Jay B. Abrams

SECOND EDITION

# Quantitative Business Valuation

## A Mathematical Approach for Today's Professionals

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## *A Mathematical Approach for Today's Professionals*

**Second Edition**

**JAY B. ABRAMS**

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To my father, Leonard Abrams, who taught me how to write. To my mother, Marilyn Abrams, who taught me mathematics. To my wife, Cindy, who believes in me. To my children, Yonatan, Binyamin, Miriam, Nechamah Leah, and Rivkah, who gave up countless Sundays with Abba (Dad) for this book.

To my great teachers, Mr. Ohshima and Christopher Hunt, who brought me to my power to make this happen. And finally, to R. K. Hiatt and Scott Deifik, who caught my mistakes and made significant contributions to the thought that permeates this book.

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# Introduction

## Nature of the Book

This is an advanced book in the science and art of valuing privately held businesses. In order to read this book, you must already have read at least one introductory book such as *Valuing a Business* (Pratt, Reilly, and Schweihs, 1996 and subsequent). Without such a background, you will be lost.

I have written this book with the professional business appraiser as my primary intended audience, though I think this book is also appropriate for attorneys who are very experienced in valuation matters, investment bankers, venture capitalists, financial analysts, and MBA students.

Throughout this book, I generally write to you, the reader, as if you are sitting next to me and we are conversing. I am writing to you as my colleague with whom I share my thinking process. I prefer a conversational tone to a more formal one.

### Uniqueness of This Book

This is a rigorous book, and it is not easy reading. However, the following unique attributes of this book make reading it worth the effort:

1. It emphasizes regression analysis of empirical data. Chapter 8, "Adjusting for Levels of Control and Marketability,"[1] contains the first regression analysis of the data related to restricted stock discounts. Chapter 9 from the first edition was a sample fractional interest discount study containing a regression analysis of the Partnership Profiles database related to secondary limited partnership market trades. In both cases, we found very significant results. We now know much of what drives (a) restricted stock discounts and (b) discounts from net asset values of the publicly registered/privately traded limited partnerships. We moved the old Chapter 9 out of this book. It is our intention eventually to publish a workbook to accompany this book—probably when we produce the third edition. In the meantime, we intend to provide the old Chapter 9 on our website, www.abramsvaluation.com, under "Books," "Quantitative Business Valuation." You will also see much empirical work in Chapter 5, "Discount Rates as a Function of Log Size," and Chapter 9, "Empirical Testing of Abrams' Valuation Theory."

[1]Chapter 7 in the first edition of the book.

2. It emphasizes quantitative skills. Chapter 3 focuses on using regression analysis in business valuation. Chapter 4, the official title of which is "Annuity Discount Factors and the Gordon Model" (and the unofficial title of which is "The Chapter that Would Not Die!") is the most comprehensive treatment of ADFs in print. For anyone wishing to use the Mercer quantitative marketability discount model, Chapter 4 contains the ADF with constant growth not included in Mercer (1997).[2] ADFs crop up in many valuation contexts. I invented several new ADFs that appear in Chapter 4 that are useful in many valuation contexts. Chapter 10 contains the first treatise on how much statistical uncertainty we have in our valuations and how value is affected when the appraiser makes various errors.
3. It emphasizes putting all the pieces of the puzzle together to present a comprehensive, unified approach to valuation that can be empirically tested and whose principles work for the valuation of billion-dollar firms and ma-and-pa firms alike. While this book contains more mathematics—a worm's-eye view, if you will—than other valuation texts, we also refocus to the bird's-eye view in this section.

## Organization

There are seven parts to this book:

1. Forecasting Cash Flows (Chapters 1 through 4)
2. Calculating Discount Rates (Chapter 5 through 7)
3. Adjusting for Control and Marketability (i.e., valuation premiums and discounts) (Chapter 8)
4. Putting It All Together (Chapters 9 and 10)
5. Litigation (Chapters 11 and 12)
6. Valuing ESOPs and Buyouts of Partners and Shareholders (Chapters 13 through 15)
7. Probabilistic Valuation Methods (Chapters 16 through 18)

The first three parts of this book follow the chronological sequence of performing a discounted cash flow, although the regression analysis material in Chapter 3 applies to market methods as well.

The fourth part is empirically testing whether my methodology in the first three parts works (i.e., yields reasonable results). Additionally, we explore (1) confidence intervals around valuation estimates and (2) what happens to the valuation when appraisers make mistakes.

The reason for moving partnership and shareholder buyouts into Part VI, the ESOP section, is they share the common intellectual problem of post-transaction dilution. While the specific topic applications differ, the intellectual problem and process to solve it are similar.

The appraisal profession is still in the relatively early stages of using probabilistic valuation methods. However, it is a topic that is rapidly growing in importance.

[2]It is possible that he included this in a later edition, but I have not verified that.

Hence we have added Chapters 17 and 18, Monte Carlo Simulation (MCS) and Real Options (RO) Analysis, to the book. Because valuing start-ups, which was Chapter 12 in the first edition, makes use of probabilistic valuation methods, it logically fits together with Chapters 17 and 18, which is why I moved it to Chapter 16 in the second edition.

I invited Dr. Johnathan Mun, author of Wiley books *Modeling Risk* and *Real Options Analysis*, in addition to many other books, to write Chapters 17 and 18. They are introductions to these two topics and to Dr. Mun's software. We intend to cover practical examples of using MCS and RO in the workbook. Since that is likely to wait to accompany the third edition of this book, in the meantime look for it on our website somewhere between June 2010 to June 2011. I encourage readers who want to develop a deep understanding of each topic to buy Dr. Mun's books and software, and watch for the workbook and updates on our website. It is simply impossible to cover these complex topics in one chapter each.

## Differences in the Chapter Numbering

I added a new chapter as Chapter 2 in the second edition. That means that Chapters 2 through 7 in the first edition are now 3 through 8, respectively. I moved Chapters 8 and 9 from the first edition to our website—eventually to appear in the workbook. Thus, Chapters 10 and 11 from the first edition are now 9 and 10 in the second edition.

Part V, the "Litigation" section, which consists of Chapters 11 and 12 in the second edition, is entirely new. "Valuing Start-Ups" moved from Chapter 12 in the first edition to Chapter 16 in this edition, as it now fits in a new section of the book, "Probabilistic Valuation Methods."

Chapter 13 has kept the same number in the second edition. Chapter 14 is new in the second edition. Chapter 14 in the first edition is now Chapter 15 in the second edition. Finally, Chapters 17 and 18 are new in the second edition.

The following two tables should help you reference between chapter numbers in the two editions. The first one is in chapter order number of the first edition, whereas the second one is in chapter order number of the second edition.

| **First Edition** | **Second Edition** |
|---|---|
| 1 | 1 |
| 2 | 3 |
| 3 | 4 |
| 4 | 5 |
| 5 | 6 |
| 6 | 7 |
| 7 | 8 |
| 8 | — |
| 9 | — |
| 10 | 9 |
| 11 | 10 |
| 12 | 16 |
| 13 | 13 |
| 14 | 15 |

The missing chapters in the second edition sequence are new to the second edition: Chapters 2, 11, 12, 14, 17, and 18.

| First Edition | Second Edition |
|---|---|
| 1 | 1 |
| NA | 2 |
| 2 | 3 |
| 3 | 4 |
| 4 | 5 |
| 5 | 6 |
| 6 | 7 |
| 7 | 8 |
| 8 | — |
| 9 | — |
| 10 | 9 |
| 11 | 10 |
| NA | 11 |
| NA | 12 |
| 13 | 13 |
| NA | 14 |
| 14 | 15 |
| 12 | 16 |
| NA | 17 |
| NA | 18 |

## Similarities and Differences in the First and Second Editions

While the intellectual content of Chapter 1, "Cash Flow: A Mathematical Derivation," is largely the same, I nevertheless made a substantial rewrite for better clarity and logical flow. In general, all chapters that were in the first edition have undergone intensive editing, even if there is no or little new material. Chapter 2, "Forecasting Cash Flow: Mathematics of the Payout Ratio," is a new chapter that did not exist in the first edition. It should help the reader in converting forecast net income to forecast cash flow.

Chapter 3 (Chapter 2 in the first edition), "Using Regression Analysis," is largely the same as in the first edition, with the important addition of regressing scaled *y*-variables (Price-to-Sales and Price-Earnings ratios) as a way to control for heteroscedasticity.

Chapter 4 (3 in the first edition), "Annuity Discount Factors and the Gordon Model," is largely the same. However, there are two new sections added: (a) Mathematical Derivation of the PS Multiple;[3] (b) The Bias in Annual (versus Monthly) Discounting Is Immaterial.

Chapter 5 (4 in the first edition) has the following new material: (a) Keeping in the Roaring Twenties and the Great Depression; (b) Ibbotson's Opinion of Outliers and the Financial Crisis of 2008; (c) Is the Equity Premium Declining?; (d) Growth

[3]The first edition had a mathematical derivation of the Price-to-Earnings (PE) ratio. Now these two topics are combined in one section.

versus Value Stocks; and (e) The Wedge between Public and Private Firm Valuations [This section is extremely important, being a reconciliation between the Ibbotson total returns equation $r = d$ (dividend yield) + $g$ (growth, i.e., capital gains) and the Gordon model.]; (f) Satisfying Revenue Ruling 59-60 is substantially different.

Chapters 6 and 7 (5 and 6, respectively, in the first edition), "Arithmetic versus Geometric Means" and "An Iterative Valuation Approach," are largely the same.

Chapter 8 (7 in the first edition), "Adjusting for Levels of Control and Marketability," is the largest chapter in the book and requires some explanation. Unlike other chapters, time pressure with the publishing schedule necessitated finishing the chapter before I would have preferred. This chapter could use another 3 to 6 months' more research. Of course, by that time, it may well be large enough to become a book by itself. When I write the third edition of this book, it is likely either that Chapter 8 will become a book by itself, or that I will split it into two or more chapters.

Table 8.1A contains new data on the Mergerstat database. Chris Mercer extended the debate that we had in the first edition into a *Business Valuation Review* article, and I responded in kind. I have added my response to this chapter, which is covered in Tables 8.20, 8.21, 8.23, and 8.24. Table 8.22 shows summary statistics of Management Planning Inc.'s 2008 restricted stock study, and Tables 8.25 through 8.27 do the same with FMV Opinions' 2008 restricted stock study.

In general, I cite and summarize new academic and professional articles and include those into our analysis. The analysis is more complex, the data conflict more, and conclusions are murkier in the second edition.

Chapters 9 and 10 (10 and 11 in the first edition), "Empirical Testing of Abrams' Valuation Theory" and "Measuring Valuation Uncertainty and Error," are largely the same. Chapters 11 and 12, "Demonstrating Expert Bias" and "Lost Inventory and Lost Profits Damage Formulas in Litigation," are new to the second edition and comprise the litigation section.

The next three chapters comprise Part 6. Chapter 13, "ESOPs: Measuring and Apportioning Dilution," is largely the same, while Chapter 14, "The Tradeoff in Selling to an ESOP versus an Outside Buyer," is new. Chapter 15 (14 in the first edition), "Buyouts of Partners and Shareholders," while covering the same topic, is completely different in the second edition. I use a different model for the effects of post-transaction dilution.

Chapter 16 (12 in the first edition), "Valuing Start-Ups," has little change to the quantitative sections. However, there is some important new research on venture capital and angel investor rates of return. Chapters 17 and 18, "Monte Carlo Risk Simulation" and "Real Options," are new.

## How to Read This Book

Because this book is more difficult than most, I have done my best to try to provide more paths through it. Chapter 5 contains a shortcut version of the chapter at the end for those who want the bottom line without all the detail. In general, I have attempted to move most of the heaviest mathematics to appendices in order to leave the bodies of the chapters more readable. Where that was not optimal, I have given instructions on which material can be safely skipped.

How you read this book depends on your quantitative skills and how much time you have available. For the reader with strong quantitative skills and abundant time, the ideal path is to read the book in its exact order, as there is a logical sequence.

Because most professionals do not have abundant time, I want to suggest another path geared for the maximum benefit from the least investment in time. The heart of the book is "Discount Rates as a Function of Log Size" and "Adjusting for Levels of Control and Marketability," Chapters 5 and 8, respectively. I recommend the time-pressed reader follow this order:

1. Chapter 4—the following sections: from the beginning through the section titled "A Brief Summary"; "Periodic Perpetuity Factors: Perpetuities for Periodic Cash Flows"; and "Relationship of the Gordon Model Multiple to the Price/Earnings and Price/Sales Ratios."
2. Chapter 5 (the log size model for calculating discount rates)
3. Chapter 8 ("Adjusting for Levels of Control and Marketability")
4. Chapter 9 (this empirically tests Chapters 5 and 9, the heart of the book)

After these chapters, you can read the remainder of the book in any order, though it is best to read each part of the book in order and, better yet, to read the entire book in order.

This book has well over 100 tables, many of them being two or three pages long. To facilitate your reading, you can go to my company's Web site, www.abramsvaluation.com, click under "Publications" (on the left), then "Books," then "Quantitative Business Valuation," and then look for the file download for the *QBV* tables in PDF format. Then print out the tables and have them handy as you read the book. Otherwise, you will spend an inordinate amount of time flipping pages back and forth.

## My Thanks to You

I thank you for investing your valuable time and money to understand my work. I sincerely hope you will greatly benefit from it.

## References

Mercer, Z. Christopher. 1997. *Quantifying Marketability Discounts: Developing and Supporting Marketability Discounts in the Appraisal of Closely Held Business Interests*. Memphis, TN: Peabody.

Pratt, Shannon P., Robert F. Reilly, and Robert P. Schweihs. 1996. *Valuing a Business: The Analysis and Appraisal of Closely Held Companies*, 3rd ed. New York: McGraw-Hill.

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Robert Reilly edited the first edition manuscript cover-to-cover. I thank Robert very much for the huge time commitment for someone else's book. Larry Kasper gave me a surprise detailed edit of the first eight chapters. I benefited much from his input and thank him profusely.

Chris Mercer also read much of the book and gave me many corrections and very useful feedback. I thank Chris very much for his valuable time, of which he gave me much.

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I also thank Bob Jones of Jones, retired from Roach & Caringella for providing me with private fractional interest sales of real estate.

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The majority of these acknowledgments apply to the first edition.

every single person, and I apologize to anyone who should be in this acknowledgment section whom I forgot. Please forgive me.

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I thank Dr. Johnathan Mun for graciously providing Chapters 17 and 18 of this book. He is a great scholar in our profession.

Finally, I thank my wife, Cindy, for holding down the fort on endless Sundays while I was working on this book.

# Quantitative Business Valuation

# PART I

# Forecasting Cash Flow

## Introduction

Part I of this book focuses on forecasting cash flows, the initial step in the valuation process. In order to forecast cash flows, it is important to:

- Precisely define the components of cash flow.
- Develop statistical tools to aid in forecasting cash flows.
- Analyze different types of annuities, which are structured series of cash flows.

## Chapter 1: Cash Flow: A Mathematical Derivation

In Chapter 1, we mathematically derive the cash flow statement as the result of creating and manipulating a series of accounting equations and identities. This may provide the appraiser with a much greater depth of understanding of how cash flows derive from and relate to the balance sheet and income statement. It may help eliminate errors made by appraisers who perform discounted cash flow analysis using shortcut or even incorrect definitions of cash flow.

## Chapter 2: Forecasting Cash Flow: Mathematics of the Payout Ratio

This chapter has extremely important practical use as a shortcut method of converting forecast net income to forecast cash flows based on a mathematical formula in the chapter. The formula measures the ratio of future capital expenditures to historical depreciation and then adds in the effect of sales growth on net working capital. It can save the valuation practitioner much time compared with the long method and alternatively can be a sanity check on the long method.

## Chapter 3: Using Regression Analysis

In Chapter 3, we demonstrate in detail:

- How appraisers can use regression analysis to forecast sales and expenses, the latter being by far the more important use of regression.

- When and why the common practice of not using more than five years of historical data to prevent using stale data may be wrong.
- How to use regression analysis in the market approach valuation methods. While this is not related to forecasting sales and expenses, it fits in with our other discussions about using regression analysis.

When using publicly traded guideline companies of widely varying sizes, ordinary least squares (OLS) regression will usually fail, as statistical error is generally proportional to the market value (size) of the guideline company. However, there are simple transformations the appraiser can make to the data that will (1) enable him or her to minimize the negative impact of differences in size and (2) still preserve the very important benefit we derive from the variation in size of the publicly traded guideline companies, as we discuss in the chapter. The final result is valuations that are more reliable, realistic, and objective.

Most electronic spreadsheets provide a least squares regression that is adequate for most appraisal needs. I am familiar with the regression tools in both Microsoft Excel and Lotus 123. Excel does a better job of presentation and offers much more comprehensive statistical feedback. Lotus 123 has one significant advantage: It can provide multiple regression analysis for a virtually unlimited number of variables, while Excel is limited to 16 independent variables. However, Lotus has lost most of its market share and is no longer widely in use.

## Chapter 4: Annuity Discount Factors and the Gordon Model

In Chapter 4, we discuss annuity discount factors (ADFs). Historically, ADFs have not been used much in business valuation. Thus, they have had relatively little importance. Their importance is growing, however, for several reasons. They can be used in:

- Calculating the present value of annuities, including those with constant growth. This application has become far more important since the Mercer "quantitative marketability discount model" requires an ADF with growth.
- Valuing intellectual property, which typically has a finite life.
- Valuing periodic expenses such as moving expenses, losses from lawsuits, and so on.
- Calculating the present value of periodic capital expenditures with growth (e.g., What is the PV of keeping one airplane of a certain class in service perpetually?).
- Calculating loan payments.
- Calculating loan principal amortization.
- Calculating the present value of a loan. This is important in calculating the cash equivalency selling price of a business, as seller financing typically takes place at less-than-market rates.
- The present value of a loan is also important in ESOP valuations.

An important addition to Chapter 4 in the second edition is developing a mathematical formula for the price-to-sales (PS) ratio. Combined with the formula we already developed for the PE ratio in the first edition, these two formulas provide important theoretical guidance as to which independent variables to consider in a market approach method, thereby reducing the probability of obtaining spurious results through data mining.

Among my colleagues in the office, I unofficially titled Chapter 4, "The Chapter that Would Not Die!!!" I edited and rewrote this chapter close to 40 times striving for perfection, the elusive and unattainable goal. It was quite a task to decide what belongs in the body of the chapter and what should be relegated to the appendix. My goal was to maximize readability by keeping the most practical formulas in the chapter and moving the least useful and most mathematical work to the appendix.

# CHAPTER 1

# Cash Flow

## *A Mathematical Derivation*

# CHAPTER 1

# Cash Flow

## *A Mathematical Derivation*

## Introduction

In 1987, the Financial Accounting Standards Board (FASB) issued Statement of Financial Accounting Standards No. 95, "Statement of Cash Flows." This standard stipulates that a statement of cash flows is required as part of a full set of financial statements for almost all business enterprises.

As an accounting student in 1972–1974, I learned the logic of the statement of cash flows by rote. My professors taught us the logic of the individual adjustments from accrual net income, but they never presented the big picture, that is, how one can derive the statement of cash flows. This chapter provides the reader with the mathematics and conceptual logic to understand how we derive cash flows. It should enable the reader to be more adept at working with cash flows in business valuations.

This chapter is intended for readers who already have a basic knowledge of accounting. Much of what follows will involve alternating between accrual and cash reporting, which can be very challenging material.

### Operating, Investing, and Financing Activities

The primary purpose of a statement of cash flows is to provide relevant information about the cash receipts and cash payments of an enterprise. We must classify these receipts and payments according to three basic types of activities—operating, investing, and financing.

**OPERATING ACTIVITIES** Operating activities involve those transactions that enter into the determination of net income. Examples of these activities are sales and purchases of goods and services and compensation of employees.

Let's define our terminology, so we are clear in our meaning. Net income on a cash basis is cash flows from operations. When we refer to net income without

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The author wishes to thank Donald Shannon, School of Accountancy, DePaul University, for his help with this chapter in the first edition of this book. The mathematical model was published in Abrams (1994). In this second edition, Abrams substantially rewrote this chapter.

qualifying the basis, we are referring to accrual basis. With *net income* the company reports its operating activities when it earns or incurs them. With c*ash flows from operations*, the company reports these activities only when it collects cash for its receivables or pays its bills.

For example, net income increases when we make a sale even though we do not collect cash. Cash flows from operations reflect the increase only when we collect the cash. Net income decreases when we receive a bill for insurance even though payment is due only in one month. Cash flows from operations reflect the decrease only when we make the payment.

**INVESTING AND FINANCING ACTIVITIES** Companies engage in numerous transactions involving cash that have no impact on the income statement. We classify these transactions as investing or financing activities. Investing activities include the acquiring of fixed assets (a.k.a. property, plant, and equipment—PP&E), and this has no income statement impact. Retiring fully depreciated fixed assets or selling them for book value also has no income statement impact.[1] Financing activities include obtaining and repaying funds from debt and equity holders and paying dividends to the owners.

## Direct versus Indirect Method

Firms can use either the direct or the indirect method as a basis for reporting cash flows from operating activities. The direct method is preferable when the information to do so exists. However, for firms with accrual-based financial statements, that information often does not exist, and the company has no choice but to employ the indirect method.

Under the direct method, the enterprise lists its major categories of cash receipts from operations, for example, receipts from product sales or consulting services and cash disbursements for inventory, wages, interest, and taxes. The difference between these receipts and disbursements is net cash flows from operations. We then subtract or add, as appropriate, cash flows from the other two types of transactions—investing and financing activities. Thus, for investing activities we subtract the cash spent for capital expenditures or cash received for selling capital equipment, and for financing activities we add cash received from borrowing or selling stock or subtract cash paid to pay off the principal of the company's loans or to repurchase company stock.

The indirect method is more laborious, as we need to make adjustments to accrual-basis accounting to compute cash-basis amounts. This entails all the work described earlier in the direct method, and in addition we need to undo various accrual entries.

Here we briefly describe the reasons why the indirect method requires additional procedures to calculate cash flow, and we follow up later with the details of how to accomplish that.

[1]This introductory comment presumes the company sells its long-lived assets for their net book values. Of course, when there are gains or losses on disposition, they *do* appear in the income statement, as does depreciation of property, plant, and equipment. We address this issue later in the chapter.

1. *Operating activities*. We cannot simply use accrual-basis sales minus expenses to compute cash from operations because they may be larger or smaller than cash-basis sales and expenses. Instead, we have to adjust their differences by adding the increase or subtracting the decrease in net working capital.
2. *Investing activities*. Accrual accounting creates entries for depreciation expense and accumulated depreciation on capital equipment purchased and retirement thereof upon the sale of the equipment. It also recognizes gain or loss on the sale of equipment, neither of which are cash basis. In the indirect method, we have to reverse out the effect of these entries.
3. *Financing activities*. Accrual accounting generates entries to record accrual and payment of interest expense as well as principal and may differ from cash-basis accounting. In the indirect method we will have to reverse any differences with cash-basis accounting.

In summary, the indirect method requires additional procedures compared to the direct method of calculating cash flow. For operations, it requires calculations of changes in current assets and liabilities, which are in the upper-left and -right sides of the balance sheet; for investing activities, it requires additional adjustments involving fixed assets (the lower-left-hand side of the balance sheet); and financing activities involve transactions in long-term debt, interest expense, and equity, which are in the middle and lower-right-hand side of the balance sheet, and non-operating expenses in the income statement.

**ANALYZING BALANCE SHEET CHANGES OVER TIME** Under the indirect method we calculate net cash flows from operations by adjusting accrual net income for changes in related asset and liability accounts. For example, let's analyze the change ($\Delta$) in accounts receivable (*AR*) from December 31, 2008, to December 31, 2009. We will denote time as $t$ and set $t = 12/31/09$ for the current balance sheet date and $t - 1 = 12/31/08$ for last year's balance sheet date. Accounts receivable on December 31, 2009, equals *AR* on 12/31/08 plus accrual-based sales in 2009 minus cash collections in 2009. Algebraically, we state this as equation (1.1):

$$AR_t = AR_{t-1} + Sales - Collections. \tag{1.1}$$

In equation (1.1), the time periods, $t$ and $t - 1$, are points in time, that is, specific days, December 31, 2009 and 2008, respectively, in our example. The sales and collections are for a span of time, that is, for the entire year 2009 in our example. These concepts of time are consistent with the balance sheet versus the income statement, where the former is a snapshot of a company at a point in time and the latter is a flow over a period of time—one year in this example. Rearranging the equation, we get

$$Collections = Sales - (AR_t - AR_{t-1}), \tag{1.2}$$

or

$$Collections = Sales - \Delta AR. \tag{1.3}$$

When accounts receivable increase, $\Delta AR$ is positive and cash collections on sales are less than accrual-based sales. The reverse is true when accounts receivable decrease.

Thus, an increase in accounts receivable indicates that *cash receipts* from sales are less than reported *revenues*. Receivables increase as a result of failing to collect all revenues reported. Therefore, in the indirect method we must subtract the increase in accounts receivable from net income to arrive at net cash flows from operations. If accounts receivable decrease instead, then we add the decrease to net income to calculate cash flow.

Parenthetically, equations (1.1) through (1.3) are equally true if we redefine the passage of time. We could define $t - 1$ as November 30, 2009. In this example, the relevant sales and collections would be during the month of November. Alternatively, we could work in quarters, in which case $t - 1$ would be September 30, 2009, and sales and collections would be for the last quarter (i.e., October through December). Thus, the equations work with different spans of time, and we need only be careful in properly defining the points in time and the spans of time and in keeping them consistent. Now we return to the previous discussion.

Let's look at a liability account. Logically, since liabilities are on the opposite side of the fundamental accounting equation, they should behave the opposite of assets; that is, increases in a liability are a source of cash rather than a use of cash. We will see that this is true.

Wages payable on December 31, 2009 ($WP_t$) equals wages payable on December 31, 2008 ($WP_{t-1}$) plus accrual-based wages minus cash payments for wages for the current year.[2] We will model the algebra in equations (1.1a) through (1.3a) parallel to the algebra for accounts receivable:

$$WP_t = WP_{t-1} + Salary\ Exp - Salaries\ (Cash). \tag{1.1a}$$

We can rearrange equation (1.1a) as equation (1.2a):

$$Salaries\ (Cash) = Salary\ Exp - (WP_t - WP_{t-1}), \tag{1.2a}$$

$$Salaries\ (Cash) = Salary\ Exp - \Delta WP. \tag{1.3a}$$

If wages payable increase from 2008 to 2009, $\Delta WP$ is positive and cash payments for salaries are less than the accrual-based salary expense. When we begin with accrual-based net income in the indirect method, we must subtract the increase in wages payable (or add the decrease) from expenses, which increases cash-basis net income. This confirms our earlier statement that an increase in a liability is a source of cash.

Usually, it is easy to follow the logic of the adjustment required to infer the cash flows associated with *any single* reported revenue or expense. However, most statements of cash flows require a number of such adjustments, which often result in confusing entanglements.

Business appraisers spend a significant part of their careers forecasting cash flows. The objective of this chapter is to improve your understanding of the statement of cash flows and its interrelationship with the balance sheet and the income statement. Hopefully, appraisers who read this chapter will be able to better understand the cash flow logic and to distinguish true cash flows from shortcut approximations thereof.

---

[2]We use the terms *wages* and *salaries* as synonymous and interchangeable.

To achieve this result, this chapter provides a mathematical derivation of the cash flow statement using the indirect method. A realistic numerical example and an intuitive explanation accompany the mathematical derivation.[3]

## The Mathematical Model

This mathematical model of the statement of cash flows involves the following process:

1. It begins with the *fundamental accounting equation*—assets equal liabilities plus capital—which is the equation of a balance sheet.
2. We then create a *dynamic* fundamental accounting equation that shows that changes in assets equal the changes in liabilities plus the changes in capital. We call this dynamic because it refers to changes over a span of time (usually a year, but it could be a month or a quarter) as opposed to quantities at a fixed point in time.
3. We go through a series of accounting definitions and algebraic substitutions, and this enables us to demonstrate how the income statement and the balance sheet affect the statement of cash flows.

Throughout this book, be careful to distinguish between equations and tables, as they have the same numbering system to describe them. Our numbering system is the chapter number, then a period, and then either the table or equation number. We generally label them as *equation* or *table*, and equations have parentheses around them.

### List of Algebraic Symbols

Following is a list of the algebraic symbols that we use in this chapter:

**List of Symbols**
**Balance Sheet**

| | | |
|---|---|---|
| *C* | = | Cash |
| *OCA* | = | Other current assets |
| *GPPE* | = | Gross property, plant, and equipment |
| *AD* | = | Accumulated depreciation |
| *NPPE* | = | Net property, plant, and equipment |
| *A* | = | Total assets |
| *CL* | = | Current liabilities |
| *LTD* | = | Long-term debt |
| *L* | = | Total liabilities |
| *CAP* | = | Capital (i.e., total stockholders' equity) |

[3]It is possible to examine in detail every conceivable type of accounting transaction and its relation to cash flow. Here we have not considered unusual transactions such as recapitalizations, the effects of accounting changes, and inventory write-downs. The author feels the additional complication of their inclusion would more than offset any benefits.

**Property, Plant, and Equipment**

| | | |
|---|---|---|
| *CAPEXP* | = | Capital expenditures |
| *DEPR* | = | Depreciation expense |
| *RETGBV* | = | Gross book value of retired property, plant, and equipment |
| *RETAD* | = | Accumulated depreciation on retired assets |
| *SALESFA* | = | Selling price of fixed assets (property, plant, and equipment) disposed or retired |

**Stockholders' Equity**

| | | |
|---|---|---|
| *NI* | = | Net income |
| *DIV* | = | Dividends paid |
| *SALSTK* | = | Sale of stock |
| *TRSTK* | = | Purchase of treasury stock |
| *OET* | = | Other equity transactions |
| *AET* | = | Additional equity transactions |

**Required Working Capital**

| | | |
|---|---|---|
| *RWC* | = | Required working capital |
| $C_{Req}$ | = | Required cash |

## The Fundamental Accounting Equation

The balance sheets for Feathers R Us for 2008 and 2009 are in Table 1.1, columns C and D. We show the changes in the balance sheet accounts from 2008 to 2009 in column E and repeat the symbols used later to refer to these accounts in mathematical expressions in column A.

**THE STATIC EQUATION** We begin with the fundamental accounting equation, which is a mathematical statement that defines a balance sheet, that is, that total assets equal total liabilities plus capital (also known as *shareholders' equity*). The balance sheet for the current year ($t$ = 2009) is in balance. Total assets equal \$3,150,000 (D14), total liabilities equal \$1,085,000 (D19), and capital equals \$2,065,000 (D26). Total liabilities plus equity also equal \$3,150,000 (D28). Equation (1.4) is the algebraic expression of the fundamental accounting equation for the current year:

$$\begin{array}{lcccc} A_t & = & L_t & + & CAP_t \\ 3{,}150{,}000 & = & 1{,}085{,}000 & + & 2{,}065{,}000 \\ \text{D14} & & \text{D19} & & \text{D26} \end{array} \tag{1.4}$$

Note that there are three rows in the equation. The top row is the algebraic equation, the middle row is the numbers in Table 1.1, and the bottom row is the cell references in the table.

Likewise, equation (1.5) is the fundamental accounting equation (balance sheet) for the preceding year, $t-1$ = 2008:

$$\begin{array}{lcccc} A_{t-1} & = & L_{t-1} & + & CAP_{t-1} \\ 2{,}800{,}000 & = & 1{,}075{,}000 & + & 1{,}725{,}000 \\ \text{C14} & & \text{C19} & & \text{C26} \end{array} \tag{1.5}$$

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | **Table 1.1** | | | |
| 2 | | **Feathers R Us** | | | |
| 3 | | **Abbreviated Balance Sheets** | | | |
| 4 | | **for Calendar Years** | | | |
| 5 | | | | | |
| 6 | | | | | **Increase** |
| 7 | ***Symbols*** | **ASSETS:** | **2008** | **2009** | **(Decrease)** |
| 8 | ***C*** | Cash | 1,125,000 | 1,500,000 | 375,000 |
| 9 | ***OCA*** | Other current assets | 875,000 | 790,000 | (85,000) |
| 10 | | Total current assets | **2,000,000** | **2,290,000** | **290,000** |
| 11 | ***GPPE*** | Gross property, plant, and equipment | 830,000 | 900,000 | 70,000 |
| 12 | ***AD*** | Accumulated depreciation | 30,000 | 40,000 | 10,000 |
| 13 | ***NPPE*** | Net property, plant, and equipment | **800,000** | **860,000** | **60,000** |
| 14 | ***A*** | Total assets = (10)+(13) | **2,800,000** | **3,150,000** | **350,000** |
| 15 | | | | | |
| 16 | | **LIABILITIES** | | | |
| 17 | ***CL*** | Current liabilities | 325,000 | 360,000 | 35,000 |
| 18 | ***LTD*** | Long-term debt | 750,000 | 725,000 | (25,000) |
| 19 | ***L*** | Total liabilities | **1,075,000** | **1,085,000** | **10,000** |
| 20 | | | | | |
| 21 | | **STOCKHOLDERS' EQUITY** | | | |
| 22 | | Capital stock | 100,000 | 150,000 | 50,000 |
| 23 | | Additional paid in capital | 200,000 | 500,000 | 300,000 |
| 24 | | Retained earnings | 1,425,000 | 1,465,000 | 40,000 |
| 25 | | Treasury stock | 0 | 50,000 | 50,000 |
| 26 | ***CAP*** | Stockholders' equity = Sum((22):(24))–(25) | **1,725,000** | **2,065,000** | **340,000** |
| 27 | | | | | |
| 28 | | Total liabilities and equity = (19)+(26) | **2,800,000** | **3,150,000** | **350,000** |

**THE DYNAMIC EQUATION** In equation (1.6), we subtract the 2008 balance sheet from the 2009 balance sheet. This shows that *the changes from one year to the next are also in balance.*

$$\begin{array}{lllll} \Delta A & = & \Delta L & + & \Delta CAP \\ 350{,}000 & = & 10{,}000 & + & 340{,}000 \\ \text{E14} & & \text{E19} & & \text{E26} \end{array} \tag{1.6}$$

Equation (1.6) is a dynamic fundamental accounting equation, while the first two equations were static; that is, equation (1.6) represents the changes in the balance sheet that occurred during the year 2009, while equations (1.4) and (1.5) represent the balance sheet at two single points in time—December 31, 2009, and 2008, respectively. Whereas the static fundamental accounting equation defines the balance sheet, the dynamic equation incorporates the income statement and defines the cash flow statement. However, it will take several more equations to see why this is true.

**SOME DETAILS OF CHANGES IN ASSETS AND LIABILITIES** We can provide some details for each of the terms in equation (1.6), although we will need to fill in more details later on. The change in total assets ($\Delta A$) consists of the changes in cash ($\Delta C$), other current assets ($\Delta OCA$), and net property, plant, and equipment ($\Delta NPPE$). Net property, plant, and equipment (*NPPE*) is gross property, plant, and equipment (*GPPE*) less the accumulated depreciation (*AD*) on these assets. As we will see in Table 1.3, the change in net property, plant, and equipment ($\Delta NPPE$) is the result of subtracting the change in accumulated depreciation from the change in gross property, plant, and equipment ($\Delta GPPE - \Delta AD$).[4]

In equation (1.7) we fill in some of the details to equation (1.6):

$$\begin{array}{lllll} \Delta A & = \Delta C & + \Delta OCA & + (\Delta GPPE & - \Delta AD) \\ 350{,}000 & = 375{,}000 & + -85{,}000 & + (70{,}000 & - 10{,}000) \\ \text{E14} & \text{E8} & \text{E9} & \text{E11} & \text{E12} \end{array} \tag{1.7}$$

Next, the change in total liabilities ($\Delta L$) consists of the change in current liabilities ($\Delta CL$) and the change in long-term debt ($\Delta LTD$):

$$\begin{array}{lll} \Delta L & = \Delta CL & + \Delta LTD \\ 10{,}000 & = 35{,}000 & + -25{,}000 \\ \text{E19} & \text{E17} & \text{E18} \end{array} \tag{1.8}$$

## Bridge to the Income Statement

The change in capital in equation (1.6) is a bridge to the income statement, since net income is the operating component—and generally the most important one—in explaining the change in capital from one year to the next. To explain the change in stockholders' equity, we need to know the company's net income, which appears in the income statement in Table 1.2.

Table 1.2 shows that Feathers R Us had net income after tax (NI) of $90,000 (B17). This explains only a portion of the change in the stockholder's equity. The total change in stockholder's equity ($\Delta CAP$) is equal to net income (NI) *and* other equity transactions (OET), which we define in equation (1.9):

$$\begin{array}{lll} \Delta CAP & = NI & + OET \\ 340{,}000 & = 90{,}000 & + 250{,}000 \\ \text{Table 1.1, E26} & \text{Table 1.2, B17} & \text{Table 1.4, G16} \end{array} \tag{1.9}$$

The OET consist of the purchase and sale of the company's stock and the payment of cash dividends.[5] We will provide a detailed description of these transactions later in our description of Table 1.4.

---

[4]We treat other long-lived assets such as intangibles and certain investments the same as property, plant, and equipment.

[5]For simplicity, we assume the company pays all dividends declared.

| | A | B |
|---|---|---|
| 1 | **Table 1.2** | |
| 2 | **Feathers R Us** | |
| 3 | **Income Statement** | |
| 4 | **for Calendar Year 2009** | |
| 5 | | |
| 6 | Sales | 1,000,000 |
| 7 | Costs of sales | 600,000 |
| 8 | Gross profit | **400,000** |
| 9 | Sales expense | 100,000 |
| 10 | General and administrative expense | 150,000 |
| 11 | Depreciation | 30,000 |
| 12 | Total expense | **280,000** |
| 13 | Operating income | **120,000** |
| 14 | Gain on sale of assets | 30,000 |
| 15 | Net income before taxes | **150,000** |
| 16 | Income taxes [1] | 60,000 |
| 17 | Net income | **90,000** |
| 18 | | |
| 19 | [1] For instructional purposes, we use a 40% tax rate even though | |
| 20 | taxable income is below the maximum corporate tax rate. | |

Substituting equations (1.7), (1.8), and (1.9) into equation (1.6) results in equation (1.10):[6]

$$\begin{aligned}
&\Delta C + \Delta OCA + (\Delta GPPE - \Delta AD)\\
&= \Delta CL + \Delta LTD + NI + OET\\
&375{,}000 + -85{,}000 + (70{,}000 - 10{,}000)\\
&= 35{,}000 + -25{,}000 + 90{,}000 + 250{,}000
\end{aligned} \tag{1.10}$$

It is in equation (1.10) that we first clearly see how the dynamic fundamental accounting equation is the interface between the income statement, balance sheet, and statement of cash flow.

## Analyzing Property, Plant, and Equipment Transactions

We put brackets around $\Delta GPPE$ and $\Delta AD$ in equation (1.10) to emphasize thinking of these terms together as a unit, as they equal $\Delta NPPE$. We can rearrange that

[6]We repeat some equations from prior pages in the footnotes for the reader's convenience.
Equation (1.6): $\Delta A = \Delta L + \Delta CAP$
Equation (1.7): $\Delta A = \Delta C + \Delta OCA + (\Delta GPPE - \Delta AD)$
Equation (1.8): $\Delta L = \Delta CL + \Delta LTD$
Equation (1.9): $\Delta CAP = NI + OET$

equation to satisfy the objective of the statement of cash flows—providing an explanation of the change in the cash balance:

$$\begin{aligned}
\Delta C \quad &= NI \quad - \Delta OCA \quad + \Delta CL \\
&- (\Delta GPPE - \Delta AD) \\
&+ \Delta LTD \quad + OET \\
375{,}000 &= 90{,}000 \quad - -85{,}000 + 35{,}000 \\
&- (70{,}000 \quad - 10{,}000) \\
&+ -25{,}000 + 250{,}000
\end{aligned} \tag{1.11}$$

Equation (1.11) provides an explanation of the \$375,000 (E8) increase in the cash balance from 2008 to 2009. However, it is still somewhat preliminary. It is best to defer our explanation until we incorporate more details into the model.

The balance sheets in Table 1.1 show that *net* property, plant, and equipment increased by \$60,000 (E13). We need a more detailed understanding of this change and can accomplish this with an analysis of property, plant, and equipment like the one in Table 1.3.

**TABLE 1.3: ANALYSIS OF PROPERTY, PLANT, AND EQUIPMENT** This analysis shows that gross property, plant, and equipment increases with capital expenditures (*CAPEXP*) and decreases with the original book value of any assets retired (*RETGBV*).

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | | | | |
| 2 | | **Table 1.3** | | | |
| 3 | | **Feathers R Us** | | | |
| 4 | | **Analysis of Property, Plant, and Equipment** | | | |
| 5 | | **for Calendar Year 2009** | | | **= (C) – (D) [1]** |
| 6 | ***Symbols*** | | ***GPPE*** | ***AD*** | ***NPPE*** |
| 7 | | | **Gross Prop,** | **Accumulated** | **Net Prop,** |
| 8 | | | **Plant and Equip** | **Depreciation** | **Plant and Equip** |
| 9 | | | | | |
| 10 | | **Balance, 2008** | 830.000 | 30,000 | 800,000 |
| 11 | | | | | |
| 12 | ***CAPEXP*** | Capital expenditures [2] | 175,000 | | 175,000 |
| 13 | ***DEPR*** | Depreciation expense [3] | | 30,000 | 30,000 |
| 14 | | Retirements | | | |
| 15 | ***RETGBV*** | Gross book value [4] | 105,000 | | 105,000 |
| 16 | ***RETAD*** | Accumulated depreciation [5] | | 20,000 | 20,000 |
| 17 | | | | | |
| 18 | | **Balance, 2009** | **900,000** | **40,000** | **860,000** |
| 19 | | | | | |
| 20 | | Change in the balance | **70,000** | **10,000** | **60,000** |
| 21 | | | | | |
| 22 | [1] Column E equals (C) – (D) for rows 10, 18, and 20, but it equals (C) + (D) for rows 12 through 16. | | | | |
| 23 | [2] *CAPEXP* adds to column C and column E. | | | | |
| 24 | [3] Depreciation expense adds to column D but subtracts from column E. | | | | |
| 25 | [4] *RETGBV* subtracts from column C and E. | | | | |
| 26 | [5] *RETAD* subtracts from column D but adds to column E. | | | | |

We restate this relationship as equation (1.12):

$$\begin{aligned} &\Delta GPPE = CAPEXP - RETGBV \\ &70{,}000 \;\; = 175{,}000 \;\; - 105{,}000 \\ &\text{C20} \qquad\quad \text{C12} \qquad\quad \text{C15} \end{aligned} \tag{1.12}$$

Similarly, accumulated depreciation increases with depreciation expense and decreases with accumulated depreciation on any assets retired. We restate this relationship as equation (1.13):

$$\begin{aligned} &\Delta AD \;\; = DEPR \;\; - RETAD \\ &10{,}000 = 30{,}000 - 20{,}000 \\ &\text{D20} \qquad \text{D13} \qquad \text{D16} \end{aligned} \tag{1.13}$$

Substituting equations (1.12) and (1.13) into equation (1.11) and rearranging the terms results in equation (1.14):[7]

$$\begin{aligned} \Delta C \quad &= NI \quad + DEPR \quad - \Delta OCA \quad + \Delta CL \\ &- CAPEXP + RETGBV - RETAD \\ &+ \Delta LTD \quad + OET \\ 375{,}000 &= 90{,}000 \quad + \mathbf{30{,}000} \quad - -85{,}000 + 35{,}000 \\ &- \mathbf{175{,}000} + \mathbf{105{,}000} - \mathbf{20{,}000} \\ &+ -25{,}000 + 250{,}000 \end{aligned} \tag{1.14}$$

The bold symbols in equation (1.14) are the symbols that changed with the substitutions described above; that is, *DEPR*, *CAPEXP*, *RETGBV*, and *RETAD* in equation (1.14) did not appear in equation (1.11). Notice that we are subtracting a decrease in other current assets of $85,000, which mathematically is the same as adding $85,000 (i.e., the double negative makes a positive number).

**GAINS AND LOSSES ON SALE OF FIXED ASSETS** Thus far, we have considered only the book value of any assets retired. Most often, the retirement or disposition of assets involves a gain or a loss (a negative gain). This gain is the difference between the selling price of the fixed assets (property, plant, and equipment) (*SALESFA*) and their net book values (*RETGBV* − *RETAD*). In this illustration the company sold its fixed assets for $115,000. They had a net book value of $85,000, producing a gain of $30,000 (Table 1.2, B14). This gain is in the income statement, but it is not cash flow.[8] Therefore, we have to subtract it from net income to calculate cash flow. We show the calculation of the gain in equation (1.15):

$$\begin{aligned} &GAIN \;\; = SALESFA - (RETGBV - RETAD) \\ &30{,}000 = 115{,}000 \;\; - (105{,}000 \;\; - 20{,}000) \end{aligned} \tag{1.15}$$

Equation (1.16) is simply a rearrangement of equation (1.15):

$$\begin{aligned} &RETGBV = SALESFA - GAIN \;\; + RETAD \\ &105{,}000 \;\; = 115{,}000 \;\; - 30{,}000 + 20{,}000 \end{aligned} \tag{1.16}$$

[7]Equation (1.11): $\Delta C = NI - \Delta OCA + \Delta CL - (\Delta GPPE - \Delta AD) + \Delta LTD + OET$

[8]Only the $115,000 sale is a cash flow.

Substituting equation (1.16) into equation (1.14) results in:[9]

$$
\begin{aligned}
\Delta C &= NI + DEPR - \Delta OCA + \Delta CL \\
&\quad - \boldsymbol{CAPEXP} + \boldsymbol{SALESFA} - \boldsymbol{GAIN} + \boldsymbol{RETAD} - RETAD \\
&\quad + \Delta LTD + OET \\
375{,}000 &= 90{,}000 + 30{,}000 - -85{,}000 + 35{,}000 \\
&\quad - 175{,}000 + \mathbf{115{,}000} - \mathbf{30{,}000} + \mathbf{20{,}000} - 20{,}000 \\
&\quad + -25{,}000 + 250{,}000
\end{aligned} \tag{1.17}
$$

After canceling the + *RETAD* and − *RETAD* terms and rearranging, equation (1.17) simplifies to:

$$
\begin{aligned}
\Delta C &= NI - GAIN + DEPR - \Delta OCA + \Delta CL \\
&\quad - CAPEXP + SALESFA \\
&\quad + \Delta LTD + OET \\
375{,}000 &= 90{,}000 - 30{,}000 + 30{,}000 - -85{,}000 + 35{,}000 \\
&\quad - 175{,}000 + 115{,}000 \\
&\quad + -25{,}000 + 250{,}000
\end{aligned} \tag{1.18}
$$

The first row of equation (1.18) represents cash flows from operating activities, which consists of making adjustments to net income, that is, adding back depreciation and other non-cash expenses, subtracting the gain on sale of fixed assets (because it is included in accrual net income but is not a source of cash), subtracting the increase in other current assets, and adding the increase in current liabilities. We will explain these adjustments in more detail later in the chapter. The second row in the equation represents cash flows from investing activities, and the third row represents a preliminary version of cash flows from financing activities.

## Equity Transactions—Dividends and Sale or Purchase of Stock

The details of the OET in equation (1.9) are also important. In this example the statement of stockholder's equity included three types of equity transactions: issuing cash dividends (DIV), selling stock (SALSTK), and buying treasury stock (TRSTK). We show these in Table 1.4.

During the year, the company paid cash dividends of $50,000 (E13), sold additional shares of stock for $350,000 (C14 + D14), and bought back treasury stock[10] for $50,000 (F15). The net effect of these three OET is a $250,000 (G16) increase in stockholder's equity. We summarize this in equation (1.19). We add the term *AET*

[9]Equation (1.14): $\Delta C = NI + DEPR - \Delta OCA + \Delta CL - CAPEXP + RETGBV - RETAD + \Delta LTD + OET$.

[10]To clarify, this is simply a transaction to buy back company stock, and we label it "treasury stock" after the fact.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | | | | | | | |
| 2 | | **Table 1.4** | | | | | |
| 3 | | **Feathers R Us** | | | | | |
| 4 | | **Statement of Stockholders' Equity** | | | | | |
| 5 | | **for Calendar Year 2009** | | | | | **= Sum((C):(F))** |
| 6–8 | | | **Capital Stock** | **Additional Paid in Capital** | **Retained Earnings** | **Treasury Stock** | **Total Shareholder Equity** |
| 9 | ***Symbols*** | **Balance 2008** | **100,000** | **200,000** | **1,425,000** | **0** | **1,725,000** |
| 10 | ***NI*** | Net income | | | 90,000 | | **90,000** |
| 11–12 | | ***Other equity transactions*** | | | | | |
| 13 | ***DIV*** | – Dividends | | | 50,000 | | **50,000** |
| 14 | ***SALSTK*** | + Sale of stock | 50,000 | 300,000 | | | **350,000** |
| 15 | ***TRSTK*** | – Purchase of stock | | | | 50,000 | **50,000** |
| 16 | ***OET*** | **Total = (14) – (13) – (15)** | **50,000** | **300,000** | **(50,000)** | **(50,000)** | **250,000** |
| 17 | | | | | | | |
| 18 | | **Balance 2009 = (9)+(10)+(16)** | **150,000** | **500,000** | **1,465,000** | **(50,000)** | **2,065,000** |

to equation (1.19) to represent *additional* equity transactions.[11]

$$\begin{aligned} OET &= SALSTK - TRSTK - DIV + AET \\ 250{,}000 &= 350{,}000 - 50{,}000 - 50{,}000 + 0 \\ G16 \quad & \quad G14 \qquad G15 \qquad G13 \end{aligned} \tag{1.19}$$

Substituting equation (1.19) into equation (1.18) results in equation (1.20):[12]

$$\begin{aligned} \Delta C &= NI - GAIN + DEPR - \Delta OCA + \Delta CL \\ &\quad - CAPEXP + SALESFA \\ &\quad + \Delta LTD + \mathbf{SALSTK} - \mathbf{TRSTK} - \mathbf{DIV} + \mathbf{AET} \\ 375{,}000 &= 90{,}000 - 30{,}000 + 30{,}000 - -85{,}000 + 35{,}000 \\ &\quad - 175{,}000 + 115{,}000 \\ &\quad + -25{,}000 + \mathbf{350{,}000} - \mathbf{50{,}000} - \mathbf{50{,}000} + \mathbf{0} \end{aligned} \tag{1.20}$$

[11]We used the term *additional* equity transactions to describe equity transactions other than the sale or purchase of the company's stock and the payment of dividends. One example of an additional equity transaction would be the contribution of property to the company in exchange for an equity interest. For analytical purposes, we could treat the increase in equity as a source of cash from financing activities and the corresponding increase in assets as a use of cash from investing activities. The net result would be an overall zero effect on cash. Normally, *noncash* transactions of this nature are not incorporated in formal statements of cash flow, but are appended in a separate schedule.

[12]Equation (1.18): $\Delta C = NI - GAIN + DEPR - \Delta OCA + \Delta CL - CAPEXP + SALESFA + \Delta LTD + OET$ Equation (1.19): $OET = SALESTK - TRSTK - DIV + AET$

We can simplify equation (1.20) to the more familiar form:

$$\begin{aligned} \Delta C &= \text{Cash flows from operating activities} \\ &+ \text{Cash flows from investing activities} \\ &+ \text{Cash flows from financing activities} \\ 375{,}000 &= 210{,}000 \\ &+ (60{,}000) \\ &+ 225{,}000 \end{aligned} \tag{1.21}$$

Equations (1.20) and (1.21) describe the conventional statement of cash flows in Table 1.5. Note that the three components of cash flow in equation (1.21) appear in Table 1.5 in D16, D21, and D28, respectively, with a total of $375,000 in D29. Total cash on December 31, 2008 and December 31, 2009 of $1,125,000 and $1,500,000 in D30 and D31 appear in Table 1.1, C8 and D8.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | **Table 1.5** | | |
| 2 | | **Feathers R Us** | | |
| 3 | | **Abbreviated** | | |
| 4 | | **Statement of Cash Flows** | | |
| 5 | | **for Calendar Year 2009** | | |
| 6 | | | | |
| 7 | ***Symbols*** | | | |
| 8 | | Cash flows from operating activities | | |
| 9 | ***NI*** | Net income | | 90,000 |
| 10 | | Adjustments to reconcile net income to | | |
| 11 | | net cash provided by operating activities: | | |
| 12 | ***GAIN*** | Gain on sale of property, plant, and equipment | (30,000) | |
| 13 | ***DEPR*** | Depreciation expense | 30,000 | |
| 14 | ***ΔOCA*** | Decrease in current assets | 85,000 | |
| 15 | ***ΔCL*** | Increase in current liabilities | 35,000 | **120,000** |
| 16 | | Net cash provided by operating activities | | **210,000** |
| 17 | | | | |
| 18 | | Cash flows from investing activities | | |
| 19 | ***CAPEXP*** | Purchase of property, plant, and equipment | (175,000) | |
| 20 | ***SALESFA*** | Sale of property, plant, and equipment | 115,000 | |
| 21 | | Net cash used by investing activities | | **(60,000)** |
| 22 | | | | |
| 23 | | Cash flows from financing activities | | |
| 24 | ***ΔLTD*** | Increase in long-term debt | (25,000) | |
| 25 | ***SALSTK*** | Sale of stock | 350,000 | |
| 26 | ***TRSTK*** | Purchase of treasury stock | (50,000) | |
| 27 | ***DIV*** | Payment of dividends | (50,000) | |
| 28 | | Net cash provided by financing activities | | **225,000** |
| 29 | | Net increase in cash | | **375,000** |
| 30 | | Cash, December 31, 2008 | | 1,125,000 |
| 31 | | Cash, December 31, 2009 | | **1,500,000** |

## Required Working Capital

Thus far, we have used the classical accounting definition of net working capital, which is current assets minus current liabilities. We filled in the details of those changes and defined cash flow as in equation (1.20). It turns out that this definition of net working capital is insufficient for valuation purposes, as some cash is required in the business for the company to be able to pay its bills. Therefore, not all cash flows are available for distribution to shareholders. We want to develop the equation for cash flows available for distribution to shareholders. We will do this in steps.

For the moment, we will define the required change in working capital as the change in current assets other than cash less the change in current liabilities, as shown in equation (1.22).[13] Note that the cell references are to Table 1.1.

$$\begin{array}{rcll} \Delta RWC & = \Delta OCA & - \Delta CL & \\ -120{,}000 & = -85{,}000 & - 35{,}000 & \\ & \text{E9} & \text{E17} & \quad (1.22) \end{array}$$

This illustration is somewhat unusual. Here, the changes in other current assets and current liabilities are *reducing* working capital. This reduction is a *source* of the cash from operating activities. (In the typical case, working capital *increases* when sales grow. In that case, the increase in working capital is a *use* of cash.)

Substituting equation (1.22) into equation (1.20) results in:[14]

$$\begin{array}{rllllll} \Delta C & = NI & - GAIN & + DEPR & - \mathbf{\Delta RWC} & & \\ & - CAPEXP & + SALESFA & & & & \\ & + \Delta LTD & + SALSTK & - TRSTK & - DIV & + AET & \\ 375{,}000 & = 90{,}000 & - 30{,}000 & + 30{,}000 & - \mathbf{-120{,}000} & & \\ & - 175{,}000 & + 115{,}000 & & & & \\ & + -25{,}000 & + 350{,}000 & - 50{,}000 & - 50{,}000 & + 0 & \quad (1.23) \end{array}$$

We can represent the first row of equation (1.23) as follows:

| Activity | Symbol | Description |
|---|---|---|
| *Operating* | *NI* | + Net income |
| | *GAIN* | − Gains (+ losses) on the sale of property, plant, and equipment |
| | *DEPR* | + Depreciation and other non-cash charges |
| | $\Delta$*RWC* | − Increases (+ decreases) in required working capital |

As mentioned previously, we subtract the gain (or add the loss) on the sale of property, plant, and equipment to compute cash flow, because it is a component of net income that does not produce cash flow.

[13]We will modify the definition in equation (1.22) later in the chapter.

[14]Equation [(1.20): $\Delta C = NI - GAIN + DEPR - \Delta OCA + \Delta CL - CAPEXP + SALESFA + \Delta LTD + SALSTK - TRSTK - DIV + AET$. Equation (1.22): $\Delta RWC = \Delta OCA - \Delta CL$. Note that $\Delta CL$ cancels out in equation (1.23).

Depreciation and other non-cash expenses *do* reduce net income, but they *do not* involve any payments during the current period. Therefore, when we use the indirect method and net income is the starting point for arriving at a firm's net cash flow, we must add back these non-cash expenses.

We will discuss the rationale for subtracting required increases (or adding decreases) in working capital at some length in the next section after introducing the components of the changes in other current assets ($\Delta OCA$) and current liabilities ($\Delta CL$).

To complete our explanation of equation (1.23), the second and third rows consist of the following:[15]

| Activity | Symbol | Description |
|---|---|---|
| *Investing* | *CAPEXP* | − Capital expenditures |
| | *SALESFA* | + Selling price of property, plant, and equipment disposed of or retired |
| *Financing* | $\Delta LTD$ | + Increases (− decreases) in long-term debt |
| | *SALSTK* | + Proceeds received from the sale of stock |
| | *TRSTK* | − Payments for treasury stock |
| | *DIV* | − Dividends |
| | *AET* | + Additional equity transactions |

**ADDING DETAIL OF THE COMPONENTS OF REQUIRED WORKING CAPITAL** Before discussing required working capital further, it will be helpful to break down changes in other current assets ($\Delta OCA$) and current liabilities ($\Delta CL$) into some typical component parts. Table 1.6 is a restatement of Table 1.1 with this additional detail in the shaded sections.

Here, other current assets consist of accounts receivable, inventory, and additional current assets. Current liabilities include accounts payable, short-term notes payable, and accrued expenses.

We treat accounts receivable, inventory, and additional current assets in the same way as other current assets. When using the indirect method, we subtract increases (add decreases) in these accounts from net income to arrive at net cash provided by operating activities.

Likewise, we treat accounts payable, short-term notes payable, and accrued expenses in the same way as current liabilities when using the indirect method. We add increases (subtract decreases) in these accounts to net income to arrive at net cash provided by operating activities.

Applying the procedures outlined in the two preceding paragraphs results in the statement of cash flows shown in Table 1.7, which is simply Table 1.5 with the addition of the shaded detail.[16]

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[15]The second row is $-\ CAPEXP + SALESFA$, and the third row is $\Delta LTD + SALSTK - TRSTK - DIV + AET$.

[16]In Table 1.7, the signs of other current assets are switched—as we did in Table 1.5—because our equation calls for subtracting $\Delta OCA$ when computing $\Delta C$.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 1.6** | | | | |
| 2 | **Feathers R Us** | | | | |
| 3 | **Balance Sheets** | | | | |
| 4 | **for Calendar Years** | | | | |
| 5 | | | | | |
| 6 | | | | | **Increase** |
| 7 | ***Symbols*** | **ASSETS:** | **2008** | **2009** | **(Decrease)** |
| 8 | ***C*** | Cash | 1,125,000 | 1,500,000 | 375,000 |
| 9 | | Accounts receivable | 100,000 | 150,000 | 50,000 |
| 10 | | Inventory | 750,000 | 600,000 | (150,000) |
| 11 | | Additional current assets | 25,000 | 40,000 | 15,000 |
| 12 | | Total current assets | **2,000,000** | **2,290,000** | **290,000** |
| 13 | ***GPPE*** | Gross property, plant, and equipment | 830,000 | 900,000 | 70,000 |
| 14 | ***AD*** | Accumulated depreciation | 30,000 | 40,000 | 10,000 |
| 15 | ***NPPE*** | Net property, plant, and equipment | **800,000** | **860,000** | **60,000** |
| 16 | ***A*** | Total assets | **2,800,000** | **3,150,000** | **350,000** |
| 17 | | | | | |
| 18 | | **LIABILITIES** | | | |
| 19 | | Accounts payable | 200,000 | 225,000 | 25,000 |
| 20 | | Short-term notes payable | 50,000 | 35,000 | (15,000) |
| 21 | | Accrued expenses | 75,000 | 100,000 | 25,000 |
| 22 | ***CL*** | Current liabilities | **325,000** | **360,000** | **35,000** |
| 23 | ***LTD*** | Long-term debt | 750,000 | 725,000 | (25,000) |
| 24 | ***L*** | Total liabilities | **1,075,000** | **1,085,000** | **10,000** |
| 25 | | | | | |
| 26 | | **STOCKHOLDERS' EQUITY** | | | |
| 27 | | Capital stock | 100,000 | 150,000 | 50,000 |
| 28 | | Additional paid in capital | 200,000 | 500,000 | 300,000 |
| 29 | | Retained earnings | 1,425,000 | 1,465,000 | 40,000 |
| 30 | | Treasury stock | 0 | 50,000 | 50,000 |
| 31 | ***CAP*** | Total stockholders' equity | **1,725,000** | **2,065,000** | **340,000** |
| 32 | | | | | |
| 33 | | Total liabilities and equity | **2,800,000** | **3,150,000** | **350,000** |

**ADJUSTING FOR REQUIRED CASH** For valuation purposes, it is important to recognize that all firms require a certain amount of cash to be kept on hand; otherwise, checks would constantly bounce. Therefore, the amount of required cash ($C_{Req}$) will not be available for dividend payments.

A good method to estimate required cash is to ask management how many days' costs and expenses it needs to be safe. For example, suppose the answer is 10 working days. Since most firms work 250 days per year, management requires 4 percent of total costs and expenses in the cash account. For example, if total cost of sales and expenses are $10 million per year, we could forecast required cash as $400,000 for the prior year, to be increased going forward by the forecast growth rate.

In equation (1.22), we defined the required change in working capital simply as the change in current assets other than cash, less the change in current liabilities.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | | **Table 1.7** | | |
| 2 | | **Feathers R Us** | | |
| 3 | | **Statement of Cash Flows—Detailed** | | |
| 4 | | **for Calendar Year 2009** | | |
| 5 | | | | |
| 6 | ***Symbols*** | Cash flows from operating activities | | |
| 7 | ***NI*** | Net Income | | 90,000 |
| 8 | | Adjustments to reconcile net income to | | |
| 9 | | net cash provided by operating activities: | | |
| 10 | ***GAIN*** | Gain on sale of property, plant, and equipment | (30,000) | |
| 11 | ***DEPR*** | Depreciation expense | 30,000 | |
| 12 | Δ | Increase in accounts receivable | (50,000) | |
| 13 | Δ | Decrease in inventory | 150,000 | |
| 14 | Δ | Increase in additional current assets | (15,000) | |
| 15 | | | | |
| 16 | Δ | Increase in accounts payable | 25,000 | |
| 17 | Δ | Decrease in short-term notes payable | (15,000) | |
| 18 | Δ | Increase in accrued expenses | 25,000 | 120,000 |
| 19 | | Net cash provided by operating activities | | **210,000** |
| 20 | | | | |
| 21 | | Cash flows from investing activities | | |
| 22 | ***CAPEXP*** | Purchase of property, plant, and equipment | (175,000) | |
| 23 | ***SALESFA*** | Sale of property, plant, and equipment | 115,000 | |
| 24 | | Net cash used by investing activities | | (60,000) |
| 25 | | | | |
| 26 | | Cash flows from financing activities | | |
| 27 | ***ΔLTD*** | Decrease in long-term debt | (25,000) | |
| 28 | ***SALSTK*** | Sale of stock | 350,000 | |
| 29 | ***TRSTK*** | Purchase of treasury stock | (50,000) | |
| 30 | ***DIV*** | Payment of dividends | (50,000) | |
| 31 | | Net cash provided by financing activities | | 225,000 |
| 32 | | Net increase in cash | | **375,000** |
| 33 | | Cash, December 31, 2008 | | 1,125,000 |
| 34 | | Cash, December 31, 2009 | | **1,500,000** |

We will now modify that definition in equation (1.24) to include the changes in the cash balance the firm must keep on hand ($20,000 in this illustration):[17]

$$\begin{aligned} \Delta RWC &= \Delta OCA - \Delta CL + \Delta C_{Req} \\ (100{,}000) &= -85{,}000 - 35{,}000 + 20{,}000 \end{aligned} \tag{1.24}$$

In equation (1.22), the $85,000 decrease in other current assets and the $35,000 increase in current liabilities gave rise to a reduction in required working capital of $120,000. After taking into consideration the $20,000 additional cash that will be required, the reduction in required working capital falls to $100,000; that is, the net

[17]Typically appraisers forecast required cash as a percentage of sales. Required cash increases (decreases) by that percentage multiplied by the increase (decrease) in sales.

addition to cash flow from the reduction in required net working capital is $20,000 less.

Using this modified definition for $\Delta RWC$ lowers the resulting cash flow to $355,000 in equation (1.23a) from the $375,000 originally shown in equation (1.23).[18] $\Delta C^*$ is the same as $\Delta C$, except that $\Delta C^*$ defines $\Delta RWC$ using equation (1.24) instead of equation (1.22).

$$\begin{aligned}
\boldsymbol{\Delta C^*} &= NI - GAIN + DEPR - \boldsymbol{\Delta RWC} \\
&\quad - CAPEXP + SALESFA \\
&\quad + \Delta LTD + SALSTK - TRSTK - DIV + AET \\
\mathbf{355{,}000} &= 90{,}000 - 30{,}000 + 30{,}000 - \mathbf{(100{,}000)} \\
&\quad - 175{,}000 + 115{,}000 \\
&\quad + -25{,}000 + 350{,}000 - 50{,}000 - 50{,}000 + 0
\end{aligned} \tag{1.23a}$$

This $355,000 amount represents the net cash flow available for dividend payments in excess of the $50,000 of dividends already paid.

Alternatively, we can add *DIV* to both sides of equation (1.23a) to show the *total amount of net cash flow available for distribution to stockholders*. That amount is $405,000, as shown in equation (1.23b):

$$\begin{aligned}
\Delta C^* + \boldsymbol{DIV} &= NI - GAIN + DEPR - \Delta RWC \\
&\quad - CAPEXP + SALESFA \\
&\quad + \Delta LTD + SALSTK - TRSTK + AET \\
\mathbf{405{,}000} &= 90{,}000 - 30{,}000 + 30{,}000 - (100{,}000) \\
&\quad - 175{,}000 + 115{,}000 \\
&\quad + -25{,}000 + 350{,}000 - 50{,}000 + 0
\end{aligned} \tag{1.23b}$$

## Analysis of the Mathematical Model

In this section, we compare our results to another definition of cash flow in valuation literature, and we make a conceptual statement of how the income statement and the statement of cash flows are both reconciliations of different parts of the balance sheet.

### Comparison to Other Cash Flow Definitions

We can summarize the definition of net cash flows available for distribution to stockholders in equation (1.23b) in the following way:

| Activity | Symbol | Description |
|---|---|---|
| *Operating* | *NI* | + Net income |
| | *GAIN* | − Gains (+ losses) on the sale of property, plant, and equipment |
| | *DEPR* | + Depreciation and other non-cash charges |
| | $\Delta RWC$ | − Increases (+ decreases) in required working capital* |

[18] $\Delta C^* = \Delta C - \Delta C_{Req}$.

| | | |
|---|---|---|
| *Investing* | *CAPEXP* | − Capital expenditures |
| | *SALESFA* | + Selling price of property, plant, and equipment disposed of or retired |
| *Financing* | $\Delta LTD$ | + Increases (− decreases) in long-term debt |
| | *SALSTK* | + Proceeds received from the sale of stock |
| | *TRSTK* | − Payments for treasury stock |
| | *AET* | + Additional equity transactions |

*After adjusting for required cash.

We compare this summary to another definition in our professional literature. For example, one group of authors (Pratt, Reilly, and Schweihs, 1996) used the following definition of net cash flow available for distribution to stockholders in their Formula 9-3 (pp. 156–157):

| Description |
|---|
| + Net income |
| + Depreciation and other non-cash charges |
| − Increases (+ decreases) in required working capital |
| − Capital expenditures |
| + Selling price of property, plant, and equipment disposed of or retired |
| + Increases (− decreases) in long-term debt |

This definition of cash flow is obviously much simpler than ours and considers only the most common types of transactions. Implicitly, this definition assumes that gains and losses on the sale of property, plant, and equipment and the selling price of property, plant, and equipment disposed of or retired are immaterial. Likewise, this definition assumes that there are no material sales or purchases of stock or additional equity transactions.

These assumptions are reasonable in a large number of cases.[19] However, it is important for the analyst to be cognizant of these underlying assumptions and beware of situations in which one or more of these assumptions are no longer reasonable. The abbreviated definition obviously is insufficient when valuing a firm that intends to raise capital in several rounds of financing or for a heavy manufacturing firm that routinely has material sales of its property, plant, and equipment.

When calculating value by capitalizing a single-period cash flow, we considerably magnify the consequences of making adjustments to the initial cash flow. It is important for the analyst to understand how these hidden assumptions might influence the amount of initial cash flow that we capitalize and how these assumptions might impact the *future* cash flows available for distribution to stockholders.

For example, if a company were to routinely sell its equipment for significant sums, the analyst would be remiss if he or she overlooked the cash flows from these sales. On the other hand, it is also important to consider the potential effect on sales and operating expenses of depleting the company's capital equipment.

[19]With respect to the proceeds from the sale of stock, it is unlikely that a firm would sell its stock in order to obtain cash for distribution to its stockholders. However, sometimes large sales of stock do occur, especially in venture-financed high-tech start-ups.

### The Income Statement and Cash Flow as Reconciliations

We can now see a conceptual similarity and difference between the income statement and the statement of cash flows. Both serve as a reconciling link between the beginning and ending balance sheets. The income statement is an accrual-based partial reconciliation between the beginning and ending balances in retained earnings,[20] and the statement of cash flows is a cash-based reconciliation between the beginning and ending cash balances.

Recall that cash flows from operating activities are the cash equivalent of the accrual-based income statement. To complete the reconciliation between the beginning and ending cash balances, the statement of cash flows (as illustrated above) must also include cash from investing and financing activities.

This explains why cash flows are much more volatile than income. Net income changes over time with revenues and expenses, while cash flow changes in response to all account changes—income, expenses, balance sheet accounts, capital expenditures, and so on. There are far more accounts affecting cash flow, so it is not surprising to find that cash flow fluctuates far more than net income.

## Summary

A clear understanding of the mathematics and accounting logic in this chapter should enhance the valuation analyst's understanding of the derivation of the statement of cash flows, how it works, and how it relates to the balance sheet and income statement. It should also make the analyst aware of the simplifying assumptions embedded in abbreviated definitions of cash flows available for distribution to stockholders. Hopefully, this awareness will result in superior valuations in those instances when it is unwarranted to make these simplifying assumptions.

## References

Abrams, Jay B. 1994. Cash flow: A mathematical derivation. *Valuation* (March 1994): 64–71.

Pratt, Shannon P., Robert F. Reilly, and Robert P. Schweihs. 1996. *Valuing a Business: The Analysis and Appraisal of Closely Held Companies, 3rd ed.* New York: McGraw-Hill.

[20]The complete reconciliation requires accounting for sales and repurchases of stock, dividends, prior-period adjustments, and occasionally certain other items.

# CHAPTER 2

# Forecasting Cash Flow

## *Mathematics of the Payout Ratio*

# CHAPTER 2

# Forecasting Cash Flow

## *Mathematics of the Payout Ratio*

## Introduction

We all have used the *discounted cash flow (DCF)* method. Most of us would agree that it is generally the best, most comprehensive, and theoretically correct valuation model available in the *income approach*. It also has an empirical reason to be the best, which is that many of us calculate our discount rates using the Ibbotson data in the SBBI annual yearbooks, which are based on publicly traded stock data. Those stock returns are cash returns—the dividend yield plus the capital gains, which can be converted to cash at any time.[1] Thus, it is consistent to discount cash flow with discount rates on cash returns. So far, everything is well and good.

### Difficulties in Forecasting Cash Flow

Well, almost; the problem is that forecasting net income is work, and forecasting net free cash flows (*net cash flows* or *cash flows*) is detailed, exacting work. Few well-adjusted people really like doing it. The most disciplined of us keep a stiff upper lip and do it—especially in the large valuation firms with clients who are willing to pay for doing it right. The American Society of Appraisers' business valuation courses teach DCF, not discount net income. Nevertheless, in the real world, as we decline in firm size, client budgets, and personal discipline, cash flow often goes by the wayside, and many of the smaller valuation firms end up discounting forecast net income, gross cash flow (net income + depreciation + amortization), EBIT, or EBITDA—and that is always inconsistent. Discounting forecast net income or

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Adapted from Abrams (2003) and Abrams (1994). The author wishes to thank Roger J. Grabowski, *Business Valuation Review*'s referee, for his insightful comments and helpful suggestions, as well as other anonymous referees.

[1]This applies equally as well for those using an ex ante approach, such as the Merrill-Lynch dividend discount model. The point is that we are still being consistent by using expected returns on cash flows (as opposed to realized historical returns—but nevertheless still on cash flows) to discount cash flows.

any of the other abovementioned measures of earning power normally leads to a guaranteed overvaluation.

Gilbert (1990) states that if you discount net income or some larger number such as gross cash flows, then you must add a premium to the discount rate, and the premium has to increase with the degree to which the measure of economic earning power exceeds net cash flows. In my opinion, he is absolutely right.

There are two problems with adding the premiums. The first problem is that almost nobody does it, even though it is common to discount forecast net income. The second problem is that there is no empirical evidence of the appropriate magnitude of the premium. In my opinion, this is reason enough to state that we should never discount forecast net income, gross cash flows, EBIT, EBITDA, or any other measure of economic earning power other than net cash flows. This brings us right back to the DCF and the need to forecast cash flows.

### Purpose of This Chapter

The main purpose of this chapter is to provide the mathematics that will simplify the mechanics of forecasting cash flow in many situations, thus making the DCF easier to do and reducing the temptation to take the shortcuts that lead to overvaluations.

## The Mathematics

In the main part of this chapter, we will use the following symbols in our mathematics:

| | |
|---|---|
| *Cap Exp* | = *CE* = Capital Expenditures. |
| *CF* | = Cash Flow, the increase or decrease in cash from one accounting period to another. |
| *Depr* | = *D* = Depreciation expense. |
| $\Delta$ | = "Delta," meaning "the change in" a balance sheet account over time. |
| *LTD* | = Long-Term Debt. |
| *NWC* | = Net Working Capital. It is the increase (or decrease) in *NWC* that is a cash flow item, not the absolute amount of *NWC*. This should include the amount of cash that the business needs to maintain to pay its bills adequately, and it should exclude excess cash that could be paid to shareholders as dividends without impairing the operations of the business. |
| *POR* | = Payout Ratio = *CF/NI*; that is, the payout ratio is the percentage of net income that the company can pay to shareholders in dividends, whether directly or disguised. Disguised dividends are excess compensation (i.e., above arm's length) paid to owners. $POR = 1 - RR$; that is, out of total net income, the percentage the owners retain for reinvestment back into the business is the retention ratio, and the remaining percentage is the payout ratio. |
| *PP&E* | = Property, Plant, and Equipment. |
| *RR* | = Retention Ratio. |

## The Cash Flow Equation

Let's begin with the complete cash flow equation:[2]

$$\text{Cash Flow} = \text{Net Income} - \text{Gain on Sale of Assets} + \text{Depreciation} - \Delta\ \text{Required Net Working Capital} - \text{Capital Expenditures} + \text{Cash Received for Sale of Fixed Assets} + \Delta\ \text{Long} - \text{Term Debt} + \text{Sale of Stock} - \text{Purchase of Treasury Stock} - \text{Dividends Paid} - \text{Additional Equity Transactions}. \quad (2.1)$$

Equation (2.1) contains many terms that are unusual items or immaterial in amount. The stock transactions generally are rare, as are dividends in private firms.[3] The cash proceeds from and the accounting gain or loss on the sale of fixed assets generally are small and can be ignored in most situations for forecasting cash flows. For practical purposes, let's work with the terms that are material and ordinary.

The one item that is regular and material, but can be treated as in or out of the cash flow equation, is increases in long-term debt. Some valuators prefer to value the firm debt-free, and one can always add in a premium for the tax-shield value of the debt afterward. In the mathematics that follows, we will keep it in the equation, but it is easy to back it out at the end. Thus, the shortcut cash flow equation is:

$$CF = NI + Depr - Cap\,Exp - \Delta NWC + \Delta LTD. \quad (2.2)$$

Another way of looking at equation (2.2) is to split the latter four terms into two pairs, each set off in parentheses, as in equation (2.2a). Also, the order of depreciation and capital expenditures is reversed, as is the sign in front of the parentheses.

$$CF = NI - (Cap\,Exp - Depr) - (\Delta NWC - \Delta LTD). \quad (2.2a)$$

Capital expenditures and depreciation is a logical unit of analysis. Today's depreciation results from capital expenditures that we made over the past several years. The amount by which capital expenditures exceeds depreciation is a subtraction from cash flow, as is the amount by which the increase in net working capital exceeds the increase in long-term debt. Another way of looking at the terms in parentheses in equation (2.2a) is that the first set deals with changes in fixed assets, which is a use of cash, while the second set deals with the changes in current assets net of current liabilities and long-term debt, which is also a use of cash.

## Defining Cash Flow through the Payout Ratio

We can derive cash flow from net income in an alternative format, that is, as a percentage adjustment to net income. It will turn out that normally this will be a much easier calculation than forecasting all the elements of cash flow, that is, depreciation, capital expenditures, and changes in net working capital and long-term debt. For

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[2]See Chapter 1 for a detailed mathematical derivation. For an earlier version of the mathematics, see "Cash Flow: A Mathematical Derivation," *Valuation*, January 1994. To download, go to www.abramsvaluation.com, select "Articles," then "Articles in .PDF."

[3]Also, since we are trying to forecast the maximum dividends the firm can pay without impairing its operations, the dividends actually paid do not matter in a DCF at the company level. They do matter in a discounted dividends model.

valuation purposes, the *payout ratio* (POR) is the portion of net income that can be distributed to owners without impairing operations.[4] The portion of net income that is required for operating and growing the business is called the *retention ratio* (RR), which equals one minus the payout ratio.

In this chapter, we will develop an exact set of formulas, equations (2.8) and (2.9) for the payout ratio and the retention ratio, respectively, that relate back to equation (2.2) for the definition of cash flow. Unfortunately, equations (2.8) and (2.9) are computationally intensive, as they require forecasting capital expenditures, depreciation, and the increase in required net working capital. This gives rise to the need for easier equations to use. Thus, the second goal is to develop an accurate formula to estimate the payout ratio.

## Payout Ratios—Exact Equations

In this series of equations, we develop an exact formula for the payout ratio. Equation (2.3) is the definition of the payout ratio:

$$CF = NI \times POR. \tag{2.3}$$

Since the left-hand sides of equations (2.2) and (2.3) are equal, their right-hand sides also must be equal. We state this in equation (2.4):

$$NI + Depr - Cap\,Exp - \Delta NWC + \Delta LTD = NI \times POR. \tag{2.4}$$

Next, we subtract $NI$ from both sides of the equation and factor $NI$ on the right-hand side:

$$Depr - Cap\,Exp - \Delta NWC + \Delta LTD = NI(POR - 1). \tag{2.5}$$

Dividing through by $NI$, we get:

$$\frac{Depr - Cap\,Exp - \Delta NWC + \Delta LTD}{NI} = POR - 1. \tag{2.6}$$

Adding 1 to both sides of the equation leads to:

$$1 + \frac{Depr - Cap\,Exp - \Delta NWC + \Delta LTD}{NI} = POR. \tag{2.7}$$

Finally, we change the plus sign on the left-hand side of the equation to a minus sign, reverse the signs of the variables in the numerator, and switch the two sides of the equation to arrive at our final solution in (2.8):

$$POR = 1 - \frac{(Cap\,Exp - Depr) + (\Delta NWC - \Delta LTD)}{NI}. \tag{2.8}$$

The net income should be a normalized net income (i.e., a long-term income base). As mentioned earlier, the retention ratio is one minus the payout ratio.

[4]In calculating the payout ratio historically, it is simply dividends paid divided by net income, regardless of whether the owner impaired operations by paying out too much in dividends. However, for valuation purposes, in forecasting ahead we consider only the dividends that can be paid without impairing operations.

Thus the retention ratio in equation (2.9) equals one minus equation (2.8):

$$RR = \frac{(Cap\,Exp - Depr) + (\Delta NWC - \Delta LTD)}{NI}. \qquad (2.9)$$

Equation (2.9) is intuitively appealing, as the greater the amount by which our capital expenditures, which is current investment, exceeds depreciation, which is our past investment, and the greater our investment in new net working capital in excess of long-term debt financing, the higher is the retention ratio.

## Developing an Estimation Formula for POR

In this section, we do the following:

1. Discuss benchmarks for payout ratios of publicly and privately held firms.
2. Develop an alternative formula for the payout ratio to make estimation easier.
3. Analyze tables that use the alternative formula to demonstrate its accuracy and to provide the specific percentage by which capital expenditures exceeds depreciation for a variety of different growth rates and equipment lives.
4. Discuss the curveballs that occur in using the alternative formula.

**BENCHMARKS FOR THE PAYOUT RATIO** We look at two different benchmarks for payout ratios. The first is the historical average payout ratios of publicly held firms, and the second is the Moskowitz-Vissing-Jorgensen (2002) (MVJ) guesstimate for privately held firms.

Ibbotson and Chen (2002) state that the dividend payout ratio for publicly held firms was 47% at the beginning of 1926 and decreased to 32% by the end of 2000. Thus, publicly traded firms now retain on average 68% of their income for cash flow and growth. Over the past 75 years, publicly held firms experienced an average growth of approximately 7% to 8%, which is much faster than private firms—certainly due to their much larger retention ratio and greater business opportunities.[5]

MVJ guesstimate an average 60% payout ratio for privately held C corporations and 80% for privately held S corporations and other non-tax entities. If you have difficulty using the payout ratio formula later in equation (2.24), then it would make sense to use their guesstimate as a benchmark. However, your clients' payout ratios may vary from 60% to 80%. MVJ emphasize that external financing is more expensive for privately held C corporations than it is for publicly held C corporations, because of their smaller size. They further wrote that the non-tax entities tend to be smaller yet, and external financing should be even more expensive for them than for the larger, privately owned C corporations. However, counterbalancing this is the likelihood that the smaller, non-tax entities probably have fewer growth

[5]According to Ibbotson and Chen (cited above), page 5, equation (6), geometric average capital gains in the public equity markets from 1926 to 2000 were 3.02% in real terms and approximately 6.2% in nominal terms. Arithmetic returns are always higher than geometric returns, and the former is the correct measure for valuation purposes. Thus, I estimate nominal capital gains of approximately 7% to 8%. Income returns were 4.28%.

opportunities than the larger firms, which is their reasoning for assuming lower retention.

It is clear from reading between the lines in their article and logically that the main determinants in the earnings retention decision are size and cost of external financing, not the form of organization. Thus, a one-person C corporation should retain as little—and, thus, pay out as much—as a sole proprietorship with no employees. I have valued no-growth clients with historical payout ratios as high as 99.8%. It is important to use common sense. The bottom line is that the higher your forecast growth rate, the lower your payout ratio should be, and vice versa.

We now proceed with the mathematics necessary to develop the alternative POR formula. There are two steps necessary to accomplish this. The first step is to develop an expression for the excess of capital expenditures over depreciation, and the second step is to develop the mathematics for the increase in net working capital and long-term debt.

**THE MATHEMATICS OF CAPITAL EXPENDITURES OVER DEPRECIATION** For simplicity, we will begin by assuming that property, plant, and equipment (PP&E) has an average five-year life. Later we will relax that assumption. We will assume the company has five machines and uses straight-line depreciation. It buys its first machine at the beginning of year 1, its second machine at the beginning of year 2, its third machine at the beginning of year 3, its fourth machine at the beginning of year 4, and its fifth machine at the beginning of year 5. At the beginning of year 6, the company retires machine #1 and buys a replacement machine for it. From then on, it always runs five machines, replacing the oldest one at the beginning of the next year.

Thus, year 5 is the first year that the company reaches a constant status; that is, there is no real growth afterward. During year 5, one-fifth of the equipment was bought at the beginning of years 1, 2, 3, 4, and 5. We will assume that the equipment cost $1,000 at the beginning of year 1, and prices increase at a rate of $g$ each year. We will for the moment assume a stagnant industry, which means it has inflationary but no real growth. Later, we will modify that assumption. Since inflation in the United States has been approximately 3% per year, we will assume $g = 3\%$.

Our procedure will be first to develop a mathematical expression for capital expenditures at the beginning of year 6. Then we will develop an expression for depreciation in year 5. Finally, we will divide the former by the latter, which will give us a ratio of the two. Later we will be able to use that to our practical advantage.

In this simple model, from year 5 and on, capital expenditures differ from the previous year's depreciation by a multiplicative factor, $CE_6 = (1 + k)\, D_5$, where normally $0 < k < 200\%$ and is typically between 6% and 20% for most businesses. Therefore, $CE_6 - D_5 = (1 + k)\, D_5 - D_5 = k\, D_5$. The percentage by which capital expenditures in year 6 exceeds depreciation expense in year 5 (or, more generally, in year $t + 1$ versus year $t$) is the ratio of the two minus 1, that is:

$$\%\,Difference = \frac{CE_6}{D_5} - 1. \tag{2.10}$$

Capital expenditures in year 6 will be the original purchase price in year 1 of $1,000 multiplied by 1 plus the growth rate to the fifth power, or:

$$CE_6 = \$1{,}000\,(1 + g)^5\,. \tag{2.11}$$

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 2.1** | | | | | | |
| 2 | **Analysis of Depreciation and Capital Expenditures** | | | | | | |
| 3 | | | | | | | |
| 4 | | 1 | 2 | 3 | 4 | 5 | 6 |
| 5 | Purchase Price of Equipment [1] | 1000 | 1030 | 1060.9 | 1092.727 | 1125.50881 | **1159.2741** |
| 6 | Depreciation of Equipment Bought Year 1 | 200 | 200 | 200 | 200 | 200.000 | |
| 7 | Depreciation of Equipment Bought Year 2 | | 206 | 206 | 206 | 206.000 | |
| 8 | Depreciation of Equipment Bought Year 3 | | | 212.18 | 212.18 | 212.180 | |
| 9 | Depreciation of Equipment Bought Year 4 | | | | 218.5454 | 218.545 | |
| 10 | Depreciation of Equipment Bought Year 5 | | | | | 225.102 | |
| 11 | Total Depreciation | | | | | **1061.827** | |
| 12 | | | | | | | |
| 13 | Growth Rate—Price of Equipment = *g* | 3% | | | | | |
| 14 | | | | | | | |
| 15 | Purchase of New Equipment—Year 6 (G5) | 1159.2741 | | | | | |
| 16 | Divide by Depreciation—Year 5 (F11) | 1061.8272 | | | | | |
| 17 | Ratio (B15/B16) | **1.092** | | | | | |
| 18 | Difference = Ratio Minus 1 = CapEquipment – Depreciation | **9.2%** | | | | | |
| 19 | **Equation [2.18]: $[5 \times g \times (1+g)^5/((1+g)^5-1)]-1$** | **9.2%** | | | | | |
| 20 | | | | | | | |
| 21 | **Sensitivity Analysis: How the Difference Varies with Changes in the Growth Rate** | | | | | | |
| 22 | | | | | | | |
| 23 | 1% | 3.0% | | | | | |
| 24 | 2% | 6.1% | | | | | |
| 25 | **3%** | **9.2%** | | | | | |
| 26 | 4% | 12.3% | | | | | |
| 27 | 5% | 15.5% | | | | | |
| 28 | 6% | 18.7% | | | | | |
| 29 | 7% | 21.9% | | | | | |
| 30 | 8% | 25.2% | | | | | |
| 31 | 9% | 28.5% | | | | | |
| 32 | 10% | 31.9% | | | | | |
| 33 | | | | | | | |
| 34 | [1] We assume we buy equipment at the beginning of each year. Thus, we replace the first piece at | | | | | | |
| 35 | the beginning of year 6. | | | | | | |

That was easy. Next we proceed to develop an expression for depreciation in year 5, which, again, generalizes to year $t$. It will be helpful to look at Table 2.1 to understand the depreciation patterns.

**Depreciation Pattern in Table 2.1** The first piece of equipment cost $1,000 (B5) at the beginning of year 1. Its depreciation will be $200 per year in years 1 through 5, which appears in B6 through F6. Since we are assuming a 3% (B13) inflation-only growth rate in the price of equipment, the second piece of equipment cost $1,030 (C5). Depreciation on it is $206 per year, which you can see in row 7.[6] Depreciation on the third piece of equipment is $212.18 per year (row 8), and so forth.

Now, let's look down column F—year 5. Depreciation in year 5 is $200 (F6) on the equipment bought at the beginning of year 1, $206 (F7) on the equipment bought at the beginning of year 2, . . . , and $225.102 (F10) on the equipment bought at the beginning of year 5. Total depreciation expense is $1,061.827 (F11). Depreciation on the equipment bought at the beginning of year $t$ is $\$200(1+g)^{t-1}$. Now, we return back to the mathematics to develop an alternative POR formula.

Equation (2.12) is the depreciation expense for year 5:

$$D_5 = \$200\left[1+(1+g)+(1+g)^2+(1+g)^3+(1+g)^4\right]. \tag{2.12}$$

[6] Table 2.1 does not show depreciation expense after year 5, even though it does continue for the second through the fifth pieces of equipment.

Multiplying equation (2.12) by $(1 + g)$ on both sides, every term on the right-hand side of the equation increments by 1 in its exponent, and we get:

$$(1+g)D_5 = \$200\left[(1+g)+(1+g)^2+(1+g)^3+(1+g)^4+(1+g)^5\right]. \quad (2.13)$$

Subtracting (2.13) from (2.12), on the right-hand side, all the intermediate terms drop out, and we get:

$$\left[1-(1+g)\right]D_5 = \$200\left[1-(1+g)^5\right]. \quad (2.14)$$

This simplifies to:

$$-gD_5 = \$200\left[1-(1+g)^5\right]. \quad (2.15)$$

Multiplying through by $-(1/g)$, we get:

$$D_5 = \$200\left[\frac{(1+g)^5-1}{g}\right]. \quad (2.16)$$

Substituting equations (2.11) and (2.16) into (2.10), the percentage by which capital expenditures in year 6 exceeds depreciation in year 5 is:

$$\frac{C_6}{D_5}-1 = \frac{\$1{,}000\,(1+g)^5}{\$200\left[\frac{(1+g)^5-1}{g}\right]}-1. \quad (2.17)$$

This simplifies to:

$$\frac{C_6}{D_5}-1 = \frac{5g\,(1+g)^5}{(1+g)^5-1}-1. \quad (2.18)$$

We can generalize the formula for any equipment life. Letting $n =$ average years of equipment life, the general formula is:

$$\frac{C_{t+1}}{D_t}-1 = \frac{ng\,(1+g)^n}{(1+g)^n-1}-1. \quad (2.19)$$

**ANALYSIS OF TABLE 2.1** Table 2.1 shows the calculation of the difference by brute force, that is, the long way, and the short way using equation (2.18), which is the same as equation (2.19), with $n = 5$. Let's look first at the brute force method.

We transfer the purchase price of the equipment at the beginning of year 6 of \$1,159.274 from G5 to B15. Then we add the depreciation in year 5 coming from each individual piece of equipment, which is in F6 through F10, and totals \$1,061.827 in F11. We transfer that to B16. In B17, we divide B15 by B16; that is, we divide the cost of new equipment in year 6 by depreciation in year 5, to calculate the ratio of 1.092. Subtracting 1 from that, the difference between capital expenditures in year 6 and depreciation expense in year 5 is 9.2% (B18).

Now we can confirm the accuracy of equation (2.18), because we use it in B19, which also equals 9.2%—the same result as the brute force method. The advantage of the formula, though, is that we can perform sensitivity analysis and see how the difference varies as the growth rate in the price of equipment varies.

Rows 23 through 32 show that sensitivity analysis. We can see that the difference of capital expenditures and the previous year's depreciation expense is 3.0% (B23)

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table 2.2** | | | | | | | |
| 2 | **How Capital Expenditures Exceeds Depreciation [1]** | | | | | | | |
| 3 | | | | | | | | |
| 4 | **Avg Annual Growth in** | **Avg Equip Life (Yrs)** | | | | | | |
| 5 | **Equipment Prices [2]** | **3** | **5** | **7** | **10** | **15** | **20** | **25** |
| 6 | 1% | 2.0% | 3.0% | 4.0% | 5.6% | 8.2% | 10.8% | 13.5% |
| 7 | 2% | 4.0% | 6.1% | 8.2% | 11.3% | 16.7% | 22.3% | 28.1% |
| 8 | 3% | 6.1% | 9.2% | 12.4% | 17.2% | 25.6% | 34.4% | 43.6% |
| 9 | 4% | 8.1% | 12.3% | 16.6% | 23.3% | 34.9% | 47.2% | 60.0% |
| 10 | 5% | 10.2% | 15.5% | 21.0% | 29.5% | 44.5% | 60.5% | 77.4% |
| 11 | 6% | 12.2% | 18.7% | 25.4% | 35.9% | 54.4% | 74.4% | 95.6% |
| 12 | 7% | 14.3% | 21.9% | 29.9% | 42.4% | 64.7% | 88.8% | 114.5% |
| 13 | 8% | 16.4% | 25.2% | 34.5% | 49.0% | 75.2% | 103.7% | 134.2% |
| 14 | 9% | 18.5% | 28.5% | 39.1% | 55.8% | 86.1% | 119.1% | 154.5% |
| 15 | 10% | 20.6% | 31.9% | 43.8% | 62.7% | 97.2% | 134.9% | 175.4% |
| 16 | | | | | | | | |
| 17 | [1] $CE_{t+1} - Depr_t, = k \quad Depr_t$, and $k$ is the factor in the table above. The formula is from equation | | | | | | | |
| 18 | (2.19). | | | | | | | |
| 19 | | | | | | | | |
| 20 | [2] You should add in real growth in your business. For example, if equipment prices increase | | | | | | | |
| 21 | an average 5% per year and you expect your sales to increase at 6%, which is 3% real growth | | | | | | | |
| 22 | above expected inflation, you should use the annual growth of 5% + 3% = 8%, that is, row 13 in the | | | | | | | |
| 23 | above table. | | | | | | | |

for a 1% growth rate, 6.1% (B24) for a 2% growth rate, 9.2% (B25 = B19),[7] and generally grows 3.2% for each additional percentage in the growth rate.[8]

**TABLE 2.2: HOW CAPITAL EXPENDITURES EXCEEDS DEPRECIATION** Table 2.2 shows the results of the general formula in equation (2.19) for a variety of assumptions of average equipment life and annual growth in equipment prices. Note that the results in column C are identical with the sensitivity analysis in Table 2.1. Also note that the percentage by which capital expenditures in year $t + 1$ exceeds depreciation in year $t$ increases as we move southeast in the table (i.e., as average equipment life and annual growth increase).

**The Meaning of the Results** Let's take a minute to understand the meaning of the results in Table 2.2. Let's start with the assumption that most businesses have an average equipment life of five years, which is a reasonable assumption. Assuming for the moment that this is true, the difference for a 3% growth rate, which is inflationary only, is 9.2% (C8). This means that in a stagnant business, we can forecast the difference between capital expenditures and depreciation expense as being 9.2% × depreciation expense. This result was a surprise to me! I always thought that a stagnant business would have capital expenditures exceeding depreciation only by inflation itself, or 3%. However, there is no substitute for rigorous analysis.

It is reasonable to expect that many businesses face real growth in their prices, not just inflation only. Thus, 5% to 7% growth in equipment prices is fairly common. At 5% annual price growth, the difference of capital expenditures and depreciation expense for an average five-year equipment life is 15.5% (C10), whereas at 7% it is 21.9% (C12). Therefore, the differences in the two can be substantial.

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[7] This equality shows the accuracy of the sensitivity analysis and is why row 25 is in bold.

[8] The difference begins to accelerate at higher growth rates. Thus, the difference is 3.3% for $g = 8\%$ and 9% and 3.4% for $g = 10\%$.

The differences are even more pronounced for longer-lived equipment. For an average seven-year equipment life, the differences are higher—and all the more so the higher is the growth rate in equipment prices. A 3% inflationary-only price growth implies a 12.4% (D8) difference, while 5% and 7% annual price increases imply differences of 21.0% (D10) and 29.9% (D12).

Some manufacturing firms may have heavy equipment with very long lives—perhaps much longer than seven years. Therefore, it is necessary to adjust the analysis to the realities of the subject company.

**HANDLING THE CURVEBALLS** There are a few curveballs that can arise in estimating the excess of capital expenditures over depreciation. The first one is the existence of fully depreciated assets, which arises when depreciable life is less than the economic life of the asset. For example, suppose your client has a large piece of equipment that cost \$1 million, which has a 10-year life, and he or she depreciated it over 5 years. In years 6 through 10, depreciation expense will be zero. We are doing our valuation as of the beginning of year 11. In this case, equation (2.19) will underestimate capital expenditures, because it will totally miss the replacement of this expensive machine. Assuming a 5% annual growth in equipment costs, we would be underestimating capital expenditures by \$1.6 million in year 11. For very expensive, long-lived equipment, it may be necessary to consider its cash flow separately from the ordinary cash flows of the business, and add its effect into the valuation separately.

The second curveball is more apparent than real. It occurs when the client uses accelerated depreciation. This causes depreciation to be higher in the earlier years and lower in the later years than straight-line depreciation.

**Table 2.3: Analysis of MACRS versus Straight-line Depreciation** For example, let's analyze Table 2.3, which shows five-year MACRS and straight-line depreciation for the same assets that appear in Table 2.1, row 5. In year 1, we buy the first piece of equipment for \$1,000 (B5). Straight-line depreciation is \$200 per year (row 8). Five-year MACRS depreciation is 150% declining balance, with a switch to straight-line in year 3, when straight-line is higher than declining balance. Year 1 MACRS is 150% × 20%[9] = 30% of the tax basis of the asset, or 30% × \$1,000 = \$300 (B6).

We subtract that from the \$1,000 purchase price, which leaves a depreciable basis of \$700 (B7) at the end of year 1. In year 1, MACRS depreciation is \$300/\$200 = 150% (B9) of straight-line. In year 2, depreciation is 30% × \$700 (the depreciable basis in B7) = \$210 (C6). The depreciable basis at the end of the year is \$700 − \$210 = 490 (B7 − C6 = C7). The 150% declining balance in year 3 would be 30% × \$490 = \$147; however, from this point on, straight-line depreciation at \$490/3 = \$163.33 (D7–F7) is higher, and we use that.

Now, let's proceed to the equipment bought in year 2. It costs \$1,030 (C5). Five-year straight-line depreciation is \$206 (row 13) per year. MACRS depreciation in year 2 for the year 2–purchased equipment is 30% × \$1,030 = 309 (C11). The depreciable basis at the end of the year is \$1,030 − \$309 = \$721 (C5 − C11 = C12). MACRS depreciation in year 3 will be 30% × \$721 = \$216.3 (D11). After that, we use straight-line depreciation for years 4 through 6 at \$168.2333 (E11, F11). (Note, we

[9]Straight-line depreciation is 20% per year for five years, so 150% DB is always 30% for five-year equipment.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 2.3** | | | | | | |
| 2 | **Analysis of Depreciation and Capital Expenditures** | | | | | | |
| 3 | | | | | | | |
| 4 | | 1 | 2 | 3 | 4 | 5 | **Total** |
| 5 | Purchase Price of Equipment | 1000 | 1030 | 1060.9 | 1092.727 | 1125.5088 | |
| 6 | MACRS Depreciation—Equipment Bought Year 1 | 300 | 210 | 163.3333 | 163.33333 | 163.33333 | **1000** |
| 7 | Depreciable Basis—End of Year | 700 | 490 | 163.3333 | 163.33333 | 163.33333 | |
| 8 | S-L Depr.—Equipment Bought Year 1 | 200 | 200 | 200 | 200 | 200 | **1000** |
| 9 | MACRS Depreciation/Straight-Line | **150%** | **NM** | **NM** | **NM** | **NM** | |
| 10 | | | | | | | |
| 11 | MACRS Depreciation—Equipment Bought Year 2 | | 309 | 216.3 | 168.23333 | 168.23333 | |
| 12 | Depreciable Basis—End of Year | | 721 | 504.7 | 336.46667 | 168.23333 | |
| 13 | S-L Depreciation of Equip. Bought Year 2 | | 206 | 206 | 206 | 206.000 | |
| 14 | Total MACRS Depreciation—Equipment Bought Years 1 and 2 | 300 | 519 | 379.633 | 331.56667 | 331.56667 | |
| 15 | Total S-L Depreciation—Equipment Bought Years 1 and 2 | 200 | 406 | 406 | 406 | 406 | |
| 16 | MACRS Depreciation/Straight-Line | **150%** | **128%** | **NM** | **NM** | **NM** | |
| 17 | | | | | | | |
| 18 | MACRS Depreciation—Equipment Bought Year 3 | | | 318.27 | 222.789 | 173.28033 | |
| 19 | Depreciable Basis—End of Year | | | 742.63 | 519.841 | 346.56067 | |
| 20 | S-L Depreciation of Equipment Bought Yr 3 | | | 212.18 | 212.18 | 212.180 | |
| 21 | Total MACRS Depreciation—Equipment Bought Years 1–3 | 300 | 519 | 697.903 | 554.35567 | 504.847 | |
| 22 | Total S-L Depreciation—Equipment Bought Years 1–3 | 200 | 406 | 618.18 | 618.18 | 618.18 | |
| 23 | MACRS Depreciation/Straight-Line | **150%** | **128%** | **113%** | **NM** | **NM** | |
| 24 | | | | | | | |
| 25 | MACRS Depreciation—Equipment Bought Year 4 | | | | 327.8181 | 229.47267 | |
| 26 | Depreciable Basis—End of Year | | | | 764.9089 | 535.43623 | |
| 27 | S-L Depreciation of Equipment Bought Yr 4 | | | | 218.5454 | 218.545 | |
| 28 | Total MACRS Depreciation—Equipment Bought Years 1–4 | 300 | 519 | 697.903 | 882.17377 | 734.31967 | |
| 29 | Total S-L Depreciation—Equipment Bought Years 1–4 | 200 | 406 | 618.18 | 836.7254 | 836.7254 | |
| 30 | MACRS Depreciation/Straight-Line | **150%** | **128%** | **113%** | **105%** | **NM** | |
| 31 | | | | | | | |
| 32 | MACRS Depreciation—Equipment Bought Year 5 | | | | | 337.65264 | |
| 33 | Depreciable Basis—End of Year | | | | | 787.85617 | |
| 34 | S-L Depreciation of Equipment Bought Yr 5 | | | | | 225.102 | |
| 35 | Total MACRS Depreciation—Equipment Bought Years 1–4 | 300 | 519 | 697.903 | 882.17377 | 1071.9723 | |
| 36 | Total S-L Depreciation—Equipment Bought Years 1–4 | 200 | 406 | 618.18 | 836.7254 | 1061.8272 | |
| 37 | MACRS Depreciation/Straight-Line | **150%** | **128%** | **113%** | **105%** | **101%** | |
| 38 | | | | | | | |
| 39 | Growth Rate—Price of Equipment = *g* | 3% | | | | | |

stop in this analysis at year 5, even though depreciation on the equipment bought in year 2 goes on to year 6.)

We subtotal straight-line depreciation in row 13 for equipment bought in years 1 and 2, and we do the same for MACRS depreciation in row 14. MACRS depreciation in year 2 is \$519 (C6 + C11 = C14), and straight-line depreciation is \$406 (C8 + C13 = C15). Thus, whereas MACRS depreciation is 150% (B9, B16) of straight-line in year 1, it is only 128% (C16) in year 2.

The analysis rolls forward in the same fashion for years 3 through 5. The final result in year 5 is that MACRS depreciation is only 1% higher than straight-line, that is, 101% (F37) of it. Thus, equation (2.19) normally should do a good job of forecasting depreciation when the firm is either stagnant or growing slowly in real terms; that is, it has reached a reasonable steady-state in its base of fixed assets.

The third curveball, which also is more apparent than real, is the effect of the policy of taking a half-year depreciation in the year of purchase and one-half year in the year of sale or retirement. The effects of this policy should average out over the long run to be the same as taking depreciation according to the month of placement in service, although it can distort the calculation for a particular year for an expensive piece of equipment. In such cases, you might have to make an adjustment to correct the distortion. Once the company has reached steady state—in this example, year 6 and on—that should not be a material issue.

## The Mathematics of the Increase in Required Net Working Capital and LT Debt

Now let's turn to the increase in required net working capital (NWC) and long-term debt (LTD). Let's make some simplifying assumptions:

- Sales grows at a constant rate, $g_s$.
- NWC and LTD grow as a constant percentage of sales.

The formula for the increase in NWC is:

$$\Delta NWC = NWC_1 - NWC_0, \tag{2.20}$$

where $NWC_0$ is last year's net working capital and $NWC_1$ is the first forecast year. However, $NWC$ grows at the rate $g_s$. Therefore, we can substitute that into (2.20), which results in:

$$\Delta NWC = [NWC_0(1 + g_s) - NWC_0] = NWC_0[(1 + g_s) - 1]. \tag{2.21}$$

This expression simplifies to:

$$\Delta NWC = NWC_0 \times g_s. \tag{2.22}$$

The mathematics of the change in long-term debt is identical to that of net working capital, although its effect on cash flow is the opposite. While an increase in net working capital is a use of cash, an increase in long-term debt is a source of cash. Thus, the only difference is that the sign in the payout ratio formula for $\Delta LTD$ is the opposite of the one for $\Delta NWC$. The formula for the change in long-term debt is in equation (2.23):

$$\Delta LTD = LTD_0 \times g_s. \tag{2.23}$$

## The Estimation Formula for the Payout Ratio

Substituting equations (2.19), (2.22), and (2.23) into (2.8), we get:

$$POR = 1 - \frac{\left[\dfrac{ng\,(1+g)^n}{(1+g)^n - 1} - 1\right] Depr_0 + [NWC_0 - LTD_0]\,g_s}{NI_1}. \tag{2.24}$$

Note that depreciation, net working capital, and long-term debt are historical amounts, with appropriate adjustments, as discussed earlier, while net income is a normalized amount. This means that if you forecast net income to be unusually high or low next year, because of a specific item that is a one-time event, it is best to calculate the payout ratio as if that item did not exist, value the firm accordingly, and then make an adjustment to the valuation at the end of the process. Otherwise, a one-year anomaly becomes forever enshrined in the valuation, causing a valuation error. Also note that net income must be positive and material in amount for this formula to work.

Assuming a reasonable 5% annual growth in equipment costs and sales and a five-year life, this simplifies to:

$$POR = 1 - \frac{(15.5\% \times Depr_0) + (NWC_0 - LTD_0) \times 5\%}{NI_1}, \tag{2.25}$$

where the 15.5% comes from Table 2.2, C10. This is a much easier calculation than equation (2.8), as it is not necessary to do the detailed forecast of capital expenditures, depreciation, and net working capital.

Let's do an example. If depreciation last year was \$50,000, required net working capital was \$250,000, long-term debt was \$50,000, and net income is \$100,000, then our estimate of the payout ratio would be:

$$\begin{aligned} POR &= 1 - \frac{(15.5\% \times 50{,}000) + \big(250{,}000 - \$50{,}000\big) \times 5\%}{\$100{,}000} \\ &= 1 - \frac{7{,}750 + 10{,}000}{\$100{,}000} = 82.25\%. \end{aligned} \tag{2.26}$$

Equation (2.26) has several very specific assumptions behind it, so it is important to modify the formula if there are any of the following four significant deviations in your fact pattern:

1. Average equipment life is not 5 years.
2. The growth rate in equipment prices (combined with real growth in the subject company) or in sales significantly differs from 5%.
3. You do not expect sales to grow at a constant rate.
4. You do not expect net working capital or long-term debt to grow as a constant percentage of sales.

Even when the immediate facts differ from these assumptions, it is still quite possible that equations (2.24) through (2.26) may be a reasonable long-term estimate. Actual cash flow frequently rises and falls in extremes from one year to the next. Therefore, historical cash flow often is not a viable basis from which to forecast a future payout ratio. If we view equations (2.24) through (2.26) as norms, they become more reasonable. While actual cash flows may vary considerably year-to-year from the average, it is reasonable to forecast the average payout ratio—unless you are able to be more accurate and forecast exact cash flows year-by-year, which is equivalent to varying the payout ratio annually according to your more specific forecast.

## Forecasting Gross Cash Flow Is Incorrect

Lerch (2001) argues for capitalizing gross cash flow. Clearly, there is a problem with that. In light of equation (2.19) and Table 2.2 in this chapter, we can see that the author's assumption (on p. 33) that depreciation equals capital expenditures is unrealistic even for a stagnant firm. Such an assumption is appropriate only for a firm in severe decline.

Imagine a firm with zero net cash flow. Such a firm would never generate any cash to pay its shareholders dividends. It is logical that this firm should have a zero fair market value—at least on an income approach. Yet capitalizing or discounting gross cash flow (or net income, for that matter) would lead to a positive valuation. Thus, net cash flow is the appropriate measure of economic earning power to capitalize or discount.

## Conclusion

In this chapter, we have developed an exact expression for the payout ratio in equation (2.8) and a good approximation formula in equation (2.24), the latter of which should be much easier to use in forecasting cash flows. This should not only save time, but increase valuation accuracy by breaking the bad habit of discounting net income (or other similar measures of economic earning power). Also, we have covered why net cash flow is the appropriate measure of economic earning power for capitalization or discounting.

## References

Abrams, Jay B. 1994. Cash Flow: A Mathematical Derivation. *Valuation* (March 1994): 64–71.

Abrams, Jay B. 2003. Cash Flow: Mathematics of the Payout Ratio. *Business Valuation Review* (June 2003): 64–76.

Gilbert, Gregory A. 1990. Discount Rates and Capitalization Rates—Where Are We? *Business Valuation Review* (December 1990): 108–113.

Ibbotson, Roger G., and Peng Chen. 2002. The Supply of Stock Market Returns. Yale ICF Working Paper No. 00-44, page 7 and Figure 4. Download at http://papers.ssrn.com/abstract=274150.

Lerch, Mary Ann. 2001. Are We Capitalizing the Right Measure of Cash Flow?, *Business Valuation Review* (September 2001): 32–34.

Moskowitz, Tobias J., and Annette Vissing-Jorgensen. 2002. The Private Equity Premium Puzzle. *American Economic Review* (September), Vol. 92, No. 4. See especially page 755, second column.

# CHAPTER 3

# Using Regression Analysis

CHAPTER 3

# Using Regression Analysis

## Introduction

*Regression analysis* is a statistical technique that estimates the mathematical relationship between causal variables, known as *independent variables*, and a dependent variable. The most common uses of regression analysis in business valuation are:

- Forecasting sales in a discounted cash flow analysis.
- Forecasting costs and expenses in a discounted cash flow analysis.
- Measuring the relationship between market capitalization (fair market value) as the dependent variable and several possible independent variables for a publicly traded guideline company valuation approach. Typical independent variables that are candidates to affect the fair market value are net income (including nonlinear transformations such as its square, square root, and logarithm), book value, the debt-to-equity ratio, and so on.

While we review some highlights of statistical theory, we are primarily focused on how to apply regression analysis to real-life appraisal assignments using standard spreadsheet regression tools. We have not attempted to provide a rigorous, exhaustive treatment on statistics and have put as much of the technical background discussion as possible into the end-of-chapter appendix to keep the body of the chapter as simple as possible. Those who want a comprehensive refresher should consult a statistics text, such as Bhattacharyya and Johnson (1977) and Wonnacott and Wonnacott (1981). We present only bits and pieces of statistics that are necessary to facilitate our discussion of the important practical issues.

To preserve readability we avoid advanced issues, as they are likely to be beyond the training of most professional business appraisers. Our focus is on the practical application of regression in business valuation, not statistics as an extreme science or an art form.

Even though you may not be familiar with the use of regression analysis, let alone nonlinear transformations of the data, the material in this chapter is not that difficult and can be very useful in your day-to-day valuation practice. We will explain all the basics you need to use this very important tool on a daily basis and will lead you step-by-step through an example, so you can use this chapter as a guide to get hands-on experience.

For those who are unfamiliar with the mechanical procedures to perform regression analysis using spreadsheets, we explain that step-by-step in the section on using regression to forecast sales.

## Forecasting Costs and Expenses

In performing a discounted cash flow analysis, an analyst forecasts sales, expenses, and changes in balance sheet accounts that affect cash flows. Frequently, analysts base their forecasts of future costs on historical averages of, or trends in, the ratio of costs as a percentage of sales.

One significant weakness of this methodology is that it ignores fixed costs, leading to undervaluation in good times and possible overvaluation in bad times. If the analyst treats all costs as variable, in good times when he or she forecasts rapid sales growth, the fixed costs should stay constant (or possibly increase with inflation, depending on the nature of the costs), but the analyst will forecast those fixed costs to rise in proportion to sales. That leads to forecasting expenses too high and income too low, which ultimately causes an undervaluation of the firm. In bad times, if one forecasts sales to be flat, then costs will be accidentally forecast correctly. If one expects sales to decline, then treating all costs as variable will lead to forecasting expenses too low and net income too high, resulting in an overvaluation.

*Ordinary least squares* (OLS) regression analysis is an excellent tool to forecast adjusted costs and expenses (which, for simplicity, we will call *adjusted costs* or *costs*) based on their historical relationship to sales. OLS produces a statistical estimate of both fixed and variable costs, which is useful in planning as well as in forecasting. Furthermore, the regression statistics produce feedback used to judge the robustness and reliability of the relationship between sales and costs.

### Adjustments to Expenses

Prior to performing regression analysis, we should analyze historical income statements to ascertain whether various expenses have maintained a consistent pattern or whether there has been a shift in the structure of a particular expense. When past data are not likely to be representative of future expectations, we make *pro forma* adjustments to historical results to model how the company would have looked if its operations in the past had conformed to the way we expect them to behave in the future. The purpose of these adjustments is to examine longstanding financial trends without the interference of obsolete information from the past.

For example, if the cost of advertising was 8% of sales for the first two years of our historical analysis, decreased to 5% for the next five years, and is expected to remain at 5% in the future, we may add back the excess 3% to net income in the first two years to reflect our future expectations. We may make similar adjustments to other expenses that have changed during the historical period or that we expect to change in the future to arrive at adjusted net income. Of course, we would have to tax affect these adjustments to calculate adjusted net income after taxes. It is also possible that it might be necessary to make incremental cash flow adjustments.

### Table 3.1A: Calculating Adjusted Costs and Expenses

Table 3.1A shows summary income statements for the years 1998 to 2007. Adjustments to pretax net income appear in rows 15 through 20. The first adjustment,

# Table 3.1A

## Adjustments to Historical Costs and Expenses

### Summary Income Statements

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 6 | | **1998** | **1999** | **2000** | **2001** | **2002** | **2003** | **2004** | **2005** | **2006** | **2007** |
| 7 | **Sales** | $250,000 | $500,000 | $750,000 | $1,000,000 | $1,060,000 | $1,123,600 | $1,191,016 | $1,262,477 | $1,338,226 | $1,415,000 |
| 8 | **Cost of Sales** | 100,000 | 250,000 | 375,000 | 500,000 | 490,000 | 505,000 | 520,000 | 535,000 | 550,000 | 600,000 |
| 9 | **S, G, and A Expenses** | 100,000 | 150,000 | 250,000 | 335,000 | 335,000 | 360,000 | 370,000 | 405,000 | 435,000 | 450,000 |
| 10 | **Operating Expenses** | 58,000 | 68,000 | 78,000 | 88,000 | 83,000 | 110,000 | 112,000 | 117,000 | 122,000 | 132,000 |
| 11 | **Other Expense** | 5,000 | 15,000 | 20,000 | 25,000 | 20,000 | 43,000 | 100,000 | 50,000 | 50,000 | 50,000 |
| 12 | **Pretax Income** | **-$13,000** | **$17,000** | **$27,000** | **$52,000** | **$132,000** | **$105,600** | **$89,016** | **$155,477** | **$181,226** | **$183,000** |
| 13 | **Pretax Profit Margin** | -5.20% | 3.40% | 3.60% | 5.20% | 12.45% | 9.40% | 7.47% | 12.32% | 13.54% | 12.93% |
| 14 | **Adjustments:** | | | | | | | | | | |
| 15 | **+ Actual Salary** | 75,000 | 80,000 | 85,000 | 130,000 | 100,000 | 100,000 | 105,000 | 107,000 | 109,000 | 111,000 |
| 16 | **+ Bonus** | 3,000 | 4,000 | 4,000 | 20,000 | 5,000 | 5,000 | 5,000 | 7,000 | 9,000 | 10,000 |
| 17 | **+ Pension** | 1,000 | 1,000 | 1,500 | 2,000 | 2,000 | 2,000 | 2,000 | 2,000 | 2,000 | 2,000 |
| 18 | **– Arm's-Length Salary [1]** | (58,015) | (60,916) | (63,961) | (67,159) | (70,517) | (74,043) | (77,745) | (81,633) | (85,714) | (90,000) |
| 19 | **Discontinued Operations [2]** | | | | | | | 55,000 | | | |
| 20 | **Moving Expense [3]** | | | | | | 20,000 | | | | |
| 21 | **Adjusted Pretax Income** | **$7,985** | **$41,084** | **$53,539** | **$136,841** | **$168,483** | **$158,557** | **$178,271** | **$189,844** | **$215,511** | **$216,000** |
| 22 | **Adjusted Pretax Profit Margin** | **3.19%** | **8.22%** | **7.14%** | **13.68%** | **15.89%** | **14.11%** | **14.97%** | **15.04%** | **16.10%** | **15.27%** |
| 23 | | | | | | | | | | | |
| 24 | **Calculation of Adjusted Costs and Expenses** | | | | | | | | | | |
| 25 | Sales | $250,000 | $500,000 | $750,000 | $1,000,000 | $1,060,000 | $1,123,600 | $1,191,016 | $1,262,477 | $1,338,226 | $1,415,000 |
| 26 | Adjusted Pretax Net Income | $7,985 | $41,084 | $53,539 | $136,841 | $168,483 | $158,557 | $178,271 | $189,844 | $215,511 | $216,000 |
| 27 | **Adjusted Costs and Expenses** | **$242,015** | **$458,916** | **$696,461** | **$863,159** | **$891,517** | **$965,043** | **$1,012,745** | **$1,072,633** | **$1,122,714** | **$1,199,000** |

[1] Arm's-length salary includes bonus and pension.

[2] A write-off for discontinued operations was an unusual one-time expense already included in other expense. We reverse it here.

[3] Moving expense is a periodic expense that occurs approximately every 10 years. For the 2003 move, we add back the $20,000 cost to pretax income and use a Periodic Perpetuity Factor to calculate an adjustment to FMV, which we apply later in the valuation process (see Chapter 4).

which appears in rows 15 and 18, converts actual salary paid—along with bonuses and pension payments—to an arm's-length salary.

The second type of adjustment is for a one-time event that is unlikely to repeat in the future. In our example, the company wrote off $55,000 for a discontinued operation in 2004. As such, we add back the write-off (H19) to pretax income, because it is not expected to recur in the future.

The third type of adjustment is for a periodic expense. We use a company move as an example, since we expect a move to occur about every 10 years.[1] In our example, the company moved in 2003, four years ago. We add back the $20,000 cost of the move in the adjustment section (G20) and treat the cost separately as a periodic perpetuity.

In Chapter 4, we develop two periodic perpetuity factors (PPF)[2] for periodic cash flows occurring every $j$ years growing at a constant rate of $g$, discounted to present value at the rate $r$, with the last cash flow having occurred $b$ years ago. Those formulas are:

$$PPF = \frac{1}{(1+r)^j - (1+g)^j} \times (1+r)^b \quad \text{PPF—end-of-year;} \tag{3.1a}$$

$$PPF = \frac{\sqrt{1+r}}{(1+r)^j - (1+g)^j} \times (1+r)^b \quad \text{PPF—midyear.} \tag{3.1b}$$

The next forecast cash flow will be the prior cash flow $\times(1+g)^j$. We assume the move occurs at the end of the year and use equation (3.1a), the end-of-year PPF. We also assume a discount rate of $r = 20\%$, moves occur every $j = 10$ years, the last move occurred $b = 4$ years ago, and the cost of moving grows at $g = 5\%$ per year. The cost of the next move, which is forecast at the end of year 6, is $\$20{,}000 \times (1 + 5\%)^{10} = \$20{,}000 \times 1.62889 = \$32{,}578$. We multiply this by the PPF, which is: $PPF = \frac{1.2^4}{1.2^{10} - 1.05^{10}} = 0.45445$ (see Chapter 4, Table 4.10, A20), which results in a present value of $14,805.

Assuming a 40% tax rate, the after-tax present value of moving costs is $\$14{,}805 \times (1 - 40\%) = \$8{,}883$. Since this is an expense, we must remember to subtract it from—not add it to—the FMV of the firm before moving expenses. For example, if we calculate a marketable minority interest FMV of $1,008,883 before moving expenses, then the marketable minority FMV would be $1 million after moving expenses. In this example, we are not adjusting from net income to cash flow, which is probably reasonable. If the period income or expense would have an impact on the balance sheet and/or capital expenditures, then we should also include that impact on the calculation if it is material.

The other possible treatment for the periodic expense, which is slightly less accurate but avoids the PPF, is to allocate the periodic expense over the applicable

[1] Losses from litigation are another type of expense that often has a periodic pattern.

[2] This is my invention to calculate the present value of a periodic cash flow that runs in perpetuity. As we mention in Chapter 4, it is a generalized Gordon model for a periodic cash flow. When sales occur every year, $j = 1$ and the left-hand terms in equations (3.1a) and (3.1b) and formulas (4.18a) and (4.19a) simplify to the familiar Gordon model multiples. The right-hand term adjusts the present value to account for the cash flow occurring $b$ years earlier than year $j$.

years—10 in this example. The appraiser who chooses this method must allocate expenses from the prior move to the years before 2003. This approach causes the regression $R^2$ to be artificially high, as the appraiser has artificially created what appears to be a perfect fixed cost. For example, suppose we allocated \$2,000 per-year moving costs to the years 2003 through 2008. If we run a regression of costs and expenses as a function of sales on those years only, $R^2$ will be overstated, as the perfect fixed cost of \$2,000 per year is merely an allocation, not the real cash flow. This approach exaggerates other regression measures. If the allocated numbers are small, however, the overstatement is also likely to be small.

Adjusted pretax income appears in row 21. Note that as a result of these adjustments, the adjusted pretax profit margin in row 22 is substantially higher than the unadjusted pretax margin in row 13.

We repeat sales (row 7) in row 25 and adjusted pretax income (row 21) in row 26. Subtracting row 26 from row 25, we arrive at adjusted costs and expenses in row 27. We use these adjusted costs and expenses in forecasting future costs and expenses using regression analysis.

## Performing Regression Analysis

Ordinary least squares regression analysis measures the linear relationship between a dependent variable and an independent variable. Its mathematical form is $y = \alpha + \beta x$, where:

$y =$ the dependent variable (in this case, adjusted costs).
$x =$ the independent variable (in this case, sales).
$\alpha =$ the true (and unobservable) $y$-intercept value, that is, fixed costs.
$\beta =$ the true (and unobservable) slope of the line, that is, variable costs.

Both $\alpha$ and $\beta$, the true fixed and variable costs of the company, are unobservable. In performing the regression, we are estimating $\alpha$ and $\beta$ from our historical analysis, and we will call our estimates:

$a =$ the estimated $y$-intercept value (estimated fixed costs).
$b =$ the estimated slope of the line (estimated variable costs).[3]

OLS estimates fixed and variable costs (the $y$-intercept and slope) by calculating the best-fit line through the data points.[4] In our case, the dependent variable ($y$) is

[3]The regression parameters $a$ and $b$ are often shown in statistical literature as $\alpha$ and $\beta$ with a circumflex (ˆ) over each letter.

[4]The interested reader should consult a statistics text for the multivariate calculus involved in calculating $a$ and $b$. Mathematically, OLS calculates the line that minimizes $S =$ the sum of the squared deviations between the actual data points, $Y_i$, and the regression estimates, $\hat{Y}_i$. (Note that $S$ in this expression is not the same as the $S$ we use later as the standard error of the $y$-estimate.) This expression becomes $S = \sum (Y_i - a - bx_i)^2$. One computes $a$ and $b$ in single-variable OLS by taking the partial derivatives of $S$ with respect to $a$ and $b$, setting those expressions to zero, and solving.

adjusted costs, and the independent variable ($x$) is sales. Sales, which is in Table 3.1A, row 7, appears in Table 3.1B as B6 to B15. Adjusted costs and expenses, Table 3.1A, row 27, appears in Table 3.1B as C6 to C15. Table 3.1B shows the regression analysis of these variables using all 10 years of data. The resulting regression yields an intercept value of \$56,770 (B33) and a (rounded) slope coefficient of 0.80 (B34). Using these results, the equation of the line becomes:

$$\text{Adjusted Costs and Expenses} = \$56{,}770 + (0.80 \times \text{Sales}).$$

The $y$-intercept, \$56,770, represents the regression's estimate of fixed costs, which is the cost of operating the business at a zero sales volume. The slope coefficient, 0.80, is the regression's estimate of variable cost per dollar of sales. This means that for every dollar of sales, there are directly related costs and expenses of \$0.80. We show this relationship graphically in Figure 3.1. The diamonds are actual data points, and the line passing through them is the regression estimate. Note how close all of the data points are to the regression line, which indicates there is a strong relationship between sales and costs.[5]

We can use this regression equation to calculate future costs once we generate a future sales forecast. Of course, to be useful, the regression equation should make common sense. For example, a negative $y$-intercept in this context would imply negative fixed costs, which makes no sense whatsoever (although in regressions involving other variables it may well make sense). Normally one should not use a result like that, despite otherwise impressive regression statistics. Instead it probably makes more sense to assume zero fixed costs, which means all costs are variable. This is the same as using the ratio of total costs to sales to forecast costs.

If the regression forecasts variable costs above \$1.00, one should be suspicious. If true, either the company must anticipate a significant decrease in its cost structure in the near future—which would invalidate applicability of the regression analysis to the future—or the company soon will be out of business. The analyst should also consider the possibility that the regression failed, perhaps because of either insufficient or incorrect data, and it may be unwise to use the results in the valuation.

## Use of Regression Statistics to Test the Robustness of the Relationship

Having determined the equation of the line, we use regression statistics to determine the strength of the relationship between the dependent and independent variable(s). We give only a brief verbal description of regression statistics here. For a more in-depth explanation, the reader should refer to a statistics book.

In an OLS regression, the *goodness of fit* of the line is measured by the degree of correlation between the dependent and independent variable, referred to as the $r$ value.[6] An $r$ value of 1 indicates a perfect direct relationship, where the independent variable explains all of the variation of the dependent variable. A value of $-1$ indicates a perfect inverse relationship. Most $r$ values fall between 1 and $-1$, but the closer to 1 (or $-1$), the better the relationship. An $r$ value of zero indicates no relationship between the variables.

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[5]We will discuss the second part of Table 3.1B later in the chapter.

[6]In statistics literature, the $r$ may be either uppercase or lowercase.

## Table 3.1B
## Regression Analysis 1998–2007

| | Actual | |
|---|---|---|
| **Year** | **Sales = *X* [1]** | **Adj. Costs = *Y* [2]** |
| 1998 | $250,000 | $242,015 |
| 1999 | $500,000 | $458,916 |
| 2000 | $750,000 | $696,461 |
| 2001 | $1,000,000 | $863,159 |
| 2002 | $1,060,000 | $891,517 |
| 2003 | $1,123,600 | $965,043 |
| 2004 | $1,191,016 | $1,012,745 |
| 2005 | $1,262,477 | $1,072,633 |
| 2006 | $1,338,226 | $1,122,714 |
| 2007 | $1,415,000 | $1,199,000 |

**SUMMARY OUTPUT**

| *Regression Statistics* | |
|---|---|
| Multiple R | 99.88% |
| R Square | 99.75% |
| Adjusted R Square | 99.72% |
| Standard Error | 16,014 |
| Observations | 10 |

**ANOVA**

| | *df* | *SS* | *MS* | *F* | *Signif F* |
|---|---|---|---|---|---|
| Regression | 1 | 8.31E+11 | 8.31E+11 | 3.24E+03 | 1.00E-11 |
| Residual | 8 | 2.05E+09 | 2.56E+08 | | |
| Total | 9 | 8.33E+11 | | | |

| | *Coef* | *Std Err* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
|---|---|---|---|---|---|---|
| Intercept [3] | 56,770 | 14,863 | 3.82 | 0.005 | 22,496 | 91,045 |
| Sales [4] | 0.8045 | 0.014 | 56.94 | 0.000 | 0.772 | 0.837 |

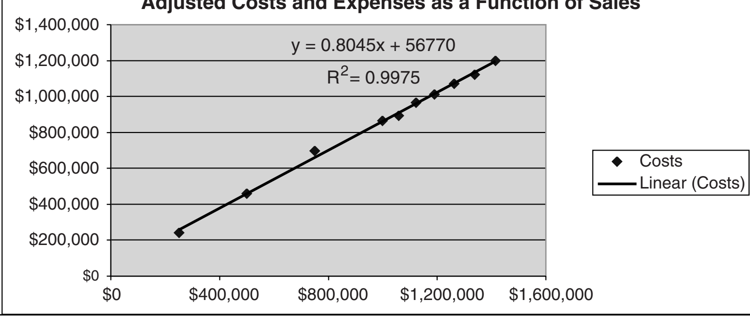
[1] From Table 3.1A row 7.

[2] From Table 3.1A row 27.

[3] Regression estimate of fixed costs.

[4] Regression estimate of variable costs.

**Figure 3.1**
**Adjusted Costs and Expenses as a Function of Sales**



In a multivariable regression equation, the multiple $R$ measures how well the dependent variable is correlated to all of the independent variables in the regression equation. Multiple $R$ measures the total amount of variation in the dependent variable that is explained by the independent variables. In our case, the value of 99.88% (B20) is very close to 1, indicating that almost all of the variation in adjusted costs is explained by sales.[7]

The square of the single or multiple $R$ value, referred to as $R$-square, $R$-squared, or $R^2$, measures the percentage of the variation in the dependent variable explained by the independent variable. It is the main measure of the goodness of fit. We obtain an $R^2$ of 99.75% (B21), which means that sales explain 99.75% of the variation in adjusted costs.

Adding more independent variables to the regression equation usually adds to $R^2$, even when there is no true causality. In statistics, this is called *spurious correlation*. The adjusted $R^2$, which is 99.72% (B22) in our example, removes the expected spurious correlation in the "gross" $R^2$:

$$Adj\,R^2 = \left(R^2 - \frac{k}{n-1}\right)\left(\frac{n-1}{n-k-1}\right),$$

where $n$ is the number of observations and $k$ is the number of independent variables (also known as *regressors*).

Although the data in Table 3.1A are fictitious, in practice I have found that regressions of adjusted costs versus sales usually give rise to $R^2$ values of 90% or more.[8]

## Standard Error of the *y*-Estimate

The standard error of the $y$-estimate is another important regression statistic that gives us information about the reliability of the regression estimate. Its formula appears later in the chapter as equation (3.7a). We can multiply the standard error of \$16,014 (B23) by 2 to calculate the 95% confidence interval for the regression estimate. Thus, we are 95% sure that the true adjusted costs are within ± \$32,028 of the regression estimate of total adjusted costs for the firms in the sample.[9] Dividing \$32,000 (rounded) by the mean of adjusted costs (approximately \$1 million) leads to a 95% confidence interval that varies by about ± 3%, or a 6% total range. Later in the chapter, we will calculate precise confidence intervals.

## The Mean of *a* and *b*

Because $a$ and $b$ are specific numbers that we calculate in a regression analysis, it is easy to lose sight of the fact that they are not simply numbers, but rather random variables. Remember that we are trying to estimate $\alpha$ and $\beta$, the true fixed and variable costs, which we will never know. If we had 20 years of financial history

[7]Although the spreadsheet labels this statistic Multiple $R$, because our example is an OLS regression, it is simply $R$.

[8]This obviously does not apply to start-ups.

[9]This is true at the sample mean of $X$, and the confidence interval widens with the distance from the mean.

for our subject company, we could take any number of combinations of years for our regression analysis. Suppose we had data for 1988–2007. We could use only the last five years, 2003–2007, or choose 2002–2005 and 2007, still keeping five years of data, but excluding 2006—although there is no good reason to do so. We could use 5, 6, 7, or more years of data. There are a large number of different samples we can draw out of 20 years of data. Each different sample would lead to a different calculation of $a$ and $b$ in our attempt to estimate $\alpha$ and $\beta$, which is one reason why $a$ and $b$ are random variables.[10] Of course, we will never be exactly correct in our estimate, and even if we were, there would be no way to know it!

Equations (3.2a) and (3.2b) state that $a$ and $b$ are unbiased estimators of $\alpha$ and $\beta$, which means that their expected values equal $\alpha$ and $\beta$. The capital $E$ is the expected value operator:

$$E(a) = \alpha \quad \text{The mean of } a \text{ is alpha.} \tag{3.2a}$$

$$E(b) = \beta \quad \text{The mean of } b \text{ is beta.} \tag{3.2b}$$

## The Variance of *a* and *b*

We want to do everything we can to minimize the variances of $a$ and $b$ in order to improve their reliability as estimators of $\alpha$ and $\beta$. If their variances are high, we cannot place much reliability on our regression estimate of costs—something we would like to avoid.

Equations (3.3) and (3.4) for the variance of $a$ and $b$ give us important insights into deciding how many years of financial data to gather and analyze. Common practice is that an appraisal should encompass five years of data. Most appraisers consider anything older than five years to be stale data, and anything less than five years insufficient. You will see that the common practice may be wrong.

The mathematical definition for the variance of $a$ is:

$$\operatorname{Var}(a) = \frac{\sigma^2}{n}, \tag{3.3}$$

where $\sigma^2$ is the true and unobservable population variance around the true regression line and $n$ = number of observations.[11]

Therefore, the variance of our estimate of fixed costs decreases with $n$, the number of years of data. If $n = 10$, the variance of our estimate of $\alpha$ is one-half of its variance if we use a sample of five years of data, and the standard deviation of our estimate is $\frac{1}{\sqrt{2}} = 71\%$ of the five-year standard deviation. Thus, doubling the number of years of data reduces the standard deviation of $a$ by $1 - 71\% = 29\%$. Thus, having more years of data may increase the reliability of our statistical estimate of fixed costs if the data are not stale, that is, out of date due to changes in the business, all else being constant.

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[10] Another reason is that $Y_i$ are random variables.

[11] Technically this is true only when the $y$-axis is placed through the mean of $x$. The following arguments are valid, however, in either case.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Table 3.2** | | | | | |
| 2 | **OLS Regression** | | | | | |
| 3 | **Example of Deviation from Mean** | | | | | |
| 4 | | | | | | |
| 5 | | | | **Variable** | | |
| 6 | | | ***Y*** | ***X*** | ***x*** | ***x²*** |
| 7 | | | | | **Deviation** | **Squared Dev.** |
| 8 | **Observation** | **Year** | **Expenses** | **Sales** | **from Mean** | **from Mean** |
| 9 | **1** | **2005** | $ 80,000 | $ 100,000 | $ (66,667) | 4,444,444,444 |
| 10 | **2** | **2006** | $ 115,000 | $ 150,000 | $ (16,667) | 277,777,778 |
| 11 | **3** | **2007** | $ 195,000 | $ 250,000 | $ 83,333 | 6,944,444,444 |
| 12 | | **Total** | | **$ 500,000** | **$ -** | **11,666,666,667** |
| 13 | | **Average** | | **$ 166,667** | | |

The variance of $b$ is equal to the population variance divided by the sum of the squared deviations from the mean of the independent variable, or:

$$\operatorname{Var}(b) = \frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}, \tag{3.4}$$

where $x_i = X_i - \overline{X}$, the deviation of the independent variable of each observation, $X_i$, from the mean, $\overline{X}$, of all its observations.

In this context, it is each year's sales minus the average of sales in the period of analysis. Since we have no control over the numerator—indeed we cannot even know it—the denominator is the only portion where we can affect the variance of $b$. Let's take a further look at the denominator.

Table 3.2 is a simple example to illustrate the meaning of $x$ versus $X$.

Expenses (column C) is our $Y$ (dependent) variable, and sales (column D) is our $X$ (independent) variable. The three years sales total \$500,000 (D12), which averages to \$166,667 (D13) per year, which is $\overline{X}$. Column E shows $x$, the deviation of each $X$ observation from the sample mean, $\overline{X}$, of \$166,667. In 2005, $x_1 = \$100{,}000 - \$166{,}667 = -\$66{,}667$ (E9). In 2006, $x_2 = \$150{,}000 - \$166{,}667 = -\$16{,}667$ (E10). Finally in 2007, $x_3 = \$250{,}000 - \$166{,}667 = \$83{,}333$ (E11). The sum of all deviations is always zero, or $\sum_{i=1}^{3} x_i = 0$ (E12).

Finally, column F shows $x^2$, the square of column E. The sum of the squared deviations, $\sum_{i=1}^{3} x_i^2$, equals \$11,666,666,667.

This squared term appears in several OLS formulas and is particularly important in calculating the variance of $b$.

When we use relatively fewer years of data, there tends to be less variation in sales. If sales are confined to a fairly narrow range, the squared deviations in the denominator of equation (3.4) are relatively small, which makes the variance of $b$ large. The opposite is true when we use more years of data. A countervailing consideration is that using more years of data may lead to a higher sample variance, which is the regression estimate of $\sigma^2$. Thus, it is difficult to say in advance how many years of data are optimal.

This means that the common practice in the industry of using only five years of data so as not to corrupt our analysis with stale data may be incorrect if there are no significant structural changes in the business and competitive environment. The number of years of available data that gives the best overall statistical output for the regression equation is the most desirable. Ideally, the analyst should experiment with different numbers of years of data and let the regression statistics—the adjusted $R^2$, $t$-statistics, and standard error of the $y$-estimate—provide the feedback to making the optimal choice of how many years of data to use.

Sometimes, prior data can truly be stale. For example, if the number of competitors in the company's geographic area doubles, this will tend to drive down prices, resulting in a decreased contribution ratio and an increase in variable costs per dollar of sales. In this case, using the old data without adjustment would distort the regression results. Nevertheless, it may be advisable in some circumstances to use some of the old data—with adjustments—in order to have enough data points for analysis. In the example of more competition in later years, it is possible to reduce the sales in the years prior to the competitive change on a pro forma basis, keeping the costs the same. The regression on this adjusted data may be more accurate than "winging it" with only two or three years of fresh data if the proper adjustments are clear.

Of course, the company's management has its view of the future. It is important for the appraiser to understand that view and consider it in his or her statistical work.

## Confidence Intervals

Constructing confidence intervals around the regression estimates $a$ and $b$ is another important step in using regression analysis. We would like to be able to make a statement that we are 95% sure that the true population coefficient (either $\alpha$ or $\beta$) is within a specific range of numbers, with our regression estimate ($a$ or $b$) at the midpoint. To calculate the range, we must use the Student's $t$-distribution, which we define in equation (3.6).

We begin with a standardized normal ($Z$) distribution. A standardized normal distribution of $b$—our estimate of $\beta$—is constructed by subtracting the mean of $b$, which is $\beta$, and dividing by its standard deviation.

$$Z = \frac{b - \beta}{\sigma / \sqrt{\sum_i x_i^2}}. \tag{3.5}$$

**THE $t$-DISTRIBUTION** Since we do not know $\sigma$, the population standard deviation, the best we can do is estimate it with $s$, the sample standard deviation. The result is the Student's $t$-distribution, or simply the $t$-distribution. Figure 3.2 shows a $z$-distribution and a $t$-distribution. The $t$-distribution is very similar to the normal ($Z$) distribution, with $t$ being slightly more spread out. The equation for the $t$-distribution is:

$$t = \frac{b - \beta}{s / \sqrt{\sum_i x_i^2}}, \tag{3.6}$$

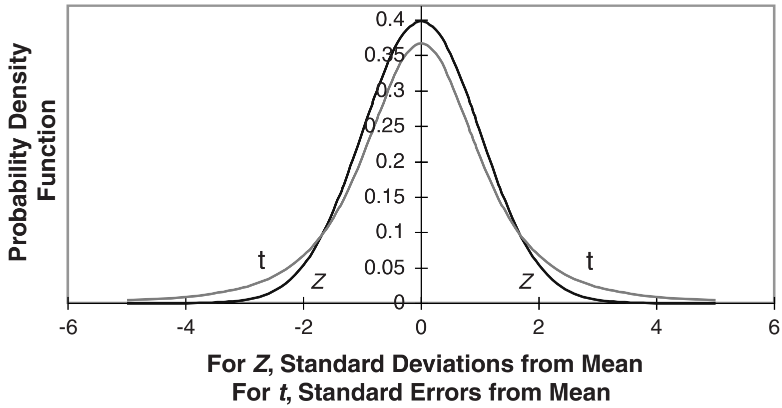



FIGURE 3.2 *Z*-Distribution versus *t*-Distribution

where the denominator is the standard error of $b$, commonly denoted as $s_b$ (the standard error of $a$ is $s_a$).

Since $\beta$ is unobservable, we have to make an assumption about it in order to calculate a $t$-distribution for it. The usual procedure is to test for the probability that, regardless of the regression's estimate of $\beta$—which is our $b$—the true $\beta$ is really zero. In statistics, this is known as the *null hypothesis*. The magnitude of the $t$-statistic is indicative of our ability to reject the null hypothesis for an individual variable in the regression equation. When we reject the null hypothesis, we are saying that our regression estimate of $\beta$ is statistically significant.

We do this by substituting in zero for $\beta$ in equation (3.6) and using $s_b$ for the denominator. This results in equation (3.6a).

$$t = \frac{b}{s_b} T\text{-Statistic to test the null hypothesis that the true } \beta = 0. \qquad (3.6a)$$

The intuition behind equation (3.6a) is as follows. Our worry is that even though the regression provides us with a positive (or negative) estimate of the slope (the $x$-coefficient), in reality it is possible that there is no relationship and the true $\beta$ is zero. We test that with the $t$-statistic. The larger the absolute value of the $t$-statistic, the less likely it is that the null hypothesis is true, which means it is more likely that our measurement of $b$ is statistically meaningful and reliable.

The $t$-statistic increases with an increase in the numerator and a decrease in the denominator. A large numerator means that the regression estimate of $\beta$, $b$, is large; that is, the regression line has a steep slope. This means that the steeper the slope of the regression line, the less worried we are that the line really should have been horizontal (i.e., a zero slope). Also the lower the standard error of $b$, the less worried we are that our results are meaningless (i.e., false).

In our example in Table 3.1B, the regression estimates variable costs, $b$, at \$0.80 (B34) per dollar of sales. With the $t$-statistic we are now asking the question, "What is the probability that variable costs are really zero?," that is, that there really is no relationship between sales and total costs.

We can construct 95% confidence intervals around our estimate, $b$, of the unknown $\beta$. This means that we are 95% sure the correct value of $\beta$

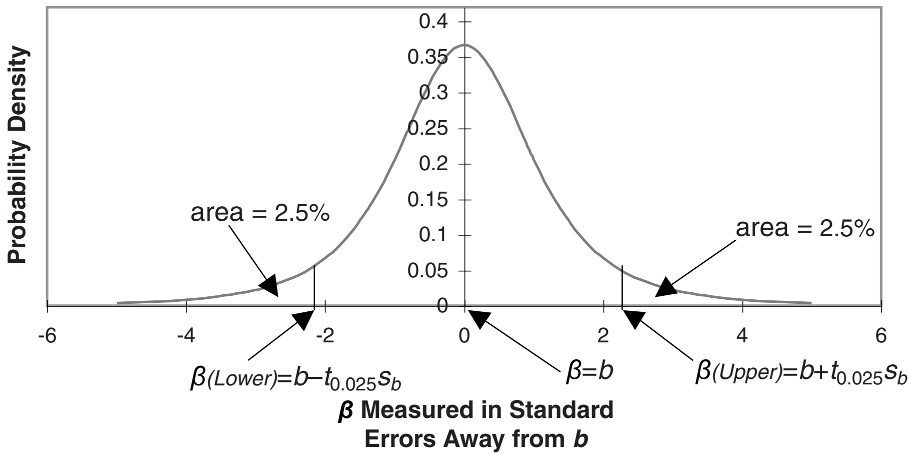



FIGURE 3.3 *T*-Distribution of 95% Confidence Interval of $\beta$ around the Estimate *b*

is in the interval described in equation (3.7).

$$\beta = b \pm t_{0.025}\ s_b \text{ Formula for 95\% confidence interval for the slope.} \quad (3.7)$$

Figure 3.3 shows a graph of the confidence interval. The graph is a $t$-distribution, with its center at $b$, our regression estimate of $\beta$. The markings on the $x$-axis are the number of standard errors below or above $b$. As mentioned before, we denote the standard error of $b$ as $s_b$. The lower boundary of the 95% confidence interval, $\beta_{Lower}$, is $b - t_{0.025}\ s_b$, and the upper boundary of the 95% confidence interval, $\beta_{Upper}$, is $b + t_{0.025}\ s_b$. We will explain the term $t_{0.025}$ in the following.

The $t$-distribution is a standard table in most statistics books. It is very important to use the 0.025 probability column in the tables for a 95% confidence interval, not the 0.05 column. The 0.025 column tells us that for the given degrees of freedom there is a 2.5% probability that the true and unobservable $\beta$ is higher than the upper end of the 95% confidence interval and a 2.5% probability that the true and unobservable $\beta$ is lower than the lower end of the 95% confidence interval (see Figure 3.3).[12] We call that term $t_{0.025,}$ which means that value in the $t$-distribution at which there is a 2.5% probability for $n$ degrees of freedom that $\beta$ is larger than the upper end of our 95% confidence interval.

**DEGREES OF FREEDOM** The degrees of freedom is equal to $n - k - 1$, where $n$ is the number of observations and $k$ is the number of independent variables. Let's try to understand this. The degrees of freedom tell us how many observations are free to take on any value, given that we have a specific measure.

We will start with a very simple example. Suppose that we have $n = 3$ observations, $x_1 = 4$, $x_2 = 6$, and $x_3 = 8$. The sum is 18, and the mean is 6. The mean has $n - 1 = 2$ degrees of freedom. This is because if we fix the sum at 18, once we have the first two observations, the last one is already determined. In other words, given

[12]It is important to be careful, as different texts will show either a one-tailed or two-tailed distribution.

that the mean is 6, only two of the three observations can vary. In this example, once we know the first two, 4 and 6, the last one must be 8 for the mean to be 6. Now let's move on to understanding degrees of freedom in regression analysis.

In Table 3.1B, we have 10 observations and one independent variable, that is, $n = 10$ and $k = 1$. The standard error of $b$ (sales) is 0.014 (C34). According to equation (3.6), in order to calculate this standard error we have to calculate the standard error of the $y$-estimate, $s$. In order to calculate $s = \$16{,}014$ (B23), we have to determine the $y$-intercept and the slope ($x$-coefficient) of the line. Since two points determine the line, there is no variance or standard deviation until we have at least three points (i.e., we lost two degrees of freedom). If we hold the standard error to be equal to \$16,014, only 8 points are free to take on any value. Once we "allow" those 8 points to take on any value, if we are trying to reverse engineer $s = \$16{,}014$, the last two points must take on specific values in order to end up with our result that $s = \$16{,}014$.

Similarly, we lose a degree of freedom for each additional regressor. If there are two independent variables, then instead of fitting a regression line through points in 2-space, we fit a regression plane through points in 3-space. There is no variance or standard error until we have 3 points to fit a plane. Only when we have at least 4 points can we calculate variance for two independent variables. In general, we must have at least $k + 1$ points to calculate variance. Subtracting that from $n$ observations, we have $n - k - 1$ degrees of freedom.

**TABLE 3.3: AN ABBREVIATED TABLE OF *t*-STATISTICS** Table 3.3 is an excerpt from a $t$-distribution table. We use the 0.025 column for a 95% confidence interval. To select the appropriate row in the table, we need to know the number of degrees of freedom. Assuming $n = 10$ observations and $k =$ one independent variable, there are eight degrees of freedom $(10 - 1 - 1)$. The $t$-statistic in Table 3.3 is 2.306 (C7). That means that we must go 2.306 standard errors below and above our regression estimate to achieve a 95% confidence interval for $\beta$. The regression itself will provide us with the standard error of $\beta$. As $n$, the number of observations, goes to infinity, the $t$-distribution becomes a $z$-distribution. When $n$ is large—over 100—the $t$-distribution is very close to a standardized normal distribution. You can

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Table 3.3** | | | |
| 2 | **Abbreviated Table of *t*-Statistics** | | | |
| 3 | | | | |
| 4 | | **Selected *t*-statistics** | | |
| 5 | **d.f.\Pr.** | 0.050 | **0.025** | 0.010 |
| 6 | 3 | 2.353 | 3.182 | 4.541 |
| 7 | **8** | 1.860 | **2.306** | 2.896 |
| 8 | 12 | 1.782 | 2.179 | 2.681 |
| 9 | 120 | 1.658 | 1.980 | 2.358 |
| 10 | Infinity | 1.645 | 1.960 | 2.326 |
| 11 | | | | |
| 12 | *Note:* We select the *t*-statistic for 8 degrees of freedom and | | | |
| 13 | a 95% single-tailed distribution. | | | |

see this in Table 3.3 in that the standard errors in row 9 are very close to those in row 10, the latter of which is equal to a standardized normal distribution.

**BACK TO TABLE 3.1B** The $t$-statistics for our regression in Table 3.1B are 3.82 (D33) and 56.94 (D34). The $p$-value, also known as the *probability* (or *prob*) value, represents the level at which we can reject the null hypothesis, which in this context is that the true and unknowable $y$-intercept and $x$-coefficient(s) are zero. In statistical hypothesis testing, the $p$-value is the probability of obtaining a result at least as extreme as the one that was actually observed, assuming that the null hypothesis is true. The lower the $p$-value, the less likely the result, assuming the null hypothesis, so the more "significant" the result, in the sense of statistical significance—one often uses $p$-values of 0.05 or 0.01, corresponding to a 5% chance or 1% of an outcome that extreme, given the null hypothesis that the true $y$-intercepts and $x$-coefficients are zero.

One minus the $p$-value is the level of statistical significance of the $y$-intercept and independent variable(s). The $p$-values of 0.005 (E33) and 0.000 (E34) mean that the $y$-intercept and slope coefficients are significant at the 99.5% and 99.9%+ levels, respectively, which means we are 99.5% sure that the true $y$-intercept is not zero and 99.9% sure that the true slope is not zero.[13]

The $F$-test is another method of testing the null hypothesis. In multivariable regressions, the $F$-statistic measures whether the independent variables as a group explain a statistically significant portion of the variation in $Y$. The $F$-statistic is $3.24 \times 10^3$ (E28) = 3,240 (rounded), which is significant at the 99.9% (1 − F28) level.

We interpret the confidence intervals as follows: There is a 95% probability that true fixed costs (the $y$-intercept) fall between \$22,496 (F33) and \$91,045 (G33). This equals \$56,770 (B33) $\pm$ (2.306 $\times$ \$14,863), where 2.306 is $t_{.025}$ and \$14,863 is the standard error of the $y$-intercept in C33. Similarly, there is a 95% probability that the true variable cost (the slope coefficient) falls between \$0.77 (F34) and \$0.84 (G34) of each dollar of sales, which is \$0.8045 (B34) $\pm$ (2.306 standard errors $\times$ 0.014 (C34)).

The denominator of equation (3.6) is called the standard error of $b$, or $s_b$. It is $s$, the standard error of the $Y$-estimate—defined in equation (3.7a)—divided by the square root of the average squared deviations of $x$, with the average determined by dividing by the degrees of freedom.

$$s = \sqrt{\sum_{i=1}^{n} \frac{(Y_i - \hat{Y}_i)^2}{n-2}}. \tag{3.7a}$$

Equation (3.7a) is the standard error of the $y$-estimate, where $\hat{Y}_i$ are the forecast (regression fitted) costs, $Y_i$ are the historical actual costs, and $n - 2 = 8$ is the degrees of freedom. The standard error of the $y$-estimate is \$16,014 (B23). We will see the components of this calculation in detail later in Appendix Table A3.1, B25 and Table 3.1B, B23. The larger the amount of scatter of the points around the regression line, the greater the standard error.

[13]For spreadsheets that do not provide $p$-values, another way of calculating the statistical significance is to look up the $t$-statistics in a Student's $t$-distribution table and find the level of statistical significance that corresponds to the $t$-statistic obtained in the regression.

**CONFIDENCE INTERVALS FOR A SPECIFIC FORECAST OF $X_0$** We have shown the variance of $a$ and $b$ and the confidence interval around $b$ in equation (3.7). Additionally, equation (3.7a) is the formula for the standard error of the $y$-estimate, which quantifies the variation of sample data. In this section, we show the confidence intervals for our forecast of any particular value of $x$, say $x_0$. In other words, while the previous formulas give us information about the regression sample and results, which are critical, it may be helpful to have confidence intervals for any application of our regression to the subject company.

Equation (3.8) is the formula for a 95% confidence interval for the mean $\mu_0, = a + (b \times x_0)$, and equation (3.9) is the formula for a 95% confidence interval for an individual $Y_0$.

$$\pm t_{0.025} s \sqrt{\frac{1}{n} + \frac{x_o^2}{\sum x_i^2}} \quad \text{95\% confidence interval for the mean forecast;} \tag{3.8}$$

$$\pm t_{0.025} s \sqrt{\frac{1}{n} + \frac{x_o^2}{\sum x_i^2} + 1} \quad \text{95\% confidence interval for a specific year's forecast.} \tag{3.9}$$

Note that $x_0$, which is the deviation of a particular $x$ observation from the mean, causes the confidence interval to increase the further the $X_0$ is from the mean.

Academic articles generally do not provide the measures in equations (3.8) and (3.9), as finance and economics professors are not interested in reporting how reliable a regression equation is for a particular choice of $x_0$. Instead, they are interested in showing how well the regression equation explains the sample data.

## Selecting the Data Set and Regression Equation

Table 3.4 is otherwise identical to Table 3.1B, except that instead of all 10 years of data, it contains only the last five years. The regression equation for the five years of data is: Adjusted Costs = \$71,252 + (\$0.79 × Sales) (Table 3.4, B27 and B28).

Examining the regression statistics, we find that the adjusted $R^2$ is 99.44% (B16), still indicating an excellent relationship. We do see a difference in the $t$-statistics for the two regressions.

The $t$-statistic for the intercept is now 1.89 (D27), indicating it is no longer significant at the 95% level, whereas it was 3.82 in Table 3.1B. Another effect of fewer data is that the 95% confidence interval for the intercept value is −\$48,485 (F27) to \$190,989 (G27), a range of \$239,475. In addition, the $t$-statistic for the slope coefficient—while still significant—has fallen from 56.94 (Table 3.1B, D34) to 26.75 (D28). The 95% confidence interval for the slope now becomes \$0.70 (F28) to \$0.89 (G28), a range that is $3\frac{1}{2}$ times greater than that in Table 3.1B and indicates much more uncertainty in the variable cost than we obtain using 10 years of data.

The standard error of the $Y$-estimate, however, decreases from \$16,014 (Table 3.1B, B23) to \$6,840 (B17). This indicates that decreasing the number of data points improves the $Y$-estimate, an opposite result from all of the preceding. Why?

Earlier we pointed out that using only a small range for the independent variable leads to a small denominator in the variance of $b$, that is, $\frac{\sigma^2}{\sum_{i=1}^{n} x_i^2}$, which leads to larger confidence intervals. However, larger data sets (using more years of data) tend to lead to a larger standard error of the $y$-estimate, $s$. As we mentioned earlier, $s = \sqrt{\frac{1}{n-2}\sum_{i=1}^{n}(Y_i - \hat{Y}_i)^2}$, where $\hat{Y}_i$ are the forecast (regression fitted) costs, $Y_i$ are the historical costs, and $n$ is the number of observations. Thus we often have a trade-off in deciding how many years of data to include in the regression. More years of data lead to better confidence intervals of $b$, but fewer years may lead to smaller standard errors of the $y$-estimate.

Table 3.4 demonstrates that you should evaluate all of the regression statistics carefully to determine whether the relationship is sufficiently strong to merit using it and which data set is best to use. Simply looking at the adjusted $R^2$ value is insufficient; all the regression statistics should be evaluated in their entirety, as an improvement in one may be counterbalanced by a deterioration in another. Therefore, if time and budget permit, it is best to test different data sets and compare all of the regression statistics to select the regression equation that represents the best overall relationship between the variables. Figure 3.4 at the bottom of Table 3.4 is a graph of the regression.

## Problems with Regression Analysis for Forecasting Costs

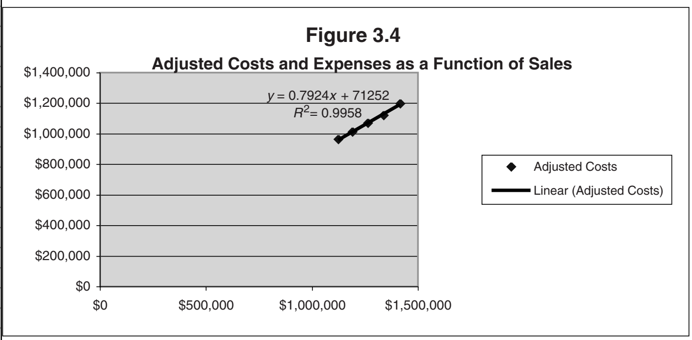
Although regression analysis is a powerful tool, its blind application can lead to serious errors. One can encounter various problems and should be cognizant of the limitations of this technique. Aside from the obvious problems of poor fit and insufficient or missing data, structural changes in the company can also invalidate the historical relationship of sales and costs.

### Insufficient or Missing Data

Insufficient data leads to wider confidence intervals in the regression and our forecasts. As mentioned previously, to optimize the regression equation it is best to examine overlapping data sets to determine which gives the best results. The fewer the observations available, the fewer degrees of freedom, which means we can have fewer independent variables.

Missing data often presents challenges, especially when working with transactional databases such as Pratt's Stats or the IBA database or with the Partnership Profiles database. Some transactions are missing data. When this occurs, there are two strategies, and it is usually best to use both. The first strategy we can take is to use the maximum number of observations, which requires using only those independent variables for which we have data for all observations. The second strategy is to maximize the number of independent variables that the analyst thinks are relevant. This requires the analyst to delete all observations that have data missing from any of the independent variables that he or she is testing. Of course if a particular independent variable proves to be statistically insignificant, then the analyst can restore observations that were deleted for lack of this independent variable.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 3.4** | | | | | | |
| 2 | **Regression Analysis 2003–2007** | | | | | | |
| 3 | | | | | | | |
| 4 | **Year** | **Sales** | **Adjusted Costs** | | | | |
| 5 | 2003 | $1,123,600 | $965,043 | | | | |
| 6 | 2004 | $1,191,016 | $1,012,745 | | | | |
| 7 | 2005 | $1,262,477 | $1,072,633 | | | | |
| 8 | 2006 | $1,338,226 | $1,122,714 | | | | |
| 9 | 2007 | $1,415,000 | $1,199,000 | | | | |
| 10 | | | | | | | |
| 11 | **SUMMARY OUTPUT** | | | | | | |
| 12 | | | | | | | |
| 13 | *Regression Statistics* | | | | | | |
| 14 | Multiple R | 99.79% | | | | | |
| 15 | R Square | 99.58% | | | | | |
| 16 | Adjusted R Square | 99.44% | | | | | |
| 17 | Standard Error | 6,840 | | | | | |
| 18 | Observations | 5 | | | | | |
| 19 | | | | | | | |
| 20 | **ANOVA** | | | | | | |
| 21 | | *df* | *SS* | *MS* | *F* | *Signif F* | |
| 22 | Regression | 1 | 3.35E+10 | 3.35E+10 | 716 | 1.15E-04 | |
| 23 | Residual | 3 | 1.40E+08 | 4.68E+07 | | | |
| 24 | Total | 4 | 3.36E+10 | | | | |
| 25 | | | | | | | |
| 26 | | *Coef* | *Std Err* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
| 27 | Intercept [1] | 71,252 | 37,624 | 1.89 | 0.15 | (48,485) | 190,989 |
| 28 | Sales [2] | 0.79 | 0.03 | 26.75 | 0.00 | 0.70 | 0.89 |
| 29 | | | | | | | |
| 30 | [1] This is the regression estimate of fixed costs. | | | | | | |
| 31 | | | | | | | |
| 32 | [2] This is the regression estimate of variable costs. | | | | | | |
| 33 | | | | | | | |




## Substantial Changes in Competition or Product/Service

Although regression analysis is applicable in most situations, substantial structural changes in a business may render it inappropriate. As mentioned previously, the appraiser can often compensate for changes in the competitive environment by making pro forma adjustments to historical sales, keeping costs the same. However, when a company changes its business, the past is less likely to be a good indicator of what may occur in the future, depending on the significance of the change.

# Using Regression Analysis to Forecast Sales

Table 3.5 is an example of using regression techniques to forecast sales. In order to do this, it must be reasonable to assume that past performance should be an

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 3.5** | | | | | | |
| 2 | **Regression Analysis of Sales as a Function of GDP [1]** | | | | | | |
| 3 | | | | | | | |
| 4 | **Year** | **GDP** | **GDP$^2$** | **Sales** | | | |
| 5 | 1988 | 5,049.6 | 25,498,460.2 | $1,000,000 | | | |
| 6 | 1989 | 5,438.7 | 29,579,457.7 | $1,090,000 | | | |
| 7 | 1990 | 5,743.0 | 32,982,049.0 | $1,177,200 | | | |
| 8 | 1991 | 5,916.7 | 35,007,338.9 | $1,259,604 | | | |
| 9 | 1992 | 6,244.4 | 38,992,531.4 | $1,341,478 | | | |
| 10 | 1993 | 6,558.1 | 43,008,675.6 | $1,442,089 | | | |
| 11 | 1994 | 6,947.0 | 48,260,809.0 | $1,528,614 | | | |
| 12 | 1995 | 7,269.6 | 52,847,084.2 | $1,617,274 | | | |
| 13 | 1996 | 7,661.6 | 58,700,114.6 | $1,706,224 | | | |
| 14 | 1997 | 8,110.9 | 65,786,698.8 | $1,812,010 | | | |
| 15 | 1998 | 8,510.7 | 72,432,014.5 | $1,929,791 | | | |
| 16 | | | | | | | |
| 17 | **SUMMARY OUTPUT** | | | | | | |
| 18 | | | | | | | |
| 19 | *Regression Statistics* | | | | | | |
| 20 | Multiple R | | 0.999 | | | | |
| 21 | R Square | | 0.998 | | | | |
| 22 | Adjusted R Square | | 0.998 | | | | |
| 23 | Standard Error | | 13,894 | | | | |
| 24 | Observations | | 11 | | | | |
| 25 | | | | | | | |
| 26 | **ANOVA** | | | | | | |
| 27 | | *df* | *SS* | *MS* | *F* | *Signif F* | |
| 28 | Regression | 2 | 9.139E+11 | 4.570E+11 | 2.367E+03 | 8.097E-12 | |
| 29 | Residual | 8 | 1,544,303,643 | 193,037,955 | | | |
| 30 | Total | 10 | 9.15482E+11 | | | | |
| 31 | | | | | | | |
| 32 | | *Coef* | *Std Err* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
| 33 | Intercept | (824,833) | 182,214 | -4.527 | 0.002 | (1,245,019) | (404,647) |
| 34 | GDP | 412.837 | 54.653 | 7.554 | 0.000 | 287 | 539 |
| 35 | GDP$^2$ | -0.011 | 0.004 | -2.645 | 0.029 | -0.020 | -0.001 |
| 36 | | | | | | | |
| 37 | [1] GDP, gross domestic product, is in billions of dollars. GDP is a proxy for the overall economy. | | | | | | |

accurate indicator of future expectations. If there are fundamental changes in the industry that render the past a poor indicator of the future, it may or may not be possible to handle that within the regression framework.

One possibility is the analyst may be able to insert a dummy variable to handle the change. For example, if there was a major change in 2006, the analyst could use a dummy variable equal to 0 for years prior to 2006 and equal to 1 for years after 2005. Another possibility is to make logical pro forma adjustments to the data. If neither of these options is possible, then regression may be useless and even quite misleading. As cautioned by Pratt, Reilly, and Schweihs (1996), blind application of regression, where past performance is the sole indicator of future sales, can be misleading and incorrect. Instead, careful analysis is required to determine whether past income-generating forces will be duplicated in the future. Nevertheless, regression analysis is often useful as a benchmark in forecasting.

In our example in Table 3.5, the primary independent variable is gross domestic product (GDP), which we show for the years 1988–1998 in billions of dollars in B5:B15 (cell references separated by a colon will be our way of indicating contiguous spreadsheet *ranges*). In range C5:C15, we show the square of GDP in billions of dollars, which is our second potential independent variable.[14] Our dependent variable is sales, which appears in D5:D15.

## Spreadsheet Procedures to Perform Regression

It is mandatory to put the variables in columns and the time periods in rows. Electronic spreadsheets will not permit you to perform regression analysis with time in columns and the variables in rows. In other words, we cannot transpose the data in Table 3.5, cells in range A4:D15, and still perform a regression analysis.

Another requirement is that all cells must contain numeric data. You cannot perform regression with blank cells or cells with alphanumeric data in them. Also, you will receive an error message if one of your independent variables is a multiple of another. For example, if each cell in range C5:C15 is three times the corresponding cell in B5:B15, then the *x* variables are perfectly collinear and the regression produces an error message.

In Microsoft Excel, the procedure to perform the regression analysis is as follows:

1. Select "Tools | Data Analysis | Regression." This will bring up a dialog box and automatically place the cursor in "Input *Y* Range."[15]
2. For the *Y* range (which is the dependent variable, sales in our example), click on the range icon with the red arrow immediately to the right. Doing so minimizes the dialog box and enables you to highlight the cell range D4:D15 with your mouse (you can also select the range without clicking on the range icon).[16] Note that we have included the label "Sales" in D4 in this range. Click again on the range icon again to return to the dialog box.
3. For the *X* range, which are the independent variables GDP and $GDP^2$ in our case, repeat the procedure in step 2 and highlight the range B4:C15.
4. Click on the box "Labels," which will put a checkmark in the box.
5. Click on "Output Range." Click on the box to the right, click on the range icon with the red arrow, and then click on A17. This tells the spreadsheet to begin the regression output at A17.
6. Click "OK."

Excel now calculates the regression and outputs the data as shown in the bottom half of Table 3.5.

The instructions for Lotus 123 are almost identical. The only differences are:

1. The command is "Range | Analyze | Regression."
2. The ranges for the dependent and independent variables should not include the label in row 4. Thus they are D5:D15 and B5:C15, respectively.
3. Lotus 123 does not compute *t*-statistics for you. To get the *t*-statistic, you will have to compute it manually by creating a formula. Divide the regression

[14]Another variation of this procedure is to substitute the square root of GDP for its square.

[15]If Data Analysis is not yet enabled in Excel, you must select add-ins and then select Analysis | ToolPak.

[16]Excel actually shows the range with dollar signs (e.g., $D$4:$D$15).

coefficient by its standard error. Unfortunately, Lotus 123 does not calculate the *p*-values either. If you want them, you will have to look up your results in a standard table of *t*-statistics. We will cover that later.

## Examining the Regression Statistics

Once again, we look at the statistical measures resulting from the regression to determine the strength of the relationship between sales and GDP. Adjusted $R^2$ is 99.8% (B22), a near-perfect relationship. The *t*-statistics for the independent variables, GDP and $GDP^2$, are 7.55 (D34) and −2.65 (D35), both statistically significant. The easiest way to determine the level of statistical significance is through the *p*-value. One minus the *p*-value is the level of statistical significance. For GDP, the *p*-value is 0.000 (E34), which is less than 0.1%. Thus GDP is statistically significant at a level greater than 100% − 0.1% = 99.9%. The square of GDP has a *p*-value of 0.029 (E35), which indicates statistical significance at the 97.1% level. We normally accept any regressor with significance greater than or equal to 95%, and we may consider accepting a regressor that is significant at the 90% to 95% level.

The standard error of the *y*-estimate (i.e., sales) is $13,894 (B23). Our approximate 95% confidence interval is ± two standard errors = ±$27,788, which is less than ±2% of the mean of sales.

In actual practice, adjusted $R^2$ for a regression of sales of mature firms is often above 90% and frequently around 98%.

## Adding Industry-Specific Independent Variables

One should also consider adding industry-specific independent variables. For example, when valuing a jeweler, we should try adding the price of gold and silver (and the nonlinear transformations, for example, squares, square roots, and logarithms) as independent variables. When valuing a firm in the oil industry, we should try using the price of a barrel of oil (and its nonlinear transformations).

When valuing a coffee producer, we would want to have not only the average price of coffee as an independent variable, but also the price of tea and perhaps even sugar. The analyst should look to the prices of the product itself, complements, and substitutes.

Once again, it is important to examine the statistical validity of the relationship and use professional judgment to determine the usefulness of the equation. Sales forecasts obtained from regression analysis can serve as a benchmark from which adjustments can be made based on qualitative factors that may influence future sales.

One should also keep in mind that just because a less quantitative method of forecasting sales does not have an embarrassingly low $R^2$ staring the analyst in the face does not mean that it is superior to the regression. It means we have no clue as to the reliability of the forecast. We should always be uncomfortable with our ignorance.

### Try Combinations of Potential Independent Variables

It is important to try as many logical combinations of independent variables as practically possible. With a statistics package, this is done automatically in using automated forward or backward regression. However, statistics packages have their drawbacks. They are not very user-friendly in communicating with spreadsheet programs, which most appraisers use in valuation analysis. Most appraisers will find the spreadsheet regression capabilities more than adequate. Nevertheless, it is often ideal to use a statistics package first to allow the automated regression to locate the best combination of independent variables and then use that combination in Excel for more attractive output and easier interface to the rest of the valuation process.

When using a spreadsheet exclusively for regression, it is important to try many combinations of logical potential independent variables in the regression process. For example, in regressing sales against both GDP and $GDP^2$, it is not unusual to find both independent variables statistically insignificant when regressed together, that is, *p*-values greater than 0.05. However, they still may be statistically significant when regressed individually. So it is important to regress sales against GDP and perform a second regression against $GDP^2$. This process becomes more complicated and time-consuming with additional candidates for independent variables.

It is also important to recognize that stretching too far in trying independent variables may yield spurious (apparently good but actually false) regressions. For example, it is possible that tea production in India by dumb luck might appear to produce statistically significant results in explaining adjusted costs and expenses, but the wise analyst will refrain from trying out independent variables that make no sense.

There is another important caveat. It has been the author's experience that using extreme[17] nonlinear independent variables such an the inverse or square can lead to erratic results—and all the more so when even one independent variable in the subject company is materially outside of the range of those in the sample. For example, our firm performs a regression of the Partnership Profiles database every year. We usually find that some combination of prior-year cash yields is statistically significant in explaining discounts from net asset value. We also often find that adding inverses of yields, that is, (1/yield), is statistically significant in the sample and improves the adjusted $R^2$ of the regression.[18] However, using inverses tends to lead to extreme results when applying the regression to our subject companies.

[17]For example, the inverse of a 1% yield is $1/0.01 = 100$, while the inverse of a 2% yield is $1/0.02 = 50$, which is a large absolute difference (although the same percentage difference). In contrast, $\ln(0.01) = -4.6$, while $\ln(0.02) = -3.9$. The percentage difference with natural logarithms between the two yields is only about 15%. Thus, logarithms tend to minimize differences and are stable nonlinear variables, while it has been the author's experience that inverses and squares are more problematic if the subject company's measure is materially outside the sample range. That does not mean that inverses and squares are always inappropriate. It is a warning to be careful about using them. One conservative way to incorporate them is to use the appropriate maximum or minimum value in the sample range for the subject company.

[18]Since the natural log of zero is undefined, when the yield is zero, we assume it was 0.001.

## Autocorrelation in Time Series Analysis

So far, both types of regression that we have discussed so far—regression of the company's costs as a function of sales and regression of sales as a function of GDP, and so forth—are time series regressions. In time series there can be problems with autocorrelation (a.k.a. *serial correlation*), which means the regression errors are correlated over time. Ideally there should be no autocorrelation, which means the regression errors are completely random. If there is autocorrelation, then the size of the regression error in one year should enable us to predict the regression error in another year.

The test for autocorrelation in the error term is the Durbin-Watson test.[19] If the Durbin-Watson indicates the presence of autocorrelation, dealing with it is very sophisticated business—beyond the scope of this chapter.

## Application of Regression Analysis to the Guideline Company (GC) Methods

Unlike the previous two applications of time series regression, regression using GCs is cross-sectional and does not have issues with autocorrelation. At its simplest level, the GC method involves the use of ratios of stock price to earnings (PE multiples), cash flow (P/CF, P/EBIT, or P/EBITDA multiples), book value (P/BV multiples), sales (P/Sales), or other measures of income, cash flow, or value.

There are two basic sources of GCs: publicly traded firms and privately traded firms, with data for the latter being available in Pratt's Stats, Done Deals, the IBA database, and BizComps. The two submethods are known as the *guideline public company method* (GPCM) and the *guideline M&A method* (GMAM). In both cases, we are looking to GCs in the same or similar business as the company. We are therefore considering what informed investors are willing to pay, adjusted for the specific circumstances of the company being valued. While the use of ratios is common in valuation, regression analysis is more sophisticated and informative, because it provides us with statistical feedback on the strength of the relationship. Pratt, Reilly, and Schweihs (1996) present a comprehensive chapter on use of the guideline company method, so we will discuss it only within the context of regression analysis.

### Table 3.6: Regression Analysis of Guideline Companies

Table 3.6 shows data from an actual guideline company analysis, with the company names disguised in column A. Column B contains the fair market values (FMVs) (market capitalization) for 11 companies, ranging from slightly over $3 million (B5) to over $150 million (B15). The average FMV is $41.3 million (B16), with a standard deviation of $44.6 million (B17). Net income (column C) averages about $5.1 million (C16), with a range of $600,000 to $16.9 million. We had to exclude companies A and B, which were outliers with price/earnings (PE) ratios over 60.

First we will briefly describe the regression results for the regression of FMV against net income (not shown in the table). The regression yields an adjusted $R^2$

[19]The Durbin-Watson test is not valid for autocorrelation in the dependent variable.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table 3.6** | | | | | | | |
| 2 | **Regression Analysis of Guideline Companies** | | | | | | | |
| 3 | | | | | | | | |
| 4 | **Company** | **FMV** | **Net Income** | **ln FMV** | **ln NI** | **1/*g*** | ***g*** | **PE Ratio** |
| 5 | C | 3,165,958 | 602,465 | 14.9680 | 13.3088 | 20.0000 | 0.0500 | 5.2550 |
| 6 | D | 6,250,000 | 659,931 | 15.6481 | 13.3999 | 10.0000 | 0.1000 | 9.4707 |
| 7 | E | 12,698,131 | 1,375,000 | 16.3570 | 14.1340 | 10.5263 | 0.0950 | 9.2350 |
| 8 | F | 24,062,948 | 2,325,000 | 16.9962 | 14.6592 | 9.0909 | 0.1100 | 10.3497 |
| 9 | G | 23,210,578 | 2,673,415 | 16.9601 | 14.7989 | 12.1951 | 0.0820 | 8.6820 |
| 10 | H | 16,683,567 | 2,982,582 | 16.6299 | 14.9083 | 20.0000 | 0.0500 | 5.5937 |
| 11 | I | 37,545,523 | 4,369,808 | 17.4411 | 15.2902 | 12.5000 | 0.0800 | 8.5920 |
| 12 | J | 46,314,262 | 4,438,000 | 17.6510 | 15.3057 | 9.3023 | 0.1075 | 10.4358 |
| 13 | K | 36,068,550 | 7,384,000 | 17.4009 | 15.8148 | 20.8333 | 0.0480 | 4.8847 |
| 14 | L | 97,482,000 | 12,679,000 | 18.3952 | 16.3555 | 9.5238 | 0.1050 | 7.6885 |
| 15 | M | 150,388,518 | 16,865,443 | 18.8287 | 16.6408 | 9.0909 | 0.1100 | 8.9170 |
| 16 | Average | **41,260,912** | **5,123,149** | **17.0251** | **14.9651** | **13.0057** | **0.0852** | **8.1004** |
| 17 | Standard Deviation | **44,558,275** | **5,233,919** | **1.1212** | **1.0814** | **4.8135** | **0.0252** | **1.9954** |
| 18 | | | | | | | | |
| 19 | | | | | | | | |
| 20 | **SUMMARY OUTPUT** | | | | | | | |
| 21 | | | | | | | | |
| 22 | *Regression Statistics* | | | | | | | |
| 23 | Multiple R | 0.998 | | | | | | |
| 24 | R Square | 0.996 | | | | | | |
| 25 | Adjusted R Square | 0.995 | | | | | | |
| 26 | Standard Error | 0.083 | | | | | | |
| 27 | Observations | 11 | | | | | | |
| 28 | | | | | | | | |
| 29 | **ANOVA** | | | | | | | |
| 30 | | *df* | *SS* | *MS* | *F* | *Signif F* | | |
| 31 | Regression | 2 | 12.517 | 6.259 | 914.637 | 0.000 | | |
| 32 | Residual | 8 | 0.055 | 0.007 | | | | |
| 33 | Total | 10 | 12.572 | | | | | |
| 34 | | | | | | | | |
| 35 | | *Coef* | *Std Err* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | |
| 36 | Intercept | 3.431 | 0.390 | 8.794 | 0.000 | 2.531 | 4.331 | |
| 37 | ln NI | 0.957 | 0.025 | 38.818 | 0.000 | 0.900 | 1.014 | |
| 38 | 1/*g* | -0.056 | 0.006 | -10.114 | 0.000 | -0.069 | -0.043 | |
| 39 | | | | | | | | |
| 40 | **Valuation** | | | | | | | |
| 41 | NI | 100,000 | 200,000 | 300,000 | 400,000 | 500,000 | 1,000,000 | |
| 42 | ln NI | 11.5129 | 12.2061 | 12.6115 | 12.8992 | 13.1224 | 13.8155 | |
| 43 | X Coefficient—NI | 0.957 | 0.957 | 0.957 | 0.957 | 0.957 | 0.957 | |
| 44 | ln NI × X Coefficient | **11.019** | **11.682** | **12.070** | **12.346** | **12.559** | **13.223** | |
| 45 | *g* | 0.05 | 0.055 | 0.06 | 0.065 | 0.07 | 0.1 | |
| 46 | 1/*g* | **20.000** | **18.182** | **16.667** | **15.385** | **14.286** | **10.000** | |
| 47 | X Coefficient—1/*g* | -0.056 | -0.056 | -0.056 | -0.056 | -0.056 | -0.056 | |
| 48 | 1/*g* × X Coefficient | **-1.120** | **-1.019** | **-0.934** | **-0.862** | **-0.800** | **-0.560** | |
| 49 | Add Intercept | 3.431 | 3.431 | 3.431 | 3.431 | 3.431 | 3.431 | |
| 50 | Total = ln FMV | **13.329** | **14.095** | **14.567** | **14.915** | **15.190** | **16.093** | |
| 51 | FMV | **$614,928** | **$1,321,816** | **$2,121,136** | **$3,001,492** | **$3,952,067** | **$9,754,515** | |
| 52 | PE Ratio | 6.149 | 6.609 | 7.070 | 7.504 | 7.904 | 9.755 | |
| 53 | | | | | | | | |
| 54 | **95% Confidence Intervals** | | | | | | | |
| 55 | 2 Standard Errors | 0.165 | | | | | | |
| 56 | $e^{2\ \text{Std Err}}$ | 1.180 | | | | | | |
| 57 | $e^{-2\ \text{Std Err}}$ | 0.848 | | | | | | |

of 94.6% and a $t$-statistic for the $x$-coefficient of 12.4, which seems to indicate a successful regression. The regression equation obtained for the complete data set is:

$$FMV = -\$1,272,335 + (8.3 \times \text{Net Income}).$$

If we were to use the regression to value a firm with net income of $100,000, it would produce a value of −$442,000. Something is wrong!

**HETEROSCEDASTICITY** The problem is that the full regression equation is:

$$FMV_i = a + b \times \text{Net Income} + u_i, \tag{3.10}$$

where $u_i$ is an error term, assumed to be normally distributed with an expected value of zero. Our specific regression equation is:

$$-\$1{,}272{,}335 + (8.3 \times \text{Net Income}) + u_i. \tag{3.11}$$

The problem is that this error term is additive and likely to be correlated to the size of the firm. When that occurs, we have a problem called *heteroscedasticity*.[20]

There are three possible solutions to the problem:

1. Use weighted least squares (WLS) instead of ordinary least squares regression. In WLS, we weight the extreme values less than the more mainstream values. This usually will not produce a usable solution for a privately held firm that is much smaller than the publicly traded guideline companies.
2. Use a log-log specification, that is, taking the log of both sides.
3. Use a scaled variable as the $y$-variable. Typical examples of this are the price-to-sales ($PS$) multiple or the price/earnings ($PE$) multiple. This is usually the most practical solution.

In using the log-log specification, we regress the natural logarithm of market capitalization as a function of the natural logarithm of net income. Its form is:

$$\ln\ FMV_i = a + b\ \ln\ NI + u_i, \quad i = \text{guideline company } 1, 2, 3, \ldots n. \tag{3.12}$$

When we take antilogs, the original equation is:

$$FMV_i = A\ NI_i^b v_i, \tag{3.13}$$

where $A = e^a$, $v_i = e^{u_i}$, $e$ is Euler's constant, and the expected value of $v_i = 1$.

In equation (3.13), the regression equation $x$-coefficient, $b$, from equation (3.12) for net income becomes an exponent to net income. If $b = 1$, then size has no scaling effect on the FMV, and we would expect price/earnings ratios to be uncorrelated to size, all other things being constant. If $b > 1$, then the price/earnings multiple should rise with net income, and the opposite is true of $b < 1$. Relating this to the log size model in Chapter 5, we would thus expect to find $b > 1$ because, over long periods of time, large firms have lower discount rates than small firms, which mean larger values relative to earnings.

Using equation (3.13), consider two identical errors of 20% for firms $i$ and $j$, where firm $i$ has net income of \$100,000 and firm $j$ has net income of \$200,000. In other words, the error terms $v_i$ and $v_j$ are both 1.2.[21] For simplicity, suppose that $b = 1$ for both firms. The same statistical error in the log of the fair market value of both firms produces an error in fair market value that is twice as large in firm $j$ as in firm $i$. This is a desirable property, as it corresponds to our intuition that large firms will tend to have larger absolute deviations from the regression-determined values. Thus, this form of regression is likely to be more successful than equation (3.10) for valuing small firms.

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[20]This is also spelled *heteroskedasticity*.

[21]This means the error terms $u_i$ and $u_j$ in equation (3.12) are equal to ln (1.2) = 0.182.

Equation (3.10) is probably fine for valuing firms of the same size as the guideline companies. When we apply equation (3.10) to various levels of net income, we find the forecast FMVs are −\$442,000, \$0 (rounded), \$2.9 million, and \$7.0 million for net incomes of \$100,000, \$154,000, \$500,000, and \$1 million. Obviously equation (3.10) works poorly at the low end. We would also have a similar, but opposite, scaling problem forecasting value for a firm with net income of \$5 billion. The additive error term restricts the applicability of equation (3.10) to subject companies of similar size to the guideline companies.

Including forecast growth as an independent variable is an important potential enhancement to the regression equation. The Internet makes it easier to obtain growth forecasts, although frequently there are no such estimates for smaller publicly traded firms.

A midyear Gordon model is the proper valuation equation for a firm with constant forecast growth:

$$FMV = CF_{t+1}\frac{\sqrt{1+r}}{(r-g)}. \tag{3.14}$$

In Chapter 5, we show that NYSE/AMEX/NASDAQ returns are negatively related to the natural logarithm of market capitalization (which can also be referred to as fair market value or size), which means that there is a nonlinear relationship between return and size. Therefore, the discount rate, $r$, in equation (3.14) impounds a nonlinear size effect. To the extent that there is a nonlinear size effect in equation (3.13), we should hopefully pick that up in the $b$ coefficient.

Note that in equation (3.14) there is a growth term, $g$, which appears in the denominator of the Gordon model multiple. Thus, it is reasonable to try $1/g$ as an additional independent variable in equation (3.13).

Continuing our description of Table 3.6, column C is net income, and columns D and E are the natural logarithms of FMV and net income. These are actual data from a real valuation. Column G is a made-up growth rate. It is not based on actual data, which were unavailable. (However, we will perform a regression using the growth rates as if they are I/B/E/S estimates.) Column F is the inverse of column G, that is, $1/g$. Thus, column D is our dependent variable, and columns E and F are our independent variables.[22]

The adjusted $R^2$ is 99.5% (B25), an excellent result. The standard error of the $y$-estimate is 0.083 (B26). The $y$-intercept is 3.43 (B36), and the $x$-coefficients for ln NI and $1/g$ are 0.95708 and −0.05602 (B37, B38), respectively.

In the valuation section of Table 3.6, we show valuations for subject companies with differing levels of net income and expected growth. Row 41 shows firms with net incomes ranging from \$100,000 to \$1 million. Row 42 is the natural log of net income.[23] We multiply that by the $x$-coefficient for net income in row 43, which produces a subtotal in row 44.

Row 45 contains our forecast of constant growth for the various subject companies. We are assuming growth of 5% per year for the \$100,000 net income firm in

[22]Electronic spreadsheets require that the independent variables be in contiguous columns.

[23]The Excel formula for B42, for example, is =ln(B41). The Lotus 123 formula would be @ln(B41).

column B, and we increase the growth estimate by 0.5% for each firm. Row 46 is 1 divided by forecast growth.

In row 47, we repeat the $x$-coefficient for $1/g$ from the regression, and row 46 × row 47 = row 48, which is another subtotal.

In row 49, we repeat the $y$-intercept from the regression. In row 50, we add rows 44, 48, and 49, the sum of which equals the natural logarithm of the forecast FMV (at the marketable minority interest level). We must then exponentiate that result (i.e., take the antilog). The Excel formula for B51 is =EXP(B50). Finally, we calculate the PE ratio in row 52 as row 51 divided by row 41.

The PE ratio rises because of the increase in the forecast growth rate across the columns. If all cells in row 45 were equal to 0.05, then the PE ratios in row 52 would actually decline going to the right across the columns. The reason for this is that the $x$-coefficient for ln NI, 0.95708 (B37), is $< 1$. This is contrary to our expectations. If B38 were greater than 1, then PE ratios would rise with firm size, holding forecast growth constant. Does this disprove the log size model? No; while all the rest of the data are real, these growth rates are not. They are made up. Also, one small sample of one industry at one point in time does not generalize to all firms at all times.

In the absence of the made-up growth rates, the actual regression yielded an adjusted $R^2$ of 93.3% and a standard error of 0.2896 (not shown).

**NINETY-FIVE PERCENT CONFIDENCE INTERVALS** We multiply the standard error in B26 by 2 = 0.165 (B55).[24] To convert the standard error of ln FMV to the standard error of FMV, we have to exponentiate the two standard errors. In B56, we raise $e$, Euler's constant, to the power of B55. Thus, $e^{0.16544} = 1.1799$, which means the high side of our 95% confidence interval is 18% higher than our estimate.[25] To calculate the low side of our 95% confidence interval, we raise $e$ to the power of two standard errors below the regression estimate. Thus B57 = $e^{-0.16544} = 0.8475$, which is approximately 15% below the regression estimate. Thus our 95% confidence interval is the regression estimate +18% and −15%. Using only the actual data that were available at the time, the same regression without $1/g$ yielded confidence intervals of the regression estimate +78% and −56%. Obviously, growth can make a huge difference. Also, without growth, the $x$-coefficient for ln NI was slightly above 1, indicating increasing PE multiples with size.

We eventually intend to cover the third method of dealing with heteroscedasticity, using scaled variables, in the workbook that should accompany the third edition of this text. Until then, look for material on our Web site.

## Summary

Regression analysis is a powerful tool for use in forecasting future costs, expenses, and sales and estimating fair market value. We should take care in evaluating and selecting the input data, however, to arrive at a meaningful answer. Similarly, we should carefully scrutinize the regression output to determine the significance of the

[24]It is 0.16544 to five decimal points.

[25]The Excel formula for B56 is =EXP(B55), and the Lotus 123 formula is @EXP(B55). Similarly, the Excel formula for B57 is =EXP(−B55), and the Lotus 123 formula is @EXP(−B55).

variables and the amount of error in the $Y$-estimate to determine whether the overall relationship is meaningful.

## References

Bhattacharyya, Gouri, K., and Richard A. Johnson. 1977. *Statistical Concepts and Methods*. New York: John Wiley & Sons.

Pratt, Shannon P., Robert F. Reilly, and Robert P. Schweihs. 1996. *Valuing a Business: The Analysis and Appraisal of Closely Held Companies, 3rd ed.* New York: McGraw-Hill.

Wonnacott, Thomas H., and Ronald J. Wonnacott. 1981. *Regression: A Second Course in Statistics*. New York: John Wiley & Sons.

# APPENDIX 3A

# The ANOVA Table (Table A3.1, Rows 28–32)

We have already discussed the importance of variance in regression analysis. The center section of Table A3.1, which is an extension of Table 3.1B, contains an *analysis of variance* (ANOVA), automatically generated by the spreadsheet. We calculate the components of ANOVA in the top portion of the table to "open up the black box" and show the reader the source of the numbers.

In D7 to D16, we calculate the regression estimate of adjusted costs using the regression equation:

$$\text{Costs} = \$56{,}770 + (0.80 \times \text{Sales})\ [\text{B35} + (\text{B36} \times \text{column B})].$$

Next, we subtract the average actual adjusted cost of $852,420 (C18) from the regression calculated costs in column D to arrive at the deviation from the mean in column E. In standard statistical notation, this is $\hat{Y} - \overline{Y}$. Note that the sum of the deviations is zero in E17, as it must be.

In column F, we square each deviation term in column E and total them in F17. This is $\sum_i (\hat{Y}_i - \overline{Y}_i)^2$. The total, $8.31 \times 10^{11}$, is known as the *sum of squares of the regression* and measures the amount of variation explained by the regression. In the absence of a regression, our best estimate of costs for any year during the 1998–2007 period is $\overline{Y}$, the mean cost. The difference between the historical mean and the regression estimate (column E) is the deviation explained by the regression and its square (column F) is the *regression sum of squares (SS)*. This term appears in C30.

In column G we calculate the difference between the actual cost ($Y$) and the calculated cost (the regression estimate, $\overline{Y}$) by subtracting the values in column D from column C. Again, the sum of the deviations is zero. We square the deviations and sum them, $\sum_i (Y_i - \hat{Y}_i)^2$, to arrive at a value of $2.05 \times 10^9$ (H17). This second sum of squares, which appears in the ANOVA table in C31, is the unexplained variation, known as the *residual sum of squares*. We calculate the corresponding mean square error term in column I by dividing the values in column H by 8 (B31), the number of degrees of freedom of the residual. The sum is $2.56 \times 10^8$ (I17), which appears in the ANOVA table in D31. Finally, we calculate the $F$-statistic

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table A3.1** | | | | | | | | |
| 2 | **Regression Analysis 1998–2007** | | | | | | | | |
| 4 | | **Actual** | | **Calculated** | **Deviation of Calc.** | **Regr Sum of** | **Deviation of Actual** | **Deviation from** | **Mean Square [6]** |
| 5 | | | | **Costs = $Y_{hat}$ [3]** | **from Mean [4]** | **Squares** | **from Calc. [5]** | **Actual Squared** | |
| 6 | **Year** | **Sales = $X$ [1]** | **Adj. Costs = $Y$ [2]** | | **= $Y_{hat} - Y_{bar}$** | **= $(Y_{hat} - Y_{bar})^2$** | **= $Y - Y_{hat}$** | **= $(Y - Y_{hat})^2$** | **= $(Y - Y_{hat})^2/8$** |
| 7 | 1998 | $250,000 | $242,015 | $257,889 | -$594,532 | 3.53E+11 | -$15,874 | 2.52E+08 | 3.15E+07 |
| 8 | 1999 | $500,000 | $458,916 | $459,007 | -$393,413 | 1.55E+11 | -$92 | 8.40E+03 | 1.05E+03 |
| 9 | 2000 | $750,000 | $696,461 | $660,126 | -$192,295 | 3.70E+10 | $36,336 | 1.32E+09 | 1.65E+08 |
| 10 | 2001 | $1,000,000 | $863,159 | $861,244 | $8,824 | 7.79E+07 | $1,915 | 3.67E+06 | 4.59E+05 |
| 11 | 2002 | $1,060,000 | $891,517 | $909,512 | $57,092 | 3.26E+09 | -$17,995 | 3.24E+08 | 4.05E+07 |
| 12 | 2003 | $1,123,600 | $965,043 | $960,677 | $108,257 | 1.17E+10 | $4,366 | 1.91E+07 | 2.38E+06 |
| 13 | 2004 | $1,191,016 | $1,012,745 | $1,014,911 | $162,491 | 2.64E+10 | -$2,166 | 4.69E+06 | 5.86E+05 |
| 14 | 2005 | $1,262,477 | $1,072,633 | $1,072,400 | $219,979 | 4.84E+10 | $233 | 5.42E+04 | 6.78E+03 |
| 15 | 2006 | $1,338,226 | $1,122,714 | $1,133,338 | $280,917 | 7.89E+10 | -$10,623 | 1.13E+08 | 1.41E+07 |
| 16 | 2007 | $1,415,000 | $1,199,000 | $1,195,101 | $342,680 | 1.17E+11 | $3,899 | 1.52E+07 | 1.90E+06 |
| 17 | **Total** | | | | **$0** | **8.31E+11** | **$0** | **2.05E+09** | **2.56E+08** |
| 18 | | **$989,032** | **$852,420 = Average (Mean) Adj Costs ($Y_{bar}$)** | | | | | | |
| 19 | **SUMMARY OUTPUT** | | | | | | | | |
| 21 | *Regression Statistics* | | | | | | | | |
| 22 | Multiple R | 0.999 | | | | | | | |
| 23 | R Square | 0.998 | | | | | | | |
| 24 | Adjusted R Square | 0.997 | | | | | | | |
| 25 | Standard Error | **16,014** | **<--- SQRT(D31) = SQRT(I17)** | | | | | | |
| 26 | Observations | 10 | | | | | | | |
| 27 | | | | | | | | | |
| 28 | **ANOVA [7]** | | | | | | | | |
| 29 | | *df* | *SS* | *MS = SS/df* | *F=MS$_{Reg}$/MS$_{Res}$* | *Signif F* | | | |
| 30 | Regression | 1 | 8.31E+11 | 8.31E+11 | 3.24E+03 | 1.00E-11 | | | |
| 31 | Residual | 8 | 2.05E+09 | 2.56E+08 | | | | | |
| 32 | Total | 9 | 8.33E+11 | | | | | | |
| 33 | | | | | | | | | |
| 34 | | *Coef* | *Std Err* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | | |
| 35 | Intercept [8] | 56,770 | 14,863 | 3.82 | 0.01 | 22,496 | 91,045 | | |
| 36 | Sales = X [9] | 0.80 | 0.01 | 56.94 | 0.00 | 0.77 | 0.84 | | |
| 38 | [a] This is an extension of Table 3.1B. | | | | | | | | |
| 40 | [1] From Table 3.1A, row 7. | | | | | | | | |
| 42 | [2] From Table 3.1A, row 27. | | | | | | | | |
| 44 | [3] Calculated costs using Costs = $56,770 + (0.80 × Sales) with sales figures in column B, that is, B35 + (B36 × column B). | | | | | | | | |
| 46 | [4] Deviation of calculated costs from average actual costs (column D – C18) = $Y_{hat} - Y_{bar}$. The square of this in column E is the explained variance, or the Regression SS. | | | | | | | | |
| 48 | [5] Deviation of actual costs from calculated costs (column C – column D). The square of this in column H is the unexplained variance, or Residual SS. | | | | | | | | |
| 50 | [6] Deviations squared /8 (degrees of freedom). Thus, column I = column H/B31 degrees of freedom. | | | | | | | | |
| 52 | [7] ANOVA Details: C30 = F17; C31 = H17; D31 = I17. | | | | | | | | |
| 54 | [8] Regression estimate of fixed costs. | | | | | | | | |
| 56 | [9] Regression estimate of variable costs. | | | | | | | | |

of $3.42 \times 10^3 = 3{,}242$ (E30) by dividing the *mean squared error* (MS)[26] of the regression by the MS of the residual (E30 = D30/D31).

The mean squared error is the sum of squares divided by the degrees of freedom. For the regression there is only one (B30) degree of freedom, as we have only one independent variable. Thus, D30 = C30/B30. The residual mean square equals $(2.05 \times 10^9) / 8 = 2.56 \times 10^8$ (C31/B31 = D31).

The $F$-statistic is the regression MS divided by the residual MS, or D30/D31 = E30 = $3.24 \times 10^3$.

The explained variation plus the unexplained variation equals the total variation. The correlation coefficient, $R^2 = \frac{\textit{Explained Variation of Y}}{\textit{Total Variation of Y}}$. In our case, the explained variation (C30) divided by the total variation (C32) is equal to 99.8%, as seen in B23.

[26]We explain MS in the next paragraph.

# CHAPTER 4

# Annuity Discount Factors and the Gordon Model

# CHAPTER 4

# Annuity Discount Factors and the Gordon Model

## Introduction

This chapter describes the derivation of *annuity discount factors* (ADFs) and the *Gordon model* (Gordon and Shapiro 1956).[1] The ADF is the present value of a finite stream of cash flows (CFs) with constant or zero growth, assuming the first cash flow equals $1.00. Thus, the actual first year's cash flow times the ADF is the present value, as of time zero, of the stream of cash flows from years 1 to $n$. Growth rates in cash flows may be positive, zero, or negative, the latter being a decline in cash flows.

The Gordon model is identical to the ADF, except that it produces the present value of perpetuity for each $1.00 of initial cash flow. The resulting present value is known as the *Gordon model multiple*. When using the Gordon model multiple, the discount rate must be larger than the constant growth rate, which is not true of the ADF.

There are several varieties of ADFs, depending on whether the cash flows:

- Are constant or grow/decline.
- Occur midyear or at the end of the year.
- Begin in the first year or at some other time.
- Occur every year or at regular, skipped intervals.
- Finish on a whole year or a fractional year.

This chapter begins with the derivation of the ADF, and later shows that the Gordon model, which is the present value of a perpetual annuity with constant growth, is simply a special case of the ADF. We will demonstrate that an ADF is actually the difference of two perpetuities.

There are several uses of ADFs, including:

- Calculating the present value of annuities. This application has become far more important because the quantitative marketability discount model (Mercer, 1997) requires an ADF with growth (see Chapter 8). While Mercer's book has an

[1] Gordon and Shapiro were preceded by Williams (1938). See also Gordon (1962).

approximation of the ADF on page 276 that appears to be fairly accurate, this chapter contains the exact formulas.

- Valuing periodic cash flows such as moving expenses, losses from lawsuits, and so on. This requires a specialized ADF called a *periodic perpetuity factor* (PPF), which we develop later in the chapter. Additionally, PPFs are useful for decisions in buying new versus used income-producing equipment (such as airplanes, ships, fleets of trucks, taxicabs, MRIs, and CT scanners) and for calculating the value of used equipment.[2]
- Calculating loan payments.
- Calculating loan principal amortization.
- Calculating the present value of a loan. This is important in calculating the correct selling price of a business, as seller financing typically takes place at less-than-market rates. The present value of a loan is also important in Employee Stock Ownership Plan (ESOP) valuation.

At first glance, this chapter appears mathematically very intensive and daunting in its use of geometric sequences. However, the primary concepts appear in equations (4.1) through (4.9), and once you understand those equations the remainder are merely special cases or slight variations on the original theme and can be easily comprehended. While the formulas look complex, we decompose them into units that behave as modular "building blocks," each of which has an intuitive explanation. You will benefit from understanding the math in the body of the chapter, as this material is useful in several areas of business valuation. Additionally, you will gain a much better understanding of the Gordon model, which appraisers often use in discounted future net income or discounted cash flow valuation.

ADFs are an area that many practitioners find difficult, leading to many mistakes. Timing errors in ADFs frequently result from the fact that the guideline company method uses the most recent *historical* earnings for calculating price/earnings (PE) multiples, whereas the Gordon model uses the first *future* period's forecast cash flow as its earnings base. Many practitioners confuse the two and use historical rather than forecast earnings as their base in a discounted cash flow or discounted future net income approach. Another common error is the use of end-of-year multiples when midyear Gordon model multiples are appropriate.

The ADF formulas given within the chapter apply only to cash flow streams that have a whole number of years associated with them. If the cash flow stream ends in a fractional year, you should use the formulas in Appendix A for ADFs with stub periods.

Unless otherwise specified, all ADF formulas are for cash flows with constant growth. At specific points in the chapter, we make the simplifying assumption that growth is zero and clearly state when that is the case. Otherwise, the reader may assume growth is constant and non-zero.

## Definitions

Let us initially consider an ADF with constant growth in cash flows, where the last cash flow occurs in period $n$. We will use the following definitions:

[2] *CFO* magazine wrote an article about this formula, "A Beautiful Find," April 2002, p. 18.

$r$ = discount rate.
$g$ = annual growth rate in cash flows.
$ADF$ = annuity discount factor.
$PV$ = present value.
$CF$ = cash flow.
$LHS$ = left-hand side of the equation.
$RHS$ = right-hand side of the equation.
$N$ = terminal year of the cash flows.
$T$ = time (which can refer to a point in time or to a year).

### Denoting Time

Timing is frequently a source of confusion. Time $t$ denotes the time period under discussion. It generally refers to a specific year.[3] Time $t$ refers to the entire year, except for two contexts that we discuss in the next paragraph. Thus, time $t$ is a span of time, not a point in time.

There are two contexts in which time $t$ means a point in time. The first occurs with the statement $t = 0$, which means the beginning of the period $t = 1$, that is, usually the beginning of the first year of cash flows. For example, if $t = 1$ represents the calendar year 2000, then $t = 0$ means January 1, 2000, the first day of $t = 1$. Usually, but not always, $t = 0$ is the valuation date. The other context in which $t$ means a point in time is when we specify either the beginning, midpoint, or end of $t$.

In business valuation, we generally assume that cash flows occur approximately evenly throughout time $t$. In present value (PV) terms, that is approximately equivalent to assuming they occur at the midpoint of time $t$.[4] Occasionally it is appropriate to assume that cash flows occur at the end of the year, which can be the case with annuities, royalties, and so on. The former is commonly known as the *midyear assumption*, while the latter is known as the *end-of-year* (or *end-year*) *assumption*.

Another important concept related to time that can be confusing is the valuation date, the point in time to which we discount the cash flows. The valuation date is rarely the same as the first cash flow. The most common valuation date in this chapter is as of time zero (i.e., $t = 0$). The cash flows usually—but not always—either begin during year 1 or occur at the end of year 1.

## ADF with End-of-Year Cash Flows

The ADF is the present value of a series of cash flows over $n$ years with constant growth, beginning with \$1 of cash flow in year 1. We multiply the first year's forecast cash flow by the ADF to arrive at the PV of the cash flow stream. For example, if the ADF is 9.367 and the first year's cash flow is \$10,000, then the PV of the annuity is $9.367 \times \$10{,}000 = \$93{,}670$.

We begin the calculation of the ADF by defining the timing and amounts of the cash flows and discounting them to their present value. Initially, for simplicity,

[3] In the context of loan amortization, periods are usually months.
[4] We cover the exact formulas at the end of this chapter.

we assume end-of-year cash flows. The PV of an annuity of \$1, paid at the end of the year for each of $n$ years is:

$$PV = \frac{\$1}{(1+r)^1} + \frac{\$1 \times (1+g)}{(1+r)^2} + \cdots + \frac{\$1 \times (1+g)^{n-1}}{(1+r)^n}. \tag{4.1}$$

Factoring out the \$1:

$$PV = \$1 \times \left[\frac{1}{(1+r)^1} + \frac{(1+g)}{(1+r)^2} + \cdots + \frac{(1+g)^{n-1}}{(1+r)^n}\right]. \tag{4.1a}$$

The ADF is the PV of the constant growth cash flows per \$1 of starting year cash flow. Dividing both sides of equation (4.1a) by \$1, the left-hand side becomes $\frac{PV}{\$1}$, which equals the ADF. Thus, equation (4.1a) simplifies to:

$$ADF = \frac{1}{(1+r)^1} + \frac{(1+g)}{(1+r)^2} + \cdots + \frac{(1+g)^{n-1}}{(1+r)^n}. \tag{4.1b}$$

The numerators in equation (4.1b) are the forecast cash flows themselves, and the denominators are the present value factors for each cash flow. As mentioned previously, the first year's cash flow in an ADF calculation is always defined as \$1. With constant growth in cash flow, each successive year is $(1 + g)$ times the previous year's cash flow, which means that the cash flow in period $n$ is $(1 + g)^{n-1}$. The cash flow is not $(1 + g)^n$, because the first year's cash flow is \$1.00, not $1 + g$. For example, if $g = 10\%$, the first year's cash flow is, by definition, \$1.00. The second year's cash flow is $1.1 \times \$1.00 = \$1.10$. The third year's cash flow is $1.1 \times \$1.10 = 1.1^2 \times \$1.00 = 1.21$. The fourth year's cash flow is $1.1^3 \times \$1.00 = \$1.331$, and so on. The denominators in equation (4.1b) discount the cash flows in the numerator to their present value.

Next, we begin a series of algebraic manipulations that will ultimately enable us to solve for the ADF and specify it in a formula. Multiplying equation (4.1b) by $\frac{1+g}{1+r}$, we get:

$$\frac{(1+g)}{(1+r)}ADF = \frac{(1+g)}{(1+r)^2} + \cdots + \frac{(1+g)^{n-1}}{(1+r)^n} + \frac{(1+g)^n}{(1+r)^{n+1}}. \tag{4.2}$$

Notice that most of the terms in equation (4.2) are identical to equation (4.1b). We next subtract equation (4.2) from equation (4.1b). All of the terms in the middle of the equation are identical and thus drop out. The only terms that remain on the RHS after the subtraction are the first term on the RHS of equation (4.1b) and the last term on the RHS of equation (4.2).

$$ADF - \frac{1+g}{1+r}ADF = \frac{1}{1+r} - \frac{(1+g)^n}{(1+r)^{n+1}}. \tag{4.3}$$

Next, we wish to simplify only the left-hand side of equation (4.3):

$$ADF - \frac{1+g}{1+r}ADF = ADF\left[1 - \frac{1+g}{1+r}\right]. \tag{4.3a}$$

Multiplying the 1 in the square brackets on the RHS of the equation by $\frac{1+r}{1+r}$, we get:

$$ADF\left[1-\frac{1+g}{1+r}\right]=ADF\left[\frac{1+r}{1+r}-\frac{1+g}{1+r}\right]=ADF\frac{(1+r)-(1+g)}{1+r}=ADF\frac{r-g}{1+r}. \tag{4.3b}$$

Substituting the last expression of equation (4.3b) into the left-hand side of equation (4.3), we get:

$$ADF\frac{(r-g)}{(1+r)}=\left[\frac{1}{(1+r)}-\frac{(1+g)^n}{(1+r)^{n+1}}\right]. \tag{4.4}$$

Multiplying both sides of the equation by $\frac{1+r}{r-g}$, we obtain:

$$ADF=\frac{(1+r)}{(r-g)}\left[\frac{1}{(1+r)}-\frac{(1+g)^n}{(1+r)^{n+1}}\right]. \tag{4.5}$$

After canceling out the $(1 + r)$, this simplifies to:

$$ADF=\frac{1}{r-g}-\left[\left(\frac{1+g}{1+r}\right)^n\frac{1}{r-g}\right] \tag{4.6}$$

ADF with growth and end-of-year cash flows.

There are three alternative ways to regroup the terms in equation (4.6) that will prove useful, which we label as equations (4.6a), (4.6b), and (4.6c). In the first alternative expression for equation (4.6), we split up the first term in the square brackets into two separate terms, placing the denominator at the far right:

$$ADF=\frac{1}{r-g}-\left[(1+g)^n\frac{1}{r-g}\,\frac{1}{(1+r)^n}\right]\quad\text{First alternative expression for (4.6).} \tag{4.6a}$$

We derive the second alternative expression by simply factoring out the $\frac{1}{r-g}$ from equation (4.6) and restate the equation as equation (4.6b). It has the advantage of being more compact than equation (4.6):

$$ADF=\frac{1}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^n\right] \tag{4.6b}$$

Second alternative expression for (4.6).

After we develop some additional results, we will be able to explain equations (4.6) through (4.6b) intuitively. In the meantime, we will make some substitutions in equation (4.6b) that will greatly simplify its form and eventually make the ADF much more intuitive.

Note that the first term on the right-hand side of equation (4.6b) is the classical Gordon model multiple, $\frac{1}{r-g}$. Let's denote it *GM*. The next substitution that will simplify the expression is to let $x=\frac{1+g}{1+r}$. Then we can restate equation (4.6b) as:

$$ADF=GM(1-x^n) \tag{4.6c}$$

Third alternative expression for (4.6).

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table 4.1** | | | | | | | | | |
| 2 | **ADFs for *n* = 3 Years** | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | **Disc Rate = *r*** | **Growth Rate = *g*** | | | | | | | | |
| 5 | | -3% | -2% | -1% | 0% | 1% | 2% | 3% | 4% | **5%** |
| 6 | 10% | 2.4177 | 2.4406 | 2.4636 | 2.4869 | 2.5102 | 2.5337 | 2.5574 | 2.5812 | 2.6052 |
| 7 | 11% | 2.3762 | 2.3985 | 2.4210 | 2.4437 | 2.4665 | 2.4895 | 2.5126 | 2.5358 | 2.5592 |
| 8 | 12% | 2.3358 | 2.3577 | 2.3797 | 2.4018 | 2.4241 | 2.4465 | 2.4691 | 2.4918 | 2.5146 |
| 9 | 13% | 2.2967 | 2.3180 | 2.3395 | 2.3612 | 2.3829 | 2.4048 | 2.4269 | 2.4490 | 2.4713 |
| 10 | 14% | 2.2587 | 2.2795 | 2.3005 | 2.3216 | 2.3429 | 2.3643 | 2.3858 | 2.4075 | 2.4293 |
| 11 | 15% | 2.2217 | 2.2421 | 2.2626 | 2.2832 | 2.3040 | 2.3249 | 2.3460 | 2.3671 | 2.3884 |
| 12 | 16% | 2.1857 | 2.2057 | 2.2257 | 2.2459 | 2.2662 | 2.2866 | 2.3072 | 2.3279 | 2.3487 |
| 13 | 17% | 2.1508 | 2.1702 | 2.1899 | 2.2096 | 2.2294 | 2.2494 | 2.2695 | 2.2898 | 2.3101 |
| 14 | 18% | 2.1168 | 2.1358 | 2.1550 | 2.1743 | 2.1937 | 2.2132 | 2.2329 | 2.2527 | 2.2726 |
| 15 | 19% | 2.0837 | 2.1023 | 2.1210 | 2.1399 | 2.1589 | 2.1780 | 2.1972 | 2.2166 | 2.2361 |
| 16 | **20%** | 2.0514 | 2.0697 | 2.0880 | 2.1065 | 2.1251 | 2.1438 | 2.1626 | 2.1815 | **2.2005** |
| 17 | 21% | 2.0201 | 2.0379 | 2.0559 | 2.0739 | 2.0921 | 2.1104 | 2.1288 | 2.1473 | 2.1659 |
| 18 | 22% | 1.9895 | 2.0070 | 2.0246 | 2.0422 | 2.0600 | 2.0779 | 2.0959 | 2.1141 | 2.1323 |
| 19 | 23% | 1.9598 | 1.9769 | 1.9941 | 2.0114 | 2.0288 | 2.0463 | 2.0639 | 2.0817 | 2.0995 |
| 20 | 24% | 1.9308 | 1.9475 | 1.9644 | 1.9813 | 1.9983 | 2.0155 | 2.0328 | 2.0501 | 2.0676 |
| 21 | 25% | 1.9025 | 1.9189 | 1.9354 | 1.9520 | 1.9687 | 1.9855 | 2.0024 | 2.0194 | 2.0365 |
| 22 | | | | | | | | | | |
| 23 | Growth Rate = *g* | 5% | | | | | | | | |
| 24 | Discount Rate = *r* | 20% | | | | | | | | |
| 25 | *x* = (1+*g*)/(1+*r*) | 0.875 | | | | | | | | |
| 26 | *n* = # Yrs | 3 | | | | | | | | |
| 27 | *GM* = 1/(*r*–*g*) | 6.6667 | | | | | | | | |
| 28 | *ADF* = *GM**(1–*x*^*n*) | **2.2005** | | | | | | | | |

## Behavior of the ADF with Growth

The ADF is inversely related to $r$ and directly related to $g$; that is, an increase in the discount rate decreases the ADF—and vice versa—while an increase in the growth rate causes an increase in the ADF—and vice versa. Rather than take partial derivatives, we will look into the intuition of why this is so.

Let's break it into its components. Looking at equation (4.6b), the first term is the Gordon model multiple. That term obviously declines with an increase in $r$ and rises with an increase in $g$.

The second term, however, behaves the opposite. An increase in $g$ causes an increase in $x$ and $x^n$ and a decrease in $1 - x^n$. An increase in $r$ has the opposite effect. Thus, $g$ and $r$ behave in an opposite fashion in the second than they do in the first term of equation (4.6c). The opposite effect of the second term is smaller when $n$ is large than when it is small because as $n$ approaches infinity, $x$ approaches zero. Thus we would expect to a greater difference between the largest and smallest ADF when $n$ is large than when it is small.

We can see this in Table 4.1. Our starting assumptions are $g = 5\%$ (B23), $r = 20\%$ (B24), and $n = 3$ years (B26). Our intermediate calculation $x = 1.05/1.20 = 0.875$ (B25). Our Gordon model multiple GM $= 1/(0.20 - 0.05) = 6.6667$ (B27), and the ADF $=$ GM $(1 - x^n) = 6.6667 \times 0.33078 = 2.2005$ (B28). The body of the table is a sensitivity analysis showing the ADF for different combinations of $r$ and $g$. Note that J16 also equals 2.2005, which demonstrates the accuracy of the formula.

Table 4.1A is identical to Table 4.1, with the only difference being that we set $n = 20$ years. The ADF is now 6.2053 (B28, J16).

You can see by scanning down the columns and across the rows that the ADF is negatively related to $r$ and positively related to $g$; that is, the ADF decreases going down the columns and increases going right across the rows. The differences

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table 4.1A** | | | | | | | | | |
| 2 | **ADFs for *n* = 20 Years** | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | **Disc Rate = *r*** | **Growth Rate = *g*** | | | | | | | | |
| 5 | | -3% | -2% | -1% | 0% | 1% | 2% | 3% | 4% | **5%** |
| 6 | 10% | 7.0705 | 7.5064 | 7.9857 | 8.5136 | 9.0959 | 9.7390 | 10.4505 | 11.2384 | 12.1121 |
| 7 | 11% | 6.6611 | 7.0553 | 7.4879 | 7.9633 | 8.4866 | 9.0632 | 9.6998 | 10.4032 | 11.1817 |
| 8 | 12% | 6.2908 | 6.6485 | 7.0401 | 7.4694 | 7.9410 | 8.4596 | 9.0307 | 9.6607 | 10.3563 |
| 9 | 13% | 5.9551 | 6.2804 | 6.6359 | 7.0248 | 7.4509 | 7.9186 | 8.4326 | 8.9983 | 9.6218 |
| 10 | 14% | 5.6496 | 5.9464 | 6.2699 | 6.6231 | 7.0094 | 7.4323 | 7.8962 | 8.4057 | 8.9660 |
| 11 | 15% | 5.3710 | 5.6424 | 5.9377 | 6.2593 | 6.6103 | 6.9939 | 7.4137 | 7.8738 | 8.3788 |
| 12 | 16% | 5.1161 | 5.3650 | 5.6351 | 5.9288 | 6.2487 | 6.5975 | 6.9784 | 7.3951 | 7.8514 |
| 13 | 17% | 4.8823 | 5.1111 | 5.3589 | 5.6278 | 5.9199 | 6.2379 | 6.5845 | 6.9628 | 7.3764 |
| 14 | 18% | 4.6674 | 4.8781 | 5.1060 | 5.3527 | 5.6203 | 5.9110 | 6.2271 | 6.5715 | 6.9472 |
| 15 | 19% | 4.4692 | 4.6639 | 4.8739 | 5.1009 | 5.3465 | 5.6128 | 5.9019 | 6.2162 | 6.5584 |
| 16 | **20%** | 4.2862 | 4.4663 | 4.6603 | 4.8696 | 5.0956 | 5.3402 | 5.6052 | 5.8928 | **6.2053** |
| 17 | 21% | 4.1166 | 4.2837 | 4.4633 | 4.6567 | 4.8652 | 5.0904 | 5.3339 | 5.5976 | 5.8836 |
| 18 | 22% | 3.9592 | 4.1145 | 4.2812 | 4.4603 | 4.6530 | 4.8608 | 5.0850 | 5.3274 | 5.5898 |
| 19 | 23% | 3.8129 | 3.9575 | 4.1124 | 4.2786 | 4.4572 | 4.6493 | 4.8562 | 5.0796 | 5.3209 |
| 20 | 24% | 3.6764 | 3.8114 | 3.9557 | 4.1103 | 4.2760 | 4.4540 | 4.6455 | 4.8517 | 5.0741 |
| 21 | 25% | 3.5490 | 3.6752 | 3.8099 | 3.9539 | 4.1081 | 4.2733 | 4.4508 | 4.6416 | 4.8470 |
| 22 | | | | | | | | | | |
| 23 | Growth Rate = *g* | 5% | | | | | | | | |
| 24 | Discount Rate = *r* | 20% | | | | | | | | |
| 25 | *x* = (1+*g*)/(1+*r*) | 0.875 | | | | | | | | |
| 26 | *n* = # Yrs | 20 | | | | | | | | |
| 27 | *GM* = 1/(*r*–*g*) | 6.6667 | | | | | | | | |
| 28 | *ADF* = *GM**(1–*x*^*n*) | **6.2053** | | | | | | | | |

between the high and low numbers are greater in Table 4.1A, because as $n$ increases, the contrary effect of $r$ and $g$ in the second term of the equation diminish toward zero, and we are left with the unambiguous relationship that the GM multiple is negatively related to $r$ and positively related to $g$, and so is the ADF.

## Special Case of ADF When $g = 0$: The Ordinary Annuity

When $g = 0$, there is no growth in cash flows, and equation (4.6) simplifies to equation (4.6d), the formula for an ordinary annuity.

$$ADF = \frac{1}{r} - \frac{1}{(1+r)^n}\,\frac{1}{r}, \quad or\ ADF = \frac{1 - \frac{1}{(1+r)^n}}{r}. \tag{4.6d}$$

$\frac{1}{r}$ is the PV of a perpetuity that is constant in nominal dollars, or a Gordon model with $g = 0$.

## Special Case When $n \to \infty$ and $r > g$: The Gordon Model

The Gordon model is a financial formula that every business appraiser knows—at least in the end-of-year form. It is the formula necessary to calculate the present value of the perpetuity with constant growth in cash flows in the terminal period (also known as the *residual* or *reversion period*), that is, from years $n + 1$ to infinity (after discounting the first $n$ years of cash flows or net income). To be valid, the growth rate must be less than the discount rate.

What few practitioners know, however, is that the Gordon model is merely a special case of the ADF. The Gordon model contains two additional assumptions that the ADF in equation (4.6) does not have:

1. The time horizon is infinite, which means that we assume that cash flows will grow at the constant rate of $g$ forever. This means that $n$, the terminal year of the cash flows, equals infinity.
2. The discount rate is greater than the growth rate, that is, $r > g$.

Since $r > g$, $\left(\frac{1+g}{1+r}\right)^n$ goes to zero as $n$ goes to infinity. Therefore, the entire term in square brackets in equation (4.6) goes to zero, which simplifies to:

$$ADF = \frac{1}{r-g} \tag{4.7}$$

Gordon model multiple, end-of-year cash flows.

Equation (4.7) is the end-of-year Gordon model multiple. In other words, the Gordon model multiple is just a special case of the ADF when $n$ equals infinity. Using this multiple, we obtain the Gordon model, with end-of-year cash flows:

$$PV = \frac{CF}{(r-g)}. \tag{4.8}$$

Another way of expressing equation (4.8) is rewriting it as:

$$PV = CF \times \left[\frac{1}{(r-g)}\right]. \tag{4.9}$$

Thus, the present value of a perpetuity with growth contains two terms conceptually:

1. $CF$, the starting year's *forecast* cash flow[5]
2. $\frac{1}{r-g}$, the Gordon model multiple, which when multiplied by the first year's forecast cash flow gives us the present value of the perpetuity

## Intuitively Understanding Equations (4.6) and (4.6a)

Now that we understand the Gordon model, we can gain deeper insight into equation (4.6). The ADF is the difference of two perpetuities. The first term, $\frac{1}{r-g}$, is the PV as of $t = 0$ of a perpetuity with cash flows starting at \$1.00 going from $t = 1$ to infinity. The second term is the PV as of $t = 0$ of a perpetuity going from $t = n + 1$ to infinity, which is explained in the next paragraph. The difference of the two is the PV as of $t = 0$ of the annuity from $t = 1$ to $n$.

Let's give an intuitive explanation of equation (4.6a). The $(1 + g)^n$ is the forecast cash flow[6] for year $(n + 1)$, which we then multiply by $\frac{1}{r-g}$, our familiar Gordon model multiple. The result is the PV as of $t = n$ of the forecast cash flows from $n + 1$ to infinity. Dividing by $(1 + r)^n$ transforms the PV as of $t = n$ to the PV as of

[5] Note that you do not use *historical* cash flow (or earnings).

[6] The first year's cash flow is 1, or $(1 + g)^0$. The second year's cash flow is $(1 + g)^1$. In general, cash flow in year $t = (1 + g)^{t-1}$.

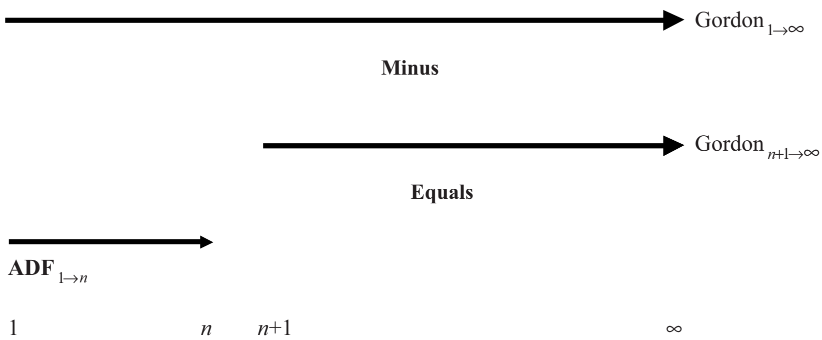


**FIGURE 4.1** **Timeline of the ADF and the Gordon Model**

$t = 0$. We subtract this from $1/(r - g)$, which is the PV of the perpetuity at $t = 0$, to yield the ADF.

## Relationship between the ADF and the Gordon Model

The relationship between the ADF and the Gordon model is so intimate that we can derive the Gordon model from the ADF and vice versa. The ADF is the difference between two Gordon models, as illustrated graphically in Figure 4.1.

In graphical terms, the top line in the figure represents the Gordon model with cash flows from $t = 1$ to infinity (our valuation date is actually time zero, which is not shown on the graph). The cash flows in the second Gordon model begin at $t = n + 1$ and continue to infinity. The difference between these two Gordon models is simply the ADF from $t = 1$ to $n$.

## Table 4.2: Demonstrating ADF Equations (4.6) through (4.6c)

Table 4.2 is the valuation of a 10-year annuity, with a discount rate of 15% and an annual growth rate of 5.1%. All assumptions appear in F24 to F28. Recall that we define $x = \frac{1+g}{1+r} = 0.8750$ (F27). If this were a perpetuity, the Gordon model multiple would be 10.101010 (F28).

We begin with a cash flow of \$1.00 at the end of year 1 (B5). Column C shows the annual growth in cash flows at 5.1%.[7] The cash flow in column B is always equal to the previous cash flow plus the growth in the current period, where Cash Flow$_t$ = Cash Flow$_{t-1}$ + Growth$_t$. Column D replicates the cash flow in column B using the formula, Cash Flow $= (1 + g)^{t-1}$, which thus provides us with a general formula for the cash flows. We multiply the cash flows in column D by the end-of-year present value factor in column E to arrive at the present value of the cash flows in column F. The sum of the present values of the 10 years of cash flows is 5.99506 (F15). This is the *brute force* method of calculating the annuity.

As we will demonstrate, equation (4.6) is a more compact and elegant solution. B20 contains the end-of-year Gordon multiple results of the first term in equation (4.6), which equals F28. This is the present value of the perpetuity of \$1.00 growing at a constant 5.1% from year 1 to infinity. In C20, we subtract the present value of

[7]We can use the same formulas for other time periods (e.g., months instead of years). Then we must use the monthly growth rate of 5.1%/12 = 0.425% instead of the annual.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Table 4.2** | | | | | |
| 2 | **ADF: End-of-Year Formula** | | | | | |
| 3 | | | | | | |
| 4 | ***t* (Yrs)** | **Cash Flow (CF)** | **Growth in CF** | **$(1+g)^{t-1}$** | **PV Factor** | **NPV** |
| 5 | **1** | 1.00000 | 0.00000 | 1.00000 | 0.86957 | 0.86957 |
| 6 | **2** | 1.05100 | 0.05100 | 1.05100 | 0.75614 | 0.79471 |
| 7 | **3** | 1.10460 | 0.05360 | 1.10460 | 0.65752 | 0.72629 |
| 8 | **4** | 1.16094 | 0.05633 | 1.16094 | 0.57175 | 0.66377 |
| 9 | **5** | 1.22014 | 0.05921 | 1.22014 | 0.49718 | 0.60663 |
| 10 | **6** | 1.28237 | 0.06223 | 1.28237 | 0.43233 | 0.55440 |
| 11 | **7** | 1.34777 | 0.06540 | 1.34777 | 0.37594 | 0.50668 |
| 12 | **8** | 1.41651 | 0.06874 | 1.41651 | 0.32690 | 0.46306 |
| 13 | **9** | 1.48875 | 0.07224 | 1.48875 | 0.28426 | 0.42320 |
| 14 | **10** | 1.56468 | 0.07593 | 1.56468 | 0.24718 | 0.38676 |
| 15 | **Totals** | | | | | **5.99506** |
| 16 | | | | | | |
| 17 | **Calculation of NPV by Formulas:** | | | | | |
| 18 | | | | | **Grand** | |
| 19 | **Time** | **1 to Infinity** | **–(*n*+1) to Infinity** | **= 1 to *n*** | **Total** | |
| 20 | **NPV** | 10.10101 | -4.10595 | 5.99506 | **5.99506** | |
| 21 | | | | | | |
| 22 | **Assumptions:** | | | | | |
| 23 | | | | | | |
| 24 | ***n* = Number of Years of Cash Flows** | | | | | **10** |
| 25 | ***r* = Discount Rate** | | | | | **15.0%** |
| 26 | ***g* = Growth Rate in Net Inc/Cash Flow** | | | | | **5.1%** |
| 27 | **$x = (1+g)/(1+r)$** | | | | | **0.9139** |
| 28 | **Gordon Model Multiple = GM = $1/(r-g)$** | | | | | **10.101010** |
| 29 | | | | | | |
| 30 | **Spreadsheet Formulas:** | | | | | |
| 31 | | | | | | |
| 32 | **B20:** | GM = 1/(*r*–*g*) | | | | |
| 33 | **C20:** | – GM**x*^*n* | | | | |
| 34 | **D20** | B20+C20 | | | | |
| 35 | **E20** | GM * (1–*x*^*n*) This is equation (4.6c) | | | | |

the perpetuity from year $n + 1$ to infinity, which equals 4.10595 and is the term in equation (4.6) in square brackets. The difference of the two perpetuities is 5.99506 (D20), which equals F15, our brute force solution. Finally, E20 is the formula for the entire equation, which equals the same 5.99506 calculated in D20 and F15, proving the validity of equation (4.6), including its components. We show the formulas for row 20 at the bottom of Table 4.2. Note that the formula in E20 is equation (4.6c).

## A Brief Summary

To help you decide whether you should read on, let's take a look at what we have covered so far, what we will cover in the remainder of the chapter, and how difficult the material will be. We have thus far derived the end-of-year ADF, examined its special cases (the Gordon model and the no-growth formula), explained the intimate relationship between the ADF and the Gordon model, explained the intuition behind the components of the ADF model, and demonstrated the accuracy of the ADF formulas with an example.

The reader now should understand the principles of ADFs and Gordon models. If you are having difficulty with the mathematics, you may wish to skip to the sections on *periodic perpetuity factors* (PPFs) and the relationship of the Gordon model to the price/earnings ratio, which are of practical significance to most readers. However, you now should understand almost everything you will need to easily comprehend the rest of the chapter. The rest of the chapter is primarily simple variations on the derivations we have done thus far.

In the remainder of the chapter, we cover:

- The midyear version of the ADF (with the same special cases of the Gordon model and $g = 0$).
- Starting periods for the cash flows that are different from year 1—which is of practical significance in discounted cash flow analysis in the calculation of the PV of the reversion.
- Calculating PPFs, which are a variation of the Gordon model for periodic expenses such as moving expense and losses from lawsuits. Additionally, PPFs are useful for decisions in buying new versus used income-producing equipment (such as airplanes, ships, fleets of trucks, taxicabs, MRIs, and CT scanners) and for calculating the value of used equipment.
- Calculating loan payments.
- Calculating the present value of loans.
- The relationship of the Gordon model to the PE multiple, the misunderstanding of which may well be the single most common source of technical error in business valuation.

## Midyear Cash Flows

Most businesses have cash flows that occur more or less evenly throughout the year. In a present value sense, this is approximately equivalent to having all cash flows occur midway through the year. Thus, in valuing most businesses, it is appropriate to use midyear cash flows rather than end-of-year cash flows.

Midyear cash flows occur six months (one half-year) earlier than end-of-year cash flows. We derive this formula in exactly the same fashion as equation (4.6). We start with equation (4.1b); however the denominators, which are the time periods by which we discount the cash flows, are one half-year less than those in equation (4.1b). We adjust for this difference by multiplying every numerator by $\sqrt{1+r}$, which has the same effect as reducing the denominators by 0.5 years. We then factor the $\sqrt{1+r}$out of the sequence, resulting in the midyear ADF, which equals $\sqrt{1+r}$ times the end-of-year ADF.

$$ADF = \frac{\sqrt{1+r}}{r-g} - \left(\frac{1+g}{1+r}\right)^n \frac{\sqrt{1+r}}{r-g} \tag{4.10}$$

Midyear ADF.

We interpret equation (4.10) in exactly the same fashion as equation (4.6). We can factor out the Gordon model multiple as before and restate equation (4.10) as equations (4.10a) and (4.10b). Note that equations (4.10a) and (4.10b) are identical to equations (4.6b) and (4.6c), respectively, except that the Gordon model multiple is midyear instead of end-of-year.

$$ADF = \frac{\sqrt{1+\mathrm{r}}}{\mathrm{r}-\mathrm{g}}\left[1-\left(\frac{1+\mathrm{g}}{1+\mathrm{r}}\right)^{n}\right] \tag{4.10a}$$

Alternative expression for (4.10).

$$ADF = GM(1-x^{n}) \tag{4.10b}$$

Second alternative expression for (4.10).

## Table 4.3: Example of Equations (4.10) through (4.10b)

Table 4.3 is identical to Table 4.2, except that here we use the midyear rather than end-of-year ADF. Note that the Gordon model multiple (GM) in B20 and F28 is 10.83213 versus 10.101010 in Table 4.2. The GM in Table 4.3 is exactly $\sqrt{1+r}$ times the GM in Table 4.2, that is, 10.1010 $\sqrt{1.15} = 10.83213$. This demonstrates the validity of equations (4.10) through (4.10b), the midyear ADF.

## Special Cases for Midyear Cash Flows: No Growth, $g = 0$

Letting $g = 0$ in equation (4.10), we obtain the following ADF for midyear cash flows with no growth:

$$ADF = \frac{\sqrt{1+r}}{r} - \frac{1}{(1+r)^{n}}\,\frac{\sqrt{1+r}}{r} \tag{4.10c}$$

Midyear ADF, no growth.

This follows the same type of logic as equation (4.6), with modification for growth being zero. The first and third terms on the RHS of equation (4.10c) are midyear Gordon models for a constant $1 cash flow. Since there is no growth of cash flows in this special case, the $(1 + g)^n$ in equation (4.10) simplifies to 1 and drops out of the equation. The $\frac{1}{(1+r)^n}$ discounts the second Gordon model term from $t = n$ back to $t = 0$; that is, it reduces the PV of the perpetuity to time zero. Again, the ADF is the difference of two perpetuities: the first one with cash flows from years 1 to infinity, less the second one with cash flows from $n + 1$ to infinity, the difference being cash flows from years 1 to $n$.

We can rewrite equation (4.10c) as equation (4.10d) by factoring out the $\frac{\sqrt{1+r}}{r}$:

$$ADF = \frac{\sqrt{1+r}}{r}\left[1-\frac{1}{(1+r)^{n}}\right] \tag{4.10d}$$

Alternate expression for (4.10c), midyear, no growth.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Table 4.3** | | | | | |
| 2 | **ADF: Midyear Formula** | | | | | |
| 3 | | | | | | |
| 4 | ***t* (Yrs)** | **Cash Flow** | **Growth in CF** | **$(1+g)^{t-1}$** | **PV Factor** | **NPV** |
| 5 | **1** | 1.00000 | 0.00000 | 1.00000 | 0.93250 | 0.93250 |
| 6 | **2** | 1.05100 | 0.05100 | 1.05100 | 0.81087 | 0.85223 |
| 7 | **3** | 1.10460 | 0.05360 | 1.10460 | 0.70511 | 0.77886 |
| 8 | **4** | 1.16094 | 0.05633 | 1.16094 | 0.61314 | 0.71181 |
| 9 | **5** | 1.22014 | 0.05921 | 1.22014 | 0.53316 | 0.65053 |
| 10 | **6** | 1.28237 | 0.06223 | 1.28237 | 0.46362 | 0.59453 |
| 11 | **7** | 1.34777 | 0.06540 | 1.34777 | 0.40315 | 0.54335 |
| 12 | **8** | 1.41651 | 0.06874 | 1.41651 | 0.35056 | 0.49658 |
| 13 | **9** | 1.48875 | 0.07224 | 1.48875 | 0.30484 | 0.45383 |
| 14 | **10** | 1.56468 | 0.07593 | 1.56468 | 0.26508 | 0.41476 |
| 15 | **Totals** | | | | | **6.42899** |
| 16 | | | | | | |
| 17 | **Calculation of NPV by Formulas:** | | | | | |
| 18 | | | | | **Grand** | |
| 19 | **Time** | **1 to Infinity** | **–(*n*+1) to Infinity** | **= 1 to *n*** | **Total** | |
| 20 | **NPV** | 10.83213 | -4.40314 | **6.42899** | **6.42899** | |
| 21 | | | | | | |
| 22 | **Assumptions:** | | | | | |
| 23 | | | | | | |
| 24 | ***n* = Number of Years of Cash Flows** | | | | | **10** |
| 25 | ***r* = Discount Rate** | | | | | **15.0%** |
| 26 | ***g* = Growth Rate in Net Inc/Cash Flow** | | | | | **5.1%** |
| 27 | ***x* = (1+*g*)/(1+*r*)** | | | | | **0.9139** |
| 28 | **Gordon Model Multiple = GM = SQRT(1+*r*)/(*r*–*g*)** | | | | | **10.83213** |
| 29 | | | | | | |
| 30 | **Spreadsheet Formulas:** | | | | | |
| 31 | | | | | | |
| 32 | **B20:** | GM = SQRT(1+*r*)/(*r*–*g*) | | | | |
| 33 | **C20:** | – GM**x*^*n* | | | | |
| 34 | **D20** | B20+C20 | | | | |
| 35 | **E20** | GM*(1–*x*^*n*) This is equation (4.10b) | | | | |

## Gordon Model

Letting $n \to \infty$ in equation (4.10) leads us to the Gordon model:

$$PV = CF\frac{\sqrt{1+r}}{r-g} \tag{4.10e}$$

Gordon model—midyear.

This can be split into the following terms: $CF \times \left[\frac{\sqrt{1+r}}{r-g}\right]$. The first term is the forecast net income for the first year, and the second term is the Gordon model multiple for a midyear cash flow.

## Starting Periods Other Than Year 1

When cash flows begin in any year other than 1, it is necessary to use a more general (and complicated) ADF formula. We will present formulas for both the end-of-year and midyear cash flows when this occurs.

## End-of-Year Formulas

In the following equations, $S$ is the *s*tarting year of the cash flows. The end-of-year ADF is:

$$ADF = \left[\frac{1}{r-g} - \left(\frac{1+g}{1+r}\right)^{n-S+1}\frac{1}{r-g}\right]\frac{1}{(1+r)^{S-1}} \tag{4.11}$$

Generalized end-of-year ADF.

Note that when $S = 1$, $n - S + 1 = n$, and equation (4.11) reduces to equation (4.6).

The intuition behind this formula is that if we are standing at point $t = S - 1$ looking at the cash flows that begin at $S$ and end at $n$, they would appear the same as if we were at $t = 0$ looking at a normal series of $n$ cash flows that begin at $t = 1$. The only difference is that there are $n$ cash flows in the latter case and $n - (S - 1) = n - S + 1$ cash flows in the former case.

The term in the square brackets is the ADF in dollars as of year $S - 1$, and the term to the right of the square brackets is the PV factor to bring it into $t = 0$ dollars.

Thus the term in square brackets, which is the PV of the cash flows at $t = S - 1$, is the usual ADF formula, except that the exponent of the second term in square brackets changes from $n$ in equation (4.6) to $n - S + 1$ in equation (4.11). If the cash flows begin in a year later than year 1, $S > 1$ and there are fewer years of cash flows from $S$ to $n$ than there are from 1 to $n$.[8] From the end of year $S - 1$ to the end of year $n$, there are $n - (S - 1) = n - S + 1$ years.

In order to calculate the PV as of $t = 0$, it is necessary to discount the cash flows $S - 1$ years using the term $\frac{1}{(1+r)^{S-1}}$. Note that at $S = 1$, the term at the right—outside the brackets—becomes 1 and effectively drops out of the equation. The exponent within the square brackets, $n - S + 1$, simplifies to $n$, and equation (4.11) simplifies to equation (4.6).

An alternative form of equation (4.11) with the Gordon model specifically factored out is:

$$ADF = \frac{1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^{n-S+1}\right]\frac{1}{(1+r)^{S-1}} \tag{4.11a}$$

Generalized end-of-year ADF—alternative form.

## Valuation Date $\neq 0$

If the valuation date is different from $t = 0$, then we do not discount by $S - 1$ years. Letting the valuation date $= v$, then we discount back to $t = S - v - 1$, the reason being that normally we discount $S - 1$ years, but in this case we will discount to $v$, not to zero. Therefore, we discount $S - 1 - v$ years, which we restate as $S - v - 1$. For example, if we want to value cash flows from $t = 23$ months to 34 months as of $t = 10$ months,[9] then we discount $23 - 10 - 1 = 12$ months, or 1 year.

[8]The converse is true for cash flows beginning in the past, where $S$ is less than 1.

[9]We actually do this in Table A4.3 in Appendix A. In the context of loan payments, cash flows are fixed, which means $g = 0$. Also, with loan payments we generally deal with time

This formula is important in calculating the reduction in principal for an amortizing loan. The formula is:

$$ADF = \left[\frac{1}{r-g} - \left(\frac{1+g}{1+r}\right)^{n-S+1}\frac{1}{r-g}\right]\frac{1}{(1+r)^{S-v-1}} \qquad (4.11b)$$

Generalized ADF—EOY—Val date = $v$.

Note that the $n - S + 1$ in the brackets remains unchanged, because there is still the same number of cash flows. (We demonstrate the accuracy of this formula in sections 2 and 3 of Table A4.3 in Appendix A.)

## Table 4.4: Example of Equation (4.11)

In Table 4.4, we begin with $1 of cash flows (C7) at $t = 3.25$ years, that is, $S = 3.25$ (G40). The discount rate is 15% (G42), and cash flows grow at 5.1% (G43). In year 4.25, cash flow grows 5.1% × $1.00 = $0.051 (B8 = G43 × C7), and is equal to the prior-year cash flow of $1.00 in C7 plus the growth in the current year in B8, for a total of $1.051 in C8. We continue in the same fashion to calculate growth in cash flows and the actual cash flows through the last year, $n = 22.25$.

In column D, we use the formula Cash Flow = $(1 + g)^{t-S}$, which duplicates the results in column C. Thus, the formula in column D is a general formula for cash flow in any period.[10]

Next, we discount the cash flows to present value. In this table, we show both a two-step and a single-step discounting process.

First, we demonstrate two-step discounting in columns E and F. Column E contains the present value (PV) factors to discount the cash flows to $t = S - 1$, the formula for which is $\frac{1}{(1+r)^{t-S+1}}$. Column F is the PV as of $t = 2.25$ years. The present value of the cash flows totals $8.43199 (F27). F28 is the PV factor, 0.73018, to discount that result back to $t = 0$ by multiplying it by F27, or $8.43199 × 0.73018 = $6.15687 (F29).

In columns G and H, we perform the same procedures, the only difference being that column G contains the PV factors to discount back to $t = 0$. Column H is the PV of the cash flows, which totals the same $6.15687 (H27), which is the same result as F29. This demonstrates that the two-step and the one-step present value calculation lead to the same results, as long as they are done properly.

B34 contains the Gordon model multiple 10.10101 for cash flows from $t = S$ (3.25) to infinity, which we can see calculated in G45. C34 is the present value as of $t = S - 1$ of the perpetuity from year $n - S + 1$ to infinity. It equals 1.66902 and is the term in equation (4.11) in square brackets after the minus sign. Subtracting C34 from B34, we get the cash flows as of $t = S - 1$ from $S$ to $n$ in D34, or $8.43199, which also equals F27, our brute force solution. Row 35 is the PV factor 0.73018 (from $t = S - 1$ to $t = 0$), and row 34 × row 35 = row 36, the PV as

measured in months, not years. To remain consistent, the discount rates must also be monthly, not annual.

[10]Note that when cash flows begin at $t = 1$, then $(1 + g)^{t-S} = (1 + g)^{t-1}$, which is the formula that describes the cash flows in column D in Tables 4.2 and 4.3. Thus, $(1 + g)^{t-S}$ is truly a general formula for the cash flow.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table 4.4** | | | | | | | |
| 2 | **ADF with Cash Flows Starting in Year 3.25** | | | | | | | |
| 3 | **End-of-Year Formula** | | | | | | | |
| 4 | | | | | | | | |
| 5 | | **Cash Flow** | | | ***t* = *S*–1** | | ***t* = 0** | |
| 6 | ***t* (Yrs)** | **Growth** | **Cash Flow** | **$(1+g)^{t-S}$** | **PV Factor** | **PV** | **PV Factor** | **PV** |
| 7 | **3.25** | NA | 1.00000 | 1.00000 | 0.86957 | 0.86957 | 0.63494 | 0.63494 |
| 8 | **4.25** | 0.05100 | 1.05100 | 1.05100 | 0.75614 | 0.79471 | 0.55212 | 0.58028 |
| 9 | **5.25** | 0.05360 | 1.10460 | 1.10460 | 0.65752 | 0.72629 | 0.48011 | 0.53032 |
| 10 | **6.25** | 0.05633 | 1.16094 | 1.16094 | 0.57175 | 0.66377 | 0.41748 | 0.48467 |
| 11 | **7.25** | 0.05921 | 1.22014 | 1.22014 | 0.49718 | 0.60663 | 0.36303 | 0.44295 |
| 12 | **8.25** | 0.06223 | 1.28237 | 1.28237 | 0.43233 | 0.55440 | 0.31568 | 0.40481 |
| 13 | **9.25** | 0.06540 | 1.34777 | 1.34777 | 0.37594 | 0.50668 | 0.27450 | 0.36997 |
| 14 | **10.25** | 0.06874 | 1.41651 | 1.41651 | 0.32690 | 0.46306 | 0.23870 | 0.33812 |
| 15 | **11.25** | 0.07224 | 1.48875 | 1.48875 | 0.28426 | 0.42320 | 0.20756 | 0.30901 |
| 16 | **12.25** | 0.07593 | 1.56468 | 1.56468 | 0.24718 | 0.38676 | 0.18049 | 0.28241 |
| 17 | **13.25** | 0.07980 | 1.64447 | 1.64447 | 0.21494 | 0.35347 | 0.15695 | 0.25810 |
| 18 | **14.25** | 0.08387 | 1.72834 | 1.72834 | 0.18691 | 0.32304 | 0.13648 | 0.23588 |
| 19 | **15.25** | 0.08815 | 1.81649 | 1.81649 | 0.16253 | 0.29523 | 0.11867 | 0.21557 |
| 20 | **16.25** | 0.09264 | 1.90913 | 1.90913 | 0.14133 | 0.26981 | 0.10320 | 0.19701 |
| 21 | **17.25** | 0.09737 | 2.00649 | 2.00649 | 0.12289 | 0.24659 | 0.08974 | 0.18005 |
| 22 | **18.25** | 0.10233 | 2.10883 | 2.10883 | 0.10686 | 0.22536 | 0.07803 | 0.16455 |
| 23 | **19.25** | 0.10755 | 2.21638 | 2.21638 | 0.09293 | 0.20596 | 0.06785 | 0.15039 |
| 24 | **20.25** | 0.11304 | 2.32941 | 2.32941 | 0.08081 | 0.18823 | 0.05900 | 0.13744 |
| 25 | **21.25** | 0.11880 | 2.44821 | 2.44821 | 0.07027 | 0.17202 | 0.05131 | 0.12561 |
| 26 | **22.25** | 0.12486 | 2.57307 | 2.57307 | 0.06110 | 0.15722 | 0.04461 | 0.11480 |
| 27 | **Pres. Value (*t* = 2.25 for Column G, *t* = 0 for Column I)** | | | | | **8.43199** | | **6.15687** |
| 28 | **Pres. Value Factor—Discount from *S* – 1 (*t* = 2.25) to 0** | | | | | **0.73018** | | |
| 29 | **Present Value (*t* = 0)** | | | | | **6.15687** | | |
| 30 | | | | | | | | |
| 31 | **Calculation of PV by Formulas:** | | | | | | | |
| 32 | | | | | **Grand** | | | |
| 33 | **Time** | ***S* to Infinity** | **–(*n*+1) to Infinity** | **= *S* to *n*** | **Total** | | | |
| 34 | ***t* = *S* – 1** | 10.10101 | -1.66902 | 8.43199 | **8.43199** | | | |
| 35 | **PV Factor** | 0.73018 | 0.73018 | 0.73018 | **0.73018** | | | |
| 36 | ***t* = 0** | 7.37555 | -1.21869 | 6.15687 | **6.15687** | | | |
| 37 | | | | | | | | |
| 38 | **Assumptions:** | | | | | | | |
| 39 | | | | | | | | |
| 40 | ***S* = Beginning Year of Cash Flows (valuation at *t* =2.25)** | | | | | | **3.25** | |
| 41 | ***n* = Ending Year of Cash Flows** | | | | | | **22.25** | |
| 42 | ***r* = Discount Rate** | | | | | | **15.0%** | |
| 43 | ***g* = Growth Rate in Net Inc/Cash Flow** | | | | | | **5.1%** | |
| 44 | ***x* = (1+*g*)/(1+*r*)** | | | | | | **0.913913** | |
| 45 | **Gordon Model Multiple = GM = [1/(*r*–*g*)]** | | | | | | **10.101010** | |
| 46 | | | | | | | | |
| 47 | **Spreadsheet Formulas:** | | | | | | | |
| 48 | | | | | | | | |
| 49 | **B34: GM Gordon Model for Years 3.25 to Infinity as of *t* = 2.25** | | | | | | | |
| 50 | **C34: –GM*(*x*^(*n*–*S*+1)) Gordon Model for Years 23.25 to Infinity as of *t* = 2.25** | | | | | | | |
| 51 | **D34: B34 + C34** | | | | | | | |
| 52 | **E34: GM*(1–*x*^(*n*–*S*+1)) Grand Total as of *t* = *S*–1 = 2.25 Years** | | | | | | | |
| 53 | **Row 35: 1/(1+*r*)^(*S*–1) Present Value Factor from *t* = *S*–1 to *t* = 0** | | | | | | | |
| 54 | **Row 36: Row 34 * Row 35** | | | | | | | |

of $t = 0$. The PV of cash flows from $S = 3.25$ to $n$, as of $t = 0$, appears in D36 as \$6.15687.

In E34, we show the grand total cash flows, as per equation (4.11). The spreadsheet formula for E34 is in A52, where GM is the Gordon model multiple. The \$8.43199 (E34) is the total of the cash flows from 3.25 to 22.25 as of $t = 2.25$ and corresponds to the term in equation (4.11) in square brackets. The PV factor 0.73018 (E35) is the term in equation (4.11) to the right of the square brackets, and the one

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table 4.5** | | | | | | | |
| 2 | **ADF with Cash Flows Starting in Year –2.00** | | | | | | | |
| 3 | **End-of-Year Formula** | | | | | | | |
| 4 | | | | | | | | |
| 5 | | **Cash Flow** | | | ***t* = *S*–1** | | ***t* = 0** | |
| 6 | ***t* (Yrs)** | **Growth** | **Cash Flow** | **$(1+g)^{t-S}$** | **PV Factor** | **PV** | **PV Factor** | **PV** |
| 7 | **-2.00** | NA | 1.00000 | 1.00000 | 0.86957 | 0.86957 | 1.32250 | 1.32250 |
| 8 | **-1.00** | 0.05100 | 1.05100 | 1.05100 | 0.75614 | 0.79471 | 1.15000 | 1.20865 |
| 9 | **0.00** | 0.05360 | 1.10460 | 1.10460 | 0.65752 | 0.72629 | 1.00000 | 1.10460 |
| 10 | **1.00** | 0.05633 | 1.16094 | 1.16094 | 0.57175 | 0.66377 | 0.86957 | 1.00951 |
| 11 | **2.00** | 0.05921 | 1.22014 | 1.22014 | 0.49718 | 0.60663 | 0.75614 | 0.92260 |
| 12 | **3.00** | 0.06223 | 1.28237 | 1.28237 | 0.43233 | 0.55440 | 0.65752 | 0.84318 |
| 13 | **4.00** | 0.06540 | 1.34777 | 1.34777 | 0.37594 | 0.50668 | 0.57175 | 0.77059 |
| 14 | **5.00** | 0.06874 | 1.41651 | 1.41651 | 0.32690 | 0.46306 | 0.49718 | 0.70425 |
| 15 | **6.00** | 0.07224 | 1.48875 | 1.48875 | 0.28426 | 0.42320 | 0.43233 | 0.64363 |
| 16 | **7.00** | 0.07593 | 1.56468 | 1.56468 | 0.24718 | 0.38676 | 0.37594 | 0.58822 |
| 17 | **8.00** | 0.07980 | 1.64447 | 1.64447 | 0.21494 | 0.35347 | 0.32690 | 0.53758 |
| 18 | **9.00** | 0.08387 | 1.72834 | 1.72834 | 0.18691 | 0.32304 | 0.28426 | 0.49130 |
| 19 | **10.00** | 0.08815 | 1.81649 | 1.81649 | 0.16253 | 0.29523 | 0.24718 | 0.44901 |
| 20 | **11.00** | 0.09264 | 1.90913 | 1.90913 | 0.14133 | 0.26981 | 0.21494 | 0.41035 |
| 21 | **12.00** | 0.09737 | 2.00649 | 2.00649 | 0.12289 | 0.24659 | 0.18691 | 0.37503 |
| 22 | **13.00** | 0.10233 | 2.10883 | 2.10883 | 0.10686 | 0.22536 | 0.16253 | 0.34274 |
| 23 | **14.00** | 0.10755 | 2.21638 | 2.21638 | 0.09293 | 0.20596 | 0.14133 | 0.31324 |
| 24 | **15.00** | 0.11304 | 2.32941 | 2.32941 | 0.08081 | 0.18823 | 0.12289 | 0.28627 |
| 25 | **16.00** | 0.11880 | 2.44821 | 2.44821 | 0.07027 | 0.17202 | 0.10686 | 0.26163 |
| 26 | **17.00** | 0.12486 | 2.57307 | 2.57307 | 0.06110 | 0.15722 | 0.09293 | 0.23910 |
| 27 | **Pres. Value (*t* = 2.25 for Column G, *t* = 0 for Column I)** | | | | | **8.43199** | | **12.82400** |
| 28 | **Pres. Value Factor—Discount from *S*–1 (*t* = –3.00) to 0** | | | | | **1.52088** | | |
| 29 | **Present Value (*t* = 0)** | | | | | **12.82400** | | |
| 30 | | | | | | | | |
| 31 | **Calculation of PV by Formulas:** | | | | | | | |
| 32 | | | | | **Grand** | | | |
| 33 | **Time** | ***S* to Infinity** | **– (*n* + 1) to Infinity** | **= *S* to *n*** | **Total** | | | |
| 34 | ***t* = *S*–1** | 10.10101 | -1.66902 | 8.43199 | **8.43199** | | | |
| 35 | **PV Factor** | 1.52088 | 1.52088 | 1.52088 | **1.52088** | | | |
| 36 | ***t* = 0** | 15.36237 | -2.53838 | 12.82400 | **12.82400** | | | |
| 37 | | | | | | | | |
| 38 | **Assumptions:** | | | | | | | |
| 39 | | | | | | | | |
| 40 | ***S* = Beginning Year of Cash Flows (valuation at *t* = –3.00)** | | | | | | **-2.00** | |
| 41 | ***n* = Ending Year of Cash Flows** | | | | | | **17.00** | |
| 42 | ***r* = Discount Rate** | | | | | | **15.0%** | |
| 43 | ***g* = Growth Rate in Net Inc/Cash Flow** | | | | | | **5.1%** | |
| 44 | **x = (1+*g*)/(1+*r*)** | | | | | | **0.913913** | |
| 45 | **Gordon Model Multiple = GM = [1/(*r*–*g*)]** | | | | | | **10.101010** | |
| 46 | | | | | | | | |
| 47 | **Spreadsheet Formulas:** | | | | | | | |
| 48 | | | | | | | | |
| 49 | **B34: GM Gordon Model for Years –2.00 to Infinity as of *t* = –3.00** | | | | | | | |
| 50 | **C34: –GM*(*x*^(*n*–*S*+1)) Gordon Model for Years 18.00 to Infinity as of *t* = – 3.00** | | | | | | | |
| 51 | **D34: B34 + C34** | | | | | | | |
| 52 | **E34: GM*(1–*x*^(*n*–*S* + 1)) Grand Total as of *t* = *S*–1 = –3.00 Years** | | | | | | | |
| 53 | **Row 35: 1/(1+*r*)^(*S*–1) Present Value Factor from *t* = *S*–1 to *t* = 0** | | | | | | | |
| 54 | **Row 36: Row 34 * Row 35** | | | | | | | |

multiplied by the other (in E36) is the entirety of equation (4.11). Note that E36 = D36 = F29 = H27, which demonstrates the validity of equation (4.11).

## Tables 4.5 through 4.7: Variations of Table 4.4 with $S < 0$, Negative Growth, and $r < g$

Tables 4.5 through 4.7 are identical to Table 4.4. The only difference is that Tables 4.5 through 4.7 have cash flows that begin in year −2, ($S = -2.00$ in G40). Additionally,

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table 4.6** | | | | | | | |
| 2 | **ADF with Cash Flows Starting in Year –2.00 with Negative Growth** | | | | | | | |
| 3 | **End-of-Year Formula** | | | | | | | |
| 4 | | | | | | | | |
| 5 | | **Cash Flow** | | | ***t* = *S*–1** | | ***t* = 0** | |
| 6 | ***t* (Yrs)** | **Growth** | **Cash Flow** | **$(1+g)^{t-S}$** | **PV Factor** | **PV** | **PV Factor** | **PV** |
| 7 | **-2.00** | NA | 1.00000 | 1.00000 | 0.86957 | 0.86957 | 1.32250 | 1.32250 |
| 8 | **-1.00** | -0.05100 | 0.94900 | 0.94900 | 0.75614 | 0.71758 | 1.15000 | 1.09135 |
| 9 | **0.00** | -0.04840 | 0.90060 | 0.90060 | 0.65752 | 0.59216 | 1.00000 | 0.90060 |
| 10 | **1.00** | -0.04593 | 0.85467 | 0.85467 | 0.57175 | 0.48866 | 0.86957 | 0.74319 |
| 11 | **2.00** | -0.04359 | 0.81108 | 0.81108 | 0.49718 | 0.40325 | 0.75614 | 0.61329 |
| 12 | **3.00** | -0.04137 | 0.76972 | 0.76972 | 0.43233 | 0.33277 | 0.65752 | 0.50610 |
| 13 | **4.00** | -0.03926 | 0.73046 | 0.73046 | 0.37594 | 0.27461 | 0.57175 | 0.41764 |
| 14 | **5.00** | -0.03725 | 0.69321 | 0.69321 | 0.32690 | 0.22661 | 0.49718 | 0.34465 |
| 15 | **6.00** | -0.03535 | 0.65785 | 0.65785 | 0.28426 | 0.18700 | 0.43233 | 0.28441 |
| 16 | **7.00** | -0.03355 | 0.62430 | 0.62430 | 0.24718 | 0.15432 | 0.37594 | 0.23470 |
| 17 | **8.00** | -0.03184 | 0.59246 | 0.59246 | 0.21494 | 0.12735 | 0.32690 | 0.19368 |
| 18 | **9.00** | -0.03022 | 0.56225 | 0.56225 | 0.18691 | 0.10509 | 0.28426 | 0.15983 |
| 19 | **10.00** | -0.02867 | 0.53357 | 0.53357 | 0.16253 | 0.08672 | 0.24718 | 0.13189 |
| 20 | **11.00** | -0.02721 | 0.50636 | 0.50636 | 0.14133 | 0.07156 | 0.21494 | 0.10884 |
| 21 | **12.00** | -0.02582 | 0.48054 | 0.48054 | 0.12289 | 0.05906 | 0.18691 | 0.08982 |
| 22 | **13.00** | -0.02451 | 0.45603 | 0.45603 | 0.10686 | 0.04873 | 0.16253 | 0.07412 |
| 23 | **14.00** | -0.02326 | 0.43277 | 0.43277 | 0.09293 | 0.04022 | 0.14133 | 0.06116 |
| 24 | **15.00** | -0.02207 | 0.41070 | 0.41070 | 0.08081 | 0.03319 | 0.12289 | 0.05047 |
| 25 | **16.00** | -0.02095 | 0.38976 | 0.38976 | 0.07027 | 0.02739 | 0.10686 | 0.04165 |
| 26 | **17.00** | -0.01988 | 0.36988 | 0.36988 | 0.06110 | 0.02260 | 0.09293 | 0.03437 |
| 27 | **Pres. Value (*t* = 2.25 for Column G, *t* = 0 for Column I)** | | | | | **4.86842** | | **7.40426** |
| 28 | **Pres. Value Factor—Discount from *S*–1 (*t* = –3.00) to 0** | | | | | **1.52088** | | |
| 29 | **Present Value (*t* = 0)** | | | | | **7.40426** | | |
| 30 | | | | | | | | |
| 31 | **Calculation of PV by Formulas:** | | | | | | | |
| 32 | | | | | **Grand** | | | |
| 33 | **Time** | ***S* to Infinity** | **– (*n*+1) to Infinity** | **= *S* to *n*** | **Total** | | | |
| 34 | ***t* = *S*–1** | 4.97512 | -0.10670 | 4.86842 | **4.86842** | | | |
| 35 | **PV Factor** | 1.52088 | 1.52088 | 1.52088 | **1.52088** | | | |
| 36 | ***t* = 0** | 7.56654 | -0.16228 | 7.40426 | **7.40426** | | | |
| 37 | | | | | | | | |
| 38 | **Assumptions:** | | | | | | | |
| 39 | | | | | | | | |
| 40 | ***S* = Beginning Year of Cash Flows (valuation at *t* = –3.00)** | | | | | | **-2.00** | |
| 41 | ***n* = Ending Year of Cash Flows** | | | | | | **17.00** | |
| 42 | ***r* = Discount Rate** | | | | | | **15.0%** | |
| 43 | ***g* = Growth Rate in Net Inc/Cash Flow** | | | | | | **-5.1%** | |
| 44 | ***x* = (1+*g*)/(1+*r*)** | | | | | | **0.825217** | |
| 45 | **Gordon Model Multiple = GM = [1/(*r*–*g*)]** | | | | | | **4.975124** | |
| 46 | | | | | | | | |
| 47 | **Spreadsheet Formulas:** | | | | | | | |
| 48 | | | | | | | | |
| 49 | **B34: GM Gordon Model for Years –2.00 to Infinity as of *t* = –3.00** | | | | | | | |
| 50 | **C34: –GM*(*x*^(*n*–*S*+1)) Gordon Model for Years 18.00 to Infinity as of *t* = –3.00** | | | | | | | |
| 51 | **D34: B34 + C34** | | | | | | | |
| 52 | **E34: GM*(1– *x*^(*n*–*S*+1)) Grand Total as of *t* = *S*–1 = –3.00 Years** | | | | | | | |
| 53 | **Row 35: 1/(1+*r*)^(*S*– 1) Present Value Factor from *t* = *S*–1 to *t* = 0** | | | | | | | |
| 54 | **Row 36: Row 34 * Row 35** | | | | | | | |

in Table 4.6 growth is a *negative* 5.1% (G43), instead of the usual positive 5.1% in the other tables.

In Table 4.7, $r < g$, so the discount rate is less than the growth rate, which is impossible for a perpetuity but acceptable for a finite annuity. Note that the Gordon model multiple is −20 (B34 and G45), which by itself would be a nonsense result. Nevertheless, it still works for a finite annuity, as the term for the cash flows

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table 4.7** | | | | | | | |
| 2 | **ADF with Cash Flows Starting in Year –2.00 with *g* > *r*** | | | | | | | |
| 3 | **End-of-Year Formula** | | | | | | | |
| 4 | | | | | | | | |
| 5 | | **Cash Flow** | | | ***t* = *S* –1** | | ***t* = 0** | |
| 6 | ***t* (Yrs)** | **Growth** | **Cash Flow** | **$(1+g)^{t-S}$** | **PV Factor** | **PV** | **PV Factor** | **PV** |
| 7 | **-2.00** | NA | 1.00000 | 1.00000 | 0.86957 | 0.86957 | 1.32250 | 1.32250 |
| 8 | **-1.00** | 0.20000 | 1.20000 | 1.20000 | 0.75614 | 0.90737 | 1.15000 | 1.38000 |
| 9 | **0.00** | 0.24000 | 1.44000 | 1.44000 | 0.65752 | 0.94682 | 1.00000 | 1.44000 |
| 10 | **1.00** | 0.28800 | 1.72800 | 1.72800 | 0.57175 | 0.98799 | 0.86957 | 1.50261 |
| 11 | **2.00** | 0.34560 | 2.07360 | 2.07360 | 0.49718 | 1.03095 | 0.75614 | 1.56794 |
| 12 | **3.00** | 0.41472 | 2.48832 | 2.48832 | 0.43233 | 1.07577 | 0.65752 | 1.63611 |
| 13 | **4.00** | 0.49766 | 2.98598 | 2.98598 | 0.37594 | 1.12254 | 0.57175 | 1.70725 |
| 14 | **5.00** | 0.59720 | 3.58318 | 3.58318 | 0.32690 | 1.17135 | 0.49718 | 1.78147 |
| 15 | **6.00** | 0.71664 | 4.29982 | 4.29982 | 0.28426 | 1.22228 | 0.43233 | 1.85893 |
| 16 | **7.00** | 0.85996 | 5.15978 | 5.15978 | 0.24718 | 1.27542 | 0.37594 | 1.93975 |
| 17 | **8.00** | 1.03196 | 6.19174 | 6.19174 | 0.21494 | 1.33087 | 0.32690 | 2.02409 |
| 18 | **9.00** | 1.23835 | 7.43008 | 7.43008 | 0.18691 | 1.38874 | 0.28426 | 2.11209 |
| 19 | **10.00** | 1.48602 | 8.91610 | 8.91610 | 0.16253 | 1.44912 | 0.24718 | 2.20392 |
| 20 | **11.00** | 1.78322 | 10.69932 | 10.69932 | 0.14133 | 1.51212 | 0.21494 | 2.29975 |
| 21 | **12.00** | 2.13986 | 12.83918 | 12.83918 | 0.12289 | 1.57786 | 0.18691 | 2.39974 |
| 22 | **13.00** | 2.56784 | 15.40702 | 15.40702 | 0.10686 | 1.64647 | 0.16253 | 2.50407 |
| 23 | **14.00** | 3.08140 | 18.48843 | 18.48843 | 0.09293 | 1.71805 | 0.14133 | 2.61294 |
| 24 | **15.00** | 3.69769 | 22.18611 | 22.18611 | 0.08081 | 1.79275 | 0.12289 | 2.72655 |
| 25 | **16.00** | 4.43722 | 26.62333 | 26.62333 | 0.07027 | 1.87070 | 0.10686 | 2.84510 |
| 26 | **17.00** | 5.32467 | 31.94800 | 31.94800 | 0.06110 | 1.95203 | 0.09293 | 2.96880 |
| 27 | **Pres. Value (*t* = –3.00 for Column G, *t* = 0 for Column I)** | | | | | **26.84876** | | **40.83361** |
| 28 | **Pres. Value Factor—Discount from *S* –1 (*t* = –3.00) to 0** | | | | | **1.52088** | | |
| 29 | **Present Value (*t* = 0)** | | | | | **40.83361** | | |
| 30 | | | | | | | | |
| 31 | **Calculation of PV by Formulas:** | | | | | | | |
| 32 | | | | | **Grand** | | | |
| 33 | **Time** | ***S* to Infinity** | **– (*n*+1) to Infinity** | **= *S* to *n*** | **Total** | | | |
| 34 | ***t* = *S*–1** | -20.00000 | 46.84876 | 26.84876 | **26.84876** | | | |
| 35 | **PV Factor** | 1.52088 | 1.52088 | 1.52088 | **1.52088** | | | |
| 36 | ***t* = 0** | -30.41750 | 71.25111 | 40.83361 | **40.83361** | | | |
| 37 | | | | | | | | |
| 38 | **Assumptions:** | | | | | | | |
| 39 | | | | | | | | |
| 40 | ***S* = Beginning Year of Cash Flows (valuation at *t* = –3.00)** | | | | | | **-2.00** | |
| 41 | ***n* = Ending Year of Cash Flows** | | | | | | **17.00** | |
| 42 | ***r* = Discount Rate** | | | | | | **15.0%** | |
| 43 | ***g* = Growth Rate in Net Inc/Cash Flow** | | | | | | **20.0%** | |
| 44 | ***x* = (1+*g*)/(1+*r*)** | | | | | | **1.043478** | |
| 45 | **Gordon Model Multiple = GM = [1/(*r*–*g*)]** | | | | | | **-20.000000** | |
| 46 | | | | | | | | |
| 47 | **Spreadsheet Formulas:** | | | | | | | |
| 48 | | | | | | | | |
| 49 | **B34: GM Gordon Model for Years –2.00 to Infinity as of *t* = –3.00** | | | | | | | |
| 50 | **C34: –GM*(*x*^(*n*–*S*+1)) Gordon Model for Years 18.00 to Infinity as of *t* = –3.00** | | | | | | | |
| 51 | **D34: B34 + C34** | | | | | | | |
| 52 | **E34: GM*(1–*x*^(*n*–*S*+1)) Grand Total as of *t* = *S*–1 = –3.00 Years** | | | | | | | |
| 53 | **Row 35: 1/(1+*r*)^(*S*–1) Present Value Factor from *t* = *S*–1 to *t* = 0** | | | | | | | |
| 54 | **Row 36: Row 34 * Row 35** | | | | | | | |

from $n + 1$ to infinity is positive and greater than the negative Gordon model multiple.[11]

In all cases, equation (4.11) performs perfectly, with D36 = E36 = F29 = H27.

[11]This is so because $\left(\frac{1+g}{1+r}\right)^n > 1$, so when we multiply that term by the GM—which is negative—the resulting term is negative and of greater magnitude than the GM itself. Since we are subtracting a larger negative from the negative GM, the overall result is a positive number.

### Special Case: No Growth, $g = 0$

Setting $g = 0$, equation (4.11) reduces to:

$$ADF = \left[\frac{1}{r} - \frac{1}{(1+r)^{n-S+1}}\frac{1}{r}\right]\frac{1}{(1+r)^{S-1}} = \frac{1}{r}\left[1 - \frac{1}{(1+r)^{n-S+1}}\right]\frac{1}{(1+r)^{S-1}} \tag{4.11c}$$

ADF: no growth.

This formula is useful in calculating loan amortization, as the reader can see in the loan amortization section of Appendix A to this chapter.

### Generalized Gordon Model

If we start with cash flows at any year other than year 1, then we have to use a *generalized* Gordon model. Letting $n \to \infty$ in equation (4.11), the end-of-year formula is:

$$PV = CF\,\frac{1}{(r-g)}\quad\frac{1}{(1+r)^{S-1}}. \tag{4.11d}$$

This is the formula for the PV of the reversion (the cash flows from $t = n + 1$ to infinity) that every appraiser uses in every discounted cash flow analysis. This is exactly what appraisers do in calculating the PV of the reversion, that is, the infinity of time that follows the discounted cash flow forecasts for the first $n$ years. For example, suppose we do a five-year forecast of cash flows in a discounted cash flow analysis and calculate its PV. We must then calculate the PV of the reversion, which is the sixth-year cash flow multiplied by the Gordon model and then discounted five years to $t = 0$, or:

$$PV = CF_6\frac{1}{r-g}\frac{1}{(1+r)^5}. \tag{4.11e}$$

The reason we discount five years and not six is that after discounting the first five years' cash flows to PV, we are standing at the end of year 5 looking at the infinity of cash flows that we forecast to occur beginning with year 6. The Gordon model requires us to use the first forecast year's cash flow, which is why we use $CF_6$ and not $CF_5$, but we still must discount the cash flows from the end of year 5, or five years. The first two terms on the right-hand side of equation (4.11d) give us the formula for the PV of the cash flows from years 6 to infinity as of the end of year 5, and the final term on the right discounts that back to $t = 0$.

### Midyear Formula

When the starting period is not $t = 0$, the midyear ADF formula is:

$$\begin{aligned} ADF &= \left[\frac{\sqrt{1+r}}{r-g} - \left(\frac{1+g}{1+r}\right)^{n-S+1}\frac{\sqrt{1+r}}{r-g}\right]\frac{1}{(1+r)^{S-1}} \\ &= \frac{\sqrt{1+r}}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^{n-S+1}\right]\frac{1}{(1+r)^{S-1}}. \end{aligned} \tag{4.12}$$

Note that at $S = 1$, the term at the right—outside the brackets—becomes 1 and effectively drops out of the equation and the exponent inside the brackets becomes $n$, which renders equation (4.12) equivalent to equation (4.10). The midyear ADF in equation (4.12) is identical to the end-of-year ADF in equation (4.11), except that we replace the two Gordon model $\sqrt{1+r}$ terms with the value 1 in the latter.

## Periodic Perpetuity Factors (PPFs): Perpetuities for Periodic Cash Flows

Thus far, all ADFs and Gordon model perpetuities have been for contiguous cash flows. In this section, we develop an equation for perpetuities with periodic cash flows that occur only at regular intervals or cycles. To my knowledge, these formulas are my own creation, and I call them *periodic perpetuity factors* (PPFs). The PPF is a generalized Gordon model multiple for periodic cash flows that may or may not be contiguous.

The example we use here arose in Chapter 3 in dealing with moving expenses. Every small-to-midsize company that is growing in real terms moves periodically. We will assume a move occurs every 10 years, although we will derive formulas that can handle any periodicity. To further simplify the initial mathematics, we will assume the last move occurred in the last historical year of analysis. Later, we will relax that assumption to handle different timing of the cash flows.

Suppose our subject company moved last year, and the move cost \$20,000. We expect to move every 10 years, and moving costs increase at $g = 5\%$ per year. The PPFs are the present values of these periodic cash flows for both midyear and end-of-year assumptions.

### The Mathematical Formulas

For every \$1.00 of *forecast* moving costs in year 10, the PV of the lifetime expected moving costs would be as follows in equation (4.13):

$$PV = \frac{1}{(1+r)^{10}} + \frac{(1+g)^{10}}{(1+r)^{20}} + \cdots + \frac{(1+g)^{\infty}}{(1+r)^{\infty}}. \tag{4.13}$$

The \$1.00 grows at rate $g$ for 10 years, and we discount it back to PV for 10 years. We follow the same pattern at 20 years, 30 years, and so on to infinity. Multiplying equation (4.13) by $\left(\frac{1+g}{1+r}\right)^{10}$, we get:

$$\left(\frac{1+g}{1+r}\right)^{10} PV = \frac{(1+g)^{10}}{(1+r)^{20}} + \frac{(1+g)^{20}}{(1+r)^{30}} + \cdots + \frac{(1+g)^{\infty}}{(1+r)^{\infty}}. \tag{4.14}$$

Subtracting equation (4.14) from equation (4.13), we get:

$$\left[1 - \left(\frac{1+g}{1+r}\right)^{10}\right] PV = \frac{1}{(1+r)^{10}}. \tag{4.15}$$

The left-hand side of equation (4.15) simplifies to $\frac{(1+r)^{10}-(1+g)^{10}}{(1+r)^{10}}PV$ . Multiplying both sides of equation (4.15) by the inverse, $\frac{(1+r)^{10}}{(1+r)^{10}-(1+g)^{10}}$, we come to:

$$PV = \frac{(1+r)^{10}}{(1+r)^{10}-(1+g)^{10}} \frac{1}{(1+r)^{10}} \tag{4.16}$$

Canceling out $(1 + r)^{10}$ in the numerator and denominator, the solution is:

$$PV = \frac{1}{(1+r)^{10}-(1+g)^{10}}. \tag{4.17}$$

We can generalize this formula to other periods of cash flows by letting cash flows occur every $j$ years. The PV of the cash flows is the same, except that we replace each 10 in equation (4.17) with a $j$ in equation (4.18). Additionally, we rename the term *PV* as *PPF*, the periodic perpetuity factor. Therefore, the PPF for $1 of payment, first occurring in year $j$, is:

$$PPF = \frac{1}{(1+r)^{j}-(1+g)^{j}} \quad \text{PPF—end-of-year.} \tag{4.18}$$

The midyear PPF is again our familiar result of $\sqrt{1+r}$ times the end-of-year PPF, or:

$$PPF = \frac{\sqrt{1+r}}{(1+r)^{j}-(1+g)^{j}} \quad \text{PPF—midyear.} \tag{4.19}$$

Note that for $j = 1$, equations (4.18) and (4.19) reduce to the Gordon model. As you will see further, the above two formulas work only if the last cash flow occurred in the immediately prior year (i.e., $t = -1$). In the section on other starting years, we generalize these two formulas to equations (4.18a) and (4.19a) to be able to handle different starting times.

## Tables 4.8 and 4.9: Examples of Equations (4.18) and (4.19)

We begin in Table 4.8 with $1.00 (B5) of moving expenses[12] that we forecast to occur in the next move, 10 years from now. The second move, which we expect to occur in 20 years, should cost $(1 + g)^{10} = \$1.62889$ (B6), assuming a 5% (D26) constant growth rate ($g$) in the cost. We discount cash flows at a 20% (D25) discount rate.

Column A shows time in 10-year increments going up to 100 years. B5 to B14 contain the forecast cash flows and are equal to $(1 + g)^{t-j}$, where $t = 10, 20, 30, \ldots, 100$ years, $g = 5.0\%$, and $j = 10$. Actually, time should continue to $t = \infty$, but at a 20% discount rate and 5% growth rate, the present value factors nullify all cash

[12]Another common periodic expense that is less predictable than moving expenses is losses from lawsuits. Rather than use the actual loss from the last lawsuit, one should use a base-level, long-run average loss, which will grow at a rate of $g$.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Table 4.8** | | | | | |
| 2 | **Periodic Perpetuity Factor (PPF)—End-of-Year Formula** | | | | | |
| 3 | | | | | | |
| 4 | ***t* (Yrs)** | **Cash Flow = $(1+g)^{t-j}$** | **PV Factor = $1/(1+r)^t$** | **PV** | **% PV** | **Cum % PV** |
| 5 | **10** | 1.00000 | 0.16151 | 0.16151 | 74% | 74% |
| 6 | **20** | 1.62889 | 0.02608 | 0.04249 | 19% | 93% |
| 7 | **30** | 2.65330 | 0.00421 | 0.01118 | 5% | 98% |
| 8 | **40** | 4.32194 | 0.00068 | 0.00294 | 1% | 100% |
| 9 | **50** | 7.03999 | 0.00011 | 0.00077 | 0% | 100% |
| 10 | **60** | 11.46740 | 0.00002 | 0.00020 | 0% | 100% |
| 11 | **70** | 18.67919 | 0.00000 | 0.00005 | 0% | 100% |
| 12 | **80** | 30.42643 | 0.00000 | 0.00001 | 0% | 100% |
| 13 | **90** | 49.56144 | 0.00000 | 0.00000 | 0% | 100% |
| 14 | **100** | 80.73037 | 0.00000 | 0.00000 | 0% | 100% |
| 15 | **Totals** | | | **0.21916** | **100%** | |
| 16 | | | | | | |
| 17 | **Calculation of PPF by Formula:** | | | | | |
| 18 | | | | | | |
| 19 | **PPF** | | | | | |
| 20 | **0.21916** | | | | | |
| 21 | | | | | | |
| 22 | **Assumptions:** | | | | | |
| 23 | | | | | | |
| 24 | ***j* = Number of Years between Moves** | | | **10** | | |
| 25 | ***r* = Discount Rate** | | | **20.0%** | | |
| 26 | ***g* = Growth Rate in Moving Costs** | | | **5.0%** | | |
| 27 | | | | | | |
| 28 | **Spreadsheet Formulas:** | | | | | |
| 29 | | | | | | |
| 30 | **A20: =1/((1+*r*)^*j*–(1+*g*)^*j*) Equation (4.18)** | | | | | |

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Table 4.9** | | | | | |
| 2 | **Periodic Perpetuity Factor (PPF)—Midyear Formula** | | | | | |
| 3 | | | | | | |
| 4 | ***t* (Yrs)** | **Cash Flow = $(1+g)^{t-j}$** | **PV Factor = $1/(1+r)^{(t-0.5)}$** | **PV** | **% PV** | **Cum % PV** |
| 5 | **10** | 1.00000 | 0.17692 | 0.17692 | 74% | 74% |
| 6 | **20** | 1.62889 | 0.02857 | 0.04654 | 19% | 93% |
| 7 | **30** | 2.65330 | 0.00461 | 0.01224 | 5% | 98% |
| 8 | **40** | 4.32194 | 0.00075 | 0.00322 | 1% | 100% |
| 9 | **50** | 7.03999 | 0.00012 | 0.00085 | 0% | 100% |
| 10 | **60** | 11.46740 | 0.00002 | 0.00022 | 0% | 100% |
| 11 | **70** | 18.67919 | 0.00000 | 0.00006 | 0% | 100% |
| 12 | **80** | 30.42643 | 0.00000 | 0.00002 | 0% | 100% |
| 13 | **90** | 49.56144 | 0.00000 | 0.00000 | 0% | 100% |
| 14 | **100** | 80.73037 | 0.00000 | 0.00000 | 0% | 100% |
| 15 | **Totals** | | | **0.24008** | **100%** | |
| 16 | | | | | | |
| 17 | **Calculation of PPF by Formula:** | | | | | |
| 18 | | | | | | |
| 19 | **PPF** | | | | | |
| 20 | **0.24008** | | | | | |
| 21 | | | | | | |
| 22 | **Assumptions:** | | | | | |
| 23 | | | | | | |
| 24 | ***j* = Number of Years between Moves** | | | **10** | | |
| 25 | ***r* = Discount Rate** | | | **20.0%** | | |
| 26 | ***g* = Growth Rate in Moving Costs** | | | **5.0%** | | |
| 27 | | | | | | |
| 28 | **Spreadsheet Formulas:** | | | | | |
| 29 | | | | | | |
| 30 | **A20: =SQRT(1+*r*)/((1+*r*)^*j*–(1+*g*)^*j*) Equation (4.19)** | | | | | |

flows after year 40.[13] Column C contains a standard present value factor, where

$$PV = \frac{1}{(1+r)^t}.$$

Column D, the present value of the cash flows, equals column B × column C. D15, the total PV, equals $0.21916 for every $1.00 of moving expenses in the next move. This is the final result using the brute force method of scheduling all the cash flows and discounting them to PV. A20 contains the formula for equation (4.18), and the result is $0.21916, which demonstrates the accuracy of the formula. Note that the formula for A20 appears in A30.

So far we have computed only the PPF. To calculate the PV of $20,000 of the previous year's moving expense growing at 5% per year and occurring every 10 years, we forecast the cost of the next move by multiplying the $20,000 by $1.05^{10} = \$32{,}577.89$. We then multiply the cost of the next move by the PPF, that is, $32,577.89 × 0.21916 (A20) = $7,139.83 before corporate taxes. Assuming a 40% tax rate, that rounds to $4,284 after tax. Since this is an expense, we must remember to subtract it from—not add it to—the value we calculated before moving expenses.[14] For example, suppose we calculated a marketable minority interest fair market value (FMV) of $1,004,284 before moving expenses. The final marketable minority FMV would be $1 million.

Column E shows the percentage of the PV contributed by each move. Seventy-four percent (E5) of the PV comes from the first move (year 10), and 19% (E6) from the second move (year 20). Column F shows the cumulative PV. The first two moves cumulatively account for 93% (F6) of the entire PV generated by all moves, and the first three moves account for 98% (F7) of the PV. Thus, in most circumstances, we need not worry about the argument that after attaining a certain size, a company tends to not move anymore. As long as it moves at least twice, the PPF will be sufficiently accurate.

Table 4.9 is identical to Table 4.8, except that it is testing equation (4.19), the midyear formula, instead of the end-of-year formula, equation (4.18). Again C20 = D15, which verifies the formula.

## Other Starting Years

Another question to address is what happens when the periodic expense occurred before the prior year. Using our moving-expense-every-10-years example, suppose the subject company last moved 4 years ago. It will be another 6 years, not 10 years, to the next move. The easiest way to handle this situation is first to value the cash flows from a point in time where we can use the ADF equations in (4.18) and (4.19) and then adjust. Thus, if we choose $t = -4$ as our temporary valuation date, all cash flows will be spaced every 10 years, and ADF formulas (4.18) and (4.19) apply. We then roll forward to $t = 0$ by multiplying the preliminary PPF by $(1 + r)^b$, where $b$

[13]Of course, at a higher growth rate and the same discount rate, it will take longer for the present value factors to nullify the growth. The converse is also true.

[14]We accomplish this by removing moving expenses from historical costs before developing our forecast of expenses (see Chapter 3).

is the number of years before $t = 0$ that the last periodic cash flow occurred. In this case, $b = 4$.

The generalized PPF formulas are:

$$PPF = \frac{1}{(1+r)^j - (1+g)^j} \times (1+r)^b \quad \text{Generalized PPF—end-of-year.} \qquad (4.18a)$$

The midyear generalized PPF is again our familiar result of $\sqrt{1+r}$ times the end-of-year generalized PPF, or:

$$PPF = \frac{\sqrt{1+r}}{(1+r)^j - (1+g)^j} \times (1+r)^b \quad \text{Generalized PPF—midyear.} \qquad (4.19a)$$

Note that for $j = 1$ the left-hand side of equations (4.18a) and (4.19a) reduces to the Gordon model, while the right-hand term is the adjustment for the next flow occurring $b$ years earlier than year $j$., that is, in year $b - j$ instead of year $j$. When $j = 1$ and $b = 0$, the entire formula simplifies to the Gordon model.

It is important to roll forward the cash flow properly. With the \$20,000 move occurring 4 years ago, our forecast of the next move is still $\$20{,}000 \times 1.05^{10} = \$32{,}578$. Whether the last move occurred 4 years ago or yesterday, the forecast cost of the next move is the same 10 years' growth. The present value, and therefore the PPF, is different for the two different moves, and that is captured in the numerator of the PPF, as we have already discussed.

Table 4.10 is identical to Table 4.8, except that the expenses occur in years 6, 16, . . . instead of 10, 20, . . . . The nominal cash flows are identical to Table 4.8, but the formula that generates them is different. In Table 4.8, the cash flows are equal to $(1 + g)^{t-j}$. In Table 4.10, the cash flows are equal to $(1 + g)^{t-j+b}$ because the cash flows still grow at the rate $g$ for 10 years from the last move, not just the 6 years to the next move.[15] However, the cash flows in Table 4.10 are discounted 6 years instead of 10 years. The PPF is \$0.45445. The calculation by formula in A20 matches the brute force calculation in D15, which demonstrates the validity of equation (4.18a).

Modifying the moving expense example in Table 4.8, the PV of all moving costs throughout time equals $\$20{,}000 \times 1.62889 \times \$0.45445 = \$14{,}805.14$. Assuming a 40% tax rate, the after-tax present value of the perpetuity of moving costs is \$8,883, compared to the \$4,284 we calculated in the discussion of Table 4.8. The present value of moving costs is higher in this example, because the first cash flow occurs in year 6 instead of year 10.

## PPFs in New-versus-Used Equipment Decisions

Another important use of PPFs is in new-versus-used equipment decisions and in valuing used income-producing equipment. Let's use a taxicab as an example. The cab company can buy a new car or a used car. Suppose a new car would last six years. It costs \$20,000 to buy a new one today, and we can model the cash flows for its six-year expected life.

[15] Of course, we could consider the formula in Table 4.8 to be $(1+g)^{t-j+b}$, with $b = 0$.

**Table 4.10**

**Periodic Perpetuity Factor (PPF)—End-of-Year—Cash Flows Begin Year 6**

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 4 | **$t$ (Yrs)** | **Cash Flow = $(1+g)^{t-j+b}$** | **PV Factor = $1/(1+r)^t$** | **PV** | **% PV** | **Cum % PV** |
| 5 | **6** | 1.00000 | 0.33490 | 0.33490 | 74% | 74% |
| 6 | **16** | 1.62889 | 0.05409 | 0.08810 | 19% | 93% |
| 7 | **26** | 2.65330 | 0.00874 | 0.02318 | 5% | 98% |
| 8 | **36** | 4.32194 | 0.00141 | 0.00610 | 1% | 100% |
| 9 | **46** | 7.03999 | 0.00023 | 0.00160 | 0% | 100% |
| 10 | **56** | 11.46740 | 0.00004 | 0.00042 | 0% | 100% |
| 11 | **66** | 18.67919 | 0.00001 | 0.00011 | 0% | 100% |
| 12 | **76** | 30.42643 | 0.00000 | 0.00003 | 0% | 100% |
| 13 | **86** | 49.56144 | 0.00000 | 0.00001 | 0% | 100% |
| 14 | **96** | 80.73037 | 0.00000 | 0.00000 | 0% | 100% |
| 15 | **Totals** | | | **0.45445** | **100%** | |

**Calculation of PPF by Formula:**

| PPF |
|---|
| **0.45445** |

**Assumptions:**

| | |
|---|---|
| **$j$ = Number of Years between Moves [1]** | **10** |
| **$r$ = Discount Rate** | **20.0%** |
| **$g$ = Growth Rate in Net Inc/Cash Flow** | **5.0%** |
| **$b$ = Number of Years from Last Cash Flow** | **4** |

**Spreadsheet Formulas:**

**A20: =(1+*r*)^*b*/((1+*r*)^*j*–(1+*g*)^*j*) Equation (4.18a)**

**Notes:**

**[1] As *j* decreases, the PV factors and the PV increase. It is possible that you will have to add additional rows above Row 15 to capture all the PV of the cash flows. Otherwise, the PV in C20 will appear to be higher than the total of the cash flows in D15.**

The cash flows will consist of the purchase of the cab, income, gasoline, maintenance, insurance, and so forth. Each expense category has its own pattern. Gas consumption is a variable expense that increases in dollars over time with the rate of increase in gas prices. Maintenance is probably low for the first two years and then begins increasing rapidly in year 3 or 4.

We can then take the NPV of the cash flows, and that represents the NPV of operating a new cab for six years. It would be nice to compare that with the NPV of operating a one-year-old cab for five years (or any other term desired). The problem is that these are different time periods. We could use the lowest common multiple of 30 years (6 years × 5 years) and run the new cab cash flows five times and the used cab cycle six times, but that is a lot of work. It is a far more elegant solution to use a PPF for the new and the used equipment. The result of those computations will be the present value of keeping one new cab and one used cab in service forever. We can then choose the one with the superior NPV.

Even though the cash flows are contiguous, which is not true in the periodic expense example, the cycle and the NPV of the cash flows are periodic. Every six years the operator buys a new cab. We can measure the NPV of the first cab as of $t = 0$. The operator buys the second cab and uses it from years 7 through 12.

Its NPV as of the end of year 6 ($t = 6$) should be the same as the NPV at $t = 0$ of the first six years' cash flows, with a growth rate for the rise in prices. If there are substantial differences in the growth rates of income versus expenses or of the different categories of expenses, then we can break the expenses into two or more subcategories and apply a PPF to each subcategory, and then add the NPVs together. Buying a new cab every six years would then generate a series of NPVs with constant growth at $t = 0, 6, 12, \ldots$ . That repeating pattern is what enables us to use a PPF to value the cash flows.

We could perform this procedure for each different vintage of used equipment, for example, buying one-year-old cabs, two-year-old cabs, and so forth. Our final comparison would be the NPV of buying and operating a single cab of each age (a new cab, one year old, two years old, etc.) forever. We then simply choose the cab life with the highest NPV.

If equipment is not income producing, we can still use the PPF to value the periodic costs in perpetuity. Then the NPV would be negative.

## ADFs in Loan Mathematics

There are four related topics that ideally all should be together with using ADFs in loan mathematics to create formulas to calculate the following: loan payments, principal amortization, the after-tax cost of a loan, and the PV of a loan when the nominal and market rates differ. We will deal with the first and the last topics in this section. Calculating the amortization of principal is mathematically very complex. To maintain readability, it will be explained, along with the related problem of calculating the after-tax cost of a loan, in Appendix A.

### Calculating Loan Payments

We can use our earlier ADF results to easily create a formula to calculate loan payments. We know that in the case of a fixed-rate amortizing loan, the principal must be equal to the PV of the payments when discounted by the *nominal* rate of the loan. We can calculate the PV of the payments using equation (4.6d) and the following definitions:

$$ADF_{Nominal} = \text{ADF at the nominal interest rate of the loan.}$$
$$ADF_{Mkt} = \text{ADF at the market interest rate of the loan.}$$

The nominal ADF is simply an end-of-year ADF with no growth. Repeating equation (4.6d), the ADF is:

$$ADF_{Nominal} = \frac{1 - \dfrac{1}{(1+r)^n}}{r},$$

where $r$ in this case is the nominal interest of the loan. If we use the market interest rate instead of the nominal rate, we get $ADF_{Mkt}$. We know that the loan payment multiplied by the nominal ADF equals the principal of the loan. Stating that as an equation:

$$Loan\ Payment \times ADF_{Nominal} = Principal. \tag{4.20}$$

Dividing both sides of the equation by $ADF_{Nominal}$, we get:

$$Loan\ Payment = \frac{Principal}{ADF_{Nominal}} = Principal \times \frac{1}{ADF_{Nominal}}. \tag{4.21}$$

## Present Value of a Loan

By definition, the PV of a loan is the loan payment times the market rate ADF:

$$PV = Loan\ Payment \times ADF_{Mkt}. \tag{4.22}$$

From equation (4.21), the loan payment is the principal divided by the nominal ADF. Substituting this into equation (4.22) gives us:

$$PV\ of\ Loan = \mathrm{Pr}incipal \times \frac{ADF_{Mkt}}{ADF_{\mathrm{Nominal}}}. \tag{4.23}$$

The intuition behind this is that the $\mathrm{Pr}incipal \times \frac{1}{ADF_{\mathrm{Nominal}}}$ is the amount of the loan payment. When we then multiply that by the $ADF_{Mkt}$, this gives us the PV of the loan.

**TABLE 4.11: EXAMPLE OF EQUATION (4.23)** Table 4.11 is an example of calculating the present value of a loan. The assumptions appear in Table 4.11 in E77 to E82. We assume a $1 million principal on a five-year loan. The loan payment, calculated using Excel's spreadsheet function, is $20,276.39 (E78) for 60 months. The annual loan rate is 8% (E79), and the monthly rate is 8%/12 months = 0.667% (E80 = E79/12). The annual market rate of interest (the discount rate) on this loan is assumed at 14% (E81), and the monthly market interest rate is 1.167% (E82 = E81/12).

Column A shows the 60 months of payments. Column B shows the monthly payment of $20,276.39 for 60 months. Columns C and D show the PV factor and the PV of each month's payment at the nominal 8% annual interest rate (0.667% monthly rate), while columns E and F show the same calculations at the market rate of 14% (1.167% monthly rate).

The present value factors in C6 to C65 total 49.31843 (C66), and present value factors in E6 to E65 total 42.97702 (E66). Note also that the PV of the loan at the nominal interest rate equals the $1 million principal (D66), as it should.

E70 is the ADF at 8% according to equation (4.6d). We show the spreadsheet formula for E70 in A86. E71 is $\frac{1}{ADF_{Nominal}}$ = $0.02027639, the amount of loan payment for each $1 of principal. We multiply that by the $1 million principal to obtain the loan payment of $20,276.39 in F71, which matches E78, as it should. In E72, we calculate the ADF at the market rate of interest, the formula for which is also equation (4.6d), merely using the 1.167% monthly interest rate in the formula, which we show in A88. In E73, we calculate the ratio of the market ADF to the nominal ADF, which is E72 divided by E70 and equals 0.871419. In F73, we multiply E73 by the $1 million principal to obtain the present value of the loan of $871,419. Note that this matches our brute force calculation in F66, as it should.

## Table 4.11
### PV of Loan with Market Rate > Nominal Rate: ADF, End-of-Year

| | | *r* = 8% | | *r* = 14% | |
|---|---|---|---|---|---|
| **Month** | **Cash Flow** | **PV Factor** | **Present Value** | **PV Factor** | **Present Value** |
| **1** | $20,276.39 | 0.99338 | $ 20,142 | 0.98847 | $ 20,043 |
| **2** | $20,276.39 | 0.98680 | $ 20,009 | 0.97707 | $ 19,811 |
| **3** | $20,276.39 | 0.98026 | $ 19,876 | 0.96580 | $ 19,583 |
| **4** | $20,276.39 | 0.97377 | $ 19,745 | 0.95466 | $ 19,357 |
| **5** | $20,276.39 | 0.96732 | $ 19,614 | 0.94365 | $ 19,134 |
| **6** | $20,276.39 | 0.96092 | $ 19,484 | 0.93277 | $ 18,913 |
| **7** | $20,276.39 | 0.95455 | $ 19,355 | 0.92201 | $ 18,695 |
| **8** | $20,276.39 | 0.94823 | $ 19,227 | 0.91138 | $ 18,480 |
| **9** | $20,276.39 | 0.94195 | $ 19,099 | 0.90087 | $ 18,266 |
| **10** | $20,276.39 | 0.93571 | $ 18,973 | 0.89048 | $ 18,056 |
| **11** | $20,276.39 | 0.92952 | $ 18,847 | 0.88021 | $ 17,848 |
| **12** | $20,276.39 | 0.92336 | $ 18,722 | 0.87006 | $ 17,642 |
| **13** | $20,276.39 | 0.91725 | $ 18,598 | 0.86003 | $ 17,438 |
| **14** | $20,276.39 | 0.91117 | $ 18,475 | 0.85011 | $ 17,237 |
| **15** | $20,276.39 | 0.90514 | $ 18,353 | 0.84031 | $ 17,038 |
| **16** | $20,276.39 | 0.89914 | $ 18,231 | 0.83062 | $ 16,842 |
| **17** | $20,276.39 | 0.89319 | $ 18,111 | 0.82104 | $ 16,648 |
| **18** | $20,276.39 | 0.88727 | $ 17,991 | 0.81157 | $ 16,456 |
| **19** | $20,276.39 | 0.88140 | $ 17,872 | 0.80221 | $ 16,266 |
| **20** | $20,276.39 | 0.87556 | $ 17,753 | 0.79296 | $ 16,078 |
| **21** | $20,276.39 | 0.86976 | $ 17,636 | 0.78382 | $ 15,893 |
| **22** | $20,276.39 | 0.86400 | $ 17,519 | 0.77478 | $ 15,710 |
| **23** | $20,276.39 | 0.85828 | $ 17,403 | 0.76584 | $ 15,529 |
| **24** | $20,276.39 | 0.85260 | $ 17,288 | 0.75701 | $ 15,349 |
| **25** | $20,276.39 | 0.84695 | $ 17,173 | 0.74828 | $ 15,172 |
| **26** | $20,276.39 | 0.84134 | $ 17,059 | 0.73965 | $ 14,997 |
| **27** | $20,276.39 | 0.83577 | $ 16,946 | 0.73112 | $ 14,824 |
| **28** | $20,276.39 | 0.83023 | $ 16,834 | 0.72269 | $ 14,654 |
| **29** | $20,276.39 | 0.82474 | $ 16,723 | 0.71436 | $ 14,485 |
| **30** | $20,276.39 | 0.81927 | $ 16,612 | 0.70612 | $ 14,318 |
| **31** | $20,276.39 | 0.81385 | $ 16,502 | 0.69797 | $ 14,152 |
| **32** | $20,276.39 | 0.80846 | $ 16,393 | 0.68993 | $ 13,989 |
| **33** | $20,276.39 | 0.80310 | $ 16,284 | 0.68197 | $ 13,828 |
| **34** | $20,276.39 | 0.79779 | $ 16,176 | 0.67410 | $ 13,668 |
| **35** | $20,276.39 | 0.79250 | $ 16,069 | 0.66633 | $ 13,511 |
| **36** | $20,276.39 | 0.78725 | $ 15,963 | 0.65865 | $ 13,355 |
| **37** | $20,276.39 | 0.78204 | $ 15,857 | 0.65105 | $ 13,201 |
| **38** | $20,276.39 | 0.77686 | $ 15,752 | 0.64354 | $ 13,049 |
| **39** | $20,276.39 | 0.77172 | $ 15,648 | 0.63612 | $ 12,898 |
| **40** | $20,276.39 | 0.76661 | $ 15,544 | 0.62879 | $ 12,749 |
| **41** | $20,276.39 | 0.76153 | $ 15,441 | 0.62153 | $ 12,602 |
| **42** | $20,276.39 | 0.75649 | $ 15,339 | 0.61437 | $ 12,457 |
| **43** | $20,276.39 | 0.75148 | $ 15,237 | 0.60728 | $ 12,313 |
| **44** | $20,276.39 | 0.74650 | $ 15,136 | 0.60028 | $ 12,171 |
| **45** | $20,276.39 | 0.74156 | $ 15,036 | 0.59336 | $ 12,031 |

(*continued*)

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Table 4.11 (cont.)** | | | | | |
| 2 | | | | | | |
| 3 | | | | | | |
| 4 | | | ***r* = 8%** | | ***r* = 14%** | |
| 5 | **Month** | **Cash Flow** | **PV Factor** | **Present Value** | **PV Factor** | **Present Value** |
| 51 | **46** | $20,276.39 | 0.73665 | $ 14,937 | 0.58651 | $ 11,892 |
| 52 | **47** | $20,276.39 | 0.73177 | $ 14,838 | 0.57975 | $ 11,755 |
| 53 | **48** | $20,276.39 | 0.72692 | $ 14,739 | 0.57306 | $ 11,620 |
| 54 | **49** | $20,276.39 | 0.72211 | $ 14,642 | 0.56645 | $ 11,486 |
| 55 | **50** | $20,276.39 | 0.71732 | $ 14,545 | 0.55992 | $ 11,353 |
| 56 | **51** | $20,276.39 | 0.71257 | $ 14,448 | 0.55347 | $ 11,222 |
| 57 | **52** | $20,276.39 | 0.70785 | $ 14,353 | 0.54708 | $ 11,093 |
| 58 | **53** | $20,276.39 | 0.70317 | $ 14,258 | 0.54077 | $ 10,965 |
| 59 | **54** | $20,276.39 | 0.69851 | $ 14,163 | 0.53454 | $ 10,838 |
| 60 | **55** | $20,276.39 | 0.69388 | $ 14,069 | 0.52837 | $ 10,714 |
| 61 | **56** | $20,276.39 | 0.68929 | $ 13,976 | 0.52228 | $ 10,590 |
| 62 | **57** | $20,276.39 | 0.68472 | $ 13,884 | 0.51626 | $ 10,468 |
| 63 | **58** | $20,276.39 | 0.68019 | $ 13,792 | 0.51030 | $ 10,347 |
| 64 | **59** | $20,276.39 | 0.67569 | $ 13,700 | 0.50442 | $ 10,228 |
| 65 | **60** | $20,276.39 | 0.67121 | $ 13,610 | 0.49860 | $ 10,110 |
| 66 | **Totals** | **$ 1,216,584** | **49.31843** | **$ 1,000,000** | **42.97702** | **$ 871,419** |
| 67 | | | | | | |
| 68 | | | | | | **× Principal** |
| 69 | | | | | **Per $1** | **of $1 Million** |
| 70 | **ADF @ 8% = C66** | | | | 49.318433 | |
| 71 | **Formula for Payment = 1/ADF** | | | | 0.02027639 | **$ 20,276.39** |
| 72 | **ADF @ 14% = E66** | | | | 42.977016 | |
| 73 | **ADF @ 14% / ADF @ 8% = F66** | | | | 0.871419 | **$ 871,419** |
| 74 | | | | | | |
| 75 | **Assumptions:** | | | | | |
| 76 | | | | | | |
| 77 | **Principal** | | | | **$ 1,000,000** | |
| 78 | **Loan Payment** | | | | **$20,276.39** | |
| 79 | **$r$ = Nominal Discount Rate—Annual** | | | | **8.0%** | |
| 80 | **$r_1$ = Nominal Discount Rate—Monthly** | | | | **0.667%** | |
| 81 | **$r_2$ = Market Discount Rate** | | | | **14.0%** | |
| 82 | **$r_3$ = Market Discount Rate** | | | | **1.167%** | |
| 83 | | | | | | |
| 84 | **Spreadsheet Formulas:** | | | | | |
| 85 | | | | | | |
| 86 | **E70: =(1–1/(1+E80)^60)/E80** | | | | | |
| 87 | **E71: =1/E70** | | | | | |
| 88 | **E72: =(1–1/(1+E82)^60)/E82** | | | | | |
| 89 | **E73: =E72/E70** | | | | | |

## Relationship of the Gordon Model to the Price/Earnings and Price/Sales Ratios

In this section, we will mathematically derive the relationship between the price/earnings (PE) ratio, price/sales (PS) multiple, and the Gordon model. The confusion between them leads to possibly more mistakes by appraisers than any other single source of mistakes. I have seen numerous reports in which the appraiser used the wrong earnings base. Understanding this section should clear the potential confusion that exists. First, we will begin with some definitions that will aid in developing the mathematics. All other definitions retain their same meaning as in the rest of the chapter.

### Definitions

$P_t$ = stock price at time $t$.
$E_{t-1}$ = *historical* earnings in the *prior* year (usually the prior 12 months).
$E_{t+1}$ = *forecast* earnings in the *upcoming* year.[16]
$g_1$ = one-year forecast growth rate in earnings, that is, $\frac{E_{t+1}}{E_{t-1}} - 1$.
$PE$ = price/earnings ratio = $\frac{P_t}{E_{t-1}}$.
$POR$ = payout ratio = 1 − retention ratio.
$PS$ = price/earnings ratio = $\frac{P_t}{S_{t-1}}$.
$RR$ = earnings retention ratio. Thus cash flow to shareholders equals $(1 - RR) \times$ earnings, where $1 - RR = POR$.

### Mathematical Derivation of the PE Multiple

We begin with the statement that the market capitalization of a publicly held firm is its fair market value, and that is equal to its PE ratio times the previous year's *historical* earnings:

$$FMV = \frac{P_t}{E_{t-1}} \times E_{t-1}. \tag{4.24}$$

We repeat equation (4.10e) as equation (4.25), with one change. We will assume that forecast cash flow to shareholders, $CF_{t+1}$, is equal to $POR \times E_{t+1}$, where $RR$ is the earnings retention ratio.[17] The *retention ratio* is the sum total of all the reconciling items between net income and cash flow (see Chapters 1 and 2).

It is important to distinguish between the amount of cash that a company could pay its shareholders and the amount that it does pay its shareholders (i.e., dividends). With publicly traded firms, it generally applies to dividends. Most privately held firms—especially C corporations—do not pay dividends explicitly. However, they may pay implicit dividends in the form of excess compensation. In the absence of implicit or explicit dividends, it is important to be careful to keep our assumptions reasonable and consistent. For example, if the company sales and cash flow grew at a rate of 20% per year for 10 years while not paying any explicit or implicit dividends, we cannot naively assume that the company will continue to grow at a 20% rate with an 80% payout ratio. Obviously, if we assume the owners will start removing significant amounts of cash out of the company, then it is unrealistic to assume that the 20% growth rate will continue.

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[16]Here we are considering $t = 0$ to be the present—a point in time, while $t - 1$ and $t + 1$ each represent a span of time, that is, the past year and the first forecast year.
[17]I wish to thank Larry Kasper for pointing out the need for this.

Now we have an expression for the FMV of the firm[18] according to the midyear Gordon model:

$$FMV = POR \times E_{t+1}\frac{\sqrt{1+r}}{(r-g)}. \tag{4.25}$$

Midyear Gordon model.

Substituting $E_{t+1} = E_{t-1}\ (1 + g_1)$ into equation (4.25), we come to:

$$FMV = POR \times E_{t-1}(1+g_1)\frac{\sqrt{1+r}}{(r-g)}. \tag{4.26}$$

The left-hand sides of equations (4.24) and (4.26) are the same. Therefore, we can equate the right-hand sides of those equations:

$$\frac{P_t}{E_{t-1}} \times E_{t-1} = POR \times E_{t-1}(1+g_1)\frac{\sqrt{1+r}}{(r-g)}. \tag{4.27}$$

$E_{t-1}$ cancels out on both sides of the equation. Additionally, we use the simpler notation *PE* for the price/earnings multiple. Thus, equation (4.27) reduces to:

$$PE = POR \times (1+g_1)\frac{\sqrt{1+r}}{r-g} \tag{4.28}$$

Relationship of PE to Gordon model multiple.

The left-hand term is the price/earnings multiple, and the right-hand term is the payout ratio times 1 plus the one-year growth rate times the midyear Gordon model multiple. In reality, investors do not expect constant growth to perpetuity. They usually have expectations of uneven growth for a few years and a vague, long-run expectation of growth thereafter that they approximate as being constant. Therefore, we should look at $g$, the perpetual growth rate in cash flow, as an average growth rate over the infinite period of time that we are modeling.

The PE ratio is a function of growth ($g$) and risk ($r$). High-growth, low-risk firms will have high PE ratios, and low-growth, high-risk firms will have low PE ratios. The payout ratio is also part of the equation, but it is less of a valuation driver than would appear on the surface. The reason for this is that POR is essentially a dividends issue, and we know from Miller and Modigliani that as a first order, dividends don't matter. Yes, there are academic articles that dispute this, but these effects are more secondary. The equation looks simple enough. Why not just increase the payout ratio to a very high percentage? The answer is that increasing the payout ratio lowers growth. So there is no magic formula to manipulate the PE multiple to the shareholders' delight. Otherwise, all firms would pay out at least 99% of their earnings, and we do not see this in the marketplace. I have seen estimates of the payout ratio for public firms ranging from one-third to 55%.

It is extremely important to be very clear that the earnings bases in the PE multiple and the Gordon model are different. The former is the immediate prior year and the latter is the first forecast year. When an appraiser develops PE multiples from guideline companies, whether publicly or privately owned, he should multiply

[18]Assuming the present value of the cash flows of the firm is its FMV. This ignores valuation discounts, an acceptable simplification in this limited context.

the PE multiple from the guideline companies (after appropriate adjustments) by the subject company's *prior*-year earnings. When using a discounted cash flow method to determine a PE, the appraiser should multiply the Gordon model by POR by the first *forecast* year's earnings. Using the wrong earnings will cause an error in the valuation by a factor of 1 plus the forecast one-year growth rate.

### Mathematical Derivation of the PS Multiple

In this section we will derive the *price-to-sales (PS)* multiple. This equation applies precisely only for a firm with no fixed costs and that has constant forecast growth perpetually. Of course, real firms in the real world differ from this theoretical construct, but the equation nevertheless is important in identifying the valuation drivers of this multiple.

We repeat equation (4.26) as equation (4.28a); however, we rename *FMV* as *P* (price), and instead of prior-year earnings we substitute prior-year sales times the profit margin.

$$P = POR \times S_{t-1} \times PM \times (1+g_1)\frac{\sqrt{1+r}}{(r-g)}. \tag{4.28a}$$

Dividing by prior year sales, we get:

$$PS = POR \times PM \times (1+g_1)\frac{\sqrt{1+r}}{(r-g)} \tag{4.28b}$$

Formula for the PS multiple.

Thus, the PS multiple is a function of the profit margin, growth, and risk, with the payout ratio mechanically part of the equation but with the same comments applying here as in the PE multiple formula. This formula and (4.28) are significant in providing us with theoretical models for performing regression analysis. They tell us which independent variables to use in our regressions. Of course, we should also look to nonlinear transformations of these variables as well, that is, their squares, logarithms, and so forth.

## The Bias in Annual (versus Monthly) Discounting Is Immaterial

In his (2001) article, Dr. Robert Trout stated that midyear discounting of annual cash flows creates a bias in the present values vis-à-vis monthly discounting. Dobner (2002) demonstrated that this is incorrect. This section of the chapter presents a discussion of the validity of using the midyear convention. Although our conclusions are similar to Dobner's, our analysis is very different. We develop exact formulas for annuity discount factors with growth for both monthly and daily cash flows. These can be useful tools for the valuation community when that level of precision is important.

The flaw in Dr. Trout's analysis is that, given compound interest, the monthly interest rate is not equal to the annual discount rate divided by 12. Dr. Trout used 12% annual interest and assumed that 1% monthly interest is equivalent. However, it is not.

Using $i$ as the periodic interest rate (monthly for equations (4.29) through (4.36), daily for equations (4.37) and (4.38)) and $r$ as the annual rate, we begin in equation (4.29) with the statement that 12 months of compounding at the equivalent monthly rate will yield the same result as the annual rate, or:

$$(1+i)^{12} = 1 + r. \tag{4.29}$$

Next we take the 12th root of both sides of the equation:

$$(1+i) = (1+r)^{\frac{1}{12}}. \tag{4.30}$$

Subtracting 1 from each side of the equation, we come to a general formula to convert any annual interest rate into a monthly interest rate:

$$i = (1+r)^{\frac{1}{12}} - 1. \tag{4.31}$$

Now let's use Dr. Trout's numbers and substitute $r = 12\%$ and solve for $i$:

$$i = 1.12^{\frac{1}{12}} - 1 = 0.9489\%. \tag{4.32}$$

Therefore, using 1% per month as the discount rate will discount the cash flows by too high of a discount rate and understate the present value of the monthly cash flows. That would cause us to come to the incorrect result that to match the monthly cash flow, we should have to discount the annual cash flows by more than one-half year.

Dr. Trout also assumed end-of-month cash flows, which puts the average timing of cash flows at 6.5 months into each year, further lowering the present value. A more accurate approximation of continuous cash flows is to assume midmonth cash flows. For proof, we merely add up the timing of end-of-month cash flows from 1 month through 12 months = sum(1,2,3,4 … 12) months = 78 months. 78 months divided by 12 equals an average timing of 6.5 months. To calculate the average timing of midmonth cash flows, we sum (0.5 months, 1.5 months, … 11.5 months) = 72 months. Dividing by 12, we arrive an average timing of 6 months into the year.

## Monthly Annuity Discount Factor Formula

In Appendix B, we develop formulas for the present value of monthly cash flows for $n$ years, with the cash flows arriving midmonth. We call these *monthly annuity discount factors*, $ADF_m$. First we present equation (B4.14) as equation (4.33):

$$ADF_m = \frac{1}{12}(1+r)^{\frac{0.5}{12}} \frac{r}{(1+r)^{\frac{1}{12}} - 1} \frac{1}{r-g}\left[1 - \left(\frac{1+g}{1+r}\right)^n\right] \tag{4.33}$$

PV of midmonth cash flows.

There are five terms in equation (4.33):

1. The first term, 1/12, is the first month's forecast cash flow, that is, 1/12 of \$1.00.
2. The second term is the midmonth correction factor. This is analogous to adding $(1 + r)^{0.5}$ in the numerator of the Gordon model to convert it from end-of-year cash flows to midyear cash flows. If we were assuming end-of-month cash flows instead of midmonth, we would delete this term from the formula.

3. The denominator of the third term is equal to $i$ from equation (4.31), the correct monthly interest rate. Thus, the third term is $r/i$, which is the ratio of the annual rate to the present value equivalent monthly interest rate. In our case, this is 12%/0.9489% = 12.6465. If we incorrectly used $i = 12\%/12 = 1\%$, then this ratio would be 12, which is just the number of months. The ratio adjusts the $ADF_m$ upward.
4. The fourth term is the end-of-year Gordon model formula.
5. The fourth term multiplied by the fifth term is the annual end-of-year ADF. The last term—the one in square brackets—converts the perpetuity of the Gordon model to a finite series of cash flows.

## Table 4.12: Present Values of Monthly Cash Flows

**ROWS 1–42** Table 4.12 contains a 36-month series of cash flows and a 3-year annual cash flow to test the accuracy of equation (4.33) and illustrate how it works. Our basic assumptions are a 12% (F47) annual discount rate ($r$) and a 6% (F49) growth rate ($g$) in cash flows. In months 1 through 12, we begin with cash flow of 1/12 of \$1.00, or \$0.08333 per month (B6–B17). The monthly present value factor is $1/(1+12\%)^{((\text{Month}-0.5)/12)}$. For example, the present value factor (PVF) for month 1 is $1/(1+12\%)^{(0.5/12)} = 0.99529$ (C6). The PVF for month 2 is $1/(1+12\%)^{(1.5/12)} = 0.98593$ (C7), and so forth . The NPV of the cash flow is column B × column C = column D.

The sum of the first 12 months' cash flows is \$1.00 (E17), as it should be, while the present value of those cash flows is \$0.94541 (F17). The next 12 months' cash flows grow by 6% (F49) to \$0.08833 (B18–B29) per month and sum to \$1.06 (E29). The present value of those cash flows is \$0.89477 (F29). The third year's cash flows sum to \$1.1236 (E41), and their present value is \$0.84683 (F41). The sum of the three years' cash flows and present values are \$3.1836 (B42, E42) and \$2.68701 (D42, F42), respectively.

**ROWS 51–63: CALCULATIONS FOR THE $ADF_m$ FORMULA** Let's look at the calculations for the $ADF_m$ formula. We begin with the end-of-year Gordon model formula, $1/(r-g) = 1/(0.12-0.06) = 1/0.06 = 16.66667$ (F52). This is the fourth term in equation (4.33). We next calculate the term that converts from a perpetuity to a finite stream of cash flows. That is the last term in equation (4.33), that is, the one in square brackets, $\left[1-\left(\frac{1+g}{1+r}\right)^n\right]$. With $g = 6\%$ (F49), $r = 12\%$ (F47), and $n = 3$ years (F53), this term equals 0.152258 (F54). Multiplying 16.66667 × 0.152258 = 2.53764 (F55), the annual ADF.

Next we multiply the first term in equation (4.33) by the third term. Also note that the denominator of the third term equals $i$ from equation (4.31). Thus, multiplying those two terms together results in $\frac{1}{12} \times \frac{r}{(1+r)^{\frac{1}{12}}-1} = \frac{r}{12i} = 1.05387$ (F56).[19]

Next we calculate the second term in equation (4.33), the midmonth correction factor, $ADF_m = (1+r)^{\frac{0.5}{12}} = 1.00473$ (F57). To calculate $ADF_m$, we multiply 2.53764 × 1.05387 × 1.00473 = 2.68701 (F55 × F56 × F57 = F58). Note that our calculation by formula in F58 agrees with the brute force calculations in D42 and F42.

---

[19]Note that if we were using simple instead of compound interest, $i$ would equal 1%, and $r/12i$ would equal 1.00.

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table 4.12** | | | | | | | | |
| 2 | **Present Values of Cash Flows: *r* = 12%, *g* = 6%** | | | | | | | | |
| 3 | | | | | | | | | |
| 4 | | **Monthly Cash Flows** | | | | | **Annual Cash Flows** | | |
| 5 | **t(Mos.)** | **Cash Flow (CF)** | **PV Factor** | **NPV** | **Sum CF** | **Sum NPV** | **CF** | **PVF** | **PV CF** |
| 6 | **1** | 0.08333 | 0.99529 | 0.08294 | | | 1.00000 | 0.94491 | 0.94491 |
| 7 | **2** | 0.08333 | 0.98593 | 0.08216 | | | 1.06000 | 0.84367 | 0.89429 |
| 8 | **3** | 0.08333 | 0.97667 | 0.08139 | | | 1.12360 | 0.75328 | 0.84638 |
| 9 | **4** | 0.08333 | 0.96749 | 0.08062 | | | **3.18360** | **2.54186** | **2.68558** |
| 10 | **5** | 0.08333 | 0.95839 | 0.07987 | | | | | |
| 11 | **6** | 0.08333 | 0.94938 | 0.07912 | | | | | |
| 12 | **7** | 0.08333 | 0.94046 | 0.07837 | | | | | |
| 13 | **8** | 0.08333 | 0.93162 | 0.07763 | | | | | |
| 14 | **9** | 0.08333 | 0.92286 | 0.07691 | | | | | |
| 15 | **10** | 0.08333 | 0.91419 | 0.07618 | | | | | |
| 16 | **11** | 0.08333 | 0.90560 | 0.07547 | | | | | |
| 17 | **12** | 0.08333 | 0.89708 | 0.07476 | 1.00000 | 0.94541 | | | |
| 18 | **13** | 0.08833 | 0.88865 | 0.07850 | | | | | |
| 19 | **14** | 0.08833 | 0.88030 | 0.07776 | | | | | |
| 20 | **15** | 0.08833 | 0.87202 | 0.07703 | | | | | |
| 21 | **16** | 0.08833 | 0.86383 | 0.07630 | | | | | |
| 22 | **17** | 0.08833 | 0.85571 | 0.07559 | | | | | |
| 23 | **18** | 0.08833 | 0.84766 | 0.07488 | | | | | |
| 24 | **19** | 0.08833 | 0.83970 | 0.07417 | | | | | |
| 25 | **20** | 0.08833 | 0.83180 | 0.07348 | | | | | |
| 26 | **21** | 0.08833 | 0.82398 | 0.07279 | | | | | |
| 27 | **22** | 0.08833 | 0.81624 | 0.07210 | | | | | |
| 28 | **23** | 0.08833 | 0.80857 | 0.07142 | | | | | |
| 29 | **24** | 0.08833 | 0.80097 | 0.07075 | 1.06000 | 0.89477 | | | |
| 30 | **25** | 0.09363 | 0.79344 | 0.07429 | | | | | |
| 31 | **26** | 0.09363 | 0.78598 | 0.07359 | | | | | |
| 32 | **27** | 0.09363 | 0.77859 | 0.07290 | | | | | |
| 33 | **28** | 0.09363 | 0.77127 | 0.07222 | | | | | |
| 34 | **29** | 0.09363 | 0.76402 | 0.07154 | | | | | |
| 35 | **30** | 0.09363 | 0.75684 | 0.07087 | | | | | |
| 36 | **31** | 0.09363 | 0.74973 | 0.07020 | | | | | |
| 37 | **32** | 0.09363 | 0.74268 | 0.06954 | | | | | |
| 38 | **33** | 0.09363 | 0.73570 | 0.06889 | | | | | |
| 39 | **34** | 0.09363 | 0.72879 | 0.06824 | | | | | |
| 40 | **35** | 0.09363 | 0.72194 | 0.06760 | | | | | |
| 41 | **36** | 0.09363 | 0.71515 | 0.06696 | 1.12360 | 0.84683 | | | |
| 42 | **Totals** | **3.18360** | | **2.68701** | **3.18360** | **2.68701** | | | |
| 43 | | | | | | | | | |
| 44 | **Assumptions:** | | | | | | | | |
| 45 | | | | | | | | | |
| 46 | ***p* = # Periods per Year** | | | | | **12** | | | |
| 47 | ***r* = Annual Discount Rate** | | | | | **12.0%** | | | |
| 48 | ***i* = (1+*r*)^(1/*p*)–1 = Periodic Interest Rate** | | | | | **0.9489%** | | | |
| 49 | ***g* = Growth Rate in Net Inc (or Cash Flow)** | | | | | **6.0%** | | | |

**PV OF MIDMONTH CASH FLOWS—ALTERNATIVE EXPRESSION** We present equation (B4.14a), which is an alternative expression for (B4.14), as (4.33a):

$$ADF_m = Midmonth\ Correction\ Factor \times ADF_{Endyear} \times \frac{r}{12i}. \tag{4.33a}$$

In F59, we calculate a midyear annuity discount factor, using equation (4.10a), repeated as equation (4.34).

$$ADF = \frac{\sqrt{1+r}}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^n\right]. \tag{4.34}$$

## Table 4.12 (cont.)

| | | F |
|---|---|---|
| 51 | **Calculations for Monthly Annuity Discount Factor Formula** | |
| 52 | Gordon Model—Endyear = 1/(*r*–*g*) | 16.66667 |
| 53 | *n* | 3 |
| 54 | Conversion from Perpetuity to ADF: 1–((1+*g*)/(1+*r*))^*n* | 0.152258 |
| 55 | ADF—Annual (F52 × F54) | **2.53764** |
| 56 | *r*/12*i* | 1.05387 |
| 57 | (1+*r*)^(0.5/12) | 1.00473 |
| 58 | ADF—Monthly (F55 × F56 × F57) | **2.68701** |
| 59 | ADF—Midyear = SQRT(1+*r*)/(*r*–*g*) × F54 | 2.68558 |
| 60 | Ratio of Midmonth ADF/MidyearADF (F58/F59) | **1.00053** |
| 61 | | |
| 62 | **Ratio of Monthly to AnnualADF—By Formula** | |
| 63 | *r*/(12×*i*)×(1+*r*)^(–5.5/12) Equation (4.35) | **1.00053** |

**Sensitivity Analysis: How the Ratio Varies with Changes in the Discount Rate**

| | Disc Rate | Ratio |
|---|---|---|
| 70 | 12% | **1.00053** |
| 71 | 14% | **1.00071** |
| 72 | 16% | **1.00091** |
| 73 | 18% | **1.00113** |
| 74 | 20% | **1.00138** |
| 75 | 22% | **1.00164** |
| 76 | 24% | **1.00192** |
| 77 | 26% | **1.00221** |
| 78 | 28% | **1.00252** |
| 79 | 30% | **1.00285** |

**Summary of Each Year's PVs: Monthly versus Annual Cash Flows**

| | Year | PV Monthly Cash Flows | PV Annual Cash Flows | Ratio Mo/Annual |
|---|---|---|---|---|
| 85 | 1 | 0.94541 | 0.94491 | **1.00053** |
| 86 | 2 | 0.89477 | 0.89429 | **1.00053** |
| 87 | 3 | 0.84683 | 0.84638 | **1.00053** |

***Note:*** **The ratio of monthly-to-annual cash flows is the same every year.**

In our example, using equation (4.34), we compute the midyear ADF as 2.68558 (F59). Dividing $ADF_m$ by $ADF$, the ratio is 1.00053 (F60). This shows that, under the previous assumptions, using the monthly instead of the annual ADF would increase the present value of cash flows by only 0.05%—an amount we can ignore.

We can develop a formula for the ratio of present values of the monthly ADF versus the annual ADF as equation (4.33) divided by equation (4.34), or:[20]

$$\frac{ADF_m}{ADF} = \frac{r}{12i}\frac{(1+r)^{\frac{0.5}{12}}}{\sqrt{1+r}} = \frac{r}{12i}(1+r)^{\frac{-5.5}{12}} = 1.00053 \quad (\text{F63} = \text{F60}). \tag{4.35}$$

Note that since $i$ is strictly a function of $r$, the ratio of monthly-to-annual ADF is strictly a function of $r$, the discount rate.

[20]In this division, we make use of the fact that as defined in equation (4.31), $i = (1 + r)^{(1/12)} - 1$.

**SENSITIVITY ANALYSIS—HOW THE RATIO VARIES WITH CHANGES IN *r*** In rows 70 through 79, we see how the ratio of monthly-to-annual ADFs varies with changes in the discount rate. The ratio increases with increases in the discount rate, but that ratio is very small at all levels. Even at venture capital–type discount rates of 30%, using monthly-versus-annual cash flows merely increases the value by less than 0.3%; that is, the ratio is 1.00285 (F79).

**PRESENT VALUE OF MONTHLY CASH FLOWS COMPARED TO PV OF ANNUAL CASH FLOWS** We transfer the present value of the sum of each year's monthly cash flows to B85–B87; that is, we transfer F17 to B85, F29 to B86, and F41 to B87.

In G6 through G8, respectively, we present the annual cash flows of $1.00, $1.06, and $1.1236, which match the annual summary of the monthly cash flows in E17, E29, and E41, respectively. We compute the annual midyear present value factors of $1/(1 + 0.12)^{0.5} = 0.94491$ (H6), $1/(1 + 0.12)^{1.5} = 0.84367$ (H7), and $1/(1 + 0.12)^{2.5} = 0.75328$ (H8). We multiply the annual cash flows by the annual midyear PVFs to compute the PV of the cash flows, which are 0.94491 (I6), 0.89429 (I7), and 0.84638 (I8). We transfer I6–I8 to C85–C87, respectively.

We then divide the annual sums of the monthly present values by the annual present values to calculate the ratio of present values year-by-year; that is, B85/C85 = 1.00053 (D85); B86/C86 = 1.00053 (D86); and B87/C87 = 1.00053 (D87). Note that the ratio in D85–D87 equals F60 and F63. Also, note that the ratio is the same every year—and it has to be so—as equation (4.35) tells us that the ratio is independent of time (and growth). The ratio depends only on the discount rate. Since this is independent of time, this also means that we would have come to the same conclusion using perpetuities, that is, the Gordon model, rather than ADFs.

**END-OF-MONTH $ADF_m$** The end-of-month $ADF_m$ is identical to the midmonth *ADF* without the midmonth conversion term. Thus, our end-of-month $ADF_m$s are:

$$ADF_m = \frac{1}{12}\frac{r}{(1+r)^{\frac{1}{12}}-1}\frac{1}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^n\right] \tag{4.36}$$

PV—end-of-month cash flows.

$$ADF_m = \frac{r}{12i} \times ADF_{Endyear} \tag{4.36a}$$

PV—end-of-month cash flows—alternative expression.

**DAILY CASH FLOWS** For daily cash flows, it is most reasonable to assume bank deposits occur at the end of the day, not midday. Therefore, unlike monthly cash flows, daily cash flows do not carry with them a midday correction. If we replace every instance of the number 12 in equations (4.36) and (4.36a) with 365 and recognize that now $i$, the periodic interest rate, means the daily interest rate in this context rather than the monthly interest rate in the earlier part of this section, the $ADF_m$ formulas convert to daily ADF formulas as follows:

$$ADF_{Daily} = \frac{1}{365}\frac{r}{(1+r)^{\frac{1}{365}}-1}\frac{1}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^n\right] \tag{4.37}$$

PV—daily cash flows.

$$ADF_{Daily} = \frac{r}{365i} \times ADF_{Endyear} \tag{4.37a}$$

PV—daily cash flows—alternative expression.

The ratio of the daily ADF to the midyear annual ADF is equal to equation (4.37a) divided by equation (4.34), the midyear annual ADF, or:

$$ADF_{Daily} = \frac{r}{365i} \times \frac{ADF_{Endyear}}{ADF_{Midyear}} = \frac{r}{365i \times \sqrt{1+r}}. \tag{4.38}$$

Table 4.13 shows the same summary calculations for daily cash flows as Table 4.12 shows for monthly cash flows. The daily ADF equals 2.68660 (F17), while the midyear annual ADF equals 2.68558 (F18). The ratio of the two is 1.00038 (F17/F18 = F19). The formula in F22 is equation (4.38), and it also equals 1.00038. Finally, the sensitivity analysis in F28 through F37 shows that the ratios of daily-to-annual ADFs are very slightly lower than the ratio of monthly-to-annual ADFs. The difference comes from using end-of-day rather than midday calculations.[21] In any case, the difference is negligible, and in all practicality, our results for daily cash flows are virtually identical to our results for monthly cash flows.

### Conclusion to Midyear Bias

Annual cash flows with midyear present value factors do not introduce any material bias in the cash flows vis-à-vis daily, weekly, or monthly cash flows. We should continue using our standard midyear present values.

## Conclusions

We can see that there is a family of annuity discount factors (ADFs), from the simplest case of an ordinary annuity to the most complicated case of an annuity with stub periods (fractional years), as discussed in Appendix A. The elements that determine which formula to use are:

- Whether the cash flows are midyear versus end-of-year.
- When the cash flows begin (year 1 versus any other time).
- Whether they occur every year, at regular, skipped intervals, or have repeating cycles.
- Whether the constant growth is zero.
- Whether there is a stub period.

For cash flows without a stub period, the ADF is the difference of two Gordon model perpetuities. The first term is the perpetuity from $S$ to infinity, where $S$ is the starting year of the cash flow. The second term is the perpetuity starting at $n + 1$ (where $n$ is the final cash flow in the annuity) going to infinity. For cash flows with

[21]We have verified the accuracy of the daily formulas and results with a spreadsheet with 730 days (2 years) of cash flows, which we have not shown in this chapter for reasons of space.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Table 4.13** | | | | | |
| 2 | **Present Values of Daily Cash Flows** | | | | | |
| 3 | | | | | | |
| 4 | **Assumptions:** | | | | | |
| 5 | | | | | | |
| 6 | **$p$ = # Periods per Year** | | | | | **365** |
| 7 | **$r$ = Annual Discount Rate** | | | | | **12.0%** |
| 8 | **$i$ = (1+$r$)^(1/$p$)–1 = Periodic Interest Rate** | | | | | **0.0311%** |
| 9 | **$g$ = Growth Rate in Net Inc (or Cash Flow)** | | | | | **6.0%** |
| 10 | | | | | | |
| 11 | **Calculations** | | | | | |
| 12 | **Gordon Model—Endyear = 1/($r$-$g$)** | | | | | 16.66667 |
| 13 | **$n$** | | | | | 3 |
| 14 | **Conversion from Perpetuity to ADF: 1–((1+$g$)/(1+$r$))^$n$** | | | | | 0.152258 |
| 15 | **ADF—Annual (F12 × F14)** | | | | | **2.53764** |
| 16 | **$r$/$pi$** | | | | | 1.05870 |
| 17 | **ADF—Daily (F15 × F16)** | | | | | **2.68660** |
| 18 | **ADF—Midyear = SQRT(1+$r$)/($r$–$g$) × F14** | | | | | 2.68558 |
| 19 | **Ratio of Daily ADF/Midyear ADF (F17/F18)** | | | | | **1.00038** |
| 20 | | | | | | |
| 21 | **Ratio of Daily to Annual ADF—By Formula** | | | | | |
| 22 | **= $r$/(($p$×$i$)×SQRT(1+$r$)) This is equation (4.38).** | | | | | **1.00038** |
| 23 | | | | | | |
| 24 | **Sensitivity Analysis: How the Ratio Varies** | | | | | |
| 25 | **with Changes in the Discount Rate** | | | | | |
| 26 | | | | | **Disc Rate** | **Ratio** |
| 27 | | | | | | |
| 28 | | | | | 12% | **1.00038** |
| 29 | | | | | 14% | **1.00054** |
| 30 | | | | | 16% | **1.00071** |
| 31 | | | | | 18% | **1.00091** |
| 32 | | | | | 20% | **1.00114** |
| 33 | | | | | 22% | **1.00138** |
| 34 | | | | | 24% | **1.00163** |
| 35 | | | | | 26% | **1.00191** |
| 36 | | | | | 28% | **1.00220** |
| 37 | | | | | 30% | **1.00251** |

a stub period, the preceding statement is true with the addition of a third term for the single cash flow of the stub period itself, discounted to PV.

While this chapter contains some complicated algebra, the focus has been on the intuitive explanation of each ADF. The most difficult mathematics have been moved to Appendix A, which contains the formulas for ADFs with stub periods and some advanced material on the use of ADFs in calculating loan amortization. ADFs are also used for practical applications in Chris Mercer's quantitative marketability discount model (see Chapter 8), periodic expenses such as moving costs and losses from lawsuits, ESOP valuation, in reducing a seller-subsidized loan to its cash equivalent price in Chapter 9 (Table 9.3), and to calculate loan payments.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 4.14** | | | | |
| 2 | **Table of ADF Equation Numbers** | | | | |
| 3 | | | | | |
| 4 | | **With Growth** | | **No Growth** | |
| 5 | **Formulas in the Chapter** | **End-of-Year** | **Midyear** | **End-of-Year** | **Midyear** |
| 6 | Ordinary ADF | (4.6) to (4.6(c)) | (4.10) to (4.10b) | (4.6(d)) | (4.10c) & (4.10d) |
| 7 | Gordon Model | (4.7) | (4.10(e)) | | |
| 8 | Starting Cash Flow not $t = 1$ | (4.11) & (4.11a) | (4.12) | (4.11c) | |
| 9 | Valuation Date = $v$ | (4.11b) | | | |
| 10 | Gordon Model for Starting CF not = 1 | (4.11d) | | | |
| 11 | Periodic Expenses | (4.18) | (4.19) | | |
| 12 | Periodic Expenses—Flexible Timing | (4.18a) | (4.19a) | | |
| 13 | Loan Payment | | | (4.21) | |
| 14 | Relationship of Gordon Model to PE | | (4.28) | | |
| 15 | Relationship of Gordon Model to PS | | (4.28b) | | |
| 16 | Monthly ADF [1] | (4.36), (4.36a) | (4.33), (4.33a) | | |
| 17 | Daily ADF [2] | (4.37), (4.37a) | | | |
| 18 | | | | | |
| 19 | | | | | |
| 20 | **Formulas in the Appendix** | | | | |
| 21 | | | | | |
| 22 | ADF with Stub Period | (A4.4) | (A4.3) | | |
| 23 | Amortization of Loan Principal | | | (A4.10) | |
| 24 | PV of Loan After-Tax | | | (A4.24) & (A4.29) | |
| 25 | | | | | |
| 26 | [1] For this ADF, read row 5 as End-of-Month and Mid-Month. | | | | |
| 27 | [2] For this ADF, read row 5 as End-of-Day. Midday has no practical meaning in this context. | | | | |

We have performed a rigorous derivation of the PE multiple and the Gordon model. This derivation demonstrates that the PE multiple equals 1 minus the earnings retention rate times 1 plus the one-year growth rate times the midyear Gordon model multiple. Further, we showed how the former uses the *prior* year's earnings, while the latter uses the first *forecast* year's earnings. Many appraisers have found that confusing, and hopefully this section of the chapter will do much to eliminate that confusion.

We also have demonstrated that the annual present value factors are substantially accurate and do not introduce a material bias vis-à-vis monthly or daily present value factors (or ADFs).

Because there are so many ADFs for different purposes and assumptions, we include Table 4.14 to point the reader to the correct ADF equations.

## References

Abrams, Jay B., and R. K. Hiatt. 2003. The Bias in Annual (vs. Monthly) Discounting Is Immaterial, Co-Authored with R. K. Hiatt, *Business Valuation Review* (September).

Dobner, Michael. 2002. Mid-Year Discounting and Seasonality Factors. *Business Valuation Review* (March): 16.

Gordon, M. J. 1962. *The Investment, Financing, and Valuation of the Corporation*, 2nd ed. Homewood, IL: Irwin.

Gordon, M. J., and E. Shapiro. 1956. Capital Equipment Analysis: The Required Rate of Profit. *Management Science* 3: 102–110.

Mercer, Z. Christopher. 1997. *Quantifying Marketability Discounts: Developing and Supporting Marketability Discounts in the Appraisal of Closely Held Business Interests*. Memphis, TN: Peabody.

Trout, Robert. 2001. Mid-Year Discounting without Bias. *Business Valuation Review* (December): 39.

Williams, J. B. 1938. *The Theory of Investment Value*. Cambridge, MA: Harvard University Press.

# APPENDIX 4A

# Mathematical Appendix

## Introduction

This appendix is an extension of the material developed in the chapter. The topics that we cover are:

- Developing ADFs for cash flows that end on a fractional year (stub period)
- Developing ADFs for loan mathematics, consisting of calculating the amortization of principal in loans and the net after-tax cost of a loan

This appendix is truly for the mathematically brave. The topics covered and formulas developed are esoteric and less practically useful than the formulas in the chapter, though the formula for the after-tax cost of a loan may be useful to some practitioners. The material in this appendix is included primarily for reference and for the mathematical gourmets and "Dirty Harrys." Nevertheless, even those not completely comfortable with the difficult mathematics can benefit from focusing on the verbal explanations before the equations and the development of the first one or two equations in the derivation of each of the formulas. The rest is just the tedious math, which can be skipped.

## The ADF with Stub Periods (Fractional Years)

We will now develop a formula to handle annuities that have stub periods, constant growth in cash flows, and cash flows that start at any time. To the best of my knowledge, I invented this formula. In this section we will assume midyear cash flows and later present the formula for end-of-year cash flows.

Let's begin with constructing a timeline of the cash flows in Figure A4.1, using the following definitions and assumptions.

### Definitions

$S =$ time (in years) until the first cash flow for end-of-year cash flows. For midyear cash flows, $S =$ end of the year in which the first cash flow occurs. In this

| Row \ Col. | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|
| 1 | Year (numeric) | 3.25 | 4.25 | 5.25 | ... | 12.25 | 12.60 |
| 2 | Year (symbolic) | $S$ | $S$+1 | $S$+2 | ... | $n$ | $z$ |
| 3 | Growth (in $) | 0 | $g$ | $g(1+g)$ | ... | $g(1+g)^{n-S-1}$ | NA |
| 4 | Cash Flow | 1 | 1+$g$ | $(1+g)^2$ | ... | $(1+g)^{n-S}$ | $p(1+g)^{n-S+1}$ |

**FIGURE A4.1 Timeline of Cash Flows**

midyear-cash-flow example, the cash flows begin at $t = 2.25$, and $S$ equals 3.25 because it is the end of the year that began at $t = 2.25$. We assume the cash flow occurs in the middle of the year, or at $S - 0.5 = 3.25 - 0.5 = 2.75$ years.

$n$ = end of the last whole year's cash flows = 12.25 years in this example.

$z$ = end of the stub period = 12.60 years.

$p$ = proportion of a full year represented by the stub period = $z - n = 12.60 - 12.25 = 0.35$ years.

$g$ = constant growth rate in cash flows = 5.1%.

$t$ = point in time—measured in years.

## The Cash Flows

We assume the cash flows occur evenly throughout each year. Thus the first cash flow of $1.00 (Figure A4.1, C4) occurs throughout year $S$,[22] which spans from $t = 2.25$ to $t = 3.25$ years. For simplicity, we denote that the cash flow is for the year ending at $t = 3.25$ years (C1). Note that for year 3.25, there is no growth in the cash flow (i.e., C3 = 0).

The following year is 4.25 (D1), or $S + 1$ (D2). The $1.00 grows at a rate of $g$ (D3), so the ending cash flow is $1 + g$ (D4). Note that the ending cash flow is equal to $(1 + g)^{t-S} = (1 + g)^{4.25-3.25}$.

For year 5.25, or $S + 2$ (E2), growth in cash flows is $g$ times the prior year's cash flow of $(1 + g)$ (D4), or $g\,(1 + g)$ (E3), which leads to a cash flow equal to the prior year's cash flow plus this year's growth, or $(1 + g) + g\,(1 + g) = (1 + g)\,(1 + g) = (1 + g)^2$ (E4). Again, the cash flow equals $(1 + g)^{t-S} = (1 + g)^{5.25-3.25}$.

For the year 6.25, or $S + 3$, which is not shown in Figure A4.1, cash flows grow by $g\,(1 + g)^2$, so cash flows are $(1 + g)^2 + g\,(1 + g)^2 = (1 + g)^2\,(1 + g) = (1 + g)^3 = (1 + g)^{t-S} = (1 + g)^{6.25-3.25}$.

We continue in this fashion through the last whole year of cash flows, which we call year $n$ (column G). In our example, $n = 12.25$ years (G1). The cash flows during year $n$ are equal to $(1 + g)^{n-S}$ (G4).

Had we completed one more full year, the cash flows would have extended to year 13.25, or year $n + 1$. If so, the cash flow would have been $(1 + g)^{n-S+1}$. However, since the stub year's cash flow is for only a partial year, the ending cash flow is multiplied by $p$—the fractional portion of the year—leading to an ending cash flow of $p(1 + g)^{n-S+1}$.[23]

[22] $S$ is for *starting* cash flow.

[23] This formula assumes growth happens annually but not continuously throughout the year. The latter would require a different formula.

It is important to recognize that there may be other ways of specifying how the partial year affects the cash flows. For example, it is possible, but very unlikely, that the cash flows can be based on a legal document that specifies that only the growth rate itself will be fractional, but the corpus of the cash flow will not diminish for the partial year. We could calculate a solution to the ADF based on that assumption—but we will not, as it is very unlikely to be of any practical use, and we have already demonstrated how to model the most likely method of splitting the cash flows in the fractional year. The point is that modeling the fractional year's cash flows depends on the agreement and/or the underlying scenario, and one should not blindly charge off into the sunset applying a formula that was developed under an assumption that does not apply in a given case.

## Discounting Periods

The first cash flow occurs during the year that spans from $t = 2.25$ to $t = 3.25$. As mentioned previously, we assume that the cash flows occur evenly throughout the year. This is tantamount to assuming all cash flows occur on average halfway through the year, that is, at year 2.75. Therefore, as of time zero, defined as $t = 0$, the first \$1 cash flow has a present value of $\frac{1}{(1+r)^{2.75}} = \frac{1}{(1+r)^{S-0.5}}$.

We will be discounting the cash flows in two stages because that will later enable us to provide a more intuitive explanation of our results. Our first discounting of cash flows will be to $t = S - 1$, the beginning of the first year of cash flows. The first year's cash flow then receives a discount of $\frac{1}{(1+r)^{0.5}}$, the second year's cash flows receive a discount of $\frac{1}{(1+r)^{1.5}}$, and so on. Thus, the denominators here are identical to those for cash flows that would begin in year 1 instead of $S$.

## The Equations

The PV of our series of cash flows as of $t = S - 1$ is:

$$PV = \frac{1}{(1+r)^{0.5}} + \frac{(1+g)}{(1+r)^{1.5}} + \cdots + \frac{(1+g)^{n-S}}{(1+r)^{n-S+0.5}} + \frac{p(1+g)^{n-S+1}}{(1+r)^{n-S+1+0.5p}}. \qquad \text{(A4.1)}$$

Note that the exponent in the denominator of the last term (the fractional year) is equal to the one before it (the last whole year) plus ½ year, to bring us to the end of year $n$, plus ½ of the fractional year, thus maintaining a midyear assumption.

We already have a solution to the PV of the whole years in the body of the chapter—equation (4.10). Thus, the PV of the entire series of cash flows as of $t = S - 1$ is equation (4.10) plus the final term in equation (A4.1), or:

$$NPV = \frac{\sqrt{1+r}}{r-g} - \left(\frac{1+g}{1+r}\right)^{n-S+1} \frac{\sqrt{1+r}}{r-g} + \frac{p(1+g)^{n-S+1}}{(1+r)^{n-S+1+0.5p}}. \qquad \text{(A4.2)}$$

The next step is to discount the PV from $t = S - 1$ to $t = 0$. We do this by multiplying by $\frac{1}{(1+r)^{S-1}}$. The result is our annuity discount factor for midyear cash flows with a stub period:

$$ADF = \left\{ \frac{\sqrt{1+r}}{r-g} - \left(\frac{1+g}{1+r}\right)^{n-S+1} \frac{\sqrt{1+r}}{r-g} + \frac{p(1+g)^{n-S+1}}{(1+r)^{n-S+1+0.5p}} \right\} \frac{1}{(1+r)^{S-1}}. \qquad \text{(A4.3)}$$

The ADF formula for end-of-year cash flows with a stub period is:

$$ADF = \left\{ \frac{1}{r-g} - \left(\frac{1+g}{1+r}\right)^{n-S+1} \frac{1}{r-g} + \frac{p(1+g)^{n-S+1}}{(1+r)^{(z-S+1)}} \right\} \frac{1}{(1+r)^{S-1}}. \qquad \text{(A4.4)}$$

The individual terms in equation (A4.4) have the same meaning as in the midyear cash flows of equation (A4.3). To easily see the derivation of the end-of-year (EOY) model from the midyear, note that an EOY model in equation (A4.1) would require the exponent in each denominator to be 0.5 years larger, which changes the $\sqrt{1+r}$ term in equation (A4.3) to 1. $\frac{1}{r-g}$ is the EOY Gordon model formula. The only other difference is the discount factor in the rightmost term in the braces of equations (A4.3) and (A4.4).[24] In the former, we discount the stub period cash flow by $(1+r)^{n-S+1+0.5p}$, while in the latter we discount by $(1+r)^{z-S+1}$.

## Tables A4.1 and A4.2: Example of Equations (A4.3) and (A4.4)

Table A4.1 is an example of the midyear ADF with a fractional year cash flow, and Table A4.2 is an example using end-of-year cash flows. Table A4.2 has the identical structure and meaning as Table A4.1—merely using end-of-year formulas rather than midyear. Therefore, we will explain only Table A4.1.

In the first part of Table A4.1, we will use a brute force method of scheduling out the cash flows, calculating their present values, and then summing them. Later, we will directly test the formulas and demonstrate that they produce the same result as the brute force method.

**BRUTE FORCE METHOD OF CALCULATING PV OF CASH FLOWS** Rows 7 through 17 in Table A4.1 are a detailed listing of the cash flows and their present values each year. The first cash flows begin in row 7 at year 2.25 and finish at $t = 3.25$, with year 2.75 as the midpoint from which we discount. We will refer to the years by the ending year; that is, the cash flow in row 7 is for the year ending at $t = 3.25$. Assumptions of the model begin in row 33.

We begin with \$1.00 of cash flow for the year ending at $t = 3.25$ (C7). Column B shows the growth in cash flows and is equal to $g = 5.1\%$ (G37) multiplied by the previous period's cash flow. In B8, the calculation is \$1.00 × 5.1% = \$0.051 (C7 × G37 = B8). The cash flow in C8 is C7 + B8, or \$1.00 + \$0.051 = \$1.051. We repeat this pattern through row 16, the last whole year's cash flow.

Column D replicates column C using the formula cash flow = $(1+g)^{t-S}$ for all cells except D17, which is the fractional year cash flow. The formula for that cell is $p(1+g)^{n-S+1}$, where multiplying by $p = 0.35$ (G38) years converts what would have been the cash flow for the whole year $n + 1$ (and would have been \$1.64447) into the fractional year cash flow of \$0.57557.[25] Note that in that formula, $n = 12.25$ years, the last whole year.

[24]Note that the term after the brackets remains unchanged, because we discount to the same starting point, $t = 0$.

[25]See A45 for the formula in the spreadsheet.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table A4.1** | | | | | | | |
| 2 | **ADF with Fractional Year** | | | | | | | |
| 3 | **Midyear Formula** | | | | | | | |
| 4 | | | | | | | | |
| 5 | | **Cash Flows** | | | ***t* = *S*–1** | | ***t* = 0** | |
| 6 | ***t* (Yrs)** | **Growth** | **Cash Flow** | **$(1+g)^{t-S}$** | **$PVF=1/(1+r)^{t-S+0.5}$** | **PV** | **$PVF=1/(1+r)^{t-0.5}$** | **PV** |
| 7 | **3.25** | NA | 1.00000 | 1.00000 | 0.93250 | 0.93250 | 0.68090 | 0.68090 |
| 8 | **4.25** | 0.05100 | 1.05100 | 1.05100 | 0.81087 | 0.85223 | 0.59208 | 0.62228 |
| 9 | **5.25** | 0.05360 | 1.10460 | 1.10460 | 0.70511 | 0.77886 | 0.51486 | 0.56871 |
| 10 | **6.25** | 0.05633 | 1.16094 | 1.16094 | 0.61314 | 0.71181 | 0.44770 | 0.51975 |
| 11 | **7.25** | 0.05921 | 1.22014 | 1.22014 | 0.53316 | 0.65053 | 0.38930 | 0.47501 |
| 12 | **8.25** | 0.06223 | 1.28237 | 1.28237 | 0.46362 | 0.59453 | 0.33853 | 0.43412 |
| 13 | **9.25** | 0.06540 | 1.34777 | 1.34777 | 0.40315 | 0.54335 | 0.29437 | 0.39674 |
| 14 | **10.25** | 0.06874 | 1.41651 | 1.41651 | 0.35056 | 0.49658 | 0.25597 | 0.36259 |
| 15 | **11.25** | 0.07224 | 1.48875 | 1.48875 | 0.30484 | 0.45383 | 0.22259 | 0.33138 |
| 16 | **12.25** | 0.07593 | 1.56468 | 1.56468 | 0.26508 | 0.41476 | 0.19355 | 0.30285 |
| 17 | **12.60** | NA | 0.57557 | 0.57557 | 0.24121 | 0.13883 | 0.17613 | 0.10137 |
| 18 | **Totals for Whole Years = 3.25 - 12.25** | | | | | **6.42899** | | **4.69432** |
| 19 | **Add Fractional Year = 12.60** | | | | | **0.13883** | | **0.10137** |
| 20 | **Grand Total (*t* = *S*–1 in Column G and *t* = 0 in Column I)** | | | | | **6.56782** | | **4.79569** |
| 21 | **Present Value Factor—Discount from *S*–1 (*t* = 2.25) to 0** | | | | | **0.73018** | | |
| 22 | **Grand Total (*t* = 0)** | | | | | **4.79569** | | |
| 23 | | | | | | | | |
| 24 | **Calculation of PV by Formulas:** | | | | | | | |
| 25 | | | | | **Grand** | | | |
| 26 | | **Whole Yrs** | **Frac Yr** | **Total** | **Total** | | | |
| 27 | ***t* = *S*–1** | 6.42899 | 0.13883 | 6.56782 | | | | |
| 28 | **PV Factor** | 0.73018 | 0.73018 | | | | | |
| 29 | ***t*=0** | 4.69432 | 0.10137 | **4.79569** | **4.79569** | | | |
| 30 | | | | | | | | |
| 31 | **Assumptions:** | | | | | | | |
| 32 | | | | | | | | |
| 33 | ***S* = Beginning Year of Cash Flows (valuation at *t* = 2.25)** | | | | | | **3.25** | |
| 34 | ***n* = Ending Year of Cash Flows—Whole Year** | | | | | | **12.25** | |
| 35 | ***z* = Ending Year of Cash Flows—Stub Year** | | | | | | **12.60** | |
| 36 | ***r* = Discount Rate** | | | | | | **15.0%** | |
| 37 | ***g* = Growth Rate in Cash Flow** | | | | | | **5.1%** | |
| 38 | ***p* = Proportion of Year in the Stub Period** | | | | | | **0.35** | |
| 39 | **Midpoint = *n* + 0.5 *p* = Midpoint of the fractional year** | | | | | | **12.425** | |
| 40 | ***x* = (1+*g*)/(1+*r*)** | | | | | | **0.913913** | |
| 41 | **Gordon Model Multiple = GM = Sqrt(1+*r*)/(*r*–*g*)** | | | | | | **10.832127** | |
| 42 | | | | | | | | |
| 43 | **Spreadsheet Formulas:** | | | | | | | |
| 44 | | | | | | | | |
| 45 | **C17, D17: *p**(1+*g*)^(*n*–*s*+1) Stub Period Cash Flow** | | | | | | | |
| 46 | **E17: 1/(1+*r*)^(*n*–*S*+1+0.5**p*) Stub Period Present Value Factor at *t* = 2.25** | | | | | | | |
| 47 | **G17: 1/(1+*r*)^(*n*+0.5**p*) Stub Period Present Value Factor for *t* = 0** | | | | | | | |
| 48 | **B27: GM*(1–*x*^(*n*–*S*+1)) ADF for Years 3.25 to 12.25 at *t* = 2.25** | | | | | | | |
| 49 | **C27: *p**(1+*g*)^(*n*–*S*+1)/(1+*r*)^(*n*–*S*+1+0.5**p*) PV of Stub Period CF at *t* = 2.25** | | | | | | | |
| 50 | **B28, C28: 1/(1+*r*)^(*S*–1) Present Value Factor at *t* = *S*–1 = 2.25** | | | | | | | |
| 51 | **E29: (GM*(1–*x*^(*n*–*S*+1))+*p**(1+*g*)^(n–*S*+1)/(1+*r*)^(*n*–*S*+1+0.5**p*))*(1/(1+*r*)^(*S*–1))** | | | | | | | |
| 52 | **Note: E29 is the formula for the Grand Total** | | | | | | | |

We show the present value factors (PVFs) and PVs of the cash flows as of $t = S - 1$ in columns E and F, respectively, and the PVFs and PVs as of $t = 0$ in columns G and H, respectively. The discount rate is 15% (G36).

Column E contains the PVFs, and its formula is[26] $PVF = \frac{1}{(1+r)^{t-S+0.5}}$. Column F is column C (or column D, as the results are identical) times column E. The only exception to the PVF formula is in E17, for the fractional year. Its formula is $PVF = \frac{1}{(1+r)^{n-S+1+0.5p}}$ (in the EOY formula, the exponent is $z - S + 1$). This formula appears in the spreadsheet at A46. The total present value at $t = 2.25$ of the cash flows from $t = 3.25$ through $t = 12.25$ is \$6.42899 (F18). The present value of

[26]The intuition behind the exponent is that we are discounting from $t$ to $S - 1$, which is equal to $t - (S - 1) = t - S + 1$ years. Using a midyear convention, we always discount from ½ year earlier than end-of-year, which reduces the exponent to $t - S + 0.5$. The 0.5 reverts to 1 in the end-of-year formula.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table A4.2** | | | | | | | |
| 2 | **ADF with Fractional Year** | | | | | | | |
| 3 | **End-of-Year Formula** | | | | | | | |
| 4 | | | | | | | | |
| 5 | | **Cash Flows** | | | ***t* = *S*–1** | | ***t* = 0** | |
| 6 | ***t* (Yrs)** | **Growth** | **Cash Flow** | **$(1+g)^{t-S}$** | **$PVF=1/(1+r)^{t-S+1}$** | **PV** | **$PVF=1/(1+r)^{t}$** | **PV** |
| 7 | **3.25** | NA | 1.00000 | 1.00000 | 0.86957 | 0.86957 | 0.63494 | 0.63494 |
| 8 | **4.25** | 0.05100 | 1.05100 | 1.05100 | 0.75614 | 0.79471 | 0.55212 | 0.58028 |
| 9 | **5.25** | 0.05360 | 1.10460 | 1.10460 | 0.65752 | 0.72629 | 0.48011 | 0.53032 |
| 10 | **6.25** | 0.05633 | 1.16094 | 1.16094 | 0.57175 | 0.66377 | 0.41748 | 0.48467 |
| 11 | **7.25** | 0.05921 | 1.22014 | 1.22014 | 0.49718 | 0.60663 | 0.36303 | 0.44295 |
| 12 | **8.25** | 0.06223 | 1.28237 | 1.28237 | 0.43233 | 0.55440 | 0.31568 | 0.40481 |
| 13 | **9.25** | 0.06540 | 1.34777 | 1.34777 | 0.37594 | 0.50668 | 0.27450 | 0.36997 |
| 14 | **10.25** | 0.06874 | 1.41651 | 1.41651 | 0.32690 | 0.46306 | 0.23870 | 0.33812 |
| 15 | **11.25** | 0.07224 | 1.48875 | 1.48875 | 0.28426 | 0.42320 | 0.20756 | 0.30901 |
| 16 | **12.25** | 0.07593 | 1.56468 | 1.56468 | 0.24718 | 0.38676 | 0.18049 | 0.28241 |
| 17 | **12.60** | NA | 0.57557 | 0.57557 | 0.23538 | 0.13548 | 0.17187 | 0.09892 |
| 18 | **Totals for Whole Years = 3.25 - 12.25** | | | | | **5.99506** | | **4.37747** |
| 19 | **Add Fractional Year = 12.60** | | | | | **0.13548** | | **0.09892** |
| 20 | **Grand Total (*t* = *S*–1 in Column F and *t* = 0 in Column H)** | | | | | **6.13054** | | **4.47640** |
| 21 | **Present Value Factor—Discount from *S*–1 (*t* = 2.25) to 0** | | | | | **0.73018** | | |
| 22 | **Grand Total (*t* = 0)** | | | | | **4.47640** | | |
| 23 | | | | | | | | |
| 24 | **Calculation of PV by Formulas:** | | | | | | | |
| 25 | | | | | **Grand** | | | |
| 26 | | **Whole Yrs** | **Frac Yr** | **Total** | **Total** | | | |
| 27 | ***t* = *S*–1** | 5.99506 | 0.13548 | 6.13054 | | | | |
| 28 | **PV Factor** | 0.73018 | 0.73018 | | | | | |
| 29 | ***t*=0** | 4.37747 | 0.09892 | **4.47640** | **4.47640** | | | |
| 30 | | | | | | | | |
| 31 | **Assumptions:** | | | | | | | |
| 32 | | | | | | | | |
| 33 | ***S* = Beginning Year of Cash Flows (valuation at *t* = 2.25)** | | | | | | **3.25** | |
| 34 | ***n* = Ending Year of Cash Flows—Whole Year** | | | | | | **12.25** | |
| 35 | ***z* = Ending Year of Cash Flows—Stub Year** | | | | | | **12.60** | |
| 36 | ***r* = Discount Rate** | | | | | | **15.0%** | |
| 37 | ***g* = Growth Rate in Cash Flow** | | | | | | **5.1%** | |
| 38 | ***p* = Proportion of Year in the Stub Period** | | | | | | **0.35** | |
| 39 | **This row is not used** | | | | | | | |
| 40 | ***x* = (1+*g*)/(1+*r*)** | | | | | | **0.913913** | |
| 41 | **Gordon Model Multiple = GM = 1/(*r*–*g*)** | | | | | | **10.101010** | |
| 42 | | | | | | | | |
| 43 | **Spreadsheet Formulas:** | | | | | | | |
| 44 | | | | | | | | |
| 45 | **C17, D17: *p**(1+*g*)^(*n*–*s*+1) Stub Period Cash Flow** | | | | | | | |
| 46 | **E17: 1/(1+*r*)^(*z*–*S*+1) Stub Period Present Value Factor at *t* = 2.25** | | | | | | | |
| 47 | **G17: 1/(1+*r*)^*z* Stub Period Present Value Factor for *t* = 0** | | | | | | | |
| 48 | **B27: GM*(1–*x*^(*n*–*S*+1)) ADF for Years 3.25 to 12.25 at *t* = 2.25** | | | | | | | |
| 49 | **C27: *p**(1+*g*)^(*n*–*S*+1)/(1+*r*)^(*z*–*S*+1) PV of Stub Period CF at *t* = 2.25** | | | | | | | |
| 50 | **B28, C28: 1/(1+*r*)^(*S*–1) Present Value Factor at *t* = *S*–1 = 2.25** | | | | | | | |
| 51 | **E29: (GM*(1–*x*^(*n*–*S*+1))+*p**(1+*g*)^(*n*–*S*+1)/(1+*r*)^(*z*–*S*+1))/(1+*r*)^(*S*–1)** | | | | | | | |
| 52 | **Note: E29 is the formula for the Grand Total** | | | | | | | |

the fractional year cash flow is \$0.13883 (F19, transferred from F17), for a total of \$6.56782 (F20). In F21, we show the present value factor of 0.73018 to discount from $t = 2.25$ to $t = 0$.[27] Multiplying F20 by F21, we come to the PV of the cash flows in F22 at $t = 0$ of \$4.79569 for each \$1.00 of starting cash flows. Thus, if our annuity were actually \$100,000 at the beginning, with all other assumptions remaining the same, the PV would be \$479,569.

Column G contains the present value factors for $t = 0$, the formula of which is the more usual $PVF = \frac{1}{(1+r)^{t-0.5}}$. When we multiply column D by column G to get column H, the latter is the PV of the cash flows as of time zero. Note that the final sum in H20 is identical to F22, as it should be.

[27] This is $1/(1+r)^{S-1} = 1/1.15^{2.25} = 0.73018$ (see formulas in A50).

So far we have come to the PV of the cash flows using the brute force method. In the next section, we will test the formulas in the preceding pages to see whether they produce the same result.

**TESTING EQUATIONS (A4.3) AND (A4.4)** B27 contains the formula for the PV of the first 10 whole years of cash flows (see A48 for the spreadsheet formula). It is the same as equation (A4.2) without the rightmost term.[28] The result of \$6.42899 in B27 matches F18, thereby demonstrating the accuracy of that portion of equation (A4.2).

C27 is calculated using the rightmost term in equation (A4.2) and comes to \$0.13883 (see A49 for the spreadsheet formula), which matches F19, thus proving that portion of the formula. The sum of the two is \$6.56782 (D27), which matches F20.

In columns B and C, row 29 is the result of multiplying row 27 by row 28, the latter of which is the present value factor to discount the cash flows from $t = 2.25$ to $t = 0$ (it is the same as F21). We total B29 and C29 to \$4.79569 (D29), which matches F22 and H20. Finally, in E29 we use the complete formula in equation (A4.3)[29] to produce the same result of \$4.79569 (see A51 for the spreadsheet formula). Thus we have demonstrated the accuracy of equation (A4.3) as a whole as well as showing how we can calculate the parts.

Table A4.2 is identical to Table A4.1, except that we use end-of-year present values, and equation (A4.4) is the relevant ADF formula. The end-of-year formula gives a grand total of 4.47640 (F22, H20, D29, and E29).

## Table A4.3: Loan Amortization

In the chapter, we demonstrated how ADFs are useful in calculating loan payments and the present value of a loan. This section on loan amortization complements the material we presented in the chapter.

The amortization of loan principal in any time period is the PV of the loan at the beginning of the period, less the PV at the end of the period.[30] While this is conceptually easy, it is a cumbersome procedure. Let's develop some preliminary results that will lead us to a more efficient way to calculate loan amortization.

### Section 1: Traditional Loan Amortization Schedule

Table A4.3 is a loan amortization schedule that is divided into three sections. Section 1 is a traditional amortization schedule for a \$1 million loan at 10% for 5 years. The loan begins on February 29, 2008 (B7), and the first payment is on March 31, 2008 (B8). During the calendar year 2008, there will be 10 payments, leaving 50 more. There will be 12 monthly payments in each of the years 2009–2012, and the final

[28]The formulas are the same; however, in the spreadsheet, we have substituted GM (Gordon multiple) for $\frac{\sqrt{1+r}}{r-g}$ and $x$ for $\frac{1+g}{1+r}$. Additionally, we factored out the GM.

[29]Just as we did for equation (A4.2), in the spreadsheet for equation (A4.3), we factored out the GM and we substituted $x$ for $\frac{1+g}{1+r}$.

[30]That is, loan amortization means the reduction in the principal owed.

two payments are in the beginning of 2013, with the final payment on February 28, 2013 (B67).

Column A is the payment number. There are 60 months of the loan, hence 60 payments. Columns D and E are the interest and principal, respectively, for the particular payment, while columns G and H are interest and principal, respectively, cumulated in calendar-year totals. Because the loan payments begin on March 31, 2008, the first year's totals in columns G and H are totals for the first 10 payments only. Column I is the present value factor (PVF) at 10%, and column J is the present value of each loan payment. Column K is the sum of the present values of the loan payments by calendar year. Note that the PVs of the loan payments sum to $1 million (J68).

## Section 2: Present Values of Yearly Loan Payment

In section 2, we calculate the present value of each year's loan payment using the ADF equation for no-growth, no stub period, and end-of-year cash flows. We could use equation (4.11b) from the chapter, but first we will simplify it further by setting $g = 0$, so equation (4.11b) reduces to:

$$ADF = \left[\frac{1}{r} - \frac{1}{(1+r)^{n-S+1}}\frac{1}{r}\right]\frac{1}{(1+r)^{S-v-1}} = \frac{1}{r}\left[1 - \frac{1}{(1+r)^{n-S+1}}\right]\frac{1}{(1+r)^{S-v-1}}. \tag{A4.5}$$

D77 through D82 list the PVs of the various calendar years' cash flows discounted to the inception of the loan, February 29, 2008. Note that these amounts exactly match those in column K of section 1, and the total is exactly $1 million—the principal of the loan—as it should be. This demonstrates the accuracy of equation (A4.5), as all amounts calculated in D77 through D82 use that equation (note that $v$, the valuation date in months since the inception of the loan, appears in row 86).

In column E, we are viewing the cash flows from January 1, 2009, that is, immediately after the last payment in 2008 and one month before the first payment in 2009. Therefore, the 2008 cash flows drop out entirely, and the PVs of the 2009–2013 cash flows increase relative to column D, because we discount the cash flows for 10 months less. The difference between the sum of the 2008 PVs discounted to February 29, 2008, and the 2009 payments discounted to January 1, 2009[31] is $1 million (D84) − $865,911 (E84) = $134,089 (E85). We follow the same procedure each year to calculate the difference in the PVs (row 85) and finally we come to a total of the reductions in PV of $1 million, in K85, which is identical with the original principal of the loan.

There are some significant numbers that repeat in southeasterly sloped diagonals in section 2. The PV of $241,675 appears in E78, F79, G80, and H81. This means that the 2009 payments as seen from the beginning of 2009 have the same PV as the 2010 payments as seen from the beginning of 2010, and so on, through 2012. Similarly, the PV of $218,767 repeats in E79, F80, and G81. The interpretation of this series is the same as before, except everything is moved back one year; that is, the

[31]Technically, we discount to the end of December 31, 2008, but in PV terms, it is easier to think of January 1, 2009.

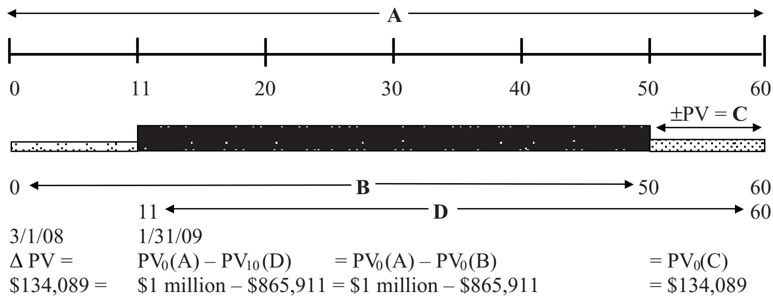



**FIGURE A4.2** Payment Schedule

2010 payments as seen from the beginning of 2009 have the same PV as the 2011 payments as seen from the beginning of 2010 and the 2012 payments as seen from the beginning of 2011.

This downward-sloping pattern gives us a clue to a more direct formula for loan amortization. At the start of the loan, we have 60 payments of $21,247. In the first calendar year, 10 payments will be made, for a total of $212,470. At the end of the first year, which effectively is the same as January 1, 2009, 50 payments will remain. The PV of the final 50 payments discounted to January 1, 2009 is the same as the PV of the first 50 payments discounted to March 1, 2008 (using March 1, 2008 synonymously with February 29, 2008 in a present value sense), because the entire timeline will have shifted by 10 months (10 payments). Therefore, the first calendar year's loan amortization can be represented by the PV of the final 10 payments discounted to March 1, 2008, as that would comprise the only difference in the two series of cash flows as perceived from their different points in time. This is illustrated graphically in Figure A4.2.

Figure A4.2 is a timeline of payments on the five-year (60-month) loan. The top portion of the figure, labeled *A*, graphically represents the entire payment schedule. In the bottom of the figure the loan is split into several pieces: payments 1 through 10, which are not labeled;[32] payments 1 through 50, labeled *B*; payments 11 through 60, labeled *D*; and payments 51 through 60, labeled *C* ($t = 50$ is the end of *B*, not the beginning of *C*).

The equation at the bottom of Figure A4.2, which we explain below in listed items 1–3, is: $\Delta \text{PV} = PV_0(A) - PV_{10}(D) = PV_0(A) - PV_0(B) = PV_0(C)$. We use the convention that the subscripts are measured in time from the start of the loan, not from the start of a period. For example, when we use the subscript 10 in $PV_{10}$(D) we do not mean the 10th month of period D, but rather the 10th month of the entire loan (i.e., the 10th month of period A). The amortization of the loan principal during any year is the change in the present value of the loan between years. That is equal to each of the following three expressions:

1. $PV_0(A) - PV_{10}(D)$: The PV at $t = 0$ of *A* (all 60 months of the loan) minus the PV at $t = 10$ of *D*, the last 50 payments of the loan. Notice that the valuation

[32] In all cases, the zero is there only as a valuation date. There are no loan payments (cash flows) that occur at zero.

dates are different, $t = 0$ versus $t = 10$. The PV at $t = 0$ of $A$ is the principal, \$1 million (Table A4.3, section 2, D84). The PV at $t = 10$ of $D$ is \$865,911 (E84). The difference of the two is the amortization of \$134,089 (E85).

2. $PV_0(A) - PV_0(B)$: The PV at $t = 0$ of $A$ (all 60 months of the loan), which is \$1 million, minus the PV at $t = 0$ of the first 50 months of the loan. The latter calculation does not appear directly in Table A4.3. However, using equation (4.6d) from the chapter with $g = 0$, $r = 0.83333\%$, and $n = 50$ periods leads to the ADF of 40.75442. Multiplying the ADF by the monthly payment of \$21,247.04 gives us the PV of $B$, which is \$865,911. The difference of the two PVs is \$134,089, the same as above. Another way of seeing this is to recognize that $PV_{10}(D)$ equals $PV_0(B)$, so subtracting either of them from $PV_0(A)$ will yield the same result.
3. $PV_0(C)$: The PV at $t = 0$[33] of $C$, payments 51–60. This is the most important of the expressions, because it is the most compact and the easiest to use. The other expressions are the difference of two formulas, whereas this one requires only a single formula. It is stated in mathematical terms in equation (A4.10). The reduction in the principal is the PV of the opposite or "mirror-image" series of cash flows working backward from the end of the loan. $PV_0(C)$ is equal to $PV_0(A) - PV_0(B)$ by definition, because looked at from $t = 0$, subtracting the first 50 payments (period B) from the entire loan (Period A) leaves the last 10 payments remaining (period C).

## Section 3: A Better Way to Calculate Loan Amortization

In section 3, we calculate the principal reduction using equation (A4.10). Let's look first at the 2008 cash flows in row 93. The amortization of principal in 2008 is equal to the PV at $t = 0$ of the *last* 10 payments of the loan. Letting $n$ (the final payment period) = 60, we want to calculate the PV of payments 51 through 60, discounted to month 0. If we let $F$ = finishing month = 10 in calendar year 2008, the formula $n - F + 1$ describes, $S^1$,[34] the starting month in our amortization formula for each $F$ in D93 through D98. The formula $n - S + 1$ describes, $F^1$, the finishing month in our amortization formula for each $S$ in C93 through C98. For 2008, $S^1 = 60 - 10$ (D93) $+ 1 = 51$, and $F^1 = 60 - 1$ (C93) $+ 1 = 60$. Thus our formulas give us the result that in calendar 2008, the amortization of principal is equal to the PV at $t = 0$ of payments 51 through 60, which is correct.

For calendar year 2009, $S^1 = 60 - 22$ (D94) $+ 1 = 39$, and $F^1 = 60 - 11$ (C94) $+ 1 = 50$. The amortization of principal in calendar 2009 is the PV at $t = 0$ of payments 39 through 50, which is also correct. Thus, the amortization of principal in any year is equal to an ADF with no growth and end-of-year cash flows that run from $n - F + 1$ to $n - S + 1$. We begin the calculation of this loan amortization ADF in equation (A4.6).

$$ADF = \frac{1}{(1+r)^{n-F+1}} + \frac{1}{(1+r)^{n-F+2}} + \cdots + \frac{1}{(1+r)^{n-S+1}}. \qquad \text{(A4.6)}$$

[33]Again, $t = 0$ does not mean the beginning of period C, but rather the beginning of the loan.
[34]$S^1$ and $F^1$ should not be confused with $S$ and $F$. $S^1$ and $F^1$ are the starting and finishing months, used in our amortization formulas that correspond to each $S$ and $F$.

Multiplying equation (A4.6) by $\frac{1}{1+r}$, we get:

$$\frac{1}{1+r}ADF = \frac{1}{(1+r)^{n-F+2}} + \frac{1}{(1+r)^{n-F+3}} + \cdots + \frac{1}{(1+r)^{n-S+1}} + \frac{1}{(1+r)^{n-S+2}}. \tag{A4.7}$$

Subtracting equation (A4.7) from equation (A4.6), we get:

$$\left[1 - \frac{1}{1+r}\right]ADF = \frac{1}{(1+r)^{n-F+1}} - \frac{1}{(1+r)^{n-S+2}}. \tag{A4.8}$$

The left-hand side of equation (A4.8) simplifies to $\frac{r}{1+r}ADF$. Multiplying both sides of equation (A4.8) by $\frac{1+r}{r}$, we come to:

$$ADF = \frac{1+r}{r}\left[\frac{1}{(1+r)^{n-F+1}} - \frac{1}{(1+r)^{n-S+2}}\right]. \tag{A4.9}$$

Canceling out the $1 + r$ in the numerator and denominator, we arrive at our final solution:

$$ADF = \frac{1}{r}\left[\frac{1}{(1+r)^{n-F}} - \frac{1}{(1+r)^{n-S+1}}\right] \tag{A4.10}$$

ADF formula for loan amortization.

We show the spreadsheet formulas in column F, rows 93 through 98. Note that we multiply the ADF in equation (A4.10) by the monthly payment in F93 through F98 to calculate the PV of the loan amortization. The term $I$ is the monthly interest rate = 10%/12 months = 0.833%, which is equivalent to $r$ in equation (A4.10).

The amortization in 2008 is $134,089 (E93), which equals:

$$ADF = \frac{1}{0.008333}\left[\frac{1}{1.008333^{60-10}} - \frac{1}{1.008333^{60-1+1}}\right]. \tag{A4.10a}$$

The amortization in 2009 is $176,309, as per E94, which equals:

$$ADF = \frac{1}{0.008333}\left[\frac{1}{1.008333^{60-22}} - \frac{1}{1.008333^{60-11+1}}\right]. \tag{A4.10b}$$

The principal amortization in E93 through E98 is equal to that in column H of section 1, which demonstrates the accuracy of equation (A4.10).

## The After-Tax Cost of a Loan

In our discussion of Table A4.3, sections 2 and 3, we came to the insight that principal amortizes in *mirror image*, and we used that understanding to develop equation (A4.10) to calculate the principal amortization over any given block of time. Now it is appropriate to present month-by-month amortization of principal, as it will enable us to develop formulas to calculate the PV of principal and interest of a loan. The primary practical application is to calculate the after-tax cost of a loan.

We begin with a month-by-month amortization. In the first month, amortization equals the PVF for the last month's payment. In the second month, amortization equals the PVF for the second-to-last month's payment, and we continue in

| | A | B | C | D | E | F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table A4.3** | | | | | | | | | | | |
| 2 | **Amortization of Principal with Irregular Starting Point** | | | | | | | | | | | |
| 4 | **SECTION 1: LOAN AMORTIZATION SCHEDULE** | | | | | | | | | | | |
| 5 | **Pmt** | | | | | | | | | **NPV** | **Annual** | **Aft-Tax** |
| 6 | **#** | **Date** | **Pmt** | **Int** | **Prin** | **Bal** | **Int** | **Prin** | **PVF** | **Pymt** | **NPV** | **Cost-Loan** |
| 7 | 0 | 02/29/08 | | | | 1,000,000 | | | 1.0000 | | | |
| 8 | 1 | 03/31/08 | 21,247 | 8,333 | 12,914 | 987,086 | | | 0.9917 | 21,071 | | 17,766 |
| 9 | 2 | 04/30/08 | 21,247 | 8,226 | 13,021 | 974,065 | | | 0.9835 | 20,897 | | 17,661 |
| 10 | 3 | 05/31/08 | 21,247 | 8,117 | 13,130 | 960,935 | | | 0.9754 | 20,725 | | 17,558 |
| 11 | 4 | 06/30/08 | 21,247 | 8,008 | 13,239 | 947,696 | | | 0.9673 | 20,553 | | 17,455 |
| 12 | 5 | 07/31/08 | 21,247 | 7,897 | 13,350 | 934,346 | | | 0.9594 | 20,383 | | 17,353 |
| 13 | 6 | 08/31/08 | 21,247 | 7,786 | 13,461 | 920,885 | | | 0.9514 | 20,215 | | 17,252 |
| 14 | 7 | 09/30/08 | 21,247 | 7,674 | 13,573 | 907,312 | | | 0.9436 | 20,048 | | 17,152 |
| 15 | 8 | 10/31/08 | 21,247 | 7,561 | 13,686 | 893,626 | | | 0.9358 | 19,882 | | 17,052 |
| 16 | 9 | 11/30/08 | 21,247 | 7,447 | 13,800 | 879,826 | | | 0.9280 | 19,718 | | 16,954 |
| 17 | 10 | 12/31/08 | 21,247 | 7,332 | 13,915 | 865,911 | 78,381 | 134,089 | 0.9204 | 19,555 | 203,048 | 16,856 |
| 18 | 11 | 01/31/09 | 21,247 | 7,216 | 14,031 | 851,880 | | | 0.9128 | 19,393 | | 16,759 |
| 19 | 12 | 02/28/09 | 21,247 | 7,099 | 14,148 | 837,732 | | | 0.9052 | 19,233 | | 16,663 |
| 20 | 13 | 03/31/09 | 21,247 | 6,981 | 14,266 | 823,466 | | | 0.8977 | 19,074 | | 16,567 |
| 21 | 14 | 04/30/09 | 21,247 | 6,862 | 14,385 | 809,081 | | | 0.8903 | 18,917 | | 16,473 |
| 22 | 15 | 05/31/09 | 21,247 | 6,742 | 14,505 | 794,576 | | | 0.8830 | 18,760 | | 16,379 |
| 23 | 16 | 06/30/09 | 21,247 | 6,621 | 14,626 | 779,951 | | | 0.8757 | 18,605 | | 16,286 |
| 24 | 17 | 07/31/09 | 21,247 | 6,500 | 14,747 | 765,203 | | | 0.8684 | 18,451 | | 16,194 |
| 25 | 18 | 08/31/09 | 21,247 | 6,377 | 14,870 | 750,333 | | | 0.8612 | 18,299 | | 16,102 |
| 26 | 19 | 09/30/09 | 21,247 | 6,253 | 14,994 | 735,339 | | | 0.8541 | 18,148 | | 16,011 |
| 27 | 20 | 10/31/09 | 21,247 | 6,128 | 15,119 | 720,220 | | | 0.8471 | 17,998 | | 15,921 |
| 28 | 21 | 11/30/09 | 21,247 | 6,002 | 15,245 | 704,974 | | | 0.8401 | 17,849 | | 15,832 |
| 29 | 22 | 12/31/09 | 21,247 | 5,875 | 15,372 | 689,602 | 78,656 | 176,309 | 0.8331 | 17,701 | 222,428 | 15,744 |
| 30 | 23 | 01/31/10 | 21,247 | 5,747 | 15,500 | 674,102 | | | 0.8262 | 17,555 | | 15,656 |
| 31 | 24 | 02/28/10 | 21,247 | 5,618 | 15,630 | 658,472 | | | 0.8194 | 17,410 | | 15,569 |
| 32 | 25 | 03/31/10 | 21,247 | 5,487 | 15,760 | 642,712 | | | 0.8126 | 17,266 | | 15,482 |
| 33 | 26 | 04/30/10 | 21,247 | 5,356 | 15,891 | 626,821 | | | 0.8059 | 17,123 | | 15,397 |
| 34 | 27 | 05/31/10 | 21,247 | 5,224 | 16,024 | 610,798 | | | 0.7993 | 16,982 | | 15,312 |
| 35 | 28 | 06/30/10 | 21,247 | 5,090 | 16,157 | 594,641 | | | 0.7927 | 16,842 | | 15,228 |
| 36 | 29 | 07/31/10 | 21,247 | 4,955 | 16,292 | 578,349 | | | 0.7861 | 16,702 | | 15,144 |
| 37 | 30 | 08/31/10 | 21,247 | 4,820 | 16,427 | 561,922 | | | 0.7796 | 16,564 | | 15,061 |
| 38 | 31 | 09/30/10 | 21,247 | 4,683 | 16,564 | 545,357 | | | 0.7732 | 16,427 | | 14,979 |
| 39 | 32 | 10/31/10 | 21,247 | 4,545 | 16,702 | 528,655 | | | 0.7668 | 16,292 | | 14,898 |
| 40 | 33 | 11/30/10 | 21,247 | 4,405 | 16,842 | 511,813 | | | 0.7604 | 16,157 | | 14,817 |
| 41 | 34 | 12/31/10 | 21,247 | 4,265 | 16,982 | 494,831 | 60,194 | 194,771 | 0.7542 | 16,024 | 201,345 | 14,737 |
| 42 | 35 | 01/31/11 | 21,247 | 4,124 | 17,123 | 477,708 | | | 0.7479 | 15,891 | | 14,657 |
| 43 | 36 | 02/28/11 | 21,247 | 3,981 | 17,266 | 460,442 | | | 0.7417 | 15,760 | | 14,579 |
| 44 | 37 | 03/31/11 | 21,247 | 3,837 | 17,410 | 443,032 | | | 0.7356 | 15,630 | | 14,501 |
| 45 | 38 | 04/30/11 | 21,247 | 3,692 | 17,555 | 425,476 | | | 0.7295 | 15,500 | | 14,423 |
| 46 | 39 | 05/31/11 | 21,247 | 3,546 | 17,701 | 407,775 | | | 0.7235 | 15,372 | | 14,346 |
| 47 | 40 | 06/30/11 | 21,247 | 3,398 | 17,849 | 389,926 | | | 0.7175 | 15,245 | | 14,270 |
| 48 | 41 | 07/31/11 | 21,247 | 3,249 | 17,998 | 371,928 | | | 0.7116 | 15,119 | | 14,194 |
| 49 | 42 | 08/31/11 | 21,247 | 3,099 | 18,148 | 353,781 | | | 0.7057 | 14,994 | | 14,119 |
| 50 | 43 | 09/30/11 | 21,247 | 2,948 | 18,299 | 335,482 | | | 0.6999 | 14,870 | | 14,045 |
| 51 | 44 | 10/31/11 | 21,247 | 2,796 | 18,451 | 317,031 | | | 0.6941 | 14,747 | | 13,971 |
| 52 | 45 | 11/30/11 | 21,247 | 2,642 | 18,605 | 298,425 | | | 0.6884 | 14,626 | | 13,898 |
| 53 | 46 | 12/31/11 | 21,247 | 2,487 | 18,760 | 279,665 | 39,799 | 215,166 | 0.6827 | 14,505 | 182,260 | 13,826 |
| 54 | 47 | 01/31/12 | 21,247 | 2,331 | 18,917 | 260,749 | | | 0.6770 | 14,385 | | 13,754 |
| 55 | 48 | 02/29/12 | 21,247 | 2,173 | 19,074 | 241,675 | | | 0.6714 | 14,266 | | 13,682 |
| 56 | 49 | 03/31/12 | 21,247 | 2,014 | 19,233 | 222,442 | | | 0.6659 | 14,148 | | 13,612 |
| 57 | 50 | 04/30/12 | 21,247 | 1,854 | 19,393 | 203,048 | | | 0.6604 | 14,031 | | 13,541 |
| 58 | 51 | 05/31/12 | 21,247 | 1,692 | 19,555 | 183,493 | | | 0.6549 | 13,915 | | 13,472 |
| 59 | 52 | 06/30/12 | 21,247 | 1,529 | 19,718 | 163,775 | | | 0.6495 | 13,800 | | 13,403 |
| 60 | 53 | 07/31/12 | 21,247 | 1,365 | 19,882 | 143,893 | | | 0.6441 | 13,686 | | 13,334 |
| 61 | 54 | 08/31/12 | 21,247 | 1,199 | 20,048 | 123,845 | | | 0.6388 | 13,573 | | 13,267 |
| 62 | 55 | 09/30/12 | 21,247 | 1,032 | 20,215 | 103,630 | | | 0.6335 | 13,461 | | 13,199 |
| 63 | 56 | 10/31/12 | 21,247 | 864 | 20,383 | 83,247 | | | 0.6283 | 13,350 | | 13,133 |
| 64 | 57 | 11/30/12 | 21,247 | 694 | 20,553 | 62,693 | | | 0.6231 | 13,239 | | 13,066 |
| 65 | 58 | 12/31/12 | 21,247 | 522 | 20,725 | 41,969 | 17,268 | 237,697 | 0.6180 | 13,130 | 164,984 | 13,001 |
| 66 | 59 | 01/31/13 | 21,247 | 350 | 20,897 | 21,071 | | | 0.6129 | 13,021 | | 12,936 |
| 67 | 60 | 02/28/13 | 21,247 | 176 | 21,071 | 0 | 525 | 41,969 | 0.6078 | 12,914 | 25,935 | 12,871 |
| 68 | | **Totals** | **1,274,823** | **274,823** | **1,000,000** | | **274,823** | **1,000,000** | | **1,000,000** | **1,000,000** | **907,368** |

that fashion. Mathematically, amortization is thus equal to:

$$Amort = \left[\frac{1}{(1+r)^n} + \frac{1}{(1+r)^{n-1}} + \frac{1}{(1+r)^{n-2}} + \cdots + \frac{1}{1+r}\right] \times Pymt \qquad \text{(A4.11)}$$

Note that this expression is the exact reverse of a simple series of cash flows that solves to an end-of-year ADF with no growth, that is, equation (4.6d) in the body of the chapter. Thus the total amortization equals equation (4.6d) × Loan Payment = Principal of the Loan. This is a rearrangement of equation (4.20). Note that one should use the nominal interest rate in this calculation.

Next we take the PV of equation (A4.11) at the nominal rate of interest (when valuing a loan at a discount rate other than the nominal rate of interest, see that discussion at the end of this chapter).

$$PV(Amort) = \left[\frac{\frac{1}{(1+r)^n}}{1+r} + \frac{\frac{1}{(1+r)^{n-1}}}{(1+r)^2} + \frac{\frac{1}{(1+r)^{n-2}}}{(1+r)^3} + \cdots + \frac{\frac{1}{1+r}}{(1+r)^n}\right] \times Pymt. \qquad \text{(A4.12)}$$

**Table A4.3 (cont.)**

**SECTION 2: SCHEDULE OF PRESENT VALUES CALCULATED BY ADF EQUATION (A4.5)**

**As Seen From The Beginning of Year**

| | A–C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|
| 76 | | **2008** | **2009** | **2010** | **2011** | **2012** | **2013** | **2014** | **Total** |
| 77 | NPV 2008 Payments [1] | 203,048 | | | | | | | |
| 78 | NPV 2009 Payments | 222,428 | 241,675 | | | | | | |
| 79 | NPV 2010 Payments | 201,345 | 218,767 | 241,675 | | | | | |
| 80 | NPV 2011 Payments | 182,260 | 198,031 | 218,767 | 241,675 | | | | |
| 81 | NPV 2012 Payments | 164,984 | 179,260 | 198,031 | 218,767 | 241,675 | | | |
| 82 | NPV 2013 Payments | 25,935 | 28,179 | 31,130 | 34,390 | 37,991 | 41,969 | | |
| 83 | NPV 2014 Payments | | | | | | | 0 | |
| 84 | **Sum NPVs—All Pymts** | **1,000,000** | **865,911** | **689,602** | **494,831** | **279,665** | **41,969** | **0** | **0** |
| 85 | **Reduction in NPV** | | **134,089** | **176,309** | **194,771** | **215,166** | **237,697** | **41,969** | **1,000,000** |
| 86 | **Valuation Date = *v*** | **0** | **10** | **22** | **34** | **46** | **58** | | |

**SECTION 3: AMORTIZATION CALCULATED AS THE PYMT × THE ADF in (A4.16)**

| | A–B | Starting Month (C) | Finishing Month (D) | Prin Amort (E) | Formulas for Principal Amortization, where: *I* = Monthly Interest = 0.833%, *n*=60 Months, Pymt=$21,247/Month |
|---|---|---|---|---|---|
| 93 | Calendar 2008 | 1 | 10 | 134,089 | PYMT*(1/*r*)*((1/(1+*r*)^(*n*–$D93)–(1/(1+*r*)^(*n*–$C93+1)))) |
| 94 | Calendar 2009 | 11 | 22 | 176,309 | PYMT*(1/*r*)*((1/(1+*r*)^(*n*–$D94)–(1/(1+*r*)^(*n*–$C94+1)))) |
| 95 | Calendar 2010 | 23 | 34 | 194,771 | PYMT*(1/*r*)*((1/(1+*r*)^(*n*–$D95)–(1/(1+*r*)^(*n*–$C95+1)))) |
| 96 | Calendar 2011 | 35 | 46 | 215,166 | PYMT*(1/*r*)*((1/(1+*r*)^(*n*–$D96)–(1/(1+*r*)^(*n*–$C96+1)))) |
| 97 | Calendar 2012 | 47 | 58 | 237,697 | PYMT*(1/*r*)*((1/(1+*r*)^(*n*–$D97)–(1/(1+*r*)^(*n*–$C97+1)))) |
| 98 | Calendar 2013 | 59 | 60 | 41,969 | PYMT*(1/*r*)*((1/(1+*r*)^(*n*–$D98)–(1/(1+*r*)^(*n*–$C98+1)))) |
| 99 | **Total** | | | **1,000,000** | |

**Assumptions:**

| | | |
|---|---|---|
| 104 | Prin | 1,000,000 |
| 105 | Int | 10.0000% |
| 106 | Int-Mo | 0.8333% |
| 107 | Years | 5 |
| 108 | Months = *n* | 60 |
| 109 | Pymt | 21,247 |
| 110 | Form-Prin | 1,000,000 |
| 111 | Start Month = *S* | 3 |
| 112 | *y* = 1/(1+*r*) [2] | 0.9917 |
| 113 | GM = 1/*r* | 120 |

**After-Tax Cost of the Loan**

| | | |
|---|---|---|
| (1–*t*) * Prin | 0.600000 | 600,000 |
| *t***n*/(1+*r*)^(*n*+1)*PYMT | 0.307368 | 307,368 |
| Total = L68 | **0.907368** | **907,368** |

**H106: (1–*t*) + [*t***n*/(1+*r*)^(*n*+1)*PYMT/*P*] Equation (A4.24a)**
**I106: (1–*t*)**P* + [*t***n*/(1+*r*)^(*n*+1)*PYMT] Equation (A4.23a)**

**[1] Formula for D77 according to (A4.5): GM*(1–*y*^($D93–$C93+1))**y*^($C93–A$86–1)*PYMT**
***n* – *S* + 1 = # months of cash flow = $D93–$C93+1, which is the ending month – beginning month +1.**
**The second exponent of *y* is *S* – *v* – 1, which is the ending month – the valuation date –1; thus it is the discounting period. This formula copies both down and across, i.e., it is the formula for all cells from D77 to I82.**
**D78 > D77 because there are 10 payments in 2008 and 12 in 2009–2012.**

**[2] Normally we would use *x* = (1+*g*)/(1+*r*) to calculate the ADF. However, since *g* = 0, *x* = *y*.**

We can move the second denominator into the first denominator, and equation (A4.12) simplifies to:

$$PV(Amort) = \left[\frac{1}{(1+r)^{n+1}} + \frac{1}{(1+r)^{n+1}} + \frac{1}{(1+r)^{n+1}} + \cdots + \frac{1}{(1+r)^{n+1}}\right] \times Pymt \quad [n \text{ terms}]. \tag{A4.13}$$

All the bracketed terms in equation (A4.13) are identical. Thus, the PV of the amortization of principal, $P$, which we denote in (A4.14) as $PV(P)$, is equal to $n \times$ any one of these terms $\times$ the loan payment.

$$PV(Amort) \equiv PV(P) = \frac{n}{(1+r)^{n+1}} \times Pymt \quad \text{PV of principal payments.} \tag{A4.14}$$

Restating equation (4.21) as equation (A4.15),

$$Pymt = \frac{P}{ADF}, \quad \text{where ADF is defined by equation (4.6d).} \tag{A4.15}$$

| | A | B | C |
|---|---|---|---|
| 1 | **Table A4.4** | | |
| 2 | **PV of Principal Amortization** | | |
| 3 | | | |
| 4 | ***r*** | **1%** | |
| 5 | ***n*** | **60** | |
| 6 | **PV(*P*)/Pmt** | **32.69997718** | |
| 7 | **Pmt / *P*** | **$0.0222444** | |
| 8 | **PV(*P*) / *P*** | **$0.7273929** | |
| 9 | **PV(*P*) / *P*** | **$0.7273929** | |
| 10 | | | |
| 11 | **Cell Formulas:** | | |
| 12 | | | |
| 13 | **B6: =*n*/(1+*r*)^(*n*+1)** | | |
| 14 | **B7: =PMT(0.01,60,–1)** | | |
| 15 | **B8: =B7*B8** | | |
| 16 | **B9: =(*n**r*)/(((1+*r*)^*n*–1)*(1+*r*))** | | |
| 17 | | | |
| 18 | | | |

Substituting equation (A4.15) into equation (A4.14), we get:

$$PV(P) = \frac{n}{(1+r)^{n+1}} \times \frac{P}{ADF}. \tag{A4.16}$$

The next section, in which we develop equations (A4.16a) and (A4.16b), is somewhat of a digression from the previous and the subsequent discussion. We do not use equations (A4.16a) and (A4.16b) in our subsequent work. However, these formulas can be useful alternative forms of equation (A4.16). Substituting in the definition of the ADF, dividing through by the principal, and solving the equation,[35] another form of equation (A4.16) is:

$$\frac{PV(P)}{P} = \frac{nr}{\left[(1+r)^n - 1\right](1+r)}. \tag{A4.16a}$$

Table A4.4 verifies the accuracy of this formula, which is my own formula, to the best of my knowledge. For a five-year (60-month) loan at 12% per year, or 1% per month (A5 and A4, respectively), the present value of the principal divided by the loan payment is 32.69997718 (B6). The formula for that cell appears in cell A13, and that formula is equation (A4.14) after dividing both sides of the equation by the payment. In B7 we show the monthly payment per dollar of loan principal, which we calculate using a standard spreadsheet financial function for a $1 loan with 60 monthly payments at 1% interest (see A14 for the formula). In B8, we multiply B6 × B7. In B9, we test equation (A4.16a), and it comes to the same answer as B8; that is, the present value of the principal is $0.7273929 per $1 of principal. That the two answers are identical demonstrates the accuracy of equation (A4.16a). Of course,

[35]We do not show the steps to the solution, as we are not using this equation in our subsequent work.

the present value of the interest on a pre-tax basis is 1 minus that, or approximately \$0.273 per \$1 of principal.

In algebraic terms, the present value of the interest portion of a loan per dollar of principal on a pre-tax basis is 1 minus equation (A4.16a), or:

$$\frac{PV(Int)}{P} = 1 - \frac{nr}{\left[(1+r)^n - 1\right](1+r)}. \tag{A4.16b}$$

Resuming our discussion after the digression in the last several paragraphs, the PV of the interest portion of the payments is simply the PV of the loan payments—which is the principal—minus the PV of the principal portion, or:

$$PV(Int) = P - PV(P). \tag{A4.17}$$

Substituting equation (A4.16) into equation (A4.17), we get:

$$PV(Int) = P - \frac{n}{(1+r)^{n+1}}\frac{P}{ADF} = P\left[1 - \frac{n}{(1+r)^{n+1}}\frac{1}{ADF}\right]. \tag{A4.18}$$

The PV of the after-tax cost of the interest portion is $(1 - t) \times$ (A4.18), where $t$ is the tax rate, or:

$$PV(Int)_{After-Tax} = (1-t)P\left[1 - \frac{n}{(1+r)^{n+1}}\frac{1}{ADF}\right]. \tag{A4.19}$$

Thus the after-tax cost of the loan, $L$, is (A4.16) plus (A4.19), or:

$$L = \frac{n}{(1+r)^{n+1}}\frac{P}{ADF} + (1-t)P\left[1 - \frac{n}{(1+r)^{n+1}}\frac{1}{ADF}\right]. \tag{A4.20}$$

Factoring terms, we get:

$$L = \frac{n}{(1+r)^{n+1}}\frac{P}{ADF}\left[1 - (1-t)\right] + (1-t)P, \tag{A4.21}$$

which simplifies to:

$$L = t\frac{n}{(1+r)^{n+1}}\frac{P}{ADF} + (1-t)P. \tag{A4.22}$$

Switching terms, our final equation for the after-tax cost of a loan is:

$$L = (1-t)P + \left[t\frac{n}{(1+r)^{n+1}}\frac{P}{ADF}\right] \tag{A4.23}$$

After-tax cost of a loan.

Alternatively, using equation (A4.15), $Loan\,Payment = \frac{P}{ADF}$, we can restate equation (A4.23) as:

$$L = (1-t)P + \left[t\frac{n}{(1+r)^{n+1}}Pymt\right] \tag{A4.23a}$$

Alternative expression—after-tax cost of loan.

Equation (A4.23) gives us the equation for the after-tax cost of a loan in dollars. We can restate equation (A4.23) to give us the after-tax cost of the loan for each

$1.00 of loan principal by dividing through by $P$.

$$\frac{L}{P} = (1 - t) + \left[ t \frac{n}{(1 + r)^{n+1}} \frac{1}{ADF} \right] \tag{A4.24}$$

After-tax cost of loan per each $1.00 of principal.

Analyzing equation (A4.24), we can see the after-tax cost of the loan is comprised of two parts:

1. The after-tax cost of the principal, as if the entire loan payment were tax-deductible, plus
2. The tax rate times the PV of the principal payments on the loan.

In item 1, we temporarily assume that both principal and interest are tax-deductible. This is actually true for ESOP loans, and the PV of an ESOP loan is item 1. To adjust item 1 upward for the lack of tax shield on the principal of ordinary loans, in item 2 we add back the tax shield included in item 1 that we do not really get. Of course, we can substitute the exact expression for ADF in equation (A4.24) to keep the solution strictly in terms of the variables $t$, $n$, and $r$.

We can derive an alternative expression for equation (A4.24) by dividing equation (A4.23a) by $P$:

$$\frac{L}{P} = (1 - t) + \left[ t \frac{n}{(1 + r)^{n+1}} \frac{Pymt}{P} \right] \tag{A4.24a}$$

Alternative expression—after-tax cost of loan/$1 of principal.

We demonstrate the accuracy of equations (A4.23a) and (A4.24a) in Table A4.3. First we compute the after-tax cost of the loan using a brute force approach. In section 1, column L is the after-tax cost of each loan payment. It is equal to: [Principal (column E) + (1 − Tax Rate) × Interest (column D)] × Present Value Factor (column I). We assume a 40% tax rate in this table. Thus, L8, the after-tax cost of the first month's loan payment, is equal to [$12,914 (E8) + (1 − 40%) × $8,333 (D8)] × 0.9917 (I8) = $17,766. The sum of the after-tax cost of the loan payments is $907,368 (L68).

We now move to section 3, F102 to I106. As we note in F108, we use equation (A4.24a) to test whether we get the same answer as the brute force approach in L68. In I104, we show the PV of the principal after tax, corresponding to item 1 above, as $600,000 (H104 is the same, but for each $1.00 of principal). In I105, we show the tax shield on the principal that we do not get at $307,368. The sum of the two is $907,368 (I106), which matches L68 and thus proves equation (A4.24a). Note that I106, which we calculate according to equation (A4.23a), shown in F109, equals $0.907368, which is the correct after-tax cost of the loan per each dollar of principal. When we multiply that by the $1 million principal, we get the correct after-tax cost of the loan in dollars, as per I106 and equation (A4.23a).

**PRESENT VALUE OF THE PRINCIPAL WHEN THE DISCOUNT RATE IS DIFFERENT FROM THE NOMINAL RATE** When valuing a loan at a discount rate, $r_1$, that is different than the nominal rate of interest, $r$, the present value of principal is as follows:

$$PV(Amort) = \left[ \frac{\frac{1}{(1+r)^n}}{1+r_1} + \frac{\frac{1}{(1+r)^{n-1}}}{(1+r_1)^2} + \frac{\frac{1}{(1+r)^{n-2}}}{(1+r_1)^3} + \cdots + \frac{\frac{1}{1+r}}{(1+r_1)^n} \right] \times Pymt. \tag{A4.25}$$

We can move the second denominator into the first to simplify the equation.

$$PV(Amort) = \left[ \frac{1}{(1+r)^n(1+r_1)} + \frac{1}{(1+r)^{n-1}(1+r_1)^2} + \cdots + \frac{1}{(1+r)(1+r_1)^n} \right] \times Pymt. \tag{A4.26}$$

Multiplying both sides by $\frac{1+r}{1+r_1}$, we get:

$$\frac{1+\mathrm{r}}{1+\mathrm{r}_1} PV(Amort) = \left[ \frac{1}{(1+r)^{n-1}(1+r_1)^2} + \frac{1}{(1+r)^{n-2}(1+r_1)^3} + \cdots + \frac{1}{(1+r_1)^{n+1}} \right] \times Pymt. \tag{A4.27}$$

Subtracting equation (A4.27) from equation (A4.26) and simplifying, we get:

$$\frac{r_1 - r}{1+r_1} PV(Amort) = \left[ \frac{1}{(1+r)^n(1+r_1)} - \frac{1}{(1+r_1)^{n+1}} \right] \times Pymt. \tag{A4.28}$$

This simplifies to:

$$PV(Amort) = \frac{1}{r_1 - r} \left[ \frac{1}{(1+r)^n} - \frac{1}{(1+r_1)^n} \right] \times Pymt. \tag{A4.29}$$

The top portion of Table A4.5 is almost identical to section 1 of Table A4.3. We use a nominal interest rate of 10% (B73) per year, which is 0.8333% (B74) per month, and a discount rate of 12% (B75) per year, or 1% (B76) per month.

We discount the principal amortization at $r_1$, the discount rate of 1%, in column F, so that column G gives us the present value of the principal (column D), which totals \$730,970 (G68). The Excel formula equivalent for equation (A4.29) appears in A81, and the result of that formula appears in D81, which matches the brute force calculation in G68, thus demonstrating the accuracy of the formula.

## Conclusion

In this mathematical appendix to the ADF chapter, we have presented:

- ADFs with stub periods (partial years) for both midyear and end-of-year
- ADFs to calculate the amortization of principal on a loan
- A formula for the after-tax PV of a loan
- Tables to demonstrate the accuracy of the various formulas

# Table A4.5

## Present Value of a Loan at Discount Rate Different than Nominal Rate

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 5 | **Pmt** | | | | | | |
| 6 | **#** | **Pmt** | **Int** | **Prin** | **Bal** | **PVF ($r_1$)** | **PV($P$)** |
| 7 | 0 | | | | 1,000,000 | 1.0000 | |
| 8 | 1 | 21,247 | 8,333 | 12,914 | 987,086 | 0.9901 | 12,786 |
| 9 | 2 | 21,247 | 8,226 | 13,021 | 974,065 | 0.9803 | 12,765 |
| 10 | 3 | 21,247 | 8,117 | 13,130 | 960,935 | 0.9706 | 12,744 |
| 11 | 4 | 21,247 | 8,008 | 13,239 | 947,696 | 0.9610 | 12,723 |
| 12 | 5 | 21,247 | 7,897 | 13,350 | 934,346 | 0.9515 | 12,702 |
| 13 | 6 | 21,247 | 7,786 | 13,461 | 920,885 | 0.9420 | 12,681 |
| 14 | 7 | 21,247 | 7,674 | 13,573 | 907,312 | 0.9327 | 12,660 |
| 15 | 8 | 21,247 | 7,561 | 13,686 | 893,626 | 0.9235 | 12,639 |
| 16 | 9 | 21,247 | 7,447 | 13,800 | 879,826 | 0.9143 | 12,618 |
| 17 | 10 | 21,247 | 7,332 | 13,915 | 865,911 | 0.9053 | 12,597 |
| 18 | 11 | 21,247 | 7,216 | 14,031 | 851,880 | 0.8963 | 12,576 |
| 19 | 12 | 21,247 | 7,099 | 14,148 | 837,732 | 0.8874 | 12,556 |
| 20 | 13 | 21,247 | 6,981 | 14,266 | 823,466 | 0.8787 | 12,535 |
| 21 | 14 | 21,247 | 6,862 | 14,385 | 809,081 | 0.8700 | 12,514 |
| 22 | 15 | 21,247 | 6,742 | 14,505 | 794,576 | 0.8613 | 12,494 |
| 23 | 16 | 21,247 | 6,621 | 14,626 | 779,951 | 0.8528 | 12,473 |
| 24 | 17 | 21,247 | 6,500 | 14,747 | 765,203 | 0.8444 | 12,452 |
| 25 | 18 | 21,247 | 6,377 | 14,870 | 750,333 | 0.8360 | 12,432 |
| 26 | 19 | 21,247 | 6,253 | 14,994 | 735,339 | 0.8277 | 12,411 |
| 27 | 20 | 21,247 | 6,128 | 15,119 | 720,220 | 0.8195 | 12,391 |
| 28 | 21 | 21,247 | 6,002 | 15,245 | 704,974 | 0.8114 | 12,370 |
| 29 | 22 | 21,247 | 5,875 | 15,372 | 689,602 | 0.8034 | 12,350 |
| 30 | 23 | 21,247 | 5,747 | 15,500 | 674,102 | 0.7954 | 12,330 |
| 31 | 24 | 21,247 | 5,618 | 15,630 | 658,472 | 0.7876 | 12,309 |
| 32 | 25 | 21,247 | 5,487 | 15,760 | 642,712 | 0.7798 | 12,289 |
| 33 | 26 | 21,247 | 5,356 | 15,891 | 626,821 | 0.7720 | 12,269 |
| 34 | 27 | 21,247 | 5,224 | 16,024 | 610,798 | 0.7644 | 12,248 |
| 35 | 28 | 21,247 | 5,090 | 16,157 | 594,641 | 0.7568 | 12,228 |
| 36 | 29 | 21,247 | 4,955 | 16,292 | 578,349 | 0.7493 | 12,208 |
| 37 | 30 | 21,247 | 4,820 | 16,427 | 561,922 | 0.7419 | 12,188 |
| 38 | 31 | 21,247 | 4,683 | 16,564 | 545,357 | 0.7346 | 12,168 |
| 39 | 32 | 21,247 | 4,545 | 16,702 | 528,655 | 0.7273 | 12,148 |
| 40 | 33 | 21,247 | 4,405 | 16,842 | 511,813 | 0.7201 | 12,128 |
| 41 | 34 | 21,247 | 4,265 | 16,982 | 494,831 | 0.7130 | 12,108 |
| 42 | 35 | 21,247 | 4,124 | 17,123 | 477,708 | 0.7059 | 12,088 |
| 43 | 36 | 21,247 | 3,981 | 17,266 | 460,442 | 0.6989 | 12,068 |
| 44 | 37 | 21,247 | 3,837 | 17,410 | 443,032 | 0.6920 | 12,048 |
| 45 | 38 | 21,247 | 3,692 | 17,555 | 425,476 | 0.6852 | 12,028 |
| 46 | 39 | 21,247 | 3,546 | 17,701 | 407,775 | 0.6784 | 12,008 |
| 47 | 40 | 21,247 | 3,398 | 17,849 | 389,926 | 0.6717 | 11,988 |
| 48 | 41 | 21,247 | 3,249 | 17,998 | 371,928 | 0.6650 | 11,968 |
| 49 | 42 | 21,247 | 3,099 | 18,148 | 353,781 | 0.6584 | 11,949 |
| 50 | 43 | 21,247 | 2,948 | 18,299 | 335,482 | 0.6519 | 11,929 |
| 51 | 44 | 21,247 | 2,796 | 18,451 | 317,031 | 0.6454 | 11,909 |
| 52 | 45 | 21,247 | 2,642 | 18,605 | 298,425 | 0.6391 | 11,890 |
| 53 | 46 | 21,247 | 2,487 | 18,760 | 279,665 | 0.6327 | 11,870 |
| 54 | 47 | 21,247 | 2,331 | 18,917 | 260,749 | 0.6265 | 11,850 |
| 55 | 48 | 21,247 | 2,173 | 19,074 | 241,675 | 0.6203 | 11,831 |
| 56 | 49 | 21,247 | 2,014 | 19,233 | 222,442 | 0.6141 | 11,811 |
| 57 | 50 | 21,247 | 1,854 | 19,393 | 203,048 | 0.6080 | 11,792 |
| 58 | 51 | 21,247 | 1,692 | 19,555 | 183,493 | 0.6020 | 11,772 |
| 59 | 52 | 21,247 | 1,529 | 19,718 | 163,775 | 0.5961 | 11,753 |
| 60 | 53 | 21,247 | 1,365 | 19,882 | 143,893 | 0.5902 | 11,734 |
| 61 | 54 | 21,247 | 1,199 | 20,048 | 123,845 | 0.5843 | 11,714 |
| 62 | 55 | 21,247 | 1,032 | 20,215 | 103,630 | 0.5785 | 11,695 |
| 63 | 56 | 21,247 | 864 | 20,383 | 83,247 | 0.5728 | 11,676 |
| 64 | 57 | 21,247 | 694 | 20,553 | 62,693 | 0.5671 | 11,656 |
| 65 | 58 | 21,247 | 522 | 20,725 | 41,969 | 0.5615 | 11,637 |
| 66 | 59 | 21,247 | 350 | 20,897 | 21,071 | 0.5560 | 11,618 |
| 67 | 60 | 21,247 | 176 | 21,071 | 0 | 0.5504 | 11,599 |
| 68 | **Total** | **1,274,823** | **274,823** | **1,000,000** | | | **730,970** |

## Assumptions:

| | A | B | C | D |
|---|---|---|---|---|
| 72 | Prin | 1,000,000 | | |
| 73 | Int | 10.0000% | | |
| 74 | Int-Mo = $r$ | 0.8333% | | |
| 75 | Int | 12.0000% | | |
| 76 | Int-Mo = $r_1$ | 1.0000% | | |
| 77 | Years | 5 | | |
| 78 | Months = $n$ | 60 | | |
| 79 | Pymt | 21,247 | | |
| 80 | Start Month=$S$ | 3 | | |
| 81 | **$(1/(r_1-r))*((1/(1+r)^n)-(1/(1+r_1)^n))*$PYMT** | | | **730,970** |

# APPENDIX 4B

# Mathematical Appendix: Monthly ADFs

In this appendix, we will develop formulas to calculate the present value of a finite series of monthly cash flows. The annual equivalent of that is known as an *annuity discount factor* (ADF). We could call this a monthly discount factor. However, since the term ADF is so well known, we will call this the monthly version of the ADF, or $ADF_m$.

In equation (B4.1), we begin with the stream of cash flows for $n$ years. We define the first year's cash flow as \$1.00. Since we are modeling this in months, the first 12 months' cash flows are \$1.00/12 = \$0.08333 ... per month. At the end of one year, the cash flow will increase by a constant growth rate of $g$. Cash flows for months 13 through 24 will be $\$0.08333 \times (1 + g)$, and cash flows for months 25 through 36 will be $\$0.08333 \times (1 + g)^2$, and so on. We discount the cash flows in the middle of each month. Thus, the present value of the monthly cash flows for $n$ years, $ADF_m$, equals:

$$ADF_m = \frac{1}{12}\left(\left[\frac{1}{(1+r)^{\frac{0.5}{12}}} + \cdots + \frac{1}{(1+r)^{\frac{11.5}{12}}}\right] + (1+g)\left[\frac{1}{(1+r)^{\frac{12.5}{12}}} + \cdots + \frac{1}{(1+r)^{\frac{23.5}{12}}}\right] + \cdots + (1+g)^{n-1}\left[\frac{1}{(1+r)^{n-\frac{11.5}{12}}} + \cdots + \frac{1}{(1+r)^{n-\frac{0.5}{12}}}\right]\right). \quad \text{(B4.1)}$$

We can multiply the numerators and denominators by $(1 + r)^{0.5/12}$, which will have the effect of increasing the exponent in the denominators by one-half month. We can then factor out the $(1 + r)^{0.5/12}$ in the numerators, with the following result:

$$ADF_m = \frac{(1+r)^{\frac{0.5}{12}}}{12}\left(\left[\frac{1}{(1+r)^{\frac{1}{12}}} + \cdots + \frac{1}{(1+r)^{1}}\right] + (1+g)\left[\frac{1}{(1+r)^{\frac{13}{12}}} + \cdots + \frac{1}{(1+r)^{2}}\right] + \cdots + (1+g)^{n-1}\left[\frac{1}{(1+r)^{n-\frac{11}{12}}} + \cdots + \frac{1}{(1+r)^{n}}\right]\right). \quad \text{(B4.2)}$$

By factoring out $(1 + r)^{t-1}$ from the denominator, where $t$ = year ($t$ = 1, 2, 3 ...), we can reduce the terms in each square bracket to identical terms:

$$ADF_m = \frac{(1+r)^{\frac{0.5}{12}}}{12}\left(\left[\frac{1}{(1+r)^{\frac{1}{12}}} + \cdots + \frac{1}{(1+r)^{1}}\right] + \frac{1+g}{1+r}\left[\frac{1}{(1+r)^{\frac{1}{12}}} + \cdots + \frac{1}{(1+r)^{1}}\right] + \cdots + \left(\frac{1+g}{1+r}\right)^{n-1}\left[\frac{1}{(1+r)^{\frac{1}{12}}} + \cdots + \frac{1}{(1+r)^{1}}\right]\right). \quad \text{(B4.3)}$$

Now, we can factor out all the terms in the square brackets.

$$ADF_m = \frac{(1+r)^{\frac{0.5}{12}}}{12}\left[\frac{1}{(1+r)^{\frac{1}{12}}} + \cdots + \frac{1}{(1+r)^1}\right]\left[1 + \left(\frac{1+g}{1+r}\right) + \cdots + \left(\frac{1+g}{1+r}\right)^{n-1}\right]. \tag{B4.4}$$

We will solve for the terms in the square brackets separately. Let's call the first one $A$ and the second one $B$.

$$A = \frac{1}{(1+r)^{\frac{1}{12}}} + \cdots + \frac{1}{(1+r)^1}. \tag{B4.5}$$

We multiply each term by $\frac{1}{(1+r)^{\frac{1}{12}}}$, which leads to:

$$\frac{1}{(1+r)^{\frac{1}{12}}}A = \frac{1}{(1+r)^{\frac{2}{12}}} + \cdots + \frac{1}{(1+r)^{\frac{13}{12}}}. \tag{B4.6}$$

Subtracting (B4.6) from (B4.5), on the right-hand side (RHS) of the equation, only the first term in (B4.5) and the last term in (B4.6) remain.

$$\left[1 - \frac{1}{(1+r)^{\frac{1}{12}}}\right]A = \frac{1}{(1+r)^{\frac{1}{12}}} - \frac{1}{(1+r)^{\frac{13}{12}}}. \tag{B4.7}$$

Simplifying the left-hand side (LHS) of (B4.7) leads to:

$$\frac{(1+r)^{\frac{1}{12}} - 1}{(1+r)^{\frac{1}{12}}}A = \frac{1}{(1+r)^{\frac{1}{12}}} - \frac{1}{(1+r)^{\frac{13}{12}}}. \tag{B4.8}$$

We multiply by the inverse of the fraction on the LHS to isolate $A$:

$$A = \frac{(1+r)^{\frac{1}{12}}}{(1+r)^{\frac{1}{12}} - 1}\left[\frac{1}{(1+r)^{\frac{1}{12}}} - \frac{1}{(1+r)^{\frac{13}{12}}}\right]. \tag{B4.9}$$

We cancel out the $(1+r)^{\frac{1}{12}}$, which leaves us with:

$$A = \frac{1}{(1+r)^{\frac{1}{12}} - 1}\left[1 - \frac{1}{(1+r)}\right]. \tag{B4.10}$$

The term in square brackets equals $r/(1+r)$. Thus, the term $A$ solves to:

$$A = \frac{1}{(1+r)^{\frac{1}{12}} - 1}\frac{r}{(1+r)}. \tag{B4.11}$$

Now we turn our attention to the term $B$, which is the rightmost term in (B4.4). $B$ is a slight variation of a traditional *ADF* with growth. In the traditional end-of-year *ADF* cash flows, the $(1+r)$ term in the denominator has an exponent that is always one higher than the $(1+g)$ term in the numerator, because we assume our first cash flow of \$1.00 occurs at the end of the first period. Thus the present values of the cash flows in the traditional *ADF* are $1/(1+r)$, $(1+g)/(1+r)^2$, and so forth. We can change $B$ into this form by multiplying all denominators in $B$ by $(1+r)$. Of course, we will have to multiply all numerators by the same term, and we can factor that out of the series. Thus, $B$ is equal to $(1 + r)$ times the end-of-year *ADF*, or:[36]

$$B = \frac{1+r}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^n\right]. \tag{B4.12}$$

Note that the numerator in the first term in the RHS of (B4.12) is $(1 + r)$ instead of 1. Substituting (B4.11) and (B4.12) into (B4.4), we get:

$$ADF_m = \frac{(1+r)^{\frac{0.5}{12}}}{12}\frac{1}{(1+r)^{\frac{1}{12}}-1}\frac{r}{(1+r)}\frac{1+r}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^n\right]. \tag{B4.13}$$

We can cancel the $1 + r$ terms and simplify to:

$$ADF_m = \frac{1}{12}(1+r)^{\frac{0.5}{12}}\frac{r}{(1+r)^{\frac{1}{12}}-1}\frac{1}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^n\right] \tag{B4.14}$$

Present value of midmonth cash flows.

There are five terms in (B4.14): The first term, 1/12, is the first month's forecast cash flow (i.e., 1/12 of $1.00). The second term is a midmonth correction factor; that is, the cash flow is that much more valuable than a series of end-of-month cash flows. In other words, if cash flows were end-of-month instead of midmonth, the ADF formula would be identical to (B4.14), except that the second term would disappear. The denominator of the third term is equal to $i$ from equation (4.31), the correct compound monthly interest rate. The fourth term multiplied by the fifth term is the annual end-of-year ADF. The fourth term is the end-of-year Gordon model multiple, and the last term—the one in square brackets—converts the perpetuity of the Gordon model to a finite series of cash flows.[37]

Thus, we can restate (B4.14) as:

$$ADF_m = \mathit{Midmonth\ Correction\ Factor} \times ADF_{Endyear} \times \frac{r}{12i} \tag{B4.14a}$$

PV of midmonth cash flows—alternative expression.

The intuition behind (B4.14a) is that the monthly annuity discount factor is primarily equal to the ordinary ADF times the last term (i.e., $r/12i$). That term is the essence of the difference in the monthly ADF and the ordinary ADF. In Dr. Trout's example, the annual ADF is 2.53764 (Table 4.12, F55). If we used simple instead of compound interest, $r$ would be equal to $12i$. He used an annual rate of $r = 12\%$, which is 12 times his monthly rate of 1%. However, using compound interest, $r/12i = 0.12/(12 \times 0.009489) = 1.05387$ (Table 4.12, F56). The only other modification is that we need to multiply that by the midmonth correction factor of 1.00473 (F57), which leads to a monthly $ADF = 2.53764 \times 1.05387 \times 1.00473 = 2.68701$ (F58).

[36]See equation (4.6b).

[37]See the section entitled, "Relationship between the ADF and the Gordon Model," earlier in this chapter, which further explains the intuition of the ADF.

# PART II

# Calculating Discount Rates

## Introduction

Part II of this book, consisting of Chapters 5, 6, and 7, deals with calculating discount rates; discounting cash flows is the second of the four steps in business valuation.

## Chapter 5: The Log Size Model

Chapter 5, the log size model, is a long chapter, with a significant amount of empirical analysis of stock market returns. Our primary finding is that returns are negatively related to the logarithm of the size of the firm. The most successful measure of size in explaining returns of publicly held stocks is market capitalization, though research by Grabowski and King that we present in the chapter shows that many other measures of size also do a fairly good job of explaining stock market returns.

In their 1999 article, Grabowski and King found the relationship of return to three underlying variables: operating margin, the logarithm of the coefficient of variation of operating margin, and the logarithm of the coefficient of variation of return on equity. This is a very important research result, and it is very important that professionals read and understand their article. Even so, their methodology is based on Compustat data, which leaves out the first 37 years of the New York Stock Exchange data. As a consequence, their standard errors are higher than my log size model, and appraisers should be familiar with both.

In this chapter, we:

- Develop the mathematics of potential log size equations.
- Analyze the statistical error in the log size equation for different time periods and determine the optimal time frame.
- Present research by Harrison that shows that the distribution of stock market returns in the eighteenth century is the same as it is in the twentieth century and discuss its implications for which twentieth-century data we should use.
- Present research on Growth versus Value stock returns and discuss implications for valuing privately held firms.
- Develop a series of equations to explain the relationship between the Ibbotson total returns equation and the Gordon model.
- Give practical examples of using the log size equation.

- Compare log size to the capital asset pricing model (CAPM) for accuracy.
- Discuss industry effects.

### Benefit of the Log Size Model

While the log size model used to save much time compared to CAPM, the availability of industry premia in the SBBI valuation yearbooks[1] levels the playing field. Log size is much more accurate for smaller firms than is either CAPM or the buildup method. Using 1926–2007 data, the log size standard error of the valuation estimate is only 27 percent as large as CAPM standard error. This means that the CAPM 95 percent confidence intervals are approximately 375 percent of the size of the log size confidence intervals.

### Appendix C: The Shortcut to Log Size

For those who prefer not to read through the research that leads to our conclusions and simply want to learn how to use the log size model, Appendix C presents a much shorter, "stripped down" version of Chapter 5. It also serves as a useful refresher for those who read Chapter 5 in its entirety but periodically wish to refresh their skills and understanding.

## Chapter 6: Arithmetic versus Geometric Mean Returns

There have been many articles in the professional literature arguing whether arithmetic or geometric mean returns are most appropriate. For valuing small businesses, the two measures can easily make a 100 percent difference in the valuation, as geometric returns are always lower than arithmetic returns (as long as returns are not identical in every period, which, of course, they are not). Most of the arguments have centered on Professor Ibbotson's famous two-period example.

The majority of Chapter 6 consists of empirical evidence that arithmetic mean returns do a better job than geometric means of explaining log size results. Additionally, we spend some time discussing a very mathematical article by Indro and Lee that argues for using a time horizon-weighted average of the arithmetic and geometric means.

## Chapter 7: An Iterative Approach for CAPM

For those who use CAPM, whether in a direct equity approach or in an invested capital approach, there is a trap into which many appraisers fall, which is producing an answer that is internally inconsistent.

Common practice is to assume a degree of leverage—usually equal to the subject company's existing or industry average leverage—assuming book value for equity. This implies an equity for the firm, which is an ex ante value of equity. The problem

[1]This is for the alter ego of CAPM, the Build-Up Model.

comes when the appraiser stops after obtaining his or her valuation estimate. This is because the calculated value of equity will almost always be inconsistent with the value of equity that is implied in the leverage assumed in the calculation of the CAPM discount rate.

In Chapter 7 we present an iterative method that solves the problem by repeating the valuation calculations until the assumed and the calculated equity are equal.

# CHAPTER 5

# Discount Rates as a Function of Log Size

# CHAPTER 5

# Discount Rates as a Function of Log Size

## Research Included in the First Edition

Historically, small companies[1] have shown higher rates of return when compared to large ones[2] over the past 82 years (Ibbotson Associates 2008). The relationship between firm size and rate of return was first published by Rolf Banz in 1981 and is now universally recognized. Accordingly, company size has been included as a variable in several models used to determine stock market returns.

Jacobs and Levy (1988) examined small firm size as one of 25 variables associated with anomalous rates of return on stocks. They found that small size was statistically significant both in single-variable and multivariate form, although size effects appear to change over time; that is, they are nonstationary. They found that the natural logarithm (log) of market capitalization was negatively related to the rate of return.

Fama and French (1993) found they could explain historical market returns well with a three-factor multiple regression model using firm size, the ratio of book equity to market equity (BE/ME), and the overall market factor $R_m - R_f$, (i.e., the equity premium). The latter factor explained overall returns to stocks across the board, but it did not explain differences from one stock to another, or more precisely, from one portfolio to another.[3]

---

Adapted and reprinted with permission from *Valuation* (August 1994): 8–24; and *The Valuation Examiner* (February/March 1997): 19–21.

[1]From 1926 to 1981, NYSE fifth quintile returns; from January 1982 to March 2001, DFA U.S. 9-10 Small Company Portfolio; from April 2001 to December 2007, DFA U.S. Micro Cap Portfolio.

[2]Based on the S&P Composite Index.

[3]The regression coefficient is essentially beta controlled for size and BE/ME. After controlling for the other two systematic variables, this beta is very close to 1 and explains only the market premium overall. It does not explain any differentials in premiums across firms or portfolios, as the variation was insignificant. In other words, this beta lacks significant explanatory power, because the major explanatory power lies in the differences in size and financial distress (growth versus value firms).

The entire variation in portfolio returns was explained by the first two factors. Fama and French found BE/ME to be the more significant factor in explaining the cross-sectional difference in returns, with firm size next; however, they consider both factors as proxies for risk. Furthermore, they state (1993)

> *Without a theory that specifies the exact form of the state variables or common factors in returns, the choice of any particular version of the factors is somewhat arbitrary. Thus detailed stories for the slopes and average premiums associated with particular versions of the factors are suggestive, but never definitive.*

Abrams (1994) showed strong statistical evidence that returns are linearly related to the natural logarithm of the value of the firm, as measured by market capitalization. He used this relationship to determine the appropriate discount rate for privately held firms. In a follow-up article, Abrams (1997) further simplified the calculations by relating the natural log of size to total return without splitting the result into the risk-free rate plus the equity premium.

Grabowski and King (1995) also described the logarithmic relationship between firm size and market return. They later (Grabowski and King, 1996) demonstrated that a similar, but weaker, logarithmic relationship exists for other measures of firm size, including the book value of common equity, five-year average net income, market value of invested capital, five-year average EBITDA, sales, and number of employees. In Grabowski and King (1999), they demonstrate a negative logarithmic relationship between returns and operating margin and a positive logarithmic relationship between returns and the coefficient of variation of operating margin and accounting return on equity. Since then, they publish their study annually in the Duff & Phelps, LLC Risk Premium Report.

The discovery that return (the discount rate) has a negative linear relationship to the natural logarithm of the value of the firm means that the value of the firm decays (i.e., decreases) exponentially with increasing rates of return. We will also show that firm value decays exponentially with the standard deviation of returns.

## Table 5.1: Analysis of Historical Stock Returns

Columns A through F in Table 5.1 contain the input data from the *Stocks, Bonds, Bills and Inflation 2008 Classic Yearbook* (Ibbotson Associates, 2008) for all of the regression analyses as well as the regression results. We use the 82-year arithmetic average returns in both regressions, from 1926 to 2007. Column A lists the NYSE/AMEX/NASDAQ divided into different groups—known as *deciles*—based on market capitalization as a proxy for size, with the largest firms in decile #1 and the smallest in decile #10.[4] Columns B through F contain market data for each decile, which is described in the following.

Note that the 82-year average market return in column B rises with each decile. The standard deviation of returns (column C) also rises with each decile. Column D shows the market capitalization of each decile near the end of 2007, with decile

[4] All of the underlying decile data in Ibbotson originate with the University of Chicago's Center for Research in Security Prices (CRSP), which also determines the composition of the deciles.

# Table 5.1

## NYSE/AMEX/NASDAQ Data by Decile and Statistical Analysis: 1926–2007

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | | [1] | [1] | [2] | [2] | [2] | = D/F | | Ln(G) | [3] | [4] | = B – K | = L$^2$ | | = B – N | = O$^2$ |
| 5 | **Variable** | ***Y*** | ***X1*** | | | | | | ***X2*** | | | | | | | |
| 6 | | **Arith Mean** | | **Recent** | | | | **% Chg** | | | | **CAPM** | | **Regr #2** | **Regr #2** | |
| 7 | **Decile** | **Return** | **Std Dev** | **Mkt Cap** | **% Cap** | **# Cos** | **Avg Cap=FMV** | **in FMV** | **Ln(FMV)** | **Beta** | **CAPM E(R)** | **Error** | **Sq Error** | **Estimate** | **Error** | **Sq Error** |
| 8 | **1** | **11.31%** | **18.91%** | 1.04E+13 | 62.34% | 167 | 62,022,860,778 | 363.7% | **24.851** | 0.91 | 11.63% | -0.31% | 0.0010% | 10.53% | 0.78% | 0.0061% |
| 9 | **2** | **13.16%** | **21.63%** | 2.33E+12 | 14.01% | 174 | 13,375,585,747 | 131.0% | **23.317** | 1.03 | 12.47% | 0.69% | 0.0048% | 12.73% | 0.43% | 0.0018% |
| 10 | **3** | **13.72%** | **23.31%** | 1.11E+12 | 6.69% | 192 | 5,789,959,375 | 50.1% | **22.479** | 1.10 | 12.97% | 0.76% | 0.0057% | 13.94% | -0.21% | 0.0005% |
| 11 | **4** | **14.07%** | **25.68%** | 7.10E+11 | 4.27% | 184 | 3,857,046,793 | 44.6% | **22.073** | 1.12 | 13.11% | 0.97% | 0.0093% | 14.52% | -0.45% | 0.0020% |
| 12 | **5** | **14.85%** | **26.49%** | 5.41E+11 | 3.26% | 203 | 2,666,994,039 | 62.9% | **21.704** | 1.16 | 13.39% | 1.46% | 0.0213% | 15.05% | -0.20% | 0.0004% |
| 13 | **6** | **15.14%** | **27.10%** | 4.11E+11 | 2.47% | 251 | 1,637,608,287 | 18.7% | **21.217** | 1.18 | 13.53% | 1.61% | 0.0258% | 15.75% | -0.61% | 0.0038% |
| 14 | **7** | **15.46%** | **29.47%** | 3.79E+11 | 2.28% | 275 | 1,379,873,309 | 80.1% | **21.045** | 1.24 | 13.95% | 1.51% | 0.0228% | 16.00% | -0.53% | 0.0028% |
| 15 | **8** | **16.58%** | **34.18%** | 2.91E+11 | 1.75% | 380 | 766,269,974 | 72.6% | **20.457** | 1.30 | 14.38% | 2.21% | 0.0488% | 16.84% | -0.26% | 0.0007% |
| 16 | **9** | **17.28%** | **36.45%** | 2.85E+11 | 1.71% | 641 | 443,897,410 | 290.6% | **19.911** | 1.35 | 14.73% | 2.55% | 0.0651% | 17.62% | -0.35% | 0.0012% |
| 17 | **10** | **20.98%** | **44.58%** | 2.02E+11 | 1.21% | 1775 | 113,636,704 | N/A | **18.549** | 1.41 | 15.15% | 5.83% | 0.3398% | 19.58% | 1.40% | 0.0195% |
| 18 | **Std Deviation** | **2.64%** | | | | **4,242** | | | | | | | | | | |
| 19 | **Value Wtd Index** | **12.47%** | **NA** | **1.66E+13** | **100.00%** | | | | | | **Totals ------------>** | | **0.5446%** | | | **0.0388%** |
| 20 | | | | | | | | | | | **Standard Error ->** | | **2.61%** | | | **0.70%** |
| 21 | | | | | | | | | | | **Std Error—CAPM / Std Error—Log Size Model** | | | | | **374.64%** |

### 1st Regression: Return = F(Std Dev. of Returns)

| | | |
|---|---|---|
| 23 | **Constant** | **5.54%** |
| 24 | **82-Yr Mean Gov Bond Yld [5]** | **5.21%** |
| 25 | **Std Err of *Y* Est** | **0.45%** |
| 26 | **R Squared** | **97.37%** |
| 27 | **Adjusted R Squared** | **97.04%** |
| 28 | **No. of Observations** | **10** |
| 29 | **Degrees of Freedom** | **8** |
| 30 | ***X* Coefficient(s)** | **33.76%** |
| 31 | **Std Err of *X*-Coef.** | **1.96%** |
| 32 | ***T*** | **17.2** |
| 33 | ***P*** | **<0.01%** |

### 2nd Regression: Return = F[LN(Mkt Capitalization)]

| | | |
|---|---|---|
| 37 | **Constant** | **46.22%** |
| 38 | **Std Err of *Y* Est. [6]** | **0.70%** |
| 39 | **R Squared** | **93.80%** |
| 40 | **Adjusted R Squared** | **93.02%** |
| 41 | **No. of Observations** | **10** |
| 42 | **Degrees of Freedom** | **8** |
| 43 | ***X* Coefficient(s)** | **-1.436%** |
| 44 | **Std Err of *X*-Coef.** | **0.131%** |
| 45 | ***T*** | **-11.0** |
| 46 | ***P*** | **<0.01%** |

### 3rd Regression: Return = F[Decile Beta]

| | | |
|---|---|---|
| 50 | **Constant** | **-4.33%** |
| 51 | **Std Err of *Y* Est** | **0.86%** |
| 52 | **R Squared** | **90.60%** |
| 53 | **Adjusted R Squared** | **89.43%** |
| 54 | **No. of Observations** | **10** |
| 55 | **Degrees of Freedom** | **8** |
| 56 | ***X* Coefficient(s)** | **16.60%** |
| 57 | **Std Err of *X*-Coef.** | **1.89%** |
| 58 | ***T*** | **878.34%** |
| 59 | ***P*** | **<0.01%** |

**Assumptions:**

| | | |
|---|---|---|
| 62 | **Lge Co Stk Returns—Arith Avg (Based on *SBBI Classic*—2008, pp. 38–39)** | 12.26% |
| 63 | **L-T Gov Bond—Arith Mean Inc Return (Based on *SBBI Classic*—2008, pp. 38–39)** | 5.21% |
| 64 | **Long Horizon Equity Premium (E62 – E63)** | **7.05%** |

Notes

[1] Derived from *SBBI Classic—2008* pp. 132–133.*

[2] *SBBI Classic—2008*, p. 140.*

[3] *SBBI Classic—2008*, p. 142.*

[4] CAPM equation: Rf + (Beta × equity premium) = 5.21% + (Beta × 7.05%), where Rf is in G63, and the equity risk premium is in G64.

[5] Average income yields derived from *SBBI Classic—2008*, pp. 38–39 and 246–247.

[6] Note that C38 = P20.

* *Source:* Morningstar, Inc.—*2008 Ibbotson® Stocks, Bonds, Bills and Inflation (SBBI) Classic Yearbook.*

#1 containing 167 firms (F8) with a total market capitalization of $10.4 trillion (D8). Market capitalization, our measure of fair market value (FMV), is the price per share times the number of shares.

Dividing column D (FMV) by column F (the number of firms in the decile), we obtain column G, the average capitalization, or the average fair market value of the firms in each decile. For example, the average company in decile #1 has an FMV of $62.023 billion (G8), while the average firm in decile #10 has an FMV of $113.637 million (G17).

Column H shows the percentage difference between each successive decile. For example, the average firm size in decile #9 ($443.9 million; G16) is 290.6% (H16) larger than the average firm size in decile #10 ($113.6 million; G17).[5] The average firm size in decile #8 is 72.6% larger (H15) than that of decile #9, and so on.

The largest gap in absolute dollars and in percentages is between decile #1 and decile #2, a difference of $48.6 billion (G8 − G9), or 363.7% (H8). Deciles #9 and #10 have the second largest difference between them in percentage terms (290.6%, per H16). Most deciles are 19% to 80% larger than the next smaller one.

The difference in return (column B) between deciles #1 and #2 is 1.8% and between deciles #9 and #10 is 3.7%,[6] while the difference between all other deciles is 1.1% or less. Thus it seems that for fairly regular percentage increases in size we see a reasonably constant drop in the average returns. This suggests a logarithmic relationship between size and return, which we investigate later and confirm.

Column I is the natural logarithm of the average FMV. The natural logarithm of FMV is the number that when used as an exponent to Euler's constant, $e$ (the natural exponent from calculus), results in the FMV. Thus, $e^{\ln \text{FMV}} = \text{FMV}$. The number $e$, like pi, is an irrational, transcendental number. Its first digits begin 2.718. . . .

The natural logarithm operates in the same way as the Richter scale—used to measure earthquakes—except that the latter works in base 10 logarithms. The principle, however, is the same. An earthquake of 7 on the Richter scale is 10 times stronger than an earthquake of 6, 100 times more powerful than an earthquake of 5, 1,000 times more powerful than an earthquake of 4, and so on. The difference in power between two earthquakes whose Richter scale measurement varies by $\Delta x$ is $10^{\Delta x}$. Thus the latter example comparing two earthquakes with a rating of 7 and 4 is a difference of 3 on the Richter scale, which means the former is $10^3 = 1{,}000$ times more powerful than the latter. Similarly, the difference in value between firms whose natural logs of average value differ by $\Delta x$ is $2.718^{\Delta x}$. An increase in the natural log by 1 means the resulting value (from taking the antilog) will be 2.718 times larger than the value whose natural log is one less. Similarly, an increase in natural log by 2 is a value $2.718^2 = 7.4$ times larger, and an increase of 3 is $2.718^3 = 20.1$ times larger than the base value.

For example, the average market capitalization for decile #10 of $113.6 million (G17) $= e^{18.5485} \cong 2.718^{18.5485}$, where the exponent is the natural log in I17. Similarly, $e^{24.8508} =$ $62.0 billion (G8), where 24.8508 (I8) is the natural log of the decile #1 average market capitalization.

---

[5]We measure this as the ratio of market caps minus 1, for example, $443.9 million/$113.6 million = 390.6% − 100% = 290.6% (G16/G17 − 1 = H16).

[6]*SBBI—2009 Classic Edition*, p. 61, notes that delisting returns are included in order to eliminate survivorship bias.

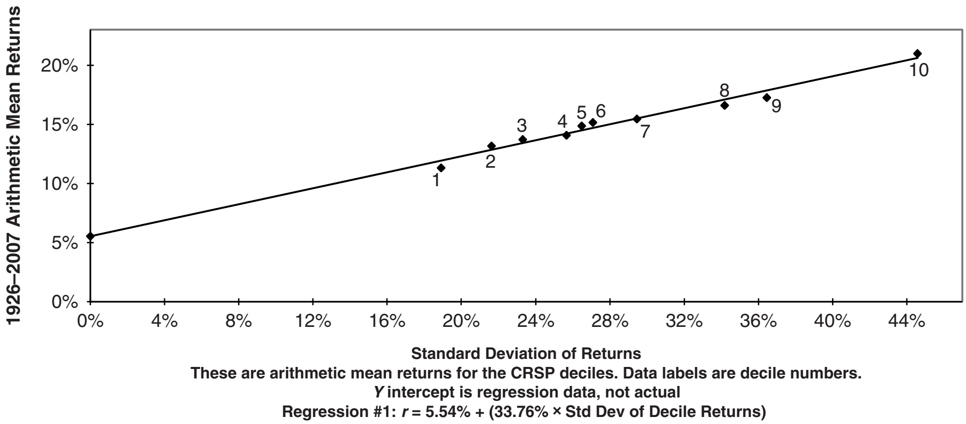



FIGURE 5.1 1926–2007 Arithmetic Mean Returns as a Function of Standard Deviation

Let's go through an example of how to generate a natural logarithm on a spreadsheet. In Microsoft Excel the formula in I8 is = ln(G8). In Lotus 123, the formula would be @ln(G8). To take the antilog (i.e., exponentiating), use the formulas: =exp(I8) in Excel® and @exp(I8) in Lotus 123.

## Regression #1: Return versus Standard Deviation of Returns for 1926–2007

Figure 5.1 is a graph of stock market returns as a function of standard deviation of returns. The nodes numbered 1 to 10 are the actual data points, with the number being the decile, and the straight line running through the points is the regression estimate. Note the strong linear relationship of the two. The deciles are in numerical order, and each successive decile is northeast of the other except for #3 to #4 and #6 to #7, which are both almost parallel. The graph tells us that as the decile number goes up—which means as size goes down—returns and risk both increase.

Of course, it is an axiom of finance that as risk increases so does return.[7] Logically, investors would never deliberately invest in one firm (or portfolio) with higher risk than another unless the expected return is also higher. It is still a relatively new observation that we can see this relationship in the size of the firms. Figure 5.1 shows this relationship graphically, and the regressions in Table 5.1 that follow demonstrate that relationship mathematically.

Regression #1 in Table 5.1 (rows 23–33) is a statistical measurement of return as a function of standard deviation of returns. The results confirm that a very strong relationship exists between historical returns and standard deviation. The regression equation is:

$$r = 5.54\% + (33.76\% \times S), \tag{5.1}$$

where $r$ = return and $S$ = standard deviation of returns.

[7]See the "Growth versus Value Stocks" section at the end of this chapter for an apparent partial exception to this.

The adjusted $R^2$ for equation (5.1) is 97.04% (C27), and the $t$-statistic of the slope is 17.2 (C32). The $p$-value is less than 0.01% (C33), which means the slope coefficient is statistically significant at the 99.9%+ level. The standard error of the estimate is 0.45% (C25), also indicating a high degree of confidence in the results obtained. Another important result is that the constant of 5.54% (C23) is the regression estimate of the long-term risk-free rate, that is, the rate of return for a no-risk (zero standard deviation) asset. The 82-year arithmetic mean income return from 1926 to 2007 on long-term government bonds is 5.21% (C24, G63).[8] Therefore, in addition to the other robust results, the regression equation does a reasonable job of estimating the risk-free rate.

**KEEPING IN THE ROARING TWENTIES AND THE GREAT DEPRESSION** In the first edition of this book we also showed regressions with data that began in 1938, as eliminating 1926–1937, which contained much of the Roaring Twenties and the Great Depression, materially reduced the volatility of returns and improved the regression results. I had consulted with economists who felt this was preferable, as they did not expect that kind of volatility to return. Indeed (Voth, 2002) attributed that magnitude of volatility to uncertainty as to whether the capitalist system would survive, as Communism was a prominent threat in people's minds. While that was still true through 2007, the Financial Crisis of 2008 is a reminder that extreme volatility still can happen, and therefore it is more logical to include all the data, accepting the higher volatility and lower confidence intervals. Table 5.1A is identical to Table 5.1, except that it contains stock market results through 2008, that is, it is from Ibbotson's 2009 yearbook. We consider this topic in more depth in the section, "Which Data to Choose?"

**LIMITATION OF REGRESSION #1 FOR PRIVATELY HELD BUSINESSES** Our goal is to calculate a discount rate. The major problem with direct application of this relationship to the valuation of privately held businesses is coming up with a reliable standard deviation of returns. Appraisers cannot directly measure the standard deviation of returns for privately held firms, since there are no objective stock prices. We can measure the standard deviation of income, and we cover that later in the chapter in our discussion of Grabowski and King (1999).

## Regression #2: Return versus Log Size

Fortunately, there is a much more practical relationship. Notice that the returns are negatively correlated with the market capitalization, that is, the fair market value of the firm. The second regression in Table 5.1 (C37 through C46) is the more useful one for valuing privately held firms. Regression #2 shows return as a function of the natural logarithm of the FMV of the firm. Regression equation (5.2) comes from C37 and C43 and is as follows:

$$r = 46.22\% - [1.436\% \times \ln(\text{FMV})]. \tag{5.2}$$

The adjusted $R^2$ is 93.0% (C40), the $t$-statistic is $-11.0$ (C45), and the $p$-value is less than 0.01% (C46), meaning that these results are statistically robust. The standard

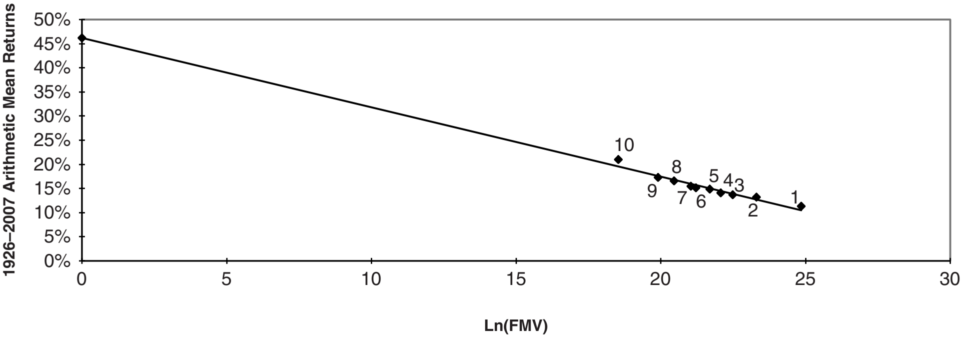
[8] *SBBI Classic 2008*, p. 142, uses this measure as the risk-free rate for CAPM.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 5.1A** | | | | | | |
| 2 | **Regression of Decile Portfolios 1926–2008 [1]** | | | | | | |
| 3 | | | | | | | |
| 4 | **Summary Output: Return as a Function of Std Deviation of Returns 1926–2008** | | | | | | |
| 5 | | | | | | | |
| 6 | *Regression Statistics* | | | | | | |
| 7 | Multiple R | 98.93% | | | | | |
| 8 | R Square | 97.87% | | | | | |
| 9 | Adjusted R Square | 97.61% | | | | | |
| 10 | Standard Error | 0.40% | | | | | |
| 11 | Observations | 10 | | | | | |
| 12 | | | | | | | |
| 13 | ANOVA | | | | | | |
| 14 | | *df* | *SS* | *MS* | *F* | *Signif F* | |
| 15 | Regression | 1 | 0.58% | 0.58% | 368.3 | 0.00% | |
| 16 | Residual | 8 | 0.01% | 0.00% | | | |
| 17 | Total | 9 | 0.60% | | | | |
| 18 | | | | | | | |
| 19 | | *Coef* | *Std Err* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
| 20 | Intercept | 4.81% | 0.53% | 9.2 | 0.00% | 3.60% | 6.03% |
| 21 | 26-08 Std Dev | 33.49% | 1.75% | 19.2 | 0.00% | 29.47% | 37.52% |
| 22 | | | | | | | |
| 23 | **Summary Output: Regression of Return as a Function of ln Mkt Cap 1926–2008** | | | | | | |
| 24 | | | | | | | |
| 25 | *Regression Statistics* | | | | | | |
| 26 | Multiple R | 97.01% | | | | | |
| 27 | R Square | 94.12% | | | | | |
| 28 | Adjusted R Square | 93.38% | | | | | |
| 29 | Standard Error | 0.66% | | | | | |
| 30 | Observations | 10 | | | | | |
| 31 | | | | | | | |
| 32 | ANOVA | | | | | | |
| 33 | | *df* | *SS* | *MS* | *F* | *Signif F* | |
| 34 | Regression | 1 | 0.56% | 0.56% | 128.0 | 0.00% | |
| 35 | Residual | 8 | 0.04% | 0.00% | | | |
| 36 | Total | 9 | 0.60% | | | | |
| 37 | | | | | | | |
| 38 | | *Coef* | *Std Err* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
| 39 | Intercept | 43.78% | 2.59% | 16.9 | 0.00% | 37.81% | 49.75% |
| 40 | Ln Size | -1.37% | 0.12% | -11.3 | 0.00% | -1.65% | -1.09% |
| 41 | | | | | | | |
| 42 | [1] Derived from *SBBI—2009 Classic Yearbook,* p. 114, Table 7–5.* *Note*: The recent market cap | | | | | | |
| 43 | data in SBBI is too low by a factor of 1,000, according to Morningstar, Inc. According | | | | | | |
| 44 | to Morningstar, Inc., the column caption should have said "in Millions," not "in Thousands." | | | | | | |
| 45 | We have made the correction. Also note that the number of companies in the deciles and the | | | | | | |
| 46 | recent market cap data are as of 9/30/2008. | | | | | | |
| 47 | | | | | | | |
| 48 | * *Source:* Morningstar, Inc.—*2009 Ibbotson® Stocks, Bonds, Bills and Inflation (SBBI) Classic* | | | | | | |
| 49 | *Yearbook.* | | | | | | |

error of the $Y$-estimate is 0.70% (C38). As discussed in Chapters 3 and 10, we can form an approximate 95% confidence interval around the regression estimate by adding and subtracting two standard errors. Thus, we can be 95% confident that the regression forecast is approximately accurate to within plus or minus $2 \times 0.70\% \cong 1.4\%$.[9]

Figure 5.2 is a graph of arithmetic mean returns over the past 82 years (1926–2007) versus the natural log of FMV. As in Figure 5.1, the numbered nodes

[9] This is true near the mean value of our data. Uncertainty increases gradually as we move from the mean.




These are arithmetic mean returns for the CRSP Deciles. Data labels are decile numbers.
*Y* intercept is regression data, not actual
Regression #2: $r = 46.22\% - [1.284\% \times \ln(\text{FMV})]$

FIGURE 5.2 1926–2007 Arithmetic Mean Returns as a Function of ln(FMV)

are the actual data for each decile, while the straight line is the regression estimate. While Figure 5.1 shows that returns are positively related to risk, Figure 5.2 shows they are negatively related to size.

## Regression #3: Return versus Beta

The third regression in Table 5.1 shows the relationship between the decile returns and the decile betas for the period 1926–2007 (C50 through C59). According to the capital asset pricing model (CAPM) equation, the $y$-intercept should be the risk-free rate, and the $x$-coefficient should be the long-run equity premium of 7.05%.[10] Instead, the $y$-intercept at −4.33% (C50) is a country mile from the historical risk-free rate of 5.21%, as is the $x$-coefficient at 16.60% (C56) from the equity premium of 7.05%, demonstrating the inaccuracy of CAPM.

While the equation we obtain is contrary to the theoretical CAPM, it does constitute an empirical CAPM, which could be used for a firm whose capitalization is at least as large as a decile #10 firm. Merely select the appropriate decile, use the beta of that decile, possibly with some adjustment, and use regression equation #3 to generate a discount rate. While it is possible to do this, it is far better to use regression #2.

We now compare the log size model to CAPM. Columns K and N show the regression estimated return for each decile using both models—column K for CAPM and N for log size. We calculated the CAPM expected return as $r = R_F + (\beta \times \text{equity premium}) = 5.21\% + (\beta \times 7.05\%)$ (column K = G63 + (column J × G64)).

Columns L and M show the error and squared error for CAPM, whereas columns N and O contain the same information for the log size model. Note that the CAPM standard error of 2.61% (M20) is 375% (P21) larger than the log size standard error of 0.70% (P20). Also note that our "long-hand" calculation of the log size standard error of 0.70% in P20 equals Excel's calculation in B38.

[10]Derived from *SBBI Classic 2008*, p. 142.

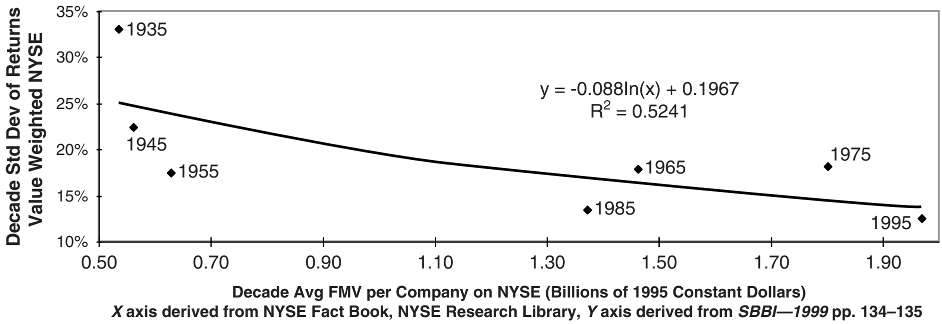



**FIGURE 5.3** Decade Standard Deviation of Returns vs. Avg. FMV per NYSE Company 1935–1995

## Table 5.1A: Regression Results through 2008

*SBBI—2009* published too late to use its results throughout this book. Thus, all other references in this book to the log size equation will cover only through 2007 data. However, we report the results here.

It is well known that the financial meltdown of 2008 produced terrible results in the markets. The decile #1 return for 2008 was −35% with a 19.5% standard deviation,[11] and the decile #10 return was −47% with a 45.0% standard deviation. The remaining decile results were in between those two extremes. Interestingly, the adjusted $R^2$ increased for both regressions. The 97.61% (B9) for the 1926–2008 data is an increase over the 97.04% adjusted $R^2$ for the 1926–2007 data in regression #1, and the 93.38% (B28) adjusted $R^2$ for the 1926–2008 data is an increase over the 93.02% adjusted $R^2$ for the 1926–2007 data in regression #2. Thus the terrible results in and of themselves do not affect the integrity of the regressions, because it is only the relative relationships of the results among the deciles that matter.

## Market Performance

Regression #1 shows that return is a linear function of risk, as measured by the standard deviation of returns. Regression #2 shows that return declines linearly with the logarithm of firm size. The logic behind this is that investors demand and receive higher returns for higher risk. Smaller firms have more volatile (risky) returns, so return is therefore negatively related to size.

Figure 5.3 shows the relationship between volatility and size, with the *y*-axis being the standard deviation of returns for the value-weighted NYSE and the *x*-axis being the average FMV per NYSE company in 1995 constant dollars in successive decades. The year adjacent to each data point is the final year of the decade; for example, 1935 encompasses 1926 to 1935. The decade average FMV (in 1995 constant dollars) has increased from slightly over \$0.5 billion to over \$1.9 billion. Therefore, we might predict from a theoretical standpoint that the standard deviation of returns should decline over time—and it seems to have done so.

[11]For 1926 through 2008.

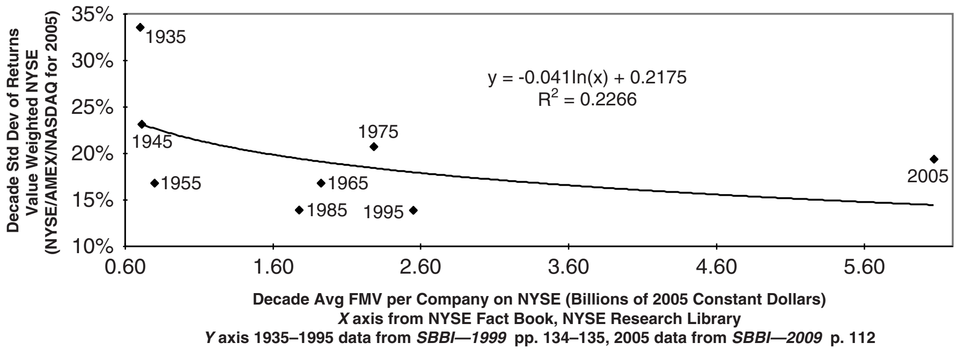



**FIGURE 5.3A** Decade Standard Deviation of Returns vs. Avg. FMV per NYSE Company 1935–2005

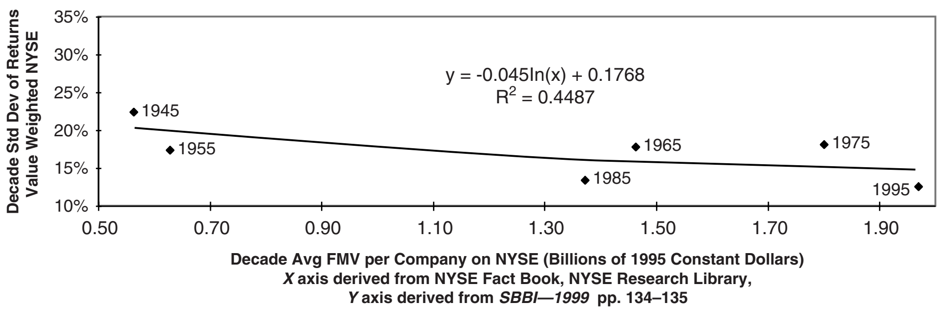



**FIGURE 5.4** Decade Standard Deviation of Returns vs. Avg. FMV per NYSE Company 1945–1995

As you can see, the standard deviation of returns per decade declines exponentially from about 33% for the decade ending in 1935 to 13% in the decade ending in 1995, for a range of 21%.[12] If we examine the major historical events that took place over time, the decade ending 1935 includes some of the Roaring Twenties and the Depression. It is no surprise that it has such a high standard deviation. Figure 5.4 is identical to Figure 5.3, except that we have eliminated the decade ending 1935 in Figure 5.4. Eliminating the most volatile decade flattens out the regression curve. The fitted curve in Figure 5.4 appears about half as steep as Figure 5.3. The standard deviation ranges from 13% to 22%, or a range of 10%,[13] versus the 21% range of Figure 5.3 and is much less curved.

However, the inclusion of the decade ending 2005 shows those relationships to be far less reliable, as we can see in Figures 5.3A and 5.4A. In Figure 5.3A, the $R^2$ (note that this is not adjusted $R^2$) declines from 52% to 23%—merely by adding the decade ending 2005.

*SBBI—2009 Valuation Yearbook* Graphs 5-10 and 5-11 also show a general decline in stock market volatility, which is consistent with the data above. Ibbotson observes that this may suggest that "we have moved into a new market regime in

[12]This is not 20% because the 33% and 13% reported are rounded numbers.
[13]This is not equal to 9%, due to rounding.

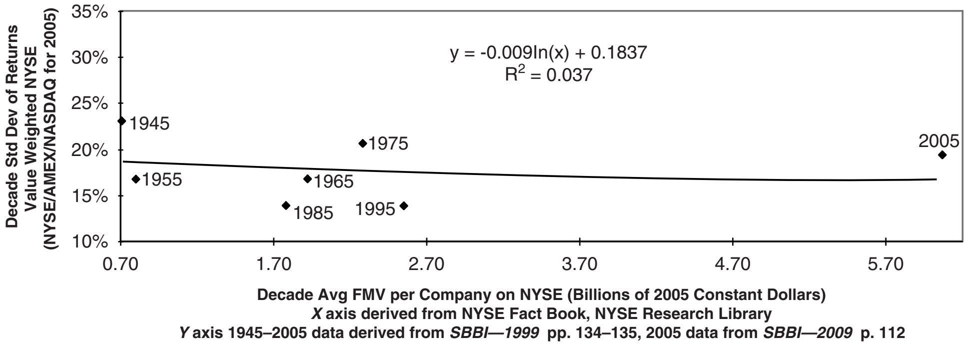



**FIGURE 5.4A** Decade Standard Deviation of Returns vs. Avg. FMV per NYSE Company 1945–2005

which stocks are less volatile and therefore require a lower risk premium than in the past." This is similar to our conclusion in the first edition of this book. However, Ibbotson continues to present arguments against this observation, which is consistent with our skepticism now.

Why the dramatic decline in the strength of the relationship? The data source for the decade ending 2005 is different. The standard deviation and size data through 1995 are NYSE only, while 1996–2005 standard deviation data also include AMEX and NASDAQ.[14] Additionally, according to the NYSE dataset for 1926–1996, the total market capitalization in 1996 was $7.3 trillion, while it was $9.2 trillion in the dataset for 1996–2005. Thus, the data are inconsistent, which reduces our ability to make inferences.

The relationship between volatility and size when viewing the market as a whole is somewhat loose, as the data points vary considerably from the fitted curve in Figure 5.3. The $R^2 = 52\%$ (45% in Figure 5.4). For the decade ending 1945, standard deviation of returns is about one-third lower than the previous decade (approximately 22% versus 33%), while average firm size is about the same. Standard deviation of returns dropped again in the decade ending 1955, with only a small increase in size. In the decade ending 1965, average firm size more than doubled in real terms, yet volatility was almost identical (we would have expected a decrease). In the decade ending 1975, firm size and volatility increased. In the decade ending 1985, both average firm size and volatility decreased significantly, which is counterintuitive, while in the final decade firm size increased from over $1.3 billion to almost $2 billion, while volatility decreased slightly.

Figure 5.5 shows the relationship of average NYSE firm return and time, with each data point being a decade.[15] The relationship is a very loose one, with $R^2 = 0.03$. However, adjusted $R^2$ is negative (not shown), and the relationship is statistically insignificant.

In summary, there appeared to be increasing efficiency of investment over time—something I described in the first edition of this book as "the same bang for less buck." The market as a whole seemed to deliver the same or better performance

[14]The 2005 FMV data, like the previous FMV data, is only for the NYSE.
[15]The 2005 data point is for NYSE/AMEX/NASDAQ.

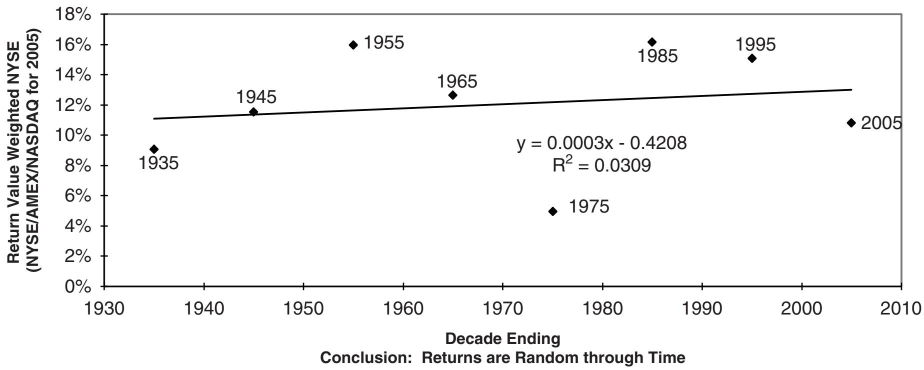



**FIGURE 5.5** Average Returns Each Decade

as measured by return experienced for risk undertaken. However, the relationship appears to have deteriorated by adding the decade ending 2005. With the Financial Crisis of 2008 it is possible that we will find the relationship will disappear, although this is difficult to say, because of the inconsistency in the data.

## Which Data to Choose?

With a total of 82 years of data on the NYSE/AMEX/NASDAQ, we must decide whether to use all of the data or some subset, and if so, which subset. In making this choice, we will consider the following sources of information:

1. Tables 5.1 through 5.2A, the statistical results of regression analyses of the different time periods of the U.S. stock markets
2. Eighteenth-century stock market returns
   a. A study (Harrison, 1998) that explores the distribution of European stock market returns
   b. Ibbotson and Brinson (1993) 201-year study
3. Ibbotson's opinion of outliers and the Financial Crisis of 2008
4. Figures 5.3–5.5
5. Academic articles on the *declining equity premium*

**TABLES 5.2 AND 5.2A: REGRESSION RESULTS FOR DIFFERENT TIME PERIODS** Nonstationary data require us to consider the possibility of removing some of the older stock market data. In Table 5.2, we repeat regressions #1 and #2 from Table 5.1 for the most recent 30, 40, 50, 60, 70, and 82 years of NYSE/AMEX/NASDAQ data. The upper table in each time period is regression #1 and the lower table is regression #2. For example, the data for regression #1 for the last 30 years appear in rows 6–8, 40 years in rows 15–17, and so on. Similarly, the data for regression #2 for 30 years appear in rows 10–12, 40 years in rows 19–21, and so on.

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | **Table 5.2** | | | | | |
| 2 | | | | **Regressions of Returns over Standard Deviation** | | | | | |
| 3 | | | | **and Log of Fair Market Value** | | | | | |
| 4 | | | | | | | | | |
| 5 | | | | **30 Year** | | | | | |
| 6 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 50.60% |
| 7 | Intercept | 11.98% | 1.27% | 9.46 | 0.00% | 9.06% | 14.89% | Adjusted R Square | 44.43% |
| 8 | Std Dev | 19.57% | 6.84% | 2.86 | 2.11% | 3.81% | 35.34% | Standard Error | 0.67% |
| 9 | | | | | | | | | |
| 10 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 84.01% |
| 11 | Intercept | 25.50% | 1.54% | 16.57 | 0.00% | 21.95% | 29.05% | Adjusted R Square | 82.02% |
| 12 | Ln(FMV) | -0.462% | 0.071% | -6.48 | 0.02% | -0.63% | -0.30% | Standard Error | 0.38% |
| 13 | | | | | | | | | |
| 14 | | | | **40 Year** | | | | | |
| 15 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 65.60% |
| 16 | Intercept | 8.81% | 1.29% | 6.85 | 0.01% | 5.84% | 11.77% | Adjusted R Square | 61.30% |
| 17 | Stdev | 22.72% | 5.82% | 3.91 | 0.45% | 9.31% | 36.14% | Standard Error | 0.72% |
| 18 | | | | | | | | | |
| 19 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 85.10% |
| 20 | Intercept | 26.61% | 1.91% | 13.94 | 0.00% | 22.21% | 31.01% | Adjusted R Square | 83.24% |
| 21 | Ln(FMV) | -0.597% | 0.088% | -6.76 | 0.01% | -0.80% | -0.39% | Standard Error | 0.47% |
| 22 | | | | | | | | | |
| 23 | | | | **50 Year** | | | | | |
| 24 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 83.66% |
| 25 | Intercept | 7.75% | 1.16% | 6.66 | 0.02% | 5.07% | 10.43% | Adjusted R Square | 81.61% |
| 26 | Std Dev | 32.42% | 5.07% | 6.40 | 0.02% | 20.74% | 44.11% | Standard Error | 0.75% |
| 27 | | | | | | | | | |
| 28 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 96.58% |
| 29 | Intercept | 35.89% | 1.39% | 25.80 | 0.00% | 32.68% | 39.10% | Adjusted R Square | 96.15% |
| 30 | Ln(FMV) | -0.967% | 0.064% | -15.03 | 0.00% | -1.12% | -0.82% | Standard Error | 0.34% |
| 31 | | | | | | | | | |
| 32 | | | | **60 Year** | | | | | |
| 33 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 84.91% |
| 34 | Intercept | 9.13% | 0.90% | 10.19 | 0.00% | 7.06% | 11.19% | Adjusted R Square | 83.03% |
| 35 | Std Dev | 26.72% | 3.98% | 6.71 | 0.02% | 17.54% | 35.90% | Standard Error | 0.56% |
| 36 | | | | | | | | | |
| 37 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 96.96% |
| 38 | Intercept | 31.13% | 1.01% | 30.77 | 0.00% | 28.79% | 33.46% | Adjusted R Square | 96.58% |
| 39 | Ln(FMV) | -0.747% | 0.047% | -15.97 | 0.00% | -0.85% | -0.64% | Standard Error | 0.25% |
| 40 | | | | | | | | | |
| 41 | | | | **70 Year** | | | | | |
| 42 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 93.04% |
| 43 | Intercept | 7.81% | 0.79% | 9.93 | 0.00% | 6.00% | 9.62% | Adjusted R Square | 92.17% |
| 44 | Std Dev | 34.00% | 3.29% | 10.34 | 0.00% | 26.42% | 41.58% | Standard Error | 0.59% |
| 45 | | | | | | | | | |
| 46 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 99.01% |
| 47 | Intercept | **41.06%** | 0.90% | 45.55 | 0.00% | 38.98% | 43.14% | Adjusted R Square | **98.88%** |
| 48 | Ln(FMV) | **-1.176%** | 0.042% | -28.22 | 0.00% | -1.27% | -1.08% | Standard Error | 0.22% |
| 49 | | | | | | | | | |
| 50 | | | | **82 Year** | | | | | |
| 51 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 97.37% |
| 52 | Intercept | 5.54% | 0.58% | 9.50 | 0.00% | 4.20% | 6.88% | Adjusted R Square | 97.04% |
| 53 | Std Dev | 33.76% | 1.96% | 17.20 | 0.00% | 29.23% | 38.29% | Standard Error | 0.45% |
| 54 | | | | | | | | | |
| 55 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | R Square | 93.80% |
| 56 | Intercept | 46.22% | 2.82% | 16.37 | 0.00% | 39.71% | 52.74% | Adjusted R Square | 93.02% |
| 57 | Ln(FMV) | -1.436% | 0.131% | -11.00 | 0.00% | -1.74% | -1.14% | Standard Error | 0.70% |

Table 5.2, rows 6–12 show regressions #1 and #2 using only the past 30 years of data (i.e., from 1969 to 1998.[16] Regression equation #1 for this period is: $r = 11.98\% + (19.57\% \times S)$ (B7, B8), and regression equation #2 is $r = 25.50\% - [0.462\% \times \ln(\text{FMV})]$ (B11 and B12).

Rows 51–53 repeat regression #1 for the same 82 years as Table 5.1. The *y*-intercept of 5.54% (B52) and the *x*-coefficient of 33.76% (B53) in Table 5.2 are

[16]The time sequence in Table 5.2 differs by two years from that in Figures 5.3 to 5.6. Whereas the latter show decades ending in 19X5 (e.g., 1945, 1955, etc.) and 2005, Table 5.2's terminal year is 2007.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 5.2A** | | | | |
| 2 | **Regression Comparison [1]** | | | | |
| 3 | | | | | |
| 4 | **Standard Errors** | | | | |
| 5 | Years | Regr #1 [2] | Regr #2 [3] | Total | Adj $R^2$ (Regr #2) [4] |
| 6 | 30 | 0.67% | 0.38% | 1.05% | 82.02% |
| 7 | 40 | 0.72% | 0.47% | 1.19% | 83.24% |
| 8 | 50 | 0.75% | 0.34% | 1.09% | 96.15% |
| 9 | 60 | 0.56% | 0.25% | 0.80% | 96.58% |
| 10 | 70 | 0.59% | 0.22% | 0.81% | 98.88% |
| 11 | 82 | 0.45% | 0.70% | 1.15% | 93.02% |
| 12 | | | | | |
| 13 | [1] Summary regression statistics from Table 5.2. | | | | |
| 14 | | | | | |
| 15 | [2] Table 5.2: I8, I17, ... | | | | |
| 16 | | | | | |
| 17 | [3] Table 5.2: I12, I21, ... | | | | |
| 18 | | | | | |
| 19 | [4] Table 5.2: I11, I20, ... | | | | |

identical to those appearing in Table 5.1 (C23 and C30, respectively). Rows 55–57 repeat regression #2 for the same period. Once again, the *y*-intercept in Table 5.2 of 46.22% (B56) and the coefficient of ln (FMV) of −1.436% (B57) match those found in Table 5.1 (C37 and C43, respectively).

Table 5.2A summarizes the key regression feedback from Table 5.2. For the six different time periods we consider, when looking at the combined standard errors for regression #1 and regression #2, both the 60-year and 70-year periods are statistically the winners. The standard error of the *y*-estimate using 60-year of data is 0.80% (D9), which is essentially the same as the 70-year standard error, 0.81% (D10). The standard error of the *y*-estimate of 1.15% (D11) using all 82 years of data is larger than these standard errors. After the 70-year standard error, the next-lowest standard error is 1.05% (D6) for 30 years of data.

Regression #2 is the more important regression for valuing privately held firms, and the 70-year standard error at 0.22% (C10) is the lowest among the six listed. The 70-year regression also has the highest adjusted $R^2$—98.88% (E10)—and it has a relatively low standard error for regression #1. Thus, it appears that the 70-year data is statistically the winner.

For regression #2, the 95% confidence intervals for the 70 years of data are smaller than they are for the other candidates. For regression #2 they are between 38.98% and 43.14% (Table 5.2, F47, G47) for the *y*-intercept—a range of 4.2%—and −1.27% to −1.08% (F48, G48) for the slope—a range of 0.19%. For 82 years of data, the range is 13% for the *y*-intercept (G56 minus F56) and 0.60% (G57 minus F57) for the slope, which is over three times larger than for the 70-year data. Thus, for the log size method, the past 70 years of data provide a tighter estimate of stock market returns than other time periods, as measured by the size of confidence intervals around the regression estimates for regression #2 and by the adjusted $R^2$.

**EIGHTEENTH-CENTURY STOCK MARKET RETURNS** Paul Harrison's article (Harrison, 1998) is a fascinating econometric study that is very advanced and extremely mathematical. The data for this study came primarily from biweekly Amsterdam stock prices published from July 1723 to December 1794 for the Dutch East India Company and a select group of English stocks that were traded in Amsterdam: the Bank of England, the English East India Company, and the South Sea Company. Harrison also examined stock prices from London spanning the eighteenth century.

Harrison found the shape of the distribution of stock price returns in the eighteenth and twentieth centuries to be very similar, although their means and standard deviations are different. The eighteenth-century returns were lower—but less volatile—than twentieth-century returns. He found the distributions to be symmetric, like a normal curve, but leptokurtic (fat tailed), which means there are more extreme events occurring than would be predicted by a normal curve. The same fundamental pattern exists in both 1725 and 1995.

Harrison remarks that clearly much has changed over the last 300 years, but, interestingly, such changes do not seem to matter in his analysis. He comments that the distribution of prices is not driven by information technology, regulatory oversight, or by the specialist—none of these existed in the eighteenth-century markets. However, what did exist in the eighteenth century bears resemblance to what exists today.

Harrison describes the following as some of the evidence for similarities in the market:

- Stock traders in the eighteenth century reacted to and affected market prices like traders today. They competed vigorously for information,[17] and the eighteenth-century markets followed a near-random walk—so much so that an entire pamphlet literature sprang up in the early eighteenth century lamenting the unpredictability of the market. Harrison said that unpredictability is a theoretical result of competition in the market.
- Eighteenth-century stock markets were informationally efficient, as shown econometrically by Neal (1990).
- The practices of eighteenth-century brokers were sophisticated. Investors early in the eighteenth century valued stocks according to their discounted stream of future dividends. Tables were published (such as Hayes, 1726) showing the appropriate discount for different interest rates and time horizons. Traders engaged in cash contracts, futures contracts, and options; they sold short, issued credit, and used "modern" investment strategies, such as forming portfolios, diversification, and hedging.

Another interesting source of very long-term stock market returns is Ibbotson and Brinson (1993).[18] The authors constructed a stock market total return going back

[17] A fascinating story that I remember from an economic history course is that Baron Rothschild, having placed men with carrier pigeons at the Battle of Waterloo, was the first nonparticipant to know the results of the battle. He first paid a visit to inform the King of the British victory, and then he proceeded to the stock market to make £100 million—many billions of dollars in today's money—a tidy sum for having insider information. He struck a blow for market efficiency. Even his method of making a fortune in the market that day is a paradigm of the extent of market efficiency then. He knew that he was being observed. He began selling, and others followed him in a panic. Later, he sent his employees to do a huge amount of buying anonymously. The markets were indeed efficient—at least they were by the end of the day!

[18] Quoted in *SBBI—2009 Valuation Edition*, page 71, footnote 5.

to 1790. Even with some uncertainty about the accuracy of the data before 1850, the real (inflation-adjusted) returns experienced by investors in three 50-year and one 51-year periods from 1790 to 1990 were statistically similar, and none of the periods differed from the 201-year average.

To all of the foregoing, I would add an observation by King Solomon, who said, "There is nothing new under the sun" (Ecclesiastes, 1:9). Also in keeping with the theme in our chapter, King Solomon became the inventor of portfolio theory when he wrote, "Divide your wealth into seven, even eight parts, for you cannot know what misfortune may occur on earth" (Ecclesiastes, 11:2).

**IBBOTSON'S OPINION OF OUTLIERS AND THE FINANCIAL CRISIS OF 2008** Ibbotson's opinion[19] is that over the very long run there are very few events that are truly outliers. The Financial Crisis of 2008 and Paul Harrison's and Ibbotson and Brinson's research seem to corroborate this. It is in the nature of the stock market for there to be periodic booms and crashes, indicating that we should use all 82 years of the U.S. stock market data.

**IS THE EQUITY PREMIUM DECLINING?** Fama and French (2002) forecast that future returns will be lower than historical returns by approximately 4%. Lettau, Ludvigson, and Wachter (2008)[20] find that long-term decline in macroeconomic risk (standard deviation in growth of nondurables and services and personal consumption expenditures) accounts for a significant portion of the decline in rates of return. They forecast a 2% decline.

Ibbotson and Chen (2003) use a supply-side model (a.k.a. *supply model*) to forecast future returns. The supply-side model is based on the idea that the productivity of corporations in the real economy generates the supply of stock market returns, and investors should expect that equity returns should be close to the long-run supply estimate.

*SBBI—2008 Valuation Yearbook* Graphs 5.13 and 5.14 update that research. Ibbotson and Chen broke historical returns into five components. Additionally, we show the *SBBI—2009* results in parentheses.

1. CPI inflation of 3.05% (3.01% for *SBBI—2009*).
2. Growth in real earnings per share of 2.14% (1.58% for *SBBI—2009*). Note the large decline for the effects of the Financial Crisis of 2008.
3. Income returns of 4.18% (4.15% for *SBBI—2009*).
4. The reinvestment return of 0.23% (0.20% for *SBBI—2009*).
5. The historical annual PE growth factor using three-year earnings of 0.67% (0.60% for *SBBI—2009*).

The result is a geometric supply of equity returns of $[(1 + 3.05\%) \times (1 + 2.14\%) - 1] + 4.18\% + 0.23\% = 9.7\%$ geometric average forecast returns (9.0% for *SBBI—*

[19] *SBBI—2009 Classic Edition*, p. 29, and *SBBI—2009 Valuation Edition*, p. 61.

[20] Even though this published in 2008, its copyright is 2007, and was written earlier. It appears that the data used by the authors stops at about the year 2002, with the majority of it being before 2000. Thus, it predates the Financial Crisis of 2008.

*2009*).[21] The only component of historical returns excluded from forecast returns is the 0.67%[22] historical annual geometric growth in the PE ratio. In Table 5.3, B32, we estimate the arithmetic annual growth at 0.80%.[23] There is no reason to forecast that it will grow in the future, as today's PE is the market's forecast of the future. Thus, Ibbotson's supply-side model implies a decline in future arithmetic returns of 0.80%. Of the various articles on the declining equity premium, I found Ibbotson and Chen's the most compelling. It has stronger theoretical appeal than Fama and French and is far simpler and more compelling than Lettau et al. Thus my conclusion is that it is appropriate to subtract 0.80% from our discount rate calculations, which we do later in Tables 5.3 and the 5.4 series.

**CONCLUSION ON DATASET** After our extensive review of the data and the academic literature, we need to conclude as to which time period is the best for our discount rate calculations. The statistical evidence in our analysis of Tables 5.1 through 5.2A and the declining volatility of returns in Figure 5.3 points to continuing with the decision we made in the first edition of this book, which is to eliminate 1926–1937 data, as the 70-year results appear the most pristine. However, the Financial Crisis of 2008 has rocked the country, and it now seems much more questionable and probably inappropriate to eliminate the volatility of the Roaring Twenties and the Great Depression.

In a personal conversation in 1998, Paul Harrison said that even with 300 years of history showing similarity in the distribution of returns, he would be inclined to label the years in question as an outlier that should probably be excluded from the regression. However, that was before September 11, 2001, and the Financial Crisis of 2008. It is a difficult call to make, but since the publication of the first edition of this book, the world at large and the financial world are more volatile. Harrison's findings of leptokurtic stock returns 300 years ago is a timely reminder now that great and terrible times are a part of the market and more so than a normal distribution would lead us to believe. Thus, while we eliminated the years 1926–1937 from the final regression in the first edition, we leave them in now and use Ibbotson's full dataset.

## Application of the Log Size Model

Equation (5.2) is the most appropriate for calculating current discount rates and will be used for the remainder of the book. In the next sections, we will use it to calculate discount rates for various firm sizes and demonstrate its use in a simplified discounted cash flow analysis.

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[21]This formula works only for geometric returns.

[22]*SBBI—2008 Valuation Yearbook*, p. 95.

[23]This calculation is an estimate based on partial data. It would have been ideal to have the geometric average annual PE growth for each decile, but the data are unavailable. Furthermore, it is likely that the arithmetic increase in PE is larger for the small firms. For simplicity, we make one single subtraction that applies to all deciles. However, it would be more accurate to calculate the ratio of arithmetic to geometric mean returns for each decile, multiply by 0.67%, subtract that amount from each decile's arithmetic mean return, and rerun the regression in Table 5.1.

| | A | B | C |
|---|---|---|---|
| 1 | **Table 5.3** | | |
| 2 | **Table of Discount Rates Based** | | |
| 3 | **on FMV: 1926–2007 SBBI Data** | | |
| 4 | | | |
| 5 | **Regression Results** | **Implied Historical** | **Implied Discount** |
| 6 | **Mktable Min FMV** | **Arithm Return [1]** | **Rate (*R*) [2]** |
| 7 | **$10,000,000,000** | **13.2%** | **12.4%** |
| 8 | **$1,000,000,000** | **16.5%** | **15.7%** |
| 9 | **$100,000,000** | **19.8%** | **19.0%** |
| 10 | **$50,000,000** | **20.8%** | **20.0%** |
| 11 | **$10,000,000** | **23.1%** | **22.3%** |
| 12 | **$5,000,000** | **24.1%** | **23.3%** |
| 13 | **$3,000,000** | **24.8%** | **24.0%** |
| 14 | **$1,000,000** | **26.4%** | **25.6%** |
| 15 | **$750,000** | **26.8%** | **26.0%** |
| 16 | **$500,000** | **27.4%** | **26.6%** |
| 17 | **$400,000** | **27.7%** | **26.9%** |
| 18 | **$300,000** | **28.1%** | **27.3%** |
| 19 | **$200,000** | **28.7%** | **27.9%** |
| 20 | **$150,000** | **29.1%** | **28.3%** |
| 21 | **$100,000** | **29.7%** | **28.9%** |
| 22 | **$50,000** | **30.7%** | **29.9%** |
| 23 | **$30,000** | **31.4%** | **30.6%** |
| 24 | **$10,000** | **33.0%** | **32.2%** |
| 25 | **$1,000** | **36.3%** | **35.5%** |
| 26 | **$1** | **46.2%** | **45.4%** |
| 27 | | | |
| 28 | Geometric Avg Annual Growth in PE [3] | **0.67%** | |
| 29 | Arithmetic Mean Returns—Value Wtd Index [4] | **12.0%** | |
| 30 | Geometric Mean Returns—Value Wtd Index [4] | **10.1%** | |
| 31 | Ratio of Arithmetic-to-Geometric Returns (B29/B30) | **1.188** | |
| 32 | Estimated Arithmetic Mean Growth in PE (B28 × B31) [5] | **0.80%** | |
| 33 | | | |
| 34 | [1] Based on constant and *x*-coefficient from Table 5.1, C39 and C45. | | |
| 35 | | | |
| 36 | [2] We subtract the estimated arithmetic average increase in PE of 0.80% in B32. | | |
| 37 | | | |
| 38 | [3] *SBBI—2008 Valuation Yearbook,* p. 95.* | | |
| 39 | | | |
| 40 | [4] *SBBI—2008 Classic Yearbook,* p. 130.** | | |
| 41 | | | |
| 42 | [5] This calculation is an estimate based on partial data. It would have been ideal to have the geometric | | |
| 43 | average annual PE growth for each decile, but the data are unavailable. Furthermore, it is likely that | | |
| 44 | the arithmetic increase in PE is larger for the small firms. For simplicity, we make one single | | |
| 45 | subtraction that applies to all deciles. However, it would be more accurate to calculate the ratio of | | |
| 46 | arithmetic to geometric mean returns for each decile, multiply by 0.67%, subtract that amount from | | |
| 47 | each decile's arithmetic mean return, and rerun the regression in Table 5.1. | | |
| 48 | | | |
| 49 | * *Source:* Morningstar, Inc.—*2008 Ibbotson® Stocks, Bonds, Bills and Inflation (SBBI) Valuation* | | |
| 50 | *Yearbook.* | | |
| 51 | | | |
| 52 | ** *Source:* Morningstar, Inc.—*2008 Ibbotson® Stocks, Bonds, Bills and Inflation (SBBI) Classic* | | |
| 53 | *Yearbook.* | | |

## Discount Rates Based on the Log Size Model

Table 5.3 shows the implied historical arithmetic rates of return in column B and the forecast equity discount rates in column C for firms of various sizes using the log size model equation (5.2). The difference in the two columns is that we subtract our estimate of the arithmetic mean growth in the PE ratio of 0.80% (B32) from each number in column B to calculate column C. For example, 13.2% − 0.8% = 12.4% (B7 − B32 = C7). This is the portion of historical returns that we do not expect to repeat in the future, per our discussion of Ibbotson's supply-side model.

Our calculation of the 0.80% begins with the geometric average annual growth in the PE multiple from 1926 to 2007 of 0.67% (B28). Unfortunately, Ibbotson does not provide the arithmetic mean annual increase in the PE multiple, so we must estimate it. The arithmetic mean return for the Value-Weighted Index of NYSE/AMEX/NASDAQ is 12.0% (B29), and the geometric mean return is 10.1% (B30). The ratio of the two is 1.188 (B31). When we multiply that by the geometric average annual growth in the PE multiple of 0.67% (B28), we get an estimated arithmetic mean annual growth in the PE multiple of 0.80% (B32). We subtract this number from each entry in column B to calculate column C.

The following discussion is based on column B, but ultimately we use column C as our final discount rate. The logic behind this is that if we expect the future to correspond exactly to the past, then column B would be our table of discount rates. However, our historical rates of return in column B contain a component measuring 0.80% that we do not expect to repeat in the future. Therefore, the final table of discount rates is column C. In the meantime, however, we proceed to explain column B.

The implied (i.e., regression calculated) historical arithmetic rate of return for a $10 billion firm is 13.2% (B7), and for a $3 million firm it is 24.8% (B13), based on 82-year arithmetic average market returns for deciles #1 to #10. The smallest firm in decile #10b is $2 million in market capitalization,[24] which interpolates to an implied discount rate of 25.6% as the average of B13 and B14. While those values and all values in between are interpolations based on the model, the discount rates for firm values below $2 million are extrapolations, as they lie outside the original dataset.

Using equation (5.2), the Excel formula for B7 is: = 0.4622 − (0.01436 * ln(A7)). In Lotus 123, the formula would be: + 0.4622 − (0.01436 * @ ln(A7)).

Regression #2 (equation (5.2)) tells us that the discount rate is a constant minus another constant multiplied by ln (FMV). Since ln (FMV) has a characteristically upward-sloping shape, as seen in Figure 5.6, subtracting a curve of that shape from a constant leads to a discount rate function that is a mirror image of Figure 5.6. Figure 5.7 is the graph of that relationship, and the reader can see that the result is a downward-sloping curve. Again, this curve depicts the rate of return, that is, the discount rate, as a function of the absolute dollar value of the firm. Note that this is not on a log scale. Since the regression equation is $r = 46.22\% - [1.436\% \times \ln(FMV)]$, we begin at the extreme left with a return of 46.22% for a firm worth $1 and subtract the fraction of the ln FMV dictated by the equation.

[24] *SBBI—2008 Valuation Yearbook*, back page.

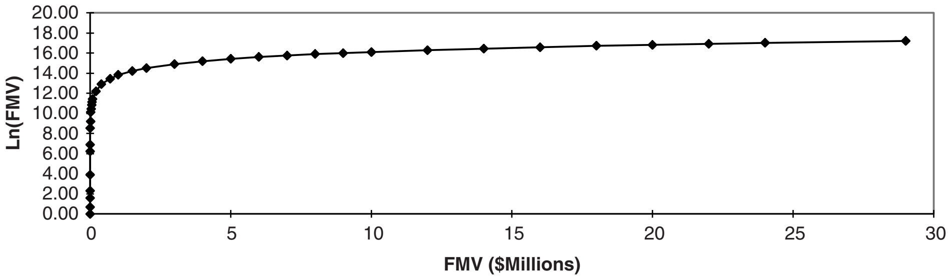
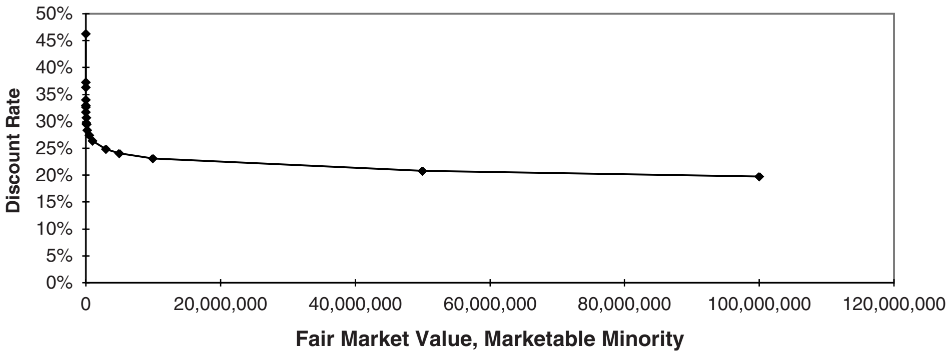



FIGURE 5.6 Natural Logarithm




**[1] This is the same as Fig. 5-2, with the *x*-axis being FMV rather than ln(FMV).**
**[2] For scaling reasons, we eliminate values above $100 million.**

FIGURE 5.7 Discount Rates as a Function of FMV

An important property of logarithms is that $\ln xy = \ln x + \ln y$.[25] Since regression equation #2 has the form $r = a + b \ln \text{FMV}$, where $a = 0.4622$ and $b = -0.01436$, we can ask how the discount rate varies with differing orders of magnitude in value. First, however, we will work through some general equations where we vary the value of the firm by a factor of $K$.

Let:

$r_1$ = the discount rate for Firm #1, whose value = $\text{FMV}_1$.
$r_2$ = the discount rate for Firm #2, whose value = $\text{FMV}_2 = K\ \text{FMV}_1$.

$$r_1 = a + b \ln \text{FMV}_1 \quad \text{Regression equation \#2 applied to Firm \#1.} \tag{5.3}$$

$$r_2 = a + b \ln(K\ \text{FMV}_1) \quad \text{Regression equation \#2 applied to Firm \#2.} \tag{5.4}$$

$$r_2 = a + b[\ln K + \ln \text{FMV}_1]. \tag{5.5}$$

$$r_2 = a + b \ln \text{FMV}_1 + b \ln K. \tag{5.6}$$

$$r_2 = r_1 + b \ln K. \tag{5.7}$$

In words, the discount rate of a firm $K$ times larger (smaller) than Firm #1 is always $|b| \ln K$ smaller (larger) than $r_1$.

---

[25] That is because $e^x \times e^y = e^{x+y}$. Taking logs of both sides of that equation is the proof.

Let's illustrate the nature of this relationship with some specific examples. First, let's examine what happens with orders of magnitude of 10. Ln 10 = 2.302585, so $b \times \ln 10 = -0.01436 \times 2.302585 = -3.3\%$. This means that if Firm #2 is 10 times larger (smaller) than Firm #1, its discount rate should be 3.3% lower (higher) than the Firm #1 discount rate. We can see this result in Table 5.3. The $10 billion firm has a discount rate of 13.2%, while the $1 billion firm has a discount rate of 16.5%, which is 3.3% higher, as it should be. The $100 million firm has a discount rate of 19.8%, which is 3.3% higher than the $1 billion firm. Because of the mathematical properties of logarithms, the same *percentage* change in FMV will always result in the same *absolute* change in the discount rate. This phenomenon is also seen in graphs containing log scales. Equal distances on a log scale are equal percentage changes, not absolute changes.

Let's try one more useful calculation—an order of magnitude 2. Ln 2 = 0.6931, so that $b \times \ln K = -0.01436 \times 0.6931 = -1\%$. Doubling (halving) the value of the firm reduces (increases) the discount rate by 1%. You can see that in going from a $100 million firm to a $50 million firm; the discount rate increased from 19.8% to 20.8%, a 1% difference (see Table 5.3, B9 and B10).

Now it is possible to construct your own table. All you need to know is your starting FMV and discount rate. The rest follows easily from the previous formulas. Also, we can easily interpolate the table. Suppose you wanted to know the discount rate for a $25 million firm. Simply start with the $50 million firm, where $r = 20.8\%$, and add 1% = 21.8%.

**NEED FOR ANNUAL UPDATING** It is important to update Tables 5.1 through 5.3 annually, as new market data become available. Additionally, it is important to be careful to match the regression equation to the year of the valuation. If the valuation assignment is retroactive and the valuation date is 2004, then one should use a regression equation for 1926–2004.

**COMPUTATION OF THE DISCOUNT RATE IS AN ITERATIVE PROCESS** In spite of the straightforwardness of these relationships, we have a problem of circular reasoning when it comes to computing the discount rate. We need FMV to obtain the discount rate, which is in turn used to discount cash flows or income to calculate the FMV! Hence, it is necessary to make sure that our initial estimate of FMV is consistent with the final result. If it is not, then we have to use the calculated FMV from the end of iteration #1 as our new assumed FMV in iteration #2. Using either equation (5.2) or Table 5.3 implies a new discount rate, which we use to value the firm. We keep repeating the process until the results are consistent.

It is extremely rare to require more than two iterations to achieve consistency in the ex ante and ex post values. The reason is that even if we guess the value of the firm incorrectly by a factor of 10, we will be only 3.3% off in our discount rate. By the time we come to the second iteration, we usually are consistent. The reason behind this is that the discount rate is based on the logarithm of the value. As we saw earlier, there is not much difference between the log of $10 billion and the log of $10 million, and multiplying that by the $x$-coefficient of −0.01436 further reduces the effects of an initial incorrect estimate of value. This is a convergent system 99% of the time with any kind of reasonable initial guess of value and even most unreasonable guesses.

The need for iteration arises because of the mathematical properties of the equations we use in valuing a firm. The simplest type of valuation is that of a firm with constant growth to perpetuity, where we simply apply the Gordon growth model ("Gordon model") to our forecast of cash flow for the coming year. For simplicity, we will use the end-of-year Gordon model formula, although it is not as accurate as the midyear formula, and we ignore the subtraction of the arithmetic average annual increase in the PE ratio.

We use the following definitions:

$CF$ = cash flow (available to equity) in year $t + 1$ (the first forecast year).
$a$ = 0.4622, the regression constant from regression #2.
$b$ = −0.01436, the $x$-coefficient from regression #2.
$V$ = fair market value (FMV) of the firm.
$r$ = the discount rate.

Using the Gordon model and ignoring valuation discounts and premiums, the FMV of the firm is:

$$V = \frac{CF}{r-g}. \tag{5.8}$$

Per equation (5.3), our log size equation for the discount rate is:

$$r = a + b \ln V. \tag{5.9}$$

Substituting equation (5.9) into (5.8), we get:

$$V = \frac{CF}{a + b \ln V - g}. \tag{5.10}$$

Equation (5.10) is a transcendental equation with no analytic solution.[26] Therefore, successive approximation is the only method of determining an answer. The simple iterative procedure in Tables 5.4A, 5.4B, and 5.4C is very easy to use and works in almost all situations.

## Practical Illustration of the Log Size Model: Discounted Cash Flow Valuations

Let's illustrate how the iterative process works with a specific example. The assumptions, formulas, and method in Tables 5.4A, 5.4B, and 5.4C are identical, except for the discount rate. Table 5.4A is a very simple discounted cash flow (DCF) analysis of a hypothetical firm. The basic assumptions appear in B30 through B35. We assume the firm had \$100,000 (B30) cash flow in 2007. We forecast annual growth rates in row 31, which we use in our calculations in row 5, and long-term growth at 4% (B33), which we use in our 2013 cash flow forecast in B10 and in calculating the Gordon model multiple in B11.[27] In B32 we make an initial and intentionally incorrect guess of a 23% discount rate.

[26] I thank my friend William Scott, Jr., a physicist, for the terminology and the definitive word that there is no analytic solution.
[27] Note that the formulas in C10 and C11 use $g$, which is the growth rate to perpetuity.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 5.4A** | | | | | | |
| 2 | **DCF Analysis Using 1926–2007 Regression Data—1st Iteration** | | | | | | |
| 3 | | | | | | | |
| 4 | **Year** | **2008** | **2009** | **2010** | **2011** | **2012** | **Total** |
| 5 | Forecast Cash Flow | $112,000 | $123,200 | $134,288 | $145,031 | $155,183 | |
| 6 | Present Value Factor | 0.9017 | 0.7331 | 0.5960 | 0.4845 | 0.3939 | |
| 7 | PV of Cash Flow | **$100,987** | **$90,314** | **$80,034** | **$70,274** | **$61,132** | **$402,741** |
| 8 | | | | | | | |
| 9 | **Calculation of Fair Market Value:** | **Amount** | **Formula** | | | | |
| 10 | Forecast Cash Flow 2013 | $161,391 | (1 + g) × F5 | | | | |
| 11 | Gordon Model Multiple | 5.8371 | SQRT (1 + *r*) / (*r* – *g*) | | | | |
| 12 | PV 2013-Infinity as of 1/1/2013 | **$942,057** | B10 × B11 | | | | |
| 13 | Present Value Factor—5 Years | 0.3552 | $1/(1 + r)^5$ | | | | |
| 14 | PV 2013-Infinity as of 1/1/2008 | **$334,620** | B12 × B13 | | | | |
| 15 | Add PV of 2008-2013 Cash Flow | 402,741 | G7 | | | | |
| 16 | FMV—Marketable Minority | **$737,360** | B14 + B15 | | | | |
| 17 | Control Premium | 294,944 | B16 × B34 | | | | |
| 18 | FMV—Marketable Control Interest | **1,032,305** | B16 + B17 | | | | |
| 19 | Disc—Lack of Marketability | (361,307) | B18 × B35 | | | | |
| 20 | Fair Market Value—Illiquid Control | **$670,998** | B18 + B19 | | | | |
| 21 | **Calc of Disc Rate—Regr Eq #2** | | | | | | |
| 22 | Ln (FMV—Marketable Minority) | 13.5108 | Ln(B16) | | | | |
| 23 | X-Coefficient (Table 5.1, C43) | -0.01436 | | | | | |
| 24 | Product | **-0.1941** | B22 × B23 | | | | |
| 25 | Constant (Table 5.1, C37) | 0.4622 | Constant—Regression #2 | | | | |
| 26 | – Annual Incr PE Ratio (5.3, B32) | -0.0080 | | | | | |
| 27 | Discount Rate (Rounded) [1] | **26%** | Sum of B24 to B26 | | | | |
| 28 | | | | | | | |
| 29 | **Assumptions:** | | | | | | |
| 30 | Base Adjusted Cash Flow | $100,000 | | | | | |
| 31 | Growth Rate in Adj Cash Flow | 12% | 10% | 9% | 8% | 7% | |
| 32 | Discount Rate = *r* | **23%** | | | | | |
| 33 | Growth Rate to Perpetuity = *g* | 4% | | | | | |
| 34 | Control Premium | 40% | | | | | |
| 35 | Discount—Lack of Marketability | 35% | | | | | |
| 37 | [1] As the assumed and the calculated discount rates (B32 and B27) are unequal, we must run an additional | | | | | | |
| 38 | iteration. | | | | | | |

The DCF analysis in rows 5 through 7 is standard and requires little explanation. The present value factors are midyear, and the value in B16 is a marketable minority interest.[28] It is this value, $737,360, which we use to compare the consistency between the assumed discount rate (the ex ante discount rate) of 23% (B32) and the calculated discount rate (the ex post discount rate) according to the log size model. The reason for this is that our regression results in Tables 5.1 and 5.1A are based on market data of publicly held (marketable) minority interests. To remain consistent in our comparison of the assumed and the calculated log size discount rate, we must use the marketable minority FMV for our calculation in B22 through B27.

We begin calculating the discount rate using the log size model in B22, where we compute ln (737,360) = 13.5108. This is the natural log of the initially computed marketable minority value of the firm. We repeat the *x*-coefficient of −0.01436 from Table 5.1, C43 in B23 and multiply B22 × B23 to calculate the product of −0.1941 in B24. To that we add the regression constant of 0.4622 (B25, transferred from Table 5.1, C37) and subtract the annual increase in the PE ratio of 0.8% (B26) from Ibbotson's supply-side model to obtain an implied (ex post) discount rate of 26% (rounded, B27).

Comparing the two discount rates—assumed and calculated—reveals that we initially assumed the discount rate too low, which means that we overvalued the firm.

[28] See Chapter 8 for explanation of the levels of value and valuation discounts and premiums.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 5.4B** | | | | | | |
| 2 | **DCF Analysis Using 1926–2007 Regression Data—2nd Iteration** | | | | | | |
| 3 | | | | | | | |
| 4 | **Year** | **2008** | **2009** | **2010** | **2011** | **2012** | **Total** |
| 5 | Forecast Cash Flow | $112,000 | $123,200 | $134,288 | $145,031 | $155,183 | |
| 6 | Present Value Factor | 0.8909 | 0.7070 | 0.5611 | 0.4454 | 0.3535 | |
| 7 | PV of Cash Flow | **$99,778** | **$87,107** | **$75,355** | **$64,590** | **$54,850** | **$381,680** |
| 8 | | | | | | | |
| 9 | **Calculation of Fair Market Value:** | **Amount** | **Formula** | | | | |
| 10 | Forecast Cash Flow | $161,391 | (1 + g) × F5 | | | | |
| 11 | Gordon Model Multiple | 5.1023 | SQRT (1 + *r*) / (*r* – *g*) | | | | |
| 12 | PV 2013-Infinity as of 1/1/2013 | **$823,457** | B10 × B11 | | | | |
| 13 | Present Value Factor—5 Years | 0.3149 | $1/(1 + r)^5$ | | | | |
| 14 | PV 2013-Infinity as of 1/1/2008 | **$259,291** | B12 × B13 | | | | |
| 15 | Add PV of 2008-2013 Cash Flow | 381,680 | G7 | | | | |
| 16 | FMV—Marketable Minority | **$640,971** | B14 + B15 | | | | |
| 17 | Control Premium | 256,388 | B16 × B34 | | | | |
| 18 | FMV—Marketable Control Interest | **897,359** | B16 + B17 | | | | |
| 19 | Disc—Lack of Marketability | (314,076) | B18 × B35 | | | | |
| 20 | Fair Market Value—Illiquid Control | **$583,284** | B18 + B19 | | | | |
| 21 | **Calc of Disc Rate—Regr Eq. #2** | | | | | | |
| 22 | Ln (FMV—Marketable Minority) | 13.3707 | Ln(B16) | | | | |
| 23 | X-Coefficient (Table 5.1, C43) | -0.01436 | | | | | |
| 24 | Product | **-0.1921** | B22 × B23 | | | | |
| 25 | Constant (Table 5.1, C37) | 0.4622 | Constant—Regression #2 | | | | |
| 26 | – Annual Incr PE Ratio (5.3, B32) | -0.0080 | | | | | |
| 27 | Discount Rate (Rounded) [1] | **26%** | Sum of B24 to B26 | | | | |
| 28 | | | | | | | |
| 29 | **Assumptions:** | | | | | | |
| 30 | Base Adjusted Cash Flow | $100,000 | | | | | |
| 31 | Growth Rate in Adj Cash Flow | 12% | 10% | 9% | 8% | 7% | |
| 32 | Discount Rate = *r* | **26%** | | | | | |
| 33 | Growth Rate to Perpetuity = *g* | 4% | | | | | |
| 34 | Control Premium | 40% | | | | | |
| 35 | Discount—Lack of Marketability | 35% | | | | | |
| 36 | | | | | | | |
| 37 | [1] As the assumed and the calculated discount rates (B32 and B27) are equal, we are consistent and do not | | | | | | |
| 38 | need an additional iteration unless we wish to add a company specific adjustment to the discount rate. | | | | | | |

We will correct that problem in Table 5.4B. In the meantime, though, we continue describing the remaining cells in the spreadsheet.

B17 through B19 contain the control premium and discount for lack of marketability, which we assume at 40% (B34) and 35% (B35), respectively. These are simple assumptions with no intent to be as realistic as possible, as we cover these topics in depth in Chapter 8. Because the assumed and calculated discount rates are not yet consistent, we ignore the specific numerical results, as they are irrelevant.

**THE SECOND ITERATION: TABLE 5.4B** We revise our discount rate to 26% (B32), which was our calculated discount rate in Table 5.4A, B27. In this case, we arrive at a marketable minority FMV of $640,971 (B16). When we perform the discount rate calculation with this value in B22 through B27, we obtain a matching discount rate of 26%, indicating that no further iterations are necessary.

**CONSISTENCY IN LEVELS OF VALUE** In calculating discount rates, it is important to be consistent in the level of fair market value that we are using. Since the log size model is based on returns from the NYSE/AMEX/NASDAQ, the corresponding values generated are on a marketable minority basis. Consequently, it is this level of value that we should use for the discount rate calculations.

Frequently, however, the marketable minority value is not the ultimate level of fair market value that we are calculating. Therefore, it is crucial to be aware

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 5.4C** | | | | | | |
| 2 | **DCF Analysis Using 1926–2007 Regression Data—3rd Iteration** | | | | | | |
| 3 | | | | | | | |
| 4 | **Year** | **2008** | **2009** | **2010** | **2011** | **2012** | **Total** |
| 5 | Forecast Cash Flow | $112,000 | $123,200 | $134,288 | $145,031 | $155,183 | |
| 6 | Present Value Factor | 0.8839 | 0.6905 | 0.5395 | 0.4215 | 0.3293 | |
| 7 | PV of Cash Flow | **$98,995** | **$85,074** | **$72,446** | **$61,126** | **$51,098** | **$368,738** |
| 8 | | | | | | | |
| 9 | **Calculation of Fair Market Value:** | **Amount** | **Formula** | | | | |
| 10 | Forecast Cash Flow | $161,391 | (1 + g) × F5 | | | | |
| 11 | Gordon Model Multiple | 4.7140 | SQRT (1 + *r*) / (*r* – *g*) | | | | |
| 12 | PV 2013-Infinity as of 1/1/2013 | **$760,802** | B10 × B11 | | | | |
| 13 | Present Value Factor—5 Years | 0.2910 | $1/(1 + r)^5$ | | | | |
| 14 | PV 2013-Infinity as of 1/1/2008 | **$221,423** | B12 × B13 | | | | |
| 15 | Add PV of 2008-2013 Cash Flow | 368,738 | G7 | | | | |
| 16 | FMV—Marketable Minority | **$590,161** | B14 + B15 | | | | |
| 17 | Control Premium | 236,064 | B16 × B34 | | | | |
| 18 | FMV—Marketable Control Interest | **826,225** | B16 + B17 | | | | |
| 19 | Disc—Lack of Marketability | (289,179) | B18 × B35 | | | | |
| 20 | Fair Market Value—Illiquid Control | **$537,046** | B18 + B19 | | | | |
| 21 | **Calc of Disc Rate-Regr Eq #2** | | | | | | |
| 22 | NA | NA | NA, as we achieved consistency in Table 5.4B | | | | |
| 23 | | | | | | | |
| 24 | | | | | | | |
| 25 | | | | | | | |
| 26 | | | | | | | |
| 27 | | | | | | | |
| 28 | | | | | | | |
| 29 | **Assumptions:** | | | | | | |
| 30 | Base Adjusted Cash Flow | $100,000 | | | | | |
| 31 | Growth Rate in Adj Cash Flow | 12% | 10% | 9% | 8% | 7% | |
| 32 | Discount Rate = *r* [1] | 28% | | | | | |
| 33 | Growth Rate to Perpetuity = *g* | 4% | | | | | |
| 34 | Control Premium | 40% | | | | | |
| 35 | Discount—Lack of Marketability | 35% | | | | | |
| 36 | | | | | | | |
| 37 | [1] We achieved consistency between the assumed and the calculated discount rates in Table 5.4B and add a | | | | | | |
| 38 | 2% company-specific premium to the discount rate. | | | | | | |

of the differing levels of FMV that occur as a result of valuation adjustments. For example, if our valuation assignment is to calculate an illiquid control interest, we will add a control premium and subtract a discount for lack of marketability from the marketable minority value.[29] Nevertheless, we use only the marketable minority level of FMV in iterating to the proper discount rate.

**TABLE 5.4C: ADDING COMPANY-SPECIFIC ADJUSTMENTS TO THE DCF ANALYSIS** The final step in our DCF analysis is performing company-specific adjustments. Let's suppose for illustrative purposes that there is only one owner of this firm. She is 62 years old and had a heart attack three years ago. The success of the firm depends to a great extent on her personal relationships with customers, which may not be easily duplicated by a new owner. Therefore, we decide to add a 2% company-specific adjustment to the 26% discount rate from Table 5.4B to reflect this situation,[30] which leads us to a 28% (B32) discount rate.

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[29]Not all authorities would agree with this statement. There is considerable disagreement on the levels of value. We cover those controversies in Chapter 8.

[30]A different approach would be to take a discount from the final value, which would be consistent with key-person-discount literature appearing in a number of articles in *Business Valuation Review* (see the BVR index for cites). Another approach is to lower our estimate of earnings to reflect our weighted average estimate of decline in earnings that would follow from a change in ownership or the decreased capacity of the existing owner, whichever is

Prior to adding a company-specific adjustment, it is important to achieve internal consistency in the ex ante and ex post marketable minority values, as we did in Table 5.4B. The remainder of the table is identical to its predecessors, except that we eliminate the ex post calculation of the discount rate in B22 through B27, since we have already achieved consistency.

It is at this point in the valuation process that the dollar amounts of our control premium and discount for lack of marketability are meaningful. Our final fair market value of $537,046 (B20) is on an illiquid control basis.

In a valuation report, it would be unnecessary to show Table 5.4A. One should show Tables 5.4B and 5.4C only.

The Table 5.4 series of examples still does not consider the material later in the chapter in the section, "The Wedge between Public and Private Firm Valuations," in which we introduce the concept of a private firm premium. Thus, the discount rate calculations in the Table 5.4 series are not the end of the story. The appraiser still needs to consider a private firm premium in addition to the company-specific premium.

## Total Return versus Equity Premium

CAPM uses an equity risk premium as one component for calculating return. The discount rate is calculated by multiplying the equity premium by beta and adding the risk-free rate. In my first article on the log size model (Abrams, 1994), I also used an equity premium in the calculation of the discount rate. Similarly, Grabowski and King (1995) used an equity risk premium in the computation of the discount rate.

The equity premium form of the log size model is:

$$r = R_F + \text{size-based equity premium.} \tag{5.11}$$

The size-based equity premium is equal to the return, as calculated by the log size model, minus the historical average risk-free rate:[31]

$$\text{Equity Premium} = a + b \ln \text{FMV} - \bar{R}_F, \tag{5.12}$$

where $\bar{R}_F$ is the historical average risk-free rate.

---

more appropriate, depending on the context of the valuation. In this example, I have already assumed that we have done that. There are opinions that one should lower earnings estimates and not increase the discount rate. It is my opinion that we should definitely increase the discount rate in such a situation, and we should also decrease the earnings estimates if that has not already been done.

[31] In CAPM, the latter term is a beta-adjusted equity risk premium, equal to ($\beta$ × equity risk premium). The equity risk premium (ERP) itself is the arithmetic average of the annual market returns in excess of the risk-free rate. Mathematically, that is $\text{ERP} = \sum_{t=1926}^{2007} \left[(r_{mt} - r_{Ft})/82\right]$, where $r$ = return, and the subscripts $m$ = market and $F$ = risk-free rate. However, we can rearrange the equation to $\text{ERP} = \sum_{t=1926}^{2007} \left[(r_{mt}/82) - (r_{Ft}/82)\right] = \bar{r}_m - \bar{r}_F$. This is appropriate for the market as a whole. To calculate a discount rate for a particular firm, in CAPM we scale the ERP up or down according to the systematic risk as measured by beta. In log size, we replace the average return on the market with the size-based return for the firm. There is no algebraic scaling, as the log size equation accomplishes the adjustment of the ERP directly by size.

Substituting equation (5.12) into (5.11), we get:

$$r = R_F + a + b \ln \text{FMV} - \bar{R}_F. \tag{5.13}$$

Rearranging terms, we get:

$$r = a + b \ln \text{FMV} + (R_F - \bar{R}_F). \tag{5.14}$$

Note that the first two terms in equation (5.14) are the sole terms included in the total return version of the log size model. Therefore, the only difference in calculation of discount rates between the two models is $R_F - \bar{R}_F$, the last two terms appearing in equation (5.14). Consequently, the total return of the log size model will exceed the equity premium version of the model whenever current bond yields exceed historical average yields and vice versa.

My second article (Abrams, 1997) eliminated the equity premium term in favor of total return because of the low correlation[32] between stock returns and bond yields for the 60 years prior to 1996, that is, for the data in the 1997 article. The actual correlation was 6.3%—an amount small enough to ignore. For 1926–2007, the correlation is down to 3.8% (Table 5.5, C90) for large cap stocks. While CAPM tells us that the discount rate should change as the risk-free rate changes, the low correlation between stock returns and bond yields gives the opposite message—hence my decision to eliminate the equity premium term and go with the simpler model form of total returns.

Bond yields were in the 2% to 3% range before 1960, under 5% until 1968, and over 7% from 1975 to 1993; in 1982 they were as high as 13%. During the 82-year period from 1938 to 2007, the low bond yields prevalent in the 1950s and 1960s are largely balanced by higher subsequent rates, resulting in little difference in the results obtained using the two models. The 82-year arithmetic mean long-term bond yield is 5.21% (Table 5.1, G63), as compared with the December 31, 2007, 20-year Treasury coupon bond yield of approximately 4.5%.[33] Thus, current yields are reasonably close to the 82-year average yields.

Therefore it is reasonable to simplify the procedure of calculating discount rates and eliminate the bifurcation of the discount rate into the risk-free rate and equity premium components.

## Adjustments to the Discount Rate

Is Table 5.3 the last word in calculating discount rates? No, but it is our starting point based on the available data. Table 5.3 is an extrapolation of NYSE/AMEX/NASDAQ data to privately held firms.

Privately held firms are generally owned by people whose investment portfolios are not well diversified. Table 5.3 was derived from stock portfolios that were

[32]The correlation of large-cap stock and bond returns is the covariance of the two divided by the standard deviation of large-cap stock returns times the standard deviation of bond returns. While the covariance depends on size, dividing by the product of the standard deviations renders correlation to be a dimensionless measure in percentage terms of the degree of relation between stock and bond returns.

[33]*SBBI—2008, Valuation Yearbook*, back page.

# Table 5.5

## Correlation of Large Stock Returns and Bond Yields

| | A | B | C |
|---|---|---|---|
| 1 | | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5–7 | | Large Co. Stocks Return [1] | LT Government Bond Income Return [2] |
| 8 | **1926** | 0.1162 | 0.0373 |
| 9 | **1927** | 0.3749 | 0.0341 |
| 10 | **1928** | 0.4361 | 0.0322 |
| 11 | **1929** | -0.0842 | 0.0347 |
| 12 | **1930** | -0.2490 | 0.0332 |
| 13 | **1931** | -0.4334 | 0.0333 |
| 14 | **1932** | -0.0819 | 0.0369 |
| 15 | **1933** | 0.5399 | 0.0312 |
| 16 | **1934** | -0.0144 | 0.0318 |
| 17 | **1935** | 0.4767 | 0.0281 |
| 18 | **1936** | 0.3392 | 0.0277 |
| 19 | **1937** | -0.3503 | 0.0266 |
| 20 | **1938** | 0.3112 | 0.0264 |
| 21 | **1939** | -0.0041 | 0.0240 |
| 22 | **1940** | -0.0978 | 0.0223 |
| 23 | **1941** | -0.1159 | 0.0194 |
| 24 | **1942** | 0.2034 | 0.0246 |
| 25 | **1943** | 0.2590 | 0.0244 |
| 26 | **1944** | 0.1975 | 0.0246 |
| 27 | **1945** | 0.3644 | 0.0234 |
| 28 | **1946** | -0.0807 | 0.0204 |
| 29 | **1947** | 0.0571 | 0.0213 |
| 30 | **1948** | 0.0550 | 0.0240 |
| 31 | **1949** | 0.1879 | 0.0225 |
| 32 | **1950** | 0.3171 | 0.0212 |
| 33 | **1951** | 0.2402 | 0.0238 |
| 34 | **1952** | 0.1837 | 0.0266 |
| 35 | **1953** | -0.0099 | 0.0284 |
| 36 | **1954** | 0.5262 | 0.0279 |
| 37 | **1955** | 0.3156 | 0.0275 |
| 38 | **1956** | 0.0656 | 0.0299 |
| 39 | **1957** | -0.1078 | 0.0344 |
| 40 | **1958** | 0.4336 | 0.0327 |
| 41 | **1959** | 0.1196 | 0.0401 |
| 42 | **1960** | 0.0047 | 0.0426 |
| 43 | **1961** | 0.2689 | 0.0383 |
| 44 | **1962** | -0.0873 | 0.0400 |
| 45 | **1963** | 0.2280 | 0.0389 |
| 46 | **1964** | 0.1648 | 0.0415 |
| 47 | **1965** | 0.1245 | 0.0419 |
| 48 | **1966** | -0.1006 | 0.0449 |
| 49 | **1967** | 0.2398 | 0.0459 |
| 50 | **1968** | 0.1106 | 0.0550 |
| 51 | **1969** | -0.0850 | 0.0595 |
| 52 | **1970** | 0.0401 | 0.0674 |
| 53 | **1971** | 0.1431 | 0.0632 |

| | A | B | C |
|---|---|---|---|
| 1 | **Table 5.5 (cont.)** | | |
| 2 | | | |
| 3 | | | |
| 4 | | | |
| 5 | | Large Co. | LT Government |
| 6 | | Stocks | Bond Income |
| 7 | | Return [1] | Return [2] |
| 54 | **1972** | 0.1898 | 0.0587 |
| 55 | **1973** | -0.1466 | 0.0651 |
| 56 | **1974** | -0.2647 | 0.0727 |
| 57 | **1975** | 0.3720 | 0.0799 |
| 58 | **1976** | 0.2384 | 0.0789 |
| 59 | **1977** | -0.0718 | 0.0714 |
| 60 | **1978** | 0.0656 | 0.0790 |
| 61 | **1979** | 0.1844 | 0.0886 |
| 62 | **1980** | 0.3242 | 0.0997 |
| 63 | **1981** | -0.0491 | 0.1155 |
| 64 | **1982** | 0.2141 | 0.1350 |
| 65 | **1983** | 0.2251 | 0.1038 |
| 66 | **1984** | 0.0627 | 0.1174 |
| 67 | **1985** | 0.3216 | 0.1125 |
| 68 | **1986** | 0.1847 | 0.0898 |
| 69 | **1987** | 0.0523 | 0.0792 |
| 70 | **1988** | 0.1681 | 0.0897 |
| 71 | **1989** | 0.3149 | 0.0881 |
| 72 | **1990** | -0.0317 | 0.0819 |
| 73 | **1991** | 0.3055 | 0.0822 |
| 74 | **1992** | 0.0767 | 0.0726 |
| 75 | **1993** | 0.0999 | 0.0717 |
| 76 | **1994** | 0.0131 | 0.0659 |
| 77 | **1995** | 0.3743 | 0.0760 |
| 78 | **1996** | 0.2307 | 0.0618 |
| 79 | **1997** | 0.3336 | 0.0664 |
| 80 | **1998** | 0.2858 | 0.0583 |
| 81 | **1999** | 0.2104 | 0.0557 |
| 82 | **2000** | -0.0911 | 0.0650 |
| 83 | **2001** | -0.1188 | 0.0553 |
| 84 | **2002** | -0.2210 | 0.0559 |
| 85 | **2003** | 0.2870 | 0.0480 |
| 86 | **2004** | 0.1087 | 0.0502 |
| 87 | **2005** | 0.0491 | 0.0469 |
| 88 | **2006** | 0.1580 | 0.0468 |
| 89 | **2007** | 0.0549 | 0.0486 |
| 90 | **Correlation** | | **0.0380** |
| 91 | **1926–2007 Avg Yields** | | **0.0521** |
| 92 | | | |
| 93 | [1] *SBBI Classic—2008,* pp. 234–235.* | | |
| 94 | | | |
| 95 | [2] *SBBI Classic—2008,* pp. 246–247.* | | |
| 96 | | | |
| 97 | * *Source:* Morningstar, Inc.—*2008 Ibbotson® Stocks,* | | |
| 98 | *Bonds, Bills and Inflation (SBBI) Classic Yearbook.* | | |

diversified in every sense except for size, as size itself was the method of sorting the deciles. In contrast, the owner of the local bar or dry cleaner is probably not well diversified, nor is the probable buyer. The appraiser should consider adding a private firm risk premium to the discount rate implied by Table 5.3 to account for that. On the other hand, a $100 million FMV firm is likely to be bought by a well-diversified buyer and may not merit increasing the discount rate.

Warren D. Miller, CFA, ASA, teaches a top-notch course to incorporate unsystematic risk into our valuations. I asked his permission to quote him in this book, and after he reviewed the above paragraph he said that the tri-level unsystematic risk framework (his SPARC system) results in adjustments from −3% to +35%. He stated that he computes his adjustments empirically and updates the range annually. He expects several Excel-based templates to accompany his book, *Value Maps: Valuation Tools that Unlock Business Wealth* (John Wiley & Sons). Adjustments of the potential magnitude that he computes deserve the serious attention of the valuation profession, as these adjustments can dwarf the choice of the baseline discount rate and almost any other valuation adjustment that we make. However, that is outside the scope of this book. It is possible that some of his unsystematic risk is included in the private firm premium that we discuss later in this chapter.

Another common adjustment to Table 5.3 discount rates would be for the depth and breadth of management of the subject company compared to other firms of the same size. In general, Table 5.3 already incorporates the size effect. No one expects a $100,000 FMV firm to have three Harvard MBAs running it, but there is still a difference between a complete one-man show and a firm with two talented people.

In general, this methodology of calculating discount rates will increase the importance of comparing the subject company to its size and industry peers via RMA Associates or Troy's Almanac data. Differences in leverage between the subject company and its RMA peers could well be another common adjustment, although it is easy to overdo this. If we suspect that an independent variable is statistically significant, we could run regressions using data from the guideline public company method and guideline M&A method to test that variable. If it is statistically significant, then it makes sense to adjust for it in a DCF. If not, then it is still possible to make an adjustment for it, but it is best to be cautious in doing so.

## Discounted Cash Flow or Net Income?

Since the market returns are based on the cash dividends and the market price at which one can sell one's stock, the discount rates obtained with the log size model should be properly applied to cash flow, not to net income. We appraisers, however, sometimes work with clients who want a quick-and-dirty valuation, and we often don't want to bother estimating cash flow. I have seen suggestions in *Business Valuation Review* (Gilbert, 1990, for example) that we can increase the discount rate and thereby apply it to net income, and that will often lead to reasonable results. Nevertheless, it is better to make an adjustment from net income based on judgment to estimate cash flow to preserve the accuracy of the discount rate. Chapters 1 and 2 cover this topic.

## Discussion of Models and Size Effects

The size effects described by Fama and French (1993), Abrams (1994, 1997), and Grabowski and King (1995) strongly suggest that the traditional one-factor CAPM model is obsolete. As Fama and French (1993, p. 54) say,

> *Many continue to use the one-factor Sharpe-Lintner model to evaluate portfolio performance and to estimate the cost of capital, despite the lack of evidence that it is relevant. At a minimum, these results here and in Fama and French (1992) should help to break this common habit.*

### CAPM

Consider the usual way we calculate discount rates using CAPM. We average the betas of many different firms in the industry, which vary considerably in size, and apply the resulting beta to a firm that is probably 0.1% to 1% of the industry average, without correction for size, and hence, risk. Ignoring the size effect corrupts the CAPM results.

This flaw also applies to the guideline public company method. The usual approach is to average price earnings multiples (and/or price cash flow multiples, etc.) for the various firms in the industry without correcting for size and apply the multiple to a small private firm. A better method is to perform a regression analysis of PE or the PS multiple with the logarithm of a size variable as part of the regression.

In equation (4.28) the PE multiple of a mature firm is the payout ratio $\times$ $(1 + g_1)$ $\times$ Gordon model multiple, where $g_1$ is the one-year growth rate and $g$ is the long-term growth rate. Thus the PE multiple should be a function of risk and growth, as Miller-Modigliani (MM) showed that we cannot affect value by manipulating the payout ratio, which is dividends.[34] We know that risk itself is related to size. Therefore it makes sense to include a size variable in the regression. I often find that size variables work best in their logarithmic form, so reasonable candidates are the logarithm of market capitalization, sales, total assets, book equity, and so on.[35] In equation (4.28b) the PS multiple of a mature firm is the payout ratio $\times$ $(1 + g_1)$ $\times$ profit margin $\times$ Gordon model multiple.

The beta used in CAPM is usually calculated by running a regression of the equity premium for an individual company versus the market premium. As previously discussed, the inability of the resulting beta to explain the size effect has called into question the validity of CAPM. An alternative method of calculating beta has been proposed that attempts to capture the size effect and better correlate with market equity returns, possibly ameliorating this problem.

**SUM BETA** Ibbotson et al. (Peterson, Kaplan, and Ibbotson, 1997) postulated that conventional estimates of beta are too low for small stocks due to the higher degree

[34]That is true under MM's assumptions of perfect capital markets, no taxes, and so forth. There are those who have some degree of disagreement with it. For our purposes, MM's statement is sufficiently accurate.

[35]Unlike the other variables, market capitalization has the disadvantage of being in both sides of the equation, so it better to use the other variables if they work almost as well.

of autocorrelation in returns exhibited by smaller firms. They calculated a beta using a multiple regression model for both the current and the prior period, which they call *sum beta*. These adjusted estimates of beta helped to account for the size effect and showed positive correlation with future returns.

This improved method of calculating betas reduces some of the downward bias in CAPM discount rates, but it still does not account for the size effect differences between the large firms in the NYSE/AMEX/NASDAQ—where even the smallest firms are larger than the majority of privately held firms that many appraisers are called on to value. Size should be an explicit variable in the model to accomplish that.

It is possible to combine the models. One could use the log size model to calculate a size premium over the average market return and add that to a CAPM calculation of the discount rate using Ibbotson's sum betas. Effectively we do that by adding or subtracting the SBBI Valuation Edition industry premia.

## The Fama-French Three-Factor Model[36]

The Fama-French (FF) cost of equity model is a multivariable regression model that uses size ("small minus big" premium ≡ SMB) and book-to-market equity ("high minus low" premium ≡ HML) in addition to beta as variables that affect market returns. While CAPM has one single factor—beta (covariance of returns divided by the market variance)—FF has three factors—covariance with the market, size, and financial distress, as measured by the ratio of book equity divided by market equity (BE/ME). There has been much research on FF since publication of the first edition of this book, and it is now a standard method of analysis in academic finance research, along with CAPM.

Michael Annin (1997) examined the model in detail and found that it does appear to correct for size, both in the long-term and short-term, over the 30-year time period tested. The cost of equity model, however, is not easy to use (Annin, 1997), and using it to determine discount rates for privately held firms is particularly problematic. Market returns are not available for these firms, rendering direct use of the model impossible.[37]

Ibbotson Associates/Morningstar publishes discount rates based on using the three-factor model in the Cost of Capital Quarterly by industry SIC code, with companies in each industry sorted from highest to lowest. Determining the appropriate percentile grouping for a privately held firm is a major obstacle, however. The Fama-French model is a superior model for calculating discount rates of publicly held firms. Lacking an objective stock price, it is more difficult to use for privately held firms. However, it is possible to make an adjustment based on the growth-versus-value-stock literature and data, which immediately follows.

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[36]The precise method of calculating beta, SMB, and HML using the three-factor model, along with the regression equation, is more fully explained in Ibbotson Associates' *Beta Book*. The SBBI Classic yearbooks also have a chapter on growth and value investing, which is Chapter 9 in the SBBI—2009 yearbook.

[37]Based on a conversation with Michael Annin.

## Growth versus Value Stocks

While this section belongs as part of the Fama-French three-factor model, it is large enough to merit its own section. The growth-versus-value-stock dichotomy is one that still has not made its way into the mainstream of literature on the valuation of private firms, yet it should. Ultimately the purpose of this discussion is to conclude how we should adjust our valuations of private firms for this phenomenon.

**DEFINING GROWTH VERSUS VALUE STOCKS** Ibbotson's SBBI[38] goes into considerable length to describe the different ways to measure growth versus value stocks, which we will not repeat. In short, value stocks are those with high BE/ME ratios. They are the "dogs" of the market—stocks that have been beaten down and are now relative bargains. The intuition behind this is that growth stocks are more glamorous and command higher stock prices than value stocks, which means higher PE multiples and a higher market-to-book value, which means a lower book-to-market value.

**DIFFERENCES IN RETURNS AND VOLATILITY** *SBBI—2009 Classic Yearbook*'s Table 8.1 shows that for 1969–2008 value stocks have higher arithmetic returns (11.4%) and lower standard deviation of returns (18.2%) than growth stocks (9.8% and 20.2%, respectively). That statement is true overall as well as in each breakdown of the market into large, mid, and small cap.

This seems to violate the risk–return trade-off and is somewhat of a puzzle. I will speculate on answers to the puzzle, but it remains a puzzle. Some of my speculation draws from the section entitled, "What Causes the Growth versus Value Phenomenon?" later in the chapter.

Value stocks are distressed, unglamorous firms that investors find unexciting. They have lower forecast growth than growth firms. The end-of-year Gordon model multiple (GMM) is $1/(r - g)$. When $g$ is small, $r - g$ will tend to be large, and the GMM will be small. An absolute 1% change in a value firm's forecast growth will have a much smaller impact on its GMM than that of a growth firm. Thus, it seems to me that the lower standard deviation of returns of value firms automatically follows from their lower forecast growth rates.

Since value firms by their nature have lower volatility of returns, aren't they less risky than growth firms, and shouldn't they have lower returns? On the surface, it would seem so. However, perhaps because they are the dogs of the market they still are more likely to go under than growth firms and therefore are riskier than growth firms, despite their lower volatility. In other words, the value firm survivors have sufficiently lower volatility that, even with a higher percentage of failures, the portfolio of value firms is still lower volatility than the portfolio of growth firms.

It is possible that skewness may account for this apparent violation of the risk–return trade-off. Growth firms are skewed right; that is, the majority of them produce low returns, but a significant minority of them produce spectacular returns. There could be a lottery effect; that is, investors may be willing to pay more for growth firms because even though most such investments do worse than investing in a portfolio of value firms, some of the growth firms offer the investor the thrill of winning the lottery, and they are willing to pay for their thrills in lower

[38]*SBBI—2009 Classic*, Chapter 8.

expected returns. Perhaps the value firms are the opposite—with returns that are skewed left. If so, on average they produce higher returns than growth firms—which the SBBI statistics confirm—but there is a significant minority of failures.

The next article that we discuss shows that the growth-versus-value phenomenon has been with us for a long time, even if we recognized it only recently.

**NINETEENTH-CENTURY BRITISH STOCK RETURNS** A working paper (Ye, Hickson, and Turner, 2009) finds that for 1,051 stocks traded in the London Stock Exchange between 1825 and 1870 value stocks[39] also have higher average returns than growth stocks. The value effect was huge. The *x*-coefficient on dividend yields was 5.46, which means that a 100% increase in dividend yield implies a 5.46% increase in monthly stock returns, which is an 89% increase in annual stock returns.

The authors also found a strong size effect. After controlling for the various stock characteristics through regression, a one-unit increase in the natural log of market capitalization (i.e., an increase of *e*, 2.718 times) is associated with 0.21% decrease in monthly stock returns, which is a 2.5% decrease in annual stock returns.

The result that stock returns in England as far back as 184 years ago demonstrate the growth-versus-value stock return and size effects is powerful evidence that they are intrinsic to investing and not a recent anomaly.

**WHAT CAUSES THE GROWTH-VERSUS-VALUE PHENOMENON?** There are different theories as to what causes a firm to be a growth or a value firm. By my review of the literature they fall into two camps. The majority of articles bear the message that value firms are those that have been beaten down by the market—too far down—either by overreaction to bad news (reported by Ibbotson), failure to anticipate mean reversion in returns (Fama-French), or because smaller firms are more subject to random noise, that is, temporary deviations of stock prices from their true values (Arnott, Hsu, Liu, and Markowitz). The other camp is that being a growth (value) firm is caused by high (low) operating leverage (Garcia-Feijoo and Jorgensen). We will begin with the articles in the first camp.

**The First Camp—Value Firms Are Dogs** Ibbotson cites Cottle, Murray, and Block (1988), followers of Graham and Dodd's *Security Analysis*, first published in 1934, who would say that value stocks are those that were once undervalued. Several academic studies have shown that the market overreacts to bad news and underreacts to good news. Value stocks are more likely to have reported bad news, which means that it is more likely that they were undervalued and will thus outperform growth stocks.

Fama and French (2007) have a more systematic explanation, even though it bears some similarity to the previous one. They say it is due to the convergence of price-to-book ratios (this is the inverse of BE/ME, i.e., ME/BE) due to mean reversion in profitability and expected returns. The market has judged growth firms as high profit and growth and value firms as the opposite. However, there are two forces that tend to erode the high profit and growth: (1) competition; and (2) that some growth firms have already exercised their most profitable growth options. Each year some growth firms cease being high profit and growth, with low costs of equity

[39]The authors use a high dividend yield as a proxy for a value stock, as book values were unavailable.

capital (expected stock returns). Therefore, price-to-book ratios of growth portfolios tend to fall with firm age. Conversely, price-to-book ratios of value portfolios tend to rise with firm age as some firms restructure, their profitability improves, and they are rewarded by the market with lower costs of equity capital and higher stock prices.

What is similar about the first two articles is that they say that the higher returns to value firms is based on market mispricing. Where they differ is that Cottle, Murray, and Block do not offer a systematic reason for the mispricing, while Fama and French do. In essence their message is that the market is giving us an opportunity to correct for our systematic errors in valuation using fundamentals, for example, a DCF. We tend to forecast that the status quo—good or bad—will continue longer than it actually does. We tend to forecast that a growth firm will remain golden for a longer period than it will and that a *schlepper*[40] will remain so forever—and neither is true, on average.

However, there is literature that contradicts Fama-French in part. Loderer and Waelchli (2009) find that performance declines with firm age. There are theoretical reasons for this. The authors create a simple and very logical equation to quantify the expected net benefits of adopting an innovation. The market periodically signals the need to innovate. The signal may or may not be accurate. The authors assume that the ability to perceive the signal declines with age, and the cost of developing and adopting the innovation increases with age, because there are more organizational rigidities to overcome. Their empirical work bears out their theoretical predictions. In general, company performance deteriorates with age. Profitability falls, costs increase, and margins decline with corporate age. Performance seems to rebound at a very old age (47 to 100 years, depending on the measure used), but few firms survive long enough to experience that, and even then, the rebound is moderate and does not overcome the robustness of a new powerhouse in the industry. The overall age effect is stronger for high-tech firms than low tech, but it affects both. This casts some doubt on Fama-French's rationale for the higher return of the value firms, as time is not on their side. However, it is not a complete refutation, as it is still possible that investors tend to be too optimistic about growth firms and too pessimistic about value firms. In fact, the article provides a strong reason why long-term rosy optimism is probably not justified for growth firms.

The results of Baker and Kennedy (2002), who find the 10-year survival rate for firms trading on the NYSE and AMEX from 1963 to 1995 to be only 61%, amplify those of Loderer and Waelchli. In any 10-year period the ratio of firms that traded all 10 years to the total that traded at all varies from 26% to 49%—basically $\frac{1}{4}$ to $\frac{1}{2}$. Of the 3,954 firms that delisted in those years, 66% were taken over and 19% were delisted due to financial distress. However, the takeover delistings show a large increase in their stock returns in the year before delisting, regaining more than the value lost in the previous nine years, while distress delistings exhibit no such increase. It is quite possible that this revives the "dog" value firms.

Arnott, Hsu, Liu, and Markowitz (2006) find that noise—a random difference between the stock price and its value—explains the growth-versus-value effect and well as the size effect. The authors show that stocks with lower prices (or price ratios) are more likely to have a negative price noise and will thus be undervalued.

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[40]A technical term. Actually this is Yiddish, loosely rendered as a dog, the living dead, and so on.

They state that their model captures the intuition that value stocks are, on average, bargains. This is similar to Fama-French, but not identical. It is on the one hand more mechanical, lacking the story of Fama-French, and on the other hand more comprehensive, explaining both of the key factors in the Fama-French three-factor model. Another observation of the authors is they find that markets are efficient on average but experience transient random inefficiency.

**The Second Camp—Operating Leverage as an Explanation** Now we discuss the second camp. Garcia-Feijoo and Jorgensen (2007) provide empirical evidence consistent with recent theoretical models (Carlson, Fisher, and Giammarino, 2004) that growth firms have higher operating leverage than value firms. Mandelker and Rhee (1984) decompose a firm's systematic risk into the degree of operating leverage (DOL), degree of financial leverage (DFL), and business risk. DOL measures a firm's reliance on fixed costs, DFL measures its reliance on debt, and business risk is the systematic risk of a firm's basic operations.

The authors measure operating leverage as the average sensitivity of the percentage deviation of EBIT from its trend relative to the percentage deviation of sales from its trend. Thus firms that are heavily invested in fixed assets—and therefore have high fixed costs—will tend to be extremely profitable when sales are high and experience low profitability or even significant losses when sales are low, whereas firms with low fixed costs will have less fluctuation in profitability.

The authors find a strong positive association between DOL—both at the firm and portfolio level—and BE/ME and average stock returns. These offer empirical support of the theoretical models in the articles mentioned earlier. They also find a positive association between size (ME) and DFL; that is, larger firms are more likely to use leverage than small firms, and BE/ME and DFL after controlling for ME. They also find a positive, but weaker association between DFL and subsequent stock returns. They suggest that if an association between BE/ME and financial distress exists, it is not a simple one and may explain why some researchers have and others have not found a positive relation between BE/ME and financial distress.

Overall, the results provide support for a risk-based explanation for the value premium that is consistent with existing theoretical models. They conclude that firm-level investment activity accounts for the value premium, that is, the firms with relatively large investments in fixed assets that are the value firms and command the value premium. It seems to me that this view is more consistent with modern portfolio theory than the first camp, as the explanation is consistent with systematic risk. In other words, in a CAPM framework, where we do not control for differences in BE/ME, we would expect that the value firms would tend to be high beta, because they are extremely sensitive to the market. However, high operating leverage causes high systematic risk, which leads to high discount rates and lower market equity and thus high BE/ME. Whereas Fama-French introduced BE/ME as a regression variable in their three-factor model, it seems that by extending the logic in our article the BE/ME factor now does the heavy lifting that previously was relegated to the market beta, thus leaving the market beta coefficient close to 1, as it should be.

**IMPLICATIONS FOR VALUING PRIVATE FIRMS** What should the valuation analyst do with this information? There are so many models with different explanations for the same phenomenon, and there is no consensus yet in academic circles.

It would make sense to make an adjustment to the discount rate if the subject company or its industry clearly falls into the growth or the value stock category. Calculating a discount rate for a publicly held firm is easy—just use the three-factor model. However, we lack the objective stock price in valuing a private firm, so we can't do that.

My first thought is that in the valuation of a private firm there is no market beating down the price of a value stock. In a DCF, we are simply forecasting cash flows and discounting to present value. The Fama-French mean reversion and the Loderer-Waelchli discovery of corporate aging certainly are a sobering splash of cold water that we should keep in mind to guard against making optimistic forecasts. Using log size to calculate the discount rate would implicitly produce an average result with respect to growth versus value firms, as the market is full of both. By this logic I would be disinclined to recommend making an adjustment for growth versus value.

The first camp's explanation of the growth firm phenomenon basically is telling us that the market is making an adjustment for market (and probably analyst) tendencies to overvalue glamour firms and undervalue the downtrodden surviving firms of yesteryear. Thus, it is telling us to watch out for biases in our cash flow forecasting and either to eliminate or to correct for them with a value or growth firm adjustment.[41]

I think that the DOL-based explanation gives us the clearest indication of a potential adjustment to the discount rate. If the subject company has lower (higher) operating leverage than its competitors, which we could measure in different ways, then it is likely to be a value (growth) company and it makes sense to add a premium (subtract from) to the discount rate.

The authors' primary method to measure DOL is very sophisticated and beyond the professional level. We could measure operating leverage as gross and/or net fixed assets as a percentage of total assets. The authors also suggest using total assets/market equity as a measure of operating leverage, but that does not work for private firms.

## Log Size Model

The log size model is a superior method to CAPM because it better correlates with historical equity returns. Therefore it enables business appraisers to dispense with CAPM altogether and use firm size as the basis for deriving a discount rate before adjustments for qualitative factors different from the norm for similarly sized companies.

In another study on stock market returns, analysts at an investment banking firm regressed P/E ratios against long-term growth rate and market capitalization. The $R^2$ values produced by the regressions were 89% for the December 1989 data and 73% for the November 1990 data. Substituting the natural logarithm of market capitalization in place of market capitalization, the same data yields an $R^2$ value

[41]The Arnott, Hsu, Liu, and Markowitz article mathematics are extremely difficult, which makes it more difficult to infer how this applies to privately held firms. Nevertheless, I do infer that my comments about Fama-French basically apply here, as well, but for statistical reasons rather than bias in forecasting.

of 91% for each dataset, a marginal increase in explanatory power for the first regression, but a significant increase in explanatory power for the second regression.

From Chapter 4, equation (4.28), the PE multiple is as follows:

$$PE = (1 - RR)(1 + g_1)\frac{\sqrt{1+r}}{r-g}.$$

Using a log size model to determine $r$, the PE multiple is equal to:

$$PE = (1 - RR)(1 + g_1)\frac{\sqrt{1 + a + b\ln(FMV)}}{a + b\ln(FMV) - g}, \qquad (5.15)$$

where $g_1$ is expected growth in the first forecast year, $RR$ is the retention ratio, $a$ and $b$ are the log size regression coefficients, and $g$ is the long-term growth rate. Looking at equation (5.15), we see that ln FMV, which is market cap for the publicly traded firms, appears twice. Thus, it is clear why using the log of market capitalization improved the $R^2$ of the investment bankers' regression.

**GRABOWSKI AND KING** Grabowski and King (1995) (GK) applied a finer breakdown of portfolio returns than was previously used to relate size to equity premiums. When they performed regressions with 31-year data for 25 and 100 portfolios (as compared to our 10), they found results similar to the equity premium form of log size model; the equity premium is a function of the negative of the log of the average market value of equity, further supporting this relationship.[42]

Grabowski and King (1996) in an update article also used other proxies for firm size to forecast the equity premium in their log size discount rate model, including sales, five-year average net income, and EBITDA. Following is a summary of their regression results sorted first by $R^2$ in descending order, then by the standard error of the $y$-estimate in ascending order. Overall, we are attempting to present their best results first.

| Measure of Size | $R^2$ | Standard Error of $Y$-Estimate |
|---|---|---|
| 1. Mkt cap—common equity | 93% | 0.862% |
| 2. Five-year average net income | 90% | 0.868% |
| 3. Market value of invested capital | 90% | 1.000% |
| 4. Five-year average EBITDA | 87% | 0.928% |
| 5. Book value—invested capital | 87% | 0.989% |
| 6. Book value—equity | 87% | 0.954% |
| 7. Number of employees | 83% | 0.726% |
| 8. Sales | 73% | 1.166% |

Note that the market value of common equity, that is, market capitalization of common equity, has the highest $R^2$ of all the measures. This is the measure that we have used in our log size model. The five-year average net income, with an $R^2$ of 90%, is the next best independent variable, superior to the market value of invested capital by virtue of its lower standard error.

[42]Grabowski and King used base 10 logarithms instead of natural logarithms.

This is a very important result. It tells us that the majority of the information conveyed in the market price of the stock is contained in net income, because both have such high $R^2$. We can also say that the majority of the information conveyed in the market price of the stock is contained in the other variables, as they are all correlated. When we use a log size model based on equity in valuing a privately held firm, we do not have the benefit of using a market-determined equity. The value will be determined primarily by the magnitude and timing of the forecast cash flows, the primary component of which is forecast net income. If we did not know that the log of net income was the primary causative variable of the log size effect, it is possible that other variables such as leverage, sales, book value, and so on could significantly impact the log size effect. If we failed to take those variables into account and our subject company's leverage varied materially from the average of the market (in each decile) as it is impounded into the log size equation, our model would be inaccurate. Grabowski and King's research eliminates this problem. Thus, we can be reasonably confident that the log size model as presented is accurate and is not missing any significant variable.

Of Grabowski and King's eight different measures of size, only market capitalization (#1) and the market value of invested capital (#3) have the circular-reasoning problem of our log size model. The other measures of size have the advantage in a log size model of eliminating the need for iteration since the discount rate equation does not depend on the market value of equity, the determination of which is the ultimate purpose of the discount rate calculation. For example, if we were to use #2, net income, we would simply insert the subject company's five-year average net income into Grabowski and King's regression equation and it would determine the discount rate. This is problematic, however, for determining discount rates for high-growth firms due to the inability to adequately capture significant future growth in sales, net income, and so on. Start-up firms in high-technology industries frequently have negative net income for the first several years due to their investment in research and development. Sales may subsequently rise dramatically once products reach the market. Therefore, five-year averages are not suitable in this situation.

Another problem with Grabowski and King's results is that their data begin only in 1963, when Compustat data were available for all companies. Thus, they are missing 1926–1962 in their results.

As mentioned in the introduction, in their (1999) article GK demonstrate a negative logarithmic relationship between returns and operating margin and a positive logarithmic relationship between returns and the coefficient of variation of operating margin and accounting return on equity.

This is their most important result so far, because it relates returns to fundamental measures of risk. Actually, it appears to me that operating margin in itself works because of its strong correlation of 0.97 to market capitalization (i.e., value). However, the coefficient of variation (COV) of operating margin and return on equity seem to be more fundamental measures of risk than size itself. In other words, it appears that size itself is a proxy for the volatility of operating margin, return on equity, and possibly other measures. Thus, we must pay serious attention to their results.

Below is a summary of their statistical results. We comment on the use of their results in the conclusion section that immediately follows the summary of their statistical results.

| Measure of Risk | $R^2$ | Standard Error of $Y$-Estimate |
|---|---|---|
| 1. Log of five-year operating margin | 76% | 1.185% |
| 2. Log COV (operating margin) | 54% | 0.957% |
| 3. Log COV (return on equity) | 54% | 0.957% |

Currently Grabowski and King publish their study annually and sell it to the valuation profession. Below are the regression statistics in their Duff & Phelps Risk Premium Report 2008.

| Measure of Size | $R^2$ [43] | Standard Error of $Y$-Estimate |
|---|---|---|
| 1. Mkt cap—common equity | 88% | 0.901% |
| 2. Five-year average net income | 88% | 0.732% |
| 3. Market value of invested capital | 88% | 0.841% |
| 4. Five-year average EBITDA | 78% | 0.992% |
| 5. Book value—invested capital | Deleted and replaced with total assets | |
| 6. Book value—equity | 85% | 0.780% |
| 7. Number of employees | 74% | 0.940% |
| 7A. Total assets[44] | 79% | 0.948% |
| 8. Sales | 72% | 0.925% |

Let's compare GK's 2008 results with their 1996 results. The $R$-squares are lower in 2008, and the standard errors are mixed, with about half of them increasing and half decreasing. On average, the regression statistics are not as good in 2008 as they were in 1996.

Now we will compare the regression statistics of the Duff & Phelps 2008 Report with Table 5.1. The $R^2$ in Table 5.1, regression #2 is 94% (C39), which is larger than Duff & Phelps' 88%, and the standard error of the $y$-estimate in Table 5.1, regression #2 is 0.70% (C38), which is smaller than Duff & Phelps' 0.901%. Thus, the advantage of having all 82 years of stock market results outweighs the benefit of having 25 (versus 10) portfolios.

In conclusion, Grabowski and King's work is very important in that it demonstrates that other measures of size can serve as effective proxies for our regression equation. It is noteworthy that the finer breakdown into 25 portfolios versus Ibbotson's 10 does not appear to have a significant impact on the reliability of the regression equation, as it did in our first edition. Their timeframe is data for 1963–2007, which is 44 years. Our 40-year results show $R^2$ of 85% (Table 5.2, I19) and our 50-year results show $R^2$ of 97% (Table 5.2, I28), for an average of 91%, while their $R^2$ was 88%—3% less.

[43]Source: Exhibit A-1, A-2, etc. Note this is not adjusted $R^2$.

[44]This measure did not appear in the earlier articles and replaces total book value of invested capital. The two are fairly similar numbers, with the only difference being that one subtracts current liabilities from total assets to equal book value of invested capital.

Grabowski and King's (1999) work is even more important. It is the first finding of the underlying variables for which size is a proxy. If Compustat data went back to 1926, as do the CRSP data, then I would recommend abandoning log size entirely in favor of their variables. As time goes on, the effect of missing the first 37 years of the stock market diminishes.

Eventually it is likely that there will not be a significant advantage to my log size model over the Duff & Phelps report. Additionally, the Duff & Phelps report has the advantage of being able to use independent variables to calculate the discount rate that lack the circularity problem. So, there are some compelling advantages to their report. However, I do not recommend abandoning my log size yet, as the $R^2$ and standard error of the $y$-estimate are better in log size in the meantime.

### Heteroscedasticity

Schwert and Seguin (1990) also found that stock market returns for small firms are higher than predicted by CAPM by using a weighted least squares estimation procedure. They suggest that the inability of beta to correctly predict market returns for small stocks is partially due to heteroscedasticity in stock returns.

*Heteroscedasticity* is the term used to describe the statistical condition that the variance of the error term is not constant. The standard assumption in an ordinary least squares (OLS) regression is that the errors are normally distributed, have constant variance, and are independent of the $x$ variable(s). When that is not true, it can bias the results. In the simplest case of heteroscedasticity, the variance of the error term is linearly related to the independent variable. This means that observations with the largest $x$ values are generating the largest errors and causing bias to the results. Using *weighted least squares* (WLS) instead of OLS will correct for that problem by weighting the largest observations the least.

In the case of CAPM, the regression is usually done in the form of excess returns to the firm as a function of excess returns in the market, or $(r_i - r_F) = \hat{\alpha} + \hat{\beta}_i(R_m - R_F)$, where the circumflex indicates the regression-determined estimate of the true $\alpha$ and $\beta$. Here we are using the historical market returns as our estimate of future returns. If everything works properly, $\hat{\alpha}$ should be equal to zero, as that would leave us with the CAPM equation by adding the risk-free rate to both sides of the equation. If there is heteroscedasticity, then when excess market returns are high, the errors will tend to be high. That is what Schwert and Seguin found.

Schwert and Seguin also discovered that after taking heteroscedasticity into account, the relationship between firm size and risk-adjusted returns is *stronger* than previously reported. They also found that the spread between the risk of small and large stocks was greater during periods of heavier market volatility (e.g., 1929–1933).

## Industry Effects

In the first edition of this book, we noted that Jacobs and Levy (1988) examined rates of return in 38 different industries by including industry as a dummy variable in their regression analyses. Only one industry (media) showed [excess] returns different from zero that were significant at the $p = 1\%$ level,[45] which the authors

[45] This means that, given the data, there is only a 1% probability that the media industry returns were the same as all other industries.

speculate was possibly related to the then-recent wave of takeovers. The higher returns to media would be relevant to a subject company only if it were a serious candidate for a takeover.

There were seven industries where [excess] returns were different from zero at the $p = 10\%$ level, but this is not persuasive, as the usual level for rejecting the null hypothesis that industry does not matter in investor returns is $p = 5\%$ or less. Thus, Jacobs and Levy's results lead to the general conclusion that industry does not matter in investor returns.[46]

Since publication of the first edition of this book, however, Ibbotson/Morningstar publishes industry adjustments in its annual SBBI series, both Classic and Valuation editions, and it is standard to make those adjustments.

## The Wedge between Public and Private Firm Valuations

In the world of publicly held firms covered by the SBBI yearbooks, the total returns, $r$, is the sum of income returns, which is dividends, plus capital gains. This is approximately equal to the dividend yield, $d$, plus the expected growth in the value of the stock, $g$, as in equation (5.16):[47]

$$r = d + g. \tag{5.16}$$

Our rate of return could be either the historical actual rate of return or our forecast future rate of return. The latter is more relevant for valuation. Therefore it is the forecast dividend yield rather than the historical yield that is appropriate in equation (5.16).

Let the dividend yield equal forecast dollar dividends, $D_{t+1}$, divided by the current stock price, $P_t$. Substituting this into equation (5.16) results in:

$$r = \frac{D_{t+1}}{P_t} + g. \tag{5.17}$$

Rearranging terms, we get:

$$\frac{D_{t+1}}{P_t} = r - g. \tag{5.18}$$

Taking reciprocals, we get:

$$\frac{P_t}{D_{t+1}} = \frac{1}{r - g}. \tag{5.19}$$

Multiplying both sides by $D_{t+1}$, we get:

$$P_t = D_{t+1} \times \frac{1}{r - g}. \tag{5.20}$$

[46]Jacobs and Levy also found an interest-rate-sensitive financial sector. They also found that macroeconomic events appear to explain some industry returns. Their example was that precious metals was the most volatile industry, and its returns were closely related to gold prices. Thus, there may be some—but not many—exceptions to the general rule of industry insignificance.

[47]It differs by changes in the PE ratio and the reinvestment yield. Also the equation is exact for geometric returns only.

If we rename dividends to cash flow and the stock price as *FMV*, the equation becomes:

$$FMV = CF_{t+1} \times \frac{1}{r-g}. \qquad (5.21)$$

Thus, Ibbotson's total return equation is our familiar Gordon model equation in disguise, albeit the end-of-year version of the Gordon model.[48]

## Private Firm Risk Is the Wedge

It has perplexed me in the past why equation (5.16) applies to public firms but does not seem to apply to private firms. I believe the reason is simple—private firms are riskier than public firms.

1. The cash flow terms in equations (5.20) and (5.21) are not identical, nor are the growth terms, $g$. In equation (5.20) cash flow (dividends) and growth are at the individual shareholder level in public firms, which are easy to measure. Private firms almost never pay formal dividends. It is occasionally possible to measure implicit dividends as being equal to excess (greater-than-arm's-length) compensation, which is difficult to measure, including the payment of personal expenses through the business, which are often unreported and difficult to detect—all the more so since valuators are not auditors. Therefore, equation (5.21) is an adaptation of equation (5.20) to accommodate the lack of explicit dividends in the private firm due to the informational uncertainties. In a private firm, much can happen to interfere with firm-level cash flows filtering down to the level of the individual shareholder. While net income can vary in both public and private firms, in private firms the retention and payout ratios (required net working capital changes and capital expenditures) are likely to vary more than in public firms. The reasons for this are that smooth dividends management is not the same priority in a private firm, and most private firms lack the easy access to debt that helps smooth cash flow volatilities.[49]
2. Public firms strive to be as transparent as possible, while private firms usually strive for the opposite to preserve competitive advantages and privacy of the owners. Transparency reduces risk to a buyer, as it promotes trust and reduces worries about negative surprises. Lack of transparency creates informational asymmetries, with the seller being knowledgeable about the business and the buyer having to perform significant due diligence in order to get to know the business. This raises the buyers' risk.
3. The quantity of information is much less for private firms.
   a. The SEC requires publication of a great deal of information about public firms. There is no such requirement with private firms.

[48]We assumed dividends come at the end of the year. If we had instead been more precise and assumed dividends occur evenly throughout the year, we would end with the midyear Gordon model.

[49]Additionally the control shareholder can divert wealth to him- or herself in a private firm, although we account for this in the discounts for lack of control and marketability. See Chapter 8.

b. The private firm has no stock analysts following it. Thus the financial and general presses publish much less about private firms.
c. The nature of public ownership creates a public relations (PR) department. The public firm has an investor relations department. There is no such built-in PR function in private firms. That would have to be created by specific efforts.

4. The quality of information is generally lower in private firms.
   a. There is no SEC scrutiny.
   b. There is no public embarrassment for having to restate earnings.
   c. Private firms tend to economize on accounting expenditures compared to public firms.
      i. While all public firms must be audited, many private firms are not.
      ii. Among those who are, private firms may choose lower-quality auditors.
      iii. Even if that is not true, it may be difficult for investors to tell quality differences in auditors, and that in itself creates risk.
      iv. The quality of the VP of finance, CFO, and controller is likely to be lower in private firms.
5. No objective stock price:
   a. Market feedback operates like a navigating system. The investing public votes with its dollars, manifested in the stock price, its opinion on company strategy and policy. That provides valuable feedback to management that the company is either on course or not and facilitates management to correct its course. The private firm lacks that important feedback mechanism and thus operates relatively more in the dark.
   b. Lacking historical stock prices, it is far more difficult to construct a stock portfolio with a private firm than public firms. It is thus more difficult and expensive to diversify away firm-specific risk.
6. Ownership in private firms is more expensive.
   a. Private firms are more difficult and expensive to appraise, thus increasing transaction costs for mergers, acquisitions, estate planning, gifting, and estate taxes.
   b. It is more difficult and expensive to sell one's stock in a private firm.
   c. It is more difficult and expensive to diversify one's position partially out of a private firm.
   d. The risk of being a minority shareholder in a private firm is greatly multiplied over the risk of being a minority shareholder in a public firm or even a control shareholder of a private firm. There is always the potential for abuse from the control shareholder, and the shareholder oppression lawsuit remedy for such abuse is inferior, more expensive, and riskier than those remedies available to a minority shareholder in a public firm (e.g., inexpensive class-action lawsuit).
   e. It is more difficult for privately held firms to get a government bailout in the event of disaster.

In the face of risks that are specific to being a private firm, equation (5.16) transforms into equation (5.22).

$$r_{Private} > d_{Private} + g_{Private}. \tag{5.22}$$

More specifically, we can restate equation (5.22) as follows:

$$r_{Private} = d_{Private} + g_{Private} + PFR + CSR, \qquad (5.23)$$

where *PFR* is a generic private firm risk that incorporates the risk differential between public firms and all private firms, while *CSR* is company-specific risk.

For example, suppose that a private firm has an expected dividend yield of 2% and growth in cash flows[50] of 5%. It is unrealistic to think that the private firm has a discount rate of 7%. As a starting point, the discount rate has to be based on the rate of return an investor could earn for a publicly traded firm of the same risk as this private firm.[51] A decile #2 private firm might have the same risk as, let's say, a decile #10 public firm.

Equations (5.22) and (5.23) show that we are using our asset pricing models—whether log size, CAPM, and so on—to calculate discount rates, not growth rates, for most privately held firms, because we cannot assume the equality of the discount rate and the dividend yield plus the growth rate. Thus, the growth rate of a private firm is rarely equal to the discount rate minus the dividend rate, almost always being lower.

It is important to keep our measures of $r$ and $g$ consistent. When we use the arithmetic rate of return, it is important to use an arithmetic forecast growth rate, not the geometric growth rate. Thus, while we appraisers are fond of calculating the compound annual rate of growth (CAGR) of sales and net income in our analysis of historical financial results, we should be using the arithmetic growth rates instead as our base.

## Measuring Private Firm Risk

Now that we have established that private firm risk exists above public firm risk, we need to measure it. In doing so we are moving into uncharted territory. As mentioned earlier, it is more difficult to create and balance portfolios with privately held firms.

We begin with the standard deviation of the 1926–2008 decile #10 portfolio of 45% (Table 5.1, C17, rounded). A finance text (Bodie, Kane, and Marcus, 1995, Figure 6.2, p. 135) shows a fully diversified portfolio as having 300 stocks in it, with a relative standard deviation of 1.00. It shows a one-stock portfolio as having a relative standard deviation of 2.50, that is, $2\frac{1}{2}$ times the standard deviation of a 300-stock portfolio. A 10-stock portfolio has a relative standard deviation of 1.26. Let's see what we can make out of these benchmarks.

We multiply the 45% standard deviation of the decile #10 portfolio by 2.5 to calculate the standard deviation of a single stock, which equals 113%. The difference in standard deviations is 68%. We multiply 68% × 34% = 23%. The 34% is the

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[50]Note that Ibbotson's definition of *growth* is growth in the stock price—capital gains—which is a definition that is unavailable to appraisers of private firms, since we do not have a market-determined stock price. Thus our next best definition of growth is that of cash flows, and we often assume that is sales growth combined with an assumed constant profit margin and retention ratio.

[51]The author thanks Scott Deifik for his observations, which I incorporated into this paragraph and the next two paragraphs.

rounded $x$-coefficient from Table 5.1, C30. It is the increases in the discount rate for each percent increase in the standard deviation of the portfolio. Table 5.1, B17 shows the decile #10 82-year return as 20.98%, and we subtract 0.8% for the average annual increase in the PE ratio, which results in a rounded return of 20% for a decile #10 portfolio. If instead the portfolio consists of only one stock, then the required return should be 20% + 23% = 43%. In other words, a rational investor should be indifferent between investing in 300 decile #10 stocks with an expected return of 20% and a single public firm with an expected return of 43%.

Instead, if we look at owning a decile #10-size private firm as part of our small portfolio as being equivalent to owning a 10-stock portfolio instead of a 300-stock portfolio, the portfolio standard deviation would be 1.26 × 45% = 57%, a 12% increase in portfolio standard deviation. Multiplying this by the $x$-coefficient of 34% we get an increase in required return of 4%. This strikes me as being a reasonable benchmark increase in the required return for investing in a private firm. This 4% calculation applies to all private firms; it is the term *PFR* (private firm risk) in equation (5.23). Of course, it is not the truth coming from Mt. Sinai, but it is a reasonable estimate.

It is possible to modify that calculation to include both private firm risk and company-specific risk. Suppose for a specific subject company we consider it equivalent to owning a 5-stock portfolio. The Bodie-Kane-Marcus table shows the adjustment factor to be 1.40. We calculate the increase as 1.40 × 45% = 63%, an 18% increase in portfolio standard deviation. Multiplying this by the $x$-coefficient of 34% we get an increase in required return of 6%. This implies the subject company's discount rate should be 20% + 6% = 26%. Of the 6% increase in the discount rate, 4% is the generic private firm risk and 2% is company-specific risk.

## Satisfying Revenue Ruling 59-60

Revenue Ruling 59-60 requires that we look at publicly traded stocks in the same industry as the subject company. In the first edition of this book, I claimed that our excellent results with the log size model to calculate a discount rate to use in a discounted cash flow method, combined with Jacobs and Levy's general finding of industry insignificance, satisfied the intent of Revenue Ruling 59-60 for small and medium firms without the need to actually perform a guideline publicly traded company method. For the moment we will follow that reasoning, and at the end of this section we will see how things have changed in the approximately 10 years since publication of the first edition.

### Strengths and Weaknesses of DCF and the Market Methods

First, however, it is worthwhile to look at the strengths and weaknesses of each of the methods that we commonly use. The DCF is an introverted valuation approach. Its strength is the ability to customize the valuation to our subject company. We do considerable financial and statistical analysis of our subject, forecast cash flows using growth rates unique to our subject, and discount them to present value at a market-determined rate, usually with some company-specific adjustments, and preferably with the generic private firm risk adjustment (per our earlier discussion). The only part of the DCF that is more outward-looking is the calculation of the discount rate.

The market methods—both public and private—are extroverted valuation methods. The majority of our efforts go to developing the mathematical relationship of value of the guideline companies to the independent variables that cause value. Then we apply those relationships to our subject company.

Thus, the DCF is best at being customized to the subject company. However, because the DCF lacks external feedback, its weakness is that it is easy enough to value a company improperly using unrealistic forecasts of sales growth, margin, and payout ratio. It is particularly difficult to value early-stage firms.

The strength of the market methods is that they are based on real valuations of real companies. Thus they do not suffer from "valuation introversion." However, they may suffer from other weaknesses: lack of comparability to the subject company, insufficient data (too few observations and/or too little information about each observation),[52] bad data,[53] and inconsistent results due to outliers.

Thus, it is best to use a DCF and both market methods when they apply. However, it is often just plain wrong to use a guideline public company method (GPCM) as a valuation method for a small business. Unless the appraiser can eliminate heteroscedasticity and control for size and risk differences in the public companies, it is often best not to use the GPCM.

## The Information in a PE Multiple and Applicability to the Subject Company

We repeat equation (4.28) from Chapter 4 to show the relationship of the PE multiple to the Gordon model.

$$PE = (1 + g_1) \times POR \times \frac{\sqrt{1+r}}{r-g} \tag{4.28}$$

Relationship of the PE multiple to the Gordon model multiple.

The PE multiple[54] of a publicly traded firm gives us information on the one-year and long-run expected growth rates and the discount rate of that firm—and nothing else. The PE multiple gives us only a combined relationship of $r$ and $g$. In order to derive either $r$ or $g$, we would have to assume a value for the other variable or calculate it according to a model.

For example, suppose we use the log size model (or any other model) to determine $r$. Then the only new information to come out of a guideline public company method is the market's estimate of $g$,[55] the growth rate of the public firm.

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[52]This is particularly a problem in the guideline M&A method. None of the databases of sales of privately held firms provide information on historical growth of sales and profitability, let alone expected growth.

[53]For one of the databases, I once found an observation with a regression standardized residual of 12 standard errors—the largest by far that I have ever seen. It does not appear in any table of $t$-statistics that I have seen. I asked one of my analysts to call the organization to report it as an error, and the spokesperson for the organization said it did not maintain the original input forms, and there was no way to verify or check the error.

[54]Included in this discussion are the variations of PE (e.g., P/CF, etc.).

[55]This is under the simplest assumption that $g_1 = g$ and that the retention ratio will remain constant.

There are much easier and less expensive ways to estimate *g* than to do a GPCM. When all the market research is finished, the appraiser still must modify *g* to be appropriate for the subject company, and its *g* is often quite different from the public companies'. So the GPCM wastes much time and accomplishes little.

Because discount rates appropriate for the publicly traded firms are much lower than are appropriate for smaller, privately held firms, using public PE multiples will lead to gross overvaluations of small and medium privately held firms. This is true even after applying a discount, which many appraisers do, typically in the 20% to 40% range—and rarely with any empirical justification.

If the appraiser uses a GPCM, then he or she should use regression analysis with a scaled variable such as the PE or PS multiple and include the logarithm of market capitalization as an independent variable. If this yields good results—a high adjusted $R^2$ and low standard error of the estimate—then this will control for size, and it is reasonable and even desirable to weigh this method heavily in the final reconciliation of value. If the regression shows the logarithm of various size measures to be statistically insignificant, then it is again reasonable to use the GPCM results in the valuation. In the absence of that, it is critical to use only public guideline companies that are approximately the same size as the subject company, which is rarely possible. When valuing a very large privately held company, where the size effect will not confound the results, it is more likely to be worthwhile to do a guideline public company method.

### Changes in the Past 10 Years

Over the past 10 years there is much better availability of data, both of publicly traded firms and private transactions. Therefore, the market valuation methods are becoming increasingly important. Whereas I argued for routinely eliminating the GPCM in the first edition of the book and didn't even consider the guideline M&A method a serious method, that is changed now.

It is still a potential danger to inappropriately use the GPCM in the valuation of a small business, and the valuation analyst must guard against that. Nevertheless, as the transactional databases continue to grow and improve, the market methods are increasingly compelling.

## Summary and Conclusions

The log size model is more accurate than CAPM for valuing privately held businesses. In the past it was also much faster and easier to use, requiring no research,[56] whereas CAPM often required considerable research of the appropriate guideline companies. Today, however, CAPM, disguised as the *build-up* method, is very easy to use, so the advantage of log size is now the accuracy.

A further danger of CAPM is not fully accounting for size differences. It is very inaccurate to apply the betas for IBM, Compaq, Apple Computer, and so on to a small start-up computer firm with $2 million in sales without carefully adjusting

[56]One needs only a single regression equation for all valuations performed within a single year.

for size differences, which may or may not be possible. The size effect drowns out any real information contained in betas, especially applying betas of large firms to small firms. The 375% (Table 5.1, P21) improvement that we found in the 0.70% (P20, C38) standard error in the log size equation versus the 2.61% (M20) standard error from the CAPM applies only to firms of the same magnitude. When applied to small firms, CAPM yields even more erroneous results, unless the appraiser compensates by blindly adding another 5% to 10% beyond the typical Ibbotson "small firm premium" and calling that a company-specific adjustment (CSA). I suspect this practice is common, but then it is not really a CSA; rather it is an outright attempt to compensate for a model that has no place being used to value small and medium firms.

Around 1994, I valued a midsize firm with $25 million in sales, $2 million in net income after taxes, and very fast growth. I used a guideline public company method—among others—and found 16 guideline companies with positive earnings in the same SIC Code. I regressed the value of the firm against net income, with "great" results—99.5% $R^2$ and high *t*-statistics. When I applied the regression equation to the subject company, the value came to −$91 million![57] I suspect that much of this scaling problem goes on with CAPM as well; many appraisers seriously overvalue small companies using discount rates appropriate only for large firms.

When using the log size model, we extrapolate the discount rate to the appropriate level for each firm that we value. There is no further need for a size adjustment. We merely need to compare our subject company to other companies of its size, not to IBM. Using Robert Morris Associates or Troy's Almanac data to compare the subject company to other firms of its size is appropriate, as those companies are often far more comparable than publicly traded firms.

Since we have already extrapolated the rate of return through the regression equation in a manner that appropriately considers the average risk of being any particular size, the relevant comparison when considering company-specific adjustments is to other companies of the same size. There is a difference between two firms with $2 million in sales volume when one is a one-man show and the other has two Harvard MBAs running it. If the former is closer to average management for a firm of that size, you should probably subtract 1% or 2% from the discount rate for the latter; if the latter is the norm, it is appropriate to add that much to the discount rate of the former—or, better yet, use Warren Miller's SPARC system. Although company-specific adjustments are subjective, they serve to further refine the discount rate obtained from discount rate calculations.

## References

Abrams, Jay B. 1994. A Breakthrough in Calculating Reliable Discount Rates. *Business Valuation Review* (August): 8–24.

[57]The magnitude problem was solved by regressing the natural log of value against the natural log of net income. That eliminated the scaling problem and led to reasonable results. That particular technique is not always the best solution, but it sometimes works beautifully. We cover this topic in more detail near the end of Chapter 3.

Abrams, Jay B. 1997. Discount Rates as a Function of Log Size and Valuation Error Measurement. *The Valuation Examiner* (February/March): 19–21.

Abrams, Jay B. 2001. *Quantitative Business Valuation*. New York: McGraw-Hill, Chapter 4.

Annin, Michael. 1997. Fama-French and Small Company Cost of Equity Calculations. *Business Valuation Review* (March): 3–12.

Arnott, Robert, Jason Hsu, Jun Liu, and Harry Markowitz. 2006. Does Noise Create the Size and Value Effects? Working paper, http://rady.ucsd.edu/faculty/directory/liu/docs/size-value.pdf or http://papers.ssrn.com/sol3/papers.cfm?abstract_id=936272.

Baker, George P., and Robert E. Kennedy. 2002. Survivorship and the Economic Grim Reaper. *Journal of Law, Economics, and Organization* 18(2): 324–361.

Banz, Rolf W. 1981. The Relationship between Returns and Market Value of Common Stocks. *Journal of Financial Economics* 9: 3–18.

Bodie, Zvi, Alex Kane, and Alan J. Marcus. 1995. *Essentials of Investments*. Richard D. Irwin (now McGraw-Hill). New York: McGraw-Hill.

Carlson, Murray, Adlai Fisher, and Ron Giammarino. 2004. Corporate Investment and Asset Price Dynamics: Implications for the Cross Section of Returns. *Journal of Finance* 59(6): 2577–2603.

Cottle, Sidney, Roger F. Murray, and Frank E. Block. 1988. *Graham and Dodd's Security Analysis*, 5th ed. New York: McGraw-Hill.

Fama, Eugene F., and Kenneth R. French. 1992. The Cross-Section of Expected Stock Returns. *Journal of Finance* (47): 427–465.

Fama, Eugene F., and Kenneth R. French. 1993. Common Risk Factors in the Returns on Stocks and Bonds. *Journal of Financial Economics* 33: 3–56.

Fama, Eugene F., and Kenneth R. French. 2002. The Equity Premium. *Journal of Finance* (April): 647–659.

Fama, Eugene F., and Kenneth R. French. 2007. The Anatomy of Value and Growth Stock Returns. *Financial Analysts Journal* 63(3) (November): 44–54.

Garcia-Feijoo, Luis, and Randy D. Jorgensen. 2007. Can Operating Leverage Be the Cause of the Value Premium? At http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1077739.

Gilbert, Gregory A. 1990. Discount Rates and Capitalization Rates: Where Are We?, *Business Valuation Review* (December): 108–113.

Grabowski, Roger, and David King. 1995. The Size Effect and Equity Returns. *Business Valuation Review* (June): 69–74.

Grabowski, Roger, and David King. 1996. New Evidence on Size Effects and Rates of Return. *Business Valuation Review* (September): 103–115.

Grabowski, Roger, and David King. 1999. New Evidence on Size Effects and Rates of Return. *Business Valuation Review* (September): 112–130.

Harrison, Paul. 1998. Similarities in the Distribution of Stock Market Price Changes between the Eighteenth and Twentieth Centuries. *Journal of Business* 71(1) (January).

Hayes, Richard. 1726. *The Money'd Man's Guide: Or, the Purchaser's Pocket-Companion*. London: W. Meadows.

Ibbotson Associates/Morningstar. 2008. *Stocks, Bonds, Bills and Inflation: 2008 Classic Yearbook*. Chicago: Morningstar.

Ibbotson, Roger G., and Gary P. Brinson. 1993. *Global Investing: The Professional's Guide to the World of Capital Markets*. New York: McGraw-Hill.

Ibbotson, Roger G., and Peng Chen. 2003. The Supply of Stock Market Returns. *Financial Analysts Journal* (January/February) 59(1): 88–98. One can download the article as it existed as Yale International Center for Finance Working Paper No. 00-44, http://icf.som.yale.edu/pdf/Supply(v5).pdf. The research is updated in *SBBI—2009, Valuation Yearbook*, 67–70.

Jacobs, Bruce I., and Kenneth N. Levy. 1988. Disentangling Equity Return Regularities: New Insights and Investment Opportunities. *Financial Analysts Journal* (May/June): 18–42.

Lettau, Martin, Sydney C. Ludvigson, and Jessica A. Wachter. 2008. The Declining Equity Premium: What Role Does Macroeconomic Risk Play? *Review of Financial Studies* (July) 1653–1687. At http://finance.wharton.upenn.edu/~jwachter/research/LettauLudvigsonWachter2008rfs.pdf.

Loderer, Claudio, and Urs Waelchli. 2009. Firm Age and Performance. SSRN working paper. At http://papers.ssrn.com/sol3/papers.cfm?abstract_id=1342248.

Mandelker, Gershon N., and S. Ghon Rhee. 1984. The Impact of the Degrees of Operating and Financial Leverage on Systematic Risk of Common Stock. *Journal of Financial and Quantitative Analysis* 19: 45–57.

Neal, L. 1990. *The Rise of Financial Capitalism: International Capital Markets in the Age of Reason*. Cambridge: Cambridge University Press.

Peterson, James D., Paul D. Kaplan, and Roger G. Ibbotson. 1997. Estimates of Small Stock Betas Are Much Too Low. *Journal of Portfolio Management* (Summer) 23, 104–111.

Schwert, G. William, and Paul J. Seguin. 1990. Heteroscedasticity in Stock Returns. *Journal of Finance*, 45: 1129–1156.

Thomas, George B., Jr. 1972. *Calculus and Analytic Geometry*. Reading, MA: Addison-Wesley.

Voth, Hans-Joachim. 2002. Stock Price Volatility and Political Uncertainty: Evidence from the Interwar Period. At http://papers.ssrn.com/sol3/papers.cfm?abstract_id=302926.

Ye, Qing, Charles R. Hickson, and John Turner. 2009. The Asset Pricing Anomalies in 19th Century Britain. SSRN working paper. At http://ssrn.com/abstract=1341328.

# APPENDIX 5A

# Automating Iteration Using Newton's Method

This appendix is optional. It is mathematically difficult and is more analytically interesting than practical. The practical reader can skip this appendix.

In this appendix, we present a numerical method for automatically iterating to the correct log size discount rate. Isaac Newton invented an iterative procedure using calculus to provide numerical solutions to equations with no analytic solution. Most calculus texts will have a section on his method (for example, see Thomas, 1972). His procedure involves making an initial guess of the solution, then subtracting the equation itself divided by its own first derivative to provide a second guess. We repeat the process until we converge to a single answer.

The benefit of Newton's method is that it will enable us to simply enter assumptions for the cash flow base and the perpetual growth, and the spreadsheet will automatically calculate the value of the firm without our having to manually go through the iterations as we did in Tables 5.4A, B, and C. Remember, some iteration process is necessary when using log size discount rates, because the discount rate is not independent of size, as it is using other discount rate models.

To use Newton's procedure, we rewrite equation (5.10) as:

$$\text{Let } f(V) = V - \left[\frac{CF}{(a + b\ln V - g)}\right] = 0. \tag{A5.1}$$

We take the first derivative of equation (A5.1), which results in:

$$f'(V) = 1 + \left[\frac{bCF}{V(a + b\ln V - g)^2}\right]. \tag{A5.2}$$

Assuming our initial guess of value is $V_0$, the formula that defines our next iteration of value, $V_1$, is:

$$V_1 = V_0 - \frac{V_0 - \dfrac{CF}{(a + b\ln V_0 - g)}}{1 + \dfrac{bCF}{V_0(a + b\ln V_0 - g)^2}}. \tag{A5.3}$$

Table A5.1 shows Newton's iterative process for the simplest valuation. In B22 we enter our initial guess of value of an arbitrary $24,000, our forecast cash flow base

| | A | B |
|---|---|---|
| 1 | **Table A5.1** | |
| 2 | **Gordon Model Valuation** | |
| 3 | **Using Newton's Iterative Process** | |
| 4 | | |
| 5 | Iteration | Value |
| 6 | *t* | *V*( *t*) |
| 7 | 0 | 24,000 |
| 8 | 1 | 17,311,063 |
| 9 | 2 | 594,875 |
| 10 | 3 | 490,645 |
| 11 | 4 | 490,080 |
| 12 | 5 | 490,080 |
| 13 | | |
| 14 | Proof of Calculation: | |
| 15 | | |
| 16 | Discount Rate | 27.40% |
| 17 | Gordon Multiple | 4.9008 |
| 18 | × CF = FMV | $490,080 |
| 19 | | |
| 20 | Parameters | |
| 21 | | |
| 22 | *V*(0) | 24,000 |
| 23 | *C F* | 100,000 |
| 24 | *g* | 7% |
| 25 | a (Table 5.1, C37) | **46.225%** |
| 26 | b (Table 5.1, C43) | **-1.436%** |
| 27 | | |
| 28 | | |
| 29 | Model Sensitivity | |
| 30 | FMV | Initial Guess = *V*(0) |
| 31 | Explodes | 300,000,000,000 |
| 32 | $ 490,080 | 200,000,000,000 |
| 33 | $ 490,080 | 24,000 |
| 34 | Explodes | 23,000 |
| 35 | | |
| 36 | Formula in Cell B8: | |
| 37 | =B7-((B7-(CF/(A+B*LN(B7)-G)))/(1+(B*CF)/(B7*(A+B*LN(B7)-G)^2))) | |
| 38 | | |
| 39 | *Note:* The above formula assumes an end-of-year Gordon model. | |
| 40 | Newton's method converges for the midyear Gordon model, but | |
| 41 | too slowly to be of practical use. | |

of \$100,000 (B23), perpetual growth $g = 7\%$ (B24), and our regression coefficients *a* and *b* (B25 and B26, which come from Table 5.1, C37 and C43, respectively).

In B7, we see our initial guess of \$24,000. The iteration #2 value of \$17,311,063 (B8) is the result of the formula for B8, which appears in B37 and is equation (A5.3).[58] B9 to B12 are simply the formula in B8 copied to the cells below.

Once we have the formula, we can value any firm with constant growth in its cash flows by simply changing the parameters in B23 to B24. Of course, we update the regression constant and slope, *a* and *b*, annually with the new SBBI yearbook.

[58] B22 (repeated in B7), our initial guess, is $V_0$ in equation (A5.3).

Rows 31 to 34 show the sensitivity of the model to the initial guess. If we guess poorly enough, the model will explode instead of converging to the right answer. For this particular set of assumptions, an initial guess of anywhere between \$24,000 and \$200 billion will converge to the right answer. Assumptions far enough above \$200 billion or below \$24,000 explode the model.

Unfortunately, the midyear Gordon model, which is more accurate, has a much more complex formula. The iterative process does converge, but much too slowly to be of any practical use. It is better to use the end-of-year Gordon model to converge to the appropriate discount rate and afterward multiply the discount rate by $\sqrt{1+r}$.

# APPENDIX 5B

# Mathematical Appendix

The purpose of this appendix is to provide the mathematics behind the log size model that might have hampered many readers had we put it in the body of the chapter. Additionally, this appendix contains some philosophical analysis of the mathematics—specifically on the nature of exponential decay function and how that relates to phenomena in physics as well as our log size model. This is intended more as intellectual observation than as required information.

We will begin with two definitions:

$r$ = return of a portfolio.
$S$ = standard deviation of returns of the portfolio.

Equation (B5.1) states that the return on a portfolio of securities (each decile is a portfolio) varies positively with the risk of the portfolio, or:

$$r = a_1 + b_1 S. \tag{B5.1}$$

This is a generalization of equation (5.1) in the chapter. This relationship is not directly observable for privately held firms. Therefore we use the next equation, which is a generalization of equation (5.2) from the chapter, to calculate expected return.

In equation (B5.1), the parameter $a_1$ is the regression estimate of the risk-free rate,[59] while the parameter $b_1$ is the regression estimate of the slope, which is the return for each unit increase of risk undertaken (i.e., the standard deviation of returns). Thus, $b_1$ is the regression estimate of the price of or the reward for taking on risk:

$$r = a_2 + b_2 \ln \ FMV, \quad b_2 < 0. \tag{B5.2}$$

Equation (B5.2) states that return decreases in a linear fashion with the natural logarithm of firm value. The parameter $a_2$ is the regression estimate of the return for a \$1 firm[60] —the valueless firm—while the parameter $b_2$ is the regression estimate of the slope, which is the return for each increase in ln FMV. Because it is negative, $b_2$ is the regression estimate of the reduction in return investors accept for investing in

[59]A zero risk asset would have no standard deviation of returns. Thus $S = 0$, and $r = a_1$.
[60]A firm worth \$1 would have ln FMV = ln \$1 = 0. Thus in equation (B5.2), for FMV = \$1, $r = a_2$.

larger firms. The terms $a_1$, $a_2$, $b_1$, and $b_2$ are all parameters determined in regression equations (5.1) and (5.2).

Using 1926–2007 stock market data, our regression estimate of $a_1 = 5.54\%$ (Table 5.1, C23), which compares well with the 82-year mean long-term government bond yield of 5.21% (C24). It would initially appear that regression #1 does a reasonable job of also providing an estimate of the risk-free return.

Focusing now on equation (B5.2), the log size equation, the 82-year regression estimate of $b_2 = -1.436\%$ (C43). Since it is negative, the parameter $b_2$ is the reduction in return that comes about from each unit increase in the natural logarithm of company value. The parameter $a_2$ is the $y$-intercept. It is the return (discount rate) for a valueless firm—more specifically, a firm with \$1 in value—as ln(\$1) = \$0.

Equating the right-hand sides of equation (B5.1) and (B5.2) and solving for $S$, we see how we are implicitly using the size of the firm as a proxy for risk.

$$S = \frac{a_2 - a_1}{b_1} + \frac{b_2}{b_1} \ln \, FMV. \tag{B5.3}$$

Since $a_2$ is the rate of return for the valueless (maximum risk) firm and $a_1$ is the regression estimate of the risk-free rate—flawed as it is—the difference between them, $a_2 - a_1$, is the equity premium for the maximum-risk firm, that is, a \$1.00 or valueless firm. Dividing by $b_1$, the price of risk (or reward) for each increment of standard deviation, we get $\frac{a_2 - a_1}{b_1}$, the standard deviation of a \$1 firm.[61] We then reduce our estimate of the standard deviation by the ratio of the relative prices of risk (the price of risk in log size divided by the price of risk in standard deviation) and multiply that ratio by the log of the size of the firm. In other words, we start with the standard deviation of the maximum-risk firm, a \$1 firm, and reduce the standard deviation by the ratio of the regression slopes[62] times the log of the value of the firm in order to calculate the standard deviation of the firm.

Rearranging equation (B5.3), we get

$$\ln \, FMV = \frac{(a_1 - a_2) + b_1 S}{b_2}. \tag{B5.4}$$

Raising both sides of the equation as powers of $e$, the natural exponent, we get:

$$FMV = e^{\frac{(a_1 - a_2) + b_1 S}{b_2}} = e^{\frac{(a_1 - a_2)}{b_2}} e^{\frac{b_1 S}{b_2}}, \tag{B5.5}$$

or

$$FMV = Ae^{kS}, \quad \text{where } A = e^{\frac{(a_1 - a_2)}{b_2}}, \; k = \frac{b_1}{b_2} < 0. \tag{B5.6}$$

Here we see that the value of the firm or portfolio declines exponentially with risk (i.e., the standard deviation).

Unfortunately, the standard deviation of most private firms is unobservable, since there are no reliable market prices. Therefore, we must solve for the value of

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[61] This is the standard deviation of a \$1 firm, because when we substitute \$1 into the right-hand term in equation (B5.3), ln \$1 = 0, and only the first term on the right side of the equation remains.

[62] $b_2/b_1$ is the ratio of the slopes of the regression lines. As $b_1$ is positive and $b_2$ is negative, $b_2/b_1$ is also negative. Each unit increase in ln FMV reduces our regression estimate of $S$.

a private firm another way. Restating equation (B5.2),

$$r = a_2 + b_2 \ln(FMV). \tag{B5.7}$$

Rearranging the equation, we get:

$$\ln\ FMV = \frac{(r - a_2)}{b_2}. \tag{B5.8}$$

Raising both sides by $e$, that is, taking the antilog, we get:

$$FMV = e^{\frac{(r-a_2)}{b_2}}, \tag{B5.9}$$

or

$$FMV = Ce^{mr}, \tag{B5.10}$$

where $C = e^{-\frac{a_2}{b_2}}$ and $m = \frac{1}{b_2}$.

This shows that the FMV of a firm or portfolio declines exponentially with the discount rate. This is reminiscent of a continuous time present value formula; in this case, though, instead of traveling through time we are traveling though expected rates of return. The same is true of equation (B5.6), where we are traveling through degree of risk.

## What Does the Exponential Relationship Mean?

Let's try to get an intuitive feel for what an exponential relationship means and why that makes intuitive sense. Equation (B5.6) shows that the fair market value of the firm is an exponentially declining function of risk, as measured by the standard deviation of returns. Repeating equation (B5.6), FMV $= Ae^{kS}$, $k < 0$. Because we find that risk itself is primarily related to the size of the firm, we come to a similar equation for size. Repeating equation (B5.10), we see that FMV $= Ce^{mr}$, $m < 0$.

In physics, radioactive minerals such as uranium decay exponentially. That means that a constant proportion of uranium decays at every moment. As the remaining portion of uranium is constantly less over time due to the radioactive decay, the amount of decay at any moment in time or during any finite time period is always less than the previous period. A graph of the amount of uranium remaining over time would be a downward-sloping curve, steep at first and increasingly shallow over time. Figures 5.3 and 5.7 are shaped like exponential decay curves.[63]

It appears the same is true of the value of firms. Instead of decaying over time, their value decays over risk. Because it turns out that both risk and the rate of return are so closely related to size, the value also decays exponentially with the market rate of return (i.e., the discount rate). The graph of exponential decay in value over risk has the same general shape as the uranium decay curve.

Imagine the largest ship in the world sailing on a moderately stormy ocean. You as a passenger hardly feel the effects of the storm. If instead you sailed on a slightly smaller ship, you would feel the storm a bit more. As we keep switching to

[63]The larger in absolute value the negative decay rate, for example, $k$ in equation (B5.6), the steeper the curve. If $k = -0.1$, the curve decreases faster than if $k = -0.5$.

increasingly smaller ships, the storm feels increasingly powerful. The smallest ship on the NYSE might be akin to a 35-foot cabin cruiser, while appraisers often have to value little paddleboats, the passengers of which would be in danger of their lives while the passengers of the General Electric boat would hardly feel the turbulence.

That is my understanding of the principle underlying the size effect. Size offers diversification of product and service. Size reduces transaction costs in proportion to the entity; for example, the proceeds of floating a $1 million stock issue after flotation costs are far less in percentage terms than floating a $100 million stock issue. Large firms have greater depth and breadth of management, and greater staying power. Even the chances of surviving bankruptcy increase with firm size. Remember Chrysler? If it were not a very big business, the government would never have jumped in to rescue it.[64] The same is true of the S&Ls. For these and other reasons, the returns of big businesses fluctuate less than small businesses, which means that the smaller the business, the greater the risk and the greater the return.

The FMV of a firm or portfolio declining exponentially with increases in the discount rate/risk is reminiscent of a continuous-time present value formula, where Present Value = Principal $\times$ $e^{-rt}$; in this case, though, instead of traveling through time we are traveling though expected rates of return/risk.

[64]I wrote this in the first edition of this book to mean when Lee Iacocca took over in the 1978, and here we are again—this time with GM, AIG Insurance, and the whole kit-and-caboodle of failed giants running into Uncle Sam's open arms!

# APPENDIX 5C

# Abbreviated Review and Use

This abbreviated version of the chapter is intended for those who simply wish to learn the model without the benefit of additional background and explanation, or those who wish to use it as a quick reference for review.

## Introduction

Historically, small companies[65] have shown higher rates of return when compared to large ones[66] over the past 82 years (Ibbotson Associates, 2008). Further investigation into this phenomenon has led to the discovery that return (the discount rate) strongly correlates with the natural logarithm of the value of the firm (firm size), which has the following implications:

- The discount rate is a linear function negatively related to the natural logarithm of the value of the firm.
- The value of the firm is an exponential decay function, decaying with the investment rate of return (the discount rate). Consequently, the value also decays in the same fashion with the standard deviation of returns.

As we have already described regression analysis in Chapter 3, we now apply these techniques to examine the statistical relationship between market returns, risk (measured by the standard deviation of returns), and company size.

## Regression #1: Return versus Standard Deviation of Returns

Columns A–F in Table 5.1 contain the input data from the *Stocks, Bonds, Bills and Inflation 2008 Classic Yearbook* (Ibbotson Associates, 2008) for all of the regression analyses as well as the regression results. We use 82-year average returns in both regressions. For simplicity, we have collapsed 820 data points (82 years × 10 deciles)

[65]From 1926 to 1981, NYSE fifth quintile returns; from January 1982 to March 2001, DFA U.S. 9-10 Small Company Portfolio; from April 2001 to December 2007, DFA U.S. Micro Cap Portfolio.
[66]Based upon the S&P Composite Index.

into 10 data points by using averages. Thus, the regressions are cross-sectional rather than time series. In column A we list Ibbotson Associates' (2008) division of the entire NYSE/AMEX/NASDAQ into 10 different divisions—known as *deciles*—based on size, with the largest firms in decile #1 and the smallest in decile #10.[67] Columns B through F contain market data for each decile, which is described in the following.

Note that the 82-year average market return in column B rises with each decile, as does the standard deviation of returns (column C). Column D shows the market capitalization, which is the price per share times the number of shares, of each decile near the end of 2007. It is also the fair market value (FMV).

Dividing column D (FMV) by column F (the number of firms in the decile), we obtain column G, the average capitalization, or the average fair market value of the firms in each decile. Column I, the last column in the table titled ln (FMV), is the natural logarithm of the average FMV.

Regression of ln (FMV) against standard deviation of returns for the period 1926–2007 (Table 5.1, C23 to C33), gives rise to the equation:

$$r = 5.54\% + (33.76\% \times S), \tag{5.1}$$

where $r$ = return and $S$ = standard deviation of returns.

The regression statistics of adjusted $R^2$ of 97.04% (C27), a $t$-statistic of the slope of 17.2 (C32), a $p$-value of less than 0.01% (C33), and the standard error of the estimate of 0.45% (C25), all indicate a high degree of confidence in the results obtained. Also, the constant of 5.54% (C23) is the regression estimate of the long-term risk-free rate, which compares favorably with the 82-year arithmetic mean income return from 1926 to 2007 on long-term government bonds of 5.21% (C24).[68]

The major problem with direct application of this relationship to the valuation of small businesses is coming up with a reliable standard deviation of returns. Appraisers cannot directly measure the standard deviation of returns for privately held firms, since there is no objective stock price. We can measure the standard deviation of income, and we covered that in our discussion in the chapter of Grabowski and King (1999).

## Regression #2: Return versus Log Size

Fortunately, there is a much more practical relationship. Notice that the returns are negatively related to the market capitalization, that is, the fair market value of the firm. The second regression in Table 5.1 (C37 to C46) is the more useful one for valuing privately held firms. Regression #2 shows return as a function of the natural logarithm of the FMV of the firm. The regression equation for the period 1926–2007 is:

$$r = 46.22\% - [1.436\% \times \ln(\text{FMV})]. \tag{5.2}$$

[67] All of the underlying decile data in Ibbotson originate with the University of Chicago's Center for Research in Security Prices (CRSP), which also determines the composition of the deciles.

[68] SBBI Classic 2008, p. 142, uses this measure as the risk-free rate for CAPM.

The adjusted $R^2$ is 93.02% (C40), the $t$-statistic is −11.0 (C45), and the $p$-value is less than 0.01% (C46), meaning that these results are statistically robust. The standard error for the $Y$-estimate is 0.70% (C38), which means that we can be 95% confident that the regression forecast is approximately accurate to within plus or minus 2 × 0.70% ≅1.4%.

## Need for Annual Updating

Table 5.1 should be updated annually, as the Ibbotson averages change, and new regression equations should be generated. This becomes more crucial when shorter time periods are used, because changes will have a greater impact on the average values. Additionally, it is important to be careful to match the regression equation to the year of the valuation. If the valuation assignment is retroactive and the valuation date is 2004, then don't use the regression equation for 1926–2008. Instead you should run your own regression on the Ibbotson data or contact the author to provide the right equation.

## Computation of Discount Rate Is an Iterative Process

In spite of the straightforwardness of these relationships, we have a problem of circular reasoning when it comes to computing the discount rate. We need the FMV to obtain the discount rate, which is in turn used to discount cash flows or income to calculate the FMV! Hence, it is necessary to make sure that our initial estimate of FMV is consistent with the final result. If it is not, then we have to keep repeating the process until the results are consistent. Fortunately, discount rates remain virtually constant over large ranges of values, so this should not be much of a problem.

## Practical Illustration of the Log Size Model: Discounted Cash Flow Valuations

Let's illustrate how the iterative process works with a specific example. The assumptions in Tables 5.4A, 5.4B and 5.4C are identical, except for the discount rate.

Table 5.4A is a very simple discounted cash flow (DCF) analysis of a hypothetical firm. The basic assumptions appear in B30–B35. We assume the firm had $100,000 cash flow in 2007. We forecast annual growth through the year 2012 in row 31 and perpetual growth at 4% (B33) thereafter. In B35 we assume a 23% discount rate.

The DCF analysis in rows 5 through 7 is standard and requires little explanation other than that the present value factors are midyear, and the value in B16 is a marketable minority interest. It is this value ($737,360) that we use to compare the consistency between the assumed discount rate of 23% (in B32) and calculated discount rate according to the log size model.

We begin calculating the discount rate using the log size model in B22, where we compute ln (737,360) = 13.5108. This is the natural log of the initially computed marketable minority value of the firm. We repeat the $x$-coefficient of −0.01436 from Table 5.1, C43 in B23 and multiply B22 × B23 to calculate the product of −0.1941 in B24. To that we add the regression constant of 0.4622 (B25, transferred from Table 5.1, C37) and subtract the annual increase in the PE ratio of 0.8% (B26) from Ibbotson's supply-side model to obtain an implied (ex-post) discount rate of 26% (rounded, B27).

Comparing the two discount rates—assumed and calculated—reveals that we initially assumed the discount rate too low, which means that we overvalued the firm. We will correct that problem in Table 5.4B. In the meantime, though, we continue describing the remaining cells in the spreadsheet.

B17 through B19 contain the control premium and discount for lack of marketability, which we assume at 40% (B34) and 35% (B35), respectively. These are simple assumptions with no intent to be as realistic as possible, as we cover these topics in depth in Chapter 8. Because the assumed and calculated discount rates are not yet consistent, we ignore the specific numerical results, as they are irrelevant.

**THE SECOND ITERATION: TABLE 5.4B** We revise our discount rate to 26% (B32), which was our calculated discount rate in Table 5.4A, B27. In this case, we arrive at a marketable minority FMV of $640,971 (B16). When we perform the discount rate calculation with this value in B22 through B27, we obtain a matching discount rate of 26%, indicating that no further iterations are necessary.

**CONSISTENCY IN LEVELS OF VALUE** In calculating discount rates, it is important to be consistent in the level of fair market value that we are using. Since the log size model is based on returns from the NYSE/AMEX/NASDAQ, the corresponding values generated are on a marketable minority basis. Consequently, it is this level of value that we should use for the discount rate calculations.

Frequently, however, the marketable minority value is not the ultimate level of fair market value that we are calculating. Therefore, it is crucial to be aware of the differing levels of FMV that occur as a result of valuation adjustments. For example, if our valuation assignment is to calculate an illiquid control interest, we will add a control premium and subtract a discount for lack of marketability from the marketable minority value.[69] Nevertheless, we use only the marketable minority level of FMV in iterating to the proper discount rate, as we must first maintain consistency in the calculation of the discount rate.

**TABLE 5.4C: ADDING COMPANY-SPECIFIC ADJUSTMENTS TO THE DCF ANALYSIS** The final step in our DCF analysis is performing company-specific adjustments. Let's suppose for illustrative purposes that there is only one owner of this firm. She is 62 years old and had a heart attack three years ago. The success of the firm depends to a great extent on her personal relationships with customers, which may not be easily duplicated by a new owner. Therefore, we decide to add a 2% company-specific adjustment to the 26% discount rate from Table 5.4B to reflect this situation,[70] which leads us to a 28% (B32) discount rate.

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[69]Not all authorities would agree with this statement. There is considerable disagreement on the levels of value. We cover those controversies in Chapter 8.

[70]A different approach would be to take a discount from the final value, which would be consistent with key-person-discount literature appearing in a number of articles in *Business Valuation Review* (see the BVR index for cites). Another approach is to lower our estimate of earnings to reflect our weighted average estimate of decline in earnings that would follow from a change in ownership or the decreased capacity of the existing owner, whichever is more appropriate, depending on the context of the valuation. In this example, I have already assumed that we have done that. There are opinions that one should lower earnings estimates and not increase the discount rate. It is my opinion that we should definitely increase the

Prior to adding a company-specific adjustment, it is important to achieve internal consistency in the *ex ante* and *ex post* marketable minority values, as we did in Table 5.4B. The remainder of the table is identical to its predecessors, except that we eliminate the *ex post* calculation of the discount rate in B22 through B27, since we have already achieved consistency.

It is at this point in the valuation process that the dollar amounts of our control premium and discount for lack of marketability are meaningful. Our final fair market value of $537,046 (B20) is on an illiquid control basis.

In a valuation report, it would be unnecessary to show Table 5.4A. One should show Tables 5.4B and 5.4C only.

The Table 5.4 series of examples still does not consider the material later in the chapter in the section, "The Wedge between Public and Private Firm Valuations," in which we introduce the concept of a private firm premium. Thus, the discount rate calculations in the Table 5.4 series are not the end of the story. The appraiser still needs to consider a private firm premium in addition to the company-specific premium.

## Total Return versus Equity Premium

CAPM uses an equity risk premium as one component for calculating return. The discount rate is calculated by multiplying the equity premium by beta and adding the risk-free rate. In my first article on the log size model (Abrams, 1994), I also used an equity premium in the calculation of discount rate. Similarly, Grabowski and King (1995) used an equity risk premium in the computation of discount rate.

I eliminated the equity premium term in my second article (Abrams, 1997) in favor of total return because of the low correlation between stock returns and bond yields for the 60 years prior to 1996, that is, for the data in the 1997 article. The actual correlation was 6.3%—an amount small enough to ignore. For 1926–2007, the correlation is down to 3.8% (Table 5.5, C90) for large cap stocks.

## Adjustments to the Discount Rate

Privately held firms are generally owned by people whose investment portfolios are not well diversified. Table 5.3 was derived from stock portfolios that were diversified in every sense except for size, as size itself was the method of sorting the deciles. In contrast, the owner of the local bar or dry cleaner is probably not well diversified, nor is the probable buyer. The appraiser should consider adding a private firm risk premium to the discount rate implied by Table 5.3 to account for that. On the other hand, a $100 million FMV firm is likely to be bought by a well-diversified buyer and may not merit increasing the discount rate.

Warren D. Miller, CFA, ASA, teaches a top-notch course to incorporate nonsystematic risk into our valuations. I asked his permission to quote him in this book, and after he reviewed the above paragraph he said that his SPARC tri-level unsystematic risk framework results in adjustments of −3% to +35%. He stated that he computes his adjustments empirically and updates them annually. Adjustments of the potential

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discount rate in such a situation, and we should also decrease the earnings estimates if that has not already been done.

magnitude that he computes deserve the serious attention of the valuation profession, as these adjustments can dwarf the choice of the baseline discount rate and almost any other valuation adjustment that we make. However, that is outside the scope of this book.[71]

Another common adjustment to Table 5.3 discount rates would be for the depth and breadth of management of the subject company compared to other firms of the same size. In general, Table 5.3 already incorporates the size effect. No one expects a $100,000 FMV firm to have three Harvard MBAs running it, but there is still a difference between a complete one-man show and a firm with two talented people.

In general, this methodology of calculating discount rates will increase the importance of comparing the subject company to its size and industry peers via RMA Associates or Troy's Almanac data. Differences in leverage between the subject company and its RMA peers could well be another common adjustment, although it is easy to overdo this. If we suspect that an independent variable is statistically significant, we could run regressions using data from the guideline public company method and guideline M&A method to test that variable. If it is statistically significant, then it makes sense to adjust for it in a DCF. If not, then it is still possible to make an adjustment for it, but it is best to be cautious in doing so.

### Discounted Cash Flow or Net Income?

Since the market returns are based on the cash dividends and the market price at which one can sell one's stock, the discount rates obtained with the log size model should be properly applied to cash flow, not to net income. We appraisers, however, sometimes work with clients who want a quick-and-dirty valuation, and we often don't want to bother estimating cash flow. I have seen suggestions in *Business Valuation Review* (Gilbert, 1990, for example) that we can increase the discount rate and thereby apply it to net income, and that will often lead to reasonable results. Nevertheless, it is better to make an adjustment from net income based on judgment to estimate cash flow to preserve the accuracy of the discount rate. Chapters 1 and 2 cover this topic.

## The Wedge between Public and Private Firm Valuations

In the world of publicly held firms covered by the SBBI yearbooks, the total returns, $r$, is the sum of income returns, which is dividends, plus capital gains. This is approximately equal to the dividend yield, $d$, plus the expected growth in the value of the stock, $g$, as in equation (5.16):[72]

$$r = d + g. \tag{5.16}$$

Our rate of return could be either the historical actual rate of return or our forecast future rate of return. The latter is more relevant for valuation. Therefore it

[71]Our discussion of generic private firm risk, however, is within the scope of this book.

[72]It differs by changes in the PE ratio and the reinvestment yield. Also the equation is exact for geometric returns only.

is the forecast dividend yield rather than the historical yield that is appropriate in equation (5.16).

Let the dividend yield equal forecast dollar dividends, $D_{t+1}$, divided by the current stock price, $P_t$. Substituting this into equation (5.16) results in:

$$r = \frac{D_{t+1}}{P_t} + g. \tag{5.17}$$

Rearranging terms, we get:

$$\frac{D_{t+1}}{P_t} = r - g. \tag{5.18}$$

Taking reciprocals, we get:

$$\frac{P_t}{D_{t+1}} = \frac{1}{r-g}. \tag{5.19}$$

Multiplying both sides by $D_{t+1}$, we get:

$$P_t = D_{t+1} \times \frac{1}{r-g}. \tag{5.20}$$

If we rename dividends to cash flow and the stock price as *FMV*, the equation becomes:

$$FMV = CF_{t+1} \times \frac{1}{r-g}. \tag{5.21}$$

Thus, Ibbotson's total return equation is our familiar Gordon model equation in disguise, albeit the end-of-year version of the Gordon model.[73]

## Private Firm Risk Is the Wedge

It has perplexed me in the past why equation (5.16) applies to public firms but does not seem to apply to private firms. I believe the reason is simple—private firms are riskier than public firms for many reasons.[74]

In the face of risks that are specific to being a private firm, equation (5.16) transforms into equation (5.22):

$$r_{Private} > d_{Private} + g_{Private}. \tag{5.22}$$

More specifically, we can restate equation (5.22) as follows:

$$r_{Private} = d_{Private} + g_{Private} + PFR + CSR, \tag{5.23}$$

where *PFR* is a generic private firm risk that incorporates the risk differential between public firms and all private firms, while *CSR* is company-specific risk.

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[73]We assumed dividends come at the end of the year. If we had instead been more precise and assumed dividends occur evenly throughout the year, we would end with the midyear Gordon model.

[74]See the chapter for the details.

For example, suppose that a private firm has an expected dividend yield of 2% and growth in cash flows[75] of 5%. It is unrealistic to think that the private firm has a discount rate of 7%. As a starting point, the discount rate has to be based on the rate of return an investor could earn for a publicly traded firm of the same risk as this private firm. A decile #2 private firm might have the same risk as, let's say, a decile #10 public firm.

Equations (5.22) and (5.23) show that we are using our asset pricing models—whether log size, CAPM, and so on—to calculate discount rates, not growth rates, for most privately held firms, because we cannot assume the equality of the discount rate and the dividend yield plus the growth rate. Thus, the growth rate of a private firm is rarely equal to the discount rate minus the dividend rate, almost always being lower.

It is important to keep our measures of $r$ and $g$ consistent. When we use the arithmetic rate of return, it is important to use an arithmetic forecast growth rate, not the geometric growth rate. Thus, while we appraisers are fond of calculating the compound annual rate of growth (CAGR) of sales and net income in our analysis of historical financial results, we should be using the arithmetic growth rates instead as our base.

## Measuring Private Firm Risk

Now that we have established that private firm risk exists above public firm risk, we need to measure it. In doing so we are moving into uncharted territory. As mentioned earlier, it is more difficult to create and balance portfolios with privately held firms.

We begin with the standard deviation of the 1926–2008 decile #10 portfolio of 45% (Table 5.1, C17, rounded). A finance text (Bodie, Kane, and Marcus, 1995, Figure 6.2, p. 135) shows a fully diversified portfolio as having 300 stocks in it, with a relative standard deviation of 1.00. It shows a one-stock portfolio as having a relative standard deviation of 2.50, that is, $2\frac{1}{2}$ times the standard deviation of a 300-stock portfolio. A 10-stock portfolio has a relative standard deviation of 1.26. Let's see what we can make out of these benchmarks.

We multiply the 45% standard deviation of the decile #10 portfolio by 2.5 to calculate the standard deviation of a single stock, which equals 113%. The difference in standard deviations is 68%. We multiply 68% × 34% = 23%. The 34% is the rounded $x$-coefficient from Table 5.1, C30. It is the increases in the discount rate for each percent increase in the standard deviation of the portfolio. Table 5.1, B17 shows the decile #10 82-year return as 20.98%, and we subtract 0.8% for the average annual increase in the PE ratio, which results in a rounded return of 20% for a decile #10 portfolio. If instead the portfolio consists of only one stock, then the required return should be 20% + 23% = 43%. In other words, a rational investor should be

[75]Note that Ibbotson's definition of *growth* is growth in the stock price—capital gains—which is a definition that is unavailable to appraisers of private firms, since we do not have a market-determined stock price. Thus our next-best definition of growth is that of cash flows, and we often assume that is sales growth combined with an assumed constant profit margin and retention ratio.

indifferent between investing in 300 decile #10 stocks with an expected return of 20% and a single public firm with an expected return of 43%.

Instead, if we look at owning a decile #10-size private firm as part of our small portfolio as being equivalent to owning a 10-stock portfolio instead of a 300-stock portfolio, the portfolio standard deviation would be 1.26 × 45% = 57%, a 12% increase in portfolio standard deviation. Multiplying this by the $x$-coefficient of 34% we get an increase in required return of 4%. This strikes me as being a reasonable benchmark increase in the required return for investing in a private firm. This 4% calculation applies to all private firms; it is the term *PFR* (private firm risk) in equation (5.23). Of course, it is not the truth coming from Mt. Sinai, but it is a reasonable estimate.

It is possible to modify that calculation to include both private firm risk and company-specific risk. Suppose for a specific subject company we consider it equivalent to owning a 5-stock portfolio. The Bodie-Kane-Marcus table shows the adjustment factor to be 1.40. We calculate the increase as 1.40 × 45% = 63%, an 18% increase in portfolio standard deviation. Multiplying this by the $x$-coefficient of 34% we get an increase in required return of 6%. This implies the subject company's discount rate should be 20% + 6% = 26%. Of the 6% increase in the discount rate, 4% is the generic private firm risk and 2% is company-specific risk.

## Satisfying Revenue Ruling 59-60

Revenue Ruling 59-60 requires that we look at publicly traded stocks in the same industry as the subject company. In the first edition of this book, I claimed that our excellent results with the log size model to calculate a discount rate to use in a discounted cash flow method, combined with Jacobs and Levy's general finding of industry insignificance, satisfied the intent of Revenue Ruling 59-60 for small and medium firms without the need to actually perform a guideline publicly traded company method (GPCM). For the moment we will follow that reasoning, and at the end of this section we will see how things have changed in the approximately 10 years since publication of the first edition.

### Strengths and Weaknesses of DCF and the Market Methods

First, however, it is worthwhile to look at the strengths and weaknesses of each of the methods that we commonly use. The DCF is an introverted valuation approach. Its strength is the ability to customize the valuation to our subject company. We do considerable financial and statistical analysis of our subject, forecast cash flows using growth rates unique to our subject, and discount them to present value at a market-determined rate, usually with some company-specific adjustments, and preferably with the generic private firm risk adjustment (per our earlier discussion). The only part of the DCF that is more outward-looking is the calculation of the discount rate.

The market methods—both public and private—are extroverted valuation methods. The majority of our efforts go to developing the mathematical relationship of value of the guideline companies to the independent variables that cause value. Then we apply those relationships to our subject company.

Thus, the DCF is best at being customized to the subject company. However, because the DCF lacks external feedback, its weakness is that it is easy enough to value a company improperly using unrealistic forecasts of sales growth, margin, and payout ratio. It is particularly difficult to value early-stage firms.

The strength of the market methods is that they are based on real valuations of real companies. Thus they do not suffer from "valuation introversion." However, they may suffer from other weaknesses: lack of comparability to the subject company, insufficient data (too few observations and/or too little information about each observation),[76] bad data,[77] and inconsistent results due to outliers.

Thus, it is best to use a DCF and both market methods when they apply. However, it is often just plain wrong to use a GPCM as a valuation method for a small business. Unless the appraiser can eliminate heteroscedasticity and control for size and risk differences in the public companies, it is often best not to use the GPCM.

## The Information in a PE Multiple and Applicability to the Subject Company

We repeat equation (4.28) from Chapter 4 to show the relationship of the PE multiple to the Gordon model:

$$PE = (1 + g_1) \times POR \times \frac{\sqrt{1+r}}{r-g} \tag{4.28}$$

Relationship of the PE multiple to the Gordon model multiple.

The PE multiple[78] of a publicly- traded firm gives us information on the one-year and long-run expected growth rates and the discount rate of that firm—and nothing else. The PE multiple gives us only a combined relationship of $r$ and $g$. In order to derive either $r$ or $g$, we would have to assume a value for the other variable or calculate it according to a model.

For example, suppose we use the log size model (or any other model) to determine $r$. Then the only new information to come out of a guideline public company method is the market's estimate of $g$,[79] the growth rate of the public firm. There are much easier and less expensive ways to estimate $g$ than to do a GPCM. When all the market research is finished, the appraiser still must modify $g$ to be appropriate for the subject company, and its $g$ is often quite different from the public companies'. So the GPCM wastes much time and accomplishes little.

Because discount rates appropriate for the publicly traded firms are much lower than are appropriate for smaller, privately held firms, using public PE multiples will

[76]This is particularly a problem in the guideline M&A method. None of the databases of sales of privately held firms provide information on historical growth of sales and profitability, let alone expected growth.

[77]For one of the databases, I once found an observation with a regression standardized residual of 12 standard errors—the largest by far that I have ever seen. It does not appear in any table of $t$-statistics that I have seen. I asked one of my analysts to call the organization to report it as an error, and the spokesperson for the organization said it did not maintain the original input forms, and there was no way to verify or check the error.

[78]Included in this discussion are the variations of PE (e.g., P/CF, etc.).

[79]This is under the simplest assumption that $g_1 = g$ and that the retention ratio will remain constant.

lead to gross overvaluations of small and medium privately held firms. This is true even after applying a discount, which many appraisers do, typically in the 20% to 40% range—and rarely with any empirical justification.

If the appraiser is set on using a GPCM, then he or she should use regression analysis and include the logarithm of market capitalization as an independent variable. This will control for size. In the absence of that, it is critical to use only public guideline companies that are approximately the same size as the subject company, which is rarely possible. When valuing a very large privately held company, where the size effect will not confound the results, it is more likely to be worthwhile to do a guideline public company method.

### Changes in the Past 10 Years

Over the past 10 years there is much better availability of data, both of publicly traded firms and private transactions. Therefore the market valuation methods are becoming increasingly important. Whereas I argued for routinely eliminating the GPCM in the first edition of the book and didn't even consider the guideline M&A method a serious method, that is changed now.

It is still a potential danger to inappropriately use the GPCM in the valuation of a small business, and the valuation analyst must guard against that. Nevertheless, as the transactional databases continue to grow and improve, the market methods are increasingly compelling.

## Summary and Conclusions

The log size model is more accurate than CAPM for valuing privately held businesses. In the past it was also much faster and easier to use, requiring no research,[80] whereas CAPM often required considerable research of the appropriate guideline companies. Today, however, CAPM, disguised as the *build-up* method, is very easy to use, so the advantage of log size is now the accuracy.

A further danger of CAPM is not fully accounting for size differences. It is very inaccurate to apply the betas for IBM, Compaq, Apple Computer, and so on to a small start-up computer firm with $2 million in sales without carefully adjusting for size differences, which may or may not be possible. The size effect drowns out any real information contained in betas, especially applying betas of large firms to small firms. The 375% (Table 5.1, P21) improvement that we found in the 0.70% (P20, C38) standard error in the log size equation versus the 2.61% (M20) standard error from the CAPM applies only to firms of the same magnitude. When applied to small firms, CAPM yields even more erroneous results, unless the appraiser compensates by blindly adding another 5% to 10% beyond the typical Ibbotson "small firm premium" and calling that a company-specific adjustment (CSA). I suspect this practice is common, but then it is not really a CSA; rather it is an outright attempt to compensate for a model that has no place being used to value small and medium firms.

[80] One needs only a single regression equation for all valuations performed within a single year.

Around 1994, I valued a midsize firm with $25 million in sales, $2 million in net income after taxes, and very fast growth. I used a guideline public company method—among others—and found 16 guideline companies with positive earnings in the same SIC Code. I regressed the value of the firm against net income, with "great" results—99.5% $R^2$ and high *t*-statistics. When I applied the regression equation to the subject company, the value came to −$91 million![81] I suspect that much of this scaling problem goes on with CAPM as well; many appraisers seriously overvalue small companies using discount rates appropriate only for large firms.

When using the log size model, we extrapolate the discount rate to the appropriate level for each firm that we value. There is no further need for a size adjustment. We merely need to compare our subject company to other companies of its size, not to IBM. Using Robert Morris Associates or Troy's Almanac data to compare the subject company to other firms of its size is appropriate, as those companies are often far more comparable than publicly traded firms.

Since we have already extrapolated the rate of return through the regression equation in a manner that appropriately considers the average risk of being any particular size, the relevant comparison when considering company-specific adjustments is to other companies of the same size. There is a difference between two firms with $2 million in sales volume when one is a one-man show and the other has two Harvard MBAs running it. If the former is closer to average management for a firm of that size, you should probably subtract 1% or 2% from the discount rate for the latter; if the latter is the norm, it is appropriate to add that much to the discount rate of the former—or, better yet, use Warren Miller's SPARC system. Although company-specific adjustments are subjective, they serve to further refine the discount rate obtained from discount rate calculations.

[81] The magnitude problem was solved by regressing the natural log of value against the natural log of net income. That eliminated the scaling problem and led to reasonable results. That particular technique is not always the best solution, but it sometimes works beautifully. We cover this topic in more detail near the end of Chapter 3.

CHAPTER 6

# Arithmetic versus Geometric Means

## *Empirical Evidence and Theoretical Issues*

CHAPTER 6

# Arithmetic versus Geometric Means

## *Empirical Evidence and Theoretical Issues*

This chapter compares the attributes of the arithmetic and geometric mean returns and presents theoretical and empirical evidence why the arithmetic mean is the proper one for use in valuation.

## Introduction

We begin with definitions of arithmetic and geometric means.

### Mathematical Definitions of Arithmetic and Geometric Means

We use the following algebraic symbols in this discussion:

$n =$ number of periods.
$r_A =$ arithmetic mean returns.
$r_G =$ geometric mean returns.
$V_0 =$ value at time 0 (beginning value).
$V_n =$ value at time $n$ (ending value).

The arithmetic mean is the simple average of the numbers in each series. We use equation (6.1) to calculate it.

$$r_A = \frac{1}{n}\sum_{t=1}^{n} r_t. \tag{6.1}$$

The geometric mean return is the compound rate of return over the period. Its formula is in equation (6.2).

$$r_G = \left[\frac{V_n}{V_0}\right]^{\frac{1}{n}} - 1. \tag{6.2}$$

Another way to represent the geometric mean return is in equation (6.3).

$$r_G = \left[\prod_{t=1}^{n}(1 + r_t)\right]^{\frac{1}{n}} - 1. \tag{6.3}$$

### Difference of Arithmetic and Geometric Means

Ibbotson[1] states there are several ways to convert from a geometric to an arithmetic average, and one of them is to assume returns are independently and lognormally distributed over time, with the following relationship:

$$r_A = r_G + \frac{\sigma^2}{2}. \tag{6.4}$$

The lognormal assumption is common. Stock returns are unlimited on the positive side, whereas returns cannot be less than −100%. The distribution of stock returns is skewed to the right, and the lognormal distribution is a standard assumption to model that. Thus, the arithmetic mean will diverge more from the geometric mean the greater is the volatility of the stock. This provides the theoretical basis for our empirical observation later in Table 6.2 that as we increase in the stock market decile number, the difference between the arithmetic and the geometric mean increases.

### Prior Literature

There have been a number of articles about the relative merits of using the arithmetic mean (AM) versus the geometric mean (GM) in valuing businesses for calculating discount rates. For many years, SBBI has taken the position that the arithmetic mean is the correct mean to use in valuation. Conversely, Allyn Joyce (1995) initiated arguments for the GM as the correct mean. Previous articles have centered on Professor Ibbotson's famous example using a binomial distribution with 50%–50% probabilities of a +30% and −10% return. Ibbotson states, "The arithmetic mean equates the expected future value with the present value; it is therefore the appropriate discount rate."[2] This is a fundamental theoretical reason for the superiority of AM. The articles critical of Ibbotson are interesting, but largely incorrect and off on a tangent. There are both theoretical and empirical reasons why the arithmetic mean is the correct one.

## Theoretical Superiority of the Arithmetic Mean

Rather than argue about Ibbotson's much-debated example, let's cite and elucidate a different quote from his book:

> *In general, the geometric mean for any time period is less than or equal to the arithmetic mean. The two means are equal only for a return series that is constant*

[1] *SBBI—2008 Valuation Edition*, p. 97.
[2] *SBBI—2008 Valuation Edition*, p. 79.

> *(i.e., the same return in every period). For a non-constant series, the difference between the two is positively related to the variability or standard deviation of the returns. For example, in Table 6.7 [the SBBI table number], the difference between the arithmetic and geometric mean is much larger for risky large company stocks than it is for nearly riskless Treasury bills.*[3]

The GM measures the magnitude of the returns as the investor starts with one portfolio value and ends with another. It does not measure the variability (volatility) of the journey, as does the AM.[4] The GM is backward-looking, while the AM is forward-looking.[5] As Mark Twain said, "Forecasting is difficult—especially into the future."

Ibbotson[6] cites another reason for using AM rather than GM, which is that when using either CAPM or a building-block approach[7] it is appropriate to use AM, because those are additive models, in which the cost of capital is the sum of its parts.

### Table 6.1: Comparison of Two Stock Portfolios

Table 6.1 contains an illustration of two differing stock series.[8] The first is highly volatile, with a standard deviation of returns of 65% (C17), while the second has a zero standard deviation. Although the arithmetic mean differs significantly for the two, both give rise to an identical geometric mean return. It makes no sense intuitively that the GM is the correct one for calculating discount rates. That would imply that both stocks are equally risky, since they have the same GM; yet no one would *really* consider stock #2 equally as risky as #1. A risk-averse investor will always pay less for #1 than for #2.

## Empirical Evidence of the Superiority of the Arithmetic Mean

Much of the remainder of this chapter is focused on empirical evidence of the superiority of the AM using the log size model. The heart of the evidence in favor of the AM can be found in Chapter 5, Table 5.1, which demonstrates that the arithmetic mean of stock market portfolio returns correlates very well (97% $R^2$) with

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[3]*SBBI—2008 Classic Edition*, p. 108.

[4]Technically it is the *difference* of the AM and GM that measures the volatility. Put another way, the AM consists of two components: the GM plus the volatility.

[5]*SBBI 1997*. *SBBI—2008 Valuation Edition*, p. 77 states GM is more appropriate for reporting past trends, which is backward-looking. On page 79 it states "the arithmetic mean equates the expected future value with the present value; [which means that it is forward-looking]; it is therefore the appropriate discount rate."

[6]*SBBI—2008 Valuation Edition*, p. 77.

[7]This would include the *build-up method*, the *Fama-French Three-Factor* model. While it is less obvious, it includes log size, as the total return is the risk-free rate plus the equity risk premium appropriate to the size firm. We simply combine the two because the correlation of bond and stock returns is very low.

[8]To be more precise, we should consider these to be portfolios. Otherwise it could be possible to diversify away some or all of the firm-specific risk.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | Table 6.1 | | | | |
| 2 | Geometric versus Arithmetic Returns | | | | |
| 3 | | | | | |
| 4 | | Stock (or Portfolio) #1 | | Stock (or Portfolio) #2 | |
| 5 | Year | Price | Annual Return | Price | Annual Return |
| 6 | 0 | $100.00 | NA | $100.00 | NA |
| 7 | 1 | $150.00 | 50.0000% | $111.61 | 11.6123% |
| 8 | 2 | $68.00 | -54.6667% | $124.57 | 11.6123% |
| 9 | 3 | $135.00 | 98.5294% | $139.04 | 11.6123% |
| 10 | 4 | $192.00 | 42.2222% | $155.18 | 11.6123% |
| 11 | 5 | $130.00 | -32.2917% | $173.21 | 11.6123% |
| 12 | 6 | $79.00 | -39.2308% | $193.32 | 11.6123% |
| 13 | 7 | $200.00 | 153.1646% | $215.77 | 11.6123% |
| 14 | 8 | $180.00 | -10.0000% | $240.82 | 11.6123% |
| 15 | 9 | $250.00 | 38.8889% | $268.79 | 11.6123% |
| 16 | 10 | $300.00 | 20.0000% | $300.00 | 11.6123% |
| 17 | Standard Deviation | | 64.9139% | | 0.0000% |
| 18 | Arithmetic Mean | | 26.6616% | | 11.6123% |
| 19 | Geometric Mean | | 11.6123% | | 11.6123% |

the standard deviation of returns (i.e., risk), as well as with the logarithm of firm size, which is related to risk. We show that the AM correlates better with risk than the GM does. Also, the dependent variable (AM returns) is consistent with the independent variable (standard deviation of returns) in the regression. The latter *is* risk, and the former is the fully risk-impounded rate of return. In contrast, the GM does not fully impound risk.

## Table 6.2: Regressions of Geometric and Arithmetic Returns for 1926–2007

Table 6.2 contains both the geometric and arithmetic means for the Ibbotson/CRSP deciles for 1926–2007 data and regressions of those returns against the standard deviation of returns and the natural logarithm of the average market capitalization of the firms in each decile. It is a repetition of Table 5.1, with the addition of the GM data.

The arithmetic mean outperforms[9] the geometric mean in regression #1, with an adjusted $R^2$ of 97.04% (C27) versus 79.16% (D27) and a $t$-statistic of 17.2 (C32) versus 5.9 (D32). In regression #2 we regress the return as a function of log size; the arithmetic mean also outperforms the geometric mean in terms of goodness of fit with the data. Its adjusted $R^2$ is 93.02% (C42) compared to 88.42% (D42) for the geometric mean. The absolute value of its $t$-statistic is 11.0 (C47), compared to 8.3 (D47) for the geometric mean. However, the geometric mean does have a lower standard error of the estimate.

[9] In other words, risk is more correlated to AM than GM.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Table 6.2** | | | | | |
| 2 | **Geometric versus Arithmetic Returns**<br>**NYSE/AMEX/NASDAQ Data by Decile and Statistical Analysis: 1926–2007** | | | | | |
| 3 | | | | | | |
| 4 | | | | | | |
| 5 | | **Geometric** | **Arithmetic** | | **Avg Cap** | |
| 6 | **Decile** | **Mean** | **Mean Return** | **Std Dev** | **= FMV [1]** | **Ln(FMV)** |
| 7 | 1 | 9.59% | 11.31% | 18.91% | $62,022,860,778 | 24.8508 |
| 8 | 2 | 10.92% | 13.16% | 21.63% | $13,375,585,747 | 23.3167 |
| 9 | 3 | 11.27% | 13.72% | 23.31% | $5,789,959,375 | 22.4794 |
| 10 | 4 | 11.12% | 14.07% | 25.68% | $3,857,046,793 | 22.0732 |
| 11 | 5 | 11.66% | 14.85% | 26.49% | $2,666,994,039 | 21.7042 |
| 12 | 6 | 11.72% | 15.14% | 27.10% | $1,637,608,287 | 21.2165 |
| 13 | 7 | 11.57% | 15.46% | 29.47% | $1,379,873,309 | 21.0453 |
| 14 | 8 | 11.76% | 16.58% | 34.18% | $766,269,974 | 20.4570 |
| 15 | 9 | 11.88% | 17.28% | 36.45% | $443,897,410 | 19.9111 |
| 16 | 10 | 13.55% | 20.98% | 44.58% | $113,636,704 | 18.5485 |
| 17 | Std Dev | 1.0% | 2.6% | | | |
| 18 | Value Wtd Index | 10.7% | 12.6% | | | |
| 19 | | | | | | |
| 20 | **Regression #1 (Based on Table 5.1): Return = f(Std Dev. of Returns)** | | | | | |
| 21 | | | | | | |
| 22 | | | **Arithmetic** | **Geometric** | | |
| 23 | | | **Mean** | **Mean** | | |
| 24 | Constant | | 5.54% | 8.19% | | |
| 25 | Std Err of *Y* Est | | 0.45% | 0.45% | | |
| 26 | R Squared | | 97.37% | 81.48% | | |
| 27 | Adjusted R Squared | | 97.04% | 79.16% | | |
| 28 | No. of Observations | | 10 | 10 | | |
| 29 | Degrees of Freedom | | 8 | 8 | | |
| 30 | *X* Coefficient(s) | | 33.76% | 11.50% | | |
| 31 | Std Err of Coef. | | 1.96% | 1.94% | | |
| 32 | *T* | | 17.2 | 5.9 | | |
| 33 | *P* | | <0.01% | 0.03% | | |
| 34 | | | | | | |
| 35 | **Regression #2 (Based on Table 5.1): Return = f[Ln(FMV)]** | | | | | |
| 36 | | | | | | |
| 37 | | | **Arithmetic** | **Geometric** | | |
| 38 | | | **Mean** | **Mean** | | |
| 39 | Constant | | 46.22% | 22.78% | | |
| 40 | Std Err of *Y* Est | | 0.70% | 0.33% | | |
| 41 | R Squared | | 93.80% | 89.70% | | |
| 42 | Adjusted R Squared | | 93.02% | 88.42% | | |
| 43 | No. of Observations | | 10 | 10 | | |
| 44 | Degrees of Freedom | | 8 | 8 | | |
| 45 | *X* Coefficient(s) | | -1.436% | -0.523% | | |
| 46 | Std Err of Coef. | | 0.131% | 0.063% | | |
| 47 | *T* | | -11.0 | -8.3 | | |
| 48 | *P* | | <0.01% | <0.01% | | |
| 49 | | | | | | |
| 50 | [1] See Table 5.1 of Chapter 5 for specific inputs and method of calcuation. | | | | | |

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Table 6.3** | | | | | |
| 2 | **The Size Effect on Discount Rates Based on the** | | | | | |
| 3 | **Arithmetic versus Geometric Means** | | | | | |
| 4 | | | | | | |
| 5 | **FMV** | **Geometric Mean [1]** | **Arithmetic Mean [2]** | **Geometric Mean – B10** | **Arithmetic Mean – B11** | **Difference (Col E – D)** |
| 6 | $20,000,000 | 14.0% | 22.1% | 13.3% | 21.3% | 8.0% |
| 7 | $300,000 | 16.2% | 28.1% | 15.5% | 27.3% | 11.8% |
| 8 | | | | | | |
| 9 | **Inputs** | | | | | |
| 10 | Geometric Avg Annual Growth in PE [3] | 0.67% | | | | |
| 11 | Estimated Arithmetic Mean Growth in PE (Chapter 5, Table 5.3, B32) | 0.80% | | | | |
| 12 | | | | | | |
| 13 | [1] Geometric mean (GM) regression equation: $r$ = 22.78% – 0.00523 × Ln (FMV) | | | | | |
| 14 | | | | | | |
| 15 | [2] Arithmetic mean (AM) regression equation: $r$ = 46.22% – 0.01436 × Ln (FMV) | | | | | |
| 16 | | | | | | |
| 17 | [3] *SBBI—2008 Valuation Yearbook,* p. 95. | | | | | |

## Table 6.3: The Size Effect on Discount Rates Based on the Arithmetic versus Geometric Means

In Table 6.3, we calculate discount rates for a $20 million (A7) firm and a $300,000 (A8) firm using the log size regression equations in columns B and C using GM and AM, respectively. In column D we subtract the geometric average annual growth in the PE of 0.67%[10] from column B, and in column E we subtract our estimate of the arithmetic mean annual growth in PE from column C.[11]

For the $20 million firm, the difference in discount rate is 8.0% (F6), and for the $300,000 FMV firm the difference in discount rates is 11.8% (F7).[12] We see a larger absolute difference for smaller firms, as shown. Thus, using the GM to compute discount rates results in overvaluation that is inversely related to size: that is, using the GM on a small firm will cause a greater overvaluation than using the GM on a large firm. This is consistent with equation (6.4) and our observation that smaller firms have more volatile returns.

## Table 6.4: Comparison of Discount Rates Derived from the Log Size Model Using Arithmetic and Geometric Means

Table 6.4 illustrates this phenomenon. We calculate discount rates using the log size model, with both the arithmetic and geometric mean regression equations from Table 6.2. Similar to the procedure that we used in Table 6.3, we first compute the log size regressions (in columns B and C), and then, in columns D and E, we subtract the PE growth percentages found in cells B10 and B11, respectively.

[10] *SBBI—2008 Valuation Yearbook*, p. 95.

[11] See Chapter 4 for details on the reason for this adjustment.

[12] The geometric mean discount rates are 37% and 43% lower than the arithmetic mean discount rates for the larger and smaller firm, respectively.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table 6.4** | | | | | | | |
| 2 | **Comparison of Discount Rates Derived from the Log Size** | | | | | | | |
| 3 | **Model Using Arithmetic and Geometric Means** | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | **Discount Rate** | | **Gordon Model Multiples** | | **Ratio** |
| 6 | **Firm Size** | **AM [1]** | **GM [2]** | **AM [3]** | **GM [3]** | **AM [4]** | **GM [4]** | **GG/AG [5]** |
| 7 | $250,000 | 28.37% | 16.28% | 27.58% | 15.61% | 5.24 | 11.19 | 213.73% |
| 8 | $1,000,000 | 26.38% | 15.55% | 25.58% | 14.88% | 5.72 | 12.06 | 210.83% |
| 9 | $25,000,000 | 21.76% | 13.87% | 20.96% | 13.20% | 7.35 | 14.78 | 200.98% |
| 10 | $50,000,000 | 20.76% | 13.51% | 19.97% | 12.84% | 7.84 | 15.53 | 198.06% |
| 11 | $100,000,000 | 19.77% | 13.15% | 18.97% | 12.48% | 8.41 | 16.38 | 194.73% |
| 12 | $500,000,000 | 17.45% | 12.30% | 16.66% | 11.63% | 10.13 | 18.75 | 185.05% |
| 13 | $10,000,000,000 | 13.15% | 10.74% | 12.35% | 10.07% | 16.68 | 25.79 | 154.64% |
| 14 | | | | | | | | |
| 15 | **Conclusion: The ratio of Gordon model multiples decreases with firm size (column H).** | | | | | | | |
| 16 | | | | | | | | |
| 17 | **Inputs** | | | | | | | |
| 18 | Geometric Avg Annual Growth in PE [6] | | | | | | 0.67% | |
| 19 | Estimated Arithmetic Mean Growth in PE (Chapter 5, Table 5.3, B32) | | | | | | 0.80% | |
| 20 | | | | | | | | |
| 21 | [1] Arithmetic mean (AM) regression equation: $r$ = 46.22% – 0.0144 × Ln (FMV) | | | | | | | |
| 22 | | | | | | | | |
| 23 | [2] Geometric mean (GM) regression equation: $r$ = 22.78% – 0.0052 × Ln (FMV) | | | | | | | |
| 24 | | | | | | | | |
| 25 | [3] Column D equals column B minus 0.80% (G19). Column E equals column C minus 0.67% (G18). | | | | | | | |
| 26 | | | | | | | | |
| 27 | [4] Gordon model multiple calculated, based on discount rates from columns D and E, assuming 6% growth | | | | | | | |
| 28 | in earnings—midyear assumption. Discount rates are not rounded in these calculations. | | | | | | | |
| 29 | | | | | | | | |
| 30 | [5] Geometric Gordon model multiple/arithmetic Gordon model multiple. | | | | | | | |
| 31 | | | | | | | | |
| 32 | [6] *SBBI—2008 Valuation Yearbook,* p. 95. | | | | | | | |

There is a dramatic difference in discount rates, especially with small firms. The log size discount rate for a $250,000 firm is 27.58% (D7) using the AM and 15.61% (E7) using the GM. Assuming a 6% growth rate, the resulting midyear Gordon model multiples are 5.24 (F7) using the AM and 11.19 (G7) using the GM.

Column H is the ratio of the Gordon model multiples using the geometric mean discount rate to the Gordon model multiples using the arithmetic mean discount rate. Dividing the 11.19 GM multiple by the 5.24 AM multiple gives us a ratio of 213.73% (H7); that is, the GM leads to a valuation that is 113.73% higher than the AM for such a small firm. Notice that the ratio declines continuously as we move down column H. The overvaluation of a $10 billion firm using the GM is 54.64%—far less than the overvaluation of the $250,000 firm. These numerical examples underscore the importance of using the arithmetic mean when valuing expected future earnings or cash flow.

Table 6.4A is identical to Table 6.4, with the only difference being that we use a more realistic assumption that the geometric growth rate is lower than the arithmetic growth rate. We use 5% for the geometric growth rate. This causes the Gordon model multiples to be higher in column G, and the ratio of GG/AG in column H is lower than it is in Table 6.4. While it reduces the magnitude of the difference of the two, it does not change the relationship or our conclusion.

**Table 6.4A**

**Comparison of Discount Rates Derived from the Log Size Model Using Arithmetic and Geometric Means (*g* = 6% for AM, 5% for GM)**

| | | | Discount Rate | | Gordon Model Multiples | | Ratio |
|---|---|---|---|---|---|---|---|
| **Firm Size** | **AM [1]** | **GM [2]** | **AM [3]** | **GM [3]** | **AM [4]** | **GM [4]** | **GG/AG [5]** |
| $250,000 | 28.37% | 16.28% | 27.58% | 15.61% | 5.24 | 10.13 | 193.58% |
| $1,000,000 | 26.38% | 15.55% | 25.58% | 14.88% | 5.72 | 10.84 | 189.50% |
| $25,000,000 | 21.76% | 13.87% | 20.96% | 13.20% | 7.35 | 12.97 | 176.48% |
| $50,000,000 | 20.76% | 13.51% | 19.97% | 12.84% | 7.84 | 13.55 | 172.79% |
| $100,000,000 | 19.77% | 13.15% | 18.97% | 12.48% | 8.41 | 14.19 | 168.68% |
| $500,000,000 | 17.45% | 12.30% | 16.66% | 11.63% | 10.13 | 15.93 | 157.15% |
| $10,000,000,000 | 13.15% | 10.74% | 12.35% | 10.07% | 16.68 | 20.70 | 124.13% |

**Conclusion: The ratio of Gordon model multiples decreases with firm size (column H).**

| **Inputs** | |
|---|---|
| Geometric Avg Annual Growth in PE [6] | 0.67% |
| Estimated Arithmetic Mean Growth in PE (Chapter 5, Table 5.3, B32) | 0.80% |

[1] Arithmetic mean (AM) regression equation: $r = 46.22\% - 0.0144 \times \text{Ln (FMV)}$

[2] Geometric mean (GM) regression equation: $r = 22.78\% - 0.0052 \times \text{Ln (FMV)}$

[3] Column D equals column B minus 0.80% (G19). Column E equals column C minus 0.67% (G18).

[4] Gordon model multiple calculated, based on discount rates from columns D and E, assuming 6% growth in earnings—midyear assumption—for AM and 5% for GM. Discount rates are not rounded in these calculations.

[5] Geometric Gordon model multiple/arithmetic Gordon model multiple.

[6] *SBBI—2008 Valuation Yearbook,* p. 95.

## Indro and Lee Article

This article (Indro and Lee, 1997) is extremely mathematical, and exceedingly difficult reading. The authors begin by citing Brealey and Myers (1991), who say that if monthly returns are identically and independently distributed, then the arithmetic average of monthly returns should be used to estimate the long-run expected return. They then cite empirical evidence that there is significant negative autocorrelation in long-term equity returns and that historical monthly returns are not independent draws from a stationary distribution. This means that high returns in one time period will tend to mean that on average there will be low returns in the next period, and vice versa. Based on this, Copeland, Koller, and Murrin (1994) argue that the geometric average is a better estimate of the long-run expected returns.

Indro and Lee show that the arithmetic and geometric means have upward and downward biases, respectively, and that a horizon-weighted average of the two is the least biased and most efficient estimator.

If the authors are correct, it would mean that there would no longer be a single discount rate. Each year's present value factor in a DCF would have its own unique weighted-average discount rate. That would also add complexity to the use of the Gordon model to calculate a residual value.

Because of the extremely difficult mathematics in the article, it was necessary to speak to academic sources to evaluate it. Professor Myers, cited earlier, did agree that long-term (five-year) returns are negatively autocorrelated, but that there are "very few data points." He had not fully read the article, is not sure of its significance, and did not have an opinion of it. Ibbotson Associates does not feel the evidence for mean reversion is that strong,[13] and on that basis, is not moved to change its opinion that the AM is the correct mean. The maximum serial correlation is 0.08 for micro-cap stocks, with all others ranging from −0.02 to 0.03.[14] This supports Ibbotson's view—all the more so, because the serial correlation for micro-cap stocks is positive and not negative, as was the evidence cited by Indro and Lee.

## References

Brealey, R. A., and Stewart C. Myers. 1991. *Principles of Corporate Finance*. New York: McGraw-Hill.

Copeland, Tom, Tim Koller, and Jack Murrin. 1994. *Valuation: Measuring and Managing the Value of Companies*. New York: John Wiley & Sons.

Ibbotson Associates/Morningstar. 2008. *Stocks, Bonds, Bills and Inflation: 2008 Valuation Yearbook*. Chicago: Ibbotson Associates/Morningstar.

Indro, Daniel C., and Wayne Y. Lee. 1997. Biases in Arithmetic and Geometric Averages as Estimates of Long-Run Expected Returns and Risk Premia. *Financial Management* 26 (4) (Winter): 81–90.

Joyce, Allyn A. 1995. Arithmetic Mean vs. Geometric Mean: The Issue in Rate of Return. *Business Valuation Review* (June): 62–68.

[13] *SBBI—2008 Valuation Edition*, p. 80.

[14] *SBBI—2008 Valuation Edition*, p. 28.

CHAPTER 7

# An Iterative Valuation Approach

# CHAPTER 7

# An Iterative Valuation Approach

## Introduction

The capital asset pricing model (CAPM) is a common model appraisers use to calculate discount rates, even though it has been supplanted by its alter ego, the build-up method, and, in my opinion, it is generally inferior to the log size model for valuing privately held businesses.

At the 1993 ASA summer conference, one of the most controversial and enjoyable sessions was titled "Invested Capital versus Equity Valuation Methods (or Direct Capital Approaches)." The proponents of the equity valuation methods—primarily using CAPM—for valuing firms argued that the weakness of the invested capital approach (ICA) is that we appraisers are not smart enough to determine the appropriate debt-to-equity ratio to use in the calculation of the weighted average cost of capital (WACC). They also argued that the wide swings in WACC caused by changes in the assumed debt/equity mix can drastically affect the calculation of fair market value (FMV). While those arguments are sound, the direct capital approach also suffers from a similar deficiency, since the appraiser must choose a debt-to-equity ratio in order to *relever beta* for the subject company. In other words, both methods suffer from essentially the same problem.

In this chapter, based on Abrams (1995), we show how using an iterative approach eliminates this deficiency in both models. After determining the market value of debt, we can assume any value for equity to get our initial debt-to-equity ratio. We calculate the first iteration of equity value using this initial ratio. After several iterations, we eventually obtain a unique solution for equity that is consistent with the last iteration of the debt-to-equity ratio and is independent of our initial choice of equity.

## Equity Valuation Method

For the equity valuation method, the iterative procedure involves the following steps:

1. Determine forecast after-tax earnings (or, preferably, the cash flow equivalent).[1] This should be normalized, that is, with owners' salaries adjusted to arm's-length for a control interest, and so on.

[1]*Note:* Cash flow is the preferable measure. In this chapter we discount earnings only for simplicity of presentation. This is also true for the invested capital approach.

2. Determine the discount rate for the company's debt and the risk-free rate of interest.
3. Determine the fair market value of interest-bearing debt. This is easy to do, and the FMV of debt will remain constant through all iterations.
4. Derive an unlevered beta using publicly traded guideline company data.
5. Determine the equity risk premium (ERP) and small company premium (SCP).
6. Specify an initial capital structure, consisting of the FMV of debt and an initial arbitrary choice of equity. You may use your best guess of the FMV of equity, or you can use the net book value of equity. Eventually, your initial guess will make no difference.

   Steps 1–6 are not repeated. The following steps are iterative:
7. Calculate a relevered beta and equity discount rate using your initial capital structure and use it to value the firm.
8. Substitute the first calculated fair market value of equity into a new capital structure, and use the new weights to calculate the next iteration of beta, equity discount rate, and FMV of equity.

Keep repeating steps 7 and 8 until you reach a steady-state value for beta, equity discount rate, and FMV of equity.

Let's illustrate this with a couple of examples.

## Table 7.1A: The First Iteration

We use a deliberately simple discounted future earnings approach in Table 7.1A to illustrate how this process works. Starting with a firm whose net income before taxes (NIBT) in 2007, the previous year, was $400,000 (cell D28), we assume a declining growth rate in income: 15% (cell B7) in 2008, 13% (cell C7) in 2009, finishing with 8% (cell F7) in 2012. We use these growth rates to forecast income in 2008–2012 (cells B6 through F6). Subtracting 40% (cell D29) for income taxes, we arrive at net income after taxes (NIAT) of $276,000 (cell B9) in 2008, rising to $407,531 (cell F9) in 2012. The bottom row of the top section is the present value of NIAT, using the calculated equity discount rate and a midyear assumption.

The valuation section begins in cell D17 with the sum of the present value of NIAT for the first five years. The next seven rows are intermediate calculations using a Gordon model with an 8% constant growth rate and the midyear assumption (cells D17–D23). Forecast income in 2013 is the 2012 net income times 1 plus the growth rate [F9 × (1 + D18) = D19 = $440,134]. The midyear Gordon model multiple, cell D20, is equal to SQRT$(1 + r) / (r - g)$ = SQRT(1 + D36)/(D36 − D18) = 8.1456. We multiply $440,134 × 8.1456 = $3,585,135 (cell D21), which is the present value of net income after year 2012 as of December 31, 2012. The present value factor for five years is 0.377146 (cell D22). Multiplying $3,585,135 × 0.377146 = $1,352,121 (cell D23), which is the present value of income after 2012 as of the valuation date, January 1, 2008.

Adding the present value of the first five years' net income of $1,055,852 (cell D17) to the present value of the net income after five years of $1,352,121 (cell D23), we arrive at our first approximation of the FMV of the equity of $2,407,973 (cell D24).

Rows 28 through 35 contain the assumptions of the model and the data necessary to lever and unlever industry average betas and calculate equity discount rates.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table 7.1A** | | | | | | | |
| 2 | **Equity Valuation Approach with Iterations Beginning with Book Equity** | | | | | | | |
| 3 | **Iteration #1** | | | | | | | |
| 4 | | | | | | | | |
| 5 | | **2008** | **2009** | **2010** | **2011** | **2012** | | |
| 6 | Net Inc before Taxes | 460,000 | 519,800 | 576,978 | 628,906 | 679,219 | | |
| 7 | Growth Rate in NIBT | 15% | 13% | 11% | 9% | 8% | | |
| 8 | Income Taxes | (184,000) | (207,920) | (230,791) | (251,562) | (271,687) | | |
| 9 | Net Inc after Taxes | 276,000 | 311,880 | 346,187 | 377,344 | 407,531 | | |
| 10 | Present Value Factor | 0.9071 | 0.7464 | 0.6141 | 0.5053 | 0.4158 | | |
| 11 | Pres Value NIAT | **$250,357** | **$232,777** | **$212,601** | **$190,675** | **$169,441** | | |
| 12 | | | | | | | | |
| 13 | | | | | | | | |
| 14 | | | | | | | | |
| 15 | | | | | | | | |
| 16 | **Final Valuation:** | | | | | | | |
| 17 | PV 2008–2012 Net Income | | | $1,055,852 | | | | |
| 18 | Constant Growth Rate in Net Income = *G* | | | 8% | | | | |
| 19 | Forecast Net Income—2013 | | | 440,134 | | | | |
| 20 | Gordon Model Mult = SQRT(1+*R*)/(*R*–*G*) | | | 8.1456 | | | | |
| 21 | Present Value—Net Inc after 2012 as of 12/31/2012 | | | **3,585,135** | | | | |
| 22 | Present Value Factor—5 Years | | | 0.377146 | | | | |
| 23 | Pres Value of Net Income after 2012 as of 1/1/08 | | | **1,352,121** | | | | |
| 24 | FMV of Equity—100% Interest | | | **$2,407,973** | | | | |
| 25 | | | | | | | | |
| 26 | | | | | | | | |
| 27 | **Assumptions:** | | | | | | | |
| 28 | Net Income before Tax—2007 | | | 400,000 | | | | |
| 29 | Income Tax Rate = ***t*** | | | 40% | | | | |
| 30 | Discount Rate—Debt: Pre-Tax | | | 10% | | | | |
| 31 | Discount Rate—Debt: After-Tax | | | 6% | | | | |
| 32 | Unlevered Beta (from F46) = $\beta_U$ | | | 0.91 | | | | |
| 33 | Risk-Free Rate = $R_f$ | | | 6% | | | | |
| 34 | Equity Risk Premium = **ERP** | | | 8% | | | | |
| 35 | Small Company Premium = **SCP** | | | 3% | | | | |
| 36 | Equity Discount Rate = ***R*** | | | **21.534%** | | | | |
| 37 | | | | | | | | |
| 38 | **Calculation of Equity Discount Rate Using Comparables** | | | | | | | |
| 39 | | | | | | **=B/[1+(1–*t*)E]** | | |
| 40 | | Equity | | | | Unlevered | | |
| 41 | | Beta | Debt | Equity | D/E | (Asset) Beta | | |
| 42 | Guideline Company #1 | 1.15 | 454,646 | 874,464 | 52.0% | 0.88 | | |
| 43 | Guideline Company #2 | 1.20 | 146,464 | 546,454 | 26.8% | 1.03 | | |
| 44 | Guideline Company #3 | 0.95 | 46,464 | 705,464 | 6.6% | 0.91 | | |
| 45 | Guideline Company #4 | 0.85 | 52,646 | 846,467 | 6.2% | 0.82 | | |
| 46 | Totals or Averages | **1.04** | **700,220** | **2,972,849** | **23.55%** | **0.91** | | |
| 47 | | | | | | | | |
| 48 | | | | | | | | |
| 49 | | | | **Capital Structure & Iterations** | | | | |
| 50 | | | | | | | | **=D24, with *R*** |
| 51 | | | Interest- | | **=C/D** | **=F46×[1+(1–*t*)E]** | **$R_f$+($\beta_L$ × ERP)+SCP** | **from col G** |
| 52 | | | Bearing | Equity Before | | Relevered | CAPM Equity | FMV |
| 53 | | *t* | Debt = ***D*** | Iteration = ***E*** | ***D*/*E*** | Beta | Disc Rate = ***R*** | Equity |
| 54 | FMV Debt, Eqty at *t*–1 | 1 | 900,000 | 750,000 | 1.20 | 1.5668 | 21.534% | 2,407,973 |

The discount rate is in cell D36, though it is calculated in cell G54 and transferred from there.

Rows 42 through 46 detail the calculation of an unlevered beta of 0.91 (cell F46) from an average of publicly traded guideline companies. In the capital structure and iterations section, row 54 shows the market value of debt and the book value of equity (our initial guess of market value) as well as the implied debt/equity ratio and relevered beta according to Hamada's formula (Hamada, 1972) in equation (7.1):[2]

$$\beta_{Levered} = \beta_{unlevered} \times \left[1 + (1 - t)\,\frac{Debt}{Equity}\right] \quad \text{Formula for levered beta.} \qquad (7.1)$$

[2]This equation is most accurate when the firm's pre-tax discount rate for debt is close to the risk-free rate.

We can solve this equation for unlevered beta as equation (7.1a):

$$\beta_{Unlevered} = \frac{\beta_{Levered}}{1 + (1 - t)\dfrac{Debt}{Equity}} \quad \text{Formula for unlevered beta.} \tag{7.1a}$$

Cell G54 is the discount rate of 21.534% for the first iteration, calculated according to the CAPM formula adjusted for size in equation (7.2):

$$R = R_f + (\beta_{levered} \times ERP) + SCP \quad \text{Formula for CAPM equity discount rate,} \tag{7.2}$$

where $R$ is the equity discount rate, $R_f$ is the risk-free rate, ERP is the equity risk premium, and SCP is the small company premium.[3]

We use this discount rate to calculate the first iteration of FMV of equity in cell H54.

## Table 7.1B: Subsequent Iterations of the First Scenario

Table 7.1B is identical to Table 7.1A, except that it contains nine iterations in the capital structure section instead of one. Also cell D36 contains the final equity discount rate from row 62.[4] We denote the iteration number as $t$, which appears in column B, rows 54–62. When $t = 1$, we obtain an equity discount rate of 21.534% (cell G54) and an FMV of the equity of $2,407,973 (cell H54), as before. This tells us that our initial guess of the FMV of the equity, which was the book value of the equity of $750,000 (cell D54), is too low.

We substitute the $2,407,973 (cell H54) first iteration of equity into the new capital structure in cell D55 to get a debt/equity ratio of 0.37 (cell E55), as seen in the second iteration of Table 7.1B. This changes the discount rate to 17.921% (cell G55). This results in the second iteration of equity value of $3,245,701 (cell H55). We use the new equity as the basis for our third iteration, which we calculate in the same fashion as the previous iteration. We follow these steps until we reach steady state in the FMV of equity, which in this case occurs in the eighth iteration, with an FMV of $3,404,686 (cell H61). We must carry out an additional iteration to know for sure that we have reached steady state, which is the purpose of iteration #9.

This spreadsheet has a macro that sets column H equal to cell D24 and then converts it to a fixed number. The purpose of doing so is that cell D24 changes in each iteration, but we need to preserve the result in column H before going on to the next iteration.

## Table 7.1C: Initial Choice of Equity Doesn't Matter

Tables 7.1B and 7.1C demonstrate that the initial choice of equity doesn't matter. Instead of choosing book equity as the starting point, in Table 7.1C we make an

[3]In the log size model, there is no single small company premium. Such a premium is unique to each size range.

[4]Actually, cell D36 takes on the value calculated in each iteration from cells G54 through G62, so the discount rate used in all the calculations changes in each iteration of the spreadsheet.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table 7.1B** | | | | | | | |
| 2 | **Equity Valuation Approach with Iterations Beginning with Book Equity** | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | **2008** | **2009** | **2010** | **2011** | **2012** | | |
| 6 | Net Inc before Taxes | 460,000 | 519,800 | 576,978 | 628,906 | 679,219 | | |
| 7 | Growth Rate in NIBT | 15% | 13% | 11% | 9% | 8% | | |
| 8 | Income Taxes | (184,000) | (207,920) | (230,791) | (251,562) | (271,687) | | |
| 9 | Net Inc after Taxes | 276,000 | 311,880 | 346,187 | 377,344 | 407,531 | | |
| 10 | Present Value Factor | 0.9228 | 0.7857 | 0.6690 | 0.5696 | 0.4850 | | |
| 11 | Pres Value NIAT | **$254,680** | **$245,045** | **$231,602** | **$214,952** | **$197,669** | | |
| 12 | | | | | | | | |
| 13 | | | | | | | | |
| 14 | | | | | | | | |
| 15 | | | | | | | | |
| 16 | **Final Valuation:** | | | | | | | |
| 17 | PV 2008–2012 Net Income | | | $1,143,949 | | | | |
| 18 | Constant Growth Rate in Net Income = *G* | | | 8% | | | | |
| 19 | Forecast Net Income—2013 | | | 440,134 | | | | |
| 20 | Gordon Model Mult = SQRT(1+*R*)/(*R*–*G*) | | | 11.4763 | | | | |
| 21 | Present Value–Net Inc after 2012 as of 12/31/2012 | | | **5,051,106** | | | | |
| 22 | Present Value Factor—5 Years | | | 0.447573 | | | | |
| 23 | Pres Value of Net Income after 2012 as of 1/1/08 | | | **2,260,738** | | | | |
| 24 | FMV of Equity—100% Interest | | | **$3,404,686** | | | | |
| 25 | | | | | | | | |
| 26 | | | | | | | | |
| 27 | **Assumptions:** | | | | | | | |
| 28 | Net Income before Tax—2007 | | | 400,000 | | | | |
| 29 | Income Tax Rate = ***t*** | | | 40% | | | | |
| 30 | Discount Rate—Debt: Pre-Tax | | | 10% | | | | |
| 31 | Discount Rate—Debt: After-Tax | | | 6% | | | | |
| 32 | Unlevered Beta (from F46) = $\beta_U$ | | | 0.91 | | | | |
| 33 | Risk-Free Rate = $R_f$ | | | 6% | | | | |
| 34 | Equity Risk Premium = **ERP** | | | 8% | | | | |
| 35 | Small Company Premium = **SCP** | | | 3% | | | | |
| 36 | Equity Discount Rate = ***R*** | | | **17.443%** | | | | |
| 37 | | | | | | | | |
| 38 | **Calculation of Equity Discount Rate Using Comparables** | | | | | | | |
| 39 | | | | | | **=B/[1+(1–*t*)E]** | | |
| 40 | | Equity | | | | Unlevered | | |
| 41 | | Beta | Debt | Equity | D/E | (Asset) Beta | | |
| 42 | Guideline Company #1 | 1.15 | 454,646 | 874,464 | 52.0% | 0.88 | | |
| 43 | Guideline Company #2 | 1.20 | 146,464 | 546,454 | 26.8% | 1.03 | | |
| 44 | Guideline Company #3 | 0.95 | 46,464 | 705,464 | 6.6% | 0.91 | | |
| 45 | Guideline Company #4 | 0.85 | 52,646 | 846,467 | 6.2% | 0.82 | | |
| 46 | Totals or Averages | **1.04** | **700,220** | **2,972,849** | **23.55%** | **0.91** | | |
| 47 | | | | | | | | |
| 48 | | | | | | | | |
| 49 | | | | | **Capital Structure & Iterations** | | | |
| 50 | | | | | | | | **=D24, with *R*** |
| 51 | | | Interest- | | **=C/D** | **=F46×[1+(1–*t*)E]** | **$R_f$+($\beta_L$×ERP)+SCP** | **from col G** |
| 52 | | | Bearing | Equity Before | | Relevered | CAPM Equity | FMV |
| 53 | | *t* | Debt = ***D*** | Iteration = ***E*** | ***D/E*** | Beta | Disc Rate = ***R*** | Equity |
| 54 | FMV Debt, Eqty at *t*–1 | 1 | 900,000 | 750,000 | 1.20 | 1.5668 | 21.534% | 2,407,973 |
| 55 | FMV Debt, Eqty at *t*–1 | 2 | 900,000 | 2,407,973 | 0.37 | 1.1152 | 17.921% | 3,245,701 |
| 56 | FMV Debt, Eqty at *t*–1 | 3 | 900,000 | 3,245,701 | 0.28 | 1.0625 | 17.500% | 3,385,037 |
| 57 | FMV Debt, Eqty at *t*–1 | 4 | 900,000 | 3,385,037 | 0.27 | 1.0562 | 17.450% | 3,402,345 |
| 58 | FMV Debt, Eqty at *t*–1 | 5 | 900,000 | 3,402,345 | 0.26 | 1.0555 | 17.444% | 3,404,409 |
| 59 | FMV Debt, Eqty at *t*–1 | 6 | 900,000 | 3,404,409 | 0.26 | 1.0554 | 17.443% | 3,404,653 |
| 60 | FMV Debt, Eqty at *t*–1 | 7 | 900,000 | 3,404,653 | 0.26 | 1.0554 | 17.443% | 3,404,682 |
| 61 | FMV Debt, Eqty at *t*–1 | 8 | 900,000 | 3,404,682 | 0.26 | 1.0554 | 17.443% | 3,404,686 |
| 62 | FMV Debt, Eqty at *t*–1 | 9 | 900,000 | 3,404,686 | 0.26 | 1.0554 | 17.443% | 3,404,686 |

arbitrary guess of $5,000,000 (cell D54) as a starting point.[5] Table 7.1C is identical to Table 7.1B, except in the initial choice of value of the equity and the intermediate

[5]For those who buy the electronic spreadsheet from the author, which is not included with this book, the steps are: (1) input your initial guess of equity in cell D54; (2) initialize the spreadsheet by pressing Control-X; (3) press Control-Z for each iteration. Every time you press Control-Z, the spreadsheet will calculate one iteration of value, as in rows 54 to 62. Repeat pressing Control-Z until you have reached steady state—that is, the value in column H is the same twice in a row.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table 7.1C** | | | | | | | |
| 2 | **Equity Valuation Approach with Iterations Beginning with Arbitrary Equity** | | | | | | | |
| 3 | | | | | | | | |
| 4 | | | | | | | | |
| 5 | | **2008** | **2009** | **2010** | **2011** | **2012** | | |
| 6 | Net Inc before Taxes | 460,000 | 519,800 | 576,978 | 628,906 | 679,219 | | |
| 7 | Growth Rate in NIBT | 15% | 13% | 11% | 9% | 8% | | |
| 8 | Income Taxes | (184,000) | (207,920) | (230,791) | (251,562) | (271,687) | | |
| 9 | Net Inc after Taxes | 276,000 | 311,880 | 346,187 | 377,344 | 407,531 | | |
| 10 | Present Value Factor | 0.9228 | 0.7857 | 0.6690 | 0.5696 | 0.4850 | | |
| 11 | Pres Value NIAT | **$254,680** | **$245,045** | **$231,602** | **$214,952** | **$197,669** | | |
| 12 | | | | | | | | |
| 13 | | | | | | | | |
| 14 | | | | | | | | |
| 15 | | | | | | | | |
| 16 | **Final Valuation:** | | | | | | | |
| 17 | PV 2008–2012 Net Income | | | $1,143,949 | | | | |
| 18 | Constant Growth Rate in Net Income = *G* | | | 8% | | | | |
| 19 | Forecast Net Income—2013 | | | 440,134 | | | | |
| 20 | Gordon Model Mult = SQRT(1+*R*)/(*R*–*G*) | | | 11.4763 | | | | |
| 21 | Present Value—Net Inc after 2012 as of 12/31/2012 | | | **5,051,106** | | | | |
| 22 | Present Value Factor—5 Years | | | 0.447573 | | | | |
| 23 | Pres Value of Net Income after 2012 as of 1/1/08 | | | **2,260,738** | | | | |
| 24 | FMV of Equity—100% Interest | | | **$3,404,686** | | | | |
| 25 | | | | | | | | |
| 26 | | | | | | | | |
| 27 | **Assumptions:** | | | | | | | |
| 28 | Net Income before Tax—2007 | | | 400,000 | | | | |
| 29 | Income Tax Rate = ***t*** | | | 40% | | | | |
| 30 | Discount Rate—Debt: Pre-Tax | | | 10% | | | | |
| 31 | Discount Rate—Debt: After-Tax | | | 6% | | | | |
| 32 | Unlevered Beta (from F46) = $\beta_U$ | | | 0.91 | | | | |
| 33 | Risk-Free Rate = $R_f$ | | | 6% | | | | |
| 34 | Equity Risk Premium = **ERP** | | | 8% | | | | |
| 35 | Small Company Premium = **SCP** | | | 3% | | | | |
| 36 | Equity Discount Rate = ***R*** | | | **17.443%** | | | | |
| 37 | | | | | | | | |
| 38 | **Calculation of Equity Discount Rate Using Comparables** | | | | | | | |
| 39 | | | | | | **=B/[1+(1−*t*)E]** | | |
| 40 | | Equity | | | | Unlevered | | |
| 41 | | Beta | Debt | Equity | D/E | Beta | | |
| 42 | Guideline Company #1 | 1.15 | 454,646 | 874,464 | 52.0% | 0.88 | | |
| 43 | Guideline Company #2 | 1.20 | 146,464 | 546,454 | 26.8% | 1.03 | | |
| 44 | Guideline Company #3 | 0.95 | 46,464 | 705,464 | 6.6% | 0.91 | | |
| 45 | Guideline Company #4 | 0.85 | 52,646 | 846,467 | 6.2% | 0.82 | | |
| 46 | Totals or Averages | **1.04** | **700,220** | **2,972,849** | **23.55%** | **0.91** | | |
| 47 | | | | | | | | |
| 48 | | | | | | | | |
| 49 | | | | **Capital Structure and Iterations** | | | | |
| 50 | | | | | | | | **=D24, with *R*** |
| 51 | | | Interest- | | **=C/D** | **=F46×[1+(1−*t*)E]** | **$R_f$+($\beta_L$×ERP)+SCP** | **from col G** |
| 52 | | | Bearing | Equity Before | | Relevered | CAPM Equity | FMV |
| 53 | | *t* | Debt = ***D*** | Iteration = ***E*** | ***D*/*E*** | Beta | Disc Rate = ***R*** | Equity |
| 54 | FMV Debt, Eqty at *t*–1 | 1 | 900,000 | 5,000,000 | 0.18 | 1.0093 | 17.074% | 3,538,676 |
| 55 | FMV Debt, Eqty at *t*–1 | 2 | 900,000 | 3,538,676 | 0.25 | 1.0499 | 17.399% | 3,420,038 |
| 56 | FMV Debt, Eqty at *t*–1 | 3 | 900,000 | 3,420,038 | 0.26 | 1.0547 | 17.438% | 3,406,499 |
| 57 | FMV Debt, Eqty at *t*–1 | 4 | 900,000 | 3,406,499 | 0.26 | 1.0553 | 17.442% | 3,404,901 |
| 58 | FMV Debt, Eqty at *t*–1 | 5 | 900,000 | 3,404,901 | 0.26 | 1.0554 | 17.443% | 3,404,712 |
| 59 | FMV Debt, Eqty at *t*–1 | 6 | 900,000 | 3,404,712 | 0.26 | 1.0554 | 17.443% | 3,404,689 |
| 60 | FMV Debt, Eqty at *t*–1 | 7 | 900,000 | 3,404,689 | 0.26 | 1.0554 | 17.443% | 3,404,687 |
| 61 | FMV Debt, Eqty at *t*–1 | 8 | 900,000 | 3,404,687 | 0.26 | 1.0554 | 17.443% | 3,404,686 |
| 62 | FMV Debt, Eqty at *t*–1 | 9 | 900,000 | 3,404,686 | 0.26 | 1.0554 | 17.443% | 3,404,686 |

iterations. The final FMV is identical. Note that it does not matter whether your initial guess is too low or too high, as Table 7.1B is too low and Table 7.1C is too high, but they both lead to the same FMV.

## Convergence of the Equity Valuation Method

While rare, it can happen that the FMV diverges instead of converges. If the method described above does not converge, an alternative is to take the average of the resulting FMV of equity and the previously assumed value as your input into column D when starting the next iteration as opposed to using just the latest iteration of equity alone. This can be done by making a small alteration to the spreadsheet.[6]

## Invested Capital Approach

Tables 7.2A and 7.2B are examples of the invested capital approach. They are very similar to Table 7.1B for the equity valuation method with the following exceptions:

1. We determine earnings before interest but after taxes (EBIBAT) as the income measure.[7] This should be normalized EBIBAT.[8]
2. We discount EBIBAT using the WACC.
3. We must subtract the market value of debt from the calculated market value of invested capital to get the market value of equity.
4. We must calculate a new WACC for each new iteration of FMV of equity.
5. We do not show the calculation of unlevered beta but will assume that it has already been calculated to be 1.05.

Let's illustrate this with a couple of examples.

### Table 7.2A: Iterations Beginning with Book Equity

Earnings before interest and taxes (EBIT) in 2007, the previous year, were $600,000 (cell D28). We assume a declining growth rate in earnings as before: 15% (cell B6) in 2008, 13% (C6) in 2009, finishing with 8% (cell F6) in 2012. We use these growth rates to forecast EBIT in 2008–2012 (cells B5–F5). Subtracting 40% (cell D29) for income taxes, we arrive at earnings before interest but after taxes (EBIBAT) of $414,000 (cell B8) in 2008, rising to $611,297 (cell F8) in 2012. The growth rates in EBIBAT are identical to those for EBIT, because we assume a constant 40% income tax. The last row of the top section is the present value of EBIBAT, using the calculated WACC as the discount rate and using a midyear assumption.

The valuation section begins in cell D15 with the sum of the present value of the first five years of EBIBAT. The next seven rows are the same intermediate calculations as in Tables 7.1A, 7.1B, and 7.1C, using a Gordon model with an 8% constant growth rate and the midyear assumption (cells D16–D21). Our final iteration of the FMV of the equity plus debt (enterprise value, or enterprise FMV) is $6,448,957 (cell D22). From this we subtract the FMV of the debt of $2,000,000 (cell D23) to arrive at the final iteration of FMV of equity of $4,448,957 (cell D24).

Let's look at the calculation of WACC for the first iteration. For this firm, we assume the FMV of interest-bearing debt is $2,000,000 (cell C43). We further

[6]Change the formula in cell D55, which previously was =H54, to =AVERAGE(D54,H54). Then copy the formula down column D.

[7]It is better to use cash flow (before interest but after taxes), but for simplicity we use EBIBAT.

[8]This does not necessarily correspond to the NIBT in Tables 7.1A, 7.1B, and 7.1C because we are dealing with a different hypothetical company.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table 7.2A** | | | | | | | | | |
| 2 | **WACC Approach with Iterations Beginning with Book Equity** | | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | **2008** | **2009** | **2010** | **2011** | **2012** | | | | |
| 5 | EBIT | 690,000 | 779,700 | 865,467 | 943,359 | 1,018,828 | | | | |
| 6 | Growth Rate in EBIT | 15% | 13% | 11% | 9% | 8% | | | | |
| 7 | Income Taxes | (276,000) | (311,880) | (346,187) | (377,344) | (407,531) | | | | |
| 8 | EBIBAT | 414,000 | 467,820 | 519,280 | 566,015 | 611,297 | | | | |
| 9 | Growth Rate—EBIBAT | 15% | 13% | 11% | 9% | 8% | | | | |
| 10 | Present Value Factor | 0.9308 | 0.8064 | 0.6986 | 0.6052 | 0.5243 | | | | |
| 11 | Pres Val—EBIBAT | **$385,341** | **$377,237** | **$362,767** | **$342,566** | **$320,523** | | | | |
| 12 | | | | | | | | | | |
| 13 | | | | | | | | | | |
| 14 | **Final Valuation:** | | | | | | | | | |
| 15 | PV 2008–2012 EBIBAT | | | $1,788,434 | | | | | | |
| 16 | Constant Growth Rate in EBIBAT = *G* | | | 8% | | | | | | |
| 17 | Forecast EBIBAT—2013 | | | 660,200 | | | | | | |
| 18 | Gordon Model Mult = SQRT(1+*R*)/(*R*–*G*) | | | 14.4646 | | | | | | |
| 19 | PV—EBIBAT after 2012 as of 1-1-2013 | | | **9,549,547** | | | | | | |
| 20 | Present Value Factor—5 Years | | | 0.488036 | | | | | | |
| 21 | PV—EBIBAT after 2012 | | | **4,660,523** | | | | | | |
| 22 | Enterprise FMV—100% Interest | | | **$6,448,957** | | | | | | |
| 23 | Less FMV of Debt | | | (2,000,000) | | | | | | |
| 24 | FMV of Equity—100% Interest | | | **$4,448,957** | | | | | | |
| 25 | | | | | | | | | | |
| 26 | | | | | | | | | | |
| 27 | **Assumptions:** | | | | | | | | | |
| 28 | EBIT—2007 | | | 600,000 | | | | | | |
| 29 | Income Tax Rate = ***t*** | | | 40% | | | | | | |
| 30 | Discount Rate—Debt: Pre-Tax | | | 10% | | | | | | |
| 31 | Discount Rate—Debt: After-Tax | | | 6% | | | | | | |
| 32 | Unlevered Beta = $\beta_u$ | | | 1.05 | | | | | | |
| 33 | Risk-Free Rate = $R_f$ | | | 6% | | | | | | |
| 34 | Equity Risk Premium = **ERP** | | | 8% | | | | | | |
| 35 | Small Company Premium = **SCP** | | | 3% | | | | | | |
| 36 | Wtd Avg Cost of Capital (WACC—Iteration *t*) | | | **15.428%** | | | | | | |
| 37 | | | | | | | | | | |
| 38 | **Capital Structure and Iterations** | | | | | | | | | |
| 39 | | | | | | | | | | |
| 40 | | | Interest- | $=J_{t-1}$ | | % Interest- | | $R_f$+($\beta_L$ × ERP)+SCP | | **=D24 at *t*** |
| 41 | | | Bearing | | | Bearing | | Equity Disc | | FMV |
| 42 | | *t* | Debt = ***D*** | Equity = ***E*** | Total | Debt | % Equity | Rate = ***R*** | WACC | Equity |
| 43 | FMV Debt, Eqty at *t*–1 | 1 | 2,000,000 | 800,000 | 2,800,000 | 71.4% | 28.6% | 30.000% | 12.857% | 7,776,091 |
| 44 | FMV Debt, Eqty at *t*–1 | 2 | 2,000,000 | 7,776,091 | 9,776,091 | 20.5% | 79.5% | 18.696% | 16.099% | 3,927,835 |
| 45 | FMV Debt, Eqty at *t*–1 | 3 | 2,000,000 | 3,927,835 | 5,927,835 | 33.7% | 66.3% | 19.966% | 15.254% | 4,599,240 |
| 46 | FMV Debt, Eqty at *t*–1 | 4 | 2,000,000 | 4,599,240 | 6,599,240 | 30.3% | 69.7% | 19.592% | 15.473% | 4,411,165 |
| 47 | FMV Debt, Eqty at *t*–1 | 5 | 2,000,000 | 4,411,165 | 6,411,165 | 31.2% | 68.8% | 19.685% | 15.416% | 4,458,814 |
| 48 | FMV Debt, Eqty at *t*–1 | 6 | 2,000,000 | 4,458,814 | 6,458,814 | 31.0% | 69.0% | 19.661% | 15.431% | 4,446,410 |
| 49 | FMV Debt, Eqty at *t*–1 | 7 | 2,000,000 | 4,446,410 | 6,446,410 | 31.0% | 69.0% | 19.667% | 15.427% | 4,449,617 |
| 50 | FMV Debt, Eqty at *t*–1 | 8 | 2,000,000 | 4,449,617 | 6,449,617 | 31.0% | 69.0% | 19.665% | 15.428% | 4,448,787 |
| 51 | FMV Debt, Eqty at *t*–1 | 9 | 2,000,000 | 4,448,787 | 6,448,787 | 31.0% | 69.0% | 19.666% | 15.428% | 4,449,002 |
| 52 | FMV Debt, Eqty at *t*–1 | 10 | 2,000,000 | 4,449,002 | 6,449,002 | 31.0% | 69.0% | 19.666% | 15.428% | 4,448,946 |
| 53 | FMV Debt, Eqty at *t*–1 | 11 | 2,000,000 | 4,448,946 | 6,448,946 | 31.0% | 69.0% | 19.666% | 15.428% | 4,448,960 |
| 54 | FMV Debt, Eqty at *t*–1 | 12 | 2,000,000 | 4,448,960 | 6,448,960 | 31.0% | 69.0% | 19.666% | 15.428% | 4,448,957 |
| 55 | FMV Debt, Eqty at *t*–1 | 13 | 2,000,000 | 4,448,957 | 6,448,957 | 31.0% | 69.0% | 19.666% | 15.428% | 4,448,957 |

temporarily assume the FMV of the equity is its book value of $800,000 (cell D43).[9] Using these two initial values as our first approximation, debt is 71.4% (cell F43) of the invested capital (cell E43) and equity is 28.6% (cell G43). We calculate the first iteration of equity discount rate of 30% in cell H43 using CAPM adjusted for small size, per equation (7.2). We calculate the WACC according to equation (7.3):

$$\begin{aligned} WACC = {} & \left[(1 - t) \times Debt\ Discount\ Rate \times \%\,Debt\right] \\ & + \left[Equity\ Discount\ Rate \times \%\,Equity\right]. \end{aligned} \tag{7.3}$$

For row 43, $WACC = \left[\left(1 - 0.4\right) \times 0.10 \times 71.4\%\right] + \left[0.30 \times 28.6\%\right] = 12.857\%$ (cell I43).[10]

---

[9]In subsequent iterations, the equity value is set to the FMV of equity (column J) of the previous iteration.

[10]There is an apparent rounding error, as the percentages of debt and equity to six decimal places are 0.714286 and 0.285714.

We discount EBIBAT at this WACC to get the FMV of equity of $7,776,091 in cell J43.[11] This iteration of equity is then transferred to cell D44, and the process is repeated. After 12 iterations we arrive at an FMV of equity of $4,448,957 (cell J54). We then confirm this value by iterating once more in row 55.

### Table 7.2B: Initial Choice of Equity Doesn't Matter

Tables 7.2A and 7.2B demonstrate that the initial choice of equity doesn't matter. Instead of choosing book equity as the starting point, in Table 7.2B we make an arbitrary guess of $10,000,000 (cell D43) as a starting point. Table 7.2B is identical to Table 7.2A, except in the initial choice of value of the equity and the intermediate iterations. The final result is identical. Note that it does not matter whether your initial guess is too low or too high: Table 7.2A is too low and Table 7.2B is too high, but they both lead to the same result.

### Convergence of the Invested Capital Approach

As with the equity valuation method, if the method described earlier does not converge, an alternative is to take the average of the resulting FMV of equity and the previously assumed value as your input into column D when starting the next iteration as opposed to just using the latest iteration of equity. This can be done by making a small alteration to the spreadsheet.

## Log Size

The log size model converges far faster than the CAPM versions of the invested capital approach or the equity valuation method. The reason is that when using logarithms to calculate the discount rate, large absolute changes in equity value cause fairly small changes in the discount rate, which is not true of CAPM.

## Summary

When using CAPM, using this iterative approach will improve appraisal accuracy and eliminate arguments over the proper leverage. One look at the difference between the beginning guess of the FMV of equity and the final FMV will show how much more accuracy can be gained. While it is true that had we guessed a number based on industry average capitalization we would have been closer, the advantage of this approach is that it obviates the need for precise initial guesses.

The iterative approach should give us the ability to get much closer answers from both the invested capital and the direct capital approaches, as long as the subject firm is sufficiently profitable. The iterative approach does not seem to work for very small firms with little profitability, but those are the firms for which you are least likely to want to bother with the extra work involved in the iterations.

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[11] As in the direct equity method, we use a macro to calculate the FMV of equity in cell D24 and convert it to a fixed value at the end of each iteration.

## Table 7.2B

### WACC Approach with Iterations Beginning with Arbitrary Guess of Equity Value

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 4 | | 2008 | 2009 | 2010 | 2011 | 2012 |
| 5 | EBIT | 690,000 | 779,700 | 865,467 | 943,359 | 1,018,828 |
| 6 | Growth Rate in EBIT | 15% | 13% | 11% | 9% | 8% |
| 7 | Income Taxes | (276,000) | (311,880) | (346,187) | (377,344) | (407,531) |
| 8 | EBIBAT | 414,000 | 467,820 | 519,280 | 566,015 | 611,297 |
| 9 | Growth Rate—EBIBAT | 15% | 13% | 11% | 9% | 8% |
| 10 | Present Value Factor | 0.9308 | 0.8064 | 0.6986 | 0.6052 | 0.5243 |
| 11 | Pres Val-EBIBAT | **$385,341** | **$377,237** | **$362,767** | **$342,566** | **$320,523** |

| | A | D |
|---|---|---|
| 14 | **Final Valuation:** | |
| 15 | PV 2008–2012 EBIBAT | $1,788,434 |
| 16 | Constant Growth Rate in EBIBAT = *G* | 8% |
| 17 | Forecast EBIBAT—2013 | 660,200 |
| 18 | Gordon Model Mult = SQRT(1+*R*)/(*R*–*G*) | 14.4646 |
| 19 | PV—EBIBAT after 2012 as of 1-1-2013 | **9,549,547** |
| 20 | Present Value Factor—5 Years | 0.488036 |
| 21 | PV—EBIBAT after 2012 | **4,660,523** |
| 22 | Enterprise FMV—100% Interest | **$6,448,957** |
| 23 | Less FMV of Debt | (2,000,000) |
| 24 | FMV of Equity—100% Interest | **$4,448,957** |

| | A | D |
|---|---|---|
| 27 | **Assumptions:** | |
| 28 | EBIT—2007 | 600,000 |
| 29 | Income Tax Rate = ***t*** | 40% |
| 30 | Discount Rate—Debt: Pre-Tax | 10% |
| 31 | Discount Rate—Debt: After-Tax | 6% |
| 32 | Unlevered Beta = $\beta_U$ | 1.05 |
| 33 | Risk-Free Rate = $R_f$ | 6% |
| 34 | Equity Risk Premium = **ERP** | 8% |
| 35 | Small Company Premium = **SCP** | 3% |
| 36 | Wtd Avg Cost of Capital (WACC—Iteration *t*) | **15.428%** |

**Capital Structure and Iterations**

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 40 | | | Interest- | $=J_{t-1}$ | | Interest- | $R_f+(\beta_L \times ERP)+SCP$ | | | **=D24 at *t*** |
| 41 | | | Bearing | | | Bearing | | Equity Disc | | FMV |
| 42 | | *t* | Debt = ***D*** | Equity = ***E*** | Total | Debt | Equity | Rate = ***R*** | WACC | Equity |
| 43 | FMV Debt, Eqty at *t*–1 | 1 | 2,000,000 | 10,000,000 | 12,000,000 | 16.7% | 83.3% | 18.408% | 16.340% | 3,761,117 |
| 44 | FMV Debt, Eqty at *t*–1 | 2 | 2,000,000 | 3,761,117 | 5,761,117 | 34.7% | 65.3% | 20.080% | 15.192% | 4,654,820 |
| 45 | FMV Debt, Eqty at *t*–1 | 3 | 2,000,000 | 4,654,820 | 6,654,820 | 30.1% | 69.9% | 19.565% | 15.489% | 4,397,731 |
| 46 | FMV Debt, Eqty at *t*–1 | 4 | 2,000,000 | 4,397,731 | 6,397,731 | 31.3% | 68.7% | 19.692% | 15.412% | 4,462,354 |
| 47 | FMV Debt, Eqty at *t*–1 | 5 | 2,000,000 | 4,462,354 | 6,462,354 | 30.9% | 69.1% | 19.659% | 15.432% | 4,445,498 |
| 48 | FMV Debt, Eqty at *t*–1 | 6 | 2,000,000 | 4,445,498 | 6,445,498 | 31.0% | 69.0% | 19.667% | 15.427% | 4,449,853 |
| 49 | FMV Debt, Eqty at *t*–1 | 7 | 2,000,000 | 4,449,853 | 6,449,853 | 31.0% | 69.0% | 19.665% | 15.428% | 4,448,725 |
| 50 | FMV Debt, Eqty at *t*–1 | 8 | 2,000,000 | 4,448,725 | 6,448,725 | 31.0% | 69.0% | 19.666% | 15.428% | 4,449,017 |
| 51 | FMV Debt, Eqty at *t*–1 | 9 | 2,000,000 | 4,449,017 | 6,449,017 | 31.0% | 69.0% | 19.666% | 15.428% | 4,448,942 |
| 52 | FMV Debt, Eqty at *t*–1 | 10 | 2,000,000 | 4,448,942 | 6,448,942 | 31.0% | 69.0% | 19.666% | 15.428% | 4,448,961 |
| 53 | FMV Debt, Eqty at *t*–1 | 11 | 2,000,000 | 4,448,961 | 6,448,961 | 31.0% | 69.0% | 19.666% | 15.428% | 4,448,956 |
| 54 | FMV Debt, Eqty at *t*–1 | 12 | 2,000,000 | 4,448,956 | 6,448,956 | 31.0% | 69.0% | 19.666% | 15.428% | 4,448,958 |
| 55 | FMV Debt, Eqty at *t*–1 | 13 | 2,000,000 | 4,448,958 | 6,448,958 | 31.0% | 69.0% | 19.666% | 15.428% | 4,448,957 |
| 56 | FMV Debt, Eqty at *t*–1 | 14 | 2,000,000 | 4,448,957 | 6,448,957 | 31.0% | 69.0% | 19.666% | 15.428% | 4,448,957 |

While CAPM generally has given way to the build-up method and log size, the practical importance of this chapter is not so much the mechanics of iterating to a steady state, but achieving complete consistency in model assumptions and results, which is important in all valuation models. The reader now should know that this should be a goal in all valuations, regardless of the model. Using the log size model we generally achieve consistency in two iterations, and the elaborate iteration model in this chapter is unnecessary. This means that when we are reviewing another appraiser's valuation, if there has been no demonstration of the consistency of the model, a light bulb should be flashing in our head to warn us to make sure that the valuation model is consistent between its assumptions and results.

## References

Abrams, Jay B. 1995. An Iterative Valuation Approach. *Business Valuation Review* (March): 26–35.

Hamada, R. S. 1972. The Effects of the Firm's Capital Structure on the Systematic Risk of Common Stocks. *Journal of Finance* 27: 435–452.

## PART III

# Adjusting for Control and Marketability

## Introduction

In the second edition, Part III consists solely of Chapters 8 (Chapter 7 in the first edition) and deals with calculating control premiums, the discount for lack of control (DLOC), and discount for lack of marketability (DLOM). We moved Chapter 8 and 9 from the first edition, which are sample valuation reports for restricted stock and fractional interest discount studies, to our Web site, www.abramsvaluation.com, "Books," "Quantitative Business Valuation." Eventually, we plan to publish them in a workbook to accompany the third edition of this book. Collectively, these topics correspond to the third and fourth steps in valuing businesses. Chapter 8 is a combination of theory and a great deal of review of academic and professional literature, while the two chapters moved to the Web site and which eventually will be in the workbook, are practical, "how-to" chapters.

## Chapter 8: Adjusting for Levels of Control and Marketability

Adjusting for levels of control and marketability are probably the most controversial topics in business valuation. As such, Chapter 8 is almost a book unto itself. It is the longest chapter in this book—so long that in future editions we will split it into two or more chapters or perhaps even to make it a book by itself. In the meantime it is somewhat unwieldy because of its length.

Chapter 8 consists of two parts. The first part primarily deals with control, and the second with marketability. I chose that order because of the one-way relationship—control affects marketability, but marketability does not affect control. The chapter begins with a comprehensive overview of the major professional articles on the topic and then proceeds to review a number of academic articles that provide insight into the issue of control.

In part 2, we review two quantitative models (other than my own): Mercer's quantitative marketability discount model (QMDM) and Kasper's bid-ask spread model. We then analyze restricted stock discounts with multiple regression analysis for two reasons. The first reason is that this is intrinsically useful in restricted stock discount studies. The second—and more important reason—is that restricted stock discounts serve as one of the components of my economic components model of DLOM, which comprises the majority of Part II. At the end of the chapter, Z. Christopher Mercer provides a rebuttal to my critique of the quantitative marketability discount model, and we go back and forth with arguments that the profession should find interesting and enlightening, and possibly somewhat confusing and frustrating as well.

## Economic Components Model

The heart of Chapter 8 is my own economic components model for DLOM, which consists of four components:

1. The economic consequences of the delay to sale experienced by all privately-held firms. I model this component using a regression analysis of restricted stock discount data published by Management Planning, Inc. in Mercer's book.[1]
2. Extra bargaining power ("monopsony power") to the buyer arising from thin markets. The academic article by Schwert contains a key finding that enables us to reliably estimate this component of DLOM.
3. Buyers' transactions costs in excess of transactions costs for publicly held stocks.
4. Sellers' transactions costs in excess of transactions costs for publicly held stocks.

We present research on the magnitude of transactions costs for both buyers and sellers with different business sizes as well as regression analyses of each. This enables us to calculate transactions costs for any business size for both buyer and seller.

Items (3) and (4) above, which we label components #3A and #3B in the chapter, occur every time the business is sold. Those fees and costs "leave the system" by being paid to outsiders such as business brokers, accountants, attorneys, and appraisers. Thus, we need to be able to calculate the present value effect of the infinite continuum of periodic transaction costs, which we do in the form of one formula for buyers' excess transactions costs and another formula for sellers' excess transactions costs.[2] This process is now vastly simplified over the process in my original *Business Valuation Review* article on the topic. We also give an example of how to calculate DLOM.

A very important test that we perform in Chapter 8 is a comparison of several models in their ability to explain the restricted stock discounts from the Management Planning, Inc. data: the Black-Scholes options pricing model (BSOPM) put formula using specifically calculated standard deviations of returns (volatility) of the public stocks, the BSOPM put using indirectly calculated (through log size equations) standard deviations, the quantitative marketability discount model (QMDM), a regression equation, and the mean discount. The regression equation was the best forecast of restricted stock discounts, with the BSOPM with directly calculated volatility a very

[1]The data have been corrected since publication in Mercer's book, and Management Planning, Inc. provided us with additional data.

[2]That is because the seller's costs on the first sale do not count in calculating DLOM, whereas buyer's costs do. In all subsequent sales of the business, both count.

close second. Both the BSOPM using indirectly calculated volatility and the QMDM were worse than the mean in forecasting discounts, with QMDM being "farthest out of the money." This is significant, because it is the first empirical test of any model to calculate restricted stock discounts.

At the end of the chapter, we present updated restricted stock summary data through 2008 from Management Planning, Inc. and through 2007 from FMV Opinions. We analyze these results in light of our MPI regression from the first edition and respond to a *Business Valuation Review* article by Mercer in 2001.

## Rough Edges

Because this chapter is so long and complex, it was the last chapter that I updated as part of publishing the second edition. As I wrote in the introduction to the book, I would have liked another three to six months of research before completing this chapter. The data are more in conflict with each other and confusing, and the conclusions more difficult and less satisfying than the first edition. But time marches on, and so must this book! Rather than sweep the new research under the rug, instead we present it, analyze it, and do the best we can to come to conclusions, as difficult and tentative as they are.

CHAPTER 8

# Adjusting for Levels of Control and Marketability

# CHAPTER 8

# Adjusting for Levels of Control and Marketability

## Introduction

Adjusting for levels of control and marketability is a complicated and very important topic. We will be discussing *control premiums* (CP), their opposite, *discount for lack of control* (DLOC), and *discount for lack of marketability* (DLOM).

Historically, these valuation adjustments have accounted for substantial adjustments in appraisal reports—often 20% to 40% of the net present value of the cash flows—and yet valuation analysts may spend little or no time calculating and explaining these adjustments.

This is a long chapter, with much data and analysis. It will be helpful to break the discussion into two parts. The first part will deal primarily with control, and the second part primarily with marketability. I say *primarily* because the two concepts are interrelated. The level of control of a business interest impacts its level of marketability. Therefore, it is logical to begin with a discussion of control. Because of the interrelationship, there are two academic articles that we will discuss in the section on control that relate more to marketability—yet they fit better in the control discussion.

## The Value of Control and Adjusting for Level of Control

We will begin our analysis of the effects of control on value by reviewing prior qualitative professional research and prior academic research. Then we will present some additional data. As we present the data, we will come to some conclusions about the magnitude of control premiums and DLOC.

### Levels of Value (LOV) Charts

The top portion of Figure 8.1 shows the traditional, one-dimensional level of values (LOV) chart.[1] The conventional wisdom represented in the traditional LOV chart

[1]The bottom portion shows Chris Mercer's modified traditional levels of value chart, which is identical to the one above, except with the addition of the strategic value. We will cover this later in the chapter.

Traditional Levels of Value Chart

| Level of Value | Adjustment Up To | Adjustment Down To |
|---|---|---|
| Control Interest | Control Premium | NA |
| Marketable Minority Interest | Reverse out DLOM | DLOC |
| Private Minority Interest | NA | DLOM |

Mercer's (Mercer 1998) Modified Traditional Levels of Value Chart

| Level of Value | Adjustment Up To | Adjustment Down To |
|---|---|---|
| Strategic Value | Value of Synergies | NA |
| Control Value | Control Premium | Eliminate Synergies |
| Marketable Minority Interest [1] | Reverse out DLOM | DLOC |
| Private Minority Interest | NA | DLOM |

[1] Often referred to in the literature as the "as-if-freely-traded-value" for private firms.

**FIGURE 8.1 Chart of Traditional Levels of Value and Mercer's (1998) Chart of Modified Traditional Levels of Value**

holds that it is appropriate to add a control premium to the marketable minority interest value. There are significant opinions to the contrary, that is, that one should not add any control premium whatsoever. Additionally, there is controversy over the appropriate magnitude of the control premium among those who do add them to the marketable minority interest value. We will cover that in greater depth later in the chapter. Of course, if the valuation method is a guideline company approach using a database of sales of privately held firms, the starting value is a private control interest, and a control premium is inappropriate.

It is extremely important to understand that the valuation adjustments in Figure 8.1 must be appropriate to the valuation method used. If we are valuing a control interest and we used a discounted cash flow analysis with discount rates calculated using New York Stock Exchange data, the resulting value would be a marketable minority interest, and a control premium must be considered.[2]

The alternative levels of value chart, depicted in Figure 8.2, is two-tiered; that is, it is a 2 × 2 chart (2 rows and 2 columns, versus the traditional chart, which is 3 × 1). It represents the four basic types of ownership interests, which are combinations of public-versus-private and control-versus-minority interest. Obviously, there are shades of gray in between the extremes. Bolotsky (1991) was the first to propound this chart, although he used it for slightly different purposes, which we discuss

[2]It is also important to make sure the measure of income is consistent with the interest valued. When valuing a control interest, it is appropriate to add back excess salaries of the owners. When valuing a minority interest that cannot force salaries lower, the add-back is inappropriate.

Two-Tiered Levels of Value Chart [1]

| | Public | Private |
|---|---|---|
| Control | × | × |
| Minority | × | × |

[1] *Note:* These are also the four basic types of ownership interests.

**FIGURE 8.2 Chart of Two-Tiered Levels of Value**

below. Much later in the chapter, we will discuss Figure 8.3, which is my own extension of Bolotsky's levels of value chart to a 4 × 2 chart.

## Mergerstat Control Premium Data

The traditional sources of control premiums are the Mergerstat and the Houlihan Lokey Howard & Zukin (HLHZ) studies.[3] In Table 8.1, columns B and C show Mergerstat's compilation of average (mean) and median five-day acquisition premiums from 1985 to 2008. The premiums were measured as $\frac{P_{Offer}}{P_{5Day}} - 1$, where the numerator is the offering price, and the denominator is the minority trading price five days before the announcement of the offer.

Note that we deliberately use the term *acquisition premium* instead of the more common term *control premium*.[4] Eventually we will distinguish between the amounts that are paid for control versus the amounts that are paid for synergies, as the latter are generally part of investment value and not fair market value.

Mean acquisition premiums have ranged from 31% to 62% (B31, B32), with the average being 42.1% (B33), the median of means being 40.9% (B34), and the standard deviation being 8.9% (B35). Median premiums have ranged from 23% to 41% (C31, C32), with the mean of the medians being 30.9% (C33), median of medians being 30.5% (C34), and standard deviation of 4.8% (C35). The standard deviation of medians is slightly over one-half that of the means.

The mean and median going-private premiums appear in columns D and E. As going-private transactions should have no element of synergy, the difference of acquisition and going-private premiums could be a measure of synergy. The differences of acquisition versus going-private premiums are in columns F and G: B − D = F, and C − E = G.

Column F is the difference of means. The minimum difference in means is −27% (F31), and the maximum difference is 16% (F32). The mean and median differences of means are 3.2% (F33) and 5.3% (F34), respectively.

Column G is the difference of medians. The minimum difference in medians is −12% (G31), and the maximum difference is 27% (G32). The mean and median differences of medians are 3% (G33) and 2% (G34), respectively. Thus, the evidence in Table 8.1 leads us to a tentative conclusion that corporate buyers are paying 2%

[3] HLHZ now owns Mergerstat, although the latter was previously owned by Merrill Lynch and the W. T. Grimm Co.
[4] Later we use the term *control premium* more loosely for familiarity.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 8.1** | | | | | | |
| 2 | **Synergies as Measured by** | | | | | | |
| 3 | **Acquisition Minus Going-Private Premiums** | | | | | | |
| 4 | | | | | | | |
| 5 | | **Acquisition Premiums [1]** | | **Going Private Prem [2]** | | **Difference = Synergy?** | |
| 6 | | **Mean** | **Median** | **Mean** | **Median** | **Mean** | **Median** |
| 7 | 1985 | 37.1% | 27.7% | 30.9% | 25.7% | 6.2% | 2.0% |
| 8 | 1986 | 38.2% | 29.9% | 31.9% | 26.1% | 6.3% | 3.8% |
| 9 | 1987 | 38.3% | 30.8% | 34.8% | 30.9% | 3.5% | -0.1% |
| 10 | 1988 | 41.9% | 30.9% | 33.8% | 26.3% | 8.1% | 4.6% |
| 11 | 1989 | 41.0% | 29.0% | 35.0% | 22.7% | 6.0% | 6.3% |
| 12 | 1990 | 42.0% | 32.0% | 34.3% | 31.6% | 7.7% | 0.4% |
| 13 | 1991 | 35.1% | 29.4% | 23.8% | 20.0% | 11.3% | 9.4% |
| 14 | 1992 | 41.0% | 34.7% | 24.8% | 8.1% | 16.2% | 26.6% |
| 15 | 1993 | 38.7% | 33.0% | 34.7% | 20.0% | 4.0% | 13.0% |
| 16 | 1994 | 41.9% | 35.0% | 41.9% | 35.0% | 0.0% | 0.0% |
| 17 | 1995 | 44.7% | 29.2% | 29.8% | 19.2% | 14.9% | 10.0% |
| 18 | 1996 | 36.6% | 27.3% | 34.8% | 26.2% | 1.8% | 1.1% |
| 19 | 1997 | 35.7% | 27.5% | 30.4% | 24.5% | 5.3% | 3.0% |
| 20 | 1998 | 40.7% | 30.1% | 29.1% | 20.4% | 11.6% | 9.7% |
| 21 | 1999 | 43.3% | 34.6% | 38.0% | 32.7% | 5.3% | 1.9% |
| 22 | 2000 | 49.2% | 41.1% | 41.9% | 38.7% | 7.3% | 2.4% |
| 23 | 2001 | 57.2% | 40.5% | 67.6% | 52.2% | -10.4% | -11.7% |
| 24 | 2002 | 59.7% | 34.4% | 86.4% | 40.0% | -26.7% | -5.6% |
| 25 | 2003 | 62.3% | 31.6% | 60.5% | 41.5% | 1.8% | -9.9% |
| 26 | 2004 | 30.7% | 23.4% | 28.9% | 17.2% | 1.8% | 6.2% |
| 27 | 2005 | 34.5% | 24.1% | 35.1% | 22.5% | -0.6% | 1.6% |
| 28 | 2006 | 31.5% | 23.1% | 31.1% | 21.1% | 0.4% | 2.0% |
| 29 | 2007 | 31.5% | 24.7% | 26.3% | 22.2% | 5.2% | 2.5% |
| 30 | 2008 | 56.5% | 36.5% | 67.5% | 36.8% | -11.0% | -0.3% |
| 31 | **Min** | **30.7%** | **23.1%** | **23.8%** | **8.1%** | **-26.7%** | **-11.7%** |
| 32 | **Max** | **62.3%** | **41.1%** | **86.4%** | **52.2%** | **16.2%** | **26.6%** |
| 33 | **Mean** | **42.1%** | **30.9%** | **38.9%** | **27.6%** | **3.2%** | **3.3%** |
| 34 | **Median** | **40.9%** | **30.5%** | **34.5%** | **25.9%** | **5.3%** | **2.2%** |
| 35 | **Std Dev** | **8.9%** | **4.8%** | **15.6%** | **9.6%** | **9.0%** | **7.6%** |
| 36 | | | | | | | |
| 37 | [1] *Mergerstat Review* —2009, Chart 1-10, p. 25 for 1999–2008. *Mergerstat Review*— | | | | | | |
| 38 | 1999, Chart 1-8, p. 23 for 1989–1998. *Mergerstat Review*—1994, figure 41, p. | | | | | | |
| 39 | 98 for 1985–1988. Mergerstat is a part of FactSet Research Systems. | | | | | | |
| 40 | | | | | | | |
| 41 | [2] *Mergerstat Review*—2009, table 1-41, p. 44 for 1999–2008. *Mergerstat Review*— | | | | | | |
| 42 | 1999, table 1-39, p. 42 for 1989–1998. For 1985–1988, *Mergerstat Review*—1994, | | | | | | |
| 43 | figure 39, p. 96. The 60.5% average going private premium in 2003 excludes an | | | | | | |
| 44 | acquisition for $1 per share that yielded a 5-day premium of 4,900%. Including that | | | | | | |
| 45 | premium, the 2003 average would have been 129.7%. | | | | | | |

to 3% more for their acquisitions than going-private transactions. Of course, not all acquirers are strategic. Some are financial buyers. This implies that strategic buyers are likely to pay more than 2% to 3% additional premiums, while financial buyers would pay less than that. Let's hold that thought until we see more evidence.

### Table 8.1A: Summary Statistics from Mergerstat Database for Different Populations

We will comment on two basic categories of the data—the control premiums and other data.

**CONTROL PREMIUMS** Table 8.1A shows summary statistics for several downloads that we made on July 21, 2009 from Mergerstat. The first population is that of strategic buyers. Column B, "CP," is the Mergerstat control premium. There are 1,398 (B21) transactions for which control premiums were available. We included only U.S. target companies. The minimum control premium was −78% (B8), and the maximum was 643% (B9). The mean control premium was 35% (B10). Rows 11 through 19 show various percentiles. The first percentile control premium is −39% (B11). The 50th percentile, which is the median, is 26% (B15). The mean control premium is significantly higher than the median, which indicates this distribution is skewed to the left. The standard deviation is 46% (B20), which is large compared to the mean and median.

Let's skip to the third population, which is all financial buyers with U.S. target companies. There are 481 (B55) transactions with control premiums available. We would think that control premiums should be higher for strategic than financial buyers, who should be able to pay more, with synergies available. Yet, the mean control premium is 37% (B44), and the median is 25% (B49)—again a left-skewed distribution. The standard deviation is 55% (B54), which is even higher than it was for strategic buyers. Thus, it seems that there is no material difference in control premiums—a surprising result. It appears that financial buyers pay an absolute 2% higher mean control premiums than strategic buyers and 1% lower control premiums as measured by the median. In other words, there is no significant difference in control premiums for strategic versus financial buyers, especially given the high standard deviations.

However, the time periods are not the same. For strategic buyers, the earliest transaction date is January 2003 (O8), while it was January 1998 for (O42) for financial buyers. Thus, it is possible that we are not comparing apples-to-apples.

Therefore, we repeated the summary analysis of financial buyers with transaction dates beginning on January 2, 2003 (O25) to match the earliest date of the strategic buyers. In this dataset, the mean control premium is 34% (B27), which is substantially the same as the 35% (B10) for strategic buyers, but the median is only 21% (B32), which is an absolute 5% less than the 26% (B15) for strategic buyers. The standard deviation is 48% (B37).

Thus, there is some weak evidence that strategic buyers pay higher control premiums than financial buyers. Given the large standard deviations, it is not so clear the 5% difference is significant—all the more so, because there is virtually no difference in the larger dataset of financial buyers. Of course, strategic buyers would not want to pay higher premiums than financial buyers. They merely theoretically can afford to pay more, which theoretically could shift their demand curve to the right and increase control premiums. Overall, this is a somewhat surprising result, as I would have expected a larger difference in the populations.

Now let's look at the other populations. Vertical buyers are a specialized type of strategic buyer. They are consolidating the chain of distribution of production, wholesaler, and retailer. There are 37 (B72) vertical buyers, which is a small population. The mean and median control premiums are both 34% (B61, B66). This is the first symmetric distribution so far. While the mean is similar to strategic and financial buyers, the median is considerably higher.

Horizontal buyers—businesses acquiring their competitors—are also a specialized type of strategic buyer. The mean and median premiums are 38% (B78) and 29%

# Table 8.1A
## Summary Statistics from Mergerstat Database for Different Populations

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | Table 8.1A | | | | | | | | | | | | | | | | | | |
| 2 | Summary Statistics from Mergerstat Database for Different Populations | | | | | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | | | | | | |
| 4 | Stats | CP | Sales | EBITDA | EBIT | Bk Value | Oper Margin | Net Margin | MV Equity | PS | PE | P/BV | TIC/EBIT | TIC/ EBITDA | Tx Date | P | %Acq | % Own Post-Tx | DLOC |
| 5 | | | | | | | | | | | | | | | | | | | |
| 6 | All Strategic Buyers USA Target as of 7/21/2009 (from Mergerstat Database) | | | | | | | | | | | | | | | | | | |
| 7 | | | | | | | | | | | | | | | | | | | |
| 8 | Min | -78% | 0 | -10,951 | -12,967 | -143,107 | (305.42) | (320.81) | 0 | - | 0.01 | - | 1.08 | 0.93 | 01/02/03 | 0 | 20 | 51 | -352% |
| 9 | Max | 643% | 51,876 | 15,392 | 14,345 | 76,870 | 4.02 | 54.54 | 2.4E+06 | 9.72 | 39.99 | 9.96 | 34.95 | 24.76 | 03/31/09 | 85,410 | 100 | 100 | 87% |
| 10 | Avg | 35% | 769 | 151 | 107 | 313 | (0.36) | (0.48) | 3,048 | 2.50 | 21.86 | 3.06 | 17.66 | 12.33 | 03/01/06 | 1,333 | 99 | 100 | 19% |
| 11 | 1%-ile | -39% | 2 | -81 | -91 | -892 | (6.84) | (8.21) | 2 | 0.04 | 2.85 | 0.07 | 2.36 | 2.00 | 01/16/03 | 3 | 61 | 90 | -63% |
| 12 | 5%-ile | -7% | 7 | -20 | -28 | -54 | (0.83) | (0.91) | 12 | 0.26 | 8.68 | 0.76 | 7.90 | 5.06 | 04/23/03 | 12 | 100 | 100 | -8% |
| 13 | 10%-ile | 1% | 11 | -7 | -13 | -2 | (0.22) | (0.27) | 22 | 0.44 | 11.44 | 1.00 | 8.99 | 6.33 | 09/22/03 | 22 | 100 | 100 | 1% |
| 14 | 25%-ile | 12% | 31 | 3 | 1 | 14 | 0.02 | (0.01) | 59 | 1.04 | 16.22 | 1.63 | 12.49 | 8.25 | 09/30/04 | 61 | 100 | 100 | 11% |
| 15 | 50%-ile | 26% | 103 | 19 | 12 | 55 | 0.11 | 0.05 | 228 | 2.22 | 21.65 | 2.53 | 16.96 | 11.75 | 04/03/06 | 224 | 100 | 100 | 21% |
| 16 | 75%-ile | 46% | 429 | 87 | 60 | 202 | 0.41 | 0.12 | 895 | 3.55 | 27.17 | 3.94 | 22.21 | 15.78 | 07/31/07 | 896 | 100 | 100 | 32% |
| 17 | 90%-ile | 79% | 1,435 | 287 | 212 | 711 | 0.59 | 0.19 | 2,872 | 4.88 | 33.04 | 6.03 | 27.77 | 19.76 | 06/09/08 | 2,854 | 100 | 100 | 44% |
| 18 | 95%-ile | 105% | 3,059 | 565 | 424 | 1,579 | 0.65 | 0.24 | 5,636 | 5.94 | 35.58 | 7.63 | 30.30 | 21.59 | 10/01/08 | 5,167 | 100 | 100 | 51% |
| 19 | 99%-ile | 186% | 14,364 | 2,753 | 2,035 | 7,949 | 0.83 | 0.36 | 17,829 | 8.46 | 39.02 | 9.53 | 33.88 | 23.95 | 01/07/09 | 17,752 | 100 | 100 | 65% |
| 20 | Std Dev | 46% | 2,823 | 847 | 809 | 4,853 | 9.16 | 9.78 | 63,865 | 1.84 | 8.09 | 2.05 | 7.05 | 5.15 | 09/17/01 | 4,673 | 6 | 3 | 28% |
| 21 | Count | 1,398 | 1,398 | 1,398 | 1,398 | 1,398 | 1,397 | 1,397 | 1,398 | 1,316 | 743 | 1,121 | 468 | 532 | 10/29/03 | 1,398 | 1398 | 1390 | 1,398 |
| 22 | | | | | | | | | | | | | | | | | | | |
| 23 | All Financial Buyers USA Target from 1/2/2003 as of 7/21/2009 | | | | | | | | | | | | | | | | | | |
| 24 | | | | | | | | | | | | | | | | | | | |
| 25 | Min | -81% | 2 | -48 | -64 | -4,708 | (3.19) | (3.32) | 0 | - | 0.02 | - | 0.00 | 0.01 | 01/02/03 | 0 | 51 | 56 | -428% |
| 26 | Max | 336% | 41,273 | 5,451 | 4,558 | 33,108 | 1.45 | 0.81 | 71,444 | 8.92 | 39.50 | 9.63 | 33.62 | 23.40 | 03/31/09 | 31,804 | 100 | 100 | 77% |
| 27 | Avg | 34% | 1,180 | 195 | 129 | 400 | 0.07 | (0.01) | 1,721 | 1.55 | 20.42 | 2.49 | 15.24 | 10.24 | 04/10/06 | 1,495 | 96 | 99 | 19% |
| 28 | 1%-ile | -17% | 3 | -24 | -46 | -1,391 | (1.05) | (1.06) | 1 | 0.00 | 0.03 | 0.00 | 1.57 | 1.04 | 01/16/03 | 2 | 54 | 76 | -21% |
| 29 | 5%-ile | -2% | 10 | -7 | -18 | -278 | (0.20) | (0.29) | 5 | 0.10 | 5.10 | 0.54 | 5.82 | 4.08 | 04/27/03 | 6 | 70 | 100 | -2% |
| 30 | 10%-ile | 2% | 17 | -1 | -5 | -54 | (0.08) | (0.11) | 13 | 0.19 | 8.20 | 0.77 | 8.24 | 5.16 | 09/13/03 | 16 | 85 | 100 | 2% |
| 31 | 25%-ile | 10% | 68 | 6 | 2 | 9 | 0.02 | (0.00) | 53 | 0.43 | 14.67 | 1.14 | 10.94 | 7.32 | 12/14/04 | 54 | 100 | 100 | 9% |
| 32 | 50%-ile | 21% | 290 | 34 | 18 | 61 | 0.07 | 0.03 | 234 | 1.08 | 19.86 | 1.79 | 14.41 | 10.02 | 08/28/06 | 240 | 100 | 100 | 17% |
| 33 | 75%-ile | 41% | 806 | 143 | 86 | 254 | 0.16 | 0.08 | 1,086 | 2.13 | 26.80 | 3.19 | 19.57 | 12.74 | 07/14/07 | 1,094 | 100 | 100 | 29% |
| 34 | 90%-ile | 69% | 3,043 | 400 | 274 | 1,003 | 0.29 | 0.13 | 3,520 | 3.29 | 32.94 | 5.39 | 23.57 | 16.04 | 02/15/08 | 3,118 | 100 | 100 | 41% |
| 35 | 95%-ile | 92% | 5,419 | 714 | 582 | 1,474 | 0.42 | 0.17 | 7,533 | 5.17 | 34.82 | 7.17 | 25.57 | 17.63 | 07/02/08 | 7,770 | 100 | 100 | 48% |
| 36 | 99%-ile | 288% | 10,765 | 2,757 | 1,694 | 4,827 | 0.66 | 0.35 | 20,762 | 7.53 | 37.53 | 8.97 | 31.01 | 21.58 | 02/01/09 | 19,277 | 100 | 100 | 74% |
| 37 | Std Dev | 48% | 3,237 | 549 | 399 | 2,158 | 0.34 | 0.31 | 5,495 | 1.58 | 8.96 | 2.03 | 6.32 | 4.36 | 08/13/01 | 3,769 | 11 | 4 | 31% |
| 38 | Count | 308 | 308 | 308 | 308 | 307 | 308 | 308 | 308 | 302 | 176 | 225 | 185 | 204 | 11/03/00 | 308 | 308 | 306 | 308 |
| 39 | | | | | | | | | | | | | | | | | | | |
| 40 | All Financial Buyers USA Target as of 7/21/2009 | | | | | | | | | | | | | | | | | | |
| 41 | | | | | | | | | | | | | | | | | | | |
| 42 | Min | -97% | 1 | -111 | -132 | -4,708 | (9.45) | (9.45) | 0 | - | 0.02 | - | 0.00 | 0.01 | 01/29/98 | 0 | 20 | 51 | -2831% |
| 43 | Max | 610% | 41,273 | 5,451 | 4,558 | 33,108 | 1.45 | 6.73 | 71,444 | 8.92 | 39.50 | 9.63 | 33.62 | 23.40 | 03/31/09 | 31,804 | 100 | 100 | 86% |
| 44 | Avg | 37% | 930 | 142 | 93 | 279 | 0.02 | (0.05) | 1,193 | 1.38 | 19.00 | 2.34 | 14.25 | 9.38 | 02/21/04 | 1,046 | 95 | 99 | 13% |
| 45 | 1%-ile | -44% | 4 | -42 | -62 | -715 | (1.99) | (2.00) | 1 | 0.00 | 0.76 | 0.05 | 1.84 | 1.03 | 03/02/98 | 2 | 51 | 64 | -80% |
| 46 | 5%-ile | -5% | 11 | -10 | -19 | -188 | (0.28) | (0.37) | 5 | 0.09 | 5.68 | 0.52 | 5.72 | 4.06 | 11/25/98 | 5 | 64 | 100 | -5% |
| 47 | 10%-ile | 2% | 20 | -1 | -5 | -41 | (0.09) | (0.12) | 12 | 0.18 | 8.95 | 0.73 | 7.11 | 4.90 | 08/13/99 | 12 | 75 | 100 | 2% |
| 48 | 25%-ile | 11% | 61 | 5 | 2 | 7 | 0.02 | (0.01) | 44 | 0.41 | 12.81 | 1.08 | 9.80 | 6.24 | 12/15/00 | 43 | 100 | 100 | 10% |
| 49 | 50%-ile | 25% | 205 | 24 | 13 | 43 | 0.07 | 0.03 | 148 | 0.90 | 17.75 | 1.78 | 12.84 | 8.50 | 08/26/04 | 139 | 100 | 100 | 20% |

# Table 8.1A (cont.)
## Summary Statistics from Mergerstat Database for Different Populations

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | Stats | CP | Sales | EBITDA | EBIT | Bk Value | Oper Margin | Net Margin | MV Equity | PS | PE | P/BV | TIC/EBIT | TIC/ EBITDA | Tx Date | P | % Acq | % Own Post-Tx | DLOC |
| 50 | 75%-ile | 50% | 558 | 91 | 51 | 153 | 0.15 | 0.07 | 576 | 1.80 | 24.60 | 2.92 | 18.61 | 12.00 | 02/01/07 | 590 | 100 | 100 | 33% |
| 51 | 90%-ile | 78% | 1,992 | 261 | 175 | 609 | 0.26 | 0.12 | 1,927 | 3.20 | 32.53 | 4.70 | 22.89 | 15.28 | 10/26/07 | 1,927 | 100 | 100 | 44% |
| 52 | 95%-ile | 111% | 3,688 | 494 | 354 | 1,185 | 0.37 | 0.16 | 5,096 | 4.34 | 34.34 | 6.55 | 25.90 | 17.38 | 05/27/08 | 4,894 | 100 | 100 | 53% |
| 53 | 99%-ile | 289% | 9,818 | 2,452 | 1,588 | 4,114 | 0.71 | 0.39 | 18,699 | 6.97 | 37.53 | 8.98 | 30.95 | 21.58 | 01/23/09 | 18,048 | 100 | 100 | 74% |
| 54 | Std Dev | 55% | 2,977 | 460 | 334 | 1,741 | 0.70 | 0.76 | 4,484 | 1.48 | 8.65 | 1.88 | 6.37 | 4.26 | 03/18/03 | 3,121 | 12 | 6 | 135% |
| 55 | Count | 481 | 481 | 481 | 481 | 479 | 481 | 481 | 481 | 474 | 274 | 356 | 299 | 331 | 04/25/01 | 481 | 481 | 479 | 481 |
| 56 | | | | | | | | | | | | | | | | | | | |
| 57 | **All Vertical Buyers USA Target as of 7/21/09** | | | | | | | | | | | | | | | | | | |
| 58 | | | | | | | | | | | | | | | | | | | |
| 59 | **Min** | **-87%** | **1** | **-98** | **-137** | **-15,256** | **(6.01)** | **(16.41)** | **14** | **0.12** | **7.79** | **0.36** | **4.92** | **0.10** | **01/22/98** | **14** | **51** | **80** | **-642%** |
| 60 | **Max** | **140%** | **15,070** | **4,128** | **3,037** | **785** | **0.24** | **0.21** | **101,003** | **6.70** | **31.57** | **6.78** | **32.37** | **21.67** | **01/14/02** | **101,003** | **100** | **100** | **58%** |
| 61 | **Avg** | **34%** | **905** | **223** | **112** | **-286** | **(0.36)** | **(0.88)** | **3,441** | **1.75** | **19.64** | **3.18** | **14.14** | **10.38** | **01/10/00** | **3,423** | **97** | **99** | **3%** |
| 62 | 1%-ile | -67% | 1 | -77 | -122 | -9,822 | (5.52) | (13.09) | 17 | 0.15 | 8.00 | 0.38 | 5.06 | 0.93 | 02/13/98 | 17 | 59 | 87 | -429% |
| 63 | 5%-ile | -26% | 4 | -37 | -76 | -119 | (2.10) | (5.38) | 30 | 0.24 | 8.85 | 0.60 | 5.62 | 4.29 | 05/14/98 | 23 | 79 | 100 | -35% |
| 64 | 10%-ile | -19% | 15 | -16 | -46 | -61 | (0.65) | (1.07) | 43 | 0.28 | 9.89 | 1.00 | 7.33 | 4.51 | 08/03/98 | 36 | 90 | 100 | -23% |
| 65 | 25%-ile | 14% | 69 | -4 | -10 | 19 | (0.18) | (0.22) | 98 | 0.46 | 15.96 | 1.54 | 9.96 | 6.90 | 06/11/99 | 95 | 100 | 100 | 12% |
| 66 | **50%-ile** | **34%** | **191** | **15** | **6** | **31** | **0.02** | **0.01** | **195** | **1.09** | **20.56** | **3.17** | **13.43** | **9.50** | **12/16/99** | **195** | **100** | **100** | **25%** |
| 67 | 75%-ile | 59% | 760 | 122 | 59 | 90 | 0.09 | 0.04 | 941 | 2.32 | 24.19 | 4.43 | 15.09 | 12.61 | 08/23/00 | 941 | 100 | 100 | 37% |
| 68 | 90%-ile | 84% | 1,669 | 311 | 193 | 532 | 0.17 | 0.07 | 2,578 | 4.64 | 28.09 | 5.69 | 22.47 | 17.90 | 06/13/01 | 2,578 | 100 | 100 | 46% |
| 69 | 95%-ile | 96% | 2,435 | 689 | 239 | 674 | 0.21 | 0.10 | 3,670 | 6.29 | 30.16 | 6.31 | 30.04 | 19.49 | 10/24/01 | 3,571 | 100 | 100 | 49% |
| 70 | 99%-ile | 129% | 10,675 | 3,371 | 2,036 | 779 | 0.24 | 0.18 | 66,418 | 6.61 | 31.29 | 6.72 | 31.91 | 21.24 | 12/29/01 | 66,240 | 100 | 100 | 56% |
| 71 | Std Dev | 42% | 2,489 | 743 | 502 | 2,541 | 1.26 | 2.98 | 16,522 | 1.85 | 7.06 | 1.88 | 7.30 | 5.30 | 01/05/01 | 16,522 | 10 | 3 | 111% |
| 72 | Count | 37 | 37 | 37 | 37 | 37 | 37 | 37 | 37 | 33 | 16 | 27 | 16 | 21 | 02/06/00 | 37 | 37 | 37 | 37 |
| 73 | | | | | | | | | | | | | | | | | | | |
| 74 | **All Horizontal Buyers USA Target as of 7/21/2009** | | | | | | | | | | | | | | | | | | |
| 75 | | | | | | | | | | | | | | | | | | | |
| 76 | **Min** | **-96%** | **0** | **-425** | **-1,023** | **-11,345** | **(733.00)** | **(740.00)** | **0** | **-** | **0.04** | **0.01** | **0.03** | **0.02** | **01/02/98** | **1** | **22** | **51** | **-2673%** |
| 77 | **Max** | **540%** | **61,831** | **19,600** | **18,803** | **21,196** | **13.26** | **22.44** | **116,707** | **10.00** | **39.89** | **9.93** | **34.37** | **24.76** | **12/31/02** | **116,705** | **100** | **100** | **84%** |
| 78 | **Avg** | **38%** | **779** | **180** | **131** | **287** | **(0.65)** | **(0.88)** | **1,487** | **2.18** | **19.96** | **2.81** | **15.09** | **10.71** | **06/16/00** | **1,479** | **99** | **100** | **11%** |
| 79 | 1%-ile | -70% | 2 | -107 | -192 | -350 | (7.01) | (10.84) | 3 | 0.07 | 2.20 | 0.27 | 2.45 | 1.85 | 01/27/98 | 3 | 60 | 94 | -235% |
| 80 | 5%-ile | -35% | 6 | -29 | -49 | -26 | (1.68) | (1.75) | 9 | 0.23 | 6.70 | 0.65 | 5.65 | 4.04 | 05/28/98 | 9 | 100 | 100 | -53% |
| 81 | 10%-ile | -13% | 11 | -13 | -20 | 2 | (0.54) | (0.59) | 16 | 0.36 | 9.32 | 0.94 | 7.67 | 5.16 | 10/15/98 | 16 | 100 | 100 | -15% |
| 82 | 25%-ile | 9% | 27 | 2 | -1 | 14 | (0.02) | (0.05) | 51 | 0.82 | 14.01 | 1.45 | 10.09 | 7.23 | 06/30/99 | 50 | 100 | 100 | 9% |
| 83 | **50%-ile** | **29%** | **90** | **16** | **11** | **49** | **0.10** | **0.04** | **167** | **1.70** | **19.32** | **2.30** | **13.77** | **9.83** | **06/21/00** | **167** | **100** | **100** | **23%** |
| 84 | 75%-ile | 57% | 346 | 69 | 51 | 152 | 0.37 | 0.11 | 707 | 2.95 | 25.83 | 3.60 | 19.03 | 13.43 | 06/14/01 | 705 | 100 | 100 | 36% |
| 85 | 90%-ile | 94% | 1,405 | 263 | 178 | 522 | 0.66 | 0.17 | 2,547 | 4.71 | 31.71 | 5.62 | 25.44 | 18.20 | 03/07/02 | 2,479 | 100 | 100 | 49% |
| 86 | 95%-ile | 124% | 3,168 | 647 | 390 | 1,171 | 0.74 | 0.22 | 5,600 | 6.01 | 34.91 | 7.17 | 28.71 | 20.70 | 07/23/02 | 5,600 | 100 | 100 | 55% |
| 87 | 99%-ile | 255% | 12,555 | 4,051 | 3,034 | 4,818 | 0.91 | 0.50 | 26,165 | 8.89 | 39.24 | 9.07 | 33.14 | 24.14 | 12/05/02 | 26,165 | 100 | 100 | 72% |
| 88 | Std Dev | 56% | 3,216 | 932 | 805 | 1,323 | 17.96 | 18.22 | 6,016 | 1.87 | 8.44 | 1.94 | 6.94 | 4.98 | 04/02/01 | 6,014 | 6 | 3 | 97% |
| 89 | Count | 1,821 | 1,821 | 1,821 | 1,821 | 1,819 | 1,819 | 1,819 | 1,821 | 1,679 | 965 | 1,466 | 626 | 725 | 12/25/04 | 1,821 | 1821 | 1821 | 1,821 |
| 90 | | | | | | | | | | | | | | | | | | | |
| 91 | **All Conglomerate Buyers USA Target as of 7/21/2009** | | | | | | | | | | | | | | | | | | |
| 92 | | | | | | | | | | | | | | | | | | | |
| 93 | **Min** | **-12%** | **6** | **-78** | **-105** | **-156** | **(4.82)** | **(7.32)** | **4** | **0.25** | **2.28** | **0.80** | **1.39** | **1.28** | **01/06/98** | **4** | **50** | **100** | **-14%** |
| 94 | **Max** | **163%** | **1,344** | **161** | **111** | **10,435** | **0.66** | **0.24** | **1,972** | **7.57** | **32.69** | **9.86** | **30.90** | **17.59** | **10/23/02** | **1,972** | **100** | **100** | **62%** |

# Table 8.1A (cont.)
## Summary Statistics from Mergerstat Database for Different Populations

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 4 | **Stats** | **CP** | **Sales** | **EBITDA** | **EBIT** | **Bk V alue** | **Oper Margin** | **Net Margin** | **MV Equity** | **PS** | **PE** | **P/BV** | **TIC/EBIT** | **TIC/ EBITDA** | **Tx Date** | **P** | **% Acq** | **% Own Post-Tx** | **DLOC** |
| 95 | **Avg** | **43%** | **205** | **19** | **7** | **439** | **(0.19)** | **(0.34)** | **336** | **2.31** | **17.96** | **3.88** | **13.09** | **9.43** | **12/20/99** | **326** | **98** | **100** | **25%** |
| 96 | 1%-ile | -12% | 7 | -68 | -92 | -153 | (4.15) | (5.88) | 5 | 0.27 | 2.99 | 0.88 | 1.79 | 1.68 | 01/31/98 | 5 | 61 | 100 | -13% |
| 97 | 5%-ile | -8% | 12 | -34 | -50 | -115 | (1.74) | (1.32) | 11 | 0.34 | 5.82 | 1.19 | 3.38 | 3.26 | 05/16/98 | 11 | 94 | 100 | -10% |
| 98 | 10%-ile | 4% | 16 | -19 | -39 | -13 | (0.40) | (0.43) | 22 | 0.38 | 8.66 | 1.25 | 5.12 | 4.76 | 09/22/98 | 22 | 98 | 100 | 4% |
| 99 | 25%-ile | 13% | 22 | -3 | -6 | 11 | (0.1 1) | (0.11) | 40 | 0.79 | 1 1.25 | 1.93 | 7.50 | 5.95 | 04/27/99 | 40 | 100 | 100 | 12% |
| 100 | **50%-ile** | **32%** | **86** | **9** | **4** | **31** | **0.08** | **0.04** | **175** | **1.60** | **18.27** | **4.00** | **12.05** | **9.39** | **01/23/00** | **173** | **100** | **100** | **24%** |
| 101 | 75%-ile | 63% | 169 | 40 | 29 | 52 | 0.18 | 0.10 | 453 | 3.09 | 25.33 | 5.23 | 19.06 | 13.07 | 06/12/00 | 406 | 100 | 100 | 39% |
| 102 | 90%-ile | 93% | 488 | 62 | 45 | 133 | 0.38 | 0.20 | 710 | 5.58 | 28.73 | 6.72 | 22.31 | 15.44 | 09/29/00 | 710 | 100 | 100 | 48% |
| 103 | 95%-ile | 110% | 1,012 | 91 | 67 | 377 | 0.44 | 0.22 | 1,465 | 5.91 | 30.46 | 7.05 | 25.96 | 16.69 | 03/16/02 | 1,465 | 100 | 100 | 52% |
| 104 | 99%-ile | 151% | 1,301 | 145 | 102 | 7,940 | 0.61 | 0.23 | 1,893 | 7.17 | 32.24 | 9.30 | 29.91 | 17.41 | 10/10/02 | 1,893 | 100 | 100 | 60% |
| 105 | Std Dev | 41% | 339 | 45 | 41 | 2,042 | 1.07 | 1.47 | 489 | 2.03 | 9.02 | 2.44 | 8.19 | 4.75 | 02/11/01 | 488 | 10 | 0 | 19% |
| 106 | Count | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 26 | 25 | 14 | 19 | 14 | 14 | 01/26/00 | 26 | 26 | 26 | 26 |

**Overall Summary of Strategic versus Financial Buyer Control Premiums**

| | | Financial | | Strategic | |
|---|---|---|---|---|---|
| 111 | | **Mean** | **Median** | **Mean** | **Median** |
| 112 | Strategic | | | 35% | 26% |
| 113 | Financial-All | 37% | 25% | | |
| 114 | Vertical | | | 34% | 34% |
| 115 | Horizontal | | | 38% | 29% |
| 116 | **Mean** | **37%** | **25%** | **35%** | **30%** |
| 117 | **Mean (Excl Vertical)** | **37%** | **25%** | **36%** | **27%** |

**Note: In this row we exclude Vertical targets, as there are only 37 observations.**

**Conclusion: Strategic buyers pay 1% to 2% more than financial buyers.**

(B83), respectively—higher than strategic and financial buyers. This includes 1,821 (B89) transactions, which is more than both strategic and financial buyers combined. This strengthens the conclusion that strategic buyers pay more than financial buyers.

Finally, there are 26 (B106) conglomerate buyers, which is a specialized type of financial buyer. The mean and median control premiums are 43% (B95) and 32% (B100), respectively—which is closer to the vertical and horizontal buyer results. However, the small number of transactions renders this result less reliable and important.

There is an overall summary of the above results in rows 110–117. Row 116 is the mean of rows 112–115, which means that it is the average of the median control premiums for strategic, vertical, and horizontal buyers. However, since there are only 37 observations for vertical buyers, the mean of just strategic and horizontal control premiums in row 117 is more meaningful.

The means of mean control premiums for financial and strategic buyers are 37% (D117) and 36% (F117), respectively—just 1% higher for strategic buyers. The mean of median control premiums is 25% (E117) for financial buyers and 27% (G117) for strategic buyers—just 2% higher for strategic buyers.

**OTHER CHARACTERISTICS OF THE MERGERSTAT DATABASE** We will find that strategic acquirers tend to acquire different types of target firms than financial buyers. Let's begin with sales. The targets that strategic buyers acquire have sales ranging from a minimum of zero (C8) to $51.9 billion (C9), with an average of $769 million (C10) and a median of $103 million (C15). Thus, this is also a left-skewed distribution, with the mean much higher than the median. Financial buyers,[5] on the other hand, acquired firms with mean sales of $1.18 billion (C27) and median sales of $290 million (C32). Thus financial buyers are acquiring firms that are 53% and 182% larger in sales than strategic buyers, as measured by means and medians, respectively. Henceforth, we will focus on the difference in medians.

Median EBITDA is $34 million (D32) for targets of financial buyers versus $19 million (D15) for targets of strategic buyers—81% higher. EBIT is 49% higher. Book value is only 11% higher, and the market value of equity (column I) is only 3% higher; it is virtually the same.

There are, however, some statistics that favor the targets of the strategic buyers. Median operating margin is 11% (G15) for targets of strategic buyers, while it is only 7% (G32) for financial buyers, with the former being 57% higher. The net margin is 64% higher. The median Price-to-Sales (PS) multiple is 106% higher for strategic targets than financial targets (2.22 in J15 versus 1.08 in J32). However, the median PE multiple is only 9% higher. Since the market value of equity is the price (i.e., the P in the PS and PE multiples),[6] and it is almost the same for strategic and financial targets, the much higher PS multiple for strategic targets is a result of the smaller sales and the larger profit margins of the strategic targets. The 9% higher median PE multiple for strategic targets is a modest difference. Since the strategic targets are smaller firms, which we generally associate with a higher discount rate, it likely indicates higher expected growth rate for the strategic targets.

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[5]From the dataset with transactions starting on 1/2/2003.

[6]Technically, column P is the price and appears to include a small amount of debt in the transaction. However, the difference is small.

Looking at the horizontal buyers, which are businesses acquiring their competitors, which is a special type of strategic buyer, the results are similar, except that median net margin is only 4% (H83), which is closer to the financial target. The PS ratio of 1.70 (J83) is between the other two, and the PE ratio of 19.32 (K83) is slightly lower than the financial targets.

**TENTATIVE CONCLUSIONS** We conclude that strategic buyers pay slightly higher control premiums than financial buyers—2% to 3% higher per our analysis in Table 8.1, and 2% more per our analysis in Table 8.1A. This does not necessarily mean that these are all the synergies from strategic acquisitions. Buyers could be retaining some of the synergies for themselves.

The majority of acquisition studies show that the sellers have significant, large, positive cumulative abnormal returns (CARs) and buyers have slightly negative or zero CARs. We later show that if buyers were paying only for control and not synergies (i.e., cash flows), their stock would decline by the amount of the control premium. Because their CARs are near zero, we can conclude that is not happening, and they are paying for synergies—and perhaps some performance improvements, as well.[7] Thus it seems that the synergies are hidden in the targets' CARs.

Maquieira, Megginson, and Nail (1998), discussed later in this chapter, find synergies of slightly less than 7% in stock-for-stock mergers. Since the target is typically a small fraction of the combined entity, the synergy as a percentage of the value of the target is large. We also conclude that the targets of strategic buyers are larger, less profitable, and slower growing than financial targets.

## Calculating the Discount for Lack of Control

The traditional calculation of discount for lack of control (DLOC) is based on the control premium. If the marketable minority FMV is $100 per share and one buys control for $140 per share, the control premium (CP) is $40 per share. In percentages, the premium is $40 per share divided by the marketable minority price of $100 per share, or $40/$100 = 40%. Going in the other direction, DLOC is the $40 premium divided by the control price of $140, or $40/$140 = 28.6%. Symbolically, DLOC = CP/(1 + CP).

In the majority of valuation assignments for business appraisers, it is necessary to remove any synergistic element in control premiums before using the above formula to calculate DLOC.

The data are confusing, and there are different ideological camps in the valuation profession as to how to conceptualize and calculate control premiums. The goal of the control section of this chapter is to present a large body of professional and academic research, arrive at a coherent explanation of the diverse data, and provide guidance and quantitative benchmarks for use in the profession.

## Prior Research—Qualitative Professional

As mentioned earlier, we examine two types of prior research: professional and academic. In this section we examine prior professional research on control premiums.

[7] We cover this later in the chapter.

The professional research itself is composed of a long series of articles that develop important valuation theory that is primarily qualitative. We will now review the main articles, which are written by Eric Nath, Chris Mercer, Michael Bolotsky, and Wayne Jankowske. Again, because control and marketability are concepts that are so intertwined, these articles also contain material relevant to adjusting for marketability.

**NATH** The original attack on the *traditional levels of value chart* (the top of Figure 8.1) came from Eric Nath (1990). Nath (1994, 1997) later clarified and slightly modified his initial position. Nath argued:

- Fewer than 4% of all public firms are taken over each year. Using an efficient markets hypothesis argument, Nath said that the LBO funds, strategic buyers, and their bankers, who collectively represent hundreds of billions of dollars scouring the market for deals, keep the market clean. Any good takeover opportunity will not last long. If there were hidden premiums in the firms, their stock prices would be rapidly bid up to that level.
- Minority shares in publicly held firms are liquid. The existence of liquidity tends to eliminate nonstrategic acquisition premiums if the companies are well managed and management communicates effectively with investors.
- The previous points lead to the conclusion that the publicly traded prices are control values and not just minority values. His major conclusion, which contradicts conventional wisdom of the three-tiered levels of value chart, is that starting from a public market–derived value, one must take both DLOM and DLOC to value a privately held minority interest. Apparently influenced by articles from Bolotsky and Jankowske, both discussed below, Nath later (1997) switched to the two-tiered levels of value structure. Doing so had no material effect on his conclusions, merely the presentation.
- Buyers are often strategically motivated, and therefore what they pay is not equivalent to FMV. Nath's evidence is that similar premiums are paid for minority interest acquisitions. Later he modified his position. Nath is concerned with whether the market of relevant buyers for a subject company is likely to consist of many strategic buyers who would participate in an auction for the company. If so, then he contends that strategic value essentially becomes fair market value. If there are not many strategic buyers, then he is still concerned that the M&A multiples may contain a strategic element in the acquisition premium, leading to overvaluation of the company unless that element is removed. He determines this by an analysis of the following three entities: the company itself, the market for firms in that industry, and the M&A databases.
- Several problems with the computations of control premiums cause them to be misleading. Mergerstat's control premium statistics exclude acquisition discounts; that is, some acquisitions occur at lower prices than the minority trading price. Other problems are that the range of premiums is enormous, and Mergerstat uses simple averages instead of market-weighted averages.
- There are at least two additional problems in using takeover prices as an indicator of value. The first is that transactions are unique and time-specific. Just because a specific buyer pays a specific premium for a particular firm, that does not mean that another buyer would pay a similar premium for a comparable firm, let alone for a much smaller and less exciting company. The second

problem, Nath contends, is that some people overpay. These points are related to the comments above on the strategic element of acquisition premiums in the M&A market.

**Summary of Nath's Position** Firms that are taken over are different from those that are not. Because of this, the public minority value is a control value and therefore we should not apply control premiums to a discounted cash flow valuation. Additionally, for a private minority interest, Nath takes both DLOM and DLOC when starting from a public market–derived value. Furthermore, Nath does not believe in taking DLOM for a control interest. Nath's reason for this position is that the M&A and business brokerage markets are very active, and that activity negates any tendency to a DLOM.

**BOLOTSKY**[8] Michael Bolotsky (1991, 1995) agrees with many of Nath's criticisms of conventional wisdom, but disagrees with his conclusions. With regard to the results of his analysis, he represents a middle position between Mercer and Nath. With regard to the theoretical underpinnings, his work is unique in that it is the first article that abandons the linear *levels of value* concept entirely and replaces it with a multifactor, multidimensional matrix of fundamental attributes. For example, he gets rid of the concept that 100% ownership value must always be somehow "higher than or equal to" minority ownership value. He contends that both Mercer and Nath are still arguing around a linear concept of going from "up here" to "down there."

Bolotsky has a comprehensive, logical framework of analysis that includes differences in ownership rights, liquidity, information access, and information reliability between the four types of ownership interests listed in Figure 8.2. Bolotsky's article is important theoretical work, and it obviously influenced subsequent articles by both Nath and Mercer. Bolotsky's article contains no empirical evidence, nor is there any attempt to quantify the implications of his framework into an economic model. The practical significance of the article is that he disagrees with Nath's conclusion that valuing a private minority interest with reference to public minority interests as a starting point requires applying both a DLOM and a DLOC.

It is significant that Bolotsky did not attempt to compress four levels of value (public-control, public-minority, private-control, and private-minority) into three, as both Nath and Mercer did.[9] Bolotsky's assertion that the more the buy side knows about the seller and the more he or she can rely on it, the higher the price, is also significant and logical.

Bolotsky characterized the public markets as a consensus opinion of value that may occasionally experience an anomalous trade, but that trade will be quickly bid back to a rational, equilibrium consensus value. He says:

> *[T]he purchase of an entire company is typically a one-time purchase of a unique item; the price that ultimately gets recorded is not the consensus opinion of the limited group of buyers and sellers for a particular entire company but is rather the winning bid, which is normally the highest bid. There is no "market"*

[8] I thank Michael Bolotsky for editing this section and helping me to correctly interpret his work.

[9] Nath stopped doing that in his December 1997 article.

> *process going on here in the sense described above for public minority blocks. It is analogous to a situation where the single anomalous trade described earlier does not get rapidly bid down to a consensus price; instead, it gets memorialized in* Mergerstat Review, *to be relied upon by valuation consultants. Clearly, relative to fair market value, there is an upwards bias in prices that represent either the highest bid, the only bid, or the bid of a buyer bringing special attributes to the table.*

Bolotsky takes a middle position as to whether the takeovers are for typical or atypical public companies, the former position being taken by Mercer and the latter by Nath. Bolotsky says that it is inappropriate to insist that unless a subject company is in play one must assume there is no control premium. He thus disagrees with Nath on that point.

Bolotsky's theoretical framework has no concept of sequential levels of value, with control value at the top, followed by minority marketable value, and nonmarketable minority value at the bottom. Rather, Bolotsky advances the concept that the value of various types of ownership interests is the result of building up the contribution to value of fundamental ownership attributes, to the degree that each attribute applies to the interest in question. In addition, Bolotsky's framework implies that, rather than *discounts* and *premiums*, there are *adjustments* for differences in ownership attributes and that the adjustment can be positive, negative, or zero. In this framework, there is nothing that mandates that a 100% ownership position will be equal to or greater in value than a public minority price; indeed, Bolotsky's framework implies that if investors in a security value liquidity and the options that liquidity provides to a greater degree than they value power, then public minority pricing for that security will exceed 100% ownership pricing.

Thus, Bolotsky says that for those public companies where we would conclude that the per-share 100% ownership value is the same amount as the public minority value, the two prices might be the same but for very different reasons. In effect, the same price for different ownership interests is resulting from the *net* of the differences in the impact of various ownership attributes on each interest. Since the concept is of the net of differences, there is no reason why the net difference in price between a 100% ownership position and a public minority position cannot be zero or even negative. Accordingly, when we state that the public minority value and the 100% ownership value are the same, we are really saying that we should apply a net value adjustment of zero. I agree with his position that there is a very important distinction in saying we are applying a net zero premium versus saying that by definition there is no premium.

Bolotsky also states that Nath's conclusion is internally flawed in that in valuing private minority interests with reference to public minority prices as a starting point we need to take both a DLOM and a DLOC. He argues that if a public minority block of shares happens to have the same per-share value as a 100% ownership interest, this does not affect the fact that the block in question is still a minority block of stock having no attributes of control over the company. He contends it would be illogical to subtract a DLOC from a block that has no attributes of control.[10] Rather,

[10]This is a very logical statement and appears to be self-evident. Nevertheless, I will disagree with this later in the chapter.

the often extreme price differentials between public and private minority interests must be explained by other ownership attributes besides control, including but not limited to differentials in relative liquidity, relative level of information availability, and relative information reliability.

Bolotsky claims—reasonably, in my opinion—that there are many public firms whose perceived 100% ownership value will be more than their minority value, but not enough more to make a tender offer worthwhile. In addition, Bolotsky's theoretical framework is the only one that can readily accommodate several market features that appear anomalous when relying on the linear levels of value framework, such as 100% ownership pricing at levels considerably below IPO pricing for many companies in today's markets.

**JANKOWSKE** Wayne Jankowske's article (Jankowske, 1991) corrects certain key errors in the articles by Nath and Bolotsky. He says one does not have to accept Nath's assertion that the marketable minority value is a control value to accept the proposition that DLOC can differ between public and private firms. He says differences in legal and contractual protection, agency costs, relative incentives, and differential economic benefits can account for differences in the public versus private DLOC.

Prevailing wisdom's assertion is that since public market prices are minority prices, we can use public guideline company prices to value private minority shares, with only DLOM necessary. Jankowske says that for that to be true, it implies that the economic disadvantage of lack of control associated with public minority shares is equal to that of private minority shares, which is unrealistic.

Conceptually, the magnitude of DLOC in guideline public prices makes no difference, whether it is 30%, or as Nath contended in his first two articles, 0%. The difference between the public and subject company's DLOC must be recognized to avoid an overvaluation.

He developed the following formula to value a private minority interest:[11]

$$\text{Additional DLOC} = [\text{FMV}_{\text{MM}}/(1 - \text{DLOC}_{\text{GC}})] \times (\text{DLOC}_{\text{SC}} - \text{DLOC}_{\text{GC}})$$

where $\text{FMV}_{\text{MM}}$ = the marketable minority fair market value.
$\text{DLOC}_{\text{GC}}$ = discount for lack of control in the public guideline companies.
$\text{DLOC}_{\text{SC}}$ = discount for lack of control in the subject company.

He gave the following example: $\text{FMV}_{\text{MM}}$, the marketable minority interest value, is \$900; $\text{DLOC}_{\text{GC}}$, the discount for lack of control implicit in the public minority stock, is 10%, and $\text{DLOC}_{\text{SC}}$, the discount for lack of control appropriate to the subject company, is 40%. His calculation of incremental discount is:

$$\frac{\$900}{(1 - 10\%)} \times (40\% - 10\%) = \$1{,}000 \times 30\% = \$300.$$

He disagrees with Bolotsky that the guideline firms must have identical shareholder attributes.

[11] I have changed his notation.

In his second article on the topic (Jankowske, 1995), he stressed that it is the economic benefits to which we must look as a justification for control premiums, not the powers that come with control. He cites the following economic benefits of control:

- Company Level
  - Performance improvements
  - Synergy
- Shareholder Level
  - Wealth transfer opportunities—the Machiavellian ability to expropriate wealth from the minority shareholders.
  - Protection of investment—the flipside of the above point is that control protects the shareholder from being exploited. This motivation is important because it relates to ambiguity avoidance in the academic literature reviewed in this chapter.
  - Liquidity—control enhances liquidity in privately held businesses.

Of the company-level advantages, the extent to which performance improvements on a stand-alone basis account for control premiums properly belongs in our calculations of fair market value. That portion accounted for by synergy is investment value and should not be included in fair market value.

**ROACH** George Roach (1998) summarized percentage acquisition premiums from a database of business sales. The premiums were measured as $\frac{P_{Announcement}}{P_{5\,Day}} - 1$, where the numerator is the announcement price, and the denominator is the minority trading price five days before the announcement of the acquisition. He also provided the premium based on the price 30 days prior. We excerpt from his Exhibit IV to our Table 8.2.

Table 8.2 lists acquisition premiums broken down by the difference between the buyers' and sellers' SIC codes. The acquisition premiums range from 37.1% (B7) to 50.8% (B8). There is no pattern to the first three premiums listed in Table 8.2. The 50+ SIC code difference level between buyers and sellers has the highest premium, which is counterintuitive. Roach found similar patterns in the results for median premiums. Additionally, while the 30-day premiums were higher than the 5-day premiums, the patterns were similar.

| | A | B |
|---|---|---|
| 1 | **Table 8.2** | |
| 2 | **Acquisition Premiums by SIC Code** | |
| 3 | | |
| 4 | **SIC Code Differences between Buyer and Seller** | **5-Day Avg Prem** |
| 5 | 0 | 37.2% |
| 6 | 1–9 | 41.0% |
| 7 | 10–48 | 37.1% |
| 8 | 50+ | 50.8% |
| 9 | | |
| 10 | *Source:* Roach, George P. 1998. "Control Premiums and Strategic | |
| 11 | Mergers." *Business Valuation Review*. June 1998, table IV, p. 47. | |

Under the assumption that acquisitions of firms in the same or almost the same SIC code are more likely to be strategic acquisitions than firms acquired in very different SIC codes, Roach's analysis appears to be strong evidence that premiums paid for strategic buys are no larger than premiums paid for financial buys. This leads to the conclusion that acquisition premiums are control premiums, not premiums for synergy.

This conclusion supports Mercer (1990) and Bolotsky in their opinion that the similarity of premiums for acquiring minority positions and control positions can be explained as acquiring *creeping control* because it appears to rule out synergy as an explanation. This is in contrast to Nath's position and Mercer (1998, 1999). This also comports with a surface reading of Tables 8.1 and 8.1A that payment for synergies is minimal, although not with our ultimate interpretation of them.

**Summary of Roach's Position** Regarding DLOC, acquisition premiums are control premiums, not premiums for synergy. The 5-day acquisition premiums range from 37.1% to 50.8%.

**MERCER (1998) AND (1999)** Mercer (1998) represents a significant change in thinking since Mercer (1990). He believes that the majority of premiums for mergers and acquisitions recorded in Houlihan Lokey Howard & Zukin's Mergerstat represent strategic premiums for synergies, which do not qualify for fair market value. He modified the traditional levels of value chart in the top of Figure 8.1 to the one below it. It is the same as the one above with the addition of a strategic value above the control value.

For sake of discussion, let's look at an end-of-year Gordon model formula to calculate value, $PV(Cash\ Flows) = \frac{NI_{t+1} \times POR}{r-g}$, where *POR* is the payout ratio, *r* is the discount rate, and *g* is the constant growth rate.

Mercer (1999) states that the discount rate is the same at the marketable minority level as it is in all levels of value above that (though not the same as the private minority level, which almost always carries a higher discount rate). The main difference in the valuation comes from the numerator, not the denominator. Control buyers, whether financial or strategic, upwardly adjust forecast cash flows. He details the types of adjustments as follows:

*Normalizing adjustments*. These adjust private company earnings to well-run public company equivalent. Mercer classifies two types of normalizing adjustments. Type 1 is to eliminate nonrecurring items and adjust for nonoperating assets. Type 2 is to adjust insider compensation to an arm's-length level, including eliminating discretionary expenses that would not exist in a public company.

*Control adjustments*. Mercer lists two types of control adjustments. Type 1 control adjustments are for what Jankowske calls performance improvements and apply to both financial buyers and strategic buyers. Mercer says these are adjustments for *improving* the [existing] earnings stream (i.e., running the company more efficiently). This could also include the volume discounts coming from the buying efficiencies achievable by being owned by a larger company that is a financial buyer. Type 2 control adjustments are for *changing* the earnings stream (i.e., running the company differently), and apply only to strategic buyers. These include consolidating G&A expenses, eliminating duplicate operations, selling more product, and enhanced negotiating power with suppliers, distributors, or customers, above and beyond that which can be achieved by a financial buyer.

Mercer is in very good company in this position. Consider Pratt (1998, p. 134):

> *The exploitation of minority shareholders is far less prevalent in public companies than in private companies, at least in larger public companies. If company cash flows are already maximized and the returns are already distributed pro rata to all shareholders, then there may be no difference between a control value and a minority value.*

Other similar opinions can be found in Ibbotson (1999), Zukin (1998), and Vander Linden (1998).

Actually, Mercer is not the originator of this position on control premiums, but he may be the person who has written the most about it. The original statement of this position came from Glass and McCarter (1995).

**Summary of Mercer (1998, 1999)** Mercer says that after taking into consideration increases in forecast cash flows from performance improvements and arm's-length salary adjustments for the control shareholder that are appropriate for a financial buyer, there are no control premiums. In the absence of information to do so, he says control premiums should be very small—no more than 10%, with the implication that it could still be as little as zero.

**SUMMARY OF PROFESSIONAL RESEARCH ON CONTROL PREMIUMS** Now we will summarize the professional research on control premiums. Tables 8.1 and 8.1A and Roach support the conclusions that acquisition premiums are not primarily for synergies, and control premiums are appropriate. The other extreme—a zero control premium—is represented by Nath and Mercer. While the original gap between Nath's and Mercer's positions on control premiums was large, the current gap is much smaller and often may not even exist. The logic of how they arrived at their opinions is different, but they would probably come to similar calculations of control premiums in the majority of circumstances.

Nath never uses control premiums. Mercer, following Glass and McCarter, now agrees that after taking into consideration increases in forecast cash flows from performance improvements and arm's-length salary adjustments that are appropriate for a financial buyer, there are no control premiums. In the absence of information to do so, he says control premiums should be very small—no more than 10%, with the implication that it could still be as little as zero. That is essentially Nath's position, that the public minority value is a control value and therefore we should not apply control premiums to a discounted cash flow valuation. Mercer rarely ever assigns control premiums unless there are identifiable increases in forecast cash flows at the control level. In that case, he would simply increase the cash flow forecast, and the control level value would increase vis-à-vis the marketable minority interest level. Nath would do the same thing, except he does not like the term *marketable minority level of value* (or its synonym, *as if freely traded*). Their terminology differs far more than their results, at least with regard to control premiums.

Regarding the discount for lack of marketability (DLOM), neither Mercer nor Nath believes in taking DLOM for a control interest—a position with which I disagree in the DLOM section of this chapter. Nath's reason for this position is that the

M&A and business brokerage markets are very active, and that activity negates any tendency to a DLOM. He makes an analogy of the real estate market to the market for companies. Rather than apply DLOM, real estate appraisers put an "expected marketing time" on their values. Mercer's main reason for opposing DLOM for control interests is that they control cash flows until a sale.

Nath always begins with a controlling owner's value, which in his view is the greater of the values obtained by the M&A markets, the public markets (reduced by the restricted stock discount that one would experience in going public), and liquidation. It is important to note that to the extent that the M&A market values contain synergies, and whenever the M&A valuation dominates, Nath's fair market values will contain synergies.

His opinion is that that is what buyers will pay, and therefore that is fair market value. There is a question as to whether that is investment value rather than fair market value. I agree that it probably is not investment value, as it is not value to a particular buyer. It is value to an entire class of buyers. If all buyers in the M&A market are strategic—which is certainly not completely true, but may be largely true is some industries—then that is what buyers would pay. With this fine distinction, it is very important to make sure that if one follows Nath's method, one must be careful that the subject company fits with the assumptions underlying Nath's logic.

I do not believe that most small firms and many midsized firms are serious candidates for the M&A market. They are business broker material, and such buyers would rarely ever be synergistic. Therefore it is imperative to be realistic about the market in which the subject company is likely to sell.

For a private minority interest, Nath takes both the discount for lack of marketability and lack of control. In conversation, he revealed his own dissatisfaction with the lack of relevant information for calculating DLOC, since there is still nothing to use other than the traditional flipside of the control premiums that he personally demolished as being valid premiums to add to a "minority value."

Mercer comes to what probably amounts to a very similar result through a different path. He does not calculate a discount for lack of control for private minority interests. Instead, he uses his quantitative marketability discount model (QMDM)—which we cover in more detail later in this chapter—to subsume any DLOC, which he feels is automatically included in his DLOM. I agree with Mercer that the QMDM includes the impact of DLOC because, in the QMDM, one must forecast the specific cash flows to the minority shareholder and discount them to present value. Thus, by using the QMDM, Mercer does not need a DLOC. Mercer's position is internally consistent.

## Prior Research—Academic

Now that we have summarized the professional literature, we will summarize the results of various academic studies relevant to our topic. The primary orientation of academic research in finance is on publicly traded stocks. It is generally not directly concerned with the issues of the valuation profession, which is focused on valuing private firms. Therefore, it is often a slightly interesting side point in an academic article that is a golden nugget for the valuation profession—if not a diamond.

There are two types of evidence of the value of control. The first type is the value of complete control. The second type deals with the value of voting rights. Voting rights do not represent control, but they do represent some degree of influence or partial control.[12]

The academic research falls into the following categories:

- The article by Schwert focuses primarily on analyzing returns in mergers and acquisitions during two periods: the runup period, which is the time before announcement of the merger, and the markup period, which is the time period after the announcement. This is significant in the context of this book primarily as providing empirical evidence that is relevant in my economic components model for the discount for lack of marketability (DLOM). It could easily belong to the DLOM part of this chapter, but I include it here with the rest of the academic articles.
- Voting rights premiums: The articles by Lease, McConnell, and Mikkelson; Megginson; Houlihan Lokey Howard & Zukin (HLHZ); and the section on international voting rights premiums all deal with the value of voting rights and provide insight on the value of control that fits in the definition of fair market value.[13]
- The articles by Bradley, Desai, and Kim and Maquieira, Megginson, and Nail are about the value of complete control. In particular, their focus is to measure the synergies in acquisitions, which is a critical piece of evidence to understand in sorting through the apparently conflicting results and opinions in the professional literature. The article by Bargeron, Schlingemann, Stulz, and Zutter looks at the difference in acquisition premiums paid by public versus private acquirers.
- The article by Menyah and Paudyal is an analysis of bid–ask spreads and is primarily related to DLOM, not control. It could also have been included in the section on marketability.

**SCHWERT (1996)** Since business appraisers calculate control premiums and discounts for lack of control from merger and acquisition (M&A) data of publicly traded firms, it is important to understand what variables drive control premiums in order to be able to properly apply them to privately held firms. Schwert's article has some important findings that are worthy of our attention.

Schwert's paper's main purpose is to examine the relationship between runups and markups in M&A pricing. The runup period is that period of time before the announcement of a merger in which the target firm's price is increasing. Schwert found that cumulative abnormal returns (CARs)[14] begin rising around 42 days before an acquisition. Thus, he defines the runup period from day −42 to day −1, with day 0 being the announcement of the merger. The markup period is from day 1 to the lesser of day 126 or delisting. The sum of the runup and markup period is the entire

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[12]Mergerstat Review does track premiums for acquisitions of minority interests, which comprises a third category of evidence. There is no academic literature of which I am aware that deals with this issue.

[13]The HLHZ article is professional rather than academic, but its topic fits better in our discussion of academic research.

[14]These are the cumulative error terms for actual returns minus market returns calculated by CAPM. It is the standard method in event studies to measure the acquisition premium.

relevant timeline of an acquisition, and the sum of their CARs is the total acquisition premium.

Schwert finds that CARs during the runup period for successful acquisitions between 1975 and 1991 average 25%, with CARs for unsuccessful acquisitions, that is, where the bidder ultimately fails to take over the target, averaging 19%.[15] After the announcement date, CARs for successful acquisitions increase to 37%, while for unsuccessful acquisitions they decrease to zero.

He discusses two opposite bidding strategies, the *substitution* hypothesis and the *markup pricing* hypothesis.

The substitution hypothesis states that each dollar of preannouncement runup reduces the post-bid markup dollar-for-dollar. The assumptions behind this hypothesis are that both the bidder and the target have private information that is not reflected in the market price of the stock and that no other bidder has valuable private information. Therefore, both the bidder and the target will ignore price movements that occur prior to and during the negotiations in setting the final deal price.

The markup pricing hypothesis is that each dollar of preannouncement runup has no impact on the post-bid markup. Thus, the preannouncement runup increases the ultimate acquisition premium dollar-for-dollar. The assumption behind this hypothesis is that both the bidder and the target are uncertain about whether movements in the market price of the target's shares reflect valuable private information of other traders. Therefore, runups in the stock price could cause both the bidder and the target to revise their valuations of the target's stock. Schwert used the example that if they suspect that another bidder may be acquiring shares, both the bidder and the target will probably revise their valuations of the target's stock upward.

The markup hypothesis reflects rational behavior of bidders and targets when they have incomplete information. A different explanation of the markup hypothesis is that of Roll (1986), who postulates that bidders are interested in taking over targets regardless of cost (the *hubris* hypothesis). This would reflect irrational behavior. Using regression analysis, Schwert finds strongly in favor of the markup hypothesis, while rejecting Roll's hubris explanation as well as the substitution hypothesis.

Had the substitution hypothesis been "the winner," this would have implied that the acquisition premiums that occur in the market would require major adjustments for calculating fair market value. It would have meant that the post-bid markups are based on private information to a particular buyer and seller, who ignore the effects of the pre-bid runup because they both believe that no other bidder has valuable private information. This would then be investment value, not fair market value. With the markup hypothesis being the winner, at least we do not have that complication.

For professional appraisers, the most important finding in Schwert's paper is the impact of competitive bidding, that is, when there is more than one bidder for a target, on the cumulative abnormal returns on the target's stock. Approximately 20% of the takeovers were competitive (312 out of 1,523), with 80% (1,211 out of 1,523) noncompetitive. Table 4 in the article shows that the presence of competitive

[15] 1,814 transactions in total, which are later reduced to 1,523 in his main sample.

bidding increases the premium paid by 12.2%.[16] This is significant evidence of the impact of competition that will have an important role to play in calculating $D_2$, the component in Abrams' economic components model of the discount for lack of marketability due to the absence of competition in thin markets. We will cover that in detail later in this chapter.

**Summary of Schwert's Findings** Regarding DLOM, in the M&A market, the presence of competitive bidding increases the premium paid by an absolute 12.2%.

**LEASE, McCONNELL, AND MIKKELSON (1983)** Lease, McConnell, and Mikkelson (LMM) examined all American companies with two classes of common stock outstanding sometime between 1940 and 1978. Both classes of common shares were entitled to identical dividends and liquidation preferences. In total, 30 companies met the criteria, although never more than 11 companies in any one year. On average, there were only seven companies in the population per year.

LMM found a statistically significant voting rights premium. They split their population into three categories. Category 1 firms had only voting and nonvoting common, with no voting preferred stock. Category 2 firms had two classes of voting common—one with superior voting rights and one with inferior voting rights. Category 3 firms had superior voting common, either nonvoting or inferior voting common, and voting preferred. Their results were as follows:

| Category | Mean Voting Rights Premium |
|---|---|
| 1 | 3.8% |
| 2 | 7.0% |
| 3 | −1.1% |

The mean voting premium of the Category 1 and 2 firms is 5.44%. There is no logical reason why Category 2 firms should have a higher voting rights premium than Category 1 firms, and the authors labeled this result "a puzzle." The relationship should have been the opposite.

There was one large outlier in Category 2. Without it, the Category 2 premium is only 1.9%. However, the authors investigated this outlier thoroughly and found no reason to exclude it from the data. It had no distinguishing characteristics.

As to the other puzzling result of a voting rights discount to the superior common shares in the presence of voting preferred stock, the authors speculate that there might be some incremental costs borne by the superior rights shares that are not borne by the inferior rights shares. However, Megginson (1990) and the HLHZ articles did not find this result. Megginson found a 23% premium for Category 3.

**Summary of LMM's Findings** Ignoring Category 3, LMM found that the mean voting rights premium of Category 1 and 2 firms was 5.44%.

[16]That is, it adds an absolute 12.2% to the premium. It does not increase the premium by 12.2%. For example, if the average premium with only one bidder is 30%, with two or more bidders it is 42.2%.

**MEGGINSON (1990)** The author analyzes 152 British firms traded on the London Stock Exchange in the 28 years from 1955 to 1982 that have at least two classes of common stock, with one class possessing superior voting (SV) power to the other, for the purpose of explaining the underlying variables that explain the voting rights premium (VRP) of the SV shares. He labels the inferior common shares those with restricted voting (RV) power. While the article does not say so, in one of many telephone conversations that I had with Professor Megginson, he said that all of the RV shares are simply nonvoting, even though he was using a more generic terminology. A minority of firms in his sample also had preferred shares. His work is a continuation of that of Lease, McConnell, and Mikkelson in a different environment.

Megginson was hoping for his regression analysis to shed light on which of three competing hypotheses explain the voting rights premium of SV shares. Ultimately, the regression results could not shed any light on the source of VRP. However, his article does provide some information to determine the magnitude of the control premium that is purely for control and not for anticipated higher cash flows.

Under the ownership structure hypothesis, there is an optimal amount of stock ownership for insiders—management and directors. If insiders hold too little SV stock, company performance can be improved by increasing insider ownership. However, if insiders own "too much" SV stock, they can become overly entrenched and immune to forced removal, lowering the value of all classes of stock in general and restricted voting (RV) stock in particular.

Under this hypothesis, the voting rights premium is positively related to insider holdings of SV shares and negatively related to their holdings of RV shares. The reason for the former is the entrenchment effect, and the reason for the latter is that the larger the percentage of RV shares owned by insiders, the more incentive they have to maximize the value of RV shares.

Some of the more interesting summary statistics from Megginson are listed below. Pay particular attention to numbers 3 and 4, as they contain the main information for our analysis below.

1. SV shares represented 38.4% of total common equity, but 94.3% of total voting power.
2. Insiders held 28.7% of SV shares (29.8% for companies with voting preferred) and 8.6% of RV shares (2.7% for companies with voting preferred).
3. The mean voting rights premium was 13.3% across all firms, 23% for firms with voting preferred, and 6% for firms that were subsidiaries of other companies.
4. Forty-three of the 152 firms (28.3%) were taken over during the sample period. In 37 out of the 43 cases, which is 86% of the 43 firms or 24.3% of the entire sample of 152 firms, the SV shares received higher prices than the RV shares by an average 27.6%.[17] The existence of significant tender offer premiums that go disproportionately to SV shares and whose timing is generally unknown could possibly explain the VRP, though Megginson feels the magnitudes of the VRP are too high to be explained by 28% premiums at unknown times.

[17]It is unclear whether the 27.6% refers to all 43 firms or just the 37 firms where the SV shares received a premium over the RV shares. Assuming the latter instead of the former changes the conclusion later in the chapter in Table 8.3, D24 from 1.4% to 1.5%.

5. His regression analysis in logarithmic form[18] with the ratio of the price of SV shares to the RV shares as the dependent variable found the percentage holdings of insiders of SV and RV shares as the only statistically significant variables. The former was positively related and the latter negatively related to the ratio of prices. Even then, the adjusted $R^2$ was only 11%.

**Summary of Megginson's Results** Megginson analyzes 152 British firms traded on the London Stock Exchange in the 28 years from 1955 to 1982. He found a mean VRP of 13.3% in the firms analyzed. Forty-three firms were taken over during the sample period. In 37 out of the 43 cases, the SV shares received higher prices than the RV shares by an average 27.6%.

**My Conclusions from the Megginson Results** The British VRP of 13.3% is significantly higher than the American VRP, which in the Lease, McConnell, and Mikkelson study is 5.4% and in the Houlihan Lokey Howard & Zukin study (which follows the section on Megginson) is 3.2%. The purpose of the next section is to determine how much of the 13.3% VRP is for the power of the vote versus the higher expected cash flows to the SV shareholders.

The analysis that follows shows that of the 13.3% VRP, 11.9% is due to higher expected cash flows to the SV shareholders, and 1.4% is being paid purely for the right to vote.

**My Analysis of the Megginson Results** This section is a detailed explanation of Table 8.3, which is my quantitative analysis of the Megginson results. The reader who wants to save time can safely skip this section and continue with the Houlihan Lokey Howard & Zukin (Much and Fagan) study.

We assume that the average holding period on the London Stock Exchange during the 1955–1982 period[19] was five years. The table begins with expected cash flows to the shareholders in rows 6 to 13, which we show in two different scenarios. In scenario #1 (columns A–F), the firm will not be acquired during the shareholder's tenure. In scenario #2 (columns H–M), the firm will be acquired during the shareholder's tenure.

The assumptions of the model are as follows: Using large-capitalization NYSE firm data from the 1999 SBBI yearbook,[20] for the years 1955–1982, total returns were 10.48% (B30), which we use as our discount rate. This broke down to a dividend yield of 3.94% (B27) and capital gains return of 6.54% (B29).

The voting rights premium is 13.3% (B28), per Megginson (1990).

When firms were acquired, we assume a 20% acquisition premium to the RV shares.[21] The final results are insensitive to the magnitude of this assumption.

The SV shares receive a premium that is 27.6% (B32) higher than the RV shares in the event of an acquisition.

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[18]The logarithm of the price variables most closely approximates a normal distribution.
[19]The period used for Megginson's analysis.
[20]London Stock Exchange data were unavailable to us. We use NYSE data as a proxy for the LSE data. According to Professor Megginson, the NYSE data should be a good proxy for the LSE. See Table 8.3, footnote [2].
[21]These data did not appear in the article and are no longer available.

The RV shareholder cash flows appear in C6 to C12. The shareholder invests \$1.00 (C6) at time zero. In year 1, he or she receives dividends of 3.94% × \$1.00 = \$0.0394 (C7). As the shares rise in price by 6.54% (B29) annually, applying the constant dividend yield is equivalent to having dividends rise by the same capital appreciation percentage of 6.54%. Thus \$0.0394 × (1 + 0.0654) = \$0.0420 (C8). As we go down the column, each year's dividend is 6.54% higher than the previous year's. The final dividend is \$0.0508 (C11). Finally, at the end of year 5, the shareholder sells for \$1.3727 (C12), which is the original investment of \$1.00 with five years of compound growth at 6.54%, or $\$1.00 \times (1 + 0.0654)^5 = \$1.3727$.

The SV share cash flows begin with a \$1.133 investment (B6), which represents the 13.3% (B28) VRP applied to the \$1.00 (C6) RV share price. The SV shareholders receive the same dividend stream as the RV shareholders, so B7 through B11 are the same as those rows in column C. At the end of year 5, the SV shareholder sells at the voting rights premium of 13.3%, that is, \$1.3727 × (1 + 0.133) = \$1.5552 [C12 × (1 + B28) = B12].

We discount the forecast cash flows at the average return of 10.48% (B30). The end-of-year present value factors at 10.48% appear in D6 to D12. Multiplying the SV and RV forecast cash flows by the present value factors leads to present values of the SV and RV forecast cash flows in E6 through E12 and F6 through F12, respectively. The totals are the net present values of the investments, which are −\$0.0221 (E13) and \$0 (F13) for SV and RV, respectively.

The analysis of scenario #2 is structured identically to that of scenario #1. The forecast cash flows in I6 through J11, which are the initial investments and the dividends, are identical to their counterparts in columns B and C. The only differences are in year 5, where we assume the firms are acquired. The acquisition amount for the RV shares is composed of two parts. The first is the five years of growth at 6.54% (B29), or $(1 + 0.0654)^5 = \$1.3727$, which is the same as C12. We then multiply that by 1 plus the assumed acquisition premium for RV shares of 20% (B31), or \$1.3727 × (1 + 0.2) = \$1.647194 (J12). The actual premium is unknown; however, a sensitivity analysis showed our final results are insensitive to this assumption within a fairly wide range around our assumption.

The SV buyout occurs at the SV-over-RV premium of 27.6% (B32), or \$1.647194 × (1 + 0.276 = \$2.101819 [J12 × (1 + B32) = I12]. The present values of the cash flows are \$0.3100 (L13) and \$0.1668 (M13) for SV and RV shares, respectively, when there is an acquisition.[22]

We now proceed to the summary of the net present values (NPVs) and begin with the no-acquisition scenario. In B17 and B18, we transfer the NPVs of −\$0.0221 and \$0 from E13 and F13 for the SV and RV shares, respectively. We then multiply those conditional FMVs by the probability of not being acquired in our assumed five-year holding period, which is 94.95% (B19) and is calculated in Table 8.3, footnote [2]. The probability-weighted NPVs for the SV and RV shares are −\$0.0210 and \$0 (B21, B22).

Next we transfer the acquisition scenario NPVs of \$0.3100 and \$0.1668 for SV and RV shares from L13 and M13 to C17 and C18, respectively. We multiply those NPVs by the probability of acquisition of 5.05% (C19), which is 1 minus the 94.95%

[22]Actually, the present values are slightly higher, as the acquisitions could take place before year 5. However, this simplification has no material impact on the outcome of the analysis.

in B19, to obtain the probability-weighted NPVs of \$0.0157 (C21) and \$0.0084 (C22) for the SV and RV shares, respectively.

We add columns B and C to obtain the probability-weighted NPVs of SV shares of −\$0.0053 (D21) and the RV shares of \$0.0084 (D22). The RV minus SV NPV difference is \$0.0137 (D23), or approximately 1.4% (D24) of the RV share price.[23]

Let's do a recap of this table, as it is very detailed. At the 10.48% (B30) discount rate, the RV shares are priced exactly right assuming there will be no acquisition; that is, they have a zero present value (F13), while they actually have a small, positive weighted average NPV of \$0.0084 (D22) after including the 5% (C19) probability of an acquisition premium. Thus, RV shares are a good buy based on expected cash flows for one with a 10.48% hurdle rate.

The SV shares, on the other hand, are a bad buy on a purely discounted cash flow basis. In the absence of an acquisition, which has a 95% (B19) probability for a five-year holding period, the NPV is −\$0.0221 (E13, transferred to B17). The positive NPV of \$0.3100 (L13, transferred to C17) in the event of an acquisition, which has only a 5% (C19) probability, is insufficient to outweigh the negative NPV absent the acquisition. Overall, the SV shares have a negative NPV of −\$0.0053 (D21). On a pure basis of NPV of forecast cash flows, the RV shares have a \$0.0137 (D23) NPV differential over the SV shares. The investor in SV shares passed up \$0.014 (rounded) of NPV to buy the vote, or 1.4% (D24) of the \$1.00 RV price. We subtract this from the average SV price of \$1.133, and \$1.119, or 11.9% of the 13.3% voting rights premium is justified by higher expected cash flows, while 1.4% of it appears to be paid for the right to vote and the marginal power that goes with it. We refer to the 1.4% as the net VRP, and to the 13.3% as the gross VRP. The net VRP is the portion of the VRP that is never regained on a present-value basis. The remaining 11.9% that is initially paid for the SV shares can be viewed as paying for the higher FMV that SV shares have over RV shares based on their higher eventual selling price.

In the middle-right section of the table, we present a sensitivity analysis of the RV-SV NPV differential. The RV-SV NPV differential rises as the fraction of the total return shifts more toward dividend yield and away from capital appreciation. For example, if capital appreciation accounted for none of the 10.48% yield, then the portion of the \$0.133 voting rights premium attributable to the power of the vote rises to \$0.0461 (I19) versus the base case.

The intuition for this result is that when returns are weighted more heavily toward dividends, the SV shares receive a lower effective dividend yield. This is because the SV shares receive the same absolute dividends as the RV shares, but they paid a higher price per share to receive them. Also, both SV and RV share prices grow more slowly, and the absolute cash value of the 27.6% SV-over-RV premium upon acquisition is less than when returns are primarily in the form of capital gains.

Table 8.3A is identical to Table 8.3, but it is for the Lease, McConnell, and Mikkelson study. We assume a 5.44% (B28) VRP, 40%[24] (B31) acquisition premium, and 0% (B32) additional acquisition premium for SV shares. The net VRP is 1.1% (D24).

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[23]This is 1.4% percent of the RV share price, since we defined the RV share price to be \$1.00.

[24]This is an assumption, as the data were unavailable. However, the final results are insensitive to the assumption.

| | A | B | C | D | E | F | G | H | I | J | K | L | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table 8.3** | | | | | | | | | | | | |
| 2 | **Analysis of Megginson Results** | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | |
| 4 | **Scenario #1: SV Shares-No Acquisition** | | | | | | | **Scenario #2: SV Shares-Acquisition** | | | | | |
| 5 | **Yr** | **SV** | **RV** | **PV Factor [1]** | **NPV SV** | **NPV RV** | | **Yr** | **SV** | **RV** | **PV Factor [1]** | **NPV SV** | **NPV RV** |
| 6 | 0 | -1.1330 | -1.0000 | 1.0000 | -1.1330 | -1.0000 | | 0 | -1.1330 | -1.0000 | 1.0000 | -1.1330 | -1.0000 |
| 7 | 1 | 0.0394 | 0.0394 | 0.9051 | 0.0357 | 0.0357 | | 1 | 0.0394 | 0.0394 | 0.9051 | 0.0357 | 0.0357 |
| 8 | 2 | 0.0420 | 0.0420 | 0.8193 | 0.0344 | 0.0344 | | 2 | 0.0420 | 0.0420 | 0.8193 | 0.0344 | 0.0344 |
| 9 | 3 | 0.0447 | 0.0447 | 0.7416 | 0.0332 | 0.0332 | | 3 | 0.0447 | 0.0447 | 0.7416 | 0.0332 | 0.0332 |
| 10 | 4 | 0.0476 | 0.0476 | 0.6712 | 0.0320 | 0.0320 | | 4 | 0.0476 | 0.0476 | 0.6712 | 0.0320 | 0.0320 |
| 11 | 5 | 0.0508 | 0.0508 | 0.6075 | 0.0308 | 0.0308 | | 5 | 0.0508 | 0.0508 | 0.6075 | 0.0308 | 0.0308 |
| 12 | 5 | 1.5552 | 1.3727 | 0.6075 | 0.9449 | 0.8340 | | 5 | 2.101819 | 1.647194 | 0.6075 | 1.2770 | 1.0008 |
| 13 | **Total** | | | | **-0.0221** | **0.0000** | | **Total** | | | | **0.3100** | **0.1668** |
| 14 | | | | | | | | | | | | | |
| 15 | | **Summary of NPVs** | | | | | | **SENSITIVITY ANALYSIS: NPV of RV-SV** | | | | | |
| 16 | | **No Acq** | **Acquisition** | **Total** | | | | | | | | | |
| 17 | SV (E13, L13) | -0.0221 | 0.3100 | | | | | **Capital** | **RV-SV** | | | | |
| 18 | RV (F13, M13) | 0.0000 | 0.1668 | | | | | **Apprec** | | | | | |
| 19 | Probabilities [2] | 94.95% | 5.05% | **100.00%** | | | | 0.00% | 0.0461 | | | | |
| 20 | Probability Wtd NPVs | | | | | | | 2.00% | 0.0371 | | | | |
| 21 | SV = (17)×(19) | -0.0210 | 0.0157 | **-0.0053** | | | | 4.00% | 0.0273 | | | | |
| 22 | RV = (18)×(19) | 0.0000 | 0.0084 | **0.0084** | | | | **6.54%** | **0.0137** | | | | |
| 23 | RV-SV (D22-D21) | | | **0.0137** | | | | 8.00% | 0.0053 | | | | |
| 24 | RV-SV (in Percent) | | | **1.4%** | | | | 10.48% | -0.0100 | | | | |
| 25 | | | | | | | | | | | | | |
| 26 | **Assumptions** | | | | | | | | | | | | |
| 27 | Dividend Yield [3] | 3.94% | | | | | | | | | | | |
| 28 | Voting Rights Prem | 0.133 | | | | | | | | | | | |
| 29 | Cap Apprec = g [3] | 6.54% | | | | | | | | | | | |
| 30 | Disc Rate = r [3] | 10.48% | | | | | | | | | | | |
| 31 | Acq Prem-RV [4] | 20% | | | | | | | | | | | |
| 32 | SV/RV Acq Prem | 27.6% | | | | | | | | | | | |
| 33 | | | | | | | | | | | | | |
| 34 | [1] Present value factors are end-of-year. Using midyear factors makes no difference in the final result to four decimal places. | | | | | | | | | | | | |
| 35 | | | | | | | | | | | | | |
| 36 | [2] Probability of acquisition: 43 Acquisitions/28 Years = 1.53571 Acq/Yr/152 Firms = 0.01010 Acq/Yr/Firm. × 5 Yrs = 5.05% Probability Acq/5Yrs/Firm | | | | | | | | | | | | |
| 37 | | | | | | | | | | | | | |
| 38 | [3] Derived from *SBBI—1999* for 1955–1982. We use the U.S. data as a proxy for U.K. data, as the latter were unavailable. | | | | | | | | | | | | |
| 39 | | | | | | | | | | | | | |
| 40 | [4] This is an assumption, as the data were unavailable. However, the final results are insensitive to the assumption. | | | | | | | | | | | | |

| | A | B | C | D | E | F | G | H | I | J | K | L | M |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table 8.3A** | | | | | | | | | | | | |
| 2 | **Analysis of American VRP Results—Lease, McConnell, and Mikkelson Results** | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | |
| 4 | **Scenario #1: SV Shares-No Acquisition** | | | | | | | **Scenario #2: SV Shares-Acquisition** | | | | | |
| 5 | **Yr** | **SV** | **RV** | **PV Factor [1]** | **PV SV** | **NPV RV** | | **Yr** | **SV** | **RV** | **PV Factor [1]** | **PV SV** | **NPV RV** |
| 6 | 0 | -1.0544 | -1.0000 | 1.0000 | -1.0544 | -1.0000 | | 0 | -1.0544 | -1.0000 | 1.0000 | -1.0544 | -1.0000 |
| 7 | 1 | 0.0394 | 0.0394 | 0.9051 | 0.0357 | 0.0357 | | 1 | 0.0394 | 0.0394 | 0.9051 | 0.0357 | 0.0357 |
| 8 | 2 | 0.0420 | 0.0420 | 0.8193 | 0.0344 | 0.0344 | | 2 | 0.0420 | 0.0420 | 0.8193 | 0.0344 | 0.0344 |
| 9 | 3 | 0.0447 | 0.0447 | 0.7416 | 0.0332 | 0.0332 | | 3 | 0.0447 | 0.0447 | 0.7416 | 0.0332 | 0.0332 |
| 10 | 4 | 0.0476 | 0.0476 | 0.6712 | 0.0320 | 0.0320 | | 4 | 0.0476 | 0.0476 | 0.6712 | 0.0320 | 0.0320 |
| 11 | 5 | 0.0508 | 0.0508 | 0.6075 | 0.0308 | 0.0308 | | 5 | 0.0508 | 0.0508 | 0.6075 | 0.0308 | 0.0308 |
| 12 | 5 | 1.4473 | 1.3727 | 0.6075 | 0.8793 | 0.8340 | | 5 | 1.921726 | 1.921726 | 0.6075 | 1.1675 | 1.1675 |
| 13 | **Total** | | | | **-0.0090** | **0.0000** | | **Total** | | | | **0.2792** | **0.3336** |
| 14 | | | | | | | | | | | | | |
| 15 | | **Summary of NPVs** | | | | | | **SENSITIVITY ANALYSIS: NPV of RV-SV** | | | | | |
| 16 | | **No Acq** | **Acquisition** | **Total** | | | | | | | | | |
| 17 | SV | -0.0090 | 0.2792 | | | | | **Capital** | **RV-SV** | | | | |
| 18 | RV | 0.0000 | 0.3336 | | | | | **Apprec** | | | | | |
| 19 | Probabilities [2] | 94.95% | 5.05% | **100.00%** | | | | 0.00% | 0.0230 | | | | |
| 20 | Probability Wtd NPVs | | | | | | | 2.00% | 0.0198 | | | | |
| 21 | SV | -0.0086 | 0.0141 | **0.0055** | | | | 4.00% | 0.0162 | | | | |
| 22 | RV | 0.0000 | 0.0169 | **0.0169** | | | | **6.54%** | **0.0113** | | | | |
| 23 | RV-SV | | | **0.0113** | | | | 8.00% | 0.0083 | | | | |
| 24 | RV-SV (in Percent) | | | **1.1%** | | | | 10.48% | 0.0027 | | | | |
| 25 | | | | | | | | | | | | | |
| 26 | **Assumptions** | | | | | | | | | | | | |
| 27 | Dividend Yield [3] | 3.94% | | | | | | | | | | | |
| 28 | Voting Rights Prem | 0.0544 | | | | | | | | | | | |
| 29 | Cap Apprec = g [3] | 6.54% | | | | | | | | | | | |
| 30 | Disc Rate = r [3] | 10.48% | | | | | | | | | | | |
| 31 | Acq Prem-Both [4] | 40% | | | | | | | | | | | |
| 32 | SV/RV Acq Prem | 0.0% | | | | | | | | | | | |
| 33 | | | | | | | | | | | | | |
| 34 | [1] Present value factors are end-of-year. Using midyear factors makes no difference in the final result to four decimal places. | | | | | | | | | | | | |
| 35 | | | | | | | | | | | | | |
| 36 | [2] Probability of acquisition is from the U.K. data. However, increasing cell C19 to 25% causes D24 to rise to only 2%. | | | | | | | | | | | | |
| 37 | | | | | | | | | | | | | |
| 38 | [3] Derived from *SBBI—1999* for 1955–1982. | | | | | | | | | | | | |
| 39 | | | | | | | | | | | | | |
| 40 | [4] This is an assumption, as the data were unavailable. However, the final results are insensitive to the assumption. | | | | | | | | | | | | |

**THE HOULIHAN LOKEY HOWARD & ZUKIN (HLHZ) STUDY** Much and Fagan (2000), of HLHZ, describe their own update of the Lease, McConnell, and Mikkelson study. The HLHZ study consists of 18 dual-class U.S. firms with identical dividend rights and liquidity preference. While this is professional rather than academic research, we include it here because it is an update of academic research and it fits in better topically.

The HLHZ study presents the VRP over a very short period of time ending with December 31, 1994.[25] In this respect, it is very different from the two previous studies, which present VRP averages over many years. The Lease, McConnell, and Mikkelson VRP results are the averages over 38 years, while the Megginson results are averages over 28 years. In contrast, the HLHZ study covers a short snippet of time.

The 260-day moving average mean and median voting rights premiums were 3.2% and 2.7%, respectively, while they were 1.5% and 1.15% for 60-day moving averages. The longer the time period, the more reliable is the result, unless there are clear trends that render older data obsolete, which is not the case here. Therefore, the 260-day moving average of 3.2% is the best measure of the VRP in this study. These are lower premiums than those in the Lease, McConnell, and Mikkelson study, although the mean VRP was monotonically increasing with the length of the moving-average time period.[26]

The authors point out anecdotally that the voting rights premium can be affected by other factors. They mentioned that until the fourth quarter of 1994, the Class A stock of Pacificare Health Systems, Inc. was included in the S&P 400 index. During this time, the Class A voting shares consistently traded at a 1.5–2.5% premium over the nonvoting shares. During the fourth quarter 1994, the Class B nonvoting stock replaced the Class A stock in the S&P 400 index. Since then, Class A traded at a 1.5% discount to the nonvoting shares. The authors conclude that the visibility of the stock, not its voting rights, accounted for its premium.

Another example they give is Playboy Enterprises, whose Class A voting shares also trade at a discount from the nonvoting shares. However, the company's largest shareholder owns over 70% of the Class A voting stock. Institutional investors are interested in liquidity and prefer to trade in the Class B stock, which has higher trading volume. The authors conclude that the liquidity difference appeared to account for the voting rights discount. Their final conclusion is that the 5.4% voting rights premium in Lease, McConnell, and Mikkelson is too high, given their more current data.

The anecdote about the liquidity difference depressing the voting rights premium is consistent with Megginson (1990), where it was far more obvious in the British markets.

**Summary of the HLHZ Study Results** This study found a VRP of 3.2%. The HLHZ study used more recent data than LMM, but it looked at a shorter time period. Also, the HLHZ study was based on 18 companies versus 30 for LMM.

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[25]In their chapter, they say as of December 31, 1994. However, I assume their use of moving averages means that it is a span of time ending on that date.

[26]The authors also presented data for 120- and 180-day moving averages. Given the reported results, it is possible that expanding the time horizon would have led to a larger VRP.

**INTERNATIONAL VOTING RIGHTS PREMIUMS STUDIES** Zingales (1995) finds that while the voting rights premium in the United States is normally small, it rises sharply in situations where control is contested, from which he infers that control shareholders receive private benefits at the expense of minority shareholders. He uses a different definition (and resulting formula) of the voting rights premium than Lease, McConnell, and Mikkelson, which takes into consideration the ratio of the differences in voting versus cash flow rights between the superior and inferior voting stock. His grand mean voting premium is 10.5% and states that part of the higher voting premium that he finds is in the difference in definition.

Maher and Andersson (1999) refer to a number of articles that deal with voting rights premiums, including Zingales (1994, 1995).

The remaining evidence in this section is from other countries, where concentrated ownership is the norm. Rydqvist (1987) finds a 6.5% voting rights premium for Sweden. Levy (1982) finds a 45.5% VRP in Israel, Horner (1988) finds a 20% VRP for Switzerland, and Zingales (1994) finds an 82% VRP on the Milan Stock Exchange. The large voting premium in Italy suggests high private benefits of control, and Zingales (1994) and Barca (1995) suggest that managers in Italy divert profits to themselves at the expense of nonvoting shareholders. Zingales also measures the average proportion of private benefits to be around 30% of the firm value. Zingales (1994) conjectures that the private benefits of control in Italy are so large because the legal system is ineffective in preventing exploitation by controlling shareholders.

**Summary of International Voting Rights Premiums Study Results** Overall, the foreign VRP seems to be in the 40% range.

**BRADLEY, DESAI, AND KIM (1988)** The authors document that successful tender offers increase the combined value of the target and acquiring firm by an average of 7.4% over the period 1963–1984. In this article, the 7.4% remains stable over the entire 22 years of analysis, which the Maquieira, Megginson, and Nail results (in the next section) do, too. However, there was a constant movement over time for the target shareholders to capture the lion's share of synergies, with the acquirer faring worse over time. Also noteworthy is that the authors present theoretical arguments why multiple-bidder contests lead to larger payments to target stockholders.

The breakdown of the 7.4% overall synergy is very important to business appraisers. The targets, who are on average 20% of the combined entity (i.e., one-fourth of the size of the bidders), experienced an average 31.8% synergistic gain, as measured by cumulative abnormal returns (CARs), and the bidders experienced a 1% synergistic gain. However, the specific results in different subperiods varied and will be significant in my synthesis and analysis later in the chapter. The acquirers had CARs of 4.1%, 1.3%, and −2.9% for July 1963–June 1968, July 1968–December 1980, and January 1981–December 1984, respectively. There is a clear downward trend in the synergistic gains of the acquirers.

They also present data showing that targets experience cumulative abnormal returns (CARs) of 9.8% from five trading days before the announcement of the first bid to five trading days after the announcement. A multiple bidding scenario increases the CARs by an absolute 13.0%, which is consistent with the result from Schwert discussed above, although not directly comparable in magnitude. Another interesting finding is that synergies were higher in multiple-bidder scenarios. As to the nature of the synergies, the authors cite work (Eckbo 1983, 1985; Stillman 1983) that indicates that the corporate acquisitions have no measurable effect on the firm's degree of market power in the economy. This is consistent with Maquieira, Megginson, and Nail's results, discussed immediately below, that the synergies are operating and not financial.

**Summary of Bradley, Desai, and Kim's Results** The authors document that successful tender offers increase the combined value of the target and acquiring firm by an average of 7.4% over the period 1963–1984. As measured by CARs, the targets, who were on average about one-fourth the size of the bidders, experienced an average 31.8% synergistic gain, and the bidders experienced a 1% synergistic gain. Looking at subperiods, the acquirers had CARs of 4.1%, 1.3%, and −2.9% for July 1963–June 1968, July 1968–December 1980, and January 1981–December 1984, respectively.

**MAQUIEIRA, MEGGINSON, AND NAIL (1998)** The authors examine wealth changes for all 1,283 publicly traded debt and equity securities in 260 pure stock-for-stock mergers. They find nonconglomerate mergers create financial synergies. They define conglomerate mergers as those mergers in which the first two digits of the SIC code of the acquirer and the target are different. They determine the SIC code by examining the primary line of business listing for each company in the relevant edition of the Moody's manual. This data source differs from Roach (1998), described earlier, and the SIC code scheme is different, which may explain their different results.

To compute the synergy from the mergers, the authors use data from two months before the merger to predict what would have been the value of the two companies (and their individual classes of equity and debt) as separate entities two months after the merger. They then added the two separate company values together to form a "predicted value" of the merged entity. From this, they subtracted the actual market valuation of the merged entity at two months after the merger, and they call this difference the *valuation prediction error* (VPE), as well as the measure of synergy.

The mean and median VPEs for common and preferred stock were 8.58% and 8.55% for nonconglomerate mergers—being statistically significant at the 1% level—while they were 3.28% and 1.98% for conglomerate mergers—being statistically insignificant. For all classes of securities, which also include convertible and non-convertible preferred stock and bonds, the mean and median net synergistic gains were 6.91% and 6.79% for nonconglomerate mergers—being statistically significant at the 1% level—while they were 3.91% and 1.25% for conglomerate mergers—being statistically insignificant. The positive VPEs in nonconglomerate mergers occur in a statistically significant 66.4% of the mergers, while a statistically insignificant 56.3% of the conglomerate mergers yield positive VPEs.

The breakdown between acquirers and targets is significant. In nonconglomerate mergers, the acquirers had mean and median VPEs of 6.14% and 4.64%, while the targets' were 38.08% and 24.33%, respectively. In conglomerate mergers, the acquirers were the only losers, with mean and median VPEs of −4.79% and −7.36%.

Maquieira, Megginson, and Nail also mention similar synergy figures provided by Lang et al. (1991), Eckbo (1992), and Berkovitch and Narayanan (1993). Another very significant conclusion of their analysis is that the stock-for-stock merger synergies are operating synergies, not financial.

The authors also report the time to complete each merger, which are interesting data and provide a benchmark for the delay-to-sale component of my economic components model, described later in the chapter. The time to complete the mergers ranged from a low of 11 months to a high of 31 months—roughly one to two and one-half years. This underestimates the time to complete a merger, as this starts from the announcement date rather than the date at which the parties first thought of the idea.

**Summary of Maquieira, Megginson, and Nail's Results** The authors examine wealth changes in 260 pure stock-for-stock mergers. In nonconglomerate mergers, the acquirers had a mean VPE of 6.14%, while they had a mean VPE of −4.79% in conglomerate mergers. Thus in nonconglomerate mergers, acquirers had roughly an 11% higher VPE than for conglomerate mergers. The only negative VPEs were for the acquirers in conglomerate mergers, who had a mean and median VPE of −4.79% and −7.36%, respectively.

**OTHER CORPORATE CONTROL RESEARCH** This section is brief. Its purpose is to present summary findings of other researchers that will ultimately add to the discussion of what business appraisers need to know about corporate control.

Franks and Harris (1989) analyze 1,445 takeovers in British stock markets using the London Share Price Database. Their findings are very similar to those in the United States discussed in the previous sections (i.e., targets capture the majority of the gains from acquisition).

Cumulative abnormal returns (CARs)[27] to the target shareholders in month 0 for single bids for which there were no revisions and no contest were 20.6%, and CARs for contested bids were 29.1%, for a differential of 8.5%. This is fairly similar, although somewhat lower than Schwert. The CARs for months −4 to +1 are much higher: 27.4% for the single bids and 46.6% for contested bids, for a differential of 19.2%, which is higher than Schwert's result. There is an interesting intermediate category of revised, but uncontested bids, which the authors say probably reflects results when the buyers are worried that another firm might compete with their initial lower bid. The CARs for this category are 28.7% in month 0 and 40.5% for months −4 to +1. These might provide interesting benchmarks for different levels of competition—both actual and potential. CARs to bidders are very low, which echo the U.S. results.

In a cross-sectional analysis of total wealth gains, multiple bidders increase the control premium by an absolute 8.44%.[28] This is also similar to Schwert's result, although slightly lower.

Harris (1994) provides an explanation of why any firm would want to be the bidder rather than the target. If target shareholders are the big winners and bidders barely break even, then why bother being a bidder? Why not wait for the other firm to be the bidder and be the target instead? The answer is that while the target's shareholders are the winners, the target's management team are the losers. Harris cites another author who cites a *Wall Street Journal* article that reported 65% of a sample of 515 target CEOs left their firms shortly after the acquisitions (less so in

[27]The authors actually use the term *Total Abnormal Returns*.
[28]See the $x$-coefficient for variable $\alpha_2$ in their Table 9.

mergers). The reward for the bidder is that the management team get to keep their jobs. The reward for target management is that the bidder pays a high price for their stock, which gives many of them plenty of time to take life graciously while looking for their next job.

**MENYAH AND PAUDYAL** This research provides a method of quantifying the bid–ask spread (BAS). Later in the chapter, we review some of the work on DLOM by Larry Kasper, which involves using an econometric equation to determine the BAS to add to the CAPM-determined discount rate before DLOM. Those interested in using Kasper's method may want to understand this research. Otherwise, this work is not used in my own models and can be skipped.

The authors study stocks on the London Stock Exchange and find the security prices, volume of transactions, risk associated with security returns, and the degree of competition among market makers explain 91% of the cross-sectional variations in bid–ask spreads (BASs) (Menyah and Paudyal, 1996).

The average inside spread[29] for liquid stocks was 0.83% before the October 1987 stock market crash. It increased to 2%, but has since declined to 0.71% by the end of 1993. The average inside spread for less liquid stocks declined from 10% to 6% over the same period. Transactions over £2000 have lower BASs.[30]

The academic literature has identified three components to the BAS: order processing, inventory adjustment, and adverse information. The authors quote Stoll (Stoll, 1978a, b), who wrote that because dealers must service their customers, they cannot maintain an optimal portfolio suitable to their risk-return strategy. Therefore, total risk, not just systematic risk—as measured by beta—is the relevant variable in determining the BAS.

Their regression equation is: $\ln \text{BAS} = -0.097 + 0.592 \ln \text{Price} + 0.649 \ln \sigma - 0.369 \ln \#\ \text{Market Makers} - 0.209 \ln \text{Volume}$. All coefficients were significant at the 5% level, except the $y$-intercept. $R^2 = 91\%$.

The BAS equation shows that in the public markets, an increase in volatility increases the BAS—and thus the "DLOM" of publicly traded stocks—while an increase in the number of market makers decreases BAS and DLOM. With privately held firms, there is no market maker (i.e., a dealer who is willing to buy and sell). Business brokers and investment bankers never take possession of the firm. The above regression is unlikely to be of any direct use to business appraisers. However, it does offer clues to us as to which variables are important in the liquidity of a business—and hence, DLOM.

**BARGERON, SCHLINGEMANN, STULZ, AND ZUTTER (2008) (BSSZ)** The authors find that mean and median CARs for public acquirers are 31.7% and 25.2%, respectively, while they are 22.2% and 18.5%, respectively, for private acquirers. The differences in means and medians are 9.5% and 6.7%, respectively, which is statistically significant

[29]The inside spread is the best bid and offer prices at which market makers are prepared to deal in specified quantities. On the London Stock Exchange they are quoted on the yellow strip of the Stock Exchange Automated Quotation (SEAQ) screens.

[30]Commission rates are also lower since 1986. In 1991, the commission rate for small trades was 2% of transaction value, while trades over £1 million incurred commissions of 0.15% and declined further in 1993 to 0.13%.

at the 99.9% level. They conclude there is an agency problem, that is, management benefits at the expense of shareholders.

**INFERENCES FROM THE ACADEMIC ARTICLES** The Bradley, Desai, and Kim results are very revealing. On average over 22 years, the acquirers actually gained, with a CAR of 1%. This means that the acquirers are not paying for control! They are paying for expected cash flows. There is no information in this article to tell us how to break down the CAR to the target between performance improvements and synergies.

However, the Maquieira, Megginson, and Nail article provides some information to enable us to do that. The VPEs for the nonconglomerate acquirers were about 11% higher than they were for the conglomerate acquirers, which suggests that synergies account for the entire premium. Bidders are approximately four times the size of the targets in their study.[31] Multiplying the VPE differential of 11% $\times$ 4 = 44% attributable to synergies—if one were to attribute the synergies completely to the target. Since the majority of acquisition premiums are smaller than 44%, this is evidence that acquisition premiums are being paid exclusively for pure synergies and not for performance improvements. However, the reality is probably less extreme. It would seem that synergies rather than performance improvements would account for the majority of acquisition premiums.

A surface reading of Tables 8.1 and 8.1A and the Roach article suggest the opposite—that the majority of the increase should be from performance improvements, since there was no pattern to the acquisition premiums by the difference in SIC codes.

The two articles used different schemes for determining a potential synergistic merger. Maquieira, Megginson, and Nail require only that the two firms be in the same two-digit SIC code. A merger of SIC codes 3600 and 3699 would be nonconglomerate, while in Roach's work it would be the equivalent of conglomerate, although he did not use that terminology. On the other hand, a merger of SIC codes 3599 and 3600 would be a conglomerate merger according to Maquieira, Megginson, and Nail, but a difference of 1 in the SIC code in Roach's work. It is logical that one can achieve synergies from combining two firms in similar but different businesses and that the two-digit SIC code scheme is better for that purpose. For our analysis, we will assume that the academic article is the more correct approach.[32] Thus, we assume that acquisition premiums are being paid exclusively for pure synergies and not for performance improvements.

Later in this chapter, our analysis of Table 8.4 fits with the "wholesale versus retail" explanation of control versus minority acquisition premiums.[33] This is

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[31]The book value of total assets of the bidding firms comprised 81.2% of total assets of both firms combined, and the targets comprised 18.8% of total assets.

[32]This is not to denigrate Roach's work, as it was very creative and is still significant evidence. I also made the same mistake in my own research.

[33]We cover this is more depth later. We need a short explanation, however, in the meantime. For publicly traded stocks, the trading volume in a particular stock produces a certain flow of buy and sell orders, usually for relatively small amounts. One wanting to buy 100% of the stock cannot buy at the current market price and must pay a premium—let's call it "paying retail"—to entice all shareholders to sell. One wanting to sell a large block will have to lower the price to "wholesale."

different from our conclusion in the first edition of the book, when the average minority-to-control acquisition premium ratio was 99.11% (K8).[34] Thus, the current Mergerstat evidence in Table 8.4 seems to point to acquisition premiums fitting with the wholesale-versus-retail explanation and not necessarily providing evidence that they are for synergies.

Back to Bradley, Desai, and Kim; in the 1981–1984 subperiod the acquirers did suffer a loss of 2.9%, as measured in CARs. This could mean that the acquiring firms were willing to suffer a net loss of 2.9% of market capitalization for the privilege of control over the target. Since targets were, on average, about one-fourth the size of the bidders, this translates to 11.6% of target value. However, 4 years out of 22 does not seem enough to strongly assert that this is a reliable control premium—let alone *the* control premium.[35]

The Maquieira, Megginson, and Nail article provides similar results. The only negative VPE was for the acquirers in conglomerate mergers, who had a mean and median VPE of −4.79% and −7.36%. Multiplying that mean and median VPE by four for the bidder-to-target size ratio leads to possible control premiums of 19.2% and 29.4% of the target's preannouncement value. Perhaps this is a subset of the market that is paying something for pure control, but it is not representative of the market as a whole. The 19% to 29% premiums may be benchmarks that we can use.

**THE DISAPPEARING "CONTROL PREMIUM"[36]** Let's consider the acquisition process. Conventional wisdom is that Company A buys control of Company B and pays, say, a 40% premium for B. Therefore, B is worth 40% more on a control basis than on a marketable minority basis. However, what happens after the acquisition? B no longer exists as an entity. It is absorbed into A, which itself is a public firm owned by a large number of minority shareholders.

How can one justify the 40% premium to the shareholders of A? Won't the minority shareholders of A lose? If it is true that A is paying purely for the control of B, then yes, they will lose, because the minority shareholders of A pay for control that they ultimately do not receive. Paying for control means that the buyer is willing to accept a lower rate of return in order to be in control of the seller, and the Bradley, Desai, and Kim results showing that the acquirers had a positive CAR do not support that contention.[37] After the acquisition, who is in control of B? The management of A is in control, not the shareholders of A. For there to be a pure value of control, it must go to the management team, who may enhance their salaries and perquisites for running a larger organization. It makes sense that if firms are paying for control anywhere it is in conglomerates, where there definitely is no synergy. That goes only so far, though, before the shareholders revolt or another firm comes along and makes a hostile takeover, booting out the management team that look after their own interests at the expense of their shareholders.

The previous analysis seems to suggest that Mercer (1998) and Glass and McCarter (1995) are correct that there is no value to control in itself. The appraiser

[34]In the data provided to me by Mergerstat, the average size of minority purchase was 38%.
[35]Even though the regression coefficient was significant at the 0.01 level.
[36]We already touched on this concept earlier. This is an elaboration.
[37]Again, with the possible exception of the 1981–1984 period, when the acquirers had a negative CAR.

should simply try to quantify the performance improvements that one can implement in the subject company, if they are relevant to the purpose of the valuation, and proceed with the discounted cash flow or guideline company valuation. The difference in the marketable minority value and the "control value" comes from the increases in cash flows that new management can produce, not from a control premium.

**THE CONTROL PREMIUM REAPPEARS** Does this mean there is no such thing as a value to control? No. It's just that we cannot find it directly in the U.S. M&A market or in the public markets, with the possible exception of the conglomerate mergers in the Maquieira, Megginson, and Nail article.

The reason for that is that there has to be one individual or a small group of individuals[38] actually in control of publicly held firms who derive psychic benefits from it for there to be a pure control premium—and there is almost no such thing in the United States.[39]

**ESTIMATING THE CONTROL PREMIUM** We begin the process of estimating the control premium by starting with the voting rights premium (VRP) data, which show that there is a value of the vote to individual shareholders. If the vote has value, then logically control must have more value—but again only to an individual who is really in control.

Our VRP analysis shows two levels of voting rights premium. The gross premiums were 5.44% and 3.2% in the United States from the Lease, McConnell, and Mikkelson study and the HLHZ study, and it was 13.3% in England, per Megginson (1990). The net premiums—meaning those paid above and beyond expected higher cash flows to the voting stock—were 1.1% to 1.4% (Tables 8.3A and 8.3, D24). For the valuation of most small and medium-size businesses, the gross VRP is the more relevant measure for reasons we will discuss shortly.

The U.S. gross VRPs average 4.3%, that is, the average of the 5.44% and 3.2% gross VRPs from the Lease, McConnell, and Mikkelson and HLHZ articles. According to Professor Megginson, we then need to add another 2% to 3%, say 2.5%, for the depressing effect on the VRP of the illiquidity of the voting shares, which brings us to 6.8%, which we round to 7%.

Control must be worth at least three to four times the value of the vote. That would place the value of control to an individual at least at 21–28%. It could easily be more. Currently, the only possible direct evidence in the United States is the conglomerate control premium of 19% to 29% based on the Maquieira, Megginson, and Nail article, which is very close to the above estimate. The VRP in Switzerland, Israel, and Milan of 20%, 45.5%, and 82%, respectively, is another indication of the value of control when minority shareholders are not well protected, again keeping in mind that those were the values of the vote, not control.

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[38]Henceforth, for ease of exposition, reference to one individual in this context will also include the possibility of meaning a small group of individuals.

[39]All the foregoing discussion excludes the well-known phenomenon of the private benefits of control, which is the value that control shareholders or management can expropriate from minority shareholders.

Another piece of data indicating the value of control is the one outlier in the Lease, McConnell, and Mikkelson study, which had a 42% VRP. Such a high VRP in the United States is probably indicative of control battles taking place and could rapidly reduce to a more normal VRP. Thus, the voting shareholders probably could not rely on being able to resell their shares at a similar premium at which they buy. That 42% premium is evidence of the value of the vote in an extreme situation when small blocks of shares would have a large impact on who has control.

The reason why the gross VRP is relevant for most businesses in the process of inferring the value of control is that as long as the buyer of a business can turn around and sell the business for the same control premium as he or she bought it, there is no loss in net present value of cash flows, other than the pure control portion, which derives from the net VRP.[40] The buyer will eventually recoup the control premium later on as a seller. That works as long as the business is small enough that its buyers will be either private individuals or private firms. If the business grows large enough to be bought by a public firm or undergoes its own IPO, then instead of recouping a control premium, the owner may receive an acquisition premium with synergies in the case of a buyout. In the case of an IPO, the company will experience an increase in value from increased marketability.[41] And whereas the control premium will be smaller, DLOM may also be smaller.

The best source of data for control premiums and DLOC for private firms in the United States will probably come from a thorough analysis of the international literature. The publicly traded firms overseas are probably better guidelines to use to understand the value of control than U.S. firms for two reasons. The first reason is that in most foreign countries—especially those outside the United Kingdom—ownership of public firms is far more concentrated than it is in the United States. The second reason is that the minority shareholders there are far more vulnerable to abuse by the control interests, which is closer to the case of privately held firms in the United States.

Unfortunately, that will have to remain as future research. In the meantime, I would suggest that the 21–28% control premium based on the gross VRP is probably reasonable to add to a marketable minority FMV—at least for small and medium firms. For large private firms, that range may still be right if a synergistic buy is likely in the future. Otherwise, it is probably more appropriate to use a smaller control premium based on the net VRP, which would be in the 3–6% range.

At this point, let's compare our control premium implied by VRPs with going-private premiums, as the latter is also a candidate for our measure of the value of control. The going-private premiums are as follows:

| | **Mean Premiums** | **Median Premiums** |
|---|---|---|
| Mean | 39% (D33) | 28% (E33) |
| Median | 35% (D34) | 26% (E34) |

[40]There is actually a second-order effect where this is not literally true. To the extent that the owner is taking implicit dividends in the form of excess salary, there is some loss in present value from this.

[41]The appraiser must consider the issue of restricted stock discounts in this case.

The mean premiums are larger than our estimate. However, the median premiums of 28% and 26% are in the upper half of our estimate. Which are more likely to be right? The going-private premiums have the advantage of being directly calculated rather than indirectly inferred, so that is one point in their favor. There is no consensus in the valuation profession in general whether medians or means are better measures of central tendency. All other things being equal, then, it would make sense to use the median, as it is consistent with the VRP-calculated control premium. I more often use means than medians, which leaves me a little dissatisfied relying solely on the consistency of the two measures.

There is other logic that convinces me that the lower measure of control is more correct. What are the motivations for going private? The management team may believe:

1. The company is underpriced in the market.
2. Removing the burdens of SEC reporting will increase profitability.
3. If the going-private transaction is a division of a public company, it can operate more efficiently without interference and the burden of overhead from the corporate people.
4. The management group and the buyout group want to be in control of the company.

Item 1 implies that the universe of going-private transactions may have a sample bias with respect to the valuation of privately held firms. The reason is that to the extent that item 1 is true, that portion of the control premium is inapplicable to the valuation of private firms, as we presume that the valuation is done correctly up to this point. Item 2 is also inapplicable to the valuation of private firms, as this represents a performance improvement to the going-private firm that is unavailable to the firm that has always been private. Therefore, that portion of going-private premiums represented by the economic efficiencies of being private also does not belong in our calculation of the value of control.

Item 3 is a performance improvement and not really the value of control itself. It represents improvements in cash flow, and thus could be a candidate for the control premium to the extent that we believe that the average going-private firm would achieve the same amount of performance improvements that an already private firm could expect with new management, but I find that very speculative.

A direct measurement of the premium associated with item 4 would be the closest to our VRP approach to calculating the value of control. However, I find it hard to believe that there is a single shareholder who is in control in the large going-private transactions recorded in Mergerstat. Who is in control of the buyout group? Management?

I think that the composition of the observed going-private premium is a mixture of all four items above and probably others of which I am unaware.[42] It is likely that some of the going-private premium is irrelevant to the valuation of private firms,

[42]This discussion is largely unchanged from the first edition, before the wholesale-versus-retail issue. Thus, we are ignoring that issue in this discussion. Introducing it would complicate the discussion and possibly make it more realistic. However, it would be unlikely to change the final conclusion.

some of it is for performance improvements that might be applicable to private firms, and some is for the value of control itself, although the latter certainly is less for going-private transactions than it is for true control of a firm by a single individual.

Let's make a wild guess as to how the four components comprise the going-private premium. Suppose each item is one-fourth of the premium, that is:

| | |
|---|---|
| Company underpriced | 8% |
| Remove SEC reporting | 8% |
| Eliminate corporate overhead | 8% |
| Control | 8% |
| Total—Mean | 32% |

If this were the true breakdown of the going-private premium, then the value of pure control would be only 8%. But, perhaps that is reasonable in a situation where control is not concentrated in a single individual but rather is spread among a few people in the buyout group and a few people in management. This would tell us fairly little about how to apply it to an already-private company.

Ultimately, I am more comfortable with the VRP inference of the value of control than the going-private premium, as it makes a clean separation of performance improvements from control. In any case, it seems clear that the mean going-private premium is probably too high as a measure of the value of control, and we should stick with the 21–28% control premium.

**DLOC** It is a bit difficult to imagine Shakespeare pondering, "To DLOC or not to DLOC. That is the question." However, the valuation profession is in serious disagreement about this question.

It is my opinion that Nath is correct in his assertion that both DLOM and DLOC are needed to reduce the marketable minority interest to a private minority interest.[43] Bolotsky disagrees with this more in form than in substance. He asks—logically enough—how can one subtract a DLOC from an interest that has no control attributes to it? That is a good question. The answer is that control matters much less in publicly held firms in the United States than it does in privately held firms. The public minority shareholder has little fear of control shareholders ruining the company or abusing the minority shareholders. Even if the public minority shareholder does have such fears, there are remedies such as class action lawsuits, takeovers, shareholder meetings, and so forth that the private minority shareholder can only wish for.

I suggest that Bolotsky's 2 × 2 levels of value (LOV) chart, as depicted in Figure 8.2, is still too simple.[44] Using his own very innovative and perceptive framework of differing shareholder attributes, it is possible to see why it may still be appropriate to subtract an incremental DLOC in valuing a private minority interest.

Figure 8.3 is my own expansion of Bolotsky's 2 × 2 levels of value chart, incorporating a strategic LOV on top. The more important addition is that I have split minority interests into well-treated and exploited. Most U.S. public minority

[43]This distinction is more important vis-à-vis Mercer's original position than it is when using his quantitative marketability discount model.

[44]In fairness, his 2 × 2 levels of value chart is a simplification of his more complicated system.

4 × 2 Levels of Value Chart

| | Public | Private |
|---|---|---|
| Strategic | × | × |
| Control | × | × |
| Minority (well-treated) | × | × |
| Minority (exploited) | × | × |

**FIGURE 8.3 Chart of 4 × 2 Levels of Value**

interests are well-treated. Most private minorities in the United States are poorly treated or, if not, may have to fear being poorly treated with a change in control ownership or a change in attitude of the existing owners. Thus, most U.S. private minorities are one row down and one column to the right of public minority interests.

The DLOC, calculated as the flipside of the control premium going from a well-treated minority to control, is insufficient to measure the lower position of an exploited minority. You saw this earlier in the chapter in the section on international voting rights premiums, where we examined the difference in the market value of voting versus nonvoting stock in international markets. When minority rights are poorly protected, the voting rights premium is as high as 82%; that is, voting stock sells for an 82% higher price than nonvoting stock. Control must be very valuable in Milan!

It often may be appropriate to use control premiums from other countries to calculate a DLOC that is appropriate for U.S. minorities. Then, one can use Jankowske's formula to make the incremental adjustment. Thus, it is my opinion that we should subtract both an incremental DLOC and DLOM from the marketable minority value to arrive at a private minority value. This is an area that requires further research.

It is important to understand that those are not eight unique and discrete cells in the figure. While public or private is an either/or concept, both the degree of control and how well-treated are the minority interests are *continuums*. Thus, there are not only eight values that one could calculate as DLOM, but an infinity of values, depending on the magnitudes.

In my correspondence with Mike Bolotsky, he agrees in substance with this view. He prefers to think in a multidimensional matrix of factors and prefers to label this something like "SEC oversight and enforcement power" instead of a control issue. Even so, I will quote from his letter to me. "In valuing private minority interests that are either poorly treated, which is typical of most, or even have reason to fear being poorly-treated, I think it is reasonable to subtract DLOC. However, we cannot learn what that is from the American public stock markets, where minority interests are well protected administratively and legally." I agree completely.

That is a research task to be done in the future. In the meantime, the above simplification works and is easier than a multifactor matrix.

What measure of control premium should we use to calculate DLOC? Starting with a marketable minority FMV, we have to decide whether we are coming down to a well-treated private minority or an exploited private minority interest. Additionally, even a well-treated private minority today may turn into a poorly treated minority tomorrow, and the fear of that alone should create a positive DLOC from the marketable minority level.

The data are not clear enough to provide a definitive answer. In the first edition, I suggested that the 40% range for the foreign VRPs and the American outlier in the Lease, McConnell, and Mikkelson study—which happen to be similar to mean American acquisition premiums by coincidence—plus some additional amount for control being more valuable than the vote—is a reasonable range from which to calculate DLOC. One caveat—if you are valuing an "exploited" minority interest and have not added back excessive salaries taken by the control shareholders, the 40+% range control premium would translate to a 28.6% DLOC,[45] which might be excessive, depending on the magnitude of excessive salary. The reason for this is that the 40+% VRP may, to some extent, represent excess salaries to holders of voting shares. Therefore if we have already accounted for it in the discounted cash flow, we do not want to double-count and take the full discount.

It is important to reiterate that I do not consider the decrease in value from a public "control" value to a marketable minority level to be DLOC. It tells us more about the wholesale-versus-retail phenomenon. It may tell us about the magnitude of synergies in acquisitions. I would not use it go from a private control interest to a private minority interest.

## Analysis and Conclusions

Trying to make sense of the oceans of research, data, and opinions is like trying to put together a giant picture puzzle without having the benefit of the picture on the cover. For a long time, it was difficult to see where some of the pieces fit, and some did not seem to fit at all. However, some coherence is beginning to form, although there are conflicting data. Overall, control premiums and discount for lack of control need more research than any other chapter in this book.

Let's begin by decomposing the acquisition premium into its three potential components. Those components are:

1. Performance improvements
2. Synergies
3. Control premium, that is, the pure value of control

**PERFORMANCE IMPROVEMENTS** Performance improvements are the additional expected cash flows from the target when the bidder runs the target more efficiently. These improvements are not due to the combination of the two companies, which is synergy. They are merely the improvements the seller could have made on its own if it had the skill to do so.

**SYNERGIES** In valuation parlance, *synergies* means the additional value that comes from combining the target with the bidder, excluding performance improvements.[46]

[45]Using the equation DLOC = P/(100% + P), where P is the control premium, we have 28.6% = 40%/(100% + 40%).

[46]However, in the academic literature cited, *synergy* is used to mean the increase in value of the combined entity regardless of the source. It is the combination of the added value

Specifically, it is that portion of the pure synergy in the control premium for which the bidder pays. We never see the portion that the bidder keeps in the Mergerstat acquisition premiums.[47]

Synergies arise from the following phenomena as a result of the combination:

1. *Increase in sales.* This can be due to:
   a. One or both parties have superior marketing and can market the other's product or service better than the other.
   b. One party's large size (or the combined entity's new large size) enables the other to compete on projects that were previously too large to take on. An example of this is that architectural projects over a certain size require bonding, and small firms simply cannot get that type of bonding. So the acquisition enables the architect to receive bonding to compete on large projects.
2. *Decrease in combined costs.* For example, the combined entity may not need all of the accounting and administrative personnel that the two individual firms had. The layoffs increase profitability.

**CONTROL PREMIUMS** As a profession we do not have a clear, consistent, and unified concept of what is a control premium. Most business appraisers think of a control premium as the value of being in control of a firm. If that were true, then there would be two implications of this, neither of which is true:

1. There should not be acquisition premiums of minority interests.
2. If the buyer's stock is publicly held, it should decline by the amount of the acquisition premium, since the individual shareholders do not have control, and the buyer paid for something the shareholders did not receive.

So, if the acquisition premiums are not for control, what are they and why do they exist?

To answer this question, we have to ask another question. What is the value of a publicly held firm? The standard answer is the market capitalization ("market cap")—price per share × the number of shares. The market capitalization is certainly the right starting point. However, it is almost impossible to buy or sell a public firm for its market capitalization. There are not enough shareholders willing to sell their shares at the current market price to allow a buyer to buy 51% of the firm, let alone all of it. There is a small supply of willing sellers and buyers at the current price, but the rest of the shareholders have various reasons why they don't want to sell. Some bought recently at a similar price and don't want to sell for a near breakeven price—especially after having spent a lot of time analyzing the stock and deciding

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from performance improvements and pure synergy. We could add valuation corrections for underpricing errors to the previous list. To the extent that bidders spot undervalued firms and pay a premium for some or all of that undervaluation, that portion does not belong in the control premium applicable to private companies, as we already presume that the valuation before discounts and premiums was done correctly.

[47]That also would be true about the portion of performance improvements that the bidder keeps.

to buy it. Others may have a buy-and-hold mentality. Thus, for various reasons, it requires a premium to get all of the shareholders to sell. It even requires a premium to entice 10% of the shareholders of a public firm to sell if that is materially above the normal daily trading volume.

Thus, we see that market capitalization is an important concept; however, it is not a real price at which any control or large minority purchase would take place.

This explains the existence of "control" premiums for minority interests and why "control" premiums for 100% interests are higher, which we see in Table 8.4. The mean of the Mergerstat mean premiums for control interests from 1990 to 2008 is 42.9% (U5), while the mean of mean premiums for minority interests is 33.7% (U6)—a difference of 9.2% (U7). The ratio of the minority/control mean premiums is 80.6% (U8), which means the buyers pay approximately $1/80\% - 1 = 25\%$ more to acquire control interests than minority interests in public firms.

Thus, what we think of as being a control premium is really a marketability phenomenon! The "control" premium for publicly held firms is really a buyer's premium for lack of marketability. However, it is somewhat dangerous to call this a marketability premium because it will be too confusing when we speak about lack of marketability, which is a different phenomenon; it takes longer and is more costly to sell private firms than interests in public firms. *Acquisition premium* is the best term, although we will sometimes continue to use the more familiar *control premium*.

There is some recognition in the professional literature of this under the terminology of a *blockage discount*. For example, see Becker and Gutzler (2000), who show supply and demand curves under various assumptions of whether there is a willing buyer and/or a willing seller for a large amount of stock, and Sonneman (2000).

Putting this in very commercial terms, it is similar to markets for physical assets; one usually buys at retail and sells at wholesale. Even in the NYSE the bid–ask spread constitutes a retail and wholesale price, although the spreads are usually quite small.

**CONCLUSION AS TO WHICH COMPONENTS BELONG IN FAIR MARKET VALUE** Now, we return to the question of which components of acquisition premiums belong in fair market value (or the other important standards of value).

**Performance Improvements** It is clear that financial buyers are willing to pay up to some or, at most, all of the value of the performance improvements, and that belongs in fair market value. If there are many probable buyers for the subject company, then it is likely they will bid the price up to include all of the performance improvements. However, if there are few potential buyers, then it is likely the buyers will not be willing to pay for all of the performance improvements.

We would not want to apply the average amount of performance improvements in the market to a subject company, as it is more appropriate to quantify the specific expected performance improvements for the subject company and add those to the forecast cash flows. This follows Mercer (1998) and Glass and McCarter (1995). Again, however, the buyer will try to pay for as little of the performance improvements as possible, so it is a matter of professional judgment as to how much of the performance improvements to add to the forecast cash flows.

## Table 8.4
## Mergerstat Mean Premiums—Control versus Minority Purchases

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O | P | Q | R | S | T | U | V |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | | | | | | | | | | | |
| 2 | | | | | | | | | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | | | | | | | | | |
| 4 | | **1990** | **1991** | **1992** | **1993** | **1994** | **1995** | **1996** | **1997** | **1998** | **1999** | **2000** | **2001** | **2002** | **2003** | **2004** | **2005** | **2006** | **2007** | **2008** | **Average** | |
| 5 | Control Interest [1] | 42.3% | 35.4% | 41.3% | 38.7% | 40.7% | 44.1% | 37.1% | 35.9% | 40.7% | 43.5% | 49.1% | 58.0% | 59.8% | 63.0% | 30.9% | 33.6% | 31.9% | 31.6% | 57.3% | **42.89%** | |
| 6 | Minority Interest [1] | 39.6% | 32.6% | 38.3% | 38.3% | 54.5% | 61.7% | 29.4% | 22.4% | 39.5% | 33.0% | 32.6% | 35.2% | 39.9% | 21.1% | 27.2% | 17.6% | 18.9% | 26.0% | 31.9% | **33.67%** | |
| 7 | Difference | **2.7%** | **2.8%** | **3.0%** | **0.4%** | **-13.8%** | **-17.6%** | **7.7%** | **13.5%** | **1.2%** | **10.5%** | **16.5%** | **22.8%** | **19.9%** | **41.9%** | **3.7%** | **16.0%** | **13.0%** | **5.6%** | **25.4%** | **9.22%** | |
| 8 | Minority/Control Interest | 93.6% | 92.1% | 92.7% | 99.0% | 133.9% | 139.9% | 79.2% | 62.4% | 97.1% | 75.9% | 66.4% | 60.7% | 66.7% | 33.5% | 88.0% | 52.4% | 59.2% | 82.3% | 55.7% | **80.56%** | |
| 9 | SYD Weights | 0.5% | 1.1% | 1.6% | 2.1% | 2.6% | 3.2% | 3.7% | 4.2% | 4.7% | 5.3% | 5.8% | 6.3% | 6.8% | 7.4% | 7.9% | 8.4% | 8.9% | 9.5% | 10.0% | **100.00%** | ← **Total** |
| 10 | SYD Difference = (7)×(9) | **0.0%** | **0.0%** | **0.0%** | **0.0%** | **-0.4%** | **-0.6%** | **0.3%** | **0.6%** | **0.1%** | **0.6%** | **1.0%** | **1.4%** | **1.4%** | **3.1%** | **0.3%** | **1.3%** | **1.2%** | **0.5%** | **2.5%** | **13.36%** | |
| 11 | SYD Min/Control = (8)×(9) | **0.5%** | **1.0%** | **1.5%** | **2.1%** | **3.5%** | **4.4%** | **2.9%** | **2.6%** | **4.6%** | **4.0%** | **3.8%** | **3.8%** | **4.6%** | **2.5%** | **6.9%** | **4.4%** | **5.3%** | **7.8%** | **5.6%** | **71.82%** | |
| 12 | | | | | | | | | | | | | | | | | | | | | | |
| 13 | # Transactions | | | | | | | | | | | | | | | | | | | | | |
| 14 | Control Interest [1] | 154 | 125 | 127 | 151 | 237 | 313 | 358 | 480 | 506 | 707 | 560 | 422 | 315 | 365 | 309 | 369 | 440 | 477 | 285 | | |
| 15 | Minority Interest [1] | 21 | 12 | 15 | 22 | 23 | 11 | 16 | 7 | 6 | 16 | 14 | 16 | 11 | 6 | 13 | 22 | 13 | 14 | 9 | | |
| 16 | | | | | | | | | | | | | | | | | | | | | | |

17 [1] *Mergerstat Review*—2009, table 1–19, p. 27 for 2008. *Mergerstat Review*—2008, table 1–19, p. 26 for 2003–2007.
18 *Mergerstat Review*—2003, table 1–19, p. 26 for 2002. *Mergerstat Review*—2002, table 1–19, p. 26 for 1999–2001.
19 *Mergerstat Review*—1999, table 1–17, p. 25 for 1994–1998. *Mergerstat Review*—1994, figure 43, p. 100 for 1990–1993.
20 Mergerstat is a part of FactSet Research Systems. The minority interest premium of 32.6% in 2000 excludes an acquisition that yielded a 5-day premium of
21 329.4%. Including that acquisition, the premium would have been 53.8% in 14 transactions.

**Synergies** Generally, synergies are part of investment value, but not fair market value, as they are buyer-specific. When a particular industry is "very hot" (i.e., perhaps one or more firms are creating "rollup" firms, and the subject company is a probable candidate), then it is possible that the relevant universe of buyers are all synergistic, in which case investment value becomes FMV.

**Control** Finally we come to the "control" portion of the acquisition premium. The answer to this question depends on the valuation method and the circumstances.

For the *guideline public company method* (GPCM), our starting point is the market capitalization of the public companies. After that, it would seem we should add the premium a buyer would have to pay to acquire 100% of the stock, that is, the "markup to retail." However, we would have to remove synergies, which often is the entire acquisition premium for strategic buyers. What about financial buyers? They seem to pay slightly lower premiums than synergistic buyers. Generally, acquirers have low negative CARs and perhaps as good as very small positive CARs. The low negative CARs may be a control premium. We saw larger negative returns in Maquieira, Megginson, and Nail (1998). If we assume a near-zero CAR, then financial buyers must be astute at making acquisitions with high performance improvements. Otherwise, their CARs would be substantially negative. So, it seems that Nath was largely right. If there is a control premium, it must come from the motivation discussed earlier of a sole shareholder or a small group who derive psychic benefit from being in control and who are willing to pay for it.

In a DCF, our starting point is a marketable minority level of value, and the same reasoning for GPCM applies here.

In the Guideline M&A method, our starting point is a private control level of value, so we obviously do not have to add a control premium. However, a more difficult question is whether we should remove the embedded control premium. The public targets presumably have an embedded premium to "pay retail," which the private firms probably do not have. Thus, it is important whenever possible to perform a regression analysis on the acquisition price and have a dummy variable to distinguish between private and public targets. We would expect a Public dummy variable to have a positive $x$-coefficient. If it is and it is statistically significant, then applying the regression equation to the subject company using a zero value for the dummy variable accomplishes that goal.

Even if it were to be demonstrable that there is an appropriate control premium, if the valuation purpose is estate tax, gift tax, divorce,[48] or other situations in which there is no real buyer banging at the door wanting to acquire the company, it often would not make sense to add a "control premium." In valuing a large company it might make sense to take a blockage discount, as one could not flood the market with that much stock without depressing the price. In other words, it would make sense to calculate the "wholesale" price (a.k.a. the exit price in FAS 157 parlance). On the other hand, if our purpose is to advise a potential buyer as to how much he

[48]Valuation for divorce is more complicated, as special standards of value or rules may apply.

or she will have to pay to buy a 100% interest in the subject company, then it may make sense to add a blockage premium.

## Discount for Lack of Marketability (DLOM)

Let's begin with the world's first measure of DLOM.

### The First Measure of DLOM in History

The Talmud[49,50] records the first measure of DLOM of which I am aware, based on the sale of Joseph. It notes that he was sold for 20 pieces of silver when the usual value of a slave is 30 pieces of silver. Thus, it concludes that when one sells cheaply—*b'zol* is the Hebrew term used in the Talmud—it means a discount of $^1/_3$. Since there was no issue of lack of control, the $^1/_3$ discount was DLOM.

### Quantitative Models of DLOM

As of the first edition, three quantitative models for calculating DLOM appeared in the professional literature: Jay Abrams' economic components model (Abrams, 1994a),[51] Z. Christopher Mercer's quantitative marketability discount model (QMDM) (Mercer, 1997), and Larry Kasper's discounted time to market model (Kasper, 1997).

Kasper (2009) published a brilliant, innovative article describing how one can use the geometric distribution to calculate the holding period for Mercer's QMDM. We will not cover this article in detail in this book. However, it can ameliorate one of my big criticisms of the QMDM by tightening up one of the loose-and-arbitrary parts of the model—the holding period.[52] In this section, we will review Mercer's and Kasper's (1997) work. In the next section, we will cover Abrams' model in greater depth.

### Mercer's Quantitative Marketability Discount Model

Mercer presents the *quantitative marketability discount model (QMDM)* in his impressive volume devoted entirely to the topic of discount for lack of marketability. His book contains much important research in the field and does an excellent job of summarizing prior research and identifying and discussing many of the important issues involved in quantifying DLOM. I consider his book mandatory reading in the field, even though I will present my own competing model that I contend is superior to the QMDM. I will not even attempt to give more than a bare summary of his work—not because it is not important, but for the opposite reason: It is too

[49]The Talmud is the Oral Law given to Moses on Mt. Sinai over 3,300 years ago. It was kept orally for about 1,800 years and finally written over 1,500 years ago.

[50]The book referring me to the exact page of Talmud is missing. My recollection is that it was in *Tractate Bava Metzia*.

[51]There is no name for the model in the article cited. I have named it since.

[52]This still leaves the increments to the company-level discount rate as being arbitrary and lacking any empirical basis.

important to be adequately represented by a summary. I therefore warn the reader that my terse summary is inadequate to understand Mercer's research.

With that *caveat* in mind, the QMDM is based on calculating the net present value of forecast cash flows to shareholders in a business entity. His key concept is that one can evaluate the additional risk of minority ownership in an illiquid business entity compared to ownership of publicly traded stock and quantify it. The appraiser evaluates a list of various factors that affect risk (Mercer, 1997, p. 323) and quantifies the differential risk of minority ownership of the private firm compared to the public firm or direct ownership of the underlying assets—whichever is appropriate—and discounts forecast cash flows to present value at the higher risk-adjusted rate of return to calculate the discount.

To simplify the calculations, Mercer usually assumes a growing annuity. He presents an approximate formula for the present value of an annuity with growth on (p. 276).[53] If one wants to use the QMDM, one improvement the appraiser can make is to use the exact annuity discount factors (ADFs) with growth that we developed in Chapter 4 and that we repeat here with the original equation numbers:

$$ADF = \frac{1}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^n\right] \quad \text{ADF with constant growth: end-of-year formula.} \tag{4.6b}$$

$$ADF = \frac{\sqrt{1+r}}{r-g}\left[1-\left(\frac{1+g}{1+r}\right)^n\right] \quad \text{ADF with constant growth: midyear formula.} \tag{4.10a}$$

Note that the first terms on the right-hand-side of equations (4.6b) and (4.10a) are the end-of-year and midyear Gordon models. As $n \to \infty$, $\left(\frac{1+g}{1+r}\right)^n \to 0$, and the annuity discount factor reduces to the Gordon model with which we are all familiar.

In his Chapter 12, Mercer reiterates his opposition to a DLOM for controlling interests from his original article (Mercer, 1994). His primary objection seems to be that the control owner has control of cash flows until he or she sells the business, at which time there is no longer DLOM. I disagree, as the ability to enjoy cash flows one day at a time and to instantaneously actualize the present value of all cash flow to perpetuity are quite different, the difference being measured by the DLOM that Mr. Mercer suggests does not exist.

In support of his belief that a DLOM is inappropriate for controlling interests, Mercer (p. 340) cites an article (Phillips and Freeman, 1995) that finds that after controlling for size, margin, and industry, privately held firms do not sell for lower multiples than publicly held firms when the buyer is another publicly held firm. There are a few problems with this study:

1. Since the buyers are all publicly held firms, once the sellers' businesses are absorbed into the buyers', there is no DLOM that applies anymore. When a privately held firm sells to a publicly held firm, ignoring any other differences such as potential synergies, there are at least two FMVs for the seller: a "floor

[53] I do not have any subsequent editions to his book, but I presume that he has incorporated the ADF with growth.

FMV," which is the FMV of the stand-alone business, including DLOM, and a "ceiling FMV," which is the FMV without DLOM. The seller should not be willing to sell below the floor FMV, and the buyer should not be willing to pay more than the ceiling FMV. An actual transaction can take place anywhere between the two, and Mergerstat will record that as the FMV. The articles by Schwert (1996) and Bradley, Desai, and Kim (1988), cited earlier in this chapter, show that the lion's share of excess returns in acquisitions goes to the seller. Thus, it is normal that the buyer pays "top dollar," which would mean that the seller would insist that the buyer forgo the DLOM, which disappears in any case after the transaction. Therefore, at a minimum, the Phillips/Freeman article's applicability is limited to privately held firms that are large enough to attract the attention of and be acquired by publicly held buyers.

2. In both regressions—the Mergerstat and the SDC database—banks show up as having different valuations than all other industries. However, the signs of the regression coefficients for banks are opposite in the two regressions. The regression of the Mergerstat database demonstrates at the 99.99% significance level that buyers pay lower multiples of sales for banks than for other industries, and the regression of the SDC database demonstrates at the 99.99% significance level that buyers pay higher multiples of sales for banks than other industries! There were several other inconsistencies in the results of the two regressions.
3. The log-log form of regression that Phillips and Freeman used can have the effect of making large variations look small. The standard errors of their regressions were very high. The standard error of the Mergerstat regression was 0.925. Two standard errors is 1.85. Exponentiating, the 95% confidence interval is approximately equal to multiplying the (value/sales) estimate by two standard errors on either side of the regression estimate. The high side of the 95% confidence interval is $e^{1.85} = 6.36$ times the regression estimate, and the low side is $e^{-1.85} = 0.157$ times the regression estimate. Let's put some specific numbers into their equation to see what the confidence intervals look like. Let's assume we are forecasting the value of the common stock as a percentage of sales for a firm with over \$100 million in value that is neither a bank, nor a private placement, nor a subsidiary. Their regression equation is $\ln(\text{Value/Sales}) = 3.242 + 0.56 \ln \text{net margin} + 0.45 \ln (1/\text{PE of the S\&P 500})$. Let's assume a 5% after-tax margin and an average PE for the S&P 500 of 15, so $1/\text{PE} = 0.067$. Then, $\ln(\text{Value/Sales}) = 3.242 + (0.56 \times \ln 0.05) + (0.45 \times \ln 0.067) = 3.242 - 1.678 - 1.219 = 0.345$. Thus, the regression estimate of $(\text{Value/Sales}) = e^{0.345} = 1.413$, or value is approximately 1.4 times sales, which seems high. If sales are \$100, then net income after taxes is \$5, which when multiplied by a PE ratio of 15 leads to a value of \$75, which implies value should be $0.75 \times$ Sales, not 1.4. The reliability of the forecast is low. The 95% confidence interval is approximately: $0.22 \times \text{Sales} < \text{Value} < 8.99 \times \text{Sales}$.
4. There were fairly few transactions with a private seller. In the Mergerstat database, private targets were 18 out of 416 transactions, and in the SDC database, private targets were 33 out of 445 targets. In total, private targets were approximately 6% of the combined databases.
5. The small number of transactions with privately held sellers is not necessarily worrisome in itself, but combined with the limitations of the results in item 1, the

inconsistent results in item 2, and the very wide confidence intervals in item 3, the results of this study are insufficient to reject DLOM for control interests of privately held firms.

**KOEPLIN, SARIN, AND SHAPIRO (2000)** Koeplin, Sarin, and Shapiro (2000) compared acquisitions of 84 domestic and 108 foreign private companies to acquisitions of the same number of public firms in the same year and industry. They found that private firms sell at discounts of 20% to 30% compared to public firms when measured by earnings multiples, with statistical significance at the 99.9% level (i.e., $p \leq 0.01$). For foreign transactions the discount rose to 45% to 55%.

Using multiples of book value, the discount declined to 7% to 18%, while it was insignificant using multiples of sales. The reason for the latter is that using multiples of earnings controls for differences in profitability, while multiples of sales does not. If the private targets in the sample had higher profit margins than the public targets, we would not expect to find a valuation discount for the private targets.

## Kasper's BAS Model

Larry Kasper (Kasper, 1997, p. 106) uses an econometric equation developed by Amihud and Mendelson (Amihud and Mendelson, 1991) to calculate the bid–ask spread (BAS) for NASDAQ stocks. Their equation is: $r = 0.006477 + 0.01012\ \beta + 0.002144 \ln \text{BAS}$, where $r$ is the *excess* monthly returns on a stock portfolio over the 90-day Treasury bill rate and the BAS is multiplied by 100 (i.e., a BAS of 25% is denominated as 25, not 0.25).

Kasper says that most business brokers would not list a business that had to be discounted more than 25%. Substituting 25 into the above equation, the excess return required for a BAS of 25% is 0.0069 per month, or approximately 8.28% per year. One would then seek out business brokers (or through IBA, Pratt's Stats, BIZCOMPS, etc.) for actual BASs. Anyone interested in using Kasper's model must read his outstanding book, as this summary is inadequate for understanding his work.

A number of differences in the environment of NASDAQ and that of privately held business can weaken the applicability of this regression equation from the former to the latter:

1. The BAS in NASDAQ compensates the dealer for actually taking possession of the stock. The dealer actually stands to gain or lose money, whereas business brokers do not.
2. It takes much longer to sell a private business than stock on NASDAQ.
3. The market for privately held firms is much thinner than it is with NASDAQ.
4. Transactions costs are far higher in privately held business than in NASDAQ.

Note that items 2 through 4 are the components of the *economic components* method, which we will cover shortly in my model. Also, the reservation in item 1 also applied in the Menyah and Paudyal results earlier in the chapter, where the BAS depends on the number of market makers. Again, business brokers are not market makers in the same sense that dealers are. Additionally, as Kasper points out, the regression coefficients will change over time. Kasper also presents a different model, the *discounted time to market* model (Kasper, 1997, pp. 103–104) that is worth reading. Neither of his models considers transactions costs or the effects of thin markets.[54]

## Restricted Stock Discounts

We will now discuss restricted stock discounts as a preparation for our general model for DLOM. First, however, we will mention the various studies of restricted stock discounts. Ten studies of sales of restricted stocks were published as of the first edition of this book.[55] The first nine studies appear in Pratt, Reilly, and Schweihs (1996, chap. 15) and Mercer (1997). In those studies, the authors did not publish the underlying data and merely presented their analysis and summary of the data. Additionally, only the Hall/Polacek study contains data beyond 1988, with theirs going through 1992.

The Management Planning study, which Mercer justifiably accorded a separate chapter and extensive commentary in his book, contains data on trades from 1980 to 1996. Thus, the Management Planning study is superior to the others in two ways: The detail of the data exists, and the data are more current than the previous studies.

**RESTRICTED STOCK STUDIES SUBSEQUENT TO THE FIRST EDITION OF THIS BOOK** Since the first edition, valuation professionals have performed the following restricted stock studies:

- Aschwald (2000) of Columbia Financial Advisors, Inc. We will cover this study as part of the MPI regression.
- Management Planning, Inc. has updated its study. MPI has provided us with summary statistics of their results, but not detail. We will provide the summary statistics later in this chapter.
- FMV Opinions (FMVO) has its own restricted stock transactional database, which it sells. FMVO was gracious enough to provide us with its data. Unfortunately, the magnitude of the database and the complexity of the data are such that we had insufficient time to arrive at satisfying results. It is our hope to continue our analysis and perhaps be able to publish some results in a workbook.

**REGRESSION OF MPI DATA**[56] We use two valuation methodologies in calculating the restricted stock discount. The first is based on my own multiple regression analysis of data collected by Management Planning, Inc. (MPI),[57] an independent valuation firm in Princeton, New Jersey. The second method involves using a Black-Scholes put option as a proxy for the discount.

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[54]That is not to say that I downgrade his book. It is brilliant and a must-read for anyone in the profession.

[55]See Mercer (1997, p. 69) for a summary of the results of the first nine studies.

[56]This analysis is unchanged since the first edition of this book.

[57]Published in Chapter 12 of Mercer (1997). I wish to thank Management Planning, Inc. (MPI) for being gracious and helpful in providing us with its data and consulting with us. In particular, Roy H. Meyers, Vice President, was extremely helpful. MPI provided us with four additional data points and some data corrections.

Table 8.5 is two pages long.[58] The first page contains data on 53 sales of restricted stock from 1980 to 1996. Column A is numbered 1 through 53 to indicate the sale number. Column C, our dependent ($Y$) variable, is the restricted stock discount for each transaction. Columns D through J are our seven statistically significant independent variables, which I have labeled $X_1, X_2, \ldots X_7$. The following is a description of the independent variables:

| # | Independent Variable |
|---|---|
| 1 | Revenues squared. |
| 2 | Shares sold—$: the discounted dollar value of the traded restricted shares. |
| 3 | Market capitalization = price per share times shares outstanding, summed for all classes of stock. |
| 4 | Earnings stability: the unadjusted $R^2$ of the regression of net income as a function of time, with time measured as years 1, 2, 3, etc. for a total of 10 years. |
| 5 | Revenue stability: the unadjusted $R^2$ of the regression of revenue as a function of time, with time measured as years 1, 2, 3, etc. for a total of 10 years. |
| 6 | Average years to sell: the weighted average years to sell by a nonaffiliate based on SEC Rule 144. I calculated the holding period for the last four issues (DPAC, UMED, NEDI, and ARCCA) based on changes in Rule 144, even though it was not effective yet, because the change was out for review at that time, and was highly likely to be accepted.[59] These last four transactions occurred near the beginning of March 1996, well after the SEC issued the exposure draft on June 27, 1995. This was approximately 14 months before the rule change went into effect at the end of April 1997. The average time to resale for the shares in these four transactions was determined based on the rule change, resulting in a minimum and maximum average holding period of 14 months and 2 years, respectively.[60] |
| 7 | Price stability: This ratio is calculated by dividing the standard deviation of the stock price by the mean of the stock price—which is the coefficient of variation of price—then multiplying by 100. MPI used the end-of-month stock prices for the 12 months prior to the transaction date. |

I regressed 30 other independent variables included in or derived from the Management Planning study, and all were statistically insignificant. I restrict our commentary to the seven independent variables that were statistically significant at the 95% level.

The second page of Table 8.5 contains the regression statistics. In regression #1 the adjusted $R^2$ is 59.47% (B9), a reasonable though not stunning result for such an analysis. This means that the regression model accounts for 59.47% of the variation in the restricted stock discounts. The other 40.53% of variation in the discounts that remains unexplained is due to two possible sources: other significant

[58]In Excel it is two pages. In the first edition, the different software of the publisher rendered this table four pages long.

[59]According to John Watson, Jr., Esq., of Latham & Watkins in Washington, D.C., the securities community knew the rule change would take place. In a telephone conversation with Mr. Watson, he said it was only a question of timing.

[60]In other words, I assumed perfect foreknowledge of when the rule change would become effective.

| | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | **Table 8.5** | | | | | | | | |
| 2 | | **Abrams Regression of Management Planning Study Data** | | | | | | | | |
| 3 | | | | | | | | | | |
| 4 | | | **Y** | **$X_1$** | **$X_2$** | **$X_3$** | **$X_4$** | **$X_5$** | **$X_6$** | **$X_7$** |
| 5 | | | | | | | | | | |
| 6 | | | **Discount** | **$Rev^2$** | **Shares Sold—$** | **Mkt Cap** | **Earn Stab** | **Rev Stab** | **AvgYrs2Sell** | **Price Stab** |
| 7 | 1 | Air Express Int'l | 0.0% | 8.58E+16 | $4,998,000 | 25,760,000 | 0.08 | 0.22 | 2.84 | 12.0 |
| 8 | 2 | AirTran Corp | 19.4% | 1.55E+16 | $9,998,000 | 63,477,000 | 0.90 | 0.94 | 2.64 | 12.0 |
| 9 | 3 | Anaren Microwave, Inc. | 34.2% | 6.90E+13 | $1,250,000 | 13,517,000 | 0.24 | 0.78 | 2.64 | 28.6 |
| 10 | 4 | Angeles Corp | 19.6% | 7.99E+14 | $1,800,000 | 16,242,000 | 0.08 | 0.82 | 2.13 | 8.4 |
| 11 | 5 | AW Computer Systems, Inc. | 57.3% | 1.82E+13 | $1,843,000 | 11,698,000 | 0.00 | 0.00 | 2.91 | 22.6 |
| 12 | 6 | Besicorp Group, Inc. | 57.6% | 1.57E+13 | $1,500,000 | 63,145,000 | 0.03 | 0.75 | 2.13 | 98.6 |
| 13 | 7 | Bioplasty, Inc, | 31.1% | 6.20E+13 | $11,550,000 | 43,478,000 | 0.38 | 0.62 | 2.85 | 44.9 |
| 14 | 8 | Blyth Holdings, Inc. | 31.4% | 8.62E+13 | $4,452,000 | 98,053,000 | 0.04 | 0.64 | 2.13 | 58.6 |
| 15 | 9 | Byers Communications Systems, Inc. | 22.5% | 4.49E+14 | $5,007,000 | 14,027,000 | 0.90 | 0.79 | 2.92 | 6.6 |
| 16 | 10 | Centennial Technologies, Inc. | 2.8% | 6.75E+13 | $656,000 | 27,045,000 | 0.94 | 0.87 | 2.13 | 35.0 |
| 17 | 11 | Chantal Pharm. Corp. | 44.8% | 5.21E+13 | $4,900,000 | 149,286,000 | 0.70 | 0.23 | 2.13 | 51.0 |
| 18 | 12 | Choice Drug Delivery Systems, Inc. | 28.8% | 6.19E+14 | $3,375,000 | 21,233,000 | 0.29 | 0.89 | 2.86 | 23.6 |
| 19 | 13 | Crystal Oil Co. | 24.1% | 7.47E+16 | $24,990,000 | 686,475,000 | 0.42 | 0.57 | 2.50 | 28.5 |
| 20 | 14 | Cucos, Inc. | 18.8% | 4.63E+13 | $2,003,000 | 12,579,000 | 0.77 | 0.87 | 2.84 | 20.4 |
| 21 | 15 | Davox Corp. | 46.3% | 1.14E+15 | $999,000 | 18,942,000 | 0.01 | 0.65 | 2.72 | 24.6 |
| 22 | 16 | Del Electronics Corp. | 41.0% | 4.21E+13 | $394,000 | 3,406,000 | 0.08 | 0.10 | 2.84 | 4.0 |
| 23 | 17 | Edmark Corp | 16.0% | 3.56E+13 | $2,000,000 | 12,275,000 | 0.57 | 0.92 | 2.84 | 10.5 |
| 24 | 18 | Electro Nucleonics | 24.8% | 1.22E+15 | $1,055,000 | 38,435,000 | 0.68 | 0.97 | 2.13 | 21.4 |
| 25 | 19 | Esmor Correctional Svces, Inc. | 32.6% | 5.89E+14 | $3,852,000 | 50,692,000 | 0.95 | 0.90 | 2.64 | 34.0 |
| 26 | 20 | Gendex Corp | 16.7% | 2.97E+15 | $5,000,000 | 55,005,000 | 0.99 | 0.71 | 2.69 | 11.5 |
| 27 | 21 | Harken Oil & Gas, Inc. | 30.4% | 7.55E+13 | $1,999,000 | 27,223,000 | 0.13 | 0.88 | 2.75 | 19.0 |
| 28 | 22 | ICN Paramaceuticals, Inc. | 10.5% | 1.50E+15 | $9,400,000 | 78,834,000 | 0.11 | 0.87 | 2.25 | 23.9 |
| 29 | 23 | Ion Laser Technology, Inc. | 41.1% | 1.02E+13 | $975,000 | 10,046,000 | 0.71 | 0.92 | 2.82 | 22.0 |
| 30 | 24 | Max & Erma's Restaurants, Inc. | 12.7% | 1.87E+15 | $1,192,000 | 31,080,000 | 0.87 | 0.87 | 2.25 | 18.8 |
| 31 | 25 | Medco Containment Svces, Inc. | 15.5% | 5.42E+15 | $99,994,000 | 561,890,000 | 0.84 | 0.89 | 2.85 | 12.8 |
| 32 | 26 | Newport Pharm. Int'l, Inc. | 37.8% | 1.10E+14 | $5,950,000 | 101,259,000 | 0.00 | 0.87 | 2.00 | 30.2 |
| 33 | 27 | Noble Roman's Inc. | 17.2% | 8.29E+13 | $1,251,000 | 11,422,000 | 0.06 | 0.47 | 2.79 | 17.0 |
| 34 | 28 | No. American Holding Corp. | 30.4% | 1.35E+15 | $3,000,000 | 79,730,000 | 0.63 | 0.84 | 2.50 | 22.1 |
| 35 | 29 | No. Hills Electronics, Inc. | 36.6% | 1.15E+13 | $3,675,000 | 21,812,000 | 0.81 | 0.79 | 2.83 | 52.7 |
| 36 | 30 | Photographic Sciences Corp | 49.5% | 2.70E+14 | $5,000,000 | 44,113,000 | 0.06 | 0.76 | 2.86 | 27.2 |
| 37 | 31 | Presidential Life Corp | 15.9% | 4.37E+16 | $38,063,000 | 246,787,000 | 0.00 | 0.00 | 2.83 | 17.0 |
| 38 | 32 | Pride Petroleum Svces, Inc. | 24.5% | 4.34E+15 | $21,500,000 | 74,028,000 | 0.31 | 0.26 | 2.83 | 18.0 |
| 39 | 33 | Quadrex Corp. | 39.4% | 1.10E+15 | $5,000,000 | 71,016,000 | 0.41 | 0.66 | 2.50 | 44.2 |
| 40 | 34 | Quality Care, Inc. | 34.4% | 7.97E+14 | $3,150,000 | 19,689,000 | 0.68 | 0.74 | 2.88 | 7.0 |
| 41 | 35 | Ragen Precision Industries, Inc. | 15.3% | 8.85E+14 | $2,000,000 | 22,653,000 | 0.61 | 0.75 | 2.25 | 26.0 |
| 42 | 36 | REN Corp-USA | 17.9% | 2.85E+15 | $53,625,000 | 151,074,000 | 0.02 | 0.88 | 2.92 | 19.8 |
| 43 | 37 | REN Corp-USA | 29.3% | 2.85E+15 | $12,003,000 | 163,749,000 | 0.02 | 0.88 | 2.72 | 36.1 |
| 44 | 38 | Rentrak Corp. | 32.5% | 1.15E+15 | $20,650,000 | 61,482,000 | 0.60 | 0.70 | 2.92 | 30.0 |
| 45 | 39 | Ryan's Family Steak Houses, Inc. | 8.7% | 1.02E+15 | $5,250,000 | 159,390,000 | 0.90 | 0.87 | 2.13 | 13.6 |
| 46 | 40 | Ryan's Family Steak Houses, Inc. | 5.2% | 1.02E+15 | $7,250,000 | 110,160,000 | 0.90 | 0.87 | 2.58 | 14.4 |
| 47 | 41 | Sahlen & Assoc., Inc. | 27.5% | 3.02E+15 | $6,057,000 | 42,955,000 | 0.54 | 0.81 | 2.72 | 26.1 |
| 48 | 42 | Starrett Housing Corp. | 44.8% | 1.11E+16 | $3,000,000 | 95,291,000 | 0.02 | 0.01 | 2.50 | 12.4 |
| 49 | 43 | Sudbury Holdings, Inc. | 46.5% | 1.39E+16 | $22,325,000 | 33,431,000 | 0.65 | 0.17 | 2.96 | 26.6 |
| 50 | 44 | Superior Care, Inc. | 41.9% | 1.32E+15 | $5,660,000 | 50,403,000 | 0.21 | 0.93 | 2.77 | 42.2 |
| 51 | 45 | Sym-Tek Systems, Inc. | 31.6% | 4.03E+14 | $995,000 | 20,550,000 | 0.34 | 0.92 | 2.58 | 13.4 |
| 52 | 46 | Telepictures Corp. | 11.6% | 5.50E+15 | $15,250,000 | 106,849,000 | 0.81 | 0.86 | 2.72 | 6.6 |
| 53 | 47 | Velo-Bind, Inc. | 19.5% | 5.51E+14 | $2,325,000 | 18,509,000 | 0.65 | 0.85 | 2.81 | 14.5 |
| 54 | 48 | Western Digital Corp. | 47.3% | 4.24E+14 | $7,825,000 | 50,417,000 | 0.00 | 0.32 | 2.64 | 22.7 |
| 55 | 49 | 50-Off Stores, Inc. | 12.5% | 6.10E+15 | $5,670,000 | 43,024,000 | 0.80 | 0.87 | 2.38 | 23.7 |
| 56 | 50 | ARC Capital | 18.8% | 3.76E+14 | $2,275,000 | 18,846,000 | 0.03 | 0.74 | 1.63 | 35.0 |
| 57 | 51 | Dense Pac Microsystems, Inc. | 23.1% | 3.24E+14 | $4,500,000 | 108,862,000 | 0.08 | 0.70 | 1.17 | 42.4 |
| 58 | 52 | Nobel Education Dynamics, Inc. | 19.3% | 1.95E+15 | $12,000,000 | 60,913,000 | 0.34 | 0.76 | 1.74 | 32.1 |
| 59 | 53 | Unimed Pharmaceuticals | 15.8% | 5.49E+13 | $8,400,000 | 44,681,000 | 0.09 | 0.74 | 1.90 | 21.0 |
| 60 | | Mean | **27.1%** | 5.65E+15 | **$9,223,226** | **$78,621,472** | **0.42** | **0.69** | **2.54** | **25.4** |
| 61 | | | | | | | | | | |
| 62 | *Source:* Management Planning, Inc. Princeton, NJ (except for "AvgYrs2Sell" and "$Rev^2$", which we derived from their data) | | | | | | | | | |

independent variables of which I (and Management Planning, Inc.) do not know, and random variation. The standard error of the *y*-estimate is 8.7% (B10 rounded). We can form approximate 95% confidence intervals around the *y*-estimate by adding and subtracting two standard errors, or 17.4%.

B20 contains the regression estimate of the *y*-intercept, and B21 through B27 contain the regression coefficients for the independent variables. The *t*-statistics are in D20 through D27. Only the *y*-intercept itself is not significant at the 95% confidence level. The market capitalization and earnings stability variables are significant at the 98% level,[61] and all the other variables are significant at the 99+% confidence level.

Note that several of the variables are similar to Grabowski and King's results (Grabowski and King, 1999) discussed in Chapter 5. They found that the coefficient

[61]The statistical significance is 1 minus the *p*-value, which are in E20 through E27.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 8.5 (cont.)** | | | | | | |
| 2 | **Abrams Regression of Management Planning Study Data** | | | | | | |
| 3 | | | | | | | |
| 4 | **Regression #1** | | | | | | |
| 5 | | | | | | | |
| 6 | *Regression Statistics* | | | | | | |
| 7 | Multiple R | 0.8058 | | | | | |
| 8 | R Square | 0.6493 | | | | | |
| 9 | Adjusted R Square | 0.5947 | | | | | |
| 10 | Standard Error | 0.0873 | | | | | |
| 11 | Observations | 53 | | | | | |
| 12 | | | | | | | |
| 13 | ANOVA | | | | | | |
| 14 | | *df* | *SS* | *MS* | *F* | *Significance F* | |
| 15 | Regression | 7 | 0.6354 | 0.0908 | 11.9009 | 1.810E-08 | |
| 16 | Residual | 45 | 0.3432 | 0.0076 | | | |
| 17 | Total | 52 | 0.9786 | | | | |
| 18 | | | | | | | |
| 19 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
| 20 | Intercept | -0.0673 | 0.1082 | -0.6221 | 0.5370 | -0.2854 | 0.1507 |
| 21 | $Rev^2$ | -4.629E-18 | 9.913E-19 | -4.6698 | 0.0000 | -6.626E-18 | -2.633E-18 |
| 22 | Shares Sold—$ | -3.619E-09 | 1.199E-09 | -3.0169 | 0.0042 | -6.035E-09 | -1.203E-09 |
| 23 | Mkt Cap | 4.789E-10 | 1.790E-10 | 2.6754 | 0.0104 | 1.184E-10 | 8.394E-10 |
| 24 | Earn Stab | -0.1038 | 0.0402 | -2.5831 | 0.0131 | -0.1848 | -0.0229 |
| 25 | Rev Stab | -0.1824 | 0.0531 | -3.4315 | 0.0013 | -0.2894 | -0.0753 |
| 26 | AvgYrs2Sell | 0.1722 | 0.0362 | 4.7569 | 0.0000 | 0.0993 | 0.2451 |
| 27 | Price Stab | 0.0037 | 8.316E-04 | 4.3909 | 0.0001 | 0.0020 | 0.0053 |
| 28 | | | | | | | |
| 29 | | | | | | | |
| 30 | | | | | | | |
| 31 | | | | | | | |
| 32 | **Regression #2 (Without Price Stability)** | | | | | | |
| 33 | | | | | | | |
| 34 | *Regression Statistics* | | | | | | |
| 35 | Multiple R | 0.7064 | | | | | |
| 36 | R Square | 0.4990 | | | | | |
| 37 | Adjusted R Square | 0.4337 | | | | | |
| 38 | Standard Error | 0.1032 | | | | | |
| 39 | Observations | 53 | | | | | |
| 40 | | | | | | | |
| 41 | ANOVA | | | | | | |
| 42 | | *df* | *SS* | *MS* | *F* | *Significance F* | |
| 43 | Regression | 6 | 0.4883 | 0.0814 | 7.6365 | 0.0000 | |
| 44 | Residual | 46 | 0.4903 | 0.0107 | | | |
| 45 | Total | 52 | 0.9786 | | | | |
| 46 | | | | | | | |
| 47 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
| 48 | Intercept | 0.1292 | 0.1165 | 1.1089 | 0.2732 | -0.1053 | 0.3637 |
| 49 | $Rev^2$ | -5.39E-18 | 1.15E-18 | -4.6740 | 0.0000 | -7.71E-18 | -3.07E-18 |
| 50 | Shares Sold—$ | -4.39E-09 | 1.40E-09 | -3.1287 | 0.0030 | -7.21E-09 | -1.57E-09 |
| 51 | Mkt Cap | 6.10E-10 | 2.09E-10 | 2.9249 | 0.0053 | 1.90E-10 | 1.03E-09 |
| 52 | Earn Stab | -0.1381 | 0.0466 | -2.9626 | 0.0048 | -0.2319 | -0.0443 |
| 53 | Rev Stab | -0.1800 | 0.0628 | -2.8653 | 0.0063 | -0.3065 | -0.0536 |
| 54 | AvgYrs2Sell | 0.1368 | 0.0417 | 3.2790 | 0.0020 | 0.0528 | 0.2208 |

of variation (in log form) of operating margin and return on equity are statistically significant in explaining stock market returns. Here we find that the stability of revenues and earnings (as well as the coefficient of variation of stock market prices) explain restricted stock discounts. Thus, these variables are significant in determining the value of the underlying companies, assuming they are marketable, and in determining restricted stock discounts when restrictions exist.

I obtained regression #2 in Table 8.5 by regressing all the independent variables in the first regression except for price stability. The adjusted $R^2$ has dropped to 43.37% (B37), indicating that regression #1 is superior when price data are available, which it generally is for restricted stock studies and is not for calculating DLOM for privately held businesses. The second regression is not recommended for the calculation of restricted stock discounts, but it will be useful in other contexts.

**USING THE PUT OPTION MODEL TO CALCULATE DLOM OF RESTRICTED STOCK** Chaffe (1993) wrote a brilliant article in which he reasoned that buying a hypothetical put option on Section 144 restricted stock would "buy" marketability, and that the cost of that put option is an excellent measure of the discount for lack of marketability of the stock. For puts, the Black-Scholes option pricing model has the following formula:

$$P = E\,N(-d_2)e^{-R_f t} - SN(-d_1),$$

where $d_1 = [\ln(S/E) + (R_f + 0.5 \times \text{variance}) \times t]/[\text{standard deviation} \times t^{0.5}]$.
$d_2 = d_1 - [\text{standard deviation} \times t^{0.5}]$.
E = exercise price.
$N()$ = cumulative normal density function.
$P$ = value of the put option.
$R_f$ = continuously compounded risk-free rate, that is, the natural logarithm of 1 plus the Treasury rate of the same term as the option.
$S$ = stock price on the valuation date.
$t$ = time remaining to expiration of the option.

We have sufficient daily price history on 13 of the stocks in Table 8.5 to derive the proper annualized standard deviation of continuously compounded returns to test Chaffe's approach.

**Annualized Standard Deviation of Continuously Compounded Returns** Table 8.6 is a sample calculation of the annualized standard deviation of continuously compounded returns for Chantal Pharmaceutical Inc. (CHTL), which is one of the 13 stocks. The purpose of this table is to demonstrate how to calculate the standard deviation. Column A shows the date, column B shows the closing price, and columns C and D show the continuously compounded returns. The sample period is just over 6 months and ends the day prior to the transaction date.

We calculate continuously compounded returns over 10-trading-day intervals for CHTL stock.[62] The reason for using 10-day intervals in our calculation instead of daily intervals is that the bid–ask spread on the stock may create apparent volatility that is not really present. This is because the quoted closing prices are from the last trade. In NASDAQ trading, one sells to a dealer at the bid price and buys at the ask price. If on successive days the last price of the day is switching randomly from a bid to an ask price and back, this can cause us to measure a considerable amount of apparent volatility that is not really there. By using 10-day intervals, we minimize this measurement error caused by the spread.

We start with the 1/31/1995 closing price in column C and the 2/7/1995 closing price in column D. For example, the 10-trading-day return from 1/31/1995 (A7) to 2/14/1995 (A9) is calculated as follows: Return = Ln(B9/B7) = ln(2.5660/2.1650) = 0.169928 (C9).

Using this methodology, we get two measures of the interval standard deviation: 0.16900 (C34) and 0.20175 (D34). To convert to the annualized standard deviation,

[62]The only exception is the return from 7/31/95 to 8/7/95, which is in D33.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Table 8.6** | | | |
| 2 | **Calculation of Continuously Compounded** | | | |
| 3 | **Standard Deviation** | | | |
| 4 | **Chantal Pharmaceutical Inc.—CHTL** | | | |
| 5 | | | | |
| 6 | **Date** | **Close** | **Interval Returns** | |
| 7 | 1/31/1995 | $2.1650 | | |
| 8 | 2/7/1995 | $2.2500 | | |
| 9 | 2/14/1995 | $2.5660 | 0.169928 | |
| 10 | 2/22/1995 | $2.8440 | | 0.234281 |
| 11 | 3/1/1995 | $2.6250 | 0.022733 | |
| 12 | 3/8/1995 | $2.9410 | | 0.033538 |
| 13 | 3/15/1995 | $2.4480 | -0.069810 | |
| 14 | 3/22/1995 | $2.5000 | | -0.162459 |
| 15 | 3/29/1995 | $2.2500 | -0.084341 | |
| 16 | 4/5/1995 | $2.0360 | | -0.205304 |
| 17 | 4/12/1995 | $2.2220 | -0.012523 | |
| 18 | 4/20/1995 | $2.1910 | | 0.073371 |
| 19 | 4/27/1995 | $2.6950 | 0.192991 | |
| 20 | 5/4/1995 | $2.6600 | | 0.193968 |
| 21 | 5/11/1995 | $2.5660 | -0.049050 | |
| 22 | 5/18/1995 | $2.5620 | | -0.037538 |
| 23 | 5/25/1995 | $2.9740 | 0.147560 | |
| 24 | 6/2/1995 | $3.3120 | | 0.256764 |
| 25 | 6/9/1995 | $5.1250 | 0.544223 | |
| 26 | 6/16/1995 | $6.0000 | | 0.594207 |
| 27 | 6/23/1995 | $5.8135 | 0.126052 | |
| 28 | 6/30/1995 | $6.4440 | | 0.071390 |
| 29 | 7/10/1995 | $6.5680 | 0.122027 | |
| 30 | 7/17/1995 | $6.6250 | | 0.027701 |
| 31 | 7/24/1995 | $8.0000 | 0.197232 | |
| 32 | 7/31/1995 | $7.1250 | | 0.072759 |
| 33 | 8/7/1995 | $7.8120 | -0.023781 | 0.092051 |
| 34 | **Interval Standard Deviation—CHTL** | | **0.16900** | **0.20175** |
| 35 | **Annualized** | | **0.84901** | **1.03298** |
| 36 | **Average of Standard Deviations** | | | **0.94099** |

we must multiply each interval standard deviation by the square root of the number of intervals that would occur in a year. The equation is as follows:

$$\sigma_{annualized} = \sigma_{interval\ returns} \times \text{SQRT}\,(\#\text{ of Interval Returns in Sample Period} \times 365 \text{ Days per Year/Days in Sample Period}).$$

For example, the sample period in column C is the time period from the close of trading on January 31, 1995 to the close of trading on August 7, 1995, or 188 days, and there are 13 calculated returns. Therefore the annualized standard deviation of returns is: $\sigma_{annualized} = 0.1690 \times \text{SQRT}(13 \times 365/188) = 0.1690 \times \text{SQRT}(25.2394) = 0.84901$ (C35 = C34 × SQRT(25.2394)). The 13 trading periods that span 188 days would become 25.2394 trading periods in one year (25.2394 = 13 × 365/188). The square root of the 25.2394 trading periods is 5.0239. We multiply the sample standard deviation of 0.1690 by 5.0239 = 0.84901 to annualize the standard deviation. Similarly

| | A | B |
|---|---|---|
| 1 | **Table 8.7** | |
| 2 | **Black-Scholes Put Option—CHTL** | |
| 3 | | |
| 4 | | |
| 5 | $S$ = Stk Price on Valuation Date | \$8.875 |
| 6 | $E$ = Exercise Price | \$8.875 |
| 7 | $t$ = Time to Expiration in Years (Table 8.5, I17) | 2.125 |
| 8 | $r$ = Risk-Free Rate [1] | 5.90% |
| 9 | stdev = Standard Deviation (Table 8.6, D36) | 0.941 |
| 10 | var = variance | 0.885 |
| 11 | $d_1$ = 1st Black-Scholes Parameter [2] | 0.777 |
| 12 | $d_2$ = 2nd Black-Scholes Parameter [3] | (0.594) |
| 13 | $N(-d_1)$ = Cum Normal Density Function | 0.219 |
| 14 | $N(-d_2)$ = Cum Normal Density Function | 0.724 |
| 15 | $P=[E\times N(-d_2)\times e^{-rt}]-S\times N(-d_1)$ | \$3.73 |
| 16 | $P/S$ | **42.0%** |
| 17 | | |
| 18 | *Note:* Values are for European options. The put option formula can be found in *Options Futures and Other* | |
| 19 | *Derivatives, 3rd Ed.* by John C. Hull, Prentice Hall, 1997, pp. 241 and 242. | |
| 20 | | |
| 21 | [1] According to the Federal Reserve Board Web site below, the 2-year Treasury rate on the transaction | |
| 22 | date, 8/8/1995, was 5.90%. This should be the continuously compounded rate of ln(1+5.90%) = 5.73%. | |
| 23 | Making this correction increases the value of the put from 42.0% to 42.2% in B16—an immaterial change. | |
| 24 | Furthermore, we could be even more accurate by interpolating to a 2.125-year Treasury rate, which | |
| 25 | would slightly decrease the value of the put and render it closer to 42.0%. | |
| 26 | http://www.federalreserve.gov/releases/H15/data/Business_day/H15_TCMNOM_Y2.txt | |
| 27 | | |
| 28 | [2] $d_1 = [\ln(S/E) + (r + 0.5 \times \text{var}) \times t] / [\text{stdev} \times t^{0.5}]$, where variance and | |
| 29 | standard deviation are expressed in annual terms. | |
| 30 | | |
| 31 | [3] $d_2 = d_1 - [\text{stdev} \times t^{0.5}]$ | |

the annualized standard deviation of returns in column D is 1.03298 (D35), and the average of the two is 0.94099 (D36).

**Calculation of the Discount** Table 8.7 is the Black-Scholes put option calculation of the restricted stock discount. We begin in B5 with $S$, the stock price on the valuation date of August 8, 1995, of \$8.875. We then assume that $E$, the exercise price, is identical (B6).

B7 is the time in years from the valuation date to marketability. According to SEC Rule 144, the shares have a two-year period of restriction before the first portion of the block can be sold. At 2.25 years, the rest can be sold. The average time to sell is 2.125 years (B7, transferred from Table 8.5, I17) for this particular block of Chantal.

B8 shows the two-year Treasury rate, which was 5.90% as of the transaction date.[63] B9 contains the annualized standard deviation of returns for CHTL of 0.941, transferred from Table 8.6, D36, while B10 is variance, merely the square of B9.

[63] As noted in Table 8.7, footnote [2], B8 should have been the continuously compounded rate $\ln(1 + 5.90\%) = 5.73\%$. Making that change has an insignificant impact on the value of the put option, increasing it from 42.0% to 42.2% in B16. However, it would have been possible to interpolate the Treasury maturity, which would have a moderating effect on the already insignificant change. Similarly, we should be using continuously compounded Treasury rates for all 13 stocks in Table 8.8; however, the error is immaterial.

B11 and B12 are the calculation of the two Black-Scholes parameters, $d_1$ and $d_2$, the formulas for which appear above. B13 and B14 are the cumulative normal density functions for $-d_1$ and $-d_2$. For example, look at B13, which is $N(-0.777) = 0.219$. This requires some explanation. The cumulative normal table from which the 0.219 came assumes the normal distribution has been standardized to a mean of zero and standard deviation of 1.[64] This means that there is a 21.9% probability that our variable is less than or equal to 0.777 standard deviations below the mean. In B14, $N(-d_2) = N(-\ (-0.594)) = N(0.594) = 0.724$, which means there is a 72.4% probability of being less than or equal to 0.594 standard deviations above the mean. For perspective, it is useful to note that since the normal distribution is symmetric, $N(0) = 0.5000$; that is, there is a 50% probability of being less than or equal to the mean, which implies there is a 50% probability of being above the mean.

In B15, we calculate the value of the put option, which is \$3.73 (B15), or 42.0% (B16) of the stock price of \$8.875 (B5). Thus, our calculation of the restricted stock discount for the Chantal block using the Black-Scholes put option model is 42.0% (B16).

**Table 8.8: Black-Scholes Put Model Results** The stock symbols in Table 8.8, column A, relate to restricted stock sale numbers 8, 11, 15, 17, 23, 31, 32, 38, and 49–53 in Table 8.5, column A. B6 through B18 show the discounts calculated using the Black-Scholes put model for the 13 stocks. The actual discounts are in column C, and the error in the put model estimate is in column D.[65] Columns E and F are the squared and absolute error, respectively. Row 19 is the mean of each column. The bottom half of the table is identical to the top half, except that we use the mean discount of 27.1% (Table 8.5, C60) as the estimated discount instead of the Black-Scholes put option model.

A comparison of the top and bottom of Table 8.8 reveals that the put option model performs much better than the mean discount of 27.1% for the 13 stocks. The put model's mean absolute error of 6.5% (F19) and mean squared error of 0.67% (E19) are much smaller than the mean absolute error of 10.1% (F38) and mean squared error of 1.28% (E38) using the MPI data mean discount as the forecast. The mean errors in D19 and D38 are not indicative of relative predictive power, since low values could be obtained even though the individual errors are high due to negative and positive errors canceling each other out.

**COMPARISON OF THE PUT MODEL AND THE REGRESSION MODEL** In order to compare the put model discount results with the regression model, we will analyze Table 8.9, which shows the calculation of discounts, using regression #1 in Table 8.5, on the 13 stocks for which price data were available.

The intercept of the regression is in B6, and the coefficients for the independent variables are in B7 through B13. The independent variables for each stock are in columns C through O, rows 7 through 13. Multiplying the variables for each stock

[64] One standardizes a normal distribution by subtracting the mean from each value and dividing by the standard deviation.

[65] The error is equal to the estimated discount minus the actual discount, or column B minus column C.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Table 8.8** | | | | | |
| 2 | **Put Model Results** | | | | | |
| 3 | | | | | | |
| 4 | | **Black-Scholes** | | | | |
| 5 | **Company** | **Put Calculation** | **Actual** | **Error** | **Error$^2$** | **Absolute Error** |
| 6 | BLYH | 32.3% | 31.4% | 0.9% | 0.0% | 0.9% |
| 7 | CHTL | 42.0% | 44.8% | -2.8% | 0.1% | 2.8% |
| 8 | DAVX | 47.5% | 46.3% | 1.2% | 0.0% | 1.2% |
| 9 | EDMK | 11.9% | 16.0% | -4.1% | 0.2% | 4.1% |
| 10 | ILT | 38.3% | 41.1% | -2.8% | 0.1% | 2.8% |
| 11 | PLFE | 23.7% | 15.9% | 7.8% | 0.6% | 7.8% |
| 12 | PRDE | 13.3% | 24.5% | -11.2% | 1.2% | 11.2% |
| 13 | RENT | 41.5% | 32.5% | 9.0% | 0.8% | 9.0% |
| 14 | FOFF | 27.2% | 12.5% | 14.7% | 2.2% | 14.7% |
| 15 | ARCCA | 36.1% | 18.8% | 17.3% | 3.0% | 17.3% |
| 16 | DPAC | 18.3% | 23.1% | -4.8% | 0.2% | 4.8% |
| 17 | NEDI | 24.6% | 19.3% | 5.3% | 0.3% | 5.3% |
| 18 | UMED | 12.9% | 15.8% | -2.9% | 0.1% | 2.9% |
| 19 | **Mean** | **28.4%** | **26.3%** | **2.1%** | **0.67%** | **6.5%** |
| 20 | | | | | | |
| 21 | | | | | | |
| 22 | **Comparison with the Mean as the Discount** | | | | | |
| 23 | | | | | | |
| 24 | **Company** | **Mean Discount** | **Actual** | **Error** | **Error$^2$** | **Absolute Error** |
| 25 | BLYH | 27.1% | 31.4% | -4.3% | 0.2% | 4.3% |
| 26 | CHTL | 27.1% | 44.8% | -17.7% | 3.1% | 17.7% |
| 27 | DAVX | 27.1% | 46.3% | -19.2% | 3.7% | 19.2% |
| 28 | EDMK | 27.1% | 16.0% | 11.1% | 1.2% | 11.1% |
| 29 | ILT | 27.1% | 41.1% | -14.0% | 2.0% | 14.0% |
| 30 | PLFE | 27.1% | 15.9% | 11.2% | 1.3% | 11.2% |
| 31 | PRDE | 27.1% | 24.5% | 2.6% | 0.1% | 2.6% |
| 32 | RENT | 27.1% | 32.5% | -5.4% | 0.3% | 5.4% |
| 33 | FOFF | 27.1% | 12.5% | 14.6% | 2.1% | 14.6% |
| 34 | ARCCA | 27.1% | 18.8% | 8.3% | 0.7% | 8.3% |
| 35 | DPAC | 27.1% | 23.1% | 4.0% | 0.2% | 4.0% |
| 36 | NEDI | 27.1% | 19.3% | 7.8% | 0.6% | 7.8% |
| 37 | UMED | 27.1% | 15.8% | 11.3% | 1.3% | 11.3% |
| 38 | **Mean** | **27.1%** | **26.3%** | **0.8%** | **1.28%** | **10.1%** |

by their respective coefficients and then adding them together with the $y$-intercept results in the regression-estimated discounts in C14 through O14.

The errors in row 16 equal the actual discounts in row 15 minus the estimated discounts in row 14. We then calculate the error squared and absolute error in rows 17 and 18, respectively.

The mean squared error of 0.57% (C20) and the mean absolute error of 6.33% (C21) are comparable but slightly better than the put model results of 0.67% and 6.5% in Table 8.8, E19 and F19, respectively. Having been able to test the put model on only 13 stocks and not the entire database of 53 reduces our ability to distinguish

| | A | B | C | D | E | F | G | H | I | J | K | L | M | N | O |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table 8.9** | | | | | | | | | | | | | | |
| 2 | **Calculation of Restricted Stock Discounts for 13 Stocks Using Regression from Table 8.5** | | | | | | | | | | | | | | |
| 3 | | | | | | | | | | | | | | | |
| 4 | | | | | | | | | | | | | | | |
| 5 | | *Coefficients* | BLYH | CHTL | DAVX | EDMK | ITL | PLFE | PRDE | RENT | FOFF | ARCCA | DPAC | NEDI | UMED |
| 6 | Intercept | -0.0673 | | | | | | | | | | | | | |
| 7 | Rev$^2$ | -4.629E-18 | 8.62E+13 | 5.21E+13 | 1.14E+15 | 3.56E+13 | 1.02E+13 | 4.37E+16 | 4.34E+15 | 1.15E+15 | 6.10E+15 | 3.76E+14 | 3.24E+14 | 1.95E+15 | 5.49E+13 |
| 8 | Shares Sold—$ | -3.619E-09 | 4,452,000 | $4,900,000 | $999,000 | $2,000,000 | $975,000 | $38,063,000 | $21,500,000 | $20,650,000 | $5,670,000 | $2,275,000 | $4,500,000 | $12,000,000 | $8,400,000 |
| 9 | Mkt Cap | 4.789E-10 | 98,053,000 | 149,286,000 | 18,942,000 | 12,275,000 | 10,046,000 | 246,787,000 | 74,028,000 | 61,482,000 | 43,024,000 | 18,846,000 | 108,862,000 | 60,913,000 | 44,681,000 |
| 10 | Earn Stab | -0.1038 | 0.04 | 0.70 | 0.01 | 0.57 | 0.71 | 0.00 | 0.31 | 0.60 | 0.80 | 0.03 | 0.08 | 0.34 | 0.09 |
| 11 | Rev Stabil | -0.1824 | 0.64 | 0.23 | 0.65 | 0.92 | 0.92 | 0.00 | 0.26 | 0.70 | 0.87 | 0.74 | 0.70 | 0.76 | 0.74 |
| 12 | Avg Yrs to Sell | 0.1722 | 2.125 | 2.125 | 2.750 | 2.868 | 2.844 | 2.861 | 2.833 | 2.950 | 2.375 | 1.633 | 1.167 | 1.738 | 1.898 |
| 13 | Price Stability | 0.0037 | 58.6 | 51.0 | 24.6 | 10.5 | 22.0 | 17.0 | 18.0 | 30.0 | 23.7 | 35.0 | 42.4 | 32.1 | 21.0 |
| 14 | **Calculated Discount** | | **42.22%** | **42.37%** | **37.67%** | **23.65%** | **26.25%** | **26.57%** | **34.43%** | **30.97%** | **15.83%** | **20.27%** | **18.68%** | **15.20%** | **18.27%** |
| 15 | Actual Discount | | 31.40% | 44.80% | 46.30% | 16.00% | 41.10% | 15.90% | 24.50% | 32.50% | 12.50% | 18.80% | 23.10% | 19.30% | 15.80% |
| 16 | Error (Actual – Calculated) | | -10.82% | 2.43% | 8.63% | -7.65% | 14.85% | -10.67% | -9.93% | 1.53% | -3.33% | -1.47% | 4.42% | 4.10% | -2.47% |
| 17 | Error Squared | | 1.17% | 0.06% | 0.75% | 0.59% | 2.21% | 1.14% | 0.99% | 0.02% | 0.11% | 0.02% | 0.20% | 0.17% | 0.06% |
| 18 | Absolute Error | | 10.82% | 2.43% | 8.63% | 7.65% | 14.85% | 10.67% | 9.93% | 1.53% | 3.33% | 1.47% | 4.42% | 4.10% | 2.47% |
| 19 | Mean Error | | -0.80% | | | | | | | | | | | | |
| 20 | Mean Squared Error | | 0.57% | | | | | | | | | | | | |
| 21 | Mean Absolute Error | | 6.33% | | | | | | | | | | | | |

which model is better. At this point, it is probably best to use an average of the results of both models when determining a discount in a restricted stock valuation.

**Empirical versus Theoretical Black-Scholes** It is important to understand that in using the Black-Scholes option pricing model (BSOPM) put for calculating restricted stock discounts, we are using it as an empirical model, not as a theoretical model. That is because buying a put on a publicly traded stock does not "buy marketability" for the restricted stock.[66] Rather, it locks in a minimum price for the restricted shares once they become marketable, while allowing for theoretically unlimited price appreciation. Therefore issuing a hypothetical put on the freely tradable stock does not accomplish the same task as providing marketability for the restricted stock, but it does compensate for the downside risk on the restricted stock during its holding period.

BSOPM has some attributes that make it a successful predictor of restricted stock discounts; it is a better forecaster than the mean discount and does almost as well as the regression of the MPI data.

The reason for BSOPM's success is that its mathematics is compatible with the underlying variable—primarily volatility—that would tend to drive restricted stock discounts. It is logical that the more volatile the restricted stock, the larger the discount, and that volatility is the single most important determinant of BSOPM results. Therefore, BSOPM is a good candidate for empirically explaining restricted stock discounts, even though that is not the original intended use of the model, nor is this scenario part of the assumptions of the model.

**COMPARISON TO THE QUANTITATIVE MARKETABILITY DISCOUNT MODEL (QMDM)** Mercer shows various examples of "Investment Risk Premium" calculations (Mercer, 1997, chapter 10). When he adds this premium to the required return on a marketable minority basis, he gets the required holding period return for a nonmarketable minority interest. Judging from his example calculations of the risk premium for other types of illiquid interests, the "investment-specific risk premium" for restricted stocks should be somewhere in the range of 1.5–5% or less.[67] This is because restricted stocks have short and well-defined holding periods. Also, the payoff at the end of the holding period is almost sure to be at the marketable minority level.

To test the applicability of QMDM to restricted stocks, we first estimate a typical marketable minority–level required return. The MPI database average market capitalization is approximately $78 million. This puts the MPI stocks in the mid-cap to small-cap category, given the dates of the transactions in the database. A reasonable expected rate of return for stocks of this size is 15% or so on a marketable minority basis.

We will assume that the stocks, given their size, were probably not paying any significant dividends. Therefore the expected growth rate equals the expected rate of return at the marketable minority level of 15%. Given the average years to liquidity of approximately 2.5 years in the dataset, we can calculate a typical restricted stock discount using QMDM.

[66] I thank R. K. Hiatt for this observation.

[67] Actually, the lower end of the range—1.5%—appears most appropriate.

Assuming a 1.5% Investment Risk Premium, and therefore a required holding period return of 16.5%, QMDM would predict the following restricted stock discount:

$$\text{Min Discount} = 1 - (\text{FV} \times \text{PVF}) = 1 - \left(1.15^{2.5} \times \frac{1}{1.165^{2.5}}\right) = 3.2\%,$$

where FV = future value of the investment and PVF = the present value factor. With a 5% "Investment Risk Premium" we have:

$$\text{Max Discount} = 1 - (\text{FV} \times \text{PVF}) = 1 - \left(1.15^{2.5} \times \frac{1}{1.20^{2.5}}\right) = 10.1\%.$$

The QMDM forecasts of restricted stock discounts thus range from 3% to 10%, with the lower end of the range appearing most appropriate, considering the examples in Mercer's Chapter 10.[68] These calculated discounts are nowhere near the average discount of 27.1% in the MPI database. This sheds doubt on the applicability of QMDM for restricted stocks and the applicability of the model in general. At least, it shows that the model does not work well for small holding periods.

I invited Chris Mercer to write a rebuttal to my analysis of the QMDM results. His rebuttal is at the end of this chapter, just before the conclusion, after which I provide my comments, as I disagree with some of his methodology.

## Abrams' Economic Components Model

The remainder of this chapter will be spent on Abrams' *economic components model* (ECM). The ECM identifies three components of the discount for lack of marketability, with the third component itself having two parts. The discounts corresponding to each component are referred to as $D_1$, $D_2$, $D_{3A}$, and $D_{3B}$. Each of these components will be described later. The origins of this model appear in Abrams (1994a) (the "original article"). While the basic structure of the model is the same, this chapter contains major revisions of that article. One of the revisions is that for greater clarity and ease of exposition, components #2 and #3 have switched places. In the original article, transactions costs was component #2 and monopsony power to the buyer due to thin markets was component #3, but in this chapter they are reversed.

**COMPONENT #1: THE DELAY TO SALE** The first component of DLOM is the economic disadvantage of the considerable time that it takes to sell a privately held business in excess of the near-instantaneous ability to sell the publicly held stocks from which we calculate our discount rates.

**Psychology** Investors don't like illiquidity. Medical and other emergencies arise in life, causing people to have to sell their assets, possibly including their businesses.

[68]The QMDM restricted stock discount is insensitive to the absolute level of the discount rate. It is sensitive only to the premium above the discount rate. For example, changing the minimum discount formula to $1 - \left(1.20^{2.5} \times \frac{1}{1.215^{2.5}}\right)$ has little impact on the QMDM result. It is the 1.5% premium that is the difference between the 20% growth and the 21.5% required return that constitutes the bulk of the QMDM discount—and, of course, the holding period.

Even without the pressure of a "fire sale," it usually takes three to six months to sell a small business and one year or more to sell a business worth $1 million or more.

The selling process may entail "dressing up the business" (i.e., tidying up the accounting records), halting the standard operating procedures of charging personal expenses to the business, and getting an appraisal. Either during or after the dress-up stage, the seller needs to identify potential buyers or engage a business broker or investment banker to do so. This is also difficult, as the most likely buyers are often competitors. If the match doesn't work, the seller is worse off, having divulged confidential information to his competitors. The potential buyers need to go through their due diligence process, which is time-consuming and expensive.

During this long process, the seller is exposed to the market. He or she would like to sell immediately, and having to wait when one wants to sell right away tries one's patience. The business environment may be better or worse when the transaction is close to consummation. It is well established in behavioral science—and, parenthetically, it is the major principle on which the sale of insurance is based—that the fear of loss is stronger than the desire for gain (Tversky and Kahneman, 1987). This creates pressure for the seller to accept a lower price in order to get on with life.

Another important finding in behavioral science that is relevant in explaining DLOM and DLOC is *ambiguity aversion* (Einhorn and Hogarth, 1986). The authors cite a paradox proposed by the psychologist Daniel Ellsberg (Ellsberg, 1961), of *Pentagon Papers* fame, known as the *Ellsberg paradox*.

Ellsberg asked subjects which of two gambles they prefer. In gamble A, the subject draws from an urn with 100 balls in it. They are red or black only, but we don't know how many of each. It could be 100 black and 0 red, 0 black and 100 red, or anything in between. The subject calls "red" or "black" before the draw, and if he or she calls it right, wins $100; otherwise, he or she gets nothing. In gamble B, the subject draws one ball from an urn that has 50 red balls and 50 black balls. Again, if the subject forecasts the correct draw, he or she wins $100 and otherwise wins nothing.

Most people are indifferent between choosing red or black in both gambles. When asked which gamble they prefer, the majority of people had an interesting response (before we proceed, ask yourself which gamble you would prefer and why). Most people prefer to draw from urn #2. This is contrary to risk-neutral logic. The finding of Ellsberg and Einhorn and Hogarth is that people dislike ambiguity and will pay to avoid it.

Ambiguity is a second-order uncertainty. It is "uncertainty about uncertainties," and it exists pervasively in our lives. Gamble B has uncertainty, but it does not have ambiguity. The return-generating process is well understood. It is a clear 50–50 gamble. Gamble A, on the other hand, is fuzzier. The return-generating process is not well understood. People feel uncomfortable with that and will pay to avoid it.

It is my opinion that ambiguity aversion probably explains much of shareholder-level discounts. As mentioned earlier in the chapter, Jankowske mentions wealth-transfer opportunities and the protection of investment as economic benefits of control. Many minority investors are exposed to the harsh reality of having their wealth transferred away. Many of those who do not experience that still have to worry about it occurring in the future. The minority investor is always in a more ambiguous position than a control shareholder.

In our regressions of the Partnership Profiles database[69] that tracks the results of trading in the secondary limited partnership markets, we find that regular cash distributions are the primary determinant of discounts from net asset value. Why would this be so? After all, there have already been appraisals of the underlying properties, and those appraisals certainly included a discounted cash flow approach to valuation.[70] If the appraisal of the properties already is considered cash flow, then why would we consider cash flow again in determining discounts? I would speculate on the following three reasons:

1. If the general partner (GP) takes greater-than-arm's-length fees for managing the property, this would not be included in the appraisal of the whole properties and would reduce the value of the limited partner (LP) interest. It is a transfer of wealth from the LP to the GP.
2. Even if the GP takes an arm's-length management fee, he or she still determines the magnitude and the timing of the distributions, which may or may not be convenient for the individual LPs.
3. LPs may fear potential actions of the GP, even if he or she never takes those actions. The LPs know only that information about the investment that the GP discloses, and may fear what the GP does not divulge—which, of course, the LPs won't know. The LPs may hear rumors of good or bad news and not know what to do about them.

The bottom line is that investors don't like ignorance, and they will pay less for investments that are ambiguous than for ones that are not—or that are, at least, less ambiguous—even if both have the same expected value.

Our paradigm for valuation is the two-parameter normal distribution, where everything depends only on expected return and expected risk. Appraisers are used to thinking of risk only as either systematic risk, measured by $\beta$, or total risk in the form of $\sigma$, the historical standard deviation of returns. The research on ambiguity avoidance adds another dimension to our concept of risk, which makes our task more difficult but affords the possibility of being more realistic.

It is also noteworthy that the magnitude of special distributions, that is, those coming from a sale or refinancing of property, was statistically insignificant.[71] Investors care only about what they feel they can count on, the regular distributions.

**Black-Scholes Options Pricing Model** As previously mentioned, one method of modeling the economic disadvantage of the period of illiquidity is to use the Black-Scholes option pricing model (BSOPM) to calculate the value of a put on the stock for the period of illiquidity. A European put, the simplest type, is the right to sell the stock

[69]This appeared as Chapter 9 in the first edition of this book. We have removed it from this book, and it will appear on our Web site and eventually in the workbook that is planned to accompany the third edition of this book.

[70]In the regression, we included a dummy variable to determine whether the discount from net asset value depended on whether the properties were appraised by the general partner or by independent appraiser. The dummy variable was statistically insignificant, meaning that the market trusts the appraisals of the general partners as much as the independent appraisers.

[71]Since then, I occasionally find special distributions to be significant.

at a specific price on a specific day. An American put is the right to sell the stock on or before the specific day. We will be using the European put.

The origins of using this method go back to David Chaffe (Chaffe, 1993), who first proposed using the BSOPM for calculating restricted stock discounts for SEC Rule 144 restricted stock. The restricted stock discounts are for minority interests of publicly held firms. There is no admixture of minority interest discount in this number, as the restricted stock studies in Pratt's Chapter 15 (Pratt, Reilly, and Schweihs, 1996) are minority interests both pre- and post-transaction.

Then Abrams (1994a) suggested that owning a privately held business is similar to owning restricted stock in that it is very difficult to sell a private firm in less than the normal due diligence time discussed above. The BSOPM is a reasonable model with which to calculate component #1 of DLOM, the delay-to-sale discount.

There is disagreement in the profession about using BSOPM for this purpose. Chapter 14 of Mercer's book (Mercer, 1997) is entitled, "Why Not the Black-Scholes Options Pricing Model Rather Than the QMDM?" Mercer's key objections to the BSOPM are:[72]

1. It requires the standard deviation of returns as an input to the model. This input is not observable in privately held companies.
2. It is too abstract and complex to meaningfully represent the thinking of the hypothetical willing investor.

Argument 2 does not matter, as the success of the model is an empirical question. Argument 1, however, turned out to be truer than I would have imagined. It is true that we cannot see or measure return volatility in privately held firms. However, there are two ways that we indirectly measured it. We combined the regression equations from regressions #1 and #2 in Table 5.1[73] to develop an expression for return volatility as a function of log size, and we performed a regression of the same data to directly develop an expression for the same. We tried using both indirect estimates of volatility as inputs to the BSOPM to forecast the restricted stock discounts in the Management Planning, Inc. data, and both approaches performed worse than using the average discount. Thus, argument 1 was an assertion that turned out to be correct.

When volatility can be directly calculated, the BSOPM is superior to using the mean and the QMDM. So, BSOPM is a competent model for forecasting when we have firm-specific volatility data, which we will not have for privately held firms.

**Others Models of Component #1** The regression equation developed from the Management Planning, Inc. data is superior to both the non-firm-specific BSOPM and the QMDM. Thus, it is, so far, the best model to measure component #1, the delay-to-sale component, as long as the expected delay to sale is one to three years. It is probably good to extend for another half-year or 1 less than or greater than the 1-to-3-year range. However, the further we move away from our data range, the less reliable it is, as it is an extrapolation.

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[72]Chapter 14 is co-authored by J. Michael Julius and Matthew R. Crow, employees at Mercer Capital.

[73]Table 4-1 from the first edition of *QBV*.

The QMDM is pure present value analysis. It has no ability to quantify volatility—other than the analyst guessing at the premium to add to the discount rate. It also suffers from being highly subjective. None of the components of the risk premium at the shareholder level can be empirically measured in any way.

Is the QMDM useless? No. It may be the best model in some scenarios. As mentioned before, one of the limitations of my restricted stock discount regression is that because the restricted stocks had so little range in time to marketability, the regression equation performs poorly when the time to marketability is substantially outside that range—above four years or below one year. Not all models work in all situations. The QMDM has its place in the toolbox of the valuation professional. It is important to understand its limitations in addition to its strengths, which are its flexibility and simplicity.

The BSOPM is based on present value analysis, but contains far more heavy-duty mathematics to quantify the probable effects of volatility on investors' potential gains or losses. While the general BSOPM did not perform well when volatility was measured indirectly, we can see by looking at the regression results that Black-Scholes has the essence of "the right idea." Two of the variables in the regression analysis are earnings stability and revenue stability. They are the $R^2$ from regressions of earnings and revenues as dependent variables against time as the independent variable. In other words, the more stable the growth of revenues and earnings throughout time, the higher the earnings and revenue stability. These are measures of volatility of earnings and revenues, which are the volatilities underlying the volatility of returns. Price stability is another of the independent variables, and that is the standard deviation of stock price divided by the mean of returns (which is the coefficient of variation of price) and then multiplied by 100.

Thus, the regression results demonstrate that using volatility to measure restricted stock discounts is empirically sound. The failure of the non-firm-specific BSOPM to quantify restricted stock discounts is a measurement problem, not a theoretical problem.[74]

An important observation regarding the MPI data is that MPI excluded start-up and developmental firms from its study. There were no firms that had negative net income in the latest fiscal year. That may possibly explain the difference in results between the average 35% discounts in most of the other studies cited in Pratt's Chapter 15 (Pratt, Reilly, and Schweihs, 1996) and MPI's results. When using my regression of the MPI data to calculate component #1, the discount for the delay to sale, for a firm without positive earnings, I would make a subjective adjustment to increase the discount. As to magnitude, we have to make an assumption. If we assume that the other studies did contain restricted stock sales of firms with negative earnings in the latest fiscal year, then it would seem that those firms should have a higher discount than the average of that study. With the average of all of them being around 33–35%, let's say for the moment that the firms with losses may have averaged 38–40% discounts, all other things being equal (see the next paragraph

[74]There is a significant difference between forecasting volatility and forecasting returns. Returns do not exhibit statistically significant trends over time, whereas volatility does (see Chapter 5). Therefore it is not surprising that using long-term averages to forecast volatility fails in the BSOPM. The market is obviously more concerned about recent than historical volatility in pricing restricted stock. That is not true about returns.

for the rationale for a higher discount). Then 38–40% minus 27% in the MPI study would lead to an upward adjustment to component #1 of 11% to 13%. That all rests on an assumption that this is the only cause of the difference in the results of the two studies. Further research is needed on this topic.

We can see the reason why firms with losses would have averaged higher discounts than those that did not in the *x*-coefficient for earnings stability in regression #2 of Table 8.5, B52, which is −0.1381.[75] This regression tells us that the market does not like volatility in earnings, which implies that the market likes stability in earnings. Logically, the market would not like earnings to be stable and negative, so investors obviously prefer stable, positive earnings. Thus we can infer from regression #2 in Table 8.5 that, all other things being equal, the discount for firms with negative earnings in the prior year must be higher than for firms with positive earnings. Ideally, we will eventually have restricted stock data on firms that have negative earnings, and we can control for that by including earnings as a regression variable.

It is also worth noting that the regression analysis results are based on the database of transactions from which we developed the regression, while the BSOPM did not have that advantage. Thus, the regression had an inherent advantage in this dataset over all other models.

**Abrams' Regression of the Management Planning, Inc. Data** As mentioned earlier in the chapter, there are two regression equations in our analysis of the MPI data. The first one includes price stability as an independent variable. This is fine for doing restricted stock studies. However, it does not work for calculating component #1 in a DLOM calculation for the valuation of a privately held firm, whether a business or a family limited partnership with real estate. In both cases, there are no objective market stock prices with which to calculate the price stability. Therefore, in those types of assignments, we use the less accurate second regression equation that excludes price stability.

Table 8.10 is an example of using regression #2 to calculate component #1, the delay to sale of DLOM, for a privately held firm. Note that "Value of Block—Post Discount" (Table 8.10, A7) is analogous to "Shares Sold—$" (Table 8.5, A50), and "FMV—100% Marketable Minority Interest" (Table 8.10, A8) is analogous to "Market Capitalization" (Table 8.5, A51). The regression coefficients are in B5–B11. We insert the subject company data in C6–C11, except for C7, which we will discuss below.

Our subject company has $6 million in revenues (which, when squared, equals $3.6 \times 10^{13}$, (C6), a 100% marketable minority interest FMV of $5 million (C8, analogous to market capitalization for the public companies in the Management Planning, Inc. data), and earnings and revenue stability of 0.45 (C9) and 0.30 (C10), respectively.[76] We estimate it will take one year to sell the interest (C11).

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[75]In Table 8.5, regression #1, earnings stability also has a negative coefficient of –0.1038 (B24).

[76]We do not explicitly show the detail of the calculations of earnings and revenue stability. Our sample Restricted Stock Discount Study, which was in Chapter 8, Table 8-1 in the first edition and which will be moved to the workbook, shows these calculations.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Table 8.10** | | | |
| 2 | **Calculation of Component #1—Delay to Sale [1]** | | | |
| 3 | | | | |
| 4 | | **Coefficients** | **Subject Co. Data** | **Discount** |
| 5 | Intercept | 0.1292 | NA | 12.9% |
| 6 | Revenues$^2$ [2] | -5.39E-18 | 3.600E+13 | 0.0% |
| 7 | Value of Block—Post-Discount [3] | -4.39E-09 | $ 4,331,435 | -1.9% |
| 8 | FMV—100% Marketable Minority Interest | 6.10E-10 | $ 5,000,000 | 0.3% |
| 9 | Earnings Stability | -0.1381 | 0.4500 | -6.2% |
| 10 | Revenue Stability | -0.1800 | 0.3000 | -5.4% |
| 11 | Average Years to Sell | 0.1368 | 1.0000 | 13.7% |
| 12 | **Total Discount** | | | **13.4%** |
| 13 | | | | |
| 14 | Value of Block—Pre-Discount [4] | $ 5,000,000 | | |
| 15 | | | | |
| 16 | [1] Based on Abrams's Regression #2 of Management Planning, Inc. data. | | | |
| 17 | | | | |
| 18 | [2] Revenues$^2$ = $6,000,000$^2$ = (6 × 10$^6$)$^2$ = 3.6 × 10$^{13}$ | | | |
| 19 | | | | |
| 20 | [3] Equal to (Value of Block—Pre-Discount) × (1 – Discount). | | | |
| 21 | | | | |
| 22 | [4] Marketable minority interest FMV. | | | |

Since we are valuing 100% of the capital stock of the firm, the value of the block of stock also has an FMV of $5 million (B14) before DLOM.[77] The regression calls for the post-discount FMV, which means we must subtract the discount. The formula in C7 is: = B14 ∗ (1 − D12); that is, the post-discount FMV equals the pre-discount FMV × (1 − Discount). However, this is a simultaneous equation, since the discount and the shares sold in dollars each depend on the other. In order to be able to calculate this, your spreadsheet should be set to allow recalculation with multiple iterations. Otherwise, you will get an error message indicating a circular reference.[78] Column D is equal to column B × column C, except for the $y$-intercept in D5, which transfers directly from B5. Adding each of the components in column D, we obtain a forecast discount of 13.4% (D12).

**Limitations of the Regression** It is possible that there may be combinations of subject company data that can lead to strange results. This is especially true when:

1. The subject company data are near the end or outside of the ranges of data in the regression of the MPI data.
2. There is very little variation in the range of the "average-time-to-sale" variable in our set. Almost all of the restricted stock could be sold between two and three years from the transaction date, which is very little variation. Only four of the 53 sales were expected to take less than two years (see below).

[77] Had we been valuing a 10% block of stock, B14 would have been $500,000.

[78] If you create your own spreadsheet and make changes to the data, the simultaneous equation is fragile, and you might easily get error messages. When that happens, you must put a simple number in C7 (e.g., $200,000), allow the spreadsheet to "recalibrate" and come back to equilibrium, and then put in the correct formula. We do not have this iterative problem with the other components of DLOM.

3. The $R^2$ is low.
4. The standard error of the $y$-estimate is fairly high—10%.

Regarding number 1, 47 of the 53 restricted stock sales in the MPI database took place before the SEC circulated its Exposure Draft on June 27, 1995[79] to amend Rules 144(d) and (k) to shorten the waiting period for selling restricted stock to one year from two years and for nonaffiliated shareholders to sell shares without restriction after two years instead of three. Two sales took place in 1995 (Esmor Correctional Services, Inc. and Chantal Pharmaceuticals Corp.), after the SEC Exposure Draft, and four sales took place in 1996 (ARC Capital, Dense Pac Microsystems, Inc., Nobel Education Dynamics, Inc., and United Pharmaceuticals). That means the market knew there was some probability that this would become law and might shorten the waiting period to sell the restricted stock it was issuing; and the later the sale, the more likely it was at the time that the Exposure Draft would become law and provide relief to the buyer of the restricted stock.

Thus, we should expect that those sales would carry lower discounts than earlier sales—and that is correct. The discounts on the 1996 sales were significantly lower than discounts on the earlier sales, all other things being equal. The discounts ranged from 16% to 23% on the 1996 sales. However, the two post–Exposure Draft 1995 sales had higher-than-average discounts, which is somewhat counterintuitive. It is true that the 1996 sales would be more affected because the relief from restrictions for the 1995 sales was more likely to have lapsed from the passage of time than for the 1996 sales, if it took a long time for the Exposure Draft to become law. Nevertheless, the two 1995 sales remain anomalies.

The average years needed to sell the stock ranged from a low of 1.2 years for Dense Pac Microsystems to 2.96 years for Sudbury Holdings, Inc., with the vast majority being between two and three years. Extrapolating this model to forecast a restricted stock discount for a sale with a restriction of 10 years, for example, leads to ridiculous results, and even more than four years is very questionable.

The coefficient for average years to sell is 0.1368 (B11), which means that for each year more (less) than the forecast we made for this subject company, the discount increases (decreases) by 13.68%, holding all else constant. Thus, if we were to forecast for a 10-year restriction, we would get a discount of 136.8%—a nonsense result.

Thus, the appraiser must exercise good judgment and common sense in using these results. Mechanically using these regression formulas in all situations can be dangerous. It may be necessary to run other regressions with the same data, using different independent variables or different transformations of the data, to accommodate valuation assignments with facts that vary considerably with those underlying these data. Another possible solution is to assume, for example, that when a particular subject company's revenues squared is beyond the maximum in the MPI database, it is equal to the maximum in the MPI database. It is possible that

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[79] Revision of Holding Period Requirements in Rule 144; Section 16(a), "Reporting of Equity Swaps and Other Derivative Securities." File No. S7-17-95, SEC Release Nos. 33-7187; 34-35896; 17 CFR Parts 230 and 241; RIN 3235-AG53. The author expresses his gratitude to John Watson, Jr., Esq., of Latham & Watkins in Washington, D.C., for providing him with a copy of the exposure draft.

it may be necessary to use the other models (i.e., BSOPM with inferred rather than explicit standard deviations or the QMDM), for more extreme situations where the regression equation is strained by extreme data. Hopefully, we will soon have much more data, as there will be increasingly more transactions subject to the relaxed Rule 144 restrictions.

**COMPONENT #2: MONOPSONY POWER** The control stockholder of a privately held firm has no guarantee at all that he or she can sell his or her firm. The market for privately held businesses is very thin. Most small and medium-size firms are unlikely to attract more than a small handful of buyers—and even then probably not more than one or two every several months—while the seller of publicly traded stock has millions of potential buyers. Just as a monopolist is a single seller who can drive up price by withholding production, a single buyer—a monopsonist—can drive down price by withholding purchase.

The presence of 100 or even 10 interested buyers is likely to drive the selling price of a business to its theoretical maximum, that is, "the right price." The absence of enough buyers may confer monopsony power to the few who are interested. Therefore, a small, unexciting business will have an additional component of the discount for lack of marketability for the additional bargaining power accruing to the buyers in thin markets.

It is easy to think that component #2 may already be included in component #1; that is, they both derive from the difficulty in selling an illiquid asset, often leading to it taking a long time to sell or a steep discount to sell in a reasonable time.

To demonstrate that they are indeed distinct components and that we are not double-counting, it is helpful to consider the hypothetical case of a very exciting privately held firm that has just discovered the cure for cancer. Such a firm would have no lack of interested buyers, yet it still is very unlikely to be sold in less than one year. In that year, other things could happen. Congress could pass legislation regulating the medical breakthrough, and the value could decrease significantly. Therefore, it would still be necessary to have a significant discount for component #1, while component #2 would be zero. It may not take longer to sell the corner dry-cleaning store, but while the first firm is virtually guaranteed to be able to sell at the highest price after its required marketing time, the dry-cleaning store will have the additional uncertainty of sale, and its few buyers would have more negotiating power than the buyers of the firm with the cure for cancer.

The results from Schwert, described earlier in the chapter, are relevant here. He found that the presence of multiple bidders for control of publicly held companies on average led to increased premiums of 12.2% compared to takeovers without competitive bidding. Based on the regression in Table 4 of his article, we assumed a typical deal configuration that would apply to a privately held firm.[80] The premium without an auction was 21.5%. Adding 12.2%, the premium with an auction was 33.7%. To calculate the discount for lack of competition, we go in the other direction, that is, 12.2% divided by 1 plus 33.7% = $0.122/(1 + 0.337) = 9.1\%$, or approximately 9%. This is a useful benchmark for $D_2$.

[80]We assume a successful purchase, a tender offer, and a cash deal.

However, it is quite possible that $D_2$ for any subject interest should be larger or smaller than 9%. It all depends on the facts and circumstances of the situation. Using Schwert's measure of the effect of multiple versus single bidders as our estimate of $D_2$ may possibly have a downward bias in that the market for the underlying minority interests in the same firms is very deep. It is only the market for control of publicly held firms that is thin. The market for privately held firms is thin for whole firms and razor thin for minority interests, which can justify a higher discount than 9%. Nevertheless, I use 9% as my standard for component #2 and increase or decrease it occasionally when circumstances call for it.

**COMPONENT #3: TRANSACTIONS COSTS** Transaction costs in selling a privately held business are substantially more than they are for selling stock in publicly traded firms. Stock in publicly traded firms can be sold with a broker's fee of as little as $7. We are valuing only incremental transaction costs, that is, the cost of selling stock in a private firm minus the cost of selling a comparable interest in a public firm.

**Table 8.11: Quantifying Transaction Costs for Buyer and Seller** Table 8.11 shows estimates of transaction costs for both the buyer and the seller for the following categories: legal, accounting, and appraisal fees (the latter split into the post-transaction, tax-based appraisal for allocation of purchase price and/or valuation of in-process R&D, and the pre-transaction "deal appraisal" to help buyer and/or seller establish the right price), the opportunity cost of internal management spending its time on the sale rather than on other company business, and investment banking (or, for small sales, business broker) fees. The first five of the categories appear in columns B through F, which we subtotal in column G, and the investment banking fees appear in column H. The reason for segregating between the investment banking fees and all the others is that the others are constantly increasing as the deal size (FMV) decreases, while investment banking fees reach a maximum of 10% and stop increasing as the deal size decreases.[81]

Rows 6 through 9 are transaction costs estimates for the buyer, while rows 13 through 16 are for the seller. Note that the buyer does not pay the investment banking fees—only the seller pays them. Rows 20 through 23 are total fees for both sides.

Note that the subtotal transaction costs (column G) are inversely related to the size of the transaction. For the buyer, they are as low as 0.23% (G6) for a $1 billion transaction and as high as 5.7% (G9) for a $1 million transaction. We summarize the total in rows 27 through 30 and include the base 10 logarithm of the sales price as a variable for regression.[82] The purpose of the regression is to allow the reader to calculate estimated subtotal transaction costs for any size transaction. We then add the forecast investment banking costs to compute to total transaction costs.

The buyer regression equation is: Buyer Subtotal Transaction Cost/Price $= 0.1531 - (0.0173 \times \log_{10}$ Price). The regression coefficients are in B48 and B49.

[81]We do not show this difference in the table.

[82]Normally we use the natural logarithm for regression. Here we chose base 10 because the logs are whole numbers and are easy to understand. Ultimately, it makes no difference which one we use in the regression. The results are identical either way.

**Table 8.11**
**Estimates of Transaction Costs [1]**

| **Buyer** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Deal Size** | **Legal [2]** | **Acctg** | **Tax Appraisal** | **Deal Appraisal [3]** | **Internal Mgt [4]** | **Subtotal** | **Inv Bank** | **Total** |
| **$1 billion** | 0.10% | 0.02% | 0.02% | 0.00% | 0.09% | 0.23% | 0.00% | **0.23%** |
| **$100 million** | 1.00% | 0.10% | 0.06% | 0.00% | 0.16% | 1.32% | 0.00% | **1.32%** |
| **$10 million** | 1.50% | 0.23% | 0.20% | 0.00% | 0.25% | 2.18% | 0.00% | **2.18%** |
| **$1 million** | 4.00% | 0.30% | 0.70% | 0.00% | 0.70% | 5.70% | 0.00% | **5.70%** |

| **Seller** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Deal Size** | **Legal [2]** | **Acctg** | **Tax Appraisal** | **Deal Appraisal [3]** | **Internal Mgt [4]** | **Subtotal** | **Inv Bank** | **Total** |
| **$1 billion** | 0.10% | 0.01% | 0.00% | 0.02% | 0.05% | 0.18% | 0.75% | **0.93%** |
| **$100 million** | 1.00% | 0.05% | 0.00% | 0.05% | 0.10% | 1.20% | 1.10% | **2.30%** |
| **$10 million** | 1.50% | 0.08% | 0.00% | 0.20% | 0.15% | 1.93% | 2.75% | **4.68%** |
| **$1 million** | 4.00% | 0.10% | 0.00% | 0.75% | 0.42% | 5.27% | 10.00% | **15.27%** |

| **Total** | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| **Deal Size** | **Legal [2]** | **Acctg** | **Tax Appraisal** | **Deal Appraisal [3]** | **Internal Mgt [4]** | **Subtotal** | **Inv Bank** | **Total** |
| **$1 billion** | 0.20% | 0.03% | 0.02% | 0.02% | 0.14% | 0.41% | 0.75% | **1.16%** |
| **$100 million** | 2.00% | 0.15% | 0.06% | 0.05% | 0.26% | 2.52% | 1.10% | **3.62%** |
| **$10 million** | 3.00% | 0.30% | 0.20% | 0.20% | 0.40% | 4.10% | 2.75% | **6.85%** |
| **$1 million** | 8.00% | 0.40% | 0.70% | 0.75% | 1.12% | 10.97% | 10.00% | **20.97%** |

**Summary for Regression Analysis-Buyer**

| **Sales Price** | **$Log_{10}$ Price** | **Subtotal** |
|---|---|---|
| $ 1,000,000,000 | 9.0 | 0.23% |
| $ 100,000,000 | 8.0 | 1.32% |
| $ 10,000,000 | 7.0 | 2.18% |
| $ 1,000,000 | 6.0 | 5.70% |

**Summary for Regression Analysis-Seller**

| **Sales Price** | **$Log_{10}$ Price** | **Subtotal** |
|---|---|---|
| $ 1,000,000,000 | 9.0 | 0.18% |
| $ 100,000,000 | 8.0 | 1.20% |
| $ 10,000,000 | 7.0 | 1.93% |
| $ 1,000,000 | 6.0 | 5.27% |

**SUMMARY OUTPUT: Buyer Subtotal Fees as a Function of $Log_{10}$ FMV**

| *Regression Statistics* | |
|---|---|
| Multiple R | 0.941762404 |
| R Square | 0.886916425 |
| Adjusted R Square | 0.830374637 |
| Standard Error | 0.009751774 |
| Observations | 4 |

ANOVA

| | *df* | *SS* | *MS* | *F* | *Significance F* |
|---|---|---|---|---|---|
| Regression | 1 | 0.001491696 | 0.001491696 | 15.68603437 | 0.058237596 |
| Residual | 2 | 0.000190194 | 9.50971E-05 | | |
| Total | 3 | 0.00168189 | | | |

| | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | *Lower 95.0%* | *Upper 95.0%* |
|---|---|---|---|---|---|---|---|---|
| Intercept | 0.1531 | 0.033069874 | 4.629591246 | 0.043626277 | 0.010811717 | 0.295388283 | 0.010811717 | 0.295388283 |
| Log10 Price | -0.0172725 | 0.004361126 | -3.960559856 | 0.058237596 | -0.036036923 | 0.001491923 | -0.036036923 | 0.001491923 |

**SUMMARY OUTPUT: Seller Subtotal Fees as a Function of $Log_{10}$ FMV**

| *Regression Statistics* | |
|---|---|
| Multiple R | 0.936972245 |
| R Square | 0.877916988 |
| Adjusted R Square | 0.816875482 |
| Standard Error | 0.009430649 |
| Observations | 4 |

ANOVA

| | *df* | *SS* | *MS* | *F* | *Significance F* |
|---|---|---|---|---|---|
| Regression | 1 | 0.00127912 | 0.00127912 | 14.38229564 | 0.063027755 |
| Residual | 2 | 0.000177874 | 8.89371E-05 | | |
| Total | 3 | 0.001456994 | | | |

| | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
|---|---|---|---|---|---|---|
| Intercept | 0.14139 | 0.031980886 | 4.421078333 | 0.04754262 | 0.00378726 | 0.27899274 |
| Log10 Price | -0.0159945 | 0.004217514 | -3.792399721 | 0.063027755 | -0.034141012 | 0.002152012 |

**Sample Forecast of Transactions Costs for $5 Million Subject Company:**

| | **FMV** | **$log_{10}$ FMV** | **X-Coeff.** | **log FMV × Coef** | **Regr. Constant** | **Forcst Subtotal** | **Inv Bank [5]** | **Forecast Total** |
|---|---|---|---|---|---|---|---|---|
| **Buyer** | $ 5,000,000 | 6.698970004 | -0.0172725 | -0.115707959 | 0.1531 | **3.7%** | 0.0% | **3.7%** |
| **Seller** | $ 5,000,000 | 6.698970004 | -0.0159945 | -0.107146676 | 0.14139 | **3.4%** | 5.0% | **8.4%** |

**Notes:**

[1] Based on interviews with investment banker Gordon Gregory, attorney David Boatwright, Esq., and Douglas Obenshain, CPA. Costs include buy and sell side. These are estimates of average costs. Actual costs vary with the complexity of the transaction.

[2] Legal fees will vary with the complexity of the transaction. An extremely complex $1 billion sale could have legal fees of as much as $5 million each for the buyer and the seller, though this is rare. Complexity increases with stock deals (or asset deals with a very large number of assets), seller "carries paper," contingent payments, escrow, tax-free (which is treated as a pooling-of-interests), etc.

[3] We are assuming the seller pays for the deal appraisal. Individual sales may vary. Sometimes both sides hire a single appraiser and split the fees, and sometimes each side has its own appraiser.

[4] Internal management costs are the most speculative of all. We estimate 6,000 hours (3 people full time for 1 year) at an average $150/hr. internal cost for the $1 billion sale, 2,000 hours @ $80 for the $100 million sale, 500 hours at $50 for the $10 million sale, and 200 hours @$35 for the $1 million sale for the buyer, and 60% of that for the seller. Actual results may vary considerably from these estimates.

[5] Ideally calculated by another regression, but this is sight-estimated. Can often use the Lehman Bros. Formula—5% for 1st $1 million, 4%, for 2nd, etc., leveling off at 1% for each $1 million.

The adjusted $R^2$ is 83% (B37), which is a good result. The standard error of the $y$-estimate is 0.9% (B38), so the 95% confidence interval around the estimate is approximately two standard errors, or ±1.8%—a very good result.

The seller regression equation is: Seller Subtotal Transaction Cost/Price = $0.1414 - (0.01599 \times \log_{10}$ Price). The regression coefficients are in B67 and B68. The adjusted $R^2$ is 82% (B56), which is also a good result. The standard error of the $y$-estimate is also 0.9% (B57), which gives us the same confidence intervals around the $y$-estimate of ±1.8%.

Rows 73 and 74 show a sample calculation of transaction costs for the buyer and seller, respectively. We estimate FMV before discounts for our subject company of \$5 million (B73, B74). The base 10 logarithm of 5 million is 6.69897 (C73, C74).[83] In D73 and D74, we insert the $x$-coefficient from the regressions, which are −0.0172725 (from B49) for the buyer and −0.0159945 (from B68) for the seller. Column C × column D = column E. F73 and F74 are repetitions of the regression constants from B48 and B67, respectively. We then add column E to column F to obtain the forecast subtotal transaction costs in G73 and G74. Finally, we add in investment banking fees of 5%[84] for the seller (the buyer doesn't pay for the investment banker or business broker) to arrive at totals of 3.7% (I73) and 8.4% (I74) for the buyer and seller, respectively.

**Component #3 Is Different from #1 and #2** Component #3, transactions costs, is different from the first two components of DLOM. For component #3, we need to explicitly calculate the present value of the occurrence of transactions costs every time the company sells. The reason is that, unlike the first two components, transactions costs are actually out-of-pocket costs that "leave the system."[85] They are paid to attorneys, accountants, appraisers, and investment bankers or business brokers. Additionally, internal management of both the buyer and the seller spend significant time on the sale to make it happen, and they often have to spend time on failed acquisitions before being successful.

We also need to distinguish between the buyer's transactions costs and the seller's costs. The reason for this is that the buyer's transactions costs are always relevant, whereas the seller's transaction costs for the immediate transaction reduce the net proceeds to the seller, but do not reduce FMV. However, before the buyer is willing to buy, he or she should be saying, "It's true, I don't care about the seller's costs. That's his or her problem. However, 10 years or so down the road when it's my turn to be the seller and I face those costs, I do care about that. To the extent that seller's costs exceed the brokerage cost of selling publicly traded stock, in 10 years my buyer will pay me less because of those costs, and therefore I must pay my seller less because of my costs as a seller in year 10. Additionally, the process goes on forever, because in year 20, my buyer becomes a seller and faces the same

[83]In other words, $10^{6.69897} = 5$ million.

[84]We could run another regression to forecast investment banking fees. This was sight estimated. One could also use a formula such as the Lehman Brothers formula to forecast investment banking fees.

[85]I thank R. K. Hiatt for the brilliant insight that the first two components of DLOM do not have this characteristic and thus do not require this additional present value calculation.

problem." Thus, we need to quantify the present value of a periodic perpetuity of buyer's transactions costs beginning with the immediate sale and sellers' transactions costs that begin with the second sale of the business.[86] In the next section, we will develop the mathematics necessary to do this.[87]

**Developing Formulas to Calculate DLOM Component #3** This section contains some difficult mathematics. Do not panic! Ultimately, we will arrive at some very usable formulas that are reasonably easy to use. It is not necessary to follow all of the mathematics that gets us there, but it is worthwhile to skim through the math to get a feel for what it means. In the Mathematical Appendix, we develop the following formulas step by step. In order to avoid presenting volumes of burdensome math in the body of the chapter, we present only "occasional snapshots" of the math—just enough to present the conclusions and convey some of the logic behind it.

For simplicity, suppose that, on average, business owners hold the business for 10 years and then sell. Every time an owner sells, he or she incurs a transaction cost of $z$—normally a percentage of the pre-discounted value. The net present value (NPV) of the cash flows to the business owner is:[88]

$$NPV = NPV_{1-10} + (1 - z)NPV_{11-\infty}. \tag{8.1}$$

Equation (8.1) states that the NPV of cash flows at year 0 to the owner is the sum of the NPV of the first 10 years' cash flows and $(1 - z)$ times the NPV of all cash flows from year 11 to infinity. Note that this excludes the seller's transaction costs for the first sale. If transaction costs are 10% every time a business sells, then $z = 10\%$ and $1 - z = 90\%$.[89] The first owner would have 10 years of cash flows undiminished by transaction costs and then pay transaction costs of 10% of the NPV at year 10 of all future cash flows.

The second owner operates the business for 10 years and then sells at year 20. He or she pays transaction costs of $z$ at year 20. The NPV of cash flows to the second owner is:

$$NPV_{11-\infty} = NPV_{11-20} + (1 - z)NPV_{21-\infty}. \tag{8.2}$$

Substituting (8.2) into equation (8.1), the NPV of cash flows to the first owner is:

$$NPV = NPV_{1-10} + (1 - z)\left[NPV_{11-20} + (1 - z)NPV_{21-\infty}\right]. \tag{8.3}$$

[86]One might think that the buyers' transactions costs are not relevant the first time because the buyer has to put in due diligence time whether or not a transaction results. In individual instances, that is true, but in the aggregate, if buyers would not receive compensation for their due diligence time, they would cease to buy private firms until the prices declined enough to compensate them.

[87]Because we are calculating the present value of the costs for more than one transaction, we use the plural term *transactions costs*.

[88]Read the en dash in the following equation's subscript text as the word *to*, that is, the NPV from one time period to another.

[89]$z$ is actually an incremental transaction cost, as we will explain later in the chapter.

This expression simplifies to:

$$NPV = NPV_{1-10} + (1-z)NPV_{11-20} + (1-z)^2 NPV_{21-\infty}. \tag{8.4}$$

We can continue on in this fashion *ad infinitum*. The final expression for NPV is:

$$NPV = \sum_{i=1}^{\infty} (1-z)^{i-1} NPV_{[10(i-1)+1]-10i}. \tag{8.5}$$

The NPV is a geometric sequence. Using a Gordon model, that is, assuming constant, perpetual growth, in the Mathematical Appendix, we show that equation (8.5) solves to:

$$NPV_{TC} = \frac{\sqrt{1+r}}{r-g}\left\{\frac{1-\left(\frac{1+g}{1+r}\right)^{10}}{1-\left[(1-z)\left(\frac{1+g}{1+r}\right)^{10}\right]}\right\}, \tag{8.6}$$

where $NPV_{TC}$ is the NPV of the cash flows with the NPV of the transactions costs (TC) that occur every 10 years removed, $g$ is the constant growth rate of cash flows, $r$ is the discount rate, and cash flows are midyear.[90] The end-of-year formula is the same, replacing the $\sqrt{1+r}$ in the numerator with the number 1.

The NPV of the cash flows without removing the NPV of transactions costs every 10 years is simply the Gordon model multiple of $\frac{\sqrt{1+r}}{r-g}$, which is identical with the first term on the right-hand side of equation (8.6). You can see this by setting $z$ to 0 and observing that the second term on the right-hand size simplifies to 1. The discount for this component is equal to:

$$D_{3B} = 1 - \frac{NPV_{TC}}{NPV}. \tag{8.7}$$

The fraction in equation (8.7) is simply the term in the large braces in equation (8.6). Thus, $D_{3B}$ simplifies to:

$$D_{3B} = 1 - \left\{\frac{1-\left(\frac{1+g}{1+r}\right)^{10}}{1-\left[(1-z)\left(\frac{1+g}{1+r}\right)^{10}\right]}\right\} = 1 - \frac{1-x^{10}}{1-(1-z)\,x^{10}}, \tag{8.8}$$

where $x = \frac{1+g}{1+r}$, $D$ is the discount, and $g < r$, which implies that $0 < x < 1$.[91]

Equation (8.8) is the formula for the discount assuming a sale every 10 years. Instead of assuming a business sale every 10 years, now we let the average years

[90] This appears as equation (A8.7) in the Mathematical Appendix.

[91] This is identical with equation (A8.10) in the Mathematical Appendix.

between sales be a variable, $j$, which leads to the generalized equation in (8.9) for sellers' transactions costs:[92]

$$D_{3B} = 1 - \left\{ \frac{1 - \left(\frac{1+g}{1+r}\right)^j}{1 - \left[(1-z)\left(\frac{1+g}{1+r}\right)^j\right]} \right\} = 1 - \frac{1-x^j}{1-(1-z)\,x^j} \quad \text{Generalized discount}$$

$\times$ formula—sellers' transactions costs. (8.9)

Using an end-of-year Gordon model assumption instead of midyear cash flows leads to the identical equation; equation (8.9) holds for both.

Analysis of partial derivatives in the Mathematical Appendix shows that the discount (i.e., $D_{3B}$), is always increasing with increases in growth ($g$) and transaction costs ($z$) and is always decreasing with increases in the discount rate ($r$) and the average number of years between sales ($j$). The converse is true as well. Decreases in the independent variables have opposite effects on $D_{3B}$ as increases do.

Equation (8.9) is the appropriate formula to use for quantifying the sellers'[93] transactions costs because it ignores the seller's transaction cost on the first sale, as discussed above. The appropriate formula for quantifying the buyers' transactions costs incorporates an initial transaction cost at time zero instead of at $t = j$. With this assumption, we would modify the above analysis by changing the $(1 - z)^{i-1}$ to $(1 - z)^i$ in equation (8.5). The transaction equivalent formula of equation (8.9) for buyers' transactions costs is:[94]

$$D_{3A} = 1 - \frac{(1-z)(1-x^j)}{1-(1-z)\,x^j} \quad \text{Generalized DLOM formula—buyers' transactions costs.} \tag{8.9a}$$

Obviously, equation (8.9a), which assumes an immediate sale, results in much larger discounts than equation (8.9), where the first sale occurs $j$ years later. Equation (8.9) constitutes the discount appropriate for sellers' transactions costs, while equation (8.9a) constitutes the discount appropriate for buyers' transactions costs. Thus, component #3 splits into #3A and #3B because we must use different formulas to value them.[95]

**A Simplified Example of Sellers' Transactions Costs** Because appraisers are used to automatically assuming that all sellers' costs merely reduce the net proceeds to the seller but have no impact on the fair market value, the concept of periodic sellers'

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[92]This is identical with equation (A8.11) in the Mathematical Appendix. Note that we use the plural possessive here, because we are speaking about an infinite continuum of sellers (and buyers).

[93]Note that we have shifted from speaking in the singular about the first seller to the plural in speaking about the entire continuum of sellers throughout infinite time. We will make the same shift in language with the buyers as well.

[94]This is identical with equation (A8.11A) in the Mathematical Appendix.

[95]It is not that buyers and sellers sit around and develop equations like (8.9) and (8.9a) and run them on their spreadsheets before making deals. One might think this complexity is silly, because real-life buyers and sellers don't do this. However, we are merely attempting to economically model their combination of ideal rationality and intuition.

costs that do affect FMV is potentially very confusing. Let's look at a very simplified example to make the concept clear.

Consider a business that will sell once at $t = 0$ for \$1,000 and once at $t =$ 10 years for \$1,500, after which the owner will run the company and eventually liquidate it. For simplicity, we will ignore buyers' transactions costs. We can model the thinking of the first buyer (at $t = 0$), as follows: "When I eventually sell in year 10, I'll have to pay a business broker 10% × \$1,500 = \$150. If I were selling publicly traded stock, I would have paid a broker's fee of 2%[96] on the \$1,500, or \$30, so the difference is \$120."

Assuming a 25% discount rate, the present value factor is 0.1074, and \$120 × 0.1074 = \$12.88 today. On a price of \$1,000, the excess transactions costs from my eventual sale are 1.288%, or approximately 1.3%. Again, it is only the incremental transactions costs that we count in our calculation of DLOM components #3A and #3B.

Formulas (8.9) and (8.9a) extend this logic to cover the infinite continuum of transactions every 10 years (or every $j$ years).

**Tables 8.12 and 8.13: Demonstrating the Accuracy of Formulas (8.9) and (8.9a)** Tables 8.12 and 8.13 demonstrate the accuracy of equations (8.9) and (8.9a), respectively. The two tables have identical structure and logic, so we will cover both of them by explaining Table 8.12.

Column A shows 100 years of cash flows. While the formulas presume perpetuities, the present value effect is so small that there is no relevant present value after year 100.

The assumptions of the model are: The discount rate is 20% (B112), the perpetual growth rate is 5% (B113), sellers' transaction costs = $z$ = 12% (B114), $x = \frac{1+g}{1+r} = \frac{1.05}{1.2} = 0.875$ (B115), and $j$, the average years between sales of the business, equals 10 years (B116).

In B7, we begin with \$1.00 of forecast cash flow in year 1. The cash flow grows at a rate of $g = 5\%$. Thus, every cash flow in column B from rows 8 to 106 equals 1.05 times the number above it. Column C is the present value factor assuming midyear cash flows at a discount rate of 20%. Column D, the present value of cash flows, equals column B × column C.

Column E is the factor that tells us how much of the cash flows from each year effectively accrues to the original owner after removing the seller's transaction costs. The buyer does not care about the seller's transaction costs, so only future sellers' transactions costs count in this calculation. In other words, the buyer cares about the transaction costs that he or she will face in 10 years when he or she sells the business. In turn, he or she knows that his or her own buyer eventually becomes a seller. Therefore, each 10 years, or more generally, each $j$ years, the cash flows that accrue to the original owner decline by a multiple of $(1 - z)$. The formula is $(1 - z)^{Int(Yr-1)}$.

Thus, for the first 10 years, 100% = 1.0000 (E7–E16) of the cash flows with respect to sellers' transaction costs remain with the original owner. For the next

[96]Ten years later, in this second edition of the book, 2% is excessive. However, the principle matters, not the specific amount, so we leave it alone.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 8.12** | | | | | | |
| 2 | **Proof of Equation (8.9)** | | | | | | |
| 3 | | | | | | | |
| 4 | | | | | **(1–*z*)^Int(Yr–1)** | **Post Tx** | |
| 5 | | **Cash** | | **PV Cash** | **=Post-Trans** | **PV Cash** | |
| 6 | **Year** | **Flow** | **PVF** | **Flow** | **Costs** | **Flow** | |
| 7 | 1 | 1.0000 | 0.912871 | 0.912871 | 1.0000 | 0.9128709 | |
| 8 | 2 | 1.0500 | 0.760726 | 0.798762 | 1.0000 | 0.7987621 | |
| 9 | 3 | 1.1025 | 0.633938 | 0.698917 | 1.0000 | 0.6989168 | |
| 10 | 4 | 1.1576 | 0.528282 | 0.611552 | 1.0000 | 0.6115522 | |
| 11 | 5 | 1.2155 | 0.440235 | 0.535108 | 1.0000 | 0.5351082 | |
| 12 | 6 | 1.2763 | 0.366862 | 0.468220 | 1.0000 | 0.4682197 | |
| 13 | 7 | 1.3401 | 0.305719 | 0.409692 | 1.0000 | 0.4096922 | |
| 14 | 8 | 1.4071 | 0.254766 | 0.358481 | 1.0000 | 0.3584807 | |
| 15 | 9 | 1.4775 | 0.212305 | 0.313671 | 1.0000 | 0.3136706 | |
| 16 | 10 | 1.5513 | 0.176921 | 0.274462 | 1.0000 | 0.2744618 | |
| 17 | 11 | 1.6289 | 0.147434 | 0.240154 | 0.8800 | 0.2113356 | |
| 18 | 12 | 1.7103 | 0.122861 | 0.210135 | 0.8800 | 0.1849186 | |
| 19 | 13 | 1.7959 | 0.102385 | 0.183868 | 0.8800 | 0.1618038 | |
| 20 | 14 | 1.8856 | 0.08532 | 0.160884 | 0.8800 | 0.1415783 | |
| 21 | 15 | 1.9799 | 0.0711 | 0.140774 | 0.8800 | 0.1238810 | |
| 22 | 16 | 2.0789 | 0.05925 | 0.123177 | 0.8800 | 0.1083959 | |
| 23 | 17 | 2.1829 | 0.049375 | 0.107780 | 0.8800 | 0.0948464 | |
| 24 | 18 | 2.2920 | 0.041146 | 0.094308 | 0.8800 | 0.0829906 | |
| 25 | 19 | 2.4066 | 0.034288 | 0.082519 | 0.8800 | 0.0726168 | |
| 26 | 20 | 2.5270 | 0.028574 | 0.072204 | 0.8800 | 0.0635397 | |
| 27 | 21 | 2.6533 | 0.023811 | 0.063179 | 0.7744 | 0.0489256 | |
| 28 | 22 | 2.7860 | 0.019843 | 0.055281 | 0.7744 | 0.0428099 | |
| 29 | 23 | 2.9253 | 0.016536 | 0.048371 | 0.7744 | 0.0374586 | |
| 30 | 24 | 3.0715 | 0.01378 | 0.042325 | 0.7744 | 0.0327763 | |
| 31 | 25 | 3.2251 | 0.011483 | 0.037034 | 0.7744 | 0.0286793 | |
| 32 | 26 | 3.3864 | 0.009569 | 0.032405 | 0.7744 | 0.0250944 | |
| 33 | 27 | 3.5557 | 0.007974 | 0.028354 | 0.7744 | 0.0219576 | |
| 34 | 28 | 3.7335 | 0.006645 | 0.024810 | 0.7744 | 0.0192129 | |
| 35 | 29 | 3.9201 | 0.005538 | 0.021709 | 0.7744 | 0.0168113 | |
| 36 | 30 | 4.1161 | 0.004615 | 0.018995 | 0.7744 | 0.0147099 | |
| 37 | 31 | 4.3219 | 0.003846 | 0.016621 | 0.6815 | 0.0113266 | |
| 38 | 32 | 4.5380 | 0.003205 | 0.014543 | 0.6815 | 0.0099108 | |
| 39 | 33 | 4.7649 | 0.002671 | 0.012725 | 0.6815 | 0.0086719 | |
| 40 | 34 | 5.0032 | 0.002226 | 0.011135 | 0.6815 | 0.0075879 | |
| 41 | 35 | 5.2533 | 0.001855 | 0.009743 | 0.6815 | 0.0066394 | |
| 42 | 36 | 5.5160 | 0.001545 | 0.008525 | 0.6815 | 0.0058095 | |
| 43 | 37 | 5.7918 | 0.001288 | 0.007459 | 0.6815 | 0.0050833 | |
| 44 | 38 | 6.0814 | 0.001073 | 0.006527 | 0.6815 | 0.0044479 | |
| 45 | 39 | 6.3855 | 0.000894 | 0.005711 | 0.6815 | 0.0038919 | |
| 46 | 40 | 6.7048 | 0.000745 | 0.004997 | 0.6815 | 0.0034054 | |
| 47 | 41 | 7.0400 | 0.000621 | 0.004373 | 0.5997 | 0.0026222 | |
| 48 | 42 | 7.3920 | 0.000518 | 0.003826 | 0.5997 | 0.0022944 | |
| 49 | 43 | 7.7616 | 0.000431 | 0.003348 | 0.5997 | 0.0020076 | |
| 50 | 44 | 8.1497 | 0.000359 | 0.002929 | 0.5997 | 0.0017567 | |
| 51 | 45 | 8.5572 | 0.0003 | 0.002563 | 0.5997 | 0.0015371 | |
| 52 | 46 | 8.9850 | 0.00025 | 0.002243 | 0.5997 | 0.0013449 | |
| 53 | 47 | 9.4343 | 0.000208 | 0.001962 | 0.5997 | 0.0011768 | |
| 54 | 48 | 9.9060 | 0.000173 | 0.001717 | 0.5997 | 0.0010297 | |
| 55 | 49 | 10.4013 | 0.000144 | 0.001502 | 0.5997 | 0.0009010 | |
| 56 | 50 | 10.9213 | 0.00012 | 0.001315 | 0.5997 | 0.0007884 | |
| 57 | 51 | 11.4674 | 0.0001 | 0.001150 | 0.5277 | 0.0006071 | |
| 58 | 52 | 12.0408 | 8.36E-05 | 0.001007 | 0.5277 | 0.0005312 | |
| 59 | 53 | 12.6428 | 6.97E-05 | 0.000881 | 0.5277 | 0.0004648 | |
| 60 | 54 | 13.2749 | 5.81E-05 | 0.000771 | 0.5277 | 0.0004067 | |
| 61 | 55 | 13.9387 | 4.84E-05 | 0.000674 | 0.5277 | 0.0003558 | |
| 62 | 56 | 14.6356 | 4.03E-05 | 0.000590 | 0.5277 | 0.0003114 | |
| 63 | 57 | 15.3674 | 3.36E-05 | 0.000516 | 0.5277 | 0.0002724 | |
| 64 | 58 | 16.1358 | 2.8E-05 | 0.000452 | 0.5277 | 0.0002384 | |
| 65 | 59 | 16.9426 | 2.33E-05 | 0.000395 | 0.5277 | 0.0002086 | |

## Table 8.12 (cont.)
## Proof of Equation (8.9)

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 4 | | | | | (1–$z$)^Int(Yr–1) | Post Tx | |
| 5 | | Cash | | PV Cash | =Post-Trans | PV Cash | |
| 6 | Year | Flow | PVF | Flow | Costs | Flow | |
| 66 | 60 | 17.7897 | 1.94E-05 | 0.000346 | 0.5277 | 0.0001825 | |
| 67 | 61 | 18.6792 | 1.62E-05 | 0.000303 | 0.4644 | 0.0001405 | |
| 68 | 62 | 19.6131 | 1.35E-05 | 0.000265 | 0.4644 | 0.0001230 | |
| 69 | 63 | 20.5938 | 1.13E-05 | 0.000232 | 0.4644 | 0.0001076 | |
| 70 | 64 | 21.6235 | 9.38E-06 | 0.000203 | 0.4644 | 0.0000941 | |
| 71 | 65 | 22.7047 | 7.81E-06 | 0.000177 | 0.4644 | 0.0000824 | |
| 72 | 66 | 23.8399 | 6.51E-06 | 0.000155 | 0.4644 | 0.0000721 | |
| 73 | 67 | 25.0319 | 5.43E-06 | 0.000136 | 0.4644 | 0.0000631 | |
| 74 | 68 | 26.2835 | 4.52E-06 | 0.000119 | 0.4644 | 0.0000552 | |
| 75 | 69 | 27.5977 | 3.77E-06 | 0.000104 | 0.4644 | 0.0000483 | |
| 76 | 70 | 28.9775 | 3.14E-06 | 0.000091 | 0.4644 | 0.0000423 | |
| 77 | 71 | 30.4264 | 2.62E-06 | 0.000080 | 0.4087 | 0.0000325 | |
| 78 | 72 | 31.9477 | 2.18E-06 | 0.000070 | 0.4087 | 0.0000285 | |
| 79 | 73 | 33.5451 | 1.82E-06 | 0.000061 | 0.4087 | 0.0000249 | |
| 80 | 74 | 35.2224 | 1.51E-06 | 0.000053 | 0.4087 | 0.0000218 | |
| 81 | 75 | 36.9835 | 1.26E-06 | 0.000047 | 0.4087 | 0.0000191 | |
| 82 | 76 | 38.8327 | 1.05E-06 | 0.000041 | 0.4087 | 0.0000167 | |
| 83 | 77 | 40.7743 | 8.76E-07 | 0.000036 | 0.4087 | 0.0000146 | |
| 84 | 78 | 42.8130 | 7.3E-07 | 0.000031 | 0.4087 | 0.0000128 | |
| 85 | 79 | 44.9537 | 6.09E-07 | 0.000027 | 0.4087 | 0.0000112 | |
| 86 | 80 | 47.2014 | 5.07E-07 | 0.000024 | 0.4087 | 0.0000098 | |
| 87 | 81 | 49.5614 | 4.23E-07 | 0.000021 | 0.3596 | 0.0000075 | |
| 88 | 82 | 52.0395 | 3.52E-07 | 0.000018 | 0.3596 | 0.0000066 | |
| 89 | 83 | 54.6415 | 2.93E-07 | 0.000016 | 0.3596 | 0.0000058 | |
| 90 | 84 | 57.3736 | 2.45E-07 | 0.000014 | 0.3596 | 0.0000050 | |
| 91 | 85 | 60.2422 | 2.04E-07 | 0.000012 | 0.3596 | 0.0000044 | |
| 92 | 86 | 63.2544 | 1.7E-07 | 0.000011 | 0.3596 | 0.0000039 | |
| 93 | 87 | 66.4171 | 1.42E-07 | 0.000009 | 0.3596 | 0.0000034 | |
| 94 | 88 | 69.7379 | 1.18E-07 | 0.000008 | 0.3596 | 0.0000030 | |
| 95 | 89 | 73.2248 | 9.83E-08 | 0.000007 | 0.3596 | 0.0000026 | |
| 96 | 90 | 76.8861 | 8.19E-08 | 0.000006 | 0.3596 | 0.0000023 | |
| 97 | 91 | 80.7304 | 6.82E-08 | 0.000006 | 0.3165 | 0.0000017 | |
| 98 | 92 | 84.7669 | 5.69E-08 | 0.000005 | 0.3165 | 0.0000015 | |
| 99 | 93 | 89.0052 | 4.74E-08 | 0.000004 | 0.3165 | 0.0000013 | |
| 100 | 94 | 93.4555 | 3.95E-08 | 0.000004 | 0.3165 | 0.0000012 | |
| 101 | 95 | 98.1283 | 3.29E-08 | 0.000003 | 0.3165 | 0.0000010 | |
| 102 | 96 | 103.0347 | 2.74E-08 | 0.000003 | 0.3165 | 0.0000009 | |
| 103 | 97 | 108.1864 | 2.29E-08 | 0.000002 | 0.3165 | 0.0000008 | |
| 104 | 98 | 113.5957 | 1.9E-08 | 0.000002 | 0.3165 | 0.0000007 | |
| 105 | 99 | 119.2755 | 1.59E-08 | 0.000002 | 0.3165 | 0.0000006 | |
| 106 | 100 | 125.2393 | 1.32E-08 | 0.000002 | 0.3165 | 0.0000005 | |
| 107 | **Totals** | | | **$ 7.3030** | | **$ 7.0030** | |
| 108 | **Discount = 1 – (F107/D107)** | | | | | **4.1%** | |
| 109 | **Discount—By Formula [1]** | | | | | **4.1%** | |

| | A | B |
|---|---|---|
| 111 | **Parameters** | |
| 112 | ***r*** | **20%** |
| 113 | ***g*** | **5%** |
| 114 | ***z*** | **12%** |
| 115 | **$x$=(1+$g$)/(1+$r$)** | **87.50%** |
| 116 | **$j$=yrs to sale** | **10** |

**Sensitivity Analysis**

| | D | E | F | G |
|---|---|---|---|---|
| 112 | | **Avg Yrs between Sales** | | |
| 113 | | **8** | **10** | **12** |
| 114 | **18%** | 7.2% | 5.1% | 3.8% |
| 115 | **20%** | 5.9% | **4.1%** | 2.9% |
| 116 | **22%** | 4.9% | 3.3% | 2.3% |

**[1] Formula for Discount: 1–((1–$x$^$j$)/((1–(1–$z$)*$x$^$j$)))**

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | | | **Table 8.13** | | | | |
| 2 | | | **Proof of Equation (8.9a)** | | | | |
| 3 | | | | | | | |
| 4 | | | | | **(1–*z*)^Int(Yr–1)** | **Post Tx** | |
| 5 | | **Cash** | | **PV Cash** | **=Post-Trans** | **PV Cash** | |
| 6 | **Year** | **Flow** | **PVF** | **Flow** | **Costs** | **Flow** | |
| 7 | 1 | 1.0000 | 0.912871 | 0.912871 | 0.8800 | 0.8033264 | |
| 8 | 2 | 1.0500 | 0.760726 | 0.798762 | 0.8800 | 0.7029106 | |
| 9 | 3 | 1.1025 | 0.633938 | 0.698917 | 0.8800 | 0.6150468 | |
| 10 | 4 | 1.1576 | 0.528282 | 0.611552 | 0.8800 | 0.5381659 | |
| 11 | 5 | 1.2155 | 0.440235 | 0.535108 | 0.8800 | 0.4708952 | |
| 12 | 6 | 1.2763 | 0.366862 | 0.468220 | 0.8800 | 0.4120333 | |
| 13 | 7 | 1.3401 | 0.305719 | 0.409692 | 0.8800 | 0.3605291 | |
| 14 | 8 | 1.4071 | 0.254766 | 0.358481 | 0.8800 | 0.3154630 | |
| 15 | 9 | 1.4775 | 0.212305 | 0.313671 | 0.8800 | 0.2760301 | |
| 16 | 10 | 1.5513 | 0.176921 | 0.274462 | 0.8800 | 0.2415264 | |
| 17 | 11 | 1.6289 | 0.147434 | 0.240154 | 0.7744 | 0.1859753 | |
| 18 | 12 | 1.7103 | 0.122861 | 0.210135 | 0.7744 | 0.1627284 | |
| 19 | 13 | 1.7959 | 0.102385 | 0.183868 | 0.7744 | 0.1423873 | |
| 20 | 14 | 1.8856 | 0.08532 | 0.160884 | 0.7744 | 0.1245889 | |
| 21 | 15 | 1.9799 | 0.0711 | 0.140774 | 0.7744 | 0.1090153 | |
| 22 | 16 | 2.0789 | 0.05925 | 0.123177 | 0.7744 | 0.0953884 | |
| 23 | 17 | 2.1829 | 0.049375 | 0.107780 | 0.7744 | 0.0834648 | |
| 24 | 18 | 2.2920 | 0.041146 | 0.094308 | 0.7744 | 0.0730317 | |
| 25 | 19 | 2.4066 | 0.034288 | 0.082519 | 0.7744 | 0.0639028 | |
| 26 | 20 | 2.5270 | 0.028574 | 0.072204 | 0.7744 | 0.0559149 | |
| 27 | 21 | 2.6533 | 0.023811 | 0.063179 | 0.6815 | 0.0430545 | |
| 28 | 22 | 2.7860 | 0.019843 | 0.055281 | 0.6815 | 0.0376727 | |
| 29 | 23 | 2.9253 | 0.016536 | 0.048371 | 0.6815 | 0.0329636 | |
| 30 | 24 | 3.0715 | 0.01378 | 0.042325 | 0.6815 | 0.0288431 | |
| 31 | 25 | 3.2251 | 0.011483 | 0.037034 | 0.6815 | 0.0252378 | |
| 32 | 26 | 3.3864 | 0.009569 | 0.032405 | 0.6815 | 0.0220830 | |
| 33 | 27 | 3.5557 | 0.007974 | 0.028354 | 0.6815 | 0.0193227 | |
| 34 | 28 | 3.7335 | 0.006645 | 0.024810 | 0.6815 | 0.0169073 | |
| 35 | 29 | 3.9201 | 0.005538 | 0.021709 | 0.6815 | 0.0147939 | |
| 36 | 30 | 4.1161 | 0.004615 | 0.018995 | 0.6815 | 0.0129447 | |
| 37 | 31 | 4.3219 | 0.003846 | 0.016621 | 0.5997 | 0.0099674 | |
| 38 | 32 | 4.5380 | 0.003205 | 0.014543 | 0.5997 | 0.0087215 | |
| 39 | 33 | 4.7649 | 0.002671 | 0.012725 | 0.5997 | 0.0076313 | |
| 40 | 34 | 5.0032 | 0.002226 | 0.011135 | 0.5997 | 0.0066774 | |
| 41 | 35 | 5.2533 | 0.001855 | 0.009743 | 0.5997 | 0.0058427 | |
| 42 | 36 | 5.5160 | 0.001545 | 0.008525 | 0.5997 | 0.0051124 | |
| 43 | 37 | 5.7918 | 0.001288 | 0.007459 | 0.5997 | 0.0044733 | |
| 44 | 38 | 6.0814 | 0.001073 | 0.006527 | 0.5997 | 0.0039142 | |
| 45 | 39 | 6.3855 | 0.000894 | 0.005711 | 0.5997 | 0.0034249 | |
| 46 | 40 | 6.7048 | 0.000745 | 0.004997 | 0.5997 | 0.0029968 | |
| 47 | 41 | 7.0400 | 0.000621 | 0.004373 | 0.5277 | 0.0023075 | |
| 48 | 42 | 7.3920 | 0.000518 | 0.003826 | 0.5277 | 0.0020191 | |
| 49 | 43 | 7.7616 | 0.000431 | 0.003348 | 0.5277 | 0.0017667 | |
| 50 | 44 | 8.1497 | 0.000359 | 0.002929 | 0.5277 | 0.0015459 | |
| 51 | 45 | 8.5572 | 0.0003 | 0.002563 | 0.5277 | 0.0013526 | |
| 52 | 46 | 8.9850 | 0.00025 | 0.002243 | 0.5277 | 0.0011835 | |
| 53 | 47 | 9.4343 | 0.000208 | 0.001962 | 0.5277 | 0.0010356 | |
| 54 | 48 | 9.9060 | 0.000173 | 0.001717 | 0.5277 | 0.0009062 | |
| 55 | 49 | 10.4013 | 0.000144 | 0.001502 | 0.5277 | 0.0007929 | |
| 56 | 50 | 10.9213 | 0.00012 | 0.001315 | 0.5277 | 0.0006938 | |
| 57 | 51 | 11.4674 | 0.0001 | 0.001150 | 0.4644 | 0.0005342 | |
| 58 | 52 | 12.0408 | 8.36E-05 | 0.001007 | 0.4644 | 0.0004674 | |
| 59 | 53 | 12.6428 | 6.97E-05 | 0.000881 | 0.4644 | 0.0004090 | |
| 60 | 54 | 13.2749 | 5.81E-05 | 0.000771 | 0.4644 | 0.0003579 | |
| 61 | 55 | 13.9387 | 4.84E-05 | 0.000674 | 0.4644 | 0.0003131 | |
| 62 | 56 | 14.6356 | 4.03E-05 | 0.000590 | 0.4644 | 0.0002740 | |
| 63 | 57 | 15.3674 | 3.36E-05 | 0.000516 | 0.4644 | 0.0002397 | |
| 64 | 58 | 16.1358 | 2.8E-05 | 0.000452 | 0.4644 | 0.0002098 | |
| 65 | 59 | 16.9426 | 2.33E-05 | 0.000395 | 0.4644 | 0.0001836 | |

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 8.13 (cont.)** | | | | | | |
| 2 | **Proof of Equation (8.9a)** | | | | | | |
| 3 | | | | | | | |
| 4 | | | | | **(1–*z*)^Int(Yr–1)** | **Post Tx** | |
| 5 | | **Cash** | | **PV Cash** | **=Post-Trans** | **PV Cash** | |
| 6 | **Year** | **Flow** | **PVF** | **Flow** | **Costs** | **Flow** | |
| 66 | 60 | 17.7897 | 1.94E-05 | 0.000346 | 0.4644 | 0.0001606 | |
| 67 | 61 | 18.6792 | 1.62E-05 | 0.000303 | 0.4087 | 0.0001237 | |
| 68 | 62 | 19.6131 | 1.35E-05 | 0.000265 | 0.4087 | 0.0001082 | |
| 69 | 63 | 20.5938 | 1.13E-05 | 0.000232 | 0.4087 | 0.0000947 | |
| 70 | 64 | 21.6235 | 9.38E-06 | 0.000203 | 0.4087 | 0.0000829 | |
| 71 | 65 | 22.7047 | 7.81E-06 | 0.000177 | 0.4087 | 0.0000725 | |
| 72 | 66 | 23.8399 | 6.51E-06 | 0.000155 | 0.4087 | 0.0000634 | |
| 73 | 67 | 25.0319 | 5.43E-06 | 0.000136 | 0.4087 | 0.0000555 | |
| 74 | 68 | 26.2835 | 4.52E-06 | 0.000119 | 0.4087 | 0.0000486 | |
| 75 | 69 | 27.5977 | 3.77E-06 | 0.000104 | 0.4087 | 0.0000425 | |
| 76 | 70 | 28.9775 | 3.14E-06 | 0.000091 | 0.4087 | 0.0000372 | |
| 77 | 71 | 30.4264 | 2.62E-06 | 0.000080 | 0.3596 | 0.0000286 | |
| 78 | 72 | 31.9477 | 2.18E-06 | 0.000070 | 0.3596 | 0.0000251 | |
| 79 | 73 | 33.5451 | 1.82E-06 | 0.000061 | 0.3596 | 0.0000219 | |
| 80 | 74 | 35.2224 | 1.51E-06 | 0.000053 | 0.3596 | 0.0000192 | |
| 81 | 75 | 36.9835 | 1.26E-06 | 0.000047 | 0.3596 | 0.0000168 | |
| 82 | 76 | 38.8327 | 1.05E-06 | 0.000041 | 0.3596 | 0.0000147 | |
| 83 | 77 | 40.7743 | 8.76E-07 | 0.000036 | 0.3596 | 0.0000128 | |
| 84 | 78 | 42.8130 | 7.3E-07 | 0.000031 | 0.3596 | 0.0000112 | |
| 85 | 79 | 44.9537 | 6.09E-07 | 0.000027 | 0.3596 | 0.0000098 | |
| 86 | 80 | 47.2014 | 5.07E-07 | 0.000024 | 0.3596 | 0.0000086 | |
| 87 | 81 | 49.5614 | 4.23E-07 | 0.000021 | 0.3165 | 0.0000066 | |
| 88 | 82 | 52.0395 | 3.52E-07 | 0.000018 | 0.3165 | 0.0000058 | |
| 89 | 83 | 54.6415 | 2.93E-07 | 0.000016 | 0.3165 | 0.0000051 | |
| 90 | 84 | 57.3736 | 2.45E-07 | 0.000014 | 0.3165 | 0.0000044 | |
| 91 | 85 | 60.2422 | 2.04E-07 | 0.000012 | 0.3165 | 0.0000039 | |
| 92 | 86 | 63.2544 | 1.7E-07 | 0.000011 | 0.3165 | 0.0000034 | |
| 93 | 87 | 66.4171 | 1.42E-07 | 0.000009 | 0.3165 | 0.0000030 | |
| 94 | 88 | 69.7379 | 1.18E-07 | 0.000008 | 0.3165 | 0.0000026 | |
| 95 | 89 | 73.2248 | 9.83E-08 | 0.000007 | 0.3165 | 0.0000023 | |
| 96 | 90 | 76.8861 | 8.19E-08 | 0.000006 | 0.3165 | 0.0000020 | |
| 97 | 91 | 80.7304 | 6.82E-08 | 0.000006 | 0.2785 | 0.0000015 | |
| 98 | 92 | 84.7669 | 5.69E-08 | 0.000005 | 0.2785 | 0.0000013 | |
| 99 | 93 | 89.0052 | 4.74E-08 | 0.000004 | 0.2785 | 0.0000012 | |
| 100 | 94 | 93.4555 | 3.95E-08 | 0.000004 | 0.2785 | 0.0000010 | |
| 101 | 95 | 98.1283 | 3.29E-08 | 0.000003 | 0.2785 | 0.0000009 | |
| 102 | 96 | 103.0347 | 2.74E-08 | 0.000003 | 0.2785 | 0.0000008 | |
| 103 | 97 | 108.1864 | 2.29E-08 | 0.000002 | 0.2785 | 0.0000007 | |
| 104 | 98 | 113.5957 | 1.9E-08 | 0.000002 | 0.2785 | 0.0000006 | |
| 105 | 99 | 119.2755 | 1.59E-08 | 0.000002 | 0.2785 | 0.0000005 | |
| 106 | 100 | 125.2393 | 1.32E-08 | 0.000002 | 0.2785 | 0.0000005 | |
| 107 | **Totals** | | | **$ 7.3030** | | **$ 6.1626** | |
| 108 | **Discount = 1 – (F107/D107)** | | | | | **15.6%** | |
| 109 | **Discount—By Formula [1]** | | | | | **15.6%** | |
| 110 | | | | | | | |
| 111 | **Parameters** | | | | **Sensitivity Analysis** | | |
| 112 | ***r*** | **20%** | | | **Avg Yrs between Sales** | | |
| 113 | ***g*** | **5%** | | | **8** | **10** | **12** |
| 114 | ***z*** | **12%** | | **18%** | 18.3% | 16.5% | 15.3% |
| 115 | ***x*=(1+*g*)/(1+*r*)** | **87.50%** | | **20%** | 17.2% | **15.6%** | 14.6% |
| 116 | ***j*=yrs to sale** | **10** | | **22%** | 16.3% | 14.9% | 14.0% |
| 117 | | | | | | | |
| 118 | **[1] Formula for Discount: 1–((1–*z*)*(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)))** | | | | | | |

10 years, years 11–20, the original owner's cash flows are reduced to $(1 - z) = 88\%$ (E17–E26) of the entire cash flow, with the 12% being lost as sellers' transaction costs to the second buyer. For years 21–30, the original owner loses another 12% of the remaining value to transaction costs for the third buyer, so the value that remains is $(1 - z)^2 = (1 - 0.12)^2 = 0.88^2 = 0.7744$ (E27–E36). This continues in the same pattern *ad infinitum*.

Column F is the post-transaction costs present value of cash flows, which is column D × column E. Thus D17 × E17 = 0.240154 × 0.8800 = 0.2113356 (F17). We sum the first 100 years' cash flows in F107, which equals $7.0030. In other words, the present value of post-transaction costs cash flows to the original owner of the business is $7.003. However, the present value of the cash flows without removing transactions costs is $7.3030 (D107). In F108, we calculate the discount as 1 − (F107/D108) = 1 − ($7.0030/$7.3030) = 4.1%.

In F109, we present the calculations according to equation (8.9), and it, too, equals 4.1%. Thus we have demonstrated that equation (8.9) is accurate.

Table 8.13 is identical to Table 8.12, except that it demonstrates the accuracy of equation (8.9a), which is the formula appropriate for buyers' transactions costs. Buyers care about their own transaction costs from the outset. Therefore, the continuum of buyers' transaction costs begins immediately. Thus, E7 to E16 equal 0.88 in Table 8.13, while they were equal to 1.00 in Table 8.12.

The discount in Table 8.13 is considerably larger—15.6%, which we calculate in F108 using the "brute force" method and in F109 using equation (8.9a). The spreadsheet formula appears in A118, as it also does in Table 8.12. Table 8.13 thus demonstrates the accuracy of equation (8.9a).

**Value Remaining Formula and the Total Discount** The fraction in Equation (8.9) is the percentage of value that remains after removing the perpetuity of transactions costs. Equation (8.10) shows the equation for the value remaining, denoted as $VR_{3B}$.

$$VR_{3B} = \frac{1 - x^j}{1 - (1 - z)\,x^j} \quad \text{Value remaining formula after subtracting sellers' costs.} \tag{8.10}$$

We can multiply all four value remaining figures for each of the four components—counting #3A and #3B as two components—and the result is the value remaining for the firm overall. The final discount is then 1 minus the value remaining for the firm overall.

Next, we will demonstrate the final calculation of DLOM.

**TABLE 8.14: SAMPLE CALCULATION OF DLOM** Table 8.14 is an example of calculating DLOM for a privately held firm with a $5 million (E18) FMV on a marketable minority basis. Column B is the pure discount of each component as calculated according to the methodology in the previous tables. Component #1, the discount due to the delay to sale, is equal to 13.4% (B9), which comes from Table 8.10, D12.

Component #2, monopsony power to the buyer, equals 9% (B10), per our discussion of Schwert's article earlier in this chapter. Component #3A, buyers' transactions costs, equals 3.7% (Table 8.11, I73) for private buyers, minus the approximately 1% brokerage fee to buy a $5 million interest in publicly traded stocks = 2.7% (B11). Component #3B, sellers' transactions costs, equals 8.4% (Table 8.11, I74) for private

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 8.14** | | | | | | |
| 2 | | | | | | | |
| 3 | **Sample Calculation of DLOM** | | | | | | |
| 4 | **Section 1: Calculation of the Discount for Lack of Marketability** | | | | | | |
| 5 | | | | | | | |
| 6 | | | | **= 1 – Col. [C]** | | | |
| 7 | | **Pure Discount** | **PV of Perpetual** | **Remaining** | | | |
| 8 | **Component** | **= *z* [1]** | **Discount [2]** | **Value** | | | |
| 9 | 1 | 13.4% | 13.4% | 86.6% | Delay to Sale—1 Yr (Table 8.10, D12) | | |
| 10 | 2 | 9.0% | 9.0% | 91.0% | Buyer's Monopsony Power—Thin Markets | | |
| 11 | 3A | 2.7% | 3.6% | 96.4% | Transactions Costs—Buyers | | |
| 12 | 3B | 7.4% | 2.4% | 97.6% | Transactions Costs—Sellers | | |
| 13 | Percent Remaining | | | 76.9% | Total % Remaining = Components 1 × 2 × 3A × 3B | | |
| 14 | **Final Discount** | | | **23.1%** | Discount = 1 – Total % Remaining | | |
| 15 | | | | | | | |
| 16 | **Section 2: Assumptions and Intermediate Calculations:** | | | | | | |
| 17 | | | | | | | |
| 18 | **FMV—Equity of Co. (before Discounts)** | | | | **$ 5,000,000** | | |
| 19 | **Discount Rate = *r* [3]** | | | | **23.0%** | | |
| 20 | **Constant Growth Rate = *g*** | | | | **7.0%** | | |
| 21 | **Intermediate Calculation: *x* = (1+*g*) / (1+*r*)** | | | | **0.8699** | | |
| 22 | **Avg # Years between Sales = *j*** | | | | **10** | | |
| 23 | | | | | | | |
| 24 | **Section 3: Sensitivity Analysis** | | | | | | |
| 25 | | | | | | | |
| 26 | | ***j* = Average Years between Sales** | | | | | |
| 27 | *j* = | **5** | **10** | **15** | **20** | | |
| 28 | Discount | 26.6% | **23.1%** | 22.0% | 21.6% | | |
| 29 | | | | | | | |
| 30 | | | | | | | |
| 31 | [1] Pure Discounts: For Component #1, Table 8.10, cell D12; for Component #2, 9% per Schwert article. For | | | | | | |
| 32 | Component #3A and #3B, Table 8.11, cells I73 and I74 – 1% for public brokerage costs. | | | | | | |
| 33 | | | | | | | |
| 34 | [2] PV of Perpetual Discount Formula: 1–(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9), used for Component #3B. | | | | | | |
| 35 | PV of Perpetual Discount Formula: 1–(1–*z*)*(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9a), used for Component #3A. | | | | | | |
| 36 | Components #1 and #2 simply transfer the pure discount. | | | | | | |
| 37 | | | | | | | |
| 38 | [3] The formula is 0.4622 – (0.01436 ln FMV) – 0.0080, based on Table 5.1, 2nd regression, adjusted for estimated | | | | | | |
| 39 | arithmetic mean yearly PE growth of 0.80% (Table 5.3, B32). | | | | | | |

buyers minus the approximately 1% brokerage fee to buy publicly traded stocks = 7.4% (B12). The reason that we subtract stock market transactions costs from the private market transactions costs is that we are using public market values as our basis of comparison (i.e., our point of reference).

Column C is the present value of the perpetual discount, which means that for components #3A and #3B, we quantify the infinite periodic transactions costs. Using equations (8.9a) for the buyers and (8.9) for the sellers, the 2.7% (B11) pure discount for buyers results in a net present value of buyers' transactions costs of 3.6% (C11), and the 7.4% (B12) pure discount for sellers results in a net present value of sellers' transactions costs of 2.4% (C12). Again, that excludes the seller's costs on the assumed sale to the hypothetical buyer at $t = 0$. The first two components, as mentioned earlier, do not repeat through time, so their perpetual discount is equal to their pure discount. Thus, C9 = B9, and C10 = B10.

Column D is the remaining value after subtracting the perpetual discount column from one, that is, column D = 1 − column C. We multiply D9 × D10 × D11 × D12 = D13 = 76.9%. The Final Discount is 1 − Remaining Value = 1 − 76.9% (D13) = 23.1% (D14).

The sensitivity analysis in section 3, row 28 of the table shows how the final discount varies with different assumptions of $j$ = the average number of years between sales. At $j = 10$ years, it appears that DLOM is more sensitive to reducing $j$

than increasing it. At $j = 5$, the discount increased from 23.1% (at $j = 10$) to 26.6% (B28), whereas it dropped only slightly for $j = 15$ and 20 to 22.0% (D28) and 21.6% (E28), respectively.

**EVIDENCE FROM THE INSTITUTE OF BUSINESS APPRAISERS** In Chapter 9, we examine data published by Raymond Miles, founder of the Institute of Business Appraisers (IBA) and apply log size discount rates and the DLOM calculations in this chapter to determine how well they explain price/earnings multiples of real-world sales of small businesses. The evidence in Chapter 9 is that within an order of magnitude, the log size model and the economic components model of DLOM perform well. Unfortunately there are a lot of data in the IBA and other databases that we need in order to be more precise. The lack of these data (e.g., forecast or at least historical growth rates) and the magnitude of personal expenses charged to the business force us to make estimates. There is too much estimating due to missing data for us to forcefully claim that the combination of the log size model, control premiums, DLOM, and DLOC as presented in this book is *the* solution to all valuation problems. Also, one could achieve the great results in Chapter 9 with other valuation assumptions. So Chapter 9 is evidence that we are in the ballpark, but it is far from proof that this is precise.

## Mercer's Rebuttal

I invited Chris Mercer to provide his rebuttal to my criticisms of the QMDM. His rebuttal follows immediately, after which I provide my counterpoints.[97]

**IS THE QUANTITATIVE MARKETABILITY DISCOUNT MODEL FLAWED?—RESPONSE BY Z. CHRISTOPHER MERCER, ASA, CFA** Jay Abrams attempts to "test the applicability" of the quantitative marketability discount model in his new book. In so doing, he uses the approximate range of "investment specific risk premiums" used in *Quantifying Marketability Discounts* of 1.5% to 5.0% and calculates, using an assumed expected rate of return for a non–dividend paying stock, implied marketability discounts over an assumed 2.5-year holding period.

He refers to the Management Planning, Inc. restricted stock study (published as Chapter 12 of *Quantifying Marketability Discounts*), and assumes a "reasonable expected rate of return for stocks of this size" of about 15% on a marketable minority interest basis. The expected rate of return is then used as a proxy for the *expected growth rate in value* factor used in the QMDM. In Abrams' calculations, there is, therefore, no differential between the required rate of return of potential investors and the expected growth rate in value.

His analysis then calculates minimum (3.2%) and maximum (10.1%) discounts based on investor-specific risk premiums in the range of 1.5% to 5.0%. Since these discounts are lower than those developed in the appraisals summarized in Chapter 10 of *Quantifying Marketability Discounts* (summarized in Table 8.18 below) and with discounts generally developed by other appraisers, Mr. Abrams suggests that the

[97] In the first edition of this book, he and I went back and forth in two complete rounds of rebuttals, and I stopped the process after his third round. We reproduce that in the following section. Note that we have adapted the table numbering to that of this chapter.

QMDM is flawed. While he has some other criticisms of *Quantifying Marketability Discounts*, I will attempt to address the threshold question he raises in this reply.

In a recent article revisiting the QMDM, I addressed Mr. Abrams' question, along with several others that have been raised since the publication of *Quantifying Marketability Discounts* in 1997.[98] The following is an excerpt from that article explaining why I believe Mr. Abrams' analysis is incorrect. I thank Mr. Abrams for this opportunity to address his criticisms.

**Expected Growth and Expected Returns** In many real-life valuation situations, there is a discrepancy between the rate of return (discount rate) implied in the valuation *of an enterprise* and the expected returns attributable to minority investors of that enterprise. There can be many sources of these differentials, several of which were noted above [in the text of the article leading to this point].

In most cases in which the QMDM is applied, there is a differential between the expected growth rate in value assumed and the required holding period return (discount rate) applied. This differential is the primary source of discounting using the QMDM. Several of my colleagues have pointed to this aspect of the QMDM. Their comments range from: (1) Mercer's Bermuda Triangle of disappearing value; to (2) there should be no difference at all; to (3) using the range of specific illiquidity discounts used in Chapter 10 of *Quantifying Marketability Discounts* (roughly 1.5% to 5.0% or so), when applied to the base equity discount rate (as a proxy for the expected growth rate), should yield much smaller marketability discounts than implied by the QMDM. Note that the essence of this third criticism [which is Mr. Abrams' criticism] is that the *differential* between the expected growth rate in value and the discount rate used would be only 1.5% to 5.0% or so in this case.

The criticisms seem to reflect a lack of understanding of the conceptual workings of the QMDM and a lack of familiarity with its consistency with existing empirical research. We can rely on market evidence from the various restricted stock studies to support the need for a differential in the expected growth rate and the required holding period return (discount) rate. The implications of two recent restricted stock studies are illustrated next, followed by a similar analysis of actual appraisals using the QMDM.

The Management Planning Study, "Analysis of Restricted Stocks of Public Companies (1980–1995)," was published, with permission of Management Planning, Inc. ("MPI"), as Chapter 12 of *Quantifying Marketability Discounts*. The median and average restricted stock discounts in the MPI study were 27.7% and 28.9%, respectively. For this analysis, we will round the average to 30%.[99] We can further assume that the typical expected holding period before the restrictions of Rule 144 were lifted was on the order of 2.5 years or two years plus a reasonable period to sell the shares into the market.

A recently published study by Bruce A. Johnson, ASA focusing on transactions in the 1991–1995 timeframe yields a smaller average restricted stock discount of

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[98] Z. Christopher Mercer, "Revisiting the Quantitative Marketability Discount Model," *Valuation Strategies,* March/April 2000.

[99] The average of the averages of the 10 restricted stock studies discussed in Chapters 2 and 12 of *Quantifying Marketability Discounts* is 31%.

**Table 8.15**

Using the MPI Study 30% Average Discount

| | | |
|---|---|---|
| Assumed Market Price of Public Entity | | $1.00 |
| Average Management Planning Discount (rounded) | 30.0% | ($0.30) |
| Assumed Purchase Price of Restricted Shares | | $0.70 |
| Holding Period until Restricted Shares Are Freely Tradable (years) | | 2.5 |

20%.[100] We will consider the implications of the Johnson study using a shorter two-year holding period (versus the MPI average of a 30% average discount and a 2.5-year holding period). Tables 8.15 and 8.16 use the MPI study and Table 8.17 uses the Johnson study to illustrate the differential between the expected growth of public companies and the discount rate imbedded in their average restricted stock pricing.

Now, we can examine a variety of assumptions about the "average" restricted stock transaction in the Management Planning study.[101] The average public price has been indexed to $1.00 per share. As a result, the average restricted stock transaction price, as indexed, is $0.70 per share.

We can estimate the implied returns that were required by investors in restricted stocks based on a variety of assumptions about the expected growth rates in value (or, the expected returns of the publicly traded stocks). For purposes of this analysis, we have assumed that the consensus expectations for the public stock returns were somewhere in the range of 0% (no expected appreciation) to 30% compounded. The most relevant portion of this range likely begins at about 10%, since stocks expected to appreciate less than that were probably not attractive for investments in their restricted shares. See Table 8.16.

Note that the implied holding period returns for the restricted stock transactions, on average, ranged from about 27% per year compounded (with value growing at 10%) to 50% per year compounded (with expected growth of 30%). As noted in Chapter 8 of *Quantifying Marketability Discounts*, the implied returns are in the range of expected venture capital returns for initial investments (not *average* venture capital returns, which include unsuccessful investments). Interestingly, the *differential* between the implied holding period returns above and the expected growth rate in values used [is] quite high, ranging from 15.3% to 20.0%.

This analysis is ex post. We do not know how the actual investment decisions were made in the transactions included in the Management Planning study or any of the restricted stock studies. But, ex post, it is clear that the investors in the "average" restricted stock transactions were, ex ante, either: (1) placing very high discount rates on their restricted stock transactions (ranging from 15% to 20% in excess of the expected returns of the public companies they were investing in); (2) questioning the consensus expectations for returns; or (3) some combination of 1 and 2.

[100]Bruce A. Johnson, "Quantitative Support for Discounts for Lack of Marketability," *Business Valuation Review*, December 1999, pp. 152–155.

[101]This analysis is for purposes of illustration only. Chapters 2 and 3 of *Quantifying Marketability Discounts* raise significant questions about reliance on averages of widely varying transactions indications for both the restricted stock and the pre-IPO studies.

**Table 8.16**

Using the MPI Study 30% Average Discount

| Assumed Expected Growth in Value (G) | Expected Future Value in 2.5 Years | Implied Return for Holding Period (R) | Annualized Incremental Return Attributable to Restricted Stock Discount ( R - G ) |
|---|---|---|---|
| 0% | $1.00 | 15.3% | 15.3% |
| 5% | $1.13 | 21.1% | 16.1% |
| 10% | $1.27 | 26.9% | 16.9% |
| 15% | $1.42 | 32.7% | 17.7% |
| 20% | $1.58 | 38.5% | 18.5% |
| 25% | $1.75 | 44.3% | 19.3% |
| 30% | $1.93 | 50.0% | 20.0% |

The Johnson study cited above focused on transactions in the 1991–1995 timeframe when the Rule 144 restriction period was still two years in length. If we assume an index price of $0.80 per share ($1.00 per share freely tradable price less the 20% average discount) and a holding period of two years (and instant liquidity thereafter) and replicate our analysis of Table 8.16, we obtain the result in Table 8.17.

Even with a shortened assumed holding period and a smaller average restricted stock discount, the implied required returns for the Johnson study are in the range of 23% to 45% for companies assumed to be growing at 10% to 30% per year. And the average differential between this calculated discount rate and the expected growth rate of the investment companies is in the range of 13.0% to 15.3%.

**Table 8.17**[1]

Using the Johnson Study 20% Average Discount

| Assumed Expected Growth in Value (G) | Expected Future Value in 2.5 Years | Implied Return for Holding Period (R) | Annualized Incremental Return Attributable to Restricted Stock Discount ( R - G ) |
|---|---|---|---|
| 0% | $1.00 | 15.3% | 15.3% |
| 5% | $1.13 | 21.1% | 16.1% |
| 10% | $1.27 | 26.9% | 16.9% |
| 15% | $1.42 | 32.7% | 17.7% |
| 20% | $1.58 | 38.5% | 18.5% |
| 25% | $1.75 | 44.3% | 19.3% |
| 30% | $1.93 | 50.0% | 20.0% |

[1] Source: Z. Christopher Mercer. This table does not match that which appeared in the first edition of *Quantitative Business Valuation*. As this is a10-year old debate as of the writing of the second edition, we are not investigating the discrepancy.

We can make several observations about the seemingly high differentials between the restricted stock investors' required returns and the expected value growth of the typical entity:

The average discounts appear to be indicative of defensive pricing.
The discounts would likely ensure at least a market return if the expected growth is not realized.
Very high implied returns are seen as expected growth increases, suggesting that high growth is viewed with skepticism.
The implied incremental returns of *R* over expected *G* are substantial at any level, suggesting that the base "cost" of 2.0 or 2.5 years of illiquidity is quite expensive.

Given varying assumptions about holding periods longer than 2.5 years and allowing for entities that pay regular dividends, we would expect some variation from the premium range found in appraisals of private company interests.

By way of comparison, we have made the same calculations for the example applications of the QMDM from Chapter 10 of *Quantifying Marketability Discounts*.

As noted in Table 8.18, the range of differences between the average required returns and the expected growth rates in value assumed in the ten appraisals was from 8.5% to 21.4%, with an average of about 13%. The table also indicates the range of other assumptions that yielded the concluded marketability discounts in the illustrations. I believe that these results, which came from actual appraisals, are generally consistent with the market evidence gleaned from the restricted stock studies above. Indeed, the premium returns required by the restricted stock investors, on average, exceed those applied in the above examples, suggesting the

**Table 8.18**

**Summary of Results of Applying the QMDM in 10 Example Appraisals**

| Example | Holding Period | Average Required Holding Period Return (R) | Expected Growth in Value Assumed (G) | (R - G) Difference | Dividend Yield | Concluded Marketability Discount |
|---|---|---|---|---|---|---|
| 1 | 5-8 years | 20.0% | 10.0% | 10.0% | 0.0% | 45.0% |
| 2 | 5-9 years | 20.5% | 4.0% | 16.5% | 8.8% | 25.0% |
| 3 | 7-15 years | 18.5% | 7.0% | 11.5% | 8.0% | 15.0% |
| 4 | 1.5-5 years | 19.5% | 7.5% | 12.0% | 0.0% | 20.0% |
| 5 | 5-10 years | 20.5% | 9.8% | 10.7% | 3.2% | 40.0% |
| 6 | 5-10 years | 18.5% | 10.0% | 8.5% | 2.1% | 25.0% |
| 7 | 5-15 years | 19.5% | 6.0% | 13.5% | 0.0% | 60.0% |
| 8 | 10-15 years | 19.5% | 5.0% | 14.5% | 10.0% | 25.0% |
| 9 | 10 years | 26.4% | 5.0% | 21.4% | 0.6% | 80.0% |
| 10 | 3-5 years | 22.5% | 6.0% | 16.5% | 0.0% | 35.0% |
| | **Averages** | 20.5% | 7.0% | 13.5% | 3.3% | 37.0% |
| | **Medians** | 19.8% | 6.5% | 12.8% | 1.4% | 30.0% |

Source: *Quantifying Marketability Discounts,* Chapter 10.

conclusions yielded conservative (i.e., relatively low) marketability discounts on average.... [section omitted].

**Conclusion** The QMDM, which is used primarily in valuing (non-marketable) minority interests of private companies, develops concrete estimates of expected growth in value of the enterprise and reasonable estimates of additional risk premia to account for risks faced by investors in non-marketable minority interests of companies. In its fully developed form, it incorporates expectations regarding distributions to assist appraisers in reaching logical, supportable, and reasonable conclusions regarding the appropriate level of marketability discounts for specific valuations.

The unpublished [and Mr. Abrams'] criticisms of the QMDM outlined above are, I believe, not correct. They do not recognize the critical distinctions that appraisers must draw between their analyses in valuing companies and valuing minority interests in those companies. And they do not consider the implications of the market evidence of required returns provided by the familiar restricted stock studies.

Marketable minority (and controlling interest) appraisals are developed based on the capitalized expected cash flows of businesses, or enterprises. Minority interests in those businesses must be valued based on consideration of the cash flows expected to be available to minority investors. The QMDM allows the business appraiser to bridge the gap between these two cash flow concepts, enterprise and shareholder, to develop reasoned and reasonable valuation conclusions at the non-marketable minority interest level.

## My Counterpoints

In responding to Mr. Mercer's rebuttal, it is clear that we will need a specific numerical example to make my criticism clear of the QMDM's inability to forecast restricted stock discounts.

Table 8.19, columns H and I, which we take from Mercer's Chapter 10, Example 1, show his calculation of the required holding period return of a minority stake for a private, closely held C corporation. The corporation is expected to grow in value by 10% each year mainly through an increase in earnings. It is not expected to pay dividends, and the majority owner is expected to retire and sell the business in five to eight years.

In columns K and L we show our own calculation of a restricted stock's required holding period return using Mercer's Example 1 as a guide. Our purpose is to show that the QMDM cannot even come close to forecasting ex ante the ex post discount rates of 27–50% from Table 8.16 that are necessary to explain restricted stock discounts using the QMDM.

We assume a non–dividend paying stock with an equivalent base equity discount rate as the stock in Mercer's example of 16.7% (row 14). It is in the investment-specific risk premiums where the restricted stock differs from the private minority shares. The restricted stock should be much easier to sell than a minority stake in a private closely held C corporation, since the ability to sell at the then-market rate in 2.5 years is guaranteed, and public minority shareholder rights are generally better protected than they are in private firms. We therefore reduce this premium for illiquidity from the premium in Mercer's example of between 1% and 2% (H18 and

| | A–F | G | H | I | J | K | L |
|---|---|---|---|---|---|---|---|
| 1 | **Table 8.19** | | | | | | |
| 2 | **QMDM Comparison of Restricted Stock** | | | | | | |
| 3 | **Discount Rate versus Mercer Example 1** | | | | | | |
| 4 | | | | | | | |
| 5 | | | **Mercer Example 1** | | | **Restricted Stock** | |
| 6 | | | **Range of Returns** | | | **Range of Returns** | |
| 7 | **Components of the Required Holding Period Return** | | Lower | Higher | | Lower | Higher |
| 8 | **Base Equity Discount Rate (Adjusted Capital Asset Pricing Model)** | | | | | | |
| 9 | Current Yield-to-Maturity Composite Long-Term Treasuries | | 6.7% | 6.7% | | 6.7% | 6.7% |
| 10 | + Adjusted Ibbotson Large Stock Premium | 6.5% | | | | | |
| 11 | × Applicable Beta Statistic | × 1 | | | | | |
| 12 | = Beta Adjusted Large Stock Premium | | 6.5% | 6.5% | | 6.5% | 6.5% |
| 13 | + Adjusted Ibbotson Small Stock Premium | | 3.5% | 3.5% | | 3.5% | 3.5% |
| 14 | **Base Equity Discount Rate** | | 16.7% | 16.7% | | 16.7% | 16.7% |
| 15 | | | | | | | |
| 16 | | | | | | | |
| 17 | **Investment-Specific Risk Premiums** | | | | | | |
| 18 | General Illiquidity of the Investment [1] | | 1.0% | 2.0% | | 0.0% | 0.0% |
| 19 | Uncertainties Related to Length of Expected Holding Period [2] | | 0.0% | 1.0% | | 0.0% | 0.0% |
| 20 | Lack of Expected Interim Cash Flows [3] | | 0.5% | 1.0% | | 0.5% | 1.0% |
| 21 | Small Shareholder Base [4] | | 0.0% | 1.0% | | 0.0% | 0.0% |
| 22 | **Range of Specific Risk Premiums for the Investment** | | **1.5%** | **5.0%** | | **0.5%** | **1.0%** |
| 23 | | | | | | | |
| 24 | **Initial Range of Required Returns** | | **18.2%** | **21.7%** | | **17.2%** | **17.7%** |
| 25 | | | | | | | |
| 26 | **Concluded Range of Required Holding Period Returns (Rounded)** | | **18.0%** | **22.0%** | | **17.0%** | **18.0%** |
| 27 | | | | | | | |
| 28–32 | [1] The restricted stock should be much easier to sell than a minority stake in a private closely held C corporation, since public minority shareholder rights are generally better protected. While it is possible that the restricted stocks should have a positive premium for this factor, they are nevertheless far more liquid than all of the private firms in Mercer's examples. If we should increase K18 and L18 to 1%, then we should increase H18 and I18 to at least 2% to 3% or probably higher yet. | | | | | | |
| 33 | | | | | | | |
| 34 | [2] Relative to the private shares, the expected holding period for the restricted stock is short and certain. | | | | | | |
| 35 | | | | | | | |
| 36–37 | [3] We assume a nondividend paying restricted stock. The example also concerned a nondividend paying C corporation. We therefore assign the same risk premium for this factor. | | | | | | |
| 38 | | | | | | | |
| 39 | [4] The restricted stock shares are shares of public corporations, which in general have large shareholder bases. | | | | | | |

I18) to 0% (K18 L18) for the restricted stock. While it is possible that the restricted stocks should have a positive premium for this factor, they are nevertheless far more liquid than all of the private firms in Mercer's examples. If we should increase K18 and L18 to, say, 1%, then we should increase H18 and I18 to at least 2–3%, respectively, or probably higher yet.

Relative to the private C corporation shares, the expected holding period for the restricted stock is short and certain. We therefore reduce the premium for holding period uncertainty from between 0% and 1% (H19 and I19) for Example 1 to 0 (K19 and L19) for the restricted shares. As both investments are expected to pay no dividends, there is no difference in the premium for lack of expected interim cash flows (row 20), although the latter experiences that lack of dividends for a far shorter and much more certain time period, which could well justify a lower premium than the former.

At this point I can digress to pose my objections to the first two factors. The term, "general illiquidity of the investment," is a very fuzzy term. It can mean almost anything. There is no empirical measure of it. Therefore, it can be almost anything that one wants it to be—which I admit has its advantages in practical application, but it's not good science. It is also unclear where general illiquidity stops and uncertainties in the holding period begin. Do they overlap? How does one prevent him- or herself from double-counting them? That is a problem with loosely defined terms.

Returning to the main train of thought, the private, closely held C corporation would have a much smaller shareholder base than the restricted stock corporations. We therefore reduce the premium for a small shareholder base from between 0% and 1% (H21 and I21) for Example 1 to 0 (K21 and L21) for the restricted stock. The total specific risk premium for the restricted stock comes to 0.5% (K22) to 1.0% (L22) versus the 1.5% (H22) to 5% (I22) for the private shares. After adding the base equity discount rates (row 14) and rounding, we arrive at a concluded range of required holding period returns of 18–22% and 17–18% (row 26) for Mercer's Example 1 and the restricted stock, respectively.

Next we need to determine the expected growth rate in value of the unrestricted marketable minority shares. Since there are no dividends, the expected growth rate must be equal to the discount rate—by definition.[102] In this example, the equity discount rate of the unrestricted marketable shares or the *base equity discount rate* is 16.7%.

Let's now calculate the QMDM discount on the restricted stock with the following assumptions:

1. A midrange (of K26 and L26) required holding period return of 17.5%
2. The 2.5-year average holding period
3. The growth rate in value of 16.7%

The calculation is as follows:

$$\text{DLOM} = 1 - (\text{FV} \times \text{PVF}) = 1 - \left(1.167^{2.5} \times \frac{1}{1.175^{2.5}}\right) = 1.7\%.$$

Assuming the correct discount is 30%, the QMDM is almost 95% too low!

## Mercer's Response

After reviewing Mr. Abrams' response to my rebuttal of his criticism of the QMDM, it is apparent that he and I continue to disagree over how the QMDM is applied in practice. The average marketability discounts in the 10 examples cited in my rebuttal of his criticism was 37%, and the median discount was 30%, not 1.7%. Mr. Abrams' mistake is in assuming that the discount rate imbedded in the pricing of a publicly traded stock is the required return of restricted stock investors. The fact that the average restricted stock discount is 30% or so indicates that investors have extracted a significant premium in return relative to the expected returns of the counterpart publicly traded securities.

What may be true "by definition" in a perpetuity calculation may well not be true for shorter holding periods. The QMDM deals, not with perpetuity calculations, but with investor assessments of expected cash flows over finite time horizons. And it makes explicit the assumptions made about the relationship between the expected growth in value of investments and the required returns of investors in those investments. I maintain that the model does, indeed, provide an excellent tool

[102]This is the discount rate applicable to marketable minority shares, not the higher discount rate applicable to illiquid shares (i.e., the required holding period return).

for estimating marketability discounts (from an estimated freely traded value) for minority interests in closely held companies.

## Problems in the QMDM and Comparison to Economic Components Model: A Response to Chris Mercer—Abrams (2002)

Chris Mercer continued the previous debate from the first edition of this book in a *Business Valuation Review* article (Mercer, 2001). This entire section is an article that I wrote in response to his article.[103]

**1. INTRODUCTION** It seems to me that healthy dialogue among practitioners is a useful tool in facilitating our growth as a profession. It is in that spirit that I wish to respond to my colleague, Chris Mercer's recent article (Mercer, 2001), wherein he asserts that my misunderstanding of his Quantitative Marketability Discount Model (QMDM) explains the disparity in my results and his in calculating the discount for lack of marketability (DLOM). Accordingly, in this article I will:

- Provide an explanation of the Economic Components Model (ECM)[104]
- Compare the theoretical underpinnings of ECM and QMDM
- Provide an empirical test of the QMDM versus the ECM
- Address logical inconsistencies in Mr. Mercer's arguments
- Compare the scope of the two models and address theoretical strengths and weaknesses

This article consists of five sections, including this introduction. In Section 2, I explain the theoretical (and some empirical) basis of the ECM. In Section 3, I provide an empirical test of the two models with restricted stock data. In Section 4, I discuss inconsistencies in the QMDM, and Section 5 is my conclusion.

**2. ECONOMIC COMPONENTS MODEL (ECM)** [I removed this section, as both editions of this book already covered the ECM. The article, however, refers to the first edition.]

**3. EMPIRICAL TEST OF THE TWO MODELS** The process of empirically testing QMDM versus ECM will involve the following steps:

- Demonstrating the mathematics of the QMDM results
- Discussing whether or not it is predictive
- Testing Mercer'*s* result in explaining the Columbia Financial Advisors, Inc. (CFAI) Study results and comparing them to the ECM calculation of the same

**The Mathematics Underlying the QMDM Calculation of the Holding Period Premium (HPP)** Mercer uses a spreadsheet to back into a 30.5% implied discount rate—required

[103]There are slight adaptations such as table and equation numbering and header formatting to accommodate this book format rather than a journal article.

[104]Since we are incorporating this article into *Quantitative Business Valuation*, which already describes the ECM, we will remove this section of the article from the book.

holding period return, $R_{HP}$, in QMDM terminology—for the Management Planning, Inc. (MPI) data. It is more instructive to solve for it analytically, which we do in equations (8.11)–(8.13).

We begin with an investment of \$1.00 at time zero. It grows at the marketable minority rate of return ($R_{mm}$) of 15% for 2.5 years (we will explain the holding period later) to $1.15^{2.5} = \$1.42$ (there is some apparent, but not real rounding error). The investor pays, on average, one dollar, less the 27.1%[105] restricted stock discount, or \$1.00 − \$.271 = \$.729. Thus an investment of \$.729 grows to \$1.42 in 2.5 years. We state that growth in equation (8.11):

$$\$0.729(1 + R_{HP})^{2.5} = \$1.42. \tag{8.11}$$

Dividing through by \$0.729, we get:

$$(1 + R_{HP})^{2.5} = \frac{\$1.42}{\$.729} = 1.945. \tag{8.12}$$

Raising both sides of (8-12) to the 0.4 power, we come to:

$$1 + R_{HP} = 1.945^{0.4} = 1.305. \tag{8.13}$$

Subtracting one from both sides of equation (8.13) leads to the solution of the 30.5% holding period return. From there, Mr. Mercer subtracted the marketable minority return ($R_{mm}$) of 15% to calculate the *HPP* of 15.5%.

**Predictive versus "Post-Dictive"** Mr. Mercer stated: "If we input an *HPP* of 15.5% into Abrams' calculations, it should be obvious that a discount of 27.1% will be achieved. The QMDM is predictive of restricted stock discounts, on average, when appropriate inputs are used."

Mr. Mercer made the mistake of assuming that which he was trying to prove. He "backed into" the 15.5% *HPP* that produced a 27.1% discount, and then he claimed that the "resulting" 27.1% discount proves the accuracy of the QMDM, since it produced a 27.1% restricted stock discount. That is not predicting the discount. It is, to coin a phrase, "post-dicting" the discount.

**Table 8.20: An Empirical Test of Predictive Ability of the Two Models** There is a way to test both the QMDM and the ECM for their predictive abilities. Since our respective books published, Aschwald (2000) published the overall results of her firm's restricted stock study in which the Section 144 minimum holding period was reduced from two years to one year.. That means non-affiliates of the company can begin selling their stock after one year according to the SEC's dribbling out rules and complete selling all their stock by the end of two years. The mean time to sell in the MPI study was 2.54 years—almost exactly halfway through the year.[106] We round to 2.5 years.

As we have no knowledge of the details of the CFAI study, we make the assumption that its population had similar characteristics to the MPI study, with

[105] *QBV* (First Edition), p. 238, C60.

[106] These are the 53 transactions reported in *Quantitative Business Valuation*, p. 238, I60.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Table 8.20** | | | |
| 2 | **Predictive Power of QMDM versus ECM** | | | |
| 3 | | | | |
| 4 | **Section 1: Calculation of QMDM Restricted Stock Discount** | | | |
| 5 | | | | |
| 6 | **Rate of Return = R** | **n = Holding Per = 2.5 Yrs** | **Value at n = $(1+R)^{2.5}$** | |
| 7 | $\mathbf{R_{mm}}$ | 15.0% | $1.418 | **Value** |
| 8 | $\mathbf{R_{HP}}$ | 30.5% | 1.945 | **Discount Factor** |
| 9 | **DLOM** | $= 1 - [(1+R_{mm})/(1+R_{HP})]^{2.5}$ | **27.1%** | **1 – (Value/Disc Factor)** |
| 10 | | | | |
| 11 | **Rate of Return = R** | **n = Holding Per = 1.5 Yrs** | **Value at n = $(1+R)^{1.5}$** | |
| 12 | $\mathbf{R_{mm}}$ | 15.0% | $1.233 | **Value** |
| 13 | $\mathbf{R_{HP}}$ | 30.5% | 1.491 | **Discount Factor** |
| 14 | **DLOM** | $= 1 - [(1+R_{mm})/(1+R_{HP})]^{1.5}$ | **17.3%** | **1 – (Value/Disc Factor)** |
| 15 | | | | |
| 16 | **Section 2: Calculation of ECM Restricted Stock Discount and** | | | |
| 17 | **Comparison of Errors in Both Models** | | | |
| 18 | | | | |
| 19 | | **ECM** | **QMDM** | |
| 20 | Avg Restricted Stock Discount—MPI | 27.1% | 27.1% | |
| 21 | Less: | | | |
| 22 | Decline of Section 144 Holding Period in Years | -1.0 | | |
| 23 | Regression Coefficient—Yrs 2 Sell [1] | 0.137 | | |
| 24 | Forecast Decline in Restricted Stock Discount | **-13.7%** | | |
| 25 | Forecast Restricted Stock Discount [2] | **13.4%** | **17.3%** | |
| 26 | Avg Restricted Stock Disc—CFAI [3] | 13.0% | 13.0% | |
| 27 | Absolute Forecast Error (Row 25 – Row 26) | **0.4%** | **4.3%** | |
| 28 | Percentage Forecast Error (Row 27 / Row 26) | **3.1%** | **32.9%** | |
| 29 | QMDM Error/ECM Error (C27 /B27) | **10.7** | | |
| 30 | | | | |
| 31 | [1] *Quantitative Business Valuation: A Mathematical Approach for Today's Professionals (QBV)*, p. 240, cell B54. To | | | |
| 32 | reconcile between the MPI and the CFAI studies, we are using the averages of the studies. Thus we use Regression 2 | | | |
| 33 | (p. 240) rather than Regression 1 (pp. 238–239) in QBV, as we do not have the average price stability for the CFAI study. | | | |
| 34 | | | | |
| 35 | [2] The ECM forecast is as calculated in B20 to B24. The QMDM forecast is from C14. | | | |
| 36 | | | | |
| 37 | [3] CFAI is Columbia Financial Advisors, Inc. Reported in *Business Valuation Update*, May 2000. | | | |

the exception of the holding period.[107] We recalculate the QMDM and the ECM restricted stock discounts in Table 8.20.

## *Section 1: Calculating the QMDM Restricted Stock Discount*

In Section 1, rows 7–9, we begin with calculating the QMDM restricted stock discount for the 2.5-year holding period. We use the marketable minority interest rate of return, $R_{mm}$, of 15% (B7) and the holding period rate of return, $R_{HP}$, of 30.5% (B8) from Mercer's article. For every \$1.00 of beginning value, the value of the enterprise should be expected to grow to $1.15^{2.5} = \$1.418$ (C7). We discount that by $1.305^{2.5} = 1.945$ (C8). The QMDM discount is equal to $1 - (1.418/1.945) = 27.1\%$. This duplicates the calculation earlier in the article.[108]

[107] There is another likely difference in the two populations. As shown in *Quantitative Business Valuation*, pp. 128–129, the standard deviation of stock market returns as a function of company size has declined exponentially over the life of the New York Stock Exchange. The transaction dates in the MPI database range from 1980 to 1996, a span of 16 years, while the CFAI study had to begin after April 27, 1997, and was published in May 2000, for a maximum span of three years, with an average transaction date approximately 9 years after the MPI study. Thus, the standard deviations of returns are very unlikely to be similar, and the interest rates prevailing during the two studies are likely to be different. This renders use of Black-Scholes less appropriate as a means to reconcile the differences in the results of the two studies.

[108] An alternative form of this calculation is Discount $= 1 - x^n$, where $x = (1+R_{mm})/(1+R_{HP})$ and $n =$ Holding Period. This is equivalent to discounting the \$1.418 at 30.5% for 2.5 years, which leads to a value of \$0.729 and a discount of 27.1%.

In rows 12–14, we redo the prior calculation using an average 1.5-year holding period instead of a 2.5-year holding period, as the latter has declined by exactly one year with the change in Rule 144. Using the same formula, the QMDM forecast restricted stock discount for a 1.5-year holding period is 17.3% (C14).

*Section 2: Calculating the ECM Restricted Stock Discount & Comparison to QMDM*

We calculate the ECM restricted stock discount in B20 to B25, beginning with the MPI study restricted discount of 27.1% (B20). In B22, we show the decrease of the average holding period of one year. In B23, we insert the regression coefficient of 0.137 for the average years to sell variable.[109] Multiplying B22 × B23 = −13.7% (B24). Adding that to B20 leads to the ECM regression forecast discount of 13.4% (B25). In C25, we repeat the QMDM forecast discount of 17.3% from C14.

In rows 26–29, we calculate the forecast errors and compare them. The CFAI average restricted stock discount was 13.0% (row 26). Subtracting row 26 from row 25 leads to our absolute forecast errors of 0.4% (B27) for ECM and 4.3% (C27) for QMDM. Dividing row 27 by row 26 produces the percentage forecast errors of 3.1% (B28) for ECM and 32.9% (C28) for QMDM. Dividing the QMDM error by the ECM error (C27/B27) shows that the QMDM error is 10.7, or almost 11 times the size of the ECM forecast error.[110]

Thus, the regression equation in the economic components model far outperformed the QMDM in its ability to predict the CFAI results. It is my claim that the disparity in model performance will be far greater for the much longer holding periods in business valuation for a variety of reasons that I will discuss later in the article.

This concludes our empirical test of the two models. In the next section, we examine inconsistencies in Mercer's use of the QMDM.

**4. INCONSISTENCIES IN THE QMDM** In my view there is an inconsistency in Mercer's logic. It is a paradox that his discount rates (holding period returns) for the Management Planning, Inc. (MPI) study firms and his Chapter 10 example firms are reversed. The former should be low and the latter high, not the other way around. Mercer's attempt to explain away this paradox suffers from its own inconsistencies.

He says that if one assumes growth rates for private firms are lower than the marketable minority interest rate of return, i.e., $G_v < R_{mm}$, that may justify using a lower holding period premium—and hence, discount rate—for private firms compared to restricted stocks. We will explore these claims in detail.

**The Discount Rates (Required Holding Period Returns) Are Reversed** Mr. Mercer has not made a satisfactory explanation as to why the average discount rate for the privately held firms in the examples in Chapter 10 of his book is 20%, while the discount rate for the MPI firms is 30.5%. Let's review the differences of the two data sets.

The MPI firms were all publicly traded, professionally managed firms, with an average market capitalization of $78 million, and a known average 2.5-year restriction before complete marketability. The holding periods were small and certain,

[109] *Quantitative Business Valuation*, p. 240, B54.
[110] C28/B28 leads to the same result.

compared to the Chapter 10 examples, where the holding periods were generally long and uncertain. Marketability at the end of the holding period for the MPI firms was guaranteed, while marketability of the example firms was uncertain. Mercer has not adequately explained why the former should have holding period premiums that are 3 to 10 times larger than the latter.

Mercer's backing into the 30.5% holding period return results is an ex post return, not an ex ante return—which he did point out on page 276 of my book. However, after making that point, he appears to have ignored its implications and forgotten it. An ex post return is not predictive, and it cannot be used when its underpinnings are so contrary to financial logic as they are here.

**Mercer's Explanation for the Inconsistency** His explanation for that inconsistency appears in footnote 14 of his article, where he states that the appraiser's judgment may dictate that the expected growth rate in value, $G_v$, may be considerably lower than the marketable minority rate of return, $R_{mm}$. He then states, "In such cases (e.g., as in the examples provided in Chapter 10 of *Quantifying Marketability Discounts*), there is no need to 'charge' the required holding period return for uncertainties related to achieving reinvestment at the enterprise discount rate. As a result, the holding period premia (HPP) used by some [a]ppraisers for expected holding periods in private company valuations may be lower than those implied by the restricted stock studies." This explanation would not survive any reasonable sanity check.

### *Flaw in the Explanation*

Let's review the concept of risk by thinking of two otherwise identical firms—one publicly held and one privately held. In finance, we think of risk as the probability distribution around our estimate of expected cash flows.

In this context, there are two components of risk. The first one is the inherent business risk of being in their particular industry and market. That would be identical for the two firms and their shareholders. The second component of risk is the overlay of the risk Mercer pointed out, that is, of being an exploited shareholder in a private firm. That increases the risk of being a private shareholder vis-à-vis a public shareholder. While abuses exist in public firms, it happens far less frequently, and there are greater remedies against this, such as class action lawsuits. The logical result is that the required holding period premium—and hence, holding period return (discount rate)—for private firms should be higher than restricted stocks, not lower.

Thus, Mercer has no logical explanation that I can perceive for the higher discount rate for the public firms with restricted stock than the private firms in his Chapter 10.

**Consistent QMDM Results with a 30.5% Discount Rate** If Mercer is correct that restricted stock of publicly traded firms with average market capitalization of $78 million and a known 2.5-year restriction before complete marketability should have discount rates of 30.5% and that the unrestricted stock has an average discount rate of around 15%, that implies that a 2.5-year period of restriction causes an average 16% (rounded) QMDM premium—well and good.

Then, imposing logical consistency, I would hazard a reasonable guess that privately held firms with expected holding periods of 8–10 years and great uncertainty as to their length and subsequent marketability should have QMDM premiums at

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 8.21** | | | | |
| 2 | **QMDM DLOM Calculations** | | | | |
| 3 | | | | | |
| 4 | Scenario | 1 | 2 | 3 | 4 |
| 5 | g = Growth Rate | 15% | 15% | 15% | 15% |
| 6 | r = Discount Rate | **16%** | **40%** | **50%** | **50%** |
| 7 | n = Number of Years (Holding Period) | **2.5** | 10 | 10 | **12** |
| 8 | x = (1+*g*)/(1+*r*) | 0.9914 | 0.8214 | 0.7667 | 0.7667 |
| 9 | $(1+g)^n$ = Value of Investment in 10 Years | $ 1.42 | $ 4.05 | $ 4.05 | $ 5.35 |
| 10 | Divide by $(1+r)^n$ to Discount to Present Value | 1.45 | 28.93 | 57.67 | 129.75 |
| 11 | $[(1+g)/(1+r)]^n = x^n$ | 0.979 | 0.140 | 0.070 | 0.041 |
| 12 | QMDM Discount = $1-x^n$ | **2%** | **86%** | **93%** | **96%** |

least 10% to 20% higher, leading to a holding period return of 40%–50% (rounded). Assuming $G_v = R_{mm}$, a 10-year holding period, and discount rates of 40% and 50%, the QMDM DLOMs are 86% (Table 8.21, C12) and 93% (D12), respectively, for an average of 90%.[111] Using a 50% discount rate and a 12-year holding period, the resulting DLOM is 96% (E12)—and we haven't calculated the discount for lack of control (DLOC) yet! Thus, if we impose rational consistency on the HPPs, then the QMDM calculation of DLOM for private firms produces extreme results.

For comparison, I included a QMDM calculation of a 2% DLOM in column B, based on a 1% HPP, which is a more appropriate HPP if Mercer's small HPPs in Chapter 10 are correct. The point is that as long as one is consistent in HPPs between the restricted stocks and private firms, the QMDM produces extreme results for either very short holding periods or very long holding periods. That is the major flaw of the model.

Other shortcomings of the QMDM vis-à-vis the economic components model are its lack of empirical data and inability to accurately quantify the effects of thin markets and transactions cost differentials between direct ownership of the underlying assets and an ownership interest in the firm.

**5. CONCLUSION** It should be clear from this article that criticisms of the QMDM in Chapter 7 of my book[112] are well founded. Mercer's attempted reconciliation does not work. Mercer's only apparent defense is to bifurcate the model and use very high discount rates ($R_{HP}$) for publicly traded firms and low discount rates for private firms.

Meanwhile, we have seen in Table 8.20 that the regression equation in the ECM outperformed the QMDM by almost 11 times in forecasting the results in the CFAI

[111] If $G_v < R_{mm}$, that would increase DLOM further. The QMDM DLOM equals $1 - x^n$, where $x = (1+g)/(1+r)$, with $g = R_{mm}$ (or $G_v$, as appropriate), and $r = R_{HP}$. The partial derivatives of the discount are as follows:

$$\frac{\partial D}{\partial g} = -nx^{n-1}\frac{1}{1+r} < 0$$

$$\frac{\partial D}{\partial r} = -nx^{n-1}\frac{-(1+g)}{(1+r)^2} > 0$$

Thus, the QMDM DLOM is negatively related to changes in $g$ and positively related to changes in $r$.

[112] Chapter 7 in the first edition, Chapter 8 in this edition.

Study, even when we allow the QMDM to "cheat" by using the ex post perfect solution from the MPI Study. When we hold the QMDM to rational consistency with the examples in Mercer's Chapter 10, then the QMDM error is over 27 times larger.[113] Also, there are substantial theoretical reasons why this gap should widen considerably with ordinary business valuation, with their longer holding periods.

As indicated at the outset of this article, my intention is to continue dialogue on these and other issues confronting our profession. We have come a long way in a relatively short time. Chris Mercer, among others, has contributed to that dialogue. In fact, we both developed our DLOM models at approximately the same time in 1994, unbeknownst to the other. Ours were the first two quantitative models to calculate DLOM, and the QMDM is certainly a substantial improvement over the pure guesswork that preceded it. Let the dialogue continue.

In fact, very infrequently, I use the QMDM as a benchmark DLOM calculation. I am most tempted to do this when the holding period is very long and there is no market. For example, if I were valuing a fractional interest in a house, if the interest is not entitled to possession or income and only would achieve liquidity upon the sale, which is not expected for over 20 years, I might be tempted to include a QMDM calculation.

## New Evidence on Restricted Stock Discounts—MPI and FMV Opinions' 2008 Studies

We have been blessed with new restricted stock data from MPI and FMV Opinions.

**MPI's 2008 RESTRICTED STOCK STUDY** MPI has updated its study with transactions through June 2008. MPI provided us with summary data in Table 8.22, but this time did not provide the detailed data for our analysis. Let's analyze Table 8.22.

**Four Holding Periods** MPI's study encompasses four holding periods:

1. *Pre-1990.* Nonaffiliates can dribble out at the beginning of each quarter according to Rule 144 at the end of year 2. At the end of year 3, they can sell all of their stock. Affiliates are always subject to the SEC dribble-out rules, although a non-officer or director can change from affiliate to nonaffiliate status by dribbling out stock until one's ownership is below the affiliate threshold.

   There is no precise definition of a nonaffiliate versus an affiliate. Kyle Vataha of FMV Opinions and Ezra Angrist of MPI both have a working definition of an affiliate to be an officer or director of the company or one who owns a certain threshold of stock in it. The threshold is commonly thought to be 10%, but can be as little as 5%. The reasoning behind this is that owning 5% to 10% of a public firm is often enough to control the firm or at least exert substantial influence.
2. *1990–April 1997.* This has the same holding period requirements as pre-1990, with the only difference being that in this period, a nonaffiliate can tack any prior ownership of the restricted stock onto the current ownership in determining the holding period. Since MPI's study is exclusively private placements, which means

[113]We begin with the CFAI result of 13.0% and subtract my 2% (actually 2.14%) calculation of the QMDM discount from Table 8.21, B12 to arrive at an error of 10.86%. We then divide that by the ECM error of 0.4% to arrive at a ratio of over 27.2 times the error.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 8.22** | | | | |
| 2 | **MPI's 2008 Study [1]** | | | | |
| 3 | **Discount by Time Period** | | | | |
| 4 | **Unregistered Restricted Stock Only** | | | | |
| 5 | | | | | |
| 6 | | | | **Discount** | |
| 7 | **Period** | **Covering** | **Observations** | **Average** | **Median** |
| 8 | 1 | Pre-1990 | 82 | 30.2% | 32.0% |
| 9 | 2 | 1990–April 1997 | 111 | 25.3% | 23.1% |
| 10 | 3 | April 1997–Feb 2008 | 244 | 20.6% | 17.3% |
| 11 | 4 | Feb 2008–June 2008 | 35 | 8.8% | 5.6% |
| 12 | **All** | | **472** | **22.5%** | **20.0%** |
| 13 | | | | | |
| 14 | [1] *Source:* Management Planning, Inc. Used with permission. According | | | | |
| 15 | to MPI, this includes negative and zero discount transactions and excludes | | | | |
| 16 | transactions with registered stock. All transactions are private placements, | | | | |
| 17 | so there is no prior ownership to affect the 144 holding period. | | | | |

that there is no prior ownership, there is no change in SEC law that would cause discounts to be different in periods 1 and 2.

3. *April 1997–February 2008*. The Rule 144 holding period for nonaffiliates declined by one year. A nonaffiliate could begin selling stock according to the dribble-out rules at the end of year 1, and could sell all remaining stock at the end of year 2. It is this one-year drop in holding period on which I relied so heavily in my article (Abrams, 2002) from the previous section of this chapter.
4. *February 2008–*. The Rule 144 holding period for nonaffiliates declined by six months in its starting point and by six months in its length of application. A nonaffiliate could begin selling stock at the end of month 6 and sell all stock at the end of month 12. We will rely on this holding period in Tables 8.23 and 8.24, which update Table 8.20.

**Analysis of Table 8.22** You can see by going down columns D and E that the average (mean) and median restricted stock discounts from the private placements have been declining in each holding period. The most dramatic drop is in period 4, where mean discounts declined by almost 12%, from 20.6% (D10) in period 3 to 8.8% (D11) in period 4. Median discounts declined by almost 12%, from 17.3% (E10) to 5.6% (E11). However, we must be a bit cautious, as the 35 (C11) transactions represent only a little over 7% of the entire dataset.

**Tables 8.23 and 8.24: A New Comparison of ECM and QMDM** We use the MPI 2008 study results from Table 8.22 to update our comparison of ECM and QMDM. Table 8.23 is an update of Table 8.20. Rows 7–8 show the QMDM results already included in Table 8.20. Row 9 is new and shows the results using the MPI 2008 study. We show the average holding period as 0.75 years, because the nonaffiliate owner of restricted stock can begin to dribble out at 6 months, again at 9 months, and then sell all remaining stock at 12 months. If the amounts sold are equal—let's say

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 8.23** | | | | |
| 2 | **Predictive Power of QMDM versus ECM [1]** | | | | |
| 3 | | | | | |
| 4 | **Section 1: Calculation of QMDM Restricted Stock Discount** | | | | |
| 5 | | | | | **=1–(C/D)** |
| 6 | **Study** | ***n* (Yrs) [2]** | **FMV = $(1+R_{mm})^n$** | **FMV = $(1+R_{HP})^n$** | **DLOM** |
| 7 | MPI-1997 | 2.50 | $1.418 | $1.945 | **27.1%** |
| 8 | Columbia Financial Advisors, Inc. | 1.50 | $1.233 | $1.491 | **17.3%** |
| 9 | MPI-2008 | 0.75 | $1.111 | $1.221 | **9.0%** |
| 10 | | | | | |
| 11 | **Section 2: Calculation of ECM Discounts and Comparison of Errors in Both Models** | | | | |
| 12 | | | | | |
| 13 | **Holding Period Reconciliation** | **2-Yr vs 1-Yr** | | **2-Yr vs 1/2 Yr** | |
| 14 | | **ECM** | **QMDM** | **ECM** | **QMDM** |
| 15 | Avg Restricted Stock Discount—MPI 1997 | 27.1% | 27.1% | 27.1% | 27.1% |
| 16 | Less: | | | | |
| 17 | Decline of Section 144 Holding Period in Years | -1.0 | | -1.75 | |
| 18 | Regression Coefficient—Yrs 2 Sell [3] | 0.137 | | 0.137 | |
| 19 | Forecast Decline in Restricted Stock Discount | **-13.7%** | | **-24.0%** | |
| 20 | Forecast Restricted Stock Discount [4] | **13.4%** | **17.3%** | **3.1%** | **9.0%** |
| 21 | Avg Restricted Stock Disc [5] | 13.0% | 13.0% | 8.8% | 8.8% |
| 22 | Absolute Value of Fcst Error = ABS[(20)–(21)] | **0.4%** | **4.3%** | **5.7%** | **0.2%** |
| 23 | | | | | |
| 24 | **Assumptions** | | | | |
| 25 | $R_{mm}$ | 15.0% | | | |
| 26 | $R_{HP}$ | 30.5% | | | |
| 27 | | | | | |
| 28–30 | [1] This table appeared as Table 1 in Abrams's article "Problems in the QMDM and Comparison to Economic Components Model: A Response to Chris Mercer," *Business Valuation Review*, June 2002. We modify it to add a comparison to MPI's 2008 study. | | | | |
| 31 | | | | | |
| 32–34 | [2] The MPI—1997 study took place when the dribble out period began at year 2, and the investor could dribble out from years 2–3 and sell at year 3, which leads to an average selling time of year 2.5. Beginning Feb. 15, 2008, the dribble out begins at 6 months and finishes at 1 year, for an average selling time of 9 months. | | | | |
| 35 | | | | | |
| 36–38 | [3] Table 7.5 (1st edition), cell B54. To reconcile between the MPI and the CFAI studies, we are using the averages of the studies. Thus we use Regression 2 rather than Regression 1, as we do not have the average price stability for the CFAI study. | | | | |
| 39 | | | | | |
| 40 | [4] The ECM forecast is as calculated in this section. The QMDM forecast is from D8 and D9. | | | | |
| 41 | | | | | |
| 42–45 | [5] Columbia Financial Advisors, Inc. (CFAI) reported its results in *Business Valuation Update,* May 2000. It appears in cells B21 and C21. The 1/2 year holding period discount is from MPI's 2008 study and appears in cells D21 and E21. MPI's new study shows an average discount of 25.3% for 1990 to April 1997 transactions; however, we use the 27.1% from its first study in our calculations. | | | | |

approximately 1% of outstanding shares—the average holding period is 9 months, or 3/4 year. Of course, if the shareholder owns more (less) than 3% of outstanding shares, then the average holding period will be longer (shorter) than 3/4 year, and there is a higher likelihood that it will be longer, since it can be shorter by only 1/4 year at the most, while it can be longer by years.

Assuming a 15% (B25) discount rate at the marketable minority shareholder level and the 30.5% (B26) discount rate for the private minority shareholder, the QMDM discount for a 0.75 (B9) year holding period is 9.0% (E9).

B15 through C22 are a repetition of Table 8.20. The final results of forecast errors of 0.4% for ECM and 4.3% for QMDM in B22 and C22 are identical to Table 8.20, row 27. Our new analysis appears in D15 through E22.

We begin with the 27.1% MPI restricted stock discount of 27.1% in D15. The average holding period in that study was 2.5 years, and we assume that the average holding period in this study is 0.75 years, although we know there is a probable bias and it should be higher, though we lack the data to make that calculation. The difference is 1.75 years, which we show as a negative number in D17. We multiply that by the *x*-coefficient for average years to sell of 0.137 (D18), which results in a

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 8.24** | | | | |
| 2 | **Predictive Power of QMDM vs. ECM—21% Discount Rate [1]** | | | | |
| 3 | | | | | |
| 4 | **Section 1: Calculation of QMDM Restricted Stock Discount** | | | | |
| 5 | | | | | **=1 – (C/D)** |
| 6 | **Study** | ***n* (Yrs) [2]** | **FMV = $(1+R_{mm})^n$** | **FMV = $(1+R_{HP})^n$** | **DLOM** |
| 7 | MPI—1997 | 2.50 | $1.418 | $1.611 | **11.9%** |
| 8 | Columbia Financial Advisors, Inc. | 1.50 | $1.233 | $1.331 | **7.3%** |
| 9 | MPI—2008 | 0.75 | $1.111 | $1.154 | **3.7%** |
| 10 | | | | | |
| 11 | **Section 2: Calculation of ECM Discounts and Comparison of Errors in Both Models** | | | | |
| 12 | | | | | |
| 13 | **Holding Period Reconciliation** | **2-Yr vs 1-Yr** | | **2-Yr vs 1/2 Yr** | |
| 14 | | **ECM** | **QMDM** | **ECM** | **QMDM** |
| 15 | Avg Restricted Stock Discount—MPI 1997 | 27.1% | 27.1% | 27.1% | 27.1% |
| 16 | Less: | | | | |
| 17 | Decline of Section 144 Holding Period in Years | -1.0 | | -1.75 | |
| 18 | Regression Coefficient—Yrs 2 Sell [3] | 0.137 | | 0.137 | |
| 19 | Forecast Decline in Restricted Stock Discount | **-13.7%** | | **-24.0%** | |
| 20 | Forecast Restricted Stock Discount [4] | **13.4%** | **7.3%** | **3.1%** | **3.7%** |
| 21 | Avg Restricted Stock Disc [5] | 13.0% | 13.0% | 8.8% | 8.8% |
| 22 | Absolute Value of Fcst Error = ABS[(20)–(21)] | **0.4%** | **5.7%** | **5.7%** | **5.1%** |
| 23 | | | | | |
| 24 | **Assumptions** | | | | |
| 25 | **$R_{mm}$** | 15.0% | | | |
| 26 | **$R_{HP}$ [6]** | 21.0% | | | |
| 27 | | | | | |
| 28 | [1] This table appeared as Table 1 in Abrams's article "Problems in the QMDM and Comparison to Economic | | | | |
| 29 | Components Model: A Response to Chris Mercer," *Business Valuation Review*, June 2002. We modify it to | | | | |
| 30 | add a comparison to MPI's 2008 study. | | | | |
| 31 | | | | | |
| 32 | [2] The MPI—1997 study took place when the dribble out period began at year 2, and the investor could dribble out | | | | |
| 33 | from years 2–3 and sell at year 3, which leads to an average selling time of year 2.5. Beginning Feb. 15, 2008, | | | | |
| 34 | the dribble out begins at 6 months and finishes at 1 year, for an average selling time of 9 months. | | | | |
| 35 | | | | | |
| 36 | [3] Table 7.5 (1st edition), cell B54. To reconcile between the MPI and the CFAI studies, we are using the averages | | | | |
| 37 | of the studies. Thus we use Regression 2 rather than Regression 1, as we do not have the average price | | | | |
| 38 | stability for the CFAI study. | | | | |
| 39 | | | | | |
| 40 | [4] The ECM forecast is as calculated in this section. The QMDM forecast is from D8 and D9. | | | | |
| 41 | | | | | |
| 42 | [5] Columbia Financial Advisors, Inc. (CFAI) reported its results in *Business Valuation Update*, May 2000. | | | | |
| 43 | It appears in cells B21 and C21. The 1/2 year holding period discount is from MPI's upcoming 2009 | | | | |
| 44 | study and appears in cells D21 and E21. MPI's new study shows an average discount of 25.3% for | | | | |
| 45 | 1990 to April 1997 transactions; however, we use the 27.1% from its first study in our calculations. | | | | |
| 46 | | | | | |
| 47 | [6] In *Quantifying Marketability Discounts* (1997), Mercer's maximum premium in his examples in Chapter 10 is | | | | |
| 48 | 6%, which we add to $R_{mm}$. | | | | |

subtraction of 24.0% in D19. Our ECM forecast restricted stock discount is 27.1% − 24.0% = 3.1% (D20). MPI's study shows an actual mean discount of 8.8%, so the ECM error is 5.7% (D22)—a poor result.

Meanwhile, the QMDM discount of 9% (E9, transferred to E20) has a forecast error of only 0.2% (E22). Is this a replay of *The Empire Strikes Back*!? Perhaps a little bit, but mostly not.

My criticisms of the QMDM still remain. My major criticism is that it is the wrong paradigm to begin with. Its focus is on the probable holding period to a liquidity event of the company—usually an acquisition or an IPO—whereas the ECM's focus is on the holding period for an orderly sale of the stock to another investor. The former may affect the latter, but I contend the latter is the right criterion.

My second criticism of the QMDM is there are no rational and consistent discount rates that produce reasonable short-term and long-term restricted stock discounts, as we saw in Table 8.21. The high private minority interest discount rates that are required to produce reasonable restricted stock discounts in the 0.75-year to 2.5-year

| | A | B | C |
|---|---|---|---|
| 1 | **Table 8.25** | | |
| 2 | **FMV Opinions' Restricted Stock Discounts [1]** | | |
| 3 | | | |
| 4 | | **Including Negative Discounts** | **Excluding Negative Discounts** |
| 5 | Average (Mean) | 20.9% | 23.5% |
| 6 | Median | 17.7% | 19.8% |
| 7 | Count | 558 | 516 |
| 8 | | | |
| 9 | [1] Of the 562 transactions, 30% included registration rights and another 30% were | | |
| 10 | ambiguous as to registration rights. Used with permission. | | |

timeframe produce discounts over 90% for holding periods of 10 years or more. If we lower the discount rates sufficiently to produce more reasonable long-run discounts, they don't work for short holding periods.

My third criticism of QMDM is that there is nothing empirical about the criteria on Mr. Mercer's list, nor of the quantitative premium for the discount. It is all subjective. My final criticism, which is partially a corollary of my third criticism, is that, taken together, the QMDM is a very loose model that can produce any discount that the appraiser wants. One cannot tell from looking at any one valuation whether the application of the QMDM is objective and unbiased. It would require looking at many appraisals by the same appraiser to see whether the appraiser is at least consistent in applying the premiums over the marketable minority holding period.

Now let's analyze why the ECM fared so poorly in this analysis.[114]

1. The final result is very sensitive to the holding period discount rate ($R_{HP}$). Table 8.24 is identical to Table 8.23, with the one change of the discount rate in B26. If we use a more normal discount rate of 21% instead of 30.5%, the QMDM is slightly better than the ECM. Thus, the QMDM final result is very sensitive to the discount rate.
2. The regression equation is 10 years old. It is likely that MPI has changed its research methodology since then. Changes in transaction selection and recording procedures (e.g., the types of transactions selected, rejection criteria, criteria for categorizing and recording data, computing estimated holding periods, etc.) affect the final outcome.
3. There are only 35 observations for period 4, which is only 7% of the entire dataset.

**FMV OPINIONS RESTRICTED STOCK DATA** Table 8.25 shows mean and median discounts for the FMV Opinions data. Based on 558 (B7) transactions, the mean discount is 20.9% (B5) for all transactions, including those with negative discounts. This is less than the MPI mean discount of 22.5% (Table 8.22, D12). Since the MPI data include negative discounts, this is the relevant comparison. The FMV mean discount excluding negative discounts is 23.5% (C5).

[114]Some of the points have been mentioned previously.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 8.26** | | | | |
| 2 | **FMV Opinions' 2008 Study [1]** | | | | |
| 3 | **Discount by Time Period** | | | | |
| 4 | | | | | |
| 5 | | | | **Discount** | |
| 6 | **Period** | **Covering** | **Observations** | **Average** | **Median** |
| 7 | 1 | Pre-1990 | 70 | 21.3% | 21.1% |
| 8 | 2 | 1990–April 1997 | 172 | 22.5% | 20.0% |
| 9 | 3 | April 1997–Feb 2008 | 316 | 20.0% | 16.3% |
| 10 | 4 | Feb 2008–June 2008 | 0 | | |
| 11 | **All** | | **558** | **20.9%** | **17.7%** |
| 12 | | | | | |
| 13 | [1] *Source:* Data from FMV Opinions. Calculations by AVGI. Used with | | | | |
| 14 | permission. This includes negative and zero discount transactions. | | | | |
| 15 | Of the 562 transactions, 30% included registration rights and another | | | | |
| 16 | 30% were ambiguous as to registration rights. | | | | |

The FMV Opinions median discount is 17.7% (B6), which also is less than the MPI median of 20.0% (Table 8.22, E12). Thus, we see that the FMV measures of central tendency are about 2.5% less than MPI. Let's investigate further.

Tables 8.26 and 8.27 include zero and negative discounts to maintain comparability with Table 8.22. In Table 8.26, we tabulate the mean and median discounts by SEC period—just as in Table 8.22. The FMV Opinions discounts in periods 1 and 2 (D7 through E8) are still significantly below those of MPI for the same periods. However, the period 3 mean and median discounts of 20.0% and 16.3% (D9 and E9, respectively) are 0.6% and 1% below MPI, respectively.

However, 30% of the FMV Opinions transactions included registration rights, and another 30% were unclear as to whether they included registration rights. Since registration rights would tend to reduce the discount, we eliminated all known and questionable transactions in Table 8.27 and show only those known not to have

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 8.27** | | | | |
| 2 | **FMV Opinions' 2008 Study [1]** | | | | |
| 3 | **Discount by Time Period** | | | | |
| 4 | **Excludes Transactions with Registration Rights** | | | | |
| 5 | | | | | |
| 6 | | | | **Discount** | |
| 7 | **Period** | **Covering** | **Observations** | **Average** | **Median** |
| 8 | 1 | Pre-1990 | 63 | 20.6% | 20.0% |
| 9 | 2 | 1990–April 1997 | 141 | 22.5% | 20.0% |
| 10 | 3 | April 1997–Feb 2008 | 21 | 14.5% | 11.1% |
| 11 | 4 | Feb 2008–June 2008 | 0 | | |
| 12 | **All** | | **225** | **21.2%** | **20.0%** |
| 13 | | | | | |
| 14 | [1] *Source:* Data from FMV Opinions. Calculations by AVGI. Used with | | | | |
| 15 | permission. This includes negative and zero discount transactions. | | | | |

registration rights. Thus, Table 8.27 is the summary of FMV Opinions data that are most comparable with MPI data. However, now there are only 225 (C12) transactions compared to 472.

The overall mean and median discounts of 21.2% and 20.0% (D12 and E12) for FMV Opinions are similar to MPI's 22.5% and 20.0% (Table 8.22, D12 and E12). However, the period-by-period comparisons are quite different. The period 1 mean and median are about 10% and 12% below the MPI figures. The period 2 mean of 22.5% (D9) is 2.8% below MPI, and the median of 20.0% (E9) is 3.1% below MPI. The mean and median discounts of 14.5% and 11.1% (D10, E10) for period 3 are about 6% below MPI. It appears that the inclusion of period 4 discounts in Table 8.22 brings MPI's overall mean and medians down close to FMV Opinions', but since the latter has no transaction in that period, it is not a fair comparison. Only the period-by-period comparison earlier in this paragraph is accurate. Taking an average of the period 1 difference of 11%, period 2 difference of 3%, and period 3 difference of 6% would lead us to conclude that FMV Opinions' discounts are about 7% below MPI's. However, the weighted average using FMV Opinions' number of transactions (calculations not shown) is a 5.5% difference, since period 2 comprises 63% of its transactions.[115]

Thus, we have two databases measuring the same phenomenon coming to different measures of central tendency. In percentage terms, the MPI and FMV Opinions' average of means for periods 1 through 3 are 25% and 19%, respectively, which means that FMV Opinions' mean discounts are 24% lower than MPI's. It would be helpful to understand what causes the difference, but we do not have any information to shed light on that.

## Conclusion

We have reviewed the professional and some of the academic literature dealing with control premiums and DLOM. My opinion is that with our current information set, we should use control premiums in the 21–28% range. We developed this as being three to four times the value of the voting rights premium adjusted to U.S. laws and for liquidity differences between voting and nonvoting stock. This measure is consistent with the mean and median of median going-private premiums of 27.6% and 25.9% (Table 8.1, E33 and E34), although it is preferable to make a clean separation of expected performance improvements, which increase the "top line," that is, cash flows, versus the pure value of control, which is represented by a reduction in the discount rate.

We reviewed three quantitative models of DLOM: Mercer's, Kasper's, and Abrams'. The QMDM was unable to provide any meaningful restricted stock discounts for the Management Planning, Inc. data, as discounting modest risk premiums for two to three years provides little variation in discount. Abrams' non-company-specific Black-Scholes option pricing model performed worse than the mean at explaining restricted stock discounts, while using BSOPM with firm-specific calculations of standard deviations was superior to the mean.

[115] Weighting by MPI's number of transactions in each period would lead to a different result.

We quantified component #2, monopsony power to the buyer, as 9%, according to Schwert's findings of a 12.2% greater premium in takeovers when there are multiple buyers than when there is only one buyer.

Finally, we quantified transactions costs separately for the buyer and the seller. The premise of fair market value is such that we ask, "What would a hypothetical buyer be willing to pay for this interest?," which means that we are presuming a first sale immediately. Buyers care about their own transaction costs, but they do not care about seller's transaction cost on the immediate transaction. However, buyers do care that in 10 years or so, they become the sellers. They therefore care about all subsequent sellers' (and buyers') transactions costs. We presented two discount formulas—equations (8.9) and (8.9a), which are appropriate for the seller and buyer, respectively, to translate the pure discount that applies to each transaction into a discount based on the present value of the infinite continuum of periodic transactions.

In Table 8.14, we applied our DLOM model to a control interest in a hypothetical private company. The result was a DLOM of 23.1%, which is a reasonable result.

Of course, the economic components model is merely a model. It is certainly imperfect, and it must be used with common sense. It is possible to obtain strange or nonsensical results, and if the appraiser is asleep at the wheel, he or she may not realize it. There is plenty of room for additional research to improve our modeling and results. Nevertheless, in my opinion, this is the most realistic and comprehensive model to date for calculating DLOM.

## References


Abrams, Jay B. 1994a. Discount for Lack of Marketability: A Theoretical Model. *Business Valuation Review* (September):132–139.

Abrams, Jay B. 1994b. A Breakthrough in Calculating Reliable Discount Rates. *Valuation* (August):8–24.

Abrams, Jay B. 2002. Problems in the QMDM and Comparison to Economic Components Model: A Response to Chris Mercer. *Business Valuation Review* (September):83.

Amihud, Y., and H. Mendelson. 1991. Liquidity, Asset Prices, and Financial Policy. *Financial Analysts Journal* (November/December):56–66.

Aschwald, Kathryn F. 2000. Restricted Stock Discounts Decline as a Result of 1-Year Holding Period. Business Valuation Update (May):1–5.

Barca, F. 1995. On Corporate Governance in Italy: Issues, Facts, and Agency. Mimeo, Bank of Italy.

Bargeron, Leonce, Frederik Schlingemann, Rene Stulz, and Chad Zutter. 2008. Why Do Private Acquirers Pay So Little Compared to Public Acquirers? *Journal of Financial Economics* 89:375–390. Also available for download as a working paper at http://www.ssrn.com/abstract=980066.

Becker, Brian, and Gary Gutzler. 2000. Should a Blockage Discount Apply? Perspectives of Both a Hypothetical Willing Buyer and a Hypothetical Willing Seller. *Business Valuation Review* (September):3–9.

Bergstrom, C., and K. Rydqvist. 1990. Ownership of Equity in Dual-Class Firms. *Journal of Banking and Finance* 14:237–253.

Berkovitch, E., and M. P. Narayanan. 1993. Motives for Takeovers: An Empirical Investigation. *Journal of Financial and Quantitative Analysis* 28:347–362.

Bolotsky, Michael J. 1991. Adjustments for Differences in Ownership Rights, Liquidity, Information Access, and Information Reliability: An Assessment of "Prevailing Wisdom" versus the "Nath Hypothesis." *Business Valuation Review* (September):94–110.

Bolotsky, Michael J. 1995. Is the "Levels of Value" Concept Still Viable? Bolotsky's Response. In Business Valuation Education Sessions, *American Society of Appraisers Convention*, June, Denver, CO, 21–34.

Bradley, M., A. Desai, and E. H. Kim. 1988. Synergistic Gains from Corporate Acquisitions and Their Division between the Stockholders of Target and Acquiring Firms. *Journal of Financial Economics* 39:3–40.

Chaffe, David B. H. 1993. Option Pricing as a Proxy for Discount for Lack of Marketability in Private Company Valuations: A Working Paper. *Business Valuation Review* (December):182–188.

Eckbo, B. E. 1983. Horizontal Mergers, Collusion, and Stockholder Wealth. *Journal of Financial Economics* 11:241–274.

Eckbo, B. E. 1985. Mergers and the Market Concentration Doctrine: Evidence from the Capital Market. *Journal of Business* 58:325–349.

Eckbo, B. E. 1992. Mergers and the Value of Antitrust Deterrence. *Journal of Finance* 47:1005–1030.

Einhorn, Hillel J., and Robin M. Hogarth. 1987. Decision Making under Ambiguity. In *Rational choice*, edited by Robin M. Hogarth and Melvin W. Reder. Chicago: University of Chicago Press, pp. 41–66.

Ellsberg, D. 1961. Risk, Ambiguity, and the Savage Axioms. *Quarterly Journal of Economics* 75:643–669.

Franks, J. R., and Julian R. Harris. 1989. Shareholder Wealth Effects of Corporate Takeovers: The U.K. Experience 1955–1985. *Journal of Financial Economics* 23:225–249.

Glass, Carla, and Mary M. McCarter. 1995. The Foundations for Minority and Control Position Adjustments. Paper presented at American Society of Appraisers' 14th Annual Advanced Business Valuation Conference, November 3, Boston, MA.

Grabowski, Roger, and David King. 1999. New Evidence on Size Effects and Rates of Return. *Business Valuation Review* (September):112–130.

Harris, Ellie G. 1994. Why One Firm Is the Target and the Other the Bidder in Single-Bidder, Synergistic Takeovers. *Journal of Business* 67(2):263–280.

Horner, M. R. 1988. The Value of the Corporate Voting Right: Evidence from Switzerland. *Journal of Banking and Finance* 12(1):69–83.

Ibbotson Associates. 1999. *Stocks, Bonds, Bills, and Inflation: 1999 Yearbook—Valuation Edition*. Chicago: The Associates, p. 119.

Jankowske, Wayne C. 1991. Valuing Minority Interests in Relation to Guideline Firms. *Business Valuation Review* (December):139–143.

Jankowske, Wayne. 1995. Frameworks for Analysis of Control Premiums. *Business Valuation Review* (March):3–10.

Johnson, Bruce A. 1999. Quantitative Support for Discounts for Lack of Marketability. *Business Valuation Review* (December):152–155.

Kasper, Larry, J. 1997. *Business Valuations: Advanced Topics*. Westport, CT: Quorum Books.

Kasper, Larry, J. 2009. The Relationship between Holding Period and Probability of Sale. *The Value Examiner* (March/April):11–28.

Koeplin, John, Atulya Sarin, and Alan C. Shapiro. 2000. The Private Company Discount. *Journal of Applied Corporate Finance* 12, no. 4 (Winter).

Lang, L. H. P., R. Stulz, and, R. A. Walkling. 1991. A Test of the Free Cash Flow Hypothesis: The Case of Bidder Returns. *Journal of Financial Economics* 29:315–336.

Lease, Ronald C., John J. McConnell, and Wayne H. Mikkelson. 1983. The Market Value of Control in Publicly-Traded Corporations. *Journal of Financial Economics* 11.

Lerch, Mary Ann. 1991. Quantitative Measures of Minority Interest Discounts. *Business Valuation Review* (March):14–20.

Lerch, Mary Ann. 1997. Yet Another Discount for Lack of Marketability Study. *Business Valuation Review* (June):70–106.

Levy, H. 1982. Economic Evaluation of Voting Power of Common Stock. *Journal of Finance* 38:79–93.

Maher, Maria, and Thomas Andersson. 1999. Corporate Governance: Effects on Firm Performance and Economic Growth. Organization for Economic Co-operation and Development. Downloadable from www.OECD.org.

Maquieira, Carlos P., William L. Megginson, and Lance Nail. 1998. Wealth Creation versus Wealth Redistributions in Pure Stock-for-Stock Mergers. *Journal of Financial Economics* 48:3–33.

Megginson, William L. 1990. Restricted Voting Stock, Acquisition Premiums, and the Market Value of Corporate Control. *The Financial Review* 25, no. 2 (May):175–198.

Menyah, Kojo, and Krishna Paudyal. 1996. The Determinants and Dynamics of Bid-Ask Spreads on the London Stock Exchange. *Journal of Financial Research* 19, no. 3 (Fall):377–394.

Mercer, Z. Christopher. 1990. Do Public Company (Minority) Transactions Yield Controlling Interest or Minority Interest Pricing Data? *Business Valuation Review* (December):123–126.

Mercer, Z. Christopher. 1994. Should "Marketability Discounts" Be Applied to Controlling Interests of Private Companies? *Business Valuation Review* (June): 55–65.

Mercer, Z. Christopher. 1997. *Quantifying Marketability Discounts: Developing and Supporting Marketability Discounts in the Appraisal of Closely Held Business Interests*. Memphis, TN: Peabody.

Mercer, Z. Christopher. 1998. The 1999 *Journal of Business Valuation: Proceedings of the Fourth Joint Business Valuation Conference of the Canadian Institute of Chartered Business Valuators and the American Society of Appraisers*, September 24 and 25, Montreal, Quebec, Canada.

Mercer, Z. Christopher. 1999. Theoretical Determinants of Value in the Context of "Levels of Value." Presentation to the American Society of Appraisers, August 23, Memphis TN, Mercer Capital. [See especially slides 17–28 and 46a–46d.]

Mercer, Z. Christopher. 2000. Revisiting the Quantitative Marketability Discount Model. *Valuation Strategies* (March/April).

Mercer, Z. Christopher. 2001. The QMDM and Estimating Required Rates of Return for Restricted Stocks of Public Companies. *Business Valuation Review* 20, no. 2 (June):5.

Much, Paul J., and Timothy J. Fagan. 2000. The Value of Voting Rights. In *Financial valuation: Business and business interests*, edited by James H. Zukin. 2000 update with cumulative index prepared by Richard C. May and Loren B. Garruto. New York: Warren, Gorham & Lamont/RIA Group, pp. U9B-1–U9B-7.

Nath, Eric. 1990. Control Premiums and Minority Interest Discounts in Private Companies. *Business Valuation Review* (June):39–46.

Nath, Eric W. 1994. A Tale of Two Markets. *Business Valuation Review* (September):107–112.

Nath, Eric W. 1997. How Public Guideline Companies Represent Control Value for a Private Company. *Business Valuation Review* (December):167–171.

Phillips, John R., and Neill Freeman. 1995. Do Privately-Held Controlling Interests Sell for Less? *Business Valuation Review* (September):102–113.

Pratt, Shannon P. 1998. *Cost of Capital: Estimations and Applications.* New York: John Wiley & Sons.

Pratt, Shannon P., Robert F. Reilly, and Robert P. Schweihs. 1996. *Valuing a Business: The Analysis and Appraisal of Closely Held Companies*, 3rd ed. New York: McGraw-Hill.

Roach, George P. 1998. Control Premiums and Strategic Mergers. *Business Valuation Review* (December):42–49.

Roll, Richard. 1986. The Hubris Hypothesis of Corporate Takeovers. *Journal of Business* 59:197–216.

Rydqvst, K. 1987. Empirical Investigation of the Voting Premium. Northwestern University. Working paper no. 35.

Schilt, James H. 1996. Discounts for Minority Interests. *Business Valuation Review* (December):161.

Schwert, G. William. 1996. Markup Pricing in Mergers and Acquisitions. *Journal of Financial Economics* 41:153–192.

Simpson, David W. 1991. Minority Interest and Marketability Discounts: A perspective. *Business Valuation Review* (March):7–13.

Sonneman, Donald. 2000. Blockage Discounts for Large Blocks of Publicly Traded Stocks: Benchmark Data to Test for Reasonableness. *Business Valuation Review* (December):201–207.

Stern, Joel. 1993. A Discussion of Mergers and Acquisitions. *Midland Corporate Finance Journal* (Summer) [cited in Jankowske 1995].

Stillman, R. 1983. Examining Antitrust Policy towards Horizontal Mergers. *Journal of Financial Economics* 11:225–240.

Stoll, H. R., 1978a. The Supply of Dealer Services in Securities Markets. *Journal of Finance* 33:1133–1151.

Stoll, H. R., 1978b. The Pricing of Security Dealer Services: An Empirical Study of NASDAQ stocks. *Journal of Finance* 33:1153–1172.

Tversky, Amos, and Daniel Kahneman. 1987. Rational Choice and the Framing of Decisions. In *Rational Choice*, ed. Robin M. Hogarth and Melvin W. Reder. Chicago: University of Chicago Press, pp. 67–94.

Vander Linden, Eric. 1998. Cost of Capital Derived from Ibbotson Data Equals Minority Value? *Business Valuation Review* (December):123–127.

Zingales, Luigi. 1994. The Value of the Voting Right: A Study of the Milan Stock Exchange. *Review of Financial Studies* 7:125–148.

Zingales, Luigi. 1995. What Determines the Value of Corporate Votes? *Quarterly Journal of Economics* (November).

Zukin, James H., ed. 1998. *Financial Valuation: Businesses and Business Interests*. 1998 update with cumulative index prepared by Martin D. Hanan and Ray A. Sheeler. New York: Warren, Gorham & Lamont/RIA Group, p. U9A-12.

# APPENDIX 8A

# Mathematical Appendix

## Developing the Discount Formulas

Initially we assume the current business owner will operate the business for 10 years, sell it, and pay transaction costs of $z$.[116] The next owner will run the business another 10 years, sell it, and pay transaction costs. We assume this pattern occurs *ad infinitum*. Of course, there will be variations from the sale every 10 years—some will sell after 1 year, others after 30 years. In the meantime, in the absence of prior knowledge, we assume every 10 years to be a reasonable estimate of the average of what will occur.

### NPV of Cash Flows with Periodic Transactions Costs Removed

The net present value (NPV) of cash flows to the existing business owner with periodic transactions costs removed is the full amount of the first 10 years' cash flows, plus $(1 - z)$ times the next 10 years' cash flows, where $z$ is the periodic transaction cost, plus $(1 - z)^2$ times the next 10 years' cash flows, and so forth. We will denote the NPV net of transactions costs (i.e., with transactions costs removed from the stream of cash flows) as $\text{NPV}_{\text{TC}}$. Equation (A8.1) computes $\text{NPV}_{\text{TC}}$ using a midyear cash flow assumption.

$$
\begin{aligned}
NPV_{TC} = & \left[\frac{1}{(1+r)^{0.5}} + \frac{(1+g)}{(1+r)^{1.5}} + \cdots + \frac{(1+g)^9}{(1+r)^{9.5}}\right] \\
& + (1-z)\left[\frac{(1+g)^{10}}{(1+r)^{10.5}} + \ldots + \frac{(1+g)^{19}}{(1+r)^{19.5}}\right] \\
& + (1-z)^2\left[\frac{(1+g)^{20}}{(1+r)^{20.5}} + \ldots + \frac{(1+g)^{29}}{(1+r)^{29.5}}\right] + \ldots . \qquad \text{(A8.1)}
\end{aligned}
$$

[116]As explained in the body of the chapter, $z$ is an incremental transaction cost. For example, when we value a small fractional ownership in a privately owned business, often our preliminary value is on a marketable minority basis. In this case $z$ would be the difference in transaction cost (expressed as a percentage) between selling a private business interest and selling publicly traded stock through a stockbroker.

Multiplying each term in equation (A8.1) by $(1 + g)/(1 + r)$, we get:

$$\begin{aligned}\frac{1+g}{1+r}NPV_{TC} &= \left[\frac{1+g}{(1+r)^{1.5}} + \ldots + \frac{(1+g)^{10}}{(1+r)^{10.5}}\right]\\ &+ (1-z)\left[\frac{(1+g)^{11}}{(1+r)^{11.5}} + \ldots + \frac{(1+g)^{20}}{(1+r)^{20.5}}\right]\\ &+ (1-z)^2\left[\frac{(1+g)^{21}}{(1+r)^{21.5}} + \ldots + \frac{(1+g)^{30}}{(1+r)^{30.5}}\right] + \ldots . \qquad \text{(A8.2)}\end{aligned}$$

Subtracting equation (A8.2) from equation (A8.1), we get:

$$\begin{aligned}[1 - \frac{1+g}{1+r}]NPV_{TC} &= \left[\frac{1}{(1+r)^{0.5}} - \frac{(1+g)^{10}}{(1+r)^{10.5}}\right] + (1-z)\left[\frac{(1+g)^{10}}{(1+r)^{10.5}} - \frac{(1+g)^{20}}{(1+r)^{20.5}}\right]\\ &+ (1-z)^2\left[\frac{(1+g)^{20}}{(1+r)^{20.5}} - \frac{(1+g)^{30}}{(1+r)^{30.5}}\right] + \ldots . \qquad \text{(A8.3)}\end{aligned}$$

Note that all terms in each sequence drop out except for the first terms in equation (A8.1) and the last terms in equation (A8.2). In equation (A8.4), we collect the positive terms from equation (A8.3) in the first set of square brackets and the negative terms from equation (A8.3) in the second set. Additionally, the left-hand side of equation (A8.3) reduces to $\frac{r-g}{1+r}NPV_{TC}$. Multiplying through by $\frac{1+r}{r-g}$, we get:

$$\begin{aligned}NPV_{TC} = \frac{1+r}{r-g}\Bigg\{&\left[\frac{1}{(1+r)^{0.5}} + (1-z)\,\frac{(1+g)^{10}}{(1+r)^{10.5}} + (1-z)^2\,\frac{(1+g)^{20}}{(1+r)^{20.5}} + \ldots\right]\\ &- \left[\frac{(1+g)^{10}}{(1+r)^{10.5}} + (1-z)\,\frac{(1+g)^{20}}{(1-r)^{20.5}} + (1-z)^2\,\frac{(1+g)^{30}}{(1+r)^{30.5}} + \ldots\right]\Bigg\}. \qquad \text{(A8.4)}\end{aligned}$$

Next we will manipulate the right-hand side of the equation only. We divide the term $\frac{1+r}{r-g}$ by $\sqrt{1+r}$, which leaves that term as $\frac{\sqrt{1+r}}{r-g}$ and we multiply all terms inside the brackets by $\sqrt{1+r}$. The latter action has the effect of reducing the exponents in the denominators by 0.5 years. Thus we get:

$$\begin{aligned}NPV_{TC} = \frac{\sqrt{1+r}}{r-g}\Bigg\{&\left[1 + (1-z)\left(\frac{1+g}{1+r}\right)^{10} + (1-z)^2\left(\frac{1+g}{1+r}\right)^{20} + \ldots\right]\\ &- \left[\left(\frac{1+g}{1+r}\right)^{10} + (1-z)\left(\frac{1+g}{1+r}\right)^{20} + (1-z)^2\left(\frac{1+g}{1+r}\right)^{30} + \ldots\right]\Bigg\}. \qquad \text{(A8.5)}\end{aligned}$$

Recognizing that each term in brackets is an infinite geometric sequence, this solves to:

$$NPV_{TC} = \frac{\sqrt{1+r}}{r-g}\left[\frac{1}{1 - \frac{(1-z)(1+g)^{10}}{(1+r)^{10}}} - \frac{\left(\frac{1+g}{1+r}\right)^{10}}{1 - \frac{(1-z)(1+g)^{10}}{(1+r)^{10}}}\right]. \qquad \text{(A8.6)}$$

Since the denominators are identical, we can combine both terms in the brackets into a single term by adding the numerators.

$$NPV_{TC} = \frac{\sqrt{1+r}}{r-g}\left[\frac{1-\left(\frac{1+g}{1+r}\right)^{10}}{1-(1-z)\left(\frac{1+g}{1+r}\right)^{10}}\right]. \tag{A8.7}$$

Letting $x = \frac{1+g}{1+r}$, this simplifies to:

$$NPV_{TC} = \frac{\sqrt{1+r}}{r-g}\left\{\frac{1-x^{10}}{1-(1-z)\,x^{10}}\right\}. \tag{A8.8}$$

## The Discount Formula

$D$, the component of the discount for lack of marketability that measures the periodic transactions costs, is 1 minus the ratio of the NPV of the cash flows net of transactions costs ($NPV_{TC}$) to the NPV without removing transactions costs (NPV). Using a midyear Gordon model formula of $\frac{\sqrt{1+r}}{r-g}$ as the NPV and substituting equation (A8.8) for $NPV_{TC}$, we come to:

$$D = 1 - \frac{NPV_{TC}}{NPV} = 1 - \frac{\frac{\sqrt{1+r}}{r-g}\left\{\frac{1-x^{10}}{1-(1-z)\,x^{10}}\right\}}{\frac{\sqrt{1+r}}{r-g}}. \tag{A8.9}$$

The term $\frac{\sqrt{1+r}}{r-g}$ cancels out, and the expression simplifies to:

$$D = 1 - \frac{1-x^{10}}{1-(1-z)\,x^{10}}, \quad \text{where } x = \frac{1+g}{1+r} \quad \text{and } g < r, \Rightarrow 0 < x < 1. \tag{A8.10}$$

Equation (A8.10) is the formula for the discount assuming a sale every 10 years. Instead of assuming a business sale every 10 years, now we let the average years between sales be a variable, $j$, which leads to the generalized equation in (A8.11):

$$D = 1 - \frac{1-x^{j}}{1-(1-z)\,x^{j}} \quad \text{Generalized discount formula—sellers' transactions costs.} \tag{A8.11}$$

In determining fair market value, we ask how much would a rational buyer pay for (and for how much would a rational seller sell) a business interest. That presumes a hypothetical sale at time zero. Equation (A8.11) is the formula appropriate for quantifying sellers' transactions costs, because the buyer does not care about the seller's costs, which means he or she will not raise the price in order to cover the seller. However, the buyer does care that 10 years down the road, he or she will be a seller, not a buyer, and the new buyer will reduce the price to cover his or her transaction costs, and so on *ad infinitum*. Thus, we want to quantify the discounts due to transactions costs for the continuum of sellers beginning with the second sale, that is, in year $j$. Equation (A8.11) accomplishes that.

Using an end-of-year Gordon model assumption instead of midyear cash flows leads to the identical equation; that is, (A8.11) holds for both.

## Buyer Discounts Begin with the First Transaction

An important variation of equation (A8.11) is to consider what happens if the first relevant transaction cost takes place at time zero instead of $t = j$, which is appropriate for quantifying the discount component due to buyers' transactions costs. With this assumption, we would modify the above analysis by inserting a $(1 - z)$ in front of the first series of bracketed terms in equation (A8.1) and increasing the exponent of all the other $(1 - z)$ terms by one. All the other equations are identical, with the $(1 - z)$ term added. Thus, the buyers' equivalent formula of equation (A8.8) is:

$$NPV_{TC} = (1 - z)\frac{\sqrt{1+r}}{r-g}\left\{\frac{1 - x^{10}}{1 - (1 - z)\,x^{10}}\right\} \text{ NPV with buyers' transactions} \times \text{costs removed.} \tag{A8.8a}$$

Obviously, equation (A8.8a) is lower than equation (A8.8), because the first relevant cost occurs 10 years earlier. The generalized discount formula equivalent of equation (A8.11) for the buyer scenario is:

$$D = 1 - \frac{(1 - z)(1 - x^j)}{1 - (1 - z)\,x^j} \text{ Generalized discount formula—buyers' transactions costs.} \tag{A8.11a}$$

We demonstrate the accuracy of equations (A8.11) and (A8.11a), which are excerpted from here and renumbered in the chapter as equations (8.9) and (8.9a), in Tables 8.12 and 8.13 in the body of the chapter.

## NPV of Cash Flows with Finite Transactions Costs Removed[117]

The previous formulas for calculating the present value of the discount component for buyers' and sellers' transactions costs are appropriate when the underlying assets have an infinite life and are accurate enough when their lives are several decades long—or, more accurately, long enough to encompass several transactions, which, of course, also depends on the average number of years between transactions ($j$).

Even if an entity with a limited life owns an asset with infinite life (e.g., real estate), there will still be an infinite continuum of transactions and related costs. The fact that most of them will occur after the termination of the entity that currently owns them is irrelevant. In this scenario, equations (A8.11) and (A8.11a) are still appropriate.

In this section we calculate a formula for the present value of a finite series of transactions costs, that is, for component #3A of DLOM for limited life assets. This section is very mathematical and will have practical significance for most readers only when the life of the entity is short (probably under 30 years) and the growth rate is close to the discount rate. Some readers will want to skip this section, perhaps noting the final equation, (A8.22). Consider this section as reference material.

[117]This section has changed substantially since the first edition.

Let's assume that a limited partnership owns a copyright with a 26-year life that sells for every $j = 10$ years. Thus, after the initial hypothetical sale, there will be sales of the fractional interest at years 10 and 20, before the copyright expires and the partnership dissolves. We begin by repeating equations (A8.1) and (A8.2) as (A8.12) and (A8.13), respectively, with the difference that the last incremental transaction cost occurs at $n = 20$ years instead of going on perpetually.

$$NPV_{TC} = \left[\frac{1}{(1+r)^{0.5}} + \frac{(1+g)}{(1+r)^{1.5}} + \ldots + \frac{(1+g)^9}{(1+r)^{9.5}}\right] + (1-z)\left[\frac{(1+g)^{10}}{(1+r)^{10.5}} + \ldots + \frac{(1+g)^{19}}{(1+r)^{19.5}}\right] + (1-z)^2\left[\frac{(1+g)^{20}}{(1+r)^{20.5}} + \ldots + \frac{(1+g)^{26}}{(1+r)^{26.5}}\right], \quad \text{(A8.12)}$$

$$\frac{1+g}{1+r}NPV_{TC} = \left[\frac{1+g}{(1+r)^{1.5}} + \ldots + \frac{(1+g)^{10}}{(1+r)^{10.5}}\right] + (1-z)\left[\frac{(1+g)^{11}}{(1+r)^{11.5}} + \ldots + \frac{(1+g)^{20}}{(1+r)^{20.5}}\right] + (1-z)^2\left[\frac{(1+g)^{21}}{(1+r)^{21.5}} + \ldots + \frac{(1+g)^{27}}{(1+r)^{27.5}}\right]. \quad \text{(A8.13)}$$

Subtracting equation (A8.13) from equation (A8.12), we get:

$$\left[1 - \frac{1+g}{1+r}\right]NPV_{TC} = \left[\frac{1}{(1+r)^{0.5}} - \frac{(1+g)^{10}}{(1+r)^{10.5}}\right] + (1-z)\left[\frac{(1+g)^{10}}{(1+r)^{10.5}} - \frac{(1+g)^{20}}{(1+r)^{20.5}}\right] + (1-z)^2\left[\frac{(1+g)^{20}}{(1+r)^{20.5}} - \frac{(1+g)^{27}}{(1+r)^{27.5}}\right]. \quad \text{(A8.14)}$$

Again, the first term of the equation reduces to $\frac{r-g}{1+r}$. We then multiply both sides by its inverse:

$$NPV_{TC} = \frac{1+r}{r-g}\left\{\left[\frac{1}{(1+r)^{0.5}} - \frac{(1+g)^{10}}{(1+r)^{10.5}}\right] + (1-z)\left[\frac{(1+g)^{10}}{(1+r)^{10.5}} - \frac{(1+g)^{20}}{(1+r)^{20.5}}\right] + (1-z)^2\left[\frac{(1+g)^{20}}{(1+r)^{20.5}} - \frac{(1+g)^{27}}{(1+r)^{27.5}}\right]\right\}. \quad \text{(A8.15)}$$

As before, we divide the first term on the right-hand side of the equation by $\sqrt{1+r}$ and multiply all terms inside the brackets by the same. This has the same effect as reducing the exponents in the denominators by 0.5 years.

$$NPV_{TC} = \frac{\sqrt{1+r}}{r-g}\left\{\left[1 - \left(\frac{1+g}{1+r}\right)^{10}\right] + (1-z)\left[\left(\frac{1+g}{1+r}\right)^{10} - \left(\frac{1+g}{1+r}\right)^{20}\right] + (1-z)^2\left[\left(\frac{1+g}{1+r}\right)^{20} - \left(\frac{1+g}{1+r}\right)^{27}\right]\right\}. \quad \text{(A8.16)}$$

Letting $y = 1 - z$ and $x = \frac{1+g}{1+r}$, equation (A8.16) becomes:

$$NPV_{TC} = \frac{\sqrt{1+r}}{r-g}\left[\left(1-x^{10}\right) + y\left(x^{10}-x^{20}\right) + y^2\left(x^{20}-x^{27}\right)\right], \tag{A8.17}$$

$$NPV_{TC} = \frac{\sqrt{1+r}}{r-g}\left[\left(1+yx^{10}+y^2x^{20}\right) - \left(x^{10}+yx^{20}\right) + y^2x^{27}\right]. \tag{A8.18}$$

Within the square brackets in equation (A8.18), there are three sets of terms, the first two being set off in parentheses. Each of them is a finite geometric sequence. Employing the geometric sequence mathematics, the first sequence solves to $\frac{1-yx^{30}}{1-yx^{10}}$. The second sequence solves to $\frac{x^{10}-y^2x^{30}}{1-yx^{10}}$. They both have the same denominator, so we can combine them, keeping in mind that the second sequence has a minus sign in front of it. Thus, equation (A8.18) simplifies to:

$$NPV_{TC} = \frac{\sqrt{1+r}}{r-g}\left[\frac{1-x^{10}+yx^{30}\left(y-1\right)}{1-yx^{10}} + y^2x^{27}\right], \tag{A8.19}$$

$$NPV_{TC} = \frac{\sqrt{1+r}}{r-g}\left\{\frac{1-x^{10}-z(1-z)x^{30}}{1-(1-z)x^{10}} + (1-z)^2x^{27}\right\}. \tag{A8.20}$$

Note that if we keep all terms to the right of the $-x^{10}$ in the numerator and the rightmost term, which is not in the fraction, equation (A8.20) reduces to equation (A8.8).

Since an ADF with growth equals the Gordon multiple times $(1 - x^n)$, the discount equals:

$$\frac{NPV_{TC}}{NPV} = \frac{\frac{\sqrt{1+r}}{r-g}\left\{\frac{1-x^{10}-z(1-z)x^{30}}{1-(1-z)x^{10}} + (1-z)^2x^{27}\right\}}{\frac{\sqrt{1+r}}{r-g}\left(1-x^{10}\right)}. \tag{A8.21}$$

As previously, the Gordon model multiple cancels. The discount for the sellers' transactions costs equals 1 minus (A8.21), or:

$$D = 1 - \frac{\frac{1-x^{10}-z(1-z)x^{30}}{1-(1-z)x^{10}} + (1-z)^2x^{27}}{\left(1-x^{10}\right)}. \tag{A8.22}$$

As is clear, this expression does not generalize easily. It does not seem worthwhile to do so. If you need such a formula, it is easier to model this in a spreadsheet, develop a discount formula specific to the timing of your valuation, or contact the author for help.

## Summary of Mathematical Analysis in Remainder of Appendix

The remainder of the appendix is devoted to calculating partial derivatives necessary to evaluate the behavior of the discount formula (A8.11). The partial derivatives of

$D$ with respect to its underlying independent variables, $g$, $r$, $z$, and $j$, give us the slope of the discount as a function of each variable. The purpose in doing so is to see how $D$ behaves as the independent variables change.

It turns out that $D$ is a monotonic function with respect to each of its independent variables. That is analytically convenient, as it means that an increase in any one of the independent variables always affects $D$ in the same direction. For example, if $D$ is monotonically increasing in $g$, that means that an increase in $g$ will always lead to an increase in $D$, and a decrease in $g$ leads to a decrease in $D$. If $D$ is monotonically increasing, there is no value of $g$ such that an increase in $g$ leads to either no change in $D$ or a decrease in $D$.

The results that we develop in the remainder of the appendix are that the discount, $D$, is monotonically increasing with $g$ and $z$ and decreasing with $r$ and $j$. The practical reader will probably want to stop here.

## Mathematical Analysis of the Discount—Calculating Partial Derivatives

We can compute an alternative form of equation (A8.11) by multiplying the numerator by $-1$ and changing the minus sign before the fraction to a plus sign. This will ease the computations of the partial derivatives of the expression.

$$D = 1 + \frac{x^j - 1}{1 - (1-z)\,x^j}, \tag{A8.23}$$

$$\frac{\partial D}{\partial x} = \frac{\left\{\left[1-(1-z)\,x^j\right] jx^{j-1}\right\} - \left\{\left(x^j - 1\right)\,\left[-(1-z)\,jx^{j-1}\right]\right\}}{\left[1-(1-z)\,x^j\right]^2}. \tag{A8.24}$$

Factoring out $jx^{j-1}$, we get:

$$\frac{\partial D}{\partial x} = \frac{jx^{j-1}\left\{\left[1-(1-z)\,x^j\right] + \left(x^j-1\right)(1-z)\right\}}{\left[1-(1-z)\,x^j\right]^2}, \tag{A8.25}$$

$$\frac{\partial D}{\partial x} = \frac{jx^{j-1}\left[1-(1-z)\,x^j + (1-z)\,x^j - (1-z)\right]}{\left[1-(1-z)\,x^j\right]^2}. \tag{A8.26}$$

Note that $-(1-z)x^j$ and $(1-z)x^j$ cancel out in the numerator. Also, the $1-(1-z) = z$. This simplifies to:

$$\frac{\partial D}{\partial x} = \frac{jx^{j-1}z}{\left[1-(1-z)\,x^j\right]^2} > 0. \tag{A8.27}$$

Since $j$, $x$, and $z$ are all positive, the numerator is positive. Since the denominator is squared, it is also positive. Therefore the entire expression is positive. The means that the discount is monotonically increasing in $x$.

We begin equation (A8.28) with a repetition of the definition of $x$ in order to compute its partial derivatives.

$$x = \frac{1+g}{1+r}. \tag{A8.28}$$

Differentiating equation (A8.28) with respect to $g$, we get:

$$\frac{\partial x}{\partial g} = \frac{(1+r)(1)}{(1+r)^2} = \frac{1}{1+r} > 0. \tag{A8.29}$$

Differentiating equation (A8.28) with respect to $r$, we get:

$$\frac{\partial x}{\partial r} = \frac{-(1+g)(1)}{(1+r)^2} = -\frac{(1+g)}{(1+r)^2} < 0. \tag{A8.30}$$

Using the chain rule, the partial derivative of $D$ with respect to $g$ is the partial derivative of $D$ with respect to $x$ multiplied by the partial derivative of $x$ with respect to $g$, or:

$$\frac{\partial D}{\partial g} = \frac{\partial D}{\partial x}\frac{\partial x}{\partial g} > 0. \tag{A8.31}$$

The first term on the right-hand side of the equation is positive by equation (A8.27), and the second term is positive by equation (A8.29). Therefore the entire expression is positive. Thus the discount is monotonically increasing in $g$. Using the chain rule again with respect to $r$, we get:

$$\frac{\partial D}{\partial r} = \frac{\partial D}{\partial x}\frac{\partial x}{\partial r} < 0. \tag{A8.32}$$

Thus the discount is monotonically decreasing in $r$. Now we make an algebraic substitution to simplify the expression for $D$ in order to facilitate calculating other partial derivatives.

Let

$$y = 1 - z, \tag{A8.33}$$

$$\frac{dy}{dz} = -1. \tag{A8.34}$$

Substituting equation (A8.33) into equation (A8.23), we get:

$$D = 1 + \frac{x^j - 1}{1 - yx^j}, \tag{A8.35}$$

$$\frac{\partial D}{\partial y} = \frac{\left(1 - x^j\right)\left(-x^j\right)}{\left(1 - yx^j\right)^2} = \frac{x^j\left(x^j - 1\right)}{\left(1 - yx^j\right)^2}, \tag{A8.36}$$

$$\frac{\partial D}{\partial z} = \frac{\partial D}{\partial y}\frac{dy}{dz} = \frac{x^j\left(x^j - 1\right)(-1)}{\left[1 - (1-z)\,x^j\right]^2} > 0. \tag{A8.37}$$

The denominator of (A8.37), being squared, is positive. The numerator is also positive, as $x^j$ is positive and less than 1, which means that $x^j - 1$ is negative, which, when multiplied by $-1$, results in a positive number. Thus the entire partial derivative is positive, which means that $D$ is monotonically increasing in $z$, the transaction costs. This result is intuitive, as it makes sense that the greater the transaction costs, the greater the discount.

Differentiating equation (A8.35) with respect to $j$, the average number of years between sales, we get:

$$\frac{\partial D}{\partial j} = \frac{\left(1 - yx^j\right) x^j \ln x - \left(x^j - 1\right)(-y)\, x^j \ln x}{\left(1 - yx^j\right)^2}. \qquad \text{(A8.38)}$$

Factoring out $x^j \ln x$, we get:

$$\frac{\partial D}{\partial j} = \frac{x^j \ln x \left(1 - yx^j + yx^j - y\right)}{\left(1 - yx^j\right)^2} = \frac{x^j \ln x \left[1 - y\right]}{\left(1 - yx^j\right)^2}. \qquad \text{(A8.39)}$$

Substituting $z$ for $1 - y$, we get:

$$\frac{\partial D}{\partial j} = \frac{x^j z \ln x}{\left[1 - (1 - z)\, x^j\right]^2} < 0. \qquad \text{(A8.40)}$$

The denominator is positive. The numerator is negative; since $x < 1$, $\ln x < 0$. Thus the discount is monotonically decreasing in $j$, the average years between sales. That is intuitive, as the less frequently businesses sell, the smaller the discount should be.

## Summary of Comparative Statics

Summarizing, the discount for periodic transaction costs is related in the following ways to its independent variables:

| Variable | Varies with Discount | Monotonically |
|---|---|---|
| $r$ | Negatively | Decreasing |
| $g$ | Positively | Increasing |
| $z$ | Positively | Increasing |
| $j$ | Negatively | Decreasing |

# PART IV

# Putting It All Together

## Introduction

Part IV of this book consists of Chapters 9 and 10. Chapter 9 empirically tests the log size and economic components models by reconciling price to cash flow (P/CF) multiples calculated using these models with P/CF multiples for groups of firms of different sizes in the Institute of Business Appraisers's (IBA) database. The results provide weak support for the two models, but missing data make it impossible to provide strong support. There is simply too much data that we need that does not exist in the IBA database or any other one of which I am aware.

In Chapter 10 we look at two issues. In the first half of the chapter, we calculate 95% confidence intervals around our valuation estimate using the log size model, assuming we forecast cash flows and adjust for control and marketability perfectly. The importance of this is to understand how much statistical uncertainty there is in our valuation estimates.

The second half of Chapter 10 is concerned with measuring the valuation errors that arise from errors in forecasting cash flow and growth rates and calculating discount rates. We look at the effects of both relative and absolute errors, and we show how the majority of these errors affect the valuation of large firms more than small firms.

Part IV does not consist of practical, hands-on, "how-to" chapters. It can be skipped by the time-pressed reader. Nevertheless, for one who wants to be well educated and familiar with important theoretical and empirical issues in valuation, these chapters are important.

# CHAPTER 9

# Empirical Testing of Abrams's Valuation Theory

# CHAPTER 9

# Empirical Testing of Abrams's Valuation Theory

## Introduction

Many appraisers have long believed that when small businesses sell, they are priced very differently than large businesses, and that the rules governing their valuation are totally different. I, too, held this opinion at one time, but this chapter is evidence—though not proof—that it is not true.

A skeptic could level the charge that the log size discount rate equation is based on a mathematical relationship that exists between returns and size of NYSE/AMEX/NASDAQ firms, but it may not apply to the universe of small and medium privately held firms. Additionally, the calculations of the transactions costs component of the discount for lack of marketability (DLOM) are based on interviews, and then quantified in an equation and extrapolated downward for small firms. Thus, it's nice in theory, but does it really work in practice?

The purpose of this chapter is to subject the log size and economic components models to empirical testing to see whether they do a good job of explaining real-world transactions of smaller businesses. Our primary data come from an article published by Raymond Miles (Miles, 1992)—which we refer to throughout this chapter as "the article"—about the relationship of size to price/earnings (PE) multiples in the Institute of Business Appraisers' (IBA) database.

### Steps in the Valuation Process

Using a simple discounted cash flow model as the valuation paradigm, valuation consists of four steps:

1. Forecast cash flows.
2. Discount to net present value.
3. Adjust for marketability or lack thereof.
4. Adjust for degree of control.

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I offer my profound thanks to Mr. Raymond Miles for his considerable help. Without his vitally important research, this chapter would be impossible. Also, Professor Haim Mendelson of Stanford University provided extremely helpful comments.

### Applying a Valuation Model to the Steps

The sales described in the article are all $1 million or less. It is a reasonable assumption that the vast majority of the small firms in the IBA transactional database are mature. The number of high-growth start-up firms in that database is likely to be small. Therefore, it is reasonable to assume a constant growth rate to perpetuity. Using a Gordon model to apply to the next year's forecast cash flows should give us a fairly accurate FMV on a marketable minority level. Using a midyear assumption, the formula is:

$$FMV = CF_{t+1} \frac{\sqrt{1+r}}{r-g}, \tag{9.1}$$

where $r$ is the discount rate, which we will estimate using the log size model, and $g$ is the constant growth rate, which we will estimate. That takes care of the first two valuation steps.

We will use the economic components model from Chapter 8 for our calculations of DLOM. We assume a control premium of 25%, which is the approximate midpoint of the 21%–28% range estimated in Chapter 8.

There are only two major principles in steps 2 and 3 of business valuation—risk and marketability—which are both functions of size. Thus, size is the overriding principle in steps 2 and 3 of the valuation process, and step 1 determines size. If value depends only on the forecast cash flows, risk, and marketability, and the latter two are in turn dependent on size, then in essence value depends only on size (and possibly control). That statement sounds like a tautology, but it is not.

This chapter is an attempt to identify the fewest, most basic principles underlying the inexact science of valuation. The remainder of this chapter covers the calculations that test the log size model and DLOM calculations.

## Table 9.1: Log Size for 1938–1986

In Table 9.1, we develop the log size equation for the years 1938–1986.[1] The reason we stop at 1986 has to do with the IBA database. The article is based on sales from 1982–1991.[2] We take 1986 as the midpoint of that range and calculate our log size equation from 1938 to 1986.

B7–B16 and C7–C16 contain the mean and standard deviation of returns for the 10 deciles for the period 1938–1986. We need to be able to regress the returns against 1986 average market capitalization for each decile. Unfortunately, those values are unavailable, and we must estimate them.

D7–D16 contain the market capitalization for the average firm in each decile for 1994, the earliest year for which decile breakdowns are available. E7–E16 are the

[1]In the first edition of this book, we used 1938 as the starting year to eliminate the highly volatile Roaring Twenties and Depression years 1926–1937. While we discontinued that practice in this edition because of the Financial Crisis of 2008, we leave this chapter largely as it was in the first edition, as the difference is unlikely to make a material difference in the final results.

[2]A footnote in the article states that in relation to Figure 1 (and I confirmed this with the author, Raymond Miles), those dates apply to the rest of the article.

| | A | B | C | D | E | F | G | H |
|---|---|---|---|---|---|---|---|---|
| 1 | **Table 9.1** | | | | | | | |
| 2 | **Log Size Equation for 1938–1986** | | | | | | | |
| 3 | **NYSE Data by Decile and Statistical Analysis: 1938–1986** | | | | | | | |
| 4 | | | | | | | | |
| 5 | | | | | **Year-End Index Values [1]** | | **[D] x [E] / [F]** | **Ln [G]** |
| 6 | **Decile** | **Mean** | **Std Dev** | **94 Mkt Cap** | **1986** | **1994** | **1986 Mkt Cap** | **Ln(Mkt Cap)** |
| 7 | **1** | **11.8%** | **15.8%** | 14,847,774,614 | 198.868 | 404.436 | 7,300,897,357 | 22.7113 |
| 8 | **2** | **14.0%** | **18.3%** | 3,860,097,544 | 434.686 | 920.740 | 1,822,371,137 | 21.3234 |
| 9 | **3** | **15.0%** | **19.7%** | 2,025,154,234 | 550.313 | 1,248.528 | 892,625,877 | 20.6097 |
| 10 | **4** | **15.8%** | **22.0%** | 1,211,090,551 | 637.197 | 1,352.924 | 570,396,575 | 20.1618 |
| 11 | **5** | **16.7%** | **23.0%** | 820,667,228 | 856.893 | 1,979.698 | 355,217,881 | 19.6882 |
| 12 | **6** | **17.1%** | **23.8%** | 510,553,019 | 809.891 | 1,809.071 | 228,566,124 | 19.2473 |
| 13 | **7** | **17.6%** | **26.4%** | 339,831,804 | 786.298 | 1,688.878 | 158,216,901 | 18.8795 |
| 14 | **8** | **19.0%** | **28.5%** | 208,098,608 | 1,122.906 | 2,010.048 | 116,253,534 | 18.5713 |
| 15 | **9** | **19.7%** | **29.9%** | 99,534,481 | 1,586.521 | 2,455.980 | 64,297,569 | 17.9790 |
| 16 | **10** | **22.7%** | **38.0%** | 33,746,259 | 6,407.216 | 6,654.508 | 32,492,195 | 17.2965 |
| 17 | | | | | | | | |
| 18 | SUMMARY OUTPUT | | | | | | | |
| 19 | | | | | | | | |
| 20 | *Regression Statistics* | | | | | | | |
| 21 | Multiple R | 0.9806 | | | | | | |
| 22 | R Square | 0.9617 | | | | | | |
| 23 | Adjusted R Square | 0.9569 | | | | | | |
| 24 | Standard Error | 0.0064 | | | | | | |
| 25 | Observations | 10 | | | | | | |
| 26 | | | | | | | | |
| 27 | ANOVA | | | | | | | |
| 28 | | *df* | *SS* | *MS* | *F* | *Significance F* | | |
| 29 | Regression | 1 | 0.0082 | 0.0082 | 200.6663 | 0.0000 | | |
| 30 | Residual | 8 | 0.0003 | 0.0000 | | | | |
| 31 | Total | 9 | 0.0085 | | | | | |
| 32 | | | | | | | | |
| 33 | | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* | |
| 34 | Intercept | 0.5352 | 0.0259 | 20.6710 | 0.0000 | 0.4755 | 0.5949 | |
| 35 | Ln(Mkt Cap) | (0.0186) | 0.0013 | (14.1657) | 0.0000 | (0.0216) | (0.0156) | |
| 36 | | | | | | | | |
| 37 | [1] SBBI, Table 7-3*, approximate income returns have been removed from the 1994 values. The adjustment was derived by | | | | | | | |
| 38 | comparing the large company stock total return indices with the capital appreciation indices for 1994 and 1986 per SBBI | | | | | | | |
| 39 | Tables B-1 and B-2. It was found that 77.4 % of the total return was due to capital appreciation. There were no capital | | | | | | | |
| 40 | appreciation indices for small company stocks. We removed (1–77.4%) of the gain in the decile index values for deciles 1 | | | | | | | |
| 41 | through 5, [(1–77.4%)/2] for deciles 6 through 8, and made no adjustment for 9 and 10. Larger stocks tend to pay larger | | | | | | | |
| 42 | dividends. | | | | | | | |
| 43 | | | | | | | | |
| 44 | * *Source:* Morningstar, Inc — *1998 Ibbotson Stocks, Bonds, Bills and Inflation (SBBI) Yearbook.* [Certain portions of this | | | | | | | |
| 45 | work were derived from copyrighted works of Roger G. Ibbotson and Rex Sinquefield.] Source: ©CRSP, University of | | | | | | | |
| 46 | Chicago. Used with permission. All rights reserved. | | | | | | | |

1986 year-end index values in Ibbotson's Table 7-4. F7–F16 are the 1994 year-end index values, with our estimate of income returns removed.[3]

Column G is our estimate of 1986 average market capitalization per firm for each decile. We calculate it as column D × column E ÷ column F. Thus, the average firm size in decile #1 for 1986 is $7.3 billion (G7), and for decile #10 it is $32.49 million (G16).

---

[3]SBBI, Table 7-4, approximate income returns have been removed from the 1994 values. The adjustment was derived by comparing the large company stock total return indices with the capital appreciation indices for 1994 and 1986 per SBBI Tables B-1 and B-2. It was found that 77.4% of the total return was due to capital appreciation. There were no capital appreciation indices for small company stocks. We removed $1 - 77.4\% = 22.6\%$ of the gain in the decile index values for deciles #1 through #5, $22.6\%/2 = 11.3\%$ for deciles #6 through #8, and made no adjustment for #9 and #10. Larger stocks tend to pay larger dividends.

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table 9.2** | | | | | | | | |
| 2 | **Reconciliation to IBA Database** | | | | | | | | |
| 3 | | | | | | | | | |
| 4 | **Part 1: IBA P/CF Multiples** | | | | | | | | |
| 5 | | | | | | | | | |
| 6 | **Mean Selling Price: Illiquid 100% Int** | **25,000** | **75,000** | **125,000** | **175,000** | **225,000** | **375,000** | **750,000** | **Avg** |
| 7 | **Mean P/E Ratio** | 1.66 | 2.11 | 2.44 | 2.74 | 3.06 | 3.44 | 4.26 | |
| 8 | **Owner's Discretionary Inc [6] / [7]** | 15,060 | 35,545 | 51,230 | 63,869 | 73,529 | 109,012 | 176,056 | |
| 9 | **Arm's-Length Salary** | 22,500 | 25,000 | 30,000 | 35,000 | 40,000 | 50,000 | 75,000 | |
| 10 | **Personal Exp Charged to Bus—Assume B33x[8]** | 1,506 | 3,555 | 5,123 | 6,387 | 7,353 | 10,901 | 17,606 | |
| 11 | **Adjusted Net Income = [8] – [9] + [10]** | (5,934) | 14,100 | 26,352 | 35,255 | 40,882 | 69,913 | 118,662 | |
| 12 | **Effective Corp. Inc Tax Rate** | 0% | 0% | 0% | 0% | 0% | 0% | 0% | |
| 13 | **Adjusted Inc Taxes** | 0 | 0 | 0 | 0 | 0 | 0 | 0 | |
| 14 | **Adj Net Inc after Tax** | (5,934) | 14,100 | 26,352 | 35,255 | 40,882 | 69,913 | 118,662 | |
| 15 | **Cash Flow/Net Income (Assumed)** | 95% | 95% | 95% | 95% | 95% | 95% | 95% | |
| 16 | **Adj Cash Flow after Tax = [14] × [15]** | (5,637) | 13,395 | 25,035 | 33,493 | 38,838 | 66,417 | 112,729 | |
| 17 | **Avg Disc to Cash Equiv Value (Table 10-3)** | 6.7% | 6.7% | 6.7% | 6.7% | 6.7% | 6.7% | 6.7% | |
| 18 | **Adj Sell Price (Illiq 100% Int) = { 1–[17] } × [6]** | 23,317 | 69,951 | 116,585 | 163,220 | 209,854 | 349,756 | 699,512 | |
| 19 | **Adjusted Price/Cash Flow Multiple = [18] / [16]** | **NM** | **5.2** | **4.7** | **4.9** | **5.4** | **5.3** | **6.2** | |
| 20 | | | | | | | | | |
| 21 | **Part 2: Log Size P/CF Multiples** | | | | | | | | |
| 22 | **Control Prem—% (1982-1991 Avg) [Note 1]** | 25% | 25% | 25% | 25% | 25% | 25% | 25% | |
| 23 | **DLOM—% (Tables 9.6, 9.6A, 9.6B, etc.)** | 9.9% | 10.1% | 10.2% | 10.4% | 10.5% | 12.4% | 18.6% | |
| 24 | **Adj Sell Price (Mkt Min)=[18]/{(1+[22]) × (1–[23])}** | 20,704 | 62,221 | 103,838 | 145,660 | 187,511 | 319,458 | 687,614 | |
| 25 | **Discount Rate = $r$ = 0.5352 – 0.0186 ln ($FMV_{Mkt\ Min}$)** | 35.0% | 33.0% | 32.0% | 31.4% | 30.9% | 29.9% | 28.5% | |
| 26 | **Growth Rate = $g$ (Assumed)** | 2.0% | 2.5% | 3.0% | 4.0% | 4.5% | 5.0% | 6.0% | |
| 27 | **Theoretical P/CF = (1+$g$) x SQRT(1+$r$)/($r$–$g$)** | 3.6 | 3.9 | 4.1 | 4.4 | 4.5 | 4.8 | 5.3 | |
| 28 | **P/CF-Illiquid Control = [27] × (1+[22]) × (1–[23])** | 4.0 | 4.4 | 4.6 | 4.9 | 5.1 | 5.3 | 5.4 | |
| 29 | **Error {1 – [28] / [19]}** | **NM** | **16.5%** | **1.7%** | **-0.1%** | **6.3%** | **0.2%** | **12.5%** | **4.1%** |
| 30 | **Absolute Error = ABS[29] [Note 2]** | **NM** | **16.5%** | **1.7%** | **0.1%** | **6.3%** | **0.2%** | **12.5%** | **4.1%** |
| 31 | **Squared Error [Note 2]** | | **2.7%** | **0.0%** | **0.0%** | **0.4%** | **0.0%** | **1.6%** | **0.4%** |
| 32 | | | | | | | | | |
| 33 | **Personal Exp = % of Owner's Discretionary Inc** | **10%** | | | | | | | |
| 34 | | | | | | | | | |
| 35 | **Sensitivity Analysis: How the error varies with personal exp** | **Cell B33** | **Error** | | | | | | |
| 36 | | | | | | | | | |
| 37 | | 2% | 17.3% | | | | | | |
| 38 | | 4% | 14.0% | | | | | | |
| 39 | | 6% | 10.7% | | | | | | |
| 40 | | 8% | 7.4% | | | | | | |
| 41 | | **10%** | **4.1%** | | | | | | |
| 42 | | | | | | | | | |
| 43 | [1] Approximate midpoint of the 21% to 28% control premium estimated in Chapter 8. | | | | | | | | |
| 44 | | | | | | | | | |
| 45 | [2] The averages are for the last 5 columns only, as the sales under $100,000 are mostly likely asset based, not income based. | | | | | | | | |

Rows 18–35 contain our regression analysis of arithmetic mean returns as a function of the logarithm of the market capitalization—exactly the same as Table 5.1, regression #2.[4] The regression equation is: $r = 0.5352 - 0.0186 \ln \text{FMV}$.[5] We use this regression equation in Table 9.2.

## Table 9.2: Reconciliation to the IBA Database

Table 9.2 is the main table in this chapter. All other tables provide details that flow into this table.

The purpose of the table is to perform two series of calculations, which make up parts 1 and 2 of the table, respectively. The first series calculates adjusted price to cash flow (P/CF) multiples for each size category of IBA database results described in the article. The second series is to calculate theoretical P/CF multiples using the

[4]This regression is different from Table 5.1, regression #2 in that it uses different data.

[5]For public firms, FMV is market capitalization, that is, price per share × number of shares.

log size equation and the DLOM methodology in Chapter 8. Ultimately we compare them, and they match reasonably well.

Unfortunately, there are a lot of data that we do not have, which will force us to make estimates. There are so many estimates in the following analysis, that we will not be able to make strong conclusions. It would be easy to manipulate the results in Table 9.2 to support different points of view. Nevertheless, it is important to proceed with the table, as we will still gain valuable insights. Additionally, it points out the deficiencies in the available information set. This is not a criticism of the IBA database. All of the other transactional databases of which I am aware suffer from the same problems. This analysis highlights the type of information that would be ideal to have in order to come to stronger conclusions.

## Part 1: IBA P/CF Multiples

We begin in row 6. The mean selling prices in row 6 are the midpoints of the corresponding range of selling prices reported in the article. Thus, B6 = \$25,000, which is the midpoint of the selling price for firms in the \$0 to \$50,000 category. At the high end, H6 = \$750,000, which is the midpoint price in the \$500,000 to \$1 million sales price category.

Row 7 is the mean PE multiple reported in the article. Note that the PE multiple constantly rises as the selling price rises. Figure 9.1 shows this relationship clearly. Row 8 is owner's discretionary income, which is row 6 divided by row 7 (i.e., $P \div PE = E$, where P is price and E is earnings).

The IBA's definition of owner's discretionary income is net income before income taxes and owner's salary. It does not conform to the arm's-length income that appraisers use in valuing businesses. Therefore, we subtract our estimate of

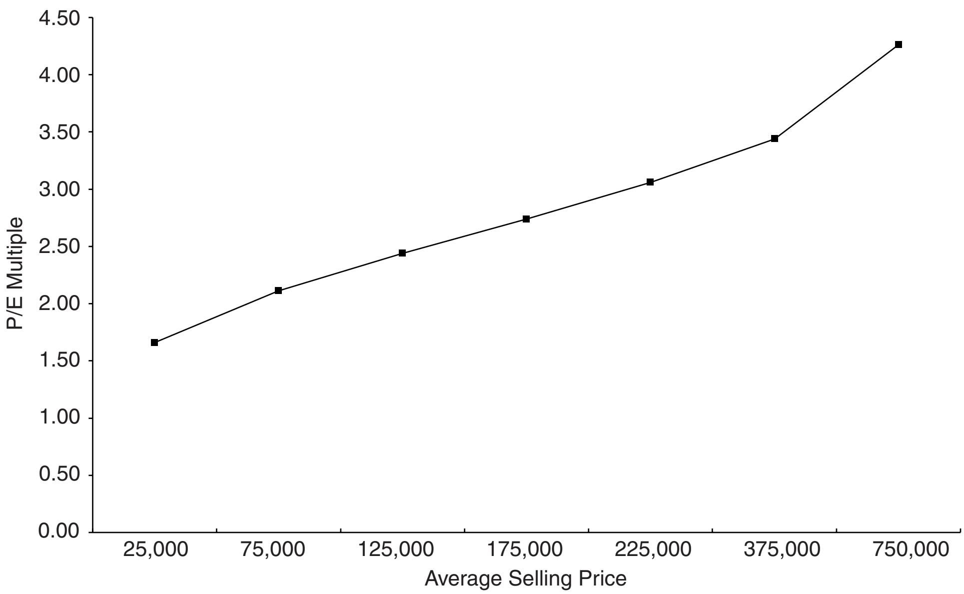


**FIGURE 9.1** PE Ratio as a Function of Size from the IBA Database

an arm's-length salary for owners from owner's discretionary income as part of our adjustments to compute arm's-length income (adjusted net income). In row 9, we have our estimate of an arm's-length salary for owners. This is purely an educated guess. Raymond Miles felt my estimates were reasonable.

In row 10, we add back personal expenses charged to the business. Unfortunately, no one I know has published data on this. I have asked many accountants for their estimates, and their answers vary wildly. Ultimately, I decided to estimate this at 10% (B33) of owner's discretionary income (row 8).

Row 11 is adjusted net income, which is row 8 − row 9 + row 10. Row 12 is an estimate of the effective corporate income tax rate. This is a judgment call. An accountant convinced me that even for the $1 million sales, the owner's discretionary income is low enough that it would not be taxed at all. His opinion was that any excess remaining over salary would be taken out of taxable income as a bonus. I acceded to his opinion, though this point is arguable—especially for the higher dollar sales. It is true that what counts here is not who the seller is, but who the buyer is. A large corporation buying a small firm would still impute corporate taxes at the maximum rate; however, only the last category is at all likely to be bought by a large firm, and even then, most buyers of $0.5 to $1 million firms are probably single individuals. Therefore, it makes sense to go with no corporate taxes, with a possible reservation in our minds about the last column.

With this zero income tax assumption, row 13 equals zero and row 14, adjusted income after taxes, equals row 11.

Next we need to convert from net income to cash flow. Again, as far as I know, the information does not exist, so we need to make reasonable assumptions. For most businesses, cash flow lags behind net income. Most of these are small businesses that sold for fairly small dollar amounts, which means that expected growth—another important missing piece of information—must be low, on average. The lower the growth, the less strain on cash flow. We assume cash flow is 95% of adjusted net income. It would be reasonable to assume this ratio is smaller for the higher-value businesses, which presumably have higher growth. We do not vary our cash flow ratio, as none of these are likely to be very high-growth businesses. Thus, all cells in row 15 equal 95%. In row 16, we multiply row 14 by row 15 to calculate adjusted after-tax cash flow.

The next step in adjusting the IBA multiples is to reduce the nominal selling price to a cash-equivalent selling price, which we calculate in Table 9.3. Exhibit 33-3 in Pratt (1993) shows a summary of sale data from Bizcomps. Businesses selling for less than $100,000 have 60% average cash down, and businesses selling for more than $500,000 have an average 58% cash down. Using 60% cash down, we assume the seller finances the 40% (Table 9.3, B11) balance for 7 years, which is 84 months (B8, C8) at 8% (B5) with a market rate of 14% (C5).

The annuity discount factor (ADF), the formula for which is $ADF = \frac{1-\frac{1}{(1+r)^n}}{r}$, is 53.3618 (C9) at the market rate of interest and 64.1593 (B9) at the nominal rate. One minus the ratio of two equals the discount to cash equivalent value if the loan is 100% financed, or $1 - \frac{53.3618}{64.1593} = 16.8\%$ (B10). We multiply this by the 40% financed (B11) to calculate the average discount to cash equivalent value of 6.7% (B12), which we transfer back to Table 9.2, row 17.

| | A | B | C |
|---|---|---|---|
| 1 | **Table 9.3** | | |
| 2 | **Proof of Discount Calculation** | | |
| 3 | | | |
| 4 | | **Nominal** | **Market** |
| 5 | *r* | 8% | 14% |
| 6 | *i* = *r*/12 | 0.6667% | 1.1667% |
| 7 | Yrs | 7 | 7 |
| 8 | *n* = Yrs × 12 | 84 | 84 |
| 9 | ADF @14%, 84 Months | 64.1593 | 53.3618 |
| 10 | Discount on Total Principle | 16.8% | |
| 11 | % Financed | 40% | |
| 12 | Discount on % Financed | **6.7%** | |

Multiplying the midpoint selling price in row 6 by 1 minus the discount to cash equivalent value in row 17 leads to an adjusted mean selling price in row 18. For example, $25,000 × (1 − 6.7%) = $23,317 [B6 × (1 − B17) = B18].

Finally, we divide row 18 by row 16 to calculate the adjusted price to cash flow (P/CF) multiple for the IBA database. In general, the P/CF multiple rises as price rises, although not always. There is no meaningful P/CF multiple in B19, because adjusted cash flow in B16 is negative. The P/CF multiples begin in C19 at 5.2 for a midpoint selling price of $75,000, then decline to 4.7 (D19) for a selling price of $125,000, and rise steadily to 6.2 (H19) for a selling price of $750,000. The only exception is that the P/CF is greater at 5.4 for the $225,000 selling price than at 5.3 for the $375,000 selling price. The first anomaly is probably not significant, because many, if not most, firms selling under $100,000 are priced based on their assets rather than their earnings capacity. The second anomaly, from P/CF of 5.4 to 5.3, is a very small reversal of the general pattern of rising P/CF multiples in the IBA database.

## Part 2: Log Size P/CF Multiples

In this section of Table 9.2, we will calculate "theoretical" P/CF multiples based on the log size model and the DLOM calculations in Chapter 8. The term *theoretical* is somewhat of a misnomer, as the calculation of both the log size equation and DLOM is empirically based. Nevertheless, we use the term for convenience.

Before we can apply the log size equation from Table 9.1, we need a marketable minority interest FMV, while the adjusted selling price (FMV) in row 18 is an illiquid control value. Therefore, we need to divide row 18 by 1 plus the control premium times 1 minus DLOM, which we do in row 24. We assume a control premium of 25% (row 22), which is the approximate midpoint of the 21%–28% range of control premiums discussed in Chapter 8.

The calculation of DLOM is unique for each size category and appears in Tables 9.6 and 9.6A–F. We will cover those tables later. In the meantime, DLOM rises steadily from 9.9% (B23) for the $25,000 selling price to 18.6% (H23) for the $750,000 selling price category.

Row 24, the marketable minority FMV, is row 18 ÷ [(1 + row 22) × (1 − row 23)]. The marketable minority values are all lower than the illiquid control values, as the control premium is much greater in magnitude than DLOM.

We calculate the log size discount rate in row 25 using the regression equation from Table 9.1.[6] It ranges from a high of 35.0% (B25) for the smallest category to a low of 28.5% (H25) for the largest category.

Next we estimate the constant growth rates that the buyers and sellers collectively implicitly forecast when they agreed on prices. It is unfortunate that none of the transactional databases that are publicly available contain even historical growth rates, let alone forecast growth rates. Therefore, we must make another estimate. We estimate growth rates to rise from 2% (B26) to 6% (H26), growing at 0.5% for each category, except the last one going from 5% to 6%. It is logical that buyers will pay more for faster-growing firms.

In row 27, we calculate a midyear Gordon model $= (1+g)\,\frac{\sqrt{1+r}}{r-g}$, with $r$ and $g$ coming from rows 25 and 26, respectively.[7] This is a marketable minority interest P/CF multiple when cash flow is expressed as the trailing year's cash flow. In row 28, we convert this to an illiquid control P/CF by doing the reverse of the procedure we performed in row 24—we multiply by 1 plus the control premium and 1 minus DLOM, that is, $P/CF_{Illiq\ Control} = P/CF_{MM} \times (1 + CP) \times (1 - DLOM)$ = row 27 × (1 + row 22) × (1 − row 23).

In row 29, we calculate the error, which is 1 minus the ratio of row 28 divided by row 19, or 1 minus the ratio of the forecast log size–based P/CF to the IBA's adjusted P/CF. Row 30 is the absolute value of the errors in row 29. The absolute values of the errors are most extreme for the low and high values of the midpoint selling price, with a 16.5% (C30) absolute error for the $75,000 midpoint selling price and a 12.5% (H30) absolute error for the $750,000 selling price, with small absolute errors in between ranging around 0.1%–6.3%. The mean error is 4.1% (I29).[8]

## Conclusion

The mean absolute error is 4.1% (I30). Rounding this to 4%, that is a very respectable result. It is evidence supporting the log size model in Chapter 5 and control premium and economic components model of DLOM in Chapter 8.

Nevertheless, as mentioned before, there are too much missing data and resulting guesswork to come to solid conclusions. The estimates are all reasonable, but one could make different reasonable estimates and come to very different results. Thus, this analysis is worthwhile evidence, but it proves nothing.

[6]Technically, we should reduce the regression output by the historical arithmetic mean yearly growth in the PE ratio. We do not do that because it is likely to be immaterial in its impact.

[7]The purpose of the (1 + $g$) term is correct for the fact that we are applying it to each dollar of prior year's cash flow and not to the customary next year's cash flow.

[8]This excludes the $75,000 mean selling price errors, as that is likely due to the sale being priced on an asset rather than an income basis. We also exclude this category in the other measures of mean error.

In the remainder of the chapter, we will describe the DLOM calculations in Tables 9.4 and 9.6, and their variations as 9.4A and 9.6A, and so on.

## Calculation of DLOM

As discussed in Chapter 8, there are three components in the economic components model to the calculation of DLOM. Components #1 and #3, the delay to sale and transactions costs components, require unique analysis for each IBA size category. Therefore, we have one spreadsheet for each of the two components for each IBA size category. Tables 9.4 and 9.6 are the calculations of components #1 and #3, respectively, for the \$25,000 midpoint selling price firm. Additionally, Table 9.6 contains the DLOM calculations. We will describe these tables in detail. Tables 9.4A and 9.6A are identical to Tables 9.4 and 9.6, with the only difference being that these are calculations for the \$75,000 selling price firms. This series continues all the way through Tables 9.4F and 9.6F for the \$750,000 selling price IBA category. Table 9.5 contains the calculations of the buyer and seller transaction costs for all size categories.

### Table 9.4: Computation of the Delay-to-Sale Component—\$25,000 Firm

Table 9.4 is identical to Table 8.10, except that we are customizing the calculation for this IBA category of firm. We begin by inserting the selling price in B16 and adjusted net income in B17. For the larger IBA categories, net income (owner's discretionary income) is positive, and we divide that by an assumed pre-tax margin of 5% in B18 to estimate sales in B19. We cannot do that for the \$25,000 sales category only, because of net losses. We estimate sales at three times the selling price, or \$75,000 (B19). The square of sales is then $\$5.625 \times 10^9$, which is calculated in B20 and transferred to C6.[9]

We insert the \$25,000 midpoint selling price in C8, B14, and B16. Here we are calculating the value of 100% of the stock, so the block value and the value of the entire firm will be identical, which is not true in the restricted stock calculations in Table 8.10.[10]

C7 is the post-discount value of the block. However, both C7 and B14 equal \$25,000. This is because the discount calculation came to zero (D12). Normally, C7 would be lower than B14.

A correlation analysis of the Management Planning data, not shown in the book, revealed that firm size and earnings and revenue stability are uncorrelated. Thus we use the averages from Table 8.5, G60 and H60 of 0.42 (C9) and 0.69 (C10), respectively.

[9]The calculations in B16 to B20 did not appear in Table 8.10, as they were unnecessary there.

[10]Technically, we should be using the marketable minority FMV rather than the illiquid control FMV in Table 9.4 (and its variants 9.4A, etc.), B14 (which also affects C7 and C8). However, we do not yet know the marketable minority FMV, as that is the point of the exercise. Even to attempt to calculate it would require multiple iterations, which would greatly complicate the analysis and add nothing, as the regression coefficients in B7 and B8 are so small that the difference is immaterial. Therefore, we use the illiquid control values.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 9.4** | | | | |
| 2 | **Calculation of Component #1—Delay to Sale—$25,000 Firm [1]** | | | | |
| 3 | | | | | |
| 4 | | **Coefficients** | **Co. Data** | **Discount** | |
| 5 | Intercept | 0.1342 | NA | 13.4% | |
| 6 | Revenues $^2$ | -5.33E-18 | 5.625E+09 | 0.0% | |
| 7 | Value of Block—Post-Discount [2] | -4.26E-09 | $ 25,000 | 0.0% | |
| 8 | FMV-Marketable Minority 100% Interest | 5.97E-10 | **$ 25,000** | 0.0% | |
| 9 | Earnings Stability [3] | -0.1376 | 0.4200 | -5.8% | |
| 10 | Revenue Stability [3] | -0.1789 | 0.6900 | -12.3% | |
| 11 | Average Years to Sell | 0.1339 | 0.2500 | 3.3% | |
| 12 | **Total Discount [4]** | | | **0.0%** | |
| 13 | | | | | |
| 14 | Value of Block—Pre-Discount [5] | $ 25,000 | | | |
| 15 | | | | | |
| 16 | Selling Price | $ 25,000 | | | |
| 17 | Adjusted Net Income | $ (5,934) | | | |
| 18 | Assumed Pre-Tax Margin | NA | | | |
| 19 | Sales | $ 75,000 | | | |
| 20 | Sales $^2$ | 5.625E+09 | | | |
| 21 | | | | | |
| 22 | | | | | |
| 23 | [1] Based on Abram's regression of Management Planning, Inc. data—Regression #2, Table 8.10. | | | | |
| 24 | | | | | |
| 25 | [2] Equal to Pre-Discount Shares Sold in dollars × (1 – Discount). B7 equals B14 only when the discount = 0%. | | | | |
| 26 | | | | | |
| 27 | [3] Earnings and Revenue stability are assumed at the averages from Table 8.5, G60 and H60, respectively, for | | | | |
| 28 | all FMVs. In the Management Planning data, a correlation analysis revealed that firm size and the stability | | | | |
| 29 | measures are uncorrelated. Therefore, we assume the same levels for all FMVs. | | | | |
| 30 | | | | | |
| 31 | [4] Total Discount = max(discount, 0), because Disc < 0 indicates the model is outside of its range of reasonability. | | | | |
| 32 | | | | | |
| 33 | [5] In our regression of the Management Planning, Inc. data, this was a marketable minority interest value. | | | | |
| 34 | This is an illiquid control value and is higher by 12% to 25% than the marketable minority value. The | | | | |
| 35 | regression coefficient relating to market capitalization in B8 is so small that the difference is immaterial, and | | | | |
| 36 | it is easier to work with the value available. | | | | |

Finally, we assume that a $25,000 firm takes only three months, or 0.25 (C11) years, to sell. Summing D5 through D11 actually results in a slightly negative discount, which does not make sense. Therefore, we use a spreadsheet formula to calculate D12 as the maximum of the sum of range D5:D11 and zero. The delay to sale component is zero for all size categories except $375,000 and $750,000. The calculations of component #1 of DLOM for those two categories appear in Tables 9.4E and 9.4F, respectively. The main reason for this is that we assume that it takes either 0.25 years or 0.33 years to sell firms under the $375,000 category, while we assume that it takes 0.5 years and 1.0 years to sell in the $375,000 and $750,000 categories, respectively (Tables 9.4E and 9.F, C11). The resulting discounts are still small in magnitude. In Table 9.4E, D12, we calculate component #1 as 1.9%, and in Table 9.4F, D12, we calculate component #1 as 8.4%.

Though we did not elect to do so here, it would be a reasonable approach to rely on our findings in Chapter 8 that the regression analysis does not work well for delays to sale of much less than a year. That being the case, it would make sense to use a different model—even something so simple as a present value—to calculate the delay to sale component for under one year. For example, if we assume a 25% discount rate, a three-month delay to sale implies a 5% discount as component #1, and a four-month delay to sale implies a 7% discount as component #1. It is important to recognize that not all models work well across all ranges of data, and sometimes circumstances force us to use different models. For simplicity in this analysis, we did not elect to use another model.

## Table 9.5: Calculation of Transaction Costs

Table 9.5 contains our calculations of transaction costs for both buyer and seller for all of the IBA size categories. Column A denotes whether the transaction costs are for buyers or sellers. Column B is the midpoint selling price of the IBA study. Column C is the base 10 logarithm of column B.

Columns D and F contain, respectively, the $x$-coefficient and the constant from the regression in Table 8.11. In column E, we multiply column C by column D. We add columns E and F together to obtain column G, which is the regression forecast of all transaction costs except for the business broker (or investment banker). Column H contains the business broker fees, which we assume at 10% for sellers and zero for buyers. Finally, column I is the grand total forecast of transaction costs for buyers and sellers by size category. Note that both buyer and seller transaction costs decline as firm size grows.

While the $10 million firm in rows 20 and 21 are outside of the scope of the IBA study, we use them later on in our own analysis to extrapolate the results that we derive from our analysis of the IBA study.

## Table 9.6: Calculation of DLOM

Table 9.6 is exactly the same format and logic as Table 8.14, which we already described in Chapter 8. B9 through B12 contain the pure discounts for the four economic components. B9, the pure discount for component #1, equals zero, and that comes from our calculation in Table 9.4, D12. B10, the pure discount for component #2, equals 9%. That is the same as it was in Table 8.14, and it comes from the Schwert article. Components 3A and 3B come from Table 9.5, I6 and I7, respectively, less a 2% brokerage cost for publicly traded stock, since we are calculating incremental costs, using publicly traded stock as our reference point. These two components are equal to 5.7% (B11) and 15.1% (B12), respectively.

As in Table 8.14, the first two components transfer from B9 and B10 to C9 and C10 directly. However, transaction costs "leave the system" with every sale. Thus, we must compute the present value of a perpetuity of transactions costs that occur every $j = 10$ years. We do so using the formulas in note [2] to the spreadsheet, which are equations (8.9) and (8.9a) from Chapter 8. The present value of all buyers' transactions costs is 6.1% (C11), and the present value of all sellers' transactions costs is 1.0% (C12). The final calculation of DLOM is 9.9% (D14).

## Tables 9.6A–F: Calculations of DLOM for Larger Firms

Tables 9.6A through 9.6F are structured and calculated identically to Table 9.6. There are five differences in the parameters, the first four of which tend to increase DLOM as firm size increases, and the last one decreases DLOM as firm size increases.

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table 9.5** | | | | | | | | |
| 2 | **Calculation of Transaction Costs** | | | | | | | | |
| 3 | **for Firms of All Sizes in the IBA Study** | | | | | | | | |
| 4 | | | | | | | | | |
| 5 | | **FMV** | **$\log_{10}$ FMV** | ***X*-Coeff.** | **log FMV × Coef** | **Regr. Constant** | **Forcst Subtotal** | **Bus. Broker** | **Forecast Total** |
| 6 | **Buyer** | $ 25,000 | 4.39794 | -0.01727 | -0.07596 | 0.15310 | **7.7%** | 0.0% | **7.7%** |
| 7 | **Seller** | $ 25,000 | 4.39794 | -0.01599 | -0.07034 | 0.14139 | **7.1%** | 10.0% | **17.1%** |
| 8 | **Buyer** | $ 75,000 | 4.87506 | -0.01727 | -0.08420 | 0.15310 | **6.9%** | 0.0% | **6.9%** |
| 9 | **Seller** | $ 75,000 | 4.87506 | -0.01599 | -0.07797 | 0.14139 | **6.3%** | 10.0% | **16.3%** |
| 10 | **Buyer** | $ 125,000 | 5.09691 | -0.01727 | -0.08804 | 0.15310 | **6.5%** | 0.0% | **6.5%** |
| 11 | **Seller** | $ 125,000 | 5.09691 | -0.01599 | -0.08152 | 0.14139 | **6.0%** | 10.0% | **16.0%** |
| 12 | **Buyer** | $ 175,000 | 5.24304 | -0.01727 | -0.09056 | 0.15310 | **6.3%** | 0.0% | **6.3%** |
| 13 | **Seller** | $ 175,000 | 5.24304 | -0.01599 | -0.08386 | 0.14139 | **5.8%** | 10.0% | **15.8%** |
| 14 | **Buyer** | $ 225,000 | 5.35218 | -0.01727 | -0.09245 | 0.15310 | **6.1%** | 0.0% | **6.1%** |
| 15 | **Seller** | $ 225,000 | 5.35218 | -0.01599 | -0.08561 | 0.14139 | **5.6%** | 10.0% | **15.6%** |
| 16 | **Buyer** | $ 375,000 | 5.57403 | -0.01727 | -0.09628 | 0.15310 | **5.7%** | 0.0% | **5.7%** |
| 17 | **Seller** | $ 375,000 | 5.57403 | -0.01599 | -0.08915 | 0.14139 | **5.2%** | 10.0% | **15.2%** |
| 18 | **Buyer** | $ 750,000 | 5.87506 | -0.01727 | -0.10148 | 0.15310 | **5.2%** | 0.0% | **5.2%** |
| 19 | **Seller** | $ 750,000 | 5.87506 | -0.01599 | -0.09397 | 0.14139 | **4.7%** | 10.0% | **14.7%** |
| 20 | **Buyer** | $ 10,000,000 | 7.00000 | -0.01727 | -0.12091 | 0.15310 | **3.2%** | 0.0% | **3.2%** |
| 21 | **Seller** | $ 10,000,000 | 7.00000 | -0.01599 | -0.11196 | 0.14139 | **2.9%** | 2.0% | **4.9%** |
| 22 | | | | | | | | | |
| 23 | ***Note:*** **Regression constants and *x*-coefficients come from Table 8.11. The $10 million firm, using a Lehman Bros. Formula, has** | | | | | | | | |
| 24 | **a 2% investment banker fee instead of a 10% business broker's fee.** | | | | | | | | |

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 9.6** | | | | | | |
| 2 | **Calculation of DLOM** | | | | | | |
| 3 | | | | | | | |
| 4 | **Section 1: Calculation of the Discount for Lack of Marketability** | | | | | | |
| 5 | | | | | | | |
| 6 | | | | **= 1 – Col. [C]** | | | |
| 7 | | **Pure Discount** | **PV of Perpetual** | **Remaining** | | | |
| 8 | **Component** | **= *z* [1]** | **Discount [2]** | **Value** | | | |
| 9 | 1 | 0.0% | 0.0% | 100.0% | Delay to Sale | | |
| 10 | 2 | 9.0% | 9.0% | 91.0% | Buyer's Monopsony Power—Thin Markets | | |
| 11 | 3A | 5.7% | 6.1% | 93.9% | Transactions Costs—Buyers | | |
| 12 | 3B | 15.1% | 1.0% | 99.0% | Transactions Costs—Sellers | | |
| 13 | Percent Remaining | | | 90.1% | Total % Remaining = Components 1 × 2 × 3A × 3B | | |
| 14 | **Final Discount** | | | **9.9%** | Discount = 1 – Total % Remaining | | |
| 15 | | | | | | | |
| 16 | **Section 2: Assumptions and Intermediate Calculations:** | | | | | | |
| 17 | | | | | | | |
| 18 | **FMV—Equity of Co. (before Discounts)** | | | | **$ 25,000** | | |
| 19 | **Discount Rate = *r* [3]** | | | | **34.7%** | | |
| 20 | **Constant Growth Rate = *g* (Table 9.2, Row 26)** | | | | **2.0%** | | |
| 21 | **Intermediate Calculation: *x* = (1+*g*) / (1+*r*)** | | | | **0.7574** | | |
| 22 | **Avg # Years between Sales = *j*** | | | | **10** | | |
| 23 | | | | | | | |
| 24 | | | | | | | |
| 25 | [1] Pure Discounts: For Component #1, Table 9.4, cell D12; for Component #2, 9% per Schwert article. For | | | | | | |
| 26 | Component #3A and #3B, Table 9.5, cells I6 and I7 – 2% for public brokerage costs. | | | | | | |
| 27 | | | | | | | |
| 28 | [2] PV of Perpetual Discount Formula: 1–(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9), used for Component #3B. | | | | | | |
| 29 | PV of Perpetual Discount Formula: 1–(1–*z*)*(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9a), used for Component #3A. | | | | | | |
| 30 | Components #1 and #2 simply transfer the pure discount. | | | | | | |
| 31 | | | | | | | |
| 32 | [3] The formula is 0.5352 – (0.0186 ln FMV), based on Table 9.1, B34 and B35. | | | | | | |

1. As firm size increases, our assumed growth rate, $g$, increases. By our analysis of the partial derivatives in the Mathematical Appendix to Chapter 8, that causes an increase in DLOM.
2. As firm size increases, the log size discount rate, $r$, decreases. By our analysis of the partial derivatives in the Mathematical Appendix to Chapter 8, that also causes an increase in DLOM.
3. As mentioned earlier, for firm sizes under $375,000, we assumed the delay to sale to be 0.33 years or less, which led to a zero discount for component #1. For the $375,000 and $750,000 firms, we assumed a one-half-year and one-year delay to sale, which led to a component #1 pure discount of 1.9% (Table 9.6E, B9) and 8.4% (Table 9.6F, B9), respectively. The latter accounts for the vast majority of the much higher DLOM for the $750,000 midpoint selling price firms. Had that been zero, like all of the others except the $375,000 firm, DLOM for the $750,000 firms would have been 13.1%—much closer to DLOM for the smaller firms.
4. We assumed a 1% broker's fee for publicly traded stocks for the $375,000 and $750,000 firms, while we assumed a 2% fee for the firms under that size. This increased the pure discount for components #3A and #3B by 1% for those two size categories, and therefore increased DLOM.

5. Transactions costs decrease as size increases. Buyers' transactions costs are 7.7% (Table 9.5, I6) for $25,000 firms and 5.2% for $750,000 firms (I18), for a difference of 2.5%. Sellers' transactions costs are 17.1% (I7) for $25,000 firms and 14.7% (I19) for $750,000 firms, for a difference of 2.4%.

Items 1 through 4 cause DLOM to increase with size, while item 5 causes DLOM to decrease with size. Looking at Table 9.2, it is clear that the first four items dominate, which causes DLOM to increase with size. This is not a result that I would have predicted before. I would have thought that, overall, DLOM decreases with size.

As mentioned earlier in this chapter, had we used a different model, it would have been possible to assign a pure discount for the delay to sale of perhaps 3%–5%. This would have narrowed the differences between DLOM for the small firms and the large ones, but we would still have come to the counterintuitive conclusion that DLOM increases with firm size.

## Calculation of DLOM for Large Firms

The preceding result begs the question of what happens to DLOM beyond the realm of small firms. To answer this question, we extend our analysis to Tables 9.4G and 9.6G.

Table 9.4G is otherwise identical to its predecessor, Table 9.4F. Since we do not have the benefit of the IBA data at this size level, we have to forecast sales in a different fashion. The calculation of component #1 is still not sensitive at this level to the square of revenues, so we can afford to be imprecise. Assuming an average PE multiple of 12.5, we divide the assumed $10 million (B16) selling price by the PE multiple to arrive at net income of $800,000 (B17). Dividing that by an assumed pre-tax margin of 5% (B18) leads to sales of $16 million (B19), which is $2.56 \times 10^{14}$ (B20, transferred to C6) when squared. That contributes only –0.1% (D6) to the calculation of the pure discount from the delay to sale component (it was 0.0% in Table 9.4F, D6).

The really significant difference in the calculation comes from D7, which is –4.0% in Table 9.4G and –0.3% in Table 9.4F. The final calculation of component #1 is 5.1% (D12) for the $10 million firm, compared to 8.4% for the $750,000 firm. Thus it seems that component #1 rises sharply somewhere between $375,000 and $750,000 firms, but then begins to decline as the size effect dominates and causes transactions costs to decline, while not adding any additional time to sell the firm.

Table 9.6G is our calculation of DLOM for the $10 million firm. Comparing it to Table 9.6F, the DLOM calculation for the $750,000 firm, the final result is 15.0% (Table 9.6G, D14) versus 18.6% (Table 9.6F, D14). Thus it appears that DLOM rises with size up to about $1 million in selling price and declines thereafter. Another factor we did not consider here that also would contribute to a declining DLOM with size is that the number of interested buyers would tend to increase with larger size, which should lower component #2—buyer's monopsony power—below the 9% from the Schwert article cited in Chapter 8.

## Table 9.4A
## Calculation of Component #1—Delay to Sale—$75,000 Firm [1]

| | A | B | C | D |
|---|---|---|---|---|
| 4 | | **Coefficients** | **Co. Data** | **Discount** |
| 5 | Intercept | 0.1342 | NA | 13.4% |
| 6 | Revenues$^2$ | -5.33E-18 | 7.952E+10 | 0.0% |
| 7 | Value of Block—Post-Discount [2] | -4.26E-09 | $ 75,000 | 0.0% |
| 8 | FMV-Marketable Minority 100% Interest | 5.97E-10 | **$ 75,000** | 0.0% |
| 9 | Earnings Stability (Assumed) | -0.1376 | 0.4200 | -5.8% |
| 10 | Revenue Stability (Assumed) | -0.1789 | 0.6900 | -12.3% |
| 11 | Average Years to Sell | 0.1339 | 0.2500 | 3.3% |
| 12 | **Total Discount [4]** | | | **0.0%** |
| 13 | | | | |
| 14 | Value of Block—Pre-Discount [5] | $ 75,000 | | |
| 15 | | | | |
| 16 | Selling Price | $ 75,000 | | |
| 17 | Adjusted Net Income | $ 14,100 | | |
| 18 | Assumed Pre-Tax Margin | 5% | | |
| 19 | Sales | $ 281,991 | | |
| 20 | Sales$^2$ | 7.95E+10 | | |

[1] Based on Abrams's regression of Management Planning, Inc. data—Regression #2, Table 8.10.

[2] Equal to Pre-Discount Shares Sold in dollars × (1 – Discount). B7 equals B14 only when the discount = 0%.

[3] Earnings and Revenue stability are assumed at the averages from Table 8.5, G60 and H60, respectively, for all FMVs. In the Management Planning data, a correlation analysis revealed that firm size and the stability measures are uncorrelated. Therefore, we assume the same levels for all FMVs.

[4] Total Discount = max(discount, 0), because Disc < 0 indicates the model is outside of its range of reasonability.

[5] In our regression of the Management Planning, Inc. data, this was a marketable minority interest value. This is an illiquid control value and is higher by 12% to 25% than the marketable minority value. The regression coefficient relating to market capitalization in B8 is so small that the difference is immaterial, and it is easier to work with the value available.

## Table 9.6A
## Calculation of DLOM

### Section 1: Calculation of the Discount for Lack of Marketability

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 6 | | | | = 1 – Col. [C] | |
| 7 | | **Pure Discount** | **PV of Perpetual** | **Remaining** | |
| 8 | **Component** | **= *z* [1]** | **Discount [2]** | **Value** | |
| 9 | 1 | 0.0% | 0.0% | 100.0% | Delay to Sale |
| 10 | 2 | 9.0% | 9.0% | 91.0% | Buyer's Monopsony Power—Thin Markets |
| 11 | 3A | 4.9% | 5.3% | 94.7% | Transactions Costs—Buyers |
| 12 | 3B | 14.3% | 1.2% | 98.8% | Transactions Costs—Sellers |
| 13 | Percent Remaining | | | 89.9% | Total % Remaining = Components 1 × 2 × 3A × 3B |
| 14 | **Final Discount** | | | **10.1%** | Discount = 1 – Total % Remaining |

### Section 2: Assumptions and Intermediate Calculations:

| | A | E |
|---|---|---|
| 18 | **FMV—Equity of Co. (before Discounts)** | **$ 75,000** |
| 19 | **Discount Rate = *r* [3]** | **32.6%** |
| 20 | **Constant Growth Rate = *g* (Table 9.2, Row 26)** | **2.5%** |
| 21 | **Intermediate Calculation: *x* = (1+*g*) / (1+*r*)** | **0.7728** |
| 22 | **Avg # Years between Sales = *j*** | **10** |

[1] Pure Discounts: For Component #1, Table 9.4A, cell D12; for Component #2, 9% per Schwert article. For Component #3A and #3B, Table 9.5, cells I8 and I9 – 2% for public brokerage costs.

[2] PV of Perpetual Discount Formula: 1–(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9), used for Component #3B.
PV of Perpetual Discount Formula: 1–(1–*z*)*(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9a), used for Component #3A.
Components #1 and #2 simply transfer the pure discount.

[3] The formula is 0.5352 – (0.0186 ln FMV), based on Table 9.1, B34 and B35.

## Table 9.4B
## Calculation of Component #1—Delay to Sale—$125,000 Firm [1]

| | Coefficients | Co. Data | Discount |
|---|---|---|---|
| Intercept | 0.1342 | NA | 13.4% |
| Revenues $^2$ [2] | -5.33E-18 | 2.778E+11 | 0.0% |
| Value of Block—Post-Discount [2] | -4.26E-09 | $ 125,000 | -0.1% |
| FMV-Marketable Minority 100% Interest | 5.97E-10 | **$ 125,000** | 0.0% |
| Earnings Stability (Assumed) | -0.1376 | 0.4200 | -5.8% |
| Revenue Stability (Assumed) | -0.1789 | 0.6900 | -12.3% |
| Average Years to Sell | 0.1339 | 0.3330 | 4.5% |
| **Total Discount [4]** | | | **0.0%** |

| | |
|---|---|
| Value of Block—Pre-Discount [5] | $ 125,000 |

| | |
|---|---|
| Selling Price | $ 125,000 |
| Adjusted Net Income | $ 26,352 |
| Assumed Pre-Tax Margin | 5% |
| Sales | $ 527,049 |
| Sales $^2$ | 2.78E+11 |

[1] Based on Abrams's regression of Management Planning, Inc. data—Regression #2, Table 8.10.

[2] Equal to Pre-Discount Shares Sold in dollars × (1 – Discount). B7 equals B14 only when the discount = 0%.

[3] Earnings and Revenue stability are assumed at the averages from Table 8.5, G60 and H60, respectively, for all FMVs. In the Management Planning data, a correlation analysis revealed that firm size and the stability measures are uncorrelated. Therefore, we assume the same levels for all FMVs.

[4] Total Discount = max(discount, 0), because Disc < 0 indicates the model is outside of its range of reasonability.

[5] In our regression of the Management Planning, Inc. data, this was a marketable minority interest value. This is an illiquid control value and is higher by 12% to 25% than the marketable minority value. The regression coefficient relating to market capitalization in B8 is so small that the difference is immaterial, and it is easier to work with the value available.

## Table 9.6B
## Calculation of DLOM

### Section 1: Calculation of the Discount for Lack of Marketability

| **Component** | **Pure Discount = *z* [1]** | **PV of Perpetual Discount [2]** | **= 1 – Col. [C] Remaining Value** | |
|---|---|---|---|---|
| 1 | 0.0% | 0.0% | 100.0% | Delay to Sale |
| 2 | 9.0% | 9.0% | 91.0% | Buyer's Monopsony Power—Thin Markets |
| 3A | 4.5% | 4.9% | 95.1% | Transactions Costs—Buyers |
| 3B | 14.0% | 1.3% | 98.7% | Transactions Costs—Sellers |
| Percent Remaining | | | 89.8% | Total % Remaining = Components 1 × 2 × 3A × 3B |
| **Final Discount** | | | **10.2%** | Discount = 1 – Total % Remaining |

### Section 2: Assumptions and Intermediate Calculations:

| | |
|---|---|
| **FMV—Equity of Co. (before Discounts)** | **$ 125,000** |
| **Discount Rate = *r* [3]** | **31.7%** |
| **Constant Growth Rate = *g* (Table 9.2, Row 26)** | **3.0%** |
| **Intermediate Calculation: $x = (1+g) / (1+r)$** | **0.7822** |
| **Avg # Years between Sales = *j*** | **10** |

[1] Pure Discounts: For Component #1, Table 9.4B, cell D12; for Component #2, 9% per Schwert article. For Component #3A and #3B, Table 9.5, cells I10 and I11 – 2% for public brokerage costs.

[2] PV of Perpetual Discount Formula: 1–(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9), used for Component #3B.
PV of Perpetual Discount Formula: 1–(1–*z*)*(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9a), used for Component #3A.
Components #1 and #2 simply transfer the pure discount.

[3] The formula is 0.5352 – (0.0186 ln FMV), based on Table 9.1, B34 and B35.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 9.4C** | | | | |
| 2 | **Calculation of Component #1—Delay to Sale—$175,000 Firm [1]** | | | | |
| 3 | | | | | |
| 4 | | **Coefficients** | **Co. Data** | **Discount** | |
| 5 | Intercept | 0.1342 | NA | 13.4% | |
| 6 | Revenues$^2$[2] | -5.33E-18 | 4.972E+11 | 0.0% | |
| 7 | Value of Block—Post-Discount [2] | -4.26E-09 | $ 175,000 | -0.1% | |
| 8 | FMV-Marketable Minority 100% Interest | 5.97E-10 | **$ 175,000** | 0.0% | |
| 9 | Earnings Stability (Assumed) | -0.1376 | 0.4200 | -5.8% | |
| 10 | Revenue Stability (Assumed) | -0.1789 | 0.6900 | -12.3% | |
| 11 | Average Years to Sell | 0.1339 | 0.3330 | 4.5% | |
| 12 | **Total Discount [4]** | | | **0.0%** | |
| 13 | | | | | |
| 14 | Value of Block—Pre-Discount [5] | $ 175,000 | | | |
| 15 | | | | | |
| 16 | Selling Price | $ 175,000 | | | |
| 17 | Adjusted Net Income | $ 35,255 | | | |
| 18 | Assumed Pre-Tax Margin | 5% | | | |
| 19 | Sales | $ 705,109 | | | |
| 20 | Sales$^2$ | 4.97E+11 | | | |
| 21 | | | | | |
| 22 | | | | | |
| 23 | [1] Based on Abrams's regression of Management Planning, Inc. data—Regression #2, Table 8.10. | | | | |
| 24 | | | | | |
| 25 | [2] Equal to Pre-Discount Shares Sold in dollars × (1 – Discount). B7 equals B14 only when the discount = 0%. | | | | |
| 26 | | | | | |
| 27 | [3] Earnings and Revenue stability are assumed at the averages from Table 8.5, G60 and H60, respectively, for | | | | |
| 28 | all FMVs. In the Management Planning data, a correlation analysis revealed that firm size and the stability | | | | |
| 29 | measures are uncorrelated. Therefore, we assume the same levels for all FMVs. | | | | |
| 30 | | | | | |
| 31 | [4] Total Discount = max(discount, 0), because Disc < 0 indicates the model is outside of its range of reasonability. | | | | |
| 32 | | | | | |
| 33 | [5] In our regression of the Management Planning, Inc. data, this was a marketable minority interest value. | | | | |
| 34 | This is an illiquid control value and is higher by 12% to 25% than the marketable minority value. The | | | | |
| 35 | regression coefficient relating to market capitalization in B8 is so small that the difference is immaterial, and | | | | |
| 36 | it is easier to work with the value available. | | | | |

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 9.6C** | | | | | | |
| 2 | **Calculation of DLOM** | | | | | | |
| 3 | | | | | | | |
| 4 | **Section 1: Calculation of the Discount for Lack of Marketability** | | | | | | |
| 5 | | | | | | | |
| 6 | | | | **= 1 – Col. [C]** | | | |
| 7 | | **Pure Discount** | **PV of Perpetual** | **Remaining** | | | |
| 8 | **Component** | **= *z* [1]** | **Discount [2]** | **Value** | | | |
| 9 | 1 | 0.0% | 0.0% | 100.0% | Delay to Sale | | |
| 10 | 2 | 9.0% | 9.0% | 91.0% | Buyer's Monopsony Power—Thin Markets | | |
| 11 | 3A | 4.3% | 4.7% | 95.3% | Transactions Costs—Buyers | | |
| 12 | 3B | 13.8% | 1.5% | 98.5% | Transactions Costs—Sellers | | |
| 13 | Percent Remaining | | | 89.6% | Total % Remaining = Components 1 × 2 × 3A × 3B | | |
| 14 | **Final Discount** | | | **10.4%** | Discount = 1 – Total % Remaining | | |
| 15 | | | | | | | |
| 16 | **Section 2: Assumptions and Intermediate Calculations:** | | | | | | |
| 17 | | | | | | | |
| 18 | **FMV—Equity of Co. (before Discounts)** | | | | **$ 175,000** | | |
| 19 | **Discount Rate = *r* [3]** | | | | **31.0%** | | |
| 20 | **Constant Growth Rate = *g* (Table 9.2, Row 26)** | | | | **4.0%** | | |
| 21 | **Intermediate Calculation: *x* = (1+*g*) / (1+*r*)** | | | | **0.7936** | | |
| 22 | **Avg # Years between Sales = *j*** | | | | **10** | | |
| 23 | | | | | | | |
| 24 | | | | | | | |
| 25 | [1] Pure Discounts: For Component #1, Table 9.4C, cell D12; for Component #2, 9% per Schwert article. For | | | | | | |
| 26 | Component #3A and #3B, Table 9.5, cells I12 and I13 – 2% for public brokerage costs. | | | | | | |
| 27 | | | | | | | |
| 28 | [2] PV of Perpetual Discount Formula: 1–(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9), used for Component #3B. | | | | | | |
| 29 | PV of Perpetual Discount Formula: 1–(1–*z*)*(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9a), used for Component #3A. | | | | | | |
| 30 | Components #1 and #2 simply transfer the pure discount. | | | | | | |
| 31 | | | | | | | |
| 32 | [3] The formula is 0.5352 – (0.0186 ln FMV), based on Table 9.1, B34 and B35. | | | | | | |

## Table 9.4D
## Calculation of Component #1—Delay to Sale—$225,000 Firm [1]

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 4 | | **Coefficients** | **Co. Data** | **Discount** | |
| 5 | Intercept | 0.1342 | NA | 13.4% | |
| 6 | Revenues$^2$[2] | -5.33E-18 | 6.685E+11 | 0.0% | |
| 7 | Value of Block—Post-Discount [2] | -4.26E-09 | $ 225,000 | -0.1% | |
| 8 | FMV-Marketable Minority 100% Interest | 5.97E-10 | **$ 225,000** | 0.0% | |
| 9 | Earnings Stability (Assumed) | -0.1376 | 0.4200 | -5.8% | |
| 10 | Revenue Stability (Assumed) | -0.1789 | 0.6900 | -12.3% | |
| 11 | Average Years to Sell | 0.1339 | 0.3330 | 4.5% | |
| 12 | **Total Discount [4]** | | | **0.0%** | |
| 13 | | | | | |
| 14 | Value of Block—Pre-Discount [5] | $ 225,000 | | | |
| 15 | | | | | |
| 16 | Selling Price | $ 225,000 | | | |
| 17 | Adjusted Net Income | $ 40,882 | | | |
| 18 | Assumed Pre-Tax Margin | 5% | | | |
| 19 | Sales | $ 817,647 | | | |
| 20 | Sales$^2$ | 6.69E+11 | | | |

[1] Based on Abrams's regression of Management Planning, Inc. data—Regression #2, Table 8.10.

[2] Equal to Pre-Discount Shares Sold in dollars × (1 – Discount). B7 equals B14 only when the discount = 0%.

[3] Earnings and Revenue stability are assumed at the averages from Table 8.5, G60 and H60, respectively, for all FMVs. In the Management Planning data, a correlation analysis revealed that firm size and the stability measures are uncorrelated. Therefore, we assume the same levels for all FMVs.

[4] Total Discount = max(discount, 0), because Disc < 0 indicates the model is outside of its range of reasonability.

[5] In our regression of the Management Planning, Inc. data, this was a marketable minority interest value. This is an illiquid control value and is higher by 12% to 25% than the marketable minority value. The regression coefficient relating to market capitalization in B8 is so small that the difference is immaterial, and it is easier to work with the value available.

## Table 9.6D
## Calculation of DLOM

### Section 1: Calculation of the Discount for Lack of Marketability

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 6 | | | | **= 1 – Col. [C]** | |
| 7 | | **Pure Discount** | **PV of Perpetual** | **Remaining** | |
| 8 | **Component** | **= *z* [1]** | **Discount [2]** | **Value** | |
| 9 | 1 | 0.0% | 0.0% | 100.0% | Delay to Sale |
| 10 | 2 | 9.0% | 9.0% | 91.0% | Buyer's Monopsony Power—Thin Markets |
| 11 | 3A | 4.1% | 4.5% | 95.5% | Transactions Costs—Buyers |
| 12 | 3B | 13.6% | 1.6% | 98.4% | Transactions Costs—Sellers |
| 13 | Percent Remaining | | | 89.5% | Total % Remaining = Components 1×2×3A×3B |
| 14 | **Final Discount** | | | **10.5%** | Discount = 1 – Total % Remaining |

### Section 2: Assumptions and Intermediate Calculations:

| | A | E |
|---|---|---|
| 18 | **FMV—Equity of Co. (before Discounts)** | **$ 225,000** |
| 19 | **Discount Rate = *r* [3]** | **30.6%** |
| 20 | **Constant Growth Rate = *g* (Table 9-2, Row 26)** | **4.5%** |
| 21 | **Intermediate Calculation: *x* = (1+*g*) / (1+*r*)** | **0.8003** |
| 22 | **Avg # Years between Sales = *j*** | **10** |

[1] Pure Discounts: For Component #1, Table 9.4D, cell D12; for Component #2, 9% per Schwert article. For Component #3A and #3B, Table 9.5, cells I14 and I15 – 2% for public brokerage costs.

[2] PV of Perpetual Discount Formula: 1–(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9), used for Component #3B.
PV of Perpetual Discount Formula: 1–(1–*z*)*(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9a), used for Component #3A.
Components #1 and #2 simply transfer the pure discount.

[3] The formula is 0.5352 – (0.0186 ln FMV), based on Table 9.1, B34 and B35.

## Table 9.4E
## Calculation of Component #1—Delay to Sale—$375,000 Firm [1]

| | A | B | C | D |
|---|---|---|---|---|
| 4 | | Coefficients | Co. Data | Discount |
| 5 | Intercept | 0.1342 | NA | 13.4% |
| 6 | Revenues $^2$ [2] | -5.33E-18 | 1.955E+12 | 0.0% |
| 7 | Value of Block—Post-Discount [2] | -4.26E-09 | $ 368,041 | -0.2% |
| 8 | FMV-Marketable Minority 100% Interest | 5.97E-10 | **$ 375,000** | 0.0% |
| 9 | Earnings Stability (Assumed) | -0.1376 | 0.4200 | -5.8% |
| 10 | Revenue Stability (Assumed) | -0.1789 | 0.6900 | -12.3% |
| 11 | Average Years to Sell | 0.1339 | 0.5000 | 6.7% |
| 12 | **Total Discount [4]** | | | **1.9%** |
| 13 | | | | |
| 14 | Value of Block—Pre-Discount [5] | $ 375,000 | | |
| 15 | | | | |
| 16 | Selling Price | $ 375,000 | | |
| 17 | Adjusted Net Income | $ 69,913 | | |
| 18 | Assumed Pre-Tax Margin | 5% | | |
| 19 | Sales | $ 1,398,256 | | |
| 20 | Sales $^2$ | 1.96E+12 | | |

[1] Based on Abrams's regression of Management Planning, Inc. data—Regression #2, Table 8.10.

[2] Equal to Pre-Discount Shares Sold in dollars × (1 – Discount). B7 equals B14 only when the discount = 0%.

[3] Earnings and Revenue stability are assumed at the averages from Table 8.5, G60 and H60, respectively, for all FMVs. In the Management Planning data, a correlation analysis revealed that firm size and the stability measures are uncorrelated. Therefore, we assume the same levels for all FMVs.

[4] Total Discount = max(discount, 0), because Disc < 0 indicates the model is outside of its range of reasonability.

[5] In our regression of the Management Planning, Inc. data, this was a marketable minority interest value. This is an illiquid control value and is higher by 12% to 25% than the marketable minority value. The regression coefficient relating to market capitalization in B8 is so small that the difference is immaterial, and it is easier to work with the value available.

## Table 9.6E
## Calculation of DLOM

### Section 1: Calculation of the Discount For Lack of Marketability

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 6 | | | | = 1 – Col. [C] | |
| 7 | | Pure Discount | PV of Perpetual | Remaining | |
| 8 | **Component** | **= *z* [1]** | **Discount [2]** | **Value** | |
| 9 | 1 | 1.9% | 1.9% | 98.1% | Delay to Sale |
| 10 | 2 | 9.0% | 9.0% | 91.0% | Buyer's Monopsony Power—Thin Markets |
| 11 | 3A | 4.7% | 5.3% | 94.7% | Transactions Costs—Buyers |
| 12 | 3B | 14.2% | 1.9% | 98.1% | Transactions Costs—Sellers |
| 13 | Percent Remaining | | | 87.6% | Total % Remaining = Components 1× 2 ×3A ×3B |
| 14 | **Final Discount** | | | **12.4%** | Discount = 1 – Total % Remaining |

### Section 2: Assumptions and Intermediate Calculations:

| | | |
|---|---|---|
| 18 | **FMV—Equity of Co. (before Discounts)** | **$ 375,000** |
| 19 | **Discount Rate = *r* [3]** | **29.6%** |
| 20 | **Constant Growth Rate = *g* (Table 9.2, Row 26)** | **5.0%** |
| 21 | **Intermediate Calculation: *x* = (1+*g*) / (1+*r*)** | **0.8100** |
| 22 | **Avg # Years between Sales = *j*** | **10** |

[1] Pure Discounts: For Component #1, Table 9.4E, cell D12; for Component #2, 9% per Schwert article. For Component #3A and #3B, Table 9.5, cells I16 and I17 – 1% for public brokerage costs.

[2] PV of Perpetual Discount Formula: 1–(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9), used for Component #3B.
PV of Perpetual Discount Formula: 1–(1–*z*)*(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9a), used for Component #3A.
Components #1 and #2 simply transfer the pure discount.

[3] The formula is 0.5352 – (0.0186 ln FMV), based on Table 9.1, B34 and B35.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 9.4F** | | | | |
| 2 | **Calculation of Component #1—Delay to Sale—$750,000 Firm [1]** | | | | |
| 3 | | | | | |
| 4 | | **Coefficients** | **Co. Data** | **Discount** | |
| 5 | Intercept | 0.1342 | NA | 13.4% | |
| 6 | Revenues$^2$ [2] | -5.33E-18 | 1.955E+12 | 0.0% | |
| 7 | Value of Block—Post-Discount [2] | -4.26E-09 | $ 686,724 | -0.3% | |
| 8 | FMV-Marketable Minority 100% Interest | 5.97E-10 | **$ 750,000** | 0.0% | |
| 9 | Earnings Stability (Assumed) | -0.1376 | 0.4200 | -5.8% | |
| 10 | Revenue Stability (Assumed) | -0.1789 | 0.6900 | -12.3% | |
| 11 | Average Years to Sell | 0.1339 | 1.0000 | 13.4% | |
| 12 | **Total Discount [4]** | | | **8.4%** | |
| 13 | | | | | |
| 14 | Value of Block—Pre-Discount [5] | $ 750,000 | | | |
| 15 | | | | | |
| 16 | Selling Price | $ 750,000 | | | |
| 17 | Adjusted Net Income | $ 69,913 | | | |
| 18 | Assumed Pre-Tax Margin | 5% | | | |
| 19 | Sales | $ 1,398,256 | | | |
| 20 | Sales$^2$ | 1.96E+12 | | | |
| 21 | | | | | |
| 22 | | | | | |
| 23 | [1] Based on Abrams's regression of Management Planning, Inc. data—Regression #2, Table 8.10. | | | | |
| 24 | | | | | |
| 25 | [2] Equal to Pre-Discount Shares Sold in dollars × (1 – Discount). B7 equals B14 only when the discount = 0%. | | | | |
| 26 | | | | | |
| 27 | [3] Earnings and Revenue stability are assumed at the averages from Table 8.5, G60 and H60, respectively, for | | | | |
| 28 | all FMVs. In the Management Planning data, a correlation analysis revealed that firm size and the stability | | | | |
| 29 | measures are uncorrelated. Therefore, we assume the same levels for all FMVs. | | | | |
| 30 | | | | | |
| 31 | [4] Total Discount = max(discount, 0), because Disc < 0 indicates the model is outside of its range of reasonability. | | | | |
| 32 | | | | | |
| 33 | [5] In our regression of the Management Planning, Inc. data, this was a marketable minority interest value. | | | | |
| 34 | This is an illiquid control value and is higher by 12% to 25% than the marketable minority value. The | | | | |
| 35 | regression coefficient relating to market capitalization in B8 is so small that the difference is immaterial, and | | | | |
| 36 | it is easier to work with the value available. | | | | |

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 9.6F** | | | | | | |
| 2 | **Calculation of DLOM** | | | | | | |
| 3 | | | | | | | |
| 4 | **Section 1: Calculation of the Discount for Lack of Marketability** | | | | | | |
| 5 | | | | | | | |
| 6 | | | | **= 1 – Col. [C]** | | | |
| 7 | | **Pure Discount** | **PV of Perpetual** | **Remaining** | | | |
| 8 | **Component** | **= z [1]** | **Discount [2]** | **Value** | | | |
| 9 | 1 | 8.4% | 8.4% | 91.6% | Delay to Sale | | |
| 10 | 2 | 9.0% | 9.0% | 91.0% | Buyer's Monopsony Power—Thin Markets | | |
| 11 | 3A | 4.2% | 4.8% | 95.2% | Transactions Costs—Buyers | | |
| 12 | 3B | 13.7% | 2.3% | 97.7% | Transactions Costs—Sellers | | |
| 13 | Percent Remaining | | | 81.4% | Total % Remaining = Components 1 × 2 × 3A × 3B | | |
| 14 | **Final Discount** | | | **18.6%** | Discount = 1 – Total % Remaining | | |
| 15 | | | | | | | |
| 16 | **Section 2: Assumptions and Intermediate Calculations:** | | | | | | |
| 17 | | | | | | | |
| 18 | **FMV—Equity of Co. (before Discounts)** | | | | **$ 750,000** | | |
| 19 | **Discount Rate = *r* [3]** | | | | **28.3%** | | |
| 20 | **Constant Growth Rate = *g* (Table 9.2, Row 26)** | | | | **6.0%** | | |
| 21 | **Intermediate Calculation: *x* = (1+*g*) / (1+*r*)** | | | | **0.8259** | | |
| 22 | **Avg # Years between Sales = *j*** | | | | **10** | | |
| 23 | | | | | | | |
| 24 | | | | | | | |
| 25 | [1] Pure Discounts: For Component #1, Table 9.4F, cell D12; for Component #2, 9% per Schwert article. For | | | | | | |
| 26 | Component #3A and #3B, Table 9.5, cells I18 and I19 – 1% for public brokerage costs. | | | | | | |
| 27 | | | | | | | |
| 28 | [2] PV of Perpetual Discount Formula: 1–(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9), used for Component #3B. | | | | | | |
| 29 | PV of Perpetual Discount Formula: 1–(1–*z*)*(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9a), used for Component #3A. | | | | | | |
| 30 | Components #1 and #2 simply transfer the pure discount. | | | | | | |
| 31 | | | | | | | |
| 32 | [3] The formula is 0.5352 – (0.0186 ln FMV), based on Table 9.1, B34 and B35. | | | | | | |

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 9.4G** | | | | |
| 2 | **Calculation of Component #1—Delay to Sale—$10 Million Firm [1]** | | | | |
| 3 | | | | | |
| 4 | | **Coefficients** | **Co. Data** | | **Discount** |
| 5 | Intercept | 0.1342 | NA | | 13.4% |
| 6 | Revenues$^2$ [2] | -5.33E-18 | 2.560E+14 | | -0.1% |
| 7 | Value of Block—Post-Discount [2] | -4.26E-09 | $ 9,489,650 | | -4.0% |
| 8 | FMV-Marketable Minority 100% Interest | 5.97E-10 | $ 10,000,000 | | 0.6% |
| 9 | Earnings Stability (Assumed) | -0.1376 | 0.4200 | | -5.8% |
| 10 | Revenue Stability (Assumed) | -0.1789 | 0.6900 | | -12.3% |
| 11 | Average Years to Sell | 0.1339 | 1.0000 | | 13.4% |
| 12 | **Total Discount [4]** | | | | **5.1%** |
| 13 | | | | | |
| 14 | Value of Block—Pre-Discount [5] | $ 10,000,000 | | | |
| 15 | | | | | |
| 16 | Selling Price | $ 10,000,000 | | | |
| 17 | Divide by P/E Multiple Assumed at 12.5 = Net Inc | $ 800,000 | | | |
| 18 | Assumed Pre-Tax Margin | 5% | | | |
| 19 | Sales | $ 16,000,000 | | | |
| 20 | Sales$^2$ | 2.56E+14 | | | |
| 21 | | | | | |
| 22 | | | | | |
| 23 | [1] Based on Abrams's regression of Management Planning, Inc. data—Regression #2, Table 8.10. | | | | |
| 24 | | | | | |
| 25 | [2] Equal to Pre-Discount Shares Sold in dollars × (1 – Discount). B7 equals B14 only when the discount = 0%. | | | | |
| 26 | | | | | |
| 27 | [3] Earnings and Revenue stability are assumed at the averages from Table 8.5, G60 and H60, respectively, for | | | | |
| 28 | all FMVs. In the Management Planning data, a correlation analysis revealed that firm size and the stability | | | | |
| 29 | measures are uncorrelated. Therefore, we assume the same levels for all FMVs. | | | | |
| 30 | | | | | |
| 31 | [4] Total Discount = max(discount, 0), because Disc < 0 indicates the model is outside of its range of reasonability. | | | | |
| 32 | | | | | |
| 33 | [5] In our regression of the Management Planning, Inc. data, this was a marketable minority interest value. | | | | |
| 34 | This is an illiquid control value and is higher by 12% to 25% than the marketable minority value. The | | | | |
| 35 | regression coefficient relating to market capitalization in B8 is so small that the difference is immaterial, and | | | | |
| 36 | it is easier to work with the value available. | | | | |

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 9.6G** | | | | | | |
| 2 | **Calculation of DLOM** | | | | | | |
| 3 | | | | | | | |
| 4 | **Section 1: Calculation of the Discount for Lack of Marketability** | | | | | | |
| 5 | | | | | | | |
| 6 | | | | **= 1 – Col. [C]** | | | |
| 7 | | **Pure Discount** | **PV of Perpetual** | **Remaining** | | | |
| 8 | **Component** | **= *z* [1]** | **Discount [2]** | **Value** | | | |
| 9 | 1 | 5.1% | 5.1% | 94.9% | Delay to Sale | | |
| 10 | 2 | 9.0% | 9.0% | 91.0% | Buyer's Monopsony Power—Thin Markets | | |
| 11 | 3A | 2.7% | 3.6% | 96.4% | Transactions Costs—Buyers | | |
| 12 | 3B | 4.4% | 1.5% | 98.5% | Transactions Costs—Sellers | | |
| 13 | Percent Remaining | | | 85.0% | Total % Remaining = Components 1 × 2 × 3A × 3B | | |
| 14 | **Final Discount** | | | **15.0%** | Discount = 1 – Total % Remaining | | |
| 15 | | | | | | | |
| 16 | **Section 2: Assumptions and Intermediate Calculations:** | | | | | | |
| 17 | | | | | | | |
| 18 | **FMV—Equity of Co. (before Discounts)** | | | | **$ 10,000,000** | | |
| 19 | **Discount Rate = *r* [3]** | | | | **23.5%** | | |
| 20 | **Constant Growth Rate = *g*** | | | | **8.0%** | | |
| 21 | **Intermediate Calculation: *x* = (1+*g*) / (1+*r*)** | | | | **0.8743** | | |
| 22 | **Avg # Years between Sales = *j*** | | | | **10** | | |
| 23 | | | | | | | |
| 24 | | | | | | | |
| 25 | [1] Pure Discounts: For Component #1, Table 9.4G, cell D12; for Component #2, 9% per Schwert article. For | | | | | | |
| 26 | Component #3A and #3B, Table 9.5, cells I20 and I21 – 0.5% for public brokerage costs. | | | | | | |
| 27 | | | | | | | |
| 28 | [2] PV of Perpetual Discount Formula: 1–(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9), used for Component #3B. | | | | | | |
| 29 | PV of Perpetual Discount Formula: 1–(1–*z*)*(1–*x*^*j*)/((1–(1–*z*)**x*^*j*)), per equation (8.9a), used for Component #3A. | | | | | | |
| 30 | Components #1 and #2 simply transfer the pure discount. | | | | | | |
| 31 | | | | | | | |
| 32 | [3] The formula is 0.5352 – (0.0186 ln FMV), based on Table 9.1, B34 and B35. | | | | | | |

## Interpretation of the Error

As mentioned earlier, the magnitude of the error in Table 9.2 is fairly small. The five right-hand columns average a 4.1% error (I29) and also a 4.1% (I30) mean absolute error. We can interpret this as a victory for the log size and economic components models—and I do interpret it that way, to some degree. However, the many assumptions that we had to make render our calculations too speculative to place great confidence in them. Our results constitute evidence that we are in the ballpark, but certainly fall short of proving that we are right.

An assumption not specifically discussed yet is the assumption that the simple midpoint of Raymond Miles' categories is the actual mean of the transactions in each category. Perhaps the mean of transactions in the \$500,000 to \$1 million category is really \$900,000, not \$750,000. Our results would be inaccurate to that extent and that would be another source of error in reconciling between the IBA PE multiples and my P/CF multiples. It does appear, though, that Table 9.2 nevertheless provides some evidence of the reasonableness of the log size and economic components models.

Amihud and Mendelson (1986) show that there is a clientele effect in investing in publicly held securities. Investors with longer investment horizons can amortize their transaction costs, which are primarily the bid–ask spread and secondarily the broker's fees,[11] over a longer period, thus reducing the transaction cost per period. Investors will thus select their investments by their investment horizons, and each security will have two components to its return: that of a zero bid–ask spread asset and a component that rewards the investor for the illiquidity that she is taking on in the form of the bid–ask spread.

Thus, investors with shorter investment horizons will choose securities with low bid–ask spreads, which also have smaller gross returns, and investors with longer time horizons will choose securities with larger bid–ask spreads and larger gross returns. Their net returns will be higher on average than those of short-term investors, because the long-term investor's securities choices will have higher gross returns to compensate them for the high bid–ask spread, which they amortize over a sufficiently long investment horizon to reduce its impact on net returns. A short-term investment in a high bid–ask spread stock would lose the benefit of the higher gross return by losing the bid–ask spread in the sale with little time over which to amortize the spread.

Investors in privately held firms usually have a very long time horizon, and the transaction costs are considerable compared to the bid–ask spreads of NYSE firms. In the economic components model, I assumed investors in privately held firms have the same estimate of $j$, the average time between sales, in addition to the other variables, growth ($g$), discount rate ($r$), and buyer's and seller's transaction costs ($z$). It is possible that there may be size-based, systematic differences in investor time horizons. If so, that would be a source of error in Table 9.2.

It is also possible that sufficiently long time horizons may predispose the buyer to forgo some of the DLOM he or she is entitled to. If DLOM should be, say, 25%, what is the likelihood of the buyer caving in and settling for 20% instead? If time

[11]Because broker's fees are relatively insignificant in publicly held securities, we will ignore them in this analysis. That is not true of business broker's fees for selling privately held firms.

horizons are $j = 10$ years, then the buyer amortizes the 5% "loss" over 10 years, which equals 0.5% per year. If $j = 20$, then the loss is only 0.25% per year. Thus, long time horizons should tend to reduce DLOM, and that is not a part of the economic components model—at least not yet. It would require further research to determine whether there are systematic relationships between firm size and buyers' time horizons.

## Conclusion

It does seem, then, that we are on our way as a profession to developing a "unified valuation theory," one with one or two major principles that govern all valuation situations. Of course, there are numerous subprinciples and details, but we are moving in the direction of a true science when we can see the underlying principles that unify all the various phenomena in our discipline.

Of course, if one asks whether valuation is a science or an art, the answer is that valuation is an art that sits on top of a science. A good scientist has to be a good artist, and valuation art without science is reckless fortunetelling.

## References

Amihud, Yakov, and Haim Mendelson. 1986. Asset Pricing and the Bid–Ask Spread. *Journal of Financial Economics* 17: 223–249.

Miles, Raymond C. 1992. Price/Earnings Ratios and Company Size Data for Small Businesses. *Business Valuation Review* (September): 135–139.

Pratt, Shannon P. 1993. *Valuing Small Businesses and Professional Practices, 2nd ed.* Burr Ridge, IL: McGraw-Hill.

# CHAPTER 10

# Measuring Valuation Uncertainty and Error

# CHAPTER 10

# Measuring Valuation Uncertainty and Error

## Introduction

This chapter describes the impact of various sources of valuation uncertainty and error on valuing large and small firms. It will also provide the reader with a greater understanding of where our analysis is most vulnerable to the effects of errors and demonstrate where appraisers need to focus the majority of their efforts.

### Differences between Uncertainty and Error

It is worthwhile to explain the differences between uncertainty and error. I developed the log size equation in Chapter 5 by regression analysis. Because the $R^2$ is less than 100%, size does not explain all of the differences in historical rates of return. Unknown variables and/or random variation explain the rest. When we calculate a 95% confidence interval, it means that we are 95% sure that the true value of the dependent variable is within the interval and 5% sure it is outside of the interval. That is the uncertainty. One does not need to make an error to have uncertainty in the valuation.

Let's suppose that for a firm of a particular size, the regression-determined discount rate is 20% and the 95% confidence interval is between 18% and 23%. It may be that the true and unobservable discount rate is also 20%, in which case we have uncertainty, but not error. On the other hand, if the true discount rate is anything other than 20%, then we have both uncertainty and error—even though we have used the model correctly. Since the true discount rate is unobservable and unknowable for publicly held firms, let alone privately held firms, we will never be certain that our model will calculate the correct discount rate—even when we use it properly. Appraiser error occurs when one makes a mistake in using the model. For the remainder of this chapter, we will use the simpler term, *error*, to mean appraiser-generated error. The first part of the chapter deals with valuation uncertainty and the second part deals with valuation error.

### Sources of Uncertainty and Error

We need only look at the discounted cash flow (DCF) valuation process in order to see the various sources of valuation uncertainty and error. The DCF process is:

- Forecast cash flows.
- Discount cash flows to present value.
- Calculate valuation premiums and discounts for degree of control and marketability.

Uncertainty is always present, and error can creep into our results at each stage of the valuation process.

## Measuring Valuation Uncertainty

In forecasting cash flows, even when regression analysis is a valid tool for forecasting both sales and costs and expenses, it is common to have fairly wide 95% confidence intervals around our sales forecasts, as we discovered in Chapter 3. Thus, we usually have a substantial degree of uncertainty surrounding the sales forecast and a typically smaller, though material, degree of uncertainty around the forecast of fixed and variable costs. As each company's results are unique, we will not focus on a quantitative measure of uncertainty around our forecast of cash flows in this chapter.[1] Instead, we will focus on quantitative measures of uncertainty around the discount rate, as that is generic.

For illustration, we use a midyear Gordon model formula, $\frac{\sqrt{1+r}}{r-g}$, as our valuation formula. Although a Gordon model is appropriate for most firms near or at maturity, this method is inapplicable to start-ups and other high-growth firms, as it presupposes that the company being valued has constant perpetual growth.

### Table 10.1: 95% Confidence Intervals

Table 10.1 contains calculations of 95% confidence intervals around the valuation that results from our calculation of discount rate. We use the regression equation for the log size model with 1926 through 2007 data from Chapter 5,[2] and we compare it with our CAPM results from Chapter 5. For purposes of this exercise, we will assume the forecast cash flows and perpetual growth rate are correct, so we can isolate the impact of the statistical uncertainty of the discount rate.

**VALUING THE HUGE FIRM** Because the log size model produces a mathematical relationship between return and size, our exploration of 95% confidence intervals

[1]In the second part of the chapter, we will explore the valuation impact of appraiser error in forecasting cash flows.

[2]In Chapter 5 we subtracted our estimate of the historical arithmetic mean yearly growth in the PE from the regression estimate. While that is the best procedure when calculating a discount rate for an actual valuation, for simplicity we omit that in this chapter, as it should have an immaterial effect on our conclusions.

# Table 10.1
## Ninety-Five Percent Confidence Intervals

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | |
| 2 | | | | | | | | | |
| 3 | | | | | | | | | |
| 4 | | **Huge Firm** | | **Large Firm** | | **Med. Firm** | | **Small Firm** | |
| 5 | **Cash Flow—CF$_{t-1}$** | 300,000,000 | | 15,000,000 | | 1,000,000 | | 100,000 | |
| 6 | ***r* (assume correct)** | 14% | | 19% | | 24% | | 27% | |
| 7 | ***g* = Constant Growth Rate** | 8% | | 7% | | 5% | | 5% | |
| 8 | **Cash Flow$_{t+1}$** | 324,000,000 | | 16,050,000 | | 1,050,000 | | 105,000 | |
| 9 | **Discount Rate Range** | | | | | | | | |
| 10 | **Log Size Model** | | | | | | | | |
| 11 | **Upper Bound = (12) + (2.306 × B46) [2]** | 15.61% | | 20.61% | | 25.61% | | 28.61% | |
| 12 | **As Calculated in Row 6 [1]** | 14.00% | | 19.00% | | 24.00% | | 27.00% | |
| 13 | **Lower Bound = (12) – (2.306 × B46) [2]** | 12.39% | | 17.39% | | 22.39% | | 25.39% | |
| 14 | **CAPM** | | | | | | | | |
| 15 | **Upper Bound = (16) + (2.306 × B47) [2]** | 20.02% | | 25.02% | | 30.02% | | 34.02% | |
| 16 | **As Calculated in Row 6 [1]** | 14.00% | | 19.00% | | 24.00% | | 28.00% | |
| 17 | **Lower Bound = (16) – (2.306 × B47) [2]** | 7.98% | | 12.98% | | 17.98% | | 21.98% | |
| 18 | **Gordon Model—Log Size [3]** | | | | | | | | |
| 19 | **Lower Bound = SQRT(1+(11))/[(11) – (7)]** | 14.1364 | | 8.0715 | | 5.4389 | | 4.8041 | |
| 20 | **Gordon—Mid = SQRT(1+(12))/[(12) – (7)]** | 17.7951 | | 9.0906 | | 5.8608 | | 5.1225 | |
| 21 | **Upper Bound = SQRT(1+(13))/[(13) – (7)]** | 24.1271 | | 10.4241 | | 6.3603 | | 5.4908 | |
| 22 | **FMV—Log Size Model [4]** | | | | | | | | |
| 23 | **Lower Bound (8) × (19)** | 4,580,184,344 | 79.4% | 129,548,161 | 88.8% | 5,710,869 | 92.8% | 504,427 | 93.8% |
| 24 | **Gordon—Mid (8) × (20)** | 5,765,622,256 | 100.0% | 145,904,025 | 100.0% | 6,153,845 | 100.0% | 537,859 | 100.0% |
| 25 | **Upper Bound (8) × (21)** | 7,817,166,444 | 135.6% | 167,306,311 | 114.7% | 6,678,335 | 108.5% | 576,533 | 107.2% |
| 26 | **Gordon Model—CAPM** | | | | | | | | |
| 27 | **Lower Bound = SQRT(1+(15))/[(15) – (7)]** | 9.1168 | | 6.2060 | | 4.5580 | | 3.9896 | |
| 28 | **Gordon—Mid = SQRT(1+(16))/[(16) – (7)]** | 17.7951 | | 9.0906 | | 5.8608 | | 4.9190 | |
| 29 | **Upper Bound = SQRT(1+(17))/[(17) – (7)]** | -6316.8439 | Explodes | 17.7643 | | 8.3660 | | 6.5031 | |
| 30 | **FMV—CAPM** | | | | | | | | |
| 31 | **Lower Bound (8) × (27)** | 2,953,855,173 | 51.2% | 99,606,891 | 68.3% | 4,785,891 | 77.8% | 418,913 | 81.1% |
| 32 | **Gordon—Mid (8) × (28)** | 5,765,622,256 | 100.0% | 145,904,025 | 100.0% | 6,153,845 | 100.0% | 516,495 | 100.0% |
| 33 | **Upper Bound (8) × (29)** | NA | NA | 285,117,421 | 195.4% | 8,784,288 | 142.7% | 682,828 | 132.2% |
| 34 | **Verify Discount Rate [5]** | | | | | | | | |
| 35 | **Add Constant (B44)** | 46.22% | | 46.22% | | 46.22% | | 46.22% | |
| 36 | **B45 × ln (FMV), FMV from (24)** | -32.28% | | -27.00% | | -22.45% | | -18.95% | |
| 37 | **Discount Rate (35) + (36)** | 13.94% | | 19.22% | | 23.77% | | 27.27% | |
| 38 | **Discount Rate—Rounded** | **14%** | | **19%** | | **24%** | | **27%** | |
| 39 | **Approx 95% Conf. Int.—Log Size +/– [6]** | | **28%** | | **13%** | | **8%** | | **7%** |
| 40 | **Approx 95% Conf. Int.—CAPM +/– [6]** | | Explodes | | **64%** | | **32%** | | **26%** |
| 41 | | | | | | | | | |
| 42 | **Assumptions:** | | | | | | | | |
| 43 | | | | | | | | | |
| 44 | **Log Size Constant (Table 5.1, C37)** | **46.22%** | | | | | | | |

# Table 10.1 (cont.)

| | A | B |
|---|---|---|
| 45 | **Log Size *X*-Coefficient (Table 5.1, C43)** | **-1.436%** |
| 46 | **Standard Error—Log Size (Table 5.1, P20)** | **0.70%** |
| 47 | **Standard Error—CAPM (Table 5.1, M20)** | **2.61%** |

**Notes:**

[1] We assume both the log size Model and CAPM arrive at the same discount rate.

[2] The lower and upper bounds of the discount rate are 2.306 standard errors below and above the discount rate estimated by the model. In a *t*-Distribution with 8 degrees of freedom, 2.306 standard errors approximately yields a 95% confidence interval.

[3] This is the Gordon model with a midyear assumption. The multiple = SQRT(1+*r*)/(*r*-*g*), where *r* is the discount rate and *g* is the perpetual growth rate. We use the lower and upper bounds of *r* to calculate our ranges. See footnote [7] for the exact calculation of the confidence intervals.

[4] FMV = Forecast Cash Flow—Next Year = $CF_{t+1}$ × Gordon Multiples

[5] Log size equation from Chapter 5, Table 4.1, 2nd Regression

[6] For simplicity of explanation, this is an approximate 95% confidence interval and is 2.306 standard errors above and below the forecast discount rate, with its effect on the valuation. This equals the mean of C22 and 1-C24.

around a valuation result necessitates separate calculations for different size firms. We begin with the largest firms and work our way down.

In Table 10.1, B5, we show last year's cash flow as \$300 million. Using the log size model, the discount rate is 14%[3] (B6), and we assume a perpetual growth rate of 8% (B7). We apply the perpetual growth rate to calculate cash flows for the first forecast year. Thus, forecast cash flow = \$300 million × 1.08 = \$324 million (B8).

In B12, we repeat the 14% discount rate. Next we form a 95% confidence interval around the 14% rate in the following manner. Regression #2 in Table 5.1 has 10 observations. The number of degrees freedom is $n - k - 1$, where $n$ is the number of observations and $k$ is the number of independent variables; thus we have 8 degrees of freedom. Using a $t$-distribution with 8 degrees of freedom, we add and subtract 2.306 standard errors to form a 95% confidence interval. The standard error of the log size equation is 0.70% (Table 5.1, P20 and C38, repeated in Table 10.1 as B46), which when multiplied by 2.306 equals 1.61%. The upper bound of the discount rate calculated by log size is 14% + 1.61% = 15.61% (B11), and the lower bound is 14% − 1.61% = 12.39% (B13).[4]

For purposes of comparison, we assume that CAPM also arrives at a 14% discount rate (B16). We multiply the CAPM standard error of 2.61% (Table 5.1, M20, repeated in Table 10.1, B47) by 2.306 standard errors, yielding ±6.02% for our 95% confidence interval. In B15, we add 6.02% to the 14% (B16) discount rate, and in B17, we subtract 6.02% from the 14% rate, arriving at upper and lower bounds of 20.02% and 7.98%, respectively.

Rows 19 to 21 show the calculations of the midyear Gordon model multiples (GMMs) for the log size model) $= \frac{\sqrt{1+r}}{r-g}$. For $r$ = 14% and $g$ = 8%, GMM = 17.7951 (B20), which we multiply by the \$324 million cash flow (B8) to come to an FMV (ignoring discounts and premiums) of \$5.77 billion (B24).

We repeat the process using 15.61% (B11), the upper bound of the 95% confidence interval for the discount rate in the GMM formula, to come to a lower bound of the GMM of 14.1364 (B19). Similarly, using a discount rate of 12.39% (B13, the lower bound of the confidence interval), the corresponding upper bound GMM formula is 24.1271 (B21). The FMVs associated with the lower and upper bound GMMs are \$4.58 billion (B23) and \$7.82 billion (B25), or 79.4% (C23) and 135.6% (C25), respectively, of our best estimate of \$5.77 billion (B24).

The average size of the 95% confidence interval around the valuation estimate is 28% (C39), which is equal to ½ × [(1 − 79.4%) + (135.6% − 1)] = ½ × [(1 − C23) + (C25 − 1)]. It is not literally true that the 95% confidence interval is the same above and below the estimate, but it is easier to speak in terms of a single number.

Row 28 shows the Gordon model multiple using a CAPM discount rate, which we assume is identical to the log size model discount rate. Using the CAPM upper and lower bound discount rates in B15 and B17, the lower and upper bounds of the 95% confidence interval for the CAPM Gordon model are 9.1168 (B27) and −6316.8439 (B29), respectively. Obviously, the latter is a nonsense result because the model exploded, and the average 95% confidence interval is infinite in this case.

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[3]Calculation of the log size discount rate is in rows 35–38. The equation is from Table 5.1.

[4]This is an approximation and is true at the mean of the distribution of Gordon model multiples. The confidence interval widens as we move away from the mean.

The Gordon model "explodes" (i.e., fails) when the growth rate exceeds the discount rate, which occurred here, as B7 is greater than B17.

We obtain the same estimate of FMV for CAPM as the log size model (B32 = B24), but look at the lower bound estimate in B31. It is $2.95 billion (rounded), or 51.2% (C31) of the best estimate, versus 79.4% (C23) for the same in the log size model. The CAPM standard error being more than three times larger creates a huge confidence interval and often leads to explosive results for very large firms.

**VALUATION ERROR IN THE OTHER SIZE FIRMS** The remaining columns in Table 10.1 have the same formulas and logic as columns B and C. The only difference is that the size of the firm varies, which implies a different discount rate—and, therefore, different 95% confidence intervals—and different growth rates. In column D, we assume the large firm had cash flows of $15 million (D5) last year, which will grow at 7% (D7). We see that the log size model has an average 95% confidence interval of ±13% (E39), and CAPM has an average 95% confidence interval of ±64% (E40).

Columns F and H are for successively smaller firms. We can see in rows 39 and 40 that valuation uncertainty declines with firm size.

The approximate 95% confidence intervals for log size are 28%, 13%, 8%, and 7% (row 39) for the huge, large, medium, and small firm, respectively. The CAPM confidence intervals also decline with firm size, but are much larger than the log size confidence intervals. For example, the CAPM small firm 95% confidence interval is ±26% (I40)—much larger than the 7% (I39) interval for the log size model.

Huge firms tend to have larger confidence intervals because they are "closer to the edge," where the growth rate approaches the discount rate.[5] Small to medium firms are "farther from the edge" and have smaller confidence intervals. The CAPM confidence intervals are much larger than the log size intervals.

Again, these confidence intervals measure only the uncertainty in the true discount rate. They do not measure any other source of uncertainty such as making adjustments to the discount rate for company-specific risk, forecasting cash flows and their growth, applying valuation adjustments for control and marketability. When we add differences in valuation methods and models and all the other sources of uncertainty and errors in valuation, it is indeed not at all surprising that professional appraisers can vary widely in their results.

## Measuring the Effects of Valuation Error

Up to now, we have focused on calculating the confidence intervals around the discount rate to measure valuation uncertainty. This uncertainty is generic to all businesses. It was also briefly mentioned that we can calculate the 95% confidence intervals around our forecast of sales, cost of sales, and expenses, though that process is unique to each firm. All of these come under the category of uncertainty. One need not make errors—and hopefully has not—in order to remain uncertain about the valuation.

[5]Smaller firms with very high expected growth will also be "close to the edge," although not as close as large firms with the same high growth rate.

In this second part of this chapter, we will consider the impact on the valuation of the appraiser making various types of errors in the valuation process. We can make some qualitative and quantitative observations using comparative static analysis common in economics.

The practical reader in a hurry may wish to skip to the conclusion section, as the analysis in the remainder of the chapter does not provide any tools that one may use directly in a valuation. However:

1. The conclusions are important in suggesting how we should allocate our time in a valuation.
2. The analysis is helpful in understanding the sensitivity of the valuation conclusion to the different variables (forecast cash flow, discount rate, and growth rate) and errors one may make in forecasting or calculating them.

## Defining Absolute and Relative Error

We will be considering errors from two different viewpoints:

1. By *variable*—we will consider errors in forecasting cash flow, discount rate, and growth rate.
2. By *type of error*, that is, absolute versus relative errors. The following examples illustrate the differences between the two:
   - *Forecasting cash flow*. If the correct cash flow forecast should have been \$1 million dollars and the appraiser incorrectly forecast it as \$1.1 million, the absolute error is \$100,000, and the relative error in the forecast is 10%.
   - *Forecasting discount and growth rates*. If the correct forecast of the discount rate is 20% and the appraiser incorrectly forecast it as 22%, his absolute forecasting error is 2% and his relative error is 10%.

We also will measure the *valuation effects* of the errors in absolute and relative terms:

- *Absolute valuation error*. We measure the absolute error of the valuation in dollars. Even if the absolute error is measured in percentages, for example, if we forecast growth too high by 2% in absolute terms, it causes an absolute valuation error that we measure in dollars. For example, a 2% absolute error in the discount rate might lead to a \$1 million overvaluation of the firm.
- *Relative valuation error*. The relative valuation error is the absolute valuation error divided by the correct valuation. This is measured in percentages. For example, if the value should have been \$5 million and it was incorrectly stated as \$6 million, there is a 16.7% overvaluation.

## The Valuation Model

We use the simplest valuation model in equation (10.1), the end-of-year Gordon model, where $V$ is the value, $r$ is the discount rate, and $g$ is the constant perpetual growth rate.

$$V = \frac{CF}{r-g} = CF\,\frac{1}{r-g} \quad \text{Gordon model—end-of-year assumption.}^{6} \tag{10.1}$$

## Dollar Effects of Absolute Errors in Forecasting Year 1 Cash Flow

We now assume the appraiser makes an absolute (dollar) error in forecasting year 1 cash flows. Instead of forecasting cash flows correctly as $CF_1$, he or she instead forecasts it as $CF_2$.[7] We define a positive forecast error as $CF_2 - CF_1 = \Delta CF > 0$. If the appraiser forecasts cash flow too low, then $CF_1 < CF_2$, and $\Delta CF < 0$.

Assuming there are no errors in calculating the discount rate and forecasting growth, the valuation error, $\Delta V$, is equal to:

$$\Delta V = \left[ CF_2 \frac{1}{(r-g)} - CF_1 \frac{1}{(r-g)} \right] = (CF_2 - CF_1)\frac{1}{(r-g)}. \tag{10.2}$$

Substituting $\Delta CF = CF_2 - CF_1$ into equation (10.2), we get:

$$\Delta V = \Delta CF \frac{1}{(r-g)}.$$

Valuation error when $r$ and $g$ are correct and CF is incorrect. (10.3)

We see that for each \$1 increase (decrease) in cash flow (i.e., $\Delta CF = 1$) the value increases (decreases) by $\frac{1}{r-g}$.[8] Assuming equal growth rates in cash flow for the moment, large firms will experience a larger increase in value in absolute dollars than small firms for each additional dollar of cash flow. The reason is that $r$ is smaller for large firms according to the log size model.[9]

If we overestimate cash flows by \$1, where $r = 0.15$, and $g = 0.09$, then value increases by $1/(0.15 - 0.09) = 1/0.06 = \$16.67$. For a small firm with $r = 0.27$ and $g = 0.05$, $1/(r - g) = 1/0.22$, implying an increase in value of \$4.55. If we overestimate cash flows by \$100,000 (i.e., $\Delta CF = 100{,}000$), we will overestimate the value of the large firm by \$1.67 million (\$100,000 × 16.67) and the small firm by \$455,000 (\$100,000 × 4.55). Here again, we find that larger firms and high-growth firms will tend to have larger valuation errors in absolute dollars; however, it turns out that the opposite is true in relative terms.

## Relative Effects of Absolute Errors in Forecasting Year 1 Cash Flow

Let's look at the relative error in the valuation (the *relative effect*) due to the absolute error in the cash flow forecast. It is equal to the valuation error in dollars divided by

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[6]For simplicity, for the remainder of this chapter we will stick to this simple equation and ignore the more proper log size expression for $r$, the discount rate, where $r = a + b \ln V$.

[7]In this context $CF_2$ does not mean year 2 cash flows. It means the wrong cash flows, while $CF_1$ are the right cash flows.

[8]It would be $\frac{\sqrt{1+r}}{r-g}$ for the more accurate midyear formula. Other differences when using the midyear formula appear in subsequent footnotes.

[9]According to CAPM, small beta firms would be more affected than large beta firms. However, there is a strong correlation between beta and firm size (see Table 5.1, Regression #3), which leads us back to the same result.

the correct valuation. If we denote the relative valuation error as $\%\Delta V$, it is equal to:

$$\%\Delta V = \frac{\Delta V}{V} \quad \text{Relative valuation error.} \tag{10.4}$$

We calculate equation (10.4) as (10.3) divided by (10.1):

$$\begin{aligned} \%Error = \frac{\Delta V}{V} &= \frac{\left[\dfrac{\Delta CF}{r-g}\right]}{\left[\dfrac{CF}{r-g}\right]} \\ &= \frac{\Delta CF}{CF} \quad \text{Relative valuation error from absolute error in CF.} \end{aligned} \tag{10.5}$$

For any given dollar error in cash flow, $\Delta CF$, the relative valuation error is greater for small firms than large firms, because the numerators are the same and the denominator in equation (10.5) is smaller for small firms than large firms.

For example, suppose the cash flow should be \$100,000 for a small firm and \$1 million for a large firm. Instead, the appraiser forecasts cash flow \$10,000 too high. The valuation error for the small firm is $\frac{\$10{,}000}{\$100{,}000} = 10\%$, whereas it is $\frac{\$10{,}000}{\$1{,}000{,}000} = 1\%$ for the large firm.[10]

## Absolute and Relative Effects of Relative Errors in Forecasting Year 1 Cash Flow

It is easy to confuse this section with the previous one, where we considered the valuation effect in relative terms of an absolute error in dollars in forecasting cash flows. In this section, we will consider an across-the-board relative (percentage) error in forecasting cash flows. If we say the error is 10%, then we incorrectly forecast the small firm's cash flow as \$110,000 and the large firm's cash flow as \$11 million. Both errors are 10% of the correct cash flow, so the errors are identical in relative terms, but in absolute dollars, the small firm error is \$10,000 and the large firm error is \$1 million. To make the analysis as general as possible, we will use a variable error of *k*% in our discussion.

A *k*% error in forecasting cash flows for both a large firm and a small firm increases value in both cases by *k*%,[11] as shown in equations (10.6) through (10.8). Let $V_1$ = the correct FMV, which is equation (10.6), and $V_2$ = the erroneous FMV, with a *k*% error in forecasting cash flows, which is shown in equation (10.7). The relative (percentage) valuation error will be $\frac{V_2}{V_1} - 1$, which we show in equation (10.8).

$$V_1 = CF\frac{1}{(r-g)}. \tag{10.6}$$

In equation (10.6), $V_1$ is the correct value, which we obtain by multiplying the correct cash flow, *CF*, by the end-of-year Gordon model multiple. Equation (10.7)

[10] This formula is identical using the midyear Gordon model, as the $\sqrt{1+r}$ appears in both numerators in equation (10.5) and cancels out.

[11] Strictly speaking, the error is really *k*, not *k*%. However, the description flows better using the percent sign after the *k*.

shows the effect of overestimating cash flows by $k$%. The overvaluation, $V_2$, equals:

$$V_2 = (1+k)\,CF\frac{1}{(r-g)} = (1+k)\,V_1. \tag{10.7}$$

$$\%\Delta V = \tfrac{V_2}{V_1} - 1 = k \quad \text{Relative effect of relative error in forecasting cash flow.} \tag{10.8}$$

Equation (10.8) shows that there is a $k$% error in value resulting from a $k$% error in forecasting year 1 cash flow, regardless of the initial firm size.[12] Of course, the error in dollars will differ. If the percentage error is large, there is a second-order effect in the log size model, as a $k$% overestimate of cash flows not only leads to a $k$% overvaluation, as we just discussed, but also will cause a decrease in the discount rate, which leads to additional overvaluation. It is also worth noting that an undervaluation works the same way. Just change $k$ to 0.9 for a 10% undervaluation instead of 1.1 for a 10% overvaluation, and the conclusions are the same.

## Absolute Errors in Forecasting Growth and the Discount Rate

A fundamental difference between these two variables, $r$ and $g$, and cash flow is that value is nonlinear in $r$ and $g$, whereas it is linear in cash flow. We will develop a formula to quantify the valuation error for any absolute error in calculating the discount rate or the growth rate, assuming cash flow is forecast correctly.

**DEFINITIONS** First, we begin with some definitions. Let:

$V_1$ = the correct value.
$V_2$ = the erroneous value.
$r_1$ = the correct discount rate.
$r_2$ = the erroneous discount rate.
$g_1$ = the correct growth rate.
$g_2$ = the erroneous growth rate.
CF = cash flow, which we will assume to be correct in this section.
$\Delta$ = the change in any value, which in our context means the error. We will consider a positive error to be when the erroneous value, discount rate, or growth rate is higher than the correct value. For example, if $g_1 = 5\%$ and $g_2 = 6\%$, then $\Delta g = g_2 - g_1 = 1\%$; if $g_1 = 6\%$ and $g_2 = 5\%$, then $\Delta g = -1\%$.

**THE MATHEMATICS** The correct valuation, according to the end-of-year Gordon model, is:

$$V_1 = \frac{CF}{r_1 - g_1} \quad \text{The correct value.} \tag{10.9}$$

[12]Again, this formula is the same with the midyear Gordon model, as the square root term cancels out.

The erroneous value is:

$$V_2 = \frac{CF}{r_2 - g_2} \quad \text{The erroneous value.} \tag{10.10}$$

The error, $\Delta V = V_2 - V_1$, equals:

$$\Delta V = \frac{CF}{r_2 - g_2} - \frac{CF}{r_1 - g_1} = CF\left[\frac{1}{r_2 - g_2} - \frac{1}{r_1 - g_1}\right]. \tag{10.11}$$

In order to have a common denominator, we multiply the first term in square brackets by $\frac{r_1 - g_1}{r_1 - g_1}$ and we multiply the second term in square brackets by $\frac{r_2 - g_2}{r_2 - g_2}$:

$$\Delta V = CF\left[\frac{(r_1 - g_1) - (r_2 - g_2)}{(r_1 - g_1)(r_2 - g_2)}\right]. \tag{10.12}$$

Rearranging the terms in the numerator, we get:

$$\Delta V = CF\left[\frac{(r_1 - r_2) - (g_1 - g_2)}{(r_1 - g_1)(r_2 - g_2)}\right]. \tag{10.13}$$

Changing signs in the numerator:

$$\Delta V = CF\left[\frac{-(r_2 - r_1) + (g_2 - g_1)}{(r_1 - g_1)(r_2 - g_2)}\right], \tag{10.14}$$

which simplifies to:

$$\Delta V = CF\left[\frac{-\Delta r + \Delta g}{(r_1 - g_1)(r_2 - g_2)}\right] \quad \text{Absolute effect of absolute error in } r \text{ or } g.^{13} \tag{10.15}$$

**EXAMPLE USING THE ERROR FORMULA** Let's use an example to demonstrate the error formula. Suppose cash flow is forecast next year at $100,000 and that the correct discount and growth rates are 20% and 5%, respectively. The Gordon model multiple is 1/(0.25 − 0.05) = 5, which leads to a valuation before discounts of $500,000. Instead, the appraiser makes an error and uses a zero growth rate. His erroneous Gordon model multiple will be 1/(0.25 − 0) = 4, leading to a $400,000 valuation. The appraiser's valuation error is $400,000 − $500,000 = −$100,000; that is, it is an undervaluation of $100,000.

Using equation (10.15), $\Delta V = \$100{,}000[\frac{0 - 0.05}{(0. - .05)(0.25 - 0)}] = 100{,}000[\frac{-0.05}{0.2 \times 0.25}] = 100{,}000 \times \frac{-0.05}{0.05} = -\$100{,}000$.

[13]When $\Delta r = 0$, then the formula using the midyear Gordon model is identical to equation (10.15), with the addition of the term $\sqrt{1+r}$ after the CF, but before the square brackets. When there is an error in the discount rate, the error formula using the midyear Gordon model is $CF[\frac{(r_1 - g_1)\sqrt{1+r_2} - (r_2 - g_2)\sqrt{1+r_1}}{(r_1 - g_1)(r_2 - g_2)}]$. The partial derivative for $g$ is similar to the discrete equation for change: $\frac{\partial V}{\partial g} = \frac{CF}{(r-g)^2}$. Since it is a partial derivative, we hold $r$ constant, which means $\Delta r = 0$, and instead of having $r_2 - g_2$, we double up on $r_1 - g_1$, which we can simplify to $r - g$. Again, these formulas are correct only when CF is forecast correctly.

**RELATIVE EFFECTS OF ABSOLUTE ERROR IN *r* AND *g*** The relative valuation error, as before, is the valuation error in dollars divided by the correct valuation, which is equation (10.15)/equation (10.9), or:

$$\%Error = \frac{\Delta V}{V} = \frac{\dfrac{CF(-\Delta r + \Delta g)}{(r_1 - g_1)(r_2 - g_2)}}{\dfrac{CF}{(r_1 - g_1)}}, \quad (10.16)$$

which simplifies to:

$$\%Error = \frac{\Delta V}{V} = \frac{-\Delta r + \Delta g}{r_2 - g_2} \quad \text{Relative effects of absolute error in } r \text{ and } g.^{14} \quad (10.17)$$

**EXAMPLE OF RELATIVE VALUATION ERROR** From the previous example, the relative valuation error is $\frac{\$400{,}000}{\$500{,}000} - 1 = -20\%$, a 20% undervaluation. Using equation (10.17), the relative error is $\frac{0-0.05}{0.25-0} = -\frac{0.05}{0.25} = -20\%$, which agrees with the previous calculation and demonstrates the accuracy of equation (10.17). It is important to be precise with the deltas, as it is easy to confuse the sign. In equation (10.17), the numerator is $-\Delta r + \Delta g$. It is easy to think that since there is a plus sign in front of $\Delta g$ we should use a positive 0.05 instead of $-0.05$. This is incorrect, because we are assuming that the appraiser's error in the growth rate itself is negative, that is, the erroneous growth rate minus the correct growth rate, $(V_2 - V_1) = 0 - 0.05 = -0.05$.

**VALUATION EFFECTS ON LARGE VERSUS SMALL FIRMS** Next, we look at the question of whether large or small firms are more affected by identical errors in absolute terms of the discount or growth rate. The numerator of equation (10.17) will be the same regardless of size. The denominator, however, will vary with size. Holding $g_2$ constant, $r_2$ will be smaller for large firms, as will $r_2 - g_2$. Thus, the relative error, as quantified in equation (10.17), will be larger for large firms than small firms, assuming equal growth rates.[15]

Table 10.2 demonstrates the above conclusion. Columns B through D show valuation calculations for the huge firm, as in Table 10.1. Historical cash flow was \$300 million (B6), and we assume a constant 8% (B7) growth rate as being correct, which leads to forecast cash flow of \$324 million (B8). Using the log size model, we get a discount rate of 14% (B9), as calculated in B14 through B17. In B10, we calculate an end-of-year Gordon model multiple of 16.6667, which differs from Table 10.1, where we were using a midyear multiple. Multiplying row 8 by row 10 produces a value of \$5.40 billion (B11).

Column C contains the erroneous valuation, where the appraiser uses a 9% growth rate (C7) instead of the correct 8% growth rate in B7. That leads to a valuation of \$6.54 billion (C11). The valuation error is \$1.14 billion (D11), which

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[14]This formula would be identical using the midyear Gordon model, as the $\sqrt{1+r}$ would appear in both numerators in equation (10.16) and cancel out.

[15]As before, this is theoretically not true in CAPM, which should be independent of size. However, in reality, $\beta$ is correlated to size.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 10.2** | | | | | | |
| 2 | **Absolute Errors in Forecasting Growth Rates** | | | | | | |
| 3 | | | | | | | |
| 4 | | **Huge Firm** | | | **Small Firm** | | |
| 5 | | **Correct** | **Erroneous** | **Error** | **Correct** | **Erroneous** | **Error** |
| 6 | **Cash Flow—$CF_{t-1}$** | 300,000,000 | 300,000,000 | | 100,000 | 100,000 | |
| 7 | **$g$ = Growth Rate** | 8% | 9% | | 8% | 9% | |
| 8 | **Cash Flow$_t$** | 324,000,000 | 327,000,000 | | 108,000 | 109,000 | |
| 9 | **Discount Rate** | 14.0% | 14.0% | | 27.0% | 27.0% | |
| 10 | **Gordon Multiple—End Year** | 16.6667 | 20.0000 | | 5.2632 | 5.5556 | |
| 11 | **FMV** | 5,400,000,000 | 6,540,000,000 | 1,140,000,000 | 568,421 | 605,556 | 37,135 |
| 12 | **Percentage Error** | | | **21.1%** | | | **6.5%** |
| 13 | **Verify Discount Rate** | | | | | | |
| 14 | **B45 × ln (FMV), FMV from (24)** | -32.19% | -32.46% | | -19.03% | -19.12% | |
| 15 | **Add Constant** | 46.22% | 46.22% | | 46.22% | 46.22% | |
| 16 | **Discount Rate** | 14.04% | 13.76% | | 27.19% | 27.10% | |
| 17 | **Rounded** | 14% | 14% | | 27% | 27% | |

is C11 − B11. Dividing the \$1.14 billion error by the correct valuation of \$5.40 billion, the valuation error is 21.1% (D12). We repeat the identical procedure with the small firm in columns E through G using the same growth rate as for the huge firm—although a higher discount rate, as is appropriate—and the valuation error is 6.5% (G12). This demonstrates the accuracy of our conclusion from equation (10.17) that equal absolute errors in the growth rate or discount rate cause larger relative valuation errors for large firms than small firms.

Let's now compare the magnitude of the effects of an error in calculating cash flow versus discount or growth rates. From equation (10.8), a 1% relative error in forecasting cash flows leads to a 1% valuation error. From equation (10.17), a 1% absolute error in forecasting growth leads to a valuation error of $\frac{0.01}{r_2 - g_2}$. Using typical values for the denominator, the valuation error will most likely be in the range of 4%–20% for each 1% error in forecasting growth (or error in the discount rate). This means we need to pay relatively more attention to forecasting growth rates and discount rates than we do to producing the first year's forecast of cash flows, and the larger the firm, the more care we should be taking in the analysis.

Also, it is clear from equations (10.15) and (10.17) that it is the net error in both $r$ and $g$ that drives the valuation error, not the error in either one individually. Using the end-of-year Gordon model, equal errors in $r$ and $g$ cancel each other out. With the more accurate midyear formula, errors in $g$ have slightly more impact on the value than errors in $r$, as an error in $r$ has opposite effects in the numerator and denominator.

### RELATIVE EFFECT OF RELATIVE ERROR IN FORECASTING GROWTH AND DISCOUNT RATES

We can investigate the impact of a $k$% relative error in estimating $g$ by substituting $(1 + k)g$ for $g_2$ in equation (10.10). Again, we denote the correct value as $V_1$ and the incorrect value as $V_2$.

$$V_2 = \frac{CF}{r - (1 + k)g}. \tag{10.18}$$

Using equations (10.9) and (10.18), the ratio of the incorrect to the correct value is $V_2/V_1$, or:

$$\frac{V_2}{V_1} = \frac{r - g}{r - (1 + k)g}. \tag{10.19}$$

The relative error in value resulting from a *relative* error in forecasting growth will be $(V_2/V_1) - 1$, or:

$$\%\,Error = \frac{r-g}{r-(1+k)g} - 1 \quad \text{Relative error in value from relative error in growth.} \tag{10.20}$$

Thus, if both a large and small firm have the same growth rate, then the lower discount rate of the large firm will lead to larger relative valuation errors in the large firm than the small firm. Note that for $k = 0$, equation (10.20) $= 0$, as it should. When $k$ is negative, which means we forecast growth too low, the result is the same—the undervaluation is greater for large firms than for small firms.

A relative error in forecasting the discount rate shifts the $(1 + k)$ in front of the $r$ in (10.20) instead of being in front of the $g$. The formula is:

$$\%\,Error = \frac{r-g}{(1+k)r-g} - 1 \quad \text{Relative error in value from relative error in } r. \tag{10.21}$$

**TABLES 10.3 THROUGH 10.3B: EXAMPLES SHOWING EFFECTS ON LARGE VERSUS SMALL FIRMS**

Table 10.3 shows the calculations for $k = 10\%$ (B38) relative error in forecasting growth. Rows 5–6 contain the discount rate and growth rate for a huge firm in column B and a small firm in column C, respectively. The end-of-year Gordon model multiples are 16.6667 (B7) and 5.2632 (C7) for the huge and the small firm, respectively. Multiplying the Gordon model multiples by the forecast cash flows in row 8 results in the correct values, $V_1$, in row 9 of \$5 billion and \$526,316, respectively.

Now let's see what happens if we forecast growth too high by 10% for each firm. Row 10 shows the erroneously high growth rate of 8.8%. Row 11 contains the new Gordon model multiples, and row 12 shows $V_2$, the incorrect values we obtain with the high growth rates. Row 13 shows the ratio of the incorrect to the correct valuation (i.e., $V_2/V_1$), and row 14 shows the relative error, $(V_2/V_1) - 1 = 15.38\%$ for the huge firm and 4.40% for the small firm.

Rows 20–36 are a sensitivity analysis that show the relative valuation errors for various combinations of $r$ and $g$ using equation (10.20), with $k = 10\%$. Note that the bolded E23 and E36 match the results in row 14, confirming the accuracy of the error formula. This verifies our observation from analysis of equation (10.20) that equal relative errors in forecasting growth will create much larger relative valuation errors for large firms than for small firms, holding growth constant. All we need do is notice that the relative errors in the sensitivity analysis decline as we move down each column, and as small firms have higher discount rates, the lower cells represent the smaller firms.

Table 10.3A is identical to Table 10.3, with the one exception that the growth rate is a negative 10% instead of a positive 10%. Table 10.3A demonstrates that, assuming identical real growth rates, forecasting growth too low also affects large firms more than small firms.

Table 10.3B is also identical to Table 10.3, except that it measures the relative valuation error arising from relative errors in calculating the discount rate. Table 10.3B uses equation (10.21) instead of equation (10.20) to calculate the error. It demonstrates that relative errors in forecasting the discount rate affect the valuation

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 10.3** | | | | | | |
| 2 | **Percent Valuation Error for 10% Relative Error in Growth** | | | | | | |
| 3 | | | | | | | |
| 4 | **Description** | **Huge Firm** | **Small Firm** | | | | |
| 5 | *r* | 14% | 27% | | | | |
| 6 | *g* | 8% | 8% | | | | |
| 7 | Gordon Model | 16.6667 | 5.2632 | | | | |
| 8 | Cash Flow | 300,000,000 | 100,000 | | | | |
| 9 | $V_1$ | 5,000,000,000 | 526,316 | | | | |
| 10 | (1 + PctError) × *g* | 8.80% | 8.80% | | | | |
| 11 | Gordon Model 2 | 19.2308 | 5.4945 | | | | |
| 12 | $V_2$ | 5,769,230,769 | 549,451 | | | | |
| 13 | $V_2 / V_1$ | **1.1538** | **1.0440** | | | | |
| 14 | $(V_2 / V_1) - 1$ | **15.38%** | **4.40%** | | | | |
| 15 | | | | | | | |
| 16 | **Sensitivity Analysis: Valuation Error for Combinations of *r* and *g*** | | | | | | |
| 17 | | | | | | | |
| 18 | | **Growth rate = *g*** | | | | | |
| 19 | **Discount Rate = *r*** | 5% | 6% | 7% | **8%** | 9% | 10% |
| 20 | 11% | 9.09% | 13.64% | 21.21% | 36.36% | 81.82% | NA |
| 21 | 12% | 7.69% | 11.11% | 16.28% | 25.00% | 42.86% | 100.00% |
| 22 | 13% | 6.67% | 9.38% | 13.21% | 19.05% | 29.03% | 50.00% |
| 23 | **14%** | 5.88% | 8.11% | 11.11% | **15.38%** | 21.95% | 33.33% |
| 24 | 15% | 5.26% | 7.14% | 9.59% | 12.90% | 17.65% | 25.00% |
| 25 | 16% | 4.76% | 6.38% | 8.43% | 11.11% | 14.75% | 20.00% |
| 26 | 17% | 4.35% | 5.77% | 7.53% | 9.76% | 12.68% | 16.67% |
| 27 | 18% | 4.00% | 5.26% | 6.80% | 8.70% | 11.11% | 14.29% |
| 28 | 19% | 3.70% | 4.84% | 6.19% | 7.84% | 9.89% | 12.50% |
| 29 | 20% | 3.45% | 4.48% | 5.69% | 7.14% | 8.91% | 11.11% |
| 30 | 21% | 3.23% | 4.17% | 5.26% | 6.56% | 8.11% | 10.00% |
| 31 | 22% | 3.03% | 3.90% | 4.90% | 6.06% | 7.44% | 9.09% |
| 32 | 23% | 2.86% | 3.66% | 4.58% | 5.63% | 6.87% | 8.33% |
| 33 | 24% | 2.70% | 3.45% | 4.29% | 5.26% | 6.38% | 7.69% |
| 34 | 25% | 2.56% | 3.26% | 4.05% | 4.94% | 5.96% | 7.14% |
| 35 | 26% | 2.44% | 3.09% | 3.83% | 4.65% | 5.59% | 6.67% |
| 36 | **27%** | 2.33% | 2.94% | 3.63% | **4.40%** | 5.26% | 6.25% |
| 37 | | | | | | | |
| 38 | Relative Error in *g* | **10%** | | | | | |
| 39 | | | | | | | |
| 42 | **Formula in B20: (which copies to the other cells in the sensitivity analysis)** | | | | | | |
| 43 | **=(($A20-B$19)/($A20-((1+$PctError)*B$19)))-1** | | | | | | |

of large firms more than the valuation of small firms, assuming identical real growth rates.

## Table 10.4: Summary of Effects of Valuation Errors

Table 10.4 summarizes the effects of the valuation errors. Each cell in the table contains three items:

1. The formula for the valuation error
2. The equation number containing the error formula
3. Whether the error is larger for large firms or for small firms, or there is no difference

The upper half of the table shows the valuation effects of absolute errors in forecasting the variables (cash flow, discount rate, and growth rate), and the lower half of the table shows the valuation effects of relative errors in forecasting the variables.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 10.3A** | | | | | | |
| 2 | **Percent Valuation Error for -10% Relative Error in Growth** | | | | | | |
| 3 | | | | | | | |
| 4 | **Description** | **Huge Firm** | **Small Firm** | | | | |
| 5 | *r* | 14% | 27% | | | | |
| 6 | *g* | 8% | 8% | | | | |
| 7 | Gordon Model | 16.6667 | 5.2632 | | | | |
| 8 | Cash Flow | 300,000,000 | 100,000 | | | | |
| 9 | $V_1$ | 5,000,000,000 | 526,316 | | | | |
| 10 | (1 + PctError) × *g* | 7.20% | 7.20% | | | | |
| 11 | Gordon Model 2 | 14.7059 | 5.0505 | | | | |
| 12 | $V_2$ | 4,411,764,706 | 505,051 | | | | |
| 13 | $V_2 / V_1$ | **0.8824** | **0.9596** | | | | |
| 14 | $(V_2 / V_1) - 1$ | **-11.76%** | **-4.04%** | | | | |
| 15 | | | | | | | |
| 16 | **Sensitivity Analysis: Valuation Error for Combinations of *r* and *g*** | | | | | | |
| 17 | | | | | | | |
| 18 | | **Growth rate = *g*** | | | | | |
| 19 | **Discount Rate = *r*** | 5% | 6% | 7% | **8%** | 9% | 10% |
| 20 | 11% | -7.69% | -10.71% | -14.89% | -21.05% | -31.03% | NA |
| 21 | 12% | -6.67% | -9.09% | -12.28% | -16.67% | -23.08% | -33.33% |
| 22 | 13% | -5.88% | -7.89% | -10.45% | -13.79% | -18.37% | -25.00% |
| 23 | **14%** | -5.26% | -6.98% | -9.09% | **-11.76%** | -15.25% | -20.00% |
| 24 | 15% | -4.76% | -6.25% | -8.05% | -10.26% | -13.04% | -16.67% |
| 25 | 16% | -4.35% | -5.66% | -7.22% | -9.09% | -11.39% | -14.29% |
| 26 | 17% | -4.00% | -5.17% | -6.54% | -8.16% | -10.11% | -12.50% |
| 27 | 18% | -3.70% | -4.76% | -5.98% | -7.41% | -9.09% | -11.11% |
| 28 | 19% | -3.45% | -4.41% | -5.51% | -6.78% | -8.26% | -10.00% |
| 29 | 20% | -3.23% | -4.11% | -5.11% | -6.25% | -7.56% | -9.09% |
| 30 | 21% | -3.03% | -3.85% | -4.76% | -5.80% | -6.98% | -8.33% |
| 31 | 22% | -2.86% | -3.61% | -4.46% | -5.41% | -6.47% | -7.69% |
| 32 | 23% | -2.70% | -3.41% | -4.19% | -5.06% | -6.04% | -7.14% |
| 33 | 24% | -2.56% | -3.23% | -3.95% | -4.76% | -5.66% | -6.67% |
| 34 | 25% | -2.44% | -3.06% | -3.74% | -4.49% | -5.33% | -6.25% |
| 35 | 26% | -2.33% | -2.91% | -3.55% | -4.26% | -5.03% | -5.88% |
| 36 | **27%** | -2.22% | -2.78% | -3.38% | **-4.04%** | -4.76% | -5.56% |
| 37 | | | | | | | |
| 38 | Relative Error in *g* | **-10.0%** | | | | | |
| 39 | | | | | | | |
| 42 | **Formula in B20: (which copies to the other cells in the sensitivity analysis)** | | | | | | |
| 43 | **=(($A20-B$19)/($A20-((1+$PctError)*B$19)))-1** | | | | | | |

In 10 of the 12 cells in the table that contain error formulas, the valuation errors are greater for large firms than for small firms. Only equation (10.5), which is the relative valuation error resulting from a dollar error in forecasting cash flows, affects small firms more than large firms. Equation (10.8), the relative valuation error resulting from a relative error in forecasting cash flows, affects both small and large firms alike. It is not surprising that the only two exceptions to the greater impact of valuation errors being on large firms come from cash flows, as value is linear in cash flows. The nonlinear relationship of value to discount rate and growth rate causes errors in those two variables to impact the valuation of large firms far more than small firms and to impact the value of both more than errors in cash flow.

Errors in forecasting growth have the greatest impact on value. Value is positively related to forecast growth. Errors in forecasting discount rates are a close second in effect,[16] though opposite in sign. Value is negatively related to discount rate. Errors in forecasting the first year's cash flow by far have the *least* impact on value.

[16] Again, this result comes from using the midyear Gordon model, not the end-of-year formula.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 10.3B** | | | | | | |
| 2 | **Percent Valuation Error for 10% Relative Error in Discount Rate** | | | | | | |
| 3 | | | | | | | |
| 4 | **Description** | **Huge Firm** | **Small Firm** | | | | |
| 5 | *r* | 14% | 27% | | | | |
| 6 | *g* | 8% | 8% | | | | |
| 7 | Gordon Model | 16.6667 | 5.2632 | | | | |
| 8 | Cash Flow | 300,000,000 | 100,000 | | | | |
| 9 | $V_1$ | 5,000,000,000 | 526,316 | | | | |
| 10 | (1 + PctError) × *r* | 15.40% | 29.70% | | | | |
| 11 | Gordon Model 2 | 13.5135 | 4.6083 | | | | |
| 12 | $V_2$ | 4,054,054,054 | 460,829 | | | | |
| 13 | $V_2 / V_1$ | **0.8108** | **0.8756** | | | | |
| 14 | $(V_2 / V_1)$ - 1 | **-18.92%** | **-12.44%** | | | | |
| 15 | | | | | | | |
| 16 | **Sensitivity Analysis: Valuation Error for Combinations of *r* and *g*** | | | | | | |
| 17 | | | | | | | |
| 18 | | **Growth rate = *g*** | | | | | |
| 19 | **Discount Rate = *r*** | **5%** | 6% | 7% | **8%** | 9% | 10% |
| 20 | 11% | -15.49% | -18.03% | -21.57% | -26.83% | -35.48% | -52.38% |
| 21 | 12% | -14.63% | -16.67% | -19.35% | -23.08% | -28.57% | -37.50% |
| 22 | 13% | -13.98% | -15.66% | -17.81% | -20.63% | -24.53% | -30.23% |
| 23 | **14%** | -13.46% | -14.89% | -16.67% | **-18.92%** | -21.88% | -25.93% |
| 24 | 15% | -13.04% | -14.29% | -15.79% | -17.65% | -20.00% | -23.08% |
| 25 | 16% | -12.70% | -13.79% | -15.09% | -16.67% | -18.60% | -21.05% |
| 26 | 17% | -12.41% | -13.39% | -14.53% | -15.89% | -17.53% | -19.54% |
| 27 | 18% | -12.16% | -13.04% | -14.06% | -15.25% | -16.67% | -18.37% |
| 28 | 19% | -11.95% | -12.75% | -13.67% | -14.73% | -15.97% | -17.43% |
| 29 | 20% | -11.76% | -12.50% | -13.33% | -14.29% | -15.38% | -16.67% |
| 30 | 21% | -11.60% | -12.28% | -13.04% | -13.91% | -14.89% | -16.03% |
| 31 | 22% | -11.46% | -12.09% | -12.79% | -13.58% | -14.47% | -15.49% |
| 32 | 23% | -11.33% | -11.92% | -12.57% | -13.29% | -14.11% | -15.03% |
| 33 | 24% | -11.21% | -11.76% | -12.37% | -13.04% | -13.79% | -14.63% |
| 34 | 25% | -11.11% | -11.63% | -12.20% | -12.82% | -13.51% | -14.29% |
| 35 | 26% | -11.02% | -11.50% | -12.04% | -12.62% | -13.27% | -13.98% |
| 36 | **27%** | -10.93% | -11.39% | -11.89% | **-12.44%** | -13.04% | -13.71% |
| 37 | | | | | | | |
| 38 | **Relative Error in *r*** | **10%** | | | | | |
| 39 | | | | | | | |
| 42 | **Formula in B20: (which copies to the other cells in the sensitivity analysis)** | | | | | | |
| 43 | **=((($A20-B$19)/(((1+$PctError)*$A20)-B$19))-1)** | | | | | | |

Another issue in valuation error in using the log size model is that while an initial error in calculating the discount rate is self-correcting using an iterative method, an error in calculating cash flows or the growth rate not only causes its own error, but also will distort the calculation of the discount rate. For example, overestimating growth, $g$, will cause an overvaluation, which will lower the discount rate beyond its proper level, which will in turn cause a second-order overvaluation. We did not see this in our comparative static analysis, because for simplicity we were working with the Gordon model multiple in the form of equation (10.1). We allowed $r$ to be an apparently independent variable instead of using its more proper, but complicated, log size form of $r = a + b \ln V$. Thus, the proper Gordon model using a log size discount rate is: $V = CF \times \frac{1}{a + b \ln V - g}$.

The secondary valuation error caused by a faulty forecast of cash flows or growth rate will be minimal, because the discount rate, as calculated using the log size model, is fairly insensitive to the error in the estimate of value. As mentioned earlier, on the surface, this would not be a source of error using CAPM, as the discount rate in CAPM does not depend on the magnitude of the subject company's cash flows. However, that is not really true, as CAPM betas are correlated to size.

**Table 10.4**

**Summary of Effects of Valuation Errors**

***Valuation Effects of Absolute Errors in the Variables [1]***

| Valuation Error | Cash Flow | Discount Rate = $r$ | Growth Rate = $g$ |
|---|---|---|---|
| Absolute ($) | $\Delta V = \Delta CF \frac{1}{(r-g)}$ [10.3] Large Firms | $\Delta V = CF\left[\frac{-\Delta r + \Delta g}{(r_1 - g_1)(r_2 - g_2)}\right]$ [10.15] Large Firms | $\Delta V = CF\left[\frac{-\Delta r + \Delta g}{(r_1 - g_1)(r_2 - g_2)}\right]$ [10.15] Large Firms |
| Relative (%) | $\frac{\Delta V}{V} = \frac{\Delta CF}{CF}$ [10.5] Small Firms | $\frac{\Delta V}{V} = \frac{-\Delta r + \Delta g}{(r_2 - g_2)}$ [10.17] [3] Large Firms | $\frac{\Delta V}{V} = \frac{-\Delta r + \Delta g}{(r_2 - g_2)}$ [10.17] [3] Large Firms |

***Valuation Effects of Relative Errors in the Variables [1]***

| Valuation Error | Cash Flow | Discount Rate = $r$ | Growth Rate = $g$ |
|---|---|---|---|
| Absolute ($) | $\Delta V = kV_1$ [2] Large Firms | [4] NA Large Firms | [4] NA Large Firms |
| Relative (%) | $\frac{V_2}{V_1} - 1 = k$ [10.8] No Difference | $\%Error = \frac{r-g}{(1+k)r - g} - 1$ [10.21] Large Firms | $\%Error = \frac{r-g}{r-(1+k)g} - 1$ [10.20] Large Firms |

**Notes**

[1] Each cell shows the formula for the valuation error, the equation number in the chapter for the formula, and whether the valuation error is larger for large firms, small firms, or there is no difference.

[2] This formula is not explicitly calculated in the chapter. We can calculate it as $V_2 - V_1 = [(1+k) V_1 - V_1] = kV_1$.

[3] While there is no difference in the magnitude of valuation errors arising from an error in $r$ or $g$ when we measure value by the end-of-year Gordon model, when we use the midyear Gordon model, errors in $g$ have slightly more impact than errors in $r$ (and much more impact than errors in cash flow).

[4] Omitted because these expressions are complex and add little to understanding the topic.

## Summary and Conclusions

We discussed valuation uncertainty in the first part of this chapter and valuation error in the second part. Using the NYSE/AMEX/NASDAQ data, the actual 95% confidence intervals around the valuation estimate for our statistical uncertainty in calculating the discount rate range from ±38% for huge firms down to ±7% for small firms, as calculated in Table 10.1, row 39. Using CAPM leads to much larger confidence intervals. Additionally, we could calculate the 95% confidence intervals around the sales and expense forecast.

Errors in forecasting the growth rate and calculating the discount rate cause much larger valuation errors than errors in forecasting the first year's cash flow. Thus, the bottom line conclusion from our analysis is that we need to be most careful in forecasting growth and discount rates because they have the most profound effect on the valuation. Usually, we spend the majority of our efforts forecasting cash flows, and it might be tempting to some appraisers to "slam dunk" the growth forecast and/or the discount rate calculation. Hopefully, the results in this chapter show that that is a bad idea.

In this chapter, we have not specifically addressed uncertainty and errors in calculating valuation discounts, but obviously one must realize that they, too, add to the overall uncertainty that we have in rendering an opinion of value. There is material in Chapter 8 relating to uncertainty in calculating restricted stock discounts, which forms part of our overall uncertainty in calculating the discount for lack of marketability.

After analysis of just the uncertainty alone in the valuation—not even considering the possibility that somewhere we have made an actual error—it is appropriate for us to display a healthy humility about our final valuation conclusions.

## Reference

Ibbotson Associates/Morningstar. 2008. *Stocks, Bonds, Bills and Inflation: 2008 Valuation Yearbook*. Chicago: Ibbotson Associates/Morningstar.

# PART V

# Litigation

## Introduction

Part V of this book consists of Chapters 11 and 12. Chapter 11 provides criteria for being a fair expert and a statistical method to measure the probability of apparent expert bias occurring at random versus being purposeful bias. It is easier to demonstrate bias in the Market Approach methods than Discounted Cash Flow, but both types of bias are possible to demonstrate and measure. As short as it is, I consider this chapter one of the most important ideas and analyses that I have ever produced.

Chapter 12 is the development of accounting formulas for use in damage calculations for manufacturers. Manufacturing accounting—known as cost accounting—is the most complex type of accounting system. Its plethora of exotic accounts, such as standard costs and their variances, can cast a huge smoke screen over the simple reality of real damages. The purpose of this chapter is to provide the logic, analysis, and the formulas to enable the practitioner to cut through the smoke and provide clear, accurate damage calculations. There are different permutations of formulas to help the practitioner work with some types of missing data[1] and still produce the correct analysis.

In some ways, Chapter 12 could belong to Part I—forecasting cash flows. It has a logical connection to Chapter 1, with its mathematical derivation of cash flow. Nevertheless, I placed it in the litigation category, as this chapter is geared for calculating economic damages, not pure cash flow.

[1]Obviously, it cannot correct for all types of missing data.

# CHAPTER 11

# Demonstrating Expert Bias

# CHAPTER 11

# Demonstrating Expert Bias

## Introduction

This chapter explores how one might measure and demonstrate expert bias. This is no easy task, and it is not always possible to achieve, even when it exists. However, when you are able to do this, you are doing the court a great service.

While we direct our comments to the litigation context, these principles and techniques apply to one's own work, whether or not the purpose of the valuation is litigation. We will focus on demonstrating bias in the discounted cash flow method and the various market approach methods.

There are three attributes of unbiased work:

1. It should be systematic.
2. It should be consistent with one's "normal" valuation methods and measurements unless there are compelling reasons not to be, in which case we must explain why.
3. It should be balanced, that is, not tilted toward an expert's client.

## Market Methods

The most blatant form of expert bias is "cherry-picking the comps" (the guideline public companies). An expert who does that violates items 1 and 3 in our attributes list.

### The Choice of Guideline Companies Should Be Systematic and Logical

The hallmark of any market method is that it should be systematic and logical. There are two accepted styles in choosing guideline companies (GCs). One is to go right away for a "sharpshooter approach," which is applicable when using guideline public companies (GPCs). We choose a small, select set of GPCs that closely match the subject company. The advantage of this approach is that it is more likely to produce a *tight fit* (i.e., great regression results)—high adjusted $R^2$ and low standard error of the $y$-estimate. The disadvantage is that the small sample size may violate a major assumption underlying regression analysis—that the regression errors are independent and normally distributed. Violating that assumption can cause regression statistics to be inflated.

The other style, which applies to both the guideline public company method and the guideline M&A method, is to choose all firms or transactions, as appropriate, in the databases to which we subscribe, choose logical criteria for elimination, and then omit all the companies or transactions based on those criteria. The usual criteria for eliminating GCs are:

1. *Eliminating firms that are in different industries or in a different niche within the industry.*
   - However, consider retaining all firms and using dummy variables to measure the statistical significance of industry niches. In my experience, often only one or two are statistically significant out of perhaps 10 or 15 niches. Using this strategy allows the expert to retain a much larger sample size, which is statistically more robust. Doing this, we have the law of large numbers on our side, and our assumption of normally distributed error terms is more likely reasonable. When the sample size is small, the validity of this assumption may be questionable.
2. *Eliminating firms with negative earnings when the subject company is profitable.*
   - However, it is often reasonable to include these firms in a regression of price-to-sales, MVIC-to-sales, or, in a pinch, MVIC-to-gross-profit multiples, as this will increase the sample size in that measure of value. Note that this step is in addition to, not instead of, item 2.
3. *Eliminating firms with sales either below one threshold or above another.*
   - However, consider regressing P/S multiples as the dependent variable and using the logarithm of sales, total assets, or book value of equity as one of the independent variables. This might eliminate the need to reject small and large firms from the sample. It also maximizes sample size, which is always a good idea.

When the opposing expert appears to use reasonable, but not ideal, criteria for eliminating potential GCs, it is probably not yet time for red lights to flash in your head. Few valuation professionals have solid grounding in statistics, and fewer still have doctorates in finance or math. However, when the opposing expert is picking GCs unsystematically, with the only obvious criterion being that it benefits his client, it is time for the bells to clang and the red lights to flash.

As an example, suppose the subject company is a fast-food restaurant. If opposing expert claims to use a certain database and seems to exhibit a pattern of choosing some but not all fast-food restaurants in it while also including some other types of restaurants, it is reasonable to wonder whether the expert is cherry-picking the observations. How do we detect and measure such a harvest?

## Calculate Summary Statistics

Start by calculating summary statistics of the database. I recommend the following summary statistics for each relevant column in the database (e.g., selling price, sales, discretionary earnings, transaction date, etc.):

1. The *minimum value*. Suppose the data exist in an Excel sheet in rows 3–200 and columns B–K. In Excel, the formula to calculate the minimum observation in column B is: =Min(B3:B200).

2. The *maximum value*. The formula is: =Max(B3:B200).
3. The *mean*. The formula is: =Average(B3:B200).
4. The *median*, which is the 50th percentile of the distribution. The formula is: =Median(B3:B200).
5. I also calculate the 5th, 10th, 25th, 75th, 90th, and 95th percentiles when the sample size is large. This gives us a good picture of where the subject company falls in the distribution.
   a. The formula for the 5th percentile is: =Percentile(B3:B200,0.05).
   b. The formula for the 95th percentile is: =Percentile(B3:B200,0.95).

   Calculate these statistics for the entire database and for the subset chosen by the opposing expert, and then compare them. For example, suppose the subject company operating profit margin of 14% is in the 98th percentile for the database as a whole, while opposing expert's GCs, chosen unsystematically, have an average profit margin of 2%. This is unlikely to pass the smell test. It is legitimate to criticize opposing expert's choice of GCs. We also should compare the percentiles of other key variables.

## Using a Binomial Distribution

Another technique to detect expert bias is to use a binomial distribution to ascertain how reasonable opposing expert's choice of GCs is. Suppose he chose price-to-sales (PS) (or MVIC-to-sales) multiples for 10 out of 50 possible fast-food restaurants, with no apparent reason or logic for rejecting the other 40. We compare the ones he chose to the ones he didn't. For instance, number the ones he chose as 1, 2 . . . 9, and 10 and the ones he rejected as 11, 12 . . . 49, and 50.

Our null hypothesis is that opposing expert is not biased. If this is true, then there should not be statistically significant differences between the PS multiples chosen and the ones ignored. We can make the following comparisons of PS multiples:

- *Compare #1 to #11 through #50*. This will produce 40 comparisons, the result of which is that #1 is either larger (L) or smaller (S) than the ones rejected.
- *Compare #2 to #11 through #50*. This will produce another 40 comparisons, results being the number of Ls or Ss.
- *Compare #3 through #10 to #11 through #50*. This will produce another 8 × 40 = 320 comparisons.

In total we will have 400 comparisons, with the results being either L or S.[1] In a binomial distribution there are only two outcomes—like heads and tails in the flip of a coin or, in this case, L or S. This distribution provides us with the probability that our outcome occurred randomly, not by design. The lower the probability that this

[1]There will be a few that are equal. If the number is small relative to the total, ignore them. However, if they are a significant percentage of the total, then we will probably fail to reject the null hypothesis that there is no expert bias. Another approach is to handle this as we do in our DCF example below, where we assume that a certain percentage of multiples will be the same, e.g., 20%, and our binomial probability that a materially lower multiple occurred randomly would be $p = 40\%$, which means that the probability that the multiple would be substantially the same or materially higher is 60%.

occurred by chance, the more likely it is that we should reject the null hypothesis and assert that opposing expert *is* biased. However, because that is a serious claim, it is a good idea to do the following first:

- Have your results reviewed by someone with a strong statistics background.
- Share your results with retaining counsel, who should depose the opposing expert before making such a claim. It is possible that opposing expert has good reasons for choosing as he did, but failed to state them in his report.

It is our intention to provide examples of this on our Web site and eventually publish them in a workbook.

It is also possible to extend the analysis under other circumstances when the opposing expert appears to have made other strange or biased choices. Suppose he chooses one or more GCs with the description "Restaurant." This exposes opposing expert to more questions and potential criticism. Since "Restaurant" is a generic description, then we can ask why he didn't choose from GCs with specific descriptions such as "White Table Cloth," "Burgers," and so forth. There are also grounds for us to extend the L-versus-S comparison to every rejected observation in the dataset.

## A Balanced DCF Valuation

Producing a balanced DCF valuation requires using hard data whenever possible. The expert should be balanced and objective when making assumptions or difficult choices. Making one "aggressive" choice does not necessarily tilt the valuation. Sometimes experts make difficult choices. We may be convinced that our estimation of the discount rate is the best one, even though reasonable experts may make a different calculation. We enter the danger zone when we make choices and/or assumptions such that each moves the valuation either up or down not just once, but every time.

Suppose there are 10 key choices in a DCF, and opposing expert made 10 aggressive assumptions that all favor his client with a high valuation. Examples are a low tax rate, high sales growth rates, high margins, high payout ratio, low discount rate, low discount for lack of control, low discount for lack of marketability, and so forth. Let's further assume that an unbiased valuation expert will make the correct assumption half of the time, will be too high a quarter of the time, and will be too low a quarter of the time by random chance. Thus, the unbiased expert will be too aggressive by random chance a quarter of the time and either correct or not aggressive enough three-quarters of the time.

### Tables 11.1 and 11.2: Binomial Distributions

Table 11.1 is a binomial distribution showing the probabilities of the different possible outcomes. Let $p$ = the probability that opposing expert will be too aggressive in each assumption by chance = 25% (B5), and $1 - p = q$ = the probability that opposing expert will be either correct or too conservative = 75% (B6).

A11 to A21 show the different possible number of times opposing expert makes an assumption too aggressively, that is, from 0 to 10. Column B is $\binom{n}{x} = \frac{n!}{x!(n-x)!}$, known as "$n$ choose $x$." It tells us how many permutations we have for each

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 11.1** | | | | | | |
| 2 | **Binomial Distribution** | | | | | | |
| 3 | ***p* = Probability of Randomly Too Aggressive = 25%** | | | | | | |
| 4 | | | | | | | |
| 5 | *p* | 25% | % Aggressive Assumption by Random Chance | | | | |
| 6 | *q* | 75% | % Accurate or Conservative Assumption | | | | |
| 7 | *n* | 10 | # Critical Assumptions in the DCF Model | | | | |
| 8 | | | | | | | |
| 9 | | | | | | | |
| 10 | ***x* = # Too Aggressive** | ***n* choose *x*** | $p^x$ | $q^{n-x}$ | $p^x \times q^{n-x}$ | **= (B) × (E)** | **Cum F** |
| 11 | 0 | 1 | 100.0% | 5.6% | 5.6% | 5.6314% | 5.6314% |
| 12 | 1 | 10 | 25.0% | 7.5% | 1.9% | 18.7712% | 24.4025% |
| 13 | 2 | 45 | 6.3% | 10.0% | 0.6% | 28.1568% | 52.5593% |
| 14 | 3 | 120 | 1.6% | 13.3% | 0.2% | 25.0282% | 77.5875% |
| 15 | 4 | 210 | 0.4% | 17.8% | 0.1% | 14.5998% | 92.1873% |
| 16 | 5 | 252 | 0.1% | 23.7% | 0.0% | 5.8399% | 98.0272% |
| 17 | 6 | 210 | 0.0% | 31.6% | 0.0% | 1.6222% | 99.6494% |
| 18 | 7 | 120 | 0.0% | 42.2% | 0.0% | 0.3090% | 99.9584% |
| 19 | 8 | 45 | 0.0% | 56.3% | 0.0% | 0.0386% | 99.9970% |
| 20 | 9 | 10 | 0.0% | 75.0% | 0.0% | 0.0029% | 99.9999% |
| 21 | 10 | 1 | 0.0% | 100.0% | 0.0% | 0.0001% | **100.0000%** |
| 22 | | | | | | **100.0000%** | |
| 23 | Formula in B11: =FACT(n)/(FACT(A11)*FACT((n-A11))) | | | | | | |

combination of $n$ and $x$. For example, B13 is 45, which we calculate as: $\binom{10}{2} = \frac{10!}{2!(10-2)!} = \frac{10!}{2!8!} = \frac{10 \times 9 \times 8!}{2!8!} = \frac{90}{2} = 45$ permutations. The aggressive scenarios could be 1 and 2, 1 and 3, 1 and 4 . . ., 1 and 10, 2 and 3, 2 and 4, . . . 2 and 10, and so on, which add to 45 unique combinations. Note that column B is symmetric around $x = 5$; that is, for $x = 4$ or 6, $\binom{n}{x} = 210$ (B15, B17), for $x = 3$ or 7, $\binom{n}{x} = 120$ (B14, B18), and so on. Also remember that 0! is defined as 1. Thus $\binom{10}{0} = \frac{10!}{0!10!} = 1$. The meaning behind this is that there is only one combination that will produce zero aggressive assumptions, which is all 10 assumptions must be reasonable or conservative.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 11.2** | | | | | | |
| 2 | **Binomial Distribution** | | | | | | |
| 3 | ***p* = Probability of Randomly Too Aggressive = 12.5%** | | | | | | |
| 4 | | | | | | | |
| 5 | *p* | 12.5% | % Aggressive Assumption by Random Chance | | | | |
| 6 | *q* | 87.5% | % Accurate or Conservative Assumption | | | | |
| 7 | *n* | 10 | # Critical Assumptions in the DCF Model | | | | |
| 8 | | | | | | | |
| 9 | | | | | | | |
| 10 | ***x* = # Too Aggressive** | ***n* choose *x*** | $p^x$ | $q^{n-x}$ | $p^x \times q^{n-x}$ | **= (B) × (E)** | **Cum F** |
| 11 | 0 | 1 | 100.0% | 26.3% | 26.3% | 26.3075576% | 26.3075576% |
| 12 | 1 | 10 | 12.5% | 30.1% | 3.8% | 37.5822252% | 63.8897828% |
| 13 | 2 | 45 | 1.6% | 34.4% | 0.5% | 24.1600019% | 88.0497847% |
| 14 | 3 | 120 | 0.2% | 39.3% | 0.1% | 9.2038102% | 97.2535949% |
| 15 | 4 | 210 | 0.0% | 44.9% | 0.0% | 2.3009526% | 99.5545475% |
| 16 | 5 | 252 | 0.0% | 51.3% | 0.0% | 0.3944490% | 99.9489965% |
| 17 | 6 | 210 | 0.0% | 58.6% | 0.0% | 0.0469582% | 99.9959547% |
| 18 | 7 | 120 | 0.0% | 67.0% | 0.0% | 0.0038333% | 99.9997880% |
| 19 | 8 | 45 | 0.0% | 76.6% | 0.0% | 0.0002054% | 99.9999934% |
| 20 | 9 | 10 | 0.0% | 87.5% | 0.0% | 0.0000065% | 99.9999999% |
| 21 | 10 | 1 | 0.0% | 100.0% | 0.0% | 0.0000001% | **100.0000000%** |
| 22 | | | | | | **100.0000000%** | |
| 23 | Formula in B11: =FACT(n)/(FACT(A11)*FACT((n-A11))) | | | | | | |

Columns C and D show the respective probabilities of obtaining $x$ number of times being too aggressive and $n - x$ times being accurate or too conservative by chance. Column E equals column C × column D and is the joint probability of both of the above conditions existing simultaneously.

Column F is column B × column E; that is, it is the joint probability multiplied by the number of permutations possible. Each entry in column F tells us the number of times we would expect $x$ occurrences of aggressive assumptions. Note that column F adds to 100% (F22). Column G is the cumulation of column F and finishes at 100% (G21). It tells us the probability of the number of aggressive assumptions being less than or equal to $x$. This column is our primary result.

Column G tells us that we should expect that under our initial assumption of 25%, an unbiased opposing expert would not have even one aggressive assumption 5.6% (G11) of the time, would have at least one aggressive assumption over 24% (G12) of the time, and at least two aggressive assumptions almost 53% (G13) of the time. As the latter is close to 50%, we should normally expect opposing expert to have at least two aggressive assumptions out of 10. However, we would expect him to have all 10 assumptions aggressive by random chance only 0.0001% (F21) of the time,[2] or about once in 10,000 times. If we can clearly demonstrate that all 10 assumptions are truly aggressive and not reasonable, it's time to cry foul. In Table 11.2 we instead assume an unbiased expert should be accurate 75% of the time, too aggressive by random chance 12.5% of the time (this is $p = 12.5\%$ in B5), and too conservative 12.5% of the time. Under these assumptions, being too aggressive is half as likely as in Table 11.1, and the probability of being too aggressive by random chance in all 10 assumptions is now 1 in 10 million (0.0000001% in F21).

However, most DCFs may have only three to five assumptions that make the most difference, and this is too small a sample to reach robust statistical conclusions. In fact, one must be careful using this approach even with a sample of 10. The good news is that using a binomial distribution to measure potential expert bias is more likely to be robust and successful in cases such as the guideline company comparison where $n = 400$ comparisons.

Another issue in DCF assumptions is that there is more subjective judgment involved in alleging that an aggressive assumption is outside the mainstream. This is another reason why it is likely to be more difficult to document expert bias with robust statistical significance in the DCF method than in the market methods.

## Summary

The hallmark of competent and objective valuation work is that it should be systematic and logical. Lacking that is fair grounds for statistically testing for expert bias. The binomial distribution is a good method for testing market approach methods and often is robust. One might use it to test for bias in the DCF, but the small number of critical assumptions and subjectivity in deciding what is a fair assumption versus an aggressive one makes it less likely to result in a credible challenge to the fairness of opposing expert.

[2] In other words, we would expect the number of aggressive assumptions to be less than or equal to 9 (A20) 99.9999% (G20) of the time.

# CHAPTER 12

# Lost Inventory and Lost Profits Damage Formulas in Litigation

# CHAPTER 12

# Lost Inventory and Lost Profits Damage Formulas in Litigation

## Introduction

Accountants and valuation analysts who do damage calculations in litigation are sometimes faced with strange and exotic damage calculations made by the other side's expert—especially in the manufacturing context, as cost accounting is more complex than accounting in retail, wholesale, or service industries. After making our way through the smoke and the mirrors, the damage calculations are fairly straightforward, based on solid accounting principles.

The most complicated situations arise when the data available to the damages expert is inadequate and cannot be improved on, because the expert was engaged after demands for documentation have been made and satisfied. This can be even more difficult when the damage occurred at a plant that is a subsidiary, and thus the data available may be a hodgepodge fraction of what we need to do our work in a simple, straightforward manner.

Let's assume that a fire started in the manufacturing plant of Sir Harvard Cucumber's firm, Cucumber Pickles (CP), and spread to his next-door-neighbor, Billabong's Boomerangs (BB), owned by Constance Billabong.

BB sued CP for a variety of different damages. We will focus on the two main conceptual categories of damages:

1. The cost of the destroyed boomerangs
2. The lost profits from lost sales of boomerangs, which come in two categories:
   a. Those from the inventory that were produced and destroyed
   b. Sales that were lost from inventory that was never produced, because the company was unable to take orders for a one-day time period during which the factory was closed or only partially operating and did not manufacture the inventory for the orders that were not taken

Thus, the destroyed boomerangs have two components—their cost (item 1) and the lost profits on them (item 2a), while the lost sales due to lack of production has only one component—item 2b.

The author wishes to thank Satoshi Kojima, CPA, for his insightful comments and his help with cost accounting.

We will develop formulas for calculating both categories of damages. Additionally, we develop separate formulas for the two types of lost profits and provide numerical examples in the tables. Tables 12.1 and 12.1a cover items 1 and 2a, while Table 12.1B covers item 2b. Table 12.1C covers an assumed change in a fact.

Rather than developing the theoretical mathematics right away, it makes for a more intuitive understanding if we first look at some tables with damage calculations. We will go through the logic and the numbers in the tables, and then we will develop general formulas for the destroyed inventory and lost profits.

## Commentary to Table 12.1: Sample Damage Calculations with $VM = \$95$

The purpose of Table 12.1 is to present a simple set of facts for lost inventory and lost profits calculations and develop formulas that quantify the damages accurately.

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 12.1** | | | | | | |
| 2 | **Sample Damage Calculations with *VM* = $95** | | | | | | |
| 3 | | | | | | | |
| 4 | | | | **Diff =** | | **Lost Boom-** | **Lost** |
| 5 | | **Without Fire** | **With Fire** | **Damage** | | **erangs** | **Profits** |
| 6 | Sales = *S* | **100** | **0** | **100** | | | **100** |
| 7 | | | | | | | |
| 8 | Variable Mfg. Costs (*VM*)** | 95 | 95 | 0 | | **95** | 95 |
| 9 | Fixed Mfg. Overhead (*FMOH*) | 10 | 10 | 0 | | | 0 |
| 10 | Total Mfg. Costs | **105** | **105** | **0** | | | **95** |
| 11 | Gross Profit | **-5** | **-105** | **100** | | | **5** |
| 12 | Selling Expenses | 10 | 0 | 10 | | | 10 |
| 13 | Gen & Admin Expenses (*G&A*) | 5 | 5 | 0 | | | 0 |
| 14 | Total SG&A Exp | **15** | **5** | **10** | | | **10** |
| 15 | Net Income | **-20** | **-110** | **90** | | | **-5** |
| 16 | | | | | | | |
| 17 | **Derivation of Total Loss Formula Assuming Boomerangs Are Destroyed and There Are Lost Profits** | | | | | | |
| 18 | | | | | | | |
| 19 | [1] *Var Mfg Cost of Lost Inv = VM* (F8) | 95 | | | | | |
| 20 | [2] *Lost Profits on Lost Inv ≡ LP = S–VM–Sell Exp* (G15) | -5 | | | | | |
| 21 | **[3] *Total Damage = Sales – Selling Exp*** | **90** | | | | | |
| 22 | | | | | | | |
| 23 | **Testing the Total Damage Formula** | | | | | | |
| 24 | | | | | | | |
| 25 | Lost Sales (D6) | 100 | | | | | |
| 26 | – Selling Exp Saved (D12) | 10 | | | | | |
| 27 | = Total Damage | **90** | | | | | |
| 28 | | | | | | | |
| 29 | **Alternate Damage Formula Based on Net Income** | | | | | | |
| 30 | | | | | | | |
| 31 | [4] *NI = S – VM – FMOH – Sell Exp. – G&A* Rearranging terms, we get: | | | | | | |
| 32 | [5] *NI* = (*S – Sell Exp.*) *– VM – FMOH – G&A* Note that the terms in parens = Damages | | | | | | |
| 33 | [6] *NI = Damages – VM – FMOH – G&A* Rearranging, we get: | | | | | | |
| 34 | **[7] *Damages = NI + VM + FMOH + G&A*** | | | | | | |
| 35 | | | | | | | |
| 36 | Net Income (B15) | -20 | | | | | |
| 37 | Variable Mfg Costs (*VM*) (B8) | 95 | | | | | |
| 38 | Fixed Mfg. OH (*FMOH*) (B9) | 10 | | | | | |
| 39 | G&A Exp (B13) | 5 | | | | | |
| 40 | Total Damages | **90** | | | | | |
| 41 | | | | | | | |
| 42 | **Formula for Lost Profits** | | | | | | |
| 43 | | | | | | | |
| 44 | [8] *Lost Profits = S – VM – Sell Exp.* | | | | | | |
| 45 | [9] Net Inc (*NI*) = *S – VM – FMOH – Sell Exp. – G&A* Subtracting equation [9] from [8], we get: | | | | | | |
| 46 | [10] *Lost Profits – NI = FMOH + G&A* Adding *NI* to both sides, we get: | | | | | | |
| 47 | **[11] *Lost Profits = NI + FMOH + G&A*** | | | | | | |
| 48 | | | | | | | |
| 49 | Net Income (B15) | -20 | | | | | |
| 50 | Fixed Overhead (B9) | 10 | | | | | |
| 51 | G&A Exp (B13) | 5 | | | | | |
| 52 | **Lost Profits** | **-5** | | | | | |
| 53 | | | | | | | |
| 54 | **** Variable Mfg. Costs (*VM*) = Direct Materials + Dir Labor + Var. Mfg. Overhead** | | | | | | |

| | A | B | C | D | E | F | G |
|---|---|---|---|---|---|---|---|
| 1 | **Table 12.1A** | | | | | | |
| 2 | **Sample Damage Calculations with *VM* = $65** | | | | | | |
| 3 | | | | | | | |
| 4 | | | | **Diff =** | | **Lost Boom-** | **Lost** |
| 5 | | **Without Fire** | **With Fire** | **Damage** | | **erangs** | **Profits** |
| 6 | Sales = *S* | 100 | 0 | 100 | | | **100** |
| 7 | | | | | | | |
| 8 | Variable Mfg. Costs (*VM*)** | 65 | 65 | 0 | | **65** | 65 |
| 9 | Fixed Mfg. Overhead (*FMOH*) | 10 | 10 | 0 | | | 0 |
| 10 | Total Mfg. Costs | **75** | **75** | **0** | | | **65** |
| 11 | Gross Profit | **25** | **-75** | **100** | | | **35** |
| 12 | Selling Expenses | 10 | 0 | 10 | | | 10 |
| 13 | Gen & Admin Expenses (*G&A*) | 5 | 5 | 0 | | | 0 |
| 14 | Total SG&A Exp. | 15 | 5 | 10 | | | **10** |
| 15 | Net Income | **10** | **-80** | **90** | | | **25** |
| 16 | | | | | | | |
| 17 | **Derivation of Total Loss Formula Assuming Boomerangs Are Destroyed and There Are Lost Profits** | | | | | | |
| 18 | | | | | | | |
| 19 | [1] *Var Mfg Cost of Lost Inv. = VM* (F8) | 65 | | | | | |
| 20 | [2] *Lost Profits on Lost Inv.≡ LP = S–VM–Sell Exp.* (G15) | 25 | | | | | |
| 21 | [3] ***Total Damage* = *Sales* – *Selling Exp.*** | **90** | | | | | |
| 22 | | | | | | | |
| 23 | **Testing the Total Damage Formula** | | | | | | |
| 24 | | | | | | | |
| 25 | Lost Sales (D6) | 100 | | | | | |
| 26 | – Selling Exp. Saved (D12) | 10 | | | | | |
| 27 | = Total Damage | **90** | | | | | |
| 28 | | | | | | | |
| 29 | **Alternate Damage Formula Based on Net Income** | | | | | | |
| 30 | | | | | | | |
| 31 | [4] *NI = S – VM – FMOH – Sell Exp. – G&A* | Rearranging terms, we get: | | | | | |
| 32 | [5] *NI* = (*S – Sell Exp.*) – *VM – FMOH – G&A* | Note that the terms in parens = Damages | | | | | |
| 33 | [6] *NI = Damages – VM – FMOH – G&A* | Rearranging, we get: | | | | | |
| 34 | [7] ***Damages* = *NI* + *VM* + *FMOH* + *G&A*** | | | | | | |
| 35 | | | | | | | |
| 36 | Net Income (B15) | 10 | | | | | |
| 37 | Variable Mfg Costs (*VM*) (B8) | 65 | | | | | |
| 38 | Fixed Mfg. OH (*FMOH*) (B9) | 10 | | | | | |
| 39 | G&A Exp (B13) | 5 | | | | | |
| 40 | Total Damages | **90** | | | | | |
| 41 | | | | | | | |
| 42 | **Formula for Lost Profits** | | | | | | |
| 43 | | | | | | | |
| 44 | [8] *Lost Profits = S – VM – Sell Exp* | | | | | | |
| 45 | [9] Net Inc. (*NI*) = *S – VM – FMOH – Sell Exp – G&A* | Subtracting equation [9] from [8], we get: | | | | | |
| 46 | [10] *Lost Profits – NI = FMOH + G&A* | Adding *NI* to both sides, we get: | | | | | |
| 47 | [11] ***Lost Profits* = *NI* + *FMOH* + *G&A*** | | | | | | |
| 48 | | | | | | | |
| 49 | Net Income (B15) | 10 | | | | | |
| 50 | Fixed Overhead (B9) | 10 | | | | | |
| 51 | G&A Exp (B13) | 5 | | | | | |
| 52 | **Lost Profits** | **25** | | | | | |
| 53 | | | | | | | |
| 54 | **** Variable Mfg. Costs (*VM*) = Direct Materials + Dir Labor + Var. Mfg. Overhead** | | | | | | |

The organization of the table is as follows. Columns B through D are the calculations of total damages. Column B shows a simple income statement without the fire, column C shows the same with the fire, and column D is the difference, which is the total damage.

Columns F and G are a decomposition of the total damages into lost inventory (boomerangs) and lost profits.

## Column B: Net Income without the Fire

Let's begin with column B, which is a simple, hypothetical income statement for BB if there were no fire. Sales (denoted as *S* in A6) are $100 (B6).[1] Variable manufacturing

[1]Obviously, in the real world the numbers would be larger. We use small numbers here for simplicity. The reader could think of these numbers as being in millions of dollars.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Table 12.1B** | | | |
| 2 | **Lost Profits Formulas and Calculations Based on *EBITDA*** | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | **Without Fire** | **With Fire** | **Diff = Damage** |
| 6 | Sales = *S* | **100** | **0** | **100** |
| 7 | | | | |
| 8 | Variable Mfg. Costs (*VM*) ** | 65 | 0 | 65 |
| 9 | Fixed Mfg. Overhead (*FMOH*) | 10 | 10 | 0 |
| 10 | Total Mfg. Costs = Cost of Sales | **75** | **10** | **65** |
| 11 | Gross Profit | **25** | **-10** | **35** |
| 12 | Selling Expenses | 10 | 0 | 10 |
| 13 | Gen & Admin. Expenses (*G&A*) | 5 | 5 | 0 |
| 14 | Total SG&A Exp. | **15** | **5** | **10** |
| 15 | Net Income | **10** | **-15** | **25** |
| 16 | | | | |
| 17 | **Damage (Lost Profits) Formula** | | | |
| 18 | Lost Sales = *S* (D6) | 100 | | |
| 19 | – Var Mfg. Costs Saved = *VM* (–D8) | -65 | | |
| 20 | – Selling Expenses Saved = *Sell Exp.* (–D12) | -10 | | |
| 21 | ***Total Damage = S – VM – Selling Exp.*** | **25** | | |
| 22 | | | | |
| 23 | **Formula for Lost Profits** | | | |
| 24 | | | | |
| 25 | [12] *Lost Profits = S – VM – Sell Exp.* | | | |
| 26 | [13] *EBITDA = S – VM – FMOH – Sell Exp. – G&A + Int + D* | Subtracting equation [13] from [12], we get: *** | | |
| 27 | [14] *LP – EBITDA = FMOH + G&A – Int – D* | Adding *EBITDA* to both sides, we get: | | |
| 28 | **[15] *LP = EBITDA + FMOH + G&A – Int – D*** | | | |
| 29 | | | | |
| 30 | **Calculation of Lost Profits per Day from Lost Sales** | | | |
| 31 | | | | |
| 32 | EBITDA (Per BB Annual Report) | (40,000,000) | | |
| 33 | FMOH—Local Plant | 7,000,000 | | |
| 34 | FMOH—Rest of U.S. | 3,000,000 | | |
| 35 | G&A Exp. (BB Inc Stmt) | 42,000,000 | | |
| 36 | Interest = I (BB I/S) | (17,000,000) | | |
| 37 | Depreciation & Amortization = D (BB I/S) | (15,000,000) | | |
| 38 | **Lost Profits—2006** | **(20,000,000)** | | |
| 39 | Divide by # Working Days/Year (5 Days/Wk × 52 Wks) | 260 | | |
| 40 | **Lost Profits per Day** | **(76,923)** | | |
| 41 | | | | |
| 42 | **** Variable Mfg. Costs (VM) = Direct Materials + Dir Labor + Var. Mfg. Overhead** | | | |
| 43 | ***** *FM* = Fixed Mfg. Costs; *D* = Depreciation & Amortization** | | | |

costs (VM), which are composed of direct materials, direct labor, and variable manufacturing overhead,[2] are \$95 (B8).[3] Fixed manufacturing overhead (FMOH) is \$10 (B9). Thus, total manufacturing overhead is \$105 (B10), with a gross profit of −\$5 (B11). Selling expenses are \$10 (B12), and general and administrative (G&A) expenses are \$5 (B13), for a total SG&A expense of \$15 (B14). BB's losses equal the negative \$5 gross profit minus SG&A expense of \$15, for a total of −\$20 (B15).

## Column C: Net Income with the Fire

Column C is net income with the fire. Sales are zero (C6) instead of \$100. For inventory that was produced and destroyed, variable and manufacturing costs and

[2]Variable manufacturing overhead (VMOH) are indirect costs such as lubricants for machinery. The lubricants cannot be traced to any particular units of output, but their use is variable and for production. In contrast, rent on production space is fixed manufacturing overhead (FMOH).

[3]We do not show the details, as it does not matter. Only the total matters.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Table 12.1C** | | | |
| 2 | **Lost Profits Formulas and Calculations** | | | |
| 3 | **When Company Paid Employees for the Day Off [1]** | | | |
| 4 | | | | |
| 5 | | **Without Fire** | **With Fire** | **Diff = Damage** |
| 6 | Sales = *S* | **100** | **0** | **100** |
| 7 | | | | |
| 8 | Variable Mfg. Costs (*VM*) ** | 65 | 35 | 30 |
| 9 | Fixed Mfg. Overhead (*FMOH*) | 10 | 10 | 0 |
| 10 | Total Mfg. Costs = Cost of Sales | **75** | **45** | **30** |
| 11 | Gross Profit | **25** | **-45** | **70** |
| 12 | Selling Expenses | 10 | 0 | 10 |
| 13 | Gen. & Admin. Expenses (*G&A*) | 5 | 5 | 0 |
| 14 | Total SG&A Exp. | **15** | **5** | **10** |
| 15 | Net Income | **10** | **-50** | **60** |
| 16 | | | | |
| 17 | **Damage (Lost Profits) Formula** | | | |
| 18 | Lost Sales = *S* (D6) | 100 | | |
| 19 | – Var Mfg. Costs Saved = *VM* (–D8) | -30 | | |
| 20 | – Selling Expenses Saved = *Sell Exp.* (–D12) | -10 | | |
| 21 | ***Total Damage = S – VM – Selling Exp.*** | **60** | | |
| 22 | | | | |
| 23 | [1] This table is identical to the first 21 rows of Table 12.1B, except C8 = $35 for Labor instead of $0. | | | |
| 24 | | | | |
| 25 | **** Variable Mfg. Costs (VM) = Direct Materials + Dir Labor + Var. Mfg. Overhead** | | | |
| 26 | ***** *FM* = Fixed Mfg. Costs; *D* = Depreciation & Amortization** | | | |

fixed manufacturing overhead are still $95 and $10 (C8 and C9). Thus, gross profit is −$105 (C11). Selling expenses are zero (C12), as the company does not incur its selling costs on the destroyed boomerangs,[4] but G&A expense of $5 (B13) remains the same. Thus, there is a $110 net loss (C15).

## Column D: The Difference Equals the Damages

Column D equals column B minus column C, and it equals the damages for lost inventory plus lost profits. BB loses $100 (D6 = B6 − C6) of sales. However, it still incurred the variable manufacturing costs and fixed overhead as before; that is, there is no damage arising from those items (D8, D9, and D10 equal zero).

BB saves $10 (D12) of selling expenses, but there is no saving (D13) on G&A expense. Thus, the total damage is $90 (D15), which is $100 of lost sales minus $10 of selling expenses saved. This also equals −$20 − (−$110) = $90 (B15 − C15 = D15).

As mentioned earlier, column D is the total damages, while columns F and G allocate the total damage between the loss of inventory and lost profits.

## Column F: Damages from the Lost Boomerangs

The value of the lost boomerang inventory is the amount of their variable manufacturing costs (direct materials and direct labor—the sum of which is called *direct costs*—and variable overhead), which totals $95 (F8, which equals B8 and C8).

[4] Later we address different possible outcomes and assumptions.

## Column G: Lost Profits

Our calculation of lost profits from lost sales begins with lost sales of \$100 (G6, from D6). If BB had sold the boomerangs, it would have incurred the \$95 (G8, which comes from B8 and C8) marginal cost of the variable manufacturing costs. Note that G8 and F8 effectively cancel each other out, as the \$95 in F8 is a positive inventory damage, while the \$95 in G8 reduces lost profits and is a negative damage. Together they add to zero, although the accounting format slightly obscures that fact in showing G8 as a positive cost.

Fixed manufacturing overhead is not a component of damages, as it would have occurred with or without the fire. Thus, G9 equals zero, and total incremental costs of the lost sales are \$95 (G8 + G9 = G10). Total incremental lost gross profit is \$5 (G6 − G10 = G11). Note that the operational word is *incremental*, as accounting gross profit is −\$5 (B11). Again, the difference between the lost profits calculation of gross profits lost and the accounting gross profits is the \$10 of fixed costs, which are a legitimate expense in the original income statement in column B, but are not damages in columns D and G.

Selling expenses saved are \$10 (G12). G&A expense, just like fixed manufacturing overhead, would have occurred with or without the fire and thus also are not damages and therefore equal zero (G13).[5] Total SG&A damages are \$10 (G12 + G13 = G14), which leads to a lost profits calculation of −\$5 (G15).

The sum of the lost boomerang inventory and lost profits calculations is total damages and is \$95 − \$5 = \$90 (F8 + G15 = D15).

**TOTAL DAMAGE FORMULA ON DESTROYED INVENTORY** Now we can derive general formulas for both categories of damages. Rows 19–21 contain the first three equations,[6] which are as follows:

$$\textit{Inventory Damage} = \textit{Variable Mfg Costs} = \textit{VM}. \tag{12.1}$$

Equation (12.1) merely states algebraically that which we have already discussed, which is that the inventory damage is the variable manufacturing costs. It is an easy mistake for plaintiff's damages expert to include fixed manufacturing overhead in his or her damages calculations, but it would be wrong.

Equation (12.2) states that lost profits equal sales minus variable manufacturing costs minus selling expenses.

$$\textit{Lost Profits} \equiv \textit{LP} = S - \textit{VM} - \textit{Sell Exp}. \tag{12.2}$$

Adding equations (12.1) and (12.2) gives us our formula for total damages on the destroyed inventory:

$$\textit{Total Damage} = \mathrm{S} - \textit{Sell Exp} = \textit{Lost Inventory} + \textit{Lost Profits on Inventory}. \tag{12.3}$$

Note that the $VM$ in equation (12.1) and the $-VM$ in equation (12.2) cancel each other out. This occurs for the reasoning mentioned earlier that the variable

[5] At the end of this article, we relax that assumption.

[6] The equation numbers in this chapter match those in the spreadsheets, except that the latter do not contain the prefix of the chapter number.

manufacturing costs are not part of total damages, as they would have occurred for inventory produced and destroyed with or without the fire. They are part of the calculation for inventory that is never produced for a lost sale, and we cover that later in the chapter.

B19 through B21 repeat the calculations above, which are equations (12.1) and (12.2), the separate components of the damage. Thus, total damages = *VM* of \$95 (B19) + lost profits of −\$5 (B20) = \$90 (B21).

Next we test the total damage formula in equation (12.3). Lost sales equal \$100 (B25, from D6) and selling expenses saved equal \$10 (B26, from D12), for a total damage of \$90 (D27). Note that this equals B21 and confirms equation (12.3).

Another important observation is that if the company has been running at a loss and the opposing expert presents only the lost inventory as the damages, this will overvalue the damages, as the loss on the sale that would have taken place without the fire will bring down the damage calculation.

**SCANTY INFORMATION** One of the big challenges in litigation is that financial statement information is often scanty. It may be provided in response to a demand before the damages expert is retained on the job, and there may be no ability to request additional financial information. Thus it is necessary to get creative and develop several different damage formulas. If the data do not exist to calculate damages according to one formula, perhaps different data exist that will enable us to calculate damages according to a different formula.

**DAMAGE FORMULAS BASED ON NET INCOME** In the next section, we develop a damage formula based on net income. Equation (12.4) is based on Table 12.1, B6 through B15. Equations (12.4) through (12.7) appear in rows 31 through 34 of Table 1. Equation (12.4) states that net income equals sales minus the sum of variable manufacturing costs, fixed manufacturing overhead, selling expenses, and G&A expense.

$$NI = S - VM - FMOH - Sell\ Exp - G\&A. \tag{12.4}$$

Rearranging equation (12.4), we get:

$$NI = (S - Sell\ Exp) - VM - FMOH - G\&A. \tag{12.5}$$

Note that the term in parentheses equals total damages,[7] per equation (12.3). We have already explained why the other three terms are not part of damages. Substituting equation (12.3) into equation (12.5), we get:

$$NI = Damages - VM - FMOH - G\&A. \tag{12.6}$$

Rearranging the terms in equation (12.6), we get:

$$Damages = NI + VM + FMOH + G\&A \tag{12.7}$$

Formula for damages based on NI.

Let's try to understand the intuition behind equation (12.7). It makes sense that net income is part of the damage calculations, and that is our starting point. However, we subtracted the three right-hand terms from sales in order to compute

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[7]In general, we use *damages* synonymously with *total damages*.

net income, yet they are not part of damages, so we must add them back to calculate damages.[8] Variable manufacturing costs (VM) from the destroyed inventory are the same with and without the fire and are thus not part of damages, as we see in D8 and the sum of F8 and G8. FMOH and G&A are also unaffected by the fire and thus not a component of damages, as we see in rows 9 and 13.

Let's elaborate on fixed manufacturing overhead (FMOH). We subtracted FMOH from sales in calculating net income, but it is not an incremental cost. Generally accepted accounting principles (GAAP) require using absorption costing for financial statements, which means subtracting fixed manufacturing overhead from net income. This is fine for presenting an income statement. However, absorption costing distorts the incremental profits analysis, which is what is relevant in damage calculations, and we have to convert to variable costing by adding back FMOH.

B36 through B40 show the components of the damage calculation per equation (12.7), and again it adds to \$90 (B40 = D15 = B21 = B27), as it should.

In the next series of equations, we will develop a formula for lost profits on the destroyed inventory based on net income. Note that these equations appear in rows 44 through 47.

We repeat equation (12.2) as equation (12.8), using LP for lost profits:

$$LP = S - VM - Sell\ Exp. \tag{12.8}$$

Next, we repeat equation (12.4) as equation (12.9):

$$NI = S - VM - FMOH - Sell\ Exp - G\&A. \tag{12.9}$$

Subtracting equation (12.9) from equation (12.8), we get:

$$LP - NI = FMOH + G\&A. \tag{12.10}$$

Rearranging the terms, we get:

$$LP = NI + FMOH + G\&A \tag{12.11}$$

Formula for lost profits based on net income.

The intuition behind this equation is that we start measuring lost profits with net income. However, *FMOH* and *G&A*, which are legitimate deductions in computing *NI*, are not part of lost profits, since they remain the same before and after the fire.[9] Therefore we must add them back in calculating lost profits. B49 through B52 show the components of the damage calculations, and total lost profits equals −\$5 (B52), which equals our calculations in G15. Thus, this calculation confirms equation (12.11).

[8]We also subtract selling expenses from sales to compute net income, but because we did not have to pay them, given the fire, they are part of damages, and therefore we do not have to adjust net income as a proxy for damages for the selling expense component.

[9]It is possible that G&A could rise because of the fire, in which case it would become a part of the damages and lost profits calculations. This could occur because of the additional administrative time required to deal with insurance and litigation. The point of this chapter is not to develop a rigid set of formulas to apply robotically in all situations. Rather it is to understand the logic and being able to adapt these equations for any curveballs thrown at us by life.

### Table 12.1A: Sample Damage Calculations with $VM = \$65$

Table 12.1A is identical to Table 12.1, except that B8 is \$65 of variable manufacturing costs instead of \$95 in Table 12.1. In this version, the company is profitable. All other calculations flow through with the same logic. Note that total damages are still \$90 (D15, B21, B27, and B40), even though gross profit and net income are positive. This demonstrates the accuracy of equations (12.3) and (12.7). Lost profits increased by the \$30 increase in VM from −\$5 in Table 12.1, G15 to \$25 in Table 12.1A, G15. This equals B52, which demonstrates the accuracy of equation (12.11).

## Table 12.1B: Lost Profits Formulas Based on EBITDA for Lost Sales on Inventory Never Produced

It happened that Billabong's Boomerangs was a wholly owned subsidiary of Sticks and Stones, Inc., a publicly traded firm. Most of the detailed financial information that we were given was for BB's local plant, and we had only summary financial statements for BB as a whole company. These summary statements were missing key data that we needed for our damage calculations. Fortunately, we were able to find a few critical pieces of information about BB in the Sticks and Stones annual report, which is publicly available information. The most important piece of information is that BB's EBITDA was −\$40 million.

Table 12.1B repeats the data from Table 12.1A for the first 21 rows for columns B through D. However, there is a conceptual difference between Tables 12.1A and 12.1B. In the former we are assuming the lost sales and the related lost profits are from the actual boomerangs that were burned, while the latter has a different twist to it. BB claimed that the fire caused an electrical outage, which caused the company's nationwide computer system to go down. Therefore, its salespeople at its national sales center did not have access to the system, and the company could not sell product for that day. Therefore, it claimed lost profits from lost sales for the entire company for one day. The accounting profits analysis of this type of lost sale is different from the lost profits on the destroyed inventory, as this inventory was never produced, and therefore the variable manufacturing costs were never incurred.

Thus, B6 through B15 and C6 are identical in Tables 1B and 1A. However, variable manufacturing costs are zero (C8) for the lost sales for the one day, which is different from the \$65 in Table 12.1A. This change flows through C10 through C15, and net income is −\$15 (C15), which is \$65 higher than the net income of −\$80 in Table 12.1A, C15. The \$65 difference also flows through D8 through D15, as the difference in net income (i.e., the damage) equals \$25 (D15), compared to \$90 (Table 12.1A, D15).

We calculate the damage formula in rows 18 through 21, and the related equation appears here:

$$\textit{Lost Profits} = S - VM - \textit{Sell Exp}. \tag{12.12}$$

Comparing this to equation (12.3), the difference is that in this damage formula, lost profits are reduced by the variable manufacturing costs, which is not true of equation (12.3). The reason for this is that for these lost sales, the inventory was never manufactured, as the plaintiff claimed that sales were made to order, and the inability to take orders for one day meant that the company never produced the boomerangs. Therefore, the lost profits are lower here at \$25 (B21) versus by \$90 (Table 12.1A, B21) by *VM* of \$65, since the company never incurred the manufacturing costs—unlike the lost profits on the destroyed inventory.

It is true that equation (12.12) is the same as equation (12.2). However, there is a significant conceptual difference. In equation (12.12) the Lost Profits is equal to the Total Damages, which is not true in the other case. In the case of inventory that was destroyed, Total Damages equal the variable cost of producing the lost inventory (i.e., the inventory damage) plus the Lost Profits. Thus, equation (12.3) = equation (12.1) + equation (12.2).

Thus, lost profits equal the \$100 of lost sales (B18, from D6), minus the sum of \$65 (B19, from D8) of variable manufacturing costs saved by not making the inventory and \$10 (B20, from D12) selling expenses saved, for a total damage of \$25 (B21).

Next, we produce an equation for EBITDA (earnings before interest, taxes, depreciation, and amortization) in equation (12.13):

$$EBITDA = S - VM - FMOH - Sell\ Exp - G\&A + Int + D. \tag{12.13}$$

Subtracting equation (12.13) from equation (12.12), we get:

$$LP - EBITDA = FMOH + G\&A - Int - D. \tag{12.14}$$

Adding EBITDA to both sides, we get:

$$LP = EBITDA + FMOH + G\&A - Int - D \tag{12.15}$$

EBITDA lost profits equation.

We base the calculations of lost profits in B32 through B40 on equation (12.15). We were able to piece together the necessary information required in equation (12.15) from Billabong's Boomerangs local manufacturing plant's income statements, the company's summary financial statement for the entire firm, and the data that were available as part of Sticks and Stones' annual statements. The lost profits for the year 2006, when the fire occurred, were negative and totaled −\$20 million (B38), which represented a loss of \$76,923 (B40) per working day.

Thus, the result is that we show that the shutting down of this unprofitable business for one day actually saved the plaintiff \$76,923 per day. Instead of Lost Profits, it was actually a Lost Loss, which is a gain for the plaintiff. Therefore the plaintiff is not entitled to damages for lost profits in this example.

## When Reality May Vary with Our Assumptions

Let's assume that, contrary to our initial assumption, BB paid its employees for the day of the fire, even though they did not do productive work that day. Table 12.1C is identical to the first 21 rows of Table 12.1B, with the only exception being that we insert \$35 in C8 for the amount of direct labor for the day instead of zero. We could make similar adjustments for selling expenses, and for excess G&A necessitated by the fire (e.g., cleanup of the site).

Reality may be even more complicated. With definitive information lacking, it may be that there was some partial benefit from the "unproductive labor." However, the important point in this article is to calculate the damage formulas and illustrate the principles by which these formulas are correct. The practitioner may be called on to make slight modifications to fit any deviation of the fact situation from the assumptions in the formulas.

## Modification of Formulas for Wholesale and Retail Businesses

Obviously, wholesale and retail businesses do not have manufacturing costs. The only categories of expenses are cost of sales, fixed overhead (nonmanufacturing, e.g., store rent), selling expenses, and G&A expenses. In the damage equations, cost of sales would take the place of variable manufacturing expense. The principles behind the formulas are the same as those in manufacturing. The actual formulas are simpler, as the accounting itself is simpler.

## Legal Treatment[10]

The legal principle is that lost profits damages are usually defined as lost net profits, which means that all costs must be deducted. For breach of contract, this means the contract price, less cost of performance—sometimes called cost of *completion*—less expenses saved as a result of plaintiff's being excused from performance by the other party's breach.[11]

Dunn (1998) states that obviously direct costs are deductible in determining lost profits. The difficult items are variable and fixed overhead.[12] The vast majority of Dunn's discussion centers around fixed overhead:

> *The question whether overhead must be deducted to reach net profits is frequently decided by default. Nobody thinks about it. The courts state only that the injured party is entitled to recover its lost profits, measured by the contract price less the cost of performance. Implicitly, cost of performance includes only direct costs attributable to the contract. The court does not discuss or focus on the question whether indirect or overhead costs that may be properly allocated to the contract are to be added to the cost of performance to reduce the lost profits recoverable. The weight of authority, however, holds that fixed overhead expenses need not be deducted from gross income to arrive at the net profit properly recoverable. Most cases that have considered the argument that fixed overhead must be allocated and deducted have rejected it.*[13]

However, there is one case that Dunn cites, *Sterling Freight Lines, Inc. v. Prairie Material Sales, Inc.*,[14] in which the court correctly stated that fixed overhead is not deductible in calculating profits, but variable overhead is deductible.

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[10]All cites are from Robert L. Dunn, *Recovery of Damages for Lost Profits, 5th ed.*,© 1998, Lawpress Corporation, Westport, CT.
[11]*Id.*, pp. 430–431.
[12]This is a summary of *Id.,* pp. 443–462.
[13]Pages 443–444. On pp. 444–445, the author cites many such cases.
[14]285 Ill App. 3d 914, 674 N.E. 2d 948 (1996), *appeal denied*, 173 Ill 2d 547, 684 N.E. 2d 1342 (1997).

On the other hand, in *Vitex Manufacturing Corp. v. Caribtex Corp.*,[15] the court started getting it right in rejecting the defendants' claim for deducting the plaintiff's fixed overhead. But then, the court fell into a logical trap of falling for full absorption accounting instead of sticking to incremental analysis:

> *. . . By the very nature of this allocation process, as the number of transaction [*sic*] over which overhead can be spread becomes smaller, each transaction must bear a greater portion or allocate [*sic*] share of the fixed overhead cost. Suppose a company has fixed overhead of $10,000 and engages in five similar transactions; then the receipts of each transaction would bear $2,000 of overhead expense. If the company is now forced to spread this $10,000 over only four transactions, then the overhead expense per transaction will rise to $2,500, significantly reducing the profitability of the four remaining transactions. Thus, where the contract is between businessmen familiar with commercial practices, as here, the breaching party should reasonably foresee that his breach will not only cause a loss of "clear" profit, but also a loss in that the profitability of other transactions will be reduced. [Citations omitted.] Therefore, this loss is within the contemplation of "losses caused and gains prevented," and overhead should be considered to be a compensable item of damage.*

Seduced by the dark side of the force! Such thinking leads to double-counting. I did not read the original case, so let's assume lost sales minus variable costs and fixed overhead were $20,000. That should be the only damage. If the court agrees to another $10,000 for the fixed overhead that gets allocated over four instead of five sales, it overly rewards the plaintiff by double-counting the $10,000. Since the overhead was fixed, the Court should leave it alone and account for the damages solely by looking at the lost income.

## Summary

This chapter presents a series of formulas for calculating damages for the cost of destroyed inventory and for lost profits. For the latter, there are separate formulas when the lost sale relates to inventory that was produced versus inventory that was not produced. This should provide a comprehensive, definitive framework for damage calculations. We also discussed how to modify these formulas for nonmanufacturing companies and for deviations in the fact patterns.

We also reviewed the court's treatment of variable and fixed overhead. In *Sterling* the court understood the correct treatment of variable overhead, while in *Vitex* it treated fixed overhead incorrectly. Hopefully this chapter is a contribution to the literature that will bring greater clarity and uniformity in litigation.

## Reference

Dunn, Robert L. 1998. *Recovery of Damages for Lost Profits*, 5th ed. Westport, CT: Lawpress Corporation.

[15] 377 F. 2d 795 (3d Cir. 1967). *See* quote in Dunn on pp. 446–447.

# PART VI

# Valuing ESOPs and Buyouts of Partners and Shareholders

## Introduction

Part VI consists of Chapters 13 through 15. Chapter 13 identifies and measures the post-transaction dilution in value that occurs in leveraged ESOPs. Chapter 14 compares the value to the owner of selling his or her shares to a C corporation ESOP versus selling as an S corporation to an outside party and provides formulas to calculate the breakeven percentage, $p*$, at which the owner is indifferent. If the owner sells a greater percentage of the firm than $p*$, then he or she is better off with the C corporation ESOP, and if he or she sells less than $p*$, it is best to sell to an outsider as an S corporation. Together these two chapters are clearly about valuing ESOPs.

Chapter 15, "Buyout of Partners and Shareholders," has nothing in itself to do with ESOP valuation. Nevertheless, we include it in Part VI, because it shares the problem of post-transaction dilution when the Company rather than individuals buys back stock or partnership interests, and the solution is similar, although the mechanics are different.

This brief description is sufficient to introduce Chapters 14 and 15. However, Chapter 13 requires a longer introduction, which follows immediately.

## Chapter 13: ESOPs—Measuring and Apportioning Dilution

ESOP valuation has generated a number of lawsuits. One of the sore points of ESOP valuation that has led to litigation is the dilution in value that the ESOP experiences after the sale. Selling stock to an ESOP that does not have the cash to pay for the stock always causes a dilution in value to the shareholders the instant the transaction takes place. Of course, it takes time for the bad news to become known, as usually the next valuation takes place one year later. Employees may be angry, feeling that they (through the ESOP) paid too much for the owner's stock. They may feel someone has pulled a fast one. This can endanger the life and health of the business.

In Chapter 13, we develop formulas to calculate the post-transaction fair market value (FMV) *before* doing the transaction. This enables the appraiser to provide accurate information to the ESOP trustee that will enable both sides to enter the transaction with both eyes open. It also demystifies the dilution in value and provides an accurate benchmark with which to measure future performance. The chapter also provides precise formulas with which the appraiser can perform the financial engineering necessary to enable the owner to reduce his or her transaction price in order to share some or all of the ESOP's dilution. While this is not common, sometimes there are benevolent owners who are sufficiently well off and concerned about their employees to do that.

# CHAPTER 13

# ESOPs

## *Measuring and Apportioning Dilution*

# CHAPTER 13

# ESOPs

## *Measuring and Apportioning Dilution*

This chapter is the result of further thought and research on my treatment of valuing ESOPs (Abrams, 1993, 1997). It not only simplifies those articles, but it goes far beyond them. Reading them is not necessary for understanding this chapter.

## Introduction

Leveraged ESOPs have bewildered and bedeviled many firms, due to a lack of understanding of the phenomenon of dilution and the ability to quantify it. Many ESOPs have soured because employees paid appraised fair market value of the stock being sold to the ESOP, only to watch the fair market value significantly decline at the next valuation because the ESOP loan was not included in the pre-transaction fair market value. As a result, employees may feel cheated and that "someone has pulled a fast one." When this happens, lawsuits sometimes follow, further lowering the value of the firm and the ESOP.

There are several types of problems relating to the dilution phenomenon:

1. The technical problem of defining and measuring the dilution in value to the ESOP *before* it happens
2. The business problem of getting the ESOP Trustee, participants, and selling owner(s) to agree on how to share the dilution
3. The technical problem of how to engineer the price to accomplish the desired goals in problem 2
4. The problem of how to communicate each of the foregoing to all of the participants so that all parties can enter the transaction with both eyes open and come away feeling that the transaction was win-win instead of win-lose

The focus of this chapter is to provide the analytical solutions to problems 1 and 3 that are necessary for resolving the business and communication problems of 2 and 4. The result is that the appraiser can include the dilution in her initial valuation

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report so that employees will not be negatively surprised when the value drops at the next annual valuation. Additionally, the appraiser can provide the technical expertise to enable the parties to share the dilution, solving problem 3. Both parties will then be fully informed beforehand, facilitating a win-win transaction.

### What You Can Skip

This chapter contains much tedious algebra. You can safely skip the section on the iterative approach, as it enhances the understanding of dilution but contains no additional formulas of practical significance.

## Definitions of *Dilution*

Two potential parties can experience dilution in stock values in ESOP transactions: the ESOP and the owner. The dilution that each experiences differs and can be easily confused.

Additionally, each party can experience two types of dilution: absolute and relative dilution. We define *absolute* dilution in the section immediately following. In this chapter we will discuss only absolute dilution. Relative dilution is more complicated because we can calculate dilution relative to more than one base. Several formulas can be developed to calculate relative dilution, but they are beyond the scope of this book. Thus, for the remainder of this chapter, *dilution* will mean absolute dilution.

### Dilution to the ESOP (Type 1 Dilution)

We define type 1 dilution as the payment to the selling owner less the post-transaction fair market value of the ESOP. This can be stated either in dollars or as a percentage of the pre-transaction value of the firm. By law, the ESOP may not pay more than fair market value to the company or to a large shareholder, though it is nowhere defined in the applicable statute whether this is pre- or post-transaction value. Case law and Department of Labor proposed regulations[1] indicate that the pre-transaction value should be used.[2]

### Dilution to the Selling Owner (Type 2 Dilution)

We define type 2 dilution as the difference in the pre-transaction fair market value of the shares sold and the price paid to the seller. Again, this can be in dollars or as a percentage of the firm's pre-transaction value. Since it is standard industry practice for the ESOP to pay the owner the pre-transaction price, type 2 dilution is virtually unknown. Those sellers who wish to reduce or eliminate dilution to the ESOP can choose to sell for less than the pre-transaction fair market value.

When the ESOP bears all of the dilution, there is only type 1 dilution. When the owner removes all dilution from the ESOP by absorbing it himself, then the selling

[1]Jared Kaplan, a lawyer with McDermott Will & Emery LLP, doubts that the regulation will ever be finalized. According to Kaplan, practitioners adhere to the regulation as if it were final, making formal finalization by the DOL superfluous.

[2]*Donovan v. Cunningham*, 716 F.2d 1467. 29 CFR 2510.3-18(b).

price and post-transaction values are equal, and there is only type 2 dilution. If the owner absorbs only part of the dilution from the ESOP, then the dilution is shared, and there is both type 1 and type 2 dilution.

As we will show in Table 13.3B and in the Mathematical Appendix, when the seller takes on a specific level of type 2 dilution, the decrease in type 1 dilution is greater than the corresponding increase in type 2 dilution.

The seller also should consider the effects of dilution on his remaining stock in the firm, but that is beyond the scope of this book.

## Defining Terms

We first define some of the terms appearing in the various equations.

Let:

$p$ = *p*ercentage of firm sold to the ESOP, assumed at 30%.

$t$ = combined federal and state corporate income *t*ax rate, assumed at 40%.

$r$ = annual loan interest *r*ate, assumed at 10%.

$i$ = monthly loan *i*nterest rate = $r/12 = 0.8333\%$ monthly.

$V_{1B}$ = pre-transaction *v*alue of 100% of the stock of the firm after discounts and premiums at the firm level but before those at the ESOP level,[3] assumed at $1,000,000, as shown in Table 13.2. The $B$ subscript means *b*efore considering the lifetime cost of initiating and maintaining the ESOP (see $E$, $e$, and $V_{jA}$ below). $V_{1B}$ does not consider the cost of the loan. This differs from $V_{jB}$, as described below.

$V_{1A}$ = same as $V_{1B}$, except this is the pre-transaction value *a*fter deducting the lifetime cost of initiating and maintaining the ESOP (see $E$, $e$, and $V_{jA}$ below), but before considering the loan. Note this differs from $V_{jA}$, where $j > 1$, where we do subtract the cost of the ESOP loan as of iteration $j - 1$.

$V_{jB}$ = value of the firm at the $j$th iteration *b*efore deducting the lifetime ESOP costs (see $E$ below), but after subtracting the net present value of the ESOP loan (see NPLV) as calculated in iteration $j - 1$ (for $j > 1$).

$V_{jA}$ = value of the firm at the $j$th iteration *a*fter deducting the lifetime ESOP costs (see immediately below) and the ESOP loan as of the $(j - 1)$st iteration.

$V_n$ = the final post-transaction value of the firm, that is, at the $n$th iteration.

$E$ = lifetime costs of initiating and running the *E*SOP. These are generally legal fees, appraisal fees, ESOP administration fees, and internal administration costs. We assume initial costs of $20,000 and annual costs of $10,000 growing at 5% each year. Table 13.1 shows a sample calculation of the lifetime costs of the ESOP as $40,000.[4]

$e$ = lifetime *E*SOP costs as a percentage of the pre-transaction value = $E/V_{1B}$ = $40,000/$1 million = 4%.

[3]In Abrams (1993), the discounts and premiums at the firm level are a separate variable. This treatment is equally accurate and is simpler.

[4]How to calculate the pre-transaction value of the firm is outside the scope of this chapter.

$D_E$ = one minus net *D*iscounts (or plus net premiums) at the *E*SOP level. This factor converts the fair market value of the entire firm on an illiquid control level ($V_{1B}$) to a fair market value (on a 100% basis) at the ESOP's level of marketability and control ($D_E V_{1B}$). If we assume that the ESOP provides complete marketability (which normally one should not, but we are doing here for didactic purposes), then to calculate $D_E$ we must merely reverse out the control premium that was applied to the entire firm (in the calculation of $V_{1B}$), which we will assume was 43%, and reverse out the discount for lack of marketability that was applied, which we will assume was 29%.[5] The result is: $D_E = [1/(1 + 43\%)] \times [1/(1 - 29\%)] = 0.7 \times 1.4 = 0.98$. In other words, the net effect of reversing out the assumed discount and premium is a 2% net discount. It could also be a net premium if the minority discount is less or the premium for marketability is higher. Also, it would require other adjustments if we were to assume that the ESOP shares are not at a marketable minority level.[6]

$L_j$ = amount of the ESOP *L*oan in iteration *j*, which equals the payment to the owner. That equals the FMV of the firm in iteration *j* multiplied by $p\,D_E$, the percentage of the firm being sold to the ESOP, multiplied again by the factor for discounts or premiums at the ESOP level. Mathematically, $L_j = p\,D_E\,V_{jA}$. *Note:* This definition applies only in the iterative approach where we are eliminating type 1 dilution.

$NPVL_j$ = after-tax, *n*et *p*resent *v*alue of the ESOP *l*oan as calculated in iteration *j*. The formula is $NPVL_j = (1 - t)\,L_j$, as explained below.

$n$ = *n*umber of iterations.

$D_1$ = type 1 dilution (dilution to the ESOP).

$D_2$ = type 2 dilution (dilution to the seller).

FMV = fair market value.

## Table 13.1: Calculation of Lifetime ESOP Costs

We begin by calculating the lifetime cost of the ESOP, including the legal, appraisal, and administration costs, which are collectively referred to throughout this chapter as the administration costs or as the lifetime ESOP costs.

The estimated annual operating costs of the ESOP in Table 13.1 are \$10,000 pre-tax (B5), or \$6,000 after-tax (B6), assuming the 40% tax rate that we previously mentioned. We assume an annual required rate of return of 25% (B7). Let's further assume ESOP administration costs will increase by 5% a year (B8). We can then calculate the lifetime value of the annual cost by multiplying the first year's cost by a Gordon model multiple (GM) using an end-of-year assumption. The GM formula is $\frac{1}{r-g}$, or $\frac{1}{0.25-0.05} = 5.000$ (B9). Multiplying 5.000 by \$6,000, we obtain a value of \$30,000 (B10).

[5]These are arbitrary assumptions chosen for mathematical ease.

[6]Since writing the articles that eventually evolved into this chapter, I have changed my opinion of the proper magnitudes of the discounts for lack of marketability and control. While my opinion of the proper parameters is different, it has no impact on the analysis.

| | A | B |
|---|---|---|
| 1 | **Table 13.1** | |
| 2 | **Calculation of Lifetime ESOP Costs** | |
| 3 | | |
| 4 | | |
| 5 | **Pretax Annual ESOP Costs** | $10,000 |
| 6 | **After-Tax Annual ESOP Costs = (1 – *t*) × Pretax** | 6,000 |
| 7 | **Required Rate of Return = *r*** | 25% |
| 8 | **Perpetual Growth of ESOP Costs = *g*** | 5% |
| 9 | **Gordon Model Multiple (End Year) = 1 / (*r* – *g*)** | 5.000 |
| 10 | **Capitalized Annual Costs** | **30,000** |
| 11 | **Initial Outlay—Pretax** | 20,000 |
| 12 | **Initial Outlay—After-Tax = (1 – *t*) × Pretax** | 12,000 |
| 13 | **Lifetime ESOP Costs** | 42,000 |
| 14 | **Lifetime ESOP Costs—Rounded To (Used in Table 13.2, B9)** | **$ 40,000** |

We next calculate the immediate costs of initiating the ESOP at time zero, which we will assume are $20,000 (B11), or $12,000 after-tax (B12). Adding $30,000 to $12,000, we arrive at a lifetime cost of $42,000 for running the ESOP (B13), which for simplicity we round off to $40,000 (B14), or 4% of the pre-transaction value of $1 million.[7] Adopting the previous definitions, $E = \$40{,}000$ and $e = 4\%$.

The previous example presumes that the ESOP is not replacing another pension plan. If the ESOP is replacing another pension plan, then it is only the incremental lifetime cost of the ESOP that we would calculate here.

## The Direct Approach

Using the direct approach, we calculate all valuation formulas directly through algebraic substitution. We will develop post-transaction valuation formulas for the following situations:

1. All dilution remains with the ESOP.
2. All dilution goes to the owner.
3. The ESOP and the owner share the dilution.

We will begin with item 1. The owner will be paid pre-transaction price, leaving the ESOP with all of the dilution in value. The following series of equations will enable us to quantify the dilution. All values are stated as a fraction of each $1 of pre-transaction value.

[7] For simplicity, we do not add a control premium and deduct a discount for lack of marketability at the firm level and then reverse that procedure at the ESOP level, as I did in Abrams (1993).

## FMV Equations—All Dilution to the ESOP (Type 1 Dilution; No Type 2 Dilution)

$$1 \quad \text{Pre-transaction value.} \tag{13.1}$$

We pay the owner the $p$% he sells to the ESOP reduced or increased by $D_E$, the net discounts or premiums at the ESOP level. For every \$1 of pre-transaction value, the payment to the owner is thus:

$$pD_E$$

Paid to owner in cash = ESOP loan. (13.1a)

$$tpD_E$$

Tax savings on ESOP loan. (13.1b)

The after-tax cost of the loan is the amount paid to the owner less the tax savings of the loan, or equation (13.1a) – (13.1b).

$$(1-t)pD_E$$

After-tax cost of the ESOP loan. (13.1c)

$$e$$

After-tax lifetime cost of the ESOP. (13.1d)

When we subtract (13.1c) plus (13.1d) from equation (13.1), we obtain the remaining value of the firm:

$$1-(1-t)pD_E-e$$

Post-transaction value of the firm. (13.1e)

Since the ESOP owns $p$% of the firm, the post-transaction value of the ESOP is $p \times D_E \times$ (13.1e):

$$pD_E-(1-t)p^2D_E^2-pD_Ee$$

Post-transaction value of the ESOP. (13.1f)

The dilution to the ESOP (type 1 dilution) is the amount paid to the owner minus the value of the ESOP's $p$% of the firm, or equation (13.1a) – (13.1f):

$$pD_E-\left[pD_E-(1-t)p^2D_E^2-pD_Ee\right]=(1-t)p^2D_E^2+pD_Ee$$

Dilution to ESOP. (13.1g)

## Table 13.2, Sections 1 and 2: Post-Transaction FMV with All Dilution to the ESOP

Now that we have established the formulas for calculating the FMV of the firm when all dilution goes to the ESOP, let's look at a concrete example in Table 13.2. The table consists of three sections. Section 1, rows 5–10, is the operating parameters of the model. Section 2 shows the calculation of the post-transaction values of the firm, ESOP, and the dilution to the ESOP according to equations (13.1e), (13.1f), and (13.1g), respectively, in rows 12–18. Rows 21–26 demonstrate the accuracy of the results, as explained below.

| | A | B | C |
|---|---|---|---|
| 1 | **Table 13.2** | | |
| 2 | **FMV Calculations: Firm, ESOP, and Dilution** | | |
| 3 | | | |
| 4 | **Section 1: Parameters** | | |
| 5 | **$V_{1B}$ = Pre-Transaction Value** | $ 1,000,000 | |
| 6 | **$p$ = Percentage of Stock Sold to ESOP** | 30% | |
| 7 | **$D_E$ = Net ESOP Discounts/Premiums** | 98% | |
| 8 | **$t$ = Tax Rate** | 40% | |
| 9 | **$E$ = ESOP Costs (Lifetime costs capitalized; Table 13.1, B14 )** | $ 40,000 | |
| 10 | **$e$ = ESOP Costs/Pre-Transaction Value = E/$V_{1B}$** | 4% | |
| 11 | | | |
| 12 | **Section 2: All Dilution to ESOP** | | |
| 13 | **$(1 - e) - (1 - t)\ p\ D_E$ = Post-Trans FMV—Firm (Eq. 13.1e)** | 0.783600 | |
| 14 | **Multiply by Pre-Trans FMV = B5 × B13 = B24** | **$ 783,600** | |
| 15 | **$p\ D_E - (1 - t)\ p^2\ D_E^2 - p\ D_E e$ = Post-Trans FMV—ESOP (Eq. 13.1f)** | 0.230378 | |
| 16 | **Multiply by Pre-Trans FMV = B5 × B15 = B25** | **$ 230,378** | |
| 17 | **$(1 - t)\ p^2\ D_E^2 + p\ D_E e$ = Dilution to the ESOP (Eq. 13.1g)** | **0.063622** | |
| 18 | **Multiply by Pre-Trans FMV = B5 × B17 = B26** | **$ 63,622** | |
| 19 | | | |
| 20 | **Proof of Section 2 Calculations:** | | |
| 21 | **Pre-Trans FMV = B5** | **$ 1,000,000** | |
| 22 | **Payment to Owner = B6 × B7 × B21** | 294,000 | |
| 23 | **After Tax Cost of Loan = (1 – B8) × B22** | 176,400 | |
| 24 | **Post-Trans FMV—Firm = B21 – B23 – B9 = B14** | 783,600 | |
| 25 | **Post-Transaction FMV of ESOP = B6 × B7 × B24 = B16** | 230,378 | |
| 26 | **Dilution to the ESOP = B22 – B25 = B18** | **$ 63,622** | |
| 27 | | | |
| 28 | **Section 3: All Dilution to Seller** | **Multiple** | **× $V_{1B}$ = FMV** |
| 29 | **$V_n = (1 - e) / [1 + (1 - t)\ p\ D]_E$ = Post-Trans FMV—Firm = B40 (Eq. 13.3n)** | 0.816049 | **$ 816,049** |
| 30 | **$L_n = p \times D_E \times V_n$ = Post-Trans FMV—ESOP (Eq. 13.3j)** | 0.239918 | **$ 239,918** |
| 31 | **Dilution to Seller = (B6 × B7) – B30 = [13.3o]** | | **5.4082%** |
| 32 | **Dilution to Seller = B5 × C31** | | **$ 54,082** |
| 33 | **Dilution to Seller = B22 – C30** | | **$ 54,082** |
| 34 | | | |
| 35 | **Proof of Calculation in C29:** | | |
| 36 | **Pre-Trans FMV = B5** | $ 1,000,000 | |
| 37 | **Payment to Owner = C30** | 239,918 | |
| 38 | **Tax Shield = t × B37** | 95,967 | |
| 39 | **After-Tax Cost of ESOP Loan = B37 – B38** | 143,951 | |
| 40 | **Post-Trans FMV—Firm = B36 – B39 – B9 = C29** | **$ 816,049** | |

Section 3 shows the calculation of the post-transaction values of the firm and the ESOP when there is no dilution to the ESOP. We will cover that part of the table later. In the meantime, let's review the numerical example in section 2.

B13 contains the results of applying equation (13.1e) using section 1 parameters to calculate the post-transaction value of the firm, which is \$0.783600 per \$1 of pre-transaction value. We multiply the \$0.783600 by the \$1 million pre-transaction value (B5) to calculate the post-transaction value of the firm = \$783,600 (B14). The post-transaction value of the ESOP according to equation (13.1f) is \$0.230378[8] (B15) × \$1 million pre-transaction value (B5) = \$230,378 (B16).

We calculate dilution to the ESOP according to equation (13.1g) as $(1 - 0.4) \times 0.3^2 \times 0.98^2 + 0.3 \times 0.98 \times 0.04 = 0.063622$ (B17). When we multiply the dilution as a percentage by the pre-transaction value of \$1 million, we get dilution of \$63,622 (B18, B26).

We now confirm these results and the formulas in rows 21–26. The payment to the owner is \$1 million × 30% × 0.98 (net of ESOP discounts/premiums) = \$294,000

[8]Which itself is equal to $p\ D_E$ × the post-transaction value of the firm, or cell B6 × B7 × B13.

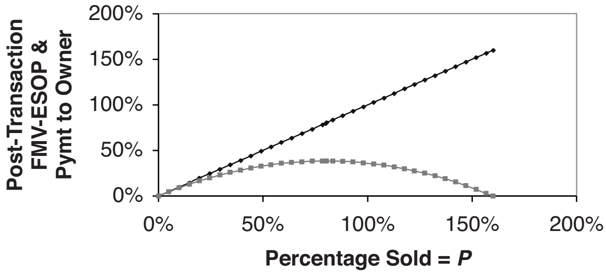


**FIGURE 13.1** Post-Transaction FMV-ESOP versus Percent Sold

(B21 × B6 × B7 = B22). The ESOP takes out a $294,000 loan to pay the owner, which the company will have to pay. The after-tax cost of the loan is (1 − $t$) multiplied by the amount of the loan, or (1 − 0.4) × $294,000 = $176,400 (B23). Subtracting the after-tax cost of the loan and the $40,000 lifetime ESOP costs from the pre-transaction value, we come to a post-transaction value of the firm of $783,600 (B24), which is identical to the value obtained by direct calculation using formula (13.1e) in B14. The post-transaction value of the ESOP is $p\ D_E$ × post-transaction FMV—firm, or 0.3 × 0.98 × $783,600 = $230,378 (B25, B16). The dilution to the ESOP is the payment to the owner minus the post-transaction value of the ESOP, or $294,000 (B22) − $230,378 (B25) = $63,622 (B26, B18). We have now demonstrated the accuracy of the direct calculations in rows 14, 16, and 18.

## The Post-Transaction Value Is a Parabola

Equation (13.1f), the formula for the post-transaction value of the ESOP, is a parabola. We can see this more easily by rewriting equation (13.1f) as $V = -D_E^2(1-t)p^2 + D_E(1-e)p$, where $V$ is the post-transaction value of the ESOP. Figure 13.1 shows this function graphically. The straight line, $p\ D_E$, is a slight modification of a simple 45° line $y = x$ (or in this case $V = p$), except multiplied by $D_E = 98\%$. This line is the payment to the owner when the ESOP bears all of the dilution. The vertical distance of the parabola (equation (13.1f)) from the straight line is the dilution of the ESOP, defined by equation (13.1g), which is itself a parabola. Figure 13.1 should actually stop where $p = 100\%$, but it has been extended merely to show the completion of the parabola, since there is no economic meaning for $p > 100\%$.

We can calculate the high point of the parabola, which is the maximum post-transaction value of the ESOP, by taking the first partial derivative of equation (13.1f) with respect to $p$ and setting the equation to zero:

$$\frac{\partial V}{\partial p} = -2(1-t)D_E^2 p + D_E(1-e) = 0. \tag{13.2}$$

This solves to $p = \frac{(1-e)}{2(1-t)\,D_E}$, or $p = 81.63265\%$. Substituting this number into equation (13.1f) gives us the maximum value of the ESOP of $V = 38.4\%$.[9] This means that if the owner sells any greater portion than 81.63265% of the firm to the

[9]We can verify this is a maximum rather than minimum value by taking the second partial derivative, $\partial^2 V/\partial p^2 = -2\,(1-t)\,D_E{}^2 < 0$, which confirms the maximum.

ESOP, he actually *decreases* the value of the ESOP, assuming a 40% tax rate and no outside capital infusions into the sale. The lower the tax rate, the more the parabola shifts to the left of the vertical line, until at $t = 0$, where most of the parabola is completed before the line.[10]

## FMV Equations—All Dilution to the Owner (Type 2 Dilution)

Let's now assume that instead of paying the owner $p\ D_E$, the ESOP pays him some unspecified amount, $x$. Accordingly, we rederive equations (13.1) through (13.1g) with that single change and label our new equations (13.3) through (13.3j).

$$1$$
Pre-transaction value. (13.3)

$$x$$
Paid to owner in cash = ESOP loan. (13.3a)

$$t\,x$$
Tax savings on ESOP loan. (13.3b)

$$(1 - t)\,x$$
After-tax cost of the ESOP loan. (13.3c)

$$e$$
After-tax ESOP cost. (13.3d)

When we subtract (13.3c) plus (13.3d) from equation (13.3), we come to the remaining value of the firm of:

$$(1 - e) - (1 - t)\,x$$
Post-transaction value of the firm. (13.3e)

Since the ESOP owns $p$% of the firm and the ESOP bears its net discount, the post-transaction value of the ESOP is $p \times D_E \times$ (13.3e), or:

$$pD_E(1 - e) - (1 - t)pD_E\,x$$
Post-transaction value of the ESOP. (13.3f)

We can eliminate dilution to the ESOP entirely by specifying that the payment to the owner, $x$, equals the post-transaction value of the ESOP, (13.3f), or:

$$x = pD_E(1 - e) - (1 - t)pD_E\,x. \tag{13.3g}$$

Moving the right term to the left side,

$$x + (1 - t)pD_E\,x = pD_E(1 - e). \tag{13.3h}$$

---

[10]This is because equation (13-1f) becomes $V = -D_E^2\ p^2 + D_E\ (1 - e)\ p$. Given our $D_E$ and $e$, $V$ is then approximately equal to $-0.92\ (p^2 - p)$. If $t = 0$, $e = 0$, and there were no discounts and premiums at the ESOP level, that is, $D_E = 1$, then the owner would be paid $p$, the post-transaction value of the firm would be $1 - p$, and the post-transaction value of the ESOP would be $p\ (1 - p)$, or $-p^2 + p$. This parabola would finish at $p = 1$. The maximum post-transaction ESOP value would be 25% at $p = 50\%$.

Factoring out $x$,

$$x\left[1 + (1 - t)pD_E\right] = pD_E(1 - e). \tag{13.3i}$$

Dividing through by $1 + (1 - t)pD_E$,

$$x = \frac{pD_E(1 - e)}{1 + (1 - t)pD_E}$$

Post-transaction FMV of ESOP, all dilution to owner. (13.3j)

Substituting equation (13.3j) into the $x$ term in equation (13.3e), the post-transaction value of the firm is:

$$(1 - e) - (1 - t)\frac{pD_E(1 - e)}{1 + (1 - t)pD_E}. \tag{13.3k}$$

Factoring out the $(1 - e)$ from both terms, we get:

$$(1 - e)\left[1 - \frac{(1 - t)pD_E}{1 + (1 - t)pD_E}\right]. \tag{13.3l}$$

Rewriting the 1 in the brackets as $\frac{1+(1-t)pD_E}{1+(1-t)pD_E}$, we obtain:

$$(1 - e)\frac{1 + (1 - t)pD_E - (1 - t)pD_E}{1 + (1 - t)pD_E}. \tag{13.3m}$$

The numerator simplifies to 1, which enables us to simplify the entire expression to:

$$\frac{1 - e}{1 + (1 - t)pD_E}$$

Post-transaction value of the firm—type 1 dilution = 0. (13.3n)

The dilution to the seller is the pre-transaction FMV of the shares sold minus the price paid, the latter of which is $p\,D_E$ × (13.3n). Thus, dilution is:

$$pD_E\left[1 - \frac{1 - e}{1 + (1 - t)pD_E}\right]. \tag{13.3o}$$

## Table 13.2, Section 3: FMV Calculations—All Dilution to the Seller

In section 3 we quantify the engineered price that eliminates all dilution to the ESOP, which according to equation (13.3n) is:

$$\$1 \text{ million} \times \frac{(1 - 0.04)}{\left[1 + (0.6) \times (0.3) \times (0.98)\right]} = \$1 \text{ million} \times 0.816049 \text{ (B29)}$$
$$= \$816{,}049 \text{ (C29)}.$$

Similarly, the value of the ESOP is: 0.3 × 0.98 × 0.816049 × \$1,000,000 = \$239,918 (C30), which is also the same amount that the owner is paid in cash. We can prove this correct as follows:

1. The ESOP borrows \$239,918 (B37, transferred from C30) to pay the owner and takes out a loan for the same amount, which the firm pays.

2. The firm gets a tax deduction, which has a net present value of its marginal tax rate multiplied by the principal of the ESOP loan, or 40% × $239,918, or $95,967 (B38), which after being subtracted from the payment to the owner leaves an after-tax cost of the payment to the owner (which is identical to the after-tax cost of the ESOP loan) of $143,951 (B39).
3. We subtract the after-tax cost of the ESOP loan of $143,951 and the $40,000 lifetime ESOP costs from the pre-transaction value of $1 million to arrive at the final value of the firm of $816,049 (B40). This is the same result as the direct calculation by formula in B29, which confirms equation (13.3n). Multiplying by $p$ $D_E$ ($0.3 \times 0.98 = 0.294$) would lead to the same result as in B30, which confirms the accuracy of equation (13.3j).

We can also confirm the dilution formulas in section 3. The seller experiences dilution equal to the normative price he would have received if he were not willing to reduce the sales price, that is, $294,000 (B22) less the engineered selling price of $239,918 (C30), or $54,082 (cell, C33). This is the same result as using a direct calculation from equation (13.3o) of 5.4082% (C31) × the pre-transaction price of $1 million = $54,082 (C32).

The net result of this approach is that the owner has shifted the entire dilution from the ESOP to himself. Thus, the ESOP no longer experiences any dilution in value. While this action is very noble on the part of the owner, in reality few owners are willing and able to do so.

## Sharing the Dilution

The direct approach also allows us to address the question of how to share the dilution. If the owner does not wish to place all the dilution on the ESOP or absorb it all personally, he can assign a portion to both parties. By subtracting the post-transaction value of the ESOP (13.3f) from the cash to the owner (13.3a), we obtain the amount of dilution. We can then specify that this dilution should be equal to a fraction $k$ of the default dilution, that is, the dilution to the ESOP when the ESOP bears all of the dilution. In our nomenclature, the post-transaction value of the ESOP – dilution to the ESOP = $k$ × (default dilution to the ESOP). Therefore,

$$k = \frac{\textit{Actual Dilution to ESOP}}{\textit{Default Dilution to ESOP}},$$

or $k$ = the % dilution remaining with the ESOP.

The reduction in dilution to the ESOP is $(1 - k)$. For example, if $k = 33\%$, the ESOP bears 33% of the dilution; the reduction in the amount of dilution borne by ESOP is 67% (from the default figure of 100%).

The left-hand side of the equation to calculate the payment to the owner when dilution is shared by both parties is the payment to the owner, $x$, minus the post-transaction FMV to the ESOP, equation (13.3f). The right-hand side of the equation is $k$ times the dilution to the ESOP, which is equation (13.1g).

$$x - \left[pD_E(1-e) - (1-t)pD_E x\right] = k\left[(1-t)p^2 D_E^2 + pD_E e\right]. \tag{13.4}$$

| | A | B |
|---|---|---|
| 1 | **Table 13.3** | |
| 2 | **Adjusting Dilution to Desired Levels** | |
| 3 | | |
| 4 | | |
| 5 | $p$ = Percentage Sold to ESOP | 30.00% |
| 6 | $D_E$ = Net Discounts at the ESOP Level | 98.00% |
| 7 | $k$ = Arbitrary Fraction of Remaining Dilution to ESOP | **66.67%** |
| 8 | $t$ = Tax Rate | 40.00% |
| 9 | $e$ = % ESOP Costs | 4.00% |
| 10 | $x$ = % to Owner = $(pD_E(1-e)+k[(1-t)(p^2D_E^2+pD_Ee)])/(1+(1-t)pD_E)$ (Eq. 13.4a) | 27.60% |
| 11 | ESOP Post-Trans = $p\,D_E[1-e-(1-t)x\,]$ (Eq. 13.3f) | 23.36% |
| 12 | Actual Dilution to ESOP = B10 – B11 | 4.24% |
| 13 | Default Dilution to ESOP: $(1-t)D_E^2\,p^2+p\,D_E\,e$ (Eq. 13.1g) | 6.36% |
| 14 | Actual/Default Dilution: [12] / [13] = $k$ = [7] | **66.67%** |
| 15 | Dilution to Owner = (B5 × B6) – B10 | **1.80%** |
| 16 | Dilution to Owner = $p\times D_E-((p\times D_E)\times(1-e)+k\times((1-t)\times D_E^2\times p^2+p\times D_E\times e))/(1+(1-t)\times p\times D_E)$ | **1.80%** |

Collecting terms, we get:

$$x\left[1+(1-t)pD_E\right]=pD_E(1-e)+k\left[(1-t)p^2D_E^2+pD_Ee\right].$$

Dividing both sides by $[1+(1-t)\,p\,D_E)$, we solve to:

$$x=\frac{pD_E(1-e)+k\left[(1-t)p^2D_E^2+pD_Ee\right]}{1+(1-t)pD_E}. \tag{13.4a}$$

In other words, equation (13.4a) is the formula for the amount of payment to the owner when the ESOP retains the fraction $k$ of the default dilution. If we let $k=0$, equation (13.4a) reduces to (13.3j), the post-transaction FMV of the ESOP when all dilution goes to the owner. When $k=1$, equation (13.4a) reduces to (13.1a), the payment to the owner when all dilution goes to the ESOP.

## Equation to Calculate Type 2 Dilution

Type 2 dilution, $D_2$, is equal to $p\,D_E$, the pre-transaction selling price adjusted for control and marketability, minus the engineered selling price, $x$. Substituting equation (13.4a) for $x$, we get:

$$D_2=pD_E-\frac{pD_E(1-e)+k[(1-t)p^2D_E^2+pD_Ee]}{1+(1-t)pD_E}. \tag{13.4b}$$

## Tables 13.3 and 13.3A: Adjusting Dilution to Desired Levels

Table 13.3 is a numerical example using equation (13.4a). We let $p=30\%$ (B5), $D_E=98\%$ (B6), $k=2/3$ (B7), $t=40\%$ (B8), and $e=4\%$ (B9). B10 is the calculation of $x$, the payment to the seller—as in equation (13.4a)—which is 27.60%. B11 is the value of the ESOP post-transaction, which we calculate according to equation (13.3f),[11] at 23.36%. Subtracting the post-transaction value of the ESOP from the payment to the owner (27.60% − 23.36%) = 4.24% (B12) gives us the amount of type 1 dilution.

[11] With $p\,D_E$ factored out.

The default type 1 dilution, where the ESOP bears all of the dilution, would be $(1-t)p^2 D_E^2 + pD_E e$, according to equation (13.1g), or 6.36% (B13). Finally, we calculate the actual dilution divided by the default dilution, or $\frac{4.24\%}{6.36\%}$, to arrive at a ratio of 66.67% (B14), or 2/3, which is the same as $k$ (B7), which confirms the accuracy of equation (13.4a). By designating the desired level of dilution to be 2/3 of the original dilution, we have reduced the dilution by 1/3, or $(1-k)$.

If we desire dilution to the ESOP to be zero, then we substitute $k = 0$ in equation (13.4a), and the equation reduces to $x = \frac{pD_E(1-e)}{1+(1-t)pD_E}$, which is identical to equation (13.3j), the post-transaction value of the ESOP when the owner bears all of the dilution. You can see that in Table 13.3A, which is identical to Table 13.3, except that we have let $k = 0$ (B7), which leads to the zero dilution, as seen in B14.

Type 2 dilution appears in Table 13.3, rows 15 and 16. The owner is paid 27.60% (B10) of the pre-transaction value for 30% of the stock of the company. He normally would have been paid 29.4% of the pre-transaction value (B5 × B6 = 0.3 × 0.98 = 29.4%). Type 2 dilution is 29.4% − 27.60% = 1.80% (B15). In B16, we calculate type 2 dilution directly using equation (13.4b). Both calculations produce identical results, confirming the accuracy of equation (13.4b). In Table 13.3A, where we let $k = 0$, type 2 dilution is 5.41% (B15 and B16).

## Table 13.3B: Summary of Dilution Trade-offs

In Table 13.3B, we summarize the dilution options that we have seen in Tables 13.2, 13.3, and 13.3A to get a feel for the tradeoffs between type 1 and type 2 dilution. In Table 13.2, where we allowed the ESOP to bear all dilution, the ESOP experienced dilution of 6.36% (Table 13.3B, B8 transferred from Table 13.2, B17). In Table 13.3, by apportioning 1/3 of the dilution to himself, the seller reduced type 1 dilution by 6.36% − 4.24% = 2.12% (Table 13.3B, D8) and undertook type 2 dilution of 1.80% (C9). The result is that the ESOP bears dilution of 4.24% (C8) and the owner bears 1.8% (C9). In Table 13.3A, we allowed the seller to bear all dilution rather than the ESOP. The seller thereby eliminated the 6.36% type 1 dilution and accepted 5.41% type 2 dilution.

| | A | B |
|---|---|---|
| 1 | **Table 13.3A** | |
| 2 | **Adjusting Dilution to Desired Levels—All Dilution to Owner** | |
| 3 | | |
| 4 | | |
| 5 | **$p$ = Percentage Sold to ESOP** | 30.00% |
| 6 | **$D_E$ = Net Discounts at the ESOP Level** | 98.00% |
| 7 | **$k$ = Arbitrary Fraction of Remaining Dilution to ESOP** | **0.00%** |
| 8 | **$t$ = Tax Rate** | 40.00% |
| 9 | **$e$ = % ESOP Costs** | 4.00% |
| 10 | **$x$ = % to Owner = $(p\,D_E(1-e) + k[(1-t)(p^2 D_E^2 + p\,D_E e)])/(1 + (1-t)\,p\,D_E)$ (Eq. 13.4a)** | 23.99% |
| 11 | **ESOP Post-Trans = $p\,D_E[(1-e-(1-t)x]$ [Eq. 13.3f]** | 23.99% |
| 12 | **Actual Dilution to ESOP = [10] – [11]** | 0.00% |
| 13 | **Default Dilution to ESOP: $(1-t)\,D_E^2\,p^2 + p\,D_E\,e$ (Eq. 13.1g)** | 6.36% |
| 14 | **Actual/Default Dilution: [12] / [13] = $k$ = [7]** | **0.00%** |
| 15 | **Dilution to Owner = (B5 × B6) – B10** | **5.41%** |
| 16 | **Dilution to Owner = $p \times D_E - ((p \times D_E) \times (1-e) + k \times ((1-t) \times D_E^2 \times p^2 + p \times D_E \times e))/(1+(1-t) \times p \times D_E)$** | **5.41%** |

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 13.3B** | | | | |
| 2 | **Summary of Dilution Trade-offs** | | | | |
| 3 | | | | | |
| 4 | | | | | |
| 5 | | **Scenario: Assignment of Dilution** | | | |
| 6 | | **100% to** | **2/3 to** | | **100% to** |
| 7 | **Dilution Type** | **ESOP** | **ESOP** | **Difference** | **Owner** |
| 8 | **1 (ESOP)** | 6.36% | 4.24% | 2.12% | 0.00% |
| 9 | **2 (Seller)** | 0.00% | 1.80% | -1.80% | 5.41% |
| 10 | **Source Table** | 14.2 | 14.3 | | 14.3A |

Judging by the results seen in Table 13.3B, it appears that when the seller takes on a specific level of type 2 dilution, the decrease in type 1 dilution is greater than the corresponding increase in type 2 dilution. This turns out to be correct in all cases, as proven in Appendix A, the Mathematical Appendix.

As mentioned in the introduction, the reader may wish to skip to the conclusion section. The following material aids in understanding dilution, but it does not contain any new formulas of practical significance.

## The Iterative Approach

We now proceed to develop formulas to measure the engineered value per share that when paid by the ESOP will eliminate dilution to the ESOP. We accomplish this by performing several iterations of calculations. Using iteration, we will calculate the payment to the owner, which becomes the ESOP loan, and the post-transaction fair market values of the firm and the ESOP.

In our first iteration, the seller pays the ESOP the pre-transaction FMV without regard for the ESOP loan. The existence of the ESOP loan then causes the post-transaction values of the firm and the ESOP to decline, which means the post-transaction value of the ESOP is lower than the pre-transaction value paid to the owner.

In our second iteration we calculate an engineered payment to the owner that will attempt to equal the post-transaction value at the end of the first iteration. In the second iteration, the payment to the owner is less than the pre-transaction price, because we have considered the ESOP loan from the first iteration in our second iteration valuation. Because the payment is lower in this iteration, the ESOP loan is lower than it is in the first iteration. We follow through with several iterations until we arrive at a steady-state value, where the engineered payment to the owner exactly equals the post-transaction value of the ESOP. This enables us to eliminate all type 1 dilution to the ESOP and shift it to the owner as type 2 dilution.

### Iteration #1

We denote the pre-transaction value of the firm before considering the lifetime ESOP administration cost as $V_{1B}$ ($B = B$efore administration costs).

$$V_{1B} = \text{Pre-transaction value.} \tag{13.5}$$

The value of the firm after deducting the lifetime ESOP costs but before considering the ESOP loan is:[12]

$$V_{1A} = V_{1B} - E = V_{1B} - V_{1B}e = V_{1B}(1 - e). \tag{13.5a}$$

The owner sells $p$% of the stock to the ESOP, so the ESOP would pay $p$ times the value of the firm. However, we also need to adjust the payment for the degree of marketability and control of the ESOP. Therefore, the ESOP pays the owner $V_{1A}$ multiplied by $p \times D_E$, or:

$$L_1 = pD_E V_{1A} = pD_E V_{1B}(1 - e). \tag{13.5b}$$

Our next step is to compute the net present value of the loan. In this chapter we greatly simplify this procedure over the more complex calculation in my original article (Abrams, 1993).[13]

The net present value of the payments of any loan discounted at the loan rate is the principal of the loan. Since both the interest and principal payments on ESOP loans are tax deductible, the after-tax cost of the ESOP loan is simply the principal of the loan multiplied by 1 minus the tax rate.[14] Therefore:

$$NPVL_1 = (1 - t)pD_E V_{1B}(1 - e). \tag{13.5c}$$

## Iteration #2

We have now finished the first iteration and are ready to begin iteration #2. We begin by subtracting equation (13.5c), the net present value of the ESOP loan, from the pre-transaction value, or:

$$V_{2B} = V_{1B} - (1 - t)pD_E V_{1B}(1 - e) = V_{1B}[1 - pD_E(1 - t)(1 - e)]. \tag{13.6}$$

We again subtract the lifetime ESOP costs to arrive at $V_{2A}$.

$$V_{2A} = V_{2B} - E. \tag{13.6a}$$

$$V_{2A} = V_{1B}[1 - pD_E(1 - t)(1 - e)] - V_{1B}e. \tag{13.6b}$$

Factoring out the $V_{1B}$, we get:

$$V_{2A} = V_{1B}[(1 - e) - pD_E(1 - t)(1 - e)]. \tag{13.6c}$$

[12]$V_{1A}$ is the only iteration of $V_{jA}$ where we do not consider the cost of the loan. For $j > 1$, we do consider the after-tax cost of the ESOP loan.

[13]You do not need to read that article to understand this chapter.

[14]One might speculate that perhaps the appraiser should discount the loan by a rate other than the nominal rate of the loan. To do so would be implicitly saying that the firm is at a suboptimal D/E (debt/equity) ratio before the ESOP loan and that increasing debt lowers the overall cost of capital. This is closer to a matter of faith than science, as there are those who argue on each side of the fence. The opposite side of the fence is covered by two Nobel Prize winners, Merton Miller and Franco Modigliani (MM), in a seminal article (Miller and Modigliani, 1958). MM's famous Proposition I states that in perfect capital markets, that is, in the absence of taxes and transactions costs, one cannot raise the value of the firm with debt. They acknowledge a secondary tax effect of debt, which I use here literally and no further, that is, adding debt increases the value of the equity only to the extent of the tax shield. Also, even if there is an optimal D/E ratio and the subject company is below it, it does not need an ESOP to borrow to achieve the optimal ratio.

Factoring out the $(1 - e)$, we then come to the post-transaction value of the firm in iteration #2 of:

$$V_{2A} = V_{1B}(1 - e)[1 - pD_E(1 - t)]. \tag{13.6d}$$

It is important to recognize that we are not double-counting $E$ (i.e., subtracting it twice). In equation (13.6), we calculate the value of the firm as its pre-transaction value minus the net present value of the loan against the firm. The latter is indirectly affected by $E$, but in each new iteration, we must subtract $E$ directly in order to count it in the post-transaction value.

The post-transaction value of the ESOP loan in iteration #2 is $p \times D_E \times$ (13.6d), or:

$$L_2 = pD_E V_{1B}(1 - e)[1 - pD_E(1 - t)]. \tag{13.6e}$$

The net present value of the loan is:

$$NPVL_2 = (1 - t)pD_E V_{1B}(1 - e)[1 - (1 - t)pD_E]. \tag{13.6f}$$

## Iteration #3

We now begin the third iteration of value. The third iteration FMV before lifetime ESOP costs is $V_{1B} - \text{NPVL}_2$, or:

$$V_{3B} = V_{1B} - (1 - t)pD_E V_{1B}(1 - e)[1 - (1 - t)pD_E]. \tag{13.7}$$

Factoring out $V_{1B}$, we have:

$$V_{3B} = V_{1B}\left\{1 - pD_E(1 - t)(1 - e)[1 - (1 - t)pD_E]\right\}. \tag{13.7a}$$

Multiplying terms, we get:

$$V_{3B} = V_{1B}[1 - pD_E(1 - t)(1 - e) + p^2D_E^2(1 - t)^2(1 - e)]. \tag{13.7b}$$

$$V_{3A} = V_{3B} - E. \tag{13.7c}$$

$$V_{3A} = V_{1B}[1 - pD_E(1 - t)(1 - e) + p^2D_E^2(1 - t)^2(1 - e) - e]. \tag{13.7d}$$

Moving the $e$ at the right immediately after the 1:

$$V_{3A} = V_{1B}[(1 - e) - pD_E(1 - t)(1 - e) + p^2D_E^2(1 - t)^2(1 - e)]. \tag{13.7e}$$

Factoring out the $(1 - e)$:

$$V_{3A} = V_{1B}(1 - e)[1 - pD_E(1 - t) + p^2D_E^2(1 - t)^2]. \tag{13.7f}$$

Note that the 1 in the square brackets $= p^0D_E^0(1 - t)^0$.

## Iteration #$n$

Continuing this pattern, it is clear that the $n$th iteration leads to the following formula:

$$V_{nA} = V_{1B}(1 - e)\sum_{j=0}^{n-1}(-1)^j p^j D_E^j(1 - t)^j. \tag{13.8}$$

This is an oscillating geometric sequence,[15] which leads to the following solutions. The ultimate post-transaction value of the firm is:

$$V_{nA} = V_{1B}\frac{1-e}{1-[-pD_E(1-t)]},$$

or, dropping the subscript $A$ and simplifying:

$$V_n = V_{1B}\frac{1-e}{1+(1-t)pD_E}$$

Post-transaction value of the firm—with type 1 dilution = 0.[16] (13.9)

Note that this is the same equation as (13.3n). We arrive at the same result from two different approaches: The post-transaction value of the ESOP is $p \times D_E \times$ the value of the firm, or:

$$L_n = V_{1B}\frac{pD_E(1-e)}{1+(1-t)pD_E}$$

Post-transaction value of the ESOP—with type 1 dilution = 0. (13.10)

This is the same solution as equation (13.3j), after multiplying by $V_{1B}$. The iterative approach solutions in equations (13.9) and (13.10) confirm the direct approach solutions of equations (13.3n) and (13.3j).

## Summary

In this chapter we have developed formulas to calculate the post-transaction values of the firm, ESOP, and the payment to the owner, both pre-transaction and post-transaction, as well as the related dilution. We also derived formulas for eliminating the dilution in each scenario individually, as well as for specifying any desired level of dilution. Additionally, we explored the trade-offs between type 1 and type 2 dilution.

### Advantages of Results

The big advantages of these results are:

1. If the owner insists on being paid at the pre-transaction value, as most will, the appraiser can now immediately calculate the dilutive effects on the value of the ESOP and report that in the initial valuation report.[17] Therefore, the employees

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[15]For the geometric sequence to work, $|\ pD_E(1-t) < 1\ |$, which will almost always be the case.

[16]The reason why the *e* term is in the numerator and not the denominator like the other terms is that the lifetime cost of the ESOP is fixed; it does not vary as a proportion of the value of the firm (or the ESOP) as that changes in each iteration.

[17]Many ESOP trustees prefer this information to remain as supplementary information outside of the report.

will be entering the transaction with both eyes open, and they will not be disgruntled and/or suspicious as to why the value, on average, declines at the next valuation. This will also provide a real benchmark to assess the impact of the ESOP itself on profitability.
2. For owners who are willing to eliminate the dilution to the ESOP or at least reduce it, this chapter provides the formulas to do so and the ability to calculate the trade-offs between type 1 and type 2 dilution.

### Function of ESOP Loan

An important by-product of this analysis is that it answers the question of what is the function of the ESOP loan. Obviously it functions as a financing vehicle, but suppose you were advising a very cash-rich firm that could fund the payment to the owner in cash. Is there any other function of the ESOP loan? The answer is yes. The ESOP loan can increase the value of the firm in two ways:

1. It can be used to shield income at the firm's highest income tax rate. To the extent that the ESOP payment is large enough to cause pre-tax income to drop to lower tax brackets, that portion shields income at lower than the marginal rate and lowers the value of the firm and the ESOP.
2. If the ESOP payment in the first year is larger than pre-tax income, then the firm cannot make immediate use of the entire tax deduction in the first year. The unused deduction will remain as a carryover, but it will suffer from a present value effect.

### Common Sense Is Required

A certain amount of common sense is required in applying these formulas. In extreme transactions such as those approaching a 100% sale to the ESOP, we need to realize that not only can tax rates change, but payments on the ESOP loan may entirely eliminate net income and reduce the present value of the tax benefit of the ESOP loan payments. In addition, the viability of the firm itself may be seriously in question, and it is possible that the appraiser will have to increase the discount rate for a post-transaction valuation. Therefore, one must use these formulas with at least two dashes of common sense.

### To Whom Should the Dilution Belong?

Appraisers almost unanimously consider the pre-transaction value appropriate, yet there has been considerable controversy on this topic. The problem is the apparent financial sleight of hand that occurs when the post-transaction value of the firm and the ESOP precipitously decline immediately after doing the transaction. On the surface, it somehow seems unfair to the ESOP. In this section, we will explore that question.

**DEFINITIONS** Let's begin to address this issue by assessing the post-transaction fair market value balance sheet. We will use the following definitions:

| Pre-Transaction | Post-Transaction |
|---|---|
| $A_1$ = assets | $A_2$ = assets = $A_1$ (assets have not changed) |
| $L_1$ = liabilities | $L_2$ = liabilities |
| $C_1$ = capital | $C_2$ = capital |

Note that the subscript 1 refers to pre-transaction and the subscript 2 refers to post-transaction.

**THE MATHEMATICS OF THE POST-TRANSACTION FAIR MARKET VALUE BALANCE SHEET** The nonmathematical reader may wish to skip or skim this section. It is more theoretical and does not result in any usable formulas.

The fundamental accounting equation representing the pre-transaction balance sheet is:

$$A_1 = L_1 + C_1$$

Pre-transaction FMV balance sheet. (13.11)

Assuming the ESOP bears all of the dilution, after the sale liabilities increase and capital decreases by the sum of the after-tax cost of the ESOP loan and the after-tax lifetime ESOP costs,[18] or:

$$C_1 \times \{(13.1c) + (13.1d)\}$$

Increase in liabilities and decrease in capital. (13.12)

As noted in the definitions, assets have not changed. Only liabilities and capital have changed.[19] Thus the post-transaction balance sheet is:

$$A_2 = \left\{L_1 + C_1\left[(1-t)pD_E + e\right]\right\} + \left\{C_1 - C_1\left[(1-t)pD_E + e\right]\right\}. \quad (13.13)$$

The first term in braces equals $L_2$, the post-transaction liabilities, and the second term in braces equals $C_2$, the post-transaction capital. Note that $A_2 = A_1$. Equation (13.13) simplifies to:

$$A_2 = \left\{L_1 + C_1\left[(1-t)pD_E + e\right]\right\} + \left\{C_1\left[1 - (1-t)pD_E - e\right]\right\} \quad (13.14)$$

Post-transaction balance sheet.

Equation (13.14) gives us an algebraic expression for the post-transaction fair market value balance sheet when the ESOP bears all of the dilution.

**ANALYZING A SIMPLE SALE** Only two aspects relevant to this discussion are unique about a sale to an ESOP: (1) tax deductibility of the loan principal, and (2) forgiveness

[18]Again, these should only be the incremental costs if the ESOP is replacing another pension plan.

[19]For simplicity, we are assuming the company hasn't yet paid any of the ESOP's lifetime costs. If it has, then that amount is a reduction in assets rather than an increase in liabilities. Additionally, the tax shield on the ESOP loan could have been treated as an asset rather than a contra-liability, as we have done for simplicity. This is not intended to be an exhaustive treatise on ESOP accounting.

of the ESOP's debt. Let's analyze a simple sale to a non-ESOP buyer and later to an ESOP buyer. For simplicity we will ignore tax benefits of all loans throughout this example.

Suppose the fair market value of all assets is \$10 million before and after the sale. Pre-transaction liabilities are zero, so capital is worth \$10 million, pre-transaction. If a buyer pays the seller personally \$5 million for one-half of the capital stock of the company, the transaction does not impact the value of the firm—ignoring adjustments for control and marketability. If the buyer takes out a personal loan for the \$5 million and pays the seller, there is also no impact on the value of the company. In both cases, the buyer owns one-half of a \$10 million firm, and it was a fair transaction.

If the corporation takes out the loan on behalf of the buyer, but the buyer ultimately has to repay the corporation, then the real liability is to the buyer, not the corporation, and there is no impact on the value of the stock—it is still worth \$5 million. The corporation is a mere conduit for the loan to the buyer.

What happens to the firm's value if the corporation takes out and eventually repays the loan? The assets are still worth \$10 million post-transaction.[20] Now there are \$5 million in liabilities, so the equity is worth \$5 million. The buyer owns one-half of a firm worth \$5 million, so his stock is worth only \$2.5 million. Was the buyer hoodwinked?

The possible confusion over value clearly arises because it is the corporation itself that is taking out the loan to fund the buyer's purchase of stock, and the corporation—not the buyer—ultimately repays the loan. By having the corporation repay the loan, the other shareholder is forgiving his half of a \$5 million loan and thus gifting \$2.5 million to the buyer.[21] Thus, the "buyer" ultimately receives a gift of \$2.5 million in the form of company stock. This is true whether the buyer is an individual or an ESOP.[22]

**DILUTION TO NON-SELLING OWNERS** When there are additional business owners who do not sell to the ESOP, they experience dilution of their interests without the benefit of getting paid. Conceptually, these owners have participated in giving the ESOP a gift by having the company repay the debt on behalf of the ESOP.

To calculate the dilution to other owners, we begin with the post-transaction value of the firm in equation (13.1e) and repeat the equation as (13.1e*). Then we will calculate the equivalent equations for the non-selling owner as we did for the ESOP in equations (13.1f) and (13.1g), and we will relabel those equations by adding an asterisk after them.

$$1 - (1 - t)pD_E - e$$

Post-transaction value of the firm (repeated). (13.1e*)

[20]There is a second-order effect of the firm being more highly leveraged and thus riskier that may affect value (and which we are ignoring here). See Chapter 16.

[21]The other half of the forgiveness is a wash—the buyer forgiving it to himself.

[22]This does not mean that an ESOP brings nothing to the table in a transaction. It does bring tax deductibility of the loan principal as well as the Section 1042 rollover.

If the non-selling shareholder owns the fraction $q$ of the outstanding stock, then his post-transaction value is:

$$q - q(1-t)pD_E - qe \qquad (13.1f^*)$$

Post-transaction value of non-selling shareholder's stock.

Finally, we calculate dilution to the non-selling shareholder as his pre-transaction value of $q$ minus the pre-transaction value in equation (13.1f*), or:

$$q\left[(1-t)pD_E + e\right] \qquad (13.1g^*)$$

Dilution to non-selling shareholder's stock.[23]

The dilution formula (13.1g*) tells us that the dilution to the non-selling shareholder is simply his ownership, $q$, multiplied by the dilution in value to the firm itself, which is the sum of the after-tax cost of the ESOP loan and the lifetime costs. Here, because we are not multiplying by the ESOP's ownership modified for its unique marketability and control attributes, we do not get the squared terms that we did in equations (13.1f) and (13.1g).

It is also important to note that equations (13.1f*) and (13.1g*) do not account for any possible increase in value the owner might experience as a result of having greater relative control of the firm. For example, if there were two 50% owners pre-transaction and one sells 30% to the ESOP, post-transaction the remaining 50% owner has relatively more control than he had before the transaction. To the extent that we might ascribe additional value to that increase in relative control, we would adjust the valuation formulas. This would mitigate the dilution in equation (13.1g*).

**LEGAL ISSUES** As already mentioned, appraisers almost unanimously consider the pre-transaction value appropriate. Also mentioned earlier in the chapter, case law and Department of Labor proposed regulations indicate the pre-transaction value is the one to be used. Nevertheless, there is ongoing controversy going back to *Farnum*, a case in which the Department of Labor withdrew before going to court, that the post-transaction value may be the most appropriate price to pay the seller.

In the previous section, we demonstrated that the ESOP is receiving a gift, not really paying anything for its stock. Therefore, there is no economic justification for reducing the payment to the owner below the pre-transaction fair market value, which is the price that the seller would receive from any other buyer. If the ESOP (or any party on its behalf) demands that it "pay" no more than post-transaction value, it is tantamount to saying, "The gift that you are giving me is not big enough."

While the dilution may belong to the ESOP, it is nevertheless an important consideration in determining the fairness of the transaction for purposes of a fairness opinion. If a bank loans $10 million to the ESOP for a 100% sale, with no recourse or personal guarantees of the owner, we would likely decide it is not a fair transaction to the ESOP and its participants. We would have serious questions about the ESOP's

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[23]One would also need to consider adjusting for each non-selling shareholder's control and marketability attributes. To do so, we would have to add a term in equation (13-1g*) immediately after the $q$. The term would be the owner's equivalent of $D_E$, except customized for his or her ownership attributes. The details of such a calculation are beyond the scope of this chapter.

probability of becoming a long-range retirement program, given the huge debt load of the company post-transaction.

**CHARITY** While the dilution technically belongs to the ESOP, I consider it my duty to inform the seller of the dilution phenomenon and how it works. While affirming the seller's right to receive fair market value undiminished by dilution, I do mention that if the seller has any charitable motivations to his or her employees—which a minority do—then voluntarily accepting some of the dilution will leave the company and the ESOP in better shape. Of course, in a partial sale it also leaves the remainder of the owner's stock at a higher value than it would have had with the ESOP bearing all of the dilution.

It is my hope that this chapter will contribute to understanding of and dealing with the phenomenon of dilution.

## References

Abrams, Jay B. 1993. An Iterative Procedure to Value Leveraged ESOPs. *Valuation* (January): 71–103.

Abrams, Jay B. 1997. ESOPs: Measuring and Apportioning Dilution. *Valuation* (June): 3–25.

Miller, Merton, and Franco Modigliani. 1958. The Cost of Capital, Corporation Finance, and the Theory of Investment. *American Economic Review* 48: 261–297.

# APPENDIX 13A

# Mathematical Appendix

The purpose of this appendix is to perform comparative static analysis, as is commonly done in economics, on the equations for dilution in the body of the chapter in order to understand the trade-offs between type 1 and type 2 dilution.

We use the same definitions in the appendix as in the chapter. Type 1 dilution is equal to the payment to the owner less the post-transaction value of the ESOP, or $x$ – (13.3f):

$$D_1 = x - [pD_E(1-e) - (1-t)pD_E x]. \tag{A13.1}$$

Factoring out the $x$,

$$D_1 = x[1 + (1-t)pD_E] - pD_E(1-e). \tag{A13.2}$$

We can investigate the impact on type 1 dilution for each \$1 change in payment to the owner by taking the partial derivative of equation (A13.2) with respect to $x$.

$$\frac{\partial D_1}{\partial x} = 1 + (1-t)pD_E > 1. \tag{A13.3}$$

Equation (A13.3) tells us that each additional dollar paid to the owner increases dilution to the ESOP by more than \$1.

A full payment to the owner (the default payment) is $p\ D_E$ for \$1 of pre-transaction value. We pay the owner $x$, and the difference of the two is $D_2$, the type 2 dilution.

$$D_2 = pD_E - x. \tag{A13.4}$$

We can investigate the impact on type 2 for each \$1 change in payment to the owner by taking the partial derivative of equation (A13.4) with respect to $x$.

$$\frac{\partial D_2}{\partial x} = -1. \tag{A13.5}$$

Type 2 dilution moves in an equal but opposite direction from the amount paid to the owner, which must be the case to make any sense. Together, equations (A13.3) and (A13.5) tell us that each additional dollar paid the owner increases the dilution to the ESOP more than it reduces the dilution to the owner. We can also see this by taking the absolute value of the ratio of the partial derivatives:

$$\frac{|\partial D_2/\partial x|}{|\partial D_1/\partial x|} = \frac{1}{1 + (1-t)pD_E} < 1. \tag{A13.6}$$

## Significance of the Results

Equation (A13.6) demonstrates that for every $1 of payment forgone by the owner, the dilution incurred by the owner will always be less than the dilution eliminated to the ESOP. The reason for this is that every $1 the owner forgoes in payment costs him $1 in type 2 dilution; yet it saves the ESOP the $1, plus it reduces the ESOP loan by $p\ D_E$ and saves the ESOP the after-tax cost of the lowered amount of the loan, or $(1 - t)\ p\ D_E$.

There appears to be some charity factor inherent in the mathematics.

Finally, we have not dealt with the fact that by the owner taking on some or all of the dilution from the ESOP loan, he increases the value of his $(1 - p)$ share of the remaining stock by reducing the dilution to it. Such an analysis has no impact on the valuation of the ESOP, but it should be considered in the decision to initiate an ESOP.

# CHAPTER 14

# The Trade-off in Selling to an ESOP versus an Outside Buyer

# CHAPTER 14

# The Trade-off in Selling to an ESOP versus an Outside Buyer

## Section 1: Introduction

C corporation ESOPs have the IRC Section 1042 rollover tax advantage, although they have the disadvantage of double taxation. S corporations—whether ESOP or non-ESOP—lack the IRC Section 1042 rollover, but have a single layer of taxation. The weight of appraisal literature shows that outside (third-party) buyers are willing to pay a premium for S corporations[1] over their C corporation counterparts—but they lose the IRC 1042 rollover. In this chapter we develop the mathematics for appraisers to assist their clients to make the right decision.

There is an additional complication, which is that the owner may sell less than 100% of the stock to the ESOP, thus losing the control premium on his or her remaining stock (unless there is a binding contract to sell 100% over time).

In this chapter, we develop the formula to determine the breakeven percentage, $p^*$, at which a 100% owner of a business who wishes to sell stock will be indifferent between selling as a C corporation to an ESOP or selling 100% of the firm as an S corporation to a third-party buyer. We show that for $p > p^*$ the ESOP is the best choice, and for $p < p^*$ the S corporation is the best choice. Each alternative has its advantages and disadvantages, and the solution to the problem involves quantifying them, setting them equal to each other, and solving for the mathematical conditions that satisfy the equation.

This chapter is organized into five sections. Section 1 is this introduction. Section 2 is a listing of the advantages and disadvantages of selling to an ESOP as a C corporation versus selling as an S corporation to a third party. Section 3 is the mathematical model and the solution for the breakeven percentage. Section 4 is a series of spreadsheets in which we test the solution to the model under a variety of assumptions, and Section 5 is the conclusion.

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[1]This applies to LLCs and other non-tax entities as well. We use *S corporations* as a simple term in this chapter to include all non-tax entities.

## Section 2: Advantages and Disadvantages of Selling to an ESOP versus a Third Party

Selling to an ESOP has the following advantages and disadvantages:

1. Assuming the company was an S corporation (or some other non-tax entity) at the outset, the company will have to switch to a C corporation for the owner to be able to take advantage of the IRC 1042 rollover. As a result, if the company were ever acquired by another C corporation, it would no longer qualify for a step-up in the tax basis of its assets through a Section 338(h)(10) election, which means the acquirer would be "stuck with" the lower basis of assets for future depreciation. Thus, S corporations carry a valuation premium over C corporations, and switching to C corporation status forgoes that premium. We discuss this further in item 5.
2. While there is controversy about this, it is common practice for appraisers to accord a lower discount for lack of marketability (DLOM) for an ESOP sale than for sale to an outside party, as the ESOP itself provides a market and the ESOP is more protected against abuse by the control shareholder.
3. After selling a portion, $p$, of the stock of the company to the ESOP, the owner will have $(1 - p)$ remaining. However, the company guarantees the ESOP loan and eventually pays the loan. Because it will pay the ESOP's loan, this creates a liability for the company the moment the transaction takes place. The post-transaction fair market value of the firm declines, because the company has a new liability that did not exist before the sale. Therefore, the fair market value of the owner's $(1 - p)$ ownership in the firm is $(1 - p) \times$ post-transaction fair market value. Thus, the owner experiences a dilution in value to the tune of $(1 - p) \times$ the decline in the post-transaction value, the latter of which is normally the after-tax cost of the ESOP loan—unless the owner sells 100% of the firm to the ESOP.
4. If the owner sells a control interest (i.e., $p > 50\%$), the remaining ownership of $(1 - p)$ in the ESOP is a minority interest. Therefore, the remaining ownership will suffer a diminution in value due to a discount for lack of control (DLOC), unless the original contract called for the follow-up sale(s) to be at a control price.
5. ESOPs are more expensive to maintain than other pension plans. The differential of the lifetime ESOP costs reduces the value of the firm and the ESOP after the sale takes place. This cost reduces the value of the $1 - p$ remaining ownership, but not the original $p$ sold to the ESOP, as that sale takes place before there is an ESOP.
6. There are several studies that provide evidence that ESOPs that combine employee ownership with participation outperform non-ESOP firms.[2] Let's use just one of them to calculate an approximate valuation increase from the greater performance of the ESOP. Kruse and Blasi (2000) found ESOP firms had higher sales growth of 2.4% over non-ESOP firms, with both higher annual employment growth and growth in sales per employee by 2.3%. Later in this chapter, in our

[2]National Center for Employee Ownership, www.nceo.org/library/corpperf.html.

discussion of Table 14.1, we will use the 2.3% higher growth to calculate the valuation advantage of ESOPs to be approximately 15%. Like item 4, this applies only to the $(1 - p)$ remaining ownership, although this item increases the value of the ESOP alternative instead of decreasing it.

7. Selling to the ESOP has the benefit of eliminating the personal capital gains tax.

Now let's list the advantages and disadvantages of remaining an S corporation and selling to a third party.

1. The S corporation has the benefit of the S corporation premium over the C corporation. Erickson and Wang (2002) found that acquiring C corporations pay 11% to 17% more for S corporations than C corporations for the ability to step up the basis of the assets according to Section 338(h)(10) of the Internal Revenue Code. Let's assume a midpoint of 15% for this analysis. In other words, we will value the company as an S corporation the same as if it were a C corporation, except that we will increase its value by a 15% premium. If the company is small enough that its relevant universe of buyers would include individuals and non-tax entities, then maintaining S corporation status eliminates corporate income tax, and it can be reasonable to assume an even higher premium.[3]
2. Selling an S corporation will leave the owner with a personal capital gains tax.

There are nonpecuniary advantages of ESOPs, for example, the ability to make a partial exit while leaving family in control, rewarding longtime employees, and so on, that are not considered in this model. Such considerations can be very important, and their lack of appearance directly in this model is not meant to downplay them. These are qualitative factors that one can apply after the analysis, or it is possible to include them in the analysis by making a decision to change the ESOP premium (EP), described later.

## Section 3: The Mathematics

Before beginning with the mathematics we need to define our terms.

### Defining Terms

We first define some of terms appearing in the various equations.

Let:

$CP$ = control premium.

$DLOC$ = discount for lack of control.

$DLOM$ = discount for lack of marketability. We are concerned with two different measures of DLOM: as an ESOP or as a non-ESOP firm. We use an ESOP subscript to indicate the former.

$e$ = lifetime ESOP costs as a percentage of the pre-transaction value = E/V1B = \$40,000/\$1 million = 4%.

[3]See Denis and Sarin (2002), where the authors find S corporation premiums of 12% to 43%.

$EP$ = ESOP premium—the factor to quantify expected higher firm performance as an ESOP.
$p$ = *p*ercentage of firm sold to the ESOP.
$t$ = combined federal and state corporate income *t*ax rate, assumed at 40%.
$E$ = lifetime costs of initiating and running the *E*SOP. These are generally legal fees, appraisal fees, ESOP administration fees, and internal administration costs.

## Three Phases of the Mathematical Analysis

We now proceed with the mathematical analysis, which has three phases:

1. We develop the equation for the total wealth of the shareholder if he or she sells to the ESOP as a C corporation. The total wealth of the shareholder will be the cash payment received from the sale, plus the value of the owner's remaining stock in the firm after the sale.
2. We calculate the price of the sale as an S corporation to be sold to a third party, less the applicable personal capital gains taxes, as that will be the total wealth of the shareholder to come from the business in that scenario.
3. We equate the two expressions for total wealth and solve for the percentage sold ($p^*$) at which the total wealth of each alternative is equal. This is the shareholder's indifference percentage.

## Calculating Total Wealth—C Corporation ESOP

Let the pre-transaction marketable minority FMV per dollar[4] for the C corporation ESOP equal the expression in equation (14.1). While it looks very strange, the reason for choosing that expression will become obvious shortly, as we arrived at this expression by working backward to produce the FMV on a private control basis of \$1.00. This facilitates easier evaluation in percentage terms of the various different adjustments to value that appear later on in the chapter in Tables 14.2A and 14.2B, where B12 equals \$10 million exactly, and all premiums to and discounts from value are easy to measure with respect to the private control value as a frame of reference.

$$\frac{1 - DLOC}{1 - DLOM_{ESOP}}$$
Pre-transaction marketable minority FMV. (14.1)

Next, let's take a discount for lack of marketability. The FMV remaining after subtracting *DLOM* for the ESOP will be $(1 - DLOM_{ESOP}) \times$ equation (14.1), or:

$$\frac{1 - DLOC}{1 - DLOM_{ESOP}} \times (1 - DLOM_{ESOP}) = 1 - DLOC$$
Private minority FMV. (14.2)

[4]Our formulas are per dollar of pre-transaction FMV, which makes it easy to apply the formulas to any FMV.

Before our next step, we will need to do some algebraic manipulation of equation (14.2). The discount for lack of control is equal to the control premium (CP) divided by 1 plus the control premium. Substituting that into equation (14.2), we get our alternative expression for the private minority FMV:

$$1 - DLOC = 1 - \frac{CP}{1+CP} = \frac{1+CP-CP}{1+CP} = \frac{1}{1+CP}$$

Private minority FMV. (14.3)

We then add a control premium of *CP*. After adding the control premium, the resulting private control FMV is one plus the control premium times equation (14.3), or:

$$\frac{1}{1+CP} \times (1+CP) = 1.00$$

Pre-transaction private control FMV. (14.4)

Thus, after the foregoing algebraic manipulations, our pre-transaction private control FMV is exactly one dollar. Next, the owner sells a proportion of the company equal to $p$ to the ESOP.

$p$ Payment to the owner per \$1.00 of pre-transaction FMV = Loan. (14.5)

The ESOP borrows $p$ from the bank. The after-tax cost of the loan is $(1 - t)p$, where $t$ is the company's marginal corporate income tax rate.

$(1-t)p$ After - tax cost of the ESOP loan. (14.6)

We subtract the after-tax cost of the ESOP loan from the \$1.00 pre-transaction FMV to determine the after-tax post-transaction value of the firm on a control basis, but before considering the lifetime differential ESOP costs.

$1-(1-t)p$ Post-transaction firm FMV—Private control basis—Pre-ESOP costs. (14.7)

Next, we compute the after-tax lifetime ESOP cost differential. Assume the following set of facts: The first-year ESOP costs are estimated at \$100,000, annual operating costs are \$50,000,[5] income taxes of 40%, a 20% discount rate, and a 5% growth rate. The end-year Gordon Model multiple is $\frac{1}{0.20-0.05} = \frac{1}{0.15} = 6.6667$. The after-tax annual operating costs are \$50,000 × (1 − 40% tax rate) = \$30,000. We multiply \$30,000 × 6.6667 = \$200,000. The first-year ESOP costs after tax are \$100,000 × 60% = \$60,000. Thus, the lifetime ESOP costs after tax are \$260,000, which is approximately 3.7% of the \$6,940,000 post-transaction FMV (Table 14.2A, B15).

If the ESOP is a new pension plan that replaces another pension plan that has a lifetime cost of 1.7% of the post-transaction FMV, then the incremental cost of the ESOP is 2% (3.7% − 1.7%) of the post-transaction FMV. If it is the first pension plan for the firm or it is implemented in addition to another plan, the benefits for which the company plans to maintain at the same level, then the entire 3.7% is the

[5]This includes legal, accounting, appraisal, and administrative costs to outsiders, as well as allocated salaries of employees who spend time administering the ESOP.

lifetime ESOP cost. For mathematical ease, let's assume the former is the case. We will denote this cost as $e = 2\%$. The post-transaction FMV after removing the lifetime ESOP cost differential appears in equation (14.8).

$$\left[1-(1-t)p\right](1-e)$$

Post-transaction FMV of the firm—Private control basis. (14.8)

If $p > 0.5$, then the remaining ownership in the firm is a minority ownership, and we have to subtract a discount for lack of control (DLOC), and we get:

$$\left[1-(1-t)p\right](1-e)(1-DLOC)$$

Post-trans. FMV of the firm—Minority basis. (14.9)

After selling $p$ of stock to the ESOP, the owner still owns $(1-p)$ of the firm. The fair market value of that on a private minority basis is:

$$\left[1-(1-t)p\right](1-p)(1-e)(1-DLOC)$$

FMV—Remaining stock in the firm. (14.10)

As mentioned in item 6 in Section 2, it has been documented that ESOP-owned firms that have a participatory employee culture outperform their non-ESOP counterparts. Let's build in a premium for the higher performance of the firm as an ESOP, and we'll call that variable *EP* (*E*SOP *P*remium).

$$\left[1-(1-t)p\right](1-p)(1-e)(1-DLOC)(1+EP). \tag{14.11}$$

The total wealth of the owner is the sum of equations (14.5) and (14.11), or:

$$p+\left[1-(1-t)p\right](1-p)(1-e)(1-DLOC)(1+EP) \tag{14.12}$$

Total wealth—ESOP sale[6]

## Calculating Total Wealth—S Corporation Sold to Third Party

Now, we will look at the alternative, which is for the owner to sell the firm in its entirety to a third party. Selling to a third party would enable the owner to maintain the company's S corporation status, which may command a valuation premium. In our example, we will assume the premium equals 15%,[7] which we denote algebraically as the variable $s$. The S corporation valuation will be $(1+s)$ times the C corporation valuation at the marketable minority level.

That also will be true at the private firm level, with one additional modification. The additional modification is that we assume that DLOM is greater for sale to a third party than it is to an ESOP. Thus, it is necessary to adjust DLOM to the level appropriate for the third party, which most appraisers consider to be higher than the DLOM for sale to an ESOP. Let $r_{DLOM}$ denote the ratio of 1 minus the DLOM for

[6]Remember, this formula requires $p > 0.5$ and that the ESOP gets a voting pass-through. If not, then the formula requires modification.

[7]In reality, this has been the subject of several academic and professional articles and is not a simple topic. In any case, the magnitude of the S corporation premium can be adjusted to any level desired, including zero, and it has no effect on the algebraic solution to this model, although it obviously will affect the calculation of the indifference percentage.

the third party to 1 minus DLOM for the ESOP. For example, suppose that DLOM for the third party is 15%[8] (Tables 14.2A and 14.2B, B31), while it is 10% (B30) for the ESOP. As discussed previously, the \$1.00 pre-transaction value already includes a 10% DLOM. If the appropriate DLOM for the third party is instead 15%, then the adjusted pre-transaction value is:

$$\$1.00 \times r_{DLOM} = \frac{1 - DLOM_{3rd\,Party}}{1 - DLOM_{DLOM}} = \frac{1 - 15\%}{1 - 10\%} = \frac{0.85}{0.9} = \$.9\overline{444} \text{ (B32)}.$$

Combining the adjustments for the S corporation premium and the greater DLOM for the sale to a private party, the pre-transaction private control FMV for sale to a third party is equal to equation (14.13).

$$\begin{array}{l} r_{DLOM}(1+s) \\ \text{FMV—Private control—S corp.} \end{array} \tag{14.13}$$

If the owner sells the entire company, he or she will pay personal capital gains taxes of $t_{pcg}$. The federal long-term capital gains tax is 15%. For our example, we will add the California rate, which is 8.4%.[9] The combined rate is 15% + 8.4% − (15% × 8.4%) = 22.14%. The after-tax net proceeds to the owner is:

$$r_{DLOM}(1+s)(1-t_{pcg}). \tag{14.14}$$

## Equating the Two Expressions of Total Wealth after Personal Taxes

We can find the breakeven percentage, $p^*$, of the firm for the owner to sell at which point he or she is indifferent by setting equation (14.12) equal to equation (14.14).

$$p + \left[1-(1-t)p\right](1-p)(1-e)(1-DLOC)(1+EP) = r_{DLOM}(1+s)(1-t_{pcg}). \tag{14.15}$$

Dividing both sides of the equation by $(1-e)(1-DLOC)(1+EP)$, we get:

$$\frac{p}{(1-e)(1-DLOC)(1+EP)} + \left[1-(1-t)p\right](1-p) = \frac{r_{DLOM}(1+s)(1-t_{pcg})}{(1-e)(1-DLOC)(1+EP)}. \tag{14.16}$$

Letting $x = \dfrac{1}{(1-e)(1-DLOC)(1+EP)}$, we can restate equation (14.16) as:

$$xp + \left[1-(1-t)p\right]\ (1-p) = \frac{r_{DLOM}(1+s)(1-t_{pcg})}{(1-e)(1-DLOC)(1+EP)}. \tag{14.17}$$

In the next three equations, we are merely manipulating the algebra to restate the equation as a quadratic that we can solve.

$$xp + (1-p+pt)(1-p) = \frac{r_{DLOM}(1+s)(1-t_{pcg})}{(1-e)(1-DLOC)(1+EP)}; \tag{14.18}$$

[8]See Abrams (2005). The 15% is an approximate average of the DLOMs in Table 8.1.

[9]This tax rate has changed since the predecessor article comprising this chapter was originally written. We leave the older tax rate in, as the point of this chapter is the methodology and the formulas, not the particular tax rate of one state.

$$xp + 1 - p + pt - p + p^2 - tp^2 = \frac{r_{DLOM}(1+s)(1-t_{pcg})}{(1-e)(1-DLOC)(1+EP)}; \quad (14.19)$$

$$(1-t)p^2 + (x+t-2)p + \left[1 - \frac{r_{DLOM}(1+s)(1-t_{pcg})}{(1-e)(1-DLOC)(1+EP)}\right] = 0. \quad (14.20)$$

Note that equation (14.20) is a quadratic, with the following parameters:

$$a = (1-t),\ b = (x+t-2),\ and\ c = 1 - \frac{r_{DLOM}(1+s)(1-t_{pcg})}{(1-e)(1-DLOC)(1+EP)}.$$

Therefore, the breakeven percentage to achieve identical results selling to an ESOP or selling as an S corporation to a third party is:

$$p^* = \frac{-(x+t-2) \pm \sqrt{(x+t-2)^2 - 4(1-t)\left[1 - \frac{r_{DLOM}(1+s)(1-t_{pcg})}{(1-e)(1-DLOC)(1+EP)}\right]}}{2(1-t)}. \quad (14.21)$$

In our calculations in Table 14.2B, we will find that only the positive sign before the square root leads to a real solution.

$$p^* = \frac{-(x+t-2) + \sqrt{(x+t-2)^2 - 4(1-t)\left[1 - \frac{r_{DLOM}(1+s)(1-t_{pcg})}{(1-e)(1-DLOC)(1+EP)}\right]}}{2(1-t)}. \quad (14.22)$$

Substituting back the definition of $x$, we can restate equation (14.22) as equation (14.23) in its most expanded form. However, we use equation (14.22) in the tables in Section 4.

$$p^* = \frac{-\left(\frac{1}{(1-e)(1-DLOC)(1+EP)} + t - 2\right) + \sqrt{\left(\frac{1}{(1-e)(1-DLOC)(1+EP)} + t - 2\right)^2 - 4(1-t)\left[1 - \frac{r_{DLOM}(1+s)(1-t_{pcg})}{(1-e)(1-DLOC)(1+EP)}\right]}}{2(1-t)} \quad (14.23)$$

## Section 4: Sample Calculations in the Tables

In this section, we show sample realistic calculations for the trade-off that Sir Maximo Bonestein, the 100% business owner, will face in his decision whether to sell to an ESOP or to an outside buyer. Our analysis, which does not attempt to address any nonfinancial considerations such as ongoing control or gift and estate planning issues, consists of Tables 14.1, 14.2A, and 14.2B.

### Table 14.1: ESOP Valuation Advantage

The purpose of this table is to calculate a benchmark for the valuation effect of the additional efficiency that some ESOP firms experience by having employees more motivated to work because of their ownership status. Kruse and Blasi (2000), cited earlier in this chapter, found a 2.4% increase in sales growth and a 2.3% higher growth in employment and growth in sales per employee. Let's use the 2.3%.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Table 14.1** | | | |
| 2 | **ESOP Valuation Advantage** | | | |
| 3 | | | | |
| 4 | Firm FMV [1] | **Variable** | **$ 1,000,000** | **$ 10,000,000** |
| 5 | Discount Rate [1] | $r$ | 26% | 22% |
| 6 | Growth Rate—Non-ESOP Firms [2] | $g_1$ | 4% | 4% |
| 7 | Growth Rate—ESOP Firms | $g_2$ | 6.30% | 6.30% |
| 8 | Gordon Model Multiple—Non-ESOP Firms [3] | $GMM_1$ | 5.1023 | 6.1363 |
| 9 | Gordon Model Multiple—ESOP Firms [3] | $GMM_2$ | 5.6980 | 7.0353 |
| 10 | Valuation Advantage of ESOP Firms (Row 9/Row 8 – 1) | | **12%** | **15%** |
| 11 | | | | |
| 12 | [1] A reasonable estimate. This uses the log size equation from Table 5.1, 2nd regression, | | | |
| 13 | adjusted for estimated arithmetic mean yearly PE growth of 0.80% (Table 5.3, B32). We | | | |
| 14 | round the results. | | | |
| 15 | | | | |
| 16 | [2] A reasonable estimate. | | | |
| 17 | | | | |
| 18 | [3] These are midyear Gordon model multiples: SQRT(1 + $r$)/($r$ – $g$). | | | |

I do not have a copy of Kruse and Blasi's Rutgers Study, and therefore I do not know the characteristics of the firms in their study, other than to know it covered only private firms. Let's assume most of the firms were between $1 million (Table 14.1, C4) and $10 million (D4) in fair market value (FMV), the calculations for which appear in columns C and D. We use the log size model regression and then subtract our estimate of arithmetic mean yearly PE growth of 0.80% (Table 5.3, B32) to determine the related discount rates, which are 26% and 22%, respectively (row 5).[10]

Let's assume a growth rate of 4% (row 6)—a reasonable estimate—for non-ESOP firms. The implied growth rate for ESOP firms is then 4% + 2.3%[11] = 6.3% (row 7). In rows 8 and 9, we calculate the midyear Gordon model multiple (GMM) = SQRT(1 + $r$)/($r$ − $g$) for the non-ESOP and ESOP firms, respectively. In row 10, we calculate the valuation advantage of the ESOP firm, which equals row 9 divided by row 8, minus 1. Using the $10 million firm as an example, the Gordon model multiple for the ESOP firm is SQRT(1 + 0.22)/(0.22 − 0.063) = 7.0353 (D9), while the GMM for the non-ESOP firm is SQRT(1 + 0.22)/(0.22 − 0.04) = 6.1363 (D8). The ESOP valuation advantage equals (7.0353/6.1363) − 1 = 15% [(D9/D8) − 1 = D10].

The ESOP advantage for the small firm is 12% (C10), and it is 15% (D10) for the large firm in this example. Since our example firm is in the $10 million FMV range, it is reasonable to use the 15% as our estimate of the ESOP valuation advantage. The lifetime ESOP cost, $e$, will vary inversely with firm size, so the net ESOP advantage should increase with firm size. However, we handle that variable separately.

## Organization of Tables 14.2A and 14.2B

Tables 14.2A and 14.2B are essentially identical, with the exception that in Table 14.2A, we assume a specific percentage sold to the ESOP in B29, while in

[10]This uses the regression from Table 5.1, 2nd regression. We round the results.
[11]This assumes all costs are perfectly variable.

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 14.2A** | | | | |
| 2 | **Shareholder Wealth Calculations—Arbitrary Percentage Sold (*p*)** | | | | |
| 3 | | | | | |
| 4 | | | **3rd Party** | | |
| 5 | | **ESOP** | **S Corp** | **Diff** | |
| 6 | **Pre-Transaction Marketable Minority FMV [1]** | 8,888,889 | 10,222,222 | | |
| 7 | **Discount-Lack of Marketability—%** | 10% | 15% | | |
| 8 | **Discount-Lack of Marketability—$** | (888,889) | (1,533,333) | | |
| 9 | **Pre-Transaction FMV—Private Minority** | 8,000,000 | 8,688,889 | | |
| 10 | **Control Premium—% = *CP*** | 25% | 25% | | |
| 11 | **Control Premium—$** | 2,000,000 | 2,172,222 | | |
| 12 | **FMV—Private Control** | 10,000,000 | 10,861,111 | | |
| 13 | **Payment to Owner = *p* × B12** | 5,100,000 | | | |
| 14 | **After-Tax Cost of Loan = (1 – t) × B13** | 3,060,000 | | | |
| 15 | **Post-Trans FMV—Private Control—Before ESOP Costs = B12 – B14** | **6,940,000** | | | |
| 16 | **After-Tax Lifetime ESOP Cost Differential = B15 × *e*** | 138,800 | | | |
| 17 | **Post-Trans FMV—Private Control = B15 – B16** | **6,801,200** | | | |
| 18 | **Discount-Lack of Control = DLOC × –B17** | (1,360,240) | | | |
| 19 | **Post-Trans FMV—Private Minority = B17 + B18** | **5,440,960** | | | |
| 20 | **ESOP Premium = B19 × *EP*** | 816,144 | | | |
| 21 | **Post-Trans FMV with ESOP Prem—Private Minority = B19 + B20** | 6,257,104 | | | |
| 22 | **Sir Maximo's Remaining Ownership = 1 – *p*** | **49.00000%** | | | |
| 23 | **Sir Maximo's Remaining Ownership—$ (B21 × B22)** | 3,065,981 | NA | | |
| 24 | Total Wealth before Personal Capital Gains Taxes (B13 + B23) | 8,165,981 | 10,861,111 | | |
| 25 | Personal Capital Gains Taxes (0 for ESOP; –C24 × $t_{pcg}$) | 0 | (2,404,650) | | |
| 26 | Total Wealth after Personal Capital Gains Taxes (Rows 24 + 25) | **8,165,981** | **8,456,461** | **290,480** | |
| 27 | | | | | |
| 28 | **Assumptions** | | | | |
| 29 | ***p* = % Sold** | 51% | | | |
| 30 | ***DLOM—ESOP*** | 10% | | | |
| 31 | ***DLOM—3rd Party*** | 15% | | | |
| 32 | **$r_{DLOM}$ = Val Adjustmt Ratio—Different DLOMs = (1 – B31)/(1 – B30)** | 0.94444444 | | | |
| 33 | ***t* = Corp Income Tax Rate** | 40% | | | |
| 34 | ***CP* = Control Premium** | 25% | | | |
| 35 | ***e* = After-Tax Lifetime ESOP Cost Differential** | 2% | | | |
| 36 | ***DLOC* = Discount for Lack of Control = *CP*/(1 + *CP*)** | 20% | | | |
| 37 | **$t_{pcg\ Fed}$ = Personal Capital Gains Tax Rate—Federal** | 15% | | | |
| 38 | **$t_{pcg\ State}$ = Personal Capital Gains Tax Rate—State (Calif)** | 8.40% | | | |
| 39 | **$t_{pcg}$ = Personal Capital Gains Tax Rate (Combined Fed & CA)** | 22.14% | | | |
| 40 | **S Corp. Premium** | 15% | | | |
| 41 | ***EP* = ESOP Premium** | 15% | | | |
| 42 | | | | | |
| 43 | **Model Parameters** | | | | |
| 44 | **$x = 1/(1 - e)(1 - DLOC)(1 + EP)$** | 1.109139 | | | |
| 45 | **$a = 1 - t$** | 0.600000 | | | |
| 46 | **$b = x + t - 2$** | (0.490861) | | | |
| 47 | **$c = 1 - [r_{DLOM}(1 + s)(1 - t_{pcg})/(1 - e)(1 - DLOC)(1 + EP)]$** | 0.062061 | | | |
| 48 | | | | | |
| 49 | [1] This is the FMV that results in a $10 million FMV at the private control level. In column C, we assume | | | | |
| 50 | an S corp. valuation = C corp + a premium equal to 15% (B40) and a higher DLOM. | | | | |
| 51 | **Sensitivity Analysis: How Total Shareholder Wealth (ESOP scenario) Changes with Different Assumptions** | | | | |
| 52 | | | | | |
| 53 | **Percentage Sold** | **ESOP Premium** | | | |
| 54 | | 0% | 5% | 10% | **15%** |
| 55 | **51%** | 7,766,070 | 7,899,374 | 8,032,677 | **8,165,981** |
| 56 | 60% | 8,007,040 | 8,107,392 | 8,207,744 | 8,308,096 |
| 57 | 66% | 8,210,022 | 8,290,524 | 8,371,025 | 8,451,526 |
| 58 | **66.18112%** | 8,216,676 | 8,296,605 | 8,376,533 | **8,456,461** |
| 59 | 67% | 8,247,146 | 8,324,503 | 8,401,860 | 8,479,217 |
| 60 | 70% | 8,364,160 | 8,432,368 | 8,500,576 | 8,568,784 |
| 61 | 80% | 8,815,360 | 8,856,128 | 8,896,896 | 8,937,664 |
| 62 | 90% | 9,360,640 | 9,378,672 | 9,396,704 | 9,414,736 |
| 63 | 100% | 10,000,000 | 10,000,000 | 10,000,000 | 10,000,000 |

Table 14.2B, we calculate the breakeven percentage, $p^*$, in B29 at which total shareholder wealth is equal in both scenarios.

For rows 6–26, column B is our calculations of the total wealth generated by a sale as a C corporation to the ESOP, while column C is our calculations of the total wealth generated by a sale as an S corporation to a third party. Rows 29–41 are our assumptions to the model. However, as already mentioned, B29 is an assumption of

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | **Table 14.2B** | | | | |
| 2 | **Shareholder Wealth Calculations—*p** Calculated** | | | | |
| 3 | | | | | |
| 4 | | | **3rd Party** | | |
| 5 | | **ESOP** | **S Corp** | | |
| 6 | **Pre-Transaction Marketable Minority FMV [1]** | 8,888,889 | 10,222,222 | | |
| 7 | **Discount-Lack of Marketability—%** | 10% | 15% | | |
| 8 | **Discount-Lack of Marketability—$** | (888,889) | (1,533,333) | | |
| 9 | **Pre-Transaction FMV-Private Minority** | 8,000,000 | 8,688,889 | | |
| 10 | **Control Premium—% = *CP*** | 25% | 25% | | |
| 11 | **Control Premium—$** | 2,000,000 | 2,172,222 | | |
| 12 | **FMV—Private Control** | 10,000,000 | 10,861,111 | | |
| 13 | **Payment to Owner = *p* × B12** | 6,618,112 | | | |
| 14 | **After-Tax Cost of Loan = (1 – t) × B13** | 3,970,867 | | | |
| 15 | **Post-Trans FMV—Private Control—Before ESOP Costs = B12 – B14** | **6,029,133** | | | |
| 16 | **After-Tax Lifetime ESOP Cost Differential = B15 × *e*** | 120,583 | | | |
| 17 | **Post-Trans FMV—Private Control = B15 – B16** | **5,908,550** | | | |
| 18 | **Discount-Lack of Control = DLOC × –B17** | (1,181,710) | | | |
| 19 | **Post-Trans FMV—Private Minority = B17 + B18** | **4,726,840** | | | |
| 20 | **ESOP Premium = B19 × *EP*** | 709,026 | | | |
| 21 | **Post-Trans FMV with ESOP Prem—Private Minority = B19 + B20** | 5,435,866 | | | |
| 22 | **Sir Maximo's Remaining Ownership = 1 – *p*** | **33.81888%** | | | |
| 23 | **Sir Maximo's Remaining Ownership—$ (B21 × B22)** | 1,838,349 | NA | | |
| 24 | Total Wealth before Personal Capital Gains Taxes (B13 + B23) | 8,456,461 | 10,861,111 | | |
| 25 | Personal Capital Gains Taxes (0 for ESOP; –C24 × $t_{pcg}$ ) | 0 | (2,404,650) | | |
| 26 | Total Wealth after Personal Capital Gains Taxes (Rows 24 + 25) | **8,456,461** | **8,456,461** | | |
| 27 | | | | | |
| 28 | **Assumptions** | | | | |
| 29 | ***p****= Breakeven %, per Equation [14.19]** | **66.18112%** | | | |
| 30 | ***DLOM—ESOP*** | 10% | | | |
| 31 | ***DLOM—3rd Party*** | 15% | | | |
| 32 | **$r_{DLOM}$ = Val Adjustmt Ratio—Different DLOMs = (1 – B31)/(1 – B30)** | 0.94444444 | | | |
| 33 | ***t* = Corp Income Tax Rate** | 40% | | | |
| 34 | ***CP* = Control Premium** | 25% | | | |
| 35 | ***e* = After-Tax Lifetime ESOP Cost Differential** | 2% | | | |
| 36 | ***DLOC* = Discount for Lack of Control = *CP*/(1 + *CP*)** | 20% | | | |
| 37 | **$t_{pcg\ Fed}$ = Personal Capital Gains Tax Rate—Federal** | 15% | | | |
| 38 | **$t_{pcg\ State}$ = Personal Capital Gains Tax Rate—State (California)** | 8.40% | | | |
| 39 | **$t_{pcg}$ = Personal Capital Gains Tax Rate (Combined Fed & CA)** | 22.14% | | | |
| 40 | **S Corp. Premium** | 15% | | | |
| 41 | ***EP* = ESOP Premium** | 15% | | | |
| 42 | | | | | |
| 43 | **Model Parameters** | | | | |
| 44 | **$x$ = 1/(1 – $e$)(1 – $DLOC$)(1 + $EP$)** | 1.109139 | | | |
| 45 | **$a$ = 1 – $t$** | 0.600000 | | | |
| 46 | **$b$ = $x$ + $t$ – 2** | (0.490861) | | | |
| 47 | **$c$ = 1 – [$r_{DLOM}$ (1 + $s$)(1 – $t_{pcg}$)/(1 – $e$)(1 – $DLOC$)(1 + $EP$)]** | 0.062061 | | | |
| 48 | | | | | |
| 49 | [1] This is the FMV that results in a $10 million FMV at the private control level. In column C, we assume | | | | |
| 50 | an S corp. valuation = C corp + a premium equal to 15% (B40) and a higher DLOM. | | | | |
| 51 | | | | | |
| 52 | **Sensitivity Analysis: How the Breakeven Percentage *p** Changes with Different Assumptions** | | | | |
| 53 | | | | | |
| 54 | **State Tax Rate** | **ESOP Premium** | | | |
| 55 | | 0% | 5% | 10% | **15%** |
| 56 | 0% | 87.79% | 87.38% | 86.92% | 86.43% |
| 57 | 1% | 86.15% | 85.66% | 85.12% | 84.51% |
| 58 | 2% | 84.47% | 83.88% | 83.24% | 82.51% |
| 59 | 3% | 82.73% | 82.05% | 81.28% | 80.41% |
| 60 | 4% | 80.94% | 80.14% | 79.23% | 78.18% |
| 61 | 5% | 79.09% | 78.15% | 77.08% | 75.82% |
| 62 | 6% | 77.17% | 76.08% | 74.80% | 73.28% |
| 63 | 7% | 75.17% | 73.90% | 72.39% | 70.53% |
| 64 | **8.4%** | 72.23% | 70.64% | 68.69% | **66.18112%** |
| 65 | 9% | 70.90% | 69.15% | 66.97% | 64.07% |
| 66 | | | | | |
| 67 | S Corp Premium | *DLOM—3rd Party* | | | |
| 68 | | 10% | 15% | **20%** | 25% |
| 69 | 0% | #NUM! | #NUM! | #NUM! | **#NUM!** |
| 70 | 5% | 51.82% | #NUM! | #NUM! | #NUM! |
| 71 | 10% | 69.87% | #NUM! | #NUM! | #NUM! |
| 72 | **15%** | 80.38% | **66.18112%** | #NUM! | #NUM! |
| 73 | 22% | 91.56% | 80.79% | 65.70% | #NUM! |
| 74 | 54% | 125.59% | 117.98% | 109.53% | 99.88% |
| 75 | | | | | |
| 76 | e = Lifetime ESOP Costs | Marginal Corporate Income Tax Rate | | | |
| 77 | | **20%** | 30% | 35% | **40.0%** |
| 78 | 1% | 75.99% | 71.77% | 68.98% | 65.47% |
| 79 | **2%** | 76.17% | 72.11% | 69.46% | **66.18112%** |
| 80 | 3% | 76.35% | 72.44% | 69.92% | 66.84% |
| 81 | 4% | 76.52% | 72.76% | 70.36% | 67.45% |
| 82 | 5% | 76.69% | 73.06% | 70.77% | 68.02% |
| 83 | 6% | 76.86% | 73.35% | 71.16% | 68.56% |

$p$, the percentage sold to the ESOP, in Table 14.2A, while it is the calculation of $p^*$, the breakeven percentage to sell to the ESOP, in Table 14.2B. Rows 44–47 are model parameters that are derived calculations. The terms $a$, $b$, and $c$ (rows 45–47) are the parameters to the quadratic formula in equation (14.20), and $x = \frac{1}{(1-e)(1-DLOC)(1+EP)}$ (row 44) is the term that we used to simplify equation (14.16) to be solvable by the quadratic formula.

## Table 14.2A: FMV Calculations for a 51% Sale to an ESOP

We begin the analysis in column B, which is the sale to the ESOP. The marketable minority FMV is $8,888,889 (B6).[12] We subtract a 10% discount for lack of marketability (DLOM) and add a 25% control premium[13] to arrive at a $10 million (B12) private control FMV before the transaction. Sir Maximo's 51% has an FMV of $5.1 million (B13).

The purpose of the next several rows is to calculate the post-transaction FMV in order to value the remaining 49%. Assuming a 40% (B33) corporate income tax rate, the after-tax cost of the ESOP loan is 1 − 40% = 60% of the amount of the loan, or 60% × $5.1 million = $3.06 million (B14), which when subtracted from the $10 million pre-transaction FMV results in a post-transaction FMV of $6,940,000 (B15).

Next we calculate the lifetime after-tax ESOP cost of 2% × $6,940,000 = $138,800 (B35 × B15 = B16), which when subtracted from B15 leaves a remaining FMV of $6,801,200 (B17).

Now we take a discount for lack of control (DLOC) of 20% (B36, which is the control premium of 25% (B34) divided by 1 plus the control premium, i.e., 0.25/1.25) to arrive the private minority FMV of $5,440,960 (B19). Then we add the 15% (B41) ESOP premium of $816,144 (B20) to arrive at a post-transaction FMV with the ESOP premium on a private-minority basis of $6,257,104 (B21).

We multiply that by Sir Maximo's 49% (B22) remaining interest, resulting in an FMV of $3,065,981 (B23). Adding that to the $5.1 million leads to a total of $8,165,981 (B13 + B23 = B24), which is our calculation of Sir Maximo's total wealth from the business if he sells a 51% share of the company to an ESOP and retains the remaining 49%. Since the sale to the ESOP provides the benefit of the IRC 1042 rollover, Sir Maximo pays no capital gains tax (B25) either on the stock already sold or on the remaining stock when it will be sold sometime in the future. Thus, FMV of Sir Maximo's cash and stock after capital gains tax is $8,165,981 (B26).

Now let's compare the ESOP alternative to sale of the company to a third party. If Sir Maximo does not sell his stock in the company to an ESOP, there is no reason to be a C corporation. Instead, the company can maintain S corporation (or any other non-tax entity form) status to eliminate double taxation.

There are a number of studies on the value effect of S versus C corporation valuation. Erickson and Wang (2007) found that acquiring C corporations pay 11% to 17% more for S corporations than for C corporations for the ability to step up the basis of the assets according to Section 338(h)(10) of the Internal Revenue Code. Let's assume a midpoint of 15% for this analysis. In other words, we will value the

[12]For simplicity, we backed into this number to result in a $10 million result in B12. B10 = B12 × 1/(0.9 × 1.25) = B12 × 0.888.

[13]In Chapter 8, I estimated the control premium at 21% to 28% for a single owner.

company as an S corporation the same as if it were a C corporation, except that we will increase its value by a 15% premium (B40). In fact, there can be reasons to assume an even higher premium, depending on the size and the circumstances of the company.[14] Abrams (2010) finds typical S corporation premiums of 9% to 20%—occasionally up to 23%—and if qualified dividends and federal capital gains tax rates increase to 40%, the S premium can go as high as 57%.

On the other hand, there are two important empirical articles on S versus C corporation valuations using the Pratt's Stats database. One (Mattson, Shannon, and Upton, 2002) finds no statistically significant S corporation premium, while the more recent one (DiGabriele, 2007) finds an 8.8% premium. Thus, there is no clear consensus yet in the profession as to the existence or magnitude of the S corporation premium, and it is important that this is a flexible parameter in the model that can be easily changed as new research emerges on the S corporation premium.

The pre-transaction FMV of the company at the public minority level, that is, before the control premium and the discount for lack of marketability (DLOM), is $10,222,222 (C6), which equals $8,888,889 × (1 + 0.15) [B6 × (1 + B40)].

We subtract the 15% DLOM (C7, transferred from B31) to arrive at the pre-transaction private minority FMV of $8,688,889 (C9). We add a 25% control premium (C10) to obtain the FMV at the private control level of $10,861,111 (C12). The ratio of the S-to-C corporation values in rows 9 and 12 are both (1 + 0.15) × 0.9444444 = 1.086111 [(1 + B40) × B32]; that is, the S corporation values are higher by 8.6111%, the combined effect of the 15% S corporation premium and the 94.444% ratio of the valuation effect of the higher S corporation DLOM compared to the ESOP DLOM.

Since there is no sale to an ESOP, no ESOP loan, and no remaining ownership in the firm, there is no analysis in column C in rows 13–23. The FMV—private control level from C12 carries through to C24, Sir Maximo's personal wealth from the business before personal capital gains taxes.

In C25, we calculate personal capital gains taxes at the combined federal and California rates (of course, the combined rate will vary by state) of 22.14% (B39). Multiplying $10,861,111 × 22.14% = $2,404,650 (C24 × B39 = C25, shown as a negative number), which is the personal capital gains that Sir Maximo will pay if he sells the company immediately as an S corporation to a third party. Subtracting that from the total wealth before personal capital gains taxes will leave him with $8,456,461 (C26).[15] That is $290,480 higher than the total wealth in the ESOP scenario (C26 − B26 = D26).

Does that mean that he should sell to a third party? That depends. There can be some non-financial reasons to sell to the ESOP. Perhaps he wants to continue to manage the company and prepare the way for his children to do the same. Perhaps the sale to an ESOP is part of a multi-faceted gift and estate planning strategy. Or, he might want the employees to have ownership in the firm. It is a value judgment as to how much one should be willing to trade in money for lifestyle and other goals, and it is not our place as business valuators to tell our clients how to live

[14]Denis and Sarin (2002) found S corporation premiums of 12% to 43%.

[15]To facilitate understanding in this already complex analysis, we have not dealt with investment banking or business broker fees, nor have we dealt with differentials in legal, accounting, and appraisal fees.

their lives. However, our goal here is to understand the financial tradeoffs, so we can communicate them to our clients.

**SENSITIVITY ANALYSIS** While it is theoretically possible to run sensitivity analyses on the variations of all 10 assumptions and inputs,[16] to keep the analysis reasonably tractable, we will focus on how the total shareholder wealth varies with changes in assumptions of the percentage sold and the ESOP premium. This analysis appears in A53 to E63. Column A shows different assumptions of the percentage sold, ranging from 51% (row 55) up to 100% (row 63), while columns B through E show the ESOP premium varying from 0% (column B) to 15% (column E).

Note that there are two numbers that appear in bold. E55 shows a total shareholder wealth of $8,165,981, which is the base case result on the previous page of Table 14.2A, that is, with the owner selling 51% of his or her stock to the ESOP, with a 15% ESOP premium. Thus, E55 equals B26, as it should. The second number that appears in bold is $8,456,461 (E58), which is the total shareholder wealth when the percentage sold rises to 66.18112% (A58), which is the calculated $p^*$ in Table 14.2B, B29.

Let's analyze the sensitivity analysis table. Moving from left to right, the total wealth increases about $130,000 for each 5% increment in the ESOP premium when $p = 51\%$ (row 55), but the greater is $p$, the less the increment. When $p = 100\%$, there is no increment; row 63 all has the same value of $10 million. That occurs because the greater the share of the business that the owner sells, the less ownership he or she has remaining in the business, and the ESOP premium benefits him or her only to the extent of the remaining ownership. When $p = 100\%$, the ESOP premium benefits only the ESOP, because the seller no longer will have any ownership in the company.

Now let's analyze the sensitivity analysis table going down. Looking at column B, the total shareholder wealth increases at an increasing rate. It starts out by increasing at about $230,000 for each 10% sold, but ends by increasing (between B62 and B63) at about $640,000. The same is true for columns C through E, although the total magnitude of the change is slightly less than that of column B. In general, the total wealth is more sensitive to changes in the percentage sold than the ESOP premium.

Also note that for all $p > p^*$ of 66.18112%, total shareholder wealth in the ESOP scenario is greater than the breakeven wealth of $8,456,461; that is, E59 through E63 are all greater than E58, which is the shareholder wealth for the 100% sale to the third party and for the 66.18112% sale to the ESOP. This demonstrates that the ESOP sale dominates the third-party sale if the owner sells more than 66.18112% to the ESOP.

Next we will proceed to Table 14.2B, where we will calculate the breakeven percentage that Sir Maximo can sell to the ESOP and achieve identical financial results with selling to a third party as an S corporation.

## Table 14.2B: Breakeven Percentage ($p^*$)

In equation (14.22), we calculated the breakeven percentage, $p^*$, which is the percentage sold to the ESOP as a C corporation at which the valuation is identical to that of the sale to the third party as an S corporation. Table 14.2B is identical to Table 14.2A, except for B29, our calculation of $p^*$, the breakeven percentage, which

[16]Of the 13 items in B29 through B41, $r_{DLOM}$, DLOC, and $t_{pcg}$ are calculations based on formulas that use other assumptions, but they are not assumptions in and of themselves.

is 66.18112%, given all the assumptions in B30 through B41. (B44–B47 are not assumptions. They are model parameters—intermediate calculations, which you can safely ignore.)

With Sir Maximo selling a higher percentage, *p*, of the company than he did in Table 14.2A, he increases his total wealth in the ESOP scenario. This is true because the control premium now applies to a larger percentage, and the dilution from the company paying the ESOP's loan and DLOC apply to a smaller percentage. As the logic of the calculations is the same in Table 14.2B as it was in Table 14.2A, we will not describe each calculation. The final valuation is $8,456,461 (B26), which equals the sale to the third party in C26.

One should note that the final valuation also equals Table 14.2A, C26, because the sale to the C corporation hasn't changed. What has changed in this table is that instead of assuming a percentage being sold to the ESOP, we have calculated the breakeven percentage for the ESOP, so it equals the third-party sale as an S corporation.

## Sensitivity Analysis: Analyzing the Mathematics

Rather than go through the torture of calculating the partial derivatives of equation (14.22), instead let's go for a more intuitive approach and evaluate how changes in the assumptions will impact the calculation of $p^*$, the breakeven percentage. An increase in the corporate tax rate ($t$), the control premium (CP) (since it applies to the entire S corporation sale, but only part of the C corporation sale), the lifetime ESOP costs ($e$), and the S corporation premium ($s$) would tend to favor the S corporation alternative and drive up $p^*$; of course, a decrease in any of those factors would decrease $p^*$. An increase in either the federal or state personal capital gains tax rates ($t_{pcg\ Fed}$ or $t_{pcg\ State}$) or the ESOP premium (EP) would favor the ESOP and drive down $p^*$, while decreases in those factors would increase $p^*$.

**TABLE 14.2B: THREE SENSITIVITY ANALYSIS TABLES** Table 14.2B has three different sensitivity analysis tables. Each of them provides the breakeven percentage sold to the ESOP with changes in different assumptions. Let's take them in order.

The first table shows how $p^*$ varies with changes in the state tax rate and the ESOP premium. The breakeven percentage is clearly much more sensitive to changes in the state tax rate than it is to the ESOP premium. The differences between row 65 and row 56 are approximately 17% to 22% in response to a 9% absolute change in state tax rates, while the differences between column B and column E are only 1% to 7% in response to a 15% absolute change in ESOP premium. Note that E64, shown in bold, is the base case of $p^* = 66.18112\%$, which also equals B29.

The second table shows how $p^*$ varies in response to changes in the S corporation premium and DLOM in the sale to a third party. About half of the cells show "#NUM!," which means that the ESOP dominates the sale to the third party at any percentage sold,[17] and there is no real solution to equation (14.22).[18] Note

[17] Or at least for any control sale, which is a basic assumption of this model. For brevity, we did not choose to model minority sales, which would require a modification of the mathematics.

[18] This is because the expression in the square root is negative, leading to complex solutions that have no interest to us.

that the #NUM!s appear more as we move southeast in the table. The ESOP sale dominates in all cases where the S corporation premium is zero (row 69), and more generally, when the S corporation premium is low and the DLOM to the third party is high—both of which would make the S corporation sale relatively less attractive. Row 74 is anomalous, as B74 through D74 are greater than 100%, which means that at a 54% ESOP premium—the theoretical maximum tax benefit of the S corporation status, according to Denis and Sarin (2002)—it would require a sale of more than 100% of the stock of the firm to the ESOP to reach a breakeven point.[19] That just means that the S corporation scenario dominates at that extreme assumption; even then, if there is a counterbalancing assumption such as a very high DLOM to the third party of 25%, then the owner is indifferent between the two scenarios if he or she sells 99.88% (E74) of the stock. Note that C72, in bold, is the base case and equals B29.

The third table shows how $p^*$ varies with changes in the lifetime ESOP costs ($e$) and the marginal corporate income tax rate ($t$). The breakeven percentage is very insensitive to changes in the lifetime ESOP costs, while it is moderately sensitive to changes in the corporate income tax rate. You can see that by noting that row 83 minus row 78 is only 1% to 3%, while column B minus column E is 8% to 11%. Note that E79 is the base case and equals B29.

## Section 5: Conclusion

The economics of a sale to an ESOP are complex. ESOPs can be very advantageous for a number of reasons—both financial and nonfinancial. The tables presented in this chapter are fairly realistic; however, real life can be even more complicated than these tables, and it is critical to understand how changes in circumstances can necessitate a change in assumptions or even a change in the model itself.

In particular, while not discussed in the chapter previously, one must be cautious about the behavior of this model in extreme situations. As $p$ approaches 100%, it is quite possible the company may not be able to use the full present value of the tax shield on the ESOP loan without additional capital. Also, the discount rate for the firm post-transaction increases as it becomes largely or completely debt financed. Thus, one must use this model with appropriate caution.

Finally, in this model we assumed a control sale in the first transaction. If the client is instead considering a minority interest sale or a multipart control sale by contract, then the model must be modified accordingly, as the current adjustments for control are incorrect for those different structures.

## References

Abrams, Jay B. 2005. *How to Value Your Business and Increase Its Potential*. New York: McGraw-Hill.

Abrams, Jay B. 2010. (Expected). *Valuation of S Corporations and Non-Tax Entities*. Hoboken, NJ: John Wiley & Sons.

[19]There is a rumor that Bernard Madoff is brokering such sales from his jail cell.

Denis, David J., and Atulya Sarin. 2002. Taxes and the Relative Valuation of S Corporations and C Corporations. *Journal of Applied Finance* (Fall/Winter): 7–16.

DiGabriele, James A. 2007. The Moderating Effects of Acquisition Premiums in Private Corporations: An Empirical Investigation of Relative S Corporation and C Corporation Valuations. http://ssrn.com/abstract= 989759. Reported in *Business Valuation Review* (Fall 2007): 85–86.

Erickson, Merle M., and Shiing-wu Wang. 2007. Tax Benefits as a Source of Merger Premiums in Acquisitions of Private Corporations. *Accounting Review* 82(2): 359–387.

Kruse, Douglas, and Joseph R. Blasi. 2000. Rutgers Study.[20]

Mattson, Michael J., Donald S. Shannon, and David E. Upton. 2002. Empirical Research Concludes S Corporation Values Same as C Corporations. *Business Valuation Update* 8, nos. 11 and 12 (November/December): 1.

[20]I do not have the proper full cite for this article. It is my recollection that the results of their article cited in this chapter appeared in an article by the National Center for Employee Ownership, but I am not certain of that.

CHAPTER 15

# Buyouts of Partners and Shareholders

# CHAPTER 15

# Buyouts of Partners and Shareholders

## Introduction

Buying out a partner[1] is intellectually related to the problem of measuring dilution in employee stock ownership plans (ESOPs), which is covered in Chapter 13. In the first edition of this book we took the approach of adapting the ESOP dilution formulas to a partner buyout. In this edition we take a simpler approach, even though we still borrow from the concepts in Chapter 13.

## Table 15.1: Pre- and Post-Transaction Valuations

Suppose you have already valued the drapery manufacturer owned by the Roth family, the Drapes of Roth.[2] There are four partners, each with a 25% share of the business: I. M. Roth, U. R. Roth, Izzy Roth, and B. Roth. The issue we want to explore is the impact on the post-transaction FMV if the three other Roths become wroth with Izzy Roth and want to buy him out. At what price should they buy?

Let's assume forecast cash flow next year is $200,000 (B13), $r = 26\%$ (B5), and forecast $g = 4\%$ (B12). Using a midyear Gordon model multiple (GMM), the GMM = 5.102 (B6), and therefore its FMV on a marketable minority interest basis is $1,020,452 (B7) pre-buyout.[3] (See Table 15.1.)

The solution to the valuation problem first depends on whether the three Roths have enough money to buy out Izzy with their personal assets. If so, then there is no impact on the value of the firm. If not, then the firm typically will take out a loan to buy out Izzy,[4] and immediately after the transaction The Drapes of Roth will

[1]There is no substantive difference in the post-transaction effects of buying out partners versus shareholders or members, so for ease of exposition we will use the term *partners* to cover all situations.

[2]I've waited for 20 years to use this pun. The least you can do is come up with a snicker or a half-hearted chortle.

[3]For simplicity we ignore adjustments for control and marketability, although they would be important in an actual transaction.

[4]It is possible for the partners to take out the loan individually and the firm would pay it indirectly by bonusing out sufficiently large salaries to cover the personal loans above and beyond their normal draw. This has no impact on the solution, as both the direct and indirect approaches will come to the same result.

| | A | B | C | D | E | F |
|---|---|---|---|---|---|---|
| 1 | **Table 15.1** | | | | | |
| 2 | **Pre- and Post-Transaction Valuations** | | | | | |
| 3 | | | | | | |
| 4 | | **Pre-Trans** | **Post-Transaction [1]** | | | |
| 5 | ***r*** | **26%** | **27%** | **28%** | **29%** | **30%** |
| 6 | GMM—Midyear | 5.102 | 4.900 | 4.714 | 4.543 | 4.385 |
| 7 | FMV | **1,020,452** | **979,950** | **942,809** | **908,625** | **877,058** |
| 8 | Loss in FMV = 1 – [(7)/(B7)] | **0.0%** | **4.0%** | **7.6%** | **11.0%** | **14.1%** |
| 9 | | | | | | |
| 10 | | | | | | |
| 11 | **Assumptions** | | | | | |
| 12 | Growth Rate = $g$ | 4% | | | | |
| 13 | $CF_{t+1}$ | 200,000 | | | | |
| 14 | Percentage Bought = $p$ | 25% | | | | |
| 15 | | | | | | |
| 16 | [1] The discount rate increases for two reasons: The firm is smaller now, which increases the | | | | | |
| 17 | discount rate through the log size effect, and financial leverage has increased. | | | | | |

have a loan payable that did not exist before the transaction. Note that this is the same concept as the effect of a loan in a sale to an ESOP, and the analysis will be similar, although simpler. We will discover that if the buyers pay the pre-transaction price they will experience dilution in their remaining value. Only if they pay the post-transaction price will their dilution equal zero; however, then the seller will experience dilution.

The immediate effect of the buyout of Izzy's stock by the company is that it lowers the company's FMV by the amount of the loan. However, it also reduces the number of shares, and it therefore does not necessarily reduce the FMV per share. We will discover these effects more precisely in our analysis of Tables 15.2 and 15.3.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Table 15.2** | | | |
| 2 | **Dilution in FMV as a Result of the Partner Buyout** | | | |
| 3 | | | | |
| 4 | Scenario [1] | 1 | 2 | 3 |
| 5 | **Pre-Transaction Discount Rate** | **26%** | **28%** | **26%** |
| 6 | Pre-Trans GMM—Midyear = SQRT(1 + $r$)/($r$ – $g$) | 5.102 | 4.714 | 5.102 |
| 7 | FMV Pre-Transaction = $CF_{t+1}$ × (6). | **1,020,452** | **942,809** | **1,020,452** |
| 8 | # Shares | 1,000,000 | 1,000,000 | 1,000,000 |
| 9 | FMV/Share for Sale = (7) / (8) | **$ 1.020** | **$ 0.943** | **$ 1.020** |
| 10 | # Shares Bought | 250,000 | 250,000 | 250,000 |
| 11 | Pymt = Loan = (9) × (10) | **255,113** | **235,702** | **255,113** |
| 12 | **Post-Transaction Discount Rate** | **26%** | **28%** | **28%** |
| 13 | FMV Post-Tx—Before Loan: B12 = B7, C12 = C7, D12 = C7 | **1,020,452** | **942,809** | **942,809** |
| 14 | FMV Post-Trans—After Loan = (13) – (11) | **765,339** | **707,107** | **687,696** |
| 15 | # Shares Post-Trans | 750,000 | 750,000 | 750,000 |
| 16 | FMV/Share Post-Trans—After Loan = (14)/(15) | **$ 1.020** | **$ 0.943** | **$ 0.917** |
| 17 | Loss in FMV/Share = (9) – (16) | **$ -** | **$ -** | **$ 0.104** |
| 18 | Loss in FMV = (15) × (17) | **$ -** | **$ -** | **$ 77,643** |
| 19 | | | | |
| 20 | **Assumptions** | | | |
| 21 | Growth Rate = $g$ | 4% | | |
| 22 | $CF_{t+1}$ | 200,000 | | |
| 23 | Percentage Bought = $p$ | 25% | | |
| 24 | | | | |
| 25 | [1] In Scenario 1 we assume the discount rate is 26% pre- and post-transaction. In Scenario 2 we | | | |
| 26 | assume the discount rate is 28% pre- and post-transaction. In Scenario 3 we assume the discount | | | |
| 27 | rate is 26% pre-transaction and 28% post-transaction. We then calculate the dilution in value to the | | | |
| 28 | remaining shareholders who buy out Izzy Roth at the pre-transaction FMV instead of the post- | | | |
| 29 | transaction FMV. | | | |

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Table 15.3** | | | |
| 2 | **Sharing the Dilution in FMV per Share** | | | |
| 3 | | | | |
| 4 | Scenario [1] | 1 | 2 | 3 |
| 5 | **Pre-Transaction Discount Rate** | **26%** | **28%** | **Avg** |
| 6 | Pre-Trans GMM—Midyear = SQRT(1 + $r$)/($r-g$). D = Avg(B,C) | 5.102 | 4.714 | **4.908** |
| 7 | FMV Pre-Transaction = $CF_{t+1}$ × (6). D = Avg(B,C) | **1,020,452** | **942,809** | **981,631** |
| 8 | # Shares | 1,000,000 | 1,000,000 | 1,000,000 |
| 9 | FMV/Share for Sale = (7) / (8). D = Avg(B,C) | **$ 1.020** | **$ 0.943** | **$ 0.982** |
| 10 | # Shares Bought | 250,000 | 250,000 | 250,000 |
| 11 | Pymt = Loan = (9) × (10) | **255,113** | **235,702** | **245,408** |
| 12 | **Post-Transaction Discount Rate** | **26%** | **28%** | **NA** |
| 13 | FMV Post-Tx—Before Loan: B12 = B7, C12 = C7, D12 = Avg(B,C) | **1,020,452** | **942,809** | **981,631** |
| 14 | FMV Post-Trans—After Loan = (13) – (11) | **765,339** | **707,107** | **736,223** |
| 15 | # Shares Post-Trans | 750,000 | 750,000 | 750,000 |
| 16 | FMV/Share Post-Trans—After Loan = (14)/(15) | **$ 1.020** | **$ 0.943** | **$ 0.982** |
| 17 | | | | |
| 18 | **Summary of Results** | **FMVs** | **Differences** | **Dilution To** |
| 19 | Pre-Transaction FMV/Share (B16) | $ 1.020 | | |
| 20 | FMV/Share Paid (D16) | $ 0.982 | **$ 0.039** | **<----Seller** |
| 21 | Post-Transaction FMV/Share (C16) | $ 0.943 | **$ 0.039** | **<----Buyer** |
| 22 | | | | |
| 23 | **Assumptions** | | | |
| 24 | Growth Rate = $g$ | 4% | | |
| 25 | $CF_{t+1}$ | 200,000 | | |
| 26 | Percentage Bought = $p$ | 25% | | |
| 27 | | | | |
| 28 | [1] In Scenario 1 we assume the discount rate is 26% pre- and post-transaction. In Scenario 2 we assume the | | | |
| 29 | discount rate is 28% pre- and post-transaction. In Scenario 3 we assume the discount rate is 26% pre- | | | |
| 30 | transaction and 28% post-transaction. We then calculate the average GMM, FMV, FMV/Share, and dilution | | | |
| 31 | to buyer and seller. Cells B5 through C16 are identical to Table 15.2. | | | |

In the meantime, however, we also know that there is a log size effect of raising the discount rate for the lower value of the firm. Additionally there is more financial risk, and you may consider increasing the discount rate for the financial leverage.[5]

The obvious question of what is the most appropriate post-transaction discount rate and valuation is not the one on which we will focus, as that is a risk assessment for the appraiser and is not particularly mysterious or difficult.

Columns C through F show the post-transaction valuation of the Drapes of Roth at discount rates of 27% to 30% (C5 to F5). The valuation decreases as we move to the right in row 7. Row 8 shows the post-transaction decline in FMV compared to the pre-transaction FMV. If the correct post-transaction discount rate is 27% (C5), then the correct FMV is $979,950 (C7), which is 4.0% (C8) lower than the pre-transaction FMV of $1,020,452 (B7). As our assumption of the correct post-transaction discount rate increases to 30% (F5), the loss in FMV increases to 14.1% (F8).

For simplicity of discussion, we ignore the subtleties of differences in the discounts for lack of control and marketability of 25% versus $33\frac{1}{3}$% interests, although in actuality the appraiser must consider those issues.

## Table 15.2: Dilution in FMV as a Result of the Partner Buyout

There are three columns in Table 15.2. Column B shows the pre-transaction valuation and the post-transaction results assuming there is no increase necessary in the

[5] In the context of the capital asset pricing model, the stock beta rises with additional financial leverage.

discount rate, that is, that it remains at 26% (B12 = B5). Column C shows the results of buying out Izzy Roth at the post-transaction valuation under the more realistic assumption that the post-transaction discount rate rises to 28% (C6), and the company pays him off at a valuation with that discount rate; that is, it computes the pre-transaction valuation using the higher, post-transaction, discount rate. Column D is a calculation of the dilution in value that results from paying Izzy Roth the pre-transaction FMV per share, but with the appropriate post-transaction discount rate being 28%.

## Scenario 1: No Increase in the Post-Transaction Discount Rate

B5 through B7 are identical to those in Table 15.1. We assume there are 1 million shares outstanding (B8), which results in an FMV per share of $1.020 (B9). The company buys Izzy's 250,000 (B10) shares at $1.020 and pays him $255,113 (B11), which we assume is financed by taking a bank loan.[6] Alternatively, we can view this case as being one in which the buyers pay the purchase price from their own money, and thus no dilution occurs.

We assume that the discount rate does not change post-transaction. Thus, it remains at 26% (B12 = B5). The post-transaction FMV before the loan,[7] $1,020,452 (B13), is thus equal to pre-transaction FMV in B7. The post-transaction FMV after the loan equals $765,339 (B14 = B13 − B11).

There are 750,000 (B15) shares post-transaction, which results in a post-transaction FMV of $1.020 (B16 = B9) per share. In other words if we make the assumption that the transaction does not affect the risk of the firm—and, hence, the discount rate—then the transaction has no impact on the FMV per share and causes no dilution (B17, B18) to the remaining partners. This is appropriate when the buyers can pay the seller from their own money rather than the company paying. Since the company paid the same price as it is worth after the transaction, dilution in value is zero (B18).

## Scenario 2: The Post-Transaction Discount Rate Increases to 28%

The analysis in column C is appropriate when the buyout does increase the discount rate, which is the more normal case. The larger the proportion ($p$) sold, the larger the impact on the post-transaction discount, as we found in Chapter 13. The company pays the seller at the post-transaction FMV. In this analysis we assume a 2% premium is appropriate, resulting in a 28% (C6) discount rate. Because the pre-transaction valuation is computed using the post-transaction discount rate, there is no dilution (C18), just as was the case for Scenario 1.

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[6]This assumption is not critical to the analysis. Most businesses do not have a quarter of their value sitting in cash, with plenty left to pay its normal expenses, but even if one does, it still has a log size effect and still probably will cause the need for the firm to borrow.

[7]By post-transaction FMV before the loan we mean the FMV of the company at the discount rate that we apply to the company post-transaction, but before accounting for the reduced FMV due to the loan.

### Scenario 3: Buy at the Pre-Transaction FMV and Measure the Dilution

In column D we quantify the dilution in FMV when the company pays the pre-transaction FMV and the post-transaction discount rate is higher than the pre-transaction discount rate. Thus, column D is a mixture of the valuations in columns B and C.

D5 through D11 contain the same amounts as the corresponding B5 through B11, because we are valuing the company using a 26% (B5, D5) discount rate. We show the pre-transaction FMV of $942,809 from C7 in D12, because that is the same as the post-transaction FMV before the loan computed at the 28% (D12) discount rate. However, the company pays the pre-transaction FMV per share of $1.02 (D9 = B9) per share, or $255,113 (D11). We subtract the $255,113 payment from the pre-loan post-transaction FMV of $942,809 (D13) to calculate the post-loan post-transaction FMV of the firm of $687,696 (D14). We divide that by 750,000 (D15) shares to calculate the post-transaction FMV of $0.917 (D16) per share, which is a loss (i.e., dilution) in value of $0.104 (D9 − D16 = D17) per share, or $77,643 (D18).

This result came about because the company paid $1.02 per share—as if the transaction would not create additional risk—but its post-loan post-transaction value is now $0.917 per share—lower than the $0.943 (C16) per share it would be if the company had paid the post-loan post-transaction value of $0.943 per share. In other words, the buyers bore the dilution in value of the sale instead of allowing it to pass to the seller. They committed the unforgivable sin of paying retail when it was only worth wholesale.

## Sharing the Dilution

Another possible transaction structure is for buyer and seller to share the dilution. Since we have structured our analysis of the effects of the transaction in terms of an increase in the discount rate post-transaction, the effects are nonlinear. Thus we cannot make use of the formulas of Chapter 13 for this type of analysis. However, it would be simple for buyer and seller to agree to an intermediate price of, let's say, approximately $0.98 per share, which is roughly halfway between B9 and C9.

The seller could object that, if the brothers allow him to sell to an outsider, he could sell for $1.02 (B9) per share and not impact the FMV of the firm. Indeed, he would be correct if there are not restrictions on transfer that would knock the FMV back down, which we have conveniently assumed away for simplicity. More precisely, the diluted values of $0.943 (C16) and $0.917 (D16) per share are not really FMV. They are value to the holders after the transaction. In reality our use of the term *FMV* in this chapter is questionable. However, this is not a treatise on FMV, and we use the familiar term for convenience.

### Definitions

$CF$ = forecast cash flow in year 1.

$d$ = per share dilution. There is dilution to the seller ($d_s$) and the buyers ($d_b$).

$g$ = growth rate of cash flows.

$FMV$ = fair market value. We use $FMV_1$ for pre-transaction and $FMV_2$ for post-transaction, where the difference is caused by different pre- and post-transaction discount rates.

$GM$ = midyear Gordon model multiple $= \frac{\sqrt{1+r}}{r-g}$. We use $GM_1$ for pre-transaction, $GM_2$ for post-transaction, and $GM_{Avg}$ for the average of the two, where the difference in the first two is caused by different pre- and post-transaction discount rates.

$p$ = percentage of the firm owned by the seller and being sold, assumed to be $1/4$ in this example.

$r$ = discount rate.

$Sh$ = number of shares.

$SP$ = selling price. We use $SP_1$ for the pre-transaction selling price and $SP$ for the actual selling price.

$x$ = per share payment to seller.

## Mathematics

Ignoring adjustments for control and marketability, the pre-transaction FMV of the firm is forecast cash flow times the Gordon model multiple.

$$FMV_1 = CF\, GM_1. \tag{15.1}$$

On a per share basis, this is:

$$FMV_1/Sh = \frac{CF\, GM_1}{Sh}. \tag{15.1a}$$

The seller owns $p = 1/4$ of the company. The pre-transaction selling price should be:

$$SP_1 = pCF\, GM_1. \tag{15.2}$$

The actual per share selling price, that is, payment to the seller, is $x$.

$$\frac{SP}{Sh} = x. \tag{15.3}$$

The dilution to the seller is the pre-transaction price per share in equation (15.1a) minus the actual payment per share in equation (15.3), or:

$$d_s = \frac{CF\, GM_1}{Sh} - x. \tag{15.4}$$

The post-transaction value per share remaining to the buyers is:[8]

$$FMV_2/Sh = \frac{(1-p)CF\, GM_2}{(1-p)Sh} = \frac{CF\, GM_2}{Sh}. \tag{15.5}$$

The dilution in per-share value to the buyers is the per-share payment, $x$, minus the post-transaction value per share, or:

$$d_b = x - \frac{CF\, GM_2}{Sh}. \tag{15.6}$$

[8]Note that after the transaction, there are only $(1-p)$ $Sh$ shares left, as we retired $p$.

It is a reasonable goal to make the buyers' and seller's dilution equal. To accomplish this, we set the right-hand sides of equations (15.4) and (15.6) equal to each other.

$$\frac{CF\,GM_1}{Sh} - x = x - \frac{CF\,GM_2}{Sh}. \qquad (15.7)$$

Simplifying, we get:

$$x = \frac{CF(GM_1 + GM_2)/2}{Sh} \quad \text{Payment to the seller that equalizes dilution.} \qquad (15.7a)$$

Note that $(GM_1 + GM_2)/2$ equals the average pre- and post-transaction Gordon model multiple. An alternative expression for the correct payment is:

$$x = \frac{CF}{Sh} \times GM_{Avg} \quad \text{Alternative expression.} \qquad (15.7b)$$

An important observation is that we must use the average Gordon model multiple (GMM) to calculate the payment that shares dilution equally between buyer and seller. We cannot simply use an average of pre- and post-transaction discount rates to calculate the GMM, because it is nonlinear in the discount rate (and the growth rate). We must decide on the appropriate post-transaction discount rate, calculate the GMM, and then average the pre- and post-transaction GMMs in our valuation to achieve sharing the dilution.

## Table 15.3: Sharing the Dilution in FMV per Share

We verify equations (15.7a) and (15.7b) in Table 15.3. B5 through C16 are identical to Table 15.2. Let's suppose that we decide that the post-transaction discount rate is 28%. We calculate the average GMM of 4.908 (D6 = average of B6 and C6). Multiplying that by our forecast cash flow of \$200,000 (B25) leads to a valuation of \$981,631 (D7). D7 is also the average of B7 and C7. We calculate the average FMV per share of \$0.982 (D9 = average of B9 and C9). Additionally, D9 = D7/D8. This is the price that we calculate for the Roth family to pay Izzy.

We multiply the average per-share price by Izzy's 250,000 shares (D10) to compute the buyout price of \$245,408 (D11 = D9 × D10). The post-transaction FMV before the loan in row 13 is a repeat of row 7. We then subtract the payment (which we assume the Roths finance with a loan) in row 11 to calculate the post-transaction FMV after the loan in row 14. There are two independent ways we can calculate the \$736,223 in D14—as D13 − D11 and as the average of B14 and C14. We divide by 750,000 (D15) shares post-transaction to calculate the FMV per share as \$0.982 (D16 = D14/D15 and also equals the average of B16 and C16).

We show a summary of results in B19 through B21. The pre-transaction FMV is \$1.020 (B19 = B16). Instead of paying the pre-transaction FMV, however, the Roths share the dilution and pay \$0.982 per share (B20 = D16). The difference of those two per-share FMVs is \$0.039 (C20 = B19 − B20) per share. This difference is the dilution to the seller, because he is taking a lower price than the pre-transaction FMV. The post-transaction FMV is \$0.943 (B21 = C16) per share. This is also a difference of \$0.039 (C21 = B20 − B21) per share and is the buyer's dilution, as the buyer paid \$0.039 per share more for the stock than it is worth post-transaction.

Both seller's and buyer's dilution being equal to $0.039 per share demonstrates the accuracy of equations (15.7a) and (15.7b).

### Effects on the Post-Transaction Discount Rate

The act of sharing the dilution between buyer and seller can affect the appropriate post-transaction discount rate. For example, in Table 15.3, Scenario 2, we assume a 28% (C5) discount rate. However, that was our assumption of the appropriate discount rate when the seller experiences all of the dilution, as the buyers pay $0.943 (C9) per share instead of $1.020 (B9). This reduces risk to the buyers. Thus it would be reasonable instead to consider using a 27% discount rate instead of 28% in C5 (and C12), and use the new result in D6.[9]

## Conclusion

In this chapter we present a simple model to handle the dilution that arises in a buyout when the company rather than the individual partners buy back stock from the seller. It draws upon the work from Chapter 13, but it is based on a different concept, which is that we can consider the impact of creating debt after the transaction by raising the post-transaction discount rate. We also developed a simple pair of formulas in equations (15.7a) and (15.7b) to share the dilution between buyer and seller.

[9]C9 would be $0.980 per share and D9 would be $1.000 per share. However, we do not show this in the spreadsheets.

# PART VII

# Probabilistic Methods

## Introduction

Part VII, which consists of Chapters 16 through 18, deals with probabilistic valuation methods.

## Chapter 16: Valuing Start-Ups

Chapter 16 covers the topic of valuing start-ups. The chapter discusses three topics. The first topic is the “First Chicago” method, which is a weighted average, multi-scenario approach to valuing start-ups. It has the benefit of breaking down the vast range of possibilities into discrete scenarios that are more credible than attempting to model all possibilities in a single scenario. Whereas almost this entire book is my own original work, the First Chicago method is based on a series of articles by Brad Fowler. It is important to understand the multi-scenario approach, not only for its own sake in valuing simple start-ups, but also as a preparation to understand the much more complex decision tree approach in the debt restructuring study.

Chapter 16 also provides an example—again based on Fowler’s work—of using a venture capital valuation approach. While this is technically a different valuation approach, we will consider it as essentially the same topic as the First Chicago approach.

The second topic in Chapter 16 is the presentation of simplification of the essential parts of an actual debt restructuring study that I performed for a client. It is an example of using an original adaptation of decision tree logic for incorporating the effects of probabilistic milestones into a spreadsheet for the valuation. In this study, the viability of the subject company, the probability of obtaining venture capital financing, its ability to survive on its own without venture capital financing, and its value depend on the outcome of four different sales milestones. The logic and structure of this analysis work well for other types of milestones such as technological (e.g., successful development), administrative (e.g., obtaining U.S. Food and Drug Administration approval), and the like.

The third topic in Chapter 16 is presenting an exponentially declining sales growth model[1] to semi-automate the process of modeling different sales growth patterns. This is a great time saver in valuing start-ups using a "top-down" approach.[2] Typically sales grow rapidly in the early years then more slowly, eventually coming to an expected constant growth rate. Rather than manually insert every year's sales growth, the appraiser can instantly change the entire sales growth pattern over $n$ years by changing the contents of four spreadsheet cells. Furthermore, it makes extensive sensitivity analysis, normally a cumbersome procedure, trivial.

## Chapters 17 and 18: Monte Carlo Simulation and Real Options

As we discussed in the introduction to the book, I invited Dr. Johnathan Mun, author of Wiley books *Modeling Risk* and *Real Options Analysis*, in addition to many other books, to write Chapters 17 and 18. They are introductions to these two topics and to Dr. Mun's software. We intend to cover practical examples of using MCS and RO on our Web site, www.abramsvaluation.com ("Books," "Quantitative Business Valuation"), and eventually in the workbook, which we expect to publish later with the third edition of this book. I encourage readers who want to develop a deep understanding of each topic to buy Dr. Mun's books and software, read the material we post on our Web site, and buy the workbook when it eventually publishes. It is simply impossible to cover these complex topics in one chapter each.

MCS in business valuation involves assigning probability distributions to certain valuation assumptions, allowing the computer to simulate the valuation thousands of times using its random number generator in accordance with the probability distribution, and tabulating the valuation results. Thus, instead of being a fixed number, value becomes a probability distribution. It is a more realistic approach to valuation, as no valuation practitioner knows any components of the valuation for sure, because we are attempting to model the future, which is never deterministic.

MCS is not only more realistic, but it also can greatly simplify the valuation of early stage firms or firms with a wide range of outcomes. Suppose a company has 10 business segments, each its own profit center. To model this with a First Chicago method would require as many as 10 segments × 4 levels of optimism per segment × 1 DCF per segment = 40 DCF analyses. That's a huge amount of work. Using MCS, we would have 10 DCFs instead of 40. MCS is a practical tool for sophisticated users and clients and a great time saver in complex valuations.

Real option valuation is more sophisticated yet. Its main use for practitioners is likely to be in valuing financial options, as not many business appraisers are sufficiently well versed in real options to use it to value businesses. However, for those who are, it is a very powerful tool.

I have three books on the topic of real option valuation. One is for academics only—way beyond practitioner level—one is below practitioner level and is a sales pitch to hire the authors. I feel a bit like Goldilocks when I say that only Dr. Mun's

[1]I thank R. K. Hiatt for developing this.

[2]This is in contrast to the "bottom up" approach, where the appraiser inserts a series of assumptions to enable one to forecast sales. For example, this might include line items such as market size, market share for the subject company, etc.

book, *Real Options Analysis*, is just right. It presents the necessary mathematical and statistical theory in a clear, well-written fashion and has many practical examples. It is *the* book for practitioners to understand RO.

I look forward to presenting applications of MCS and RO on our Web site and eventually in the workbook.

# CHAPTER 16

# Valuing Start-Ups

# CHAPTER 16

# Valuing Start-Ups

## Issues Unique to Start-Ups

A number of issues fairly unique to valuing start-ups arise chiefly from the uncertainty associated with new ventures. This uncertainty usually necessitates a more complex, multiple scenario analysis known as the *First Chicago approach* and requires more creativity on the part of the appraiser than other, more routine assignments.[1] In this chapter we also present a much shorter, easier valuation method for start-ups known as the *venture capital pricing approach*.

Many new ventures have sequential events ("milestones") that may or may not occur, and the valuation depends on the probabilities of the occurrence of these milestones. Often, in order for event $n$ to occur, event $(n - 1)$ must occur—but it may or may not. When valuing such firms, we often combine the First Chicago approach with *decision tree analysis* to arrive at a credible fair market value. This is a much more complex task than the First Chicago approach by itself. The most common types of milestones are sales, financing, technical, and regulatory, the latter two being universal in the valuation of pharmaceutical and biotechnology firms.

Another issue is that start-ups typically have a pattern of rapid sales growth followed by declining sales growth rates, finally reaching some steady-state growth rate. Performing sensitivity analysis can be cumbersome when the appraiser manually enters sales growth rates under a number of different scenarios.

## Organization of the Chapter

This chapter addresses these issues in three parts. Part 1 consists of the First Chicago approach of forecasting multiple scenarios, each with its own discounted cash flow analysis. We produce a conditional FMV for each scenario and then calculate a weighted average FMV based on VC industry research that specifies the probabilities of each scenario coming to fruition. We also include the venture capital pricing approach in Part 1, as it is short and simple.

Part 2 consists of using a very sophisticated decision tree analysis to value an early-stage firm for the purpose of deciding whether to restructure its debt (the "debt

[1]Two more sophisticated approaches are using *Monte Carlo simulation* and *real options*, which are excellent solutions but beyond the scope of this chapter.

restructuring study"). The success or failure of the firm depends on the outcome of a sequence of four events, which will impact the decision. This came from an actual valuation assignment.

Part 3 consists of a mathematical technique to streamline the process of forecasting sales for a start-up. We call the technique the *exponentially declining sales growth model*. This model enables the user to generate a realistic, exponentially declining sales pattern over the life of the product/service with ease and greatly simplifies and facilitates sensitivity analysis, as it eliminates or at least greatly reduces the need to manually insert sales growth percentages in spreadsheets.

## Part 1: First Chicago Approach

Start-ups are much riskier ventures than mature businesses. Because of a lack of sales history and often a lack of market information, a number of widely varying scenarios are plausible, and the range of outcomes is much wider and more unpredictable than that of mature businesses.

In a discounted cash flow (DCF) analysis, the forecast cash flows are supposed to be the weighted average cash flows, with the appraiser having considered the full range of possible outcomes. However, it is difficult to do this with such a wide range of possible outcomes. Instead, typically the appraiser, investment banker, or venture capitalist uses the usually optimistic forecast of the client—perhaps downplayed somewhat—and discounts that to present value at a very high rate, around 50% to 75%.

Thus, a more traditional single-scenario DCF analysis to calculate fair market value is not only more difficult to perform, but is also far more subject to criticism by parties with different interests. Short of using Monte Carlo simulation—a complex approach that we cover in Chapter 17—it is virtually impossible to accurately portray the cash flows in a single scenario. Instead, the best solution is to use a *multiple-scenario approach* known as the First Chicago approach. I name the typical scenarios: *very optimistic* (the "grand slam home run"), *optimistic* (the "home run"), *conservative* (the "single"), and *pessimistic* (the "strikeout").

According to James Plummer (Plummer, 1987), Stanley C. Golder (Golder, 1986) was the originator of the First Chicago approach, named after First Chicago Ventures, a spinoff of First Chicago Bank's Equity Group. In 1980, he founded the venture firm Golder, Thoma, and Cressey. James Plummer actually gave the name to the First Chicago approach. In valuation journals, Bradley Fowler wrote the original literature on the First Chicago approach (Fowler, 1989, 1990, 1996).

### Discounting Cash Flow Is Preferable to Net Income

While discounting forecast cash flow is always preferable to discounting forecast net income, it is even more important to use cash flow in valuing start-ups than it is in mature firms. This is because cash is far more likely to run out in a start-up than in a mature firm. When that happens, the firm is forced either to take on new investment, which dilutes existing shareholders' ownership in the company, or to go out of business. In both cases, using a discounted future net income approach will lead to a serious overvaluation.

## Capital Structure Changes

Start-ups tend to have somewhat frequent changes in capital structure. Investment often occurs in several tranches, usually with new rounds of convertible preferred stock, often with different terms than previous rounds of preferred stock. This complicates the value calculations, because one must be very careful about whose equity one is measuring. Each round of investment dilutes existing equity, and it is easy to measure the wrong equity portion if one is not careful.

## Venture Capital Rates of Return

*Venture capitalists* (VCs) price companies by determining the present value of forecast cash flow. One method of valuation is to discount an optimistic forecast of FMV at the required rate of return. Required rates of return for VC vary directly with the stage of the company, with start-ups being the riskiest, hence requiring rates of return of 50% to 75% (Plummer, 1987).

Fowler (1990) cites a survey published by Venture Economics covering 200 companies that indicated that 40% of VC investments lost money, 30% proceeded sideways or were classified as "the living dead," 20% returned 2 to 5 times invested capital, 8% returned 5 to 10 times, and 2% returned greater than 10 times the investment. In a follow-up article (Fowler, 1996), he refers to comments made by Professor Stewart Myers of MIT in his November 1995 address to the American Society of Appraisers confirming that 70% to 80% of VC investments are failures, whereas 20% to 30% are big winners. In addition, Professor Myers observed that the overall internal rate of return (IRR) for successful VC partnerships was approximately 25%.[2] Note that the IRR is a geometric average return.

The 25% rate of return is consistent with a *Wall Street Journal* article (Pacelle, 1999) that cites Venture Economics as a source that venture capital firms returned an average 27.4% over the past 5 years, although they returned only 15.1% over the past 20 years. From this, we can calculate the first 15 years' (roughly 1979–1993) compound average return as 11.27%.[3] That is a very low return for VC firms. It is comparable to NYSE decile #1 firm long-run returns. I would attribute that low return to two factors:

1. That period was the infancy of the VC industry, and the early entrants faced a steep learning curve.
2. That period included two severe recessions.

It is not reasonable to expect VC investors to be happy with a 15% return long run. The five-year average of 27.4% is more in line with the risk undertaken.

[2]He also mentioned that the simple average VC project return was 1%. He said the difference in returns is due to the skewness in the distribution that comes from the venture capitalists quickly identifying and pulling the plug on the losers; that is, they do not continue to fund the bad projects. Thus, the bad projects have the least investment.

[3]The equation is: $(1 + r_{15})^{15}(1 + 0.274)^5 = (1 + 0.151)^{20}$, which solves to $r_{15} = 11.27\%$.

As to "batting averages," a reasonable synthesis of this information is that 2% of VC investments are grand slams, 8% are home runs, 20% are moderately successful, and 70% are worthless or close to it.

**RECENT EVIDENCE**[4] DeGennaro and Dwyer (DD) report on returns to angel investors and summarize significant prior literature on angel and VC investing. We will summarize their more interesting data and conclusions after we define these terms.[5]

An angel investor or angel (also known as a business angel or informal investor) is an affluent individual who provides capital for a business start-up, usually in exchange for convertible debt or ownership equity. A small but increasing number of angel investors organize themselves into angel groups or angel networks to share research and pool their investment capital. Angels typically invest their own funds, unlike venture capitalists, who manage the pooled money of others in a professionally-managed fund. VCs typically invest at a later stage than angels.

DD cites Wiltbank et al. (2009), who find that formal VCs invested less than 2% of the total capital in seed-stage companies during the "past 10 years," which presumably should be from 1999 to 2008. Wiltbank and Boeker (2007) report that according to the Angel Capital Education Foundation, about 10,000 accredited investors belong to 265 angel groups as of 2007. The primary focus of DD's article is analysis of the Angel Investor Performance Project (AIPP),[6] which produced the newest and most extensive database available on angel investments.

Cochrane (2005) reports on VC returns from 1987 through June 2000 and finds expected "proportional returns"[7] of 50% per year. He concludes that VC investments and the smallest NASDAQ stocks have roughly similar returns and volatilities during his sample period.

Wiltbank and Boeker (2007) report the AIPP database contains data from 86 angel groups with 539 total investors who made 3,097 investments, of which 1,137 achieved exits. There were only 603 useable investments, as many of them were missing data. Of the 603, 434 are exited investments and 169 are not.

Wiltbank and Boeker (2007) report an internal rate of return (IRR) of 27% per year, which is very similar to and just slightly higher than Stewart Myers' 25% for VC investments.

Following are some interesting AIPP statistics of angel investment:

- The mean and median investments are \$155,000 and \$49,000, respectively, which are small by VC standards. Cochrane (2005) reports a mean investment of \$6.7 million per tranche.
- The mean and median "cash-out" amounts are \$477,486 and \$40,833, respectively.
- The "base multiple" is the cash-out amount divided by the investment. The equally weighted mean and median base multiples for all investments—exited

[4]This section is entirely based on DeGennaro and Dwyer (2009).
[5]Definitions are from Wikipedia.
[6]Data at www.kauffman.org/aipp.
[7]I interpret this to mean returns that are weighted by the amount of investment as opposed to equally weighted returns across all investments.

and not—are 8.31 and 0, with a range from 0 to 1,333! For exited investments, the mean and median base multiples are 11.54 and 0.97. This means that the majority of angel investments fare poorly, but the winners are big winners.

- Almost one-third of the investments return nothing, and over one-half return no more than their investment. About 15% of the angel investments return at least five times their investment.
- The global multiple for the full sample of 603 projects—the sum of all cash inflows divided by the sum of all cash outflows—is only 2.24 for all investments and 2.64 for exited investments. This is a weighted-average-based multiple and is only about one-quarter of the 8.31 and 11.54 reported above for the equally weighted investments. The explanation for this difference is that there were a few small investments with enormous returns.
- The average and median exited project lasts 3.6 and 3.0 years.
- Here are some statistics about the angel investors:
  - The average angel investor has been making angel investments for 11.3 years.
  - Angel investors invest a mean and median 13.2% and 10% of their individual wealth in angel investments.
  - They spend a mean and median 65.5 and 15 hours in due diligence per investment.
- Of the 434 exits, 121 ended in failure, another company bought 188 of them, other investors bought 21, and 57—almost one-quarter—end in IPO.
- Estimated returns:
  - DD compute a 97% equally weighted IRR and a 33% value-weighted expected IRR and find these numbers similar to other reports. Note the latter number is an expected return, not an experienced return, although they may be similar.
  - DD warn that these returns are likely to be materially higher than returns for angel investors who are not part of organized groups.

We conclude that the subsequent research tends to support our earlier conclusions based on Professor Myers. He found VCs had a 25% geometric return, while DD find angels have a 33% IRR, which also is a geometric return. Thus the arithmetic return for angel investments would be higher—probably close to 40%. It is logical that angel investments, being earlier stage than VC, should have higher rates of return. We do not yet know the effect of the U.S. and world financial crisis on VC and angel returns. That remains to be seen.

## Table 16.1: Example of the First Chicago Approach

In Table 16.1, we use the percentages presented earlier for weighting the four different scenarios, very optimistic, optimistic, conservative, and pessimistic, respectively.

Initially, we perform discounted cash flow calculations to determine the conditional FMV of the subject company under the different scenarios. Typical venture capital rates of return include the discount for lack of marketability (DLOM) and discount for lack of control (DLOC). This tends to obscure the discount rate, DLOM, and DLOC. The appropriate discount rate using the First Chicago approach begins with the average success rate of approximately 25% reported by Professor Myers.

| | A | B | C | D |
|---|---|---|---|---|
| 1 | **Table 16.1** | | | |
| 2 | **First Chicago Method** | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | **Conditional FMV [1]** | **Probability [2]** | **Wtd FMV** |
| 6 | **Very Optimistic Scenario** | **$ 130,000,000** | **2%** | **$ 2,600,000** |
| 7 | **Optimistic Scenario** | **50,000,000** | **8%** | **4,000,000** |
| 8 | **Conservative Scenario** | **10,000,000** | **20%** | **2,000,000** |
| 9 | **Pessimistic Scenario** | **0** | **70%** | **-** |
| 10 | **Weighted Average FMV** | | **100%** | **$ 8,600,000** |
| 11 | | | | |
| 12 | [1] Individual discounted cash flow analyses are the source for the numbers in this column. | | | |
| 13 | | | | |
| 14 | [2] Based on the VC rates discussed in the chapter. | | | |

The 25%, however, is a geometric average rate of return. We should estimate an increment to add in order to estimate the arithmetic rate of return.[8] In Table 6.4, we show arithmetic and geometric mean rates of return from log size model regressions of the 1926–2007 NYSE/AMEX/NASDAQ data for different size firms.

For a firm of $1 million FMV, the regression forecast arithmetic and geometric returns, rounded to the nearest percent, are 26.4% and 15.6%, respectively, for a differential of 11% (rounded). For a firm of $25 million FMV, the regression forecast arithmetic and geometric returns, rounded to the nearest percent, are 21.8% and 13.9%, respectively, for a differential of 8%. For a $100 million FMV, the differential is 6.5%, which we round to 7%. We note that the difference of the arithmetic and geometric returns decreases with size, which is logical, as large size is correlated with lower volatility, which should reduce the difference of the AM and GM returns.

We can add the size-based differential to estimate the arithmetic average rate of return to use for our discount rate of 25% + 7% to 11% = 32% to 36%.[9] For mathematical ease we round to a 30% discount rate, but in practice it is best to use an appropriate rate in the range above.

Column B of Table 16.1 lists the conditional FMVs obtained from discounted cash flow analyses using different sets of assumptions. In the very optimistic scenario, we forecast outstanding performance of the company, with a resulting FMV of $130,000,000 (B6). B7 and B8 display the FMVs resulting from optimistic and conservative forecasts, respectively. In the pessimistic scenario, we assume the company fails completely, resulting in zero value. When valuing a general partnership interest, which has unlimited liability, the appraiser should consider the possibility of negative value.

Column C lists the probability associated with each scenario. These are derived directly from the empirical probabilities of VC success discussed above. We calculate the weighted FMV in column D by multiplying the conditional FMV in column B by its associated probability in column C and summing the results. Thus, in this example, the weighted average FMV is $8,600,000 (D10).

[8] I confirmed this in a telephone conversation with Professor Myers.

[9] Fowler's article did not address this adjustment.

## Advantages of the First Chicago Approach

Three major advantages of the First Chicago approach are:

1. It reduces the uncertainty associated with a single FMV by allowing for several scenarios representing differing levels of success of the company.
2. It breaks down the huge range of potential outcomes into bite-size chunks; the individual scenarios are credible and plausible when performed carefully.
3. It makes the appraiser's probability distribution of outcomes explicit. In doing so, it has two additional advantages: (a) If the client agrees with the conditional FMVs of each scenario but for some reason feels the probabilities are not representative of the subject company's chances, it is an easy exercise for the client to weight the probabilities differently and adjust the valuation herself. This is particularly important when the assignment is to provide existing shareholders with information to negotiate with funding sources. If both sides accept the scenario valuations, it is usually easy for them to come to terms by agreeing on the probabilities of the outcomes, which they can easily do without the appraiser; and (b) it protects the appraiser. When the appraiser shows a final weighting of the conditional FMVs multiplied by their probabilities to calculate the FMV and shows the probability of total failure as, say, 70%, it can protect the appraiser from a disgruntled investor in the event the company fails. The appraiser has clearly communicated the high probability of investors losing all their money, despite the fact that the FMV may be very high—and hopefully is—due to the large values in the upper 30% of probable outcomes.

Therefore the First Chicago approach is normally the preferred method of valuation of start-ups. It is also useful in valuing existing firms that are facing radically different outcomes that are hard to forecast. For example, I used it once to assist warring shareholders who wanted one side to buy out the other in a four-year-old company (the "Company"). The firm was profitable and had grown rapidly, but there were several major uncertainties that were impossible to credibly consider with accuracy in a single DCF scenario. The uncertainties were as follows:

- There was much customer turnover in the prior year, even though there was healthy growth.
- If one of the shareholders left the firm, sales might suffer greatly for two or three years and even endanger the Company.
- There were regulatory issues that could have a dramatic impact on the Company.
- Profit margins were highly variable in the past four years and could have been affected by regulation.

Collectively, these uncertainties made a single-scenario forecast of sales growth and profitability very difficult. Despite considerable partisanship by the shareholders, who often actively lobbied for changes in the DCF analyses, the First Chicago approach enabled us to credibly model the different paths the Company could take and quantify the valuation implications of that. Ultimately, we presented them with the valuation of the different scenarios and our estimates of the probabilities, and the weighted average of the product of the two constituted our estimate of FMV. We also explained that they could change their subjective weighting of probabilities

| | A | B |
|---|---|---|
| 1 | **Table 16.2** | |
| 2 | **VC Pricing Approach [1]** | |
| 3 | | |
| 4 | | |
| 5 | Assumed Cash Out—5 Yrs @ 12 × Earnings | $ 23,200,000 |
| 6 | Present Value Factor—5 Years @ 45% ROI | 0.1560 |
| 7 | Present Value—Rounded | **$ 3,619,000** |
| 8 | | |
| 9 | [1] *Source:* Bradley Fowler, "What Do Venture Capital Pricing Method Tell about | |
| 10 | Valuation of Closely Held Firms?" *Business Valuation Review*, June 1989, page 77. | |

of outcomes, thus changing the FMV. Ultimately, they worked out an arrangement without any further need of our help.

### Discounts for Lack of Marketability and Control

Finally, it is important to mention that venture capitalists typically have more control and possibly marketability than most other investors. When valuing the interests of other investors, the appraiser must add the incremental discounts for lack of control and marketability that apply to the specific interests; that is, an arm's-length investor would typically require a higher rate of return on smaller interests than the 30% that the VC expects.

## Venture Capital Valuation Approach

In this approach, the appraiser estimates net earnings at cash-out time, often at year 5 or 6. He or she then estimates a PE multiple, and multiplies the two to estimate the cash-out.

In Table 16.2, we use Fowler's (1989) numbers, with minor changes in the presentation. Fowler assumed year 5 net income of $1,936,167 and multiplied it by a PE multiple of 12 to calculate the year 5 cash-out at $23.2 million (B5), rounded.

He then used a 45% rate of return to discount cash flows, based on industry statistics he presented in the article, which we repeat in the next section. The present value factor at 45% for five years is 0.156, and the present value of the company is then $3,619,000 (B7), after rounding.

### Venture Capital Rates of Return

Fowler (1989) cited rates of return from two different studies. Plummer (1987) found that the required rates of return (ROR), which included discounts for lack of control (DLOC) and discounts for lack of marketability (DLOM), were:

| Stage of Development of Company | Required Rate of Return |
|---|---|
| Seed-capital stage | 50–75% |
| 1st stage | 40–60% |
| 2nd stage | 35–50% |
| 3rd stage | 30–50% |
| 4th stage | 30–40% |

Morris (1988, p. 55) wrote that VCs are looking for the following rates of return:

| Stage of Development of Company | Required Rate of Return |
|---|---|
| Seed-capital stage | 50%+ |
| 2nd stage | 30–40% |

### Summary of the VC Approach

The VC approach is a valid valuation approach, though certainly less analytically precise than the First Chicago approach. Nevertheless, it is used by venture capitalists, and it serves as a quick-and-dirty valuation method, on the one hand, and as a useful alternative approach, on the other.

This concludes Part 1 of this chapter. Part 2 is a complex decision tree analysis combined with multiscenario valuation.

## Part 2: Debt Restructuring Study

Early-stage technology-based companies often find themselves in financial hot water. They incur large expenses for years during the development of a new product. Consequently, they run short of funds and often require the infusion of venture capital, which may or may not occur. In the following example—which is based on an actual assignment, with names and numbers changed—the subject company (the "Company") has several possible events that can impact the probability of obtaining venture capital as well as surviving as a firm without venture capital (i.e., bootstrapping to success).

### Background

The Company and its former parent (the "parent") share a nearly identical set of shareholders—well over 100. The president is the major shareholder of the firm, with effective but not absolute control. The parent had lent the Company $1 million to get started as a spinoff, but the debt would be coming due in four years, and the Company would have no way of paying it off.

The parent proposed the following restructuring of the debt:

1. The parent would convert the debt into $400,000 of convertible preferred stock—and part of the valuation exercise was to determine how many shares of preferred stock that would be. There would be no preferred dividends, but the parent would have a liquidation preference.
2. The president would have to relinquish a certain number of his shares in the parent back to the parent, which had a ready buyer for the shares.

In return for relinquishing his shares to the parent, the president wants the Company to issue 1.3 million new shares to him. The board of directors wants an independent appraisal to determine whether the transaction is favorable to the other shareholders. This example, however, is typical of the types of decisions faced by start-up firms in their quest for adequate funding. More importantly, the statistical approach we use in this valuation is applicable to the valuation of many start-ups, regardless of industry.

## Key Events

The Company president, Mr. Smith, has identified a sequence of four key events that could occur, and each one of them increases the Company's ability to obtain venture capital financing as well as to successfully bootstrap the firm without VC financing. The events are sequentially dependent; that is, event #1 is necessary, but not sufficient for event #2 to occur. Events #1, #2, and #3 must occur in order for #4 to occur. These events are:

*Event #1:* The Company sells its product to company #1. The conditional probability of this event occurring is 75% (Table 16.3, B11).
*Event #2:* The Company sells its product to company #2. The conditional probability of this happening, assuming event #1 occurs, is 90% (B12).
*Event #3:* The Company sells its product to company #3, which has a 60% (B13) conditional probability (i.e., assuming event #2 occurs).
*Event #4:* The Company sells its product to company #4. If the Company sells its product to company #3, then it has an 80% (B14) probability of selling it to company #4.

While these four events are all potential sales, the statistical process involved in this analysis is generic. The four events could just as easily be a mixture of technology milestones, rounds of financing, regulatory, sales, and other events.

## Decision Trees and Spreadsheet Calculations

Our analysis begins as decision trees, which appear in Figures 16.1 and 16.2. However, careful analysis leads to our being able to mathematically generalize the decision tree calculations and transform them into expressions that we can calculate in a spreadsheet. This has tremendous computational advantage, which is not very apparent in a four-milestone analysis. Increase the number of milestones to 20, and the decision tree becomes very unwieldy to present, let alone to calculate, while the spreadsheet is easy. The discussions over the next few pages ultimately culminate in the development of equations (16.3) through (16.6). The equations provide the blueprint for the structure of the calculations in Table 16.3.

## Table 16.3: Statistical Calculation of FMV

Table 16.3 is a statistical calculation of the FMV of the common shares of the Company owned by the existing minority shareholders, based on the probabilities of the different events occurring and the results of DCF analyses of several different scenario outcomes.

# Table 16.3
# Statistical Calculation of Fair Market Value

## Section 1A: Weighted Average Values Assuming Venture Capital Scenario and Debt Restructure with Parent

| Row | A | B | C | D | E | F | G | H | I | J |
|---|---|---|---|---|---|---|---|---|---|---|
| 6–7 | | | Cum Product [B] | | 1 – [D] | Cumulative Product [E] | [C]×[D]×[$F_{n-1}$] | 1 – VC% | [G] × B18 × [H] | [1 – Min] × [I] |
| 8–10 | Event | Conditional Probability of Sale | Cumulative Joint Probability of Sale | Venture Cap Conditional Probability | Prob No VC = 1 – VC Cond. Probability | Cum. No VC | Prob of VC | Current Shareholders % Own | Current Shareholders FMV—Control | Current Shareholders FMV—Minor |
| 11 | #1: Company makes sale #1 | 75.000% | 75.000% | 50.000% | 50.000% | 50.000% | 37.500% | 50.000% | $18,750,000 | $14,062,500 |
| 12 | #3: Company makes sale #2 | 90.000% | 67.500% | 60.000% | 40.000% | 20.000% | 20.250% | 60.000% | $12,150,000 | $9,112,500 |
| 13 | #3: Company makes sale #3 | 60.000% | 40.500% | 70.000% | 30.000% | 6.000% | 5.670% | 70.000% | $3,969,000 | $2,976,750 |
| 14 | #4: Company makes sale #4 | 80.000% | 32.400% | 100.000% | 0.000% | 0.000% | 1.944% | 85.000% | $1,652,400 | $1,239,300 |
| 15 | Totals | | | | | | 65.364% | | $36,521,400 | $27,391,050 |

| Row | A | B |
|---|---|---|
| 17 | **Assumptions** | |
| 18 | FMV—VC Scenario | $100,000,000 |
| 19 | Minority Interest Discount (assumed) | 25% |

## Section 1B: Bootstrap Scenario Assuming Debt Restructuring with Parent

| Row | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 23 | | | Cum Product [B] | | 1 – [D] | Cum Prod. [E] | $P$[S$i$\|$i$,–($i$+1)] | [C]×[F]×{1–[$B_{t+1}$]}×[G] | Note [1] | [H] × [I] | [1 – Min] × [J] |
| 26–28 | Event | Conditional Probability | Cumulative Joint Probability | Venture Cap Conditional Probability | Prob No VC = 1 – VC Cond. Probability | Cum. No VC | Bootstrap Conditional Probability | Prob of Survival/No-VC | Conditional FMV | Wtd Avg FMV = FMV—Control | Current Shareholders FMV—Minor |
| 29 | #1: Company makes sale #1 | 75.000% | 75.000% | 50.000% | 50.000% | 50.000% | 30.000% | 1.125% | 15,286,460 | $171,973 | $128,980 |
| 30 | #3: Company makes sale #2 | 90.000% | 67.500% | 60.000% | 40.000% | 20.000% | 35.000% | 1.890% | 15,464,845 | 292,286 | 219,214 |
| 31 | #3: Company makes sale #3 | 60.000% | 40.500% | 70.000% | 30.000% | 6.000% | 75.000% | 0.365% | 15,732,422 | 57,345 | 43,009 |
| 32 | #4: Company makes sale #4 | 80.000% | 32.400% | 100.000% | 0.000% | 0.000% | 90.000% | 0.000% | 16,000,000 | 0 | 0 |
| 33 | Totals | | | | | | | 3.380% | | $521,603 | $391,202 |

## Section 2: No Debt Restructure with Parent

| Row | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 37 | #1: Company makes sale #1 | 75.000% | 75.000% | 0.000% | 100.000% | 100.000% | 30.000% | 2.250% | 7,286,460 | $163,945 | $122,959 |
| 38 | #3: Company makes sale #2 | 90.000% | 67.500% | 0.000% | 100.000% | 100.000% | 35.000% | 9.450% | 7,464,845 | 705,428 | 529,071 |
| 39 | #3: Company makes sale #3 | 60.000% | 40.500% | 0.000% | 100.000% | 100.000% | 75.000% | 6.075% | 7,732,422 | 469,745 | 352,308 |
| 40 | #4: Company makes sale #4 | 80.000% | 32.400% | 0.000% | 100.000% | 100.000% | 90.000% | 29.160% | 8,000,000 | 2,332,800 | 1,749,600 |
| 41 | Totals | | | | | | | 46.935% | | $3,671,918 | $2,753,938 |

| Row | A | B | C |
|---|---|---|---|
| 43 | **Assumptions** | Restructure | No Restructure |
| 44 | = Adjusted FMV—Bootstrap | $16,000,000 | $8,000,000 |
| 45 | Minority Interest Discount (assumed) | 25% | |

# Table 16.3 (cont.)

## Section 3: Calculation of FMV per Share

| Row | A | B: Restructure — Venture Capital | C: Restructure — Bootstrap | D: Restructure — Total | F: No Restructure: Investor % = 33.33% |
|---|---|---|---|---|---|
| 53 | **Sec 1: Venture Capital Scenario** | $27,391,050 | $391,202 | $27,782,252 | $2,753,938 |
| 54 | **Calculation of Fully Diluted Shares:** | | | | |
| 55 | **Original Shares** | 1,000,000 | 1,000,000 | | 1,000,000 |
| 56 | **Options:** | | | | |
| 57 | **200,000 @ $0.50 per share [2]** | 200,000 | 200,000 | | 200,000 |
| 58 | **66,667shares @ $0.75 per share** | 66,667 | 0 | | 0 |
| 59 | **100,000 shares @ $1.00 per share** | 100,000 | 0 | | 0 |
| 60 | **Preferred Stock Conversion (B89) [3]** | 9,624 | 0 | | 0 |
| 61 | **Total Option Shares** | 376,290 | 200,000 | | 200,000 |
| 62 | **Original Shares Plus Options** | 1,376,290 | 1,200,000 | | 1,200,000 |
| 63 | **Proposed Issuance To President** | 1,300,000 | 1,300,000 | | 0 |
| 64 | **Shares To Outside Investors [4]** | 0 | 0 | | 600,000 |
| 65 | **Fully-Diluted Shares [5]** | **2,676,290** | **2,500,000** | | **1,800,000** |
| 66 | **Fully-Diluted FMV/Share—Post Transaction** | **$10.235** | **$0.156** | **$10.391** | **$1.530** |

## Section 4: Year *t*+4 Investor Percentage Taken

| Row | A | B: Control FMVs |
|---|---|---|
| 71 | Yr *t*+4 FMV—40% Disc Rate—Control Basis | $8,000,000 |
| 72 | Less: Discount for Lack of Control—% (assumed) | -25.0% |
| 73 | Less: Discount for Lack of Control—$ | ($2,000,000) |
| 74 | Yr *t*+4 FMV—40% Discount Rate—Minority Basis | $6,000,000 |
| 75 | Percentage Required For $2 Million Investment | **33.3%** |

**Notes:**

[1] Column I Calculations: Beginning with FMV for Event #4, we subtract $750,000 for not reaching each of Events #4 and #3 and $500,000 for not reaching Event #2. All previous numbers are tax effected and present valued.

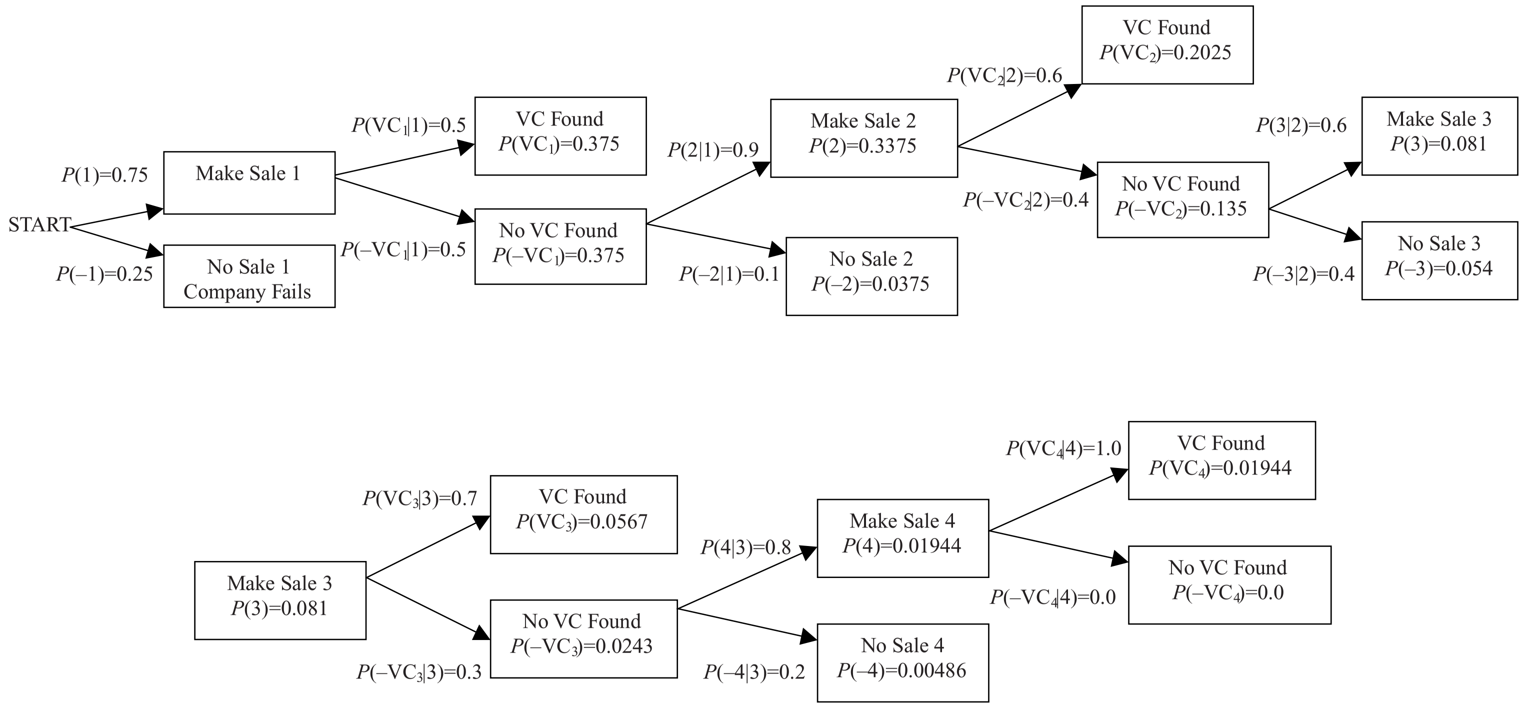
[2] Only the 200,000 shares are applicable in all scenarios. The remaining options apply only to the VC Scenario.

[3] Assume 4 to 1 preferred-to-common conversion ratio, per CFO, as follows:

| Row | A | B |
|---|---|---|
| 86 | Preferred Stock—Stated Value | $400,000 |
| 87 | FMV Per Share of Common (D66) | $10.391 |
| 88 | Multiply by 4 | $41.56 |
| 89 | Convert To # Common Shares (B86/B88) (To B60) | **9,624** |

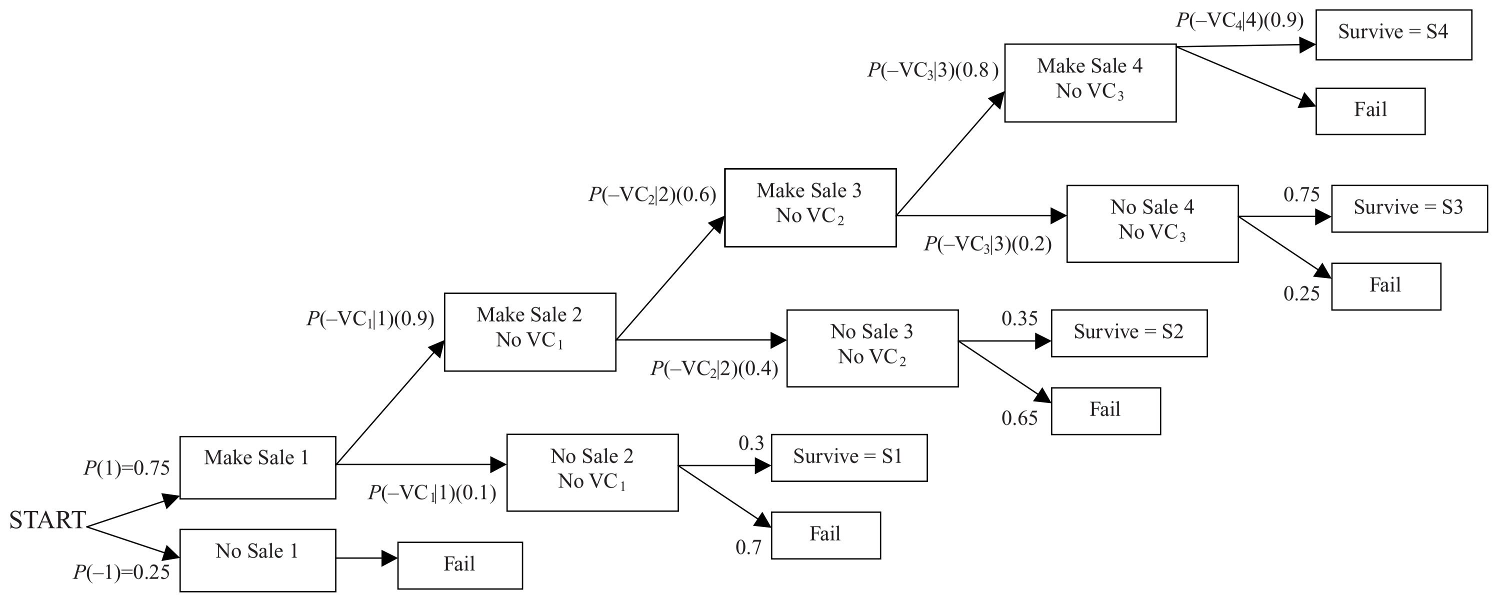
[4] In the Bootstrap—No Restructure Scenario, the Company falls $1 million short of cash and owes $1 million to the parent. We assume it will have to take on $2M investment for 33% of the stock. See section 4.

[5] Actually, fully-diluted shares will be more, as will FMV when VC shares are included. In section 1A, columns H and I, we calculated the FMV of the current shareholders' shares, which is simpler than using actual FMV and wtd avg shares.




Many ofthe probabilities in this figure appear in Table 16-3, Section 1A, Columns B, D, and G.
Also $P(-VC_1|1)$ is equivalent to $[1 - P(VC_1|1)]$ in the text and $P(-2|1) = [1 - P(2|1)]$ etc.

**FIGURE 16.1 Decision Tree for Venture Capital Funding**




Note: $P(-VC_1|1)$ is equivalent to $[1 - P(VC_1|1)]$ in the text.

**FIGURE 16.2** Decision Tree for Bootstrapping Assuming Debt Restructure and No Venture Capital

**ORGANIZATION** The table is divided into four sections. In the first two sections, 1A and 1B, the Company does restructure its debt with the parent. Section 1A is the calculation of the probability-weighted contribution to the FMV of the current shareholders' shares when the Company is successful in obtaining venture capital. Several possible combinations of events can lead to this outcome, and we identify the probabilities and payoffs of each combination in order to calculate the FMV of the common stock owned by the existing minority shareholders. Section 1B is the probability-weighted equivalent of section 1A when the Company is not successful in obtaining venture capital and instead attempts to bootstrap its way to success. The total of sections 1A and 1B is the FMV of current shareholders' shares assuming the Company restructures the debt.

Section 2 is an analysis of the combination of events in which the Company does not restructure its debt with the parent. Section 3 is a summary of the FMVs under the different scenarios and contains calculations of the per-share values. This is the *bottom line* of the valuation assignment. We describe the purpose of section 4 later in the chapter.

**TABLE 16.3, SECTION 1A: VENTURE CAPITAL SCENARIO** In section 1A, the primary task is to determine the probability of receiving VC[10] funding. Once we have accomplished that, it is simple to determine the contribution to FMV from the VC scenario.

Figure 16.1 is a diagram of the decision tree for section 1A. We begin by noting that there is a 75% probability of making sale #1 and a 25% probability of not making sale #1, in which case the Company fails. We denote the former as $P(1) = 75\%$ and the latter as $P(-1) = 25\%$. We denote the conditional probabilities of subsequent sales as $P(j \mid j-1)$, where $j$ is the sale number. For example, $P(2 \mid 1)$ is the conditional probability of making sale #2, given that the Company already made sale #1. The probability of making sale #2 is the probability of making sale #1 multiplied by the conditional probability of making sale #2, given that the Company makes sale #1, or: $P(2) = P(1) \times P(2 \mid 1) = 0.75 \times 0.9 = 0.675$. Also note that $P(1)$ is the same as $P(1 \mid 0)$, since there is no sale zero.

**Probability of VC Financing after Sale #1** If the Company makes sale #1, there is a 50% conditional probability of receiving VC funding at that time. We denote that event as $VC_1$, which means receiving VC funding after sale #1, but before sale #2 is attempted,[11] and we denote its conditional probability of occurrence as $P(VC_1 \mid 1)$, the probability of VC funding after sale #1, given that sale #1 occurs. The probability of receiving VC funding after the first sale is the conditional probability of the first sale occurring times the conditional probability of VC funding, given the sale.[12] The statistical statement is: $P(VC_1) = P(1) \times P(VC_1 \mid 1)$, where $P(1)$ is the probability of making sale #1. Thus $P(VC_1) = 0.75 \times 0.5 = 0.375$.

We denote the conditional probability of failure to obtain VC funding after sale #1 as $P(-VC_1 \mid 1) = 1 - P(VC_1 \mid 1) = 0.5$. Thus the absolute probability of not

[10] *VC* stands for venture capital and for venture capitalist. The context should make it clear which meaning is appropriate.

[11] From now on, when we say "after sale $i$," we also mean "but before the Company attempts sale $i + 1$."

[12] For the first sale, the conditional probability and the absolute probabilities are identical.

receiving VC financing after sale #1 is $P(-\text{VC}_1) = P(1) \times P(-\text{VC}_1 \,|\, 1) = 0.75 \times 0.5 = 0.375$, which is the same result as $P(\text{VC}_1)$. This occurs because the conditional probability of obtaining venture capital, given that the Company makes the first sale, is 50%. At any other probability, $P(\text{VC}_1 \,|\, 1) \neq P(-\text{VC}_1 \,|\, 1)$. These statements generalize for sale $i$, $i = 1, 2, 3, 4$.

**Probability of VC Financing after Sale #2** Let's move on to the next step in our analysis—sale #2 and the probability of VC funding after it. If the Company receives VC after sale #1, we have already quantified that earlier. Our task in this iteration is to quantify the probability of VC funding if it did not come after sale #1 but does come after sale #2. Thus, the chain of events we are quantifying in this round is: sale #1 $\rightarrow -\text{VC}_1 \rightarrow$ sale #2 $\rightarrow \text{VC}_2$; that is, the Company makes sale #1, doesn't receive venture capital, makes sale #2, and then receives venture capital.

The probability of obtaining VC funding after sale #2 is:

$$\begin{aligned} P(\text{VC}_2) &= P(1) \times [1 - P(\text{VC}_1 \,|1)] \times P(2\,|1) \times P(\text{VC}_2 \,|2) \\ &= 0.75 \times (1 - 0.5) \times 0.9 \times 0.6 = 0.2025. \end{aligned} \tag{16.1}$$

Note that the conditional probability of VC financing, given that the Company makes sale #2, $P(\text{VC}_2 \,|\, 2) = 0.6$, compared to 0.5 after sale #1. In general, it makes sense that the conditional probability of receiving VC financing rises with each new key sale.

We can rearrange equation (16.1) as:

$$P(\text{VC}_2) = P(1) \times P(2|\,1) \times [1 - P(\text{VC}_1|\,1)] \times P(\text{VC}_2|\,2). \tag{16.2}$$

In other words, the probability of obtaining VC financing after sale #2 is the cumulative joint probability of making both sale #1 and sale #2 times the conditional probability of not obtaining VC funding after sale #1 times the conditional probability of obtaining VC funding after sale #2.

**Generalizing to Probability of VC Financing after Sale #*k*** We can generalize the probability of obtaining VC funding after sale #*k* as:[13]

$$P(VC_k) = \left( \prod_{i=1}^{k} P(i|i-1) \prod_{j=0}^{k-1} \left[1 - P(VC_j|j)\right] \right) P(VC_k|k). \tag{16.3}$$

Equation (16.3) states that the probability of obtaining venture capital financing after sale #*k* is the cumulative joint probability of sale #*k* occurring times the cumulative joint probability of having been refused VC financing through sale #$(k - 1)$ times the conditional probability of receiving VC financing after sale #*k*.

Finally, the total probability of obtaining VC financing is the sum of equation (16.3) across all $n$ sales, where $n = 4$ in this example:

$$P(VC) = \sum_{k=1}^{n} \left[ \prod_{i=1}^{k} P(i|i-1) \prod_{j=0}^{k-1} \left[1 - P(VC_j|j)\right] \right] P(VC_k|k). \tag{16.4}$$

[13]Of course, $P(1\,|\,0) \equiv P(1)$, as the former has no meaning. Also, in the first iteration of equation (16.3), when $j = 0$, the term $P(\text{VC}_j \,|\, j)$ is the cumulative probability of receiving VC financing from sale #0, which is a zero probability. Thus $1 - P(\text{VC}_j \,|\, j)$ goes to 1.0, as it should.

**Explanation of Table 16.3, Section 1A** Column A lists the sales events described earlier, and column B lists their associated conditional probabilities in B11 through B14; that is, $P(1) = 75\%$ (B11), $P(2|1) = 90\%$ (B12), and so on. Column C is the cumulative joint probability, which is just the cumulation of the conditional probabilities. For example, the cumulative joint probability of making sale #4 is $P(1) \times P(2|1) \times P(3|2) \times P(4|3) = 75\% \times 90\% \times 60\% \times 80\% = 32.4\%$ (C14), where the conditional probabilities we multiply by each other are in B11 through B14. C11 through C14 represent the term $\prod_{i=1}^{n} P(i|i-1)$ in equations (16.3) and (16.4).

Column D is the president's forecast of the conditional probability of obtaining VC financing. Each conditional probability is $P(VC_j|j)$, that is, the probability of obtaining VC financing after sale #*j*, given that the Company makes sale #*j*, but before attempting sale #$j + 1$. Every subsequent sale increases the probability of obtaining venture capital beyond the level of the previous event. The conditional probability of VC financing rises from 50% (D11) after sale #1 to 60%, 70%, and 100% for sales #2, #3, and #4, respectively (D12 through D14).

Column E, the conditional probability of not receiving VC financing after each sale, is 1 minus column D. Column F is the cumulative product of column E. It is the $\prod_{j=0}^{k-1} \left[1 - P(VC_j|j)\right]$ in equation (16.3) when we use the cumulation of the previous sale. For example, the probability of obtaining VC financing after the sale to company #4 is the cumulative joint probability of making sale #4, which is 32.4% (C14) × the cumulative joint probability of not having obtained VC financing after the first *three* sales, which is 6% (F13) × the conditional probability of making sale #4, which is 100% (D14) = 1.944% (G14).

Finally, the probability of obtaining VC financing, according to equation (16.4), is 65.364% (G15), the sum of column G. The FMV of the Company, if it obtains VC financing, is $100 million (B18), which we determined with a DCF analysis.

Column H is 1 minus the percentage that Mr. Smith estimates the venture capital firm would take in the Company's stock. After sale #1, he estimates the venture capitalist would take 50%, leaving 50% (H11) to the existing shareholders after the conditional transaction. If the Company makes the sale to company #2, it will be in a stronger bargaining position, and Mr. Smith estimates the venture capitalist would take 40% of the Company, leaving 60% (H12) to existing shareholders after the transaction. If the Company makes the sale to company #3, then he estimates the venture capitalist would take 30% of the Company, leaving 70% (H13) to the existing shareholders after the transaction. Finally, if the Company makes the sale to company #4, then he estimates the venture capitalist would take 15% of the Company, leaving 85% (H14) to the existing shareholders after the transaction.

Columns I and J are the FMVs of the current shareholders' shares on a control and minority basis resulting from obtaining venture capital financing. Later on, we will add in the current shareholders' FMV from bootstrapping the Company to come to a total current shareholders' FMV for the debt restructure option. Column I is the control value FMV and is obtained by multiplying the probability of obtaining VC financing in column G times the $100 million FMV of the Company if it receives VC financing (B18) times column H, the current shareholder ownership percentages. Column J is the FMV on a minority interest basis, which is column I times 1 minus the minority interest discount of 25.0% (B19), the magnitude of which is an arbitrary assumption in this analysis. The total FMVs of current shareholder shares are $36,521,400 (I15) and $27,391,050 (J15) on a control and minority basis, respectively.

The final equation describing the FMV is:[14]

$$FMV(VC) = \left\{ \sum_{k=1}^{n} \left( \prod_{i=1}^{k} P(i|i-1) \prod_{j=0}^{k-1} \left[1 - P(VC_j|j)\right] \right) P(VC_k|k) \times SH\%_k \right\} \times \$100\ million. \tag{16.5}$$

In words, the contribution to FMV from the VC scenario is the sum of the probabilities of obtaining VC, which we quantified in equation (16.4), times the $100 million FMV of the Company, assuming it is VC financed.

## SECTION 1B: THE BOOTSTRAP SCENARIO ASSUMING DEBT RESTRUCTURING WITH PARENT

Bootstrapping occurs when the Company fails to attract venture capital, but still manages to stay in business. The bootstrap scenario includes both success and failure at its attempts to bootstrap. Figure 16.2 shows the decision tree for the bootstrap scenario.

The pattern of events is that the Company can make the sale or not in each iteration. After each sale, it might get VC financing or it might not. In section 1B, we are not interested in the nodes on the decision tree where the Company receives VC financing, as we have already quantified that in section 1A. Thus, we do not show those nodes. Nevertheless, it is important to account for the probabilities of obtaining VC financing, because if we don't, we will be double-counting that portion of the time that the Company could finance through a VC or bootstrap successfully. The Company can't do both at the same time. Thus, we remove the statistical probability of overlap. We accomplish that by multiplying all probabilities by $[1 - P(VC_i \mid i)]$ for all relevant $i$, where $i$ is the sale number (also the iteration number).

If the Company does not make the sale, then it has a probability of survival and failure. We denote the survival after its last sale as $Sj$, where $j$ is the sale number. The conditional probability of survival after its last sale is $P[Sj \mid j, -(j+1)]$. For example, if the Company makes sale #3, does not make sale #4, and survives, we denote that as $S3$, and its conditional probability of occurrence is $P(S3 \mid 3, -4)$, which reads, "the probability of Company long-term survival, given that it made sale #3, but does not make sale #4." If the Company makes the next sale, then we repeat the iteration, incrementing the sale number.

Without going through all of the step-by-step analysis we did for the VC scenario, the FMV of the bootstrap scenario is:

$$FMV(Bootstrap) = \sum_{j=1}^{n} \left[ \prod_{i=1}^{j} P(i|i-1)[1 - P(VC_i|i)] \right] (1 - P(j+1|j)P[Sj|j, -(j+1)]FMV(Sj). \tag{16.6}$$

Let's use the first iteration as an example. The probability of making sale #1 is 0.75. There is a 0.5 probability of obtaining VC financing if the Company makes

[14]The term *SH%* is the percentage ownership of the current shareholders after VC financing.

sale #1, so there is also a 0.5 probability of not obtaining VC financing, that is, $[1 - P(VC_i \mid i)] = 0.5$. In order to terminate at $S1$, the Company must make sale #1 and fail to make sale #2, which means we multiply by $[1 - P(2 \mid 1)]$, which is equal to 1 minus the conditional probability of making sale #2 = 1 − 0.9 (B30) = 0.1. The probability of survival if the Company makes sale #1 but stops there is 0.30 (G29). Thus, $P(S1) = P(1) \times [1 - P(VC_1 \mid 1)] \times [1 - P(2 \mid 1)] \times P(S1 \mid 1, -2) = 0.75 \times (1 - 0.5) \times (1 - 0.9) \times 0.3 = 1.125\%$ (H29).[15]

Column I is the conditional FMV of the Company at each respective event level. This is different than in section 1, where the FMV is the same regardless of stage. The reason is that in section 1A, the sole objective is obtaining venture capital funding, which will enable the Company to sell to the world. The lost profits on the "key sales" not made are immaterial compared to the $100 million FMV. In contrast, in section 1B each sale is significant relative to the total value and adds to the value of the Company.[16]

In section 1B, we begin with a conditional FMV of $16,000,000 (B44, repeated in I32). That value contains an implicit assumption that the Company makes it to event #4, the sale to company #4. At each level before that, we subtract the net present value of the after-tax profits[17] from the sale that does not occur; we work our way backward up this column. We assume pretax profits of $750,000 for the sales in events #3 and #4 and $500,000 for event #2. The numbers are then tax-effected and discounted to present value. If the Company does not make it to event #1, this model assumes the Company fails entirely and has a zero value.

Column J is the contribution to the FMV of the Company on a control basis coming from the bootstrap scenario and is simply column H times column I, which totals $521,603 (J33).

Column K is the same value as column J, except that it is a minority interest conditional FMV. The discount for minority interest is 25%, which appears in B45. On a minority interest basis, the bootstrap scenario FMV is $391,202 (K33).

**SECTION 2: NO-RESTRUCTURE SCENARIO** The final scenario is the no-restructure-with-parent scenario. Section 2 is identical to section 1B, except:

- Column F, the probability of not obtaining venture capital financing, is 100% by definition for all four events in section 2, since the president informs us that a VC will not finance the Company as long as it still has the parent's debt on the books.
- Column I is calculated identically to section 1B, except that the baseline FMV as calculated by DCF analysis is $8 million (C44, repeated in I40) for the no-restructure scenario instead of $16 million (B44, repeated in I32).

Columns J and K in section 2 are the same as in section 1B, except that there are no values originating from the venture capital scenario that have to be removed.

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[15]Note that for the last milestone, $1 - P(n + 1 \mid n)$ must be equal to 1, since the probability of making the $(n + 1)$st sale is zero.

[16]The sales actually do affect the values in section 1A, but their impact is immaterial relative to the much larger total value, which is not true in the bootstrap scenarios.

[17]To be more precise, we would also include the related cash flow effects.

**SECTION 3: FMVs PER SHARE UNDER VARIOUS RESTRUCTURE SCENARIOS** In section 3 we calculate the fully diluted FMV per share post-transaction under the various scenarios.

**Venture Capital Scenario** The conditional FMV of the Company on a minority interest basis from the venture capital scenario is $27,391,050 (B53, transferred from J15). The Company currently has 1,000,000 shares of common stock outstanding, as appears in B55, C55, and F55. Rows 57 through 59 show employee stock options. Row 57 shows outstanding options for 200,000 shares at $0.50 per share. These options are *in the money*, and we assume they will be exercised. That would result in $100,000 being paid to the Company, which is included in the DCF analysis and is therefore already incorporated into the $27,391,050 value. These 200,000 additional shares are taken into account in all of the valuation scenarios.

Rows 58 and 59, however, are for options that are granted but could be exercised only if the Company does the restructure and obtains VC financing.[18] Mr. Johnson says that if the Company does obtain VC financing, it will issue 66,667 options with a $0.75 exercise price this year (B58) and 100,000 options (B59) at a $1.00-per-share exercise price next year. Again, the cash inflows from exercise of the options are already included in the DCF analysis.

In the restructure scenario, the parent receives $400,000 of preferred stock, which can be converted to common if the Company goes public or gets acquired. Otherwise, it serves only to increase the liquidation preference, as preferred dividends will never be paid. Therefore, the dividends, which are not tax deductible, do not appear in any of the cash flows. We presume in the venture capital scenario that the probability of going public or being acquired is significant and that preferred will convert. According to Mr. Johnson, a reasonable conversion ratio is 4 to 1. In note 3 to section 3, the $400,000 is divided by four times the fully diluted FMV of $10.391 per share (D66, repeated in B87), or $41.56 (B88) per share, resulting in an estimated conversion to common shares of 9,624 (B89, transferred to B60). This calculation is a simultaneous equation and requires the use of multiple iterations on the spreadsheet. The number of converted shares depends on the fair market value per common share, but the FMV per common share depends on the number of preferred shares.

The total option shares are 376,290 (B61), including the assumed conversion of preferred in the venture capital scenario. In B63 we show the proposed issuance of 1.3 million shares to the president. Adding the 1,000,000 original shares, 376,290 option granted shares, and the 1.3 million new shares, we come to 2,676,290 (B65) fully diluted shares in the venture capital scenario. Dividing the $27,391,050 FMV by 2,676,290 shares, we arrive at the FMV per share of $10.235 (B66) for the venture capital scenario.

Next, we consider the bootstrap portion of the restructure scenario. We begin with the $391,202 (K33) FMV as calculated in section 1B and repeat it in C53. Again, this is the portion of bootstrap value from which venture capital is excluded.

In this scenario, the fully diluted shares are the same as in the venture capital scenario, except that the 66,667, 100,000 and 9,624 shares in rows 58 through 60 are zero in this case. There are 1,200,000 shares (C62) in this scenario before issuing the

[18]The Company cannot obtain VC financing without restructuring its debt.

1.3 million, and 2,500,000 (C65) shares after doing so. Dividing $391,202 by 2,500,000 shares, we come to an FMV in this scenario of $0.156 (C66) per share. Adding the per-share values together, we come to $10.235 + $0.156 = $10.391 (B66 + C66 = D66) as the weighted average conditional FMV of the restructure scenario.

**No-Restructure Scenario** The name of this scenario is somewhat of a misnomer. It means that the Company does not restructure its debt with the parent. At the onset of this assignment there was no way to know this, but restructuring of debt would eventually be required. The discounted cash flow analysis leads to the conclusion that the Company is unlikely to be able to generate enough cash to pay off the parent's note by its due date of December 31, 2000[19]—even though the forecast shows profits. Therefore, the Company has two choices: Become insolvent and undergo liquidation or restructure later and undergo a distress sale of equity approximately one year before the note becomes due.

The second choice obviously leads to a higher value for the shareholders, as it preserves the cash flows, even though some of them will be diverted to the new investor. Accordingly, we ran a DCF analysis to the fiscal year ending closest to the due date of the note. That value is $8,000,000 and appears in C44.

The subtotal number of shares is 1,200,000 (F62) before the new investor. Since there is no restructure with the parent in this scenario, the shares issued to the president is zero here (F63). In section 4, we calculate that the new investor will demand one-third of the Company post-transaction (see description below). That implies the investor will demand 600,000 shares (F64), which will bring the total shares to 1,800,000 (F65). Dividing $2,753,938 (K41, repeated in F53) by 1,800,000 shares leads to a value of $1.530 (F66) per share for the no-restructure scenario (this should more appropriately be called "restructure later").

## Conclusion

Thus, the restructure is preferable by an FMV per share of $10.391 − $1.530 = $8.861 per share (= D66 − F66).

**TABLE 16.3, SECTION 4: YEAR $t + 4$ INVESTOR PERCENTAGE** A future restructure would be a more distressed one than the current one. The discounted cash flow analysis indicates that the Company would be short of cash to pay off the note. With two years gone by, it is very likely the Company would lose the possibility of becoming the market leader and would more likely be an "also-ran." Also, it would be a far more highly leveraged firm without the restructure. Therefore, it would be a higher-risk firm in the year $t + 4$, which dictates using a higher discount rate than the other scenarios. The result is a value of $8,000,000 (C44, repeated as B71) before the minority interest discount.

Subtracting the $2 million (B73) minority interest discount leaves us with an FMV of $6 million (B74). In the DCF, we determined the Company would need a $2 million investment by a new investor, who would require taking one-third (B75) of the Company. This percentage is used in section 3, F52 in the no-restructure calculations, as discussed earlier.

[19]The analysis was done in 1996.

## Part 3: Exponentially Declining Sales Growth Model

When forecasting yearly sales for a start-up, the appraiser ideally has a *bottom-up* forecast based on a combination of market data and reasonable assumptions. Sometimes those data are not available to us, and even when they are available, it is often beneficial to use a *top-down* approach that is based on reasonable assumptions of sales growth rates. In this section, we present a model for forecasting sales of a start-up or early-stage company that semi-automates the process of forecasting sales and can be easily manipulated for sensitivity analysis. The other choice is to insert sales growth rates manually for, say, 10 years, print out the spreadsheet with that scenario, change all 10 growth rates, and repeat the process for valuation of multiple scenarios. Life is too short.

One such sales model that has intuitive appeal is the exponentially declining sales growth rate model, presented in Table 16.4. In the model we have a peak growth rate ($P$), which decays with a decay rate constant ($k$) to a final growth rate ($G$). The mathematics may look a little difficult, but it is not necessary to understand the math in order to benefit from using the model.

The top of Table 16.4 is a list of the parameters of the model. In the example, the final sales growth rate ($G$) is set at 6% (E6), and the additional growth rate ($A$) is calculated to be 294% (E7). The additional growth rate ($A$) is the difference between the peak growth rate ($P$), which is set at 300% (E8), and the final sales growth rate of 6%. Next we have the decay rate constant ($k$), which is set at 0.50 (E9). The larger the decay rate constant, the faster the sales growth rate will decline to the final growth rate. Finally, we have year 1 forecast sales of 100 (E10). All the variables are specified by the model user with the exception of the additional growth rate ($A$), which depends on $P$ and $G$.

Example #1 shows the forecast sales growth rates (row 17) and sales (row 18) using the previously specified variables for a case where the sales growth rate declines after year 2. We have no sales growth rate in year 1, because we assume there are no prior-year sales. The expression for the sales growth rate $= G + Ae^{-k(t-2)}$, for all $t$ greater than or equal to 2, where $t$ is expressed in years. For year 2, the sales growth rate is $G + Ae^{-k(2-2)} = G + A = 6\% + 294\% = 300\%$ (C17), which is our specified peak growth rate $P$. Year 3 growth is $G + Ae^{-k(3-2)} = 6\% + 294\% \times e^{-0.5\times 1} = 184\%$ (D17). Year 4 growth is $G + Ae^{-k(4-2)} = 6\% + 294\% \times e^{-0.5\times 2} = 114\%$ (E17), and so on. To calculate yearly sales, we simply multiply the previous year's sales by 1 plus the forecast growth rate.

Example #1A is identical to example #1, except that we have changed the decay rate constant ($k$) from 0.50 to 0.30. Notice how reducing $k$ slows the decay in the sales growth rate. In example #2, we present a case of the peak growth rate ($P$) occurring in a general future year $f$, where we have chosen the future year to be year 4. The model user specifies the growth rates prior to year $f$ (we have chosen 100% and 200% in years 2 and 3, respectively). The growth rates for year $f$ and later are $G + Ae^{-k(t-f)}$. As you can see, the growth rates from years 4 through 10 in this example are identical to the growth rates from years 2 through 8 in example #1.

Figures 16.3 and 16.4 are graphs that show the sales forecasts from examples #1 and #1A extended to 28 years. The slower decay rate of 0.3 in Figure 16.4 (versus 0.5 in Figure 16.3) leads to much faster growth. After 28 years, sales are close to \$450,000 versus \$38,000. Changing one single parameter can give the analyst a great deal of

| | A | B | C | D | E | F | G | H | I | J | K |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **Table 16.4** | | | | | | | | | | |
| 2 | **Sales Model with Exponentially Declining Growth Rate Assumption** | | | | | | | | | | |
| 3 | | | | | | | | | | | |
| 4 | | | | | | | | | | | |
| 5 | **Variable Name** | | | **Symbol** | **Value** | **Specified/Calculated** | | | | | |
| 6 | **Final Growth Rate** | | | ***G*** | 6% | Specified | | | | | |
| 7 | **Additional Growth Rate** | | | ***A*** | 294% | Calculated | | | | | |
| 8 | **Peak Growth Rate** | | | ***P*** | **300%** | Specified | | | | | |
| 9 | **Decay Rate** | | | ***k*** | 0.50 | Specified | | | | | |
| 10 | **First Year's Sales** | | | **Sales1** | 100 | Specified | | | | | |
| 11 | | | | | | | | | | | |
| 12 | | | | | | | | | | | |
| 13 | **Example # 1 — Sales growth rate declines after year 2** | | | | | | | | | | |
| 14 | Yearly growth = $G + Ae^{-k(t-2)}$ for all $t$ greater than or equal to 2 | | | | | | | | | | |
| 15 | | | | | | | | | | | |
| 16 | **Year** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
| 17 | **Growth** | N/A | 300% | 184% | 114% | 72% | 46% | 30% | 21% | 15% | 11% |
| 18 | **Sales** | 100 | 400 | 1,137 | 2,436 | 4,179 | 6,093 | 7,929 | 9,566 | 10,989 | 12,240 |
| 19 | | | | | | | | | | | |
| 20 | | | | | | | | | | | |
| 21 | **Example # 1A — Changing the decay rate (*k*) from 0.50 to 0.30 slows the decline in the sales growth rate** | | | | | | | | | | |
| 22 | | | | | | | | | | | |
| 23 | **Year** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
| 24 | **Growth** | N/A | 300% | 224% | 167% | 126% | 95% | 72% | 55% | 42% | 33% |
| 25 | **Sales** | 100 | 400 | 1,295 | 3,463 | 7,810 | 15,194 | 26,072 | 40,307 | 57,237 | 75,937 |
| 26 | | | | | | | | | | | |
| 27 | | | | | | | | | | | |
| 28 | **Example # 2 — Sales growth rate declines after future year *f*** | | | | | | | | | | |
| 29 | Sales growth rate = $G + Ae^{-k(t-f)}$, for all $t$ greater than or equal to $f$, where sales growth rate declines after future year $f$ and | | | | | | | | | | |
| 30 | the peak sales growth ($P$) occurs in year $f$. Growth through year $f$ is to be specified by model user. The following is an | | | | | | | | | | |
| 31 | example with year $f = 4$, and decay rate $k = 0.5$. | | | | | | | | | | |
| 32 | | | | | | | | | | | |
| 33 | **Year** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** |
| 34 | **Growth** | N/A | 100% | 200% | 300% | 184% | 114% | 72% | 46% | 30% | 21% |
| 35 | **Sales** | 100 | 200 | 600 | 2,400 | 6,824 | 14,613 | 25,077 | 36,559 | 47,575 | 57,393 |
| 36 | | | | | | | | | | | |
| 37 | **Formula in Cell C17: = G+A*EXP(-k*(C16-2))** | | | | | | | | | | |

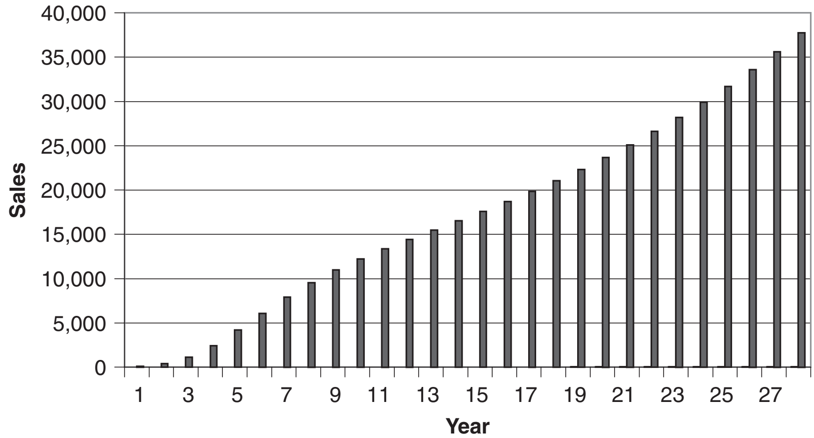


FIGURE 16.3 **Sales Forecast (Decay Rate = 0.5)**

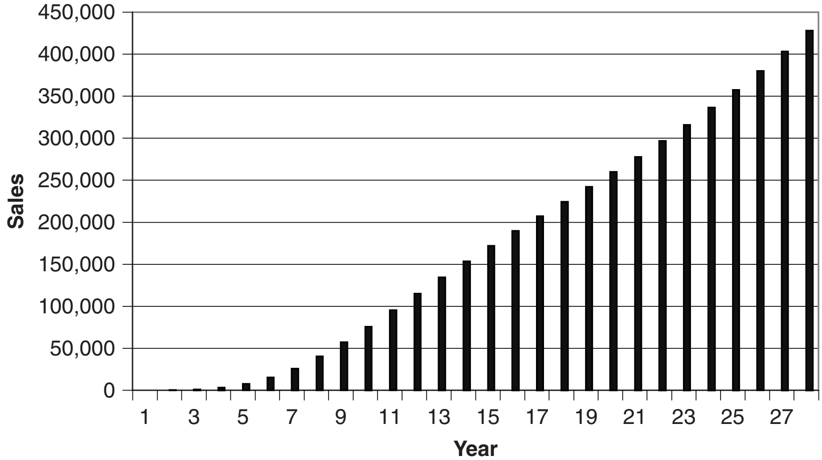


FIGURE 16.4 **Sales Forecast (Decay Rate = 0.3)**

control over the sales forecast. When sensitivity analysis is important, we can control the decline in sales growth simply by using different numbers in E9, the decay rate. This is not only a nice timesaver, but it can lead to more accurate forecasts, as many phenomena in life have exponential decay (or growth), for example, the decay of radiation, population of bacteria, and so forth.

## References

Cochrane, John H. 2005. The Risk and Return of Venture Capital. *Journal of Financial Economics* 75(January): 3–52.

DeGennaro, Ramon P., and Gerald P. Dwyer, Jr. 2009. Expected Returns to Angel Investors. http://ssrn.com/abstract=1343124; January.

Fowler, Bradley A. 1989. What Do Venture Capital Pricing Methods Tell about Valuation of Closely Held Firms? *Business Valuation Review* (June): 73–79.

Fowler, Bradley A. 1990. Valuation of Venture Capital Portfolio Companies—And Other Moving Targets. *Business Valuation Review* (March): 13–17.

Fowler, Bradley A. 1996. Venture Capital Rates of Return Revisited. *Business Valuation Review* (March): 13–16.

Golder, Stanley C. 1986. Structuring and Pricing the Financing. In *Pratt's Guide to Venture Capital Sources*, 10th ed., edited by Stanley E. Pratt and Jane K. Morris. Wellesley Hills, MA: Venture Economics.

Morris, Jane K. 1988. In *Pratt's Guide to Venture Capital Sources*, 12th ed., edited by Jane K. Morris. Wellesley Hills, MA: Venture Economics, p. 55.

Pacelle, Mitchell. 1999. Venture Firms Dethroning Buyout Kings. *Wall Street Journal* 7(June): C1.

Plummer, James L. 1987. *QED Report on Venture Capital Financial Analysis*. Palo Alto, CA: QED Research, Inc. See especially 2-7–2-10 and 6-2–6-13.

Pratt, Stanley E., and Jane K. Morris. 1986. *Guide to Venture Capital Sources*. Wellesley Hills, MA: Venture Economics.

Wiltbank, Robert, and Warren Boeker. 2007. Returns to Angel Investors in Groups. Ewing Marion Kauffman Foundation and Angel Capital Education Foundation. November 2007.

Wiltbank, Robert, Stuart Read, Nicholas Dew, and Saras D. Sarasvathy. 2009. Prediction and Control under Uncertainty: Outcomes in Angel Investing. *Journal of Business Venturing*. 24(2): 116–133.

# CHAPTER 17

# Monte Carlo Risk Simulation


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# CHAPTER 17

# Monte Carlo Risk Simulation


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In this chapter, we will go over the basics of Monte Carlo Risk Simulation, and in particular, the use of the Risk Simulator software. As the concepts of risk simulation are vast and sometimes get technical, we present merely the tip of the iceberg in this chapter. For more details, case studies, example models, free training videos, and so forth, please visit my Web site at www.realoptionsvaluation.com or www.risksimulator.com and click on the ***Downloads*** link for these free models and videos plus to obtain the Risk Simulator software showcased in this chapter.

As mentioned before, this chapter touches only the tip of the iceberg in terms of risk analysis as applied to advanced valuation methodologies. It lacks, for example, basic applications such as scenario analysis, overlay charts, linear and nonlinear pairwise correlations among input assumptions, more advanced forecasting techniques such as ARIMA or GARCH models for valuing volatility, or stochastic processes like Brownian motion, mean-reversion and jump-diffusion processes for forecasting and valuing stocks and other liquid commodities, as well as optimization techniques of finding the best allocation of a portfolio of assets or investments subject to internal and external constraints.

You can refer to my books—especially the two that are recommended for valuation experts: *Modeling Risk: Applying Monte Carlo Simulation, Real Options, Portfolio Optimization and Stochastic Forecasting* (MR) (John Wiley & Sons, 2006), and *Real Options Analysis: Second Edition* (ROA) (John Wiley & Sons, 2006), for more details, step-by-step instructions, theoretical constructs, case studies and applications of risk simulation, strategic real options analysis, exotic and financial options valuation, forecasting (basic-to-advanced techniques), portfolio optimization, and other analytical techniques. Nonetheless, this chapter is sufficient to get the reader started in using the basic applications in Risk Simulator.

## What Is Monte Carlo Risk Simulation?

*Monte Carlo simulation*, named for the famous gambling capital of Monaco, is a very potent methodology. For the practitioner, simulation opens the door for solving difficult and complex but practical problems with great ease. Perhaps the most famous early use of Monte Carlo simulation was by the Nobel physicist Enrico Fermi (sometimes referred to as the father of the atomic bomb) in 1930, when he used a random method to calculate the properties of the newly discovered neutron. Monte Carlo methods were central to the simulations required for the Manhattan Project, while in the 1950s Monte Carlo simulation was used at Los Alamos for early work relating to the development of the hydrogen bomb, and became popularized in the fields of physics and operations research. The Rand Corporation and the U.S. Air Force were two of the major organizations responsible for funding and disseminating information on Monte Carlo methods during this time, and today there is a wide application of Monte Carlo simulation in many different fields, including engineering, physics, research and development, business, and finance.

Simplistically, Monte Carlo simulation creates artificial futures by generating thousands and even hundreds of thousands of sample paths of outcomes and analyzes their prevalent characteristics. In practice, Monte Carlo simulation methods are used for risk analysis, risk quantification, sensitivity analysis, and prediction. An alternative to simulation is the use of highly complex stochastic closed-form mathematical models. For analysts in a company, taking graduate-level advanced math and statistics courses is just not logical or practical. A brilliant analyst would use all available tools at his or her disposal to obtain the same answer in the easiest and most practical way possible. And in all cases, when modeled correctly, Monte Carlo simulation provides similar answers to the more mathematically elegant methods. In addition, there are many real-life applications where closed-form models do not exist and the only recourse is to apply simulation methods. So, what exactly is Monte Carlo simulation and how does it work?

Today, fast computers have made possible many complex computations that were seemingly intractable in past years. For scientists, engineers, statisticians, managers, business analysts, and others, computers have made it possible to create models that simulate reality and aid in making predictions, one of which is used in simulating real systems by accounting for randomness and future uncertainties through investigating hundreds and even thousands of different scenarios. The results are then compiled and used to make decisions. This is what Monte Carlo simulation is all about.

Monte Carlo simulation in its simplest form is a random number generator that is useful for forecasting, estimation, and risk analysis. A simulation calculates numerous scenarios of a model by repeatedly picking values from a user-predefined *probability distribution* for the uncertain variables and using those values for the model. As all those scenarios produce associated results in a model, each scenario can have a forecast. *Forecasts* are events (usually with formulas or functions) that you define as important outputs of the model.

Think of the Monte Carlo simulation approach as picking golf balls out of a large basket repeatedly with replacement. The size and shape of the basket depend on the distributional *input assumption* (e.g., a normal distribution with a mean of 100 and a standard deviation of 10, versus a uniform distribution or a triangular distribution) where some baskets are deeper or more symmetrical than others, allowing certain balls to be pulled out more frequently than others. The number of balls pulled repeatedly depends on the number of *trials* simulated. For a large model with multiple related assumptions, imagine the large model as a very large basket, wherein many baby baskets reside. Each baby basket has its own set of colored golf balls that are bouncing around. Sometimes these baby baskets are linked with each other (if there is a *correlation* between the variables), forcing the golf balls to bounce in tandem, whereas in other, uncorrelated cases, the balls are bouncing independently of one another. The balls that are picked each time from these interactions within the model (the large basket) are tabulated and recorded, providing a *forecast output* result of the simulation.

## Comparing Simulation with Traditional Analyses

Figure 17.1 illustrates some traditional approaches used to deal with uncertainty and risk. The methods include performing sensitivity analysis, scenario analysis, and probabilistic scenarios. The next step is the application of Monte Carlo simulation, which can be seen as an extension to the next step in uncertainty and risk analysis. Figure 17.1A shows a more advanced use of Monte Carlo simulation for forecasting. The examples in Figure 17.1A show how simulation can be really complicated depending on its use.

## Running a Monte Carlo Simulation Using Risk Simulator

This section illustrates some sample step-by-step instructions to create and run a Monte Carlo simulation model using Risk Simulator. For more details, please refer to the software user manual, view some free getting-started videos online at www.realoptionsvaluation.com, or review some of my books as described previously. It is further assumed that you have successfully followed the instructions at the end of this chapter to install Risk Simulator. If so, when you start Excel, you will see the Risk Simulator icons and menu item. (See Figure 17.2.)

To run a simulation in your existing Excel model, the following steps have to be performed:

1. Start a new simulation profile or open an existing profile.
2. Define input assumptions in the relevant cells.
3. Define output forecasts in the relevant cells.
4. Run the simulation.
5. Interpret the results.

If desired, and for practice, open the example file called ***Basic Simulation Model*** and follow along with the examples below on creating a simulation. The example file can be found by starting Excel, and clicking on ***Risk Simulator | Example Models | 02 Basic Simulation Model.*** Of course, feel free to review each of the 23 sample models available in this submenu, and each model has an Information worksheet with details of the model as well as step-by-step instructions on how to run the required analysis.

## Point Estimates, Sensitivity Analysis, Scenario Analysis, Probabilistic Scenarios and Simulations

### Point Estimates

| | | |
|---|---|---|
| Unit Sales | 10 | |
| Unit Price | $10 | 10 units x $10 per unit |
| Total Revenue | $100 | |
| Unit Variable Cost | $5 | |
| Fixed Cost | $20 | $20 Fixed + ($5 x 10) Variable |
| Total Cost | $70 | |
| Net Income | $30 | $100 - $70 |

This is a simple example of a Point Estimate approach. The issues that arise may include the risk of how confident you are in the unit sales projections, the sales price and variable unit cost.

Since the bottom line Net Income is the key financial performance indicator here, an uncertainty in future sales volume will be impounded into the Net Income calculation. How much faith do you have on your calculation based on a simple point estimate?

Recall the Flaw of Average example where a simple point estimate could yield disastrous conclusions.

### Sensitivity Analysis

Here, we can make unit changes to the variables in our simple model to see the final effects of such a change. Looking at the simple example, we know that only Unit Sales, Unit Price and Unit Variable Cost can change. This is since Total Revenues, Total Costs and Net Income are calculated values while Fixed Cost is assumed to be fixed and unchanging, regardless of the amount of sales units or sales price. Changing these three variables by one unit shows that from the original $40, Net Income has now increased $5 for Unit Sales, increased $10 for Unit Price and decreased $10 for Unit Variable Cost.

| | | | | | |
|---|---|---|---|---|---|
| Unit Sales | 11 (Change 1 unit) | Unit Sales | 10 | Unit Sales | 10 |
| Unit Price | $10 | Unit Price | $11 (Change 1 unit) | Unit Price | $10 |
| Total Revenue | $110 | Total Revenue | $110 | Total Revenue | $100 |
| Unit Variable Cost | $5 | Unit Variable Cost | $5 | Unit Variable Cost | $6 (Change 1 unit) |
| Fixed Cost | $20 | Fixed Cost | $20 | Fixed Cost | $20 |
| Total Cost | $75 | Total Cost | $70 | Total Cost | $80 |
| Net Income | $35 (Up $5) | Net Income | $40 (Up $10) | Net Income | $20 (Down $10) |

Hence, we know that Unit Price has the most positive impact on the Net Income bottom line and Unit Variable Cost the most negative impact. In terms of making assumptions, we know that additional care must be taken when forecasting and estimating these variables. However, we still are in the dark concerning which sensitivity set of results we should be looking at or using.

### Scenario Analysis

In order to provide an added element of variability, using the simple example above, you can perform a Scenario Analysis, where you would change values of key variables by certain units given certain assumed scenarios. For instance, you may assume three economic scenarios where unit sales and unit sale prices will vary. Under a good economic condition, unit sales go up to 14 at $11 per unit. Under a nominal economic scenario, units sales will be 10 units at $10 per unit. Under a bleak economic scenario, unit sales decrease to 8 units but prices per unit stays at $10.

| | Good Economy | | Average Economy | | Bad Economy |
|---|---|---|---|---|---|
| Unit Sales | 14 | Unit Sales | 10 | Unit Sales | 8 |
| Unit Price | $11 | Unit Price | $10 | Unit Price | $10 |
| Total Revenue | $154 | Total Revenue | $100 | Total Revenue | $80 |
| Unit Variable Cost | $5 | Unit Variable Cost | $5 | Unit Variable Cost | $5 |
| Fixed Cost | $20 | Fixed Cost | $20 | Fixed Cost | $20 |
| Total Cost | $90 | Total Cost | $70 | Total Cost | $60 |
| Net Income | $64 | Net Income | $30 | Net Income | $20 |

Looking at the Net Income results, we have $64, $30 and $20. The problem here is, the variation is too large. Which condition do I think will most likely occur and which result do I use in my budget forecast for the firm? Although Scenario Analysis is useful in ascertaining the impact of different conditions, both advantageous and adverse, the analysis provides little insight to which result to use.

### Probabilistic Scenario Analysis

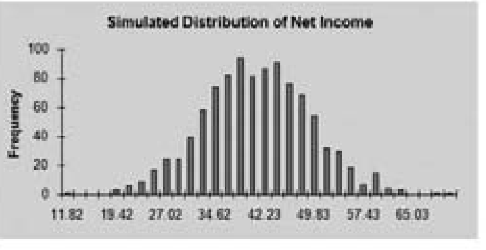
| | Probability | Net Income |
|---|---|---|
| Good Economy | 35% | $64.00 |
| Average Economy | 40% | $30.00 |
| Bad Economy | 25% | $20.00 |
| EMV | | $39.40 |

We can always assign probabilities that each scenario will occur, creating a Probabilistic Scenario Analysis and simply calculate the Expected Monetary Value (EMV) of the forecasts. The results here are more robust and reliable than a simple scenario analysis since we have collapsed the entire range of potential outcomes of $64, $30 and $20 into a single expected value. This value is what you would expect to get on average.

### Simulation Analysis

| | |
|---|---|
| Unit Sales | 10 ? |
| Unit Price | $10 ? |
| Total Revenue | $100 |
| Unit Variable Cost | $5 ? |
| Fixed Cost | $20 |
| Total Cost | $70 |
| Net Income | $30 |

Looking at the original model, we know that through Sensitivity Analysis, Unit Sales, Unit Price and Unit Variable Cost are three highly uncertain variables. We can then very easily simulate these three unknowns thousands of times (based on certain distributional assumptions) to see what the final Net Income value looks like.




By performing the simulation thousands of times, we essentially perform thousands of sensitivity analysis and scenario analysis given different sets of probabilities. These are all set in the original simulation assumptions (types of probability distributions, the parameters of the distributions and which variables to simulate).

The results calculated from the simulation output can then be interpreted as follows:

| | |
|---|---|
| Average | $40.04 |
| Median | $39.98 |
| Mode | $46.63 |
| Standard Deviation | $8.20 |
| 95th Confidence | Between $56.16 and $24.09 |

Discussions about types of distributional assumptions to use and the actual simulation approach will be discussed later.

FIGURE 17.1 Point Estimates, Sensitivity Analysis, Scenario Analysis, Probabilistic Scenarios, and Simulations

**Conceptualizing the Lognormal Distribution**

A Simple Simulation Example

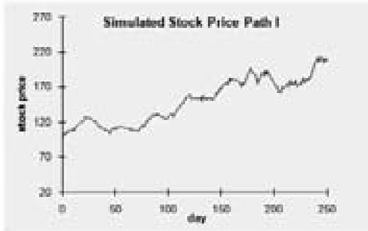
We need to perform many simulations to obtain a valid distribution

| | |
|---|---|
| Mean | 15% |
| Sigma | 30% |
| Timing | Daily |
| Starting Value | 100 |

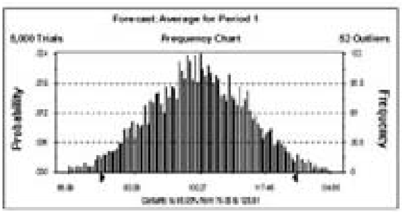
Here we see the effects of performing a simulation of stock price paths following a Geometric Brownian Motion model for daily closing prices. Three sample paths are seen here. In reality, thousands of simulations are performed and their distributional properties are analyzed. Frequently, the average closing prices of these thousands of simulations are analyzed, based on these simulated price paths.

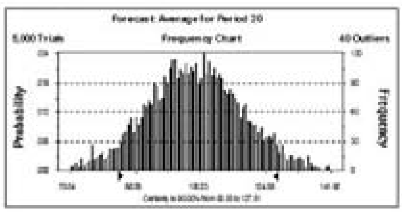
| time days | normal deviates | value simulated |
|---|---|---|
| 0 | NA | 100.0000 |
| 1 | 0.0873 | 100.2259 |
| 2 | -0.4320 | 99.4675 |
| 3 | -0.1389 | 99.2652 |
| 4 | -0.4583 | 98.4649 |
| 5 | 1.7807 | 101.9095 |
| 6 | -1.4406 | 99.2212 |
| 7 | -0.5577 | 98.2357 |
| 8 | 0.5277 | 99.2838 |
| 9 | -0.4844 | 98.4345 |
| 10 | -0.2307 | 98.0634 |
| 11 | 0.8588 | 99.7532 |
| 12 | 2.1195 | 83.9088 |
| 13 | -1.9756 | 100.1461 |
| 14 | 1.3734 | 102.8517 |
| 15 | -0.8790 | 101.2112 |
| 16 | -0.7610 | 99.8203 |
| 17 | 0.3168 | 100.4024 |
| 18 | -0.0511 | 100.4452 |
| 19 | 0.0653 | 100.6301 |
| 20 | -0.6073 | 99.5368 |
| 21 | 0.6900 | 100.9091 |
| 22 | -0.7012 | 99.6353 |
| 23 | 1.4784 | 102.5312 |
| 24 | -0.9195 | 100.8184 |
| 25 | -0.3343 | 100.2411 |
| 26 | -2.3395 | 96.9465 |
| 27 | -1.7831 | 92.8103 |
| 28 | -0.3247 | 92.2958 |
| 29 | 0.5053 | 93.2409 |
| 30 | 0.0386 | 93.3652 |
| 247 | 1.0418 | 100.9205 |
| 248 | -0.7052 | 99.6388 |
| 249 | 0.1338 | 99.9521 |
| 250 | 0.0451 | 100.0978 |

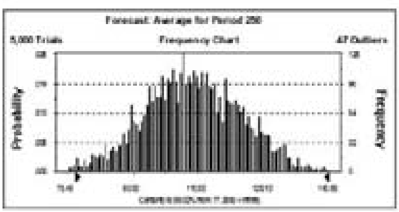
Rows 31 through 246 have been hidden to conserve space.






The thousands of simulated price paths are then tabulated into probability distributions. Here are three sample price paths at three different points in time, for periods 1, 20, and 250. There will be a total of 250 distributions for each time period, which corresponds to the number of trading days a year.

We can also analyze each of these time-specific probability distribution and calculate relevant statistically valid confidence intervals for decision-making purposes.


We can then graph out the confidence intervals together with the expected values of each forecasted time period.

Notice that as time increases, the confidence interval widens since there will be more risk and uncertainty as more time passes.

FIGURE 17.1A Conceptualizing the Lognormal Distribution

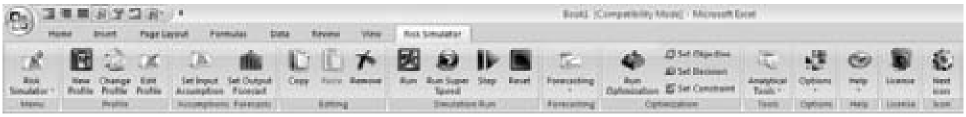


FIGURE 17.2 Risk Simulator Icons in Excel

## Starting a New Simulation Profile

To start a new simulation, you will first need to create a simulation profile. A *simulation profile* contains a complete set of instructions on how you would like to run a simulation, that is, all the assumptions, forecasts, run preferences, and so forth. Having profiles facilitates creating multiple scenarios of simulations. That is, using the same exact model, several profiles can be created, each with its own specific simulation properties and requirements. The same person can create different test scenarios using different distributional assumptions and inputs or multiple persons can test their own assumptions and inputs on the same model.

1. Start Excel and create a new or open an existing model (you can use the Basic Simulation Model example to follow along).
2. Click on ***Risk Simulator*** and select ***New Simulation Profile***.
3. Specify a title for your simulation as well as all other pertinent information (Figure 17.3).

*Title*. Specifying a simulation title allows you to create multiple simulation profiles in a single Excel model. This means that you can now save different simulation scenario profiles within the same model without having to delete existing assumptions or changing them each time a new simulation scenario is required. You can always change the profile's name later (***Risk Simulator*** | ***Edit Profile***).

*Number of trials*. This is where the number of simulation trials required is entered. That is, running 1,000 trials means that 1,000 different iterations of outcomes based on the input assumptions will be generated. You can change this as desired

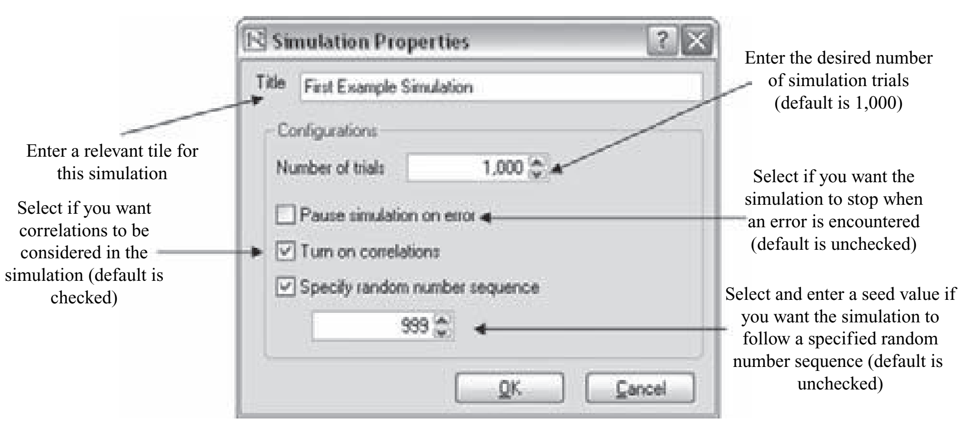



FIGURE 17.3 New Simulation Profile

but the input has to be positive integers. The default number of runs is 1,000 trials. You can use precision and error control to automatically help determine how many simulation trials to run (see the section on precision and error control for details).

*Pause on simulation error.* If checked, the simulation stops every time an error is encountered in the Excel model. That is, if your model encounters a computation error (e.g., some input values generated in a simulation trial may yield a divide-by-zero error in one of your spreadsheet cells), the simulation stops. This is important to help audit your model to make sure there are no computational errors in your Excel model. However, if you are sure the model works, then there is no need for this preference to be checked.

*Turn on correlations.* If checked, correlations between paired input assumptions will be computed. Otherwise, correlations will all be set to zero and a simulation is run assuming no cross-correlations between input assumptions. As an example, applying correlations will yield more accurate results if indeed correlations exist, and will tend to yield a lower forecast confidence if negative correlations exist. After turning on correlations here, you can later set the relevant correlation coefficients on each assumption generated (see the section on correlations for more details).

*Specify a random number sequence.* Simulation by definition will yield slightly different results every time a simulation is run. This is by virtue of the random number generation routine in Monte Carlo simulation and is a theoretical fact in all random number generators. However, when making presentations, sometimes you may require the same results (especially when the report being presented shows one set of results and during a live presentation you would like to show the same results being generated, or when you are sharing models with others and would like the same results to be obtained every time); then check this preference and enter in an initial seed number. The seed number can be any positive integer. Using the same initial seed value, the same number of trials, and the same input assumptions, the simulation will always yield the same sequence of random numbers, guaranteeing the same final set of results.

Note that once a new simulation profile has been created, you can come back later and modify these selections. In order to do this, make sure that the current active profile is the profile you wish to modify, otherwise, click on ***Risk Simulator | Change Simulation Profile***, select the profile you wish to change and click ***OK*** (Figure 17.4 shows an example where there are multiple profiles and how to activate a selected profile). Then, click on ***Risk Simulator | Edit Simulation Profile*** and make the required changes. You can also duplicate or rename an existing profile.

## Defining Input Assumptions

The next step is to set input assumptions in your model. Note that assumptions can only be assigned to cells without any equations or functions, that is, typed-in numerical values that are inputs in a model, whereas output forecasts can only be assigned to cells with equations and functions (i.e., outputs of a model). Recall that assumptions and forecasts cannot be set unless a simulation profile already exists. Do the following to set new input assumptions in your model:

- Make sure a Simulation Profile exists, or open an existing profile, or start a new profile (***Risk Simulator | New Simulation Profile***).

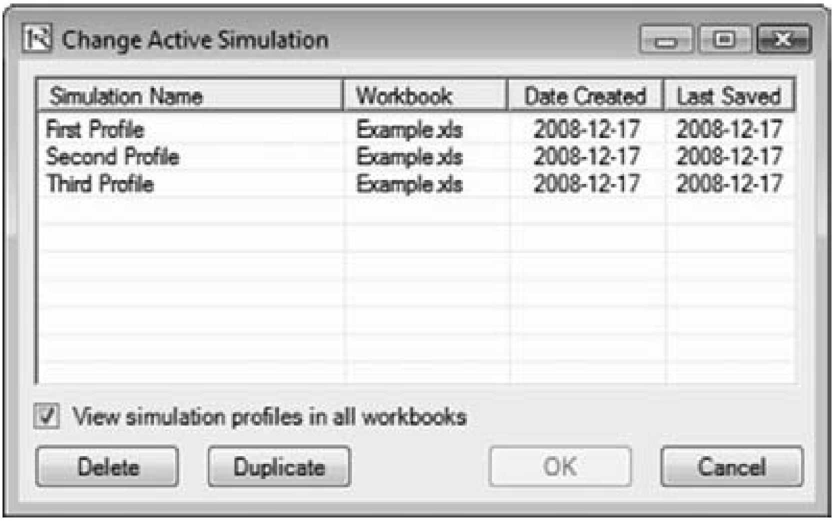



FIGURE 17.4 Change Active Simulation

- Select the cell you wish to set an assumption on (e.g., cell G8 in the Basic Simulation Model example).
- Click on ***Risk Simulator | Set Input Assumption*** or click on the fourth icon in the Risk Simulator icon toolbar.
- Select the relevant distribution you want and enter the relevant distribution parameters and hit ***OK*** to insert the input assumption into your model (Figure 17.5).

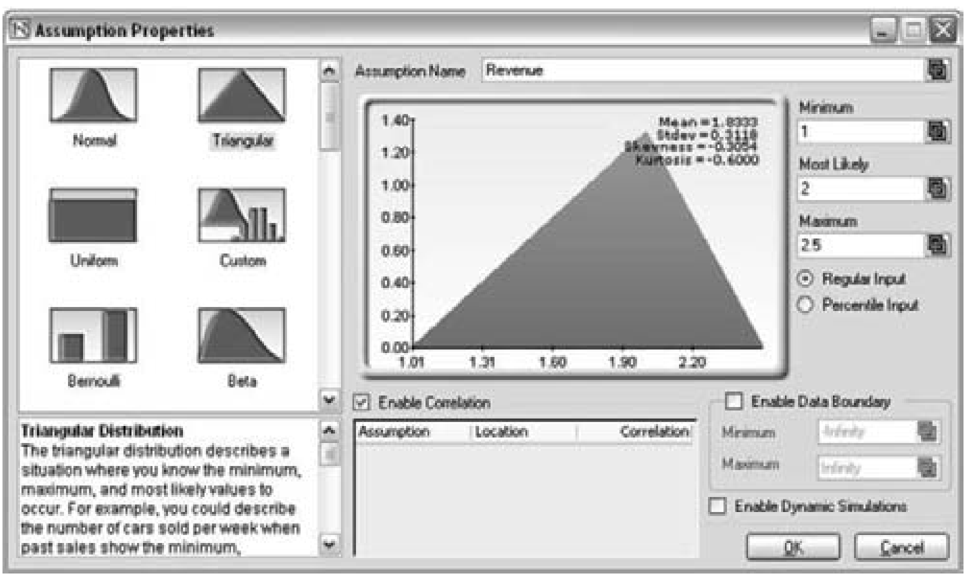



FIGURE 17.5 Setting an Input Assumption

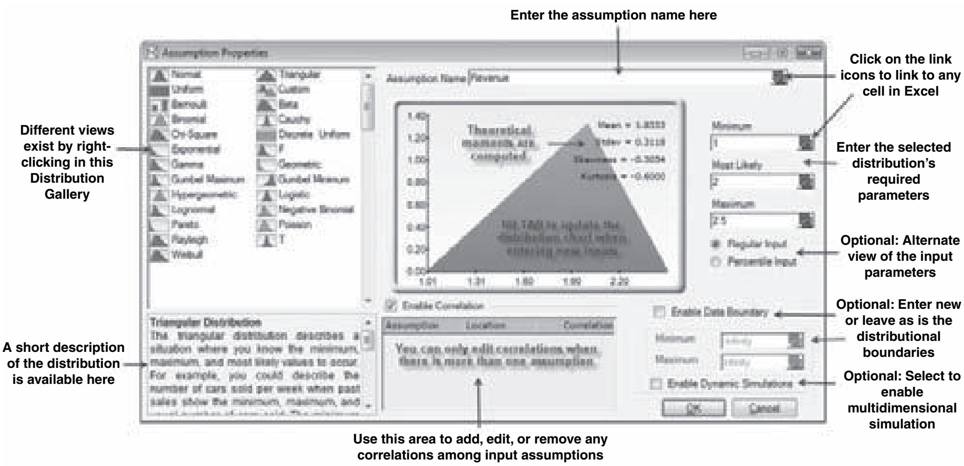



FIGURE 17.6 Assumption Properties

Notice that in the Assumption Properties, there are several key areas worthy of mention. Figure 17.6 shows the different areas:

*Assumption name.* This is an optional area to allow you to enter in unique names for the assumptions to help track what each of the assumptions represent. Good modeling practice is to use short but precise assumption names.

*Distribution gallery.* This area to the left shows all of the different distributions available in the software. To change the views, right-click anywhere in the gallery and select large icons, small icons, or list. There are over two dozen distributions available.

*Input parameters.* Depending on the distribution selected, the required relevant parameters are shown. You may either enter the parameters directly or link them to specific cells in your worksheet. Hard-coding or typing the parameters is useful when the assumption parameters are assumed not to change. Linking to worksheet cells is useful when the input parameters need to be visible or are allowed to be changed (click on the link icon to link an input parameter to a worksheet cell).

*Enable data boundary.* These are typically not used by the average analyst but exist for truncating the distributional assumptions. For instance, if a normal distribution is selected, the theoretical boundaries are between negative infinity and positive infinity. However, in practice, the simulated variable exists only within some smaller range and this range can then be entered to truncate the distribution appropriately.

*Correlations.* Pairwise correlations can be assigned to input assumptions here. If assumptions are required, remember to check the *Turn on Correlations* preference by clicking on ***Risk Simulator | Edit Simulation Profile***.[1] Note that you can either truncate a distribution or correlate it to another assumption but not both.

---

[1]See the discussion on correlations in *Modeling Risk*, pp. 100–104 for more details about assigning correlations and the effects correlations will have on a model.

*Short descriptions.* These exist for each of the distributions in the gallery. The short descriptions explain when a certain distribution is used as well as the input parameter requirements.[2]

*Regular input and percentile input.* This option allows the user to perform a quick due diligence test of the input assumption. For instance, when setting a normal distribution with some mean and standard deviation inputs, you can click on the percentile input to see what the corresponding 10th and 90th percentiles are.

*Enable dynamic simulation.* This option is unchecked by default, but if you wish to run a multidimensional simulation (i.e., if you link the input parameters of the assumption to another cell that is itself an assumption, you are simulating the inputs, or simulating the simulation), then remember to check this option. Dynamic simulation will not work unless the inputs are linked to other changing input assumptions.

> If you are following along with the example, continue by setting another assumption on cell G9. This time use the Uniform distribution with a minimum value of 0.9 and a maximum value of 1.1. Then, proceed to defining the output forecasts in the next step.

## Defining Output Forecasts

The next step is to define output forecasts in the model. Forecasts can only be defined on output cells with equations or functions. The following describes the set forecast process:

1. Select the cell you wish to set an assumption on (e.g., cell G10 in the Basic Simulation Model example).
2. Click on ***Risk Simulator*** and select ***Set Output Forecast*** or click on the fifth icon on the Risk Simulator icon toolbar (Figure 17.7).
3. Enter the relevant information and click ***OK***.

Figure 17.7 illustrates the set forecast properties:

*Forecast name.* Specify the name of the forecast cell. This is important because when you have a large model with multiple forecast cells, naming the forecast cells individually allows you to access the right results quickly. Do not underestimate the importance of this simple step. Good modeling practice is to use short but precise assumption names.

*Forecast precision.* Instead of relying on a guesstimate of how many trials to run in your simulation, you can set up precision and error controls. When an error-precision combination has been achieved in the simulation, the simulation will pause and inform you of the precision achieved, making the number of simulation trials

[2]See *Modeling Risk*, pp. 107–128, "Understanding Probability Distributions for Monte Carlo Simulation," for details on each distribution type available in the software.

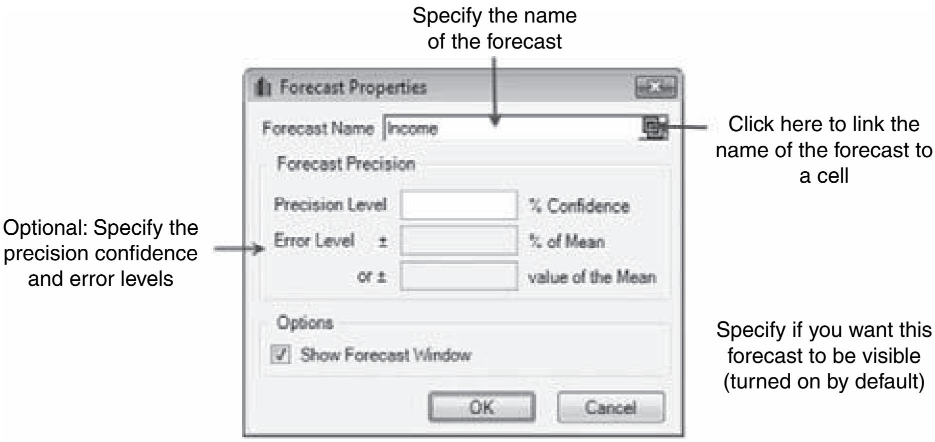



FIGURE 17.7 Set Output Forecast

an automated process and not requiring guesses on the required number of trials to simulate. Review the section on error and precision control for more specific details.[3]

*Show forecast window.* Allows the user to show or not show a particular forecast window. The default is always to show a forecast chart.

## Run Simulation

If everything looks right, simply click on ***Risk Simulator*** | ***Run Simulation*** or click on the Run icon (the 9th icon on the Risk Simulator toolbar) and the simulation will proceed. You may also reset a simulation after it has run to rerun it (***Risk Simulator*** | ***Reset Simulation*** or the 12th icon on the toolbar), or to pause it during a run. Also, the *step* function (***Risk Simulator*** | ***Step Simulation*** or the 11th icon on the toolbar) allows you to simulate a single trial, one at a time, useful for educating others on simulation; that is, you can show that at each trial, all the values in the assumption cells are being replaced and the entire model is recalculated each time.

## Interpreting the Forecast Results

The final step in Monte Carlo simulation is to interpret the resulting forecast charts. Figures 17.8 to 17.15 show the forecast chart and the corresponding statistics generated after running the simulation. Typically, the following are important in interpreting the results of a simulation:

*Forecast chart.* The forecast chart shown in Figure 17.8 is a probability histogram that shows the frequency counts of values occurring in the total number of trials simulated. The vertical bars show the frequency of a particular $x$ value occurring out of the total number of trials, while the cumulative frequency (smooth line) shows the total probabilities of all values at and below $x$ occurring in the forecast.

[3]*Modeling Risk*, pp. 104–106.

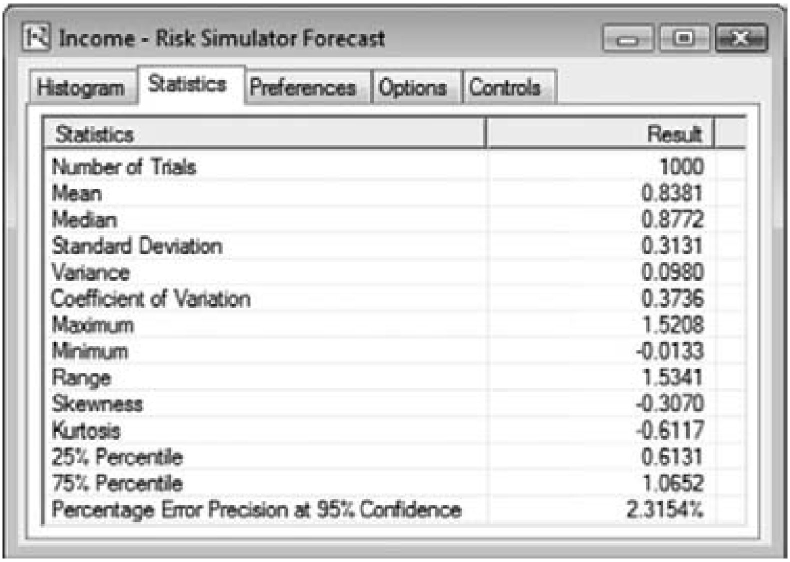

FIGURE 17.8 Forecast Chart

*Forecast statistics*. The forecast statistics shown in Figure 17.9 summarize the distribution of the forecast values in terms of the four moments of a distribution.[4] You can rotate between the histogram and statistics tab by depressing the space bar.

*Preferences*. The preferences tab in the forecast chart allows you to change the look and feel of the charts. For instance, if ***Always on Top*** is selected, the forecast charts will always be visible regardless of what other software are running on your computer. ***Histogram Resolution*** allows you to change the number of bins of the histogram, anywhere from 5 bins to 100 bins. Also, the ***Data Update*** section allows you to control how fast the simulation runs versus how often the forecast chart is updated. That is, if you wish to see the forecast chart updated at almost every trial, this will slow down the simulation as more memory is being allocated to updating the chart versus running the simulation. This is merely a user preference and in no way changes the results of the simulation, just the speed of completing the simulation. To further increase the speed of the simulation, you can minimize Excel while the simulation is running, thereby reducing the memory required to visibly update the Excel spreadsheet and freeing up the memory to run the simulation. The ***Clear All*** and ***Minimize All*** controls all the open forecast charts.

*Options*. This forecast chart option allows you to show all the forecast data or to filter in/out values that fall within some specified interval you choose, or within some standard deviation you choose. Also, the precision level can be set here for this specific forecast to show the error levels in the statistics view. See the section on error and precision control for more details. *Show the Following Statistics* is a

[4]See *Modeling Risk*, pp. 34–48 for more details on what some of these statistics mean.




| Statistics | Result |
|---|---|
| Number of Trials | 1000 |
| Mean | 0.8381 |
| Median | 0.8772 |
| Standard Deviation | 0.3131 |
| Variance | 0.0980 |
| Coefficient of Variation | 0.3736 |
| Maximum | 1.5208 |
| Minimum | -0.0133 |
| Range | 1.5341 |
| Skewness | -0.3070 |
| Kurtosis | -0.6117 |
| 25% Percentile | 0.6131 |
| 75% Percentile | 1.0652 |
| Percentage Error Precision at 95% Confidence | 2.3154% |

FIGURE 17.9 Forecast Statistics

user preference if the mean, median, first quartile, and fourth quartile lines (25th and 75th percentiles) should be displayed on the forecast chart.

*Controls*. This tab has all the functionalities in allowing you to change the type, color, size, zoom, tilt, 3D and other things in the forecast chart, as well as provide overlay charts (PDF, CDF), and runs distributional fitting on your forecast data (see the Data Fitting sections for more details on this methodology).

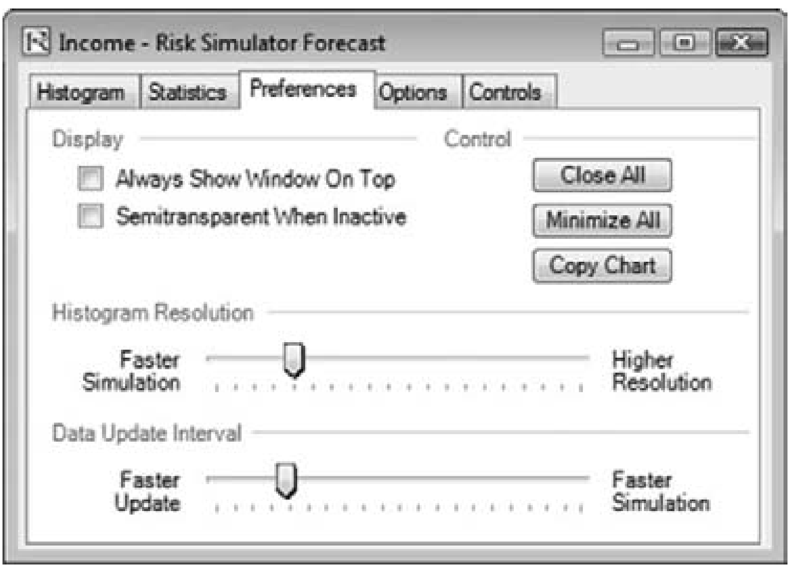



FIGURE 17.10 Forecast Chart Preferences

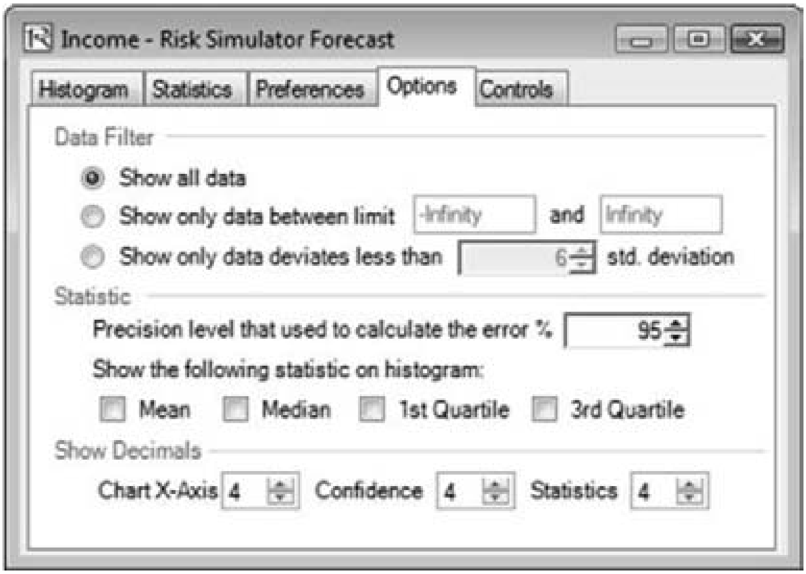



FIGURE 17.11 Forecast Chart Options

## Using Forecast Charts and Confidence Intervals

In forecast charts, you can determine the probability of occurrence, called *confidence intervals*. That is, given two values, what are the chances that the outcome will fall between these two values? Figure 17.12 illustrates that there is a 90% probability


FIGURE 17.12 Forecast Chart Two-Tail Confidence Interval


FIGURE 17.13 Forecast Chart One-Tail Confidence Interval

that the final outcome (in this case, the level of income) will be between \$0.2781 and \$1.3068. The two-tailed confidence interval can be obtained by first selecting Two-Tail as the type, entering the desired certainty value (e.g., 90) and hitting TAB on the keyboard. The two computed values corresponding to the certainty value will then be displayed. In this example, there is a 5% probability that income will be below \$0.2781 and another 5% probability that income will be above \$1.3068. That is, the two-tailed confidence interval is a symmetrical interval centered on the median or 50th percentile value. Thus, both tails will have the same probability.

Alternatively, a one-tail probability can be computed. Figure 17.13 shows a Left-Tail selection at 95% confidence (i.e., choose Left-Tail as the type, enter 95 as the certainty level, and hit TAB on the keyboard). This means that there is a 95% probability that the income will be below \$1.3068 or a 5% probability that income will be above \$1.3068, corresponding perfectly with the results seen in Figure 17.12.

In addition to evaluating what the confidence interval is (i.e., given a probability level and finding the relevant income values), you can determine the probability of a given income value. For instance, what is the probability that income will be less than \$1? To do this, select the Left-Tail probability type, enter 1 into the value input box, and hit TAB. The corresponding certainty will then be computed (in this case, there is a 64.80% probability that income will be below \$1).

For the sake of completeness, you can select the Right-Tail probability type and enter the value 1 in the value input box, and hit TAB. The resulting probability indicates the right-tail probability past the value 1, that is, the probability of income exceeding \$1 (in this case, we see that there is a 35.20% probability of income exceeding \$1).

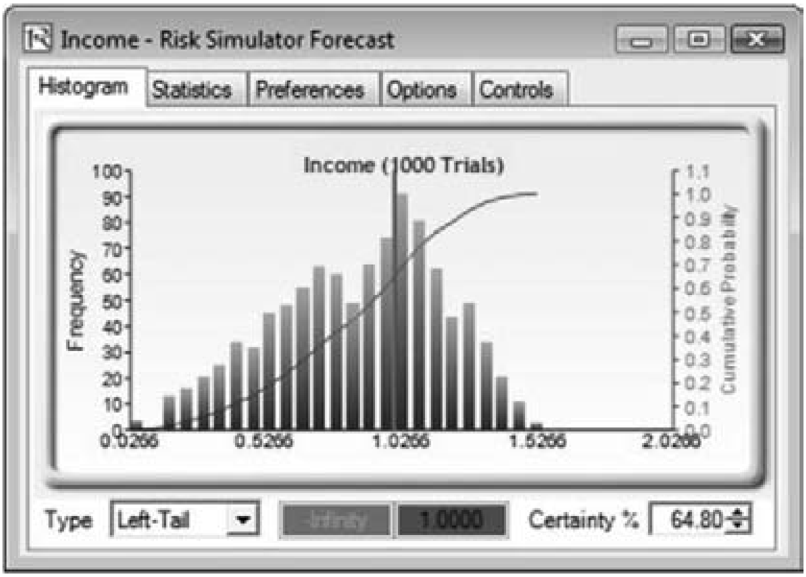



FIGURE 17.14 Forecast Chart Probability Evaluation: Left-Tail

The forecast window is resizable by clicking on and dragging the bottom-right corner of the forecast window. Finally, it is always advisable that before rerunning a simulation, the current simulation should be reset (***Risk Simulator | Reset Simulation***). Remember that you will need to hit TAB on the keyboard to update the chart and results when you type in the certainty values or right- and left-tail values.

## Tornado and Sensitivity Tools in Simulation

### Theory

One powerful simulation tool is *tornado analysis*—it captures the static impacts of each variable on the outcome of the model; the tool automatically perturbs each variable in the model a preset amount, captures the fluctuation in the model's forecast or final result, and lists the resulting perturbations ranked from the most significant to the least. Figures 17.16 through 17.22 illustrate the application of a tornado analysis. For instance, Figure 17.16 is a sample discounted cash flow model where the input assumptions in the model are shown. The question is, What are the critical success drivers that affect the model's output the most? That is, what really drives the net present value of $96.63, or which input variables impact this value the most?

The tornado chart tool can be obtained through ***Risk Simulator | Tools | Tornado Analysis***. To follow along the first example, open the ***Tornado and***


FIGURE 17.15 **Forecast Chart Probability Evaluation: Right-Tail**

***Sensitivity Charts (Linear)*** file in the examples folder. Figure 17.17 shows this sample model where cell G6 containing the net present value is chosen as the target result to be analyzed. The target cell's precedents in the model are used in creating the tornado chart. Precedents are all the input and intermediate variables that affect the outcome of the model. For instance, if the model consists of A = B + C, and where C = D + E, then B, D, and E are the precedents for A (C is not a precedent as it is only an intermediate calculated value). Figure 17.17 also shows the testing range of each precedent variable used to estimate the target result. If the precedent variables are simple inputs, then the testing range will be a simple perturbation based on the range chosen (e.g., the default is ±10%). Each precedent variable can be perturbed at different percentages if required. A wider range is important as it is better able to test extreme values rather than smaller perturbations around the expected values. In certain circumstances, extreme values may have a larger, smaller, or unbalanced impact (e.g., nonlinearities may occur where increasing or decreasing economies of scale and scope creep in for larger or smaller values of a variable) and only a wider range will capture this nonlinear impact.

## Procedure

Use the following steps to create a tornado analysis:

1. Select the single output cell (i.e., a cell with a function or equation) in an Excel model (e.g., cell G6 is selected in our example).
2. Select ***Risk Simulator | Tools | Tornado Analysis***.

| | | | |
|---|---|---|---|
| *Base Year* | 2005 | *Sum PV Net Benefits* | ***$1,896.63*** |
| *Market Risk-Adjusted Discount Rate* | 15.00% | *Sum PV Investments* | ***$1,800.00*** |
| *Private-Risk Discount Rate* | 5.00% | *Net Present Value* | ***$96.63*** |
| *Annualized Sales Growth Rate* | 2.00% | *Internal Rate of Return* | 18.80% |
| *Price Erosion Rate* | 5.00% | *Return on Investment* | 5.37% |
| *Effective Tax Rate* | 40.00% | | |

| | 2005 | 2006 | 2007 | 2008 | 2009 |
|---|---|---|---|---|---|
| Product A Avg Price/Unit | $10.00 | $9.50 | $9.03 | $8.57 | $8.15 |
| Product B Avg Price/Unit | $12.25 | $11.64 | $11.06 | $10.50 | $9.98 |
| Product C Avg Price/Unit | $15.15 | $14.39 | $13.67 | $12.99 | $12.34 |
| Product A Sale Quantity ('000s) | 50.00 | 51.00 | 52.02 | 53.06 | 54.12 |
| Product B Sale Quantity ('000s) | 35.00 | 35.70 | 36.41 | 37.14 | 37.89 |
| Product C Sale Quantity ('000s) | 20.00 | 20.40 | 20.81 | 21.22 | 21.65 |
| **Total Revenues** | ***$1,231.75*** | ***$1,193.57*** | ***$1,156.57*** | ***$1,120.71*** | ***$1,085.97*** |
| Direct Cost of Goods Sold | $184.76 | $179.03 | $173.48 | $168.11 | $162.90 |
| **Gross Profit** | ***$1,046.99*** | ***$1,014.53*** | ***$983.08*** | ***$952.60*** | ***$923.07*** |
| Operating Expenses | $157.50 | $160.65 | $163.86 | $167.14 | $170.48 |
| Sales, General and Admin. Costs | $15.75 | $16.07 | $16.39 | $16.71 | $17.05 |
| **Operating Income (EBITDA)** | ***$873.74*** | ***$837.82*** | ***$802.83*** | ***$768.75*** | ***$735.54*** |
| Depreciation | $10.00 | $10.00 | $10.00 | $10.00 | $10.00 |
| Amortization | $3.00 | $3.00 | $3.00 | $3.00 | $3.00 |
| **EBIT** | ***$860.74*** | ***$824.82*** | ***$789.83*** | ***$755.75*** | ***$722.54*** |
| Interest Payments | $2.00 | $2.00 | $2.00 | $2.00 | $2.00 |
| **EBT** | ***$858.74*** | ***$822.82*** | ***$787.83*** | ***$753.75*** | ***$720.54*** |
| Taxes | $343.50 | $329.13 | $315.13 | $301.50 | $288.22 |
| **Net Income** | ***$515.24*** | ***$493.69*** | ***$472.70*** | ***$452.25*** | ***$432.33*** |
| Noncash: Depreciation Amortization | $13.00 | $13.00 | $13.00 | $13.00 | $13.00 |
| Noncash: Change in Net Working Capital | $0.00 | $0.00 | $0.00 | $0.00 | $0.00 |
| Noncash: Capital Expenditures | $0.00 | $0.00 | $0.00 | $0.00 | $0.00 |
| **Free Cash Flow** | ***$528.24*** | ***$506.69*** | ***$485.70*** | ***$465.25*** | ***$445.33*** |
| | | | | | |
| **Investment Outlay** | ***$1,800.00*** | | | | |
| | | | | | |
| ***Financial Analysis*** | | | | | |
| **Present Value of Free Cash Flow** | ***$528.24*** | ***$440.60*** | ***$367.26*** | ***$305.91*** | ***$254.62*** |
| **Present Value of Investment Outlay** | ***$1,800.00*** | ***$0.00*** | ***$0.00*** | ***$0.00*** | ***$0.00*** |
| **Net Cash Flows** | ($1,271.76) | $506.69 | $485.70 | $465.25 | $445.33 |

FIGURE 17.16 Sample Discounted Cash Flow Model

3. Review the precedents and rename them as appropriate (renaming the precedents to shorter names allows a more visually pleasing tornado and spider chart) and click *OK*.

Alternatively, click on ***Use Cell Address*** to apply cell locations as the variable names.

## Results Interpretation

Figure 17.18 shows the resulting tornado analysis report, which indicates that capital investment has the largest impact on net present value (NPV), followed by tax rate, average sale price, quantity demanded of the product lines, and so forth.

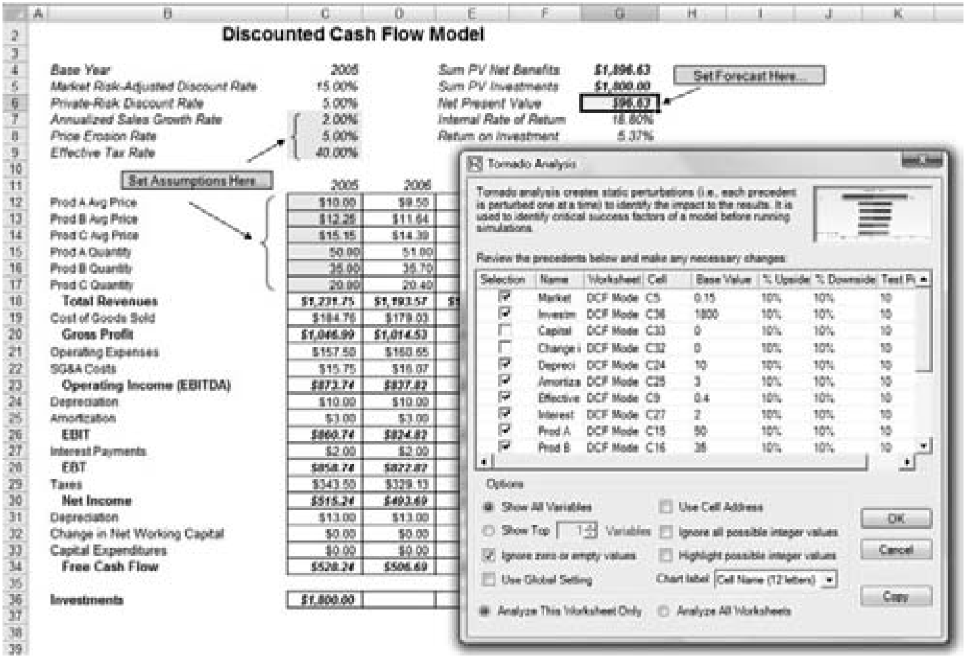



FIGURE 17.17 Running Tornado Analysis

The report contains four distinct elements:

1. The *statistical summary* lists the procedure performed.
2. The *sensitivity table* (Figure 17.19) shows the starting NPV base value of \$96.63 and how each input is changed (e.g., Investment is changed from \$1,800 to \$1,980 on the upside with a +10% swing, and from \$1,800 to \$1,620 on the downside with a −10% swing). The resulting upside and downside values on NPV are −\$83.37 and \$276.63, with a total change of \$360, making it the variable with the highest impact on NPV. The precedent variables are ranked from the highest impact to the lowest impact.
3. The *spider chart* (Figure 17.20) illustrates these effects graphically. The $y$-axis is the NPV target value while the $x$-axis depicts the percentage change on each of the precedent values (the central point is the base case value at \$96.63 at 0% change from the base value of each precedent). Positively sloped lines indicate a positive relationship or effect, whereas negatively sloped lines indicate a negative relationship (e.g., investment is negatively sloped, which means that the higher the investment level, the lower the NPV). The absolute value of the slope indicates the magnitude of the effect computed as the percentage change in the result given a percentage change in the precedent (a steep line indicates a higher impact on the NPV $y$-axis given a change in the precedent $x$-axis).
4. The *tornado chart* (Figure 17.21) illustrates the results in another graphical manner, where the highest impacting precedent is listed first. The $x$-axis is the NPV value with the center of the chart being the base case condition. Green

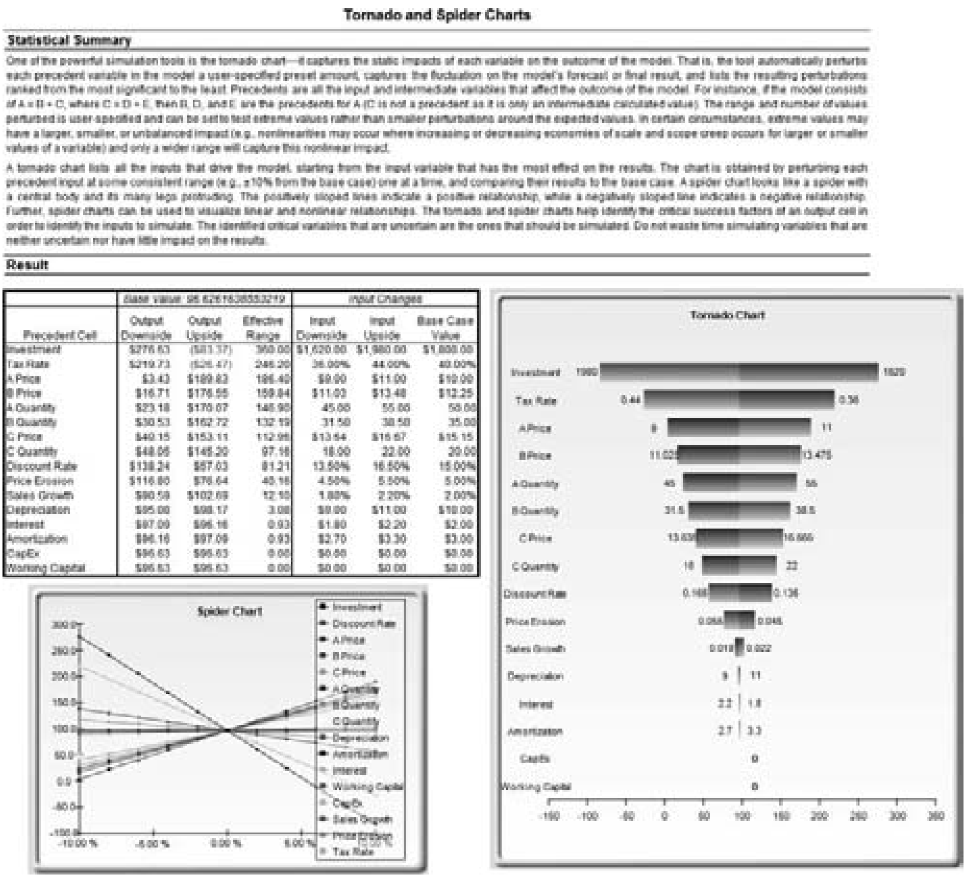



FIGURE 17.18 Tornado Analysis Report

| | Base Value: 96.6261638553219 | | | Input Changes | | |
|---|---|---|---|---|---|---|
| Precedent Cell | Output Downside | Output Upside | Effective Range | Input Downside | Input Upside | Base Case Value |
| Investment | $276.63 | ($83.37) | 360.00 | $1,620.00 | $1,980.00 | $1,800.00 |
| Tax Rate | $219.73 | ($26.47) | 246.20 | 36.00% | 44.00% | 40.00% |
| A Price | $3.43 | $189.83 | 186.40 | $9.00 | $11.00 | $10.00 |
| B Price | $16.71 | $176.55 | 159.84 | $11.03 | $13.48 | $12.25 |
| A Quantity | $23.18 | $170.07 | 146.90 | 45.00 | 55.00 | 50.00 |
| B Quantity | $30.53 | $162.72 | 132.19 | 31.50 | 38.50 | 35.00 |
| C Price | $40.15 | $153.11 | 112.96 | $13.64 | $16.67 | $15.15 |
| C Quantity | $48.05 | $145.20 | 97.16 | 18.00 | 22.00 | 20.00 |
| Discount Rate | $138.24 | $57.03 | 81.21 | 13.50% | 16.50% | 15.00% |
| Price Erosion | $116.80 | $76.64 | 40.16 | 4.50% | 5.50% | 5.00% |
| Sales Growth | $90.59 | $102.69 | 12.10 | 1.80% | 2.20% | 2.00% |
| Depreciation | $95.08 | $98.17 | 3.08 | $9.00 | $11.00 | $10.00 |
| Interest | $97.09 | $96.16 | 0.93 | $1.80 | $2.20 | $2.00 |
| Amortization | $96.16 | $97.09 | 0.93 | $2.70 | $3.30 | $3.00 |
| CapEx | $96.63 | $96.63 | 0.00 | $0.00 | $0.00 | $0.00 |
| Working Capital | $96.63 | $96.63 | 0.00 | $0.00 | $0.00 | $0.00 |

FIGURE 17.19 Sensitivity Table

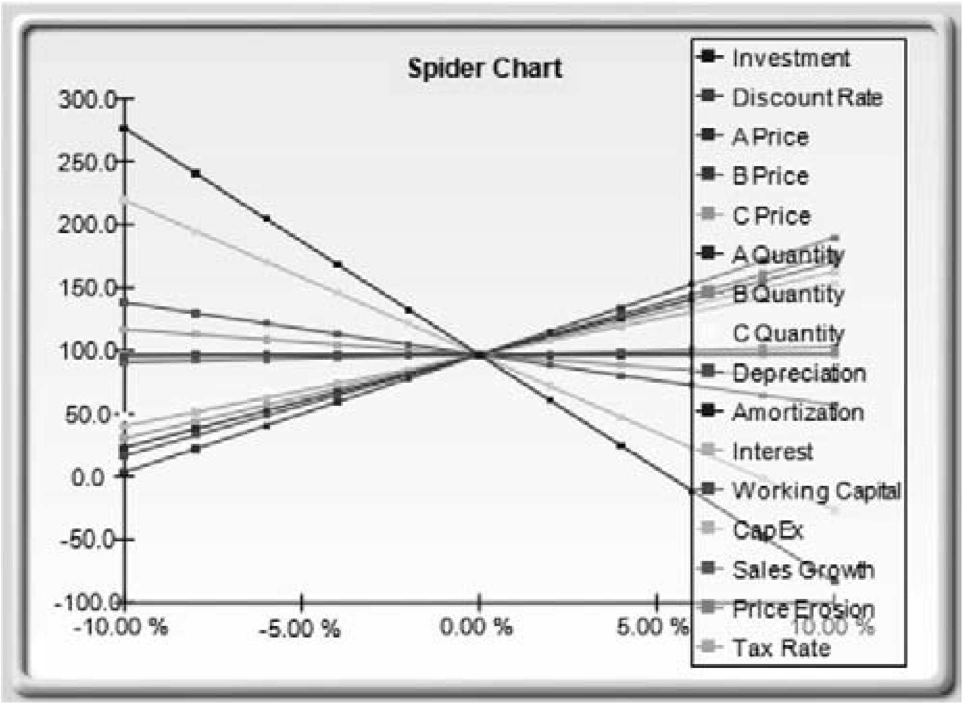



FIGURE 17.20 Spider Chart

(lighter) bars in the chart indicate a positive effect while red (darker) bars indicate a negative effect.[5] Therefore, for investments, the darker bars on the right side indicate a negative effect of investment on higher NPV—in other words, capital investment and NPV are negatively correlated. The opposite is true for price and quantity of products A to C (their lighter bars are on the right side of the chart).

## Notes

Remember that tornado analysis is a static sensitivity analysis applied on each input variable in the model—that is, each variable is perturbed individually and the resulting effects are tabulated. This makes tornado analysis a key component to execute before running a simulation. One of the very first steps in risk analysis is where the most important impact drivers in the model are captured and identified. The next step is to identify which of these important impact drivers are uncertain. These uncertain impact drivers are the critical success drivers of a project, where the results of the model depend on these critical success drivers. These variables are the ones that should be simulated. Do not waste time simulating variables that are neither uncertain nor have little impact on the results. Tornado charts assist in

[5]The colors show up on screen; however, they show up as shades of gray in this book.

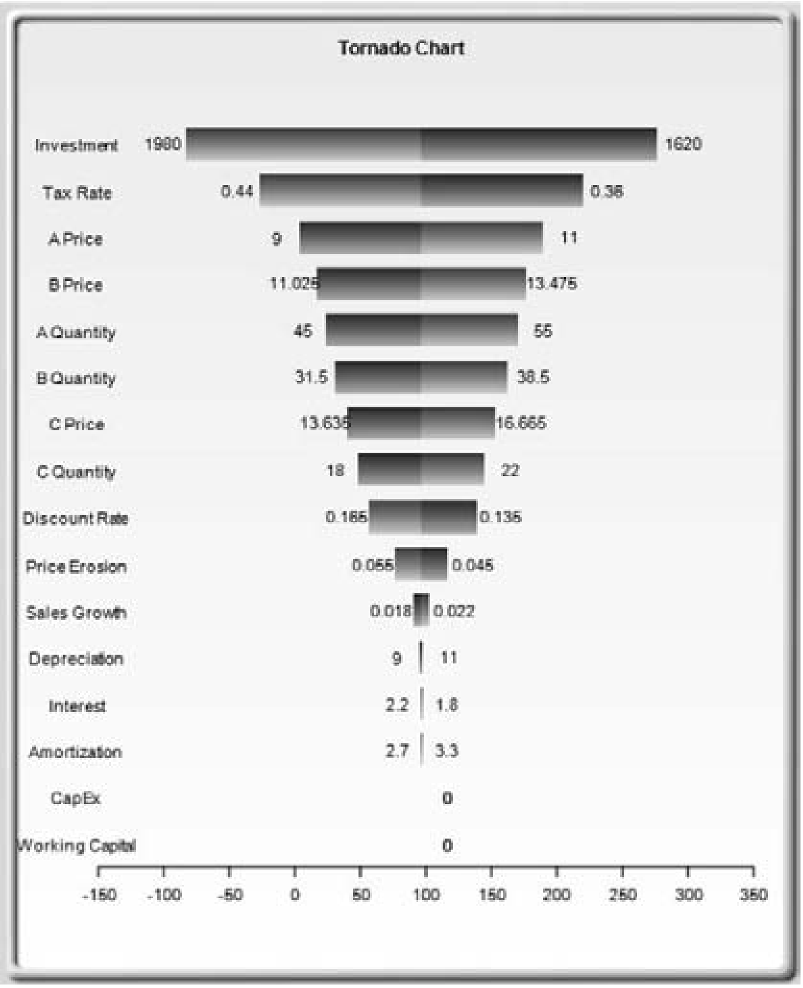



FIGURE 17.21 Tornado Chart

identifying these critical success drivers quickly and easily. Following this example, it might be that price and quantity should be simulated, assuming that the required investment and effective tax rate are both known in advance and unchanging.

Although the tornado chart is easier to read, the spider chart is important to determine whether there are any nonlinearities in the model. For instance, Figure 17.22 shows another spider chart where nonlinearities are fairly evident (the lines on the graph are not straight but curved). The example model used is ***Tornado and Sensitivity Charts (Nonlinear)***, which applies the Black-Scholes option pricing model. Such nonlinearities cannot be ascertained from a tornado chart and may be important information in the model or provide decision makers important insight into the model's dynamics. For instance, in this Black-Scholes model, the fact that stock price and strike price are nonlinearly related to the option value is important to know. This characteristic implies that option value will not increase or decrease

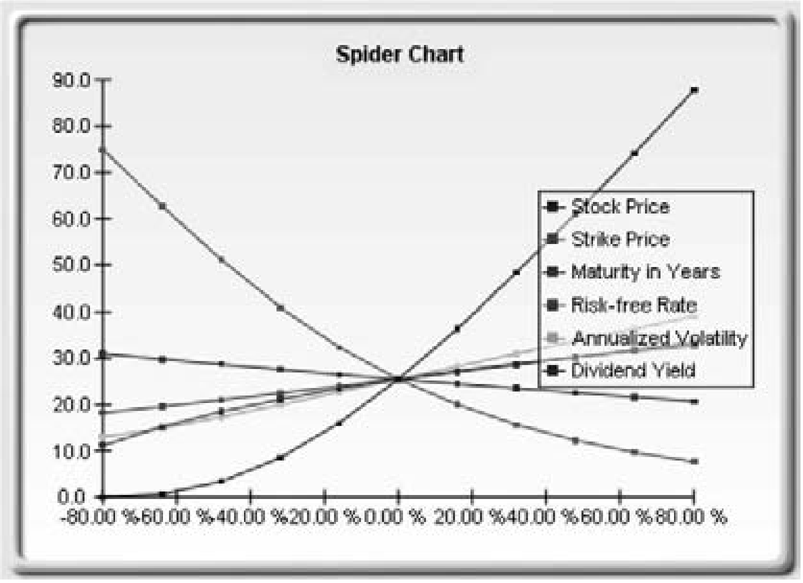



FIGURE 17.22 Nonlinear Spider Chart

proportionally to the changes in stock or strike price, and that there might be some interactions between these two prices as well as other variables. As another example, an engineering model depicting nonlinearities might indicate that a particular part or component, when subjected to a high enough force or tension, will break. Clearly, it is important to understand such nonlinearities.

## Sensitivity Analysis

### Theory

A related feature is *sensitivity analysis*. While tornado analysis (tornado charts and spider charts) applies static perturbations before a simulation run, sensitivity analysis applies dynamic perturbations created after the simulation run. Tornado and spider charts are the results of static perturbations, meaning that each precedent or assumption variable is perturbed a preset amount one at a time, and the fluctuations in the results are tabulated. In contrast, sensitivity charts are the results of dynamic perturbations in the sense that multiple assumptions are perturbed simultaneously and their interactions in the model and correlations among variables are captured in the fluctuations of the results

Tornado charts therefore identify which variables drive the results the most and hence are suitable for simulation, whereas sensitivity charts identify the impact to the results when multiple interacting variables are simulated together in the model.

This effect is clearly illustrated in Figure 17.23. Notice that the ranking of critical success drivers is similar to the tornado chart in the previous examples. However,

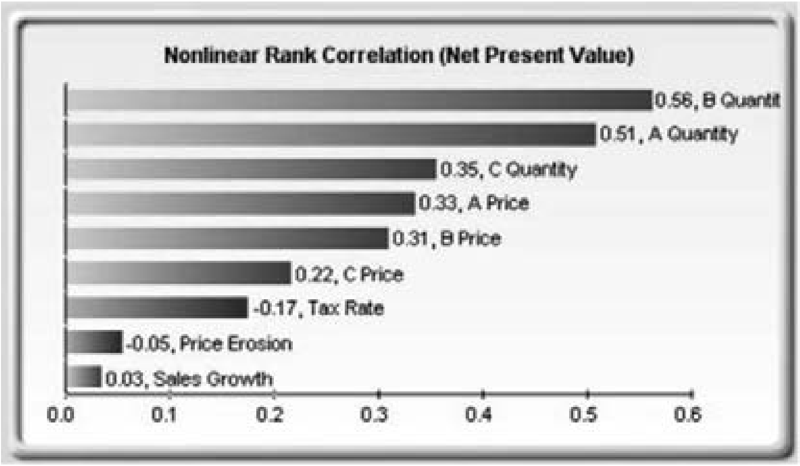



FIGURE 17.23 **Sensitivity Chart without Correlations**

if correlations are added between the assumptions, Figure 17.24 shows a very different picture. Notice, for instance, that price erosion had little impact on NPV, but when some of the input assumptions are correlated, the interaction that exists between these correlated variables makes price erosion have more impact. Note that tornado analysis cannot capture these correlated dynamic relationships. Only after a simulation is run will such relationships become evident in a sensitivity analysis.

A tornado chart's pre-simulation critical success factors will therefore sometimes be different than a sensitivity chart's post-simulation critical success factors. The post-simulation critical success factors should be the ones that are of interest as these more readily capture the model precedents' interactions.

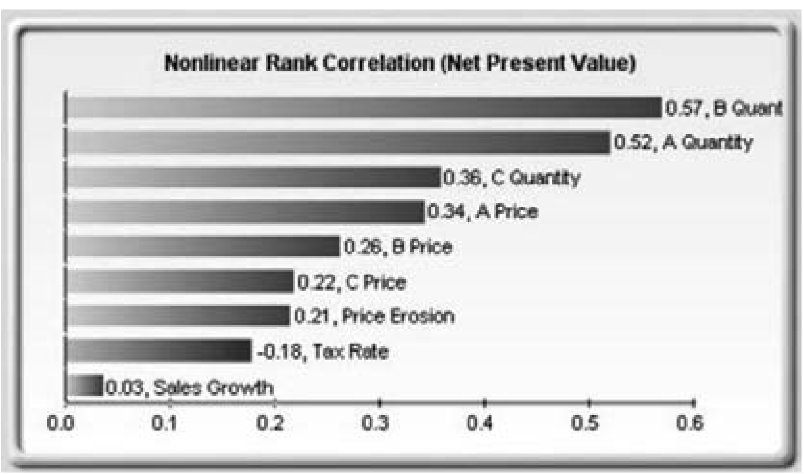



FIGURE 17.24 **Sensitivity Chart with Correlations**

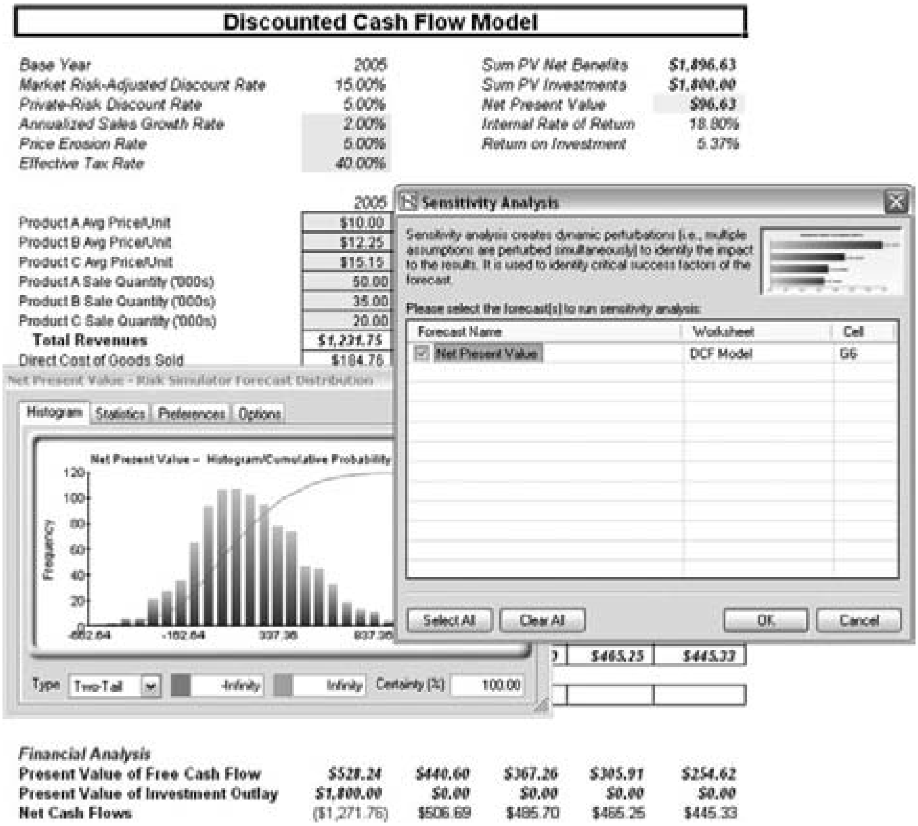



FIGURE 17.25 Running Sensitivity Analysis

## Procedure

Use the following steps to create a sensitivity analysis:

1. Open or create a model, define assumptions and forecasts, and run the simulation—the example here uses the ***Tornado and Sensitivity Charts (Linear)*** file.
2. Select ***Risk Simulator*** | ***Tools*** | ***Sensitivity Analysis***.
3. Select the forecast of choice to analyze and click ***OK*** (Figure 17.25).

Note that sensitivity analysis cannot be run unless assumptions and forecasts have been defined and a simulation has been run.

## Results Interpretation

The results of the sensitivity analysis comprise a report and two key charts. The first is a nonlinear rank correlation chart (Figure 17.26) that ranks from highest to lowest the assumption-forecast correlation pairs. These correlations are nonlinear and nonparametric, making them free of any distributional requirements (i.e., an

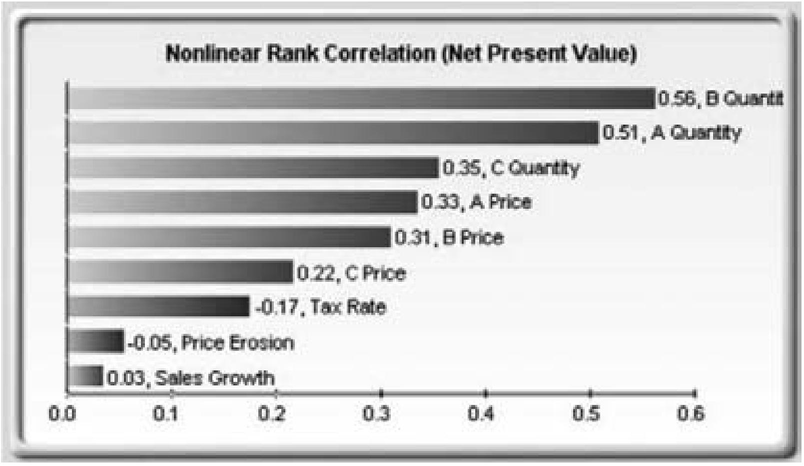



FIGURE 17.26 Rank Correlation Chart

assumption with a Weibull distribution can be compared to another with a Beta distribution).

The results from this chart are fairly similar to that of the tornado analysis seen previously (of course, without the capital investment value, which we decided was a known value and hence was not simulated), with one special exception. Tax rate was relegated to a much lower position in the sensitivity analysis chart (Figure 17.26) as compared to the tornado chart (Figure 17.21). This is because, by itself, the tax rate will have a significant impact, but once the other variables are interacting in the model, it appears that tax rate has less of a dominant effect.[6] This example proves that performing sensitivity analysis after a simulation run is important to ascertain whether there are any interactions in the model and whether the effects of certain variables still hold.

The second chart (Figure 17.27) illustrates the percent variation explained; that is, of the fluctuations in the forecast, how much of the variation can be explained by each of the assumptions after accounting for all the interactions among variables? Notice that the sum of all variations explained is usually close to 100%.[7]

## Notes

Tornado analysis is performed before a simulation run, whereas sensitivity analysis is performed after a simulation run. Spider charts in tornado analysis can consider nonlinearities, whereas rank-correlation charts in sensitivity analysis can account for nonlinear and distributional-free conditions.

---

[6]This is because tax rate has a smaller distribution as historical tax rates tend not to fluctuate too much, and also because tax rate is a straight percentage value of the income before taxes, where other precedent variables have a larger effect on NPV

[7]Sometimes other elements impact the model, but they cannot be captured here directly, and if correlations exist, the sum may sometimes exceed 100% due to the interaction effects that are cumulative.

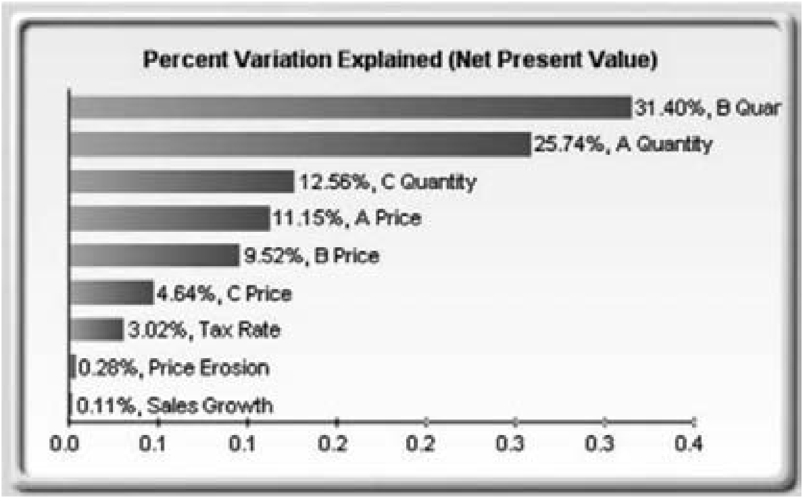



FIGURE 17.27 Contribution to Variance Chart

## Distributional Fitting: Single Variable and Multiple Variables

### Theory

Another powerful simulation tool is *distributional fitting*, that is, which distribution does an analyst or engineer use for a particular input variable in a model? What are the relevant distributional parameters? If no historical data exist, then the analyst must make assumptions about the variables in question. One approach is to use the Delphi method, where a group of experts are tasked with estimating the behavior of each variable. For instance, a group of mechanical engineers can be tasked with evaluating the extreme possibilities of a spring coil's diameter through rigorous experimentation or guesstimates. These values can be used as the variable's input parameters (e.g., uniform distribution with extreme values between 0.5 and 1.2). When testing is not possible (e.g., market share and revenue growth rate), management can still make estimates of potential outcomes and provide the best-case, most-likely case, and worst-case scenarios, whereupon a triangular or custom distribution can be created.

However, if reliable historical data are available, distributional fitting can be accomplished. Assuming that historical patterns hold and that history tends to repeat itself, then historical data can be used to find the best-fitting distribution with its relevant parameters to better define the variables to be simulated. Figures 17.28 through 17.30 illustrate a distributional-fitting example. The following illustration uses the ***Data Fitting*** file in the examples folder.

### Procedure

Use the following steps to perform a distributional fitting model:

1. Open a spreadsheet with existing data for fitting (e.g., use the Data Fitting example file).

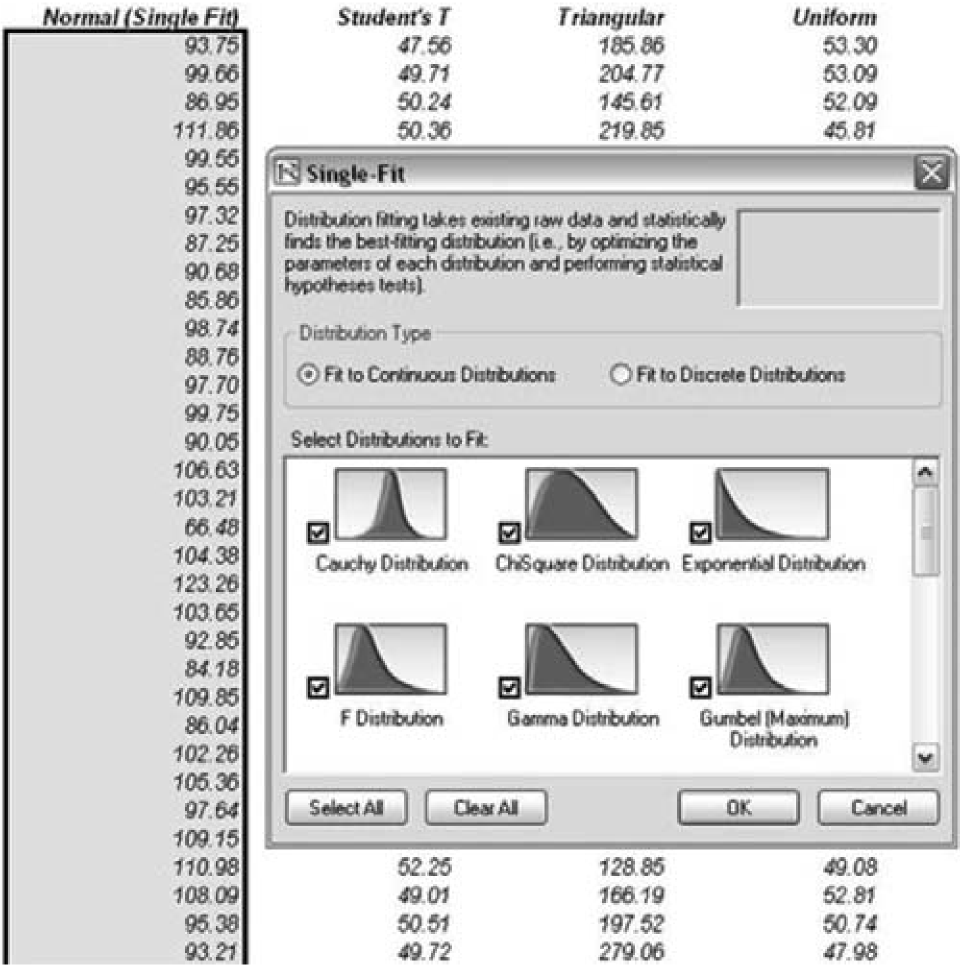



FIGURE 17.28 Single Variable Distributional Fitting

2. Select the data you wish to fit, not including the variable name (data should be in a single column with multiple rows).
3. Select ***Risk Simulator*** | ***Tools*** | ***Distributional Fitting (Single-Variable)***.
4. Select the specific distributions you wish to fit to or keep the default where all distributions are selected and click ***OK*** (Figure 17.28).
5. Review the results of the fit, choose the relevant distribution you want, and click ***OK*** (Figure 17.29).

## Results Interpretation

The null hypothesis ($H_0$) being tested is such that the fitted distribution is the same distribution as the population from which the sample data to be fitted come. Thus, if the computed *p*-value is lower than a critical alpha level (typically 0.10 or 0.05), then the distribution is the wrong distribution. Conversely, the higher the

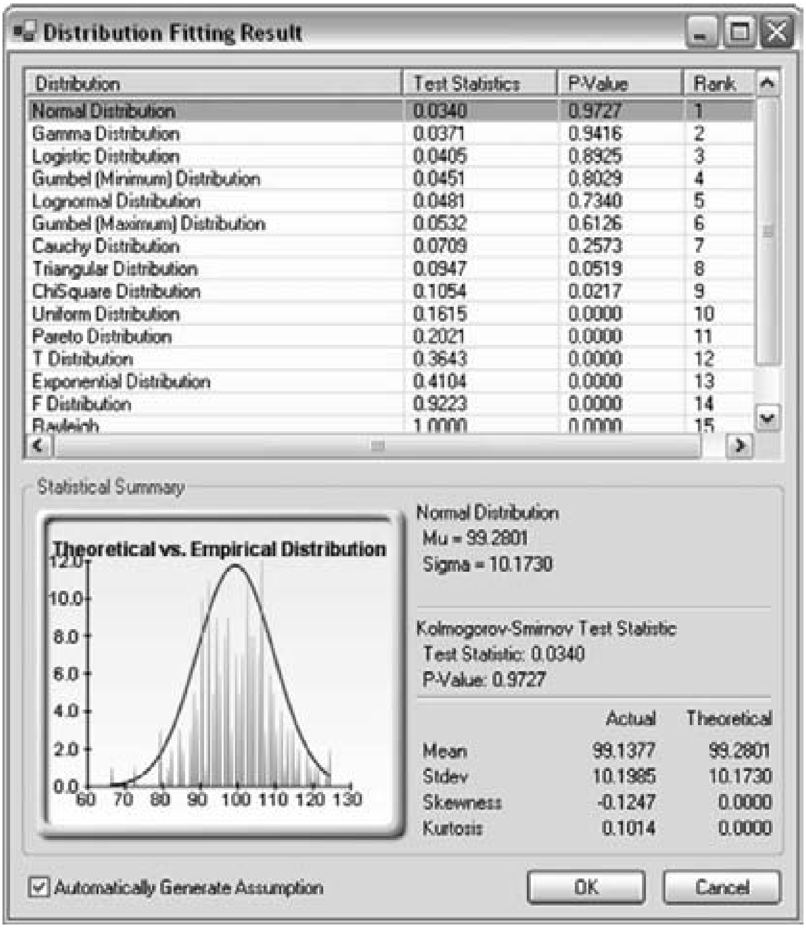



**FIGURE 17.29** Distribution Fitting Result

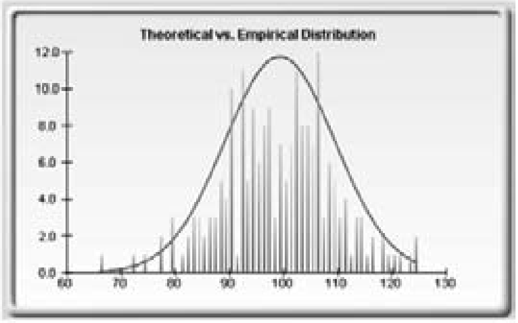
*p*-value, the better the distribution fits the data. Roughly, you can think of *p*-value as a percentage explained; that is, if the *p*-value is 0.9727 (Figure 17.29), then setting a normal distribution with a mean of 99.28 and a standard deviation of 10.17 explains about 97.27% of the variation in the data, indicating an especially good fit. The data was from a 1,000-trial simulation in Risk Simulator based on a normal distribution with a mean of 100 and a standard deviation of 10. Because only 1,000 trials were simulated, the resulting distribution is fairly close to the specified distributional parameters, and in this case, about a 97.27% precision.

Both the results (Figure 17.29) and the report (Figure 17.30) show the test statistic, *p*-value, theoretical statistics (based on the selected distribution), empirical statistics (based on the raw data), the original data (to maintain a record of the data used), and the assumption complete with the relevant distributional parameters (i.e., if you selected the option to automatically generate assumption and if a simulation profile already exists). The results also rank all the selected distributions and how well they fit the data.

**Single Variable Distributional Fitting**

**Statistical Summary**

| | | |
|---|---|---|
| *Fitted Assumption* | 99.14 | |
| *Fitted Distribution* | **Normal Distribution** | |
| *Mu* | 99.28 | |
| *Sigma* | 10.17 | |
| *Kolmogorov-Smirnov Statistic* | 0.03 | |
| *P-Value for Test Statistic* | 0.9737 | |
| | *Actual* | *Theoretical* |
| *Mean* | 99.14 | 99.28 |
| *Standard Deviation* | 10.20 | 10.17 |
| *Skewness* | -0.12 | 0.00 |
| *Excess Kurtosis* | 0.10 | 0.00 |




*Original Fitted Data*

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 93.75 | 99.66 | 86.95 | 111.86 | 99.55 | 95.55 | 97.32 | 87.25 | 90.68 | 85.86 | 98.74 | 88.76 | 97.70 |
| 99.75 | 90.05 | 106.63 | 103.21 | 66.48 | 104.38 | 123.26 | 103.65 | 92.85 | 84.18 | 109.85 | 86.04 | 102.26 |
| 105.36 | 97.64 | 109.15 | 110.98 | 108.09 | 95.38 | 93.21 | 83.86 | 100.17 | 110.17 | 103.72 | 120.52 | 95.09 |
| 115.18 | 83.64 | 90.23 | 92.44 | 92.37 | 92.70 | 110.81 | 72.67 | 104.23 | 96.47 | 121.15 | 94.92 | 77.26 |
| 103.45 | 96.75 | 93.91 | 101.91 | 124.14 | 90.95 | 107.13 | 92.02 | 96.43 | 96.35 | 88.30 | 108.40 | 113.50 |
| 101.40 | 104.72 | 102.43 | 113.59 | 124.15 | 109.24 | 105.34 | 104.57 | 97.83 | 94.39 | 116.19 | 84.66 | 101.17 |
| 106.13 | 107.17 | 95.83 | 106.67 | 92.42 | 79.64 | 94.15 | 106.00 | 113.45 | 92.63 | 94.51 | 93.05 | 96.19 |
| 100.85 | 83.34 | 111.82 | 118.12 | 87.17 | 103.66 | 106.93 | 82.45 | 102.74 | 86.82 | 106.68 | 89.61 | 94.56 |
| 101.34 | 91.32 | 102.02 | 82.51 | 104.46 | 84.72 | 105.05 | 108.40 | 106.59 | 109.43 | 92.49 | 94.52 | 94.00 |
| 105.92 | 88.13 | 96.41 | 101.45 | 79.93 | 89.68 | 102.91 | 114.95 | 92.58 | 94.05 | 107.90 | 111.05 | 90.58 |
| 97.09 | 105.44 | 94.95 | 102.55 | 77.41 | 108.53 | 90.54 | 100.41 | 106.83 | 99.63 | 79.72 | 89.32 | 116.30 |
| 98.27 | 101.73 | 90.84 | 74.45 | 102.24 | 103.34 | 96.51 | 114.55 | 93.94 | 106.29 | 102.95 | 112.73 | 98.09 |
| 108.20 | 105.80 | 106.48 | 102.88 | 104.93 | 103.00 | 99.10 | 108.52 | 101.31 | 88.17 | 90.62 | 96.53 | 106.03 |
| 109.12 | 104.23 | 90.34 | 95.12 | 102.03 | 100.00 | 118.17 | 99.06 | 81.89 | 104.29 | 92.68 | 114.89 | 102.49 |
| 119.21 | 106.20 | 88.26 | 92.45 | 105.15 | 103.79 | 100.84 | 95.19 | 85.10 | 97.25 | 87.65 | 97.58 | 111.44 |
| 99.52 | 89.83 | 97.86 | 90.96 | 97.14 | | | | | | | | |

FIGURE 17.30 Single-Variable Distributional Fitting Report

## Fitting Multiple Variables

For fitting multiple variables, the process is fairly similar to fitting individual variables. However, the data should be arranged in columns (i.e., each variable is arranged as a column) and all the variables are fitted. The same analysis is performed when fitting multiple variables as when single variables are fitted. The difference here is that only the final report will be generated and you do not get to review each variable's distributional rankings. If the rankings are important, run the single-variable fitting procedure instead, on one variable at a time.

## Procedure

1. Open a spreadsheet with existing data for fitting.
2. Select the data you wish to fit (data should be in multiple columns with multiple rows).
3. Select ***Risk Simulator*** | ***Tools*** | ***Distributional Fitting (Multi-Variable)***.
4. Review the data, choose the types of distributions you want to fit to, and click ***OK***.

## Notes

Notice that the statistical ranking methods used in the distributional fitting routines are the *chi-square* test and *Kolmogorov-Smirnov* test. The former is used to test discrete distributions and the latter continuous distributions. Briefly, it is a hypothesis test coupled with the maximum likelihood procedure. An internal optimization routine is used to find the best-fitting parameters on each distribution tested, and the results are ranked from the best fit to the worst fit.

There are other distributional fitting tests, such as the Anderson-Darling, Shapiro-Wilks, and so forth. However, these tests are very sensitive parametric tests and are highly inappropriate in Monte Carlo simulation distribution-fitting routines when different distributions are being tested. Due to their parametric requirements, these tests are most suited for testing normal distributions and distributions with normal-like behaviors (e.g., binomial distribution with a high number of trials and symmetrical probabilities) and will provide less accurate results when performed on nonnormal distributions. Take great care when using such parametric tests. The Kolmogorov-Smirnov and chi-square tests employed in Risk Simulator are nonparametric and semiparametric in nature and are better suited for fitting normal and nonnormal distributions.

## Getting the Risk Simulator Software

Please follow the instructions below to obtain an extended trial license software.

1. Make sure your computer has Windows XP, Vista, or later, as well as Excel 2003, 2007, or later, .NET Framework 2.0 or later (most newer computers and Vista computers have this preinstalled), administrative rights to install software (all personal computers have administrative rights by default, but some corporate computers may have limited rights as they are locked by stringent IT standards, which means you will require the assistance of your company's IT department to install the software for you).
2. Visit www.risksimulator.com or www.realoptionsvaluation.com and click on the ***Downloads*** link. Here you can view all the free getting-started videos, and obtain free case studies, study materials, and software applications. You can try out other software applications available on this page, but the extended trial license offer from this book is only for Risk Simulator. Please scroll down to the Risk Simulator software, review the system requirements, and then download and install the software.
3. After installation, start Excel and you will see the Risk Simulator toolbar and menu. You now have 10 days to play with this software. To get the extended license, download it from www.realoptionsvaluation.com/attachments/abramsmun.zip and unzip the license file and save it to your computer. Start Excel, click on ***Risk Simulator***, ***License***, ***Install License***, and then browse to the license file you just unzipped. It should take merely a few seconds and you are now licensed for another month. If you wish, you can also purchase a permanent license from the same Web site by clicking on ***Purchase***.

# CHAPTER 18

# Real Options


## Dr. Johnathan Mun

*Founder and CEO of Real Options Valuation, Inc.*

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## Part 1: Introduction to Real Options

### Real Options: Theory and Practice

This chapter provides a cursory look at and quick introduction to real options analysis. It explains why merely running simulations, forecasting, and optimization is not sufficient in a comprehensive risk management paradigm. That is, time-series forecasting and Monte Carlo simulation are used for *identifying*, *predicting*, and *quantifying* risks. The question that should be asked is, what next? Quantifying and understanding risk is one thing, but turning this information into *actionable intelligence* is another. Real options analysis, when applied appropriately, allows you to *value* risk, creating strategies to *mitigate* risk, and to position yourself to *take advantage* of risk. It is highly recommended that you refer to *Real Options Analysis: Tools and Techniques, Second Edition* (Johnathan Mun, John Wiley & Sons, 2006), in order to learn more about the theoretical as well as pragmatic step-by-step computational details of real options analysis.

**WHAT ARE REAL OPTIONS?** In the past, corporate investment decisions were cut and dried. Buy a new machine that is more efficient, make more products costing a certain amount, and, if the benefits outweigh the costs, execute the investment. Hire a larger pool of sales associates, expand the current geographical area, and, if the marginal increase in forecast sales revenues exceeds the additional salary and implementation costs, start hiring. Need a new manufacturing plant? Show that the construction costs can be recouped quickly and easily by the increase in revenues the plant will generate through new and improved products, and the initiative is approved.

However, real-life business conditions are a lot more complicated. Your firm decides to go with an e-commerce strategy, but multiple strategic paths exist. Which path do you choose? What options do you have? If you choose the wrong path, how do you get back on the right track? How do you value and prioritize the paths that exist? You are a venture capitalist firm with multiple business plans to consider. How do you value a start-up firm with no proven track record? How do you structure a mutually beneficial investment deal? What is the optimal timing to a second or third round of financing?

Business conditions are fraught with uncertainty and risks. These uncertainties hold with them valuable information. When uncertainty becomes resolved through the passage of time, managers can make the appropriate midcourse corrections through a change in business decisions and strategies. Real options incorporate this learning model, akin to having a strategic road map, whereas traditional analyses that neglect this managerial flexibility will grossly undervalue certain projects and strategies.

Real options are useful not only in valuing a firm through its strategic business options, but also as a strategic business tool in capital investment decisions. For instance, should a firm invest millions in a new e-commerce initiative? How does a firm choose among several seemingly cashless, costly, and unprofitable information-technology infrastructure projects? Should a firm indulge its billions in a risky research and development initiative? The consequences of a wrong decision can be disastrous or even terminal for certain firms. In a traditional discounted cash flow model, these questions cannot be answered with any certainty. In fact, some of the answers generated through the use of the traditional discounted cash flow model are flawed because the model assumes a static, one-time decision-making process, whereas the real options approach takes into consideration the strategic managerial options certain projects create under uncertainty and management's flexibility in exercising or abandoning these options at different points in time, when the level of uncertainty has decreased or has become known over time.

The real options approach incorporates a learning model, such that management makes better and more informed strategic decisions when some levels of uncertainty are resolved through the passage of time. The discounted cash flow analysis assumes a static investment decision and assumes that strategic decisions are made initially with no recourse to choose other pathways or options in the future. To create a good analogy of real options, visualize it as a strategic road map of long and winding roads with multiple perilous turns and branches along the way. Imagine the intrinsic and extrinsic value of having such a road map or global positioning system when navigating through unfamiliar territory, as well as having road signs at every turn to guide you in making the best and most informed driving decisions. Such a strategic map is the essence of real options.

The answer to evaluating such projects lies in real options analysis, which can be used in a variety of settings, including pharmaceutical drug development, oil and gas exploration and production, manufacturing, start-up valuation, venture capital investment, information technology infrastructure, research and development, mergers and acquisitions, e-commerce and e-business, intellectual capital development, technology development, facility expansion, business project prioritization, enterprise-wide risk management, business unit capital budgeting, licenses, contracts, intangible asset valuation, and the like. The following section illustrates some business cases and how real options can assist in identifying and capturing additional strategic value for a firm.

**THE REAL OPTIONS SOLUTION IN A NUTSHELL** Simply defined, real options is a systematic approach and integrated solution using financial theory, economic analysis, management science, decision sciences, statistics, and econometric modeling in applying options theory in valuing real physical assets, as opposed to financial assets, in a dynamic and uncertain business environment where business decisions are flexible in the context of strategic capital investment decision making, valuing investment opportunities, and project capital expenditures.

Real options are crucial in the following:

- Identifying different corporate investment decision pathways or projects that management can navigate given highly uncertain business conditions
- Valuing each of the strategic decision pathways and what it represents in terms of financial viability and feasibility
- Prioritizing these pathways or projects based on a series of qualitative and quantitative metrics
- Optimizing the value of strategic investment decisions by evaluating different decision paths under certain conditions or using a different sequence of pathways that can lead to the optimal strategy
- Timing the effective execution of investments and finding the optimal trigger values and cost or revenue drivers
- Managing existing or developing new optionalities and strategic decision pathways for future opportunities

**ISSUES TO CONSIDER** Strategic options do have significant intrinsic value, but this value is realized only when management decides to execute the strategies. Real options theory assumes that management is logical and competent and that management acts in the best interests of the company and its shareholders through the maximization of wealth and minimization of risk of losses. For example, suppose a firm owns the rights to a piece of land that fluctuates dramatically in price. An analyst calculates the volatility of prices and recommends that management retain ownership for a specified time period, where within this period there is a good chance that the price of real estate will triple. Therefore, management owns a call option, an *option to wait* and defer sale for a particular time period. The value of the real estate is therefore higher than the value that is based on today's sale price. The difference is simply this option to wait. However, the value of the real estate will not command the higher value if prices do triple but management decides not to execute the option to sell. In that case, the price of real estate goes back to its original levels after the specified period, and then management finally relinquishes its rights.

> Strategic optionality value can be obtained only if the option is executed; otherwise, all the options in the world are worthless.

Was the analyst right or wrong? What was the true value of the piece of land? Should it have been valued at its explicit value on a deterministic case where you know what the price of land is right now, and therefore this is its value? Or should it include some types of optionality where there is a good probability that the price of land could triple in value, and hence, the piece of land is truly worth more than it is now and should therefore be valued accordingly? The latter is the real options view.

The additional strategic optionality value can be obtained only if the option is executed; otherwise, all the options in the world are worthless. This idea of *explicit* versus *implicit* value becomes highly significant when management's compensation is tied directly to the actual performance of particular projects or strategies.

To further illustrate this point, suppose the price of the land in the market is currently $10 million. Further, suppose that the market is highly liquid and volatile and that the firm can easily sell off the land at a moment's notice within the next 5 years, the same amount of time the firm owns the rights to the land. If there is a 50% chance the price will increase to $15 million and a 50% chance it will decrease to $5 million within this time period, is the property worth an expected value of $10 million? If the price rises to $15 million, management should be competent and rational enough to execute the option and sell that piece of land immediately to capture the additional $5 million premium. However, if management acts inappropriately or decides to hold off selling in the hopes that prices will rise even farther, the property value may eventually drop back down to $5 million. Now, how much is this property really worth? What if there happens to be an *abandonment option*? Suppose there is a perfect counterparty to this transaction who decides to enter into a contractual agreement whereby, for a contractual fee, the counterparty agrees to purchase the property for $10 million within the next five years, regardless of the market price and executable at the whim of the firm that owns the property. Effectively, a safety net has been created whereby the minimum floor value of the property has been set at $10 million (less the fee paid). That is, there is a limited downside but an unlimited upside, as the firm can always sell the property at market price if it exceeds the floor value. Hence, this strategic *abandonment option* has increased the value of the property significantly. Logically, with this *abandonment option* in place, the value of the land with the option is definitely worth more than $10 million. The real options approach seeks to value this additional inherent flexibility. Real options analysis allows the firm to determine how much this safety downside insurance or abandonment option is worth (i.e., what is the fair market value of the contractual fee to obtain the option), the optimal trigger price (i.e., what price will make it optimal to sell the land), and the optimal timing (i.e., what is the optimal amount of time to hold onto the land).

**IMPLEMENTING REAL OPTIONS ANALYSIS** First, it is vital to understand that real options analysis is *not* a simple set of equations or models. It is an *entire decision-making process* that enhances the traditional decision analysis approaches. It takes tried-and-true financial analytics and evolves it to the next step by pushing the envelope of analytical techniques. In addition, it is vital to understand that 50% of the value in real options analysis is simply thinking about it. Another 25% of the value comes from the number-crunching activities, while the final 25% comes from the results interpretation and explanation to management. Several issues should be considered when attempting to implement real options analysis:

- *Tools.* The correct tools are important. These tools must be more comprehensive than initially required because analysts will grow into them over time. Do not be restrictive in choosing the relevant tools. Always provide room for expansion. Advanced tools will relieve the analyst of detailed model building and let him or her focus instead on 75% of the value—thinking about the problem and interpreting the results. Chapter 3 of *Real Options Analysis* illustrates a quick getting-started in using the Real Options Super Lattice Solver (SLS) software and how even complex and customized real options problems can be solved with great ease. There are significant amounts of theory required to truly understand the use of these software tools; therefore, it is highly recommended that you visit www.realoptionsvaluation.com and click on the ***Downloads*** link to watch some free getting-started videos on using Risk Simulator (for running risk-based Monte Carlo simulation to obtain volatilities and forecast values to use as inputs in the real options analysis models) as well as the Real Options SLS software, to download the latest software trial versions, and to download free videos and other relevant materials.

- *Resources.* The best tools in the world are useless without the relevant human resources to back them up. Tools do not eliminate the analyst, but enhance the analyst's ability to effectively and efficiently execute the analysis. The right people with the right tools will go a long way. Because there are only a few true real options experts in the world who truly understand the theoretical underpinnings of the models as well the practical applications, care should be taken in choosing the correct team. A team of real options experts is vital in the success of the initiative. A company should consider building a team of in-house experts to implement real options analysis and to maintain the ability for continuity, training, and knowledge transfer over time. Knowledge and experience in the theories, implementation, training, and consulting are the core requirements of this team of individuals. This is why training is vital. For instance, the CRA certification program provides analysts and managers the opportunity to immerse themselves in the theoretical and real-life applications of simulation, forecasting, optimization, and real options (see www.realoptionsvaluation.com for details).
- *Senior management buy-in.* The analysis buy-in has to be top-down, where senior management drives the real options analysis initiative. A bottom-up approach, where a few inexperienced junior analysts try to impress the powers that be, will fail miserably.

**INDUSTRY LEADERS EMBRACING REAL OPTIONS** Industries using real options as a tool for strategic decision making started with oil and gas and mining companies and later expanded into utilities, biotechnology, and pharmaceuticals, and now into telecommunications and high-tech, and across all industries. The following examples relate how real options have been or should be used in various companies.

**Automobile and Manufacturing Industry** In the automobile and manufacturing arena, General Motors (GM) applies real options to create *switching options* in producing its new series of autos. This option is essentially to use a cheaper resource over a given period of time. GM holds excess raw materials and has multiple global vendors for similar materials with excess contractual obligations above what it projects as necessary. The excess contractual cost is outweighed by the significant savings of switching vendors when a certain raw material becomes too expensive in a particular region of the world. By spending the additional money in contracting with vendors and meeting their minimum purchase requirements, GM has essentially paid the premium on purchasing a *switching option,* which is important especially when the price of raw materials fluctuates significantly in different regions around the world. Having an option here provides the holder with a hedging vehicle against pricing risks.

**Computer Industry** In the computer industry, HP-Compaq used to forecast sales in foreign countries months in advance. It then configured, assembled, and shipped the highly specifically configured printers to these countries. However, given that demand changes rapidly and forecast figures are seldom correct, the preconfigured printers usually suffer the higher inventory holding cost or the cost of technological obsolescence. HP-Compaq can create an *option to wait* and defer making any decisions too early through building assembly plants in these foreign countries. Parts can then be shipped and assembled in specific configurations when demand is known, possibly weeks in advance rather than months in advance. These parts can be shipped anywhere in the world and assembled in any configuration necessary, while excess parts are interchangeable across different countries. The premium paid on this option is building the assembly plants, and the upside potential is the savings in making wrong demand forecasts.

**Airline Industry** In the airline industry, Boeing spends billions of dollars and several years to decide whether a certain aircraft model should even be built. Should the wrong model be tested in this elaborate strategy, Boeing's competitors may gain a competitive advantage relatively quickly. Because so many technical, engineering, market, and financial uncertainties are involved in the decision-making process, Boeing can conceivably create an *option to choose* through parallel development of multiple plane designs simultaneously, knowing very well the increasing cost of developing multiple designs simultaneously with the sole purpose of eliminating all but one in the near future. The added cost is the premium paid on the option. However, Boeing will be able to decide which model to abandon or continue when these uncertainties and risks become known over time. Eventually, all the models will be eliminated save one. This way, the company can hedge itself against making the wrong initial decision and benefit from the knowledge gained through parallel development initiatives.

**Oil and Gas Industry** In the oil and gas industry, companies spend millions of dollars to refurbish their refineries and add new technology to create an *option to switch* their mix of outputs among heating oil, diesel, and other petrochemicals as a final product, using real options as a means of making capital and investment decisions. This option allows the refinery to switch its final output to one that is more profitable based on prevailing market prices, to capture the demand and price cyclicality in the market.

**Telecommunications Industry** In the telecommunications industry, in the past, companies like Sprint and AT&T installed more fiber-optic cable and other telecommunications infrastructure than other companies in order to create a *growth option* in the future by providing a secure and extensive network, and to create a high barrier to entry, providing a first-to-market advantage. Imagine having to justify to the board of directors the need to spend billions of dollars on infrastructure that will not be used for years to come. Without the use of real options, this decision would have been impossible to justify.

**Utilities Industry** In the utilities industry, firms have created an *option to execute* and an *option to switch* by installing cheap-to-build inefficient energy generator *peaker* plants to be used only when electricity prices are high and to shut down when prices are low. The price of electricity tends to remain constant until it hits a certain capacity utilization trigger level, when prices shoot up significantly. Although this occurs infrequently, the possibility still exists, and by having a cheap standby plant, the firm has created the option to turn on the switch whenever it becomes necessary, to capture this upside price fluctuation.

**Real Estate Industry** In the real estate arena, leaving land undeveloped creates an option to develop at a later date at a more lucrative profit level. However, what is the optimal wait time or the optimal trigger price to maximize returns? In theory, one can wait for an infinite amount of time, and real options provide the solution for the optimal timing and optimal price trigger value.

**Pharmaceutical Research and Development Industry** In pharmaceutical research and development initiatives, real options can be used to justify the large investments in what seems to be cashless and unprofitable under the discounted cash flow method but actually creates *compound expansion options* in the future. Under the myopic lens of a traditional discounted cash flow analysis, the high initial investment of, say, a billion dollars in research and development may return a highly uncertain projected few million dollars over the next few years. Management will conclude under a net present value analysis that the project is not financially feasible. However, a cursory look at the industry indicates that research and development is performed everywhere. Hence, management must see an intrinsic strategic value in research and development. How is this intrinsic strategic value quantified? A real options approach would optimally time and spread the billion-dollar initial investment into a multiple-stage investment structure. At each stage, management has an *option to wait* and see what happens as well as the *option to abandon* or the *option to expand* into the subsequent stages. The ability to defer cost and proceed only if situations are permissible creates value for the investment.

**High-Tech and e-Business Industry** In e-business strategies, real options can be used to prioritize different e-commerce initiatives and to justify those large initial investments that have an uncertain future. Real options can be used in e-commerce to create incremental investment stages compared to a large one-time investment (invest a little now; wait and see before investing more) as well as create *options to abandon* and other future growth options.

**Mergers and Acquisitions** In valuing a firm for acquisition, you should not only consider the revenues and cash flows generated from the firm's operations but also the strategic options that come with the firm. For instance, if the acquired firm does not operate up to expectations, an *abandonment option* can be executed where it can be sold for its intellectual property and other tangible assets. If the firm is highly successful, it can be spun off into other industries and verticals, or new products and services can be eventually developed through the execution of an *expansion option*. In fact, in mergers and acquisitions, several strategic options exist. For instance, a firm acquires other entities to enlarge its existing portfolio of products or geographic locations or to obtain new technology (*expansion option*), or to divide the acquisition into many smaller pieces and sell them off, as in the case of a corporate raider (*abandonment option*); or it merges to form a larger organization due to certain synergies, and immediately lays off many of its employees (*contraction option*). If the seller does not value its real options, it may be leaving money on the negotiation table. If the buyer does not value these strategic options, it is undervaluing a potentially highly lucrative acquisition target.

* * *

All of these cases where the high cost of implementation with no apparent payback in the near future seems foolish and incomprehensible in the traditional discounted cash flow sense are fully justified in the real options sense when taking into account the strategic options the practice creates for the future, the uncertainty of the future operating environment, and management's flexibility in making the right choices at the appropriate time.

**WHAT THE EXPERTS ARE SAYING** The trend in the market is quickly approaching the acceptance of real options, as can be seen from the following sample publication excerpts:

According to an article in *Bloomberg Wealth Manager* (November 2001):

*Real options provide a powerful way of thinking and I can't think of any analytical framework that has been of more use to me in the past five years that I've been in this business.*

According to a *Wall Street Journal* article (February 2000):

*Investors who, after its IPO in 1997, valued only Amazon.com's prospects as a book business would have concluded that the stock was significantly overpriced and missed the subsequent extraordinary price appreciation. Though assessing the value of real options is challenging, without doing it an investor has no basis for deciding whether the current stock price incorporates a reasonable premium for real options or whether the shares are simply overvalued.*

*CFO Europe* (July/August 1999) cites the importance of real options in that:

*A lot of companies have been brainwashed into doing their valuations on a one-scenario discounted cash flow basis and sometimes our recommendations are not what intuition would suggest, and that's where the real surprises come from—and with real options, you can tell exactly where they came from.*

According to a *BusinessWeek* article (June 1999):

*The real options revolution in decision making is the next big thing to sell to clients and has the potential to be the next major business breakthrough. Doing this analysis has provided a lot of intuition you didn't have in the past and . . . as it takes hold, it's clear that a new generation of business analysts will be schooled in options thinking. Silicon Valley is fast embracing the concepts of real options analytics, in its tradition of fail fast so that other options may be sought after.*

In *Products Financiers* (April 1999):

*Real options is a new and advanced technique that handles uncertainty much better than traditional evaluation methods. Because many managers feel that uncertainty is the most serious issue they have to face, there is no doubt that this method will have a bright future as any industry faces uncertainty in its investment strategies.*

A *Harvard Business Review* article (September/October 1998) hits home:

*Unfortunately, the financial tool most widely relied on to estimate the value of a strategy is the discounted cash flow, which assumes that we will follow a predetermined plan regardless of how events unfold. A better approach to valuation would incorporate both the uncertainty inherent in business and the active decision making required for a strategy to succeed. It would help executives to think strategically on their feet by capturing the value of doing just that—of managing actively rather than passively and real options can deliver that extra insight.*

This chapter provides a novel approach to applying real options to answering these issues and more. In particular, a real options framework is presented. It takes into account managerial flexibility in adapting to ever-changing strategic, corporate, economic, and financial environments over time as well as the fact that in the real business world opportunities and uncertainty exist and are dynamic in nature. This chapter provides a real options process framework to identify, justify, time, prioritize, value, and manage corporate investment strategies under uncertainty in the context of applying real options.

The recommendations, strategies, and methodologies outlined here are not meant to replace traditional discounted cash flow analysis but to complement it when the situation and the need arise. The entire analysis could be done, or parts of it could be adapted to a more traditional approach. In essence, the process methodology outlined starts with traditional analyses and continues with value- and insight-adding analytics, including Monte Carlo simulation, forecasting, real options analysis, and portfolio optimization. The real options approach outlined is not the only viable alternative nor will it provide a set of infallible results. However, if utilized correctly with the traditional approaches, it may lead to a set of more robust, accurate, insightful, and plausible results. The insights generated through real options analytics provide significant value in understanding a project's true strategic value.

**CRITICISMS, CAVEATS, AND MISUNDERSTANDINGS IN REAL OPTIONS** Before embarking on a real options analysis, analysts should be aware of several caveats. The following five requirements need to be satisfied before a real options analysis can be run:

1. *A financial model must exist.* Real options analysis requires the use of an existing discounted cash flow model, as real options build on the existing tried-and-true approaches of current financial modeling techniques. If a model does not exist, it means that strategic decisions have already been made and no

financial justifications are required, and hence, there is no need for financial modeling or real options analysis.

2. *Uncertainties must exist.* Otherwise, the option value is worthless. If everything is known for certain in advance, then a discounted cash flow model is sufficient. In fact, when volatility (a measure of risk and uncertainty) is zero, everything is certain, the real options value is zero, and the total strategic value of the project or asset reverts to the net present value in a discounted cash flow model.
3. *Uncertainties must affect decisions when the firm is actively managing the project and these uncertainties must affect the results of the financial model.* These uncertainties will then become risks, and real options can be used to hedge the downside risk and take advantage of the upside uncertainties.
4. *Management must have strategic flexibility or options to make midcourse corrections when actively managing the projects.* Otherwise, do not apply real options analysis when there are no options or management flexibility to value.
5. *Management must be smart enough and credible enough to execute the options when it becomes optimal to do so.* Otherwise, all the options in the world are useless unless they are executed appropriately, at the right time, and under the right conditions.

There are also several criticisms against real options analysis. It is vital that the analyst understands what they are and what the appropriate responses are, prior to applying real options.

- *Real options analysis is merely an academic exercise and is not practical in actual business applications.* Nothing is further from the truth. Although it was true in the past that real options analysis was merely academic, many corporations have begun to embrace and apply real options analysis. Also, its concepts are very pragmatic and with the use of the Real Options *Super Lattice Solver* software, even very difficult problems can be easily solved, as will become evident later in the next few chapters. This chapter and software have helped bring the theoretical a lot closer to practice. Firms are using it and universities are teaching it. It is only a matter of time before real options analysis becomes part of normal financial analysis.
- *Real options analysis is just another way to bump up and incorrectly increase the value of a project to get it justified.* Again, nothing is further from the truth. If a project has significant strategic options but the analyst does not value them appropriately, he or she is leaving money on the table. In fact, the analyst will be incorrectly undervaluing the project or asset. Also, one of the foregoing requirements states that one should never run real options analysis unless strategic options and flexibility exist. If they do not exist, then the option value is zero, but if they do exist, neglecting their valuation will grossly and significantly underestimate the project's or asset's value.
- *Real options analysis ends up choosing the highest-risk projects as the higher the volatility, the higher the option value.* This criticism is also incorrect. The option value is zero if no options exist. However, if a project is highly risky and has high volatility, then real options analysis becomes more important. That is, if a project is strategic but is risky, then you'd better incorporate, create, integrate, or obtain strategic real options to reduce and hedge the downside risk and

take advantage of the upside uncertainties. Therefore, this argument is actually heading in the wrong direction. It is not that real options will overinflate a project's value; but for risky projects, you should create or obtain real options to reduce the risk and increase the upside, thereby increasing the total strategic value of the project. Also, although an option value is always greater than or equal to zero, sometimes the cost to obtain a certain option may exceed its benefit, making the entire strategic value of the option negative, although the option value itself is always zero or positive.

So, it is incorrect to say that real options will always increase the value of a project or that only risky projects are selected. People who make these criticisms do not truly understand how real options work. However, having said that, real options analysis is just another financial analysis tool, and the old axiom of garbage in, garbage out still holds. But if care and due diligence are exercised, the analytical process and results can provide highly valuable insights. In fact, I believe that 50% (rounded, of course) of the challenge and value of real options analysis is simply *thinking about it*. Understanding that you have options, or obtaining options to hedge the risks and take advantage of the upside, and to think in terms of strategic options, is half the battle. Another 25% of the value comes from actually running the analysis and obtaining the results. The final 25% of the value comes from being able to explain it to management, to your clients, and to yourself, such that the results become actionable, and not merely another set of numbers.

## Part 2: Traditional Valuation Approaches

### Introduction

This section begins with an introduction to the traditional analysis, namely, the discounted cash flow model. It showcases some of the limitations and shortcomings through several examples. Specifically, traditional approaches underestimate the value of a project by ignoring the value of its flexibility. Some of these limitations are addressed in greater detail, and potential approaches to correct these shortcomings are also addressed. Further improvements in the areas of more advanced analytics are discussed, including the potential use of Monte Carlo simulation, real options analysis, and portfolio resource optimization.

### The Traditional Views

*Value* is defined as the single time-value discounted number that is representative of all future net profitability. In contrast, the market price of an asset may or may not be identical to its value. (*Assets*, *projects*, and *strategies* are used interchangeably.) For instance, when an asset is sold at a significant bargain, its price may be somewhat lower than its value, and one would surmise that the purchaser has obtained a significant amount of value. The idea of valuation in creating a fair market value is to determine the price that closely resembles the true value of an asset. This true value comes from the physical aspect of the asset as well as the nonphysical, intrinsic or intangible aspect of the asset. Both aspects have the capabilities of generating extrinsic monetary or intrinsic strategic value. Traditionally, there are three mainstream approaches to valuation, namely, the market approach, the income approach, and the cost approach.

## Market Approach

The market approach looks at comparable assets in the marketplace and their corresponding prices and assumes that market forces will tend to move the market price to an equilibrium level. It is further assumed that the market price is also the fair market value after adjusting for transaction costs and risk differentials. Sometimes a market-, industry-, or firm-specific adjustment is warranted, to bring the comparables closer to the operating structure of the firm whose asset is being valued. These approaches could include common-sizing the comparable firms—performing quantitative screening using criteria that closely resemble the firm's industry, operations, size, revenues, functions, profitability levels, operational efficiency, competition, market, and risks.

## Income Approach

The income approach looks at the future potential profit or free-cash-flow-generating potential of the asset and attempts to quantify, forecast, and discount these net free cash flows to a present value. The cost of implementation, acquisition, and development of the asset is then deducted from this present value of cash flows to generate a net present value. Often, the cash flow stream is discounted at a firm-specified hurdle rate, at the weighted average cost of capital, or at a risk-adjusted discount rate based on the perceived project-specific risk, historical firm risk, or overall business risk.

## Cost Approach

The cost approach looks at the cost a firm would incur if it were to replace or reproduce the asset's future profitability potential, including the cost of its strategic intangibles, if the asset were to be created from the ground up. Although the financial theories underlying these approaches are sound in the more traditional deterministic view, they cannot be reasonably used in isolation when analyzing the true strategic flexibility value of a firm, project, or asset.

## Other Approaches

Other approaches used in valuation, more appropriately applied to the valuation of intangibles, rely on quantifying the economic viability and economic gains the asset brings to the firm. There are several well-known methodologies to intangible-asset valuation, particularly in valuing trademarks and brand names. These methodologies apply the combination of the market, income, and cost approaches just described.

The first method compares pricing strategies and assumes that by having some dominant market position by virtue of a strong trademark or brand recognition—for instance, Coca-Cola—the firm can charge a premium price for its product. Hence, if we can find market comparables producing similar products, in similar markets, performing similar functions, and facing similar market uncertainties and risks, the price differential would then pertain exclusively to the brand name. These comparables are generally adjusted to account for the different conditions under which the firms operate. This price premium per unit is then multiplied by the projected quantity of sales, and the outcome after performing a discounted cash flow analysis will be the residual profits allocated to the intangible. A similar argument can be set forth in using operating profit margin in lieu of price per unit. Operating profit before taxes is used instead of net profit after taxes because it avoids the problems of comparables having different capital structure policies or carryforward net operating losses and other tax-shield implications.

Another method uses a common-size analysis of the profit-and-loss statements between the firm holding the asset and market comparables. This takes into account any advantage from economies of scale and economies of scope. The idea here is to convert the income statement items as a percentage of sales, and balance sheet items as a percentage of total assets. In addition, in order to increase comparability, the ratio of operating profit to sales of the comparable firm is then multiplied by the asset-holding firm's projected revenue structure, thereby eliminating the potential problem of having to account for differences in economies of scale and scope. This approach uses a percentage of sales, return on investment, or return on asset ratio as the common-size variable.

## Practical Issues Using Traditional Valuation Methodologies

The traditional valuation methodology relying on a discounted cash flow series does not get at some of the intrinsic attributes of the asset or investment opportunity. Traditional methods assume that the investment is an all-or-nothing strategy and do not account for managerial flexibility that exists such that management can alter the course of an investment over time when certain aspects of the project's uncertainty become known. One of the value-added components of using real options is that it takes into account management's ability to create, execute, and abandon strategic and flexible options.

There are several potential problem areas in using a traditional discounted cash flow calculation on strategic optionalities. These problems include undervaluing an asset that currently produces little or no cash flow, the non-constant nature of the weighted average cost of capital discount rate through time, the estimation of an asset's economic life, forecast errors in creating the future cash flows, and insufficient tests for plausibility of the final results. Real options, when applied using an options theoretical framework, can mitigate some of these problematic areas. Otherwise, financial profit-level metrics, such as NPV or internal rate of return (IRR), will be skewed and not provide a comprehensive view of the entire investment value. However, the discounted cash flow model does have its merits.

### DISCOUNTED CASH FLOW ADVANTAGES

- Clear, consistent decision criteria for all projects.
- Same results regardless of risk preferences of investors.
- Quantitative, decent level of precision and economically rational.

- Not as vulnerable to accounting conventions (depreciation, inventory valuation, etc).
- Factors in the time value of money and risk structures.
- Relatively simple, widely taught, and widely accepted.
- Simple to explain to management: "If benefits outweigh the costs, do it!"

In reality, an analyst should be aware of several issues prior to using discounted cash flow models, as shown in Table 18.1. The most important aspects include the business reality that risks and uncertainty abound when decisions have to be made and that management has the strategic flexibility to make and change decisions as these uncertainties become known over time. In such a stochastic world, using

Table 18.1 **Disadvantages of DCF: Assumptions versus Realities**

| DCF Assumptions | Realities |
|---|---|
| Decisions are made now, and cash flow streams are fixed for the future. | Uncertainty and variability in future outcomes. Not all decisions are made today as some may be deferred to the future, when uncertainty becomes resolved. |
| Projects are "mini-firms," and they are interchangeable with whole firms. | With the inclusion of network effects, diversification, interdependencies, and synergy, firms are portfolios of projects and their resulting cash flows. Sometimes projects cannot be evaluated as stand-alone cash flows. |
| Once launched, all projects are passively managed. | Projects are usually actively managed through project lifecycle, including checkpoints, decision options, budget constraints, etc. |
| Future free cash flow streams are all highly predictable and deterministic. | It may be difficult to estimate future cash flows as they are usually stochastic and risky in nature. |
| Project discount rate used is the opportunity cost of capital, which is proportional to nondiversifiable risk. | There are multiple sources of business risks with different characteristics, and some are diversifiable across projects or time. |
| All risks are completely accounted for by the discount rate. | Firm and project risk can change during the course of a project. |
| All factors that could affect the outcome of the project and value to the investors are reflected in the DCF model through the NPV or IRR. | Because of project complexity and so-called externalities, it may be difficult or impossible to quantify all factors in terms of incremental cash flows. Distributed, unplanned outcomes (e.g., strategic vision and entrepreneurial activity) can be significant and strategically important. |
| Unknown, intangible, or immeasurable factors are valued at zero. | Many of the important benefits are intangible assets or qualitative strategic positions. |

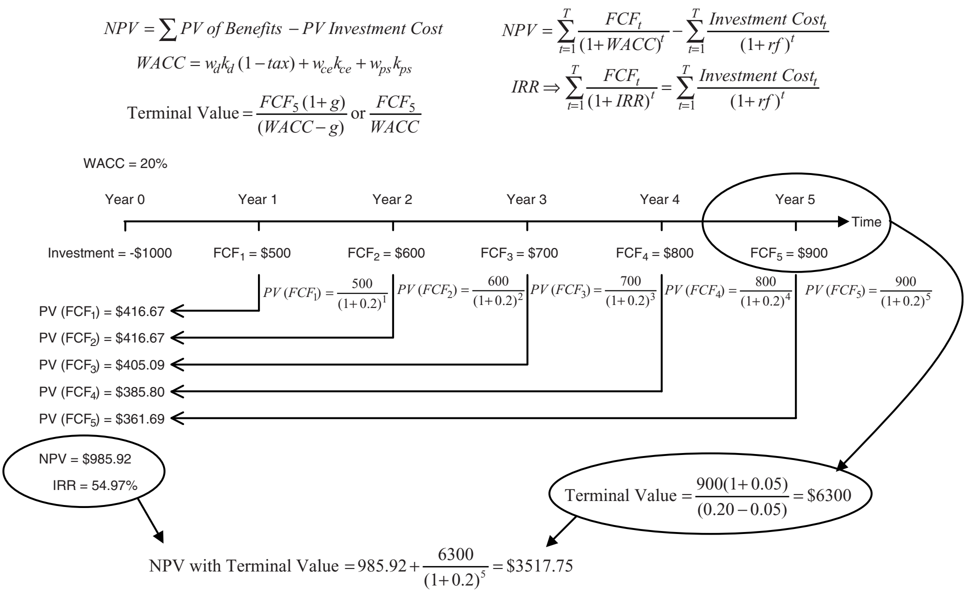


**FIGURE 18.1** **Applying Discounted Cash Flow Analysis**

deterministic models like the discounted cash flow may potentially grossly underestimate the value of a particular project. A deterministic discounted cash flow model assumes at the outset that all future outcomes are fixed. If this is the case, then the discounted cash flow model is correctly specified as there would be no fluctuations in business conditions that would change the value of a particular project. In essence, there would be no value in flexibility. However, the actual business environment is highly fluid, and if management has the flexibility to make appropriate changes when conditions differ, then there is indeed value in flexibility, a value that will be grossly underestimated using a discounted cash flow model.

Figure 18.1 shows a simple example of applying discounted cash flow analysis. Assume that there is a project that costs $1,000 to implement at year 0 that will bring in the following projected positive cash flows in the subsequent five years: $500, $600, $700, $800, and $900. These projected values are simply subjective best-guess forecasts on the part of the analyst. As can be seen in Figure 18.1, the timeline shows all the pertinent cash flows and their respective discounted present values. Assuming that the analyst decides that the project should be discounted at a 20% risk-adjusted discount rate using a weighted average cost of capital (WACC), we calculate the NPV to be $985.92 and a corresponding IRR of 54.97%.[1] Furthermore, the analyst assumes that the project will have an infinite economic life and assumes a long-term growth rate of cash flows of 5%. Using the Gordon constant growth model, the analyst calculates the terminal value of the project's cash flow at year 5 to be $6,300. Discounting this figure for five years at the risk-adjusted discount rate

[1]The NPV is simply the sum of the present values of future cash flows less the implementation cost. The IRR is the implicit discount rate that forces the NPV to be zero. Both calculations can be easily performed in Excel using its "NPV()" and "IRR()" functions.

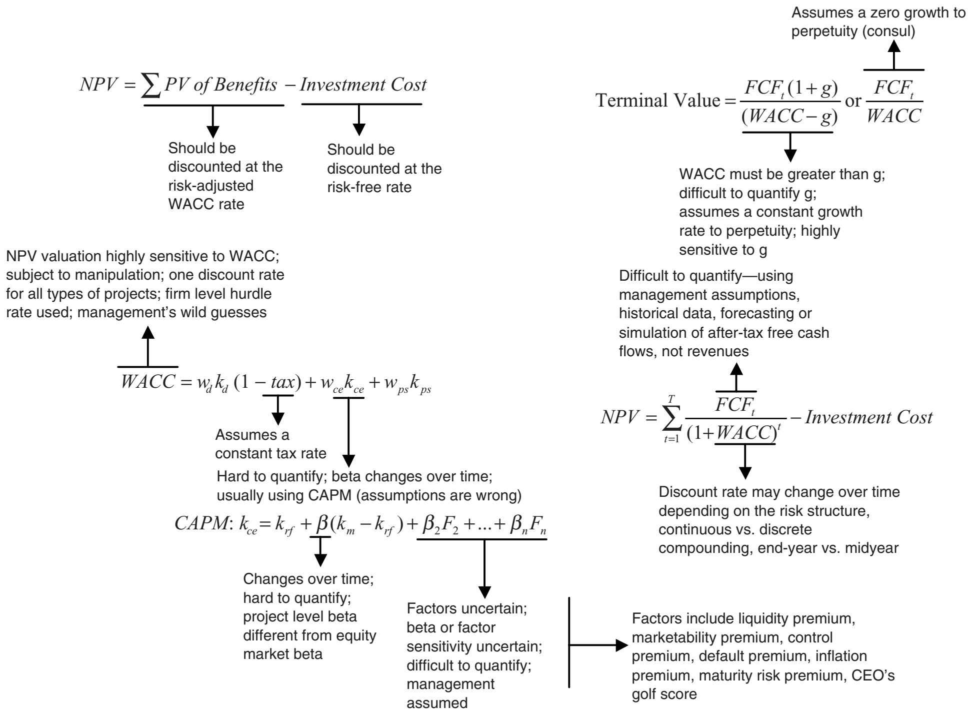


**FIGURE 18.2** **Shortcomings of Discounted Cash Flow Analysis**

and adding it to the original NPV yields a total NPV with terminal value of $3,517.75. The calculations can all be seen in Figure 18.1, where we further define $w$ as the weights, $d$ for debt, $ce$ for common equity and $ps$ for preferred stocks, $FCF$ as the free cash flows, $tax$ as the corporate tax rate, $g$ as the long-term growth rate of cash flows, and $rf$ as the risk-free rate.

Even with a simplistic discounted cash flow model like this, we can see the many shortcomings of using a discounted cash flow model that are worthy of mention. Figure 18.2 lists some of the more noteworthy issues. For instance, the NPV is calculated as the present value of future net free cash flows (benefits) less the present value of implementation costs (investment costs). However, in many instances, analysts tend to discount both benefits and investment costs at a single identical market risk-adjusted discount rate, usually the WACC. This, of course, is flawed.

> Variables with market risks should be discounted at a market risk-adjusted rate, which is higher than the risk-free rate, which is used to discount variables with private risks.

The benefits should be discounted at a market risk-adjusted discount rate like the WACC, but the investment cost should be discounted at a reinvestment rate similar to the risk-free rate. Cash flows that have market risks should be discounted at the market risk-adjusted rate, while cash flows that have private risks should be discounted at the risk-free rate because the market will compensate the firm only for taking on the market risks but not private risks. It is usually assumed that the benefits are subject to market risks (because benefit free cash flows depend on market demand, market prices, and other exogenous market factors) while investment costs depend on internal private risks (such as the firm's ability to complete building a project in a timely fashion or the costs and inefficiencies incurred beyond what is projected). On occasion, these implementation costs may also be discounted at a rate slightly higher than a risk-free rate, such as a money-market rate or at the opportunity cost of being able to invest the sum in another project yielding a particular interest rate. Suffice it to say that benefits and investment costs should be discounted at different rates if they are subject to different risks. Otherwise, discounting the costs at a much higher market risk-adjusted rate will reduce the costs significantly, making the project look as though it were more valuable than it actually is.

The discount rate that is used is usually calculated from a WACC, capital asset-pricing model (CAPM), multiple asset-pricing theory (MAPT), arbitrage pricing theory (APT), set by management as a requirement for the firm, or as a hurdle rate for specific projects.[2] In most circumstances, if we were to perform a simple discounted cash flow model, the most sensitive variable is usually the discount rate. The discount rate is also the most difficult variable to correctly quantify. Hence, this leaves the discount rate to potential abuse and subjective manipulation. A target NPV value can be obtained by simply massaging the discount rate to a suitable level.

In addition, certain input assumptions required to calculate the discount rate are also subject to question. For instance, in the WACC, the input for cost of common equity is usually derived using some form of the CAPM. In the CAPM, the infamous beta ($\beta$) is extremely difficult to calculate. In financial assets, we can obtain beta through a simple calculation of the covariance between a firm's stock prices and the market portfolio, divided by the variance of the market portfolio. Beta is then a sensitivity factor measuring the co-movements of a firm's equity prices with respect to the market. The problem is that equity prices change every few minutes! Depending on the timeframe used for the calculation, beta may fluctuate wildly. In addition, for nontraded physical assets, we cannot reasonably calculate beta this way. Using a firm's tradable financial assets' beta as a proxy for the beta on a project within a firm that has many other projects is ill advised.

There are risk-and-return diversification effects among projects as well as investor psychology and overreaction in the market that are not accounted for. There are also other more robust asset-pricing models that can be used to estimate a project's discount rate, but they require great care. For instance, the APT models are built on the CAPM and have additional risk factors that may drive the value of the discount rate. These risk factors include maturity risk, default risk, inflation risk, country risk, size risk, nonmarketable risk, control risk, minority shareholder risk, and others. Even the firm's CEO's golf score can be a risk hazard (e.g., rash decisions may be made after a bad game or bad projects may be approved after a hole-in-one, believing in a lucky streak). The issue arises when one has to decide

[2]See *Real Options Analysis*, Appendix 2B for a more detailed discussion on discount rate models.

which risks to include and which not to include. This is definitely a difficult task, to say the least.[3]

> The methods to find a relevant discount rate include using a weighted average cost of capital (WACC), capital asset-pricing model (CAPM), arbitrage pricing theory (APT), multifactor asset-pricing theory (MAPT), comparability analysis, management assumptions, and a firm- or project-specific hurdle rate.

One other widely used method is that of comparability analysis. By gathering publicly available data on the trading of financial assets by stripped-down entities with similar functions, markets, risks, and geographical locations, analysts can then estimate the beta (a measure of systematic risk) or even a relevant discount rate from these comparable firms. For instance, an analyst who is trying to gather information on a research-and-development effort for a particular type of drug can conceivably gather market data on pharmaceutical firms performing research and development only on similar drugs, existing in the same market, and having the same risks. The median or average beta value can then be used as a market proxy for the project currently under evaluation. Obviously, there is no silver bullet, but if an analyst were diligent enough, he or she could obtain estimates from these different sources and create a better estimate. Monte Carlo simulation is most preferred in situations like these. The analyst can define the relevant simulation inputs using the range obtained from the comparable firms and simulate the discounted cash flow model to obtain the range of relevant variables (typically the NPV and IRR).

Now that you have the relevant discount rate, the free cash flow stream should then be discounted appropriately. Herein lies another problem: forecasting the relevant free cash flows and deciding whether they should be discounted on a continuous basis or a discrete basis, versus using end-of-year or midyear conventions. Free cash flows should be net of taxes, with the relevant noncash expenses added back.[4] Because free cash flows are generally calculated starting with revenues and proceeding through direct cost of goods sold, operating expenses, depreciation expenses, interest payments, taxes, and so forth, there is certainly room for mistakes to compound over time.

Forecasting cash flows several years into the future is often very difficult and may require the use of fancy econometric regression modeling techniques, time-series analysis, management hunches, and experience. A recommended method is not to create single-point estimates of cash flows at certain time periods but to use Monte Carlo simulation and assess the relevant probabilities of cash flow events. In addition, because cash flows in the distant future are certainly riskier than in the near future, the relevant discount rate should also change to reflect this. Instead of using a single discount rate for all future cash flow events, the discount rate should incorporate

[3]A multiple regression or principal component analysis can be performed but probably with only limited success for physical assets as opposed to financial assets because there are usually very little historical data available for such analyses.

[4]*Real Options Analysis*, Appendix 2A provides details on calculating free cash flows from financial statements.

the changing risk structure of cash flows over time. This can be done by either weighing the cash flow streams' probabilistic risks (standard deviations of forecast distributions) or using a stepwise technique of adding the maturity risk premium inherent in U.S. Treasury securities at different maturity periods. This bootstrapping approach allows the analyst to incorporate what the market experts predict the future market risk structure looks like. That is, discount the cash flows twice: once for time value of money, and once for risk. This way, changes in risk structure and risk-free rate can be adjusted accordingly over time.

Finally, the issue of *terminal value* is of major concern for anyone using a discounted cash flow model. Several methods of calculating terminal values exist, such as the Gordon constant growth model (GGM), zero-growth perpetuity consul, and the supernormal growth models. The GGM is the most widely used, where at the end of a series of forecast cash flows, the GGM assumes that cash flow growth will be constant through perpetuity. The GGM is calculated as the free cash flow at the end of the forecast period multiplied by a relative growth rate, divided by the discount rate less the long-term growth rate. Shown in Figure 18.2, we see that the GGM breaks down when the long-term growth rate exceeds the discount rate. This growth rate is also assumed to be fixed, and the entire terminal value is highly sensitive to this growth rate assumption. In the end, the value calculated is highly suspect because a small difference in growth rates will mean a significant fluctuation in value. Perhaps a better method is to assume some types of growth curves in the free cash flow series. These growth curves can be obtained through some basic time-series analysis as well as using more advanced assumptions in stochastic modeling. Nonetheless, we see that even a well-known, generally accepted and applied discounted cash flow model has analytical restrictions and problems. These problems are rather significant and can compound over time, creating misleading results. Great care should be taken when performing such analyses.

Due to limited space in this book, please refer to my *Modeling Risk* and *Real Options Analysis, Second Edition* (John Wiley & Sons, 2006) books for details on theory and hands-on applications of Monte Carlo risk simulation, real options, and portfolio optimization. These new analytical methods address some of the issues discussed earlier. However, it should be stressed that these new analytics do not provide the silver bullet for valuation and decision making. They provide value-added insights, and the magnitude of insights and value obtained from these new methods depend solely on the type and characteristic of the project under evaluation.

The applicability of traditional analysis versus the new analytics across a time horizon is depicted in Figure 18.3. During the shorter time period, holding everything else constant, the ability for the analyst to predict the near future is greater than when the period extends beyond the historical and forecast periods. This is because the longer the horizon, the harder it is to fully predict all the unknowns, and hence, management can create value by being able to successfully initiate and execute strategic options.

The traditional and new analytics can also be viewed as a matrix of approaches as seen in Figure 18.4, where the analytics are segregated by analytical perspective and type. With regard to perspective, the analytical approach can be either a top-down or a bottom-up approach. A top-down approach implies a higher focus on macro variables than on micro variables. The level of granularity from the macro to micro levels includes starting from the global perspective, and working through

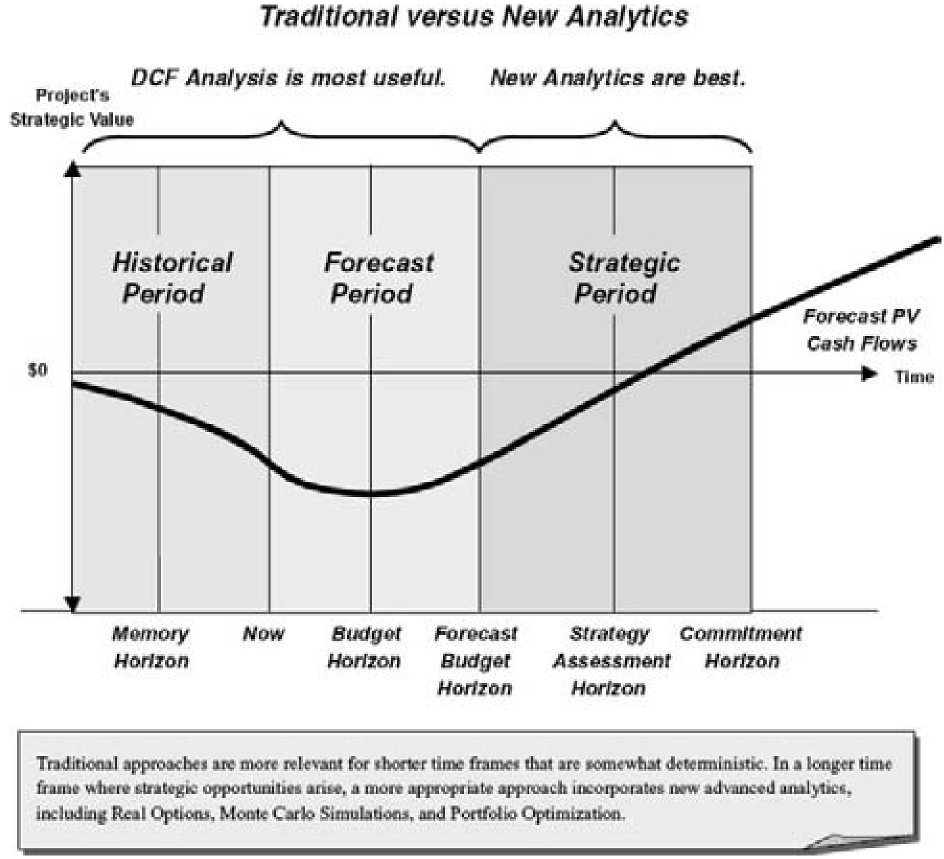


FIGURE 18.3 Using the Appropriate Analysis

market or economic conditions, impact on a specific industry, and more specifically, the firm's competitive options. At the firm level, the analyst may be concerned with a single project and the portfolio of projects from a risk management perspective. At the project level, detail focus will be on the variables impacting the value of the project.

**REAL OPTIONS VERSUS FINANCIAL OPTIONS** Real options apply financial options theory in analyzing real or physical assets. Therefore, there are certainly many similarities between financial and real options. However, there are key differences, as listed in Figure 18.5. For example, financial options have short maturities, usually expiring in several months. Real options have longer maturities, usually expiring in several years, with some exotic-type options having an infinite expiration date. The underlying asset in financial options is the stock price as compared to a multitude of other business variables in real options. These variables may include free cash flows, market demand, commodity prices, and so forth. Thus, when applying real options analysis to analyzing physical assets, we have to be careful in discerning what the underlying variable is because the volatility measures used in options modeling pertain to the underlying variable. In financial options, due to insider trading

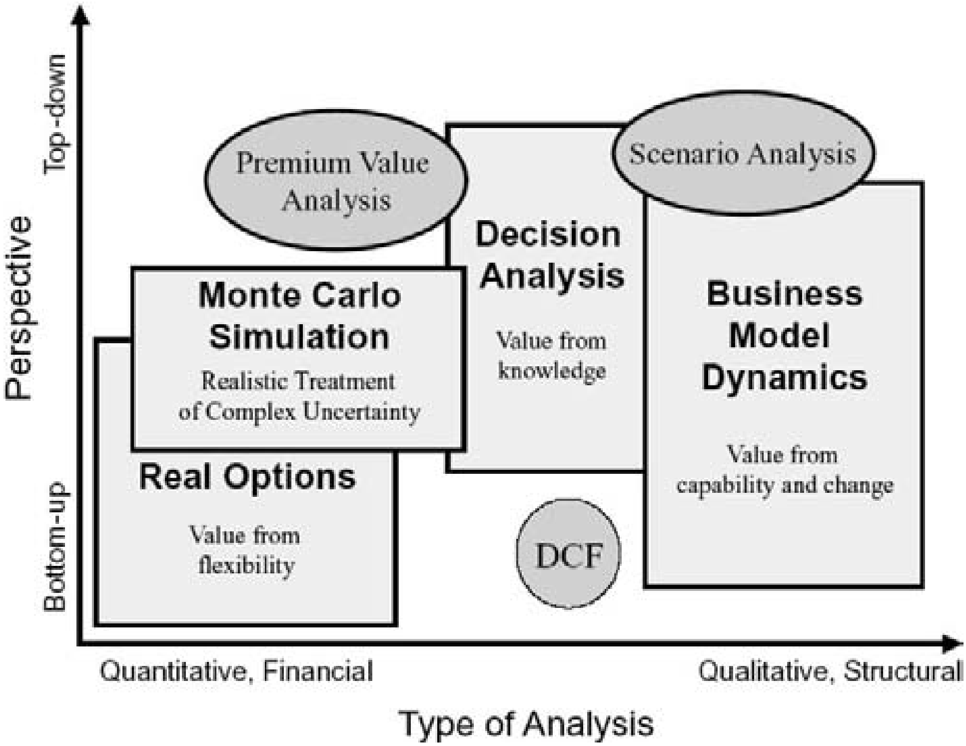


FIGURE 18.4 An Analytical Perspective

FINANCIAL OPTIONS

- Short maturity, usually in months.
- Underlying variable driving its value is equity price or price of a financial asset.
- Cannot control option value by manipulating stock prices.
- Values are usually small.
- Competitive or market effects are irrelevant to its value and pricing.
- Have been around and traded for more than three decades.
- Usually solved using closed-form partial differential equations and simulation/variance reduction techniques for exotic options.
- Marketable and traded security with comparables and pricing info.
- Management assumptions and actions have no bearing on valuation.

REAL OPTIONS

- Longer maturity, usually in years.
- Underlying variables are free cash flows, which in turn are driven by competition, demand, management.
- Can increase strategic option value by management decisions and flexibility.
- Major million and billion dollar decisions.
- Competition and market drive the value of a strategic option.
- A recent development in corporate finance within the last decade.
- Usually solved using closed-form equations and binomial lattices with simulation of the underlying variables, not on the option analysis.
- Not traded and proprietary in nature, with no market comparables.
- Management assumptions and actions drive the value of a real option.

FIGURE 18.5 Financial Options versus Real Options regulations, options holders cannot, at least in theory, manipulate stock prices to their advantage. However, in real options, because certain strategic options can be created by management, their decisions can increase the value of the project's real options. Financial options have relatively less value (measured in tens or hundreds of dollars per option) than real options (thousands, millions, or even billions of dollars per strategic option).

Financial options have been traded for several decades, but the real options phenomenon is only a recent development, especially in the industry at large. Both types of options can be solved using similar approaches, including closed-form solutions, partial-differential equations, finite-differences, binomial lattices, and simulation; but industry acceptance for real options has been in the use of binomial lattices. This is because binomial lattices are much more easily explained to and accepted by management, because the methodology is much simpler to understand. Chapters 6 to 9 of *Real Options Analysis, Second Edition* (Johnathan Mun, John Wiley & Sons, 2006) provide step-by-step details on how to create and solve binomial and multinomial lattices. Finally, financial options models are based on market-traded securities and visible asset prices making their construction easier and more objective. Real options tend to be based on non-market-traded assets, and financially traded proxies are seldom available. Hence management assumptions are key in valuing real options and relatively less important in valuing financial options. Given a particular project, management can create strategies that will provide themselves with options in the future. The value of these options can change depending on how they are constructed.

In several basic cases, real options are similar to financial options. Figure 18.6 shows the payoff charts of a call option and a put option. On all four charts, the

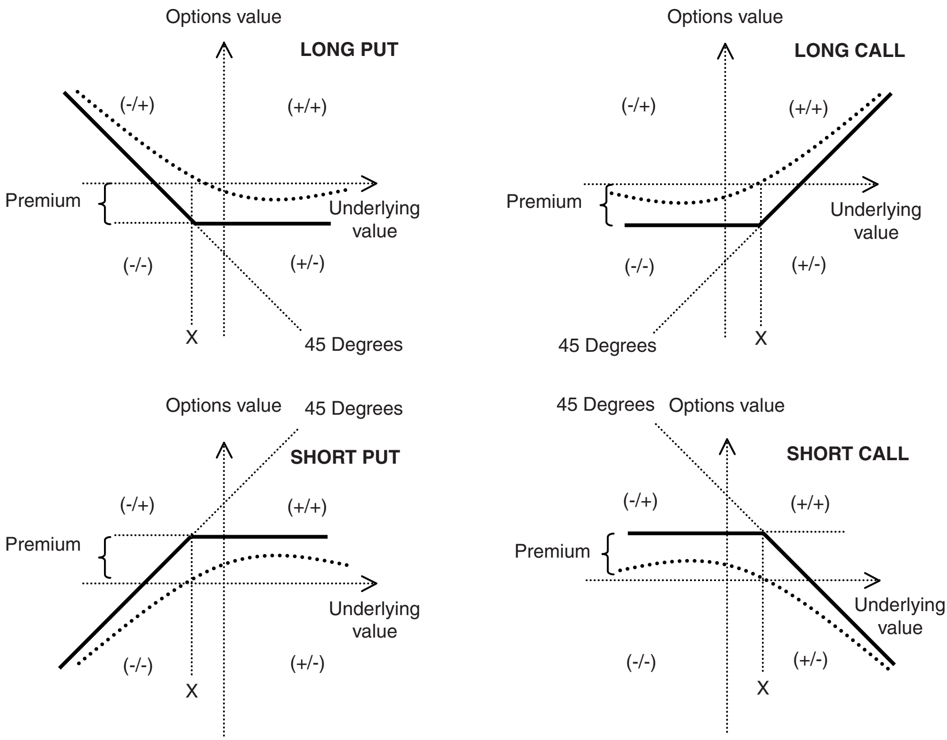


FIGURE 18.6 Option Payoff Charts

vertical axes represent the value of the strategic option, and the horizontal axes represent the value of the underlying asset. The kinked bold line represents the payoff function of the option at termination, effectively the project's net present value, because at termination, maturity effectively becomes zero and the option value reverts to the net present value (underlying asset less implementation costs). The dotted curved line represents the payoff function of the option prior to termination, where there is still time before maturity and hence uncertainty still exists and option value is positive. The curved line is the net present value, including the strategic option value. Both lines effectively have a horizontal floor value, which is effectively the premium on the option, where the maximum value at risk is the premium or cost of obtaining the option, indicating the option's maximum loss as the price paid to obtain it.

The position of a long call or the buyer and holder of a call option is akin to an *expansion option*. This is because an expansion option usually costs something to create or set up, which is akin to the option's premium or purchase price. If the underlying asset does not increase in value over time, the maximum losses incurred by the holder of this expansion option will be the cost of setting up this option (e.g., market research cost). When the value of the underlying asset increases sufficiently above the strike price (denoted $X$ in the charts), the value of this expansion option increases. There is unlimited upside to this option, but the downside is limited to the premium paid for the option. The breakeven point is where the bold line crosses the horizontal axis, which is equivalent to the strike price plus the premium paid.

The long put option position or the buyer and holder of a put option is akin to an *abandonment option*. This is because an abandonment option usually costs something to create or set up, which is akin to the option's premium or purchase price. If the value of the underlying asset does not decrease over time, the maximum losses incurred by the holder of this abandonment option will be the cost of setting up this option (seen as the horizontal bold line equivalent to the premium). When the value of the underlying asset decreases sufficiently below the strike price (denoted $X$ in the charts), the value of this abandonment option increases. The option holder will find it more profitable to abandon the project currently in existence. There is unlimited upside to this option but the downside is limited to the premium paid for the option. The breakeven point is where the bold line crosses the horizontal axis, which is equivalent to the strike price less the premium paid.[5]

The short positions or the writer and seller on both calls and puts have payoff profiles that are horizontal reflections of the long positions. That is, if you overlay both a long and short position of a call or a put, it becomes a zero-sum game. These short positions reflect the side of the issuer of the option. For instance, if the expansion and contraction options are based on some legally binding contract, the counterparty issuer of the contract would hold these short positions.

## Part 3: Application: Real Options SLS Software

Now that you are confident with the applicability of real options, it is time to move on and use the Real Options *Super Lattice Solver* software in the enclosed

[5]In an abandonment option, there is usually a maximum to the salvage value; thus, the payoff function may actually look like a put but with a limit cap on the upside.

CD-ROM. To get started using the software, visit www.realoptionsvaluation.com and click on the ***Downloads*** link to view some free getting-started videos on advanced risk analysis and real options modeling techniques using Risk Simulator and Real Options SLS software applications, plus to download the trial software versions and sample models. The use of software-based models allows the analyst to apply a consistent, well-tested, and replicable set of models. It reduces computational errors and allows the user to focus more on the process and problem at hand rather than on building potentially complex and mathematically intractable models. This chapter provides a good starting point with an introduction to the *Super Lattice Solver* software. For more details on using the software, consult the user manual, whereas for more technical, theoretical, and practical details of real options analysis, consult *Real Options Analysis: Tools and Techniques, Second Edition* (John Wiley & Sons, 2005). The materials covered in this chapter assume that the reader is sufficiently well versed in the basics of real options analytics.

The enclosed CD-ROM has a 30-day trial version of the *Super Lattice Solver* and Risk Simulator software. For professors, please contact the author for complimentary semester-long licenses for you and your students for installation in computer labs if this text and associated software are used in an entire class. The remainder of this chapter and relevant examples require the use of these software applications. To install the *Super Lattice Solver* software, insert the CD and wait for the setup program to start. If it does not start automatically, browse the content of the CD and double-click on the ***CDAutorun.exe*** file and follow the simple onscreen instructions. You must be connected to the Internet before you can download and install the latest version of the software. Click on ***Install the Super Lattice Solver*** software. When prompted, enter the following user name and license key for a 30-day trial of the SLS software:

Name: **30 Day License**
License Key: **513C-27D2-DC6B-9666**

Another license key is required to permanently unlock and use the software, and the license can be purchased by going to www.realoptionsvaluation.com. After successfully installing the software, verify that the installation was successful by clicking on and making sure that the following folder exists: ***Start | Programs | Real Options Valuation | Real Options Super Lattice Solver***. Note that the SLS software will work on most international Windows operating systems but requires a quick change in settings by clicking on ***Start | Control Panel | Regional and Language Options.*** Select ***English (United States).*** This is required because the numbering convention is different in foreign countries (e.g., one thousand dollars and fifty cents is written as 1,000.50 in the United States versus 1.000,50 in certain European countries).

## Introduction to the Real Options Super Lattice Solver Software

The Real Options Super Lattice Software comprises several modules, including the Single Super Lattice Solver (SLS), Multiple Super Lattice Solver (MSLS), Multinomial Lattice Solver (MNLS), SLS Excel Solution, and SLS Functions. These modules are highly powerful and customizable binomial and multinomial lattice solvers and can be used to solve many types of options (including the three main families of options: real options, which deals with physical and intangible assets; financial options, which deals with financial assets and the investments of such assets; and employee stock options, which deals with financial assets provided to employees within a corporation). This text illustrates some sample real options, financial options, and employee stock options applications that users will encounter most frequently. The following are the modules in the Real Options Super Lattice software:

- The *SLS* is used primarily for solving options with a *single underlying asset* using binomial lattices. Even highly complex options with a single underlying asset can be solved using the SLS. The types of options solved include American, Bermudan, and European options to abandon, choose, contract, defer, execute, expand, wait, as well as any customized combinations of these options with changing inputs.
- The *MSLS* is used for solving options with *multiple underlying assets* and sequential compound options with *multiple phases* using binomial lattices. Highly complex options with multiple underlying assets and phases can be solved using the MSLS. The types of options solved include multiple-phased stage-gate sequential compound options, simultaneous compound options, switching options, multiple-asset chooser options, and customized combinations of phased options with all the option types solved using the SLS module described above.
- The *MNLS* uses *multinomial lattices* (trinomial, quadranomial, pentanomial) to solve specific options that cannot be solved using binomial lattices. The options solved include mean-reverting, jump-diffusion, and rainbow options.
- The *SLS Excel Solution* implements the SLS and MSLS computations within the Excel environment, allowing users to access the SLS and MSLS functions directly in Excel. This feature facilitates model building, formula and value linking and embedding, as well as running simulations, and provides the user with sample templates to create such models.
- The *SLS functions* are additional real options and financial options models accessible directly through Excel. This module facilitates model building, linking and embedding, and running simulations.

The SLS software is created by the author and accompanies the materials presented at different training courses on real options, simulation, employee stock options valuation, and Certified Risk Analyst (CRA) programs taught by the author. While the software and its models are based on my books, the training courses cover the real options subject matter in more depth, including the solution of sample business cases and the framing of real options of actual cases. It is highly suggested that the reader familiarize him- or herself with the fundamental concepts of real options in Chapters 6 and 7 of *Real Options Analysis, Second Edition* (John Wiley & Sons, 2005) prior to attempting an in-depth real options analysis using the software. Note that the first edition of *Real Options Analysis: Tools and Techniques* (published in 2002) shows the *Real Options Analysis Toolkit* software, an older precursor to my *Super Lattice Solver*. The *Super Lattice Solver version 1.1* supersedes the *Real Options*

*Analysis Toolkit* by providing the following enhancements, and is introduced in this second edition:

- All inconsistencies, computation errors, and bugs fixed and verified
- Allowance of changing input parameters over time (customized options)
- Allowance of changing volatilities over time
- Incorporation of Bermudan (vesting and blackout periods) and customized options
- Flexible modeling capabilities in creating or engineering your own customized options
- General enhancements to accuracy, precision, and analytical prowess

As the creator of both the *Super Lattice Solver* and *Real Options Analysis Toolkit* software, I would suggest that the reader focus on using the *Super Lattice Solver* as it provides many powerful enhancements and analytical flexibility over its predecessor, the older, less powerful and less flexible *Real Options Analysis Toolkit* software.

## Single Asset Super Lattice Solver (SLS)

Figure 18.7 illustrates the SLS module. After installing the software, the user can access the SLS by clicking on ***Start* | *Programs* | *Real Options Valuation* |**

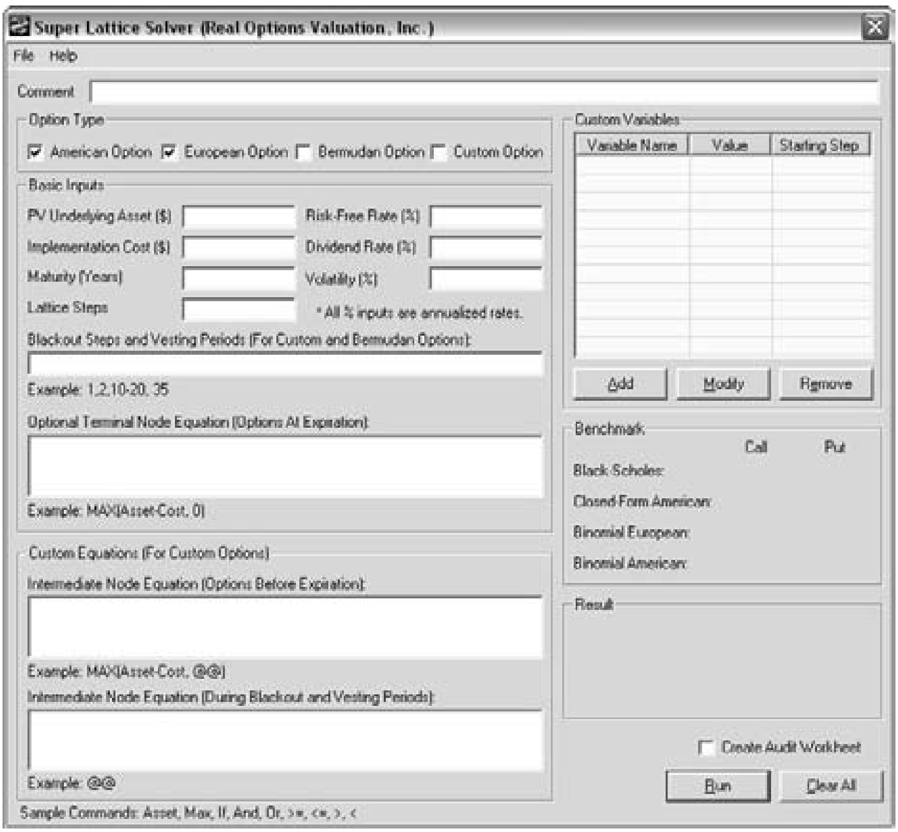


FIGURE 18.7 Single Super Lattice Solver (SLS)

***Real Options Super Lattice Solver | Single Super Lattice Solver***. The SLS has several sections: Option Type, Basic Inputs, Custom Equations, Custom Variables, Benchmark, Result, and Create Audit Worksheet.

**SLS EXAMPLES** To help you get started, several simple examples are in order. A simple European call option is computed in this example using SLS. To follow along, start this example file by selecting ***Start | Programs | Real Options Valuation | Real Options Super Lattice Solver | Sample Files | Plain Vanilla Call Option I***. This example file will be loaded into the SLS software as seen in Figure 18.8. The starting PV Underlying Asset or starting stock price is $100, and the Implementation Cost or strike price is $100 with a 5-year maturity. The annualized risk-free rate of return is 5%, and the historical, comparable, or future expected annualized volatility is 10%. Click on ***RUN*** (or Alt-R) and a 100-step binomial lattice is computed and the results indicate a value of $23.3975 for both the European and American call options. Benchmark values using Black-Scholes and Closed-Form American approximation models as well as standard plain-vanilla Binomial American and Binomial European Call and Put Options with 1,000-step binomial lattices are also computed. Notice

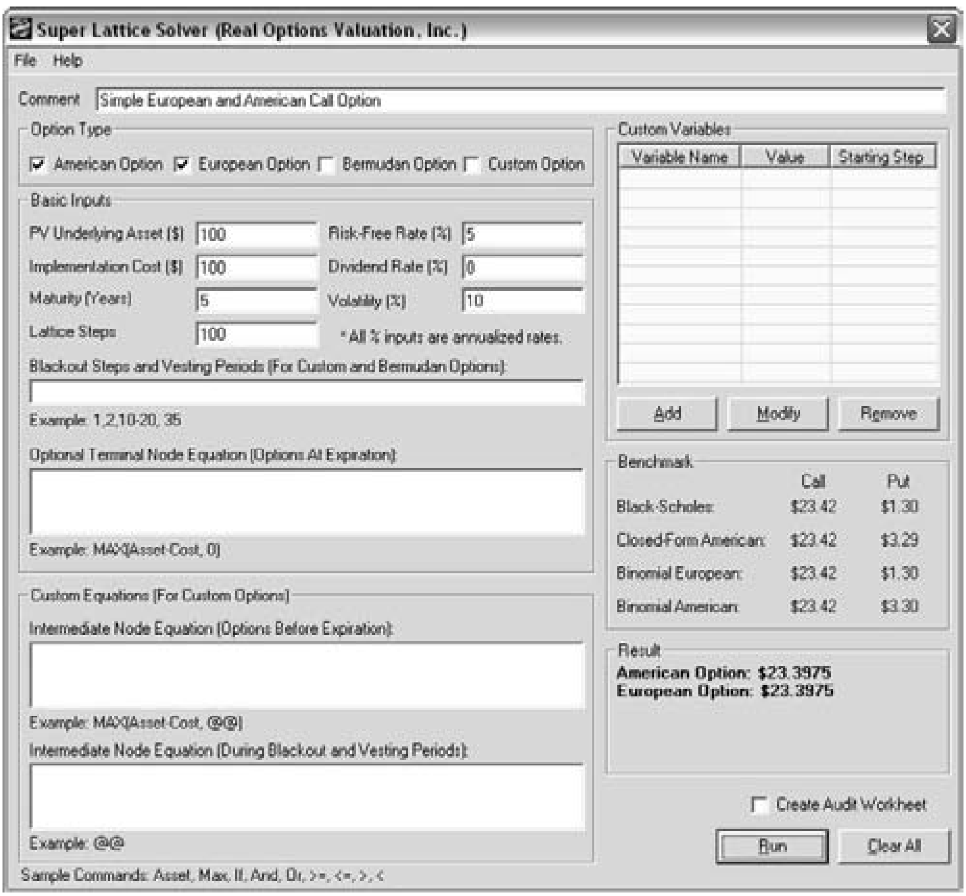


**FIGURE 18.8** SLS Results of a Simple European and American Call Option

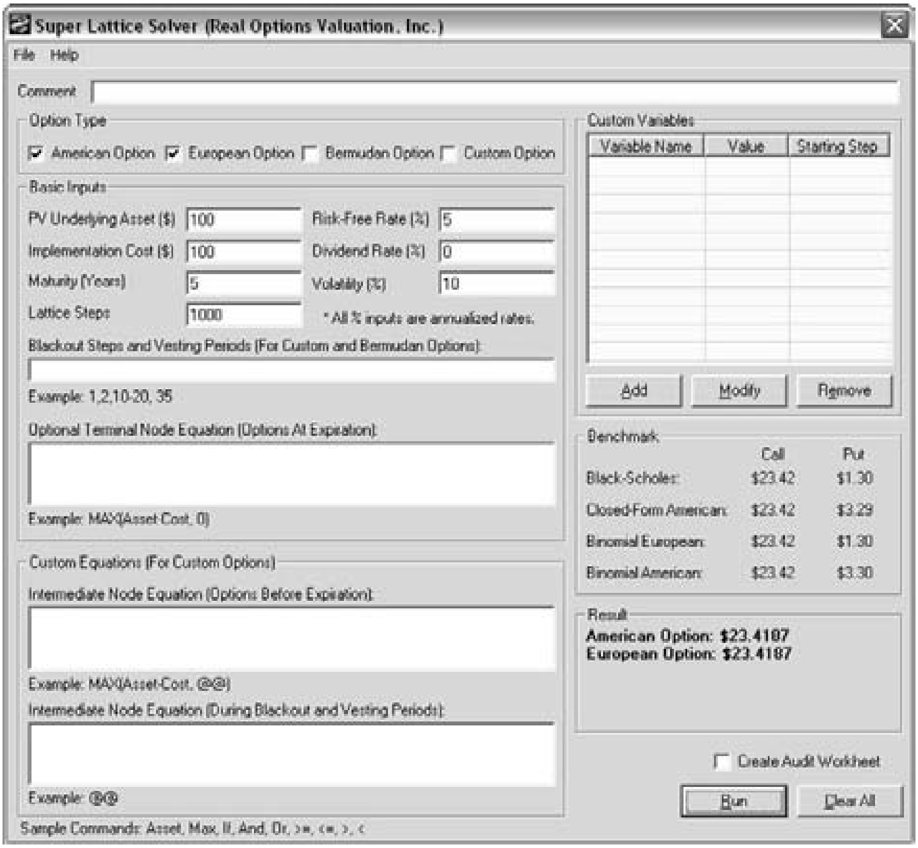


**FIGURE 18.9** SLS Comparing Results with Benchmarks

that only the American and European options are selected and the computed results are for these simple, plain-vanilla American and European call options.

The benchmark results use both closed-form models (Black-Scholes and Closed-Form Approximation models) and 1,000-step binomial lattices on plain-vanilla options. You can change the steps to ***1000*** in the basic inputs section to verify that the answers computed are equivalent to the benchmarks as seen in Figure 18.9. Notice that, of course, the values computed for the American and European options are identical to each other and identical to the benchmark values of $23.4187, as it is never optimal to exercise a standard plain-vanilla call option early if there are no dividends. Be aware that the higher the number of lattice steps, the longer it takes to compute the results. It is advisable to start with fewer lattice steps to make sure the analysis is robust and then progressively increase lattice steps to check for results convergence.

Alternatively, you can enter Terminal and Intermediate Node Equations for a call option to obtain the same results. Notice that using 100 steps and creating your own Terminal Node Equation of *Max(Asset-Cost,0)* and Intermediate Node Equation of *Max(Asset-Cost,@@)* will yield the same answer. When entering your own equations, make sure that Custom Option is first checked.

When entering your own equations, make sure that Custom Option is first checked.

Figure 18.10 illustrates how the analysis is done. The example file used in this illustration is ***Plain Vanilla Call Option III***. Notice that the value $23.3975 in Figure 18.10 agrees with the value in Figure 18.8. The Terminal Node Equation is the computation that occurs at maturity, whereas the Intermediate Node Equation is the computation that occurs at all periods prior to maturity, and is computed using backward induction. The symbol "@@" represents "keeping the option open," and is often used in the Intermediate Node Equation when analytically representing the fact that the option is not executed but kept open for possible future execution. Therefore, in Figure 18.10, the Intermediate Node Equation *Max(Asset-Cost,@@)* represents the profit maximization decision of either executing the option or leaving it open for possible future execution. In contrast, the Terminal Node Equation of *Max(Asset-Cost,0)* represents the profit maximization decision at maturity of either executing the option if it is in-the-money or allowing it to expire worthless if it is at-the-money or out-of-the-money.

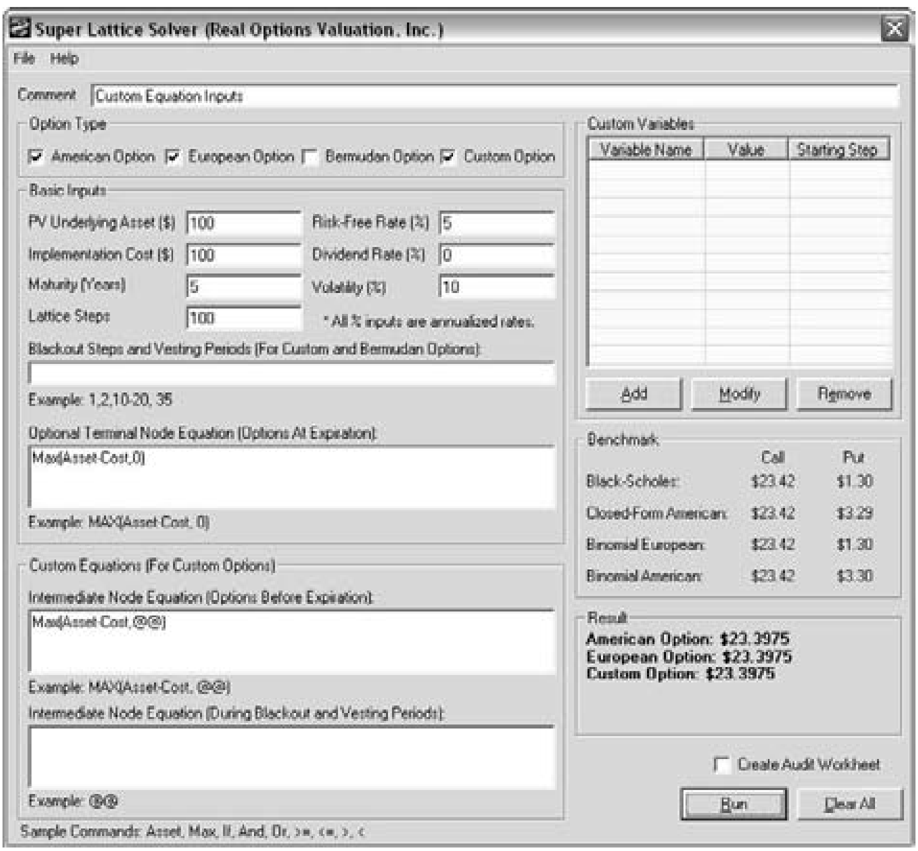


FIGURE 18.10 Custom Equation Inputs

**Option Valuation Audit Sheet**

| Assumptions | | Intermediate Computations | |
|---|---|---|---|
| PV Asset Value ($) | $100.00 | Stepping Time (dt) | 0.0500 |
| Implementation Cost ($) | $100.00 | Up Step Size (up) | 1.0226 |
| Maturity (Years) | 5.00 | Down Step Size (down) | 0.9779 |
| Risk-free Rate (%) | 5.00% | Risk-neutral Probability | 0.5504 |
| Dividends (%) | 0.00% | | |
| Volatility (%) | 10.00% | **Results** | |
| Lattice Steps | 100 | Lattice Result | 23.40 |
| Option Type | European | | |

| | |
|---|---|
| Terminal Equation | Max(Asset-Cost,0) |
| Intermediate Equation | @@ |
| Intermediate Equation (Blackouts) | @@ |

**Underlying Asset Lattice**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | 125.06 |
| | | | | | | | | | 122.29 | |
| | | | | | | | | 119.59 | | 119.59 |
| | | | | | | | 116.94 | | 116.94 | |
| | | | | | | 114.36 | | 114.36 | | 114.36 |
| | | | | | 111.83 | | 111.83 | | 111.83 | |
| | | | | 109.36 | | 109.36 | | 109.36 | | 109.36 |
| | | | 106.94 | | 106.94 | | 106.94 | | 106.94 | |
| | | 104.57 | | 104.57 | | 104.57 | | 104.57 | | 104.57 |
| | 102.26 | | 102.26 | | 102.26 | | 102.26 | | 102.26 | |
| 100.00 | | 100.00 | | 100.00 | | 100.00 | | 100.00 | | 100.00 |
| | 97.79 | | 97.79 | | 97.79 | | 97.79 | | 97.79 | |
| | | 95.63 | | 95.63 | | 95.63 | | 95.63 | | 95.63 |
| | | | 93.51 | | 93.51 | | 93.51 | | 93.51 | |
| | | | | 91.44 | | 91.44 | | 91.44 | | 91.44 |
| | | | | | 89.42 | | 89.42 | | 89.42 | |
| | | | | | | 87.44 | | 87.44 | | 87.44 |
| | | | | | | | 85.51 | | 85.51 | |
| | | | | | | | | 83.62 | | 83.62 |
| | | | | | | | | | 81.77 | |
| | | | | | | | | | | 79.96 |

**Option Valuation Lattice**

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | 45.33 |
| | | | | | | | | | 42.81 | |
| | | | | | | | | 40.35 | | 39.96 |
| | | | | | | | 37.97 | | 37.58 | |
| | | | | | | 35.68 | | 35.27 | | 34.87 |
| | | | | | 33.43 | | 33.04 | | 32.64 | |
| | | | | 31.27 | | 30.88 | | 30.49 | | 30.09 |
| | | | 29.18 | | 28.80 | | 28.41 | | 28.02 | |
| | | 27.10 | | 26.79 | | 26.41 | | 26.02 | | 25.64 |
| | 25.25 | | 24.87 | | 24.49 | | 24.11 | | 23.73 | |
| 23.40 | | 23.03 | | 22.65 | | 22.28 | | 21.90 | | 21.52 |
| | 21.26 | | 20.90 | | 20.53 | | 20.16 | | 19.79 | |
| | | 19.22 | | 18.86 | | 18.50 | | 18.14 | | 17.77 |
| | | | 17.28 | | 16.93 | | 16.58 | | 16.22 | |
| | | | | 15.45 | | 15.10 | | 14.76 | | 14.41 |
| | | | | | 13.71 | | 13.38 | | 13.05 | |
| | | | | | | 12.09 | | 11.77 | | 11.45 |
| | | | | | | | 10.58 | | 10.27 | |
| | | | | | | | | 9.19 | | 8.89 |
| | | | | | | | | | 7.91 | |
| | | | | | | | | | | 6.74 |

FIGURE 18.11 SLS Generated Audit Worksheet

In addition, you can create an Audit Worksheet in Excel to view a sample 10-step binomial lattice by checking the box ***Create Audit Worksheet***. For instance, loading the example file ***Plain Vanilla Call Option I*** and selecting the box creates a worksheet as seen in Figure 18.11. A couple of items on this audit worksheet are noteworthy:

- The audit worksheet generated will show the first 10 steps of the lattice, regardless of how many you enter. That is, if you enter 1,000 steps, the first 10 steps will be generated. If a complete lattice is required, simply enter 10 steps in the

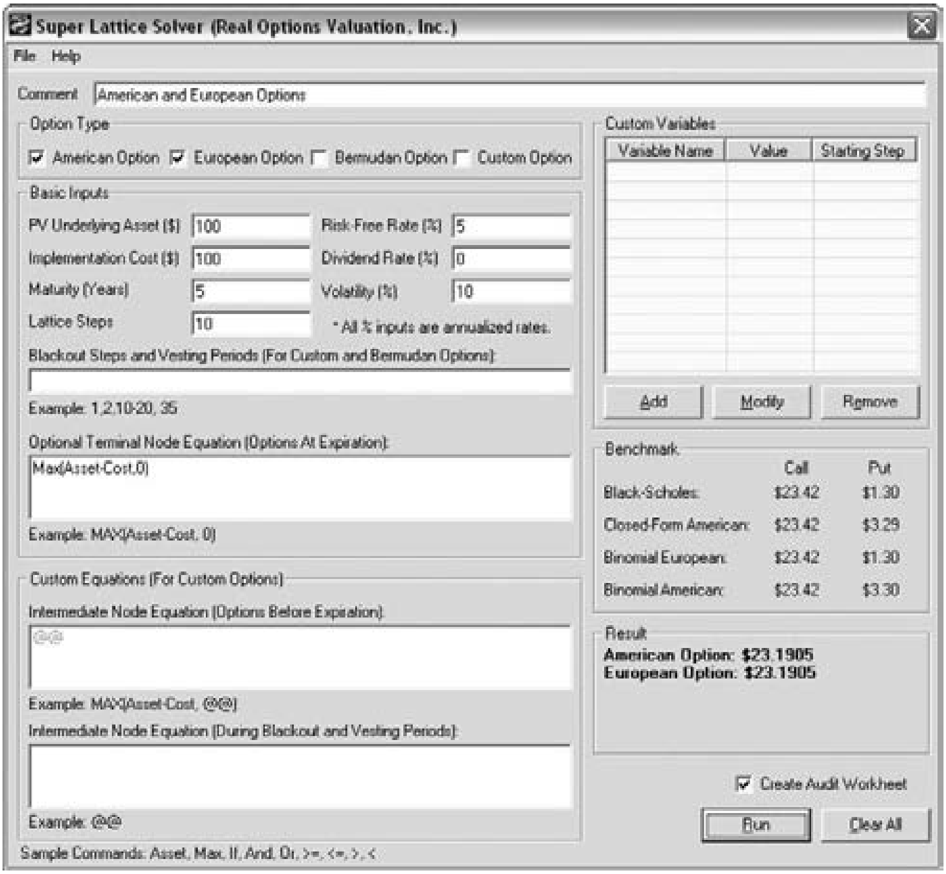


**FIGURE 18.12** SLS Results with a 10-Step Lattice

SLS and the full 10-step lattice will be generated instead. The Intermediate Computations and Results are for the Super Lattice, based on the number of lattice steps entered, and not based on the 10-step lattice generated. To obtain the Intermediate Computations for 10-step lattices, simply rerun the analysis inputting 10 as the lattice steps. This way, the audit worksheet generated will be for a 10-step lattice, and the results from SLS will now be comparable (Figure 18.12).

- The worksheet only provides values as it is assumed that the user was the one who entered in the terminal and intermediate node equations, hence there is really no need to recreate these equations in Excel again. The user can always reload the SLS file and view the equations or print out the form if required (by clicking on ***File*** | ***Print***).

The software also allows you to save or open analysis files. That is, all the inputs in the software will be saved and can be retrieved for future use. The results will not be saved because you may accidentally delete or change an input and the results will no longer be valid. In addition, rerunning the super lattice computations will only take a few seconds, and it is advisable for you to always rerun the model when opening an old analysis file.

You may also enter in Blackout Steps. These are the steps on the super lattice that will have different behaviors than the terminal or intermediate steps. For instance, you can enter ***1000*** as the lattice steps, and enter ***0-400*** as the blackout steps, and some Blackout Equation (e.g., @@). This means that for the first 400 steps, the option holder can only keep the option open. Other examples include entering: ***1, 3, 5,*** and ***10*** if these are the lattice steps where blackout periods occur. You will have to calculate the relevant steps within the lattice where the blackout exists. For instance, if the blackout exists in years 1 and 3 on a 10-year, 10-step lattice, then steps 1, 3 will be the blackout dates. This blackout step feature comes in handy when analyzing options with holding periods, vesting periods, or periods where the option cannot be executed. Employee stock options have blackout and vesting periods, and certain contractual real options have periods during which the option cannot be executed (e.g., cooling-off periods, or proof of concept periods).

If equations are entered into the Terminal Node Equation box and American, European, or Bermudan Options are chosen, the Terminal Node Equation you entered will be the one used in the super lattice for the terminal nodes. However, for the intermediate nodes, the American option assumes the same Terminal Node Equation plus the ability to keep the option open; the European option assumes that the option can only be kept open and not executed; while the Bermudan option assumes that during the blackout lattice steps, the option will be kept open and cannot be executed. If you also enter the Intermediate Node Equation, the Custom Option should first be chosen (otherwise you cannot use the Intermediate Node Equation box). The Custom Option result uses all the equations you have entered in Terminal, Intermediate, and Intermediate during Blackout sections.

The Custom Variables list is where you can add, modify, or delete custom variables, the variables that are required beyond the basic inputs. For instance, when running an abandonment option, you need the salvage value. You can add this in the Custom Variables list, provide it a name (a variable name must be a single word), the appropriate value, and the starting step when this value becomes effective. For example, if you have multiple salvage values (i.e., if salvage values change over time), you can enter the same variable name (e.g., *salvage*) several times, but each time, its value changes and you can specify when the appropriate salvage value becomes effective. For instance, in a 10-year, 100-step super lattice problem where there are two salvage values—$100 occurring within the first 5 years and increasing to $150 at the beginning of Year 6—you can enter two salvage variables with the same name, $100 with a starting step of 0, and $150 with a starting step of 51. Be careful here as Year 6 starts at step 51 and not 61. That is, for a 10-year option with a 100-step lattice, we have: Steps 1–10 = Year 1; Steps 11–20 = Year 2; Steps 21–30 = Year 3; Steps 31–40 = Year 4; Steps 41–50 = Year 5; Steps 51–60 = Year 6; Steps 61–70 = Year 7; Steps 71–80 = Year 8; Steps 81–90 = Year 9; and Steps 91–100 = Year 10. Finally, incorporating *0* as a blackout step indicates that the option cannot be executed immediately.

## Multiple Super Lattice Solver (MSLS)

The MSLS is an extension of the SLS in that the MSLS can be used to solve options with multiple underlying assets and multiple phases. The MSLS allows the user to enter multiple underlying assets as well as multiple valuation lattices (Figure 18.13).

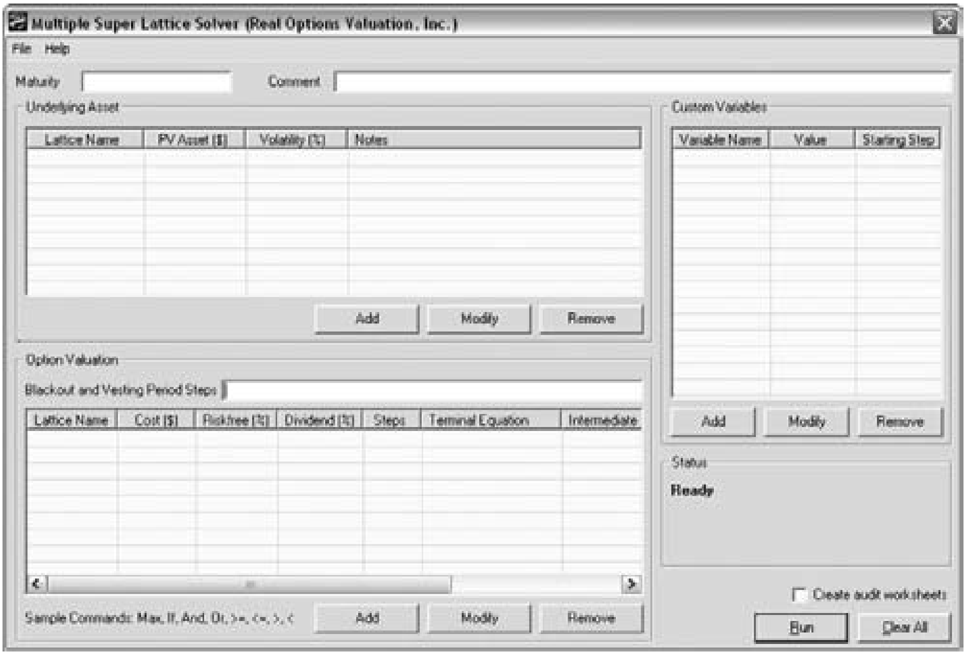


FIGURE 18.13 **Multiple Super Lattice Solver**

These valuation lattices can call to user-defined custom variables. Some examples of the types of options that the MSLS can be used to solve include:

- Sequential Compound Options (two-, three-, and multiple-phased sequential options)
- Simultaneous Compound Options (multiple assets with multiple simultaneous options)
- Chooser and Switching Options (choosing among several options and underlying assets)
- Floating Options (choosing between calls and puts)
- Multiple Asset Options (3D binomial option models)

The MSLS software has several areas including a *Maturity* and *Comment* area. The Maturity value is a global value for the entire option, regardless of how many underlying or valuation lattices exist. The comment field is for your personal notes describing the model you are building. There is also a *Blackout and Vesting Period Steps* section and a *Custom Variables* list similar to the SLS. The MSLS also allows you to create Audit Worksheets.

To illustrate the power of the MSLS, a simple illustration is in order. Click on ***Start | Programs | Real Options Valuation | Real Options Super Lattice Solver | Sample Files | MSLS—Two Phased Sequential Compound Option***. Figure 18.14 shows the MSLS example loaded. In this simple example, a single underlying asset is created with two valuation phases.

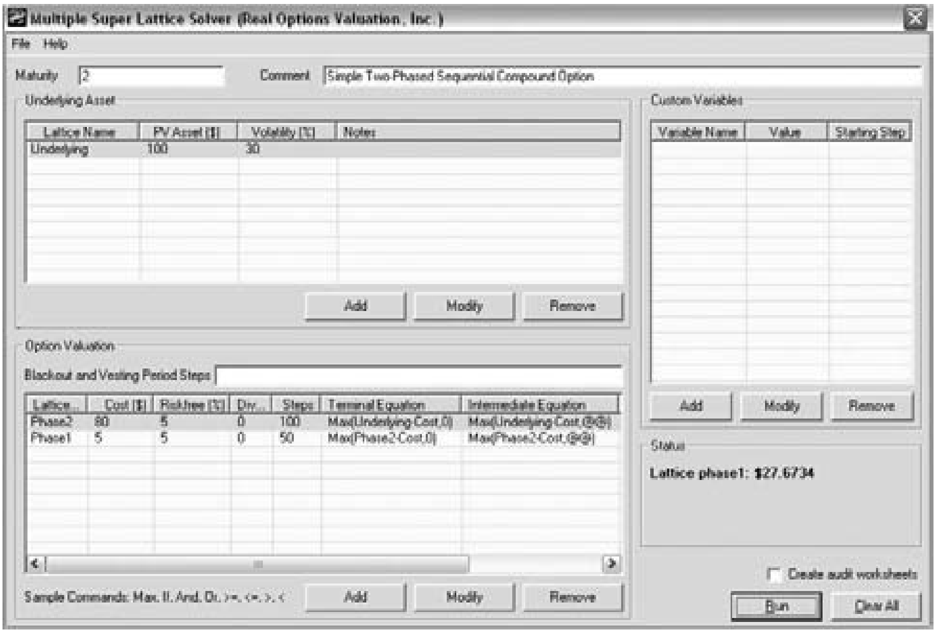


FIGURE 18.14 MSLS Solution to a Simple Two-Phased Sequential Compound Option

The strategy tree for this option is seen in Figure 18.15. The project is executed in two phases—the first phase within the first year costs $5 million, while the second phase occurs within two years but only after the first phase is executed, and costs $80 million, both in present value dollars. The PV Asset of the project is $100 million (NPV is therefore $15 million), and faces 30% volatility in its cash flows. The computed strategic value using the MSLS is $27.67 million, indicating that there is a $12.67 million in option value. That is, spreading out and staging the investment into two phases has significant value (an expected value of $12.67 million, to be exact).

## Multinomial Lattice Solver (MNLS)

The *Multinomial Lattice Solver* (MNLS) is another module of the Real Options Valuation's *Super Lattice Solver* software. The MNLS applies multinomial lattices—where

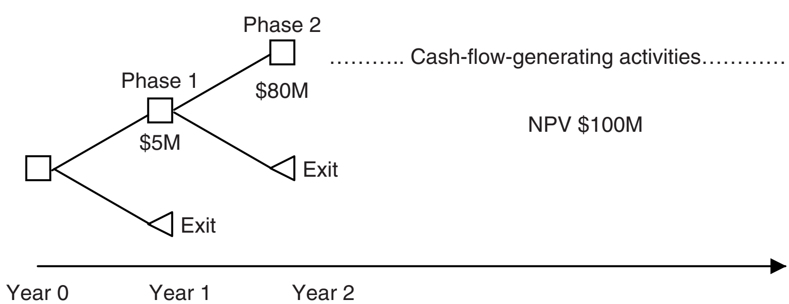


FIGURE 18.15 Strategy Tree for Two-Phased Sequential Compound Option

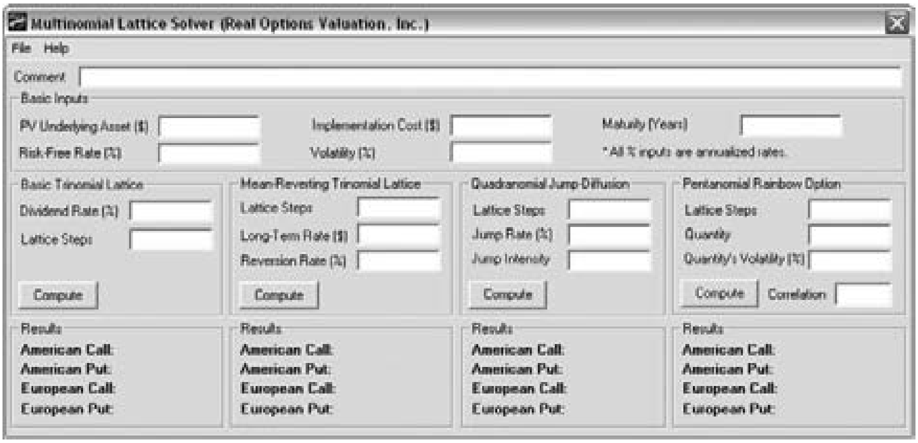


**FIGURE 18.16** Multinomial Lattice Solver

multiple branches stem from each node—such as trinomials (three branches), quadranomials (four branches), and pentanomials (five branches). Figure 18.16 illustrates the MNLS module. The module has a Basic Inputs section, where all of the common inputs for the multinomials are listed. Then, there are four sections with four different multinomial applications complete with the additional required inputs and results for both American and European call and put options.

Figure 18.17 shows an example call and put option computation using trinomial lattices. To follow along, open the example file ***MNLS—Simple Calls and Puts using Trinomial Lattices***. Note that the results shown in Figure 18.17 using a 50-step lattice are equivalent to the results shown in Figure 18.8 using a 100-step binomial lattice. In fact, a trinomial lattice or any other multinomial lattice provides identical answers to the binomial lattice at the limit, but convergence is achieved faster at lower steps. To illustrate, Table 18.2 shows how the trinomial lattice of a certain set of input assumptions yields the correct option value with fewer steps than it takes

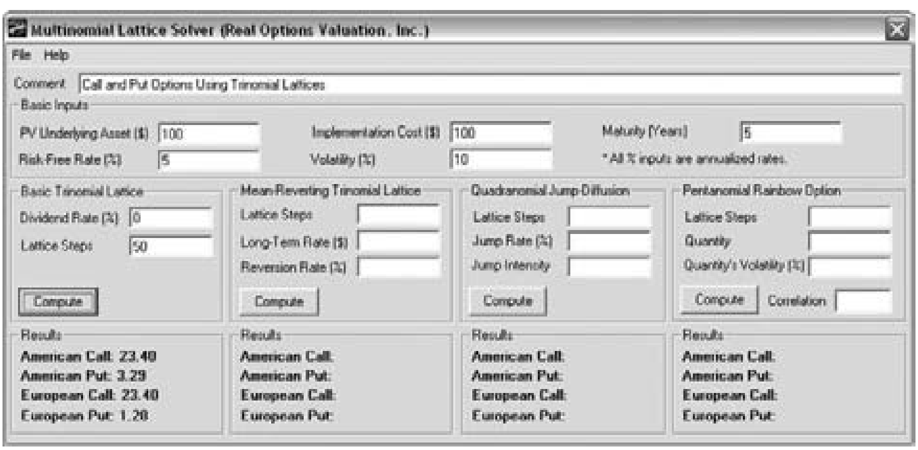


**FIGURE 18.17** A Simple Call and Put Using Trinomial Lattices

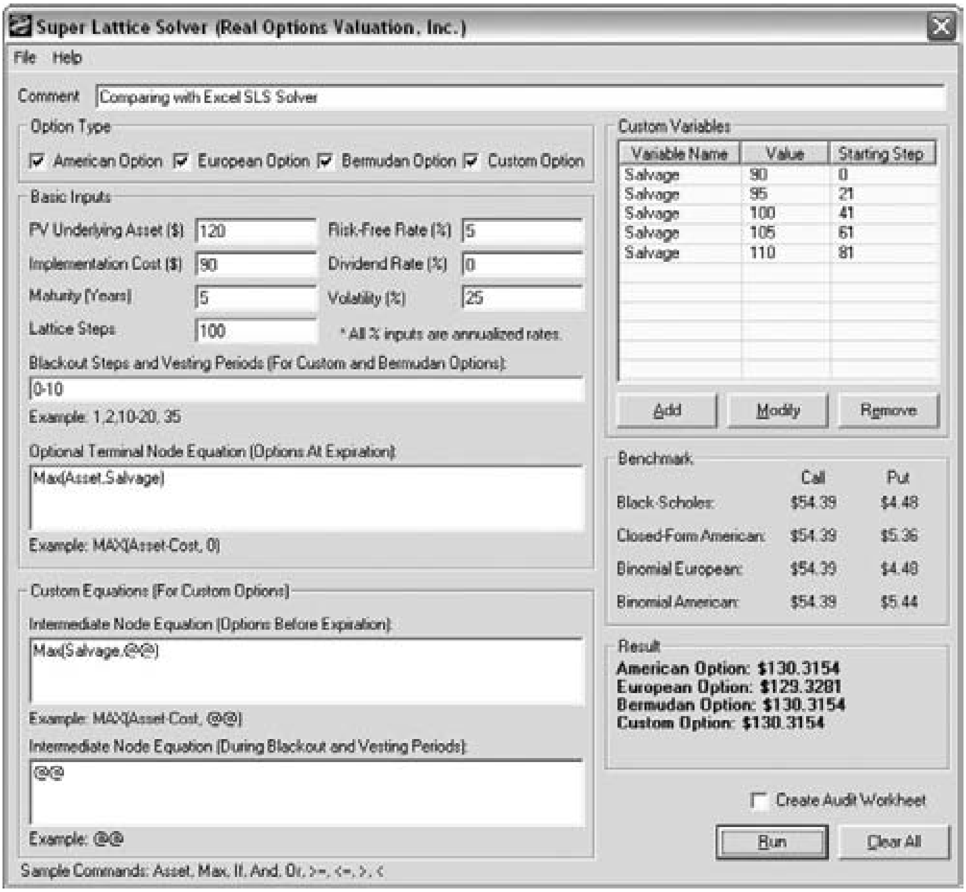


**FIGURE 18.18** Customized Abandonment Option Using SLS

for a binomial lattice. Because both yield identical results at the limit but trinomials are much more difficult to calculate and take a longer computation time, the binomial lattice is usually used instead. However, a trinomial is required only under one special circumstance: when the underlying asset follows a mean-reverting process.

With the same logic, quadranomials and pentanomials yield identical results as the binomial lattice with the exception that these multinomial lattices can be used to solve the following different special limiting conditions:

- *Trinomials.* Results are identical to binomials and are most appropriate when used to solve mean-reverting underlying assets.

**Table 18.2** Binomial versus Trinomial Lattices

| Steps | 5 | 10 | 100 | 1,000 | 5,000 |
|---|---|---|---|---|---|
| Binomial Lattice | $30.73 | $29.22 | $29.72 | $29.77 | $29.78 |
| Trinomial Lattice | $29.22 | $29.50 | $29.75 | $29.78 | $29.78 |

**SUPER LATTICE SOLVER (SINGLE ASSET)**

| | |
|---|---|
| *Option Type* | 0 |
| *PV Underlying Asset* | $120.00 |
| *Annualized Volatility* | 25.00% |
| *Maturity (Years)* | 5.00 |
| *Implementation Cost* | $0.00 |
| *Risk-Free Rate* | 5.00% |
| *Dividend Yield* | 0.00% |
| *Lattice Steps* | 100 |
| *Terminal Equation* | MAX(Asset, Salvage) |
| *Intermediate Equation* | MAX(Salvage, @@) |
| *Intermediate Equation During Blackout* | @@ |
| *Blackout Steps* | 0-10 |
| | |
| ***Super Lattice Solver Result*** | ***$130.3154*** |

**Custom Variables List**

| *Variable Name* | *Value* | *Starting Steps* |
|---|---|---|
| *Salvage* | *90.00* | *0* |
| *Salvage* | *95.00* | *21* |
| *Salvage* | *100.00* | *41* |
| *Salvage* | *105.00* | *61* |
| *Salvage* | *110.00* | *81* |
| | | |
| | | |
| | | |
| | | |
| | | |

Note: This is the Excel version of the Super Lattice Solver, useful when running simulations or when linking to and from other spreadsheets.
Use this sample spreadsheet for your models. You can simply click on File, Save As to save as a different file and start using the model.
For the option type, set 0 = American, 1 = European, 2 = Bermudan, 3 = Custom
The function used is: *SLSSingle*

FIGURE 18.19 **Customized Abandonment Option Using SLS Excel Solution**

- *Quadranomials.* Results are identical to binomials and are most appropriate when used to solve options whose underlying assets follow jump-diffusion processes.
- *Pentanomials.* Results are identical to binomials and are most appropriate when used to solve two underlying assets that are combined, called *rainbow* options (e.g., price and quantity are multiplied to obtain total revenues, but price and quantity each follows a different underlying lattice with its own volatility but both underlying parameters could be correlated to one another).

## SLS Excel Solution (SLS, MSLS, and Changing Volatility Models in Excel)

The SLS software also allows you to create your own models in Excel using customized functions. This functionality is important because certain models may require linking from other spreadsheets or databases, or run certain Excel macros and functions, or certain inputs need to be simulated, or inputs may change over the course of modeling your options. This Excel compatibility allows you the flexibility to innovate within the Excel spreadsheet environment. Specifically, the sample worksheet included in the software solves the SLS, MSLS, and Changing Volatility model.

To illustrate, Figure 18.18 shows a Customized Abandonment Option solved using SLS. The same problem can be solved using the *SLS Excel Solution* by clicking on ***Start | Programs | Real Options Valuation | Real Options Super Lattice Solver | SLS Excel Solution***. The sample solution is seen in Figure 18.19. Notice the same results using the SLS versus the ***SLS Excel Solution*** file. You can use the template provided by simply clicking on ***File | Save As*** in Excel and use the new file for your own modeling needs.

Similarly, the MSLS can also be solved using the SLS Excel Solver. Figure 18.20 shows a complex multiple-phased sequential compound option solved using the SLS Excel Solver. The results shown here are identical to the results generated from the MSLS module (example file: ***MSLS—Multiple Phased Complex Sequential***

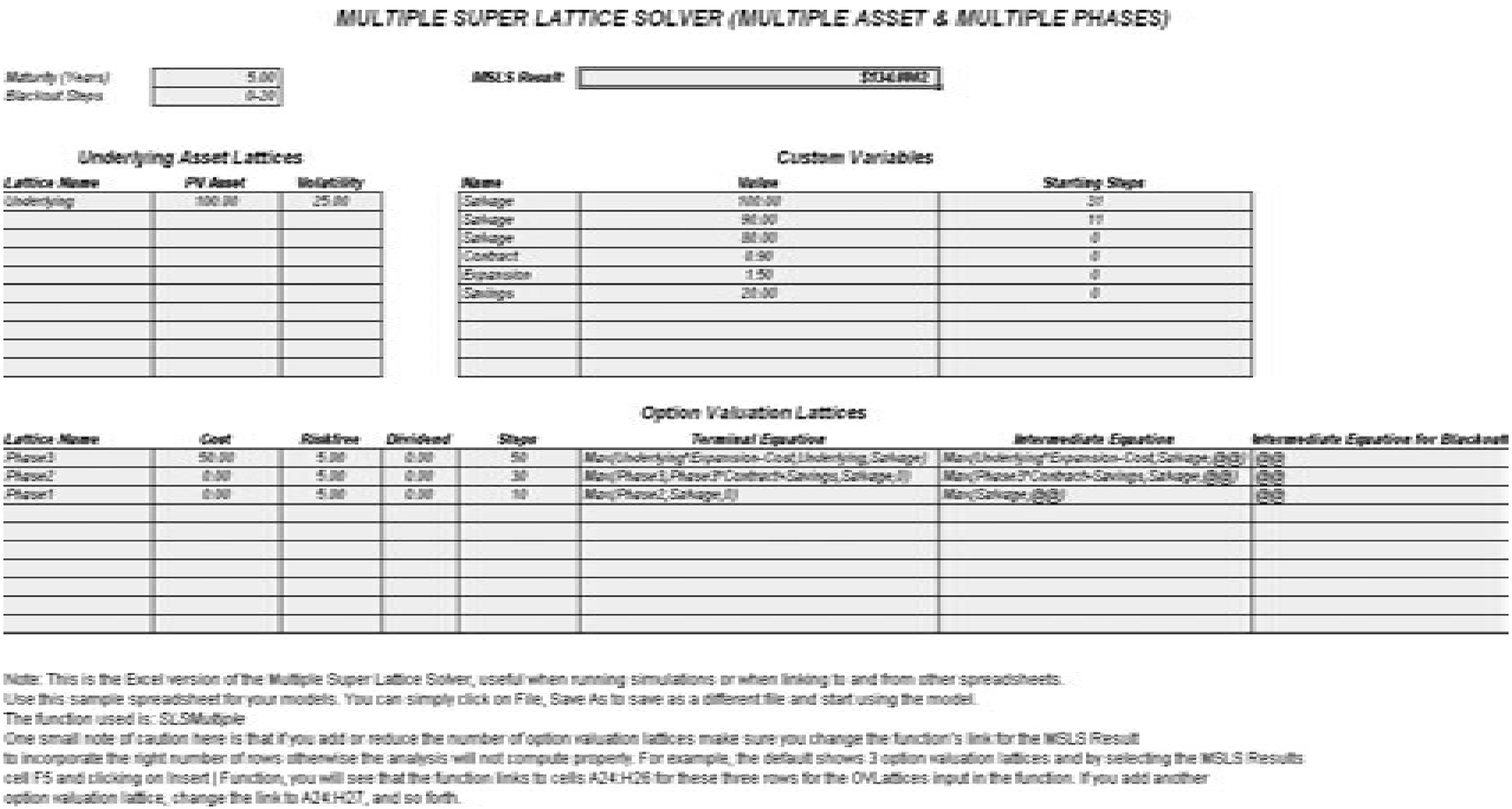


FIGURE 18.20 Complex Sequential Compound Option Using SLS Excel Solver

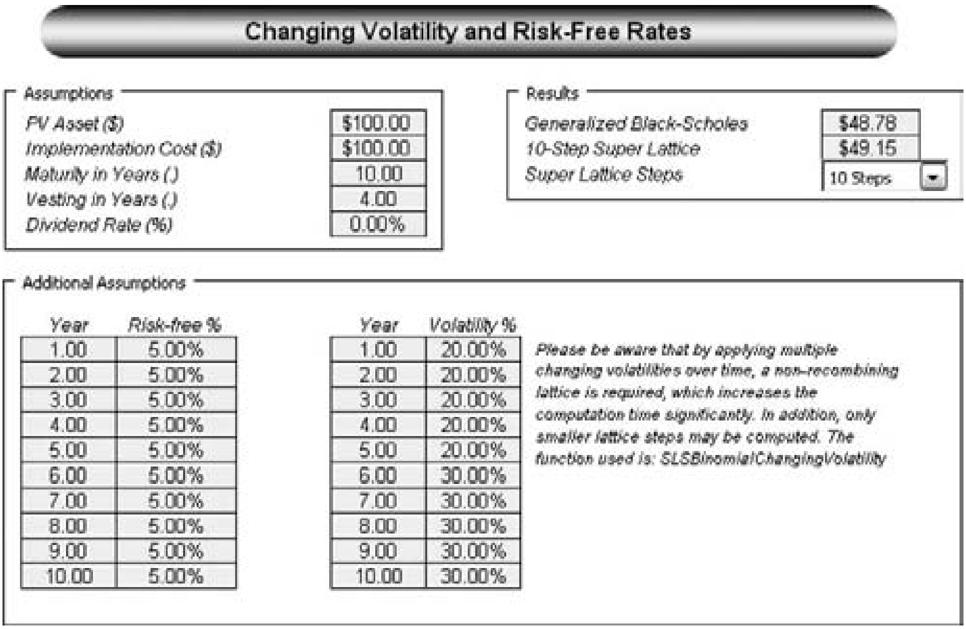


**FIGURE 18.21** **Changing Volatility and Risk-Free Rate Option**

***Compound Option***). One small note of caution here is that if you add or reduce the number of option valuation lattices, make sure you change the function's link for the *MSLS Result* to incorporate the right number of rows; otherwise, the analysis will not compute properly. For example, the default shows three option valuation lattices, and by selecting the *MSLS Results* cell in the spreadsheet and clicking on ***Insert | Function***, you will see that the function links to cells A24:H26 for these three rows for the *OVLattices* input in the function. If you add another option valuation lattice, change the link to cells A24:H27, and so forth. You can also leave the list of custom variables as is. The results will not be affected if these variables are not used in the custom equations.

Finally, Figure 18.21 shows a Changing Volatility and Changing Risk-Free Rate Option. In this model, the volatility and risk-free yields are allowed to change over time and a non-recombining lattice is required to solve the option. In most cases, it is recommended that you create option models without the changing volatility term structure because getting a single volatility is difficult enough, let alone a series of changing volatilities over time. If different volatilities that are uncertain need to be modeled, run a Monte Carlo simulation using the Risk Simulator software on volatilities instead. This model should be used only when the volatilities are modeled robustly and the volatilities are rather certain and change over time. The same advice applies to a changing risk-free rate term structure.

## SLS Functions

The software also provides a series of SLS functions that are directly accessible in Excel. To illustrate its use, start the SLS functions by clicking on ***Start | Programs |***

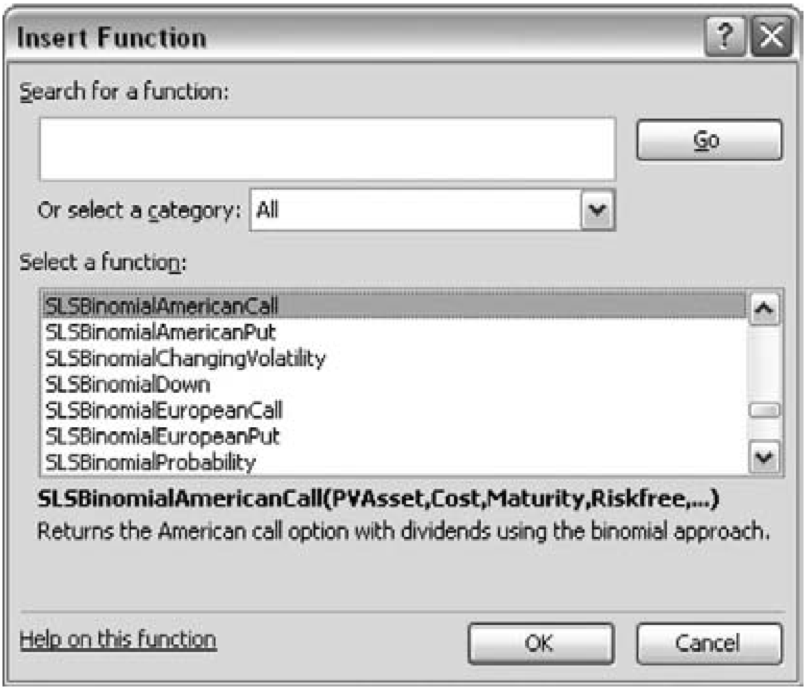


FIGURE 18.22 Excel's Equation Wizard

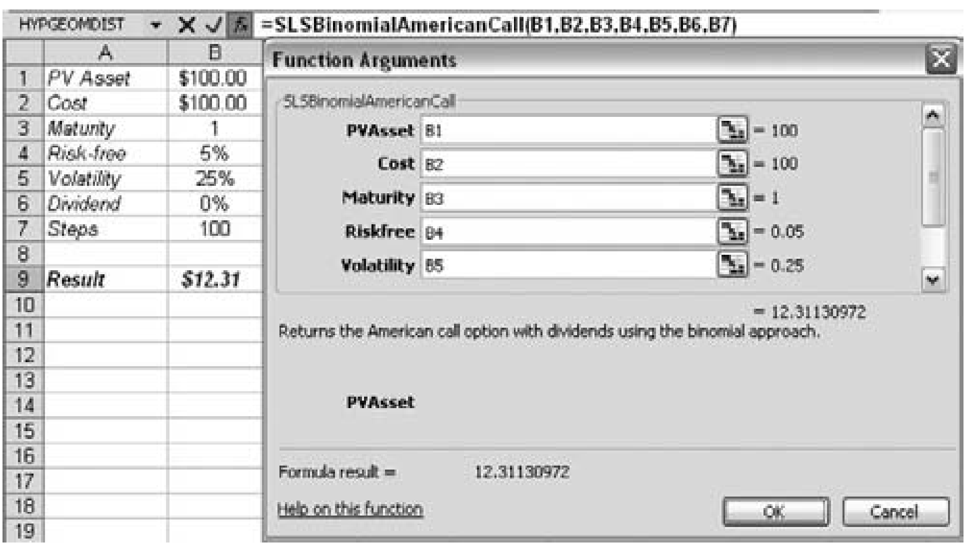


FIGURE 18.23 Using SLS Functions in Excel

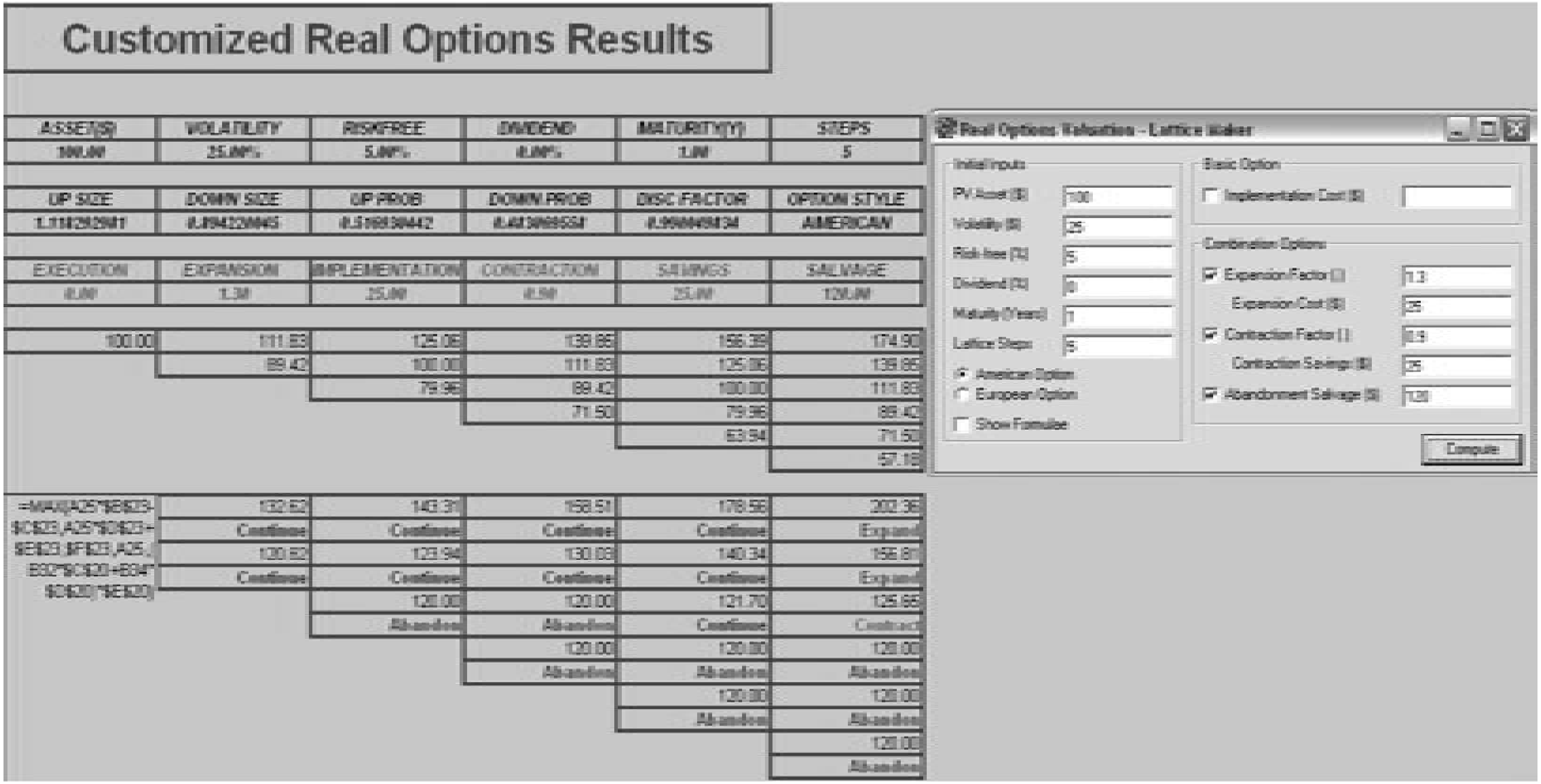


FIGURE 18.24 Lattice Maker

***Real Options Valuation | Real Options Super Lattice Solver | SLS Functions***, and Excel will start. When in Excel, you can click on the function wizard icon or simply select an empty cell and click on ***Insert | Function***. While in Excel's equation wizard, either select the ***All*** category or ***Real Options Valuation***, the name of the company that developed the software. Here you will see a list of SLS functions (with SLS prefixes) that are ready for use in Excel. Figure 18.22 shows the Excel equation wizard.

Suppose you select the first function, *SLSBinomialAmericanCall*, and hit ***OK***. Figure 18.23 shows how the function can be linked to an existing Excel model. The values in cells B1 to B7 can be linked from other models or spreadsheets, or can be created using Excel's Visual Basic for Applications (VBA) macros, or can be dynamic and changing as in when running a simulation. Another quick note of caution here is that certain SLS functions require many input variables, and Excel's equation wizard can show only five variables at a time. Therefore, remember to scroll down the list of variables by clicking on the vertical scroll bar to access the rest of the variables.

## Lattice Maker

Finally, the full version of the software comes with an advanced binomial *Lattice Maker* module. This Lattice Maker is capable of generating binomial lattices and decision lattices with visible formulas in an Excel spreadsheet. Figure 18.24 illustrates an example option generated using this module. The illustration shows the module inputs (you can obtain this module by clicking on ***Start | Programs | Real Options Valuation | Real Options Super Lattice Solver | Lattice Maker***) and the resulting output lattice. Notice that the visible equations are linked to the existing spreadsheet, which means this module will come in handy when running Monte Carlo simulations or when used to link to and from other spreadsheet models. The results can also be used as a presentation and learning tool to peep inside the analytical black box of binomial lattices. Last, but not least, a decision lattice with specific decision nodes indicating expected optimal times of execution of certain options is also available in this module. The results generated from this module are identical to those generated using the SLS and Excel functions, but this has the added advantage of a visible lattice (lattices of up to 200 steps can be generated using this module). You are now equipped to start using the SLS software in building and solving real options, financial options, and employee stock options problems.

# Glossary

**ADF (annuity discount factor):** The present value of a finite stream of cash flows for every beginning $1 of cash flow. See Chapter 4.

**control premium:** The additional value inherent in the control interest as contrasted to a minority interest that reflect its power of control.[1] However, our analysis in Chapter 8 shows that the true nature of a control premium is much less clear than appears on the surface.

**CARs (cumulative abnormal returns):** A measure used in academic finance articles to measure the excess returns an investor would have received over a particular time period if he or she were invested in a particular stock. This is typically used in control and takeover studies, where stockholders are paid a premium for being taken over. Starting some time period before the takeover (often five days before the first announced bid, but sometimes a longer period), the researchers calculate the actual daily stock returns for the target firm and subtract out the expected market returns (usually calculated using the firm's beta and applying it to overall market movements during the time period under observation). The excess actual return over the capital asset pricing model-determined expected market return is called an "abnormal return." The cumulation of the daily abnormal returns over the time period under observation is the CAR. The term CAR(–5, 0) means the CAR calculated from five days before the announcement to the day of announcement. The CAR(–1, 0) is a control premium, although Mergerstat generally uses the stock price five days before announcement rather than one day before announcement as the denominator in its control premium calculation. However, the CAR for any period other than (–1, 0) is not mathematically equivalent to a control premium.

**DLOC (discount for lack of control):** An amount or percentage deducted from a pro rata share of the value of 100% of an equity interest in a business, to reflect the absence of some or all of the powers of control.[2]

**DLOM (discount for lack of marketability):** An amount or percentage deducted from an equity interest to reflect lack of marketability.[3]

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[1]American Society of Appraisers. "Definitions," *Business Valuation Standards* (2005).
[2]*Ibid*.
[3]*Ibid*.

**economic components model:** Abrams's model for calculating DLOM based on the interaction of discounts from four economic components. This model consists of four components: the measure of the economic impact of the delay-to-sale, monopsony power to buyers, and incremental transactions costs to both buyers and sellers. See the second half of Chapter 8.

**discount rate:** The rate of return on investment that would be required by a prudent investor to invest in an asset with a specific level risk. Also, a rate of return used to convert a monetary sum, payable or receivable in the future, into present value.[4]

**fractional interest discount:** The combined discounts for lack of control and marketability

**g:** The constant growth rate in cash flows or net income used in the ADF, Gordon model, or present value factor.

**Gordon model:** Present value of a perpetuity with growth. The end-of-year Gordon model formula is $\frac{1}{r-g}$, and the midyear formula is $\frac{\sqrt{1+r}}{r-g}$. See Chapter 4.

**log size model:** Abrams's model to calculate discount rates as a function of the logarithm of the value of the firm. See Chapter 5.

**markup:** The period after an announcement of a takeover bid in which stock prices typically rise until a merger or acquisition is made (or until it falls through).

**ordinary least squares (OLS) regression analysis:** A statistical technique that minimizes the sum of the squared deviations between a dependent variable and one or more independent variables and provides the user with a $y$-intercept and $x$-coefficients, as well as feedback such as $R^2$ (explained variation/total variation) $t$-statistics, $p$-values, etc.

**NPV (net present value) of cash flows:** Same as PV, but usually includes a subtraction for an initial cash outlay.

**PPF (periodic perpetuity factor):** A generalization formula invented by Abrams that is the present value of regular, but noncontiguous cash flows that have constant growth to perpetuity. The end-of-year PPF is equal to: $PPF = \frac{(1+r)^b}{(1+r)^j-(1+g)^j}$, and the midyear PPF is equal to $PPF = \frac{\sqrt{1+r}(1+r)^b}{(1+r)^j-(1+g)^j}$, where $r$ is the discount rate, $b$ is the number of years (before) since the last occurrence of the cash flow, and $j$ is the number of years between cash flows. See Chapter 4.

**PV (present value of cash flows):** The value in today's dollars of cash flows that occur in different time periods.

**present value factor:** Equal to the formula $\frac{1}{(1+r)^n}$, where $n$ is the number of years from the valuation date to the cash flow and $r$ is the discount rate. For business valuation, $n$ should usually be midyear, that is, $n = 0.5, 1.5, \ldots$

[4] *Ibid.*

**QMDM (quantitative marketability discount model):** Model for calculating DLOM for minority interests.[5]

***r*:** The discount rate.

**runup:** The period before a formal announcement of a takeover bid in which one or more bidders are either preparing to make an announcement or speculating that someone else will.

[5]Z. Christopher, Mercer, *Quantifying Marketability Discounts: Developing and Supporting Marketability Discounts in the Appraisal of Closely-Held Business Interests* (Memphis, TN: Peabody, 1997)

# About the Author

**Jay B. Abrams,** ASA, CPA, MBA, is a nationally known authority in valuing privately held businesses. He has published numerous seminal articles in the American Society of Appraisers's *Valuation Journal*, *Business Valuation Review*, *The Valuation Examiner,* and *The Practical Accountant*.

Mr. Abrams is the President of Abrams Valuation Group, Inc. in North Hollywood, California. He was a Project Manager at Arthur D. Little Valuation, Inc. in Woodland Hills, California, where he performed the valuations of Columbia Pictures, Dr. Pepper, Purex, MCO Geothermal, VSA, and many other large firms.

Mr. Abrams has several inventions to his name, many of which are discussed in this work. In 1992, he published a formula and algorithm that solved a 500-year-old problem—how to pinpoint an accounting transposition error. He has invented more than 150 mathematical formulas and several valuation models:

- The Economic Components Model for calculating Discount for Lack of Marketability
- The formulas to calculate the Present Value lost due to periodic Transactions Costs
- The Log Size Model for calculating discount rates
- Abrams Table of and Formula for Accounting Transposition Errors
- Formulas for.
  - Calculating the discount for lack of marketability from periodic transactions costs
  - Periodic Perpetuity Factors, a generalized Annuity Discount Factor for non-contiguous cash flows
  - Valuing Leveraged ESOPs and calculating dilution
  - Annuity Discount Formulas for cash flows with a fractional year
  - Present Value of Step Variable Costs
  - And many others

Mr. Abrams has an MBA in finance from the University of Chicago, where he also took graduate courses in the Department of Economics. He received his B.S. in Business Administration from California State University, Northridge, where he received the Arthur Young Outstanding Accounting Student Award in 1972.

Mr. Abrams has spoken in a variety of different professional and public forums about valuing privately held businesses, including the 1998 Conference of the National Association of Valuation Analysts; the 1996 International Conference of the

American Society of Appraisers, in Toronto; Anthony Robbins's Mastery University; and National Center for Employee Ownership Annual Conference. He has taught business valuation as continuing legal education and at the University of California at San Diego Extension and has been a guest lecturer at the University of Southern California and California State University, Northridge.

Mr. Abrams lives in North Hollywood, California, with his wife and five children.

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