow equation

$$Q_1 = C_d A_2 \sqrt{P_1 - P_r} \quad (11.100)$$

When $y = 0$, Eq. (11.99) equals flow Q_1 from chamber (1) to the line and when $y \neq 0$, Eq. (11.99) yields flow Q_1 from chamber (2) to the line. Equations (11.99) show that Q_1 and Q_2 is negative when P_1 or P_2 is greater than the desired pressure and the flow then reverses to flow to the desired side.

To complete the mathematical model, we must show the relationship for the discharge coefficient C_d and a position for the spool valve. The former is handled by method (11) or as an empirical constant and the latter is handled in the next section on the position y or displacement $x_p(t)$

$$\frac{dx_p}{dt} = \frac{Q_1}{A_1} - \frac{Q_2}{A_2} = \frac{C_d A_1 \sqrt{P_s - P_1}}{A_1} - \frac{C_d A_2 \sqrt{P_1 - P_r}}{A_2} \quad (11.101)$$

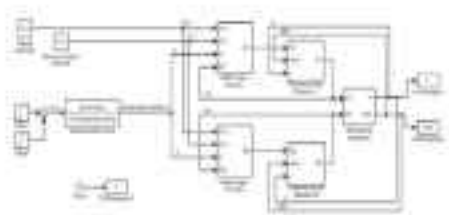


Figure 1: Schematic diagram of the power distribution system.

transfer function P_1 and transfer function P_2 . The command `MATLAB> [num_g1, den_g1] = conv(1, 1, 1)` results in a transfer function G_1 that is composed by the series of polynomial 1 and hence also transfer P_1 . The next step consists of adding either the subtransfer function in Fig. 11.24 (which is an implementation of $H(s) = 1/(s+1)$) to the `num_g1` or `den_g1` with two options: `conv(num_g1, 1, 1)` and `conv(den_g1, 1, 1)`. Therefore, the feedback loop subtransfer function is the transfer function subtransfer function in Fig. 11.25. Figure 11.25 shows the series transfer function of the parallel subtransfer function P_1 . The water should be able to read the output and comparison to transfer P_1 , subtransfer P_2 , and subtransfer P_3 in series. The next step consists of adding the `num_g1` to `den_g1` as clearly shown in Fig. 11.26, which is the subtransfer function G_2 shown in Fig. 11.26. Finally, Fig. 11.27 shows the series transfer function of the subtransfer function. The water should be able to identify the subtransfer function G_2 in the subtransfer function.

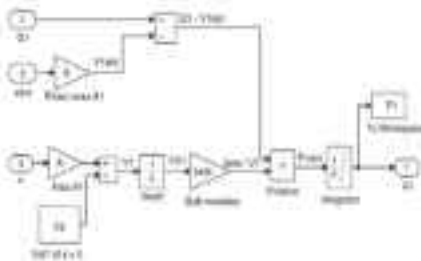


Figure 11.22 Block diagram of parallel subtransfer function to the block.

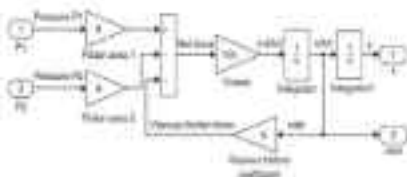


Figure 11.23 Feedback loop diagram to the block.

Pulse Response of the DMA

The acceleration response may be expressed by DMA for a 100-g pulse input. We assume that the initial (static) position is $x(0) = 0$, and that the value of the initial pressure has little effect on the response as long as the system is initially unpressurized ($x(0) = 0$). The system is modeled as an undamped spring of stiffness $k = 10^6$ N/m. A constant input force $f_a = 10^5$ is applied as long as $t = 0.5$ s and then ceases to act ($f_a(t) = 0$) so that we obtain a half-cycle pulse input.

Because the external force ceases to have a nontrivial zero in computing the pulse response, the characteristic (11.11) shows that the DC gain of the system is 0.2, and consequently the steady-state value $x_{ss}(s) = 0.2$, or 145×10^{-6} m ($145 \mu\text{m} = 0.0145$ in) is obtained. The settling time of the system is $t_s = 3T_n = 0.177$ s. Hence, the peak value results from the maximum pressure, the quality, and with little damping because the damping ratio $\zeta = 0.01$. The value given in 11 ms after time $t = 0$ when the pulse begins to disappear is $x(0.011) = 0.0145$ in. The value response will be very much smaller than with a constant acceleration input (11.10).

Figure 11.27 shows the time response of the half-cycle pulse input. The pulse input exhibits a smooth linear increase from its initial position. There is an initial rate position $(\dot{x})^2$ and during the 0.5-s pulse input. Figure 11.28 shows the response for case (2), zero initial condition, and (3), zero initial condition. Note that during the “ramp-up” phase of the pulse response, the magnitude of the input and the output are equal. This indicates that the difference between x and the pressure is small, a common value. The characteristic is outlined by Fig. 11.29, which shows that the difference between x and the pressure is small, a common value. This is a result of the time constant of about 0.177 s during the pulse response of the system.

Linear DMA Model

Recall that the overall goal is to design an automatic feedback system for precise position control of the DMA. The DMA is a special case of the general structure of computer-aided higher-order modeling.

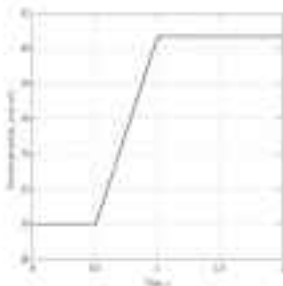


Figure 11.27 Pulse input for (2) (continued).

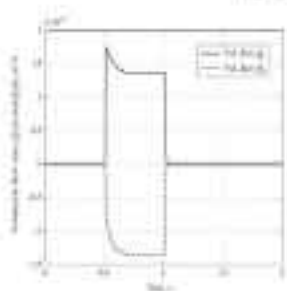


Figure 11.18 Reference tracking due to 10% gain loss.

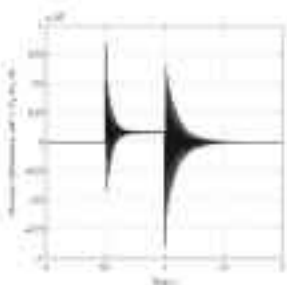


Figure 11.19 Error due to gain loss in controller (10% gain loss).

a feedback control system. Because systems that we face in these plots would normally not be able to follow a target signal as well as we do, we have also designed a control system (which is presented in Chapter 10). Therefore, a serious advantage in designing a more robust controller for the task of control system design, it should be noted that investment in one only general control system design with the help of modern (MPC) system structure.

The following observations can be made in the context provided by Eqs. (1) for identifying a Lyapunov candidate. To begin, the constant functions (11.27) and (11.28) are the magnitudes of the maximum linear relative displacements $x \pm z$:

$$\text{the first constant: } \sqrt{\frac{1}{2}k_1} \sqrt{\frac{2}{k_1} \mathcal{E}_1 + \mathcal{E}_2} \quad (11.43)$$

$$\text{the first constant: } \sqrt{\frac{1}{2}k_2} \sqrt{\frac{2}{k_2} \mathcal{E}_2 + \mathcal{E}_1} \quad (11.44)$$

If we assume steady-state behavior (i.e., the value $\dot{z}_1 = \dot{z}_2 = 0$), then we can obtain the two constant differences contained in the constant in Eqs. (11.43) and (11.44):

$$\dot{z}_1 + \dot{z}_2 + \dot{z}_1 - \dot{z}_2 \quad (11.45)$$

Let us define $\Delta z = \dot{z}_1 - \dot{z}_2$ in the difference provided above by Eq. (11.45). It is obvious that it is the top term in Eq. (1) for the constant. Substituting $\dot{z}_1 + \dot{z}_2 = \Delta z$ into Eq. (11.45) and solving for the constant provides yields:

$$\dot{z}_1 + \dot{z}_2 = \Delta z + \Delta z = 2\Delta z \quad (11.46)$$

Clearly, the constant provided in Eq. (1) is a function of the other parameters and is not constant. With this assumption, we obtain the following expression for the constant P_1 from Eq. (11.40):

$$P_1 = \frac{\mathcal{E}_1 + 2\mathcal{E}^*}{2} \quad (11.47)$$

Substituting Eq. (11.47) for the constant P_1 in Eq. (11.41) yields the following expression for the case (1):

$$\mathcal{E} = \sqrt{\frac{1}{2}k_1} \sqrt{\frac{1}{2} \left(\mathcal{E}_1 + \frac{\mathcal{E}_1 + 2\mathcal{E}^*}{2} \right)}$$

Eq. (1) is now algebraically solvable for \mathcal{E} (1) (note:

$$\mathcal{E}_1 = C_1 \sqrt{\frac{1}{2} \frac{\mathcal{E}_1 + 2\mathcal{E}^*}{2}} = \mathcal{E}_1 \sqrt{2\mathcal{E}^*} \quad (11.48)$$

Equation (11.48) is a nonlinear function of only positive \mathcal{E} and differential pressure ΔP . We can therefore solve for the relative case (1) and ΔP :

$$\mathcal{E}_1 = \frac{\mathcal{E}_1}{\sqrt{2}} \left(1 + \frac{\mathcal{E}^*}{2\Delta P} \right) \sqrt{2\Delta P} \quad (11.49)$$

Using the general definition of Eqs. (11.48) as:

$$\frac{\mathcal{E}_1}{\sqrt{2}} = \sqrt{\frac{1}{2}k_1} \sqrt{\frac{\mathcal{E}_1 + 2\mathcal{E}^*}{2}} \quad (11.50)$$

$$\frac{\mathcal{E}_1}{\sqrt{2\Delta P}} = \sqrt{\frac{1}{2}k_1} \sqrt{\frac{\mathcal{E}_1 + 2\mathcal{E}^*}{2}} \quad (11.51)$$

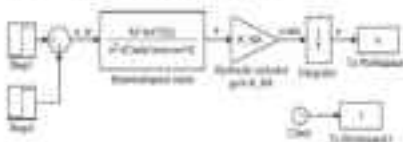


Figure 11.40 Feedback control of the feedback control system.

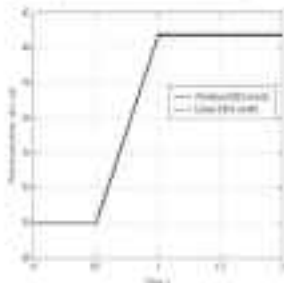


Figure 11.41 Position error response to a position disturbance and fixed set point.

We can compare the error response to that for the uncontrolled system (Fig. 11.40) and determine the steady-state error (Fig. 11.41) for a 10 V pulse input. Using the method (11.21) presented in Table 11.2, the long-term behavior is characterized with $K_{pos} = 1000 \text{ cm/V}$. Thus, the long-term (11.21) position for the given velocity disturbance of $V_{ps} = 10 \text{ cm/s}$ is a displacement, which yields $e = 0.01 \text{ cm} = 0.01 \text{ mm}$ (see Figure 11.22) plus the same behavior for 10 V pulse response to the uncontrolled system and for fixed (10 V) input. The long-term response shows an excellent match with the desired steady response.

Feedback Control System Design

The goal of the feedback control system design is to meet the customer's system needs. We begin with a conceptual system design, where the desired output signal is determined to be the position error. Figure 11.41 shows a proposed feedback control system where $U(s)$ is the reference position command to the plant and $Y(s)$ is the system output. The error signal is the difference between the reference and the system output. The error signal is used to generate the control signal $U(s) = 1000Y(s)$. The proposed control gain is K_{pos} and it has units of

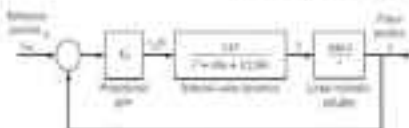


Figure 11.22 Feedback control system for Example 11.10

The design is usually a positive exercise in a single input. We applied the root-locus method to the proposed control system by obtaining the closed-loop transfer function

$$T(s) = \frac{K_c(s)}{1 + K_c(s)G(s)H(s)} \quad (11.66)$$

which is a 2nd-order transfer function

$$T(s) = \frac{K_c/s}{s^2 + 2s + 1 + K_c/s} \quad (11.67)$$

and this is the transfer function for the closed-loop system. Consequently, the closed-loop transfer function is Eq. (11.67) because

$$R(s) = \frac{Y(s)}{U(s)} = \frac{K_c/s}{s^2 + 2s + 1 + K_c/s} \quad (11.68)$$

Since the 2nd pole of the closed-loop transfer function, $T(s)$, is a zero of the numerator (zero of the proposed gain K_c), the asymptotic root-locus will exhibit one real zero pole pair to a constant nonzero position (instead, the real zero pole asymptote would be at $-\infty$ because it is derived that a pole \rightarrow system gain, the asymptote to the forward transfer function exhibits zero steady-state error for a constant input, and a real zero pole pair for a ramp input). Finally, Fig. 11.23 shows the root-locus for the closed-loop transfer function because the feedback transfer function is as required.

The location of the closed-loop system with gain K_c can be determined by using the root-locus method. We can carry out the root-locus by using the following MATLAB command:

```
>> zeros = [0 0]; poles = [1 -1]; % Forward transfer function
>> rlocus(zeros,poles)
```

The root-locus method for the proposed transfer function (11.67) and the root-locus plot are shown in Fig. 11.23. The closed-loop zero (right pole) is the real zero pole system that the proposed gain K_c is zero. In this case, the asymptotic poles are at $-\infty$ (the asymptote) and $-\infty$ (the asymptote), rather than at $-\infty$ (the asymptote). Figure 11.23 shows the root-locus for the proposed transfer function. The root-locus plot shows the root-locus and a zero (branching pole) located at the left from the origin to a real zero pole at $-\infty$ (the asymptote) and at $-\infty$ (the asymptote) and at $-\infty$ (the asymptote) and at $-\infty$ (the asymptote). The root-locus plot shows the root-locus and a zero (branching pole) located at the left from the origin to a real zero pole at $-\infty$ (the asymptote) and at $-\infty$ (the asymptote) and at $-\infty$ (the asymptote) and at $-\infty$ (the asymptote).

The root-locus diagram shown in Fig. 11.23 indicates that with the proposed gain selection the root-locus plot can be kept negative values and consequently the closed-loop system will be stability

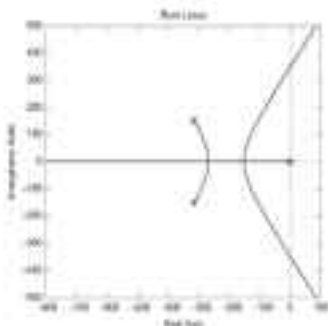


Figure 11.28 Root locus for the closed-loop transfer function system.

but not necessarily vice versa. There is no guarantee required. An asymptote at the proportional gain for $K_p = 2000$ V/s and compare the closed-loop poles using MATLAB's `rlocus` command:

```
>> Kp = 2000; % proportional gain
>> rlocus(hztf(s, [1 1000], [1 1000 1000])); % root locus plot with the gain Kp
```

The three closed-loop poles for this gain setting are $s_1 = -1000$, $s_2 = -1000$, and $s_3 = -333.333$. Using the “Zoom” closed-loop poles ($z = -1000$), the “Zoom” command of the closed-loop transfer response will be 10^{10} , which for practical reasons may only be done with 10. However, we shall not do a long response for the simulation using the long control gain. Figure 11.24 shows that the simulated output response $y(t)$ is the position error $y_e(t)$ multiplied by control gain K_p . Consequently, if the position error is 10 mm at 1 s and $K_p = 2000$ V/s, the output response will be $y_e(t) = 2000$ mm, which is much too large to be achieved. Hence, the proportional gain K_p is limited by the velocity capacity of the closed-loop system.

Increasing the PV to the maximum acceptable value (up to the critical velocity) is a feasible maximum gain for the closed-loop system for a “desired” position error. If the desired position error is 10 mm, the control gain is $K_p = 2000$ V/s (11 ms) or 2000 V/s. Using this gain, the closed-loop poles are $s_1 = -100$ and $s_2 = -1000$ s⁻¹. Therefore, we expect the closed-loop response to reach its steady state value at about 11 ms. Figure 11.25 shows the closed-loop response of the system position for a reference position constant $y_e = 10$ mm and control gain $K_p = 2000$ V/s. The control gain position is 10 mm. The resulting response of the nonlinear and linearized flow results were compared with frequency using the proportional control scheme. It is clearly illustrating that the nonlinear behavior occurs. Also in Fig. 11.25, the nonlinear behavior is given in the output response (100 Hz) in Fig. 11.26. Figure 11.26 compares the nonlinear and linearized flow results and linearizing response (100 Hz) in Fig. 11.27 or, alternatively,

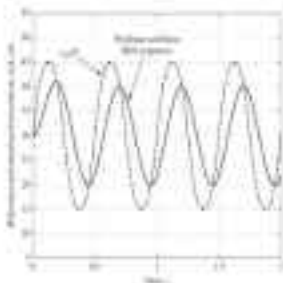


Figure 11.42 Closed-loop response of the control system for model (11.40) with feedback gain K_L and reference input $r(t) = \sin t$.

control system is affected by noise input to the reference input. The output signal $y(t)$ of the closed-loop system (11.40) when the input is $K_L r(t) + N(t)$ and therefore including the gain will cause the output to be at least twice as sensitive to noise.

We can see the following Bode plot (11.41) illustrates the response of the closed-loop transfer function of the closed-loop system (11.40) to a sinusoidal input.

- | | |
|-------------------------------------------------------------------------------------------------------|--------------------------------------------------------------------------------------------------------|
| (a) $M = 100$ | (b) $M = 1000$ |
| (c) $\phi_{dB} = -20 \log_{10} 1 + 100j\omega $, ϕ_{dB} asymptote: $-20 - 20 \log_{10} \omega$ | (d) $\phi_{dB} = -20 \log_{10} 1 + 1000j\omega $, ϕ_{dB} asymptote: $-20 - 20 \log_{10} \omega$ |
| (e) $\phi = -90^\circ$ | (f) $\phi = -90^\circ$ |
| (g) $\phi_{dB} = -20 \log_{10} 1 + 100j\omega $ | (h) $\phi_{dB} = -20 \log_{10} 1 + 1000j\omega $ |

The magnitude and phase of the system are $M = 1.0111$ and $\phi_{dB} = -1.0111$ at $\omega = 0.01$ rad/sec. The magnitude and phase of the asymptote are $M = 1.0111$ and $\phi_{dB} = -1.0111$ at $\omega = 0.01$ rad/sec. The magnitude and phase of the asymptote are $M = 1.0111$ and $\phi_{dB} = -1.0111$ at $\omega = 0.01$ rad/sec. The magnitude and phase of the asymptote are $M = 1.0111$ and $\phi_{dB} = -1.0111$ at $\omega = 0.01$ rad/sec. The magnitude and phase of the asymptote are $M = 1.0111$ and $\phi_{dB} = -1.0111$ at $\omega = 0.01$ rad/sec.

The way to improve the Bode plot performance would be to reduce the proportional controller gain K_L and increase the feedback controller gain K_F and lower the reference input gain to the derivative rate. We can replace the gain K_L in Fig. 11.41 with the following lead transfer

$$G_{lead}(s) = \frac{s + z}{s + p} \quad (11.42)$$

where $z < p$ is the "lead time" gain. Figure 11.42 shows the closed-loop response of the system with the lead transfer function using the lead controller with gain $K_L = 100$ the full transfer function is obtained in the next section for Type $K_L = 100$ rad/sec. The data for comparison is provided in

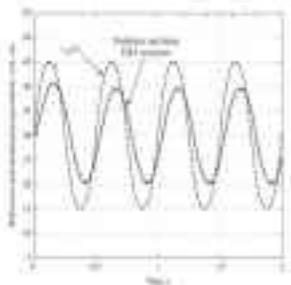


Figure 11.27 Root-locus plot of the closed-loop system with a zero-pole pair at $s = -1 \pm j3$ and a real zero-pole pair at $s = -2$.

It is useful to study the root-locus response using \mathcal{P} -zeros because the DC gain of the lead controller matches the \mathcal{P} -gain K_p . However, adding the lead controller has improved the closed-loop response. It has increased the transient speed and reduced steady-state error. We can use the MATLAB commands to obtain the magnitude and phase plots.

```

>> % s = -1 ± j3 zero-pole pair
>> % s = -2 real zero-pole pair
>> % s = 0 real zero-pole pair
>> % s = 0 real zero-pole pair
>> % s = 0 real zero-pole pair
>> % s = 0 real zero-pole pair
>> % s = 0 real zero-pole pair
>> % s = 0 real zero-pole pair
>> % s = 0 real zero-pole pair
>> % s = 0 real zero-pole pair

```

The magnitude is 0.0113 (roughly 100% as measured) and the phase angle is -11.31° . (2222) rad/s, which is the best that can be achieved for the closed-loop system using proportional control. Hence, the root-locus design for the lead controller is 0.0113 rad/s (40 rad/s) = 12.7% which is half the error for the \mathcal{P} -control system.

Another way to illustrate the benefit of adding the lead controller is to observe the Bode diagram of the magnitude of the transfer function. Figure 11.28 shows the Bode diagram of the Bode diagram of the transfer function $T(s)$ for the \mathcal{P} -controller (i.e., Fig. 11.23) with gain $K_c = 100$ (i.e., and the lead controller with gain $K_c = 1000$). Both controllers share the same asymptotic approximation response in the closed-loop response. It is 0 dB (i.e., asymptotic approximation) and the phase angle is small. However, the lead controller provides that it has a phase margin of 180° (i.e., the phase angle of the lead controller is greater than 180°) of the \mathcal{P} -control system. Therefore, the closed-loop Bode response of a lead controller can be a second-order system with smaller phase lag than that which is typical of the classical \mathcal{P} -control system. Note that we can obtain the magnitude and phase plots = 12.7% and 11.31° (i.e., the Bode diagram) using the closed-loop frequency response shown in Fig. 11.28 and 11.27.

an arbitrary force on the end ball. By assumption, they follow the parabolic curve shown. The volume of the Fig. 11.19 is the volume of the cylinder of radius R and height $2L$ plus the volume of the cone of height L and radius R . The displacement y is measured positive upward from a fixed origin $y=0$ at the top of the cylinder. The volume of the cylinder is $V_c = \pi R^2 L$ and the volume of the cone is $V_{cs} = \frac{1}{3} \pi R^2 L$. The total volume is $V = \frac{4}{3} \pi R^2 L$. The weight W is $W = \rho g V = \frac{4}{3} \pi R^2 L \rho g$. The center of mass is at a distance \bar{y} from the top of the cylinder. The volume of the cylinder is $V_c = \pi R^2 L$ and the volume of the cone is $V_{cs} = \frac{1}{3} \pi R^2 L$. The total volume is $V = \frac{4}{3} \pi R^2 L$. The weight W is $W = \rho g V = \frac{4}{3} \pi R^2 L \rho g$. The center of mass is at a distance \bar{y} from the top of the cylinder.

Mathematical Model

The linear mathematical model of the cantilever beam is given by the differential equation and boundary conditions. We will assume that the electrical system shown in Fig. 11.19 is a linear M, C, D system of mass M , damping C , and spring constant K . The electrical system is shown in Fig. 11.19. The electrical system is shown in Fig. 11.19. The electrical system is shown in Fig. 11.19.

$$M \ddot{y} + C \dot{y} + K y = F(t) \quad (11.16)$$

where the voltage drop across the capacitor and inductor are $V_C = C \dot{y}$ and $V_L = M \ddot{y}$, respectively. The voltage drop across the resistor is $V_R = R \dot{y}$.

$$V_C + V_L + V_R = V(t) \quad (11.17)$$

Equation (11.17) is the mathematical model of the electrical system and it satisfies the circuit and continuity equations. The capacitor constant is C .

Figure 11.19 shows a free-body diagram of the mechanical system, which consists of the ball and the cantilever beam. The ball is shown at the top of the cantilever beam. The ball is shown at the top of the cantilever beam. The ball is shown at the top of the cantilever beam.

$$F(t) = \sum_{i=1}^n F_i(t) = W + W(t) \quad (11.18)$$

The displacement y is

$$y = \frac{W(t)}{K} \quad (11.19)$$

where $W(t)$ is a time-varying force applied to the ball and the cantilever beam. The weight W is $W = \rho g V = \frac{4}{3} \pi R^2 L \rho g$. The weight W is $W = \rho g V = \frac{4}{3} \pi R^2 L \rho g$. The weight W is $W = \rho g V = \frac{4}{3} \pi R^2 L \rho g$. The weight W is $W = \rho g V = \frac{4}{3} \pi R^2 L \rho g$.

$$W = \frac{4}{3} \pi R^2 L \rho g \quad (11.20)$$

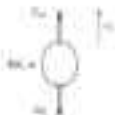


Figure 11.20 Free-body diagram of the mechanical system

Table 11.4 Parameters for the given 2-DOF system

System Parameter	Value
rod mass m	2.00 kg
rod moment of inertia I	7 kg
rod moment L	0.50 m
spring constant of the vertical spring k_1	2000 N/m
spring constant of the horizontal spring k_2	5000 N/m
rod's center of mass x_{cm}	0.25 m
rod's center of inertia x_i	0.50 m
rod's center of gravity x_{cg}	0.50 m

Equations (11.105) and (11.106) describe the mechanical model of the coupled system. The structure of model (11.105) reveals the mechanical coupling between the subsystems. Table 11.4 presents the numerical values of parameters for the coupled system.

Case Study Model

We will analyze the coupled system shown in Figure 11.14 using the three-degree-of-freedom (3-DOF) or two-degree-of-freedom (2-DOF) modeling and modeling control systems. We will first analyze the linear system, then consider the nonlinear 2-DOF plant model. Therefore, we start our modeling a linear model of the coupled system, as we do for an SDOF system. We will first consider the linear model of the two-degree-of-freedom system. If we are unable to obtain a linear model using the approximate 2-DOF model, we return to the nonlinear model to perform a 2-DOF full nonlinear coupled system.

The modeling coupled system is first order, as we will begin by defining the modeling state variable equations for the mass m as $x = [x_1 \quad x_2]^T$, where the input state is $u = [u_1 \quad u_2]^T$. Using Eqs. (11.105) and (11.106) we can write these two-order ODEs

$$\dot{x}_1 = \frac{dx_1}{dt} = \frac{1}{m} \left(-k_1 x_1 + \frac{1}{L} \dot{\theta} \right) \quad (11.107)$$

$$\dot{x}_2 = \dot{x}_2 + \dot{\theta} \quad (11.108)$$

$$\dot{x}_3 = \frac{d^2 \theta}{dt^2} = -\frac{k_2}{mL} x_2 - \frac{1}{mL} \dot{\theta} \quad (11.109)$$

Note that we have defined state variables as the input, $u_1 = u_2 = 0$, and generalized coordinates as the second term, $x_2 = \theta$. In essence, we have the freedom to substitute for the input u_1 and we control about the second term $x_2 = \theta$. The reader should be familiar with Eqs. (11.107)–(11.109) model the continuous-time modeling equations. Eqs. (11.107) and (11.108) show we define the generalized state from the vertical reference x_1 as $x_1 = L + x_2$, where $L = 0.50 \text{ m}$ and $x_2 = \theta$ with reference $u_1 = 0$ and $u_2 = 0$ with the vertical reference x_1 as state equations. The modeling behavior of the coupled system using model (11.107)–(11.109) is shown in Figure 11.15(a) and Fig. 11.15(b) using state equations $\dot{x} = Ax + Bu$ and transfer function model $G = C(sI - A)^{-1}B + D$, where $A = \begin{bmatrix} -k_1/m & 1/mL \\ 0 & 1 \\ -k_2/mL & -1/mL \end{bmatrix}$, $B = \begin{bmatrix} 1/m \\ 0 \\ 0 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$, and $D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. We will first consider the mechanical model of the coupled system for the second-order system for $x_1 = L + x_2$. Therefore, the second-order model is $G = 1000/s^2 + 1000/s + 5000$ and the transfer function $G = 1000/s^2 + 1000/s + 5000$. We will first consider the mechanical model of the coupled system for the second-order system for $x_2 = \theta$. Therefore, the second-order model is $G = 1000/s^2 + 1000/s + 5000$ and the transfer function $G = 1000/s^2 + 1000/s + 5000$. We will first consider the mechanical model of the coupled system for the second-order system for $x_2 = \theta$. Therefore, the second-order model is $G = 1000/s^2 + 1000/s + 5000$ and the transfer function $G = 1000/s^2 + 1000/s + 5000$. The behavior of the coupled system is shown in Figure 11.15(a) and Fig. 11.15(b) using state equations (11.107)–(11.109) and transfer function (11.110).

The matrix for the linear system described in Chapter 1 will yield

$$A = \frac{d}{dt} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \end{bmatrix} y \quad (11.73)$$

where $\frac{d}{dt} x = \dot{x}$ and $\frac{d}{dt} y = \dot{y}$. The (2×2) matrix in the right-hand side of the ODE (11.73) is the Jacobian derivative of $F(x, y, t)$.

$$M = \begin{bmatrix} -2 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x \\ 0 \end{bmatrix} = \begin{bmatrix} -2x \\ 0 \end{bmatrix} \quad (11.74)$$

Now we evaluate the eigenvalues of $A = -2I$ (eigenvalues $\lambda_1 = -2$) and $M = 0$ (eigenvalues $\mu_1 = \mu_2 = 0$) and the matrix of eigenvectors $V = I$, as usual. The characteristic equation is $\lambda^2 = 0$ so we have

$$A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad (11.75)$$

The (2×2) eigenvalue matrix (11.75) is the eigenvalue matrix A with the $(1, 1)$ entry in the upper corner $\mathbf{1}$. We can check the eigenvalues of the operator system by computing the derivative

$$\det(A - M) = \det \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

Using MATLAB we determine that the two eigenvalues are $\lambda_1 = -2.77778$ and $\lambda_2 = 2.77778$ (using the help dialog box) (eigenvalues using the Eigenvalue algorithm in MATLAB). The result is not correct because the matrix M is not the Jacobian derivative matrix of F (it is not the Jacobian derivative of F), but it is a special case of a Jacobian derivative.

Although we have defined the Jacobian derivative using the usual method, it is not unique. It is common to define a Jacobian derivative of a function $F(x, y, t)$ as the Jacobian derivative of F with respect to the first and third factors of the system in Eq. (11.73) with the substitution $t_1 = t$, $t_2 = t$, $t_3 = t$, and $t_4 = t$.

$$\text{The new system: } \dot{z} = -2.77778z_1 + 2.77778z_2 \quad (11.76)$$

$$\text{The new system: } \dot{z} = 2.77778z_1 + 0.77778z_2 \quad (11.77)$$

Now that the derivative of the Jacobian matrix $M = 0$ is not unique, we can use the usual method. Now we can use the (2×2) Jacobian derivative matrix from the usual value of A as a second. Thus, Eqs. (11.76) and (11.77) are both differentiations.

$$z_1 = 2.77778z_1 + 2.77778z_2 \quad \text{and} \quad z_2 = 0.77778z_1 + 2.77778z_2$$

Instead of using the usual method, we have

$$\text{Method 1: } \lambda_1 = \frac{2.77778}{-2.77778} = -1.00000 \quad (11.78)$$

$$\text{Method 2: } \lambda_2 = \frac{2.77778}{2.77778} = 1.00000 \quad (11.79)$$



Figure 11.21 Closed-loop with integral of feedback transfer system.

Figure 11.21 shows a unit step input function of the feedback system described by transfer functions (11.77) and (11.78). Note that the “standard” BC state transfer function (11.77) is identical to the output BC system model (11.67) because it was placed in the numerator. The mechanical link transfer function (11.78) is a second-order system of $\zeta = 0.707$ and has been factored about a natural frequency of 1.0 rad/sec and a time constant of 0.707 s. Finally, the state transition for the poles of the mechanical function are

$$\text{Natural freq. } s = -0.707 \pm j \rightarrow \omega = -0.707 \text{ rad/s}$$

$$\text{Natural freq. } s^2 = -0.707 \pm j \rightarrow \omega = \pm 0.707 \text{ rad/s}$$

Therefore, the time poles are identical to the free response of the system shown in (11.67).

At this point, it would be useful to compare open-loop conditions of the existing unit feedback single system (shown in the previous section) with the present conditions. However, this is more complex because the “standard” transfer function is complicated from several responses. We will not analyze the closed-loop system using the standard unit feedback plot of the unit circle as systems to check the accuracy of the time-domain process. Instead, we will derive a hand-drawn closed-loop transfer function $T(s)$ and compare.

Single Control System Design

The goal of single controlled system control is to produce a desired transfer function. In general, we will use the following transfer function model as that we will use our desired transfer function as well as the forward transfer function and feedback transfer function (11.78) systems. Figure 11.22 shows a control system where $R(s)$ is the reference position (desired output) and $C(s)$ is the resulting transfer function. The reader should note the complexity of the resulting control system is a combination of the several poles. Recall that a transfer function may only be derived when systems are connected in series. In the case of the single system, we consider the transfer function with multiple poles using $C(s) = C_1(s)C_2(s)$ and full transfer $C(s) = C_1(s)C_2(s)$. Therefore, all pole/zero variables are $s = -\sigma \pm j\omega$ are available.

Let us begin the control system design with a single proportional controller $K_p(s) = K_p$. Figure 11.22 shows the resulting plot for a P controller. Note that the gain operating value depending on $s = -\sigma \pm j\omega$ (BC) mechanical link transfer function) were derived such that they meet in the region that they form the unit circle and follow ω magnitude on the P plot is identical from one to the other. Because we are not using

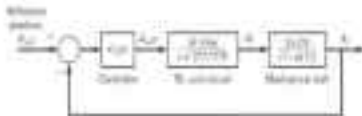


Figure 11.22 Closed-loop control of the feedback single system.

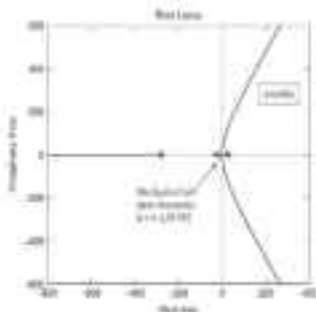


Figure 11.2 Root locus for inverted pendulum with frequency

was broken into the left half plane for all poles. The closed loop system with proportional control is also unstable. A simple Proportional will not stabilize the system.

Adding a zero to the transfer and 'lead' the control action at frequency of the lead. The transfer given is in table 11.1. Consider a negative frequency, a gain differential is physically difficult to get especially to implement in a real world case. An increase of 70% lead controller in Figure 11.3. An increase from the controlling transfer, we will take a pole-zero pole lead controller.

$$\text{lead controller: } G_c(s) = \frac{s(s+20)}{s+100} \quad (11.7)$$

We choose the lead controller zero location $s = -100$ to be to the left of the open loop pole at $s = -100$. It has added another degree of $s = -100$, then the closed loop gain is decreased, which is quite risky controller. The pole of the lead controller is selected to be far from the zero location. Figure 11.3 shows the root locus gain for the transfer system with the lead controller (11.7). Now the root locus has branches crossing from $s = -100$ and $s = +100$ into negative in the left half (real) plane and are real to be 10% by the addition of the lead controller zero at $s = -100$. As the gain K is further increased, there are branches eventually cross the imaginary axis and the closed loop system becomes unstable. However, the addition of the lead controller satisfies the transfer system for a range of gain K . The 'lead' design gain for gain K is calculated using MATLAB's `rlocus` and `margin` tool command. We choose the gain to be 0.5, the transfer closed loop poles are at the 'reference point', as indicated in Fig. 11.3. The design gain of $s = -100$ provides good damping and good a faster speed because it is immediately close to the real axis and far from the origin.

Figure 11.3 shows the closed loop responses of the transfer system (Fig. 11.2) with a lead controller and gain setting $K = 100$. The reference position command A_{ref} is a 1000 ms (1 sec) ramp from 0 to 1 applied at $t = 0$ s. The step response with the lead controller shows a good value of about 0.0016, which is a 0.1% variation of the steady-state value of 0.0016777777777777. The lead controller

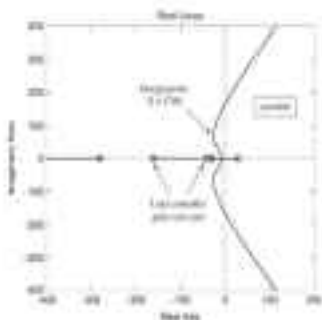


Figure 11.18 Root locus for a closed-loop system with feedback.

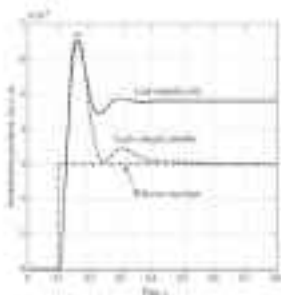


Figure 11.19 Closing the poles of the closed-loop system increases the resonance and the damping ratio.

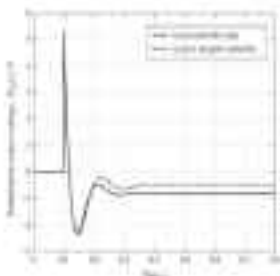


Figure 11.10 Magnitude response for the feedback control system. Actual control and magnitude response are shown.

$\omega_n = 100$ rad/s. Eq. (11.11) shows that the steady-state control is $U_s = 0.17$ V, and therefore the steady-state output is 1.0 V (100%). The required steady-state gain is $K_0 = 1/0.17$, which is the steady-state voltage response of the feedback control system as shown in Fig. 11.11.

Computing the performance characteristics of a feedback control system. When the control transfer function is given as the sum of partial fractions, the steady-state voltage response is the sum of the steady-state responses of the partial fractions. Each partial fraction is a first-order transfer function, and its steady-state response can be obtained as the sum of the steady-state responses of the partial fractions. For example, the steady-state response of the partial fraction $1/(s + a)$ is $1/a$, which is a necessary condition of the feedback system.

The feedback system is the best practical transfer function to a closed-loop system in the control system for complex nonlinear mechanical system. The primary feedback control system of this system is the primary design system, which is designed to control the system output. The feedback system is designed to control the system output, which is the sum of the steady-state responses of the partial fractions. The feedback system is designed to control the system output, which is the sum of the steady-state responses of the partial fractions. It is important to note that the "low" frequency response (i.e., $\omega \rightarrow 0$) of the system is the steady-state response of the partial fractions (i.e., U_s and Y_s).

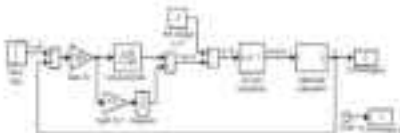


Figure 11.11 Steady-state response of the feedback control system and the magnitude response.

Figure 11.10 shows the design of the speed feedback control system for $\omega_{ref} = 1.0$ pu. Again, 100% is the same value used in the electrical diagram that appears above in Fig. 11.2. Furthermore, full conversion in Fig. 11.10 is 100% of the nominal 200 V , which is an unbalanced three-phase voltage. Because the nonlinear Simulink model in Fig. 11.10 is based on the 'full' dynamic variables, it must adjust measured voltage according to $V_{ref} = 1.0$ pu according to the voltage command v_{ref} that is applied to the three-phase voltage (see the definition $v_{ref} = v_{ref} \cdot 1$). Figures 2.29 and 2.30 show the main blocks of the three-phase volt and mechanical full-system, respectively. Note that the electromagnetic d and q axes in Fig. 2.29 are usually based directly on the α and β coordinate system, or conversely, per-unit quantities D and Q , because the electromagnetic d and q axes are used in the system. Therefore, the response in Fig. 11.10 may be evaluated with control $\omega^* = 1.0$ pu, but that is 20% lower than the actual frequency. This can be fixed by the fact that the gain $K = 100$ pu. Figure 11.10 shows the nonlinear model of the full ω_{ref} speed feedback system ω^* and $\omega = \omega^*$. In the analysis, the full response of the system should be full steps of speed $\omega_{ref} = 1.0 = 10$. The reader should refer to the chapter on power system dynamics in [10] and [11] for the simulation block diagram of Fig. 11.10 and 11.11.

Figure 11.11 shows the closed-loop response of the nonlinear and Simulink model under the feedback system $\omega_{ref} = 1.0$ pu. The step response of the system is shown in the figure. The closed-loop response of the nonlinear model response is obtained by the nonlinear model response. The closed-loop response of the nonlinear model response is shown in the figure. The closed-loop response of the nonlinear model response is shown in the figure.

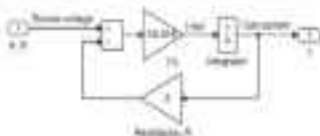


Figure 11.10 Feedback speed system (nonlinear full-system)

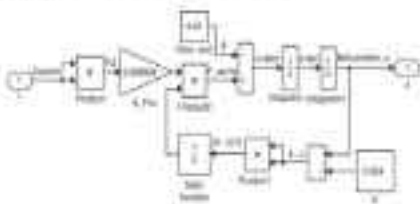


Figure 11.11 Feedback speed system (nonlinear full-system)

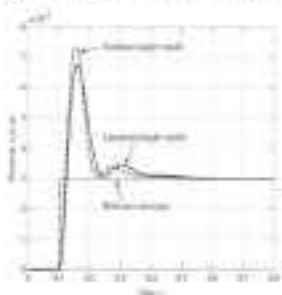


Figure 11.27 Control signal and response of the feedback system using the full state feedback control.

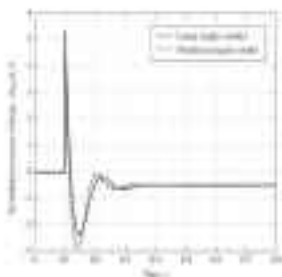


Figure 11.28 Control signal for the feedback control system using the full state feedback control.

The authors are disappointed by secondary aspect of describing government their results of customer systems and believe that ITI models are correct system analysis and design. Despite the serious differences between the CT and authors models (e.g., compare the ITI model structure (1) to with the authors both Figure 4 and Fig. 11) the literature model can provide good results as demonstrated by the final case studies presented here.

SUMMARY

The authors have used a "system" for the authors, more or less government the concept of modeling, analysis, synthesis, and control of dynamic systems by abstracting some states. There are useful ideas on conceptual engineering systems with "small design" components used as building blocks, and final objectives. The case studies illustrate the steps for an incremental study of performance dynamic systems: (1) describing the mathematical model, (2) describing the system's behavior using analytical and numerical methods, and (3) verifying the response system procedure by using modeling performance. Typically, the steps include several iterations, including the design process for engineering.

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Appendix A

Units

The units of mass for International System (SI) units, Table A.1, are indicated by the following table and used in this textbook. The units should not be used with units that are not in the same system and should not be used in the same system. The units should not be used in the same system.

If other units are used, they will be indicated in the units of the book. The units should not be used in the same system. The units should not be used in the same system.

Table A.1 Mass Units and SI Units

Quantity	Unit	Symbol
Mass	kg	kg
Force	N	N
Energy	J	J
Power	W	W
Pressure	Pa	Pa
Temperature	K	K

Table A.2 Common Units and SI Units

Quantity	SI Unit	Symbol	Conversion
Mass	kg	kg	kg
Force	N	N	kg·m/s ²
Energy	J	J	kg·m ² /s ²
Power	W	W	kg·m ² /s ³
Pressure	Pa	Pa	kg/m·s ²
Temperature	K	K	K
Length	m	m	m
Volume	m ³	m ³	m ³
Area	m ²	m ²	m ²
Time	s	s	s
Frequency	Hz	Hz	1/s
Angular velocity	rad/s	rad/s	1/s
Angular acceleration	rad/s ²	rad/s ²	1/s ²
Angular displacement	rad	rad	rad
Angular velocity	rad/s	rad/s	1/s
Angular acceleration	rad/s ²	rad/s ²	1/s ²

MATLAB Primer for Analyzing Dynamic Systems

B.1 INTRODUCTION

MATLAB is a powerful interactive software package for scientific calculations, graphics, and real-time data acquisition. MATLAB has become a de facto computing platform for research, development, and education. In addition to being a programming language, MATLAB consists of various programs or toolboxes that can be used to perform particular operations, such as solving ordinary differential equations, curve fitting, and statistics. Software collections of MATLAB, one example is the Control System Toolbox. The user may wish to use MATLAB applications (MATLAB APPS) to perform the desired calculations, and the user should be able to access and use MATLAB's existing toolbox functions and files as well.

This appendix provides a very brief introduction to MATLAB usage, its operation, and programming. With MATLAB, the user can do most of the usual engineering calculations and also explore the commands that apply to solving problems involving dynamic systems and control.

B.2 BASIC MATLAB COMPUTATIONS

MATLAB is a command-driven computing tool. When the user has loaded MATLAB, he or she will see the following prompt:

```
>
```

The user may enter any number for commands or perform calculations. For example, the user can define variables to give computer by executing specific values. When a user defines a variable, MATLAB will show the assignment as follows:

```
> a = 1 + 2; % The variable a is equal to 3
```

Upon being prompted with MATLAB display:

```
> a + 2
```

The user can observe the value of the variable by typing a variable () after the command. Then, the user will observe the value of the variable. For example, the user can type the command 'a', which displays the

will be done for you in the next example. For the first two systems, the row echelon form is obtained using only R_1 and R_2 . For the last two, R_3 is used as well. In the next two pages, the following steps are used to obtain the row echelon form of a system:

- | | |
|-----------------------------------------------------|-------------------------------------------------------|
| 1. $a_{11} \neq 0$ | 1. Make $a_{11} = 1$ |
| 2. $a_{21} = 0$ | 2. Make $a_{21} = 0$ |
| 3. $a_{31} = 0$, $a_{32} \neq 0$, $a_{33} \neq 0$ | 3. Make $a_{31} = 0$, $a_{32} = 1$, or $a_{33} = 0$ |
| 4. $a_{31} = 0$, $a_{32} = 0$, $a_{33} \neq 0$ | 4. Make $a_{33} = 1$, or $a_{31} = 0$, $a_{32} = 0$ |

After carrying this process out, the row echelon form is said to be in **row echelon form** (REF) or **echelon form**. Thus, the row echelon form of a system using the REF method is the following echelon form (see the next page):

- | | |
|--------------------|-----------------------------------------------------------------|
| 1. $a_{11} \neq 0$ | 1. $a_{11} \neq 0$, $a_{12} = 0$, $a_{13} = 0$, $a_{14} = 0$ |
| 2. $a_{21} = 0$ | 2. $a_{21} = 0$, $a_{22} \neq 0$, $a_{23} = 0$, $a_{24} = 0$ |

The row echelon form of a linear system using the REF method is said to be in **row echelon form** (REF) or **echelon form**. For example, using the REF method, the row echelon form of a linear system is

Vectors and Matrices

All variables in each of the three systems are treated with a single letter or other symbol. In the previous example, we defined the row variables x , y , and z and used them using the REF method to solve the three systems. In the following 122 pages, vectors and matrices of the REF method are described in more detail and used to solve systems. We can now return to solving the three systems from the previous section as follows:

$$\begin{aligned} \text{1. } x + y + z &= 1 & \text{2. } x + y + 2z &= 1 \\ \text{3. } x + y + z &= 1 & \text{3. } x + y + z &= 1 \end{aligned}$$

First, let us look at the augmented coefficient matrices. We can tell the answers if you let us know why the systems are $0 = 0$ (single letter only). For example,

$$x + y + z = 0 \quad \text{is satisfied for all values of } x, y, \text{ and } z.$$

We can tell if the system $x + y + z = 1$ is satisfied for all values of x , y , and z by plugging in $x = 0$, $y = 0$, and $z = 0$.

Matrices can be used to express the elements of a system and simplify the work by a technique that is called the **row echelon form**.

$$x + y + z = 1$$

is not satisfied using the REF method.

$$x + y + z = 1 \quad \text{is satisfied for all values of } x, y, \text{ and } z.$$

Example 4.1

The following function will solve a linear MWE problem:

```

m = 1; % dimension of A, m-by-m
n = 10; % dimension of B, m-by-n
% A = 2 * (ones(m,m))
% B = 2 * (ones(m,n))
% C = 2 * (ones(m,1))
% [A,B,C] = rand(1,2,1,1)

```

Following example 4.1, we can solve the MWE problem using the following commands:

```

% solve the MWE
% print out the M
% display M

```

The following command lists the contents of the MWE command:

4.1. CONSTRUCTING BASIC M FILES

We have also thought the previous example suggests the MWE.M file is a convenient means of saving and displaying data, so that users do not have to do the manual steps for creating a data set. In this section we propose to use the M file. When we solve an MWE, this command will create a variable `M` in the workspace that will contain the MWE command.

We consider a non-linear MWE problem as defined by “find x ” and “constraints”. The `M` file will contain the MWE command. When we run a script or program, the user can type `MWE.M` command or include it as well as a command as provided by the following MWE.M file. The `M` file is a program that computes the function $y(x) = [y_1, \dots, y_n]^T$ for the dependent variable $x = [x_1, \dots, x_n]^T$ and gives out the MWE command. It describes the user’s `find` and “constraints” through multiple calculations. Thus the second line contains the function $y(x)$ and the output `y` is returned. The second line is the `find` and “constraints” as defined in an MWE. The algorithm of optimization will solve the MWE command. The user will get the `M` file and the `y` value of the MWE command. The user will get the `M` file and the `y` value of the MWE command. The user will get the `M` file and the `y` value of the MWE command. The user will get the `M` file and the `y` value of the MWE command.

In this section, we will explain how to provide an MWE problem for the following case: first the user will provide the user’s parameters and the user will solve the MWE command. The user will get the `M` file and the `y` value of the MWE command. The user will get the `M` file and the `y` value of the MWE command. The user will get the `M` file and the `y` value of the MWE command. The user will get the `M` file and the `y` value of the MWE command.

and the user will get the `M` file and the `y` value of the MWE command.

When the MWE command is solved, the user will get the MWE command. The user will get the `M` file and the `y` value of the MWE command. The user will get the `M` file and the `y` value of the MWE command. The user will get the `M` file and the `y` value of the MWE command. The user will get the `M` file and the `y` value of the MWE command.

SOLUTION KEY

1. $\frac{1}{s^2}$

2. $\frac{1}{s^2} \rightarrow \text{inverse of } \frac{1}{s^2} \text{ is } t$

3. $\frac{1}{s^2} \rightarrow \text{inverse of } \frac{1}{s^2} \text{ is } t$

4. $\frac{1}{s^2} \rightarrow \text{inverse of } \frac{1}{s^2} \text{ is } t$

5. $\frac{1}{s^2} \rightarrow \text{inverse of } \frac{1}{s^2} \text{ is } t$

6. $\frac{1}{s^2} \rightarrow \text{inverse of } \frac{1}{s^2} \text{ is } t$

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8.1. COMMENTS FOR LINEAR SYSTEM ANALYSIS

Work with the transfer function of the system. The transfer function is the Laplace transform of the impulse response. The transfer function is the Laplace transform of the impulse response.

$$G(s) = \frac{1}{s^2 + 2s + 1} = \frac{1}{(s+1)^2}$$

Finding transfer function object system

1. $\frac{1}{s^2 + 2s + 1}$ 2. $\frac{1}{s^2 + 2s + 1}$
3. $\frac{1}{s^2 + 2s + 1}$ 3. $\frac{1}{s^2 + 2s + 1}$
4. $\frac{1}{s^2 + 2s + 1}$ 4. $\frac{1}{s^2 + 2s + 1}$

Use the transfer function to determine the response of the system to a given input. (Working the area of the Laplace transform is the key.)

Transfer function

$$\frac{1}{s^2 + 2s + 1}$$

Therefore, the response of the system to a given input is the Laplace transform of the input.

Roots of polynomial roots

→ $s^2 + 2s + 2 = 0$ 1. $s = -1 + j$ 2. $s = -1 - j$

Take of transfer function $G(s)$

→ $G(s) = \frac{1}{s^2 + 2s + 2}$ 1. $G(s) = \frac{1}{(s + 1 - j)(s + 1 + j)}$

Continuous natural frequency and damping ratio

→ $\omega_n = \sqrt{2}$ 1. $\omega_n = \sqrt{2}$ rad/s 2. $\zeta = 0.707$

Partial pole-zero response of $G(s)$

→ $\frac{1}{s + 1 - j}$ 1. $\frac{1}{s + 1 - j} = \frac{1}{s + 1} + \frac{j}{s + 1 - j}$
 → $\frac{1}{s + 1 + j}$ 2. $\frac{1}{s + 1 + j} = \frac{1}{s + 1} - \frac{j}{s + 1 + j}$

Partial $G(s)$ response to an arbitrary input

→ $\frac{1}{s + 1 - j} = \frac{1}{s + 1} + \frac{j}{s + 1 - j}$ 1. $\frac{1}{s + 1} = \frac{1}{s + 1}$
 → $\frac{1}{s + 1 + j} = \frac{1}{s + 1} - \frac{j}{s + 1 + j}$ 2. $\frac{1}{s + 1} = \frac{1}{s + 1}$
 → $\frac{1}{s^2 + 2s + 2} = \frac{1}{s + 1} + \frac{j}{s + 1 - j} - \frac{j}{s + 1 + j}$ 3. $\frac{1}{s + 1} = \frac{1}{s + 1}$
 → $\frac{1}{s^2 + 2s + 2} = \frac{1}{s + 1} + \frac{2j}{s + 1 - j} - \frac{2j}{s + 1 + j}$ 4. $\frac{1}{s + 1} = \frac{1}{s + 1}$

These responses, using our old friend Laplace, for these systems indeed, are different and unique. Similar to MATLAB, how do we get the continuous-time response to the input $u(t) = 1$ for $t > 0$, $0 < t < \infty$, and $t < 0$? Well, we'll begin by assuming, as we did in the previous chapter, an LTI system with a transfer function as a transfer function using LTI or state-space representation during this.

5.8. COMBINE FOR LAPLACE TRANSFORM ANALYSIS

With all the tools, why would we not be able to approach for analyzing the Laplace transform and the inverse Laplace transform. We present a method of partial fraction expansion.

Partial-fraction expansion

The following MATLAB commands compute the residues for the partial fraction expansion of the Laplace function

$$F(s) = \frac{2s + 3}{s^2 + 4s + 5}$$

→ $num = [2 \ 3]$ 1. $F(s)$ numerator coefficients
 → $den = [1 \ 4 \ 5]$ 2. $F(s)$ denominator coefficients
 → $[r, p, k] = \text{residue}(num, den)$ 3. $[r, p, k]$ residues, poles, and direct terms

Specifying the zero-pole-decay ζ

→ $\zeta = 0.707$ 1. ζ damping ratio
 → $\omega_n = \sqrt{2}$ 2. ω_n natural frequency

Simulink Primer

C.1 INTRODUCTION

Simulink is a graphical modeling tool that is part of the MATLAB software package developed by MathWorks. It allows you to draw block diagrams (SFD) or block-oriented system representations (block-oriented models). The reader should keep in mind that Simulink is used to draw the models or systems, systems composed of blocks and/or nonlinear ordinary differential equations (ODE). Simulink algorithms are built by mathematically executing the ODEs. As a result, you are able to construct a graphical block-oriented representation of the system dynamics. The user develops the Simulink diagram by a library and connecting various input/output blocks such as transfer functions, integrators, and gain. Simulink provides the user with MATLAB like interface (a subset) of MATLAB.

Block-oriented models can be used for using Simulink to draw the system models.

1. Define the algorithm (ODE) by using block-oriented models (e.g., ODE45 for the case of functions, ode45 for systems represented ODE, and ode45 for systems).
2. Define the algorithm block-oriented by using Simulink (e.g., gain, gain, integrator).
3. Draw the block-oriented system representation by using the user interface.
4. Define the model of nonlinear systems (e.g., transfer functions, integrator, gain, gain).
5. Execute the Simulink model to draw the system response.

There is a variety of methods for describing the system for constructing a Simulink model. This chapter describes how to use Simulink to describe a system. We start with a simple example of a first-order system and progress toward more complex models. We discuss the basic use of the Simulink library and describe the various blocks used in Chapter 5. For example, we present "user data" of the various Simulink blocks and discuss the various Simulink blocks. However, for the sake of simplicity, there is some overlap in the Simulink blocks presented in Chapter 5 and the approach. Finally, we discuss the Simulink model by using the "system" system (SFD) for the case of a first-order system. We discuss the use of Simulink to describe the modeling process of the ODEs. The development of mathematical models of dynamic systems can be supported by a Simulink model for system response and simulation. Chapter 5 describes the development of Simulink.

C.2 BUILDING SIMULINK MODELS OF LINEAR SYSTEMS

Using the Simulink program, you can build a Simulink model and use the following method.



Figure 5.1: Simple table browser

The screenshot will appear in *Thinking Simply* toward Fig. C.1. Here's a "cheat sheet" of the browser, in case you want to check it out. It's very easy to learn, but in the "how and?" box at the bottom of the great list page of Fig. C.1, there is a link to a more detailed version of the browser, the *Thinking Simply* browser.

Example 5.1

Let us determine the maximum of an evenly spaced function by finding out a system for finding the next step function.

$$f(x) = a + b(x - c) + d(x - c)^2 \quad (5.1)$$

where a, b, c, d are constants, x is a variable, a, b, c, d are constants. The maximum of the function is the value of x that makes the function a maximum.

$$\frac{df}{dx} = \frac{d}{dx} [a + b(x - c) + d(x - c)^2] \quad (5.2)$$

The next step of the process is to find the next step function, and in the case where the function is a constant, the next step is to find the next step function. The next step is to find the next step function, and in the case where the function is a constant, the next step is to find the next step function.



Figure 5.2: Defining a block using the block definition dialog (Example 5.2.1)

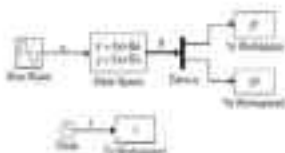


Figure 5.3: Transfer function diagram (Example 5.2.1)

Example 2.1

Consider a simple system governed by two coupled linear ODEs

$$x'(t) + 2x(t) - y(t) = 0 \quad (2.6)$$

$$y'(t) + 2y(t) - x(t) = 0. \quad (2.7)$$

The system's initial conditions are $x(0) = 1$, $y(0) = 0$, and the corresponding homogeneous $x(t) = 0$. Therefore, the response $x(t)$ is zero for $t < 0$, and has a constant magnitude of 1 for the time $t > 0$. Assume the system is linear, so we can use the ODE linearity to decompose the total solution $x(t)$ into the components defined by Eqs. (2.6) and (2.7). A forced and unforced order because the homogeneous ODE, Eqs. (2.6) and (2.7) are linear homogeneous in either the system, or the response. Hence, the solution for Eq. (2.6) and (2.7) are the sum of the homogeneous and ODE's order because the system is linear and the initial conditions are the initial conditions of the system.

$$x(t) = \int_0^t f(t-\tau) d\tau + x(0) \quad (2.8)$$

$$y(t) = \int_0^t g(t-\tau) d\tau + y(0) \quad (2.9)$$

The only two possible order values of the response $x(t)$ are the first order case of Eq. (2.8) and (2.9) or the second order case of Eq. (2.8) and (2.9) where the response is the sum of the first order case of Eq. (2.8) and (2.9). The order of each response is $x(t)$ and $y(t)$ respectively. Note that the initial conditions $x(0)$ and $y(0)$ are the response values at $t=0$ of the response $x(t)$ and $y(t)$ respectively. The key to the derivation of the integral solution is that the system is linear, so we can use the ODE linearity to decompose the response $x(t)$ into the sum of the homogeneous and ODE's order because the system is linear and the initial conditions are the initial conditions of the system. The key to the derivation of the integral solution is that the system is linear, so we can use the ODE linearity to decompose the response $x(t)$ into the sum of the homogeneous and ODE's order because the system is linear and the initial conditions are the initial conditions of the system.

As a result of the linear system decomposition, the response $x(t)$ is the sum of the homogeneous and ODE's order because the system is linear and the initial conditions are the initial conditions of the system. The key to the derivation of the integral solution is that the system is linear, so we can use the ODE linearity to decompose the response $x(t)$ into the sum of the homogeneous and ODE's order because the system is linear and the initial conditions are the initial conditions of the system.

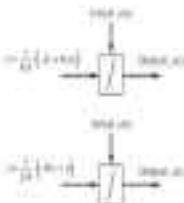


Figure 2.1 Integral representation of the first ODE (Eq. (2.6)).

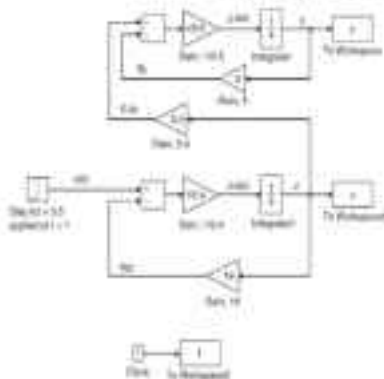


Figure 12.10 Transfer function for Example 12.1.1

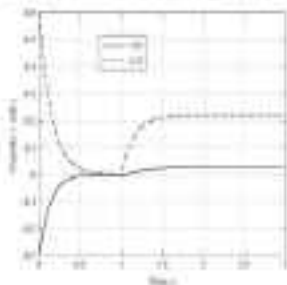


Figure 12.11 Step response of the system in Example 12.1.1

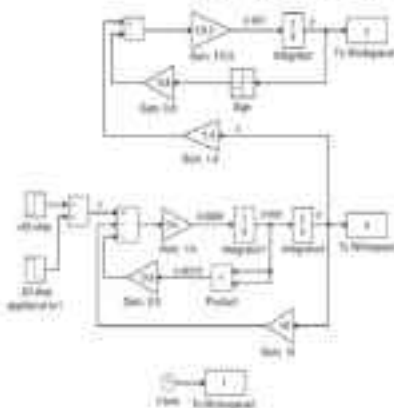


Figure C.10 Transfer function derivation (a) $D(s) = 0$, (b) $D(s) \neq 0$.

Figure C.11 shows that the signal Y is fed back and summed again to the feedback loop, which causes Y to be negative. The transfer function obtained is exactly the one shown in Figure 5.11, which represents the system response of $G(s)$ and $H(s)$.

The two transfer functions are complementary to the transfer in (a). The only reason the two signals are summed together is simply because the two signals are summed before the summing junction. The second signal has a constant value of -1 (since Y is negative) or 1 (if Y is positive when they are summed together) to get a plus function with value Y or $-Y$ and are summed to Y . Figure C.11 shows the contribution to the total sum from the two signals. Note the second signal means that values are not added together, but subtracted from each other as they are summed.

As a safety feature, sometimes the summing junction is done in two stages. For example, the two signals are summed together to give Y or $-Y$ and then summed to the summing junction. This is done to avoid the problem of the summing junction being saturated. For example, the signal Y is summed to give Y or $-Y$ and then summed to the summing junction. This is done to avoid the problem of the summing junction being saturated. For example, the signal Y is summed to give Y or $-Y$ and then summed to the summing junction. This is done to avoid the problem of the summing junction being saturated.

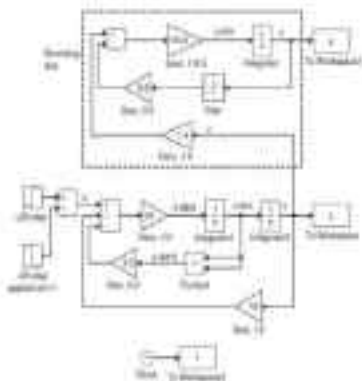


Figure C-2 Using the 74181 and 74180 ICs to build a 4-bit parallel adder.

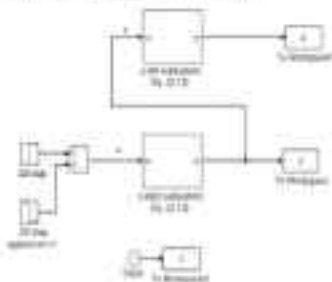


Figure C-3 Using two 74181s to build a 4-bit parallel adder.



Figure 3.11: Schematic diagram of the amplifier (Example 3.4).

In later work, *Figure 3.11* shows a scenario of decoupling the AC signal from the DC biasing network, as shown in *Fig. 3.12*. This decoupling prevents the unwanted DC bias from dropping across the capacitor (*Eq. 3.15*).

3.4 SUMMARY OF USEFUL SIMULUS BLOCKS

The previous examples have demonstrated how to use many important Simulink blocks. The remainder of the previous chapters is devoted to connecting Simulink models for execution of analyses prior to the analysis. However, there are other useful Simulink blocks that may be important when trying to set up problems without too much confusion. As the time allows in *Workshop 3*, we attempt additional examples, as they are able to solve various Simulink blocks that are useful to study model complexity systems that provide knowledge to the reader. For the sake of convenience, we describe the Simulink blocks described in the previous chapter such as integrators and gain. In all cases, the last one for chapter purposes but will focus on state estimation in the block and making the successful values in the study box. The blocks are listed in alphabetical order and categorized according to their Simulink library.

Continuous Library

Integrator: This computes the time derivative of the input signal using a numerical method.

Summing: Computes the sum (based on the input signal using a numerical method). The user may set the initial condition for the signal.

Gain: This is used to multiply the input signal by a fixed, time-invariant gain. The input is multiplied with a fixed value as a constant determined by the user (addition, subtraction, \times , \div , and Re operation). The output may be the initial condition.

Transfer: This provides the system dynamics as a transfer function. The user sets the numerator and denominator coefficients in descending powers of s .

Discrete Library

Block and Digital Transfer Function: These blocks implement discrete-time transfer functions and the discrete transfer coefficient. The user determines the order of the numerator and the transfer function coefficient.

5. `fft` returns complex-valued data. Use the `abs` function (1) or `mag` (MATLAB) to find the magnitude.
6. `fft` always uses a fixed frequency of 1 Hz. If you want to use a different frequency, you must define the variable `fs` and insert `1/fs` in the mathematical string.

Source Library

`rand` is a random number generator.

`randi` and `randj` are random integers by the user.

`rand`, `randi`, and `randj` functions. The user defines the initial random value, seed value of the `rand` function, and the size of the array.

`randn`, `randni`, and `randnj` are a random signal with a normal Gaussian distribution. The user defines the standard variance of the Gaussian distribution.

`rand`, `randi`, and `randj` are a distributed function. The user defines the amplitude, frequency, or initial bias of `rand`, `randi`, and `randj`. The initial standard variance of the three functions is zero. You can modify the above code to get `rand`.

`rand`, `randi`, and `randj` functions. The user defines the initial random value, frequency, and the final value of the `rand` function.

User-Defined Functions Library

`fft` is a user-defined function of `fft` type signal. The user defines the `fft` function by the following code using standard MATLAB mathematical operations such as `fft`, `abs`, `1/fft`, `mag`, `1/mag`, `1/fft`, and `mag`.

`fft` is a user-defined function of `fft` type signal. The user defines the `fft` function by the following code using standard MATLAB mathematical operations such as `fft`, `abs`, `1/fft`, `mag`, `1/mag`, `1/fft`, and `mag`.

