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Optimal Methods for Ill-Posed Problems

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Introduction

Many problems of mathematical physics arising in applications are not well-posed in the sense of Hadamard [26, 27], i. e., they do not satisfy the three conditions of well-posedness: the existence of a solution, the solution uniqueness, and the solution continuous dependence on the initial data. Therefore, traditional methods, reduced to the inversion of the problem operator, cannot be used to solve such problems, which have been called ill-posed problems. For a long time mathematicians have been taking little interest in these problems, denying their practical value.

The practical value of such problems was for the first time pointed out by A. N. Tikhonov in his well-known paper [96]. In addition, in the mentioned paper Tikhonov formulated the concept of a conditionally well-posed problem, which played an important role in the development of the theory of such problems and their applications.

The issues of posing ill-posed problems and developing special methods for their solutions were also addressed in papers, such as those by A. N. Tikhonov [96–98], M. M. Lavrentiev [41–44], and V. K. Ivanov [29–32], that fundamentally contributed to this field of research. This theory was further developed by A. N. Tikhonov, M. M. Lavrentiev, and V. K. Ivanov, as well as their students and followers V. Ya. Arsenin, A. L. Ageev, A. B. Bakushinskii, A. L. Buhgeim, G. M. Vainikko, F. P. Vasiliev, V. V. Vasin, V. A. Vinokurov, A. V. Goncharskii, V. B. Glasko, A. R. Danilin, A. M. Denisov, E. V. Zakharov, V. I. Dmitriev, S. I. Kabanikhin, A. S. Leonov, O. A. Liskovets, I. V. Melnikova, L. D. Menikhes, V. A. Morozov, A. I. Prilepko, V. G. Romanov, V. N. Strakhov, V. P. Tanana, A. M. Fedotov, G. V. Khromova, A. V. Chechkin, and A. G. Yagola and many other mathematicians [1–11, 102–113], [13–17, 23–25, 114–117], [46–49], [55, 56, 58–63], [20, 67–87, 89–95, 99–101, 118], and [38]. To date the theory of ill-posed problems has become one of the main trends in modern applied mathematics. It is widely used in a constantly growing number of new technological applications.

The current state of the theory of ill-posed problems is described in the well-known monographs by M. M. Lavrentiev [43], A. N. Tikhonov and V. Ya. Arsenin [99], R. Lattes and J. L. Lions [40], V. K. Ivanov, V. V. Vasin, and V. P. Tanana [28], V. A. Morozov [62], M. M. Lavrentiev, V. G. Romanov, and S. P. Shishatskii [45], O. A. Liskovets [51], V. P. Tanana [80, 95], V. V. Vasin and A.L. Ageev [111], G. M. Vainikko [103], A. S. Leonov [48], A. N. Tikhonov, A. S. Leonov, and A. G. Yagola [101], A. M. Fedotov [20], A. N. Tikhonov, A. V. Goncharskii, V. V. Stepanov, and A. G. Yagola [100], S. I. Kabanikhin [34–37], and many other researchers. A large number of monographs show the maturity of this branch of mathematics. Abroad a significant contribution to this theory has been made by the following mathematicians: J. N. Franklin [22], J. Gullum [12], K. Miller [57], D. L. Phillips [64], A. Melkman and C. Micchelli [54], R. Lattes and J. L. Lions [40], H. W. Engl, M. Hanke, and A. Neubauer [19], and many others.

Among the important characteristics of the methods for solving ill-posed problems, one can name their accuracy, which is controlled by error estimates for these

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methods. These estimates allow for comparing different methods, as well as developing optimal and near-optimal methods.

The issues related to the development and studies of optimal methods for solving ill-posed problems were investigated by V. K. Ivanov, V. V. Vasin, and V. P. Tanana [28], V. P. Tanana [80], and V. P. Tanana, M. A. Rekant, and S. I. Yanchenko [95]. As this theory has been rapidly developing over the recent decades and new important facts and applications of the theory to the solution of practical problems have been revealed, a new book to cover this gap was to be written.

It should be noted that, in dealing with the existence and uniqueness of the classical solutions for the direct heat conduction problem addressed in Section 5.1.1, we could have just referred to the great books by Arsenin [7] and Vladimirov [117]. However, to ensure a complete and smooth narration these issues are considered in detail in the corresponding sections of the current book. The obtained formulas are further used to study the solution methods of the direct problem for $t \to \infty$.

This book is based on lecture notes covering the course on the theory of ill-posed problems that has been delivered by the authors to the students majoring in Applied Mathematics and Informatics within the master program at the Chelyabinsk State University and South Ural State University over the past decade.

Contents

1 Modulus of continuity of the inverse operator and methods for solving ill-posed problems

1.1 Modulus of continuity and its properties

1.1.1 Problem posing

Let $\mathbb{U}$, $\mathbb{F}$, and $\mathbb{V}$ be Banach spaces, let A be an injective linear bounded operator that maps $\mathbb{U}$ into $\mathbb{F}$ and has an unbounded inverse operator, let B be a linear bounded operator that maps $\mathbb{V}$ into $\mathbb{U}$, $M_r = B\overline{S}_r$, where $\overline{S}_r = \{v : v \in \mathbb{V},\ \|v\| \le r\}$, and let $N_r = AM_r$. Consider the following operator equation:

$$Au = f, \quad u \in \mathbb{U}, f \in \mathbb{F}. \tag{1.1}$$

Definition 1.1. A set M_r is called the class of correctness for equation (1.1), if the restriction $A^{-1}_{N_r}$ of the operator A^{-1} to the set N_r is uniformly continuous on N_r.

Lemma 1.1. *In order for the set M_r to be the class of correctness of equation* (1.1), *it is necessary and sufficient for the restriction $A^{-1}_{N_r}$ of the operator A^{-1} to the set N_r to be continuous at zero.*

Proof. The necessity is obvious.

Sufficiency. Since $A^{-1}_{N_r}$ is continuous at zero, for any $\varepsilon > 0$ there is $\delta > 0$ such that for any $f \in N_r$ and $\|f\| < \delta$ it follows that

$$\|A^{-1}f\| < \frac{\varepsilon}{2}.$$

Hence, for any f_1 and $f_2 \in N_r$ such that $\|f_1 - f_2\| < \delta$ it follows that

$$-f_2 \in N_r, \quad \frac{f_1 - f_2}{2} \in N_r \quad \text{and} \quad \left\|\frac{f_1 - f_2}{2}\right\| < \delta,$$

whence

$$\left\|A^{-1}\left(\frac{f_1 - f_2}{2}\right)\right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \|A^{-1}f_1 - A^{-1}f_2\| < \varepsilon.$$

The lemma is thereby proved. □

Now following [33], define functions $\omega_1(\tau, r)$ and $\omega(\tau, r)$ as follows:

$$\omega_1(\tau, r) = \sup\{\|u_1 - u_2\| : u_1, u_2 \in M_r,\ \|Au_1 - Au_2\| \le \tau\}, \tag{1.2}$$

$$\omega(\tau, r) = \sup\{\|u\| : u \in M_r,\ \|Au\| \le \tau\}, \tag{1.3}$$

where $r > 0$ and $\tau > 0$.

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Corollary 1.1. *If $\omega(\tau, r) \to 0$ for $\tau \to 0$, then the set M_r is the class of correctness.*

It follows from (1.3) by Lemma 1.1.

Lemma 1.2. *Let the functions $\omega_1(\tau, r)$ and $\omega(\tau, r)$ be defined by formulas* (1.2) *and* (1.3). *Then they are related as follows:*

$$\omega_1(\tau, r) = \omega(\tau, 2r).$$

Proof. Let u_1 and u_2 belong to the set M_r and let

$$\|Au_1 - Au_2\| \le \tau. \tag{1.4}$$

Then $u_1 - u_2 \in M_{2r}$ and from (1.4) it follows that

$$\|u_1 - u_2\| \le \omega(\tau, 2r). \tag{1.5}$$

From (1.5) we have

$$\omega_1(\tau, r) \le \omega(\tau, 2r). \tag{1.6}$$

In the reverse direction, let $u \in M_{2r}$ and $\|Au\| \le \tau$. Then assuming

$$u_1 = u/2 \quad \text{and} \quad u_2 = -u/2,$$

we deduce that u_1 and u_2 belong to the set M_r and $\|Au_1 - Au_2\| \le \tau$. Thus,

$$\omega_1(\tau, r) \ge \omega(\tau, 2r). \tag{1.7}$$

The proof of the lemma follows from (1.6) and (1.7). □

Lemma 1.3. *Let $k \ge 0$. Then the following equation holds:*

$$\omega(k\tau, kr) = k\omega(\tau, r).$$

Proof. For $k = 0$ the lemma is obvious. Let $k > 0$ and $\tau \ge r\|AB\|$. Then $k\tau \ge kr\|AB\|$. From (1.3) it follows that

$$\omega(\tau, r) = r\|AB\| \tag{1.8}$$

and

$$\omega(k\tau, kr) = kr\|AB\|. \tag{1.9}$$

From (1.8) and (1.9) it follows that $\omega(k\tau, kr) = k\omega(\tau, r)$.

Let $k > 0$ and $\tau < r\|AB\|$. Then from $u \in M_r$ and $\|Au\| \le \tau$ it follows that $ku \in M_{kr}$ and $\|A(ku)\| \le k\tau$. Thus,

$$k\omega(\tau, r) \le \omega(k\tau, kr). \tag{1.10}$$

In the reverse direction, let $u \in M_{kr}$ and $\|Au\| \le k\tau$. Then $u/k \in M_r$ and $\|A(u/k)\| \le \tau$, that is,

$$\frac{1}{k}\omega(k\tau, kr) \le \omega(\tau, r)$$

or

$$\omega(k\tau, kr) \le k\omega(\tau, r). \tag{1.11}$$

The assertion of the lemma follows from (1.10) and (1.11). □

We formulate an obvious lemma.

Lemma 1.4. *The function $\omega(\tau, r)$ does not decrease on τ and r.*

Lemma 1.5. *If $M_1 = B\overline{S}_1$ is the class of correctness for equation* (1.1)*, then for any $r \ge 0$ the set $M_r = B\overline{S}_r$ is the class of correctness for equation* (1.1)*.*

Proof. The case where $r = 0$ is obvious. Assume that $r > 0$. Then it follows from Lemma 1.2 that

$$\omega_1(\tau, 1 + r) = \omega(\tau, 2 + 2r).$$

It follows from Lemma 1.4 that

$$\omega(\tau, 2 + 2r) \le \omega\big((1 + r)\tau, 2 + 2r\big) \tag{1.12}$$

and it follows from Lemma 1.3 that

$$\omega\big((1 + r)\tau, 2 + 2r\big) = (1 + r)\omega(\tau, 2). \tag{1.13}$$

Since $\omega_1(\tau, 1) \longrightarrow 0$ for $\tau \longrightarrow 0$, by Lemma 1.2, (1.12), and (1.13) the assertion of the lemma is proved. □

Lemma 1.6. *If the set $M_1 = B\overline{S}_1$ is the class of correctness for equation* (1.1)*, then $\omega(\tau, r) \in C([0, \infty) \times [0, \infty))$.*

Proof. Assume that $\tau_n \to \tau$ and $r_n \to r$, where $\tau > 0$ and $r > 0$. Let us introduce the numbers

$$k_n = \max(c_n, d_n), \quad k'_n = \min(c'_n, d'_n),$$

where

$$c_n = \frac{\tau + |\tau_n - \tau|}{\tau}, \quad c'_n = \frac{\tau - |\tau_n - \tau|}{\tau} \tag{1.14}$$

and

$$d_n = \frac{r + |r_n - r|}{r}, \quad d'_n = \frac{r - |r_n - r|}{r}. \tag{1.15}$$

Then it follows from Lemmas 1.3 and 1.4 and from (1.14) and (1.15) that

$$k_n'\omega(\tau, r) \le \omega(\tau_n, r_n) \le k_n\omega(\tau, r). \tag{1.16}$$

Since

$$\lim_{n\to\infty} k_n = \lim_{n\to\infty} k_n' = 1,$$

the assertion of the lemma for $\tau > 0$ and $r > 0$ follows from (1.16).

If $r = 0$, then it follows from (1.3) that $\omega(\tau, r) = 0$.

Let

$$r_n \to 0, \quad \tau_n \to \tau, \quad \tau \ge 0.$$

Then from (1.3) it follows that

$$\omega(\tau_n, r_n) \le r_n\|B\| \tag{1.17}$$

and from (1.17) it follows that

$$\omega(\tau_n, r_n) \to 0 \quad \text{for } n \to \infty.$$

Now, let

$$\tau_n \to 0 \quad \text{and} \quad r_n \to r, \quad r \ge 0.$$

Then there exists a number $\bar{r} \ge 0$ such that for any n

$$r_n \le \bar{r}. \tag{1.18}$$

For any n we introduce a set $\overline{M}_n$, defined as follows:

$$\overline{M}_n = \{u : u \in B\overline{S}_{\bar{r}},\ \|Au\| \le \tau_n\}, \tag{1.19}$$

where

$$\overline{S}_{\bar{r}} = \{v : v \in \mathbb{V},\ \|v\| \le \bar{r}\}.$$

Since for any n the set $\overline{M}_n$ defined by formula (1.19) is bounded, there exists an element $\bar{u}_n \in \overline{M}_n$ such that

$$\|\bar{u}_n\| \ge \frac{1}{2}\sup\{\|u\| : u \in \overline{M}_n\}. \tag{1.20}$$

It follows from Lemma 1.5 and (1.19) that

$$A\bar{u}_n \to 0 \quad \text{for } n \to \infty. \tag{1.21}$$

It follows from (1.21) and Lemmas 1.1 and 1.5 that

$$\overline{u}_n \to 0 \quad \text{for } n \to \infty \tag{1.22}$$

and it follows from (1.20) and (1.22) that

$$\sup\{\|u\| : u \in \overline{M}_n\} \to 0 \quad \text{at } n \to \infty. \tag{1.23}$$

Since

$$\sup\{\|u\| : u \in \overline{M}_n\} = \omega(\tau_n, \overline{r}),$$

it follows from Lemma 1.4 and (1.18) that

$$\omega(\tau_n, r_n) \le \omega(\tau_n, \overline{r}). \tag{1.24}$$

Then it follows from (1.23) and (1.24) that $\omega(\tau_n, r_n) \to 0$ at $n \to \infty$.

The lemma is thereby proved. □

Definition 1.2. The bounded linear operator Q, mapping the Hilbert space $\mathbb{H}$ into itself, is called isometric if for any $u \in \mathbb{H}$

$$\|Qu\| = \|u\|.$$

Definition 1.3. The isometric operator Q is called unitary if its range of values $R(Q)$ coincides with $\mathbb{H}$.

Lemma 1.7. *If A is an injective bounded linear operator mapping the space $\mathbb{H}$ into itself and its range of values $R(A)$ is everywhere dense on $\mathbb{H}$, then we have a polar decomposition for A as follows:*

$$A = Q\overline{A},$$

where Q is a unitary operator, A^* is the conjugated operator A, and $\overline{A} = \sqrt{A^*A}$.

Proof. The proof follows from the theorem formulated in [66] on p. 325. □

Assume that the injective bounded linear operators A and B have the everywhere dense ranges of values $R(A)$ and $R(B)$, where the ranges of values $R(A)$ and $R(B)$ are everywhere dense on $\mathbb{H}$. Then by Lemma 1.7 for the operators A and B there exist polar decompositions $A = Q\overline{A}$ and $B = \overline{B}P$, where Q and P are unitary operators, $\overline{A} = \sqrt{A^*A}$, and $\overline{B} = \sqrt{BB^*}$. In addition, assume that the spectrum $\mathrm{Sp}(\overline{A})$ of the operator $\overline{A}$ coincides with the interval $[0, \|A\|]$ and $\overline{B} = G(\overline{A})$, where $G(\sigma)$ is a strictly increasing and continuous function on the interval $[0, \|A\|]$ such that $G(0) = 0$. Consider the following equation:

$$rG(\sigma)\sigma = \tau, \quad \sigma \in [0, \|A\|]. \tag{1.25}$$

It follows from (1.25) that, if $0 < \tau < rG(\|A\|)\|A\|$, then this equation has a unique solution $\overline{\sigma}(\tau) = \psi(\frac{\tau}{r})$, where $\psi(x)$ is the inverse function of $G(\sigma)\sigma$. It follows from the inverse function theorem that $\psi \in C[0, G(\|A\|)\|A\|]$ and $\psi(0) = 0$. Thus,

$$\overline{\sigma}(\tau) \to 0 \quad \text{for } \tau \to 0. \tag{1.26}$$

Denote by $\overline{\omega}(\tau, r)$ the function defined by the formula

$$\overline{\omega}(\tau, r) = \sup\{\|u\| : u \in \overline{B}\,\overline{S}_r,\ \|\overline{A}u\| \le \tau\}. \tag{1.27}$$

Lemma 1.8. *Under the above-formulated conditions, we have*

$$\overline{\omega}(\tau, r) = \omega(\tau, r).$$

Proof. Let $u \in M_r$ and $\|Au\| \le \tau$. Then there exists $v \in \mathbb{H}$ such that

$$u = Bv \quad \text{and} \quad \|v\| \le r.$$

Since $B = \overline{B}P$, there exists an element $v_1 \in \mathbb{H}$ such that $v = Pv_1$. Thus,

$$u = \overline{B}v_1, \tag{1.28}$$

where $\|v_1\| \le r$. It follows from $A = Q\overline{A}$ that

$$\|\overline{A}u\| = \|Q^{-1}Au\| \le \|Q^{-1}\|\|Au\| = \|Au\| \le \tau. \tag{1.29}$$

From (1.28) and (1.29) it follows that

$$\omega(\tau, r) \le \overline{\omega}(\tau, r). \tag{1.30}$$

In the reverse direction, it follows from $u \in \overline{B}S_r$ that there exists an element $\overline{v} \in \mathbb{H}$ such that $\|\overline{v}\| \le r$ and $u = \overline{B}\overline{v}$. Since

$$\|\overline{A}u\| \le \tau \quad \text{and} \quad \overline{A} = Q^{-1}A,$$

we have

$$\|Au\| = \|Q\overline{A}u\| \le \|Q\|\|\overline{A}u\| = \|\overline{A}u\| \le \tau. \tag{1.31}$$

Thus, it follows from (1.31) that

$$\overline{\omega}(\tau, r) \le \omega(\tau, r). \tag{1.32}$$

The assertion of the lemma follows from inequalities (1.30) and (1.32). □

Lemma 1.9. *Let*

$$A = Q\overline{A} \quad \text{and} \quad B = \overline{B}P, \quad \text{where } \overline{A} = \sqrt{A^*A},\ \overline{B} = \sqrt{BB^*},$$

and P and Q are unitary operators. In addition,

$$\overline{B} = G(\overline{A}),$$

where $G(\sigma)$ *is a strictly increasing function continuous over the interval* $[0, \|\overline{A}\|]$ *such that* $G(0) = 0$. *Also,* $\tau < r\|A\| \cdot \|B\|$. *Then we have* $\omega(\tau, r) = rG[\overline{\sigma}(\tau)]$, *where* $\overline{\sigma}(\tau)$ *is the solution of equation* (1.25).

Proof. Let ε be a sufficiently small positive number and let $\overline{\sigma}(\tau)$ be the solution of equation (1.25). Then select a natural number n_0 such that

$$rG[\overline{\sigma}(\tau)] - rG\left[\frac{n_0 - 1}{n_0}\overline{\sigma}(\tau)\right] < \varepsilon \tag{1.33}$$

and consider the space $\mathbb{H}_0$ defined by the formula

$$\mathbb{H}_0 = E_{\overline{\sigma}(\tau)}\mathbb{H} - E_{\frac{n_0-1}{n_0}\overline{\sigma}(\tau)}\mathbb{H}, \tag{1.34}$$

where $\{E_\sigma : 0 \le \sigma \le \|A\|\}$ is a partition of unity generated by the operator $\overline{A}$ [52] (p. 336).

Let $\overline{M}_r = \overline{B}\,\overline{S}_r$, $v_0 \in \mathbb{H}_0$, and

$$\|v_0\| = r. \tag{1.35}$$

Then it follows from (1.35) that

$$u_0 = \overline{B}v_0 \in \overline{M}_r. \tag{1.36}$$

Since $u_0 \in H_0$, from (1.33)–(1.35) we deduce

$$\|u_0\| \ge rG[\overline{\sigma}(\tau)] - \varepsilon. \tag{1.37}$$

As $u_0, \overline{A}u_0 \in \mathbb{H}_0$ and the function $G(\sigma)$ strictly increases, it follows from (1.33) and (1.34) that

$$\|\overline{A}u_0\| \le rG[\overline{\sigma}(\tau)]\overline{\sigma}(\tau) = \tau. \tag{1.38}$$

From (1.36) and (1.38) it follows that

$$\|u_0\| \le \overline{\omega}(\tau, r) \tag{1.39}$$

and from (1.37) and (1.39) it follows that

$$\overline{\omega}(\tau, r) \ge rG[\overline{\sigma}(\tau)] - \varepsilon.$$

Due to the arbitrariness of ε we have

$$\overline{\omega}(\tau, r) \geq rG[\overline{\sigma}(\tau)]. \tag{1.40}$$

Let us prove the inequality in the reverse direction. For this purpose, represent the space $\mathbb{H}$ as the orthogonal sum

$$\mathbb{H} = \mathbb{H}_1 + \mathbb{H}_2 \tag{1.41}$$

of the subspaces

$$\mathbb{H}_1 = E_{\overline{\sigma}(\tau)}\mathbb{H} \quad \text{and} \quad \mathbb{H}_2 = (E - E_{\overline{\sigma}(\tau)})\mathbb{H}.$$

The theorem proved in [52] (p. 336) shows that the subspaces $\mathbb{H}_1$ and $\mathbb{H}_2$ are invariant for the operators $\overline{A}$ and $\overline{B}$. It follows from the notions that $u_0 \in \overline{M}_r$ and

$$\|\overline{A}u_0\| \leq \tau \tag{1.42}$$

that there exists an element $v_0 \in \mathbb{H}$, such that

$$\|v_0\| \leq r \tag{1.43}$$

and

$$u_0 = \overline{B}v_0. \tag{1.44}$$

Using (1.41), represent the element v_0 as the orthogonal sum

$$v_0 = v_1 + v_2, \tag{1.45}$$

where $v_i = \mathrm{pr}(v_0, \mathbb{H}_i)$, $i = 1, 2$. Let $r_1 = \|v_1\|$ and $r_2 = \|v_2\|$. Then from (1.43) and (1.45) it follows that

$$r_1^2 + r_1^2 \leq r^2. \tag{1.46}$$

From the invariance of the spaces $\mathbb{H}_1$ and $\mathbb{H}_2$ for the operator $\overline{B}$ and (1.44) it follows that $u_0 = u_1 + u_2$ and

$$u_i = Bv_i \in \mathbb{H}, \quad i = 1, 2. \tag{1.47}$$

From the invariance of the spaces $\mathbb{H}_1$ and $\mathbb{H}_2$ for the operator $\overline{A}$ it follows that

$$\overline{A}u_i \in \mathbb{H}_i, \quad i = 1, 2. \tag{1.48}$$

From (1.42), (1.47), and (1.48) it follows that

$$\|\overline{A}u_i\| \leq \frac{r_i}{r}\tau, \quad i = 1, 2. \tag{1.49}$$

Since $G(\sigma)$ is strictly increasing, it follows from (1.47) that

$$\|u_1\| \le r_1 G[\overline{\sigma}(\tau)] \tag{1.50}$$

and it follows from (1.49) that

$$\|u_2\| \le \frac{r_2\tau}{r\overline{\sigma}(\tau)}. \tag{1.51}$$

Since

$$r_2 G[\overline{\sigma}(\tau)]\overline{\sigma}(\tau) = \frac{r_2}{r}\tau, \tag{1.52}$$

it follows from (1.51) and (1.52) that

$$\|u_2\| \le r_2 G[\overline{\sigma}(\tau)]. \tag{1.53}$$

From (1.46), (1.47), (1.50), and (1.53) it follows that

$$\|u_0\| \le rG[\overline{\sigma}(\tau)]. \tag{1.54}$$

Due to the arbitrariness of u_0 on (1.42)–(1.44) and (1.54), it follows that

$$\overline{\omega}(\tau, r) \le rG[\overline{\sigma}(\tau)] \tag{1.55}$$

and from (1.40) and (1.55) it follows that

$$\overline{\omega}(\tau, r) = rG[\overline{\sigma}(\tau)]. \tag{1.56}$$

The assertion of the lemma follows from Lemma 1.8 and (1.56). □

Lemma 1.10. *Under the conditions to be met by the operators A and B, formulated in Lemma* 1.9*, the set $M_r = B\overline{S}_r$ is the class of correctness for equation* (1.1).

Proof. Since $G \in C[0, |A|]$, as (1.26) $\overline{\sigma}_1(\tau) \to 0$ for $\tau \to 0$, where $\overline{\sigma}_1(\tau)$ is the solution of the equation $2rG(\sigma)\sigma = \tau$, we have

$$G(\overline{\sigma}_1(\tau)) \to 0 \quad \text{for } \tau \to 0. \tag{1.57}$$

From (1.57) and Lemma 1.9, it follows that

$$\omega(\tau, 2r) \to 0 \quad \text{for } \tau \to 0. \tag{1.58}$$

It follows from (1.58) and Lemma 1.4 that

$$\omega_1(\tau, r) \to 0 \quad \text{for } \tau \to 0. \tag{1.59}$$

The assertion of the lemma follows from (1.59) and Corollary 1.1. □

Let us strengthen Lemma 1.4.

Lemma 1.11. *Let $\overline{B} = G(\overline{A})$, where the function $G(\sigma) \in C[0, \|A\|]$ is strictly increasing over this interval, and let $G(0) = 0$. Then, if $0 < \tau < r\|AB\|$, the function $\omega(\tau, r)$ is strictly increasing on τ and r.*

Proof. It follows from Lemma 1.9 that

$$\omega(\tau, r) = rG[\overline{\sigma}(\tau)], \tag{1.60}$$

where $\overline{\sigma} = \psi(\tau/r)$ and $\psi(x)$ is the inverse function of $G(\sigma)\sigma$.

It follows from the inverse function theorem that the function $\overline{\sigma}(\tau)$ strictly increases on τ and, consequently, by (1.60) $\omega(\tau, r)$ strictly increases on τ.

To prove that the function $\omega(\tau, r)$ is strictly increasing on r, we write

$$r = \frac{\tau r}{\tau} = \left[G\left[\psi\left(\frac{\tau}{r}\right)\right]\psi\left(\frac{\tau}{r}\right)\right]^{-1}\tau. \tag{1.61}$$

From (1.60) and (1.61) it follows that

$$\omega(\tau, r) = \frac{\tau G[\psi(\frac{\tau}{r})]}{G[\psi(\frac{\tau}{r})]\psi(\frac{\tau}{r})} = \frac{\tau}{\psi(\frac{\tau}{r})}. \tag{1.62}$$

Since the function $\psi(\frac{\tau}{r})$ strictly decreases on r, it follows from (1.62) that the function $\omega(\tau, r)$ strictly increases on r.

The lemma is thereby proved. □

Note that long before the paper [33] was published, in his famous monograph [43] M. M. Lavrent'ev introduced the concept of the modulus of continuity $\omega(\tau)$ and used it to estimate the errors of the methods for solving operator equations of the first kind.

Since the concept of the modulus of continuity defined by M. M. Lavrent'ev differed from the concept of the modulus of continuity $\omega(\tau, r)$, used by V. K. Ivanov, it is appropriate to compare these concepts. The following definition of the modulus of continuity is given in [43] (p. 11).

Let $M = BS_1$, where B is a linear completely continuous operator mapping a Hilbert space $\mathbb{H}$ into itself.

Further the function $\omega(\tau)$ is introduced that satisfies the following conditions:

1. $\omega(\tau)$ is a continuous non-decreasing function and $\omega(0) = 0$;
2. for any $u \in M$ satisfying the inequality $\|Au\| \le \tau$, we have the following inequality:

$$\|u\| \le \omega(\tau). \tag{1.63}$$

From Lemma 1.4 and Lemma 1.6 and from the fact that M_1 is the class of correctness it follows that the function $\omega(\tau, 1)$ defined by formula (1.3) is a special case of the function $\omega(\tau)$ suggested by M. M. Lavrent'ev.

Compare the following functions.

Lemma 1.12. *Let* $\omega(\tau, r)$ *be defined by formula* (1.3) *and let* $\omega(\tau)$ *be defined by formula* (1.63). *Then for any* $\tau \geq 0$ *the following relation holds:*

$$\omega(\tau, 1) \leq \omega(\tau).$$

Proof. The case where $\tau = 0$ is obvious, since $\omega(0, 1) = \omega(0) = 0$.

Let $\tau > 0$. Assume the contrary, i. e., there exists $\tau_0 > 0$ such that

$$\omega(\tau_0, 1) > \omega(\tau_0). \tag{1.64}$$

Denote the difference $\omega(\tau_0, 1) - \omega(\tau_0)$ by d. Then it follows from (1.3) and (1.64) that there exists an element $u_0 \in B\overline{S}_1$ such that $\|Au_0\| \leq \tau$ and

$$\|u_0\| > \omega(\tau_0, 1) - \frac{d}{4} \geq \omega(\tau_0) + \frac{d}{4} > \omega(\tau_0),$$

which contradicts (1.63).

The lemma is thereby proved. □

It follows from Lemma 1.12 that the function $\omega(\tau, 1)$ is minimal among all possible functions $\omega(\tau)$, i. e., for any $\tau \geq 0$ it follows that $\omega(\tau, 1) \leq \omega(\tau)$.

Now find the connection between the functions $\omega(\tau, 1)$ and $\omega(\tau, r)$, where $r > 0$.

Lemma 1.13. *If the functions* $\omega(\tau, 1)$ *and* $\omega(\tau, r)$ *are defined by formula* (1.3) *and* $r > 0$, *then the following equation holds:*

$$\omega(\tau, r) = r\omega(\tau/r, 1).$$

Proof. The assertion of this lemma follows from Lemma 1.3. □

Thus, the function $\omega(\tau, 1)$ is a special case of the function $\omega(\tau)$ suggested by M. M. Lavrent'ev and is minimal of all possible variants of the function $\omega(\tau)$.

1.2 The concept of the method for solving an ill-posed problem

As in Section 1.1, $\mathbb{U}$, $\mathbb{F}$, and $\mathbb{V}$ are Banach spaces, A is an injective bounded linear operator mapping the space $\mathbb{U}$ into $\mathbb{F}$ that has an unlimited inverse operator, B is a bounded linear operator mapping $\mathbb{V}$ into $\mathbb{U}$, and $M_r = B\overline{S}_r$. We formulate the ill-posed problem of finding an approximate solution to equation (1.1) as follows.

Assume that for $f = f_0$ there exists an exact solution u_0 of equation (1.1), which belongs to the set M_r, but the exact value of the right-hand side f_0 is unknown. Instead, a certain approximation $f_\delta \in \mathbb{F}$ and error level $\delta > 0$ are given such that $\|f_\delta - f_0\| \leq \delta$. Using the initial data M_r, f_δ, δ of the problem, it is required to find the approximate solution u_δ of equation (1.1) and estimate its deviation from the exact solution u_0.

Definition 1.4. We will call the family of operators $\{T_\delta : 0 < \delta \le \delta_0\}$ an approximate solution method for equation (1.1) over the set M_r, if for any $\delta \in (0, \delta_0]$ the operator T_δ continuously maps the space $\mathbb{F}$ into $\mathbb{U}$ and $T_\delta f_\delta \to u_0$ for $\delta \to 0$ is uniform over the set M_r if $\|f_\delta - Au_0\| \le \delta$.

Let M_r be the class of correctness and let $\{T_\delta : 0 < \delta \le \delta_0\}$ be an approximate solution method for equation (1.1) on this class. Then for any $\delta \in (0, \delta_0]$ introduce a quantitative characteristic of the accuracy of this method over the set M_r. We have

$$\Delta_\delta[T_\delta] = \sup_{u, f_\delta} \{\|u - T_\delta f_\delta\| : u \in M_r,\ \|Au - f_\delta\| \le \delta\}. \tag{1.65}$$

Lemma 1.14. *Let* $\{T_\delta : 0 < \delta \le \delta_0\}$ *be an approximate solution method for equation* (1.1) *and let* $\omega(\delta, r)$ *be the modulus of continuity of the inverse operator at zero defined by formula* (1.3). *Then the following estimate holds:*

$$\Delta_\delta[T_\delta] \ge \omega(\delta, r).$$

Proof. Let ε be a sufficiently small positive number. Then from (1.2) it follows that there exist elements $u_1, u_2 \in M_r$ such that

$$\|u_1 - u_2\| \ge \omega_1(2\delta, r) - \varepsilon \tag{1.66}$$

and

$$\|Au_1 - Au_2\| \le 2\delta. \tag{1.67}$$

If

$$\bar{f}_\delta = (Au_1 + Au_2)/2,$$

it follows from (1.67) that

$$\|Au_1 - \bar{f}_\delta\| \le \delta \quad \text{and} \quad \|Au_2 - \bar{f}_\delta\| \le \delta. \tag{1.68}$$

From (1.68) it follows that

$$\max\{\|u_1 - T_\delta \bar{f}_\delta\|, \|u_2 - T_\delta \bar{f}_\delta\|\} \ge \frac{\|u_1 - u_2\|}{2}. \tag{1.69}$$

From (1.66) and (1.69) it follows that

$$\max\{\|u_1 - T_\delta \bar{f}_\delta\|, \|u_2 - T_\delta \bar{f}_\delta\|\} \ge \frac{1}{2}\omega_1(2\delta, r) - \varepsilon \tag{1.70}$$

and from (1.65) it follows that

$$\Delta_\delta[T_\delta] \ge \max\{\|u_1 - T_\delta \bar{f}_\delta\|, \|u_2 - T_\delta \bar{f}_\delta\|\}. \tag{1.71}$$

The assertion of the lemma follows from Lemma 1.2, (1.70), and (1.71). □

Denote by $C[\mathbb{F},\mathbb{U}]$ the set of all operators continuously mapping the space $\mathbb{F}$ into $\mathbb{U}$ and denote by $\Delta_\delta^{\text{opt}}$ the quantity defined by

$$\Delta_\delta^{\text{opt}} = \inf\{\Delta_\delta(P) : P \in C[\mathbb{F},\mathbb{U}]\},$$

where

$$\Delta_\delta = \sup_{u,f_\delta} \{\|u - Pf_\delta\| : u \in M_r, \|f_\delta - Au\| \le \delta\}.$$

Definition 1.5. The method $\{T_\delta^{\text{opt}} : 0 < \delta \le \delta_0\}$ will be called optimal on the class M_r, if for any $\delta \in (0,\delta_0]$

$$\Delta_\delta[T_\delta^{\text{opt}}] = \Delta_\delta^{\text{opt}}.$$

Definition 1.6. The method $\{\overline{T}_\delta : 0 < \delta \le \delta_0\}$ will be called optimal-by-order on the class M_r, if there exists a number $K > 1$ such that for any $\delta \in (0,\delta_0]$

$$\Delta_\delta[\overline{T}_\delta] \le K\Delta_\delta^{\text{opt}}.$$

It follows from Lemma 1.14 that for any $\delta \in (0,\delta_0]$

$$\Delta_\delta^{\text{opt}} \ge \omega(\delta, r). \tag{1.72}$$

2 Lavrent'ev methods for constructing approximate solutions of linear operator equations of the first kind

2.1 On the accuracy of the Lavrent'ev method with the regularization parameter chosen based on the Strakhov scheme

This method is borrowed from [43]. It is based on substituting the operator equation (1.1) by the family of operator equations of the second kind, depending on the parameter $\alpha > 0$. By applying different schemes to choose the regularization parameter α, we will get different methods. Below we present the optimal Lavrent'ev method.

Let

$$\mathbb{Z} = \mathbb{F} = \mathbb{C} = \mathbb{H},$$

where $\mathbb{H}$ is a Hilbert space, operators A and B are injective, and the ranges of values $R(A)$ and $R(B)$ of the operators A and B are everywhere dense on H. Then by Lemma 1.7 for the operators A and B there exist polar decompositions

$$A = Q\overline{A} \quad \text{and} \quad B = \overline{B}P,$$

where P and Q are unitary operators,

$$\overline{A} = \sqrt{A^*A}, \quad \text{and} \quad \overline{B} = \sqrt{BB^*}.$$

In addition, assume that the spectrum $\mathrm{Sp}(\overline{A})$ of the operator $\overline{A}$ coincides with the segment $[0, \|A\|]$ and

$$\overline{B} = G(\overline{A}), \tag{2.1}$$

where the function

$$G(\sigma) \in C[0, \|A\|] \cap C^1(0, \|A\|), \quad G(0) = 0,$$

and for any $\sigma \in (0, \|A\|)$

$$G'(\sigma) > 0.$$

Assume that the class of correctness M_r is of the form

$$M_r = \overline{B}\,\overline{S}_r, \tag{2.2}$$

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where

$$\overline{S}_r = \{v : v \in \mathbb{H},\ \|v\| \le r\}.$$

From Lemmas 1.8 and 1.9, it follows that the set M_r, defined by formulas (2.1) and (2.2), is the class of correctness for equation (1.1) and the modulus of continuity $\omega(\tau, r)$ of the inverse operator $\overline{A}^{-1}$ on the set $N_r = \overline{A}M_r$ is calculated by the formula

$$\omega(\tau, r) = rG[\overline{\sigma}(\tau)], \quad \tau < r\|A\|\|B\|, \tag{2.3}$$

where $\overline{\sigma}(\tau)$ is a solution of the equation

$$rG(\sigma)\sigma = \tau. \tag{2.4}$$

Using Lemma 1.7, equation (1.1) can be substituted with the following equivalent equation:

$$\overline{A}u = g, \tag{2.5}$$

where

$$\overline{A} = \sqrt{A^*A}, \quad g = Q^*f,$$

and the set of M_r is defined by formulas (2.1) and (2.2).

Assume that for $g = g_0 \in \mathbb{H}$ there exists the exact solution u_0 of equation (2.5), which belongs to the set M_r, but the exact value of the right-hand side g_0 is not known. Instead, a certain approximation $g_\delta \in \mathbb{H}$ and error level $\delta > 0$ are given, such that

$$\|g_\delta - g_0\| \le \delta.$$

Using the initial data M_r, g_δ, and δ it is required to find the approximate solution u_δ of equation (2.5) and estimate its deviation from the exact solution.

The Lavrent'ev method described in [43] (p. 14) uses the regularizing family of operators $\{R_\alpha : 0 < \alpha \le \alpha_0\}$, acting from $\mathbb{H}$ into $\mathbb{H}$ and defined by the formula

$$R_\alpha = \overline{B}(\overline{C} + \alpha E)^{-1}, \quad \alpha \in (0, \alpha_0], \tag{2.6}$$

where $\overline{C} = \overline{A}\,\overline{B}$.

Define the approximate solution u_δ^α by the formula

$$u_\delta^\alpha = R_\alpha g_\delta. \tag{2.7}$$

We will now estimate the deviation $\|u_\delta^\alpha - u_0\|$ of the approximate solution u_δ^α from the exact solution u_0. We have

$$\begin{aligned}\|u_\delta^\alpha - u_0\| \le \sup\{\|u_\delta^\alpha - u_0^\alpha\| : u_0 \in M_r,\ \|g_\delta - Au_0\| \le \delta\}\\ + \sup\{\|u_0^\alpha - u_0\| : u_0 \in M_r\},\end{aligned} \tag{2.8}$$

where

$$u_0^\alpha = R_\alpha g_0.$$

From (2.8) it follows that

$$\left\|u_\delta^\alpha - u_0\right\| \le \|R_\alpha\|\delta + \sup_{\|v_0\|\le r} \|R_\alpha \overline{C} v_0 - \overline{B} v_0\|. \tag{2.9}$$

We will then define the value of the regularization parameter $\overline{\alpha}(\delta)$ in formula (2.7) by the method of V. N. Strakhov [72], from the condition

$$\inf\left\{\|R_\alpha\|\delta + \sup_{\|v_0\|\le r} \|R_\alpha \overline{C} v_0 - \overline{B} v_0\|\right\}. \tag{2.10}$$

Lemma 2.1. *For any $\alpha > 0$, the operator R_α, defined by formula* (2.6), *is bounded and*

$$\|R_\alpha\| = \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \alpha}.$$

Proof. As

$$\|R_\alpha\|^2 = \sup_{\|g\|\le 1} \|R_\alpha g\|^2 \tag{2.11}$$

and

$$\|R_\alpha g\|^2 = (R_\alpha g, R_\alpha g), \tag{2.12}$$

keeping in mind that R_α is a self-adjoint operator, it follows from (2.11) and (2.12) that

$$\|R_\alpha\|^2 = \sup_{\|g\|\le 1} (R_\alpha^2 g, g). \tag{2.13}$$

From (2.6) and (2.13) it follows that

$$\|R_\alpha\|^2 = \sup_{\|g\|\le 1} (\overline{B}^2 [\overline{C} + \alpha E]^{-2} g, g). \tag{2.14}$$

Let $\{E_\sigma : 0 \le \sigma \le \|A\|\}$ be the spectral decomposition of the unity E, generated by the operator $\overline{A}$. Then from (2.6) it follows that

$$R_\alpha^2 g = \int_0^{\|A\|} \frac{G^2(\sigma)}{[G(\sigma)\sigma + \alpha]^2} dE_\sigma g \tag{2.15}$$

and from (2.14) and (2.15) it follows that

$$\|R_\alpha\|^2 = \sup_{\|g\|\le 1} \int_0^{\|A\|} \frac{G^2(\sigma)}{[G(\sigma)\sigma + \alpha]^2} d(E_\sigma g, g). \tag{2.16}$$

Given (2.16), we get

$$\|R_\alpha\|^2 \le \sup_{0\le\sigma\le\|A\|} \frac{G^2(\sigma)}{[G(\sigma)\sigma+\alpha]^2} \sup_{\|g\|\le 1} \int_0^{\|A\|} d(E_\sigma g, g) \tag{2.17}$$

and from (2.17) it follows that

$$\|R_\alpha\|^2 \le \sup_{0\le\sigma\le\|A\|} \frac{G^2(\sigma)}{[G(\sigma)\sigma+\alpha]^2}. \tag{2.18}$$

Since the function

$$\frac{G^2(\sigma)}{[G(\sigma)\sigma+\alpha]^2}$$

is continuous on $[0,\|A\|]$, there exists the value $\overline{\sigma} \in [0,\|A\|]$ such that

$$\frac{G^2(\overline{\sigma})}{[G(\overline{\sigma})\overline{\sigma}+\alpha]^2} = \sup_{0\le\sigma\le\|A\|} \frac{G^2(\sigma)}{[G(\sigma)\sigma+\alpha]^2}. \tag{2.19}$$

From relations (2.18) and (2.19) and from the fact that $\overline{\sigma}$ is a point on the spectrum of the operator $\overline{A}$ it follows that the lemma is proved. □

Lemma 2.2. *For any $\alpha > 0$ and $r > 0$ we have the following relation:*

$$\sup_{\|v\|\le r} \|R_\alpha \overline{C} v - \overline{B} v\| = r\alpha \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma+\alpha}.$$

Proof. As

$$\overline{B}(\overline{C}+\alpha E)^{-1}\overline{C}v - \overline{B}v = -\alpha\overline{B}(\overline{C}+\alpha E)^{-1}v, \tag{2.20}$$

from (2.6) and (2.20) it follows that

$$\|R_\alpha \overline{C} v - \overline{B} v\| = \alpha\left\|\overline{B}(\overline{C}+\alpha E)^{-1}v\right\|. \tag{2.21}$$

If $v \ne 0$, then from (2.21) it follows that

$$\|R_\alpha \overline{C} v - \overline{B} v\| = \alpha\|v\| \left\|\overline{B}(\overline{C}+\alpha E)^{-1}\frac{v}{\|v\|}\right\|. \tag{2.22}$$

Since

$$\sup_{\|v\|\le r} \|R_\alpha \overline{C} v - \overline{B} v\| = \alpha \sup_{0<\|v\|\le r} \left\|\overline{B}(\overline{C}+\alpha E)^{-1}v\right\|,$$

from (2.22) it follows that

$$\sup_{\|v\|\le r} \|R_\alpha \overline{C} v - \overline{B} v\| \le r\alpha \sup_{\|w\|\le 1} \left\|\overline{B}(\overline{C}+\alpha E)^{-1}w\right\|. \tag{2.23}$$

From (2.6) and (2.23) it follows that

$$\sup_{\|v\|\leq r} \|R_\alpha \overline{C} v - \overline{B} v\| = r\alpha \|R_\alpha\|. \tag{2.24}$$

From (2.24) and Lemma 2.1 it follows that the lemma is proved. □

From the relation (2.9) and Lemmas 2.1 and 2.2 it follows that

$$\|u_0 - R_\alpha g_\delta\| \leq (r\alpha + \delta) \max_{0\leq\sigma\leq\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \alpha}. \tag{2.25}$$

Now consider the equation

$$rG(\sigma)\sigma = \delta. \tag{2.26}$$

From the properties of the function $G(\sigma)$, it follows that, if $\delta < rG(\|A\|)\|A\|$, equation (2.26) has the unique solution $\overline{\sigma}(\delta)$.

Theorem 2.1. *Let the function*

$$G(\sigma) \in C[0, \|A\|] \cap C^1(0, \|A\|),$$

where for any $\sigma \in (0, \|A\|)$,

$$G'(\sigma) > 0,$$

$G^2(\sigma)/G'(\sigma)$ *increases, let* $G(0) = 0$, $\delta < rG(\|A\|)\|A\|$, $\overline{\sigma}(\delta)$ *be the solution of equation* (2.26), *and let*

$$\overline{\alpha}(\delta) = \frac{G^2(\overline{\sigma}(\delta))}{G'(\overline{\sigma}(\delta))}.$$

Then

$$\Delta_\delta(R_{\overline{\alpha}(\delta)}) \leq rG(\overline{\sigma}(\delta)).$$

Proof. Let u_0 be an arbitrary element of the set M_r and let

$$\|g_\delta - \overline{A} u_0\| \leq \delta.$$

Then from formula (2.25) it follows that

$$\|u_0 - R_{\overline{\alpha}(\delta)} g_\delta\| \leq (r\overline{\alpha}(\delta) + \delta) \max_{0\leq\sigma\leq\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)}. \tag{2.27}$$

We will now calculate

$$\max\left\{ \frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)} : 0 \leq \sigma \leq \|A\| \right\}.$$

To do this, we differentiate the function

$$\frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)}.$$

We have

$$\left[\frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)}\right]' = \frac{\overline{\alpha}(\delta)G'(\sigma) - G^2(\sigma)}{[G(\sigma)\sigma + \overline{\alpha}(\delta)]^2}. \tag{2.28}$$

To determine the maximum, it is sufficient to investigate the behavior of the numerator on the right-hand side of equality (2.28). We thus find that for $\sigma < \overline{\sigma}(\delta)$

$$\frac{G^2(\overline{\sigma}(\delta))}{G'(\overline{\sigma}(\delta))}G'(\sigma) - G^2(\sigma) > 0. \tag{2.29}$$

For $\sigma = \overline{\sigma}(\delta)$,

$$\frac{G^2(\overline{\sigma}(\delta))}{G'(\overline{\sigma}(\delta))}G'(\sigma) - G^2(\sigma) = 0 \tag{2.30}$$

and for $\sigma > \overline{\sigma}(\delta)$,

$$\frac{G^2(\overline{\sigma}(\delta))}{G'(\overline{\sigma}(\delta))}G'(\sigma) - G^2(\sigma) < 0. \tag{2.31}$$

From relations (2.29)–(2.31), it follows that

$$\max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)} = \frac{G(\overline{\sigma}(\delta))}{G(\overline{\sigma}(\delta))\overline{\sigma}(\delta) + \overline{\alpha}(\delta)} \tag{2.32}$$

and from (2.26), (2.27), and (2.32), it follows that

$$\|u_0 - R_{\overline{\alpha}(\delta)g_\delta}\| \le rG(\overline{\sigma}(\delta)). \tag{2.33}$$

Due to the arbitrariness of the elements u_0 and g_δ, the assertion of the theorem follows from relation (2.33). □

Corollary 2.1. *Let, for any* $\sigma \in (0, \|A\|)$,

$$G'(\sigma) > 0,$$

$G^2(\sigma)/G'(\sigma)$ *increase, let* $\delta < rG(\|A\|)\|A\|$, *and let* $\overline{\sigma}(\delta)$ *be the solution of equation* (2.26). *Let*

$$\overline{\alpha}(\delta) = G^2(\overline{\sigma}(\delta))/G'(\overline{\sigma}(\delta)).$$

Then the method

$$\{R_{\overline{\alpha}(\delta)} : 0 < \delta \le \delta_0\}$$

defined by formula (2.6) *is optimal on the set* M_r.

This result was published in [88].

Corollary 2.2. *Let, for any* $\sigma \in (0, \|A\|)$,

$$G'(\sigma) > 0,$$

$G^2(\sigma)/G'(\sigma)$ *increase and let* $\overline{\sigma}(\delta)$ *be the solution of equation* (2.26). *Then for any* $\delta \in (0, rG(\|A\|)\|A\|)$

$$\Delta_\delta^{\text{opt}} = rG(\overline{\sigma}(\delta)).$$

Now consider the method

$$\{R_{\overline{\alpha}(\delta)} : 0 < \delta \le \delta_0\}$$

on the class of correctness M_r, defined by the function

$$G(\sigma) = \sigma^p, \quad p > 0.$$

Corollary 2.3. *If* $G(\sigma) = \sigma^p$, $p > 0$, *then*

$$\overline{\sigma}(\delta) = \left(\frac{\delta}{r}\right)^{\frac{1}{p+1}}, \quad \overline{\alpha}(\delta) = \frac{\delta}{pr}, \quad \text{and} \quad \Delta_\delta^{\text{opt}} = r^{\frac{1}{p+1}}\delta^{\frac{p}{p+1}}.$$

2.2 On the accuracy of the Lavrent'ev method with the choice of the regularization parameter based on the Lavrent'ev scheme

This method is described in [42]. It uses the regularizing family of operators $\{R_\alpha : \alpha > 0\}$ defined by formula (2.6) and it differs from the method described in the previous section of this chapter in that the value of the regularization parameter $\hat{\alpha}(\delta)$ in formula (2.7) is defined by

$$\|R_\alpha\|\delta = \sup_{\|v_0\|\le r} \|R_\alpha \overline{C} v_0 - \overline{B} v_0\|. \tag{2.34}$$

In what follows we assume that the operators $\overline{A}$ and $\overline{B}$ satisfy the conditions given in Section 2.1.

Lemma 2.3. *If* $\hat{\alpha}(\delta)$ *is defined by equation* (2.34), *then*

$$\hat{\alpha}(\delta) = \frac{\delta}{r}.$$

Proof. From Lemmas 2.1 and 2.2 it follows that

$$\|R_\alpha\| = \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{\sigma G(\sigma)+\alpha} \tag{2.35}$$

and

$$\sup_{\|v\|\le r} \|R_\alpha \overline{C} v - \overline{B} v\| = r\alpha \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{\sigma G(\sigma)+\alpha}. \tag{2.36}$$

Thus, the assertion of the lemma follows from formulas (2.34)–(2.36). □

From Lemma 2.3 and Corollary 2.3 it follows that the methods

$$\{R_{\overline{\alpha}(\delta)} : 0 < \delta \le \delta_0\} \quad \text{and} \quad \{R_{\hat{\alpha}(\delta)} : 0 < \delta \le \delta_0\},$$

described in the first and second sections of the current chapter, are, generally speaking, different. In more detail, for

$$G(\sigma) = \sigma^p, \quad p > 0,$$

we have

$$R_{\overline{\alpha}(\delta)} = R_{\hat{\alpha}(\delta)} \quad \text{at } p = 1$$

and

$$R_{\overline{\alpha}(\delta)} \ne R_{\hat{\alpha}(\delta)} \quad \text{at } p \ne 1.$$

We will now estimate from above the accuracy of the method

$$\{R_{\hat{\alpha}(\delta)} : 0 < \delta \le \delta_0\}$$

and we will prove that the method is optimal-by-order.

As defined in the previous paragraph,

$$R_\alpha g = \overline{B}(\overline{C}+\alpha E)^{-1} g, \quad \alpha \in (0,\alpha_0], \quad \text{and} \quad \overline{C} = \overline{A}\cdot\overline{B}, \quad \hat{\alpha}(\delta) = \frac{\delta}{r}.$$

Thus, the approximate solution u_δ of equation (1.1) is defined by

$$u_\delta^{\hat{\alpha}(\delta)} = R_{\hat{\alpha}(\delta)} g_\delta.$$

We will now estimate the accuracy of the method

$$\{R_{\hat{\alpha}(\delta)} : 0 < \delta \le \delta_0\}$$

on the class M_r. For this we need to prove two lemmas.

Lemma 2.4. *If the regularizing family of the operators* $\{R_\alpha : \alpha > 0\}$ *is defined by formula* (2.6) *and* $\alpha_1 \in (0, \alpha_2)$, *then*

$$\|R_{\alpha_1}\| > \|R_{\alpha_2}\|.$$

Proof. Since by Lemma 2.1

$$\|R_\alpha\| = \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{\sigma G(\sigma) + \alpha}, \tag{2.37}$$

from $\alpha_1 < \alpha_2$ it follows that for any $\sigma \in (0, \|A\|]$

$$\frac{G(\sigma)}{\sigma G(\sigma) + \alpha_1} > \frac{G(\sigma)}{\sigma G(\sigma) + \alpha_2}. \tag{2.38}$$

The assertion of the lemma follows from (2.37) and (2.38). □

Lemma 2.5. *Let* $G(\sigma) \in C[0, \|A\|] \cap C^1(0, \|A\|)$ *and*

$$\Phi(\sigma, \alpha) = \frac{\alpha G(\sigma)}{G(\sigma)\sigma + \alpha}.$$

Then for any $\sigma \in [0, \|A\|]$ *the function* $\Phi(\sigma, \alpha)$ *is* α*-non-decreasing.*

This result was published in [72].

Proof. To prove the lemma we calculate the α-derivative $\Phi'(\sigma, \alpha)$ of the function $\Phi(\sigma, \alpha)$. We have

$$\Phi'_\alpha(\sigma, \alpha) = \frac{\sigma G^2(\sigma)}{[\sigma G(\sigma) + \alpha]^2}. \tag{2.39}$$

From (2.39) it follows that for any $\sigma \in [0, \|A\|]$

$$\Phi'_\alpha(\sigma, \alpha) \ge 0.$$

The lemma is thereby proved. □

Lemma 2.6. *Let*

$$G(\sigma) \in C[0, \|A\|] \cap C^1(0, \|A\|)$$

and for any $\sigma \in (0, \|A\|)$

$$G'(\sigma) > 0, \quad G(0) = 0,$$

let $G^2(\sigma)/G'(\sigma)$ *increase, and let* $\alpha_1 \in (0, \alpha_2]$.

Then

$$\alpha_1 \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{\sigma G(\sigma) + \alpha_1} \le \alpha_2 \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{\sigma G(\sigma) + \alpha_2}.$$

Proof. Since the function $\Phi(\sigma,\alpha)$ is σ-continuous on $[0,\|A\|]$ for $\alpha>0$, for any $\alpha>0$ there exists $\overline{\sigma}(\alpha)\in[0,\|A\|]$ such that

$$\Phi(\overline{\sigma}(\alpha),\alpha)=\max_{0\le\sigma\le\|A\|}\Phi(\sigma,\alpha). \tag{2.40}$$

Thus, from (2.40) and by Lemma 2.4, we have

$$\max_{0\le\sigma\le\|A\|}\Phi(\sigma,\alpha_1)=\Phi(\overline{\sigma}(\alpha_1),\alpha_1)\le\Phi(\overline{\sigma}(\alpha_1),\alpha_2)\le\max_{0\le\sigma\le\|A\|}\Phi(\sigma,\alpha_2) \tag{2.41}$$

and the assertion of the lemma follows from (2.41) . □

Theorem 2.2. *Let the function*

$$G(\sigma)\in C[0,\|A\|]\cap C^1(0,\|A\|),$$

let for all $\sigma\in(0,\|A\|)$,

$$G'(\sigma)>0,$$

$G^2(\sigma)/G'(\sigma)$ *increase, let* $G(0)=0$, $\delta<rG(\|A\|)\|A\|$, *let* $\overline{\sigma}(\delta)$ *be the solution of equation* (2.26), *and let* $\hat{\alpha}(\delta)$ *be the solution of equation* (2.34). *Then*

$$\Delta_\delta(R_{\hat{\alpha}(\delta)})\le 2rG(\overline{\sigma}(\delta)).$$

Proof. Let u_0 be an arbitrary element from the set M_r and $\|g_\delta-\overline{A}u_0\|\le\delta$. Then, if $u_0=\overline{B}v_0$

$$\|u_0-R_{\hat{\alpha}(\delta)}g_\delta\|\le\|R_{\hat{\alpha}(\delta)}\|\delta+\sup_{\|v_0\|\le r}\|R_{\hat{\alpha}(\delta)}\overline{C}v_0-\overline{B}v_0\|. \tag{2.42}$$

Since from formulas (2.35) and (2.36) it follows that

$$\|R_{\hat{\alpha}(\delta)}\|=\max_{0\le\sigma\le\|A\|}\frac{G(\sigma)}{G(\sigma)\sigma+\hat{\alpha}(\delta)}$$

and

$$\sup_{\|v\|\le r}\|R_{\hat{\alpha}(\delta)}\overline{C}v-\overline{B}v\|=r\hat{\alpha}(\delta)\max_{0\le\sigma\le\|A\|}\frac{G(\sigma)}{G(\sigma)\sigma+\hat{\alpha}(\delta)},$$

by formula (2.42) we get

$$\Delta_\delta(R_{\hat{\alpha}(\delta)})\le\delta\max_{0\le\sigma\le\|A\|}\frac{G(\sigma)}{G(\sigma)\sigma+\hat{\alpha}(\delta)}+r\hat{\alpha}(\delta)\max_{0\le\sigma\le\|A\|}\frac{G(\sigma)}{G(\sigma)\sigma+\hat{\alpha}(\delta)}. \tag{2.43}$$

Consider the value of the parameter $\overline{\alpha}(\delta)$ defined by the formula

$$\overline{\alpha}(\delta)=\frac{G^2(\overline{\sigma}(\delta))}{G'(\overline{\sigma}(\delta))},$$

where $\overline{\sigma}(\delta)$ is the solution of equation (2.26). We consider three cases.

First case: $\hat{\alpha}(\delta) = \overline{\alpha}(\delta)$
Then from formula (2.43) it follows that

$$\Delta_\delta(R_{\hat{\alpha}(\delta)}) \le \left(r\overline{\alpha}(\delta) + \delta\right) \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)} \tag{2.44}$$

and by Theorem 2.1 and formula (2.44) we get

$$\Delta_\delta(R_{\hat{\alpha}(\delta)}) \le rG\left(\overline{\sigma}(\delta)\right). \tag{2.45}$$

Second case: $\hat{\alpha}(\delta) < \overline{\alpha}(\delta)$
Then from (2.36) it follows that

$$r\hat{\alpha}(\delta) \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \hat{\alpha}(\delta)} \le r\overline{\alpha}(\delta) \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)}. \tag{2.46}$$

From formula (2.32) it follows that

$$r\overline{\alpha}(\delta) \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)} = r\overline{\alpha}(\delta)\frac{G(\overline{\sigma}(\delta))}{G(\overline{\sigma}(\delta))\overline{\sigma}(\delta) + \overline{\alpha}(\delta)}. \tag{2.47}$$

From (2.26) it follows that

$$G(\overline{\sigma}(\delta))\overline{\sigma}(\delta) = \frac{\delta}{r}. \tag{2.48}$$

Since

$$\frac{\overline{\alpha}(\delta)}{\delta/r + \overline{\alpha}(\delta)} < 1,$$

from (2.47) and (2.48) it follows that

$$r\overline{\alpha}(\delta) \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \overline{\alpha}(\delta)} \le rG\left(\overline{\sigma}(\delta)\right) \tag{2.49}$$

and from (2.46) and (2.49) it follows that

$$\Delta_\delta(R_{\hat{\alpha}(\delta)}) \le 2rG\left(\overline{\sigma}(\delta)\right). \tag{2.50}$$

Third case: $\hat{\alpha}(\delta) > \overline{\alpha}(\delta)$
Then from Lemma 2.4 it follows that

$$\|R_{\hat{\alpha}(\delta)}\| \le \|R_{\overline{\alpha}(\delta)}\|. \tag{2.51}$$

Since from (2.35) it follows that

$$\|R_{\hat{\alpha}(\delta)}\| = \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \hat{\alpha}(\delta)}, \tag{2.52}$$

from (2.27), (2.51), and (2.52) it follows that

$$\delta \max_{0\le\sigma\le\|A\|} \frac{G(\sigma)}{G(\sigma)\sigma + \hat{\alpha}(\delta)} \le rG(\overline{\sigma}(\delta)) \tag{2.53}$$

and from (2.34), (2.43), and (2.51) we have

$$\Delta_\delta(R_{\hat{\alpha}(\delta)}) \le 2rG(\overline{\sigma}(\delta)). \tag{2.54}$$

The theorem is thereby proved. □

Corollary 2.4. *Let the function*

$$G(\sigma) \in C[0, \|A\|] \cap C^1(0, \|A\|),$$

let for any $\sigma \in (0, \|A\|)$,

$$G'(\sigma) > 0,$$

$G^2(\sigma)/G'(\sigma)$ *increase, let* $G(0) = 0$, $\delta < rG(\|A\|)\|A\|$, *and let* $\hat{\alpha}(\delta)$ *be the solution of equation* (2.34). *Then the Lavrent'ev method*

$$\{R_{\hat{\alpha}(\delta)} : 0 < \delta \le \delta_0\}$$

defined by formulas (2.34) *and* (2.6) *is optimal-by-order on the class* M_r *and we have the following estimate:*

$$\Delta_\delta(R_{\hat{\alpha}}(\delta)) \le 2\Delta_\delta^{\text{opt}}.$$

Proof. The proof of the corollary follows from Theorem 2.2 and Lemma 2.2. □

2.3 Application of the method to the solution of the inverse Cauchy problem for the heat conduction equation

2.3.1 Posing the direct Cauchy problem for the heat conduction equation

Consider the equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad -\infty < x < \infty,\ t \in (0,T],\ T > 0. \tag{2.55}$$

Assume that the solution $u(x,t) \in C\{(-\infty,\infty) \times [0,T]\}$ for any $t \in (0,T]$ and

$$u(x,t),\ u_x'(x,t),\ u_{xx}''(x,t) \in L_1(-\infty,\infty) \cap L_2(-\infty,\infty).$$

There exists a function $\chi(x) \in L_1(-\infty, \infty)$ such that almost for any $t \in (0, T]$

$$|u_t'(x,t)| \le \chi(x).$$

In addition, for $t = 0$

$$u(x,0) = v_0(x), \tag{2.56}$$
$$v_0(x) \in W_2^2(-\infty,\infty) \cap W_1^2(-\infty,\infty).$$

Then the existence and uniqueness of the generalized solution of problem (2.55), (2.56), which can be found using the Fourier transform, follows from [39] (p. 407).

2.3.2 Posing the inverse Cauchy problem for the heat conduction equation

Consider equation (2.55) and assume that

$$u(x,T) = f(x), \tag{2.57}$$

where $f(x) \in C(-\infty,\infty) \cap L_2(-\infty,\infty)$.

In addition, for

$$f(x) = f_0(x)$$

there exists

$$v_0(x) \in W_2^2(-\infty,\infty) \cap W_2^1(-\infty,\infty), \quad \|v_0(x)\|_{L_2} \le r,$$

for which there exists the generalized solution $u(x,t)$ of problem (2.55), (2.56), such that

$$u(x,T) = f_0(x). \tag{2.58}$$

However, $f_0(x)$ is unknown. Instead, we know $f_\delta(x) \in L_2(-\infty,\infty)$ and $\delta > 0$ such that

$$\|f_\delta - f_0\|_{L_2} \le \delta. \tag{2.59}$$

It is required to find the function $u_\delta(x) \in L_2(-\infty,\infty)$ and estimate its deviation $\|u_\delta - u_0\|_{L_2}$ from the function $u_0(x)$, using the initial data f_δ, δ, and r. We have

$$u_0(x) = u(x,t_0), \quad t_0 \in (0,T).$$

The function $u(x,t)$ is the generalized solution of the direct problem (2.55), (2.56). To solve this problem, we will use the Fourier transform, defined by the formula

$$F[u(x,t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x,t) e^{-i\lambda x} dx = \hat{u}(\lambda,t), \quad \lambda \in \mathbb{R}. \tag{2.60}$$

The operator A, defined by equality (2.60), maps the space $L_2(-\infty,\infty) \cap L_1(-\infty,\infty)$ into $L_2(-\infty,\infty)$ and because the Plancherel theorem [39] is isometric, that is, after the application of the Fourier transform to equation (2.55), the equality

$$\|Fu\|_{L_2} = \|u\|_{L_2}$$

will be reduced to the ordinary differential equation

$$\frac{d\hat{u}(\lambda,t)}{dt} = -\lambda^2\hat{u}(\lambda,t), \quad -\infty < \lambda < \infty,\ t \in (0,T]. \tag{2.61}$$

From (2.56) it follows that

$$\hat{u}(\lambda,0) = \hat{v}(\lambda), \quad \lambda \in \mathbb{R}, \tag{2.62}$$

where $\hat{v}(\lambda) = F[u(x,0)]$. From (2.57) it follows that

$$\hat{u}(\lambda,T) = \hat{f}(\lambda), \quad \lambda \in \mathbb{R}, \tag{2.63}$$

where $\hat{f}(\lambda) = F[f(x)]$.

The function

$$\hat{u}(\lambda) = \hat{u}(\lambda,t_0)$$

must be found.

Thus, from (2.61)–(2.63) it follows that

$$A\hat{u}(\lambda) = e^{-\lambda^2(T-t_0)}\hat{u}(\lambda) = \hat{f}(\lambda), \quad \lambda \in \mathbb{R}, \tag{2.64}$$

$$\hat{u}(\lambda) = B\hat{v}(\lambda) = e^{-\lambda^2 t_0} \cdot \hat{v}(\lambda). \tag{2.65}$$

Applying the Lavrent'ev method to problem (2.64), (2.65), we define its approximate solution by the formula

$$\hat{u}_\delta^\alpha(\lambda) = \frac{e^{-\lambda^2 t_0}}{e^{-\lambda^2 T} + \alpha}\hat{f}_\delta(\lambda), \quad \alpha > 0. \tag{2.66}$$

Note that the function $G(\sigma)$ defining the operator $B = G(A)$ is defined parametrically as follows:

$$\begin{cases} \sigma = e^{-\lambda^2(T-t_0)}, \\ G(\sigma) = e^{-\lambda^2 t_0}. \end{cases} \tag{2.67}$$

From (2.67) it follows that

$$G(\sigma) = \sigma^{\frac{t_0}{T-t_0}}. \tag{2.68}$$

Then from Lemma 2.3 it follows that

$$\overline{\alpha}(\delta) = \frac{\delta t_0}{r[T - t_0]} \tag{2.69}$$

and

$$\sup_{\hat{u}_0, \hat{f}_\delta} \{\|\hat{u}_\delta^{\overline{\alpha}(\delta)} - \hat{u}_0\| : \hat{u} \in BS_r,\ \|A\hat{u}_0 - \hat{f}_\delta\| \le \delta\} = r^{\frac{T-t_0}{T}} \delta^{\frac{t_0}{T}}. \tag{2.70}$$

Applying the inverse Fourier transform F^{-1} to the function $\hat{u}_\delta^{\overline{\alpha}(\delta)}(\lambda)$, we obtain the solution of problem (2.55)–(2.59)

$$u_\delta(x) = \mathrm{Re}[F^{-1}[\hat{u}_\delta^{\overline{\alpha}(\delta)}(\lambda)]],$$

for which, from formula (2.70) and by the Plancherel theorem, we have the following estimate:

$$\|u_\delta(x) - u_0(x)\|_{L_2} \le r^{\frac{T-t_0}{T}} \delta^{\frac{t_0}{T}}.$$

3 Tikhonov regularization method

This method was proposed and justified in the well-known papers by A. N. Tikhonov in 1963 [97, 98] that drew attention of mathematicians to this direction of research and caused the intensive development of the theory of ill-posed problems.

3.1 A linear version of the Tikhonov regularization method

Let $\mathbb{U}$, $\mathbb{F}$, and $\mathbb{V}$ be Hilbert spaces, let A be a linear, injective and bounded operator mapping $\mathbb{U}$ into $\mathbb{F}$, and let B be a linear bounded operator mapping $\mathbb{V}$ into $\mathbb{U}$.

Consider the operator equation (1.1) and

$$Au = f, \quad u \in \mathbb{U}, f \in \mathbb{F}.$$

Assume that for $f = f_0$ there exists an accurate solution u_0 of equation (1.1) that belongs to the range of values $R(B)$ of the operator B though f_0 is not known. Instead, given are an element $f_\delta \in \mathbb{F}$ and error level $\delta > 0$ such that

$$\|f_\delta - f_0\| \le \delta. \tag{3.1}$$

It is required to find the approximate solution $u_\delta \in \mathbb{U}$ of equation (1.1) using the initial data (f_δ, δ) and estimate the value $\|u_\delta - u_0\|$, assuming $u_0 \in M_r = B\overline{S}_r$. The Tikhonov regularization method consists of reducing the problem of the approximate solution of operator equation (1.1) to the variational problem

$$\inf\{\|Cv - f_\delta\|^2 + \alpha\|v\|^2 : v \in \mathbb{V}\}, \tag{3.2}$$

where $\alpha > 0$, $C = AB$.

Lemma 3.1. *For any values $\alpha > 0$ and $f_\delta \in \mathbb{F}$ the variational problem* (3.2) *is solvable.*

Proof. Consider a minimizing sequence $\{v_n\} \subset \mathbb{V}$ such that for $n \longrightarrow \infty$

$$\|Cv_n - f_\delta\|^2 + \alpha\|v_n\|^2 \longrightarrow \inf\{\|Cv - f_\delta\|^2 + \alpha\|v\|^2 : v \in \mathbb{V}\}. \tag{3.3}$$

The boundedness of the sequence $\{v_n\}$ follows from (3.3) and the weak precompactness of this sequence follows from its boundedness. Thus, there exists a subsequence $\{v_{n_k}\}$ such that

$$v_{n_k} \xrightarrow{\text{ne}} \hat{v} \quad \text{for } k \longrightarrow \infty. \tag{3.4}$$

It follows from (3.4) that

$$Cv_{n_k} - f_\delta \xrightarrow{\text{ne}} C\hat{v} - f_\delta \quad \text{for } k \longrightarrow \infty. \tag{3.5}$$

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From (3.4) and (3.5) according to the property of weak limit norm we obtain

$$\alpha\|\hat{v}\|^2 \leq \varliminf_{k\to\infty} \alpha\|v_{n_k}\|^2 \tag{3.6}$$

and

$$\|C\hat{v} - f_\delta\|^2 \leq \varliminf_{k\to\infty} \|Cv_{n_k} - f_\delta\|^2. \tag{3.7}$$

By the termwise summation of (3.6) and (3.7) and using (3.3) we obtain

$$\|C\hat{v} - f_\delta\|^2 + \alpha\|\hat{v}\|^2 \leq \inf\{\|Cv - f_\delta\|^2 + \alpha\|v\|^2 : v \in \mathbb{V}\}. \tag{3.8}$$

Since this cannot be smaller, it follows from (3.8) that

$$\|C\hat{v} - f_\delta\|^2 + \alpha\|\hat{v}\|^2 = \inf\{\|Cv - f_\delta\|^2 + \alpha\|v\|^2 : v \in \mathbb{V}\},$$

and $\hat{v}$ belongs to the solutions of the variational problem (3.2). The lemma is thereby proved. □

Note. In [28] it is shown that Lemma 3.1 is true under the condition of reflexivity of the space $\mathbb{V}$.

Lemma 3.2. *The solution of the variation problem* (3.2) *is unique.*

Proof. Assume the contrary, i. e., that there exist two points $\hat{v}_1, \hat{v}_2 \in \mathbb{V}$ such that $\hat{v}_1 \neq \hat{v}_2$ and

$$\|C\hat{v}_1 - f_\delta\|^2 + \alpha\|\hat{v}_1\|^2 = \|C\hat{v}_2 - f_\delta\|^2 + \alpha\|\hat{v}_2\|^2 = \inf_{v\in\mathbb{V}}\{\|Cv - f_\delta\|^2 + \alpha\|v\|^2\}. \tag{3.9}$$

It follows from (3.9) that, if we assume

$$\hat{v} = \frac{\hat{v}_1 + \hat{v}_2}{2},$$

then

$$\begin{aligned}&\|C\hat{v} - f_\delta\|^2 + \alpha\|\hat{v}\|^2 \\ &\quad\leq \frac{1}{2}(\|C\hat{v}_1 - f_\delta\|^2 + \alpha\|\hat{v}_1\|^2) + \frac{1}{2}(\|C\hat{v}_2 - f_\delta\|^2 + \alpha\|\hat{v}_2\|^2).\end{aligned} \tag{3.10}$$

Since by (3.9) this cannot be smaller, it follows from (3.10) that

$$\begin{aligned}&\|C\hat{v} - f_\delta\|^2 + \alpha\|\hat{v}\|^2 \\ &\quad= \frac{1}{2}(\|C\hat{v}_1 - f_\delta\|^2 + \alpha\|\hat{v}_1\|^2) + \frac{1}{2}(\|C\hat{v}_2 - f_\delta\|^2 + \alpha\|\hat{v}_2\|^2).\end{aligned} \tag{3.11}$$

As spaces $\mathbb{V}$ and $\mathbb{F}$ are Hilbert spaces we get

$$\alpha\|\hat{v}\|^2 \leq \alpha\frac{\|\hat{v}_1\|^2 + \|\hat{v}_2\|^2}{2} \tag{3.12}$$

and

$$\|C\hat{v} - f_\delta\|^2 \le \frac{1}{2}\|C\hat{v}_1 - f_\delta\|^2 + \frac{1}{2}\|C\hat{v}_2 - f_\delta\|^2. \tag{3.13}$$

Taking into account (3.11)–(3.13) we obtain

$$\left\|\frac{\hat{v}_1 + \hat{v}_2}{2}\right\|^2 = \frac{\|\hat{v}_1\|^2 + \|\hat{v}_2\|^2}{2}. \tag{3.14}$$

Since

$$\left\|\frac{\hat{v}_1 + \hat{v}_2}{2}\right\|^2 = \frac{1}{4}\|\hat{v}_1\|^2 + \frac{1}{4}\|\hat{v}_2\|^2 + \frac{1}{2}(\hat{v}_1, \hat{v}_2), \tag{3.15}$$

it follows from (3.14) and (3.15) that

$$2(\hat{v}_1, \hat{v}_2) = \|\hat{v}_1\|^2 + \|\hat{v}_2\|^2. \tag{3.16}$$

It follows from (3.16) that

$$\|\hat{v}_1 - \hat{v}_2\|^2 = \|\hat{v}_1\|^2 + \|\hat{v}_2\|^2 - 2(\hat{v}_1, \hat{v}_2) = 0,$$

i. e., $\hat{v}_1 = \hat{v}_2$, which contradicts the assumption. The lemma is thereby proved. □

Note. It is shown in [28] that Lemma 3.2 is true under the condition of reflexivity and strict convexity of the space $\mathbb{V}$. Let P_α be an operator acting from $\mathbb{F}$ into $\mathbb{V}$ mapping the element $f_\delta \in \mathbb{F}$ into the solution $\hat{v}_\delta^\alpha$ of the variational problem (3.2).

Lemma 3.3. *Let P_α be an operator mapping a space $\mathbb{F}$ into $\mathbb{V}$ and defined as above. Then for any $\alpha > 0$ the operator P_α is continuous over the space $\mathbb{F}$.*

Proof. Assume the contrary. Then there could be found a number $\varepsilon_0 > 0$, element $f_\delta \in \mathbb{F}$, and sequence $\{f_\delta(n)\} \subset \mathbb{F}$, such that

$$f_\delta(n) \to f_\delta \quad \text{for } n \to \infty$$

and for any n

$$\left\|\hat{v}_\delta^\alpha(n) - \hat{v}_\delta^\alpha\right\| \ge \varepsilon_0, \tag{3.17}$$

where $\hat{v}_\delta^\alpha$ is the solution of the variational problem (3.2) and $\hat{v}_\delta^\alpha$ is the solution of the variational problem

$$\inf\{\left\|Cv - f_\delta(n)\right\|^2 + \alpha\|v\|^2 : v \in \mathbb{V}\}. \tag{3.18}$$

It follows from (3.18) that for any n the relation

$$\left\|C\hat{v}_\delta^\alpha(n) - f_\delta(n)\right\|^2 + \alpha\left\|\hat{v}_\delta^\alpha(n)\right\|^2 \le \left\|C\hat{v}_\delta^\alpha - f_\delta(n)\right\|^2 + \alpha\left\|\hat{v}_\delta^\alpha\right\|^2 \tag{3.19}$$

is true. Without loss of generality it will follow from relation (3.19) that

$$\lim_{n\to\infty} \left\|C\hat{v}_\delta^\alpha(n) - f_\delta(n)\right\|^2 + \alpha\left\|\hat{v}_\delta^\alpha(n)\right\|^2 \leq \left\|C\hat{v}_\delta^\alpha - f_\delta\right\|^2 + \alpha\left\|\hat{v}_\delta^\alpha\right\|^2 \tag{3.20}$$

and it follows from (3.18) that

$$\left\|C\hat{v}_\delta^\alpha - f_\delta\right\|^2 + \alpha\left\|\hat{v}_\delta^\alpha\right\|^2 = \inf\{\left\|Cv - f_\delta(n)\right\|^2 + \alpha\|v\|^2 : v \in \mathbb{V}\}. \tag{3.21}$$

Thus, the boundedness of the sequence $\{\hat{v}_\delta^\alpha(n)\}$ follows from (3.20) and the weak pre-compactness of this sequence follows from its boundedness. Without loss of generality, we say that

$$\hat{v}_\delta^\alpha(n) \xrightarrow{\text{ne}} \hat{v} \quad \text{for } n \to \infty. \tag{3.22}$$

Since the operator C is linear and bounded, from (3.22) it follows that

$$C\hat{v}_\delta^\alpha(n) - f_\delta(n) \xrightarrow{\text{ne}} C\hat{v} - f_\delta \quad \text{for } n \to \infty. \tag{3.23}$$

Without loss of generality, from (3.22) and (3.23) it follows that

$$\|\hat{v}\| \leq \lim_{n\to\infty} \left\|\hat{v}_\delta^\alpha(n)\right\| \tag{3.24}$$

and

$$\|C\hat{v} - f_\delta\| \leq \lim_{n\to\infty} \left\|\hat{v}_\delta^\alpha(n) - f_\delta(n)\right\|. \tag{3.25}$$

From (3.24) and (3.25) it follows that

$$\|C\hat{v} - f_\delta\|^2 + \alpha\|\hat{v}\|^2 \leq \left\|C\hat{v}_\delta^\alpha - f_\delta(n)\right\|^2 + \alpha\left\|\hat{v}_\delta^\alpha\right\|^2. \tag{3.26}$$

Since there cannot be less than the infimum, from (3.21) and (3.26) it follows that

$$\|C\hat{v} - f_\delta\|^2 + \alpha\|\hat{v}\|^2 = \inf\{\|Cv - f_\delta\|^2 + \alpha\|v\|^2 : v \in \mathbb{V}\}. \tag{3.27}$$

From relations (3.21) and (3.27), by Lemma 3.2 it follows that

$$\hat{v} = \hat{v}_\delta^\alpha \tag{3.28}$$

and it follows from (3.22), (3.23), and (3.28) that

$$\hat{v}_\delta^\alpha(n) \xrightarrow{\text{ne}} \hat{v}_\delta^\alpha \tag{3.29}$$

and

$$C\hat{v}_\delta^\alpha(n) - f_\delta(n) \xrightarrow{\text{ne}} C\hat{v}_\delta^\alpha - f_\delta.$$

It follows from (3.24), (3.25), and (3.28) that

$$\alpha\|\hat{v}_\delta^\alpha\|^2 \leq \lim_{n\to\infty} \alpha\|\hat{v}_\delta^\alpha(n)\|^2 \tag{3.30}$$

and

$$\|C\hat{v}_\delta^\alpha - f_\delta\|^2 \leq \lim_{n\to\infty} \|C\hat{v}_\delta^\alpha(n) - f_\delta(n)\|^2. \tag{3.31}$$

Summing termwise (3.30) and (3.31) we obtain

$$\|C\hat{v}_\delta^\alpha - f_\delta\|^2 + \alpha\|\hat{v}_\delta^\alpha\|^2 \leq \lim_{n\to\infty}\{\|C\hat{v}_\delta^\alpha(n) - f_\delta(n)\|^2 + \alpha\|\hat{v}_\delta^\alpha(n)\|^2\}. \tag{3.32}$$

It follows from (3.20) and (3.32) that

$$\|C\hat{v}_\delta^\alpha - f_\delta\|^2 + \alpha\|\hat{v}_\delta^\alpha\|^2 = \lim_{n\to\infty}\{\|C\hat{v}_\delta^\alpha(n) - f_\delta(n)\|^2 + \alpha\|\hat{v}_\delta^\alpha(n)\|^2\}. \tag{3.33}$$

From (3.30), (3.31), and (3.33) it follows that

$$\|\hat{v}_\delta^\alpha\| = \lim_{n\to\infty} \|\hat{v}_\delta^\alpha(n)\| \tag{3.34}$$

and

$$\|C\hat{v} - f_\delta\| = \lim_{n\to\infty} \|C\hat{v}_\delta^\alpha(n) - f_\delta(n)\|.$$

Since space $\mathbb{V}$ is Hilbert space, it follows from (3.29) and (3.34) that

$$\hat{v}_\delta^\alpha(n) \to \hat{v}_\delta^\alpha \quad \text{for } n \to \infty. \tag{3.35}$$

Relation (3.35) contradicts (3.17) and proves the lemma. □

It follows from Lemmas 3.1–3.3 that the variational problem (3.2) is well-posed according to Hadamard. We further define the approximate solution u_δ of equation (1.1) by the formulas

$$u_\delta = \hat{u}_\delta^{\alpha(\delta)}, \tag{3.36}$$

where

$$\hat{u}_\delta^{\alpha(\delta)} = B\hat{v}_\delta^{\alpha(\delta)},$$

$\hat{v}_\delta^\alpha$ is the solution of the variational problem (3.2), and

$$\alpha\delta = \delta^2. \tag{3.37}$$

It follows from Lemmas 3.1–3.3 that, if T_δ is the operator acting from the space $\mathbb{F}$ into $\mathbb{U}$ and it is defined by formulas (3.36) and (3.37), then, if it maps the problem initial data (f_δ, δ) into the approximate solution u_δ of equation (1.1), by Lemma 3.3 the operator T_δ is continuous over the space $\mathbb{F}$. Estimate the error $\Delta_\delta[T_\delta]$ of the operator T_δ.

Theorem 3.1. *Assume that $M_r = B\overline{S}_r$, $u_0 \in M_r$, and u_δ is defined by formulas* (3.36) *and* (3.37). *Then the following estimate is true:*

$$\|u_\delta - u_0\| \le \begin{cases} 2\sqrt{1+r^2}\omega(\delta, r) & \text{for } r \ge 1, \\ 2\sqrt{1+(\frac{1}{r})^2}\omega(\delta, r) & \text{for } r < 1. \end{cases}$$

Proof. Since $u_0 \in M_r$, there exists $v_0 \in \mathbb{V}$ such that $u_0 = Bv_0$ and $\|v_0\| \le r$.

Thus, it follows from (3.20) and (3.32) that

$$\|v_\delta\|^2 \le \frac{1}{\delta^2}\|Cv_0 - f_\delta\|^2 + \|v_0\|^2, \tag{3.38}$$

where $v_\delta = B^{-1}u_\delta$. It follows from

$$\|Cv_0 - f_\delta\|^2 = \|Au_0 - f_\delta\|^2 \le \delta^2, \quad \|v_0\|^2 \le r^2,$$

and (3.38) that

$$\|v_\delta\| \le \sqrt{1+r^2}. \tag{3.39}$$

It follows from (3.2) that

$$\|Cv_\delta - f_\delta\|^2 \le \delta^2 + \delta^2\|v_0\|^2 \le \delta^2(1+r^2),$$

i. e.,

$$\|Cv_\delta - f_\delta\| \le \delta\sqrt{1+r^2}. \tag{3.40}$$

It follows from (3.40) that

$$\|Au_\delta - Au_0\| \le 2\delta\sqrt{1+r^2} \tag{3.41}$$

and it follows from (3.39) that

$$u_\delta, u_0 \in B\overline{S}_{\sqrt{1+r^2}}. \tag{3.42}$$

Thus, it follows from (1.2), (3.41), and (3.42) that

$$\|u_\delta - u_0\| \le \omega_1(2\delta\sqrt{1+r^2}, \sqrt{1+r^2}) \tag{3.43}$$

and it follows from Lemma 1.2 and (3.43) that

$$\|u_\delta - u_0\| \le \omega(2\delta\sqrt{1+r^2}, 2\sqrt{1+r^2}). \tag{3.44}$$

It follows from Lemma 1.3 and (3.44) that

$$\|u_\delta - u_0\| \le 2\sqrt{1+r^2}\omega(\delta, 1) \tag{3.45}$$

and the assertion of the theorem follows from (3.45). □

Since in Theorem 3.1 u_0 is any element from M_r and f_δ is any element from $\mathbb{F}$ such that

$$\|f_\delta - Au_0\| \le \delta,$$

it follows from (1.65) and (3.29) that for any $\delta \in (0, \delta_0]$ the following relation is true:

$$\Delta_\delta[T_\delta] \le 2\sqrt{1 + r^2}\omega(\delta, 1). \tag{3.46}$$

The following theorem follows from Lemma 1.14 and estimate (3.46).

Theorem 3.2. *Assume that all conditions of Theorem* 3.1 *are satisfied and a set* $M_r = B\overline{S}_r$ *is the correctness class for equation* (1.1). *Then the method* $\{T_\delta : 0 < \delta \le \delta_0\}$ *is optimal-by-order for the class* M_r *and for any* $\delta \in (0, \delta_0]$ *the following estimate is true:*

$$\Delta_\delta[T_\delta] \le 2\sqrt{1 + \left[\max\left(r, \frac{1}{r}\right)\right]^2}\,\Delta_\delta^{\text{opt}}.$$

The proof of this theorem follows from Lemma 1.14 and Theorem 3.1.

Note that the optimality-by-order for the method $\{T_\delta : 0 < \delta \le \delta_0\}$ and the error estimate (3.46) for this method, unlike for other methods, have been obtained without the assumption of commutativity of the operators $\overline{A}$ and $\overline{B}$, where

$$\overline{A} = \sqrt{A^*A} \quad \text{and} \quad \overline{B} = \sqrt{BB^*}.$$

3.2 A study of the variational problem (3.2) with a parameter α selected based on the residual principle

The application of the residual principle for the selection of the regularization parameter when using the Tikhonov method was first justified for differential-operator equations in the paper by I. N. Dombrovskaya [18] in 1964. A more substantial justification of this principle as related to solving operator equations of the first kind was done in the papers by V. A. Morozov [59] and V. K. Ivanov [31] in 1966. Assume that all conditions of Lemma 3.3 are satisfied, i. e., $\mathbb{U}$, $\mathbb{F}$, and $\mathbb{V}$ are Hilbert spaces, A is an injective linear unbounded operator mapping $\mathbb{U}$ into $\mathbb{F}$ with the set of values $R(A)$ which is dense everywhere in $\mathbb{F}$, and B is a linear bounded operator mapping the space $\mathbb{F}$ into $\mathbb{U}$ with the set of values $R(B)$ which is dense everywhere in $\mathbb{U}$. Consider the variational problem (3.2). We write

$$\inf\{\|Cv - f_\delta\|^2 + \alpha\|v\|^2 : v \in \mathbb{V}\},$$

where $\alpha > 0$ and $C = AB$.

Select a regularization parameter $\alpha = \alpha(f_\delta, \delta)$ for the variational problem (3.2) from the equation

$$\|C\hat{v}_\delta^\alpha - f_\delta\|^2 = \delta^2, \tag{3.47}$$

where $\hat{v}_\delta^\alpha$ is the solution of the variational problem (3.2). Introduce a function $\varphi_\delta(\alpha)$, defined by the formula

$$\varphi_\delta(\alpha) = \|C\hat{v}_\delta^\alpha - f_\delta\|^2, \quad \alpha \in (0, \infty), \tag{3.48}$$

where $f_\delta \in F$ and $\hat{v}_\delta^\alpha$ is the solution of problem (1.2). We now get down to the justification of the residual principle (3.47).

Lemma 3.4. *Let $\alpha > 0$ and $\{\alpha_n\} \subset (0, \infty)$ and let $\hat{v}_\delta^\alpha$ and $\hat{v}_\delta^{\alpha_n}$ be the solutions of problem* (3.2) *for α and α_n respectively. Then*

$$\hat{v}_\delta^{\alpha_n} \longrightarrow \hat{v}_\delta^\alpha \quad \textit{for } \alpha_n \longrightarrow \alpha.$$

Proof. Assume the contrary. Then there exist a number $\varepsilon_0 > 0$ and a subsequence $\{\alpha_{n_k}\}$, such that for any k

$$\|\hat{v}_\delta^{\alpha_{n_k}} - \hat{v}_\delta^\alpha\| \geq \varepsilon_0. \tag{3.49}$$

It follows from the definition of the solution $\hat{v}_\delta^{\alpha_{n_k}}$ that for any k

$$\|C\hat{v}_\delta^{\alpha_{n_k}} - f_\delta\|^2 + \alpha_{n_k}\|\hat{v}_\delta^{\alpha_{n_k}}\|^2 \leq \|C\hat{v}_\delta^\alpha - f_\delta\|^2 + \alpha_{n_k}\|\hat{v}_\delta^\alpha\|^2. \tag{3.50}$$

It follows from (3.50) that

$$\overline{\lim_{k\longrightarrow\infty}}\{\|C\hat{v}_\delta^{\alpha_{n_k}} - f_\delta\|^2 + \alpha_{n_k}\|\hat{v}_\delta^{\alpha_{n_k}}\|^2\} \leq \|C\hat{v}_\delta^\alpha - f_\delta\|^2 + \alpha\|\hat{v}_\delta^\alpha\|^2 \tag{3.51}$$

and the boundedness of the sequence $\{\hat{v}_\delta^{\alpha_{n_k}}\}$ follows from (3.51). Thus, the sequence $\{\hat{v}_\delta^{\alpha_{n_k}}\}$ is weakly precompact. Without loss of generality we say that

$$\hat{v}_\delta^{\alpha_{n_k}} \overset{\text{ne}}{\longrightarrow} \tilde{v} \quad \text{for } k \longrightarrow \infty \tag{3.52}$$

and, due to the linearity and boundedness of the operator C,

$$C\hat{v}_\delta^{\alpha_{n_k}} \overset{\text{ne}}{\longrightarrow} C\tilde{v} \quad \text{for } k \longrightarrow \infty. \tag{3.53}$$

By the property of the weak limit norm, it follows from (3.52) and (3.53) that

$$\|C\tilde{v} - f_\delta\|^2 + \alpha\|\tilde{v}\|^2 \leq \underline{\lim}_{k\to\infty}\{\|C\hat{v}_\delta^{\alpha_{n_k}} - f_\delta\|^2 + \alpha_{n_k}\|\hat{v}_\delta^{\alpha_{n_k}}\|^2\}. \tag{3.54}$$

It follows from (3.51) and (3.54) that

$$\|C\tilde{v} - f_\delta\|^2 + \alpha\|\tilde{v}\|^2 \le \left\|C\hat{v}_\delta^\alpha - f_\delta\right\|^2 + \alpha\left\|\hat{v}_\delta^\alpha\right\|^2. \tag{3.55}$$

Since $\hat{v}_\delta^\alpha$ is the solution of problem (3.2), the left-hand side of (3.55) cannot be smaller and, therefore, it follows from (3.55) that

$$\|C\tilde{v} - f_\delta\|^2 + \alpha\|\tilde{v}\|^2 = \left\|C\hat{v}_\delta^\alpha - f_\delta\right\|^2 + \alpha\left\|\hat{v}_\delta^\alpha\right\|^2. \tag{3.56}$$

Due to the uniqueness of the solution of problem (3.2), by Lemma 3.2, it follows from (3.56) that

$$\tilde{v} = \hat{v}_\delta^\alpha. \tag{3.57}$$

It follows from (3.52) and (3.57) that

$$\hat{v}_\delta^{\alpha_{n_k}} \xrightarrow{\text{ne}} \hat{v}_\delta^\alpha \tag{3.58}$$

and it follows from (3.51) and (3.54) that

$$\left\|C\hat{v}_\delta^\alpha - f_\delta\right\|^2 + \alpha\left\|\hat{v}_\delta^\alpha\right\| = \lim_{k\longrightarrow\infty}\left\{\left\|C\hat{v}_\delta^{\alpha_{n_k}} - f_\delta\right\|^2 + \alpha_{n_k}\left\|\hat{v}_\delta^{\alpha_{n_k}}\right\|^2\right\}. \tag{3.59}$$

It follows from (3.58) that

$$\left\|\hat{v}_\delta^\alpha\right\| \le \varliminf_{k\to\infty}\left\|\hat{v}_\delta^{\alpha_{n_k}}\right\| \tag{3.60}$$

and without loss of generality it follows from (3.59) and (3.60) that

$$\left\|\hat{v}_\delta^{\alpha_{n_k}}\right\| \longrightarrow \left\|\hat{v}_\delta^\alpha\right\|, \quad \text{for } k \longrightarrow \infty. \tag{3.61}$$

Thus, it follows from (3.58) and (3.61) that

$$\hat{v}_\delta^{\alpha_{n_k}} \longrightarrow \hat{v}_\delta^\alpha, \quad \text{for } k \longrightarrow \infty,$$

which contradicts (3.49) and proves the lemma. □

It follows from Lemma 3.4 that the function $\varphi_\delta(\alpha)$ defined by (3.48) is continuous for any value $\alpha > 0$.

Lemma 3.5. *Let all the conditions of this paragraph be satisfied. Then*

$$\lim_{\alpha\longrightarrow 0}\varphi_\delta(\alpha) = 0 \quad \textit{and} \quad \lim_{\alpha\longrightarrow\infty}\varphi_\delta(\alpha) = \|f_\delta\|^2.$$

Proof. Since

$$\overline{R(C)} = F, \quad \text{for any } \varepsilon > 0$$

there could be found a point $\bar{v}_0 \in \mathbb{V}$, such that

$$\|C\bar{v} - f_\delta\|^2 < \frac{\varepsilon}{2}. \tag{3.62}$$

Then having selected the value $\overline{\alpha} > 0$, such as

$$\bar{\alpha}\|\bar{v}_0\|^2 < \frac{\varepsilon}{2}, \tag{3.63}$$

for any $\alpha \le \overline{\alpha}$ it will follow from relations (3.62) and (3.63) that

$$\varphi_\delta(\alpha) = \left\|C\hat{v}_\delta^\alpha - f_\delta\right\|^2 \le \|C\bar{v}_0 - f_\delta\|^2 + \alpha\|\bar{v}_0\|^2 < \varepsilon,$$

i. e.,

$$\varphi_\delta(\alpha) \longrightarrow 0 \quad \text{for } \alpha \longrightarrow 0.$$

We will now prove that

$$\varphi_\delta \longrightarrow \|f_\delta\|^2 \quad \text{for } \alpha \longrightarrow \infty.$$

Since for any $\alpha > 0$ it follows from

$$\left\|C\hat{v}_\delta^\alpha - f_\delta\right\|^2 + \alpha\left\|\hat{v}_\delta^\alpha\right\|^2 \le \|C0 - f_\delta\|^2 + \alpha\|0\|^2 = \|f_\delta\|^2$$

that

$$\alpha\left\|\hat{v}_\delta^\alpha\right\|^2 \le \|f_\delta\|^2,$$

for any $\varepsilon > 0$ there exists a value $\overline{\alpha} = \frac{\|f_\delta\|^2}{\varepsilon^2}$ such that for $\alpha > \overline{\alpha}$

$$\left\|\overline{v}_\delta^\alpha\right\| < \varepsilon. \tag{3.64}$$

It follows from (3.64) that

$$\hat{v}_\delta^\alpha \longrightarrow 0 \quad \text{for } \alpha \longrightarrow \infty \quad \text{and} \quad \varphi_\delta(\alpha) \longrightarrow \|f_\delta\|^2.$$

The lemma is thereby proved. □

It follows from Lemmas 3.4 and 3.5 that, if $\|f_\delta\| > \delta$, then there exists such value $\alpha(f_\delta, \delta)$, for which the solution $\hat{v}_\delta^{\alpha(f_\delta,\delta)}$ of problem (3.2) satisfies the equation

$$\left\|C\hat{v}_\delta^{\alpha(f_\delta,\delta)} - f_\delta\right\|^2 = \delta^2. \tag{3.65}$$

3.3 Residual method

The residual method was first used by Phillips [64] in 1962 to solve applied problems. Then this method was further developed in the well-known paper by V. K. Ivanov [32] in 1966. In 1972 V. V. Vasin found in [108] a connection between the residual method and Tikhonov's regularization method.

Let $\mathbb{U}$, $\mathbb{F}$, and $\mathbb{V}$ be Hilbert spaces, let A be a linear injective and bounded operator mapping $\mathbb{U}$ into $\mathbb{F}$, and let B be a linear bounded operator mapping $\mathbb{V}$ into $\mathbb{U}$. In addition assume that the set of values $R(A)$ of the operator A is everywhere dense in $\mathbb{F}$ and the set of values $R(B)$ of the operator B is everywhere dense in $\mathbb{U}$.

Like in the first paragraph, assume that for $f = f_0$ there exists an exact solution u_0 of equation (1.1), which belongs to the set $R(B)$ though f_0 is unknown. Instead, an element $f_\delta \in \mathbb{F}$ and an error level $\delta > 0$ are given such that

$$\|f_\delta - f_0\| \le \delta. \tag{3.66}$$

It is required to find an approximate solution $u_\delta \in \mathbb{U}$ of equation (1.1) by the initial data (f_δ, δ) and, assuming that $u_0 \in M_r = B\overline{S}_r$, estimate the value $\|u_\delta - u_0\|$.

The residual method consists of reducing the given problem to the variational problem

$$\inf\{\|v\|^2 : v \in \mathbb{V},\ \|Cv - f_\delta\| \le \delta\}, \tag{3.67}$$

where $C = AB$.

Lemma 3.6. *For any values $\delta > 0$ and $f_\delta \in \mathbb{F}$, the variational problem* (3.67) *is solvable.*

Proof. Let

$$\Omega_\delta = \{v : v \in \mathbb{V},\ \|Cv - f_\delta\| \le \delta\}.$$

Then it follows from $\delta > 0$ and $\overline{R(C)} = F$ that

$$\Omega_\delta \ne \emptyset.$$

Thus, the numerical set

$$K_\delta = \{\|v\|^2 : v \in \Omega_\delta\}$$

is non-empty and bounded from below by the number 0.

If $\|f_\delta\| \le \delta$, then $0 \in \Omega_\delta$ is the unique solution of the variational problem (3.67).

If $\|f_\delta\| > \delta$, then from the boundedness from below of the set K_δ it follows that the lower bound exists.

We have

$$\inf\{\|v\|^2 : v \in \mathbb{V},\ \|Cv - f_\delta\| \le \delta\}.$$

By the definition of the lower bound it follows that there exists a minimizing sequence $\{v_n\} \subset \Omega_\delta$ such that

$$\|v_n\|^2 \longrightarrow \inf\{\|v\|^2 : v \in \Omega_\delta\} \quad \text{for } n \longrightarrow \infty. \tag{3.68}$$

The boundedness of the sequence $\{v_n\}$ follows from (3.68) and its weak precompactness follows from the Hilbertness of the space $\mathbb{V}$.

Thus, there exists a subsequence $\{v_{n_k}\}$ such that

$$v_{n_k} \overset{\text{ne}}{\longrightarrow} \hat{v} \quad \text{for } k \longrightarrow \infty, \tag{3.69}$$

where $\hat{v} \in \mathbb{V}$.

Since C is a linear bounded operator, it follows from (3.69) that

$$Cv_{n_k} \overset{\text{ne}}{\longrightarrow} C\hat{v} \quad \text{for } k \longrightarrow \infty \tag{3.70}$$

and it follows from (3.70) that

$$Cv_{n_k} - f_\delta \overset{\text{ne}}{\longrightarrow} C\hat{v} - f_\delta \quad \text{for } k \longrightarrow \infty. \tag{3.71}$$

From (3.71), by the property of the weak limit norm, we get

$$\|C\hat{v} - f_\delta\| \le \varliminf_{k\to\infty} \|Cv_{n_k} - f_\delta\| \tag{3.72}$$

and, due to the fact that for any k, $v_{n_k} \in \Omega_\delta$, and, consequently,

$$\|Cv_{n_k} - f_\delta\| \le \delta,$$

according to (3.72), we obtain

$$\hat{v} \in \Omega_\delta. \tag{3.73}$$

By the property of the weak limit norm it follows from relation (3.69) that

$$\|\hat{v}\|^2 \le \varliminf_{k\to\infty} \|v_{n_k}\|^2. \tag{3.74}$$

It follows from relations (3.68), (3.73), and (3.74) that $\hat{v}$ is the solution of problem (3.67).

The lemma is thereby proved. □

Note. This lemma is proved in [28] under the condition that the space $\mathbb{V}$ is reflexive and that $\mathbb{U}$ and $\mathbb{F}$ are Banach spaces.

In addition to problem (3.67), consider the problem

$$\inf\{\|v\|^2 : v \in \mathbb{V},\ \|Cv - f_\delta\| = \delta\}. \tag{3.75}$$

Lemma 3.7. *If* $\|f_\delta\| > \delta$, *then problems* (3.67) *and* (3.75) *are equivalent.*

Proof. In order not to check the resolvability of problem (3.75), let us prove that any of the solutions of problem (3.67) is a solution of problem (3.75).

Assume the contrary, i. e., that there exists a point $\hat{v} \in \mathbb{V}$ such that $\|C\hat{v} - f_\delta\| < \delta$ and

$$\|\hat{v}\|^2 = \inf\{\|v\|^2 : v \in \Omega_\delta\} \tag{3.76}$$

and consider the numerical function $\varphi(\lambda)$ defined by the formula

$$\varphi(\lambda) = \big\|C(\lambda\hat{v}) - f_\delta\big\|, \quad \lambda \geq 0. \tag{3.77}$$

It follows from (3.77) that the function $\varphi(\lambda)$ is continuous and that

$$\varphi(1) = \|C\hat{v} - f_\delta\| < \delta. \tag{3.78}$$

Then it follows from (3.78) that there exists $\varepsilon_0 > 0$ such that, for any value λ satisfying the condition $|\lambda - 1| < \varepsilon_0$, the following inequality is true:

$$\varphi(\lambda) < \delta. \tag{3.79}$$

Thus, it follows from (3.79) that

$$\varphi\left(1 - \frac{\varepsilon_0}{2}\right) = \left\|C\left[\left(1 - \frac{\varepsilon_0}{2}\right)\hat{v}\right] - f_\delta\right\| < \delta$$

and, consequently,

$$\left(1 - \frac{\varepsilon_0}{2}\right)\hat{v} \in \Omega_\delta$$

and

$$\left(1 - \frac{\varepsilon_0}{2}\right)^2 \|\hat{v}\|^2 < \|\hat{v}\|^2,$$

which contradicts the fact that $\hat{v}$ is the solution of problem (3.67).

Thus, $\|C\hat{v} - f_\delta\| = \delta$ and $\hat{v}$ is the solution of problem (3.75). The fact that the solution of problem (3.75) is the solution of problem (3.67) is proved in the same way.

The lemma is thereby proved. □

Lemma 3.8. *If* $\|f_\delta\| > \delta$, *then the solution of the variational problem* (3.67) *is unique.*

Proof. Assume the contrary. Then there exist points $\hat{v}_1$ and $\hat{v}_2 \in \Omega_\delta$ such that $\hat{v}_1 \neq \hat{v}_2$ and

$$\|\hat{v}_1\|^2 = \|\hat{v}_2\|^2 = \inf\{\|v\|^2 : v \in \Omega_\delta\}. \tag{3.80}$$

Let

$$\hat{v} = \frac{\hat{v}_1 + \hat{v}_2}{2}.$$

Then it follows from relation (3.80) that

$$\|\hat{v}\|^2 \le \inf\{\|v\|^2 : v \in \Omega_\delta\}. \tag{3.81}$$

Since it follows from Lemma 3.7 that

$$\|C\hat{v}_1 - f_\delta\| = \delta \quad \text{and} \quad \|C\hat{v}_2 - f_\delta\| = \delta,$$

by strict convexity of the Hilbert space $\mathbb{F}$ it follows that

$$\|C\hat{v} - f_\delta\| < \delta. \tag{3.82}$$

It follows from (3.82) that there exists a number $\varepsilon_0 > 0$ such that

$$\left\| C\left[\left(1 - \frac{\varepsilon_0}{2}\right)\hat{v}\right] - f_\delta \right\| < \delta$$

and, consequently,

$$\left(1 - \frac{\varepsilon_0}{2}\right)\hat{v} \in \Omega_\delta. \tag{3.83}$$

Then from (3.81) and (3.83) it follows that

$$\left(1 - \frac{\varepsilon_0}{2}\right)^2 \|\hat{v}\|^2 < \inf\{\|v\|^2 : v \in \Omega_\delta\}. \tag{3.84}$$

Relation (3.84) contradicts the assumption about the existence of two different solutions of problem (3.67) and thus proves the lemma. □

Note. In [28], Lemma 3.8 is proved under the condition of reflexivity and strict convexity of the space $\mathbb{V}$ and the condition that $\mathbb{U}$ are $\mathbb{F}$ Banach spaces.

We further denote the solution of problem (3.67) by v_δ and, simultaneously with problem (3.67), consider the problem

$$\inf\{\|v\|^2 : v \in \mathbb{V},\ \|Cv - f_\delta(n)\| \le \delta\}, \tag{3.85}$$

where

$$f_\delta(n) \in \mathbb{F} \quad \text{and} \quad \|f_\delta(n)\| > \delta.$$

From Lemmas 3.6–3.8 it follows that there exists a unique solution $v_\delta(n)$ of problem (3.85) and that the condition

$$\|Cv_\delta(n) - f_\delta(n)\| = \delta \tag{3.86}$$

is satisfied.

Lemma 3.9. *If $\|f_\delta\| > \delta$ and for any n*

$$\|f_\delta(n)\| > \delta \quad \text{while } f_\delta(n) \longrightarrow f_\delta \quad \text{for } n \longrightarrow \infty,$$

then

$$v_\delta(n) \longrightarrow v_\delta \quad \text{for } n \longrightarrow \infty.$$

Proof. Assume the contrary, i. e., $v_\delta(n)$ does not converge to v_δ for $n \longrightarrow \infty$. Then there exist a number $\varepsilon_0 > 0$ and subsequence $\{n_k\}$ such that for any k

$$\|v_\delta(n_k) - v_\delta\| \geq \varepsilon_0. \tag{3.87}$$

Since $\overline{R(C)} = \mathbb{F}$, there exists a point $v_0 \in \mathbb{V}$ such that

$$\|Cv_0 - f_\delta\| \leq \frac{\delta}{2}. \tag{3.88}$$

It follows from

$$f_\delta(n_k) \longrightarrow f_\delta \quad \text{for } k \longrightarrow \infty$$

that there exists a number k_1 such that for any $k \geq k_1$

$$\|f_\delta(n_k) - f_\delta\| < \frac{\delta}{2}. \tag{3.89}$$

Let $f_0 = Cv_0$. Then for any $k \geq k_1$, by (3.89), it follows that

$$\|f_0 - f_\delta(n_k)\| \leq \|f_0 - f_\delta\| + \|f_\delta - f_\delta(n_k)\| \leq \delta. \tag{3.90}$$

It follows from (3.90) that for any $k \geq k_1$

$$\|Cv_0 - f_\delta(n_k)\| \leq \delta \tag{3.91}$$

and it follows from (3.91) that for any $k \geq k_1$

$$\|v_\delta(n_k)\| \leq \|v_0\|. \tag{3.92}$$

It follows from (3.92) that the sequence $\{v_\delta(n_k)\}$ is weakly precompact and one can select its subsequence $\{v_\delta(n_{k_l})\}$ such that

$$v_\delta(n_{k_l}) \overset{\text{ne}}{\longrightarrow} \hat{v} \quad \text{for } l \longrightarrow \infty. \tag{3.93}$$

It follows from (3.93) that

$$Cv_\delta(n_{k_l}) - f_\delta \overset{\text{ne}}{\longrightarrow} C\hat{v} - f_\delta \quad \text{for } l \longrightarrow \infty \tag{3.94}$$

and it follows from (3.94), by the property of the weak limit norm, that

$$\|C\hat{v} - f_\delta\| \le \varliminf_{l\to\infty} \|Cv_\delta(n_{k_l}) - f_\delta\|.$$

Taking into account that for any l

$$\|Cv_\delta(n_{k_l}) - f_\delta\| \le \delta + \|f_\delta(n_{k_l}) - f_\delta\|,$$

where

$$\|f_\delta(n_{k_l}) - f_\delta\| \longrightarrow 0 \quad \text{for } n \longrightarrow \infty,$$

we obtain

$$\|C\hat{v} - f_\delta\| \le \delta. \tag{3.95}$$

It follows from (3.95) that

$$\|\hat{v}\| \ge \|v_\delta\|. \tag{3.96}$$

Introduce a sequence $\{\hat{v}_l\}$, defined by the formula

$$\hat{v}_l = \gamma_l \hat{v} + (1-\gamma_l)v_0, \tag{3.97}$$

where $\gamma_l \in [0,1]$, and satisfying the condition

$$\|C\hat{v}_l - f_\delta\| = \delta - \|f_\delta(n_{k_l}) - f_\delta\|. \tag{3.98}$$

It follows from (3.97) and (3.98) that for any l

$$\begin{aligned}\|C\hat{v}_l - f_\delta\| &\le \gamma_l\|C\hat{v} - f_\delta\| + (1-\gamma_l)\|Cv_0 - f_\delta\| \\ &\le \gamma_l\delta + (1-\gamma_l)\frac{\delta}{2} = (1+\gamma_l)\frac{\delta}{2}.\end{aligned} \tag{3.99}$$

Since

$$\|f_\delta(n_{k_l}) - f_\delta\| \longrightarrow 0 \quad \text{for } l \longrightarrow \infty,$$

it follows from (3.98) and (3.99) that $\gamma_l \to 1$ and it follows from (3.97) that

$$\hat{v}_l \longrightarrow \hat{v} \quad \text{for } l \longrightarrow \infty. \tag{3.100}$$

It follows from (3.100) that

$$\|\hat{v}_l\| \longrightarrow \|\hat{v}\| \quad \text{for } l \longrightarrow \infty \tag{3.101}$$

and it follows from the definition of $v_\delta(n_{k_l})$ and Lemma 3.7 that for any l

$$\|Cv_\delta(n_{k_l}) - f_\delta\| \geq \delta - \|f_\delta(n_{k_l}) - f_\delta\|. \tag{3.102}$$

It follows from relations (3.98) and (3.102) that for any l

$$\|v_\delta(n_{k_l})\| \leq \|\hat{v}_l\|. \tag{3.103}$$

It follows from (3.101) and (3.103) that

$$\|\hat{v}\| \geq \overline{\lim_{l \longrightarrow \infty}} \|v_\delta(n_{k_l})\| \tag{3.104}$$

and it follows from (3.93) that

$$\|\hat{v}\| \leq \underline{\lim}_{l \to \infty} \|v_\delta(n_{k_l})\|. \tag{3.105}$$

It follows from (3.96), (3.104), and (3.105) that

$$\|\hat{v}\| = \|v_\delta\| \tag{3.106}$$

and it follows from (3.95) and (3.106), by Lemma 3.8, that

$$v_\delta = \hat{v}. \tag{3.107}$$

Thus, it follows from (3.93) and (3.107) that

$$v_\delta(n_{k_l}) \overset{\text{ne}}{\longrightarrow} v_\delta \quad \text{for } l \longrightarrow \infty \tag{3.108}$$

and it follows from (3.104)–(3.106) that

$$\|v_\delta(n_{k_l})\| \longrightarrow \|v_\delta\| \quad \text{for } l \longrightarrow \infty. \tag{3.109}$$

It follows from (3.108) and (3.109) that

$$v_\delta(n_{k_l}) \longrightarrow v_\delta \quad \text{for } l \longrightarrow \infty,$$

which contradicts (3.87) and thereby proves the lemma. □

Lemma 3.10. *If*

$$\|f_\delta\| \leq \delta \quad \textit{and} \quad f_\delta(n) \longrightarrow f_\delta \quad \textit{for } n \longrightarrow \infty,$$

then

$$v_\delta(n) \longrightarrow v_\delta \quad \textit{for } n \longrightarrow \infty.$$

Proof. As was mentioned above, if $\|f_\delta\| \leq \delta$, then problem (3.67) has the unique solution $v_\delta = 0$. We consider two cases.

First case

Assume that

$$\|f_\delta\| < \delta \quad \text{and} \quad f_\delta(n) \longrightarrow f_\delta \quad \text{for } n \longrightarrow \infty.$$

Then there exists a number N such that for any $n \geq N$ we have the inequality

$$\|f_\delta(n)\| < \delta.$$

Thus, for any $n \geq N$, $v_\delta(n) = 0$ and for this case the lemma is proved.

Second case

Assume that $\|f_\delta\| = \delta$ and assume for any n

$$\|f_\delta(n)\| \geq \delta \quad \text{and} \quad f_\delta(n) \longrightarrow f_\delta \quad \text{for } n \longrightarrow \infty.$$

Without loss of generality, we take for any n, $\|f_\delta(n)\| > \delta$. Then for any n the corresponding solution of problem (3.85) is $v_\delta(n) \neq 0$.

Since

$$f_\delta(n) \longrightarrow f_\delta \quad \text{for } n \longrightarrow \infty,$$

there exists a number N_1 such that for any $n \geq N_1$

$$\|f_\delta(n) - f_\delta\| < \frac{\delta}{2}. \tag{3.110}$$

It follows from $\overline{R(C)} = \mathbb{F}$ that there exists a number $v_0 \in \mathbb{V}$ such that

$$\|Cv_0 - f_\delta\| < \frac{\delta}{2}. \tag{3.111}$$

Introduce a sequence $\{v_0(n)\}$ defined by the formula

$$v_0(n) = \lambda_n v_0, \tag{3.112}$$

where for any n, $\lambda_n > 0$ and

$$\|Cv_0(n) - f_\delta\| = \delta - \|f_\delta(n) - f_\delta\|. \tag{3.113}$$

Without loss of generality we set $n \geq N_1$.

It follows from (3.112) and (3.113) that for any n

$$\|Cv_0(n) - f_\delta\| \leq \frac{\delta}{2} + (1 - \lambda_n)\frac{\delta}{2} \tag{3.114}$$

and it follows from (3.113) and (3.114) that

$$\frac{\delta}{2} + (1 - \lambda_n)\frac{\delta}{2} \geq \delta - \|f_\delta(n) - f_\delta\|, \tag{3.115}$$

where for any n

$$\lambda_n > 0 \quad \text{and} \quad \|f_\delta(n) - f_\delta\| \longrightarrow 0 \quad \text{for } n \longrightarrow \infty.$$

Thus, it follows from (3.115) that $\lambda_n \longrightarrow 0$ for $n \longrightarrow \infty$, whence

$$v_0(n) \longrightarrow 0 \quad \text{for } n \longrightarrow \infty. \tag{3.116}$$

Since for any n

$$\|Cv_\delta(n) - f_\delta\| \geq \delta - \|f_\delta(n) - f_\delta\|, \tag{3.117}$$

it follows from (3.113), (3.117), and (3.85) that for any n

$$\|v_\delta(n)\| \leq \|v_0(n)\|. \tag{3.118}$$

It follows from (3.116) and (3.118) that

$$v_\delta(n) \longrightarrow 0 \quad \text{for } n \longrightarrow \infty.$$

The lemma is thereby proved. □

It follows from Lemmas 3.6–3.10 that the variational problem (3.67) is well-posed according to Hadamard.

Theorem 3.3. *Let*

$$\overline{R(C)} = \mathbb{F} \quad \textit{and} \quad \|f_\delta\| > \delta.$$

Then the variational problem (3.67) *is equivalent to the variational problem* (3.2) *with the regularization parameter* α *satisfying equation* (3.47).

Proof. Let $\hat{v}_\delta^{\alpha(f_\delta,\delta)}$ be a solution of problem (3.2), (3.47). Then

$$\|C\hat{v}_\delta^{\alpha(f_\delta,\delta)} - f_\delta\|^2 = \delta^2 \tag{3.119}$$

and

$$\begin{aligned} &\|C\hat{v}_\delta^{\alpha(f_\delta,\delta)} - f_\delta\|^2 + \alpha(f_\delta,\delta)\|\hat{v}_\delta^{\alpha(f_\delta,\delta)}\|^2 \\ &\quad = \delta^2 + \alpha(f_\delta,\delta)\|\hat{v}_\delta^{\alpha(f_\delta,\delta)}\|^2 \\ &\quad \leq \inf_v\{\delta^2 + \alpha(f_\delta,\delta)\|v\|^2 : v \in \mathbb{V},\ \|Cv - f_\delta\| = \delta\}. \end{aligned} \tag{3.120}$$

Since it follows from Lemma 3.7 that

$$\inf\{\|v\|^2 : v \in \mathbb{V},\ \|Cv - f_\delta\| = \delta\} = \|v_\delta\|^2,$$

where v_δ is a solution of the variational problem (3.67), it follows from (3.120) that

$$\|v_\delta\| \le \|\hat{v}_\delta^{\alpha(f_\delta,\delta)}\|. \tag{3.121}$$

If we assume that $\|v_\delta\| < \|\hat{v}_\delta^{\alpha(f_\delta,\delta)}\|$, then

$$\|Cv_\delta - f_\delta\|^2 + \alpha(f_\delta,\delta)\|v_\delta\|^2 = \delta^2 + \alpha(f_\delta,\delta)\|v_\delta\|^2 < \delta^2 + \alpha(f_\delta,\delta)\|\hat{v}_\delta^{\alpha(f_\delta,\delta)}\|^2,$$

which contradicts the definition of the solution $\hat{v}_\delta^{\alpha(f_\delta,\delta)}$ of the variational problem (3.2) for $\alpha = \alpha(f_\delta,\delta)$.

Thus,

$$\|v_\delta\| = \|\hat{v}_\delta^{\alpha(f_\delta,\delta)}\| \tag{3.122}$$

and it follows from (3.119), (3.122), and Lemmas 3.7, and 3.8 that $v_\delta = \hat{v}_\delta^{\alpha(f_\delta,\delta)}$.

We move on to the inverse direction. Let v_δ be a solution of problem (3.67), let $\alpha(f_\delta,\delta)$ be a solution of equation (3.47), and let $\hat{v}_\delta^{\alpha(f_\delta,\delta)}$ be a solution of problem (3.2) for $\alpha = \alpha(f_\delta,\delta)$.

Then it follows from Lemma 3.7 that

$$\|v_\delta\|^2 = \inf\{\|v\|^2 : v \in \mathbb{V},\ \|Cv - f_\delta\| = \delta\}$$

and it follows from (3.119) that

$$\|C\hat{v}_\delta^{\alpha(f_\delta,\delta)} - f_\delta\| = \delta.$$

Thus,

$$\|v_\delta\| \le \|\hat{v}_\delta^{\alpha(f_\delta,\delta)}\|.$$

Assume that

$$\|v_\delta\| < \|\hat{v}_\delta^{\alpha(f_\delta,\delta)}\|.$$

Then

$$\|Cv_\delta - f_\delta\|^2 + \alpha(f_\delta,\delta)\|v_\delta\|^2 = \delta^2 + \alpha(f_\delta,\delta)\|v_\delta\|^2 < \delta^2 + \alpha(f_\delta,\delta)\|\hat{v}_\delta^{\alpha(f_\delta,\delta)}\|^2,$$

which contradicts the definition of the solution $\hat{v}_\delta^{\alpha(f_\delta,\delta)}$ of the variational problem (3.2) for $\alpha = \alpha(f_\delta,\delta)$.

Consequently,

$$\|v_\delta\| = \|\hat{v}_\delta^{\alpha(f_\delta,\delta)}\|$$

and

$$\|Cv_\delta - f_\delta\|^2 + \alpha(f_\delta,\delta)\|v_\delta\|^2 = \delta^2 + \alpha(f_\delta,\delta)\|v_\delta\|^2$$
$$= \|C\hat{v}_\delta^{\alpha(f_\delta,\delta)} - f_\delta\|^2 + \alpha(f_\delta,\delta)\|\hat{v}_\delta^{\alpha(f_\delta,\delta)}\|^2. \tag{3.123}$$

Since it follows from Lemma 3.2 that problem (3.2) has a unique solution, it follows from (3.123) that $v_\delta = \hat{v}_\delta^{\alpha(f_\delta,\delta)}$.

The theorem is thereby proved. □

The residual method is defined by the operator family $\{T_\delta : 0 < \delta \leq \delta_0\}$ mapping $\mathbb{F}$ into $\mathbb{U}$ and defined by the formula

$$T_\delta f_\delta = Bv_\delta, \quad f_\delta \in \mathbb{F}, \quad Bv_\delta \in \mathbb{U}, \tag{3.124}$$

where v_δ is the solution of problem (3.67).

If follows from Lemmas 3.6 and 3.8–3.10 that for any $\delta \in (0,\delta_0]$ the operator T_δ continuously maps the space $\mathbb{F}$ into $\mathbb{U}$.

Define the approximate solution u_δ of equation (1.1) by the formula $u_\delta = T_\delta f_\delta$.

We will now estimate the accuracy of the residual method $\Delta_\delta[T_\delta]$

$$\{T_\delta : 0 < \delta \leq \delta_0\}$$

over the set $M_r = B\overline{S_r}$ defined by formula (1.65) for any $\delta \in (0,\delta_0]$. We have

$$\Delta_\delta[T_\delta] = \sup_{u,f_\delta}\{\|u - T_\delta f_\delta\| : u \in M_r,\ \|Au - f_\delta\| \leq \delta\}.$$

For this purpose, estimate the deviation $\|u_\delta - u_0\|$ of the approximate solution u_δ of equation (1.1) from the accurate solution u_0.

Theorem 3.4. *Let* $u_0 \in M_r$,

$$\|f_\delta - Au_0\| \leq \delta, \quad \textit{and} \quad u_\delta = T_\delta f_\delta.$$

Then

$$\|u_\delta - u_0\| \leq 2\omega(\delta, r).$$

Proof. Since $u_0 \in M_r$, there exists $v_0 \in \mathbb{V}$ such that

$$\|v_0\| \leq r. \tag{3.125}$$

By $u_\delta = Bv_\delta$, where v_δ is the solution of problem (3.67), it follows that

$$\|v_\delta\| \le \|v_0\| \tag{3.126}$$

and

$$\|Au_\delta - f_\delta\| = \delta. \tag{3.127}$$

It follows from (3.125) and (3.126) that

$$u_\delta \in M_r \tag{3.128}$$

and it follows from (3.127) that

$$\|Au_0 - Au_\delta\| \le 2\delta. \tag{3.129}$$

It follows from (3.128) and (3.129) that

$$\|u_\delta - u_0\| \le \omega_1(2\delta, r). \tag{3.130}$$

It follows from (3.130) and Lemmas 1.2 and 1.3 that

$$\|u_\delta - u_0\| \le 2\omega(\delta, r).$$

The theorem is thereby proved. □

It follows from Theorem 3.4 that for any $\delta \in (0, \delta_0]$

$$\Delta_\delta[T_\delta] \le 2\omega(\delta, r). \tag{3.131}$$

The following theorem follows from Lemma 1.14 and formula (3.131).

Theorem 3.5. *The residual method $\{T_\delta : 0 < \delta \le \delta_0\}$ is optimal-by-order on the class of solutions M_r and for any $\delta \in (0, \delta_0]$ the following estimate holds true*

$$\Delta_\delta[T_\delta] \le 2\Delta_\delta^{\text{opt}}$$

3.4 The error estimate for the Tikhonov regularization method with parameter *α*, selected by the residual principle

Assume that all conditions of Lemma 3.3 are satisfied in this paragraph, i. e., $\mathbb{U}$, $\mathbb{F}$, and $\mathbb{V}$ are Hilbert spaces, A is an injective linear operator mapping $\mathbb{U}$ into $\mathbb{F}$ with the set of values $R(A)$ everywhere dense in $\mathbb{F}$, and B is a linear bounded operator mapping $\mathbb{V}$ into $\mathbb{U}$ with the set $R(B)$ everywhere dense in $\mathbb{U}$.

Theorem 3.6. *Let* $0 < \delta < \|f_\delta\|$. *Then there exists a unique value of the parameter* α *that satisfies equation* (3.47).

Proof. It follows from Lemmas 3.4 and 3.5 that there exists a value of the parameter $\alpha(f_\delta, \delta)$ satisfying the equation

$$\|C\hat{v}_\delta^{\alpha(f_\delta,\delta)} - f_\delta\| = \delta^2, \tag{3.132}$$

where v_δ^α is a solution of the variational problem (3.2).

We now move on to the proof of the solution uniqueness for equation (3.47). For this purpose, consider the contrary. Then we find two different solutions $\overline{\alpha}_1$ and $\overline{\alpha}_2$ of equation (3.47). Denote the solutions of problem (3.2) for these values by $\hat{v}_\delta^{\overline{\alpha}_1}$ and $\hat{v}_\delta^{\overline{\alpha}_2}$.

Let v_δ be a solution of problem (3.67). Then it follows from Theorem 3.3 that

$$\hat{v}_\delta^{\overline{\alpha}_1} = v_\delta \quad \text{and} \quad \hat{v}_\delta^{\overline{\alpha}_2} = v_\delta,$$

i. e.,

$$\hat{v}_\delta^{\overline{\alpha}_1} = \hat{v}_\delta^{\overline{\alpha}_2}. \tag{3.133}$$

It follows from (3.133), Lemma 3.7, and Theorem 3.3 that

$$\delta^2 + \overline{\alpha}_2\|\hat{v}_\delta^{\overline{\alpha}_2}\|^2 = \min_\lambda\{\|\lambda C\hat{v}_\delta^{\overline{\alpha}_2} - f_\delta\|^2 + \overline{\alpha}_2\lambda^2\|\hat{v}_\delta^{\overline{\alpha}_2}\|^2\} \tag{3.134}$$

and

$$\delta^2 + \overline{\alpha}_1\|\hat{v}_\delta^{\overline{\alpha}_1}\|^2 = \min_\lambda\{\|\lambda C\hat{v}_\delta^{\overline{\alpha}_2} - f_\delta\|^2 + \overline{\alpha}_1\lambda^2\|\hat{v}_\delta^{\overline{\alpha}_2}\|^2\}. \tag{3.135}$$

In formulas (3.134) and (3.135) the minimum is achieved for $\lambda = 1$.

Since

$$\begin{aligned}&\|\lambda C\hat{v}_\delta^{\overline{\alpha}_2} - f_\delta\|^2 + \overline{\alpha}_2\lambda^2\|\hat{v}_\delta^{\overline{\alpha}_2}\|^2 \\ &\quad = \lambda^2\|C\hat{v}_\delta^{\overline{\alpha}_2}\|^2 - 2\lambda(C\hat{v}_\delta^{\overline{\alpha}_2}, f_\delta) + \|f_\delta\|^2 + \overline{\alpha}_2\lambda^2\|\hat{v}_\delta^{\overline{\alpha}_2}\|^2\end{aligned} \tag{3.136}$$

and

$$\begin{aligned}&\|\lambda C\hat{v}_\delta^{\overline{\alpha}_2} - f_\delta\|^2 + \overline{\alpha}_1\lambda^2\|\hat{v}_\delta^{\overline{\alpha}_2}\|^2 \\ &\quad = \lambda^2\|C\hat{v}_\delta^{\overline{\alpha}_2}\|^2 - 2\lambda(C\hat{v}_\delta^{\overline{\alpha}_2}, f_\delta) + \|f_\delta\|^2 + \overline{\alpha}_1\lambda^2\|\hat{v}_\delta^{\overline{\alpha}_2}\|^2,\end{aligned} \tag{3.137}$$

having λ-differentiated expressions (3.136) and (3.137) and having set the values of the derivatives for $\lambda = 1$ to be zero, we obtain

$$\|C\hat{v}_\delta^{\overline{\alpha}_2}\|^2 - (C\hat{v}_\delta^{\overline{\alpha}_2}, f_\delta) + \overline{\alpha}_2\lambda^2\|\hat{v}_\delta^{\overline{\alpha}_2}\|^2 = 0 \tag{3.138}$$

and

$$\|C\hat{v}_\delta^{\overline{\alpha}_2}\|^2 - (C\hat{v}_\delta^{\overline{\alpha}_2}, f_\delta) + \overline{\alpha}_1\lambda^2\|\hat{v}_\delta^{\overline{\alpha}_2}\|^2 = 0. \tag{3.139}$$

By subtracting termwise equality (3.139) from (3.138) we obtain

$$(\overline{\alpha}_2 - \overline{\alpha}_1)\|\hat{v}_\delta^{\overline{\alpha}_2}\|^2 = 0. \tag{3.140}$$

Since $\overline{\alpha}_1 \neq \overline{\alpha}_2$, it follows from (3.140) that $\hat{v}_\delta^{\overline{\alpha}_2} = 0$ and, due to Theorem 3.3 and $v_\delta = 0$, this contradicts the condition $\delta < \|f_\delta\|$.

The theorem is thereby proved. □

It follows from Lemma 3.3 and Theorem 3.3 that, if

$$\mathbb{V}^* = \mathbb{V} \quad \text{and} \quad \mathbb{F}^* = \mathbb{F},$$

then the Tikhonov regularization method with the parameter α, selected by the residual principle (3.2), is defined by the equation

$$\overline{T}_\delta f_\delta = \begin{cases} B[C^*C + \alpha(f_\delta, \delta)E]^{-1}C^*f_\delta & \text{for } \|f_\delta\| > \delta, \\ 0 & \text{for } \|f_\delta\| \le \delta, \end{cases} \tag{3.141}$$

where $C = AB$, C^* is the operator adjoint with C, $\alpha(f_\delta, \delta)$ is the solution of equation (3.47),

$$\|C\hat{v}_\delta^\alpha - f_\delta\|^2 = \delta^2,$$

and

$$\hat{v}_\delta^\alpha = [C^*C + \alpha(f_\delta, \delta)E]^{-1}C^*f_\delta.$$

Let $\Delta_\delta[\overline{T}_\delta]$ be the accuracy estimate for the method $\{\overline{T}_\delta : 0 < \delta \le \delta_0\}$, defined by (3.75). Then

$$\Delta_\delta[\overline{T}_\delta] = \sup_{u, f_\delta}\{\|u - \overline{T}_\delta f_\delta\| : u \in M_r,\ \|Au - f_\delta\|\},$$

where

$$M_r = B\overline{S}_r, \quad \overline{S}_r = \{v : v \in \mathbb{V},\ \|v\| \le r\},$$

and $\omega(\delta, r)$ is the modulus of continuity at zero of the inverse operator A^{-1} on the set $N_r = AM_r$. It follows from Theorems 3.3 and 3.4.

Theorem 3.7. *Under the conditions defined above, for the method* $\{\overline{T}_\delta : 0 < \delta \le \delta_0\}$, *defined by formula* (3.141), *the following estimate is true:*

$$\Delta_\delta[T_\delta] \le 2\omega(\delta, r).$$

3.5 On solving an inverse problem in solid state physics with the Tikhonov regularization method

Following [50], note that, at sufficiently low temperatures, many macroscopic systems behave thermodynamically as an ideal gas of certain "quasi-particles" (elementary excitations), obeying Bose statistics. The energy spectrum of such a system is determined by the spectrum of quasi-particles, i. e., by the number of quasi-particles levels $n(\varepsilon)d\varepsilon$ on the energy interval $d\varepsilon$.

Recovering the phonon density of states $n(\varepsilon)$, it is important to find the characteristic structure, since it is this structure that defines many physical properties of crystals.

3.5.1 Setting of the problem

The relationship between the energy spectrum of a Bose system and its temperature-dependent heat capacity is described by the integral equation of the first kind [50]

$$Sn(\varepsilon) = \int_0^\infty S\left(\frac{\varepsilon}{\theta}\right)\frac{\varepsilon}{\theta}n(\varepsilon)\frac{d\varepsilon}{\varepsilon} = \frac{C(\theta)}{\theta}, \quad 0 \le \theta < \infty, \tag{3.142}$$

where

$$S(x) = \frac{x^2}{2\sinh^2(\frac{x}{2})},$$

$C(\theta)$ is the heat capacity of the system $\theta = kT$, T is the absolute temperature, k is a constant defined by the system, and $n(\varepsilon)$ is the spectral density (see [4]).

Denote by $\mathbb{H}$ a real space of the functions $f(x)$ measurable on $[0,\infty)$ with the norm defined by the formula

$$\|f(x)\|^2_{\mathbb{H}} = \int_0^\infty |f(x)|^2\frac{dx}{x}. \tag{3.143}$$

Note that the integral in formula (3.143) is understood in the sense of Lebesgue.

Assume that for

$$\frac{C(\theta)}{\theta} = \frac{C_0(\theta)}{\theta} \in \mathbb{H}$$

there exists an exact solution $n_0(\varepsilon) \in \mathbb{H}$ of equation (3.142), which is unique and satisfies the relation

$$n_0(\varepsilon) \in G_r, \tag{3.144}$$

where

$$G_r = \left\{ n(\varepsilon) : n(\varepsilon) \in \mathbb{H}, \int_0^\infty \frac{n^2(\varepsilon)}{\varepsilon} d\varepsilon + \int_0^\infty [n'(\varepsilon)]^2 \varepsilon d\varepsilon \le r^2 \right\}, \tag{3.145}$$

where $n'(\varepsilon)$ is the derivative of the function $n(\varepsilon)$, but instead of the exact value of the right-hand side $\frac{C_0(\theta)}{\theta}$ of equation (3.142) we know a certain approximation $\frac{C_\delta(\theta)}{\theta} \in \mathbb{H}$ and an error level $\delta > 0$ such that

$$\left\| \frac{C_\delta(\theta)}{\theta} - \frac{C_0(\theta)}{\theta} \right\|_{\mathbb{H}} \le \delta. \tag{3.146}$$

It is required to find the solution $n_\delta(\varepsilon) \in \mathbb{H}$ of problem (3.142)–(3.146) and estimate its deviation $\|n_\delta(\varepsilon) - n_0(\varepsilon)\|_{\mathbb{H}}$ from the exact solution $n_0(\varepsilon)$ of equation (3.142) in the metrics of the space $\mathbb{H}$.

If we assume that $\frac{C(\theta)}{\theta}$ and $n(\varepsilon) \in \mathbb{H}$, then equation (3.142) becomes an ill-posed problem.

3.5.2 Tikhonov regularization method

The Tikhonov regularization method (see [97]) for the approximate solution of equation (3.142) consists of reducing it to the variational problem

$$\inf\left\{ \int_0^\infty \left[\int_0^\infty S(\varepsilon/\theta) \frac{\varepsilon}{\theta} n(\varepsilon) \frac{d\varepsilon}{\varepsilon} - \frac{C_\delta(\theta)}{\theta} \right]^2 \frac{d\theta}{\theta} \right.$$
$$\left. + \alpha \int_0^\infty \frac{n^2(\varepsilon)}{\varepsilon} d\varepsilon + \alpha \int_0^\infty [n'(\varepsilon)]^2 \cdot \varepsilon d\varepsilon : n(\varepsilon) \in \mathbb{H}^1[0,\infty) \right\}, \tag{3.147}$$

where $\mathbb{H}^1[0,\infty)$ is a Hilbert space defined by the norm

$$\|n(\varepsilon)\|^2_{\mathbb{H}^1[0,\infty)} = \int_0^\infty \frac{n^2(\varepsilon)}{\varepsilon} d\varepsilon + \int_0^\infty [n'(\varepsilon)]^2 \cdot \varepsilon d\varepsilon, \quad \alpha > 0.$$

It follows from Lemmas 3.1 and 3.2 that for any function $\frac{C(\theta)}{\theta} \in \mathbb{H}$ there exists a unique solution n_δ^α of the variational problem (3.147).

To find the value of the regularization parameter α in the problem (3.147), we use the residual principle (3.47) that is reduced to the solution of the equation

$$\int_0^\infty \left[\int_0^\infty S\left(\frac{\varepsilon}{\theta}\right) \cdot \frac{\varepsilon}{\theta} \cdot n_\delta^\alpha(\varepsilon) \frac{d\varepsilon}{\varepsilon} - \frac{C_\delta(\theta)}{\theta} \right]^2 \frac{d\theta}{\theta} = \delta^2 \tag{3.148}$$

with respect to α.

It follows from Lemmas 3.4 and 3.5 that, if the condition

$$\int_0^\infty \left[\frac{C_\delta(\theta)}{\theta}\right]^2 \frac{d\theta}{\theta} > \delta^2$$

is satisfied, then equation (3.148) has the unique solution $\overline{\alpha}(C_\delta, \delta)$.

Define the approximate solution $n_\delta(\varepsilon)$ of equation (3.142) by the formula

$$n_\delta(\varepsilon) = n_\delta^{\overline{\alpha}(C_\delta,\delta)}(\varepsilon)$$

and define the corresponding regularization method by the family of operators $\{R_\delta : 0 < \delta \le \delta_0\}$ continuously mapping $\mathbb{H}$ into $\mathbb{H}$, defined by the formula

$$R_\delta\left[\frac{C_\delta(\theta)}{\theta}\right] = \begin{cases} n_\delta(\varepsilon), & \|\frac{C_\delta(\theta)}{\theta}\|_{\mathbb{H}} > \delta, \\ 0, & \|\frac{C_\delta(\theta)}{\theta}\|_{\mathbb{H}} \le \delta. \end{cases} \tag{3.149}$$

3.5.3 Error estimation for the method $\{R_\delta : 0 < \delta \le \delta_0\}$ defined by (3.149) on the class of solutions G_r

Define the error estimate for the method $\{R_\delta : 0 < \delta \le \delta_0\}$ by the family of functionals $\{\Delta_\delta(R_\delta) : 0 < \delta \le \delta_0\}$ defined by formula (1.65) as follows:

$$\Delta_\delta(R_\delta) = \sup\left\{\left\|R_\delta\left(\frac{C_\delta(\theta)}{\theta}\right) - n_0(\varepsilon)\right\|_{\mathbb{H}} : n_0(\varepsilon) \in G_r, \left\|S \cdot n_0(\varepsilon) - \frac{C_\delta(\theta)}{\theta}\right\|_{\mathbb{H}} \le \delta\right\}. \tag{3.150}$$

Denote by $\omega(\delta, r)$ the modulus of continuity at zero of the operator S^{-1} on the set $S[G_r]$ as follows:

$$\omega(\delta, r) = \sup\{\|n(\varepsilon)\| : n(\varepsilon) \in G_r, \|Sn(\varepsilon)\| \le \delta\|\}. \tag{3.151}$$

For the quantities $\Delta_\delta(R_\delta)$ and $\omega(\delta, r)$ in Theorem 3.7 we obtain the estimate

$$\Delta_\delta(R_\delta) \le 2\omega(\delta, r), \quad 0 < \delta \le \delta_0, \tag{3.152}$$

where $\omega(\delta, r)$ is defined by (3.151) and $\Delta_\delta(R_\delta)$ is defined by formula (3.150).

3.5.4 Estimation of the modulus of continuity $\omega(\delta, r)$ defined by formula (3.151)

Make the following substitution of variables in (3.142):

$$\varepsilon = e^t \quad \text{and} \quad \theta = e^\tau, \ -\infty < t, \ \tau < \infty. \tag{3.153}$$

Then the operator S is reduced to the convolution-type operator A. We have

$$Au(t) = \int_{-\infty}^{\infty} K(\tau - t)u(t)dt, \quad -\infty < t,\ \tau < \infty, \tag{3.154}$$

$$u(t) = n(e^t), \quad \kappa(x) = \frac{e^{-3x}}{2\sinh^2(\frac{e^{-x}}{2})}.$$

In addition, $u(t)$ and $Au(t) \in L_2(-\infty, \infty)$.

Note that after the substitution (3.153) the class of correctness G_r defined by formula (3.145) will move towards M_r. We have

$$M_r = \left\{ \|u(t)\|_{L_2} : u(t) \in W_2^1(-\infty, \infty), \int_{-\infty}^{\infty} u^2(t)dt + \int_{-\infty}^{\infty} |u'(t)|^2 dt \le r^2 \right\}. \tag{3.155}$$

Now define the modulus of continuity at zero of the operator A^{-1} on the set $N_r = AM_r$ by

$$\overline{\omega}(\delta, r) = \sup\{\|u(t)\|_{L_2} : u(t) \in M_r,\ \|Au(t)\|_{L_2} \le \delta\}. \tag{3.156}$$

Lemma 3.11. *Let $\omega(\delta, r)$ be defined by formula* (3.151) *and let $\overline{\omega}(\delta, r)$ be defined by formula* (3.156). *Then the following equality is true:*

$$\overline{\omega}(\delta, r) = \omega(\delta, r).$$

3.5.5 Estimation of the modulus of continuity $\overline{\omega}(\delta, r)$ defined by formula (3.156)

Assuming that $u(t) \in L_1(-\infty, \infty) \cap L_2(-\infty, \infty)$ and define the Fourier transform F as follows:

$$F[u(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(t)e^{ipt}dt. \tag{3.157}$$

It follows from the Plancherel theorem that the transform F is isometric on the space $L_2(-\infty, \infty)$. To distinguish a complex space from a real space, denote it by $\overline{L}_2(-\infty, \infty)$.

Thus, the operator F, defined by formula (3.157), will isometrically map the set $L_1(-\infty, \infty) \cap L_2(-\infty, \infty)$ into the space $L_2(-\infty, \infty)$ in the metrics $\overline{L}_2(-\infty, \infty)$.

Since the space $L_1(-\infty, \infty)$ is dense in $L_2(-\infty, \infty)$, extend the operator F by continuity onto the whole space $L_2(-\infty, \infty)$. Denote this extension by $\overline{F}$.

Now the operator F maps isometrically the space $L_2(-\infty, \infty)$ into $\overline{L}_2(-\infty, \infty)$. We will further denote the image of the operator $\overline{F}$ by Y and note that Y will be the subspace $\overline{L}_2(-\infty, \infty)$.

After the transformation $\overline{F}$ the operator A will be reduced to the following:

$$\hat{A}\hat{u}(p) = \hat{K}(p)\hat{u}(p), \quad \hat{u}(p) \in Y, \quad \hat{A}\hat{u}(p) \in \overline{L}_2(-\infty,\infty), \tag{3.158}$$

where

$$\hat{u}(p) = \overline{F}[u(t)].$$

Since $K(x) \in L_1(-\infty,\infty)$,

$$\hat{K}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} K(x)e^{ixp}\,dx$$

and from the form of the function $K(x)$ it will follow that

$$\hat{K}(p) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{e^{-(2-ip)x} \cdot e^{-x}}{(\coth(e^{-x}) - 1)}\,dx = -\sqrt{\frac{2}{\pi}} \int_{-\infty}^{\infty} \frac{e^{-(2-ip)x} \cdot e^{e^{-x}}}{(e^{e^{-x}} - 1)^2}\,d(e^{-x}).$$

Substituting $z = e^{-x}$ in the last expression, we obtain

$$\hat{K}(p) = -\sqrt{\frac{2}{\pi}} \int_{\infty}^{0} \frac{z^{(2-ip)} \cdot e^{z}}{(e^{z} - 1)^2}\,dz = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} \frac{z^{(2-ip)} \cdot e^{z}}{(e^{z} - 1)^2}\,dz.$$

Partially integrating the last expression we obtain

$$\hat{K}(p) = \frac{(2-ip)\sqrt{2}}{\sqrt{\pi}} \int_{0}^{\infty} \frac{z^{(2-ip)-1}}{e^{z} - 1}\,dz.$$

Using the properties of gamma and zeta functions [118] (p. 79),

$$\Gamma(s)\zeta(s) = \int_{0}^{\infty} \frac{z^{s-1}}{e^{z} - 1}\,dz,$$

we obtain

$$\hat{K}(p) = \sqrt{\frac{2}{\pi}}(2-ip)\Gamma(2-ip)\zeta(2-ip) = \sqrt{\frac{2}{\pi}}\Gamma(3-ip)\zeta(2-ip), \tag{3.159}$$

where $\Gamma(z)$ is the Euler gamma function and $\zeta(z)$ is the Riemann zeta function.

To estimate from below the function

$$|\hat{K}(p)| \quad \text{for } p \longrightarrow \infty,$$

we will give some well-known properties of the gamma function formulated in [118] (pp. 16 and 19). We write

$$\Gamma(z+1) = z\Gamma(z), \tag{3.160}$$

$$\overline{\Gamma}(z) = \Gamma(\overline{z}), \tag{3.161}$$

where $\overline{\Gamma}(z)$ is conjugated with $\Gamma(z)$, $\overline{z}$ is conjugated with z, and

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}. \tag{3.162}$$

It thus follows from (3.160) that

$$\left|\Gamma(3-ip)\right| = \sqrt{p^2+1}\sqrt{p^2+4}\left|\Gamma(1-ip)\right| \tag{3.163}$$

and it follows from (3.161) and (3.162) that

$$\left|\Gamma(1-ip)\right| = \sqrt{\frac{\pi p}{\sinh \pi p}}. \tag{3.164}$$

It follows from (3.163) and (3.164) that for any $p \geq 2$ the following estimate is true:

$$\left|\Gamma(3-ip)\right| \geq \sqrt{2\pi} e^{-\frac{\pi}{2}p}. \tag{3.165}$$

We will now estimate from below the modulus of the Riemann zeta function $|\zeta(2-ip)|$. Since

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{3.166}$$

it follows from (3.166) that

$$\zeta(2-ip) = \sum_{n=1}^{\infty} \frac{e^{ip\ln n}}{n^2}. \tag{3.167}$$

Taking into account that $|e^{ip\ln n}| = 1$, from (3.167) we obtain

$$\left|\zeta(2-ip)\right| \leq 1 - \sum_{n=2}^{\infty} \frac{1}{n^2} \geq \frac{1}{3}. \tag{3.168}$$

Thus, from (3.165) and (3.168) it follows that for $p \geq 2$ the following estimate from below is true:

$$\left|\hat{K}(p)\right| \geq \frac{2}{3} e^{-\frac{\pi}{2}p}. \tag{3.169}$$

Now consider the extension $\hat{A}_1$ of the operator $\hat{A}$, defined by formula (3.158), onto the whole space $\overline{L}_2(-\infty,\infty)$. We have

$$\hat{A}_1\hat{u}(p) = \hat{K}(p)\hat{u}(p), \quad \hat{u}(p), \hat{A}_1\hat{u}(p) \in \overline{L}_2(-\infty,\infty). \tag{3.170}$$

Consider a set $\hat{M}_r \subset \overline{L}_2(-\infty,\infty)$ defined by the formula

$$\hat{M}_r = \left\{\hat{u}(p) : \hat{u}(p), p\hat{u} \in \overline{L}_2(-\infty,\infty), \int_{-\infty}^{\infty}(1+p^2)|\hat{u}(p)|^2 dp \le r^2\right\}. \tag{3.171}$$

From (3.155) and (3.171) it follows that

$$\overline{F}[M_r] \subset \hat{M}_r. \tag{3.172}$$

Consider moduli of continuity at zero defined by the formulas

$$\hat{\omega}(\delta,r) = \sup\{\|\hat{u}(p)\|_{\overline{L}_2} : \hat{u}(p) \in \overline{F}[M_r],\ \|\hat{A}\hat{u}(p)\|_{\overline{L}_2} \le \delta\}, \tag{3.173}$$

$$\hat{\omega}_1(\delta,r) = \sup\{\|\hat{u}(p)\|_{\overline{L}_2} : \hat{u}(p) \in \overline{F}[M_r],\ \|\hat{A}_1\hat{u}(p)\|_{\overline{L}_2} \le \delta\}. \tag{3.174}$$

It follows from the unitary transformation $\overline{F}$ and formulas (3.154), (3.156), (3.158), and (3.173) that

$$\hat{\omega}(\delta,r) = \omega(\delta,r). \tag{3.175}$$

It follows from (3.158), (3.170), and (3.172)–(3.174) that

$$\hat{\omega}_1(\delta,r) \ge \hat{\omega}(\delta,r). \tag{3.176}$$

Thus, it follows from (3.175) and (3.176) that

$$\hat{\omega}(\delta,r) \le \hat{\omega}_1(\delta,r). \tag{3.177}$$

For the sake of convenience substitute the operator $\hat{A}_1$ defined by formula (3.170) by the inverse operator $\hat{A}_1^{-1}$, which we denote by $\hat{T}_1$. We have

$$\hat{T}_1\hat{f}(p) = \hat{A}_1^{-1}\hat{f}(p), \quad \hat{f}(p) \in R(\hat{A}_1),\ \hat{T}_1\hat{f}(p) \in \overline{L}_2, \tag{3.178}$$

where $R(\hat{A}_1)$ is the value range of the operator $\hat{A}_1$.

Define the set $\hat{M}_r$ defined by formula (3.172) with the operator B as follows:

$$B\hat{u}(p) = \sqrt{1+p^2}\hat{u}(p), \quad \hat{u}(p), B\hat{u}(p) \in \overline{L}_2(-\infty,\infty), \tag{3.179}$$

$$\hat{M}_r = B^{-1}\overline{S}_r, \tag{3.180}$$

where

$$\overline{S}_r = \{\hat{u}(p) : \hat{u}(p) \in \overline{L}_2(-\infty, \infty),\ \|\hat{u}(p)\|_{\overline{L}_2} \le r\}.$$

On the set $\overline{L}_2(-\infty, \infty)$ introduce the set $\hat{N}_r$ defined by the formula

$$\hat{N}_r^1 = T_1^{-1}(\hat{M}_r). \tag{3.181}$$

Then it follows from (3.171), (3.174), and (3.178)–(3.181) that

$$\hat{\omega}_1(\delta, r) = \sup\{\|\hat{T}_1\hat{f}(p)\| : \hat{f}(p) \in \hat{N}_r^1,\ \|\hat{f}(p)\|_{\overline{L}_2} \le \delta\}. \tag{3.182}$$

We continue with the estimation of the modulus of continuity $\hat{\omega}_1(\delta, r)$ defined by (3.182).

For this purpose consider the operator $\hat{T}$ acting from $\overline{L}_2(-\infty, \infty)$ into $\overline{L}_2(-\infty, \infty)$ defined by the formula

$$\hat{T}\hat{f}(p) = g(p)\hat{f}(p), \tag{3.183}$$

where

$$\begin{aligned} &g(p) \in C(-\infty, \infty), \quad g(-p) = g(p), \quad g(0) > 0, \\ &\lim_{p\to\infty} g(p) = \infty, \quad \text{and} \quad g(p) \text{ increases on } [0, \infty). \end{aligned} \tag{3.184}$$

Define by $\hat{\omega}_2(\delta, r)$ the modulus of continuity at zero of the operator $\hat{T}$ on the set $\hat{N}_r = \hat{T}^{-1}(\hat{M}_r)$ and let $\hat{M}_r$ be defined by (3.180). Then consider the equation

$$\frac{r}{\sqrt{1+p^2}} = g(p)\delta. \tag{3.185}$$

If $g(0)\delta < r$, then equation (3.185) has a unique positive root $\overline{p}$.

It follows from Lemma 1.11 that

$$\hat{\omega}_2(\delta, r) = \frac{r}{\sqrt{1+\overline{p}^2}}. \tag{3.186}$$

Assume that the operator $\hat{T}_1$ is defined by formulas (3.170) and (3.178). In addition $\hat{T}$ is defined by formula (3.183).

Then the following lemma is true.

Lemma 3.12. *If $g(p)$ satisfies* (3.184) *and there exists $p_0 \ge 0$ such that for any $p \ge p_0$ we have*

$$|\hat{K}(p)|^{-1} \le g(p),$$

then, if

$$g(p_0)\delta < \frac{r}{\sqrt{1+p_0^2}},$$

the following estimate is true:

$$\hat{\omega}_1(\delta, r) \le \hat{\omega}_2(\delta, r).$$

We will now use Lemma 3.12 to estimate the accuracy of the method $\{R_\delta : 0 < \delta \le \delta_0\}$. It follows from (3.169) that for $p \ge 2$

$$\left|\hat{K}(p)\right|^{-1} \le \frac{3}{2} e^{\frac{\pi}{2}p}. \tag{3.187}$$

Thus, it follows from (3.152), (3.175), (3.176), (3.186), and (3.187) and from Lemma 3.12 that, if

$$\delta_0 = \frac{2re^{-\pi}}{3\sqrt{5}}$$

for the method $\{R_\delta : 0 < \delta \le \delta_0\}$, then by (3.152) and (3.177) the following estimate is true:

$$\Delta_\delta(R_\delta) = \frac{2r}{\sqrt{1 + \frac{1}{\pi^2}\ln^2(\frac{2r}{3\delta})}}.$$

4 Projection-regularization method

4.1 Posing of the problem of unbounded operator values and the projection-regularization method

4.1.1 Posing of the problem

Let $\mathbb{U}$, $\mathbb{F}$, and $\mathbb{V}$ be Hilbert spaces, let T be a closed linear operator with the domain $D(T) \subset \mathbb{F}$ and the range $R(T) \subset \mathbb{U}$, and let B be an injective linear unbounded operator with the domain $D(B) \subset \mathbb{U}$ and the range $R(B) \subset \mathbb{V}$. Assume that the set $D(T)$ is dense in $\mathbb{F}$, $R(B)$ is dense in $\mathbb{V}$, and $R(T) \cap D(B)$ is dense in $\mathbb{U}$.

Denote by M_r the set defined by the formula

$$M_r = \{u : u \in R(T) \cap D(B),\ \|Bu\| \leqslant r\}. \tag{4.1}$$

Consider the problem of finding the value Tf_0 of the operator T at the point $f_0 \in D(T)$, where

$$Tf = u. \tag{4.2}$$

Assume that for $f = f_0$ the element $u_0 = Tf_0$ belongs to the set M_r, but the exact value of f_0 is unknown. Instead, the element $f_\delta \in \mathbb{F}$ and the error level $\delta > 0$ are given, such that

$$\|f_\delta - f_0\| \leq \delta. \tag{4.3}$$

Using the a priori information f_δ, δ, and M_r it is required to find the approximate solution $u_\delta \in \mathbb{U}$ of problem (4.2) and estimate its deviation $\|u_\delta - u_0\|$ from the exact solution u_0.

4.1.2 Basic notions

Definition 4.1. A set M_r is called the class of correctness for problem (4.2), if the restriction of the operator T on the set $T^{-1}(M_r)$ is uniformly continuous.

Following [44] we will call the problem of finding the unbounded operator T a conditionally well-posed problem if we know the class of correctness M_r, to which the exact value u_0 of the operator T belongs.

Definition 4.2. A family $\{T_\delta : 0 < \delta \leq \delta_0\}$ of linear bounded operators T_δ, mapping the space $\mathbb{F}$ into $\mathbb{U}$, is called the linear method of solving problem (4.2) if

$$\Delta_\delta[T_\delta] \to 0 \quad \text{for } \delta \to 0,$$

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where

$$\Delta_\delta[T_\delta] = \sup\{\|T_\delta f_\delta - Tf_0\| : f_0 \in T^{-1}(M_r),\ \|f_\delta - f_0\| \le \delta\} \quad [84].$$

One of the ways of posing the linear method consists of using the regularizing family of the operators $\{T_\alpha : \alpha > 0\}$.

Definition 4.3. A family $\{T_\alpha : 0 \le \alpha < \alpha_0\}$ of linear bounded operators T_α, mapping a space $\mathbb{F}$ into $\mathbb{U}$, is called a family regularizing the operator T if for any $f \in D(T)$

$$T_\alpha f \longrightarrow Tf \quad \text{for } \alpha \longrightarrow \alpha_0.$$

Definition 4.4. A regularizing family $\{T_\alpha : 0 \le \alpha < \alpha_0\}$ is called a family uniformly regularizing the operator T over the set M_r, if

$$\omega(\alpha) \longrightarrow 0 \quad \text{for } \alpha \longrightarrow \alpha_0,$$

where

$$\omega(\alpha) = \sup\{\|T_\alpha f_0 - Tf_0\| : Tf_0 \in M_r\} \quad [94].$$

Consider the equation

$$\omega(\alpha) = \|T_\alpha\|\delta. \tag{4.4}$$

In [94] it is proved that, if

$$\omega(\alpha) \in C[0,\alpha_0), \quad \|T_\alpha\| \in C[0,\alpha_0), \quad \omega(\alpha), \|T_\alpha\|^{-1} \longrightarrow 0 \quad \text{at } \alpha \longrightarrow 0,$$
$$\delta \in (0,\delta_0], \quad \text{and} \quad \omega(0) > \|T_0\|\delta_0,$$

then equation (4.4) has a solution $\alpha = \alpha(\delta)$. If equation (4.4) has multiple solutions, then any of the solutions can be used.

Consider the linear method $\{T_\delta : 0 < \delta \le \delta_0\}$ of solving problem (4.2) and a function $\Delta(\delta) : \delta \in (0,\delta_0]$ such that $\Delta(\delta) \to 0$ for $\delta \to 0$.

Assume that there exists a number $\overline{b} > 0$ such that for any $\delta \in (0,\delta_0]$ the relation

$$\Delta_\delta[T_\delta] \le \overline{b}\Delta(\delta) \tag{4.5}$$

is true.

Then the value $\overline{b}\Delta(\delta)$ is called the error estimate for the method $\{T_\delta : 0 < \delta \le \delta_0\}$ on the set M_r. If there exists a number $\overline{b}_1 > 0$ such that for any $\delta \in (0,\delta_0]$

$$\Delta_\delta[T_\delta] \ge \overline{b}_1\Delta(\delta),$$

then the error estimate (4.5) is called accurate-by-order.

We will now consider a family of linear bounded operators $\{T_\alpha : 0 \le \alpha < \alpha_0\}$ uniformly regularizing the operator T over the set M_r and define the function $\mu(\delta)$ as follows:

$$\mu(\delta) = \inf\{\Delta_\delta[T_\alpha] : 0 \le \alpha < \alpha_0\},$$

where

$$\Delta_\delta[T_\alpha] = \sup\{\|T_\alpha f_\delta - Tf_0\| : f_0 \in T^{-1}(M_r),\ \|f_\delta - f_0\| \le \delta\}.$$

Then we will call the dependence $\alpha = \alpha(\delta)$ quasi-optimal if there exists a number $\overline{b}_2 > 0$ such that for any $\delta \in (0, \delta_0]$

$$\Delta_\delta[T_{\alpha(\delta)}] \le \overline{b}_2\mu(\delta).$$

Denote by $B[\mathbb{F} \text{ in } \mathbb{U}]$ a space of linear bounded operators mapping $\mathbb{F}$ into $\mathbb{U}$ and by $\Delta_\delta^{\text{opt}}$ the value

$$\Delta_\delta^{\text{opt}} = \inf\{\Delta_\delta[P] : P \in B[\mathbb{F}, \mathbb{U}]\},$$

where

$$\Delta_\delta[P] = \sup\{\|Tf_0 - Pf_\delta\| : f_0 \in T^{-1}(M_r),\ \|f_\delta - f_0\| \le \delta\}.$$

Definition 4.5. A method $\{T_\delta^{\text{opt}} : 0 < \delta \le \delta_0\}$ is called optimal on a class M_r, if for any $\delta \in (0, \delta_0]$

$$\Delta_\delta[T_\delta^{\text{opt}}] = \Delta_\delta^{\text{opt}}.$$

Definition 4.6. A method $\{\overline{T}_\delta : 0 < \delta \le \delta_0\}$ is called optimal-by-order on a class M_r, if there exists a number κ such that for any $\delta \in (0, \delta_0]$

$$\Delta_\delta[\overline{T}_\delta] \le \kappa\Delta_\delta^{\text{opt}}.$$

Following [33] we define the modulus of continuity at zero of the operator T restriction on the set $T^{-1}(M_r)$ as follows:

$$\omega(\tau, r) = \sup\{\|Tf\| : f \in T^{-1}(M_r),\ \|f\| \le \tau\}.$$

It is known [28] that $\Delta_\delta^{\text{opt}} \ge \omega(\delta, r)$.

Let

$$\mathbb{U} = \mathbb{F} = \mathbb{V} = \mathbb{H},$$

where $\mathbb{H}$ is a Hilbert space and T and B are injective closed linear operators in $\mathbb{H}$, satisfying the following properties:

$$\overline{D(T)} = \overline{D(B)} = \overline{R(T)} = \overline{R(B)} = \mathbb{H}, \tag{4.6}$$

where $\overline{D(T)}$, $\overline{D(B)}$ are closures in $\mathbb{H}$ of the corresponding domains $D(T)$ and $D(B)$ of the operators T and B, while $\overline{R(T)}$ and $\overline{R(B)}$ are the closures of the corresponding value ranges of said operators.

From the theorem proved in [66] (p. 325) it follows that for the operators T and B there hold polar decompositions, where

$$B = \overline{B}P \quad \text{and} \quad T = Q\overline{T},$$

where

$$\overline{B} = \sqrt{BB^*}, \quad \overline{T} = \sqrt{T^*T},$$

while P and Q are unitary operators.

In addition let

$$\overline{B} = G(\overline{T}), \tag{4.7}$$

where the spectrum

$$\mathrm{Sp}(\overline{T}) = [a, \infty),$$

$G(\sigma) \in C^1[a, \infty)$, and for any $\sigma \in [a, \infty)$

$$G'(\sigma) > 0, \quad \lim_{\sigma\to\infty} G(\sigma) = \infty.$$

Consider the equation

$$\sigma G(\sigma) = \frac{r}{\tau}, \tag{4.8}$$

that has the unique solution $\overline{\sigma}(\tau, r)$ if $\frac{r}{\tau} > aG(a)$. From [72] it follows that under the above conditions

$$\omega(\tau, r) = \frac{r}{G(\overline{\sigma}(\tau, r))}, \quad \Delta_\delta^{\text{opt}} = \omega(\tau, r).$$

4.1.3 Projection-regularization method

Assume that the function $G(\sigma)$ in formula (4.7) where $G(\sigma)$ is strictly increasing is continuous over $[a, \infty)$ such that

$$\lim_{\sigma\to\infty} G(\sigma) = \infty.$$

Then the problem of finding the values of the operator T, (4.2), can be substituted by the equivalent problem

$$\overline{T}g = u, \tag{4.9}$$

where $g = Q^* f$, and the set M_r can be defined by the formula

$$M_r = \{u : u \in D(\overline{B}),\ \|\overline{B}u\| \le r\}. \tag{4.10}$$

Assume that it is required to define the value of Tg_0 that belongs to M_r, but the exact value g_0 is not known. Instead, we have a certain approximation $g_\delta \in \mathbb{H}$ and error level $\delta > 0$ such that

$$\|g_\delta - g_0\| \le \delta.$$

Using the initial data of M_r, g_δ, and δ it is required to define the approximate value of u_δ for problem (4.9) and to estimate the deviation u_δ from u_0.

The projection-regularization method [28] uses a regularizing set of operators $\{\overline{T}_\alpha : a \le \alpha < \infty\}$, acting from $\mathbb{H}$ into $\mathbb{H}$ defined by the formula

$$\overline{T}_\alpha g = \int_a^\alpha \sigma dE_\sigma g, \quad \alpha \in [a, \infty), \tag{4.11}$$

where $\{E_\sigma : a \le \sigma < \infty\}$ is the spectral decomposition of the unit E, generated by the operator $\overline{T}$.

We will define the approximate solution of problem (4.9) by the formula

$$u_\delta^\alpha = \overline{T}_\alpha g_\delta. \tag{4.12}$$

Now select the parameter $\alpha = \alpha(\delta)$ in formula (4.12).

For this purpose consider

$$\|u_\delta^\alpha - u_0\|^2 = \|\overline{T}_\alpha g_\delta - u_0\|^2. \tag{4.13}$$

It follows from (4.13) that

$$\|u_\delta^\alpha - u_0\|^2 = \|u_\delta^\alpha - u_0^\alpha\|^2 + \|u_0^\alpha - u_0\|^2 + 2(u_\delta^\alpha - u_0^\alpha, u_0^\alpha - u_0), \tag{4.14}$$

where $u_0^\alpha = \overline{T}_\alpha g_0$.

Since

$$\mathbb{H} = \mathbb{H}_\alpha + \mathbb{H}_\alpha^\perp, \quad \text{where } \mathbb{H}_\alpha = E_\alpha \mathbb{H},$$

and since it follows from (4.11) and (4.12) that $u_0^\alpha - u_0 \in \mathbb{H}_\alpha$, $u_\delta^\alpha - u_0^\alpha \in \mathbb{H}_\alpha^\perp$, we have

$$(u_\delta^\alpha - u_0^\alpha, u_0^\alpha - u_0) = 0.$$

Thus, it follows from (4.14) that

$$\|u_\delta^\alpha - u_0\|^2 = \|u_\delta^\alpha - u_0^\alpha\|^2 + \|u_0^\alpha - u_0\|^2. \tag{4.15}$$

Now we introduce the following quantities:

$$\Delta(\alpha,\delta) = \sup\{\|\overline{T}_\alpha g_\delta - \overline{T} g_0\| : g_0 \in \overline{T}^{-1}(M_r),\ \|g_\delta - g_0\| \le \delta\}, \tag{4.16}$$

$$\Delta_1(\alpha) = \sup\{\|\overline{T}_\alpha g_0 - \overline{T} g_0\| : g_0 \in \overline{T}^{-1}(M_r)\}, \tag{4.17}$$

and

$$\Delta_2(\alpha,\delta) = \sup\{\|\overline{T}_\alpha g_\delta - \overline{T}_\alpha g_0\| : g_0 \in \overline{T}^{-1}(M_r),\ \|g_\delta - g_0\| \le \delta\}. \tag{4.18}$$

Then it follows from (4.15)–(4.18) that

$$\Delta^2(\alpha,\delta) \le \Delta_1^2(\alpha) + \Delta_2^2(\alpha,\delta). \tag{4.19}$$

It follows from (4.18) that

$$\Delta_2(\alpha,\delta) \le \|\overline{T}_\alpha\|\delta. \tag{4.20}$$

It follows from (4.19) and (4.20) that

$$\Delta^2(\alpha,\delta) \le \Delta_1^2(\alpha) + \|\overline{T}_\alpha\|^2\delta^2. \tag{4.21}$$

Lemma 4.1. *We have the equality* $\|\overline{T}_\alpha\| = \alpha$.

Proof. It follows from (4.11) that $\|\overline{T}_\alpha\| \le \alpha$, but since α belongs to the spectrum $\mathrm{Sp}(\overline{T}_\alpha)$ of the operator $\overline{T}_\alpha$, we have $\|\overline{T}_\alpha\| = \alpha$. □

Lemma 4.2. *We have the equality*

$$\Delta_1(\alpha) = \frac{r}{G(\alpha)}.$$

Proof. It follows from (4.17) that

$$\Delta_1^2(\alpha) = \sup_{v_0}\left\{\int_\alpha^\infty G^{-2}(\sigma)d(E_\sigma v_0, v_0) : \|v_0\| \le r\right\}. \tag{4.22}$$

It follows from (4.22) and from the properties of the function $G(\sigma)$ that

$$\Delta_1^2(\alpha) \le \frac{1}{G^2(\alpha)} \sup \int_\alpha^\infty d(E_\sigma v_0, v_0) \le \frac{r^2}{G^2(\alpha)}. \tag{4.23}$$

Since $G^{-2} \in C[\alpha,\infty)$, for any $\varepsilon > 0$ there exists $\mu > 0$ such that, for any σ such that $0 \le \sigma - \alpha \le \mu$, we have

$$0 \le G^{-2}(\alpha) - G^{-2}(\sigma) \le \frac{\varepsilon}{r^2}. \tag{4.24}$$

It follows from (4.24) that there exists an element $\overline{v}_0 \in (E_{\alpha+\mu} - E_\alpha)H$ such that $\|\overline{v}_0\| = r$ and

$$\|\overline{B}^{-1}\overline{v}_0\|^2 \geq \frac{r^2}{G^2(\alpha)} - \varepsilon. \tag{4.25}$$

Since $\|\overline{B}^{-1}\overline{v}_0\|^2 \leq \Delta_1^2(\alpha)$, it follows from (4.25) that $\Delta_1^2(\alpha) \geq \frac{r^2}{G^2(\alpha)} - \varepsilon$ and due to the arbitrariness of ε

$$\Delta_1^2(\alpha) \geq \frac{r^2}{G^2(\alpha)}. \tag{4.26}$$

From relations (4.23) and (4.26) it follows that the lemma is proved. □

Thus, it follows from (4.21) and Lemmas 4.1 and 4.2 that

$$\Delta^2(\alpha, \delta) \leq \frac{r^2}{G^2(\alpha)} + \delta^2\alpha^2. \tag{4.27}$$

We will now obtain a reverse inequality. For this purpose we will use the fact that $G^{-2}(\sigma) \in C[a, \infty)$ and $\sigma^2 \in C[a, \infty)$, whence for any $\varepsilon > 0$ there exists $\mu_1 > 0$ such that for any σ satisfying the condition $0 \leq \sigma - \alpha \leq \mu_1$ it follows that

$$0 \leq G^{-2}(\alpha) - G^{-2}(\sigma) \leq \frac{\varepsilon}{2r^2}. \tag{4.28}$$

Similarly, for any σ such that $0 \leq \alpha - \sigma \leq \mu_1$ it follows that

$$\alpha^2 - \sigma^2 \leq \frac{\varepsilon}{\delta^2}. \tag{4.29}$$

It follows from (4.28) that there exists an element

$$\overline{v}_0 \in (E_{\alpha+\mu_1} - E_\alpha)H \quad \text{and} \quad \|\overline{v}_0\| = r$$

such that for the element

$$\overline{u}_0 = B^{-1}\overline{v}_0 \quad \text{and} \quad \overline{u}_0^\alpha = \overline{T}^\alpha \overline{T}^{-1} \overline{B}^{-1} \overline{v}_0$$

we have the relation

$$\|\overline{u}_0^\alpha - \overline{u}_0\|^2 \geq \frac{r^2}{G^2(\alpha)} - \frac{\varepsilon}{2}. \tag{4.30}$$

Similarly, there exists an element $\delta\overline{g} \in (E_\alpha - E_{\alpha-\mu_1})H$ and $\|\delta\overline{g}\| = \delta$ such that for the elements

$$\overline{g}_\delta = \overline{T}^{-1}\overline{u}_0 + \delta\overline{g} \quad \text{and} \quad \overline{u}_\delta^\alpha = \overline{T}_\alpha \overline{g}_\delta$$

we have the following relation:

$$\|\overline{u}_\delta^\alpha - \overline{u}_0^\alpha\|^2 \geq \alpha^2\delta^2 - \frac{\varepsilon}{2}. \tag{4.31}$$

It follows from (4.15), (4.30), and (4.31) that

$$\|\overline{u}_\delta^\alpha - \overline{u}_0\|^2 \geq \frac{r^2}{G^2(\alpha)} + \delta^2\alpha^2 - \varepsilon \tag{4.32}$$

and it follows from (4.16) and (4.32) that

$$\Delta^2(\alpha, \delta) \geq \frac{r^2}{G^2(\alpha)} + \delta^2\alpha^2 - \varepsilon. \tag{4.33}$$

Due to the arbitrariness of ε it follows from (4.33) that

$$\Delta^2(\alpha, \delta) \geq \frac{r^2}{G^2(\alpha)} + \delta^2\alpha^2 \tag{4.34}$$

and it follows from (4.27) and (4.34) that

$$\Delta^2(\alpha, \delta) = \frac{r^2}{G^2(\alpha)} + \delta^2\alpha^2. \tag{4.35}$$

We will define the regularization parameter $\overline{\alpha} = \overline{\alpha}(\delta)$ from the equation

$$\alpha G(\alpha) = \frac{r}{\delta}. \tag{4.36}$$

It follows from the properties of the function $G(\alpha)$ that for $\frac{r}{\delta} > aG(a)$ equation (4.36) has the unique solution $\overline{\alpha}(\delta)$.

Thus, the regularizing family $\{\overline{T}_\alpha : \alpha \geq a\}$ of the linear bounded operators $\overline{T}_\alpha$, defined by formula (4.11), and the dependence $\overline{\alpha} = \overline{\alpha}(\delta)$, defined by equation (4.36), give the method $\{\overline{T}_{\overline{\alpha}(\delta)} : 0 < \delta < r/aG(a)\}$ of projection regularization and for this method we have the exact error estimate

$$\Delta_\delta[\overline{P}_{\overline{\alpha}(\delta)}] = \frac{\sqrt{2}r}{G(\overline{\alpha}(\delta))}. \tag{4.37}$$

Theorem 4.1. *If $G(\sigma) \in C^1[a, \infty)$, for any $\sigma \in [a, \infty)$*

$$G'(\sigma) > 0 \quad \text{and} \quad G(\sigma) \to \infty \quad \text{for } \sigma \to \infty,$$

then, if $\frac{r}{\delta} > aG(a)$, the projection-regularization method $\{\overline{T}_{\alpha(\delta)} : 0 < \delta < r/aG(a)\}$, defined by formulas (4.11) and (4.36), is optimal-by-order with the constant $\sqrt{2}$ and for this method we have the exact error estimate

$$\Delta_\delta[\overline{T}_{\overline{\alpha}_\delta}] = \sqrt{2}\Delta_\delta^{\text{opt}}.$$

The proof of the theorem follows from relations (4.8), (4.36), and (4.37).

4.2 Isometry of the Fourier transform on the space $L_2[0,\infty)$

Let $f(t) \in L_1(-\infty,\infty)$. Then the Fourier transform $\hat{f}(\tau)$ is defined by the formula

$$\hat{f}(\tau) = F[f(t)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\tau t} dt, \quad \tau \in \mathbb{R}. \tag{4.38}$$

It is well known that

$$\hat{f}(\tau) \in C_0(-\infty,\infty) \quad \text{and} \quad |\hat{f}(\tau)| \le \int_{-\infty}^{\infty} |f(t)| dt.$$

Thus, the operator F, defined by formula (4.38), is a linear bounded operator mapping the space $L_1(-\infty,\infty)$ into $C_0(-\infty,\infty)$. In addition, the inverse operator F^{-1} is defined by the formula

$$f(t) = F^{-1}[\hat{f}(\tau)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\tau) e^{it\tau} d\tau$$

and is a linear unbounded operator acting from the space $C_0(-\infty,\infty)$ into $L_1(-\infty,\infty)$.

If the function $f(t) \in L_2(-\infty,\infty)$, then the Fourier transform F of this function in the sense of definition (4.38) generally speaking is meaningless. Using the well-known Plancherel theorem (see [39] (p. 412)), it is possible to extend the Fourier transform F to the space $L_2(-\infty,\infty)$.

Let $L_2(-\infty,\infty)$ be a complex space.

Theorem 4.2 (Plancherel). *For any function $f(t) \in L_2(-\infty,\infty)$ for any N the integral*

$$g_N(\tau) = \frac{1}{\sqrt{2\pi}} \int_{-N}^{N} f(t) e^{-it\tau} dt$$

belongs to the space $L_2(-\infty,\infty)$. For $N \longrightarrow \infty$ the sequence of the functions $g_N(\tau)$ converges in the metrics of the space $L_2(-\infty,\infty)$ to a certain limit $g(\tau)$ and

$$\int_{-\infty}^{\infty} |g(\tau)|^2 d\tau = \int_{-\infty}^{\infty} |f(t)|^2 dt.$$

This function $g(\tau)$ is called the Fourier transform of the function $f(t) \in L_2(-\infty,\infty)$. If the function $f(t)$ also belongs to $L_1(-\infty,\infty)$, then the corresponding function $g(\tau)$ coincides with the Fourier transform of the function $f(t)$ in the sense of definition (4.38).

Thus, from the Plancherel theorem it follows that the Fourier transform of F, extended onto the space $L_2(-\infty,\infty)$, maps this space into the space $L_2(-\infty,\infty)$, in the isometric way.

Let

$$\overline{H} = L_2[0,\infty) + iL_2[0,\infty)$$

over the field of complex numbers and let $L_2[0,\infty)$ be a real space. Assume that $f(t) \in L_2[0,\infty) \cap L_1[0,\infty)$ and define the Fourier transform of F, acting from the space $L_2[0,\infty)$ into $\overline{H}$, by the formula

$$\hat{f}(\tau) = F[f(t)] = \frac{1}{\sqrt{\pi}} \int_0^\infty f(t)e^{-i\tau t}dt, \quad \tau \geq 0. \tag{4.39}$$

Lemma 4.3. *The operator F, defined by formula* (4.39) *and acting from the space $L_2[0,\infty)$ into $\overline{H}$, is isometric.*

Proof. Let $f(t) \in L_2[0,\infty) \cap L_1[0,\infty)$. Extend this function to the negative semi-axis assuming that

$$f(t) = 0 \quad \text{at } t < 0. \tag{4.40}$$

Thus, $f(t) \in L_2(-\infty,\infty) \cap L_1(-\infty,\infty)$. Denote by $\bar{f}(\tau)$ the Fourier transform of the following function $f(t)$:

$$\bar{f}(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)e^{-it\tau}d\tau, \quad -\infty < \tau < \infty. \tag{4.41}$$

It follows from the Plancherel theorem that

$$\|\bar{f}(\tau)\|_{L_2(-\infty,\infty)} = \|f(t)\|_{L_2[0,\infty)}. \tag{4.42}$$

It follows from (4.40) and (4.41) that

$$\bar{f}(\tau) = \begin{cases} \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)e^{-it\tau}dt, & \tau \geq 0, \\ \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t)e^{it|\tau|}dt, & \tau < 0. \end{cases} \tag{4.43}$$

It follows from (4.43) that

$$\|\bar{f}(\tau)\|^2_{L_2(-\infty,\infty)} = \int_0^\infty |\bar{f}(\tau)|^2 d\tau + \int_0^\infty |\overline{\bar{f}(\tau)}|^2 d\tau, \tag{4.44}$$

where $\overline{\bar{f}(\tau)}$ is a function conjugate with $\bar{f}(\tau)$.

Since for any $\tau \geq 0$

$$|\overline{\bar{f}(\tau)}|^2 = |\bar{f}(\tau)|^2,$$

we obtain from (4.44) that

$$\|\bar{f}(\tau)\|^2_{L_2(-\infty,\infty)} = 2\int_0^\infty |\bar{f}(\tau)|^2 d\tau. \tag{4.45}$$

It follows from (4.42) that

$$\|f(t)\|^2_{L_2[0,\infty)} = \|\bar{f}(\tau)\|^2_{L_2(-\infty,\infty)} \tag{4.46}$$

and it follows from (4.39), (4.41), and (4.45) that

$$\|\bar{f}(\tau)\|^2_{L_2(-\infty,\infty)} = \|\hat{f}(\tau)\|^2_{L_2[0,\infty)}. \tag{4.47}$$

The assertion of the lemma follows from (4.46) and (4.47). □

It follows from Lemma 4.3 that the transformation of F can be expanded by continuity over the whole of the space $L_2[0,\infty)$. It will then isometrically map the space $L_2[0,\infty)$ into $\overline{H}$.

5 Inverse heat exchange problems

5.1 A study of the inverse boundary-value problem for the heat conduction equation with a constant coefficient

5.1.1 Problem posing

Let a thermal process be described by the equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1,\ t > 0, \tag{5.1}$$

where the solution $u(x,t) \in C([0,1]\times[0,\infty)) \cap C^{2,1}((0,1)\times(0,\infty))$ satisfies the following initial and boundary conditions:

$$u(x,0) = 0, \quad 0 \le x \le 1, \tag{5.2}$$

$$u(0,t) = h(t), \quad t \ge 0, \tag{5.3}$$

and

$$\frac{\partial u(1,t)}{\partial x} + \kappa u(1,t) = 0, \quad \kappa > 0,\ t \ge 0, \tag{5.4}$$

where

$$h(t) \in C^2[0,\infty), \quad h(0) = h'(0) = 0. \tag{5.5}$$

Also, let there exist a number $t_0 > 0$ such that for any $t \ge t_0$

$$h(t) = 0. \tag{5.6}$$

5.1.2 A study of the smoothness of the function $u(x,t)$

Let us make the substitution

$$v(x,t) = u(x,t) + \left[\frac{\kappa}{\kappa+1}x - 1\right]h(t). \tag{5.7}$$

Then

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} + \left[\frac{\kappa}{\kappa+1}x - 1\right]h'(t), \quad 0 < x < 1,\ t > 0, \tag{5.8}$$

$$v(x,0) = 0, \quad 0 \le x \le 1, \tag{5.9}$$

$$v(0,t) = 0, \quad t \ge 0, \tag{5.10}$$

$$v'_x(1,t) + \kappa v(1,t) = 0, \quad t \ge 0. \tag{5.11}$$

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The solution of problem (5.8)–(5.11) is as follows:

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t) \sin \lambda_n x, \tag{5.12}$$

where λ_n are positive solutions of the equation

$$\tan \lambda = -\frac{\lambda}{\kappa}, \tag{5.13}$$

$$\int_0^1 \sin^2 \lambda_n x dx = \frac{2\lambda_n - \sin 2\lambda_n}{4\lambda_n}, \tag{5.14}$$

and

$$v_n(t) = 2b_n \int_0^t e^{-\lambda_n^2(t-\tau)} h'(\tau) d\tau, \tag{5.15}$$

where

$$b_n = -\frac{4}{2\lambda_n - \sin 2\lambda_n}. \tag{5.16}$$

By partially integrating the right-hand side of equation (5.15) and taking into account (5.5), we obtain

$$v_n(t) = \frac{2b_n}{\lambda_n^2} \left[h'(t) - \int_0^t e^{-\lambda_n^2(t-\tau)} h''(\tau) d\tau \right]. \tag{5.17}$$

Lemma 5.1. *Let* $u(x,t)$ *be a solution of problem* (5.1)–(5.4), *defined by formulas* (5.12)–(5.16). *Then*

$$u(x,t) \to 0 \quad \textit{for } t \to 0$$

is uniform over the interval $[0,1]$.

Proof. Let us denote by r_1 the number defined by the formula

$$r_1 = \max_{t \in [0,t_0]} \left(|h(t)| + |h'(t)| + |h''(t)| \right). \tag{5.18}$$

The following estimate is true for the general term of series (5.12):

$$|v_n(t) \sin \lambda_n x| \le |v_n(t)|. \tag{5.19}$$

It follows from (5.16) and (5.17) that

$$|v_n(t)| \le \frac{8}{\lambda_n^2 (2\lambda_n - 1)} \left[|h'(t)| + \max_{0 \le \tau \le t} e^{-\lambda_n^2(t-\tau)} \int_0^t |h''(\tau)| d\tau \right]. \tag{5.20}$$

Since

$$\max_{0\le\tau\le t} e^{-\lambda_n^2(t-\tau)} \le 1,$$

it follows from (5.18) and (5.20) that

$$|v_n(t)| \le \frac{16 r_1 t_0}{\lambda_n^2(2\lambda_n - 1)}. \tag{5.21}$$

If follows from (5.13) that for any n

$$\lambda_n = \frac{2n+1}{2}\pi + \mu_n, \tag{5.22}$$

where

$$\mu_n \to +0 \quad \text{for } n \to \infty. \tag{5.23}$$

It follows from (5.22) and (5.23) that there exist numbers c_1 and $c_2 > 0$ such that for any n

$$c_1(n+1) \le \lambda_n \le c_2(n+1). \tag{5.24}$$

It follows from (5.19), (5.21), and (5.24) that there is a number $c_3 > 0$ such that for any n

$$|v_n(t)\sin\lambda_n x| \le \frac{c_3}{(n+1)^3}. \tag{5.25}$$

Since the series $\sum_{n=0}^{\infty}(n+1)^{-3}$ converges, according to the Weierstrass criterion series (5.12) converges uniformly over the band $[0,1]\times[0,\infty)$. Thus, it follows from the theorem on passage to the limit under the series sign that

$$v(x,t) \to 0 \quad \text{at } t \to 0 \quad (\text{uniformly in } [0,1]) \tag{5.26}$$

and the assertion of the lemma follows from (5.7) and (5.25). □

It follows from Lemma 5.1 and relations (5.7), (5.12), (5.25), and (5.26) that

$$u(x,t) \in C([0,1]\times[0,\infty)). \tag{5.27}$$

In order to study the continuity of the function $v'_x(x,t)$, consider the series composed of the first-order derivatives of the summands of series (5.12). We write

$$\sum_{n=1}^{\infty} \lambda_n v_n(t)\cos\lambda_n x. \tag{5.28}$$

It follows from (5.19), (5.20), and (5.24) that there is a number $c_4 > 0$ such that for any values of $x \in [0,1]$, $t \geq 0$, and n

$$|\lambda_n v_n(t) \cos \lambda_n x| \leq \frac{c_4}{(n+1)^2}. \tag{5.29}$$

From the convergence of the series $\sum_{n=1}^{\infty}(n+1)^{-2}$ and relation (5.29), by the Weierstrass criterion it follows that series (5.28) converges uniformly over the band $[0,1] \times [0,\infty)$.

Thus, it follows from Theorem 7, proved in [21] (p. 476), that for any values of $x \in [0,1]$ and $t > 0$ the following equation is true:

$$v'_x(x,t) = \sum_{n=1}^{\infty} \lambda_n v_n(t) \cos \lambda_n x.$$

Note that the function $v'_x(x,t)$ is extendable by continuity to the interval $t = 0$, so we have

$$\bar{v}'_x(x,t) \in C([0,1] \times [0,\infty)). \tag{5.30}$$

It follows from (5.7) and (5.30) that

$$\bar{u}'_x(x,t) \in C([0,1] \times [0,\infty)), \tag{5.31}$$

where for any $t > 0$ and $0 < x \leq 1$

$$\bar{u}'_x(x,t) = u'_x(x,t).$$

Consider the series composed of the second-order derivatives of the summands of series (5.12). We write

$$-\sum_{n=1}^{\infty} \lambda_n^2 v_n(t) \sin \lambda_n x. \tag{5.32}$$

It follows from (5.24) and (5.29) that there is a number $c_5 > 0$ such that for any values of $x \in [0,1]$, $t \geq 0$, and n

$$|\lambda_n^2 v_n(t) \sin \lambda_n x| \leq \frac{c_5}{n+1}. \tag{5.33}$$

It follows from (5.33) that for any $t > 0$ series (5.32) converges in the metric of the space $L_2[0,1]$.

Since the operator $\frac{d^2}{dx^2}$, defined on the class of functions

$$D = \{\varphi(x) : \varphi, \varphi', \varphi'' \in L_2[0,1],\ \varphi(0) = \varphi'(1) + \kappa\varphi(1) = 0\},$$

is closed in the space $L_2[0,1]$, it follows from the uniform convergence of series (5.28) and the convergence of series (5.32) in the space $L_2[0,1]$ that for any $t > 0$

$$\frac{\partial^2 v(x,t)}{\partial x^2} = -\sum_{n=1}^{\infty} \lambda_n^2 v_n(t) \sin \lambda_n x \quad \text{almost everywhere.} \tag{5.34}$$

It follows from (5.7) and (5.34) that for any $t > 0$

$$u''_{xx}(x,t) \in L_2[0,1]. \tag{5.35}$$

Let us get down to a more detailed study of the continuity of the function $u''_{xx}(x,t)$ on the band $(0,1] \times (0,\infty)$. We prove the following lemma for this purpose.

Lemma 5.2. *For any $\varepsilon > 0$ the series*

$$\sum_{n=1}^{\infty} \frac{\sin \lambda_n x}{2\lambda_n - \sin 2\lambda_n}$$

converges uniformly over the interval $[\varepsilon, 1]$.

Proof. First, transform equation (5.13) as follows:

$$\sin\lambda + \frac{\lambda}{\kappa}\cos\lambda = 0. \tag{5.36}$$

It follows from (5.36) that

$$\left(1+\frac{\lambda^2}{\kappa^2}\right)^{-\frac{1}{2}} \cdot \sin\lambda + \frac{\lambda}{\kappa}\left(1+\frac{\lambda^2}{\kappa^2}\right)^{-\frac{1}{2}} \cdot \cos\lambda = 0. \tag{5.37}$$

Let us denote

$$\sin\alpha = \left(1+\frac{\lambda^2}{\kappa^2}\right)^{-\frac{1}{2}} \quad \text{and} \quad \cos\alpha = \frac{\lambda}{\kappa}\left(1+\frac{\lambda^2}{\kappa^2}\right)^{-\frac{1}{2}}. \tag{5.38}$$

It follows from (5.37) and (5.38) that

$$\cos(\lambda - \alpha) = 0. \tag{5.39}$$

Given that $\lambda > 0$ and $\operatorname{tg}\lambda < 0$, from (5.39) it follows that

$$\lambda - \alpha = \frac{\pi}{2} + \pi n. \tag{5.40}$$

From (5.38) and (5.40) it follows that

$$\sin\left[\lambda_n - \left(\frac{\pi}{2} + \pi n\right)\right] = \left(1+\frac{\lambda_n^2}{\kappa^2}\right)^{-\frac{1}{2}},$$

where

$$\lambda_n = \left(\frac{\pi}{2} + \pi n\right) + \alpha_n \tag{5.41}$$

and, given (5.13),

$$\alpha_n \to +0 \quad \text{at } n \to \infty. \tag{5.42}$$

From (5.37), (5.41), and (5.42) it follows that for any n

$$\sin \alpha_n \leq \frac{1}{\kappa \pi n}. \tag{5.43}$$

It follows from (5.41)–(5.43) that

$$\left| \sin \lambda_n x - \sin\left(\frac{\pi}{2} + \pi n\right) x \right| \leq 2 \sin \frac{\alpha_n}{2} \leq \frac{2}{\kappa \pi n}, \tag{5.44}$$

$$\frac{\sin \lambda_n x}{2\lambda_n - \sin 2\lambda_n} = \frac{\sin \lambda_n x}{2\lambda_n} + \left(\frac{\sin \lambda_n x}{2\lambda_n - \sin 2\lambda_n} - \frac{\sin \lambda_n x}{2\lambda_n} \right),$$

and

$$\left| \frac{\sin(\frac{\pi}{2} + \pi n)x}{2\lambda_n - \sin 2\lambda_n} - \frac{\sin(\frac{\pi}{2} + \pi n)x}{2\lambda_n} \right| \leq \frac{1}{\lambda_n(\lambda_n - 1)}. \tag{5.45}$$

Let us denote

$$\varphi_n(x) = \left(\sin \lambda_n x - \sin\left(\frac{\pi}{2} + \pi n\right) x \right)$$

and

$$\overline{\psi}_n(x) = \left(\frac{\sin(\frac{\pi}{2} + \pi n)x}{2\lambda_n - \sin 2\lambda_n} - \frac{\sin(\frac{\pi}{2} + \pi n)x}{2\lambda_n} \right).$$

Then

$$\frac{\sin \lambda_n x}{2\lambda_n - \sin 2\lambda_n} = \frac{\sin(\frac{\pi}{2} + \pi n)x}{2\lambda_n} + \frac{\varphi_n(x)}{2\lambda_n - \sin 2\lambda_n} + \overline{\psi}_n(x) \tag{5.46}$$

and the assertion of the lemma follows from (5.44)–(5.46). □

Lemma 5.3. *For any $\varepsilon > 0$ the series*

$$\sum_{n=1}^{\infty} \lambda_n^2 v_n(t) \sin \lambda_n x$$

converges uniformly over the band $[\varepsilon, 1] \times [0, \infty)$.

Proof. The following estimate follows from (5.6) and (5.18):

$$\left| \int_0^t e^{-\lambda_n^2 (t-\tau)} h''(\tau) d\tau \right| \leq \frac{r_1}{\lambda_n^2}. \tag{5.47}$$

It follows from (5.24) and (5.47) that the series

$$\sum_{n=1}^{\infty}\left[\int_0^t e^{-\lambda_n^2(t-\tau)}h''(\tau)d\tau\right]\frac{\sin\lambda_n x}{2\lambda_n - \sin 2\lambda_n} \tag{5.48}$$

converges uniformly over the band $[0,1]\times[0,\infty)$.

Since from (5.16) and (5.17) it follows that for any n

$$\lambda_n^2 v_n(t)\sin\lambda_n x = \frac{8h'(t)\sin\lambda_n x}{2\lambda_n - \sin 2\lambda_n} - 8\left[\int_0^t e^{-\lambda_n^2(t-\tau)}h''(\tau)d\tau\right]\frac{\sin\lambda_n x}{2\lambda_n - \sin 2\lambda_n}, \tag{5.49}$$

it follows from Lemma 5.2 and relations (5.48) and (5.49) that the series

$$\sum_{n=1}^{\infty}\lambda_n^2 v_n(t)\sin\lambda_n x$$

converges uniformly over the band $[\varepsilon,1]\times[0,\infty)$.

The lemma is thereby proved. □

From Lemma 5.3 it follows that for any $x\in(0,1)$ and $t>0$

$$v''_{xx} = -\sum_{n=1}^{\infty}\lambda_n^2 v_n(t)\sin\lambda_n x. \tag{5.50}$$

In addition, from Lemma 5.3 and (5.50) it follows that $v''_{xx}(x,t)$ is extendable by continuity up to $t=0$. Let us denote this extension by $\bar{v}''_{xx}(x,t)$.

Then

$$\bar{v}''_{xx}(x,t)\in C\big((0,1]\times[0,\infty)\big) \tag{5.51}$$

and it follows from (5.7) and (5.51) that

$$\bar{u}''_{xx}(x,t)\in C\big((0,1]\times[0,\infty)\big), \tag{5.52}$$

where for any $t>0$ and $0<x<1$

$$\bar{u}''_{xx}(x,t) = u''_{xx}(x,t).$$

Note that the proof of formulas (5.27), (5.31), and (5.52) can be obtained from a corollary of the theorems given in [7] and [117].

Let $t_1\ge t_0$ and $\Phi(t)\in C[0,t_1]$.

Then from (5.31) and (5.52) it follows that

$$\int_0^{t_1} u'_x(x,t)\Phi(t)dt = \frac{\partial}{\partial x}\left[\int_0^{t_1} u(x,t)\Phi(t)dt\right] \tag{5.53}$$

and

$$\int_0^{t_1} u''_{xx}(x,t)\Phi(t)dt = \frac{\partial^2}{\partial x^2}\left[\int_0^{t_1} u(x,t)\Phi(t)dt\right]. \tag{5.54}$$

For the complete justification of the applicability of the Fourier transform with respect to t over the half-line $[0,\infty)$ it is necessary to extend formulas (5.53) and (5.54) to the case where $t_1 = \infty$. For this purpose let us study the decrease rate of the functions

$$u(x,t), \quad u'_x(x,t) \quad \text{and} \quad u''_{xx}(x,t) \quad \text{for } t \to \infty.$$

5.1.3 A study of the decrease rate of the functions $u(x,t)$, $u'_x(x,t)$ and $u''_{xx}(x,t)$ for $t \to \infty$

Consider an auxiliary problem that uses the condition of (5.6). We have

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1,\ t \geq t_0, \tag{5.55}$$

$$u(x,t_0) = u_0(x), \quad 0 \leq x \leq 1, \tag{5.56}$$

$$u(0,t) = 0, \quad t \geq t_0, \tag{5.57}$$

and

$$\frac{\partial u(1,t)}{\partial x} + \kappa u(1,t) = 0, \quad t \geq t_0. \tag{5.58}$$

It follows from (5.6), (5.27), (5.31), and (5.35) that

$$u_0(x) \in W_2^2[0,1], \quad u_0(0) = 0, \quad u'_0(1) + \kappa u_0(1) = 0. \tag{5.59}$$

The solution of problem (5.55)–(5.58) is as follows:

$$u(x,t) = \sum_{n=1}^{\infty} u_n e^{-\lambda_n^2(t-t_0)} \sin \lambda_n x, \tag{5.60}$$

where λ_n are defined by formula (5.13) and

$$u_n = \frac{4}{2\lambda_n - \sin 2\lambda_n} \int_0^1 u_0(x) \sin \lambda_n x dx. \tag{5.61}$$

By partially integrating the right-hand side of equation (5.61), we obtain

$$u_n = -\frac{4}{\lambda_n(2\lambda_n - \sin 2\lambda_n)} \int_0^1 u''_0(x) \sin \lambda_n x dx. \tag{5.62}$$

It follows from (5.59) and (5.62) that there exists a number $c_6 > 0$ such that for any n

$$|u_n| \leq \frac{c_6}{\lambda_n^2}. \tag{5.63}$$

It follows from (5.60) and (5.63) that for any $t \geq t_0 + 1$

$$\left|u(x,t)\right| \leq c_6 \sum_{n=1}^{\infty} \lambda_n^{-2} e^{-\lambda_n^2 (t-t_0)}, \tag{5.64}$$

$$\left|u_x'(x,t)\right| \leq c_6 \sum_{n=1}^{\infty} \lambda_n^{-1} e^{-\lambda_n^2 (t-t_0)}, \tag{5.65}$$

and

$$\left|u_{xx}''(x,t)\right| \leq c_6 \sum_{n=1}^{\infty} e^{-\lambda_n^2 (t-t_0)}. \tag{5.66}$$

Since

$$e^{-\lambda_n^2 (t-t_0)} = e^{-\lambda_n^2} \cdot e^{-\lambda_n^2 (t-t_0-1)} \tag{5.67}$$

and it follows from (5.24) that

$$e^{-\lambda_n^2} \leq \left[e^{c_1^2}\right]^{-n}, \tag{5.68}$$

it follows from (5.55) and (5.64)–(5.68) that there exists a number $c_7 > 0$ such that for any $t \geq t_0 + 2$

$$\sup_{x \in [0,1]} \left\{\left|u(x,t)\right|, \left|u_x'(x,t)\right|, \left|u_t'(x,t)\right|, \left|u_{xx}''(x,t)\right|\right\} \leq c_7 e^{-(t-t_0-1)}. \tag{5.69}$$

From (5.31), (5.52)–(5.54), and (5.69) by the theorem proved in [119] (p. 417) the following theorem arises.

Theorem 5.1. *Let $\Phi(t) \in C[0,\infty)$ and let $\Phi(t)$ be limited over this half-line. Then the following relations are true:*

$$\int_0^{\infty} u_x'(x,t)\Phi(t)dt = \frac{\partial}{\partial x}\left[\int_0^{\infty} u(x,t)\Phi(t)dt\right]$$

and

$$\int_0^{\infty} u_{xx}''(x,t)\Phi(t)dt = \frac{\partial^2}{\partial x^2}\left[\int_0^{\infty} u(x,t)\Phi(t)dt\right].$$

Lemma 5.4. *Let $u(x,t)$ be a solution of problem* (5.1)–(5.4). *Then the following relations are true:*

$$\lim_{x\to 0}\int_0^\infty |u(x,t)-h(t)|dt = \lim_{x\to 1}\int_0^\infty |u(x,t)-u(1,t)|dt$$
$$= \lim_{x\to 1}\int_0^\infty |u'_x(x,t)-u'_x(1,t)|dt = 0.$$

Proof. It follows from (5.27) and (5.31) that for any $t \ge 0$

$$\lim\nolimits_{x\to 0} u(x,t) = h(t), \quad \lim\nolimits_{x\to 1} u(x,t) = u(1,t), \quad \text{and}$$
$$\lim\nolimits_{x\to 1} u'_x(x,t) = u'_x(1,t). \tag{5.70}$$

Let the number $c_8 > 0$ be defined by the formula

$$c_8 = \max\{|u(x,t)| + |u'_x(x,t)| : 0 \le x \le 1,\ 0 \le t \le t_0+2\}.$$

Then let us denote by $g(t)$ the function defined by the formula

$$g(t) = \begin{cases} c_8, & 0 \le t \le t_0+2, \\ c_7 e^{-(t-t_0-1)}, & t > t_0+2. \end{cases}$$

Since

$$\int_0^\infty |g(t)|dt < \infty$$

and for any $t \ge 0$

$$|u(x,t)| \le g(t), \quad |u'_x(x,t)| \le g(t),$$

given (5.70), by the Lebesgue theorem on the passage to the limit under the integral sign, the assertion of the lemma is proved. □

5.2 On the accuracy estimation of the approximate solution of an inverse boundary-value problem for a heat conduction equation with a constant coefficient

5.2.1 Posing of the inverse problem

Let the thermal process be described by the equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2}, \quad 0 < x < 1,\ t > 0, \tag{5.71}$$

where the solution $u(x,t) \in C([0,1]\times[0,\infty))\cap C^{2,1}((0,1)\times(0,\infty))$ satisfies the following initial and boundary conditions:

$$u(x,0) = 0, \quad 0 \le x \le 1, \tag{5.72}$$

$$u(0,t) = h(t), \quad t \ge 0, \tag{5.73}$$

and

$$\frac{\partial u(1,t)}{\partial x} + \kappa u(1,t) = 0, \quad \kappa > 0, \; t \ge 0, \tag{5.74}$$

where

$$h(t) \in C^2[0,\infty), \quad h(0) = h'(0) = 0, \tag{5.75}$$

and there exists a number $t_0 > 0$ such that for any $t \ge t_0$

$$h(t) = 0. \tag{5.76}$$

Assume that the function $h(t)$ is unknown and should be defined and that the temperature $f(t)$ of the rod corresponding to this process is measured at the point $x_1 \in (0,1)$. We have

$$u(x_1,t) = f(t), \quad t \ge 0. \tag{5.77}$$

5.2.2 Reducing problem (5.71)–(5.73), (5.77) to the problem of calculating unbounded operator values

Let the set M_r be defined by the formula

$$M_r = \left\{ h(t) : h(t) \in L_2[0,\infty), \int_0^\infty |h(t)|^2 dt + \int_0^\infty |h'(t)|^2 dt \le r^2 \right\}, \tag{5.78}$$

where $h'(t)$ is the derivative of the function $h(t)$ and r is a known positive number. Then assume that for $f(t) = f_0(t)$, from condition of (5.77), there exists a function $h_0(t)$ belonging to the set M_r, but the exact value of the function $f_0(t)$ is unknown. Instead, a certain approximating function $f_\delta(t) \in L_2[0,\infty) \cap L_1[0,\infty)$ and a number $\delta > 0$ such that

$$\|f_\delta - f_0\|_{L_2} \le \delta \tag{5.79}$$

are given. It is required to find an approximate solution $h_\delta(t)$ of problem (5.71)–(5.73), (5.77) and to estimate the deviation $\|h_\delta - h_0\|_{L_2}$ of the approximate solution h_δ from the exact solution h_0, using f_δ, δ, and M_r.

Let

$$\overline{H} = L_2[0,\infty) + iL_2[0,\infty)$$

over the field of complex numbers and let F be an operator mapping $L_2[0,\infty)$ into $\overline{H}$ defined by the formula

$$F[h(t)] = \frac{1}{\sqrt{\pi}} \int_0^\infty h(t)e^{-i\tau t}dt, \quad \tau \ge 0. \tag{5.80}$$

If follows from Theorem 5.1 and Lemma 4.3 that the transformation F is applicable for the solution of equation (5.71). Thus, reduce equation (5.71) to the equation

$$\frac{\partial^2 \hat{u}(x,\tau)}{\partial x^2} = i\tau\hat{u}(x,\tau), \quad x \in (0,1),\ \tau \ge 0, \tag{5.81}$$

where

$$\hat{u}(x,\tau) = F[u(x,t)].$$

It follows from (5.74) and (5.77) that

$$\frac{\partial \hat{u}(1,\tau)}{\partial x} + \kappa\hat{u}(1,\tau) = 0, \quad \tau \ge 0, \tag{5.82}$$

and

$$\hat{u}(x_1,\tau) = \hat{f}(\tau), \quad \tau \ge 0, \tag{5.83}$$

where

$$\hat{f}(\tau) = F[f(t)].$$

It follows from Lemma 5.4 that the solution $\hat{u}(x,\tau)$ of problem (5.81)–(5.83) is continuous over the band $[0,1] \times [0,\infty)$. The solution of problem (5.81) is as follows:

$$\hat{u}(x,\tau) = A(\tau)e^{\mu_0 x\sqrt{\tau}} + B(\tau)e^{-\mu_0 x\sqrt{\tau}}, \tag{5.84}$$

where

$$\mu_0 = \frac{1}{\sqrt{2}}(1+i)$$

and $A(\tau)$ and $B(\tau)$ are arbitrary functions. It follows from (5.82)–(5.84) that

$$\hat{u}(0,\tau) = \frac{\cosh\mu_0\sqrt{\tau} + (\mu_0\sqrt{\tau})^{-1}\kappa\sinh\mu_0\sqrt{\tau}}{\cosh\mu_0(1-x_1)\sqrt{\tau} + (\mu_0\sqrt{\tau})^{-1}\kappa\sinh\mu_0(1-x_1)\sqrt{\tau}}\hat{f}(\tau), \quad \tau \ge 0. \tag{5.85}$$

Let us denote the denominator of the right-hand side of formula (5.85) by $\psi(\tau)$. We write

$$\psi(\tau) = \cosh\mu_0(1-x_1)\sqrt{\tau} + (\mu_0\sqrt{\tau})^{-1}\kappa\sinh\mu_0(1-x_1)\sqrt{\tau}.$$

Lemma 5.5. *Let $\kappa \le \frac{1}{2}$. Then there exists a number $c_1 > 0$ such that for any $\tau \ge 0$*

$$|\psi(\tau)| \ge c_1.$$

Proof. Since

$$\begin{aligned}\operatorname{Re}[\psi(\tau)] = &\left\{\cos(1-x_1)\sqrt{\frac{\tau}{2}}\left[\sqrt{\frac{2\kappa^2}{\tau}}\sinh(1-x_1)\sqrt{\frac{\tau}{2}} + \coth(1-x_1)\sqrt{\frac{\tau}{2}}\right]\right.\\ &\left. + \sqrt{\frac{2\kappa^2}{\tau}}\coth(1-x_1)\sqrt{\frac{\tau}{2}}\sin(1-x_1)\sqrt{\frac{\tau}{2}}\right\},\end{aligned} \tag{5.86}$$

$$\begin{aligned}\operatorname{Im}[\psi(\tau)] = &\left\{\sin(1-x_1)\sqrt{\frac{\tau}{2}}\left[\sqrt{\frac{2\kappa^2}{\tau}}\cosh(1-x_1)\sqrt{\frac{\tau}{2}} + \sinh(1-x_1)\sqrt{\frac{\tau}{2}}\right]\right.\\ &\left. - \sqrt{\frac{2\kappa^2}{\tau}}\sinh(1-x_1)\sqrt{\frac{\tau}{2}}\cos(1-x_1)\sqrt{\frac{\tau}{2}}\right\},\end{aligned} \tag{5.87}$$

it follows from (5.86) that, if

$$0 \le (1-x_1)\sqrt{\frac{\tau}{2}} \le \frac{\pi}{3}, \quad \cos(1-x_1)\sqrt{\frac{\tau}{2}} \ge \frac{1}{2}$$

and

$$|\psi(\tau)| \ge \cos(1-x_1)\sqrt{\frac{\tau}{2}}\coth(1-x_1)\sqrt{\frac{\tau}{2}} \ge \frac{1}{2}. \tag{5.88}$$

If

$$\frac{\pi}{3} \le (1-x_1)\sqrt{\frac{\tau}{2}} \le \frac{\pi}{2}, \quad \text{then } \sin(1-x_1)\sqrt{\frac{\tau}{2}} \ge \frac{1}{2}$$

and from (5.86) it follows that

$$|\psi(\tau)| \ge \sqrt{\frac{2\kappa^2}{\tau}}\sin(1-x_1)\sqrt{\frac{\tau}{2}}\coth(1-x_1)\sqrt{\frac{\tau}{2}} \ge \frac{(1-x_1)\kappa}{\pi}\coth\frac{\pi}{3}. \tag{5.89}$$

If

$$\frac{\pi}{2} \le (1-x_1)\sqrt{\frac{\tau}{2}} \le \frac{3\pi}{4}, \quad \sin(1-x_1)\sqrt{\frac{\tau}{2}} \ge \frac{\sqrt{2}}{2}.$$

It follows from (5.86) that

$$|\psi(\tau)| \ge \frac{(1-x_1)2\sqrt{2}}{3\pi}\kappa\coth\frac{\pi}{2}. \tag{5.90}$$

If

$$\frac{3\pi}{4} \le (1-x_1)\sqrt{\frac{\tau}{2}} \le \pi, \quad -\cos(1-x_1)\sqrt{\frac{\tau}{2}} \ge \frac{\sqrt{2}}{2}.$$

It follows from (5.87) that

$$\left|\psi(\tau)\right| \geq \frac{(1-x_1)\sqrt{2}}{2\pi}\kappa \sinh \frac{3\pi}{4}. \tag{5.91}$$

Thus, it follows from (5.88)–(5.91) that there exists a number $c_2 > 0$ such that, for any $\tau \in [0, \frac{2\pi^2}{(1-x_1)^2}]$,

$$\left|\psi(\tau)\right| \geq c_2.$$

Since

$$\kappa \leq \frac{1}{2} \quad \text{and} \quad \left|\psi(\tau)\right| \geq \left|\cosh \mu_0(1-x_1)\sqrt{\tau}\right| - \frac{\kappa}{\sqrt{\tau}}\left|\sinh \mu_0(1-x_1)\sqrt{\tau}\right|,$$

it is easy to verify the existence of a number $c_3 > 0$ such that for any $\tau \geq \frac{2\pi^2}{(1-x_1)^2}$

$$\left|\psi(\tau)\right| \geq c_3. \tag{5.92}$$

The assertion of the lemma follows from (5.88) and (5.92). □

Since the functions

$$\cosh \mu_0 \sqrt{\tau} + (\mu_0 \sqrt{\tau})^{-1}\kappa \sinh \mu_0 \sqrt{\tau}$$

and

$$\cosh \mu_0(1-x_1)\sqrt{\tau} + (\mu_0 \sqrt{\tau})^{-1}\kappa \sinh \mu_0(1-x_1)\sqrt{\tau}$$

are continuous over $[0,\infty)$, the function continuity follows from Lemma 5.5. We have

$$\frac{\cosh \mu_0 \sqrt{\tau} + (\mu_0 \sqrt{\tau})^{-1}\kappa \sinh \mu_0 \sqrt{\tau}}{\cosh \mu_0(1-x_1)\sqrt{\tau} + (\mu_0 \sqrt{\tau})^{-1}\kappa \sinh \mu_0(1-x_1)\sqrt{\tau}}$$

over this half-line. Thus, for any $\overline{\tau} > 0$ there is a number $c_{\overline{\tau}} > 0$ such that $\tau \in [0,\overline{\tau}]$ and

$$\left|\frac{\coth \mu_0 \sqrt{\tau} + (\mu_0 \sqrt{\tau})^{-1}\kappa \sinh \mu_0 \sqrt{\tau}}{\cosh \mu_0(1-x_1)\sqrt{\tau} + (\mu_0 \sqrt{\tau})^{-1}\kappa \sinh \mu_0(1-x_1)\sqrt{\tau}}\right| \leq c_{\overline{\tau}}. \tag{5.93}$$

Let us denote $\hat{u}(0,\tau)$ by $\hat{h}(\tau)$ and transform formula (5.85) as follows:

$$\hat{h}(\tau) = \frac{\frac{\sqrt{\tau}}{\sqrt{\tau+i\kappa^2}} \cosh \mu_0 \sqrt{\tau} + \frac{\kappa}{\mu_0 \sqrt{\tau+i\kappa^2}} \sinh \mu_0 \sqrt{\tau}}{\frac{\sqrt{\tau}}{\sqrt{\tau+i\kappa^2}} \cosh \mu_0(1-x_1)\sqrt{\tau} + \frac{\kappa}{\mu_0 \sqrt{\tau+i\kappa^2}} \sinh \mu_0(1-x_1)\sqrt{\tau}}\hat{f}(\tau), \tag{5.94}$$

$\tau \geq 0$. Let $\beta(\tau)$ be defined by the formula

$$\sinh \beta(\tau) = \frac{\kappa}{\mu_0 \sqrt{\tau + i\kappa^2}}. \tag{5.95}$$

It follows from the features of the function *Arsh* proved in [65] (pp. 84–86) that this function maps the complex plane $\mathbb{C}$, from which the rays $1 \le y < \infty$ and $-\infty < y \le -1$ have been removed into the band $-\frac{\pi}{2} < v < \frac{\pi}{2}$. Thus, it follows from (5.95) that there exists a function $\beta(\tau)$ that satisfies relation (5.95). Besides, it follows from (5.95) that

$$\beta(\tau) \to 0 \quad \text{for } \tau \to \infty \tag{5.96}$$

and it follows from (5.94) that

$$\hat{h}(\tau) = \cosh[\mu_0\sqrt{\tau} + \beta(\tau)] \cdot \cosh^{-1}[\mu_0(1 - x_1)\sqrt{\tau} + \beta(\tau)]\hat{f}(\tau). \tag{5.97}$$

Let us define the operator (5.97), using the formula T, assuming that

$$T\hat{f}(\tau) = \cosh[\mu_0\sqrt{\tau} + \beta(\tau)] \cdot \cosh^{-1}[\mu_0(1 - x_1)\sqrt{\tau} + \beta(\tau)]\hat{f}(\tau) \tag{5.98}$$

and

$$D(T) = \{\hat{f}(\tau) : \hat{f}(\tau) \in \overline{H} \text{ and } T\hat{f}(\tau) \in \overline{H}\}. \tag{5.99}$$

It follows from (5.98) and (5.99) that the operator T is linear, unbounded, and closed. We have

$$T\hat{f}(\tau) = \hat{h}(\tau). \tag{5.100}$$

Let

$$\hat{h}_0(\tau) = T\hat{f}_0(\tau), \quad \hat{f}_0(\tau) = F[f_0(t)], \quad \hat{f}_\delta(\tau) = F[f_\delta(t)].$$

Then it follows from formula (5.79) that

$$\|\hat{f}_\delta - \hat{f}_0\|_{\overline{H}} \le \delta. \tag{5.101}$$

Let us denote by $\hat{M}_r$ a set from $\overline{H}$ such that $\hat{M}_r \supset F[M_r]$ and

$$\hat{M}_r = \left\{\hat{h}(\tau) : \hat{h}(\tau) \in \overline{H}, \int_0^\infty (1 + \tau^2)|\hat{h}(\tau)|^2 d\tau \le r^2\right\}. \tag{5.102}$$

It follows from $h_0(t) \in M_r$ that

$$\hat{h}_0(\tau) \in \hat{M}_r. \tag{5.103}$$

5.2.3 Solving problem (5.100)–(5.103)

Lemma 5.6. *For any* $\varepsilon > 0$ *there exists a number* $\tau_\varepsilon > 0$ *such that for any* $\tau \geq \tau_\varepsilon$

$$\left(1 - \frac{\varepsilon}{4+\varepsilon}\right) e^{x_1\sqrt{\frac{\tau}{2}}} \leq \frac{|\cosh[\mu_0\sqrt{\tau} + \beta(\tau)]|}{|\cosh[\mu_0(1-x_1)\sqrt{\tau} + \beta(\tau)]|} \leq \left(1 + \frac{\varepsilon}{4+\varepsilon}\right) e^{x_1\sqrt{\frac{\tau}{2}}}. \tag{5.104}$$

Proof. Since

$$\beta(\tau) = \beta_1(\tau) + i\beta_2(\tau),$$

we have

$$\left|\cosh[\mu_0\sqrt{\tau} + \beta(\tau)]\right| = \sqrt{\cosh^2\left[\sqrt{\frac{\tau}{2}} + \beta_1(\tau)\right] - \sin^2\left[\sqrt{\frac{\tau}{2}} + \beta_2(\tau)\right]}$$

and

$$\left|\cosh[\mu_0(1-x_1)\sqrt{\tau} + \beta(\tau)]\right| = \sqrt{\sinh^2\left[(1-x_1)\sqrt{\frac{\tau}{2}} + \beta_1(\tau)\right] + \cos^2\left[(1-x_1)\sqrt{\frac{\tau}{2}} + \beta_2(\tau)\right]}.$$

Hence

$$\frac{|\cosh[\mu_0\sqrt{\tau} + \beta(\tau)]|}{|\cosh[\mu_0(1-x_1)\sqrt{\tau} + \beta(\tau)]|} \leq \frac{\cosh[\sqrt{\frac{\tau}{2}} + \beta_1(\tau)]}{\sinh[(1-x_1)\sqrt{\frac{\tau}{2}} + \beta_1(\tau)]} \tag{5.105}$$

and

$$\frac{\cosh[\sqrt{\frac{\tau}{2}} + \beta_1(\tau)]}{\sinh[(1-x_1)\sqrt{\frac{\tau}{2}} + \beta_1(\tau)]} = \frac{e^{\sqrt{\frac{\tau}{2}}+\beta_1(\tau)}[1 + e^{-\sqrt{2\tau}-2\beta_1(\tau)}]}{e^{(1-x_1)\sqrt{\frac{\tau}{2}}+\beta_1(\tau)}[1 + e^{-(1-x_1)\sqrt{2\tau}-2\beta_1(\tau)}]}. \tag{5.106}$$

Since it follows from (5.96) that

$$\beta(\tau) \to 0 \quad \text{for } \tau \to \infty,$$

it follows from (5.105) and (5.106) that for any $\mu > 0$ there is $\tau_1 > 0$ such that for any $\tau \geq \tau_1$

$$\sup\{e^{-\sqrt{2\tau}-2\beta_1(\tau)}, e^{-(1-x_1)\sqrt{2\tau}-2\beta_1(\tau)}\} < \mu. \tag{5.107}$$

It follows from (5.107) that

$$\tau_1 = \frac{1}{2}\ln^2\frac{1}{\mu}.$$

Thus, it follows from (5.105)–(5.107) that for any $\tau \geq \tau_1$

$$\frac{|\cosh[\mu_0\sqrt{\tau}+\beta(\tau)]|}{|\cosh[\mu_0(1-x_1)\sqrt{\tau}+\beta(\tau)]|} \leq \frac{1+\mu}{1-\mu}e^{x_1\sqrt{\frac{\tau}{2}}}. \tag{5.108}$$

Similarly to (5.105) it can be shown that

$$\frac{|\cosh[\mu_0\sqrt{\tau}+\beta(\tau)]|}{|\cosh[\mu_0(1-x_1)\sqrt{\tau}+\beta(\tau)]|} \geq \frac{\sinh[\sqrt{\frac{\tau}{2}}+\beta_1(\tau)]}{\cosh[(1-x_1)\sqrt{\frac{\tau}{2}}+\beta_1(\tau)]} \tag{5.109}$$

and

$$\frac{\sinh[\sqrt{\frac{\tau}{2}}+\beta_1(\tau)]}{\cosh[(1-x_1)\sqrt{\frac{\tau}{2}}+\beta_1(\tau)]} = \frac{e^{\sqrt{\frac{\tau}{2}}+\beta_1(\tau)}[1-e^{-\sqrt{2\tau}-2\beta_1(\tau)}]}{e^{(1-x_1)\sqrt{\frac{\tau}{2}}+\beta_1(\tau)}[1+e^{-(1-x_1)\sqrt{2\tau}-2\beta_1(\tau)}]}. \tag{5.110}$$

It follows from (5.107), (5.109), and (5.110) that for any $\tau \geq \tau_1$

$$\frac{|\cosh[\mu_0\sqrt{\tau}+\beta(\tau)]|}{|\cosh[\mu_0(1-x_1)\sqrt{\tau}+\beta(\tau)]|} \geq \frac{1-\mu}{1+\mu}e^{x_1\sqrt{\frac{\tau}{2}}}. \tag{5.111}$$

It is easy to show that, if we assume

$$\mu = \frac{\varepsilon}{8+3\varepsilon},$$

then the assertion of the lemma follows from (5.108) and (5.111). □

Consider two complex-valued functions $\psi_1(\tau)$ and $\psi_2(\tau) \in C[a,\infty)$ such that

$$|\psi_i(\tau)| \to \infty \quad \text{for } \tau \to \infty,\ i = 1,2.$$

Let us introduce operators T_1 and T_2, acting from the complex space $L_2[a,\infty)$ into themselves and defined by the formulas

$$T_i f(\tau) = \psi_i(\tau) f(\tau), \quad i = 1,2. \tag{5.112}$$

Let M_r be the class of correctness on $L_2[a,\infty)$, defined by formula (4.1). We further assume that T_i are injective and we denote by $\omega^i(\delta,r)$ the corresponding moduli of continuity of the operators T_i on the class of correctness M_r. We write

$$\omega^i(\delta,r) = \sup\{\|T_i f\| : f \in T_i^{-1}(M_r),\ \|f\| \leq \delta\}. \tag{5.113}$$

Lemma 5.7. *Let T_i be the operators defined by formulas* (5.112) *and* (5.113) *and for any $\tau \in [a,\infty)$*

$$|\psi_1(\tau)| \leq |\psi_2(\tau)|.$$

Then $\omega^1(\delta,r) \leq \omega^2(\delta,r)$.

The assertion of the lemma directly follows from the definition of the modulus of continuity $\omega^i(\delta, r)$ (see (5.113)). To study and solve problem (5.100)–(5.103) let us split it into two problems. The first of these problems is well-posed while the operator of the second problem satisfies conditions (5.104). Thus, the first of the problems is as follows:

$$T^1\hat{f}^1(\tau) = \frac{\cosh[\mu_0\sqrt{\tau} + \beta(\tau)]}{\cosh[\mu_0(1-x_1)\sqrt{\tau} + \beta(\tau)]}\hat{f}^1(\tau) = \hat{h}^1(\tau), \quad 0 \le \tau \le \tau_\varepsilon, \tag{5.114}$$

where τ_ε is described in Lemma 5.6,

$$\hat{f}^1(\tau) = \hat{f}(\tau) \quad \text{for } 0 \le \tau \le \tau_\varepsilon,$$

and

$$\hat{h}^1(\tau) = \hat{h}(\tau) \quad \text{given } 0 \le \tau \le \tau_\varepsilon.$$

It follows from Lemma 5.5 and relations (5.94)–(5.96) that for $\kappa \le \frac{1}{2}$ the function

$$\frac{\cosh[\mu_0\sqrt{\tau} + \beta(\tau)]}{\cosh[\mu_0(1-x_1)\sqrt{\tau} + \beta(\tau)]}$$

is continuous over the interval $[0, \tau_\varepsilon]$. It follows from (5.114) that the operator T^1 is bounded on the space

$$\overline{H}_1 = L_2[0, \tau_\varepsilon] + iL_2[0, \tau_\varepsilon]$$

and there exists a number $c_\varepsilon > 0$ such that

$$\|T^1\| \le c_\varepsilon. \tag{5.115}$$

The second problem is a problem of calculating the values of the unbounded operator T^2 defined by the formula

$$T^2\hat{f}^2(\tau) = \frac{\cosh[\mu_0\sqrt{\tau} + \beta(\tau)]}{\cosh[\mu_0(1-x_1)\sqrt{\tau} + \beta(\tau)]}\hat{f}^2(\tau) = \hat{h}^2(\tau), \tag{5.116}$$

where $\tau \ge \tau_\varepsilon$,

$$\hat{f}^2(\tau) = \hat{f}(\tau) \quad \text{given } \tau \ge \tau_\varepsilon,$$

and

$$\hat{h}^2(\tau) = \hat{h}(\tau) \quad \text{given } \tau \ge \tau_\varepsilon,$$

over the space

$$\overline{H}_2 = L_2[\tau_\varepsilon, \infty) + iL_2[\tau_\varepsilon, \infty).$$

To solve problem (5.116) let us use the family of operators $\{T_\alpha^2 : \alpha > \tau_\varepsilon\}$ defined by the formula

$$T_\alpha^2 \hat{f}^2(\tau) = \begin{cases} T^2 \hat{f}^2(\tau), & \tau_\varepsilon \le \tau \le \alpha, \\ 0, & \tau > \alpha. \end{cases} \tag{5.117}$$

Define the approximate value $\hat{h}_\delta^{2,\alpha}(\tau)$ of problem (5.116) by the formula

$$\hat{h}_\delta^{2,\alpha}(\tau) = T_\alpha^2 \hat{f}_\delta^2(\tau), \quad \tau \ge \tau_\varepsilon. \tag{5.118}$$

To select the regularization parameter $\overline{\alpha} = \overline{\alpha}(\delta, r)$ in formula (5.118), let us use the condition

$$\hat{h}_0^2(\tau) \in \hat{M}_r^2, \tag{5.119}$$

where

$$\hat{M}_r^2 = \left\{ \hat{h}^2(\tau) : \int_{\tau_\varepsilon}^{\infty} (1+\tau^2) |\hat{h}^2(\tau)|^2 d\tau \le r^2 \right\}. \tag{5.120}$$

It follows from (4.35) and (5.116)–(5.119) that

$$\begin{aligned} &\sup\{\|T_\alpha^2 \hat{f}_\delta^2(\tau) - T^2 \hat{f}_0^2(\tau)\|^2 : \hat{f}_0^2(\tau) \in [T^2]^{-1}(\hat{M}_r^2),\ \|\hat{f}_\delta^2 - \hat{f}_0^2\| \le \delta\} \\ &= \Delta_1^2(\alpha) + \|T_\alpha^2\|^2 \delta^2, \end{aligned} \tag{5.121}$$

where $[T^2]^{-1}$ is the inverse of the operator T^2 and

$$\Delta_1(\alpha) = \sup\{\|T_\alpha^2 \hat{f}_0^2(\tau) - T^2 \hat{f}_0^2(\tau)\| : \hat{f}_0^2(\tau) \in [T^2]^{-1}(\hat{M}_r^2)\}. \tag{5.122}$$

Let us now move on to estimating $\|T_\alpha^2\|$.

Lemma 5.8. *Under the above-formulated conditions the following relations are true:*

$$\left(1 - \frac{\varepsilon}{4+\varepsilon}\right) e^{x_1 \sqrt{\alpha/2}} \le \|T_\alpha^2\| \le \left(1 + \frac{\varepsilon}{4+\varepsilon}\right) e^{x_1 \sqrt{\alpha/2}}, \quad \alpha \ge \tau_\varepsilon.$$

Proof. By the definition of the operator norm we have

$$\|T_\alpha^2\| = \sup_{\tau_\varepsilon \le \tau \le \alpha} \frac{|\cosh[\mu_0 \sqrt{\tau} + \beta(\tau)]|}{|\cosh[\mu_0 (1 - x_1) \sqrt{\tau} + \beta(\tau)]|}. \tag{5.123}$$

The assertion of the lemma follows from (5.123) and Lemma 5.6. □

Let

$$\omega^2(\alpha) = \sup\left\{ \int_{\alpha}^{\infty} |\hat{h}_0^2(\tau)|^2 \tau : \hat{h}_0^2(\tau) \in \hat{M}_r^2 \right\}. \tag{5.124}$$

Then it follows from (5.120), (5.122), and (5.124) that

$$\Delta_1^2(\alpha) = \omega^2(\alpha). \tag{5.125}$$

It follows from (5.120) that, if $\hat{h}_0^2(\tau) \in \hat{M}_r^2$, then

$$\int_{\tau_\varepsilon}^{\infty} (1+\tau^2)\left|\hat{h}_0^2(\tau)\right|^2 d\tau \le r^2. \tag{5.126}$$

It follows from (5.124) and (5.126) that

$$\omega^2(\alpha) = \frac{r^2}{1+\alpha^2}. \tag{5.127}$$

Since

$$\Delta_\delta[T_\alpha^2] = \sup\{\|T_\alpha^2\hat{f}_\delta^2(\tau) - T^2\hat{f}_0^2(\tau)\| : \hat{f}_0^2(\tau) \in [T^2]^{-1}(\hat{M}_r^2),\ \|\hat{f}_\delta^2 - \hat{f}_0^2\| \le \delta\}, \tag{5.128}$$

it follows from (4.35) and (5.128) that

$$\Delta_\delta^2[T_\alpha^2] = \frac{r^2}{1+\alpha^2} + \|T_\alpha^2\|^2\delta^2, \tag{5.129}$$

while it follows from Lemma 5.8 and (5.129) that

$$\begin{aligned} &\frac{r^2}{1+\alpha^2} + \delta^2\left(1 - \frac{\varepsilon}{4+\varepsilon}\right)^2 e^{2x_1\sqrt{\alpha/2}} \\ &\quad \le \Delta_\delta^2[T_\alpha^2] \le \frac{r^2}{1+\alpha^2} + \delta^2\left(1 + \frac{\varepsilon}{4+\varepsilon}\right)^2 e^{2x_1\sqrt{\alpha/2}}. \end{aligned} \tag{5.130}$$

Choose the regularization parameter $\overline{\alpha} = \overline{\alpha}(\delta, r)$ in formula (5.118) from the condition

$$\frac{r}{\sqrt{1+\alpha^2}} = e^{x_1\sqrt{\alpha/2}}\delta. \tag{5.131}$$

Let us denote by $\alpha = \alpha(\delta, r)$ the value of the regularization parameter taken from the equation

$$\frac{r}{\sqrt{1+\alpha^2}} = \|T_\alpha^2\|\delta. \tag{5.132}$$

To obtain the final error estimate of the approximate value, let us introduce two more values of the regularization parameter

$$\overline{\alpha}_1 = \overline{\alpha}_1(\delta, r) \quad \text{and} \quad \overline{\alpha}_2 = \overline{\alpha}_2(\delta, r),$$

selected respectively from the equations

$$\frac{r}{\sqrt{1+\alpha^2}} = \left(1 - \frac{\varepsilon}{4+\varepsilon}\right)e^{x_1\sqrt{\alpha/2}}\delta, \tag{5.133}$$

$$\frac{r}{\sqrt{1+\alpha^2}} = \left(1+\frac{\varepsilon}{4+\varepsilon}\right)e^{\chi_1\sqrt{\alpha/2}}\delta. \tag{5.134}$$

It follows from 5.8 and (5.128)–(5.134) that there exists $\delta_\varepsilon > 0$ such that

$$\overline{\alpha}_2(\delta_\varepsilon, r) \geq \alpha_\varepsilon$$

and consequently for any $\delta < \delta_\varepsilon$

$$\overline{\alpha}_2(\delta, r) \leq \overline{\alpha}(\delta, r) \leq \overline{\alpha}_1(\delta, r), \tag{5.135}$$

$$\overline{\alpha}_2(\delta, r) \leq \alpha(\delta, r) \leq \overline{\alpha}_1(\delta, r). \tag{5.136}$$

It follows from (4.37), (5.129), and (5.132) that

$$\Delta_\delta[T^2_{\alpha(\delta,r)}] = \sqrt{2}\|T^2_{\alpha(\delta,r)}\|\delta. \tag{5.137}$$

Similarly, it follows from Lemma 5.8 and relations (5.131)–(5.134) that for any $\delta < \delta_\varepsilon$

$$\sqrt{2}\delta\left(1-\frac{\varepsilon}{4+\varepsilon}\right)e^{\chi_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}} \leq \Delta_\delta[T^2_{\alpha(\delta,r)}] \leq \sqrt{2}\delta\left(1+\frac{\varepsilon}{4+\varepsilon}\right)e^{\chi_1\sqrt{\frac{\overline{\alpha}_2(\delta,r)}{2}}} \tag{5.138}$$

and

$$\sqrt{2}\delta\left(1-\frac{\varepsilon}{4+\varepsilon}\right)e^{\chi_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}} \leq \Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}] \leq \sqrt{2}\delta\left(1+\frac{\varepsilon}{4+\varepsilon}\right)e^{\chi_1\sqrt{\frac{\overline{\alpha}_2(\delta,r)}{2}}}. \tag{5.139}$$

Theorem 5.2. *For any $\delta \in (0, \delta_\varepsilon)$ the following relations are true:*

$$\left(1-\frac{\varepsilon}{2}\right)\Delta_\delta[T^2_{\overline{\alpha}_1(\delta,r)}] \leq \Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}] \leq \left(1+\frac{\varepsilon}{2}\right)\Delta_\delta[T^2_{\overline{\alpha}_1(\delta,r)}].$$

Proof. We have

$$\Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}] \leq \Delta_\delta[T^2_{\overline{\alpha}_1(\delta,r)}] + \left|\Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}] - \Delta_\delta[T^2_{\overline{\alpha}_1(\delta,r)}]\right| \tag{5.140}$$

and

$$\Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}] \geq \Delta_\delta[T^2_{\overline{\alpha}_1(\delta,r)}] - \left|\Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}] - \Delta_\delta[T^2_{\overline{\alpha}_1(\delta,r)}]\right|. \tag{5.141}$$

It follows from (5.140) and (5.141) that to prove the theorem it is sufficient to estimate $|\Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}] - \Delta_\delta[T^2_{\overline{\alpha}_1(\delta,r)}]|$. It follows from (5.133), (5.134), and (5.135) that

$$\begin{aligned}&\left|\Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}] - \Delta_\delta[T^2_{\overline{\alpha}_1(\delta,r)}]\right| \\ &\quad \leq \sqrt{2}\left(1+\frac{\varepsilon}{4+\varepsilon}\right)e^{\chi_1\sqrt{\frac{\overline{\alpha}(\delta,r)}{2}}}\delta - \sqrt{2}\left(1-\frac{\varepsilon}{4+\varepsilon}\right)e^{\chi_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta\end{aligned} \tag{5.142}$$

and it follows from (5.135) and (5.142) that

$$\begin{aligned}&\left|\Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}]-\Delta_\delta[T^2_{\overline{\alpha}_1(\delta,r)}]\right|\\&\le\sqrt{2}\left(1+\frac{\varepsilon}{4+\varepsilon}\right)e^{x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta-\sqrt{2}\left(1-\frac{\varepsilon}{4+\varepsilon}\right)e^{x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta.\end{aligned}\tag{5.143}$$

It follows from (5.143) that

$$\left|\Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}]-\Delta_\delta[T^2_{\overline{\alpha}_1(\delta,r)}]\right|\le\sqrt{2}\frac{\varepsilon}{2}e^{x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta.\tag{5.144}$$

The proof of the theorem follows from (5.140), (5.141), and (5.144). □

Theorem 5.3. *For the method* $\{T^2_{\overline{\alpha}(\delta,r)}:0<\delta\le\delta_\varepsilon\}$, *defined by formulas* (5.117) *and* (5.131), *the following accurate-by-order error estimate is true:*

$$\sqrt{2}\left(1-\frac{\varepsilon}{2}\right)e^{x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta\le\Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}]\le\sqrt{2}\left(1+\frac{\varepsilon}{2}\right)e^{x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta.$$

Proof. It follows from Theorem 5.2 and relations (5.129) and (5.135) that for any $\delta\in(0,\delta_\varepsilon]$

$$\Delta^2_\delta[T^2_{\overline{\alpha}(\delta,r)}]\le\frac{r^2}{1+\overline{\alpha}^2(\delta,r)}+\left(1+\frac{\varepsilon}{2}\right)e^{2x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta^2\tag{5.145}$$

and

$$\Delta^2_\delta[T^2_{\overline{\alpha}(\delta,r)}]\ge\frac{r^2}{1+\overline{\alpha}_1^2(\delta,r)}+\left(1-\frac{\varepsilon}{2}\right)e^{2x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta^2.\tag{5.146}$$

It follows from (5.131), (5.134), (5.145), and (5.146) that

$$\Delta^2_\delta[T^2_{\overline{\alpha}(\delta,r)}]\le\left(1+\frac{\varepsilon}{2}\right)^2e^{2x_1\sqrt{\frac{\overline{\alpha}(\delta,r)}{2}}}\delta^2+\left(1+\frac{\varepsilon}{2}\right)^2e^{2x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta^2\tag{5.147}$$

and

$$\Delta^2_\delta[T^2_{\overline{\alpha}(\delta,r)}]\ge\left(1-\frac{\varepsilon}{2}\right)^2e^{2x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta^2+\left(1-\frac{\varepsilon}{2}\right)^2e^{2x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta^2.\tag{5.148}$$

It follows from (5.135) that

$$e^{2x_1\sqrt{\frac{\overline{\alpha}(\delta,r)}{2}}}<e^{2x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\tag{5.149}$$

and from (5.147) and (5.149) that

$$\Delta^2_\delta[T^2_{\overline{\alpha}(\delta,r)}]\le\left(1+\frac{\varepsilon}{2}\right)^2e^{2x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta^2+\left(1+\frac{\varepsilon}{2}\right)^2e^{2x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta^2.\tag{5.150}$$

The assertion of the theorem follows from (5.148) and (5.150). □

Theorem 5.4. *The solution method* $\{T^2_{\overline{\alpha}(\delta,r)} : 0 < \delta \le \delta_\varepsilon\}$, *for problem* (5.116), *defined by formulas* (5.117) *and* (5.131), *is optimal-by-order on the class* $\hat{M}^2_r$ *and for this method the following error estimate is true:*

$$\Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}] \le \sqrt{2}(1+\varepsilon)\Delta^{\text{opt}}_\delta.$$

Proof. It follows from Lemmas 5.6 and 5.7 that

$$\omega^1(\delta,r) \le \omega^2(\delta,r), \tag{5.151}$$

where

$$\omega^2(\delta,r) = \sup\{\|T^2\hat{f}^2(\tau)\| : \hat{f}^2(\tau) \in [T^2]^{-1}(\hat{M}^2_r),\ \|\hat{f}^2(\tau)\| \le \delta\}$$

and

$$\omega^1(\delta,r) = \sup\left\{\left\|\left(1-\frac{\varepsilon}{4+\varepsilon}\right)e^{x_1\sqrt{\frac{\tau}{2}}}\hat{f}^2(\tau)\right\| : \hat{f}^2(\tau) \in \left(1-\frac{\varepsilon}{4+\varepsilon}\right)^{-1}e^{-x_1\sqrt{\frac{\tau}{2}}}(\hat{M}^2_r),\right.$$
$$\left.\|\hat{f}^2(\tau)\| \le \delta\right\}. \tag{5.152}$$

It follows from (4.16), (5.102), and (5.152) that

$$\omega^2(\delta,r) = \frac{r}{\sqrt{1+\overline{\alpha}_1^2(\delta,r)}}, \tag{5.153}$$

where $\overline{\alpha}_1(\delta,r)$ is defined by equation (5.133). It follows from (5.133) and (5.153) that

$$\omega^1(\delta,r) = \left(1-\frac{\varepsilon}{4+\varepsilon}\right)e^{x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta. \tag{5.154}$$

Since

$$\Delta^{\text{opt}}_\delta \ge \omega^1(\delta,r), \tag{5.155}$$

from (5.151), (5.154), and (5.155) we have

$$\Delta^{\text{opt}}_\delta \ge \left(1-\frac{\varepsilon}{4+\varepsilon}\right)e^{x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta. \tag{5.156}$$

The assertion of the lemma follows from Theorem 5.3 and relation (5.156). □

Since it follows from relation (5.133) that

$$e^{x_1\sqrt{\frac{\overline{\alpha}_1(\delta,r)}{2}}}\delta = \left(1+\frac{\varepsilon}{4}\right)\frac{r}{\sqrt{1+\overline{\alpha}_1^2(\delta,r)}}, \tag{5.157}$$

it follows from Theorem 5.3 that for $\delta \le \delta_\varepsilon$

$$\Delta_\delta[T^2_{\overline{\alpha}(\delta,r)}] \le \sqrt{2}\left(1+\frac{\varepsilon}{2}\right)^2 \frac{r}{\sqrt{1+\overline{\alpha}_1^2(\delta,r)}}. \tag{5.158}$$

In order to find the asymptotics of estimate (5.158), consider the following two equations:

$$e^{x_1\sqrt{\frac{\alpha}{2}}} = \frac{r}{\delta} \tag{5.159}$$

and

$$e^{2x_1\sqrt{\frac{\alpha}{2}}} = \frac{r}{\delta}. \tag{5.160}$$

Let us denote by (5.159) and (5.160) the solutions of equations $\hat{\alpha}_1(\delta,r)$ and $\hat{\alpha}_2(\delta,r)$. Then it follows from (5.133), (5.159), and (5.160) that for sufficiently low values of δ the following relations are true:

$$\hat{\alpha}_2(\delta,r) \le \overline{\alpha}_1(\delta,r) \le \hat{\alpha}_1(\delta,r). \tag{5.161}$$

It follows from (5.159) and (5.160) that

$$\hat{\alpha}_1(\delta,r) = \frac{2}{x_1^2}\ln^2\frac{r}{\delta} \quad \text{and} \quad \hat{\alpha}_2(\delta,r) = \frac{1}{2x_1^2}\ln^2\frac{r}{\delta}$$

and it follows from (5.161) that

$$\overline{\alpha}_1(\delta,r) \sim \ln^2\delta \quad \text{given } \delta \to 0. \tag{5.162}$$

From ratio (5.162) the following theorem arises.

Theorem 5.5. *For any $r > 0$ there exist numbers*

$$c_1(r), c_2(r) > 0 \quad \textit{and} \quad \delta_1 \in (0,\delta_\varepsilon)$$

such that for any $\delta \in (0,\delta_1)$ the following estimates are true:

$$c_1(r)\ln^2\delta \le \sqrt{1+\overline{\alpha}_1^2(\delta,r)} \le c_2(r)\ln^2\delta.$$

We further denote the solution of problem (5.114) by

$$h^1_\delta(\tau) = T^1\hat{f}^1_\delta(\tau). \tag{5.163}$$

It follows from (5.115) and (5.163) that

$$\|\hat{h}^1_\delta(\tau) - \hat{h}^1_0(\tau)\| \le c_\varepsilon\delta, \tag{5.164}$$

where

$$\hat{h}_0^1(\tau) = T^1 \hat{f}_0^1(\tau).$$

Define the solution of problem (5.100)–(5.103) by the formula

$$\hat{h}_\delta(\tau) = \hat{h}_\delta^1(\tau) + \hat{h}_\delta^{2,\overline{\alpha}(\delta,r)}(\tau). \tag{5.165}$$

Then it follows from relations (5.158), (5.164), and (5.165) that

$$\|\hat{h}_\delta(\tau) - \hat{h}_0(\tau)\| \le \sqrt{2}\left(1 + \frac{\varepsilon}{2}\right)^2 \frac{r}{\sqrt{1 + \overline{\alpha}_1^2(\delta, r)}} + c_\varepsilon \delta. \tag{5.166}$$

Note that the function $\hat{h}_\delta(\tau)$, defined by formula (5.165), may be defined in a different way by introducing a family of regularization operators $\{T_\alpha : \alpha > 0\}$, defined by the formula

$$T_\alpha \hat{f}(\tau) = \begin{cases} T\hat{f}(\tau), & 0 \le \tau \le \alpha, \\ 0, & \tau > \alpha. \end{cases} \tag{5.167}$$

Then

$$\hat{h}_\delta(\tau) = T_\alpha \hat{f}_\delta(\tau). \tag{5.168}$$

If we select the value of the regularization parameter $\overline{\alpha}(\delta, r)$ in formula (5.168) from the condition

$$\frac{r}{\sqrt{1 + \alpha^2}} = e^{\chi_1 \sqrt{\frac{\alpha}{2}}} \delta, \tag{5.169}$$

then, for the solution $\hat{h}_\delta^{\overline{\alpha}(\delta,r)}(\tau)$ of problem (5.100)–(5.103), the following estimate is true:

$$\|\hat{h}_\delta(\tau) - \hat{h}_0(\tau)\| \le \sqrt{2}\left(1 + \frac{\varepsilon}{2}\right)^2 \frac{r}{\sqrt{1 + \overline{\alpha}_1^2(\delta, r)}} + c_\varepsilon \delta. \tag{5.170}$$

It follows from Theorem 5.5 that there is $\delta_0 < \delta_\varepsilon$ such that for any $\delta < \delta_0$

$$c_\varepsilon \delta < \sqrt{2} \cdot \frac{\varepsilon^2}{2} \cdot \frac{r}{\sqrt{1 + \overline{\alpha}_1^2(\delta, r)}}. \tag{5.171}$$

Then the following theorem arises from relations (5.170) and (5.171).

Theorem 5.6. *The solution method* $\{T_{\overline{\alpha}(\delta,r)} : 0 < \delta < \delta_0\}$ *for problem* (5.100)–(5.103) *is optimal-by-order on the class* $\hat{M}_r$ *and the following estimate is true:*

$$\Delta_\delta[T_{\overline{\alpha}(\delta,r)}] \leq \sqrt{2}(1+\varepsilon+\varepsilon^2)\frac{r}{\sqrt{1+\overline{\alpha}_1^2(\delta,r)}}.$$

This estimate is accurate-by-order.

Now consider a subspace $\overline{H}_0$, defined by the formula

$$\overline{H}_0 = F[L_2[0,\infty)],$$

and denote by $\overline{h}_\delta(\tau)$ the element defined by the formula

$$\overline{h}_\delta(\tau) = \mathrm{pr}[\hat{h}_\delta(\tau); \overline{H}_0].$$

Since $\hat{h}_0(\tau) \in \overline{H}_0$, it follows from (5.170) that

$$\|\overline{h}_\delta(\tau) - \hat{h}_0(\tau)\| \leq \sqrt{2}\left(1+\frac{\varepsilon}{2}\right)^2 \frac{r}{\sqrt{1+\overline{\alpha}_1^2(\delta,r)}} + c_\varepsilon\delta. \tag{5.172}$$

Finally, let us define the solution of $h_\delta(t)$ of the inverse problem (5.71)–(5.73), (5.77) by the formula

$$h_\delta(t) = \begin{cases} F^{-1}[\overline{h}_\delta(\tau)], & t \in [0,t_0], \\ 0, & 0 < t,\ t > t_0, \end{cases} \tag{5.173}$$

where F^{-1} is the inverse of the operator F. It follows from (5.172) and (5.173) that for $h_\delta(t)$ the following estimate is true:

$$\|h_\delta(t) - h_0(t)\| \leq \sqrt{2}\left(1+\frac{\varepsilon}{2}\right)^2 \frac{r}{\sqrt{1+\overline{\alpha}_1^2(\delta,r)}} + c_\varepsilon\delta. \tag{5.174}$$

It follows from (5.174) that there exists a number $d > 0$ such that for any $\delta \in (0,\delta_0)$ the following relation is true:

$$\|h_\delta(t) - h_0(t)\| \leq d \cdot r \ln^{-2}\delta.$$

5.3 A study of the solution to a direct boundary-value problem for the heat conduction equation with a variable coefficient

5.3.1 Problem posing

Let $a(x) \in C^2[0,1]$, $a(x) \leq 0$, and let a thermal process be described by the equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + a(x)u(x,t), \quad 0 < x < 1,\ t > 0, \tag{5.175}$$

where the solution $u(x,t) \in C([0,1]\times[0,\infty)) \cap W_2^{2,1}([0,1]\times[0,\infty))$ satisfies the following initial and boundary conditions:

$$u(x,0) = 0, \quad 0 \le x \le 1, \tag{5.176}$$

$$u(0,t) = 0, \quad t \ge 0, \tag{5.177}$$

$$u(1,t) = h(t), \quad t \ge 0, \tag{5.178}$$

where

$$h(t) \in C^2[0,\infty), \quad h(0) = h'(0) = 0, \tag{5.179}$$

and where there exists a number $t_0 > 0$ such that for any $t \ge t_0$

$$h(t) = 0. \tag{5.180}$$

5.3.2 A study of the smoothness for the function $u(x,t)$

Consider the substitution

$$v(x,t) = u(x,t) - xh(t). \tag{5.181}$$

Then

$$\frac{\partial v(x,t)}{\partial t} = \frac{\partial^2 v(x,t)}{\partial x^2} + a(x)v(x,t) + a(x)xh(t) - xh'(t), \quad x \in (0,1), \quad t > 0, \tag{5.182}$$

$$v(x,0) = 0, \quad 0 \le x \le 1, \tag{5.183}$$

$$v(0,t) = 0, \quad t \ge 0, \tag{5.184}$$

$$v(1,t) = 0, \quad t \ge 0. \tag{5.185}$$

The solution of problem (5.182)–(5.185) is as follows:

$$v(x,t) = \sum_{n=1}^{\infty} v_n(t)\psi_n(x), \tag{5.186}$$

where

$$v_n(t) = b_n \int_0^t e^{\lambda_n(t-\tau)} h(\tau)d\tau - c_n \int_0^t e^{\lambda_n(t-\tau)} h'(\tau)d\tau, \tag{5.187}$$

$\{\lambda_n\}$ is a sequence of eigenvalues of the corresponding Sturm–Liouville problem, and $\{\psi_n(x)\}$ is the corresponding sequence of eigenfunctions of the following problem:

$$b_n = \int_0^1 xa(x)\psi_n(x)dx, \tag{5.188}$$

$$c_n = \int_0^1 x\psi_n(x)dx. \tag{5.189}$$

From the theorem posed in [53] (p. 37) it follows that there exist positive numbers d_1 and d_2 such that for any n

$$-d_1n^2 \leq \lambda_n \leq -d_2n^2 \tag{5.190}$$

and from the theorem posed in [53] (pp. 15–16) it follows that the system $\{\psi_n(x)\}$ of eigenfunctions is orthonormal and complete on the space $L_2[0,1]$.

Thus, from (5.188) and (5.189) it follows that

$$\sum_{n=1}^{\infty} b_n^2 < \infty, \tag{5.191}$$

$$\sum_{n=1}^{\infty} c_n^2 < \infty. \tag{5.192}$$

Partially integrating the right-hand side of equation (5.187) and taking into account (5.179) we obtain

$$v_n(t) = -\frac{b_n}{\lambda_n}\left[h(t) - \int_0^t e^{\lambda_n(t-\tau)}h'(\tau)d\tau\right] + \frac{c_n}{\lambda_n}\left[h'(t) - \int_0^t e^{\lambda_n(t-\tau)}h''(\tau)d\tau\right]. \tag{5.193}$$

Let

$$d_3 = \max_{t\in[0,t_0]}(|h(t)| + |h'(t)| + |h''(t)|). \tag{5.194}$$

Then, by (5.179), (5.180), and (5.192)–(5.194) for any values of n and $T > 0$ the following relations are true:

$$\int_0^T\int_0^1 v_n^2(t)\psi_n^2(x)dxdt \leq \frac{2Td_3^2[1+t_0]^2}{\lambda_n^2}[b_n^2 + c_n^2], \tag{5.195}$$

$$\int_0^T\int_0^1 \lambda_n^2 v_n^2(t)\psi_n^2(x)dxdt \leq 2Td_3^2(1+t_0)^2[b_n^2 + c_n^2]. \tag{5.196}$$

It follows from (5.186), (5.191), (5.192), (5.195), and (5.196) that

$$v(x,t) \in C([0,1]\times[0,T]), \tag{5.197}$$

$$\frac{\partial^2 v(x,t)}{\partial x^2} + a(x)v(x,t) \in L_2([0,1]\times[0,T]). \tag{5.198}$$

From (5.197) and (5.198) it follows that

$$\frac{\partial^2 v(x,t)}{\partial x^2} \in L_2([0,1] \times [0,T]). \tag{5.199}$$

Let

$$U_N(x,t) = \sum_{n=1}^{N} v_n(t)\psi_n(x). \tag{5.200}$$

Then from (5.197), (5.198), and (5.200) it follows that

$$U_N(x,t) \longrightarrow v(x,t) \quad \text{in the metrics } C([0,1] \times [0,T]) \tag{5.201}$$

and

$$\frac{\partial^2 U_N(x,t)}{\partial x^2} + a(x)U_N(x,t) \longrightarrow \frac{\partial^2 v(x,t)}{\partial x^2} + a(x)v(x,t) \tag{5.202}$$

in the metrics of the space $L_2([0,1] \times [0,T])$.

Lemma 5.9. *Let $\Phi(t) \in C[0,T]$. Then the following formula is true:*

$$\int_0^T \Phi(t)[v''_{xx}(x,t) + a(x)v(x,t)]dt = \frac{\partial^2}{\partial x^2}\int_0^T \Phi(t)v(x,t)dt + a(x)\int_0^T \Phi(t)v(x,t)dt.$$

Proof. From (5.200) it follows that

$$\begin{aligned}
&\int_0^T \Phi(t)\left[\frac{\partial^2 U_N(x,t)}{\partial x^2} + a(x)U_N(x,t)\right]dx \\
&\quad = \int_0^T \frac{\partial^2 U_N(x,t)}{\partial x^2}\Phi(t)dt + a(x)\int_0^T U_N(x,t)\Phi(t)dt \\
&\quad = \frac{\partial^2}{\partial x^2}\left[\int_0^T U_N(x,t)\Phi(t)dt\right] + a(x)\int_0^T U_N(x,t)\Phi(t)dt.
\end{aligned} \tag{5.203}$$

If $G(x,t) \in L_2([0,1] \times [0,T])$, then the operator B, defined by the formula

$$BG(x,t) = \int_0^T G(x,t)dt,$$

continuously maps the space $L_2([0,1] \times [0,T])$ into $L_2[0,1]$ [83].

Thus, the assertion of the lemma follows from (5.201)–(5.203). □

5.3.3 Justification of the method of integral transforms with respect to t as applied to solving problem (5.175)

Lemma 5.10. *Let $u(x,t)$ be the solution of problem* (5.175)–(5.178). *Then for any $t > 0$*

$$u(x,t) \in W_2^2[0,1].$$

Proof. It follows from (5.179), (5.180), and (5.193) that there exists a number $d_4 > 0$ such that for any values of $t > 0$ and n

$$|\lambda_n v_n(t)| \le d_4 \sqrt{b_n^2 + c_n^2}. \tag{5.204}$$

Since the system of eigenfunctions $\{\psi_n(x)\}$ of the operator $\frac{d^2}{dx^2} + a(x)$ is orthonormalized on the space $L_2[0,1]$, from (5.186), (5.191), (5.192), and (5.204) it follows that for any $t > 0$

$$\frac{\partial^2 u(x,t)}{\partial x^2} + a(x)u(x,t) \in L_2[0,1]. \tag{5.205}$$

Since $u(x,t) \in C[0,1]$, from (5.205) it follows that for any $t > 0$

$$\frac{\partial^2 u(x,t)}{\partial x^2} \in L_2[0,1]. \tag{5.206}$$

Taking into account that

$$\frac{\partial u(x,t)}{\partial x} = \frac{\partial u(0,t)}{\partial x} + \int_0^x \frac{\partial^2 u(\xi,t)}{\partial \xi^2} d\xi,$$

by (5.206) we obtain for any $t > 0$

$$\frac{\partial u(x,t)}{\partial x} \in W_2^1[0,1]. \tag{5.207}$$

Similarly, the assertion of the lemma follows from (5.207). □

Lemma 5.11. *Let $\{\psi_n(x)\}$ be a system of eigenfunctions of the corresponding Sturm–Liouville problem. Then there exists a number $d_5 > 0$ such that for any n*

$$\max_{0 \le x \le 1} |\psi_n(x)| \le d_5 n^2.$$

Proof. Since from the theorem in [53] it follows that for any n

$$\psi_n(x) \in C^2[0,1] \quad \text{and} \quad \psi_n(0) = \psi_n(1) = 0,$$

there exists a point $a_n \in [0,1]$ such that

$$\psi_n'(a_n) = 0.$$

Thus,

$$|\psi_n'(x)| = \left| \int_{a_n}^{x} \psi_n''(\xi)d\xi \right| \le \int_0^1 |\psi_n''(\xi)|d\xi \le \left[\int_0^1 |\psi_n''(\xi)|^2 d\xi \right]^{\frac{1}{2}}. \tag{5.208}$$

From

$$\frac{d^2\psi_n(x,t)}{dx^2} - |a(x)|\psi_n(x) = \lambda_n\psi_n(x)$$

it follows that

$$\left| \frac{d^2\psi_n(x)}{dx^2} \right| \le \left[|\lambda_n| + \max_{0\le x\le 1} |a(x)| \right] |\psi_n(x)|. \tag{5.209}$$

From (5.209) it follows that there exists a number $d_6 > 0$ such that

$$\left| \frac{d^2\psi_n(x)}{dx^2} \right| \le d_6|\lambda_n||\psi_n(x)|. \tag{5.210}$$

From

$$|\psi_n(x)| = \left| \int_0^x \psi_n'(\xi)d\xi \right| \le \int_0^1 |\psi_n'(\xi)|d\xi$$

it follows that

$$|\psi_n(x)| \le \max_{0\le x\le 1} |\psi_n'(x)| \tag{5.211}$$

and the assertion of the lemma follows from (5.208), (5.210), and (5.211). □

Now consider an auxiliary problem that uses condition (5.180). We write

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + a(x)u(x,t), \quad x \in (0,1),\ t \ge t_0, \tag{5.212}$$

$$u(x,t_0) = u_0(x), \quad 0 \le x \le 1, \tag{5.213}$$

$$u(0,t) = u(1,t) = 0, \quad t \ge t_0. \tag{5.214}$$

From Lemma 5.10 it follows that

$$u_0(x) \in W_2^2[0,1], \tag{5.215}$$

$$u_0(0) = u_0(1) = 0. \tag{5.216}$$

The solution of problem (5.212)–(5.214) is as follows:

$$u(x,t) = \sum_{n=1}^{\infty} u_n e^{\lambda_n(t-t_0)}\psi_n(x), \tag{5.217}$$

where λ_n and $\psi_n(x)$ have been defined before and

$$u_n = \int_0^1 u_0(x)\psi_n(x)dx. \tag{5.218}$$

Since

$$\int_0^1 u_0(x)\psi_n(x)dx = \frac{1}{\lambda_n}\left[\int_0^1 u_0(x)\psi_n''(x)dx + \int_0^1 a(x)u_0(x)\psi_n(x)dx\right], \tag{5.219}$$

it follows from (5.218) that for any n

$$u_n = \frac{p_n + q_n}{\lambda_n}, \tag{5.220}$$

where

$$p_n = \int_0^1 u_0''(x)\psi_n(x)dx \tag{5.221}$$

and

$$q_n = \int_0^1 a(x)u_0(x)\psi_n(x)dx. \tag{5.222}$$

Since the system $\{\psi_n(x)\}$ is orthonormalized on the space $L_2[0,1]$, from (5.215), (5.221), and (5.222) it follows that

$$\sum_{n=1}^{\infty}(p_n + q_n)^2 < \infty. \tag{5.223}$$

It follows from (5.223) that

$$p_n + q_n \longrightarrow 0 \quad \text{for } n \longrightarrow \infty. \tag{5.224}$$

It follows from relation (5.217) that

$$|u(x,t)| \le \sum_{n=1}^{\infty} |u_n(t)| e^{\lambda_n(t-t_0)} |\psi_n(x)| \tag{5.225}$$

and

$$\left|\frac{\partial^2 u(x,t)}{\partial x^2} + a(x)u(x,t)\right| \le \sum_{n=1}^{\infty} |\lambda_n||u_n| e^{\lambda_n(t-t_0)} |\psi_n(x)|. \tag{5.226}$$

From Lemma 5.11 and relations (5.220) and (5.224) it follows that there exists a number $d_7 > 0$ such that for any n

$$|\lambda_n||u_n|e^{\lambda_n(t-t_0)}\left|\psi_n(x)\right| \le d_7|\lambda_n|e^{\lambda_n(t-t_0)}. \tag{5.227}$$

Since

$$e^{\lambda_n(t-t_0)} = e^{\lambda_n}e^{\lambda_n(t-t_0-1)},$$

it follows from (5.190) that for $t \ge t_0 + 2$

$$d_7|\lambda_n|e^{\lambda_n(t-t_0)} \le d_7\frac{|\lambda_n|}{e^{|\lambda_n|}}e^{-d_2(t-t_0-1)}. \tag{5.228}$$

From the convergence of the series

$$\sum_{n=1}^{\infty}\frac{|\lambda_n|}{e^{|\lambda_n|}}$$

and relation (5.228) it follows that there exists a number $d_8 > 0$ such that for any $t \ge t_0 + 2$

$$d_7\sum_{n=1}^{\infty}|\lambda_n|e^{\lambda_n(t-t_0)} \le d_8 e^{-d_2(t-t_0-1)}. \tag{5.229}$$

Thus, from (5.226), (5.227), and (5.229) it follows that for $t \ge t_0 + 2$

$$\sup_{x\in[0,1]}\left|\frac{\partial^2 u(x,t)}{\partial x^2} + a(x)u(x,t)\right| \le d_8 e^{-d_2(t-t_0-1)}. \tag{5.230}$$

It follows from (5.225), (5.226), and (5.230) that there exists a number $d_9 > 0$ such that for $t \ge t_0 + 2$

$$\sup_{x\in[0,1]}\left|u(x,t)\right| \le d_9 e^{-d_2(t-t_0-1)}. \tag{5.231}$$

It follows from (5.230) and (5.231) that there exists a number $d_{10} > 0$ such that for $t \ge t_0 + 2$

$$\sup_{x\in[0,1]}\left|\frac{\partial^2 u(x,t)}{\partial x^2}\right| \le d_{10} e^{-d_2(t-t_0-1)}. \tag{5.232}$$

Since

$$u'_x(x,t) = \int\limits_{x_0(t)}^{x} u''_{xx}(\xi,t)d\xi, \quad 0 \le x_0(t) \le 1,$$

where

$$u'_x(x_0(t), t) = 0, \quad \text{for any } x \in [0, 1] \text{ and } t \ge t_0$$

we obtain

$$|u'_x(x, t)| \le \sup_{x \in [0,1]} |u''_{xx}(x, t)|. \tag{5.233}$$

It follows from (5.232) and (5.233) that for $t \ge t_0 + 2$

$$\sup_{x \in [0,1]} |u'_x(x, t)| \le d_8 e^{-d_2(t-t_0-1)}. \tag{5.234}$$

Lemma 5.12. *Let* $u(x, t)$ *be a solution of problem* (5.175)–(5.178) *and let* $\Phi(t)$ *be a bounded function continuous over* $[t_0 + 2, \infty)$. *Then the following formula is correct:*

$$\int_{t_0+2}^{\infty} \Phi(t)[u''_{xx}(x, t) + a(x)u(x, t)]dt$$
$$= \frac{\partial^2}{\partial x^2} \int_{t_0+2}^{\infty} \Phi(t)u(x, t)dt + a(x) \int_{t_0+2}^{\infty} \Phi(t)u(x, t)dt.$$

Proof. It follows from (5.199) that the function $u'_x(x, t)$ is measurable and from (5.234) and the notion that

$$\int_{t_0+2}^{\infty} e^{-d^2(t-t_0-1)}dt < \infty$$

it follows that

$$\int_{t_0+2}^{\infty} \Phi(t)u'_x(x, t)dt = \frac{\partial}{\partial x}\left[\int_{t_0+2}^{\infty} \Phi(t)u(x, t)dt\right]. \tag{5.235}$$

From (5.199), (5.232), and (5.235) it follows that

$$\int_{t_0+2}^{\infty} \Phi(t)u''_{xx}(x, t)dt = \frac{\partial^2}{\partial x^2}\left[\int_{t_0+2}^{\infty} \Phi(t)u(x, t)dt\right] \tag{5.236}$$

and the assertion of the lemma follows from (5.236). □

From Lemmas 5.9 and 5.12 follows the following theorem.

Theorem 5.7. *Let* $u(x,t)$ *be the solution of problem* (5.175)–(5.178) *and let* $\Phi(t)$ *be a bounded function that is continuous over* $[0,\infty)$. *Then the following formula is true:*

$$\int_0^\infty \Phi(t)[u''_{xx}(x,t) + a(x)u(x,t)]dt$$
$$= \frac{\partial^2}{\partial x^2}\int_0^\infty \Phi(t)u(x,t)dt + a(x)\int_0^\infty \Phi(t)u(x,t)dt.$$

Lemma 5.13. *Let* $u(x,t)$ *be the solution of problem* (5.175)–(5.178). *Then the following relations are correct:*

$$\lim_{x\longrightarrow 0}\int_0^\infty |u(x,t)|dt = \lim_{x\longrightarrow 1}\int_0^\infty |u(x,t) - h(t)|dt = 0.$$

Proof. It follows from (5.181) and (5.197) that for any $t \geq 0$

$$\lim_{x\longrightarrow 0} u(x,t) = 0, \qquad \lim_{x\longrightarrow 1} u(x,t) = h(t). \tag{5.237}$$

Denote by $g(t)$ the function defined by the formula

$$g(t) = \begin{cases} d_{11}, & 0 \leq t \leq t_0 + 2, \\ d_9 e^{-d_2(t-t_0-1)}, & t > t_0 + 2. \end{cases}$$

Since

$$\int_0^\infty |g(t)|dt < \infty$$

and for any $t \geq 0$

$$|u(x,t)| \leq g(t),$$

the assertion of the lemma will follow by the Lebesgue theorem from (5.237). □

5.4 On estimating the approximate accuracy of a solution to the inverse boundary-value problem for the heat conduction equation with a variable coefficient

5.4.1 Problem posing

Let $a(x) \in C^2[0,1]$, $a(x) \leq 0$, and let a thermal process be described by the equation

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + a(x)u(x,t), \quad 0 < x < 1,\ t > 0, \tag{5.238}$$

where the solution $u(x,t) \in C([0,1]\times[0,\infty)) \cap W_2^{2,1}([0,1]\times[0,\infty))$ satisfies the following initial and boundary conditions:

$$u(x,0) = 0, \quad 0 \le x \le 1, \tag{5.239}$$

$$u(0,t) = 0, \quad t \ge 0, \tag{5.240}$$

$$u(1,t) = h(t), \quad t \ge 0, \tag{5.241}$$

where

$$h(t) \in C^2[0,\infty), \quad h(0) = h'(0) = 0, \tag{5.242}$$

and there exists a number $t_0 > 0$ such that for any $t \ge t_0$

$$h(t) = 0. \tag{5.243}$$

Assume that the function $h(t)$ is unknown and must be defined. Instead, at the point $x_1 \in (0,1)$ the temperature $f(t)$ of the rod corresponding to this process is measured, so we have

$$u(x_1,t) = f(t), \quad t \ge 0. \tag{5.244}$$

Let the set M_r be defined by the formula

$$M_r = \left\{ h(t) : h(t) \in L_2[0,\infty), \int_0^\infty |h(t)|^2 dt + \int_0^\infty |h'(t)|^2 dt \le r^2 \right\}, \tag{5.245}$$

where $h'(t)$ is the derivative of the function $h(t)$ and r is a known positive number. Then assume that for $f(t) = f_0(t)$, being a part of condition (5.244), there exists a function $h_0(t)$, that belongs to the set M_r, but the function $f_0(t)$ is unknown. Instead, the approximate function $f_\delta(t) \in L_2[0,\infty) \cap L_1[0,\infty)$ and number $\delta > 0$ are given such that

$$\|f_\delta - f_0\|_{L_2} \le \delta. \tag{5.246}$$

Using f_δ, δ, and M_r, it is required to define an approximate solution $h_\delta(t)$ of problem (5.238)–(5.241), (5.244) and estimate the deviation $\|h_\delta - h_0\|_{L_2}$ of the approximate solution h_δ from the exact solution h_0.

Let

$$\overline{H} = L_2[0,\infty) + iL_2[0,\infty)$$

over the field of complex numbers and let F be an operator mapping $L_2[0,\infty)$ into $\overline{H}$, defined by the formula

$$F[h(t)] = \frac{1}{\sqrt{\pi}} \int_0^\infty h(t) e^{-i\tau t} dt, \quad \tau \ge 0,\ h(t) \in L_2[0,\infty). \tag{5.247}$$

The proof that the operator F is isometric is given in Lemma 4.3.

It follows from Lemma 5.13 and Theorem 5.7 that the transformation F is applicable to solving problem (5.238).

Applying transformation F to equation (5.238) we obtain

$$\frac{\partial^2 \hat{u}(x,\tau)}{\partial x^2} + a(x)\hat{u}(x,\tau) = i\tau\hat{u}(x,\tau), \quad x \in (0,1),\ \tau \geq 0, \tag{5.248}$$

where

$$\hat{u}(x,\tau) = F[u(x,t)].$$

It follows from (5.241) and (5.244) that

$$\hat{u}(0,\tau) = 0, \quad \tau \geq 0, \tag{5.249}$$

$$\hat{u}(x_1,\tau) = i\hat{f}(\tau), \quad \tau \geq 0, \tag{5.250}$$

where

$$\hat{f}(\tau) = F[f(t)].$$

It follows from (5.197) that the solution $\hat{u}(x,\tau)$ of problem (5.248)–(5.250) is continuous on the band $[0,1] \times [0,\infty)$.

From the general solution of the ordinary linear differential equation of the second order, it follows that the solution $\hat{u}(x,\tau)$ of problem (5.248)–(5.250) is defined by the formula

$$\hat{u}(x,\tau) = l(\tau)e(x,\tau), \quad x \in [0,1],\ \tau \geq 0, \tag{5.251}$$

where $l(\tau)$ is a certain function and $e(x,\tau)$ is the solution of problem (5.248), (5.249), satisfying the condition

$$e'_x(0,\tau) = 1.$$

Using the condition (5.250) define the function $l(\tau)$ by the formula

$$l(\tau) = \frac{i\hat{f}(\tau)}{e(x_1,\tau)}, \quad \tau \geq 0. \tag{5.252}$$

By (5.251) and (5.252),

$$\hat{u}(1,\tau) = i\hat{f}(\tau)e^{-1}(x_1,\tau)e(1,\tau), \quad \tau \geq 0. \tag{5.253}$$

Lemma 5.14. *The function $l(\tau)$ is continuous on the half-line $[0,\infty)$.*

Proof. Since $\hat{f}(\tau)$ and $e(x_1,\tau)$ are continuous on the half-line $[0,\infty)$, to prove the theorem it is sufficient to make sure that

$$e(x_1,\tau) \neq 0 \quad \text{for any } \tau \geq 0.$$

Assume the contrary, i. e., that there exists a number $\tau_0 \geq 0$ such that

$$e(x_1, \tau_0) = 0. \tag{5.254}$$

Then taking into account (5.254), consider the space

$$H_0 = L_2[0, x_1] + iL_2[0, x_1]$$

over the field of complex numbers and an operator A_1, acting from H_0 into H_0 which is defined by the formula

$$A_1 u(x) = \frac{d^2 u(x)}{dx^2} + a(x)u(x), \quad u \in D(A_1), \tag{5.255}$$

where

$$D(A_1) = \{u : u, A_1 u \in H_0,\ u(0) = u(x_1) = 0\}. \tag{5.256}$$

It follows from (5.255) and (5.256) that the operator A_1 is negatively defined and self-adjoint. Therefore, there exists a number $\overline{\lambda}_1 < 0$ such that the spectrum

$$\operatorname{Sp}(A_1) \subset (-\infty, \overline{\lambda}_1].$$

Since

$$A_1 e(x, \tau_0) = i\tau_0 e(x, \tau_0),$$

we know

$$e(x, \tau_0) = 0 \quad \text{for } x \in [0, x_1] \quad \text{and} \quad e'_x(1, \tau_0) = 0,$$

which contradicts the definition of the function $e(x, \tau)$. The lemma is thereby proved. □

Let

$$\lambda = \sqrt{\tau} \quad \text{and} \quad e_1(x, \lambda) = e(x, \tau).$$

Then the function $e_1(x, \lambda)$ will satisfy the integral equation

$$e_1(x, \lambda) = \frac{\sinh \mu_0 x \lambda}{\mu_0 \lambda} - \int_0^x \frac{\sinh \mu_0 (x - \xi)\lambda}{\mu_0 \lambda} a(\xi) e_1(\xi, \lambda) d\xi, \tag{5.257}$$

where

$$\mu_0 = \frac{1}{\sqrt{2}}(1 + i), \quad x \in [0, 1],\ \lambda \geq 0.$$

Lemma 5.15. *Let $a(x) \in C^2[0,1]$. Then there exists a number $\lambda_1 > 0$ such that for any $\lambda \geq \lambda_1$ the following inequalities are true:*

$$\frac{2}{3}\frac{|\sinh\mu_0 x\lambda|}{\lambda} \leq |e_1(x,\lambda)| \leq \frac{4}{3}\frac{|\sinh\mu_0 x\lambda|}{\lambda}.$$

Proof. Let

$$\varepsilon(x,\lambda) = \frac{\mu_0\lambda}{\sinh\mu_0 x\lambda} e_1(x,\lambda).$$

Then from (5.257) it follows that

$$\varepsilon(x,\lambda) = 1 - \frac{1}{\mu_0\lambda}\int_0^x \frac{\sinh\mu_0(x-\xi)\lambda \sinh\mu_0\xi\lambda}{\sinh\mu_0 x\lambda} a(\xi)\varepsilon(\xi,\lambda)d\xi. \tag{5.258}$$

Since

$$\left|\frac{\sinh\mu_0(x-\xi)\lambda \sinh\mu_0\xi\lambda}{\sinh\mu_0 x\lambda}\right| = 1 + o(1) \quad \text{for } \lambda \longrightarrow \infty,$$

from (5.258) it follows that there exists a number $\lambda_1 > 0$ such that for any $\lambda \geq \lambda_1$ the following inequality is correct:

$$\left|\frac{1}{\mu_0\lambda}\int_0^x \frac{\sinh\mu_0(x-\xi)\lambda \sinh\mu_0\xi\lambda}{\sinh\mu_0 x\lambda} a(\xi)d\xi\right| \leq \frac{1}{4}. \tag{5.259}$$

We will search for a solution of equation (5.258) in the form of the series

$$\varepsilon(x,\lambda) = \sum_{k=0}^{\infty} \varepsilon_k(x,\lambda), \tag{5.260}$$

where $\varepsilon_0(x,\lambda) = 1$ and

$$\varepsilon_{k+1}(x,\lambda) = -\frac{1}{\mu_0\lambda}\int_0^x \frac{\sinh\mu_0(x-\xi)\lambda \sinh\mu_0\xi\lambda}{\sinh\mu_0 x\lambda} a(\xi)\varepsilon_k(\xi,\lambda)d\xi. \tag{5.261}$$

According to (5.259)–(5.261), for any values of k, $\lambda \geq \lambda_1$, and $x \in [0,1]$

$$|\varepsilon_k(x,\lambda)| \leq 4^{-k}. \tag{5.262}$$

From (5.260)–(5.262) it follows that for any values of $x \in [0,1]$ and $\lambda \geq \lambda_1$

$$2/3 = 1 - \sum_{k=1}^{\infty} 4^{-k} \leq |\varepsilon(x,\lambda)| \leq \sum_{k=0}^{\infty} 4^{-k}.$$

Thus, for any values of $x \in [0,1]$ and $\lambda \geq \lambda_1$ we have

$$\frac{2}{3}\frac{|\sinh\mu_0 x\lambda|}{\lambda} \leq |e_1(x,\lambda)| \leq \frac{4}{3}\frac{|\sinh\mu_0 x\lambda|}{\lambda}.$$

The lemma is thereby proved. □

Since

$$|\sinh \mu_0 x\lambda| = e^{\frac{x\lambda}{\sqrt{2}}}(1 + o(1)) \quad \text{at } \lambda \longrightarrow \infty,$$

from Lemma 5.15 it follows that there exists a number $\lambda_2 \geq \lambda_1$ such that for any values of $\lambda \geq \lambda_2$ and $x \in [0, 1]$

$$\frac{1}{3}\frac{e^{\frac{x\lambda}{\sqrt{2}}}}{\lambda} \leq |e_1(x, \lambda)| \leq \frac{8}{3}\frac{e^{\frac{x\lambda}{\sqrt{2}}}}{\lambda}. \tag{5.263}$$

Denote by L the operator acting from the space $\overline{H}$ into $\overline{H}$ defined by the formula

$$L\hat{f}(\tau) = i\frac{e(1, \tau)}{e(x_1, \tau)}\hat{f}(\tau),$$

where $e(x, \tau)$ is defined by formula (5.251).

Further, without changing the notation extend the operator L to the maximum, i. e., assume that

$$D(L) = \left\{\hat{f}(\tau) : \hat{f}(\tau) \in \overline{H} \text{ and } i\frac{e(1, \tau)}{e(x_1, \tau)}\hat{f}(\tau) \in \overline{H}\right\} \tag{5.264}$$

and

$$L\hat{f}(\tau) = i\frac{e(1, \tau)}{e(x_1, \tau)}\hat{f}(\tau), \quad \tau \geq 0. \tag{5.265}$$

From Lemma 5.15 and relations (5.264)–(5.265) it follows that the operator L is linear and unbounded.

Denote $\hat{u}(1, \tau)$ by $\hat{h}(\tau)$, where

$$\hat{h}(\tau) = F[h(t)].$$

Write problem (5.253) as a problem of calculating values of the unbounded operator L as follows:

$$\hat{h}(\tau) = L\hat{f}(\tau), \quad \tau \geq 0, \hat{f}(\tau) \in D(L). \tag{5.266}$$

Let $\hat{M}_r \supset F[M_r]$, where M_r is defined by formula (5.245). Then

$$\hat{M}_r = \left\{\hat{h}(\tau) : \hat{h}(\tau) \in \overline{H}, \int_0^\infty (1 + \tau^2)|\hat{h}(\tau)|^2 d\tau \leq r^2\right\}. \tag{5.267}$$

Let

$$\hat{f}_0(\tau) = F[f_0(t)] \quad \text{and} \quad \hat{f}_\delta(\tau) = F[f_\delta(t)].$$

Then from condition (5.246) it follows that

$$\|\hat{f}_\delta(\tau) - \hat{f}_0(\tau)\|_{\overline{H}} \le \delta, \tag{5.268}$$

where

$$\hat{f}_0(\tau) \in D(L) \quad \text{and} \quad \hat{h}_0(\tau) = L\hat{f}_0(\tau)$$

satisfy the condition

$$\hat{h}_0(\tau) \in \hat{M}_r. \tag{5.269}$$

By using the a priori information $\hat{f}_\delta(\lambda)$, δ and conditions (5.268) and (5.269) it is required to define the approximate value $\hat{h}_\delta(\lambda)$ of the operator L and estimate its error $\|\hat{h}_\delta - \hat{h}_0\|$.

5.4.2 Calculation of the approximate values of the operator *L*

Split problem (5.266)–(5.269) into two problems. The first problem is

$$\hat{h}^1(\tau) = L^1\hat{f}^1(\tau), \quad 0 \le \tau \le \lambda_2^2, \tag{5.270}$$

where

$$\hat{h}^1(\tau) = \hat{h}(\tau) \quad \text{under } \tau \in [0, \lambda_2^2],$$
$$\hat{f}^1(\tau) = \hat{f}(\tau) \quad \text{under } \tau \in [0, \lambda_2^2],$$

and

$$L^1\hat{f}^1(\tau) = L\hat{f}(\tau) \quad \text{under } \tau \in [0, \lambda_2^2].$$

Since from Lemma 5.14 it follows that the function $\frac{e(1,\tau)}{e(x_1,\tau)}$ is continuous on the half-line $[0,\infty)$, there exists a number d_{12} such that for any $\tau \in [0, \lambda_2^2]$

$$\left|\frac{e(1,\tau)}{e(x_1,\tau)}\right| \le d_{12}. \tag{5.271}$$

Problem (5.270), (5.271) is a problem of calculating values of the bounded operator. From relation (5.271) it follows that problem (5.271) is well-posed on the space

$$\overline{H}_1 = L_2[0, \lambda_2^2] + iL_2[0, \lambda_2^2].$$

The second problem is a problem of calculating values of the unbounded operator L^2 on the space

$$\overline{H}_2 = L_2[\lambda_2^2, \infty) + iL_2[\lambda_2^2, \infty).$$

We have

$$\hat{h}^2(\tau) = L^2\hat{f}^2(\tau), \quad \tau \ge \lambda_2^2. \tag{5.272}$$

To solve problem (5.272) we use the family $\{L_\alpha^2 : \alpha \ge \lambda_2^2\}$ of linear bounded operators L_α^2, mapping the space $\overline{H}_2$ into $\overline{H}_2$ and defined by the formula

$$L_\alpha^2\hat{f}^2(\tau) = \begin{cases} L^2\hat{f}^2(\tau), & \tau \le \alpha, \\ 0, & \tau > \alpha. \end{cases} \tag{5.273}$$

We define the approximate value $\hat{h}_\delta^{2,\alpha}(\tau)$ of the operator L^2 by the formula

$$\hat{h}_\delta^{2,\alpha}(\tau) = L_\alpha^2\hat{f}_\delta^2(\tau), \quad \tau \ge \lambda_2^2. \tag{5.274}$$

Then

$$\|\hat{h}_\delta^{2,\alpha}(\tau) - \hat{h}_0^2(\tau)\| \le \|\hat{h}_\delta^{2,\alpha}(\tau) - \hat{h}_0^{2,\alpha}(\tau)\| + \|\hat{h}_0^{2,\alpha}(\tau) - \hat{h}_0^2(\tau)\|. \tag{5.275}$$

Since

$$\|\hat{h}_0^{2,\alpha}(\tau) - \hat{h}_0^2(\tau)\|^2 \le \int_\alpha^\infty |\hat{h}_0(\tau)|^2 d\tau, \quad \hat{h}_0(\tau) \in \hat{M}_r. \tag{5.276}$$

It follows from (5.267) and (5.269) that

$$\int_\alpha^\infty |\hat{h}_0(\tau)|^2 d\tau \le \frac{1}{1+\alpha^2}\int_\alpha^\infty (1+\tau^2)|\hat{h}_0(\tau)|^2 d\tau \le \frac{r^2}{1+\alpha^2}. \tag{5.277}$$

It follows from (5.276) and (5.277) that

$$\|\hat{h}_0^{2,\alpha}(\tau) - \hat{h}_0^2(\tau)\| \le \frac{r}{\sqrt{1+\alpha^2}}. \tag{5.278}$$

It follows from (5.268) and (5.274) that

$$\|\hat{h}_\delta^{2,\alpha}(\tau) - \hat{h}_0^{2,\alpha}(\tau)\| \le \|L_\alpha^2\|\delta. \tag{5.279}$$

Since it follows from (5.270)–(5.273) that

$$\|L_\alpha^2\| = \max_{\lambda_2^2 \le \tau \le \alpha} \frac{|e(1,\tau)|}{|e(x_1,\tau)|}, \tag{5.280}$$

by (5.275) and (5.278)–(5.280) we obtain

$$\|\hat{h}_\delta^{2,\alpha}(\tau) - \hat{h}_0^2(\tau)\| \le \frac{r}{\sqrt{1+\alpha^2}} + \delta \max_{\lambda_2^2 \le \tau \le \alpha} \frac{|e(1,\tau)|}{|e(x_1,\tau)|}. \tag{5.281}$$

It follows from (5.263) and (5.280) that

$$\frac{1}{16}e^{(1-x_1)\sqrt{\alpha/2}} \le \|L^2_\alpha\| \le 16e^{(1-x_1)\sqrt{\alpha/2}}. \tag{5.282}$$

Since it follows from relation (5.276) that

$$\sup_{\hat{h}_0\in\hat{M}_r} \|\hat{h}^{2,\alpha}_0(\tau) - \hat{h}^2_0(\tau)\|^2 = \sup_{\hat{h}_0\in\hat{M}_r} \int_\alpha^\infty |\hat{h}_0(\tau)|^2 d\tau,$$

it follows from (5.277) and (5.278) that

$$\sup_{\hat{h}_0\in\hat{M}_r} \|\hat{h}^{2,\alpha}_0 - \hat{h}^2_0\| = \frac{r}{\sqrt{1+\alpha^2}}. \tag{5.283}$$

If the value of the parameter $\overline{\alpha} = \overline{\alpha}(\delta)$ in formula (5.274) is selected from the equation

$$\frac{16r}{\sqrt{1+\alpha^2}} = \delta e^{(1-x_1)\sqrt{\alpha/2}}, \tag{5.284}$$

then it follows from (5.281) and (5.284) that

$$\|\hat{h}^{2,\overline{\alpha}(\delta)}_\delta - \hat{h}^2_0\| \le \frac{2r}{\sqrt{1+\overline{\alpha}^2(\delta)}}. \tag{5.285}$$

Since the functions $\sqrt{1+\alpha^2}$ and $e^{(1-x_1)\sqrt{\alpha/2}} \in C[\lambda^2_2, \infty)$ are strictly increasing, it follows from Theorem 1 proved in [90] that estimate (5.285) is accurate-by-order, i. e., there exists a number $d_{13} > 0$ such that for sufficiently small values of δ the following relation is correct:

$$\sup\{\|\hat{h}^{2,\overline{\alpha}(\delta)}_\delta - \hat{h}^2_0\| : \hat{h}^2_0 \in \hat{M}_r, \ \|\hat{f}^2_\delta - \hat{f}^2_0\| \le \delta\} \ge d_{13}(1+\overline{\alpha}^2(\delta))^{-\frac{1}{2}}.$$

It follows from Theorem 2 proved in [90] that the method $\{L^2_{\overline{\alpha}(\delta)} : 0 < \delta \le \delta_0\}$, defined by formulas (5.273) and (5.284), will be optimal-by-order on the class $\hat{M}_r$, i. e., there exists a number $d_{14} > 0$ such that for sufficiently small values of δ the following relation is correct:

$$\frac{2r}{\sqrt{1+\overline{\alpha}^2(\delta)}} \le d_{14} \sup\{\|L^2\hat{f}^2(\tau)\| : \|\hat{f}^2\| \le \delta, \ L^2\hat{f}^2 \in \hat{M}_r\}.$$

Now, alongside with equation (5.284), consider the following two equations:

$$e^{(1-x_1)\sqrt{\frac{\alpha}{2}}} = \frac{16r}{\delta}, \tag{5.286}$$

$$e^{2(1-x_1)\sqrt{\frac{\alpha}{2}}} = \frac{r}{16\delta}. \tag{5.287}$$

Denote the solution of equations (5.286) and (5.287) by $\overline{\alpha}_1(\delta)$ and $\overline{\alpha}_2(\delta)$, respectively. Then

$$\overline{\alpha}_1(\delta) = \frac{2}{(1-x_1)^2}\ln^2\frac{16r}{\delta} \quad \text{and} \quad \overline{\alpha}_2(\delta) = \frac{1}{2(1-x_1)^2}\ln^2\frac{r}{16\delta}.$$

There exists $\alpha_1 > \lambda_2^2$ such that for $\alpha \geq \alpha_1$ the following relations will be correct:

$$e^{(1-x_1)\sqrt{\frac{\alpha}{2}}} \leq \sqrt{1+\alpha^2}e^{(1-x_1)\sqrt{\alpha/2}} \leq e^{2(1-x_1)\sqrt{\frac{\alpha}{2}}}. \tag{5.288}$$

Therefore, from (5.284) and (5.286)–(5.288) it will follow that for $\alpha \geq \alpha_1$

$$\overline{\alpha}_2(\delta) \leq \overline{\alpha}(\delta) \leq \overline{\alpha}_1(\delta). \tag{5.289}$$

Thus, it will follow from (5.289) that

$$\overline{\alpha}(\delta) \sim \ln^2\delta \quad \text{under } \delta \longrightarrow 0. \tag{5.290}$$

It follows from (5.290) that there exists a number $d_{14} > 0$ such that for sufficiently small values of δ the following estimate is true:

$$\|\hat{h}_\delta^{2,\overline{\alpha}(\delta)} - \hat{h}_0^2\| \leq d_{14}\ln^{-2}\delta. \tag{5.291}$$

We will define the solution of problem (5.270) by the formula

$$\hat{h}_\delta^1(\tau) = L^1\hat{f}_\delta^1(\tau), \quad 0 \leq \tau \leq \lambda_2^2. \tag{5.292}$$

It follows from (5.271) and (5.292) that

$$\|\hat{h}_\delta^1(\tau) - \hat{h}_0^1(\tau)\| \leq d_{12}\delta. \tag{5.293}$$

We define the final solution $\hat{h}_\delta(\tau)$ of problem (5.266)–(5.269) by the formula

$$\hat{h}_\delta(\tau) = \begin{cases} \hat{h}_\delta^1(\tau), & 0 \leq \tau \leq \lambda_2^2, \\ \hat{h}_\delta^{2,\overline{\alpha}(\delta)}(\tau), & \tau \geq \lambda_2^2. \end{cases} \tag{5.294}$$

It follows from (5.291), (5.293), and (5.294) that there exists a number $d_{15} > 0$ such that for sufficiently small values of δ

$$\|\hat{h}_\delta(\tau) - \hat{h}_0(\tau)\| \leq d_{15}\ln^{-2}\delta. \tag{5.295}$$

Now consider the subspace $\overline{H}_0$, defined by the formula

$$\overline{H}_0 = F[L_2[0,\infty)],$$

and denote by $\overline{h}_\delta(\tau)$ an element defined by the formula

$$\overline{h}_\delta(\tau) = \text{pr}(\hat{h}_\delta(\tau), \overline{H}_0).$$

Since $\hat{h}_0(\tau) \in \overline{H}_0$, from (5.295) it follows that

$$\|\overline{h}_\delta(\tau) - \hat{h}_0(\tau)\| \le d_{15} \ln^{-2} \delta. \tag{5.296}$$

Finally, we define the solution $h_\delta(t)$ of the inverse problem (5.238)–(5.240), (5.244) by the formula

$$h_\delta(t) = F^{-1}[\overline{h}_\delta(\tau)]. \tag{5.297}$$

It follows from (5.296) and (5.297) that

$$\|h_\delta(t) - h_0(t)\| \le d_{15} \ln^{-2} \delta.$$

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