Mei Symmetries of Lagrangian of the Minkowski Spacetime

Aqsa Mughal
Regn.00000331133

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Supervised by: Prof. Dr Tooba Feroze

Department of Mathematics

School of Natural Sciences
National University of Sciences and Technology
H-12, Islamabad, Pakistan

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Name of Supervisor: Prof. Tooba Feroze

Date: Oct 18, 2024.

Signature (HoD): ____________

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Examination Committee Members

1. Name: PROF. IBRAR HUSSAIN Signature: ____________

2. Name: DR. ADNAN ASLAM Signature: ____________

Supervisor's Name: PROF. TOOBA FEROZE Signature: ____________

Head of Department

18-10-2024
Date

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Dedication

I would like to dedicate this thesis to *me and my parents.*

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All praise is due to **Allah Almighty**, the Most Merciful, who has been my strength and guide through all the hurdles and challenges and blessed me with the courage to complete this thesis. I am deeply grateful for the teachings of Hazrat Muhammad (PBUH), which have inspired me.

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Abstract

This thesis explores Mei symmetries related to the Lagrangian of Minkowski space-time. The study begins by analyzing the criteria for Mei symmetries, followed by applying the Minkowski space-time Lagrangian to derive the Euler Lagrange equations and the associated determining equations. Through solving these determining equations, a total of thirteen Mei symmetries are identified for the given Lagrangian. Additionally, the research reviews Lie point symmetries and the Noether symmetries of the same Lagrangian, but precisely in this case, the Mei symmetries and the Noether symmetries are a subset of the Lie point symmetries. A verification procedure is conducted to confirm the accuracy of the obtained results.

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Chapter 1

Introduction

This chapter provides an overview of differential equations (DEs) and offers examples of the historical development of symmetries in ordinary differential equations (ODEs) and partial differential equations (PDEs). It introduces the basic concepts, definitions, Lie point symmetries, Noether symmetries, and Mei symmetries. It also explains how to find these symmetries step-by-step, using straightforward explanations and practical examples to illustrate the process.

1.1 Historical Overview

The evolution of DEs, a fundamental tool for understanding dynamic processes, has been shaped by the contributions of many brilliant minds over the centuries. These equations find applications in various fields, such as physics, engineering, economics, and biology, making them crucial to our understanding of the world. Although pinpointing a singular origin is difficult, key moments and figures have been recognized as foundational in developing this critical field.

In 1671, Sir Isaac Newton laid the foundation for calculus, a significant development in the history of mathematics, which he documented in his work "The Method of Fluxions and Infinite Series" [1]. However, it was not until 1693 that Gottfried Wilhelm Leibniz formulated a solution to the first differential equation. This solution not just change but revolutionize the course of mathematics, and the true potential of Newton's insights was realized. This marked a crucial turning point in the advancement of differential equations.

Jacob and Johann Bernoulli extended Leibniz's work and in 1695, introduced a technique for solving a particular type of first-order ODE known as the Bernoulli equation, which further advanced the field. The Bernoulli equation is significant not just for its mathematical elegance, but also for its practical applications. It is solved using a simple substitution, making it a powerful tool for solving various problems in physics and engineering, thereby demonstrating the real-world impact of mathematical concepts [2].

Leonhard Euler made substantial contributions across various branches of mathematics, including infinitesimal calculus, trigonometry, geometry, number theory, and algebra. His extensive work often involved using power series to solve differential equations, and he introduced Euler's identity and formula, which remain cornerstones in mathematics today, underscoring the enduring impact of his work. Euler collaborated with Joseph-Louis Lagrange to establish the calculus of variations, leading to the development of the Euler-Lagrange equation [3]. Euler's work continues to shape the field of mathematics, demonstrating the lasting influence of his contributions.

As the study of differential equations progressed, a new generation of mathematicians emerged, each making significant contributions that furthered the understanding and application of these equations. Figures such as Joseph-Louis Lagrange, Pierre-Simon Laplace, Adrien-Marie Legendre, Joseph Fourier, Friedrich Bessel, Augustin-Louis Cauchy, Rudolf Lipschitz, Bernhard Riemann, Carl Friedrich Gauss, Emmy Noether, and George David Birkhoff all played crucial roles in advancing the discipline. Their collaborative efforts have not only shaped the field of mathematics but also continue to inspire and guide us today, serving as a testament to the power of human intellect and determination in the pursuit of knowledge [4].

The contributions of Evariste Galois have profoundly impacted the solution of polynomial equations by using group theory, while Marius Sophus Lie's work applied group theory to differential equations. Lie's approach used symmetries to show how important it is to understand transformations and their corresponding generators. This insight facilitates the discovery of patterns and innovative solutions in equation solving [5].

1.2 Preliminaries

The Lie symmetry method focuses on identifying symmetries of DEs and using these symmetries to transform the DEs into reduced order DEs. To effectively apply Lie's theory, it is crucial to grasp the foundational concepts and terminology associated with his approach.

1.2.1 Lie Point Transformations and Infinitesimal Generators

Definition 1.2.1. It is a transformation that transforms any point (x, y) into a new point $(\tilde{x}, \tilde{y})$

$$\tilde{x} = \tilde{x}(x, y), \quad \tilde{y} = \tilde{y}(x, y). \tag{1.1}$$

where, x and y are independent and dependent variables, respectively. Point transformations with parameter dependence must be included in symmetry considerations.

1.2.2 Lie Group Transformations with a Single Parameter

Definition 1.2.2. A single or one-parameter group of Lie point transformations consists of transformations that rely on a single parameter $\varepsilon \in \mathbb{R}$, i.e.

$$\tilde{x} = \tilde{x}(x, y, \varepsilon), \quad \tilde{y} = \tilde{y}(x, y, \varepsilon). \tag{1.2}$$

This group satisfies the properties of closure, inverse, and identity. Setting $\varepsilon = 0$ results in the identity transformation.

$$\tilde{x}(x, y, 0) = x, \quad \tilde{y}(x, y, 0) = y. \tag{1.3}$$

Translation

$$(\tilde{x}, \tilde{y}, \tilde{z}) \longmapsto (x + \varepsilon, y + 2\varepsilon, z - \varepsilon). \tag{1.4}$$

Rotation

$$(\tilde{x}, \tilde{y}, \tilde{z}) \longmapsto (x \cos\varepsilon - z \sin\varepsilon, y, x \sin\varepsilon + z \cos\varepsilon). \tag{1.5}$$

Scaling

$$(\tilde{x}, \tilde{y}) = (e^{\varepsilon} x,\ e^{\varepsilon} y). \tag{1.6}$$

These transformations rely on one parameter and satisfy Lie group axioms. However, the reflection transformation

$$\tilde{x} = -x, \quad \tilde{y} = -y, \tag{1.7}$$

does not belong to a single-parameter group of Lie point transformations, though it is still a point transformation [6].

Symmetry transformations transform one DE solution into another, preserving its essential characteristics. For instance, given ODE

$$V = V\left(x, y, y', \ldots, y^{(m)}\right) = 0, \tag{1.8}$$

applying a Lie point transformation,

$$V = V(\tilde{x}, \tilde{y}, \tilde{y}', \ldots, \tilde{y}^{(m)}) = 0. \tag{1.9}$$

shows that the transformation is symmetric as it does not alter the form of the equation.

1.2.3 Infinitesimal Generators

Applying the Taylor series to eq (1.2) about $\varepsilon = 0$ provides an infinitesimal representation of the Lie point transformation [6].

$$\begin{aligned} \tilde{x} &= x + \varepsilon \left(\frac{\partial \tilde{x}}{\partial \varepsilon} \right) \Bigg|_{\varepsilon=0} + O\left(\varepsilon^2\right), \\ \tilde{y} &= y + \varepsilon \left(\frac{\partial \tilde{y}}{\partial \varepsilon} \right) \Bigg|_{\varepsilon=0} + O\left(\varepsilon^2\right). \end{aligned} \tag{1.10}$$

Writing

$$\left.\frac{\partial \tilde{x}}{\partial \varepsilon}\right|_{\varepsilon=0} = \xi(x, y), \quad \left.\frac{\partial \tilde{y}}{\partial \varepsilon}\right|_{\varepsilon=0} = \eta(x, y), \tag{1.11}$$

the symmetry generator for the transformation is established as

$$\mathbf{X} = \xi(x, y) \frac{\partial}{\partial x} + \eta(x, y) \frac{\partial}{\partial y}. \tag{1.12}$$

Consider the following examples to illustrate transformations and their generators Translational group

$$\tilde{x} = x, \quad \tilde{y} = y + \varepsilon, \tag{1.13}$$

We have,

$$\xi(x,y) = \left.\frac{\partial \tilde{x}}{\partial \varepsilon}\right|_{\varepsilon=0} = 0, \quad \eta(x,y) = \left.\frac{\partial \tilde{y}}{\partial \varepsilon}\right|_{\varepsilon=0} = 1, \tag{1.14}$$

and the generator is

$$\mathbf{X} = \frac{\partial}{\partial y}. \tag{1.15}$$

Now consider another example, for eq (1.6) we have,

$$\xi(x,y) = \left.\frac{\partial \tilde{x}}{\partial \varepsilon}\right|_{\varepsilon=0} = x, \quad \eta(x,y) = \left.\frac{\partial \tilde{y}}{\partial \varepsilon}\right|_{\varepsilon=0} = y, \tag{1.16}$$

and the generator is

$$\mathbf{X} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}. \tag{1.17}$$

1.2.4 Lie Point Transformations for M-Parameter Group

A group of transformations may depend on multiple parameters. This implies that, unlike eq (1.2), transformation can be written as

$$\tilde{x} = \tilde{x}\left(x, y, \varepsilon_M\right), \quad \tilde{y} = \tilde{y}\left(x, y, \varepsilon_M\right), \quad M = 1, \ldots, m. \tag{1.18}$$

If all these parameters satisfy the group axioms independently of each other, then these Lie point transformations form an M-parameter group (G_M) [7]. For every parameter ε_M of the M-parameter Lie point transformation group, an infinitesimal generator can be constructed as

$$\mathbf{X}_M = \xi_M\frac{\partial}{\partial x} + \eta_M\frac{\partial}{\partial y}, \tag{1.19}$$

where the infinitesimals are described as

$$\xi_M(x,y) = \left.\frac{\partial \tilde{x}}{\partial \varepsilon_M}\right|_{\varepsilon_M=0}, \eta_M(x,y) = \left.\frac{\partial \tilde{y}}{\partial \varepsilon_M}\right|_{\varepsilon_M=0}. \tag{1.20}$$

Example 1.2.3. Consider a Lie point transformation group [8] characterized by three parameters

$$\tilde{x} = x + a, \quad \tilde{y} = ye^{b} + c. \tag{1.21}$$

We need to determine the infinitesimal generator of these transformations. Suppose the infinitesimal transformations where a, b, and c are very small, denoted as $\varepsilon\alpha, \varepsilon\beta$ and $\varepsilon\gamma$ respectively and ε is a small parameter.
Transformation given by eq (1.21) become

$$\tilde{x} = x + \varepsilon\alpha, \quad \tilde{y} = ye^{\varepsilon\beta} + \varepsilon\gamma. \tag{1.22}$$

Expanding and neglecting terms of order ε^2 and higher, we obtain

$$\tilde{x} = x + \varepsilon\alpha, \quad \tilde{y} = y(1 + \varepsilon\beta) + \varepsilon\gamma. \tag{1.23}$$

Translation in x (parameter a)

$$\tilde{x} = x + \varepsilon\alpha \tag{1.24}$$

Thus,

$$\xi_1(x, y) = 1, \quad \eta_1(x, y) = 0. \tag{1.25}$$

Associated generator is

$$\mathbf{X}_1 = \frac{\partial}{\partial x}. \tag{1.26}$$

Scaling in y (parameter b)

$$\tilde{y} \approx y + \varepsilon y\beta, \tag{1.27}$$

$$\xi_2(x, y) = 0, \quad \eta_2(x, y) = y. \tag{1.28}$$

The associated generator is

$$\mathbf{X}_2 = y\frac{\partial}{\partial y}. \tag{1.29}$$

Translation in y (parameter c)

$$\tilde{y} = y + \varepsilon\gamma, \tag{1.30}$$

$$\xi_3(x, y) = 0, \quad \eta_3(x, y) = 1. \tag{1.31}$$

Now the associated generator is

$$\mathbf{X}_3 = \frac{\partial}{\partial y}. \tag{1.32}$$

1.2.5 Extended Lie Point Transformations and their Generators

It becomes necessary to extend Lie point transformation in eq (1.2) to its derivatives $y^{(m)}, \quad m = 1, 2, \ldots n$, so that eq (1.8) may undergo these transformations. The extension is as follows

$$\begin{aligned} \tilde{x} &= x + \varepsilon\xi(x, y) + O\left(\varepsilon^2\right), \\ \tilde{y} &= y + \varepsilon\eta(x, y) + O\left(\varepsilon^2\right), \\ \tilde{y}' &= y' + \varepsilon\eta^{(1)}\left(x, y, y'\right) + O\left(\varepsilon^2\right), \\ &\vdots \\ \tilde{y}^{(m)} &= y^{(m)} + \varepsilon\eta^{(m)}\left(x, y, y', \ldots, y^{(m)}\right) + O\left(\varepsilon^2\right), \end{aligned} \tag{1.33}$$

where

$$\begin{aligned} \eta^{(1)} &= \frac{\partial \tilde{y}'}{\partial \varepsilon}|_{\varepsilon=0}, \\ \eta^{(2)} &= \frac{\partial \tilde{y}''}{\partial \varepsilon}|_{\varepsilon=0}, \\ &\vdots \\ \eta^{(m)} &= \frac{\partial \tilde{y}^{m}}{\partial \varepsilon}|_{\varepsilon=0}. \end{aligned} \tag{1.34}$$

For finding prolongation coefficients, put eq (1.33) in eq (1.8), we get

$$\tilde{y}' = \frac{D_x\left(\tilde{y}\right)}{D_x\left(\tilde{x}\right)} = \frac{dy + \varepsilon d\eta + \ldots\ldots}{dx + \varepsilon d\xi + \ldots\ldots}, \tag{1.35}$$

$$= y' + \varepsilon\left((d\eta/dx) - y'(d\xi/dx)\right) + \ldots\ldots \tag{1.36}$$

where D_x represents the total derivative

$$D_x = \frac{d}{dx} = \frac{\partial}{\partial x} + y'\frac{\partial}{\partial y} + y''\frac{\partial}{\partial y'} + y'''\frac{\partial}{\partial y''} + \ldots\ldots.. \tag{1.37}$$

On comparing $\tilde{y}'$ from eq (1.36) to $\tilde{y}'$ from eq (1.33), we get

$$\eta^{(1)} = D_x\eta - y'D_x\xi \tag{1.38}$$

Likewise, for $\tilde{y}^m$, we get

$$\tilde{y}^m = y^{(m)} + \varepsilon\left(D_x\eta^{(m-1)} - y^m D_x\xi\right) + O\left(\varepsilon^2\right). \tag{1.39}$$

When comparing $\tilde{y}^m$ from eq (1.39) to $\tilde{y}^m$ in eq (1.33), we find

$$\eta^{(m)} = D_x \eta^{(m-1)} - y^{(m)} D_x \xi. \tag{1.40}$$

and the extended form of an infinitesimal generator is

$$\mathbf{X}^{[m]} = \mathbf{X} + \eta^{(1)} \frac{\partial}{\partial y'} + \ldots + \eta^{(m)} \frac{\partial}{\partial y^{(m)}}. \tag{1.41}$$

where $\mathbf{X}$ is given by the eq (1.12).

Furthermore, the first two prolongations of η can be computed by putting eq (1.37) into eq (1.40)

$$\eta^{(1)} = \eta_x + y' \left(\eta_y - \xi_x\right) - y'^2 \xi_y, \tag{1.42}$$

$$\eta^{(2)} = \eta_{xx} + y' \left(2\eta_{xy} - \xi_{xx}\right) + y'^2 \left(\eta_{yy} - 2\xi_{xy}\right) - y'^3 \xi_{yy} + y'' \left(\eta_y - 2\xi_x - 3y' \xi_y\right). \tag{1.43}$$

Example 1.2.4. Consider a generator

$$\mathbf{X} = y \frac{\partial}{\partial y}, \tag{1.44}$$

For finding a second-order prolonged generator, from eq (1.44), we can deduce that $\xi = 0$ and $\eta = y$. Using (1.42) and (1.43),

$$\implies \eta^{(1)} = y^{'} \; and \; \eta^{(2)} = y^{''}. \tag{1.45}$$

As a result, the second-order extended generator is

$$\mathbf{X}^{[2]} = y \frac{\partial}{\partial y} + y^{'} \frac{\partial}{\partial y'} + y^{''} \frac{\partial}{\partial y''}. \tag{1.46}$$

1.2.6 Lie Point Symmetries of ODEs

Consider a set of symmetry transformations that depends on at least one parameter. These are Lie point symmetries, a term borrowed from the Norwegian mathematician Sophus Lie [6]. Our focus now shifts to the ODE (1.8) which admits a group of symmetries with generator $\mathbf{X}$ for which

$$\mathbf{X}^{[m]} V \Big|_{V=0} = 0, \tag{1.47}$$

holds, where $\mathbf{X}^{[m]}$ represents the mth prolongation of an infinitesimal generator as defined in eq (1.41).

Example 1.2.5. Suppose for a DE [9]

$$y^{'} = \frac{-y+2x}{x}, \tag{1.48}$$

admitting a generator

$$\mathbf{X} = \frac{1}{y+2x}\frac{\partial}{\partial x}. \tag{1.49}$$

From eq. (1.49)

$$\xi = \frac{1}{y+2x}, \; \eta = 0. \tag{1.50}$$

Now,

$$\xi_x = \frac{-2}{(y+2x)^2}, \tag{1.51}$$

Prolonging η as follows

$$\Longrightarrow \; \eta^{(1)} = \frac{2}{(y+2x)^2}\; y^{'}, \tag{1.52}$$

$$\mathbf{X}^{[1]} = \frac{1}{y+2x}\;\frac{\partial}{\partial x} + \frac{2}{(y+2x)^2}\; y^{'}\;\frac{\partial}{\partial y^{'}}. \tag{1.53}$$

and,

$$\mathbf{X}^{[1]}V|_{V=0} = \left(\frac{1}{y+2x}\;\frac{\partial}{\partial x} + \frac{2}{(y+2x)^2}\; y^{'}\;\frac{\partial}{\partial y^{'}}\right)\left(y^{'} + \frac{y}{x} - 2\right) = 0. \tag{1.54}$$

So, the given ODE (1.48) admits the symmetry.

Example 1.2.6. Consider

$$y^{''} = 0, \tag{1.55}$$

admitting a generator

$$\mathbf{X} = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y}, \tag{1.56}$$

from (1.56), we have that

$$\xi = xy \ ; \ \eta = y^2, \tag{1.57}$$

For prolonging generator in (1.56) up to the order of DE given in (1.55). Using definition (1.42) and (1.43) we obtain

$$\eta^{(1)} = yy^{'} - 2yy^{'2}, \tag{1.58}$$

and

$$\eta^{(2)} = -3xy^{'}y^{''}, \tag{1.59}$$

As a result, the prolonged second-order generator is

$$\mathbf{X}^{[2]} = xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (yy^{'} - 2yy^{'2})\frac{\partial}{\partial y^{'}} - 3xy^{'}y^{''}\frac{\partial}{\partial y^{''}}, \tag{1.60}$$

and

$$\begin{aligned} \mathbf{X}^{[2]}V|_{V=0} &= \left(xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} + (yy^{'} - 2yy^{'2})\frac{\partial}{\partial y^{'}} - 3xy^{'}y^{''}\frac{\partial}{\partial y^{''}}\right)\left(y^{''}\right) \\ &= -3xy^{'}y^{''}(1) = 0 \text{ as } y^{''} = 0, \end{aligned} \tag{1.61}$$

so, the ODE in (1.55) admits the symmetry generator in eq (1.56).

1.2.7 General Procedure of Finding Symmetries

To identify Lie point symmetries [10] of the DE (1.8), we aim to determine ξ and η. Without the loss of generality of the DE (1.8), and considering that many DEs naturally arise as linear equations with the highest derivative, we begin with the $y^{(m)} = \omega(x, y, y', y'', \ldots, y^{(m-1)})$ for DE. The symmetry condition [6] can be written as follows

$$\mathbf{X}^{[m]}\omega = \left(\mathbf{X} + \eta'\frac{\partial}{\partial y'} + \eta''\frac{\partial}{\partial y''} + \ldots + \eta^{(m-1)}\frac{\partial}{\partial y^{(m-1)}}\right)\omega = \eta^{(m)}, \tag{1.62}$$

Example 1.2.7. Suppose for a second order DE [9]

$$y^{''} = x^{-5}y^2, \tag{1.63}$$

Using the conditions given in (1.62) for finding the symmetries of (1.63)

$$\Longrightarrow \eta^{(2)} = \mathbf{X}^{[2]} w. \tag{1.64}$$

Now, the right-hand side (RHS) of (1.64) becomes

$$\eta^{(2)} = \left(\xi \frac{\partial}{\partial x} + \eta \frac{\partial}{\partial y} + \eta^{(1)} \frac{\partial}{\partial y'} + \eta^{(2)} \frac{\partial}{\partial y''} \right) w. \tag{1.65}$$

Using eq (1.43) in (1.65) we get

$$\eta_{xx} + y^{'}(2\eta_{xy} - \xi_{xx}) + y^{'2}(\eta_{yy} - 2\xi_{xy}) - y^{'3}\xi_{yy} + y^{''}(\eta_y - 2\xi_x - 3y^{'}\xi_y) = \xi w_x + \eta w_y. \tag{1.66}$$

Putting $y^{''} = x^{-5}y^{-2}$ in (1.66), we get

$$\begin{aligned} &\eta_{xx} + y^{'}(2\eta_{xy} - \xi_{xx}) + y^{'2}(\eta_{yy} - 2\xi_{xy}) - y^{'3}\xi_{yy} + (x^{-5}y^{-2})(\eta_y - 2\xi_x - 3y^{'}\xi_y) \\ &= \xi(-5x^{-6}y^2) + \eta(2yx^{-5}). \end{aligned} \tag{1.67}$$

By comparing coefficients of various powers of $y^{'}$ and solving further, we obtain

$$\xi(x, y) = (C_1 x + C_2)x. \tag{1.68}$$

$$\eta(x, y) = (C_1 x + 3C_2)y. \tag{1.69}$$

with C_1 and C_2 as constants. The symmetry generator for eq (1.63) can be expressed as

$$\mathbf{X} = (C_1 x^2 + C_2 x)\frac{\partial}{\partial x} + (C_1 xy + 3C_2 y)\frac{\partial}{\partial y}, \tag{1.70}$$

for each $C_i = 1$ and $C_j = 0$ we get the following two symmetries

$$\mathbf{X}_1 = x^2 \frac{\partial}{\partial x} + xy \frac{\partial}{\partial y}, \quad \mathbf{X}_2 = x \frac{\partial}{\partial x} + 3y \frac{\partial}{\partial y}. \tag{1.71}$$

Example 1.2.8. Consider a second order DE [11]

$$y^{''} = 3y^{'} - 2y, \tag{1.72}$$

Using condition (1.62), we get

$$\eta^{(2)} = -2\eta + 3\eta^{1}, \tag{1.73}$$

using (1.43) and (1.72), we get

$$\eta_{xx} + y^{'}(2\eta_{xy} - \xi_{xx}) + y^{'^2}(\eta_{yy} - 2\xi_{xy}) - y^{'^3}\xi_{yy} + (3y^{'} - 2y)\left(\eta_y - 2\xi_x - 3y^{'}\xi_y\right)$$
$$+ 2\eta - 3\eta_x - 3y^{'}\eta_y + 3y^{'}\xi_x + 3y^{'^2}\eta_y = 0. \tag{1.74}$$

By comparing coefficients of various powers of $y^{'}$ we obtain the following system of PDEs

$$\begin{aligned}
(y^0) &: \eta_{xx} - 2y\eta_y + 4y\xi_x + 2\eta - 3\eta_x = 0,\\
(y^{'}) &: 2\eta_{xy} - \xi_{xx} - 6\xi_x + 6y\eta_y = 0,\\
(y^{'^2}) &: \eta_{yy} - 2\xi_{xy} - 9\xi_y + 3\eta_y = 0,\\
(y^{'^3}) &: \xi_{yy} = 0.
\end{aligned} \tag{1.75}$$

Solving above system we get,

$$\xi(x, y) = C_1e^{-x}y + C_4e^{-x} + C_2e^{-2x}y + C_5e^{x} + C_3, \tag{1.76}$$

and

$$\eta(x, y) = 2C_1y^2e^{-x} + C_4e^{-x}y + C_2y^2e^{-2x} + C_1e^{2x} + 2C_5ye^{x} + C_8e^{x} + C_6y. \tag{1.77}$$

Using (1.76) and (1.77) as there are 8 arbitrary constants so the generator can be expressed as

$$\mathbf{X} = \left(C_1e^{-x}y + C_4e^{-x} + C_2e^{-2x}y + C_5e^{x} + C_3\right)\frac{\partial}{\partial x}$$
$$+ \left(2C_1y^2e^{-x} + C_4e^{-x}y + C_2y^2e^{-2x} + C_1e^{2x} + 2C_5ye^{x} + C_8e^{x} + C_6y\right)\frac{\partial}{\partial y}. \tag{1.78}$$

for each $C_i = 1, C_j = 0$ we get the following symmetries [12]

$$\begin{aligned}
\mathbf{X}_1 &= e^{-x}y\frac{\partial}{\partial x} + 2y^2e^{-x}\frac{\partial}{\partial y}, & \mathbf{X}_2 &= e^{-2x}y\frac{\partial}{\partial x} + y^2e^{-2x}\frac{\partial}{\partial y},\\
\mathbf{X}_3 &= \frac{\partial}{\partial x}, & \mathbf{X}_4 &= e^{-x}\frac{\partial}{\partial x} + ye^{-x}\frac{\partial}{\partial y},\\
\mathbf{X}_5 &= e^{x}\frac{\partial}{\partial x} + 2ye^{x}\frac{\partial}{\partial y}, & \mathbf{X}_6 &= y\frac{\partial}{\partial y},\\
\mathbf{X}_7 &= e^{2x}\frac{\partial}{\partial y}, & \mathbf{X}_8 &= e^{x}\frac{\partial}{\partial y}.
\end{aligned}$$

1.3 Lie Algebras

A Lie algebra L is a type of vector space that includes a bilinear product $[\,,]$: $L \times L \to L$ meeting the following criteria [7]

1. $[\mathbf{X}_m, \mathbf{X}_n] = C^p_{mn}\mathbf{X}_p, \quad \forall \quad \mathbf{X}_m, \mathbf{X}_n$ and $\mathbf{X}_p \in L$,

2. $[\mathbf{X}_m, \mathbf{X}_n] = -[\mathbf{X}_n, \mathbf{X}_m]$,

3. $[\mathbf{X}_m, [\mathbf{X}_n, \mathbf{X}_p]] + [\mathbf{X}_p, [\mathbf{X}_m, \mathbf{X}_n]] + [\mathbf{X}_n, [\mathbf{X}_p, \mathbf{X}_m]] = 0 \quad \forall \quad \mathbf{X}_m, \mathbf{X}_n, \mathbf{X}_p \in L.$

whereas, the coefficients C^p_{mn}, where m, n, p=1,2,$\cdots$,r are called structure constants. The Lie bracket $[\,,]$ or the commutator relation for any two symmetry generators $\mathbf{X}_m, \mathbf{X}_n \in L$ is defined as

$$[\mathbf{X}_m, \mathbf{X}_n] = \mathbf{X}_m\mathbf{X}_n - \mathbf{X}_n\mathbf{X}_m.$$

Example 1.3.1. Consider the generators in Example (1.2.3)

$$\mathbf{X}_1 = \frac{\partial}{\partial x}, \mathbf{X}_2 = y\frac{\partial}{\partial y}, \mathbf{X}_3 = \frac{\partial}{\partial y}.$$

Corresponding Lie Algebra is

$$[\mathbf{X}_1, \mathbf{X}_3] = 0, [\mathbf{X}_1, \mathbf{X}_2] = 0, [\ \mathbf{X}_2, \mathbf{X}_3] = -\frac{\partial}{\partial y} = -\mathbf{X}_3.$$

For detailed discussion, one may refer to [13–15].

1.4 Lagrangian-Based Systems

In the study of classical mechanics, physical systems are often described using a system of second-order DEs. These equations are crucial for describing the behavior of the physical systems. A commonly used notation in this context is $\dot{q}^k = \frac{dx^k}{dt}$, where t is time and x^k are generalized coordinates. A general system of second-order DEs now takes the form [6]

$$\ddot{x}^k = \omega^k(t, x^j, \dot{x}^j), \quad j, k = 1, \ldots, N, \tag{1.79}$$

which can be related to a linear PDE as follows

$$\boldsymbol{A}f = \left(\frac{\partial}{\partial t} + \dot{x}^k \frac{\partial}{\partial x^k} + \omega^k(t, x^j, \dot{x}^j)\frac{\partial}{\partial \dot{x}^i}\right) f = 0. \tag{1.80}$$

Solving this PDE provides solutions ϕ^k, which correspond to the first integrals of the system, similar to the approach taken with ODEs. By considering small perturbations or infinitesimal transformations of the time and generalized coordinates

$$t^* = t + \varepsilon\xi(t, x^j), \quad x^{*k} = x^k + \varepsilon\eta^k(t, x^j), \tag{1.81}$$

We can examine the effects of these transformations on the system given by eq (1.79) In the context of these transformations, the generator and its first prolongation are

$$\mathbf{X} = \xi(t, x^j)\frac{\partial}{\partial t} + \eta^k(t, x^j)\frac{\partial}{\partial x^k} \tag{1.82}$$

$$\mathbf{X}^{[1]} = \mathbf{X} + \dot{\eta}^k(t, x^j, \dot{x}^j)\frac{\partial}{\partial \dot{x}^k}, \tag{1.83}$$

where the term $\dot{\eta}^k$ is defined as

$$\dot{\eta}^k = \frac{d\eta^k}{dt} - \dot{x}^k\frac{d\xi}{dt}. \tag{1.84}$$

By continuing this process, further extensions, or prolongations, $\mathbf{X}^{[m]}$, can be derived. If the commutation relation

$$[\mathbf{X}, \boldsymbol{A}] = \lambda \boldsymbol{A} \tag{1.85}$$

is satisfied, the system's symmetries can be identified. Once these symmetries are known, each can be linked to a corresponding first integral, providing insights into the system's conserved quantities.

The **Lagrangian** function, L, is central to this approach and is given by the difference between the K.E (T) and P.E (V)

$$L(t, x^j, \dot{x}^j) = T - V. \tag{1.86}$$

To connect the system's symmetries with its first integrals, the system must be representable by an action, N, defined as

$$N = \int_{t_a}^{t_b} L(t, x^k, \dot{x}^k)\, dt. \tag{1.87}$$

From this action principle, we derive the **Lagrange equations**, which govern the dynamics of the system

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^k}\right) - \frac{\partial L}{\partial x^k} = 0. \tag{1.88}$$

These equations are fundamental in describing the motion of systems in classical mechanics, offering a unified approach to understanding the dynamics of various physical systems.

1.4.1 The Noether Symmetries

Noether symmetries are those infinitesimal symmetry generators that leave a Lagrangian invariant. An infinitesimal transformation generator denoted as $\mathbf{X}$, is considered a Noether symmetry generator if it adheres to the condition

$$\mathbf{X}^{[1]}L + L\mathbf{A}\xi = \mathbf{A}B(t, x^k), \tag{1.89}$$

where $B(t, x^k)$ represents a gauge function, and $\mathbf{A}$ is an operator defined by

$$\mathbf{A} = \frac{\partial}{\partial t} + \dot{x}^k \frac{\partial}{\partial x^k}. \tag{1.90}$$

Here, $\mathbf{X}^{[1]}$ refers to the first order prolonged generator, as specified in eq (1.83). In the case where $B(t, x^k) = 0$, Noether symmetries coincide with variational symmetries [16]. But if $B(t, x^k)$ is non-zero, then it will be termed Noether Gauge symmetry.

1.4.2 Relationship between Lie and Noether Symmetries

To better understand the connection between Noether symmetries and the first integrals associated with each Noether symmetry, a conserved quantity [17]

$$\phi = \xi \left[\dot{x}^\alpha L_{\dot{x}^\alpha} - L\right] - \eta^\alpha L_{\dot{x}^\alpha} + B\left(t, x^\beta\right), \tag{1.91}$$

satisfying requirement $\mathbf{X}\phi = 0$ may be found.

Noether symmetries always form a subalegbra of the Lie point symmetries [13].

For further details, see the discussions in [18] and [19].

Example 1.4.1. Consider the Lagrangian

$$L = \frac{1}{2}(\dot{x}^2 + \dot{y}^2), \tag{1.92}$$

This Lagrangian represents the K.E of the system, with the first two terms corresponding to the K.E (T) in the x and y directions. In this case, P.E (V) is zero.
From eq (1.88) we get,

$$\ddot{x} = 0, \quad \ddot{y} = 0. \tag{1.93}$$

Using eq (1.89), (1.90) and (1.92) and we get

$$\dot{x}\left(\eta^1_t + \dot{x}\eta^1_x - \frac{1}{2}\dot{x}\xi_t - \dot{x}^2\xi_x + \dot{y}\eta^1_y - \dot{x}\dot{y}\xi_y\right) + \dot{y}\left(\eta^2_t + \dot{x}\eta^2_x + \dot{y}\eta^2_y - \frac{1}{2}\dot{y}\xi_t - \dot{y}^2\xi_y - \dot{x}\dot{y}\xi_x\right)$$
$$+ 2\left[\dot{x}^2 + \dot{y}^2\left(\xi_t + \dot{x}\xi_x + \dot{y}^2\xi_y\right)\right] = B_t + \dot{x}B_x + \dot{y}B_y. \tag{1.94}$$

On comparing coefficients of various powers of $\dot{x}, \dot{y}$ and their multiples we have,

$$\begin{aligned}
(constant) &: B_t = 0,\\
(\dot{x}) &: \eta^1_t - B_x = 0,\\
(\dot{x}^2) &: 2\eta^1_x - \xi_t = 0,\\
(\dot{x}^3) &: \xi_x = 0,\\
(\dot{y}) &: \eta^2_t - B_y = 0,\\
(\dot{y}^2) &: 2\eta^2_y - \xi_t = 0,\\
(\dot{x}\dot{y}) &: \eta^1_y + \eta^2_x = 0,\\
(\dot{x}^2\dot{y}) &: \xi_y = 0.
\end{aligned} \tag{1.95}$$

From $\dot{x}^3$ and $\dot{x}^2\dot{y}$, we get

$$\xi = \xi(t). \tag{1.96}$$

$B_t = 0$ yields

$$B = B(x, y). \tag{1.97}$$

Solving above system of eq (1.95), (1.96) and (1.97)

$$\begin{aligned}
\xi &= k_1t^2 + k_2t + k_3,\\
\eta^1 &= t((k_1x + k_5y + k_6) + k_2x + k_8y + k_9),\\
\eta^2 &= t((k_3y - k_5x + k_7) + k_4y - k_8x + k_{10}),\\
\text{and } B &= \frac{1}{2}k_1x^2 + k_5y + k_6x.
\end{aligned}$$

As a result, generators look like this

$$\begin{aligned}\mathbf{X}^{[1]} &= (k_1t^2+k_2t+k_3)\frac{\partial}{\partial t} + (t((k_1x+k_5y+k_6)+k_2x+k_8y+k_9))\frac{\partial}{\partial x}\\ &+ (t((k_3y-k_5x+k_7)+k_4y-k_8x+k_{10}))\frac{\partial}{\partial y},\end{aligned} \tag{1.98}$$

Now for each $k_i = 1$ and $k_j = 0$, we have the following Noether symmetries

$$\begin{aligned}
\mathbf{X}_1 &= t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x}, & \mathbf{X}_2 &= t\frac{\partial}{\partial t} + x\frac{\partial}{\partial x},\\
\mathbf{X}_3 &= \frac{\partial}{\partial t} + ty\frac{\partial}{\partial y}, & \mathbf{X}_4 &= y\frac{\partial}{\partial y},\\
\mathbf{X}_5 &= ty\frac{\partial}{\partial x} - tx\frac{\partial}{\partial x}, & \mathbf{X}_6 &= t\frac{\partial}{\partial x},\\
\mathbf{X}_7 &= t\frac{\partial}{\partial y}, & \mathbf{X}_8 &= y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y},\\
\mathbf{X}_9 &= \frac{\partial}{\partial x}, & \mathbf{X}_{10} &= \frac{\partial}{\partial y}.
\end{aligned} \tag{1.99}$$

Now, to understand the relation between Noether and Lie symmetries, we will find Lie symmetries.

Using the condition given in (1.62) and eq (1.43) for finding the symmetries of (1.93), we get the following

$$\begin{aligned}
\xi &= C_1t^2 + C_3yt + \frac{1}{2}C_5x + C_{14}t + C_{13}x + C_{12}y + C_{15},\\
\eta &= \frac{1}{2}C_5x^2 + x(C_1t + C_3y + C_6) + C_2t + C_4y + C_7,\\
\psi &= C_3y^2 + C_1yt + \frac{1}{2}C_5xy + C_{10}y + C_8t + C_9x + C_{11},
\end{aligned}$$

Hence the generator looks like

$$\begin{aligned}
\mathbf{X} &= (C_1t^2 + C_3yt + \frac{1}{2}C_5x + C_{14}t + C_{13}x + C_{12}y + C_{15})\frac{\partial}{\partial t}\\
&+ (\frac{1}{2}C_5x^2 + x(C_1t + C_3y + C_6) + C_2t + C_4y + C_7)\frac{\partial}{\partial x}\\
&+ (C_3y^2 + C_1yt + \frac{1}{2}C_5xy + C_{10}y + C_8t + C_9x + C_{11})\frac{\partial}{\partial y},
\end{aligned} \tag{1.100}$$

Now for each $C_i = 1$ and $C_j = 0$, we have the following Lie symmetries

$$\begin{aligned}
\mathbf{X}_1 &= t^2\frac{\partial}{\partial t} + tx\frac{\partial}{\partial x} + ty\frac{\partial}{\partial y}, & \mathbf{X}_2 &= t\frac{\partial}{\partial x}, \\
\mathbf{X}_3 &= ty\frac{\partial}{\partial t} + xy\frac{\partial}{\partial x} + y^2\frac{\partial}{\partial y} & \mathbf{X}_4 &= y\frac{\partial}{\partial x}, \\
\mathbf{X}_5 &= x\frac{\partial}{\partial t} + x^2\frac{\partial}{\partial x} + xy\frac{\partial}{\partial y}, & \mathbf{X}_6 &= x\frac{\partial}{\partial x}, \\
\mathbf{X}_7 &= \frac{\partial}{\partial x}, & \mathbf{X}_8 &= t\frac{\partial}{\partial y}, \\
\mathbf{X}_9 &= x\frac{\partial}{\partial y}, & \mathbf{X}_{10} &= y\frac{\partial}{\partial y} \\
\mathbf{X}_{11} &= \frac{\partial}{\partial y}, & \mathbf{X}_{12} &= y\frac{\partial}{\partial t} \\
\mathbf{X}_{13} &= x\frac{\partial}{\partial t}, & \mathbf{X}_{14} &= t\frac{\partial}{\partial t} \\
\mathbf{X}_{15} &= \frac{\partial}{\partial t}.
\end{aligned} \tag{1.101}$$

Hence from eq (1.99) and eq (1.101) it is evident that Noether symmetries form a subalgebra of the Lie symmetries.

1.4.3 Mei Symmetries

In 2000, Mei formally established the concept of form invariance, alternatively referred to as Mei symmetry. This symmetry principle asserts that the dynamical functions, including the Lagrangian, within a mechanical system's equations of motion remain invariant under an infinitesimal transformation, thereby continuing to satisfy the original equations. Like Noether symmetry, Mei symmetry results in the emergence of conserved quantities, referred to as Mei conserved quantities, which significantly impact the study of mechanical systems.

Establishing and applying a specific methodological framework [20] to identify Mei symmetries is not just a choice but a necessity. This structured approach is of the utmost importance, as it is fundamental for a comprehensive understanding of the dynamics of mechanical systems and guides researchers in their quest for knowledge.

Consider an infinitesimal transformation group characterized by a single parameter given by eq (1.81). The associated infinitesimal generator is defined as

$$\mathbf{X} = \xi \frac{\partial}{\partial t} + \eta^k \frac{\partial}{\partial x^k}. \tag{1.102}$$

As a result of the transformation described in eq (1.81), the Lagrangian presented in eq (1.87) is modified to become

$$\begin{aligned} L^* &= L^* \left(t^*, x^{*k}, \dot{x}^{*k}\right) \\ &= L^* \left(t + \varepsilon\xi, x^k + \varepsilon\eta^k, \frac{\dot{x}^k + \varepsilon\dot{\eta}^k}{1 + \varepsilon\dot{\xi}}\right). \end{aligned} \tag{1.103}$$

The Taylor series expansion of eq (1.103) about $\varepsilon = 0$ gives

$$L^* = L\left(t, x^k, \dot{x}^k\right) + \varepsilon \mathbf{X}^{[1]} L + O\left(\varepsilon^2\right), \tag{1.104}$$

where

$$\mathbf{X}^{[1]} = \mathbf{X} + \left(\dot{\eta}^k - \dot{\xi}\dot{x}^k\right) \frac{\partial}{\partial \dot{x}^k}, \tag{1.105}$$

The Euler-Lagrange equation can be expressed as

$$E_k(L) = 0, \tag{1.106}$$

where E_k denotes the Euler operator

$$E_k = \frac{d}{dt}\frac{\partial}{\partial \dot{x}^k} - \frac{\partial}{\partial x^k}. \tag{1.107}$$

If eq (1.106) stays the same when the modified Lagrangian L^* from eq (1.104) is substituted instead of the original Lagrangian, expressed as

$$E_k\left(L^*\right) = 0, \tag{1.108}$$

This invariance defines the Mei symmetries associated with the Lagrangian. Consequently, we can outline the procedure for identifying Mei symmetries [21–24].

Procedure for Identifying Mei Symmetries

If the infinitesimals ξ and η satisfy

$$E_k\left[\mathbf{X}^{[1]} L\right] = 0, \quad k = 1, \ldots, m \tag{1.109}$$

then the resultant is recognized as the Mei symmetry of the Lagrangian. Before applying this method to identify Mei symmetries, it is essential to explore the relationship between Mei symmetries and Noether symmetries, as understanding this connection is vital for determining Mei-conserved quantities and Noether-conserved quantities.

1.4.4 Relationship of Noether and Mei Symmetries

To explain this relationship, we start by presenting an important theorem [25].

Theorem 1. If the Mei symmetry of the Lagrangian $L = L(t, x^k, \dot{x}^k)$ and the infinitesimals ξ and η^k of the gauge function $B(t, x^k, \dot{x}^k)$ satisfy the equation

$$\boldsymbol{X}^{[1]} L\dot{\xi} + \boldsymbol{X}^{[1]}\left(\boldsymbol{X}^{[1]} L\right) + z(t)\frac{\partial\left(\boldsymbol{X}^{[1]} L\right)}{\partial x^k}\dot{x}^k\dot{\xi} + \dot{B} = 0 \tag{1.110}$$

then the Mei symmetry can lead to a new conserved quantity

$$\phi_1 = \frac{\partial\left(\boldsymbol{X}^{[1]} L\right)}{\partial \dot{x}}\eta^i + \left(\boldsymbol{X}^{[1]} L - \frac{\partial\left(\boldsymbol{X}^{[1]} L\right)}{\partial \dot{x}}\dot{x} - z(t)\frac{\partial\left(\boldsymbol{X}^{[1]} L\right)}{\partial t}\right)\xi + B \tag{1.111}$$

This theorem helps in the construction of a relationship between the Noether and Mei symmetries.

Consider the integral function

$$N(x) = \int_{t_1}^{t_2} \boldsymbol{X}^{[1]} L\left(L\left(t, x^k(t), \dot{x}^k(t)\right)\right) dt \tag{1.112}$$

with boundary conditions $x^k(t)\big|_{t=a} = x^k(a)$ and $x^k(t)\big|_{t=b} = x^k(b)$ where $k = 1, \ldots, n$. A similar form to equation (1.109) can be derived from the Lagrange equations of motion. Additionally, since Noether symmetry corresponds to the invariance of the action, then if

$$N^*\left(x^*\right) = N(x) \tag{1.113}$$

remains valid under infinitesimal transformations, and this invariance characterizes Noether symmetry. There exists a boundary function associated with $b\left(t, x^k, \dot{x}^k\right)$ for ξ and η, such that

$$\frac{\partial\left(\boldsymbol{X}^{[1]} L\right)}{\partial t}\xi + \frac{\partial\left(\boldsymbol{X}^{[1]} L\right)}{\partial x^k}\eta^k + \frac{\partial\left(\boldsymbol{X}^{[1]} L\right)}{\partial \dot{x}^k}\left(\dot{\eta}^k - \dot{x}^k\dot{\xi}\right) + \boldsymbol{X}^{[1]} L\dot{\xi} = -\dot{B} \tag{1.114}$$

We obtain the same equation as in (1.110), which is referred to as the Noether identity for the problem (1.112). From this, the Noether first integral, or conserved quantity, can be derived, coinciding with the result in (1.111). From equations (1.109) and (1.114), it is evident that Mei symmetry generally differs from Noether symmetry.

The methods of Lie point symmetries and Noether symmetries have advanced significantly over time, finding applications in solving various problems. However, despite these advancements, substantial research into Mei symmetries and their applications remains ongoing. The primary objective of this study is to identify Mei symmetries associated with a particular Lagrangian, as outlined in chapter 2. A more detailed discussion is available in reference [21].

Chapter 2

Mei Symmetries for the Lagrangian of Minkowski Spacetime

This chapter offers a brief overview of Minkowski spacetime, an essential foundation for the subsequent detailed analysis of its symmetries.

2.1 Minkowski Spacetime

Minkowski spacetime, first introduced by Hermann Minkowski in 1908, is a crucial concept in the theory of special relativity. This framework seamlessly merges space and time into a unified, four-dimensional structure. Unlike the complex, curved spacetimes encountered in general relativity, such as the Kerr metric, Minkowski spacetime is characterized by its flatness, making it mathematically more straightforward. This flat structure facilitates the application of Lorentz transformations, which are essential for accurately describing the behavior of particles and light at relativistic speeds. As a result, Minkowski spacetime plays a fundamental role in enlightening us about the dynamics of objects moving close to the speed of light [26].

2.2 Noether and Lie Symmetries of Lagrangian of Minkowski Spacetime: A Review

The Noether symmetries of the Lagrangian of Minkowski spacetime are explored in this section [27].

To start we express the Lagrangian for Minkowski spacetime as follows

$$L = \dot{t}^2 - \dot{r}^2 - r^2\dot{\theta}^2 - r^2 \sin^2\theta\dot{\phi}^2. \tag{2.1}$$

According to the [28], Lagrangian of Minkowski spacetime is associated with a 17-dimensional Lie algebra, spanned by its symmetry generators mentioned in the Table 3.1. Additionally, the geodesic equations corresponding to the Lagrangian presented in eq (2.1) are

$$\ddot{t} = 0, \tag{2.2}$$

$$\ddot{r} - r\left(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2\right) = 0, \tag{2.3}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0, \tag{2.4}$$

$$\ddot{\phi} + \frac{2}{r}\dot{r}\dot{\phi} + 2\cot\theta\dot{\theta}\dot{\phi} = 0. \tag{2.5}$$

According to [29], there are 35 Lie symmetries for Minkowski spacetime. These symmetries are outlined in chapter 3, where they are compared with Mei and Noether symmetries. The findings of this comparison are presented in detail in Table 3.1,3.2, and 3.3.

Subsequently, we find the Mei symmetries for the Lagrangian described in eq (2.1) to analyze their relationship with the corresponding Lie and Noether symmetries.

2.3 Mei Symmetries of Minkowski Spacetime

Consider the method for the Mei symmetries as defined in eq (1.109), i.e.

$$E_k\left[\mathbf{X}^{[1]}L\right] = 0, \quad k = 1, \ldots, m \tag{2.6}$$

Here L is the Lagrangian defined in (2.1) whereas $E_k = \frac{d}{ds}\frac{\partial}{\partial\dot{x}^k} - \frac{\partial}{\partial x^k}$ is the Euler operator and the first-order extended infinitesimal generator is

$$\mathbf{X}^{[1]} = \xi\frac{\partial}{\partial s} + \eta^k\frac{\partial}{\partial x^k} + \dot{\eta}^k\frac{\partial}{\partial\dot{x}^k}, \tag{2.7}$$

Where $\dot{\eta}^k$ is defined in (1.84), using independent variable as s, it can also be written as

$$\dot{\eta}^k = \left(\frac{\partial}{\partial s} + \dot{x}^b\frac{\partial}{\partial x^b}\right)\eta^k - \dot{x}^k\left(\frac{\partial}{\partial s} + \dot{x}^b\frac{\partial}{\partial x^b}\right)\xi. \tag{2.8}$$

In the Lagrangian (2.1), t, r, θ, ϕ are dependent variables, s is the independent variable. Substituting $b = 1, 2, 3, 4$ in (2.8) for t, r, θ, ϕ respectively and substituting it in (2.7), hence eq (2.7) now becomes

$$\mathbf{X}^{[1]} = \xi\frac{\partial}{\partial s} + \eta^1\frac{\partial}{\partial t} + \eta^2\frac{\partial}{\partial r} + \eta^3\frac{\partial}{\partial \theta} + \eta^4\frac{\partial}{\partial \phi} + \dot{\eta}^1\frac{\partial}{\partial \dot{t}} + \dot{\eta}^2\frac{\partial}{\partial \dot{r}} + \dot{\eta}^3\frac{\partial}{\partial \dot{\theta}} + \dot{\eta}^4\frac{\partial}{\partial \dot{\phi}} \tag{2.9}$$

As defined in eq (2.9), the generator is applied to the Lagrangian specified in eq (2.1), yielding a transformed expression

$$\mathbf{X}^{[1]}L = -2r\dot{\theta}^2\eta^2 - 2r\sin^2\theta\dot{\phi}^2\eta^2 + 2\dot{t}\dot{\eta}^1 - 2\dot{r}\dot{\eta}^2 - 2r^2\dot{\theta}\dot{\eta}^3 - 2r^2\sin^2\theta\dot{\phi}\dot{\eta}^4. \tag{2.10}$$

For $x^1 = t$, eq (2.6) yields

$$\left[\frac{d}{ds}\frac{\partial}{\partial \dot{t}} - \frac{\partial}{\partial t}\right]\left[\mathbf{X}^{[1]}L\right] = 0. \tag{2.11}$$

Using eq (2.10) in eq (2.11) and substituting equations (2.2) to (2.5). By comparing the powers of $(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$, a set of determining equations is derived.

$$(constant): \quad \eta^1_{ss} = 0, \tag{2.12}$$

$$(\dot{t}): \quad \xi_{ss} - \eta^1_{st} = 0, \tag{2.13}$$

$$(\dot{r}): \quad \eta^1_{sr} = 0, \tag{2.14}$$

$$(\dot{\theta}): \quad \eta^1_{s\theta} = 0, \tag{2.15}$$

$$(\dot{\phi}): \quad \eta^1_{s\phi} = 0, \tag{2.16}$$

$$(\dot{t}^2): \quad \eta^1_{tt} - 4\xi_{st} = 0, \tag{2.17}$$

$$(\dot{r}^2): \quad \eta^1_{rr} = 0, \tag{2.18}$$

$$(\dot{\theta}^2): \quad \eta^1_{\theta\theta} + r\eta^1_r = 0, \tag{2.19}$$

$$(\dot{\phi}^2): \quad \eta^1_{\phi\phi} + r\eta^1_r\sin^2\theta + \eta^1_\theta\cos\theta\sin\theta = 0, \tag{2.20}$$

$$(\dot{t}\dot{r}): \quad \eta^1_{tr} - 2\xi_{sr} = 0, \tag{2.21}$$

$$(\dot{t}\dot{\theta}): \quad \eta^1_{t\theta} - 2\xi_{s\theta} = 0, \tag{2.22}$$

$$(\dot{t}\dot{\phi}): \quad \eta^1_{t\phi} - 2\xi_{s\phi} = 0, \tag{2.23}$$

$$(\dot{r}\dot{\theta}): \quad \eta^1_\theta - r\eta^1_{r\theta} = 0, \tag{2.24}$$

$$(\dot{r}\dot{\phi}): \quad \eta^1_\phi - r\eta^1_{r\phi} = 0, \tag{2.25}$$

$$(\dot{\theta}\dot{\phi}): \quad \eta^1_{\theta\phi} - \eta^1_\phi \cot\theta = 0, \tag{2.26}$$

$$(\dot{t}^3): \quad \xi_{tt} = 0, \tag{2.27}$$

$$(\dot{t}^2\dot{r}): \quad \xi_{tr} = 0, \tag{2.28}$$

$$(\dot{t}^2\dot{\theta}): \quad \xi_{t\theta} = 0, \tag{2.29}$$

$$(\dot{t}^2\dot{\phi}): \quad \xi_{t\phi} = 0, \tag{2.30}$$

$$(\dot{t}\dot{r}^2): \quad \xi_{rr} = 0, \tag{2.31}$$

$$(\dot{t}\dot{\theta}^2): \quad r\xi_r + \xi_{\theta\theta} = 0, \tag{2.32}$$

$$(\dot{t}\dot{r}\dot{\theta}): \quad r\xi_{r\theta} - \xi_\theta = 0, \tag{2.33}$$

$$(\dot{t}\dot{r}\dot{\phi}): \quad r\xi_{r\phi} - \xi_\phi = 0, \tag{2.34}$$

$$(\dot{t}\dot{\theta}\dot{\phi}): \quad \xi_{\theta\phi} - \xi_\phi \cot\theta = 0, \tag{2.35}$$

$$(\dot{t}\dot{\phi}^2): \quad r\xi_r \sin^2\theta + \xi_{\phi\phi} + \xi_\theta \sin\theta\cos\theta = 0. \tag{2.36}$$

When $x^2 = r$ is substituted into eq (2.6), we get

$$\left[\frac{d}{ds}\frac{\partial}{\partial \dot{r}} - \frac{\partial}{\partial r}\right]\left[\mathbf{X}^{[1]}L\right] = 0. \tag{2.37}$$

Using eq. (2.10) into eq. (2.37) and subsequently simplify the resulting expression by putting equations (2.2) to (2.5). On comparing the coefficients and powers of $(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$, yields some identical equations as sub-equations (2.27) - (2.36). The remaining equations are listed below

$$(constant): \quad \eta^2_{ss} = 0, \tag{2.38}$$

$$(\dot{t}): \quad \eta^2_{st} = 0, \tag{2.39}$$

$$(\dot{r}): \quad \xi_{ss} - \eta^2_{sr} = 0, \tag{2.40}$$

$$(\dot{\theta}): \quad r\eta^3_s - \eta^2_{s\theta} = 0, \tag{2.41}$$

$$(\dot{\phi}): \quad \eta^2_{s\phi} - r\eta^4_s \sin^2\theta = 0, \tag{2.42}$$

$$(\dot{t}^2): \quad \eta^2_{tt} = 0, \tag{2.43}$$

$$(\dot{r}^2): \quad \eta^2_{rr} - 4\xi_{sr} = 0, \tag{2.44}$$

$$(\dot{\theta}^2): \quad \eta^2_{\theta\theta} - \eta^2 + r\eta^2_r - 2r\eta^3_\theta = 0, \tag{2.45}$$

$$(\dot{\phi}^2): \quad \eta^2_{\phi\phi} - \eta^2 \sin^2\theta + \eta^2_\theta \cos\theta \sin\theta - 2r\eta^3 \cos\theta \sin\theta$$

$$- 2r\eta^4_\phi \sin^2\theta + r\sin^2\theta \eta^2_{0,r} = 0, \tag{2.46}$$

$$(\dot{t}\dot{r}): \quad \eta^2_{tr} - 2\xi_{st} = 0, \tag{2.47}$$

$$(\dot{t}\dot{\theta}): \quad \eta^2_{t\theta} - r\eta^3_t = 0, \tag{2.48}$$

$$(\dot{t}\dot{\phi}): \quad \eta^2_{t\phi} - r\sin^2\theta \eta^4_t = 0, \tag{2.49}$$

$$(\dot{r}\dot{\theta}): \quad r\eta^2_{r\theta} - 2r\xi_{s\theta} - r^2\eta^3_r - \eta^2_\theta = 0, \tag{2.50}$$

$$(\dot{r}\dot{\phi}): \quad r\eta^2_{r\phi} - 2r\xi_{s\phi} - r^2\eta^4_r \sin^2\theta - \eta^2_\phi = 0, \tag{2.51}$$

$$(\dot{\theta}\dot{\phi}): \quad \eta^2_{\theta\phi} - \eta^2_\phi \cot\theta - r\eta^3_\phi - r\eta^4_\theta \sin^2\theta = 0. \tag{2.52}$$

When $x^3 = \theta$ is substituted into eq (2.6), we get

$$\left[\frac{d}{ds}\frac{\partial}{\partial\dot{\theta}} - \frac{\partial}{\partial\theta}\right]\left[\mathbf{X}^{[1]}L\right] = 0. \tag{2.53}$$

Putting eq. (2.10) into eq. (2.53) and simplifying the resulting expression using equations (2.2) through (2.5). Upon examining the coefficients and powers of $(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$, a significant redundancy is identified, where some equations are identical to the subequations (2.27) - (2.36). The remaining determining equations are listed below

$$(constant): \quad \eta^3_{ss} = 0, \tag{2.54}$$

$$(\dot{t}): \quad \eta^3_{st} = 0, \tag{2.55}$$

$$(\dot{r}): \quad \eta^3_s + r\eta^3_{sr} = 0, \tag{2.56}$$

$$(\dot{\theta}): \quad \eta^2_s + r\eta^3_{s\theta} - r\xi_{ss} = 0, \tag{2.57}$$

$$(\dot{\phi}): \quad \eta^4_s \sin\theta\cos\theta - \eta^3_{s\phi} = 0, \tag{2.58}$$

$$(\dot{t}^2): \quad \eta^3_{tt} = 0, \tag{2.59}$$

$$(\dot{r}^2): \quad 2\eta^3_r + r\eta^3_{rr} = 0, \tag{2.60}$$

$$(\dot{\theta}^2): \quad r^2\eta^3_r + 2\eta^2_\theta + r\eta^3_{\theta\theta} - 4r\xi_{s\theta} = 0, \tag{2.61}$$

$$(\dot{\phi}^2): \quad 2\cos\theta\sin\theta\eta^4_\phi - r\sin^2\theta\eta^3_r - \eta^3_{\phi\phi}$$
$$- \sin^2\theta\eta^3 - \sin\theta\cos\theta\eta^2_\theta + \cos^2\theta\eta^3 = 0, \tag{2.62}$$
$$(\dot{t}\dot{r}): \quad r\eta^3_{tr} + \eta^3_t = 0, \tag{2.63}$$
$$(\dot{t}\dot{\theta}): \quad \eta^2_t - 2r\xi_{st} + r\eta^3_{t\theta} = 0, \tag{2.64}$$
$$(\dot{t}\dot{\phi}): \quad \eta^4_t \sin\theta\cos\theta - \eta^3_{t\phi} = 0, \tag{2.65}$$
$$(\dot{r}\dot{\phi}): \quad \eta^4_r \cos\theta\sin\theta - \eta^3_{r\phi} = 0, \tag{2.66}$$
$$(\dot{r}\dot{\theta}): \quad \eta^2_r - 2r\xi_{sr} + r\eta^3_{r\theta} - \frac{1}{r}\eta^2 = 0, \tag{2.67}$$
$$(\dot{\theta}\dot{\phi}): \quad r\cos\theta\sin\theta\eta^4_\theta + \eta^2_\phi - 2r\xi_{s\phi}$$
$$+ r\eta^3_{\theta\phi} - r\cot\theta\eta^3_\phi = 0. \tag{2.68}$$

For last variable $x^4 = \phi$, eq (2.6) returns

$$\left[\frac{d}{ds}\frac{\partial}{\partial\dot{\phi}} - \frac{\partial}{\partial\phi}\right]\left[\mathbf{X}^{[1]}L\right] = 0. \tag{2.69}$$

It is simplified by using eq (2.10) in eq (2.69) and substituting equations (2.2) to (2.5), then setting to zero the coefficients of $(\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi})$ and their powers produce some similar equations (2.27) - (2.36). The remaining equations are

$$(constant): \quad \eta^4_{ss} = 0, \tag{2.70}$$
$$(\dot{t}): \quad \eta^4_{st} = 0, \tag{2.71}$$
$$(\dot{r}): \quad \eta^4_s + r\eta^4_{sr} = 0, \tag{2.72}$$
$$(\dot{\theta}): \quad \sin\theta\eta^4_{s\theta} + \cos\theta\eta^4_s = 0, \tag{2.73}$$
$$(\dot{\phi}): \quad \eta^2_s + r\eta^4_{s\phi} - r\xi_{ss} + r\cot\theta\eta^3_s = 0, \tag{2.74}$$
$$(\dot{t}^2): \quad \eta^4_{tt} = 0, \tag{2.75}$$
$$(\dot{r}^2): \quad r\eta^4_{rr} + 2\eta^4_r = 0, \tag{2.76}$$
$$(\dot{\theta}^2): \quad \eta^4_{\theta\theta} + 2\cot\theta\eta^4_\phi + r\eta^4_r = 0, \tag{2.77}$$
$$(\dot{\phi}^2): \quad \eta^4_{\phi\phi} - 4\xi_{s\phi} + \frac{2}{r}\eta^2_\phi + 2\eta^3_\phi\cot\theta$$
$$+ \eta^4_\theta\cos\theta\sin\theta + r\sin^2\theta\eta^4_r = 0, \tag{2.78}$$

$$(\dot{t}\dot{r}): \quad \eta^4_t + r\eta^4_{tr} = 0, \tag{2.79}$$

$$(\dot{t}\dot{\theta}): \quad \sin\theta\eta^4_{t\theta} + \cos\theta\eta^4_t = 0, \tag{2.80}$$

$$(\dot{t}\dot{\phi}): \quad 2r\xi_{st} + \eta^2_t + r\eta^4_{t\phi} + r\cot\theta\eta^3_t = 0, \tag{2.81}$$

$$(\dot{r}\dot{\theta}): \quad \sin\theta\eta^4_{r\theta} + \sin\theta\cos\theta\eta^4_r = 0, \tag{2.82}$$

$$(\dot{r}\dot{\phi}): \quad \eta^2_r + r\eta^4_{r\phi} - 2r\xi_{sr} - \eta^2 + r\eta^3_r\cot\theta = 0, \tag{2.83}$$

$$(\dot{\theta}\dot{\phi}): \quad \eta^4_{\theta\phi} - 2\xi_{s\theta} + \frac{1}{r}\eta^2_\theta - \eta^3 - \eta^3\cot^2\theta + \eta^3_\theta\cot\theta = 0. \tag{2.84}$$

We now solve the above system of PDEs to determine values of $\xi, \eta^1, \eta^2, \eta^3$, and η^4. From eq (2.27) we can write

$$\xi = a_1(s,r,\theta,\phi)\,t + a_2(s,r,\theta,\phi)\,. \tag{2.85}$$

Now using eq (2.28)

$$\xi_{tr} = 0, \tag{2.86}$$

Eq (2.85) implies

$$\xi_t = a_1(s,r,\theta,\phi)\,, \tag{2.87}$$

and

$$\xi_{tr} = a_{1,r} = 0.$$

This implies

$$a_1 = b_1(s,\theta,\phi)\,. \tag{2.88}$$

Differentiating (2.87) w.r.t. θ,

$$\xi_{t\theta} = a_{1,\theta} = 0. \tag{2.89}$$

Now (2.88) implies

$$a_1 = b_2(s,\phi)\,. \tag{2.90}$$

Using eq. (2.90) in eq. (2.85)

$$\xi = b_2(s,\phi)\,t + a_2(s,r,\theta,\phi)\,. \tag{2.91}$$

Using eq. (2.31) in (2.91), it implies

$$\xi_r = a_{2,r}, \tag{2.92}$$

$$\xi_{rr} = a_{2,rr} = 0,$$

which further implies

$$a_2 (s, r, \theta, \phi) = c_1 (s, \theta, \phi) r + c_2 (s, \theta, \phi) . \tag{2.93}$$

Putting eq. (2.93) in eq. (2.91) we get

$$\xi = b_2 (s, \phi) t + c_1 (s, \theta, \phi) r + c_2 (s, \theta, \phi) . \tag{2.94}$$

Differentiating (2.94) w.r.t. r and θ

$$\xi_r = c_1 (s, \theta, \phi) , \tag{2.95}$$

$$\xi_\theta = c_{1,\theta}\; r + c_{2,\theta},$$

$$\xi_{r\theta} = c_{1,\theta}. \tag{2.96}$$

Putting eq. (2.95) and eq. (2.96) in eq. (2.33) we get

$$rc_{1,\theta} - rc_{1,\theta} - c_{2,\theta} = 0,$$

$$c_{2,\theta} = 0,$$

which implies

$$c_2 (s, \theta, \phi) = d_1 (s, \phi) . \tag{2.97}$$

Using eq. (2.97) in eq. (2.94)

$$\xi = b_2 (s, \phi) t + c_1 (s, \theta, \phi) r + d_1 (s, \phi) . \tag{2.98}$$

Now to find ξ_r and $\xi_{\theta\theta}$ From (2.98) we have

$$\xi_{\theta\theta} = rc_{1,\theta\theta}. \tag{2.99}$$

Using eq. (2.32) and putting eq. (2.95) and eq. (2.99) we get

$$rc_1 (s, \theta, \phi) + rc_{1,\theta\theta} = 0, \tag{2.100}$$

which further implies

$$c_1\left(s,\theta,\phi\right)+c_{1,\theta\theta}=0, \tag{2.101}$$

and its solution is given by

$$c_1=e_1(s,\phi)\ \cos\theta+e_2(s,\phi)\ \sin\theta. \tag{2.102}$$

Putting eq (2.102) in eq (2.98),

$$\xi=b_2\left(s,\phi\right)t+re_1(s,\phi)\ \cos\theta+re_2(s,\phi)\ \sin\theta+d_1\left(s,\phi\right). \tag{2.103}$$

Differentiating eq. (2.103) w.r.t. ϕ

$$\xi_\phi=tb_{2,\phi}+r\cos\theta\ e_{1,\phi}+r\sin\theta\ e_{2,\phi}+d_{1,\phi}, \tag{2.104}$$

and

$$\xi_{r\phi}=\cos\theta e_{1,\phi}+\sin\theta e_{2,\phi}. \tag{2.105}$$

Putting eq. (2.104) and eq. (2.105) in eq (2.34) we get

$$tb_{2,\phi}+d_{1,\phi}=0. \tag{2.106}$$

Now in eq. (2.106) b_2 and d_1 are independent of t. So, on comparing coefficients of t

$$\begin{aligned} t: b_{2,\phi}=0 \implies b_2=f_1(s),\\ t^0: d_{1,\phi}=0 \implies d_1=f_2(s). \end{aligned} \tag{2.107}$$

Now, putting eq. (2.107) in eq. (2.103) we have

$$\xi=tf_1(s)+r\cos\theta e_{1,\phi}+r\sin\theta e_{2,\phi}+f_2(s). \tag{2.108}$$

Differentiating eq. (2.108) w.r.t ϕ and then θ,

$$\begin{aligned} \xi_\phi&=re_{1,\theta}\cos\theta+re_{2,\phi}\sin\theta,\\ \xi_{\theta\phi}&=-re_{1,\phi}\sin\theta+re_{2,\phi}\cos\theta. \end{aligned} \tag{2.109}$$

Using eq. (2.109) in eq. (2.35) we get,

$$-re_{1,\phi}(\sin\theta+\cot\theta\cos\theta)=0, \tag{2.110}$$

$$e_1(s,\phi) = g_1(s). \tag{2.111}$$

Eq. (2.108) becomes

$$\xi = tf_1(s) + r\cos\theta g_1(s) + r\sin\theta e_{2,\phi} + f_2(s). \tag{2.112}$$

Differentiating eq. (2.112) w.r.t r, θ and ϕ

$$\begin{aligned}
\xi_r &= \cos\theta g_1(s) + \sin\theta e_2(s,\phi), \\
\xi_\theta &= -r\sin\theta g_1(s) + r\cos\theta e_2(s,\phi, \\
\xi_\phi &= r\sin\theta e_{2,\phi}, \\
\xi_{\phi\phi} &= r\sin\theta e_{2,\phi\phi}.
\end{aligned} \tag{2.113}$$

Using eqs in (2.113) in eq. (2.36) we get,

$$\begin{aligned}
&[r\cos\theta g_1(s) + r\sin\theta e_2(s,\phi)]\sin^2\theta + r\sin\theta e_{2,\phi,\phi} \\
&+ (-r\sin\theta g_1(s) + r\cos\theta e_2(s,\phi))\cos\theta\sin\theta = 0.
\end{aligned} \tag{2.114}$$

Simplifying further we get

$$e_{2,\phi\phi} + e_2(s,\phi) = 0, \tag{2.115}$$

and its solution is given by

$$e_2 = h_1(s)\cos\phi + h_2(s)\sin\phi. \tag{2.116}$$

Using eq. (2.116) in eq. (2.112)

$$\xi = tf_1(s) + r\cos\theta g_1(s) + r\sin\theta\,(h_1(s)\cos\phi + h_2(s)\sin\phi) + f_2(s). \tag{2.117}$$

From eq (2.12)

$$\eta^1_{ss} = 0 \Rightarrow \eta^1_s = a_3(t,r,\theta,\phi) \Rightarrow \eta^1 = a_3(t,r,\theta,\phi)s + a_4(t,r,\theta,\phi). \tag{2.118}$$

From eq (2.14), (2.15), (2.16) we realize that a_3 must only be the function of 't '.

$$\Rightarrow \eta^1 = a_3(t)s + a_4(t,r,\theta,\phi). \tag{2.119}$$

From eq (2.18) we have,

$$\Rightarrow \eta^1_{rr} = a_{4,rr} = 0, \tag{2.120}$$

which implies

$$a_4 = b_3(t,\theta,\phi)r + b_4(t,\theta,\phi). \tag{2.121}$$

Now eq (2.119) becomes

$$\eta^1 = a_3(t)s + b_3(t,\theta,\phi)r + b_4(t,\theta,\phi). \tag{2.122}$$

Differentiating eq (2.122) w.r.t. θ and r we get

$$\eta^1_\theta = rb_{3,\theta} + b_{4,\theta}, \tag{2.123}$$

$$\eta^1_{r\theta} = b_{3,\theta}. \tag{2.124}$$

Put eq (2.123) and (2.124) in (2.24) we get

$$b_{4,\theta} = 0, \tag{2.125}$$

and which implies

$$b_4 = c_3(t,\phi). \tag{2.126}$$

Eq (2.122) simplifies to

$$\eta^1 = a_3(t)s + b_3(t,\theta,\phi)r + c_3(t,\phi). \tag{2.127}$$

Differentiating eq (2.127) w.r.t ϕ and r we get

$$\eta^1_\phi = rb_{3,\phi} + c_{3,\phi}, \tag{2.128}$$

and

$$\eta^1_{r\phi} = b_{3,\phi}. \tag{2.129}$$

Substitute eq (2.128) and (2.129) in (2.25)

$$c_{3,\phi} = 0, \tag{2.130}$$

$$c_3 = c_4(t). \tag{2.131}$$

Putting eq (2.131) in (2.127)

$$\eta^1 = a_3(t)s + b_3(t, \theta, \phi)r + c_4(t). \tag{2.132}$$

Differentiating eq (2.132) with θ & r

$$\eta^1_\theta = rb_{3,\theta}, \tag{2.133}$$

$$\eta^1_r = b_3(t, \theta, \phi), \tag{2.134}$$

$$\eta^1_{\theta\theta} = rb_{3,\theta\theta}. \tag{2.135}$$

By substituting eq (2.134) and (2.135) in (2.19) and simplifying further we get

$$b_3 = d_3(t, \phi) \cos\theta + d_4(t, \phi) \sin\theta. \tag{2.136}$$

Now, put eq (2.136) in (2.132)

$$\eta^1 = a_3(t)s + r\cos\theta d_3(t, \phi) + r\sin\theta d_4(t, \phi) + c_4(t). \tag{2.137}$$

Differentiating eq (2.137) w.r.t ϕ and then θ

$$\eta^1_\phi = r\cos\theta d_{3,\phi} + r\sin\theta d_{4,\phi}, \tag{2.138}$$

$$\eta^1_{\theta\phi} = -r\sin\theta d_{3,\phi} + r\cos\theta d_{4,\phi}. \tag{2.139}$$

Now, putting eq (2.138) and (2.139) in (2.26) and simplifying further, we get

$$d_{3,\phi} = 0, \tag{2.140}$$

$$d_3 = e_3(t). \tag{2.141}$$

Eq (2.137) simplifies to

$$\eta^1 = a_3(t)s + r\cos\theta e_3(t) + r\sin\theta d_4(t, \phi) + c_4(t). \tag{2.142}$$

Differentiating eq (2.142) w.r.t. r, θ, ϕ

$$\eta^1_r = \cos\theta e_3(t) + \sin\theta d_4(t, \phi), \tag{2.143}$$

$$\eta^1_\theta = -r\sin\theta e_3(t) + r\cos\theta d_4(t, \phi), \tag{2.144}$$

$$\eta^1_\phi = r\sin\theta d_{4,\phi}, \tag{2.145}$$

$$\eta^1_{\phi\phi} = r\sin\theta d_{4,\phi\phi}. \tag{2.146}$$

Now, putting (2.143) to (2.146) in (2.20) and simplifying we get

$$d_{4,\phi\phi} + d_4(t,\phi) = 0, \tag{2.147}$$

which implies

$$d_4 = e_4(t)\cos\phi + e_5(t)\sin\phi. \tag{2.148}$$

Puting (2.148) in (2.142)

$$\eta^1 = a_3(t)s + r\cos\theta e_3(t) + rsin\theta\cos\phi e_4(t) + r\sin\theta\sin\phi e_5(t) + c_4(t). \tag{2.149}$$

Now eq (2.38) implies

$$\eta^2_s = a_5(t,r,\theta,\phi), \tag{2.150}$$

and

$$\eta^2 = a_5(t,r,\theta,\phi)s + a_6(t,r,\theta,\phi). \tag{2.151}$$

Differentiating eq (2.151) w.r.t. s and t

$$\eta^2_s = a_5(t,r,\theta,\phi), \tag{2.152}$$

$$\eta^2_{st} = a_{5,t}. \tag{2.153}$$

which implies

$$a_5 = b_5(r,\theta,\phi). \tag{2.154}$$

Eq (2.151) simplifies to

$$\eta^2 = b_5(r,\theta,\phi)s + a_6(t,r,\theta,\phi). \tag{2.155}$$

Taking the double derivative of eq (2.155) w.r.t. t

$$\eta^2_{tt} = a_{6,tt}, \tag{2.156}$$

which further reduces to

$$a_6 = b_6(r,\theta,\phi)t + b_7(r,\theta,\phi). \tag{2.157}$$

Substituting (2.157) in (2.158) we get

$$\eta^2 = b_5(r,\theta,\phi)s + b_6(r,\theta,\phi)t + b_7(r,\theta,\phi). \tag{2.158}$$

Similarly, using eq (2.54) we get

$$\eta^3 = a_7(t, r, \theta, \phi)s + a_8(t, r, \theta, \phi). \tag{2.159}$$

After differentiating (2.159) w.r.t. s and t, then putting in eq (2.55) we have

$$\eta^3_{st} = a_{7,t} = 0, \tag{2.160}$$

and which, after further simplifications, gives us the following

$$a_7 = b_8(r, \theta, \phi). \tag{2.161}$$

After putting (2.161) in eq (2.159) we have

$$\eta^3 = b_8(r, \theta, \phi)s + a_8(t, r, \theta, \phi). \tag{2.162}$$

Taking the double derivative of (2.162) and putting in (2.59), we get

$$\eta^3_{tt} = a_{8,tt} = 0, \tag{2.163}$$

which simplifies to

$$a_8 = c_5(r, \theta, \phi)t + c_6(r, \theta, \phi). \tag{2.164}$$

Now eq (2.162) reduces to

$$\eta^3 = b_8(r, \theta, \phi)s + c_5(r, \theta, \phi)t + c_6(r, \theta, \phi). \tag{2.165}$$

Differentiating eq (2.165) with s and r and putting in (2.56) we get

$$b_8(r, \theta, \phi) + rb_{8,r} = 0, \tag{2.166}$$

which on simplification, gives us the following

$$b_8 = \frac{c_7(\theta, \phi)}{r}. \tag{2.167}$$

Now putting eq (2.167) in (2.165)

$$\eta^3 = \frac{1}{r}c_7(\theta, \phi)s + c_5(r, \theta, \phi)t + c_6(r, \theta, \phi). \tag{2.168}$$

Differentiating eq (2.168) w.r.t r and t and putting in eq (2.63) we have

$$rc_{5,r} + c_5(r,\theta,\phi) = 0. \tag{2.169}$$

On simplification, we get

$$c_5 = \frac{c_8(\theta,\phi)}{r}. \tag{2.170}$$

Putting c_5 in (2.168)

$$\eta^3 = \frac{1}{r}c_7(\theta,\phi)s + \frac{1}{r}c_8(\theta,\phi)t + c_6(r,\theta,\phi). \tag{2.171}$$

Differentiating (2.171) with r and substituting in (2.60)

$$rc_{6,rr} + 2c_{6,r} = 0, \tag{2.172}$$

which simplifies to

$$c_6(r,\theta,\phi) = c_9(\theta,\phi) + \frac{c_{10}(\theta,\phi)}{r}. \tag{2.173}$$

Putting (2.173) in (2.171)

$$\eta^3 = \frac{1}{r}c_7(\theta,\phi)s + \frac{1}{r}c_8(\theta,\phi)t + c_9(\theta,\phi) + \frac{1}{r}c_{10}(\theta,\phi). \tag{2.174}$$

Similarly, simplifying (2.70) we have

$$\eta^4_0 = a_9(t,r,\theta,\phi)s + a_{10}(t,r,\theta,\phi). \tag{2.175}$$

After performing similar operations, eq (2.71) reduces to

$$\eta^4_{st} = a_{9,t} = 0, \tag{2.176}$$

which gives us

$$a_9 = b_q(r,\theta,\phi). \tag{2.177}$$

Substitute (2.177) in (2.175)

$$\eta^4 = b_q(r,\theta,\phi)s + a_{10}(t,r,\theta,\phi). \tag{2.178}$$

Similarly (2.75) gives us

$$\eta^4_{tt} = a_{10,tt} = 0, \tag{2.179}$$

$$a_{10}(t,r,\theta,\phi) = c_{11}(r,\theta,\phi)t + c_{12}(r,\theta,\phi). \tag{2.180}$$

Now (2.172) implies

$$\eta^4 = b_9(r,\theta,\phi)s + c_{11}(r,\theta,\phi)t + c_{12}(r,\theta,\phi). \tag{2.181}$$

Differentiate (2.181) w.r.t. s and r and putting in (2.72) we have

$$b_9(r,\theta,\phi) + rb_{9,r} = 0, \tag{2.182}$$

$$b_9 = \frac{1}{r}c_{13}(\theta,\phi). \tag{2.183}$$

Now putting (2.183) in (2.181) we get

$$\eta^4 = \frac{1}{r}c_{13}(\theta,\phi)s + c_{11}(r,\theta,\phi)t + c_{12}(r,\theta,\phi). \tag{2.184}$$

Differentiating (2.184) w.r.t. t and r and putting in (2.79) we get

$$c_{11}(r,\theta,\phi) + rc_{11,r} = 0, \tag{2.185}$$

which gives us

$$c_{11} = \frac{1}{r}c_{14}(\theta,\phi). \tag{2.186}$$

Putting (2.186) in (2.184) we get

$$\eta^4 = \frac{1}{r}c_{13}(\theta,\phi)s + \frac{1}{r}c_{14}(\theta,\phi)t + c_{12}(r,\theta,\phi). \tag{2.187}$$

Differentiating (2.187) w.r.t. s and θ and putting in (2.73) we get

$$c_{13} = \frac{1}{\sin\theta}c_{15}(\phi). \tag{2.188}$$

Putting (2.188) in (2.187)

$$\eta^4 = \frac{c_{15}(\phi)s}{r\sin\theta} + \frac{1}{r}c_{15}(\theta,\phi)t + c_{12}(r,\theta,\phi). \tag{2.189}$$

Now differentiate (2.189) w.r.t t and θ and putting in (2.80)

$$c_{15} = \frac{1}{\sin\theta}c_{16}(\phi), \tag{2.190}$$

Now (2.189) simplifies to

$$\eta^4 = \frac{1}{r\sin\theta}c_{15}(\phi)s + \frac{1}{r\sin\theta}c_{16}(\phi)t + c_{12}(r,\theta,\phi). \tag{2.191}$$

Differentiating (2.191) w.r.t. r and putting in (2.76), after simplification we get

$$rc_{12,rr} + 2c_{12,r}(\theta, \phi) = 0, \tag{2.192}$$

Solving it further gives us

$$c_{12}(r, \theta, \phi) = c_{17}(\theta, \phi) + \frac{1}{r}c_{18}(\theta, \phi). \tag{2.193}$$

Eq. (2.191) simplifies to

$$\eta^4 = \frac{1}{r\sin\theta}c_{15}(\phi)s + \frac{1}{r\sin\theta}c_{16}(\phi)t + c_{17}(\theta, \phi) + \frac{c_{18}(\theta, \phi)}{r}. \tag{2.194}$$

Differentiating eq (2.149) w.r.t. t and eq (2.117) with s and t and putting in eq (2.17)

$$a_{3,tt}s + r\cos\theta e_{3,tt} + r\sin\theta\cos\phi e_{4,tt} + r\sin\theta\sin\phi e_{5,tt} + c_{4,tt} = 0. \tag{2.195}$$

Now again differentiating eq (2.149) w.r.t t and r, and eq (2.117) with s and r and putting in eq (2.21) we have

$$\cos\theta e_{3,t} + \sin\theta\cos\phi e_{4,t} + r\sin\theta\sin\phi e_{5,t} = 2(\cos\theta g_{1,s} + \sin\theta\cos\phi h_{1,s} + \sin\theta\sin\phi h_{2,s}). \tag{2.196}$$

Similarly eq (2.22) gives us

$$\sin\theta e_{3,t} - \cos\theta\cos\phi e_{4,t} - \sin\theta\sin\phi e_{5,t} = 2(\sin\theta g_{1,s} + \cos\theta\cos\phi h_{1,s} + \cos\theta\sin\phi h_{2,s}). \tag{2.197}$$

Also, eq (2.23) simplifies to

$$\cos\phi e_{5,t} - \sin\phi e_{4,t} = 2(\cos\phi h_{2,s} - \sin\phi h_{1,s}), \tag{2.198}$$

Using eq (2.41) we have

$$c_7(\theta, \phi) = b_{5,\theta}. \tag{2.199}$$

Therefore eq (2.174) can be written as

$$\eta^3 = \frac{1}{r}b_{5,\theta}s + \frac{1}{r}c_8(\theta, \phi)t + c_9(\theta, \phi) + \frac{1}{r}c_{10}(\theta, \phi). \tag{2.200}$$

Using eq (2.47) we have

$$b_{6,r} = 2f_{1,s}, \tag{2.201}$$

Differentiating it w.r.t. s we have

$$f_{1,ss} = 0, \tag{2.202}$$

which implies

$$f_1 = F_1 s + F_2. \tag{2.203}$$

Hence eq (2.117) simplifies to

$$\xi = stF_1 + tF_2 + r\cos\theta g_1(s) + r\sin\theta\,(h_1(s)\cos\phi + h_2(s)\sin\phi) + f_2(s). \tag{2.204}$$

Similarly eq (2.13) gives us

$$a_{3,t} = f_{1,ss}t + r\cos\theta g_{1,ss} + r\sin\theta\cos\phi h_{1,ss} + r\sin\theta\sin\phi h_{2,ss} + f_{2,ss}, \tag{2.205}$$

Differentiating it w.r.t. t we get

$$a_{3,tt} = 0, \tag{2.206}$$

which gives us

$$a_3 = A_1 t + A_2. \tag{2.207}$$

Using eq (2.206) in eq (2.195) we have

$$\cos\theta e_{3,tt} + \sin\theta\cos\phi e_{4,tt} + \sin\theta\sin\phi e_{5,tt} + \frac{c_{4,tt}}{r} = 0. \tag{2.208}$$

Differentiating eq (2.196) w.r.t t and using it in eq (2.208) we have

$$c_4 = C_1 t + C_2. \tag{2.209}$$

Using eq (2.207) and eq (2.209) and hence eq (2.142) simplifies to

$$\eta^1 = A_1 st + A_2 s + r\cos\theta e_3(t) + r sin\theta\cos\phi e_4(t) + r\sin\theta\sin\phi e_5(t) + C_1 t + C_2. \tag{2.210}$$

Also eq (2.42) reduces to

$$b_{5,\phi} = \sin\theta c_{15}(\phi), \tag{2.211}$$

Putting eq (2.210) in eq (2.194) we have

$$\eta^4 = \frac{1}{r\sin^2\theta} b_{5,\phi} s + \frac{1}{r\sin\theta} c_{16}(\phi)t + c_{17}(\theta,\phi) + \frac{c_{18}(\theta,\phi)}{r}. \tag{2.212}$$

Similarly (2.48) gives us

$$c_8(\theta,\phi) = b_{6,\theta}. \tag{2.213}$$

which further reduces eq (2.200) to

$$\eta^3 = \frac{s}{r}b_{5,\theta} + \frac{t}{r}b_{6,\theta} + c_9(\theta,\phi) + \frac{1}{r}c_{10}(\theta,\phi). \tag{2.214}$$

Now making use of eq (2.49) we have

$$c_{16}(\phi) = \frac{1}{\sin\theta}b_{6,\phi}. \tag{2.215}$$

Eq (2.212) simplifies to

$$\eta^4 = \frac{s}{r\sin^2\theta}b_{5,\phi} + \frac{t}{r\sin^2\theta}b_{6,\phi} + c_{17}(\theta,\phi) + \frac{c_{18}(\theta,\phi)}{r}. \tag{2.216}$$

Now differentiating eq (2.198) w.r.t s we have

$$h_{1,ss} = \cot\phi h_{2,ss}, \tag{2.217}$$

Using eq (2.217) and also differentiating eq (2.196) and eq (2.197) w.r.t s we have

$$g_{1,ss} = 0, \tag{2.218}$$

Also,

$$h_{2,ss} = 0, \tag{2.219}$$

which further implies

$$g_1 = G_1 s + G_2. \tag{2.220}$$

and

$$h_2 = H_1 s + H_2. \tag{2.221}$$

Now eq (2.204) simplifies to

$$\xi = stF_1 + tF_2 + rs\cos\theta G_1 + r\cos\theta G_2 + r\sin\theta$$
$$(s\cos\phi\cot\phi H_1 + \cos\phi H_3 + s\sin\phi H_1 + \sin\phi H_2) + f_2(s). \tag{2.222}$$

Differentiating eq (2.45) w.r.t s we have

$$rb_{5,r} - b_{5,\theta\theta} - b_5(r,\theta,\phi) = 0, \tag{2.223}$$

Making use of eq (2.57) and eq (2.74) we have

$$b_{5,\theta\theta} = \cot\theta b_{5,\theta}, \tag{2.224}$$

Similarly using eq (2.58) we have

$$b_{5,\theta\phi} = \cot\theta b_{5,\phi}. \tag{2.225}$$

Using eq (2.223) and eq (2.224) we have

$$b_5(r,\theta,\phi) = ri_1(\phi) + \cos\theta i_2(\phi). \tag{2.226}$$

Using eq (2.225) and eq (2.57) we have

$$b_5(r,\theta,\phi) = B_1\cos\theta. \tag{2.227}$$

Differentiating above eq w.r.t r and putting in eq (2.40) we have

$$f_2 = F_3 s + F_4. \tag{2.228}$$

Now eq (2.222) gives us

$$\begin{aligned}\xi &= stF_1 + tF_2 + rs\cos\theta G_1 + r\cos\theta G_2 + r\sin\theta \\ &(s\cos\phi\cot\phi H_1 + \cos\phi H_3 + s\sin\phi H_1 + \sin\phi H_2) + F_3 s + F_4.\end{aligned} \tag{2.229}$$

Now differentiating (2.196) and (2.197) w.r.t. θ and after subtracting them we get

$$e_4(t) = c_{19}. \tag{2.230}$$

and

$$e_5(t) = c_{20}. \tag{2.231}$$

Now using eq (2.230) and eq (2.231) in eq (2.208) we have

$$e_3(t) = c_{21}t + c_{22}. \tag{2.232}$$

Using eq (2.17) we get

$$F_1 = 0. \tag{2.233}$$

Now putting eq (2.230), eq (2.231) and eq (2.233) we can write eq (2.229) as

$$\begin{aligned}\xi = tF_2 + rs\cos\theta G_1 + r\cos\theta G_2 + r\sin\theta\,(s\cos\phi\cot\phi H_1 \\ + \cos\phi H_3 + s\sin\phi H_1 + \sin\phi H_2\ ) + F_3 s + F_4.\end{aligned} \tag{2.234}$$

Now using eq (2.21) and eq (2.22) we get

$$c_{21} = G_1 = H_1 = 0. \tag{2.235}$$

and therefore eq (2.232) becomes

$$e_3(t) = c_{22}. \tag{2.236}$$

Hence (2.234) becomes

$$\xi = tF_2 + F_3 s + F_4 + r\cos\theta G_2 + r\sin\theta\left(\cos\phi H_3 + \sin\phi H_2\right). \tag{2.237}$$

Similary eq (2.149) can be written as

$$\eta^1 = stA_1 + sA_2 + C_1 t + C_2 + r\cos\theta c_{22} + rsin\theta\left(\cos\phi c_{19} + \sin\phi c_{20}\right). \tag{2.238}$$

Similarly eq (2.13) gives us

$$A_1 = 0. \tag{2.239}$$

Hence eq (2.238) simplifies to

$$\eta^1 = sA_2 + C_1 t + C_2 + r\cos\theta c_{22} + rsin\theta\left(\cos\phi c_{19} + \sin\phi c_{20}\right). \tag{2.240}$$

and by putting eq (2.227) in eq (2.158) it becomes

$$\eta^2 = sB_1\cos\theta + b_6(r,\theta,\phi)t + b_7(r,\theta,\phi). \tag{2.241}$$

Similarly eq (2.214) and eq (2.216) becomes

$$\eta^3 = -\frac{s}{r}B_1\cos\theta + \frac{t}{r}b_{6,\theta} + c_9(\theta,\phi) + \frac{1}{r}c_{10}(\theta,\phi). \tag{2.242}$$

and

$$\eta^4 = \frac{t}{r\sin^2\theta} b_{6,\phi} + c_{17}(\theta,\phi) + \frac{c_{18}(\theta,\phi)}{r}. \tag{2.243}$$

Now using eq (2.65) we get

$$b_{6,\theta\phi} = \cot\theta b_{6,\phi}. \tag{2.244}$$

Making use of eq (2.212), eq (2.241) and eq(2.244) in eq (2.66) we have

$$c_{18}(\theta,\phi) = \csc\theta\sec\theta c_{10,\phi}. \tag{2.245}$$

Now eq (2.243) becomes

$$\eta^4 = \frac{t}{r}\csc^2\theta b_{6,\phi} + c_{17}(\theta,\phi) + \frac{\csc\theta\sec\theta c_{10,\phi}}{r}. \tag{2.246}$$

Using eq (2.47) we have

$$b_{6,r} = 0, \tag{2.247}$$

Using eq (2.81) we have

$$b_{6,\theta\theta} + \cot\theta b_{6,\theta} + \csc^2\theta b_{6,\phi\phi} = 0, \tag{2.248}$$

Now using (2.244), (2.247) and (2.248) we have using eq (2.47) we have

$$b_6(r,\theta,\phi) = 0. \tag{2.249}$$

Using (2.57) we have

$$B_1 = 0. \tag{2.250}$$

Using (2.249), (2.250) in (2.241), (2.242) and (2.246) we have the following

$$\eta^2 = b_7(r,\theta,\phi). \tag{2.251}$$

$$\eta^3 = c_9(\theta,\phi) + \frac{1}{r}c_{10}(\theta,\phi). \tag{2.252}$$

$$\eta^4 = c_{17}(\theta,\phi) + \frac{\csc\theta\sec\theta c_{10,\phi}}{r}. \tag{2.253}$$

Using (2.44) gives us

$$b_{7,rr} = 0, \tag{2.254}$$

Now making use of eq (2.50) gives us

$$rb_{7,r\theta} - b_{7,\theta} + c_{10}(\theta, \phi) = 0, \tag{2.255}$$

Simplify eq (2.51) we have

$$rb_{7,r\phi} + \tan\theta c_{10,\phi} - b_{7,\phi} = 0, \tag{2.256}$$

Now use eq (2.67)

$$rb_{7,r} - c_{10,\theta} - b_7(r, \theta\phi) = 0, \tag{2.257}$$

Similarly eq (2.83) simplifies to

$$b_{7,r} - \frac{1}{r}\csc\theta\sec\theta c_{10,\phi\phi} - b_7(r,\theta,\phi) - \frac{1}{r}\cot\theta c_{10}(\theta,\phi) = 0, \tag{2.258}$$

Now simultaneously solving eq (2.254) - eq (2.258) we get

$$b_7(r,\theta,\phi) = 0. \tag{2.259}$$

and

$$c_{10}(\theta,\phi) = 0. \tag{2.260}$$

Using both values we reduce eq (2.251) to eq (2.253) as

$$\eta^2 = 0. \tag{2.261}$$

$$\eta^3 = c_9(\theta,\phi). \tag{2.262}$$

$$\eta^4 = c_{17}(\theta,\phi). \tag{2.263}$$

Now using eq (2.45) we have

$$c_{9,\theta} = 0, \tag{2.264}$$

Eq (2.52) gives

$$c_{9,\phi} + \sin^2\theta c_{17,\theta} = 0, \tag{2.265}$$

Eq (2.46) simplifies to

$$\cos\theta c_9(\theta,\phi) + \sin\theta c_{17,\phi} = 0, \tag{2.266}$$

Eq (2.62) gives

$$2\cos\theta\sin\theta c_{17,\phi} - c_{9,\phi\phi} - \sin^2\theta c_9(\theta,\phi) + \cos^2\theta c_9(\theta,\phi) = 0. \tag{2.267}$$

Making use of eq (2.68) we have

$$r\cos\theta\sin\theta c_{17,\theta} + rc_{9,\theta\phi} - r\cot\theta c_{9,\phi} = 0. \tag{2.268}$$

Using eq (2.77) we get

$$c_{17,\theta\theta} + 2\cot\theta c_{17,\phi} = 0, \tag{2.269}$$

Also, making use of eq (2.78) we have

$$c_{17,\phi\phi} + 2\cot\theta c_{9,\phi} + \sin\theta\cos\theta c_{17,\theta} = 0, \tag{2.270}$$

and eq (2.84) gives

$$c_{17,\theta\phi} - c_9(\theta,\phi) - \cot^2\theta c_9(\theta,\phi) = 0, \tag{2.271}$$

Simultaneously solving eq (2.264) and eq (2.271) we have

$$c_9(\theta,\phi) = 0. \tag{2.272}$$

and

$$c_{17}(\theta,\phi) = c_{23}. \tag{2.273}$$

Both of the above equations reduce eq (2.262) and eq (2.263) to

$$\eta^3 = 0. \tag{2.274}$$

$$\eta^4 = c_{23}. \tag{2.275}$$

We have found all of the necessary infinitesimals and if we assume

$$(F_2, F_3, F_4, G_2, H_2, H_3, A_2, C_1, C_2, c_{19}, c_{20}, c_{22}, c_{23}) = (c_1, c_2, c_3, c_4, \ldots, c_{13}).$$

Hence we can write

$$\xi = c_8 s + r\left(c_{12}\sin\phi + c_{13}\cos\phi\right)\sin\theta + c_{11} r\cos\theta + c_9 t + c_{10}, \tag{2.276}$$

$$\eta^1 = c_1 s + r\left(c_5\sin\phi + c_6\cos\phi\right)\sin\theta + c_4 r\cos\theta + c_2 t + c_3, \tag{2.277}$$

$$\eta^2 = 0, \tag{2.278}$$

$$\eta^3 = 0, \tag{2.279}$$

$$\eta^4 = c_7. \tag{2.280}$$

Hence, the symmetry generator (2.9) can be written as

$$\begin{aligned}\mathbf{X}^{[1]} &= (c_8 s + r\,(c_{12}\sin\phi + c_{13}\cos\phi)\sin\theta + c_{11} r\cos\theta + c_9 t + c_{10})\frac{\partial}{\partial s}\\ &+(c_1 s + r\,(c_5\sin\phi + c_6\cos\phi)\sin\theta + c_4 r\cos\theta + c_2 t + c_3)\frac{\partial}{\partial t}\\ &+c_7\frac{\partial}{\partial\phi}.\end{aligned} \tag{2.281}$$

For all $c_k = 0$, where $k = 1, 2, 3 \ldots 13$, we get 13 Mei symmetries:

$$\begin{aligned}
&\mathbf{X}_1 = s\frac{\partial}{\partial t}, && \mathbf{X}_2 = t\frac{\partial}{\partial t}, && \mathbf{X}_3 = \frac{\partial}{\partial t},\\
&\mathbf{X}_4 = r\cos\theta\frac{\partial}{\partial t}, && \mathbf{X}_5 = r\sin\phi\sin\theta\frac{\partial}{\partial t}, && \mathbf{X}_6 = r\cos\phi\sin\theta\frac{\partial}{\partial t},\\
&\mathbf{X}_7 = \frac{\partial}{\partial\phi}, && \mathbf{X}_8 = s\frac{\partial}{\partial s}, && \mathbf{X}_9 = t\frac{\partial}{\partial s},\\
&\mathbf{X}_{10} = \frac{\partial}{\partial s}, && \mathbf{X}_{11} = r\cos\theta\frac{\partial}{\partial s}, && \mathbf{X}_{12} = r\sin\phi\sin\theta\frac{\partial}{\partial s},\\
&\mathbf{X}_{13} = r\cos\phi\sin\theta\frac{\partial}{\partial s}.
\end{aligned}$$

The 13 symmetries obtained are the desired Mei Symmetries. All 13 Mei symmetries are the subset of Lie point symmetries discussed in detail in chapter 3. Also, all of them are closed under the Lie bracket. The commutator relation among these vector fields is given on the next page.

$[\mathbf{X_m},\mathbf{X_n}]$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	$\mathbf{X}_5$	$\mathbf{X}_6$	$\mathbf{X}_7$	$\mathbf{X}_8$	$\mathbf{X}_9$	$\mathbf{X}_{10}$	$\mathbf{X}_{11}$	$\mathbf{X}_{12}$	$\mathbf{X}_{13}$
$\mathbf{X_1}$	0	$\mathbf{X}_1$	0	0	0	0	0	-$\mathbf{X}_1$	$\mathbf{X}_8 - \mathbf{X}_2$	-$\mathbf{X}_3$	-$\mathbf{X}_4$	-$\mathbf{X}_5$	-$\mathbf{X}_6$
$\mathbf{X_2}$	-$\mathbf{X}_1$	0	-$\mathbf{X}_3$	-$\mathbf{X}_4$	-$\mathbf{X}_5$	-$\mathbf{X}_6$	0	0	$\mathbf{X}_9$	0	0	0	0
$\mathbf{X_3}$	0	$\mathbf{X}_3$	0	0	0	0	0	0	$\mathbf{X}_{10}$	0	0	0	0
$\mathbf{X_4}$	0	$\mathbf{X}_4$	0	0	0	0	0	0	$\mathbf{X}_{11}$	0	0	0	0
$\mathbf{X_5}$	0	$\mathbf{X}_5$	0	0	0	0	-$\mathbf{X}_6$	0	$\mathbf{X}_{12}$	0	0	0	0
$\mathbf{X_6}$	0	$\mathbf{X}_6$	0	0	0	0	-$\mathbf{X}_5$	0	$\mathbf{X}_{13}$	0	0	0	0
$\mathbf{X_7}$	0	0	0	0	$\mathbf{X}_6$	$\mathbf{X}_5$	0	0	0	0	0	$\mathbf{X}_{13}$	-$\mathbf{X}_{12}$
$\mathbf{X_8}$	$\mathbf{X}_1$	0	0	0	0	0	0	0	-$\mathbf{X}_9$	-$\mathbf{X}_{10}$	-$\mathbf{X}_{11}$	-$\mathbf{X}_{12}$	-$\mathbf{X}_{13}$
$\mathbf{X_9}$	$\mathbf{X}_2 - \mathbf{X}_8$	-$\mathbf{X}_9$	-$\mathbf{X}_{10}$	-$\mathbf{X}_{11}$	-$\mathbf{X}_{12}$	-$\mathbf{X}_{13}$	0	$\mathbf{X}_9$	0	0	0	0	0
$\mathbf{X_{10}}$	$\mathbf{X}_3$	0	0	0	0	0	0	$\mathbf{X}_{10}$	0	0	0	0	0
$\mathbf{X_{11}}$	$\mathbf{X}_4$	0	0	0	0	0	0	$\mathbf{X}_{11}$	0	0	0	0	0
$\mathbf{X_{12}}$	$\mathbf{X}_5$	0	0	0	0	0	-$\mathbf{X}_{13}$	$\mathbf{X}_{12}$	0	0	0	0	0
$\mathbf{X_{13}}$	$\mathbf{X}_6$	0	0	0	0	0	$\mathbf{X}_{12}$	$\mathbf{X}_{13}$	0	0	0	0	0

Table 2.1: Commutator table for the Mei Symmetries

2.4 Verification of Mei Symmetries

We can verify whether the obtained symmetries satisfy the criteria for Mei symmetries as outlined in eq (2.6). Using the computed values of the infinitesimals, as provided in equations (2.276)- (2.280), we express eq (2.10) for $\mathbf{X}^{[1]}L$ as follows

$$\begin{aligned}\mathbf{X}^{[1]}L =& 2\dot{t}\Big[c_1 + \dot{t}c_2 + \dot{r}\left(\sin\phi\sin\theta c_5 + \cos\phi\sin\theta c_6 + \cos\theta c_4\right)\\ &+ \dot{\theta}r\left(\sin\phi\cos\theta c_5 + \cos\phi\cos\theta c_6 - \sin\theta c_4\right)\\ &+ \dot{\phi}r\left(\cos\phi\sin\theta c_5 - \sin\phi\sin\theta c_6\right) - \dot{t}\dot{r}\left(\sin\phi\sin\theta c_{12}\right)\\ &+ \left(\cos\phi\sin\theta c_{13} + \cos\theta c_{11}\right) + \dot{t}c_8 - \dot{t}^2 c_9\end{aligned}$$

$$
\begin{aligned}
&- \dot{t}\dot{\theta}r\left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right)\\
&- \dot{t}\dot{\phi}r\left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\Big]\\
&- 2\dot{r}\Big[\dot{r}^2\left(\sin\phi\sin\theta c_{12} + \cos\phi\sin\theta c_{13} + \cos\theta c_{11}\right)\\
&+ \dot{r}\dot{\theta}r\left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right)\\
&+ \dot{r}c_8 + \dot{t}\dot{r}c_9 + \dot{r}\dot{\phi}r\left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\Big]\\
&- 2r^2\dot{\theta}\Big[\dot{\theta}c_8 - \dot{t}\dot{\theta}c_9\\
&- \dot{r}\dot{\theta}\left(\sin\phi\sin\theta c_{12} + \cos\phi\sin\theta c_{13} + \cos\theta c_{11}\right)\\
&- \dot{\theta}^2 r\left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right)\\
&- \dot{\theta}\dot{\phi}r\left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\Big]\\
&- 2r^2\sin^2\theta\dot{\phi}\Big[\dot{\phi}c_8 + \dot{t}\dot{\phi}c_9\\
&+ \dot{r}\dot{\phi}\left(\sin\phi\sin\theta c_{12} + \cos\phi\sin\theta c_{13} + \cos\theta c_{11}\right)\\
&+ \dot{\theta}\dot{\phi}r\left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\\
&+ \dot{\phi}^2 r\left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\Big]
\end{aligned}
\tag{2.282}
$$

We apply the Euler operator to each dependent variable one at a time, following the guidelines outlined in eq (2.6). For $x^1 = t$ criteria given by eq. (2.6) gives

$$
\left(\frac{d}{ds}\frac{\partial}{\partial\dot{t}} - \frac{\partial}{\partial t}\right)\left(\mathbf{X}^{[1]}L\right) = 0. \tag{2.283}
$$

Using eq. (2.282), the left hand side (LHS) of eq. (2.283) gives

$$
\begin{aligned}
&- 4\dot{t}\dot{r}\dot{\theta}\left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right) + 2\dot{\phi}^2\left(-r\sin\phi\sin\theta c_5 - r\cos\phi\sin\theta c_6\right)\\
&+ 2\dot{\phi}^2\left(r\sin\phi\sin^3\theta c_5 + r\cos\phi\sin^3\theta c_6 + r\cos\theta\sin\theta c_4\right)\\
&+ 2\dot{\phi}^2\left(r\sin\phi\sin\theta\cos^2\theta c_5 + r\cos\phi\sin\theta\cos^2\theta c_6 - r\sin^2\theta\cos\theta c_4\right)\\
&- 4\dot{t}\dot{r}\dot{\theta}\left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right) - 2\dot{t}\dot{\phi}^2\left(-r\sin\phi\sin\theta c_{12} - r\cos\phi\sin\theta c_{13}\right)\\
&- 2\dot{t}\dot{\phi}^2\left(r\sin\phi\sin^3\theta c_{12} + r\cos\phi\sin^3\theta c_{13} + r\cos\theta\sin\theta c_{11}\right)\\
&- 2\dot{t}\dot{\phi}^2\left(r\sin\phi\sin\theta\cos^2\theta c_{12} + r\cos\phi\sin^2\theta\cos^2\theta c_{13} - r\sin^2\theta\cos\theta c_{11}\right) = 0.
\end{aligned}
\tag{2.284}
$$

This indicates the criterion is satisfied when $x^1 = t$. Now, for $x^2 = r$ criterion given by eq. (2.6) produce

$$\left(\frac{d}{ds}\frac{\partial}{\partial \dot{r}} - \frac{\partial}{\partial r}\right)\left(\mathbf{X}^{[1]}L\right) = 0. \tag{2.285}$$

Using eq. (2.282) into eq. (2.285) yields

$$\begin{aligned}
&\frac{d}{ds}\Big[6\dot{r}^2 \sin\phi \sin\theta c_{12} + 6\dot{r}^2 \cos\phi \sin\theta c_{13} + 6\dot{r}^2 \cos\theta c_{11} \\
&\quad + 4\dot{r}\dot{\theta}r \sin\phi \cos\theta c_{12} + 4\dot{r}\dot{\theta}r \cos\phi \sin\theta c_{13} - 4\dot{r}\dot{\theta}r \sin\theta c_{11} + 4\dot{r}c_8 \\
&\quad + 4\dot{t}\dot{r}c_9 + 4\dot{r}\dot{\phi}r \cos\phi \sin\theta c_{12} - 4\dot{r}\dot{\phi}r \sin\phi \sin\theta c_{13} + 2\dot{t} \sin\phi \sin\theta c_5 \\
&\quad + 2\dot{t} \cos\phi \sin\theta c_6 - 2\dot{t}^2 \sin\phi \sin\theta c_{12} - 2\dot{t}^2 \cos\phi \sin\theta c_{13} - 2\dot{t}^2 \cos\theta c_{11} \\
&\quad + 2\dot{\theta}^2 r^2 \sin\phi \sin\theta c_{12} + 2\dot{\theta}^2 r^2 \cos\phi \sin\theta c_{13} + 2\dot{\theta}^2 r^2 \cos\theta c_{11} \\
&\quad + 2\dot{\phi}^2 r^2 \sin\phi \sin^3\theta c_{12} + 2\dot{\phi}^2 r^2 \cos\phi \sin^3\theta c_{13} + 2\dot{\phi}^2 r^2 \sin^2\theta \cos\theta c_{11}\Big] \\
&\quad - \frac{\partial}{\partial r}\Big[2\dot{t}\dot{\phi}r\left(\sin\phi \cos\theta c_5 + \cos\phi \cos\theta c_6 - \sin\theta c_4\right) \\
&\quad + 2\dot{t}\dot{\phi}r\left(\cos\phi \sin\theta c_5 - \sin\phi \sin\theta c_6\right) \\
&\quad - 2\dot{t}^2\dot{\theta}r\left(\sin\phi \cos\theta c_{12} + \cos\phi \cos\theta c_{13} - \sin\theta c_{11}\right) \\
&\quad - 2\dot{t}^2\dot{\phi}r\left(\cos\phi \sin\theta c_{12} - \sin\phi \sin\theta c_{13}\right) \\
&\quad + 2r^3\dot{\theta}^3\left(\sin\phi \cos\theta c_{12} + \cos\phi \cos\theta c_{13} - \sin\theta c_{11}\right) \\
&\quad + 2r^3\dot{\theta}^2\dot{\phi}\left(\cos\phi \sin\theta c_{12} - \sin\phi \sin\theta c_{13}\right) \\
&\quad + 2\dot{\theta}\dot{\phi}^2 r^3 \sin^2\theta\left(\cos\phi \sin\theta c_{12} - \sin\phi \sin\theta c_{13}\right) \\
&\quad + 2\dot{\phi}^3 r^3 \sin^2\theta\left(\cos\phi \sin\theta c_{12} - \sin\phi \sin\theta c_{13}\right)\Big].
\end{aligned} \tag{2.286}$$

This implies

$$\begin{aligned}
&2\dot{\phi}r^2\left(\cos\phi \sin^3\theta c_{12} - \sin\phi \sin^3\theta c_{13}\right) + 4\dot{\phi}^3 r^2\left(\cos\phi \sin^3\theta c_{12}\right. \\
&\left. - \sin\phi \sin^3\theta c_{13}\right) - 6\dot{\phi}^3 r^2\left(\cos\phi \sin^3\theta c_{12} - \sin\phi \sin^3\theta c_{13}\right) \\
&+ 4\dot{r}^2\dot{\theta} \sin\phi \cos\theta c_{12} + 6\dot{r}^2\dot{\theta} \sin\phi \cos\theta c_{12} - 8\dot{r}^2\dot{\theta} \sin\phi \cos\theta c_{12} \\
&- 2\dot{r}^2\dot{\theta} \sin\phi \cos\theta c_{12} - 6\dot{\theta}^3 r^2\left(\sin\phi \cos\theta c_{12} + \cos\phi \cos\theta c_{13} - \sin\theta c_{11}\right) \\
&+ 2\dot{\theta}^3 r^2\left(\sin\phi \cos\theta c_{12} + \cos\phi \cos\theta c_{13} - \sin\theta c_{11}\right)
\end{aligned}$$

$$
\begin{aligned}
&+4\dot{\theta}^3 r^2 \left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right) \\
&+4\dot{r}\dot{\phi}^2 r \left(\sin\phi\sin^3\theta c_{12} + \cos\phi\sin^3\theta c_{13} + \sin^2\theta\cos\theta c_{11}\right) \\
&-4\dot{r}\dot{\phi}^2 r \left(\sin\phi\sin\theta c_{12} + \cos\phi\sin\theta c_{13}\right) \\
&+12\dot{r}\dot{\phi}^2 r \left(\sin\phi\sin^3\theta c_{12} + \cos\phi\sin^3\theta c_{13} + \sin^2\theta\cos\theta c_{11}\right) \\
&+4\dot{r}\dot{\phi}^2 r \left(\sin\phi\sin\theta\cos^2\theta c_{12} + \cos\phi\sin^2\theta\cos\theta c_{13} - \sin\theta\sin^2\theta\cos\theta c_{11}\right) \\
&-8\dot{r}\dot{\phi}^2 r \left(\sin\phi\sin^3\theta c_{12} + \cos\phi\sin^3\theta c_{13}\right) \\
&+12\dot{r}\dot{\phi}^2 r \left(\sin\phi\sin\theta c_{12} + \cos\phi\sin\theta c_{13} + \cos\theta c_{11}\right) \\
&-8\dot{r}\dot{\theta}^2 r \left(\sin\phi\sin\theta c_{12} + \cos\phi\sin\theta c_{13}\right) + \cos\theta c_{11} \\
&+4\dot{r}\dot{\theta}^2 r \left(\sin\phi\sin\theta c_{12} + \cos\phi\sin\theta c_{13} + \cos\theta c_{11}\right) \\
&-4\dot{r}\dot{\theta}^2 r \left(\sin\phi\sin\theta c_{12} + \cos\phi\cos\theta c_{13} + \cos\theta c_{11}\right) \\
&+2\dot{t}^2\dot{\theta} \left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right) \\
&-2\dot{t}^2\dot{\theta} \left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right) \\
&+2\dot{t}^2\dot{\phi} \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right) \\
&-2t^2\dot{\phi} \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right) \\
&+4\dot{r}\dot{\theta}\dot{\phi} \left(\cos\phi\cos\theta c_{12} - \sin\phi\cos\theta c_{13}\right) \\
&+4\dot{r}\dot{\theta}\dot{\phi} \left(\cos\phi\cos\theta c_{12} - \sin\phi\cos\theta c_{13}\right) \\
&-8\dot{r}\dot{\theta}\dot{\phi} \left(\cos\phi\cos\theta c_{12} - \sin\phi\cos\theta c_{13}\right) = 0. \qquad (2.287)
\end{aligned}
$$

It approaches zero, thus confirming that the criterion is satisfied for $x^2 = r$. Similarly, for $x^3 = \theta$, the criterion becomes

$$
\left(\frac{d}{ds}\frac{\partial}{\partial\dot{\theta}} - \frac{\partial}{\partial\theta}\right)\left(\mathbf{X}^{[1]}(L)\right) = 0. \qquad (2.288)
$$

Solving the LHS, we get

$$
\begin{aligned}
&8\dot{t}\dot{r}\dot{\theta} r c_9 - 8\dot{t}\dot{r}\dot{\theta} r c_9 + 8\dot{r}\dot{\theta} r c_8 - 8\dot{r}\dot{\theta} r c_8 \\
&+18\dot{r}\dot{\theta}^2 r^2 \left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right) \\
&+4\dot{r}\dot{\theta}^2 r^2 \left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right) \\
&-4\dot{r}\dot{\theta}^2 r^2 \left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right) \\
&-2\dot{r}\dot{\theta}^2 r^2 \left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right)
\end{aligned}
$$

$$
\begin{aligned}
&- 24\dot{r}\dot{\theta}^2 r^2 \left(\sin\phi\cos\theta c_{12} + \cos\phi\cos\theta c_{13} - \sin\theta c_{11}\right)\\
&+ 12\dot{r}\dot{\theta}\dot{\phi} r^2 \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\\
&+ 4\dot{r}\dot{\theta}\dot{\phi} r^2 \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\\
&- 8\dot{r}\dot{\theta}\dot{\phi} r^2 \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right) = 0.
\end{aligned}
\tag{2.289}
$$

Hence, it also holds for $x^3 = \theta$. Also, for $x^4 = \phi$

$$
\left(\frac{d}{ds}\frac{\partial}{\partial\dot{\phi}} - \frac{\partial}{\partial\phi}\right)\left(\mathbf{X}^{[1]}(L)\right) = 0 \tag{2.290}
$$

Now the LHS gives

$$
\begin{aligned}
&- 6\dot{\phi}^3 r^3 \sin^2\theta \left(\sin\phi\sin\theta c_{12} + \cos\phi\sin\theta c_{13}\right)\\
&+ 4\dot{\phi}^3 r^3 \sin^2\theta \left(\sin\phi\sin\theta c_{12} + \cos\phi\sin\theta c_{13}\right)\\
&+ 2\dot{\phi}^3 r^3 \sin^2\theta \left(\sin\phi\sin\theta c_{12} + \cos\phi\sin\theta c_{13}\right)\\
&+ 8\dot{r}\dot{\phi} r \sin^2\theta c_8 - 8\dot{r}\dot{\phi} r \sin^2\theta c_8\\
&+ 8\dot{t}\dot{r}\dot{\phi} r \sin^2\theta c_9 - 8\dot{t}\dot{r}\dot{\phi} r \sin^2\theta c_9\\
&+ 2\dot{r}^3 \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\\
&- 2\dot{r}^3 \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\\
&+ 2\dot{t}\dot{r} \left(\cos\phi\sin\theta c_5 - \sin\phi\sin\theta c_6\right)\\
&- 2\dot{t}\dot{r} \left(\cos\phi\sin\theta c_5 - \sin\phi\sin\theta c_6\right)\\
&- 2\dot{t}^2\dot{r} \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\\
&+ 2\dot{t}^2\dot{r} \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\\
&+ 2\dot{t}\dot{\theta} r \left(\cos\phi\cos\theta c_5 - \sin\phi\cos\theta c_6\right)\\
&- 2\dot{t}\dot{\theta} r \left(\cos\phi\cos\theta c_5 - \sin\phi\cos\theta c_6\right)\\
&+ 6\dot{r}\dot{\theta}^2 r^2 \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\\
&- 2\dot{r}\dot{\theta}^2 r^2 \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\\
&+ 4\dot{r}\dot{\theta}^2 r^2 \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)\\
&- 8\dot{r}\dot{\theta}^2 r^2 \left(\cos\phi\sin\theta c_{12} - \sin\phi\sin\theta c_{13}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+ 2\dot{\theta}^3 r^3 \left(\cos\phi\cos\theta c_{12} - \sin\phi\cos\theta c_{13}\right) \\
&- 2\dot{\theta}^3 r^3 \left(\cos\phi\cos\theta c_{12} - \sin\phi\cos\theta c_{13}\right) \\
&+ 8\dot{\theta}\dot{\phi} r^2 \sin\theta\cos\theta c_8 - 8\dot{\theta}\dot{\phi} r^2 \sin\theta\cos\theta c_8 \\
&+ 8\dot{t}\dot{\theta}\dot{\phi} r^2 \sin\theta\cos\theta c_9 - 8\dot{t}\dot{\theta}\dot{\phi} r^2 \sin\theta\cos\theta c_9 = 0.
\end{aligned} \tag{2.291}
$$

This indicates that the eq (2.6) criterion is also satisfied for $x^4 = \phi$. Therefore, it is established that the Lagrangian in eq (2.1) has 13 Mei symmetries.

Chapter 3

Summary

Analyzing symmetry and conserved quantities play a crucial role in mathematics and mechanics. A contemporary approach to determining conserved quantities in mechanical systems invloves Noether symmetries, which are characterized by invariances in the Lagrangian under small transformations. In the last decade, significant advancements have been achieved in studying Lie point symmetries [30] and Noether symmetries [31].

One of the significant developments in this area is the concept of Mei symmetry, also known as form invariance, was introduced by Mei [19]. Like Lie point and Noether symmetries, Mei symmetry states that the dynamical functions involved in a mechanical system, such as the Lagrangian, remain valid under infinitesimal transformations, continuing to satisfy the original equations of motion. This exploration of Mei symmetries enriches our theoretical understanding and has practical implications in Minkowski spacetime.

Recently, Mei symmetries for the Lagrangian corresponding to Schwarzschild and Kerr metric [32] have been studied. This thesis obtains the Mei symmetries for Lagrangian corresponding to Minkowski spacetime. Minkowski spacetime, the simplest solution to Einstein's field equations, represents a flat, four-dimensional spacetime without any gravitational effects. It is a special case of spacetime, characterized by its flat surface and used to describe the geometry of space and time in the absence of mass and energy.

Chapter one provides a comprehensive exploration of significant development in

DEs developments throughout their rich history including ODEs. It thoroughly examines symmetry groups related to point transformations and their infinitesimal generators. The method of Lie point symmetry is critically analyzed and applied to various well-known DEs. The Lie algebras and Lie brackets associated with basic symmetry generators are also assessed, ensuring a thorough understanding of the subject matter.

Building upon the definition of the Lagrangian, this work introduces Noether symmetries and Mei symmetries, outlining the conditions necessary for each. Furthermore, it explains the relation between Lie symmetry and Noether symmetry [18] and between Noether symmetry and Mei symmetry [25], using historical context to enrich the discussion and provide a deeper understanding of the subject matter. Chapter one also investigates the Mei symmetries and the method for finding the Mei Symmetries.

Chapter two begins by reviewing the Noether and Lie symmetries related to this Lagrangian, drawing upon insights from the research outlined in [27] and [29] respectively. Following that, the main task of finding the Mei symmetries related to the Lagrangian of Minkowski Spacetime is carried out. Using the Mei symmetries criteria, the infinitesimal generator is prolonged, and the system of determining equations for all dependent variables is obtained.

Having established the system, we proceed to solve it to find the values of the infinitesimals $\xi, \eta^1, \eta^2, \eta^3$, and η^4. Our findings reveal that two of these infinitesimals, i.e., η^2, η^3, equal zero, while the other three depend on thirteen arbitrary constants, ultimately identifying thirteen Mei symmetries.

Lie point symmetries, Noether symmetries, and Mei symmetries relevant to Minkowski spacetime are listed as

Table 3.1: Lie Point Symmetries

$L_1 = rs\cos\theta\frac{\partial}{\partial s} + rt\cos\theta\frac{\partial}{\partial t} + r^2\cos\theta\frac{\partial}{\partial r}$,
$L_2 = rs\sin\phi\sin\theta\frac{\partial}{\partial s} + rt\sin\theta\sin\phi\frac{\partial}{\partial t} + r^2\sin\phi\sin\theta\frac{\partial}{\partial r}$,
$L_3 = rs\sin\phi\sin\theta\frac{\partial}{\partial s} + rt\sin\theta\cos\phi\frac{\partial}{\partial t} + r^2\sin\phi\sin\theta\frac{\partial}{\partial r}$,
$L_4 = r\cos\theta\frac{\partial}{\partial s}$,
$L_5 = r\sin\phi\sin\theta\frac{\partial}{\partial s}$,
$L_6 = r\sin\theta\cos\phi\frac{\partial}{\partial s}$,
$L_7 = st\frac{\partial}{\partial s} + t^2\frac{\partial}{\partial t} + rt\frac{\partial}{\partial r}$,
$L_8 = t\frac{\partial}{\partial s}$,
$L_9 = \frac{1}{2}s^2\frac{\partial}{\partial s} + \frac{1}{2}st\frac{\partial}{\partial t} + \frac{1}{2}s\frac{\partial}{\partial r}$,
$L_{10} = s\frac{\partial}{\partial s}$,
$L_{11} = \frac{\partial}{\partial s}$,
$L_{12} = -t\sin\theta\cos\phi\frac{\partial}{\partial r} - \frac{t}{r}\cos\phi\frac{\partial}{\partial\theta} + \frac{t}{r\sin\theta}\sin\phi\frac{\partial}{\partial\phi}$,
$L_{13} = t\sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{t}{r}\sin\phi\cos\theta\frac{\partial}{\partial\theta} + \frac{t}{r\sin\theta}\cos\phi\frac{\partial}{\partial\phi}$,
$L_{14} = -s\sin\theta\cos\phi\frac{\partial}{\partial r} - \frac{s}{r}\cos\phi\frac{\partial}{\partial\theta} + \frac{s}{r\sin\theta}\sin\phi\frac{\partial}{\partial\phi}$,
$L_{15} = s\sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{s}{r}\sin\phi\cos\theta\frac{\partial}{\partial\theta} + \frac{s}{r\sin\theta}\cos\phi\frac{\partial}{\partial\phi}$,
$L_{16} = -\sin\theta\cos\phi\frac{\partial}{\partial r} - \frac{1}{r}\cos\phi\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\sin\phi\frac{\partial}{\partial\phi}$,
$L_{17} = \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{1}{r}\sin\phi\cos\theta\frac{\partial}{\partial\theta} + \frac{1}{r\sin\theta}\cos\phi\frac{\partial}{\partial\phi}$,
$L_{18} = -r\sin 2\theta\cos\phi\frac{\partial}{\partial r} + \cos 2\theta\cos\phi\frac{\partial}{\partial\theta} + \sin\phi\cot\theta\frac{\partial}{\partial\phi}$,
$L_{19} = r\sin 2\theta\cos\phi\frac{\partial}{\partial r} + \left(\sin\phi(2\cos^2\theta + 1)\right)\frac{\partial}{\partial\theta} + \cot\theta\cos\phi\frac{\partial}{\partial\phi}$,
$L_{20} = \frac{\partial}{\partial\phi}$,
$L_{21} = \left(\frac{r}{2}\cos 2\theta + 2\sin^2\theta\sin^2\phi\right)\frac{\partial}{\partial r} - \sin\theta\cos\theta\cos 2\phi\frac{\partial}{\partial\theta} + \sin 2\phi\frac{\partial}{\partial\phi}$,
$L_{22} = 2r\sin^2\theta\cos\phi\sin\phi\frac{\partial}{\partial r} + \sin\theta\cos\theta\sin 2\phi\frac{\partial}{\partial\theta} + \cos 2\phi\frac{\partial}{\partial\phi}$,
$L_{23} = s\frac{\partial}{\partial t}$,
$L_{24} = t\frac{\partial}{\partial t}$,
$L_{25} = \frac{\partial}{\partial t}$,
$L_{26} = r\cos\theta\frac{\partial}{\partial t}$,
$L_{27} = r\sin\phi\sin\theta\frac{\partial}{\partial t}$,
$L_{28} = r\sin\theta\cos\phi\frac{\partial}{\partial t}$,
$L_{29} = -t\cos\theta\frac{\partial}{\partial r} + \frac{t}{r}\sin\theta\frac{\partial}{\partial\theta}$,
$L_{30} = -t\cos\theta\frac{\partial}{\partial r} + \frac{t}{r}\sin\theta\frac{\partial}{\partial\theta}$,
$L_{31} = \cos\theta\frac{\partial}{\partial r} + \frac{1}{r}\sin\theta\frac{\partial}{\partial\theta}$,
$L_{32} = -\frac{r}{2}\cos 2\theta\frac{\partial}{\partial r} + \sin\theta\frac{\partial}{\partial\theta}$,
$L_{33} = -r\sin\theta\sin\phi\cos\theta\frac{\partial}{\partial r} + \sin^2\theta\sin\phi\frac{\partial}{\partial\theta}$,
$L_{34} = -r\sin\theta\cos\theta\cos\phi\frac{\partial}{\partial r} + \sin^2\theta\cos\phi\frac{\partial}{\partial\theta}$,
$L_{35} = \frac{1}{2}\frac{\partial}{\partial r}$.

Table 3.2: Noether Symmetries

$$N_1 = \frac{\partial}{\partial t},$$
$$N_2 = \cos\phi\frac{\partial}{\partial \theta} - \cot\theta\sin\phi\frac{\partial}{\partial \phi},$$
$$N_3 = \frac{\partial}{\partial \phi},$$
$$N_4 = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\csc\theta\cos\phi}{r}\frac{\partial}{\partial \phi},$$
$$N_5 = \sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial \theta} + \frac{\csc\theta\cos\phi}{r}\frac{\partial}{\partial \phi},$$
$$N_6 = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta},$$
$$N_7 = r\sin\theta\cos\phi\frac{\partial}{\partial t} + t\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\csc\theta\sin\phi}{r}\frac{\partial}{\partial \phi}\right),$$
$$N_8 = r\sin\phi\sin\theta\frac{\partial}{\partial t} + t\left(\sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\csc\theta\cos\phi}{r}\frac{\partial}{\partial \phi}\right),$$
$$N_9 = r\cos\theta\frac{\partial}{\partial t} + t\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right),$$
$$N_{10} = \frac{\partial}{\partial s},$$
$$N_{11} = s\frac{\partial}{\partial s} + \frac{1}{2}\left(t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}\right),$$
$$N_{12} = s\frac{\partial}{\partial t},$$
$$N_{13} = s\left(\sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\csc\theta\cos\phi}{r}\frac{\partial}{\partial \theta}\right),$$
$$N_{14} = s\left(\sin\phi\sin\theta\frac{\partial}{\partial r} + \frac{\sin\phi\cos\theta}{r}\frac{\partial}{\partial \theta} + \frac{\csc\theta\cos\phi}{r}\frac{\partial}{\partial \phi}\right),$$
$$N_{15} = s\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right),$$
$$N_{16} = \sin\phi\frac{\partial}{\partial \theta} + \cot\theta\cos\phi\frac{\partial}{\partial \phi},$$
$$N_{17} = \frac{1}{2}s\left(s\frac{\partial}{\partial s} + t\frac{\partial}{\partial t} + r\frac{\partial}{\partial r}\right).$$

Table 3.3: Mei Symmetries

$$M_1 = s\frac{\partial}{\partial t},$$
$$M_2 = t\frac{\partial}{\partial t},$$
$$M_3 = \frac{\partial}{\partial t},$$
$$M_4 = r\cos\theta\frac{\partial}{\partial t},$$
$$M_5 = r\sin\theta\sin\phi\frac{\partial}{\partial t},$$
$$M_6 = r\cos\phi\sin\theta\frac{\partial}{\partial t},$$
$$M_7 = \frac{\partial}{\partial \phi},$$
$$M_8 = s\frac{\partial}{\partial s},$$
$$M_9 = t\frac{\partial}{\partial s},$$
$$M_{10} = \frac{\partial}{\partial s},$$
$$M_{11} = r\cos\theta\frac{\partial}{\partial s},$$
$$M_{12} = r\sin\theta\sin\phi\frac{\partial}{\partial s},$$
$$M_{13} = r\cos\phi\sin\theta\frac{\partial}{\partial s}.$$

The findings indicate that, within Minkowski spacetime, Mei symmetries are a subset of the broader Lie symmetries. However, this is not true in general as a recent study [33] indicates that not every Mei symmetry is essentially a Lie symmetry. Hence Lie symmetries are the superset of symmetries. In the case of Minkowski spacetime, Mei and Noether symmetries are part of Lie symmetries. No direct relationship between Noether and Mei symmetries was found in the case of Minkowski spacetime. Finally, the Mei symmetries derived in this study are verified.

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