Strongly Reciprocally $(\mathfrak{p}, \mathfrak{h})$-Convex Functions And Some Inequalities

By

Farah Jamil

(Registration No: 00000402935)

Supervisor: **Dr. Matloob Anwar**

A thesis submitted in partial fulfillment of the requirements

for the degree of Master of Science in Mathematics

School of Natural Sciences

National University of Sciences and Technology (NUST)

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LIST OF SYMBOLS and ABBREVIATIONS

$\mathcal{R}$	The set of real numbers
$\mathcal{R}^n$	n-dimensional real space
$\mathcal{R}^+$	The set of positive real numbers
$\mathcal{I}$	Subset of $\mathcal{R}$
epi$\mathfrak{F}$	Epigraph of the Function $\mathfrak{F}$
$\mathfrak{F}'_-$	Left derivative of the function $\mathfrak{F}$
$\mathfrak{F}'_+$	Right derivative of the function $\mathfrak{F}$
SR($\mathfrak{p}\mathfrak{h}$)	Strongly reciprocally $(\mathfrak{p}, \mathfrak{h})$-convex function
SX($\mathfrak{h}, \mathcal{I}$)	$\mathfrak{h}$-convex function
SV($\mathfrak{h}, \mathcal{I}$)	$\mathfrak{h}$-concave function
Q($\mathcal{I}$)	Godunova–Levin function
$\mathrm{K}_{\mathfrak{s}}^2$	$\mathfrak{s}$-convex function in the second sense
P($\mathcal{I}$)	$\mathfrak{p}$-convex function
$\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}$	A mapping defined on integrable functions $\mathfrak{F}$ and $\mathfrak{w}$
$\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}$	A mapping defined on integrable functions $\mathfrak{F}$ and $\mathfrak{w}$

Abstract

This thesis is a detailed study of convex functions, focusing on $\mathfrak{h}$-convex, $\mathfrak{p}$-convex, and strongly reciprocally $(\mathfrak{p}, \mathfrak{h})$-convex functions of higher order cases. The main objective of this research is to enhance both the understanding and analysis of these generalized convex functions, with a focus on their connections to established mathematical inequalities, such as the Hermite-Hadamard inequality, Fejér inequality, and fractional integral inequalities. The foundational concept of convexity, its basic properties, and applications are reexamined. The utility of convex functions in numerous areas ranging from theoretical research to practical applications of pure and applied sciences including, economic models, mathematical analysis, and optimization problems is highlighted. The role of inequality theory is also pointed out for convex functions. The classical convexity is extended to strongly reciprocally $(\mathfrak{p}, \mathfrak{h})$-convex functions of higher order instances, enabling a deeper insight into how convex functions can be applied to mathematical inequalities, such as the Hermite-Hadamard inequality, the Fejér inequality, and fractional integral inequalities are explored. A novel mapping $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ for $\mathfrak{h}$-convex functions is explored along a series of useful results. These results are formulated through lemmas and propositions, which are then utilized to derive some generalized Fejér-type inequalities and an improved variant of Hermite-Hadamard inequalities. A second mapping, $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ for

$\mathfrak{h}$-convex function is introduced, with the further contribution of some results that leads to the derivation of a significant theorem. These findings contribute new tools for mathematical analysis and significantly broaden the scope of known results to generalize convex functions and inequalities.

Chapter 1

Introduction

Convexity plays a key role in mathematical analysis, due to its natural and robust problem-solving approach in different areas of science such as optimization, economics, physics, and engineering. Its simple yet rich properties make it an essential component of study across the broad spectrum of scientific fields.

Before delving into my research work, this chapter includes a general background of convexity and convex functions, providing readers with a glimpse into modern approaches to convex analysis, advancement, extensions, and generalizations, illustrating how these concepts are applied across different fields. Finally, we discuss some well-known inequalities for convex functions, such as Hermite-Hadamard inequality and Fejér-type inequality.

History contains many concepts that have been essential in the development of mathematics. Of these, convexity stands out as one of the exceptional concepts, serving as the foundation of numerous theories and scientific developments. Its origins can be traced back to ancient Archimedes and Greece, where Euclidean geometry examined convex figures and their properties, see [1] and [2] for details. The significance of these was recognized by many famous mathematicians, leading to further exploration and development. The subsequent mathematicians contributed innumerable extensions and generalizations of the initially developed concepts, which became solutions to many problems across multiple science disciplines. The systematic study of convexity began at the end of the nineteenth century, with contributions from O. Hölder (1889) [3], O. Stolz (1893) [4], Ch. Hermite (1883) [5] and J. Hadamard (1893) [6]. J.L.W.V. Jensen (1905, 1906) [7, 8] was among the first to recognize its value and he made a structured analysis of convexity and provided its modern definition.

Convexity constitutes two fundamental concepts: convex sets, and convex functions. Below, I have defined these concepts with some properties.

1.1 Convex Set

Definition 1.1.1. *[9] For a non-empty set $\mathcal{I} \subseteq \mathcal{R}$ to be convex, consider any two distinct points $\alpha_1, \alpha_2 \in \mathcal{I}$ and ζ in $[0, 1]$ we have*

$$\alpha_1\zeta + (1 - \zeta)\alpha_2 \in \mathcal{I}. \tag{1.1}$$

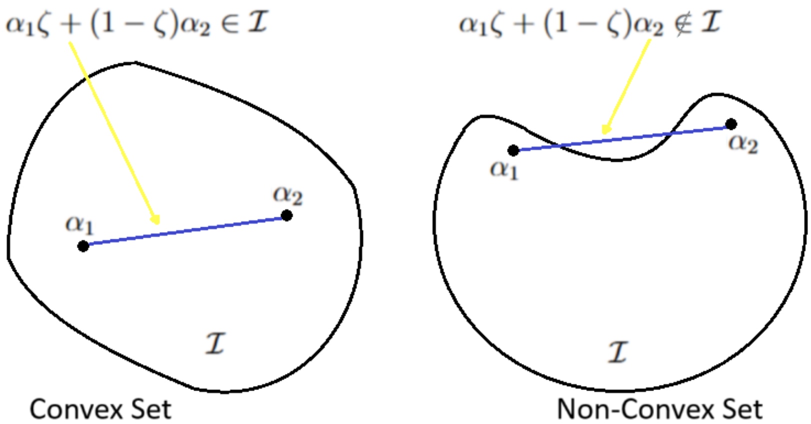

Figure 1.1: Convex set vs Non-convex set

In Figure 1.1 we can see the key differences between a convex set and a non-convex set. It can be viewed geometrically, every point on the line between any two distinct points in the set $\mathcal{I}$ is also contained in the set, i.e. $\forall \alpha_1, \alpha_2 \in \mathcal{I}$ and $\zeta \in [0,1]$, all the points on the line between α_1 and α_2 can be written as $\mathfrak{z} = \alpha_1\zeta + (1-\zeta)\alpha_2 \in \mathcal{I}$, hence we can say that $\mathcal{I}$ contains no indentation or holes. Every real interval can be expressed as a convex combination of any two distinct points within its interval, that is each real interval is a convex set. The result of the intersection between any collection of convex sets is again a convex set. Similarly, convexity is preserved between the addition of two convex sets, and the scalar multiplication of a convex set is just like a vector space.

1.2 Convex Function

Let's come to the main idea of a classical convex function over a real interval.

Definition 1.2.1. *[9] Consider a function $\mathfrak{F} : \mathcal{I} = [\alpha_1, \alpha_2] \to \mathcal{R}$, where domain of the convex function $\mathfrak{F}$ is a convex set $\emptyset \neq \mathcal{I} \subseteq \mathcal{R}$. The function $\mathfrak{F}$ is termed as a convex function if it satisfies the inequality*

$$\mathfrak{F}\big(\zeta\mathfrak{a} + (1-\zeta)\mathfrak{b}\big) \leq \zeta\mathfrak{F}(\mathfrak{a}) + (1-\zeta)\mathfrak{F}(\mathfrak{b}), \quad \forall \zeta \in [0,1], \tag{1.2}$$

where $\mathfrak{a}, \mathfrak{b} \in [\alpha_1, \alpha_2]$ and $\mathfrak{a} \neq \mathfrak{b}$.

If inequality (1.2) is reversed (i.e. $\leq$ is changed with $\geq$), then it leads to the function being concave. Likewise, if inequality (1.2) $\leq$ is modified with $<$, then $\mathfrak{F}$ becomes strictly convex function.

For a particular case, we obtain a midpoint convex function for value $\zeta = \frac{1}{2}$ which is a well-known midpoint convex function as follows:

$$\mathfrak{F}\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) \leq \frac{\mathfrak{F}(\mathfrak{a})+\mathfrak{F}(\mathfrak{b})}{2} \qquad \forall \mathfrak{a}, \mathfrak{b} \in \mathcal{I}.$$

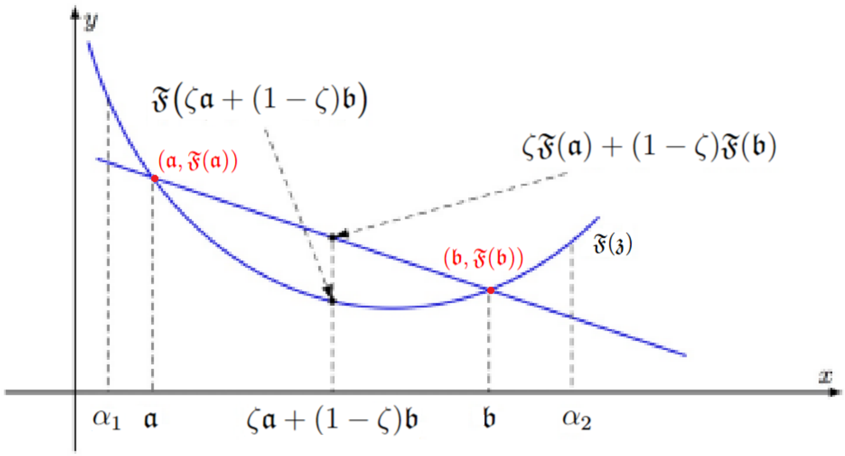

Figure 1.2: Convex function

The geometric interpretation of the equation in (1.2) is straightforward. We can see in Figure 1.2 the line segment joined by two distinct points $(\mathfrak{a}, \mathfrak{F}(\mathfrak{a}))$ and $(\mathfrak{b}, \mathfrak{F}(\mathfrak{b}))$ always lies either on or above the curve of the function $\mathfrak{F}$ within the interval from $\mathfrak{a}$ to $\mathfrak{b}$.

Example 1.2.1. Below are a few examples of convex function $\mathfrak{F}\colon \mathcal{R} \to \mathcal{R}$

- Linear Function: $\mathfrak{F}(x) = \mathfrak{a}x + b$.
- Quadratic functions: $\mathfrak{F}(x) = \mathfrak{a}x^2 + \mathfrak{b}x + c$, where $\mathfrak{a} > 0$.
- Power function: $\mathfrak{F}(x) = x^n$, $\forall n \geq 1$.
- Absolute value function: $\mathfrak{F}(x) = |x|$.

- Exponential function: $\mathfrak{F}(x) = e^{\mathfrak{b}x}$, where $\mathfrak{b}$ is a constant.

Definition 1.2.2 (Epigraph of a function)**.** *[9] An epigraph is a set of points positioned either on or above the graph of the function* $\mathfrak{F} : \mathcal{I} \subseteq \mathcal{R} \to \mathcal{R}$ *defined as:*

$$epi\mathfrak{F} = \{(\mathfrak{a}, \mathfrak{t}) \in \mathcal{I} \times \mathcal{R} : \mathfrak{t} \geq \mathfrak{F}(\mathfrak{a})\}.$$

In the following graph, we can visualize the epigraph as a collection of all those points that belong to the colored region above the graph of the function $\mathfrak{F}$.

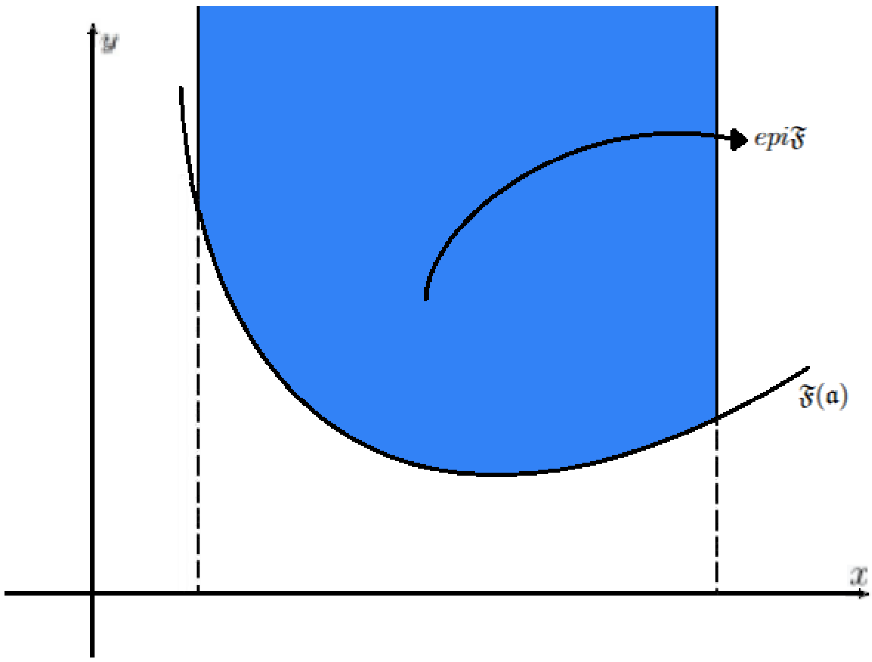

Figure 1.3: Epigraph of the function $\mathfrak{F}$

1.2.1 Properties of Convex Function

- If the second derivative of the function $\mathfrak{F}$ is non-negative for every point in domain $\mathcal{I}$ then it is a convex function.

- Due to the nature of the convex function, every local minimum within its domain is guaranteed to be a global minimum.
- A function $\mathfrak{F}$ is known as convex if and only if the epigraph of $\mathfrak{F}$ is a convex set.

1.3 Continuity and Differentiability of Convex functions

Continuity and differentiability are essential in the mathematical analysis of convex function to understand its behavior and practical use.

1.3.1 Boundedness

Considering a finite convex function $\mathfrak{F}$ on $\mathcal{I} = [\mathfrak{a}, \mathfrak{b}]$ is bounded above by $\mathrm{M} = \max\big(\mathfrak{F}(\mathfrak{a}), \mathfrak{F}(\mathfrak{b})\big)$, any point $\mathfrak{z} \in \mathcal{I}$ can be written as $\mathfrak{z} = \zeta\mathfrak{a} + (1-\zeta)\mathfrak{b}$,we obtain the following relation:

$$\mathfrak{F}(\mathfrak{z}) \leqslant \zeta\mathfrak{F}(\mathfrak{a}) + (1-\zeta)\mathfrak{F}(\mathfrak{b}) \leqslant \zeta\mathrm{M} + (1-\zeta)\mathrm{M} = \mathrm{M}.$$

Moreover, it has a lower bound that is determined by expressing any point in the form: $\frac{\mathfrak{a}+\mathfrak{b}}{2} + t$. then

$$\mathfrak{F}\big(\frac{\mathfrak{a}+\mathfrak{b}}{2}\big) \leqslant \frac{1}{2}\mathfrak{F}\big(\frac{\mathfrak{a}+\mathfrak{b}}{2} + t\big) + \frac{1}{2}\mathfrak{F}\big(\frac{\mathfrak{a}+\mathfrak{b}}{2} - t\big),$$

or

$$\mathfrak{F}\big(\frac{\mathfrak{a}+\mathfrak{b}}{2} + t\big) \leqslant 2\mathfrak{F}\big(\frac{\mathfrak{a}+\mathfrak{b}}{2}\big) - \mathfrak{F}\big(\frac{\mathfrak{a}+\mathfrak{b}}{2} - t\big).$$

As M is the upper bound, we have $-\mathfrak{F}[\frac{(\mathfrak{a}+\mathfrak{b})}{2} - t] \geq -\mathrm{M}$, so

$$\mathfrak{F}(\frac{\mathfrak{a}+\mathfrak{b}}{2} + t) \leqslant 2\mathfrak{F}(\frac{\mathfrak{a}+\mathfrak{b}}{2}) - \mathrm{M} = \mathrm{m}.$$

Therefore, convex functions are continuous on the interior of their domain, but continuity at the boundary points is not guaranteed, as the function might display upward jumps.

1.3.2 Continuity and differentiability

The definition of convex functions ensures its smoothness and continuity. Moreover, it is evident, from the geometrical interpretation and its property of having a non-negative second derivative. The following proposition highlights the geometric significance of the convex function:

Proposition 1. [10] Consider $\mathfrak{F}$ be a convex function on $(\mathfrak{a}, \mathfrak{b})$. If $\mathfrak{a} < \mathrm{c} < \mathrm{u} < \mathrm{d} < \mathfrak{b}$, and the corresponding points $C = (\mathrm{c}, \mathfrak{F}(\mathrm{c}))$, $U = (\mathrm{u}, \mathfrak{F}(\mathrm{u}))$, $D = (\mathrm{d}, \mathfrak{F}(\mathrm{d}))$, we have the following relation between the slopes:

$$slopeCU = \frac{\mathfrak{F}(\mathrm{u}) - \mathfrak{F}(\mathrm{c})}{\mathrm{u} - \mathrm{c}} \leq slopeCD = \frac{\mathfrak{F}(\mathrm{d}) - \mathfrak{F}(\mathrm{c})}{\mathrm{d} - \mathrm{c}} \leq slopeDU = \frac{\mathfrak{F}(\mathrm{d}) - \mathfrak{F}(\mathrm{u})}{\mathrm{d} - \mathrm{u}}.$$

Theorem 1.3.1. *(see [11]) Consider an open set $\mathcal{I} \neq 0$ in $\mathcal{R}$ that is a convex set. Any convex function $\mathfrak{F} : \mathcal{I} \to \mathcal{R}$ must be continuous throughout the domain $\mathcal{I}$.*

Theorem 1.3.2. *Every convex function $\mathfrak{F} : \mathcal{I} \to \mathcal{R}$ is continuous on an open set $\mathcal{I}$, if*

$$\mathfrak{F}'_+(\mathfrak{a}) \leqslant \mathfrak{F}'_-(\mathfrak{b}), \quad \mathfrak{a} \leq \mathfrak{b}, \tag{1.3}$$

where, the limit

$$\mathfrak{F}'_+(\mathfrak{a}) = \lim_{\mathfrak{a}_0 \to \mathfrak{a}^+} \frac{\mathfrak{F}(\mathfrak{a}) - \mathfrak{F}(\mathfrak{a}_o)}{\mathfrak{a} - \mathfrak{a}_o}, \quad \mathfrak{a} \in \mathcal{I},$$

exist, we say it right derivative of $\mathfrak{F}$ *at* $\mathfrak{a}$, *similarly if limit*

$$\mathfrak{F}'_-(\mathfrak{b}) = \lim_{\mathfrak{b}_0 \to \mathfrak{b}^-} \frac{\mathfrak{F}(\mathfrak{b}) - \mathfrak{F}(\mathfrak{b}_o)}{\mathfrak{b} - \mathfrak{b}_o}, \quad \mathfrak{b} \in \mathcal{I},$$

exist, we say it left derivative of $\mathfrak{F}$ *at* $\mathfrak{b}$.

Below is an example of a discontinuous convex function at a specific point but continuous otherwise.

Example 1.3.1. Consider a convex function defined as

$$\mathfrak{F}(\mathrm{u}) = \begin{cases} 0 & \text{if u} < 0, \\ u^2 & \text{if u} \geq 0. \end{cases}$$

Another approach to check the continuity of a convex function is the Lipschitz condition [10, 12].

Theorem 1.3.3. *Consider a convex function* $\mathfrak{F} : \mathcal{I} \to \mathcal{R}$ *where* $\mathcal{I}$ *is a non-empty set,* $\mathfrak{F}$ *is considered to be Lipschitz continuous with any real number* $\mathbf{l} \geq 0$ *if it satisfies the following condition:*

$$|\mathfrak{F}(\mathfrak{a}) - \mathfrak{F}(\mathfrak{b})| \leq \mathbf{l}||\mathfrak{a} - \mathfrak{b}|| \qquad \forall \mathfrak{a}, \mathfrak{b} \in \mathcal{I}.$$

The differentiability properties of convex functions depend on the nature of the function, some convex functions are differentiable but some are not. Examine derivatives for the best possible results regarding left and right derivatives. Considering a convex function $\mathfrak{F} : \mathcal{I} \to \mathcal{R}$ has left and right derivatives at any point $\mathfrak{a} \in \mathcal{I}$ as under

$$\mathfrak{F}'_{-}(\mathfrak{a}) = \lim_{\mathfrak{a}_0 \to \mathfrak{a}^-} \frac{\mathfrak{F}(\mathfrak{a}) - \mathfrak{F}(\mathfrak{a}_o)}{\mathfrak{a} - \mathfrak{a}_o}, \tag{1.4}$$

$$\mathfrak{F}'_{+}(\mathfrak{a}) = \lim_{\mathfrak{a}_0 \to \mathfrak{a}^+} \frac{\mathfrak{F}(\mathfrak{a}) - \mathfrak{F}(\mathfrak{a}_o)}{\mathfrak{a} - \mathfrak{a}_o}. \tag{1.5}$$

Theorem 1.3.4. *Let $\mathfrak{F} : \mathcal{I} \to \mathcal{R}$ be a convex function where $\mathcal{I}$ is an open set, then $\mathfrak{F}$ is increasing if and only if $\mathfrak{F}'_{-}(\mathfrak{a})$ and $\mathfrak{F}'_{+}(\mathfrak{a})$ exist.*

Theorem 1.3.5. *[1] Consider a convex function $\mathfrak{F} : \mathcal{I} \to \mathcal{R}$. Then $\mathfrak{F}$ is continuous on an open set $\mathcal{I}$ and it has left and right derivatives on every point of $\mathcal{I}$. Additionally, for any two points $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ we have following relation*

$$\mathfrak{F}'_{-}(\mathfrak{a}) \leq \mathfrak{F}'_{+}(\mathfrak{a}) \leq \mathfrak{F}'_{-}(\mathfrak{b}) \leq \mathfrak{F}'_{+}(\mathfrak{b}),$$

where $\mathfrak{a} \leq \mathfrak{b}$. In particular, both $\mathfrak{F}'_{-}$ and $\mathfrak{F}'_{+}$ are increasing on $\mathcal{I}$.

Here are a few examples of differentiable and non-differentiable convex functions.

Example 1.3.2. let $\mathfrak{F}$ be a convex function

- $\mathfrak{F}(\eta) = \eta^2$ is differentiable on whole domain.
- $\mathfrak{F}(\eta) = e^{\eta}$ is differentiable on whole domain.
- $\mathfrak{F}(\eta) = |\eta|$ is not differentiable at $\eta = 0$.
- $\mathfrak{F}(\eta) = \max(0, \eta)$ is not differentiable at $\eta = 0$.
- $$\mathfrak{F}(\eta) = \begin{cases} 0 & \text{if } \eta < 0, \\ \eta & \text{if } \geq 0. \end{cases}$$

The function $\mathfrak{F}$ is not differentiable at $\eta = 0$.

Theorem 1.3.6. *[10] Let $\mathfrak{F} : \mathcal{I} \to \mathcal{R}$ be a twice differentiable function. The function $\mathfrak{F}$ is convex if and only if $\forall \mathfrak{a} \in \mathcal{I},\ \mathfrak{F}''(\mathfrak{a}) \geq 0$.*

1.3.3 Convex functions: Applications, Extensions, and Generalizations

In the above sections, we have explored a detailed analysis of the convex function, highlighting its fundamental properties, continuity, and differentiability details. Due to these attributes, the convex function became pivotal across various applications (see [1]). Such as in optimization it ensures the local minima as global minima so that it can be used to model real-world optimization problems, in economics it is used to model utility and production functions for better efficiency and resource allocation. It aids in control systems and signal processing in engineering, while in machine learning it helps to support vector machines, constructing effective loss functions and neural networks. Most importantly, it has many applications within mathematics such as in calculus of variations, where it helps to formulate and solve complicated problems that are required to minimize or maximize specified functionals, it has multiple applications in geometry and functional analysis, and it contributes in development and refinement of various mathematical concepts and theorems. Indeed, it contributes useful insights in many scientific studies due to its adaptability through extensions and generalizations further enhances its utility across diverse research areas. Some notable generalizations are $\mathfrak{p}-$convex functions [13], $\mathfrak{s}$trongly convex functions [14], $\mathfrak{h}-$convex functions [15], $\mathfrak{s}-$convex functions [16], $\mathfrak{m}-$convex functions [17], $\mathbf{log}-$convex functions [18], $\eta-$convex functions [19], $\phi-$convex functions [20], $\mathbf{k}-$convex functions [21], many other generalizations have been explored (see [22]), and further developments and extensions are still underway.

1.4 Some remarkable Inequalities

All the time, inequalities have played a remarkable role in many scientific studies. Very famous triangular inequality and isoperimetric inequality were discovered by the ancient Greeks. In the eighteenth century, eminent scientists such as Gauss, Cauchy, and Chebyshev provided modern mathematical inequalities formalization and applications. Over the last few years, inequalities have become powerful problem-solving tools, widely applied across various fields, including mathematical analysis, physics, optimization, statistics, and other scientific studies (see [22]). Notable inequalities like Hölder's inequality, the power mean inequality, and Jensen's inequality are among the most common and significant in the study of convex functions, which are established within the subset of real numbers. Several integral inequalities, including the Hermite-Hadamard inequality, Fejér-type inequality, Ostrowski inequality, and Gauss inequality, are formulated for certain functions within convex analysis.

Some fundamental inequalities related to the theory of convex functions are subjected below.

1.4.1 Jensen's inequality

Jensen's inequality is among the most significant inequalities in mathematical analyses for convex functions. This was first proved in 1906 by Johan Ludwig Jensen [23, 8].

Consider a convex function $\mathfrak{F} : \mathcal{I} \subseteq \mathcal{R} \to \mathcal{R}$, Jensen's inequality states that:

$$\mathfrak{F}\left(\sum_{i=1}^{\mathrm{m}} \mathfrak{p}_i \mathfrak{a}_i\right) \leq \sum_{i=1}^{\mathrm{m}} \mathfrak{p}_i \mathfrak{F}(\mathfrak{a}_i), \tag{1.6}$$

where $\mathfrak{a}_i \in \mathcal{I}$, $\mathfrak{p}_i \geq 0$ with $\sum_{i=1}^{\mathrm{m}} \mathfrak{p}_i = 1$ and $i = 1, \ldots, m$. This inequality shows the behavior of the convex function, the function evaluates at the weighted average is always less or equal to the weighted average of the function value on every point of the interval.

It has become an inspiration for many mathematicians and scientists. They extended this inequality in the form of different generalizations and inequalities i.e. reverse Jensen's inequality, Jensen's inequality for Expectation with Convex functions, Hölder's inequality, and Minkowski's inequality. It has many applications in numerous fields such as statistics, mathematics, physics, optimization, and economics.

1.4.2 Hermite-Hadamard inequality

The classical Hermite-Hadamard inequality was established by Hermite in 1881 [5], and Later, in 1893 [6] Hadamard explored the same inequality, yet; he was not familiar with Hermite's result. Let $\mathfrak{F} : \mathcal{I} \subseteq \mathcal{R} \to \mathcal{R}$, $\mathcal{I} \neq \emptyset$ be a convex function, for $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$, the inequality holds

$$\mathfrak{F}\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) \leq \int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{F}(\mathrm{x}) d\mathrm{x} \leq \frac{\mathfrak{F}(\mathfrak{a}) + \mathfrak{F}(\mathfrak{b})}{2}. \tag{1.7}$$

Hermite-Hadamard inequality estimates, the integral average over a compact interval of a convex function and establishes upper and lower bounds for the function's value at the midpoint and endpoint of the interval. Due to these valuable properties, many researchers have made significant results and generalizations in different domains of mathematics. These generalizations helped to investigate many problems developed in applied mathematics, analyses, optimization, and economics. Given below is a weighted version of Hermite-Hadamard inequality.

1.4.3 Fejér-type inequality

In 1906, Lipót Fejér introduced a generalized version of Hermite-Hadamard inequality while studying trigonometric polynomials. This famous integral inequality is known as Fejér's inequalities [24], where a convex function along the concept of non-negative symmetric weights is involved. Consider a convex function $\mathfrak{F} : [\alpha_1, \alpha_2] = \mathcal{I} \subseteq \mathcal{R} \to \mathcal{R}$ and weight $\mathfrak{w} : [\alpha_1, \alpha_2] \to \mathcal{R}$, Fejér's inequality states that:

$$\mathfrak{F}\left(\frac{\mathfrak{a}+\mathfrak{b}}{2}\right) \int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{w}(\mathrm{x}) d\mathrm{x} \leq \int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{F}(\mathrm{x})\mathfrak{w}(\mathrm{x}) d\mathrm{x} \leq \frac{\mathfrak{F}(\mathfrak{a}) + \mathfrak{F}(\mathfrak{b})}{2} \int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{w}(\mathrm{x}) d\mathrm{x}, \tag{1.8}$$

where, $\mathfrak{w}$ is non-negative, integrable weight such that $\int_{\mathfrak{a}}^{\mathfrak{b}} \mathfrak{w}(x)dx > 0$, also it is symmetric with respect to $\frac{\mathfrak{a}+\mathfrak{b}}{2}$ implies that, $\mathfrak{w}$ satisfies $\mathfrak{w}(x) = \mathfrak{w}(\mathfrak{a} + \mathfrak{b} - x)$ $\forall x \in [\alpha_1, \alpha_2]$. For $\mathfrak{w}(x) = 1$ it becomes Hermite-Hadamard inequality in equation (1.7).

This contribution had a lasting impact on mathematical analysis. It has numerous applications and generalizations in many fields i.e. numerical methods, approximation theory, statistics, discrete mathematics, and beyond, see [25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35] for more detail.

Chapter 2

Strongly Reciprocally (p,h)-Convex Function

2.1 Preliminaries

Over the past century, the concept of convex function has been extended in various dimensions, either by modifying the originally established inequalities or generalizing them to abstract spaces. To establish a strong foundation for the later chapters, this section introduces fundamental definitions and preliminary concepts related to strongly reciprocally $(\mathfrak{p}, \mathfrak{h})$-convex functions.

Definition 2.1.1 ($\mathfrak{p}$-convex set [13]). *A set $\mathcal{I} = [\alpha_1, \beta_1] \subseteq \mathcal{R} \setminus \{0\}$ is a $\mathfrak{p}$-convex set if*

$$\left(\zeta \mathfrak{a}^{\mathfrak{p}} + (1-\zeta)\mathfrak{b}^{\mathfrak{p}}\right)^{\frac{1}{\mathfrak{p}}} \in \mathcal{I}, \tag{2.1}$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ and $\zeta \in [0,1]$, where $\mathfrak{p} = 2\mathbf{u} + 1$ or $\mathfrak{p} = \frac{\mathbf{d}}{\mathbf{c}}, \mathbf{d} = 2\mathbf{v} + 1,\ \mathbf{c} = 2\mathbf{w} + 1,$ and $\mathbf{u}, \mathbf{v}, \mathbf{w} \in N$.

Definition 2.1.2 (𝔭-convex function [13]). *Let* $\mathcal{I} = [\alpha_1, \beta_1] \subseteq \mathcal{R} \setminus \{0\}$ *be a* $\mathfrak{p}$*-convex set. A function* $\mathfrak{F} : \mathcal{I} = [\alpha_1, \beta_1] \subseteq \mathcal{R}$ *is* $\mathfrak{p}$*-convex if*

$$\mathfrak{F}\big((\zeta\mathfrak{a}^{\mathfrak{p}} + (1-\zeta)\mathfrak{b}^{\mathfrak{p}})^{\frac{1}{\mathfrak{p}}}\big) \leq \zeta\mathfrak{F}(\mathfrak{a}) + (1-\zeta)\mathfrak{F}(\mathfrak{b}), \tag{2.2}$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ *and* $\zeta \in [0,1]$.

Definition 2.1.3 (**H**armonic convex set [36]). *A set* $\mathcal{I} = [\alpha_1, \beta_1] \subseteq \mathcal{R}\setminus\{0\}$ *is harmonic convex if*

$$\frac{\mathfrak{a}\mathfrak{b}}{\zeta\mathfrak{a} + (1-\zeta)\mathfrak{b}} \in \mathcal{I}, \tag{2.3}$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ *and* $\zeta \in [0,1]$.

Definition 2.1.4 (**H**armonic convex function [36]). *Let* $\mathcal{I} = [\alpha_1, \beta_1] \subseteq \mathcal{R} \setminus \{0\}$ *be a harmonic convex set. A function* $\mathfrak{F} : \mathcal{I} = [\alpha_1, \beta_1] \subseteq \mathcal{R}$ *is harmonic convex if*

$$\mathfrak{F}\Big(\frac{\mathfrak{a}\mathfrak{b}}{\zeta\mathfrak{a} + (1-\zeta)\mathfrak{b}}\Big) \leq (1-\zeta)\mathfrak{F}(\mathfrak{a}) + \zeta\mathfrak{F}(\mathfrak{b}), \tag{2.4}$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ *and* $\zeta \in [0,1]$.

Definition 2.1.5 (**S**trongly convex function [37]). *A function* $\mathfrak{F} : \mathcal{I} = [\alpha_1, \beta_1] \subseteq \mathcal{R}$ *is a strongly convex function with modulus* $\chi \geq 0$ *on* $\mathcal{I}$ *if*

$$\mathfrak{F}\big(\zeta\mathfrak{a} + (1-\zeta)\mathfrak{b}\big) \leq \zeta\mathfrak{F}(\mathfrak{a}) + (1-\zeta)\mathfrak{F}(\mathfrak{b}) - \chi\zeta(1-\zeta)(\mathfrak{b}-\mathfrak{a})^2, \tag{2.5}$$

for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ *and* $\zeta \in [0,1]$.

Definition 2.1.6 (**S**trongly 𝔭-convex function [38]). *A function* $\mathfrak{F} : \mathcal{I} = [\alpha_1, \beta_1] \subseteq \mathcal{R}$ *is* *strongly* $\mathfrak{p}$*-convex if*

$$\mathfrak{F}\big((\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}})^{\frac{1}{\mathfrak{p}}}\big)\leq\zeta\mathfrak{F}(\mathfrak{a})+(1-\zeta)\mathfrak{F}(\mathfrak{b})-\chi\zeta(1-\zeta)(\mathfrak{b}^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}})^{2}, \tag{2.6}$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$ *and* $\zeta\in[0,1]$.

Definition 2.1.7 ($\mathfrak{p}$**-H**armonic convex set [39])**.** *A set* $\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}\setminus\{0\}$ *is a* $\mathfrak{p}$*-harmonic convex set if*

$$\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\in\mathcal{I}, \tag{2.7}$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$ *and* $\zeta\in[0,1]$.

Definition 2.1.8 ($\mathfrak{p}$**-H**armonic convex function [39])**.** *Let* $\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}\setminus\{0\}$ *be a* $\mathfrak{p}$*-harmonic convex set. A function* $\mathfrak{F}:\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}$ *is* $\mathfrak{p}$*-harmonic convex if*

$$\mathfrak{F}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\leq(1-\zeta)\mathfrak{F}(\mathfrak{a})+\zeta\mathfrak{F}(\mathfrak{b}), \tag{2.8}$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$ *and* $\zeta\in[0,1]$.

Definition 2.1.9 (**S**trongly reciprocally convex function [40])**.** *Let* $\mathcal{I}$ *be an interval, and let* $\chi\in(0,\infty)$. *A function* $\mathfrak{F}:\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}$ *is strongly reciprocally convex with modulus* χ *on* $\mathcal{I}$ *if*

$$\mathfrak{F}\left(\frac{\mathfrak{a}\mathfrak{b}}{\zeta\mathfrak{a}+(1-\zeta)\mathfrak{b}}\right)\leqslant(1-\zeta)\mathfrak{F}(\mathfrak{a})+\zeta\mathfrak{F}(\mathfrak{b})-\chi\zeta(1-\zeta)\left(\frac{1}{\mathfrak{a}}-\frac{1}{\mathfrak{b}}\right)^{2}, \tag{2.9}$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$ *and* $\zeta\in[0,1]$.

Definition 2.1.10 (**S**trongly reciprocally $\mathfrak{p}$-convex function [41])**.** *Let* $\mathcal{I}$ *be a* $\mathfrak{p}$*-convex set, and let* $\chi\in(0,\infty)$. *A function* $\mathfrak{F}:\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}$ *is strongly reciprocally*

$\mathfrak{p}$-*convex function with modulus* χ *on* $\mathcal{I}$ *if*

$$\mathfrak{F}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right) \leqslant (1-\zeta)\mathfrak{F}(\mathfrak{a})+\zeta\mathfrak{F}(\mathfrak{b})-\chi\zeta(1-\zeta)\left(\frac{1}{\mathfrak{a}^{\mathfrak{p}}}-\frac{1}{\mathfrak{b}^{\mathfrak{p}}}\right)^{2}, \qquad (2.10)$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$ *and* $\zeta\in[0,1]$.

Definition 2.1.11 ($\mathfrak{h}$-convex function [15]). *Let* $\mathcal{I},\mathfrak{h}:\zeta=[\alpha_1,\beta_1]\to\mathcal{R}$ *be non-negative functions. Then* $\mathfrak{F}$ *is* $\mathfrak{h}$-*convex if*

$$\mathfrak{F}\big(\zeta\mathfrak{a}+(1-\zeta)\mathfrak{b}\big)\leq\mathfrak{h}(\zeta)\mathfrak{F}(\mathfrak{a})+\mathfrak{h}(1-\zeta)\mathfrak{F}(\mathfrak{b}), \qquad (2.11)$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$ *and* $\zeta\in[0,1]$.

Definition 2.1.12 (($\mathfrak{p},\mathfrak{h}$)-convex function [42]). *Let* $\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}\setminus\{0\}$ *be a* $\mathfrak{p}$-*convex set. A function* $\mathfrak{F}:\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}$ *is* ($\mathfrak{p},\mathfrak{h}$)-*convex function if* $\mathfrak{F}$ *is non-negative and*

$$\mathfrak{F}\Big(\big(\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}\big)^{\frac{1}{\mathfrak{p}}}\Big)\leq\mathfrak{h}(\zeta)\mathfrak{F}(\mathfrak{a})+\mathfrak{h}(1-\zeta)\mathfrak{F}(\mathfrak{b}), \qquad (2.12)$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$ *and* $\zeta\in[0,1]$.

Definition 2.1.13 (higher-order **S**trongly convex [43]). *Let* $\mathcal{I}$ *be an interval, and let* $\chi\in(0,\infty)$. *A function* $\mathfrak{F}:\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}$ *is higher-order strongly convex with modulus* χ *on* $\mathcal{I}$ *if*

$$\mathfrak{F}\big(\zeta\mathfrak{a}+(1-\zeta)\mathfrak{b}\big)\leq\zeta\mathfrak{F}(\mathfrak{a})+(1-\zeta)\mathfrak{F}(\mathfrak{b})-\chi\phi_1(\zeta)\|\mathfrak{a}-\mathfrak{b}\|^{\mathfrak{l}}, \qquad (2.13)$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$ *and* $\zeta\in[0,1]$ *and* $\mathfrak{l}\geq1$, *where* $\phi_1(\zeta)=\zeta(1-\zeta)$.

Definition 2.1.14 (**S**trongly reciprocally ($\mathfrak{p},\mathfrak{h}$)-convex function [44]). *Let* $\mathcal{I}$ *be an interval, and let* $\chi\in(0,\infty)$. *A function* $\mathfrak{F}:\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}$ *is strongly reciprocally*

$(\mathfrak{p},\mathfrak{h})$-convex with modulus χ on $\mathcal{I}$ if

$$\mathfrak{F}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right) \leqslant \mathfrak{h}(1-\zeta)\mathfrak{F}(\mathfrak{a})+\mathfrak{h}(\zeta)\mathfrak{F}(\mathfrak{b})-\chi\zeta(1-\zeta)\left(\frac{1}{\mathfrak{a}^{\mathfrak{p}}}-\frac{1}{\mathfrak{b}^{\mathfrak{p}}}\right)^{2}, \tag{2.14}$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$ and $\zeta\in[0,1]$.

Definition 2.1.15 (**S**trongly reciprocally $(\mathfrak{p},\mathfrak{h})$-convex function of higher-order [44])**.** *Let $\mathcal{I}$ be an interval, and let $\chi\in(0,\infty)$. A function $\mathfrak{F}:\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}$ is strongly reciprocally $(\mathfrak{p},\mathfrak{h})$-convex of higher-order with modulus χ on $\mathcal{I}$ if*

$$\mathfrak{F}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right) \leqslant \mathfrak{h}(1-\zeta)\mathfrak{F}(\mathfrak{a})+\mathfrak{h}(\zeta)\mathfrak{F}(\mathfrak{b})-\chi\phi_1(\zeta)\left\|\frac{1}{\mathfrak{a}^{\mathfrak{p}}}-\frac{1}{\mathfrak{b}^{\mathfrak{p}}}\right\|^{\mathfrak{l}}, \tag{2.15}$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$, $\zeta\in[0,1]$, and $\mathfrak{l}\geq 1$, where $\phi_1(\zeta)=\zeta(1-\zeta)$.

Remark 1. Inserting $\mathfrak{l}=2$ into Definition 2.1.15 with $\phi_1(\zeta)$ as above, we obtain Definition 2.1.14. Similarly, inserting $\mathfrak{l}=2$ and $\mathfrak{h}(\zeta)=\zeta$ into Definition 2.1.15,we obtain Definition 2.1.10, and $\mathfrak{l}=2$, $\mathfrak{h}(\zeta)=\zeta$ and $\mathfrak{p}=1$, Definition 2.1.15, reduces to Definition 2.1.9.

As we know, $\mathcal{R}$ is a normed space with the standard modulus norm applied. Thus for any $\mathfrak{a}\in\mathcal{R}$,

$$\|\mathfrak{a}\|=|\mathfrak{a}|. \tag{2.16}$$

Using (2.16), inequality 2.1.15 can be expressed as

$$\mathfrak{F}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right) \leqslant \mathfrak{h}(1-\zeta)\mathfrak{F}(\mathfrak{a})+\mathfrak{h}(\zeta)\mathfrak{F}(\mathfrak{b})-\chi\phi_1(\zeta)\left|\frac{1}{\mathfrak{a}^{\mathfrak{p}}}-\frac{1}{\mathfrak{b}^{\mathfrak{p}}}\right|^{\mathfrak{l}}, \tag{2.17}$$

for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$, $\zeta\in[0,1]$, and $\mathfrak{l}\geq 1$, where $\phi_1(\zeta)=\zeta(1-\zeta)$.

2.2 Fundamental Results

Onward in this section, we explore an interesting domain called strongly reciprocally $(\mathfrak{p},\mathfrak{h})$-convex functions of higher order, which is named as $SR(\mathfrak{p}\mathfrak{h})$ for the rest of this thesis. We obtained some useful results by performing simple algebraic operations on $SR(\mathfrak{p}\mathfrak{h})$.

The addition of two functions from $SR(\mathfrak{p}\mathfrak{h})$ is obtained in the following proposition.

Proposition 2. [44] Consider two function from $SR(\mathfrak{p}\mathfrak{h})$, $\mathfrak{F}$,$G:\mathcal{I}\to\mathcal{R}$ with modulus χ on $\mathcal{I}$. Then $\mathfrak{F}$,$G:\mathcal{I}\to\mathcal{R}$ is also in $SR(\mathfrak{p}\mathfrak{h})$ with modulus χ^* on $\mathcal{I}$, where $\chi^*=2\chi$.

Proof. We begin by the definition provided:

$$\begin{aligned}\mathfrak{F}{+}G\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right) &= \mathfrak{F}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)+G\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\\ &\leq \mathfrak{h}(\zeta)\mathfrak{F}(\mathfrak{a})+\mathfrak{h}(1-\zeta)\mathfrak{F}(\mathfrak{b})-\chi\phi_1(\zeta)\left\|\frac{1}{\mathfrak{a}^{\mathfrak{p}}}-\frac{1}{\mathfrak{b}^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\\ &\quad+\mathfrak{h}(\zeta)G(\mathfrak{a})+\mathfrak{h}(1-\zeta)G(\mathfrak{b})-\chi\phi_1(\zeta)\left\|\frac{1}{\mathfrak{a}^{\mathfrak{p}}}-\frac{1}{\mathfrak{b}^{\mathfrak{p}}}\right\|^{\mathfrak{l}},\end{aligned} \tag{2.18}$$

which is further simplified as

$$\begin{aligned}&=\mathfrak{h}(\zeta)(\mathfrak{F}{+}G)(\mathfrak{a})+\mathfrak{h}(1-\zeta)(\mathfrak{F}+G)(\mathfrak{b})-2\chi\phi_1(\zeta)\left\|\frac{1}{\mathfrak{a}^{\mathfrak{p}}}-\frac{1}{\mathfrak{b}^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\\ &=\mathfrak{h}(\zeta)(\mathfrak{F}+G)(\mathfrak{a})+\mathfrak{h}(1-\zeta)(\mathfrak{F}+G)(\mathfrak{b})-\chi^*\phi_1(\zeta)\left\|\frac{1}{\mathfrak{a}^{\mathfrak{p}}}-\frac{1}{\mathfrak{b}^{\mathfrak{p}}}\right\|^{\mathfrak{l}},\end{aligned}$$

where $\chi^*=2\chi,\chi\geq 0$ and $\phi_1(\zeta)=\zeta(1-\zeta)$. Hence the proof is established. □

The result for scalar multiplication in $SR(\mathfrak{p}\mathfrak{h})$ is achieved in following proposition.

Proposition 3. [44] Considering a function in $SR(\mathfrak{ph})$ $\mathfrak{F}:\mathcal{I}\rightarrow\mathcal{R}$ with modulus $\chi\geq 0$. Then for any $\lambda\geq 0$, $\lambda\mathfrak{F}:\mathcal{I}\rightarrow\mathcal{R}$ is also in $SR(\mathfrak{ph})$ with modulus ψ^{*} on $\mathcal{I}$, where $\psi^{*}=\lambda\chi$.

Proof. Consider $\lambda\geq 0$. Therefore, by given definition of $\mathfrak{F}$ we have

$$\begin{aligned}\lambda\mathfrak{F}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)&=\lambda\left(\mathfrak{F}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\right)\\&\leq\lambda\left[\mathfrak{h}(\zeta)\mathfrak{F}(\mathfrak{a})+\mathfrak{h}(1-\zeta)\mathfrak{F}(\mathfrak{b})-\chi\phi_{1}(\zeta)\left\|\frac{1}{\mathfrak{b}^{\mathfrak{p}}}-\frac{1}{\mathfrak{a}^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\right]\\&=\mathfrak{h}(\zeta)\lambda\mathfrak{F}(\mathfrak{a})+\mathfrak{h}(1-\zeta)\lambda\mathfrak{F}(\mathfrak{b})-\lambda\chi\phi_{1}(\zeta)\left\|\frac{1}{\mathfrak{b}^{\mathfrak{p}}}-\frac{1}{\mathfrak{a}^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\\&=\mathfrak{h}(\zeta)\lambda\mathfrak{F}(\mathfrak{a})+\mathfrak{h}(1-\zeta)\lambda\mathfrak{F}(\mathfrak{b})-\psi^{*}\phi_{1}(\zeta)\left\|\frac{1}{\mathfrak{b}^{\mathfrak{p}}}-\frac{1}{\mathfrak{a}^{\mathfrak{p}}}\right\|^{\mathfrak{l}},\end{aligned}$$

where $\psi^{*}=\lambda\chi,\chi\geq 0,$ and $\phi_{1}(\zeta)=\zeta(1-\zeta)$. Hence the proof is established. □

Proposition 4. [44] Considering $\mathfrak{F}_{\mathfrak{i}}:\mathcal{I}\rightarrow\mathcal{R}$, $1\leq\mathfrak{i}\leq\mathfrak{n}$, be in $SR(\mathfrak{ph})$ with modulus χ. Then for $\lambda_{\mathfrak{i}}\geq 0$, $1\leq\mathfrak{i}\leq\mathfrak{n}$, the function $\mathfrak{F}:\mathcal{I}\rightarrow\mathcal{R}$, where $\mathfrak{F}=\sum_{i=1}^{n}\lambda_{i}\mathfrak{F}i$, is also in $SR(\mathfrak{ph})$ with modulus $\gamma\geq 0$, where $\gamma=\sum_{i=1}^{n}\lambda_{i}\chi$.

Proof. Let $\mathcal{I}$ be a $\mathfrak{p}$-harmonic convex set. Then for all $\mathfrak{a},\mathfrak{b}\in\mathcal{I}$ ans $\zeta\in[0,1]$, we have

$$\begin{aligned}\mathfrak{F}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)&=\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\lambda_{\mathfrak{i}}\mathfrak{F}_{\mathfrak{i}}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}}+(1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\\&\leq\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\lambda_{\mathfrak{i}}\left[\mathfrak{h}(\zeta)\mathfrak{F}_{\mathfrak{i}}(\mathfrak{a})+\mathfrak{h}(1-\zeta)\mathfrak{F}_{\mathfrak{i}}(\mathfrak{b})-\chi\phi_{1}(\zeta)\left\|\frac{1}{\mathfrak{b}^{\mathfrak{p}}}-\frac{1}{\mathfrak{a}^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\right]\\&=\mathfrak{h}(\zeta)\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\lambda_{i}\mathfrak{F}_{\mathfrak{i}}(\mathfrak{a})+\mathfrak{h}(1-\zeta)\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\lambda_{\mathfrak{i}}\mathfrak{F}_{\mathfrak{i}}(\mathfrak{b})-\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\lambda_{\mathfrak{i}}\left[\chi\phi_{1}(\zeta)\left\|\frac{1}{\mathfrak{b}^{\mathfrak{p}}}-\frac{1}{\mathfrak{a}^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\right]\\&=h(\zeta)\mathfrak{F}(\mathfrak{a})+\mathfrak{h}(1-\zeta)\mathfrak{F}(\mathfrak{b})-\gamma\phi_{1}(\zeta)\left\|\frac{1}{\mathfrak{b}^{\mathfrak{p}}}-\frac{1}{\mathfrak{a}^{\mathfrak{p}}}\right\|^{\mathfrak{l}},\end{aligned}$$

where $\gamma=\sum_{\mathfrak{i}=1}^{\mathfrak{n}}\lambda_{\mathfrak{i}}\chi$. Hence this is the required result. □

Proposition 5. [44] Considering $\mathfrak{F}_{\mathfrak{i}} : \mathcal{I} \to \mathcal{R}$, $1 \leq \mathfrak{i} \leq \mathfrak{n}$ be in $SR(\mathfrak{p}\mathfrak{h})$ with modulus χ. Then $\mathfrak{F} = \max\{\mathfrak{F}_{\mathfrak{i}}, \mathfrak{i} = 1, 2, \ldots, \mathfrak{n}\}$ is also in $SR(\mathfrak{p}\mathfrak{h})$ with modulus χ.

Proof. Let $\mathcal{I}$ be a $\mathfrak{p}$-harmonic convex set. Then for all $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ and $\zeta \in [0, 1]$, we have

$$\begin{aligned}
&\mathfrak{F}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}} + (1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right) \\
&= \max\left\{\mathfrak{F}_{\mathfrak{i}}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}} + (1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right), \mathfrak{i} = 1, 2, 3, \ldots, \mathfrak{n}\right\} \\
&= \mathfrak{F}_{\mathfrak{c}}\left(\left(\frac{\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\zeta\mathfrak{a}^{\mathfrak{p}} + (1-\zeta)\mathfrak{b}^{\mathfrak{p}}}\right)\right) \\
&\leq \mathfrak{h}(\zeta)\mathfrak{F}_{\mathfrak{c}}(\mathfrak{a}) + \mathfrak{h}(1-\zeta)\mathfrak{F}_{\mathfrak{c}}(\mathfrak{b}) - \chi\phi_1(\zeta)\left\|\frac{1}{\mathfrak{b}^{\mathfrak{p}}} - \frac{1}{\mathfrak{a}^{\mathfrak{p}}}\right\|^{\mathfrak{l}} \\
&= \mathfrak{h}(\zeta)\max\left\{\mathfrak{F}_{\mathfrak{i}}(\mathfrak{a})\right\} + \mathfrak{h}(1-\zeta)\max\left\{\mathfrak{F}_{\mathfrak{i}}(\mathfrak{b})\right\} - \chi\phi_1(\zeta)\left\|\frac{1}{\mathfrak{b}^{\mathfrak{p}}} - \frac{1}{\mathfrak{a}^{\mathfrak{p}}}\right\|^{\mathfrak{l}} \\
&= \mathfrak{h}(\zeta)\mathfrak{F}(\mathfrak{a}) + \mathfrak{h}(1-\zeta)\mathfrak{F}(\mathfrak{b}) - \chi\phi_1(\zeta)\left\|\frac{1}{\mathfrak{b}^{\mathfrak{p}}} - \frac{1}{\mathfrak{a}^{\mathfrak{p}}}\right\|^{\mathfrak{l}}.
\end{aligned} \tag{2.19}$$

Hence the proof is established. □

2.3 Hermite-Hadamard type inequality

In this section, we have proved a generalized form of the Hermite-Hadamard type for the function $SR(\mathfrak{p}\mathfrak{h})$.

Theorem 2.3.1. *[44] Let $\mathcal{I} \subset \mathcal{R} \setminus \{0\}$ be an interval. If $\mathfrak{F} : \mathcal{I} \to \mathcal{R}$ be in $SR(\mathfrak{p}\mathfrak{h})$ with modulus $\chi \geq 0$ and $\mathfrak{F} \in \mathfrak{L}[x, y]$, then for $\mathfrak{h}(\frac{1}{2}) \neq 0$, we have*

$$\begin{aligned}
&\frac{1}{2\mathfrak{h}(\frac{1}{2})}\left[\mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}+\chi\phi_1\left(\frac{1}{2}\right)\left|\frac{y^{\mathfrak{p}}-x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{\mathfrak{l}}\left[\frac{1-(-1)^{2\mathfrak{l}+1}}{2(\mathfrak{l}+1)}\right]\right]\\
&\quad\leq \frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y \frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a} \qquad (2.20)\\
&\quad\leq \int_0^1\left[\mathfrak{h}(1-\zeta)\mathfrak{F}(x)+\mathfrak{h}(\zeta)\mathfrak{F}(y)\right]d\zeta-\chi\left|\frac{y^{\mathfrak{p}}-x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{\mathfrak{l}}\int_0^1\phi_1(\zeta)\,d\zeta.
\end{aligned}$$

Proof. Substituting $\zeta=\frac{1}{2}$ in the Definition 2.1.15. We have

$$\mathfrak{F}\left(\left(\frac{2\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\leqslant \mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}(\mathfrak{a})+\mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}(\mathfrak{b})-\chi\phi_1\left(\frac{1}{2}\right)\left\|\frac{1}{\mathfrak{a}^{\mathfrak{p}}}-\frac{1}{\mathfrak{b}^{\mathfrak{p}}}\right\|^{\mathfrak{l}}. \qquad (2.21)$$

Let $\mathfrak{a}=[(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}})^{\frac{1}{\mathfrak{p}}}]$ and $\mathfrak{b}=[(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta y^{\mathfrak{p}}+(1-\zeta)x^{\mathfrak{p}}})^{\frac{1}{\mathfrak{p}}}]$. Integrating 2.21 with respect to ζ over $[0,1]$, we have

$$\begin{aligned}
\mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}&\leq \mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)+\mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)x^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\\
&\quad-\chi\phi_1\left(\frac{1}{2}\right)\left|\frac{y^{\mathfrak{p}}-x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{\mathfrak{l}}|1-2\zeta|^{\mathfrak{l}},\\
\int_0^1\mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}d\zeta&\leq\int_0^1\mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)d\zeta\\
&\quad+\int_0^1\mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta y^{\mathfrak{p}}+(1-\zeta)x^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)d\zeta\\
&\quad-\chi\phi_1\left(\frac{1}{2}\right)\left|\frac{y^{\mathfrak{p}}-x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{\mathfrak{l}}\int_0^1|1-2\zeta|^{\mathfrak{l}}\,d\zeta,\\
\mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}&\leq 2\mathfrak{h}\left(\frac{1}{2}\right)\frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y\frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}-\chi\phi_1\left(\frac{1}{2}\right)\left|\frac{y^{\mathfrak{p}}-x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{\mathfrak{l}}\left[\frac{1-(-1)^{2\mathfrak{l}+1}}{2(\mathfrak{l}+1)}\right],
\end{aligned}$$

and

$$\mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}} + \chi\phi_1\left(\frac{1}{2}\right)\left|\frac{y^{\mathfrak{p}}-x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{\mathfrak{l}}\left[\frac{1-(-1)^{2\mathfrak{l}+1}}{2(\mathfrak{l}+1)}\right]$$
$$\leq 2\mathfrak{h}\left(\frac{1}{2}\right)\frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y \frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a},$$
$$\frac{1}{2\mathfrak{h}(\frac{1}{2})}\left[\mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}} + \chi\phi_1\left(\frac{1}{2}\right)\left|\frac{y^{\mathfrak{p}}-x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{\mathfrak{l}}\left[\frac{1-(-1)^{2\mathfrak{l}+1}}{2(\mathfrak{l}+1)}\right]\right]$$
$$\leq \frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y \frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a},$$

the required left side of inequality (2.20) is established.

Furthermore, in the right side of inequality (2.20), setting $\mathfrak{a} = x$ and $\mathfrak{b} = y$ in Definition 2.1.15 gives

$$\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{tx^{\mathfrak{p}}+(1-t)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right) \leqslant \mathfrak{h}(1-\zeta)\mathfrak{F}(x)+\mathfrak{h}(\zeta)\mathfrak{F}(y)-\chi\phi_1(\zeta)\left\|\frac{1}{x^{\mathfrak{p}}}-\frac{1}{y^{\mathfrak{p}}}\right\|^{\mathfrak{l}}. \tag{2.22}$$

Integrating (2.22) with respect to ζ over [0,1], we get

$$\int_0^1 \mathfrak{F}\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{tx^{\mathfrak{p}}+(1-t)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]d\zeta \leq \int_0^1 \mathfrak{h}(1-\zeta)\mathfrak{F}(x)\,d\zeta + \int_0^1 \mathfrak{h}(\zeta)\mathfrak{F}(y)\,d\zeta$$
$$-\chi\left|\frac{y^{\mathfrak{p}}-x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{\mathfrak{l}}\int_0^1 \phi_1(\zeta)\,d\zeta$$

and

$$\frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y \frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a} \leq \int_0^1 \mathfrak{h}(1-\zeta)\mathfrak{F}(x)+\mathfrak{h}(\zeta)\mathfrak{F}(y)\,d\zeta - \chi\left|\frac{y^{\mathfrak{p}}-x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{\mathfrak{l}}\int_0^1 \phi_1(\zeta)\,d\zeta,$$

the required right side of inequality (2.20) is established here. Thus, the proof is now complete. □

Remark 2. 1. Setting $\mathfrak{l} = 2$, $h(\zeta) = \zeta$, and $\mathfrak{p} = 1$, with $\phi_1(\zeta) = \zeta(1-\zeta)$ into Theorem 2.3.1, We get the Hermite–Hadamard inequality for **S**trongly reciprocally convex function; (see [40], Theorem 3.1)

2. Setting $\mathfrak{l} = 2$, $\mathfrak{h}(\zeta) = \zeta$, $\mathfrak{p} = 1$, and $\chi = 0$ with $\phi_1(\zeta) = \zeta(1-\zeta)$ into Theorem 2.3.1 we obtain the Hermite-Hadamard inequality for harmonic convex functions; (see [40], Theorem 2.4).

2.4 Fejér-type inequality

Now, for the functions belonging to $SR(\mathfrak{p}\mathfrak{h})$ we will develop Fejér-type inequality.

Theorem 2.4.1. *[44] Considering $\mathcal{I} \subset \mathcal{R} \setminus \{0\}$ be an interval. If $\mathfrak{F} : \mathcal{I} \to \mathcal{R}$ is in SR($\mathfrak{p}\mathfrak{h}$) with modulus $\chi \geq 0$, then for $\mathfrak{h}(\frac{1}{2}) \neq 0$, we have*

$$\begin{aligned}
&\frac{1}{2\mathfrak{h}(\frac{1}{2})}\Bigg[\mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\int_x^y \frac{\mathfrak{w}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a} \\
&\qquad + \frac{\chi}{|x^{\mathfrak{p}}y^{\mathfrak{p}}|^{\mathfrak{l}}}\phi_1\left(\frac{1}{2}\right)\int_x^y \frac{|2x^{\mathfrak{p}}y^{\mathfrak{p}} - (x^{\mathfrak{p}}+y^{\mathfrak{p}})\mathfrak{a}^{\mathfrak{p}}|^{l}\mathfrak{w}(\mathfrak{a})}{|\mathfrak{a}^{\mathfrak{p}}|^{\mathfrak{l}}\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}\Bigg] \\
&\leq \int_x^y \frac{\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a} \qquad (2.23)\\
&\leq \big[\mathfrak{F}(x)+\mathfrak{F}(y)\big]\int_x^y \mathfrak{h}\left(\frac{x^{\mathfrak{p}}(y^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}})}{x^{\mathfrak{p}}(y^{\mathfrak{p}}-x^{\mathfrak{p}})}\right)\frac{\mathfrak{w}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a} \\
&\qquad - \chi\left\|\frac{y^{\mathfrak{p}}-x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\int_x^y \phi_1\left(\frac{x^{\mathfrak{p}}(y^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}})}{\mathfrak{a}^{\mathfrak{p}}(y^{\mathfrak{p}}-x^{\mathfrak{p}})}\right)\frac{\mathfrak{w}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}
\end{aligned}$$

for $x, y \in \mathcal{I}$ with $x \leq y$ and $\mathfrak{F} \in \mathfrak{L}[x,y]$, where $\mathfrak{w} : \mathcal{I} \to \mathcal{R}$ is a non-negative integrable function satisfying

$$\mathfrak{w}\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\mathfrak{a}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}} = \mathfrak{w}\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right].$$

Proof. Substituting $\zeta = \frac{1}{2}$ into Definition 2.1.15 yields

$$\mathfrak{F}\left(\left(\frac{2\mathfrak{a}^{\mathfrak{p}}\mathfrak{b}^{\mathfrak{p}}}{\mathfrak{a}^{\mathfrak{p}}+\mathfrak{b}^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right) \leqslant \mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}(\mathfrak{a}) + \mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}(\mathfrak{b}) - \chi\phi_1\left(\frac{1}{2}\right)\left\|\frac{1}{\mathfrak{a}^{\mathfrak{p}}} - \frac{1}{\mathfrak{b}^{\mathfrak{p}}}\right\|^{\mathfrak{l}}. \qquad (2.24)$$

Let $\mathfrak{a} = [(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}})^{\frac{1}{\mathfrak{p}}}]$ and $\mathfrak{b} = [(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta y^{\mathfrak{p}}+(1-\zeta)x^{\mathfrak{p}}})^{\frac{1}{\mathfrak{p}}}]$. Integrating (2.24) with respect to ζ over $[0,1]$, we have

$$\mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}} \leq \mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right) + \mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta y^{\mathfrak{p}}+(1-\zeta)x^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)$$
$$- \chi\phi_1\left(\frac{1}{2}\right)\left|\frac{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}} - \frac{\zeta y^{\mathfrak{p}}+(1-\zeta)x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{l}.$$

Since $\mathfrak{w}$ is a non-negative symmetric and integrable function, we have

$$\begin{aligned}
&\mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\\
&\leq \mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\\
&\quad+ \mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta y^{\mathfrak{p}}+(1-\zeta)x^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\\
&\quad- \chi\phi_1\left(\frac{1}{2}\right)\left|\frac{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}} - \frac{\zeta y^{\mathfrak{p}}+(1-\zeta)x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{\mathfrak{l}}\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right).
\end{aligned} \tag{2.25}$$

Integrating inequality (2.25) with respect to ζ over $[0,1]$, we have

$$\begin{aligned}
&\int_0^1 \mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)d\zeta\\
&\leq \int_0^1 \mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)d\zeta\\
&\quad+ \int_0^1 \mathfrak{h}\left(\frac{1}{2}\right)\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta y^{\mathfrak{p}}+(1-\zeta)x^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)d\zeta\\
&\quad- \chi\phi_1\left(\frac{1}{2}\right)\int_0^1\left|\frac{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}} - \frac{\zeta y^{\mathfrak{p}}+(1-\zeta)x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right|^{\mathfrak{l}}\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)d\zeta,
\end{aligned}$$

and

$$\mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\int_x^y \frac{\mathfrak{w}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a} + \frac{\chi}{|x^{\mathfrak{p}}y^{\mathfrak{p}}|^{\mathfrak{l}}}\phi_1\left(\frac{1}{2}\right)\int_x^y \frac{|2x^{\mathfrak{p}}y^{\mathfrak{p}}-(x^{\mathfrak{p}}+y^{\mathfrak{p}})\mathfrak{a}^{\mathfrak{p}}|^{\mathfrak{l}}\mathfrak{w}(\mathfrak{a})}{|\mathfrak{a}^{\mathfrak{p}}|^{\mathfrak{l}}\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}$$
$$\leq 2\mathfrak{h}\left(\frac{1}{2}\right)\int_x^y \frac{\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a},$$
$$\frac{1}{2\mathfrak{h}(\frac{1}{2})}\left(\mathfrak{F}\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\int_x^y \frac{\mathfrak{w}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}\right.$$
$$\left.+\frac{\chi}{|x^{\mathfrak{p}}y^{\mathfrak{p}}|^{\mathfrak{l}}}\phi_1\left(\frac{1}{2}\right)\int_x^y \frac{|2x^{\mathfrak{p}}y^{\mathfrak{p}}-(x^{\mathfrak{p}}+y^{\mathfrak{p}})\mathfrak{a}^{\mathfrak{p}}|^{\mathfrak{l}}\mathfrak{w}(\mathfrak{a})}{|\mathfrak{a}^{\mathfrak{p}}|^{\mathfrak{l}}\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}\right)$$
$$\leq \int_x^y \frac{\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a},$$

this is the required left side if inequality (2.23).

Further, for the right side of inequality (2.23), let $\mathfrak{a}=x$ in Definition 2.1.15 we get

$$\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right) \leqslant \mathfrak{h}(1-\zeta)\mathfrak{F}(x)+\mathfrak{h}(\zeta)\mathfrak{F}(y)-\chi\phi_1(\zeta)\left\|\frac{1}{x^{\mathfrak{p}}}-\frac{1}{y^{\mathfrak{p}}}\right\|^{\mathfrak{l}}.$$

Since $\mathfrak{w}$ is a non negative symmetric and integrable function, we have

$$\mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)$$
$$\leqslant \mathfrak{h}(1-\zeta)\mathfrak{F}(x)\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)+\mathfrak{h}(\zeta)\mathfrak{F}(y)\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)$$
$$-\chi\phi_1(\zeta)\left\|\frac{1}{x^{\mathfrak{p}}}-\frac{1}{y^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right). \tag{2.26}$$

Integrating inequality (2.26) with respect to ζ over $[0,1]$, we obtain

$$\int_0^1 \mathfrak{F}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)w\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)d\zeta$$
$$\leqslant \int_0^1 \mathfrak{h}(1-\zeta)\mathfrak{F}(x)\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)d\zeta$$

$$+\int_0^1 \mathfrak{h}(\zeta)\mathfrak{F}(y)\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)d\zeta$$
$$-\chi\int_0^1 \phi_1(\zeta)\left\|\frac{1}{x^{\mathfrak{p}}}-\frac{1}{y^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\mathfrak{w}\left(\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right)d\zeta,$$

and

$$\int_x^y \frac{\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a} \leq \big(\mathfrak{F}(x)+\mathfrak{F}(y)\big)\int_x^y \mathfrak{h}\left(\frac{x^{\mathfrak{p}}(y^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}})}{\mathfrak{a}^{\mathfrak{p}}(y^{\mathfrak{p}}-x^{\mathfrak{p}})}\right)\frac{\mathfrak{w}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}$$
$$-\chi\left\|\frac{y^{\mathfrak{p}}-x^{\mathfrak{p}}}{x^{\mathfrak{p}}y^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\int_x^y \phi_1\left(\frac{x^{\mathfrak{p}}(y^{\mathfrak{p}}-\mathfrak{a}^{\mathfrak{p}})}{\mathfrak{a}^{\mathfrak{p}}(y^{\mathfrak{p}}-x^{\mathfrak{p}})}\right)\frac{\mathfrak{w}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a},$$

this is the required right side of inequality (2.23). This completes the proof. □

Remark 3. (i) Setting $\mathfrak{l}=2$, $\mathfrak{h}(\zeta)=\zeta$, and $\mathfrak{p}=1$ with $\phi_1(\zeta)=\zeta(1-\zeta)$ into Theorem 2.4.1, we get Fejér-type inequality for **S**trongly reciprocally convex functions; (see [40], Theorem 3.7).

(ii) In the same fashion the insertion of $\mathfrak{l}=2$ and $\mathfrak{h}(\zeta)=\zeta$ with $\phi_1(\zeta)=\zeta(1-\zeta)$ into Theorem 2.4.1 produces Fejér-type inequality for **S**trongly reciprocally $\mathfrak{p}$-convex functions; (see [41], Theorem 3.5).

2.5 Fractional integral inequalities

In the past years, Fractional integral inequalities have gained significant popularity and importance in many fields of science and engineering, to study see [29, 45, 46, 47]. Now we establish some fractional integral inequalities for functions with derivatives in $SR(\mathfrak{p}\mathfrak{h})$. To obtain results of our desired type, we need the following lemma, which can be found in [48] Lemma 2.1.

Lemma 2.5.1. *[48] Let $\mathfrak{F}:\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}$ be a differentiable function on the interior*

$\mathcal{I}^o$ of $\mathcal{I}$. If $\mathfrak{F}' \in \mathfrak{L}[x,y]$ and $\lambda \in [0,1]$, then

$$(1-\lambda)\mathfrak{F}\left[\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right] + \lambda\left(\frac{\mathfrak{F}(x+\mathfrak{F}(y)}{2}\right) - \frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y \frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}$$
$$= \frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\left[\int_0^{\frac{1}{2}}(2\zeta-\lambda)\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]d\zeta\right.$$
$$\left.+\int_{\frac{1}{2}}^1(2\zeta-2+\lambda)\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]d\zeta\right]. \tag{2.27}$$

Theorem 2.5.2. *[44] Let $\mathcal{I} = [\alpha_1, \beta_1] \subseteq \mathcal{R}\backslash\{0\}$ be a $\mathfrak{p}$-harmonic convex set, and let $\mathfrak{F} : \mathcal{I} \to \mathcal{R}$ be a differentiable function on the interior $\mathcal{I}^o$ of $\mathcal{I}$. If $\mathfrak{F}' \in \mathfrak{L}[x,y]$ and $|\mathfrak{F}'|^{\mathfrak{q}}$ is a* **S***trongly reciprocally $(\mathfrak{p},\mathfrak{h})$-convex function of higher order on $\mathcal{I}$, $\mathfrak{q} \geq 1$, and $\lambda \in [0,1]$, then*

$$\left|(1-\lambda)\mathfrak{F}\left[\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right] + \lambda\left(\frac{\mathfrak{F}(x)+\mathfrak{F}(y)}{2}\right) - \frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y \frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}\right|$$
$$\leq \frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\Big[C_1(\mathfrak{p},x,y)^{1-\frac{1}{\mathfrak{q}}}\big[C_3(\mathfrak{p},x,y)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}} + C_5(\mathfrak{p},x,y)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}}$$
$$+ C_7(\mathfrak{p},x,y)\chi\big]^{\frac{1}{\mathfrak{q}}} + C_2(\mathfrak{p},y,x)^{1-\frac{1}{\mathfrak{q}}}\big[C_6(\mathfrak{p},y,x)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}} + C_4(\mathfrak{p},y,x)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}}$$
$$+ C_8(\mathfrak{p},y,x)\chi\big]^{\frac{1}{\mathfrak{q}}}\Big], \tag{2.28}$$

where

$$C_1(\mathfrak{p},x,y) = \int_0^{\frac{1}{2}} |2\zeta-\lambda|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}} d\zeta, \tag{2.29}$$

$$C_2(\mathfrak{p},y,x) = \int_{\frac{1}{2}}^1 |2\zeta-2+\lambda|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}} d\zeta, \tag{2.30}$$

$$C_3(\mathfrak{p},x,y) = \int_0^{\frac{1}{2}} \mathfrak{h}(1-\zeta)|2\zeta-\lambda|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}} d\zeta, \tag{2.31}$$

$$C_4(\mathfrak{p},y,x)=\int_{\frac{1}{2}}^{1}\mathfrak{h}(\zeta)|2\zeta-2+\lambda|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}d\zeta, \tag{2.32}$$

$$C_5(\mathfrak{p},x,y)=\int_{0}^{\frac{1}{2}}\mathfrak{h}(\zeta)|2\zeta-\lambda|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}d\zeta, \tag{2.33}$$

$$C_6(\mathfrak{p},y,x)=\int_{\frac{1}{2}}^{1}\mathfrak{h}(1-\zeta)|2\zeta-2+\lambda|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}d\zeta, \tag{2.34}$$

$$C_7(\mathfrak{p},x,y)=-\int_{0}^{\frac{1}{2}}\phi_1(\zeta)|2\zeta-\lambda|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\left\|\frac{1}{y^{\mathfrak{p}}}-\frac{1}{x^{\mathfrak{p}}}\right\|^{\mathfrak{l}}d\zeta, \tag{2.35}$$

$$C_8(\mathfrak{p},y,x)=-\int_{\frac{1}{2}}^{1}\phi_1(\zeta)|2\zeta-2+\lambda|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\left\|\frac{1}{y^{\mathfrak{p}}}-\frac{1}{x^{\mathfrak{p}}}\right\|^{\mathfrak{l}}d\zeta. \tag{2.36}$$

Proof. Applying Lemma 2.5.1, we have

$$\begin{aligned}
&\left|(1-\lambda)\mathfrak{F}\left[\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]+\lambda\left(\frac{\mathfrak{F}(x)+\mathfrak{F}(y)}{2}\right)-\frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y\frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}\right|\\
&\quad\leq\frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\left[\int_0^{\frac{1}{2}}\left|(2\zeta-\lambda)\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\right|\left|\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]\right|d\zeta\right.\\
&\qquad\left.+\int_{\frac{1}{2}}^{1}\left|(2\zeta-2+\lambda)\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\right|\left|\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]\right|d\zeta\right].
\end{aligned}$$

Now applying power mean inequality,

$$\begin{aligned}
&\leq\frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\left[\left(\int_0^{\frac{1}{2}}\left|(2\zeta-\lambda)\right|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}d\zeta\right)^{1-\frac{1}{\mathfrak{q}}}\right.\\
&\quad\times\left(\int_0^{\frac{1}{2}}\left|(2\zeta-\lambda)\right|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\left|\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]\right|^{\mathfrak{q}}d\zeta\right)^{\frac{1}{\mathfrak{q}}}\\
&\quad+\left(\int_{\frac{1}{2}}^{1}\left|(2\zeta-2+\lambda)\right|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}d\zeta\right)^{1-\frac{1}{\mathfrak{q}}}\\
&\quad\left.\times\left(\int_{\frac{1}{2}}^{1}\left|(2\zeta-2+\lambda)\right|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\left|\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]\right|^{\mathfrak{q}}d\zeta\right)^{\frac{1}{\mathfrak{q}}}\right].
\end{aligned}$$

Since $|\mathfrak{F}'(\mathfrak{a})|^{\mathfrak{q}}$ is in $SR(\mathfrak{p}\mathfrak{h})$, now we've

$$
\begin{aligned}
&\leq \frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\Bigg[\Bigg(\int_0^{\frac{1}{2}}\big|(2\zeta-\lambda)\big|\bigg(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\bigg)^{1+\frac{1}{\mathfrak{p}}}d\zeta\Bigg)^{1-\frac{1}{\mathfrak{q}}}\\
&\quad\times\Bigg(\int_0^{\frac{1}{2}}\big|(2\zeta-\lambda)\big|\bigg(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\bigg)^{1+\frac{1}{\mathfrak{p}}}\bigg[\mathfrak{h}(1-\zeta)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}}+h(\zeta)\big|\mathfrak{F}(y)\big|^{\mathfrak{q}}\\
&\quad-\chi\phi_1(\zeta)\bigg\|\frac{1}{y^{\mathfrak{p}}}-\frac{1}{x^{\mathfrak{p}}}\bigg\|^{\mathfrak{l}}\bigg]d\zeta\Bigg)^{\frac{1}{\mathfrak{q}}}+\Bigg(\int_{\frac{1}{2}}^1\big|(2\zeta-2+\lambda)\big|\bigg(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\bigg)^{1+\frac{1}{\mathfrak{p}}}d\zeta\Bigg)^{1-\frac{1}{\mathfrak{q}}}\\
&\quad\times\Bigg(\int_{\frac{1}{2}}^1\big|(2\zeta-2+\lambda)\big|\bigg(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\bigg)^{1+\frac{1}{\mathfrak{p}}}\bigg[\mathfrak{h}(1-\zeta)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}}+h(\zeta)\big|\mathfrak{F}'(b_1)\big|^{\mathfrak{q}}\\
&\quad-\chi\phi_1(\zeta)\bigg\|\frac{1}{y^{\mathfrak{p}}}-\frac{1}{x^{\mathfrak{p}}}\bigg\|^{l}\bigg]d\zeta\Bigg)^{\frac{1}{\mathfrak{q}}}\Bigg]\\
&=\frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\big[C_1(\mathfrak{p},x,y)^{1-\frac{1}{\mathfrak{q}}}\big[C_3(\mathfrak{p},x,y)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}}+C_5(\mathfrak{p},x,y)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}}\\
&\quad+C_7(\mathfrak{p},x,y)\chi\big]^{\frac{1}{\mathfrak{q}}}+C_2(\mathfrak{p},y,x)^{1-\frac{1}{\mathfrak{q}}}\big[C_6(\mathfrak{p},y,x)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}}+C_4(\mathfrak{p},y,x)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}}\\
&\quad+C_8(\mathfrak{p},y,x)\chi\big]^{\frac{1}{\mathfrak{q}}}\big],
\end{aligned}
$$

hence, this is required result. □

Remark 4. In Theorem 2.5.2, inserting $h(\zeta)=\zeta,\chi=0$, and $l=2$ with $\phi_1(\zeta)=\zeta(1-\zeta)$, we obtain Theorem 2.2 in reference [48].

Now substituting $\mathfrak{q}=1$, Theorem 2.5.2 reduces to the following result.

Corollary 2.5.3. *[44] Let $\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}\backslash\{0\}$ be a $\mathfrak{p}$-harmonic convex set, and let $\mathfrak{F}:\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}\backslash\{0\}\to\mathcal{R}$ be a differentiable function on the interior $\mathcal{I}^o$ of $\mathcal{I}$. If $\mathfrak{F}'\in\mathfrak{L}[x,y]$, $|\mathfrak{F}'|^q$ is in $SR(\mathfrak{p}\mathfrak{h})$ on $\mathcal{I}$, and $\lambda\in[0,1]$, then*

$$\left|(1-\lambda)\mathfrak{F}\left[\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]+\lambda\left(\frac{\mathfrak{F}(x)+\mathfrak{F}(y)}{2}\right)-\frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y\frac{\mathfrak{F}(x)}{x^{1+\mathfrak{p}}}\,dx\right|$$
$$\leq\frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\big[\big(C_3(\mathfrak{p},x,y)+C_6(\mathfrak{p},y,x)\big)\big|\mathfrak{F}'(x)\big| \tag{2.37}$$
$$+\big(C_5(\mathfrak{p},y,x)+C_4(\mathfrak{p},x,y)\big)\big|\mathfrak{F}'(y)\big|+\big(C_7(\mathfrak{p},x,y)+C_8(\mathfrak{p},y,x)\big)\chi\big],$$

where C_3, C_4, C_5, C_6, C_7 *and* C_8 *are given by* (2.31)-(2.36).

Remark 5. In Corollary 2.5.3 inserting $\mathfrak{h}(\zeta)=\zeta,\chi=0,$ and $\mathfrak{l}=2$ with $\phi_1(\zeta)=\zeta(1-\zeta)$, we obtain Corollary 2.3 in reference [48].

Theorem 2.5.4. *[44] Let* $\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}\backslash\{0\}$ *be a* $\mathfrak{p}$*-harmonic convex set, and let* $\mathfrak{F}:\mathcal{I}=[\alpha_1,\beta_1]\subseteq\mathcal{R}\backslash\{0\}\to\mathcal{R}$ *be a differentiable function on the interior* $\mathcal{I}^o$ *of* $\mathcal{I}$*. If* $\mathfrak{F}'\in\mathfrak{L}[x,y]$ *is* **S***trongly reciprocally* $(\mathfrak{p},\mathfrak{h})$*-convex function of higher order on* $\mathcal{I}$*,* $\mathfrak{r},\mathfrak{q}>1$*,* $\frac{1}{\mathfrak{r}}+\frac{1}{\mathfrak{q}}=1$*, and* $\lambda\in[0,1]$*, then*

$$\left|(1-\lambda)\mathfrak{F}\left(\left[\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right]^{\frac{1}{\mathfrak{p}}}\right)+\lambda\left(\frac{\mathfrak{F}(x)+\mathfrak{F}(y)}{2}\right)-\frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y\frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}\right|$$
$$\leq\frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\times\left(\frac{\lambda^{\mathfrak{r}+1}+(1-\lambda)^{\mathfrak{r}+1}}{2(\mathfrak{r}+1)}\right)^{\frac{1}{\mathfrak{r}}}\big[\big(C_9(\mathfrak{q},\mathfrak{p};x,y)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}}$$
$$+C_{11}(\mathfrak{q},\mathfrak{p};x,y)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}}+C_{13}(\mathfrak{q},\mathfrak{p};x,y)\chi\big)^{\frac{1}{\mathfrak{q}}} \tag{2.38}$$
$$+\big(C_{12}(\mathfrak{q},\mathfrak{p};y,x)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}}+C_{10}(\mathfrak{q},\mathfrak{p};y,x)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}}$$
$$+C_{14}(\mathfrak{q},\mathfrak{p};y,x)\chi\big)^{\frac{1}{\mathfrak{q}}}\big],$$

where

$$C_9(\mathfrak{q},\mathfrak{p};x,y)=\int_0^{\frac{1}{2}}\mathfrak{h}(1-\zeta)\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\mathfrak{q}+\frac{\mathfrak{q}}{\mathfrak{p}}}d\zeta, \tag{2.39}$$
$$C_{10}(\mathfrak{q},\mathfrak{p};y,x)=\int_{\frac{1}{2}}^1\mathfrak{h}(\zeta)\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\mathfrak{q}+\frac{\mathfrak{q}}{\mathfrak{p}}}d\zeta, \tag{2.40}$$

$$C_{11}(\mathfrak{q},\mathfrak{p};x,y) = \int_0^{\frac{1}{2}} \mathfrak{h}(\zeta)\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\mathfrak{q}+\frac{\mathfrak{q}}{\mathfrak{p}}} d\zeta, \tag{2.41}$$

$$C_{12}(\mathfrak{q},\mathfrak{p};y,x) = \int_{\frac{1}{2}}^{1} \mathfrak{h}(1-\zeta)\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\mathfrak{q}+\frac{\mathfrak{q}}{\mathfrak{p}}} d\zeta, \tag{2.42}$$

$$C_{13}(\mathfrak{q},\mathfrak{p};x,y) = -\int_0^{\frac{1}{2}} \phi_1(\zeta)\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\mathfrak{q}+\frac{\mathfrak{q}}{\mathfrak{p}}} \left\|\frac{1}{y^{\mathfrak{p}}}-\frac{1}{x^{\mathfrak{p}}}\right\|^{\mathfrak{l}} d\zeta, \tag{2.43}$$

$$C_{14}(\mathfrak{q},\mathfrak{p};y,x) = -\int_{\frac{1}{2}}^{1} \phi_1(\zeta)|2\zeta-2+\lambda|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\mathfrak{q}+\frac{\mathfrak{q}}{\mathfrak{p}}} \left\|\frac{1}{y^{\mathfrak{p}}}-\frac{1}{x^{\mathfrak{p}}}\right\|^{\mathfrak{l}} d\zeta. \tag{2.44}$$

Proof. Using Lemma 2.5.1, we have

$$\begin{aligned}
&\left|(1-\lambda)\mathfrak{F}\left(\left[\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right]\right)^{\frac{1}{\mathfrak{p}}} + \lambda\left(\frac{\mathfrak{F}(x)+\mathfrak{F}(y)}{2}\right) - \frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y \frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}\right| \\
&\quad\leq \frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\Bigg[\int_0^{\frac{1}{2}}\left|(2\zeta-\lambda)\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\right|\left|\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]\right| d\zeta \\
&\qquad + \int_{\frac{1}{2}}^{1}\left|(2\zeta-2+\lambda)\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\right|\left|\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]\right| d\zeta\Bigg].
\end{aligned}$$

Using Hölder's integral inequality, we get

$$\begin{aligned}
&\leq \frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\Bigg[\left(\int_0^{\frac{1}{2}}\left|(2\zeta-\lambda)\right|^{\mathfrak{r}} d\zeta\right)^{\frac{1}{\mathfrak{r}}} \\
&\quad\times\left(\int_0^{\frac{1}{2}}\left|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]\right|^{\mathfrak{q}} d\zeta\right)^{\frac{1}{\mathfrak{q}}} \\
&\quad+\left(\int_{\frac{1}{2}}^{1}\left|(2\zeta-2+\lambda)\right|^{\mathfrak{r}} d\zeta\right)^{\frac{1}{\mathfrak{r}}} \\
&\quad\times\left(\int_{\frac{1}{2}}^{1}\left|\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{1+\frac{1}{\mathfrak{p}}}\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]\right|^{\mathfrak{q}} d\zeta\right)^{\frac{1}{\mathfrak{q}}}\Bigg] \\
&= \frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\Bigg[\left(\int_0^{\frac{1}{2}}\left|(2\zeta-\lambda)\right|^{\mathfrak{r}} d\zeta\right)^{\frac{1}{\mathfrak{r}}} \\
&\quad\times\left(\int_0^{\frac{1}{2}}\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\mathfrak{q}+\frac{\mathfrak{q}}{\mathfrak{p}}}\left|\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]\right|^{\mathfrak{q}} d\zeta\right)^{\frac{1}{\mathfrak{q}}}
\end{aligned}$$

$$+\left(\int_{\frac{1}{2}}^{1}\left|(2\zeta-2+\lambda)\right|^{\mathfrak{r}}d\zeta\right)^{\frac{1}{\mathfrak{r}}}\times\left(\int_{\frac{1}{2}}^{1}\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\mathfrak{q}+\frac{\mathfrak{q}}{\mathfrak{p}}}\left|\mathfrak{F}'\left[\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]\right|^{\mathfrak{q}}d\zeta\right)^{\frac{1}{\mathfrak{q}}}\Bigg].$$

Since $|\mathfrak{F}'(\mathfrak{a})|^{\mathfrak{q}}$ is in $SR(\mathfrak{p}\mathfrak{h})$, we have

$$\begin{aligned}
&\leq\frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\Bigg[\left(\int_{0}^{\frac{1}{2}}\left|(2\zeta-\lambda)\right|^{\mathfrak{r}}d\zeta\right)^{\frac{1}{\mathfrak{r}}}\left(\int_{0}^{\frac{1}{2}}\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\mathfrak{q}+\frac{\mathfrak{q}}{\mathfrak{p}}}\right.\\
&\quad\times\left.\left[\mathfrak{h}(1-\zeta)\left|\mathfrak{F}'(x)\right|^{\mathfrak{q}}+\mathfrak{h}(\zeta)\left|\mathfrak{F}'(y)\right|^{\mathfrak{q}}-\chi\phi_1(\zeta)\left\|\frac{1}{y^{\mathfrak{p}}}-\frac{1}{x^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\right]d\zeta\right)^{\frac{1}{\mathfrak{q}}}\\
&\quad+\left(\int_{\frac{1}{2}}^{1}\left|(2\zeta-2+\lambda)\right|^{\mathfrak{r}}d\zeta\right)^{\frac{1}{\mathfrak{r}}}\left(\int_{\frac{1}{2}}^{1}\left(\frac{x^{\mathfrak{p}}y^{\mathfrak{p}}}{\zeta x^{\mathfrak{p}}+(1-\zeta)y^{\mathfrak{p}}}\right)^{\mathfrak{q}+\frac{\mathfrak{q}}{\mathfrak{p}}}\right.\\
&\quad\times\left.\left[\mathfrak{h}(1-\zeta)\left|\mathfrak{F}'(x)\right|^{\mathfrak{q}}+\mathfrak{h}(\zeta)\left|(y)\right|^{\mathfrak{q}}-\chi\phi_1(\zeta)\left\|\frac{1}{y^{\mathfrak{p}}}-\frac{1}{x^{\mathfrak{p}}}\right\|^{\mathfrak{l}}\right]d\zeta\right)^{\frac{1}{\mathfrak{q}}}\Bigg]\\
&\leq\frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\Bigg[\left(\int_{0}^{\frac{1}{2}}\left|(2\zeta-\lambda)\right|^{\mathfrak{r}}d\zeta\right)^{\frac{1}{\mathfrak{r}}}\\
&\quad\times\left(C_9(\mathfrak{q},\mathfrak{p};x,y)\left|\mathfrak{F}'(x)\right|^{\mathfrak{q}}+C_{11}(\mathfrak{q},\mathfrak{p};x,y)\left|\mathfrak{F}'(y)\right|^{\mathfrak{q}}+C_{13}(\mathfrak{q},\mathfrak{p};x,y)\chi\right)^{\frac{1}{\mathfrak{q}}}\\
&\quad+\left(\int_{0}^{\frac{1}{2}}\left|(2\zeta-2+\lambda)\right|^{\mathfrak{r}}d\zeta\right)^{\frac{1}{\mathfrak{r}}}\\
&\quad\times\left(C_{12}(\mathfrak{q},\mathfrak{p};y,x)\left|\mathfrak{F}'(x)\right|^{\mathfrak{q}}+C_{10}(\mathfrak{q},\mathfrak{p};y,x)\left|\mathfrak{F}'(y)\right|^{\mathfrak{q}}+C_{14}(\mathfrak{q},\mathfrak{p};y,x)\chi\right)^{\frac{1}{\mathfrak{q}}}\Bigg]\\
&\leq\frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\times\left(\frac{\lambda^{\mathfrak{r}+1}+(1-\lambda)^{\mathfrak{r}+1}}{2(\mathfrak{r}+1)}\right)^{\frac{1}{\mathfrak{r}}}\Big[\big(C_9(\mathfrak{q},\mathfrak{p};x,y)\left|\mathfrak{F}'(x)\right|^{\mathfrak{q}}+C_{11}(\mathfrak{q},\mathfrak{p};x,y)\left|\mathfrak{F}'(y)\right|^{\mathfrak{q}}\\
&\quad+C_{13}(\mathfrak{q},\mathfrak{p};x,y)\chi\big)^{\frac{1}{\mathfrak{q}}}+\left(C_{12}(\mathfrak{q},\mathfrak{p};y,x)\left|\mathfrak{F}'(x)\right|^{\mathfrak{q}}+C_{10}(\mathfrak{q},\mathfrak{p};y,x)\left|\mathfrak{F}'(y)\right|^{\mathfrak{q}}+C_{14}(\mathfrak{q},\mathfrak{p};y,x)\chi\right)^{\frac{1}{\mathfrak{q}}}\Big],
\end{aligned}$$

hence we get the required result. □

Remark 6. In Theorem 2.5.4 inserting $\mathfrak{h}(\zeta)=\zeta$, $\chi=0$, and $\mathfrak{l}=2$ with $\phi_1(\zeta)=\zeta(1-\zeta)$, we obtain Theorem 2.5 in reference [48].

For $\lambda=0$, Theorem 2.5.4 reduces to the following result.

Corollary 2.5.5. *[44] Let $\mathcal{I}=[\alpha_1,\beta_1]\subset\mathcal{R}\backslash\{0\}$ be a $\mathfrak{p}$-harmonic convex set, and let*

$\mathfrak{F} : \mathcal{I} = [\alpha_1, \beta_1] \subseteq \mathcal{R}\backslash\{0\} \rightarrow \mathcal{R}$ *be a differentiable function on the interior* $\mathcal{I}^o$ *of* $\mathcal{I}$*. If* $\mathfrak{F}' \in \mathfrak{L}[x, y]$ *is in* $SR(\mathfrak{p}\mathfrak{h})$ *on* $\mathcal{I}$*,* $\mathfrak{r}, \mathfrak{q} > 1$*,* $\frac{1}{\mathfrak{r}} + \frac{1}{\mathfrak{q}} = 1$*, and* $\lambda \in [0, 1]$*, then*

$$\begin{aligned} &\left|\mathfrak{F}\left[\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right] - \frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y \frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}\right| \\ &\le \frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})} \times \left(\frac{1}{2(\mathfrak{r}+1)}\right)^{\frac{1}{\mathfrak{r}}} \Big[\big(C_9(\mathfrak{q},\mathfrak{p};x,y)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}} \\ &+ C_{11}(\mathfrak{q},\mathfrak{p};x,y)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}} + C_{14}(\mathfrak{q},\mathfrak{p};x,y)\chi\big)^{\frac{1}{\mathfrak{q}}} \\ &+ \big(C_{12}(\mathfrak{q},\mathfrak{p};y,x)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}} + C_{10}(\mathfrak{q},\mathfrak{p};y,x)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}} \\ &+ C_{14}(\mathfrak{q},\mathfrak{p};y,x)\chi\big)^{\frac{1}{\mathfrak{q}}}\Big], \end{aligned} \tag{2.45}$$

where $C_9, C_{10}, C_{11}, C_{12}, C_{14}$ *and* C_{14} *are given by (2.39)-(2.44).*

Remark 7. In Corollary 2.5.5 inserting $\mathfrak{h}(\zeta) = \zeta, \chi = 0$, and $\mathfrak{l} = 2$ with $\phi_1(\zeta) = \zeta(1-\zeta)$, we obtain Corollary 3.6 in reference [48].

Theorem 2.5.4 reduces to the following result by putting $\lambda = 1$.

Corollary 2.5.6. *[44] Let* $\mathcal{I} = [\alpha_1, \beta_1] \subset \mathcal{R}\backslash\{0\}$ *be a* $\mathfrak{p}$*-harmonic convex set, and let* $\mathfrak{F} : \mathcal{I} = [\alpha_1, \beta_1] \subset \mathcal{R}\backslash\{0\} \rightarrow \mathcal{R}$ *be a differentiable function on the interior* $\mathcal{I}^o$ *of* $\mathcal{I}$*. If* $\mathfrak{F}' \in \mathfrak{L}[x, y]$*,* $|\mathfrak{F}'|^{\mathfrak{q}}$ *is a* **S***trongly reciprocally* $(\mathfrak{p}, \mathfrak{h})$*-convex function of higher order on* $\mathcal{I}$*,* $\mathfrak{r}, \mathfrak{q} > 1$*,* $\frac{1}{\mathfrak{r}} + \frac{1}{\mathfrak{q}} = 1$*, and* $\lambda \in [0, 1]$*, then*

$$\begin{aligned} &\left|\frac{\mathfrak{F}(x)+\mathfrak{F}(y)}{2} - \frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_x^y \frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}\right| \\ &\le \frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})} \times \left(\frac{1}{2(\mathfrak{r}+1)}\right)^{\frac{1}{\mathfrak{r}}} \Big[\big(C_9(\mathfrak{q},\mathfrak{p};x,y)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}} \\ &+ C_{11}(\mathfrak{q},\mathfrak{p};x,y)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}} + C_{14}(\mathfrak{q},\mathfrak{p};x,y)\chi\big)^{\frac{1}{\mathfrak{q}}} \\ &+ \big(C_{12}(\mathfrak{q},\mathfrak{p};y,x)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}} + C_{10}(\mathfrak{q},\mathfrak{p};y,x)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}} + C_{14}(\mathfrak{q},\mathfrak{p};y,x)\chi\big)^{\frac{1}{\mathfrak{q}}}\Big], \end{aligned}$$

where $C_9, C_{10}, C_{11}, C_{12}, C_{13}$ and C_{14} are given by (2.39)-(2.44).

Remark 8. In Corollary 2.5.6 inserting $\mathfrak{h}(\zeta) = \zeta, \chi = 0$, and $\mathfrak{l} = 2$ with $\phi_1(\zeta) = \zeta(1-\zeta)$, we obtain Corollary 3.6 in reference [48].

For $\lambda = \frac{1}{3}$, Theorem 2.5.4 reduces to the following result.

Corollary 2.5.7. *[44] Let $\mathcal{I} = [\alpha_1, \beta_1] \subset \mathcal{R}\backslash\{0\}$ be a $\mathfrak{p}$-harmonic convex set, and let $\mathfrak{F} : \mathcal{I} = [\alpha_1, \beta_1] \subset \mathcal{R}\backslash\{0\} \to \mathcal{R}$ be a differentiable function on the interior $\mathcal{I}^o$ of $\mathcal{I}$. If $\mathfrak{F}' \in \mathfrak{L}[x, y]$, $|\mathfrak{F}'|^q$ is a* S*trongly reciprocally $(\mathfrak{p}, \mathfrak{h})$-convex function of higher order on $\mathcal{I}$, $\mathfrak{r}, \mathfrak{q} > 1$, $\frac{1}{\mathfrak{r}} + \frac{1}{\mathfrak{q}} = 1$, and $\lambda \in [0, 1]$, then*

$$\begin{aligned}
&\left|\frac{1}{6}\left[\mathfrak{F}(x) + 4\mathfrak{F}\left[\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}} + y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right] + \mathfrak{F}(y)\right] - \frac{p(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}} - x^{\mathfrak{p}}}\int_x^y \frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}\right| \\
&\leq \frac{(y^{\mathfrak{p}} - x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})} \times \left(\frac{1 + 2^{\mathfrak{r}+1}}{6.3^{\mathfrak{r}}(\mathfrak{r}+1)}\right)^{\frac{1}{\mathfrak{r}}} \Big[\Big(C_9(\mathfrak{q}, \mathfrak{p}; x, y)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}} + C_{11}(\mathfrak{q}, \mathfrak{p}; x, y)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}} \\
&\quad + C_{14}(\mathfrak{q}, \mathfrak{p}; x, y)\chi\Big)^{\frac{1}{\mathfrak{q}}} + \Big(C_{12}(\mathfrak{q}, \mathfrak{p}; y, x)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}} + C_{10}(\mathfrak{q}, \mathfrak{p}; y, x)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}} + C_{14}(\mathfrak{q}, \mathfrak{p}; y, x)\chi\Big)^{\frac{1}{\mathfrak{q}}}\Big],
\end{aligned}$$

where $C_9, C_{10}, C_{11}, C_{12}, C_{13}$ and C_{14} are given by (2.39)-(2.44).

Remark 9. In Corollary 2.5.7 inserting $\mathfrak{h}(\zeta) = \zeta, \chi = 0$, and $\mathfrak{l} = 2$ with $\phi_1(\zeta) = \zeta(1-\zeta)$, we obtain Corollary 3.7 in reference [48].

For $\lambda = \frac{1}{2}$, Theorem 2.5.4 reduces to the following result.

Corollary 2.5.8. *[44] Let $\mathcal{I} = [\alpha_1, \beta_1] \subset \mathcal{R}\backslash\{0\}$ be a $\mathfrak{p}$-harmonic convex set, and let $\mathfrak{F} : \mathcal{I} = [\alpha_1, \beta_1] \subset \mathcal{R}\backslash\{0\} \to \mathcal{R}$ be a differentiable function on the interior $\mathcal{I}^o$ of $\mathcal{I}$. If $\mathfrak{F}' \in \mathfrak{L}[x, y]$, $|\mathfrak{F}'|^{\mathfrak{q}}$ is a* S*trongly reciprocally $(\mathfrak{p}, \mathfrak{h})$-convex function of higher order on $\mathcal{I}$, $\mathfrak{r}, \mathfrak{q} > 1$, $\frac{1}{\mathfrak{r}} + \frac{1}{\mathfrak{q}} = 1$, and $\lambda \in [0, 1]$, then*

$$\left|\frac{1}{4}\left[\mathfrak{F}(x)+2\mathfrak{F}\left[\left(\frac{2x^{\mathfrak{p}}y^{\mathfrak{p}}}{x^{\mathfrak{p}}+y^{\mathfrak{p}}}\right)^{\frac{1}{\mathfrak{p}}}\right]+\mathfrak{F}(y)\right]-\frac{\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}{y^{\mathfrak{p}}-x^{\mathfrak{p}}}\int_{x}^{y}\frac{\mathfrak{F}(\mathfrak{a})}{\mathfrak{a}^{1+\mathfrak{p}}}\,d\mathfrak{a}\right|$$
$$\leq\frac{(y^{\mathfrak{p}}-x^{\mathfrak{p}})}{2\mathfrak{p}(x^{\mathfrak{p}}y^{\mathfrak{p}})}\times\left(\frac{2}{4.2^{\mathfrak{r}}(\mathfrak{r}+1)}\right)^{\frac{1}{\mathfrak{r}}}\big[\big(C_9(\mathfrak{q},\mathfrak{p};x,y)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}}$$
$$+C_{11}(\mathfrak{q},\mathfrak{p};x,y)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}}+C_{14}(\mathfrak{q},\mathfrak{p};x,y)\chi\big)^{\frac{1}{\mathfrak{q}}}$$
$$+\big(C_{12}(\mathfrak{q},\mathfrak{p};y,x)\big|\mathfrak{F}'(x)\big|^{\mathfrak{q}}+C_{10}(\mathfrak{q},\mathfrak{p};y,x)\big|\mathfrak{F}'(y)\big|^{\mathfrak{q}}+C_{14}(\mathfrak{q},\mathfrak{p};y,x)\chi\big)^{\frac{1}{\mathfrak{q}}}\big],$$

where $C_9, C_{10}, C_{11}, C_{12}, C_{13}$ *and* C_{14} *are given by (2.39)-(2.44).*

Remark 10. In Corollary 2.5.8, inserting $\mathfrak{h}(\zeta)=\zeta, \chi=0$, and $\mathfrak{l}=2$ with $\phi_1(\zeta)=\zeta(1-\zeta)$, we obtain Corollary 3.8 in reference [48].

Chapter 3

Main Results

This chapter began with defining $\mathfrak{h}$-convex function, Godunova–Levin function, $\mathfrak{s}$-convex function in the second sense, and $\mathfrak{p}$-convex function. In Section 3.2 we discussed a newly defined mapping $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ for $\mathfrak{h}$-convex function, see [32], a couple of basic results in the form of lemma and proposition are made. In Section 3.2.1 we utilize these results to establish a new generalized Fejér-type inequality. Afterward, with some modifications, we construct a generalized form of Hermite-Hadamard inequality in Section 3.2.2. In the end, we talked about the most recent mapping that has been defined as $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ for $\mathfrak{h}$-convex function.

3.1 Introduction

As, $\mathfrak{h}$-convex function is also known as $\mathrm{SX}(\mathfrak{h}, \mathcal{I})$, the classes such as Godunova–Levin functions known as $Q(\mathcal{I})$ [49], $\mathfrak{s}$-convex functions in the second sense known as $K_{\mathfrak{s}}^2$ [50], and $\mathfrak{p}$-convex functions known as $P(\mathcal{I})$ [51]. Here we have some basic definitions of the Godunova–Levin function and $\mathfrak{s}$-convex function.

Definition 3.1.1 (Godunova–Levin function)**.** *Consider a non-negative convex function* $\mathfrak{F} : \mathcal{I} \subseteq \mathcal{R} \to \mathcal{R}$ *and for all* $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$ *and* $\zeta \in (0,1)$*, we have the following inequality:*

$$\mathfrak{F}\big(\zeta\mathfrak{a} + (1-\zeta)\mathfrak{b}\big) \leq \frac{\mathfrak{F}(\mathfrak{a})}{\zeta} + \frac{\mathfrak{F}(\mathfrak{b})}{1-\zeta}.$$

Definition 3.1.2 ($\mathfrak{s}$-convex function)**.** *Consider a non-negative convex function* $\mathfrak{F} : \mathcal{I} \subseteq \mathcal{R} \to \mathcal{R}$ *and for all* $\mathfrak{a}, \mathfrak{b} \in \mathcal{I}$*,* $\mathfrak{s} \in (0,1]$*, and* $\zeta \in [0,1]$*, we have the following inequality:*

$$\mathfrak{F}\big(\zeta\mathfrak{a} + (1-\zeta)\mathfrak{b}\big) \leq \zeta^{\mathfrak{s}}\mathfrak{F}(\mathfrak{a}) + (1-\zeta)^{\mathfrak{s}}\mathfrak{F}(\mathfrak{b}).$$

From the Definition (2.1.11), we consider $\mathfrak{h}(\zeta) = \zeta$, then all non-negative convex functions belong to $SX(\mathfrak{h}, \mathcal{I})$. If the above inequality is reversed $\mathfrak{F}$ is said to be $\mathfrak{h}$-concave or $\mathfrak{F} \in$SV$(\mathfrak{h}, \mathcal{I})$. Moreover, all non-negative concave functions belongs to SV$(\mathfrak{h}, \mathcal{I})$ for $\mathfrak{h}(\zeta) = \zeta$. We can included $Q(\mathcal{I})$, $K^2_{\mathfrak{s}}$, and $P(\mathcal{I})$ in the class of $\mathfrak{h}$-convex functions. If $\mathfrak{h}(\zeta) = \frac{1}{\zeta}$, $\mathfrak{h}(\zeta) = 1$, and $\mathfrak{h}(\zeta) = \zeta^s$, where $s \in (0,1)$, then $Q(\mathcal{I}) = SX(\mathfrak{h}, \mathcal{I})$, $P(\mathcal{I}) \subseteq SX(\mathfrak{h}, \mathcal{I})$, and $K^2_{\mathfrak{s}} \subseteq SX(\mathfrak{h}, \mathcal{I})$, respectively.
Onward in this thesis, the function $\mathfrak{h}$ is considered integrable on $[0,1]$.

3.2 The mapping $\mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}}(x)$

Consider two real numbers $\kappa_1 < \kappa_2$, consider integrable functions $\mathfrak{F} : [\kappa_1, \kappa_2] = \mathcal{I} \to \mathcal{R}$ and $\mathfrak{w} : [\kappa_1, \kappa_2] = \mathcal{I} \to \mathcal{R}^+ \cup \{0\}$. Define the mapping $\mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}}(x) : [0,1] \to \mathcal{R}$ as

$$\mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}}(x) = \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a},$$

where

$$C_x(\mu,\nu) = \min\big\{\mu(x), \nu(x)\big\}, \qquad D_x(\mu,\nu) = \max\big\{\mu(x), \nu(x)\big\},$$

for $x \in [0,1]$ we define $\mu : [0,1] \to [\kappa_1, \kappa_2]$ and $\nu(x) : [0,1] \to [\kappa_1, \kappa_2]$ as

$$\mu(x) = x\kappa_2 + (1-x)\kappa_1, \qquad \nu(x) = x\kappa_1 + (1-x)\kappa_2$$

Note that, for $\mathfrak{w} = 1$ in $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ we have

$$\mathbb{M}_{\mathfrak{F}}^{1}(x) = \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\, d\mathfrak{a},$$

The following lemma will be useful to us often.

Lemma 3.2.1. *[32] Consider two functions $\mathfrak{F} : [\kappa_1, \kappa_2] \to \mathcal{R}$ and $\mathfrak{w} : [\kappa_1, \kappa_2] \to \mathcal{R}^+ \cup \{0\}$. Also, for any $\mathfrak{s} \in [0,1]$, define the $\phi_{\mathfrak{s}} : [\kappa_1, \kappa_2] \times [\kappa_1, \kappa_2] \to [\kappa_1, \kappa_2]$ as $\phi_{\mathfrak{s}}(\mathfrak{a}, \mathfrak{b}) = \mathfrak{s}\mathfrak{a} + (1-\mathfrak{s})\mathfrak{b}$ for $\mathfrak{a}, \mathfrak{b} \in [\kappa_1, \kappa_2]$. Then for all $x \in [0,1]$,*

(i)

$$C_x(\mu,\nu) + D_x(\mu,\nu) = \mu(x) + \nu(x) = \kappa_1 + \kappa_2.$$

(ii)

$$D_x(\mu,\nu) - C_x(\mu,\nu) = |\mu(x) - \nu(x)| = |1-2x|(\kappa_2 - \kappa_1).$$

(iii)

$$\mathfrak{F}(C_x(\mu,\nu)) + \mathfrak{F}(D_x(\mu,\nu)) = [\mathfrak{F} \circ \mu](x) + [\mathfrak{F} \circ \nu](x).$$

(iv)

$$\phi_{\mathfrak{s}}(C_x(\mu,\nu), D_x(\mu,\nu)) + \phi_{\mathfrak{s}}(D_x(\mu,\nu), C_x(\mu,\nu)) = \kappa_1 + \kappa_2.$$

If $\mathfrak{w}$ is symmetric on $[\kappa_1, \kappa_2]$ with respect to $\frac{\kappa_1+\kappa_2}{2}$, then:

(v) It is symmetric on the interval $[C_x(\mu,\nu), D_x(\mu,\nu)]$ with respect to $\frac{\kappa_1+\kappa_2}{2}$,

(vi) We have the following identities:

$$[\mathfrak{w} \circ \phi_{\mathfrak{s}}]\big(C_x(\mu,\nu), D_x(\mu,\nu)\big) = [\mathfrak{w} \circ \phi_{\mathfrak{s}}]\big(D_x(\mu,\nu), C_x(\mu,\nu)\big)$$
$$= [\mathfrak{w} \circ \phi_{\mathfrak{s}}]\big(\mu(x), \nu(x)\big) = [\mathfrak{w} \circ \phi_{\mathfrak{s}}]\big(\nu(x), \mu(x)\big),$$

(vii) We have the following integral inequalities:

$$\int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} = \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{w}(\mathfrak{a})\, d\mathfrak{a}, \qquad \int_{C_x(\mu,\nu)}^{\frac{\kappa_1+\kappa_2}{2}} \mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} = \int_{\frac{\kappa_1+\kappa_2}{2}}^{D_x(\mu,\nu)} \mathfrak{w}(\mathfrak{a})\, d\mathfrak{a}.$$

Proof. The proofs of (i), (iii), and (iv) are straight-forward, and for (ii), we can utilize the following identities

$$\min\{\mathfrak{a}, \mathfrak{b}\} = \frac{\mathfrak{a} + \mathfrak{b} - |\mathfrak{b} - \mathfrak{a}|}{2},$$

and

$$\max\{\mathfrak{a}, \mathfrak{b}\} = \frac{\mathfrak{a} + \mathfrak{b} + |\mathfrak{b} - \mathfrak{a}|}{2}.$$

For (v), suppose $\kappa_1 \leq C_x(\mu,\nu) \leq \frac{\kappa_1+\kappa_2}{2} \leq D_x(\mu,\nu) \leq \kappa_2$, and so for $\mathfrak{a} \in [C_x(\mu,\nu), D_x(\mu,\nu)]$, from (i) we have

$$\mathfrak{w}\big(C_x(\mu,\nu) + D_x(\mu,\nu) - \mathfrak{a}\big) = \mathfrak{w}(\kappa_1 + \kappa_2 - \mathfrak{a}) = \mathfrak{w}(\mathfrak{a}),$$

since, $\mathfrak{w}$ is symmetric with respect to $\frac{\kappa_1+\kappa_2}{2}$. For (vi), we must consider the following equalities:

$$[\mathfrak{w} \circ \phi_{\mathfrak{s}}]\big(\nu(x), \mu(x)\big) = \mathfrak{w}\Big(\mathfrak{s}\big(x\kappa_2 + (1-x)\kappa_1\big) + (1-\mathfrak{s})\big(x\kappa_1 + (1-x)\kappa_2\big)\Big)$$
$$= \mathfrak{w}\Big(x\big(\mathfrak{s}\kappa_2 + (1-\mathfrak{s})\kappa_1\big) + (1-x)\big(\mathfrak{s}\kappa_1 + (1-\mathfrak{s})\kappa_2\big)\Big)$$

$$= \mathfrak{w}\big((1-x)\big(\mathfrak{s}\kappa_2 + (1-\mathfrak{s})\kappa_1\big) + x\big(\mathfrak{s}\kappa_1 + (1-\mathfrak{s})\kappa_2\big)\big)$$
$$= \mathfrak{w}\big(\mathfrak{s}\big(x\kappa_1 + (1-x)\kappa_2\big) + (1-\mathfrak{s})\big(x\kappa_2 + (1-x)\kappa_1\big)\big)$$
$$= [\mathfrak{w} \circ \phi_{\mathfrak{s}}]\big(\mu(x), \nu(x)\big).$$

Finally, it is sufficient to use (i) and the change of variable $u = \kappa_1 + \kappa_2 - \mathfrak{a}$ for (vii). □

By Lemma 3.2.1 we have some basic properties for the mapping $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ below.

Proposition 6. [32] Consider two functions $\mathfrak{F} : [\kappa_1, \kappa_2] \to \mathcal{R}$ and $\mathfrak{w} : [\kappa_1, \kappa_2] \to \mathcal{R}^+ \cup \{0\}$. Then:

(i)

$$\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x) = \mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(1-x) \qquad \forall x \in [0,1],$$

which shows that $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ is symmetric on $[\kappa_1, \kappa_2]$ with respect to $\frac{1}{2}$.

(ii) For symmetric $\mathfrak{w}$ on $[\kappa_1, \kappa_2]$ with respect to $\frac{\kappa_1+\kappa_2}{2}$ and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\big|\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)\big| \leq \|\mathfrak{F}\|_p \|\mathfrak{w}\|_q.$$

Also, if $C_x(\mu, \nu) = \mu(x)$, then

$$\big|\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)\big| \leq \left(\frac{1}{2}\right)^{\frac{1}{q}} \big[x(\kappa_2 - \kappa_1)\big]^{\frac{1}{p}} \|\mathfrak{w}\|_q \|\mathfrak{F}\|_\infty,$$

and if $C_x(\mu, \nu) = \nu(x)$, then

$$\big|\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)\big| \leq \left(\frac{1}{2}\right)^{\frac{1}{q}} \big[(1-x)(\kappa_2 - \kappa_1)\big]^{\frac{1}{p}} \|\mathfrak{w}\|_q \|\mathfrak{F}\|_\infty.$$

(iii) Let, the function $(\mathfrak{F}\mathfrak{w})(\mathfrak{a}) = \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})$ is convex on $[\kappa_1, \kappa_2]$. If $C_x(\mu, \nu) =$

$\nu(x)$ for some $x \in [0,1)$, then the function $\frac{\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)}{1-x}$ is convex. Also, if $C_x(\mu,\nu) = \mu(x)$ for some $x \in (0,1]$, then the function $\frac{\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)}{x}$ is convex.

(iv) Considering $\mathfrak{F}$ and $\mathfrak{w}$ are two continuous functions on $[\kappa_1,\kappa_2]$. If $\mathfrak{F}$ is non-negative (non-positive) on $[\kappa_1,\kappa_2]$, then the function $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ is increasing (decreasing) on $[0,\frac{1}{2})$ and is decreasing (increasing) on $(\frac{1}{2},0]$. Also, $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ has a relative extreme point at $x = \frac{1}{2}$. If $\mathfrak{w} \not\equiv 0$, then corresponding to any $\mathfrak{a} \in [\kappa_1,\kappa_2] \setminus \{\frac{\kappa_1+\kappa_2}{2}\}$ satisfying

$$\mathfrak{F}(\mathfrak{a}) + \mathfrak{F}(\kappa_1+\kappa_2-\mathfrak{a}) = 0,$$

there exists a critical point for $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$.

Proof. (i) This follows from the facts $\mu(1-x) = \nu(x)$ and $\nu(1-x) = \mu(x)$.

(ii) Since $C_x(\mu,\nu) \le \frac{\kappa_1+\kappa_2}{2} \le D_x(\mu,\nu)$, using the statements (iii) and (vii) in Lemma 3.2.1 and Hölder's inequality, we have the following inequalities:

$$\begin{aligned}
&\left|\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)\right| \\
&\le \left(\int_{\kappa_1}^{C_x(\mu,\nu)} \left|\mathfrak{F}(\mathfrak{a})\right|^p d\mathfrak{a}\right)^{\frac{1}{p}} \left(\int_{\kappa_1}^{C_x(\mu,\nu)} \left|\mathfrak{w}(\mathfrak{a})\right|^q d\mathfrak{a}\right)^{\frac{1}{q}} \\
&\quad + \left(\int_{D_x(\mu,\nu)}^{\kappa_2} \left|\mathfrak{F}(\mathfrak{a})\right|^p d\mathfrak{a}\right)^{\frac{1}{p}} \left(\int_{D_x(\mu,\nu)}^{\kappa_2} \left|\mathfrak{w}(\mathfrak{a})\right|^q d\mathfrak{a}\right)^{\frac{1}{q}} \\
&= \left(\int_{\kappa_1}^{C_x(\mu,\nu)} \left|\mathfrak{w}(\mathfrak{a})\right|^q d\mathfrak{a}\right)^{\frac{1}{q}} \left[\left(\int_{\kappa_1}^{C_x(\mu,\nu)} \left|\mathfrak{F}(\mathfrak{a})\right|^p d\mathfrak{a}\right)^{\frac{1}{p}} + \left(\int_{D_x(\mu,\nu)}^{\kappa_2} \left|\mathfrak{F}(\mathfrak{a})\right|^p d\mathfrak{a}\right)^{\frac{1}{p}}\right] \\
&\le \frac{1}{2}\left(\int_{\kappa_1}^{C_x(\mu,\nu)} \left|\mathfrak{w}(\mathfrak{a})\right|^q d\mathfrak{a}\right)^{\frac{1}{q}} \left[\left(\int_{\kappa_1}^{C_x(\mu,\nu)} \left|\mathfrak{F}(\mathfrak{a})\right|^p d\mathfrak{a}\right)^{\frac{1}{p}} + \left(\int_{D_x(\mu,\nu)}^{\kappa_2} \left|\mathfrak{F}(\mathfrak{a})\right|^p d\mathfrak{a}\right)^{\frac{1}{p}}\right].
\end{aligned}$$

Thus (ii) is proved.

(iii) We prove the first part. Considering the changes of variable $\mathfrak{a} = x\kappa_1 + (1-x)u$ and $\mathfrak{a} = x\kappa_2 + (1-x)u$ in two integrals of $\mathcal{H}(x) = \frac{\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)}{1-x}$, we obtain that

$$\mathcal{H}(x) = \int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big(x\kappa_1 + (1-x)\mathfrak{a}\big)\, d\mathfrak{a} + \int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big(x\kappa_2 + (1-x)\mathfrak{a}\big)\, d\mathfrak{a}.$$

Now for $x_1, x_2 \in [0,1)$ and non-negative r, s with $r+s=1$, we have

$$\begin{aligned}
\mathcal{H}(rx_1 + sx_2) &= \int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big((rx_1+sx_2)\kappa_1 + \big(1-(rx_1+sx_2)\big)\mathfrak{a}\big)\, d\mathfrak{a} \\
&\quad + \int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big((rx_1+sx_2)\kappa_2 + \big(1-(rx_1+sx_2)\big)\mathfrak{a}\big)\, d\mathfrak{a} \\
&= \int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big(r\big(x_1\kappa_1 + (1-x_1)\mathfrak{a}\big)\big)\, d\mathfrak{a} + \int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big(s\big(x_2\kappa_1 + (1-x_2)\mathfrak{a}\big)\big)\, d\mathfrak{a} \\
&\quad + \int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big(r\big(x_1\kappa_2 + (1-x_1)\mathfrak{a}\big)\big)\, d\mathfrak{a} + \int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big(s\big(x_2\kappa_2 + (1-x_2)\mathfrak{a}\big)\big)\, d\mathfrak{a} \\
&\leq r\left[\int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big(x_1\kappa_1 + (1-x_1)\mathfrak{a}\big)\, d\mathfrak{a} + \int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big(x_1\kappa_2 + (1-x_1)\mathfrak{a}\big)\, d\mathfrak{a}\right] \\
&\quad + s\left[\int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big(x_2\kappa_1 + (1-x_2)\mathfrak{a}\big)\, d\mathfrak{a} + \int_{\kappa_1}^{\kappa_2} (\mathfrak{F}\mathfrak{w})\big(x_2\kappa_2 + (1-x_2)\mathfrak{a}\big)\, d\mathfrak{a}\right] \\
&= r\mathcal{H}(x_1) + s\mathcal{H}(x_2).
\end{aligned}$$

(iv) Using the fact that $\mathfrak{w}$ is symmetric on $[\kappa_1, \kappa_2]$ with respect to $\frac{\kappa_1+\kappa_2}{2}$ and the Leibniz integral rule, we obtain the following result:

$$\frac{1}{\kappa_2 - \kappa_1} \frac{d\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}}{dx}(x) = \begin{cases} [\mathfrak{w} \circ \mu](x)\{[\mathfrak{F} \circ \mu](x) + [\mathfrak{F} \circ \nu](x)\}, & x \in [0, \frac{1}{2}), \\ -[\mathfrak{w} \circ \nu](x)\{[\mathfrak{F} \circ \mu](x) + [\mathfrak{F} \circ \nu](x)\}, & x \in (\frac{1}{2}, 1). \end{cases}$$

□

Proposition 7. Let $\mathfrak{F} : [\kappa_1, \kappa_2] \to \mathcal{R}$, $G : [\kappa_1, \kappa_2] \to \mathcal{R}$, and $\mathfrak{w} : [\kappa_1, \kappa_2] \to \mathcal{R}^+ \cup \{0\}$ be integrable functions. The mappings $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ and $\mathbb{M}_{g}^{\mathfrak{w}}(x)$ for two real numbers $\kappa_1 \leq \kappa_2$. Define the mapping $\mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}}(x) : [0,1] \to \mathbb{R}$ as:

$$\mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}}(x) = \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a},$$

and $\mathbb{M}^{\mathfrak{w}}_{g}(x) : [0,1] \to \mathcal{R}$ as

$$\mathbb{M}^{\mathfrak{w}}_{g}(x) = \int_{\kappa_1}^{C_x(\mu,\nu)} g(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} g(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a}.$$

Then $\mathfrak{F} + g : [\kappa_1, \kappa_2] \to \mathcal{R}$ is also integrable function.

Proof.

$$\begin{aligned}
\mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}+g}(x) &= \int_{\kappa_1}^{C_x(\mu,\nu)} (\mathfrak{F}+g)(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} (\mathfrak{F}+g)(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} [(\mathfrak{F}(\mathfrak{a}) + g(\mathfrak{a})]\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} [(\mathfrak{F}(\mathfrak{a}) + g(\mathfrak{a})]\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} [\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a}) + g(\mathfrak{a})\mathfrak{w}(\mathfrak{a})]\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} [\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a}) + g(\mathfrak{a})\mathfrak{w}(\mathfrak{a})]\, d\mathfrak{a} \\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{\kappa_1}^{C_x(\mu,\nu)} g(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} g(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&\quad + \int_{\kappa_1}^{C_x(\mu,\nu)} g(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} g(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}}(x) + \mathbb{M}^{\mathfrak{w}}_{g}(x).
\end{aligned}$$

Hence proved. □

Proposition 8. Let $\mathfrak{F} : [\kappa_1, \kappa_2] \to \mathcal{R}$, and $\mathfrak{w} : [\kappa_1, \kappa_2] \to \mathcal{R}^+ \cup \{0\}$ be two integrable functions. Then for any $\lambda \geq 0$, $\lambda\mathfrak{F} : [\kappa_1, \kappa_2] \to \mathcal{R}$, is also integrable function.

Define the mapping $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x) : [0,1] \to \mathbb{R}$ as:

$$\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x) = \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a}.$$

Proof. Let $\lambda \geq 0$,

Considering the mapping $\mathbb{M}_{\lambda\mathfrak{F}}^{\mathfrak{w}}(x)$

$$\begin{aligned}
\mathbb{M}_{\lambda\mathfrak{F}}^{\mathfrak{w}}(x) &= \int_{\kappa_1}^{C_x(\mu,\nu)} (\lambda\mathfrak{F}(\mathfrak{a}))\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} (\lambda\mathfrak{F}(\mathfrak{a}))\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} \lambda\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \lambda\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \lambda \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \lambda \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \lambda \left[\int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \right] = \lambda\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x).
\end{aligned}$$

Hence proved. □

Proposition 9. The mapping $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ for two real numbers $\kappa_1 \leq \kappa_2$ integrable functions $\mathfrak{F}_i : [\kappa_1, \kappa_2] \to \mathbb{R}$, and $\mathfrak{w} : [\kappa_1, \kappa_2] \to \mathcal{R}^+ \cup \{0\},\ 1 \leq i \leq n$. For $\lambda_i \geq 0,\ 1 \leq i \leq n$ the function $\mathfrak{F} : [0,1] \to \mathbb{R}$ where $\mathfrak{F} = \sum_{i=1}^{n} \lambda_i \mathfrak{F}_i$ is also integrable function.

Proof. Considering the mapping $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$

$$\begin{aligned}
\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x) &= \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} \sum_{i=1}^{n} \lambda_i \mathfrak{F}_i(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \sum_{i=1}^{n} \lambda_i \mathfrak{F}_i(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} \big[\lambda_1\mathfrak{F}_1(\mathfrak{a}) + \lambda_2\mathfrak{F}_2(\mathfrak{a}) + \cdots + \lambda_n\mathfrak{F}_n(\mathfrak{a})\big]\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \big[\lambda_1\mathfrak{F}_1(\mathfrak{a}) + \lambda_2\mathfrak{F}_2(\mathfrak{a}) + \cdots + \lambda_n\mathfrak{F}_n(\mathfrak{a})\big]\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a}
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\kappa_1}^{C_x(\mu,\nu)} \big[\lambda_1\mathfrak{F}_1(\mathfrak{a})\mathfrak{w}(\mathfrak{a}) + \lambda_2\mathfrak{F}_2(\mathfrak{a})\mathfrak{w}(\mathfrak{a}) + \cdots + \lambda_n\mathfrak{F}_n(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\big]\, d\mathfrak{a} \\
&\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \big[\lambda_1\mathfrak{F}_1(\mathfrak{a})\mathfrak{w}(\mathfrak{a}) + \lambda_2\mathfrak{F}_2(\mathfrak{a})\mathfrak{w}(\mathfrak{a}) + \cdots + \lambda_n\mathfrak{F}_n(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\big]\, d\mathfrak{a} \\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} \lambda_1\mathfrak{F}_1(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{\kappa_1}^{C_x(\mu,\nu)} \lambda_2\mathfrak{F}_2(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \cdots + \int_{\kappa_1}^{C_x(\mu,\nu)} \lambda_n\mathfrak{F}_n(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \lambda_1\mathfrak{F}_1(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \lambda_2\mathfrak{F}_2(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \cdots + \int_{D_x(\mu,\nu)}^{\kappa_2} \lambda_n\mathfrak{F}_n(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \lambda_1 \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}_1(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \lambda_2 \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}_2(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, dx + \cdots + \lambda_n \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}_n(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&\quad + \lambda_1 \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}_1(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \lambda_2 \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}_2(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, dx + \cdots + \lambda_n \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}_n(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \lambda_1 \left[\int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}_1(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}_1(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \right] \\
&\quad + \lambda_2 \left[\int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}_2(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}_2(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, dx \right] \\
&\quad + \cdots + \lambda_n \left[\int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}_n(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}_n(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \right] \\
&= \lambda_1 \mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}_1}(x) + \lambda_2 \mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}_2}(x) + \cdots + \lambda_n \mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}_n}(x) \\
&= \sum_{i=1}^{n} \lambda_i \mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}_i}(x).
\end{aligned}
$$

Hence proved. □

Proposition 10. Let $\mathfrak{F}_i : [\kappa_1, \kappa_2] \to \mathcal{R},\ 1 \le i \le n$ and $\mathfrak{w} : [\kappa_1, \kappa_2] \to \mathcal{R}^+ \cup \{0\}$ be integrable functions. Then $\mathfrak{F} = \max \mathfrak{F}_i, i = 1, 2, \ldots, n$ is also integrable function.

Proof.

$$
\begin{aligned}
\mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}}(x) &= \int_{\kappa_1}^{C_x(\mu,\nu)} \max \mathfrak{F}_i(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \max \mathfrak{F}_i(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}_c(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}_c(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} = \mathbb{M}^{\mathfrak{w}}_{\mathfrak{F}_c}(x).
\end{aligned}
$$

Hence proved. □

3.2.1 A New Generalized form of Fejér's inequality

Here is a new and improved generalization of Fejér's inequality, extending its applicability to the newly defined mapping $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ for $\mathfrak{h}$-convex functions. This will broaden the scope and provide a more flexible tool for analysis. This is also known as Hermite-Hadamard-Fejér's inequality for an $\mathfrak{h}$-convex function. Once the derivation is done, we developed several results in the form of corollaries and remarks, for deeper insights into the inequality's structure.

Theorem 3.2.2. *Considering* $\mathfrak{h}$*-convex function* $\mathfrak{F} : [\kappa_1, \kappa_2] = \mathcal{I} \to \mathcal{R}$ *and* $\mathfrak{w} : [\kappa_1, \kappa_2] \to \mathcal{R}^+ \cup \{0\}$ *such that* $\mathfrak{w}$ *is symmetric with respect to* $\frac{\kappa_1+\kappa_2}{2}$*. For all* $x \in [0,1]$*, we have the following inequality:*

$$\begin{aligned}
&\frac{1}{2\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&\leq \int_{\kappa_1}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} - \mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x) \\
&\leq \frac{|\nu(x)-\mu(x)|[\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)}{(\mu(x)-\nu(x))}\int_{\nu(x)}^{\mu(x)} \mathfrak{h}\left(\frac{\mathfrak{a}-\nu(x)}{\mu(x)-\nu(x)}\right)\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\
&= \frac{|\nu(x)-\mu(x)|[\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)}{(\nu(x)-\mu(x))}\int_{\mu(x)}^{\nu(x)} \mathfrak{h}\left(\frac{\mathfrak{a}-\mu(x)}{\nu(x)-\mu(x)}\right)\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a}.
\end{aligned} \tag{3.1}$$

Proof. For any $\rho \in [C_x(\mu,\nu), D_x(\mu,\nu)]$ exists $k \in (0,1)$ such that $\rho = kC_x(\mu,\nu) + k_1D_x(\mu,\nu)$, $k_1 = 1-k$. From the definition of an $\mathfrak{h}$-convex function, we have

$$\begin{aligned}
&\mathfrak{F}(kC_x(\mu,\nu)+k_1D_x(\mu,\nu))\mathfrak{w}(kC_x(\mu,\nu)+k_1D_x(\mu,\nu)) \\
&\leq \big(\mathfrak{h}(k)\mathfrak{F}\big(C_x(\mu,\nu)+\mathfrak{h}(k_1)\mathfrak{F}(D_x(\mu,\nu))\big)\mathfrak{w}(kC_x(\mu,\nu)+k_1D_x(\mu,\nu)) \\
&= \mathfrak{h}(k)\mathfrak{F}\big(C_x(\mu,\nu)\mathfrak{w}(kC_x(\mu,\nu)+k_1D_x(\mu,\nu))+\mathfrak{h}(k_1)\mathfrak{F}(D_x(\mu,\nu))\mathfrak{w}(kC_x(\mu,\nu)+k_1D_x(\mu,\nu)),
\end{aligned} \tag{3.2}$$

and

$$\mathfrak{F}(k_1C_x(\mu,\nu)+kD_x(\mu,\nu))\mathfrak{w}\big(k_1C_x(\mu,\nu)+kD_x(\mu,\nu)\big)$$
$$\leq \big(\mathfrak{h}(k_1)\mathfrak{F}\big(C_x(\mu,\nu)\big)+\mathfrak{h}(k)\mathfrak{F}\big(D_x(\mu,\nu)\big)\big)\mathfrak{w}(k_1C_x(\mu,\nu)+kD_x(\mu,\nu))$$
$$=\mathfrak{h}(k_1)\mathfrak{F}\big(C_x(\mu,\nu)\big)\mathfrak{w}(k_1C_x(\mu,\nu)+kD_x(\mu,\nu))+\mathfrak{h}(k)\mathfrak{F}\big(D_x(\mu,\nu)\big)\mathfrak{w}(k_1C_x(\mu,\nu)+kD_x(\mu,\nu)). \tag{3.3}$$

After adding (3.2) and (3.3), and integrating with respect to k over $[0,1]$, we obtain

$$\begin{aligned}
&\int_0^1 \mathfrak{F}\big(kC_x(\mu,\nu)+k_1D_x(\mu,\nu)\big)\mathfrak{w}\big(kC_x(\mu,\nu)+k_1D_x(\mu,\nu)\big)\,dk \\
&\quad+\int_0^1 \mathfrak{F}\big(k_1C_x(\mu,\nu)+kD_x(\mu,\nu)\big)\mathfrak{w}\big(k_1C_x(\mu,\nu)+kD_x(\mu,\nu)\big)\,dk \\
&\leq \int_0^1 \Big[\mathfrak{h}(k)\mathfrak{F}\big(C_x(\mu,\nu)\big)\mathfrak{w}\big(kC_x(\mu,\nu)+k_1D_x(\mu,\nu)\big) \\
&\quad+\mathfrak{h}(k_1)\mathfrak{F}\big(D_x(\mu,\nu)\big)\mathfrak{w}\big(kC_x(\mu,\nu)+k_1D_x(\mu,\nu)\big)\Big]\,dk \\
&\quad+\int_0^1 \Big[\mathfrak{h}(k_1)\mathfrak{F}\big(C_x(\mu,\nu)\big)\mathfrak{w}\big(k_1C_x(\mu,\nu)+kD_x(\mu,\nu)\big) \\
&\quad+\mathfrak{h}(k)\mathfrak{F}\big(D_x(\mu,\nu)\big)\mathfrak{w}\big(k_1C_x(\mu,\nu)+kD_x(\mu,\nu)\big)\Big]\,dk.
\end{aligned}$$

Considering left hand side of the inequality and substituting $\mathfrak{a}=kC_x(\mu,\nu)+k_1D_x(\mu,\nu)$, $\mathfrak{a}=k_1C_x(\mu,\nu)+kD_x(\mu,\nu)$ in the first and second integral respectively, we get

$$\begin{aligned}
&\frac{1}{C_x(\mu,\nu)-D_x(\mu,\nu)}\int_{D_x(\mu,\nu)}^{C_x(\mu,\nu)}\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}+\frac{1}{D_x(\mu,\nu)-C_x(\mu,\nu)}\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)}\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} \\
&=\frac{2}{D_x(\mu,\nu)-C_x(\mu,\nu)}\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)}\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a},
\end{aligned} \tag{3.4}$$

solving the right-hand side of the inequality

$$\begin{aligned}
&\int_0^1 \mathfrak{F}\big(C_x(\mu,\nu)\big)\big[\mathfrak{h}(k)\mathfrak{w}\big(kC_x(\mu,\nu)+k_1D_x(\mu,\nu)\big)+\mathfrak{h}(k_1)\mathfrak{w}\big(k_1C_x(\mu,\nu)+kD_x(\mu,\nu)\big)\big]\,dk\\
&\quad+\int_0^1 \mathfrak{F}\big(D_x(\mu,\nu)\big)\big[\mathfrak{h}(k_1)\mathfrak{w}\big(kC_x(\mu,\nu)+k_1D_x(\mu,\nu)\big)+\mathfrak{h}(k)\mathfrak{w}\big(k_1C_x(\mu,\nu)+D_x(\mu,\nu)\big)\big]\,dk\\
&=2\mathfrak{F}\big(C_x(\mu,\nu)\big)\int_0^1 \mathfrak{h}(\mathfrak{s})\mathfrak{w}\big(\mathfrak{s}C_x(\mu,\nu)+(1-\mathfrak{s})D_x(\mu,\nu)\big)\,d\mathfrak{s}\\
&\quad+2\mathfrak{F}\big(D_x(\mu,\nu)\big)\int_0^1 \mathfrak{h}(\mathfrak{s})\mathfrak{w}\big((1-\mathfrak{s})C_x(\mu,\nu)+\mathfrak{s}D_x(\mu,\nu)\big)\,d\mathfrak{s}\\
&=2\big[\mathfrak{F}\big(C_x(\mu,\nu)\big)+\mathfrak{F}\big(D_x(\mu,\nu)\big)\big]\int_0^1 \mathfrak{h}(\mathfrak{s})\mathfrak{w}(\mathfrak{s}C_x(\mu,\nu)+(1-\mathfrak{s})D_x(\mu,\nu))\,d\mathfrak{s}\\
&=2\big[\mathfrak{F}\big(C_x(\mu,\nu)\big)+\mathfrak{F}\big(D_x(\mu,\nu)\big)\big]\int_0^1 \mathfrak{h}(\mathfrak{s})\big[\mathfrak{w}\circ\phi_{\mathfrak{s}}\big]\big(C_x(\mu,\nu),D_x(\mu,\nu)\big)\,d\mathfrak{s},
\end{aligned} \tag{3.5}$$

here we use the symmetry of the weight $\mathfrak{w}$. Now combining and simplifying the results obtained in equation (3.4) and (3.5)

$$\begin{aligned}
&\frac{1}{D_x(\mu,\nu)-C_x(\mu,\nu)}\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\le \big[\mathfrak{F}\big(C_x(\mu,\nu)\big)+\mathfrak{F}\big(D_x(\mu,\nu)\big)\big]\int_0^1 \mathfrak{h}(\mathfrak{s})[\mathfrak{w}\circ\phi_{\mathfrak{s}}](C_x(\mu,\nu),D_x(\mu,\nu))\,d\mathfrak{s},
\end{aligned} \tag{3.6}$$

Let $\mathfrak{F}$ be an $\mathfrak{h}$-convex function

$$\mathfrak{F}(ju+(1-j)v))\le \mathfrak{h}(j)\mathfrak{F}(u)+\mathfrak{h}(1-j)\mathfrak{F}(v).$$

Now substituting $j=\frac{1}{2}$, $u=kC_x(\mu,\nu)+(1-k)D_x(\mu,\nu)$ and $v=(1-k)C_x(\mu,\nu)+kD_x(\mu,\nu)$, simplifying the equation

$$\begin{aligned}
&\mathfrak{F}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\\
&\le \mathfrak{h}(\frac{1}{2})\big[\mathfrak{F}(kC_x(\mu,\nu)+(1-k)D_x(\mu,\nu)+\mathfrak{F}((1-k)C_x(\mu,\nu)+kD_x(\mu,\nu)\big].
\end{aligned}$$

Now, multiply $\mathfrak{w}\big(kC_x(\mu,\nu)+(1-k)D_x(\mu,\nu)\big)=\mathfrak{w}\big((1-k)C_x(\mu,\nu)+kD_x(\mu,\nu)\big)$ and integrate with respect to k over $[0,1]$

$$\begin{aligned}
&\mathfrak{F}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\int_0^1 \mathfrak{w}\big(kC_x(\mu,\nu)+(1-k)D_x(\mu,\nu)\big)\,dk\\
&\le \mathfrak{h}(\frac{1}{2})\bigg[\int_0^1 \mathfrak{F}\big(kC_x(\mu,\nu)+(1-k)D_x(\mu,\nu)\big)\mathfrak{w}\big(kC_x(\mu,\nu)+(1-k)D_x(\mu,\nu)\big)\,dk\\
&\quad+\int_0^1 \mathfrak{F}\big((1-k)C_x(\mu,\nu)+kD_x(\mu,\nu)\big)\mathfrak{w}\big((1-k)C_x(\mu,\nu)+kD_x(\mu,\nu)\big)\,dk\bigg],
\end{aligned}$$

after some suitable substitutions, we obtain

$$\begin{aligned}
&\mathfrak{F}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\frac{1}{D_x(\mu,\nu)-C_x(\mu,\nu)}\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\le \frac{2\mathfrak{h}(\frac{1}{2})}{D_x(\mu,\nu)-C_x(\mu,\nu)}\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a},
\end{aligned}$$

or

$$\frac{1}{2\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} \le \int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}. \tag{3.7}$$

Now combine equation (3.6) and (3.7), we obtain

$$\begin{aligned}
&\frac{1}{2\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\le \int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\le \big[D_x(\mu,\nu)-C_x(\mu,\nu)\big]\big[\mathfrak{F}\big(C_x(\mu,\nu)\big)+\mathfrak{F}\big(D_x(\mu,\nu)\big)\big]\\
&\quad\times\int_0^1 \mathfrak{h}(s)[\mathfrak{w}\circ\phi_{\mathfrak{s}}](C_x(\mu,\nu),D_x(\mu,\nu))\,ds.
\end{aligned} \tag{3.8}$$

By the definition of mapping $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}$ for all $x \in [0,1]$ we get the following identity

$$\int_{\kappa_1}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} - \int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} = \mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x). \tag{3.9}$$

Now using equation (3.9) and (ii), (iii), (v), and (vi) statements of Lemma 3.2.1 we get the following result

$$\begin{aligned}
&\frac{1}{2\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\le \int_{\kappa_1}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} - \mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)\\
&\le |\nu(x)-\mu(x)|\big([\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)\big)\int_0^1 \mathfrak{h}(s)[\mathfrak{w}\circ\phi_{\mathfrak{s}}](\mu(x),\nu(x))\,ds\\
&= |\nu(x)-\mu(x)|\big([\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)\big)\int_0^1 \mathfrak{h}(s)[\mathfrak{w}\circ\phi_{\mathfrak{s}}](\nu(x),\mu(x))\,ds.
\end{aligned} \tag{3.10}$$

Applying the change of variable $b = \phi_{\mathfrak{s}}\big(\mu(x),\nu(x)\big)$ or $b = \phi_{\mathfrak{s}}\big(\nu(x),\mu(x)\big)$ in the last two integrals in (3.10) and consider that

$$\frac{1}{\mu(x)-\nu(x)}\int_{\nu(x)}^{\mu(x)} \mathfrak{h}\left(\frac{\mathfrak{a}-\nu(x)}{\mu(x)-\nu(x)}\right)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} = \frac{1}{\nu(x)-\mu(x)}\int_{\mu(x)}^{\nu(x)} \mathfrak{h}\left(\frac{\mathfrak{a}-\mu(x)}{\nu(x)-\mu(x)}\right)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}. \tag{3.11}$$

Finally, by using equations (3.10) and (3.11) we get the desired result. □

The above theorem is valid in general. Furthermore, the following corollary holds for more specific values.

Corollary 3.2.3. *Considering integrable functions defined as* $\mathfrak{h} : [0, \max 1, \kappa_2 - \kappa_1]$, $\mathfrak{F} : [\kappa_1, \kappa_2] \to \mathcal{R}$ *and* $\mathfrak{w} : [\kappa_1, \kappa_2] \to \mathcal{R}^+ \cup \{0\}$, $\mathfrak{w} \ge 0$, *where* $\mathfrak{h}$ *is multiplicative or super multiplicative,* $\mathfrak{F}$ *is non-negative* $\mathfrak{h}$*-convex function, and* $\mathfrak{w}$ *is symmetric with respect* *to* $\frac{\kappa_1+\kappa_2}{2}$ *and* $\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{w}(x)\,dx > 0$. *Then*

$$\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq B\int_{\kappa_1}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}, \tag{3.12}$$

where $B = \min\left\{\frac{2h(\frac{1}{2})}{\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}}, \frac{\int 0^{\frac{D_x(\mu,\nu)-C_x(\mu,\nu)}{2}} \mathfrak{h}(u)\mathfrak{w}(u+\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2})\,du}{\int_{C_x(\mu,\nu)}^{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}\int_{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}^{D_x(\mu,\nu)} \mathfrak{h}(v-u)\mathfrak{w}(v)\mathfrak{w}(u)\,dvdu}\right\}$ *and* $\int_{C_x(\mu,\nu)}^{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}\int_{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}^{D_x(\mu,\nu)} \mathfrak{h}(v-u)\mathfrak{w}(v)\mathfrak{w}(u)\,dvdu \neq 0$, $\mathfrak{h}(u) \neq 0$ *for* $u \neq 0$.

Proof. Let $\mathfrak{h}$ be super-multiplicative $\mathfrak{h}(u) \neq 0$ for $u \neq 0$. Then $\mathfrak{h}(u) > 0$ for $u > 0$ we have $u, v \in [\kappa_1, \kappa_2]$ such that $\kappa_1 \leq C_x(\mu,\nu) \leq u < \frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2} < v \leq D_x(\mu,\nu) \leq \kappa_2$ we have

$$\frac{\kappa_1+\kappa_2}{2} = \frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2} = \left(\frac{v-\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}{v-u}\right)u + \left(\frac{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}-u}{v-u}\right)v.$$

Denote $\delta = \frac{v-\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}{v-u} > 0$. Then $\gamma = 1-\delta = \frac{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}-u}{v-u}$ and $\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2} = \delta u + \gamma v$, and $\mathfrak{F}(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}) = \mathfrak{F}(\delta u + \gamma v) \leq \mathfrak{h}(\delta)\mathfrak{F}(u) + \mathfrak{h}(\gamma)\mathfrak{F}(v)$. Since $\mathfrak{h}$ is super multiplicative, we have $\mathfrak{h}(\delta) = \mathfrak{h}\left(\frac{v-\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}{v-u}\right) \leq \frac{\mathfrak{h}(v-\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2})}{\mathfrak{h}(v-u)}$ similarly $\mathfrak{h}(\gamma) \leq \frac{\mathfrak{h}(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}-u)}{\mathfrak{h}(v-u)}$. So, when $\mathfrak{F} > 0$ we have

$$\mathfrak{F}\left(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\right) \leq \frac{\mathfrak{h}(v-\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2})}{\mathfrak{h}(v-u)}\mathfrak{F}(u) + \frac{\mathfrak{h}(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}-u)}{\mathfrak{h}(v-u)}\mathfrak{F}(v),$$

multiplying by $\mathfrak{h}(v-u)$ on both sides

$$\begin{aligned}&\mathfrak{h}(v-u)\mathfrak{F}\left(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\right)\\&\leq \mathfrak{h}\left(v-\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\right)\mathfrak{F}(u) + \mathfrak{h}\left(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}-u\right)f(v).\end{aligned} \tag{3.13}$$

This inequality continues to hold when $\mathfrak{h}$ is multiplicative, regardless the sign of $\mathfrak{F}$. Multiplying equation (3.13) with $\mathfrak{w}(u)$ and integrating over the interval $[C_x(\mu,\nu), \frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}]$ with respect to du, and we then multiply with $\mathfrak{w}(v)$ and integrate over interval $[\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}, D_x(\mu,\nu)]$ with respect to dv we get

$$\begin{aligned}
&\mathfrak{F}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\int_{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}^{D_x(\mu,\nu)}\bigg(\int_{C_x(\mu,\nu)}^{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}\mathfrak{h}(v-u)\mathfrak{w}(u)\,du\bigg)\mathfrak{w}(v)\,dv \\
&\le \int_{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}^{D_x(\mu,\nu)}\mathfrak{h}\bigg(v-\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\mathfrak{w}(v)\,dv\int_{C_x(\mu,\nu)}^{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}\mathfrak{F}(u)\mathfrak{w}(u)\,du \\
&\quad + \int_{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}^{D_x(\mu,\nu)}\mathfrak{F}(v)\mathfrak{w}(v)\,dv\int_{C_x(\mu,\nu)}^{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}\mathfrak{h}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}-u\bigg)\mathfrak{w}(u)\,du.
\end{aligned}$$

On the right hand side we apply the substitution $v-\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}=t$ in the first integral and $\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}-u=t$ in the second integral of the sum, we get

$$\begin{aligned}
&\mathfrak{F}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\int_{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}^{D_x(\mu,\nu)}\int_{C_x(\mu,\nu)}^{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}\mathfrak{h}(v-u)\mathfrak{w}(u)\mathfrak{w}(v)\,dudv \\
&\le \int_0^{\frac{D_x(\mu,\nu)-C_x(\mu,\nu)}{2}}\mathfrak{h}(t)\mathfrak{w}\bigg(t+\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\,dy\int_{C_x(\mu,\nu)}^{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}\mathfrak{F}(u)\mathfrak{w}(u)\,du \\
&\quad + \int_0^{\frac{D_x(\mu,\nu)-C_x(\mu,\nu)}{2}}\mathfrak{h}(t)\mathfrak{w}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}-t\bigg)\,dt\int_{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}^{D_x(\mu,\nu)}\mathfrak{F}(v)\mathfrak{w}(v)\,dv \\
&= \int_0^{\frac{D_x(\mu,\nu)-C_x(\mu,\nu)}{2}}\mathfrak{h}(t)\mathfrak{w}\bigg(t+\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\,dt.\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)}\mathfrak{F}(u)\mathfrak{w}(u)\,du,
\end{aligned}$$

where, in the first we apply the symmetry property of the function $\mathfrak{w}$ over the interval $[C_x(\mu,\nu), D_x(\mu,\nu)]$, i.e. $\mathfrak{w}(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}-t) = w(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}+t)$ for $t\in[0,\frac{D_x(\mu,\nu)-C_x(\mu,\nu)}{2}]$. Now applying Lemma 3.2.1 and substituting $u=\mathfrak{a}$ on the right side of the inequality respectively, we get the desired result

$$\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \leq \frac{\int_0^{\frac{D_x(\mu,\nu)-C_x(\mu,\nu)}{2}} \mathfrak{h}(t)\mathfrak{w}\big(t+\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\,dt}{\int_{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}^{D_x(\mu,\nu)} \int_{C_x(\mu,\nu)}^{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}} \mathfrak{h}(v-u)\mathfrak{w}(u)\mathfrak{w}(v)\,dudv} \times \int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}$$

or we can write

$$\begin{aligned}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) &\leq \frac{\int_0^{\frac{D_x(\mu,\nu)-C_x(\mu,\nu)}{2}} \mathfrak{h}(u)\mathfrak{w}\big(u+\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\,du}{\int_{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}^{D_x(\mu,\nu)} \int_{C_x(\mu,\nu)}^{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}} \mathfrak{h}(v-u)\mathfrak{w}(u)\mathfrak{w}(v)\,dudv} \times \int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} \\ &= B\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}.\end{aligned}$$

□

Remark 11. (i) Inequality (3.1) is reversed if $\mathfrak{F}$ is $\mathfrak{h}$-concave function in the theorem (3.2.2).

(ii) If $\mathfrak{h}$ is sub multiplicative, $\int_{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}}^{D_x(\mu,\nu)} \int_{C_x(\mu,\nu)}^{\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}} \mathfrak{h}(v-u)\mathfrak{w}(u)\mathfrak{w}(v)\,dudv \neq 0$, $\mathfrak{h} \geq 0$, and if $\mathfrak{F}$ is $\mathfrak{h}$-concave function then the inequality (3.12) is reversed, where constant B changes from min $\to$ max.

Here is another way of presenting the above generalization of Fejér's inequality.

Corollary 3.2.4. *Suppose that $\mathfrak{F} : [\kappa_1,\kappa_2] \to \mathcal{R}$ is non-negative integrable $\mathfrak{h}$-convex function and $\mathfrak{w} : [\kappa_1,\kappa_2] \to \mathcal{R}^+ \cup \{0\}$ is also integrable function and symmetric with respect to $\frac{\kappa_1+\kappa_2}{2}$. For all $x \in [0,1]$, we have the following inequality:*

$$\begin{aligned}&\frac{1}{2\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} \\ &\leq \int_{\kappa_1}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} - \mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x) \\ &\leq \frac{[\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)}{2}\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} H\left(\frac{D_x(\mu,\nu)-\mathfrak{a}}{D_x(\mu,\nu)-C_x(\mu,\nu)}\right)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} \\ &= \frac{[\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)}{2}\int_{D_x(\mu,\nu)}^{C_x(\mu,\nu)} H\left(\frac{\mathfrak{a}-C_x(\mu,\nu)}{D_x(\mu,\nu)-C_x(\mu,\nu)}\right)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}.\end{aligned} \tag{3.14}$$

Proof. Consider $\mathfrak{F}$ and $\mathfrak{h}$-convex function and some statements in Lemma 3.2.1

$$\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)=\mathfrak{F}\left(\frac{\phi_{\mathfrak{s}}\big(C_x(\mu,\nu),D_x(\mu,\nu)\big)+\phi_{\mathfrak{s}}\big(D_x(\mu,\nu),C_x(\mu,\nu)\big)}{2}\right)$$
$$\leq \mathfrak{h}\left(\frac{1}{2}\right)\big[\mathfrak{F}\circ\phi_{\mathfrak{s}}\big(C_x(\mu,\nu),D_x(\mu,\nu)\big)+\mathfrak{F}\circ\phi_{\mathfrak{s}}\big(D_x(\mu,\nu),C_x(\mu,\nu)\big)\big]$$
$$\leq \mathfrak{h}\left(\frac{1}{2}\right)H(\mathfrak{s})\big[\mathfrak{F}(C_x(\mu,\nu)+\mathfrak{F}\big(D_x(\mu,\nu)\big)\big]$$
$$= \mathfrak{h}\left(\frac{1}{2}\right)H(\mathfrak{s})\big([\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)\big),$$

where $H(\mathfrak{s}) = \mathfrak{h}(\mathfrak{s}) + \mathfrak{h}(1-\mathfrak{s})$, $s \in [0,1]$. Using the symmetric property of the $\mathfrak{h}$-convex function, we have

$$\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} H\left(\frac{D_x(\mu,\nu)-\mathfrak{a}}{D_x(\mu,\nu)-C_x(\mu,\nu)}\right)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} = 2\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{h}\left(\frac{D_x(\mu,\nu)-\mathfrak{a}}{D_x(\mu,\nu)-C_x(\mu,\nu)}\right)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}.$$

Multiplying by $[\mathfrak{w}\circ\phi_{\mathfrak{s}}]\big(C_x(\mu,\nu),D_x(\mu,\nu)\big)$ for all $\mathfrak{s}, x \in [0,1]$ and then integrating with respect to $\mathfrak{s} \in [0,1]$ we get required generalization of Fejér's inequality given in Equation (3.14). □

Remark 12. Thus, it is clear that (3.1) and (3.14) are equivalent, and the only change is in the manner of presentation and the resulting consequences. These generalizations of Fejér's inequality for $\mathfrak{h}$-convex function provided in these inequalities are valuable contributions to the literature.

Remark 13. Substituting $\mathfrak{h}(\mathfrak{s}) = \mathfrak{s}^\eta$ ($\eta \in (0,1]$), $\mathfrak{h}(\mathfrak{s}) = \frac{1}{\mathfrak{s}}$ ($s \in (0,1)$), $\mathfrak{h}(\mathfrak{s}) = 1$, and $\mathfrak{h}(\mathfrak{s}) = \mathfrak{s}$ in (3.1) and (3.14), we obtain generalizations of Fejér's inequality for $\mathfrak{s}$-convex functions in the second sense, Godunova-Levin functions, $\mathfrak{p}$-convex functions, and convex functions, respectively.

Now, taking particular value $\mathfrak{h}(\mathfrak{s}) = \mathfrak{s}$ in (3.1) and (3.14), we have the inequalities as below, respectively

$$
\begin{aligned}
\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)}\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} &\leq \int_{\kappa_1}^{\kappa_2}\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}-\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)\\
&\leq \frac{[\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)}{|\nu(x)-\mu(x)|}\int_{\mu(x)}^{\nu(x)}\big(\mathfrak{a}-\mu(x)\big)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&= \frac{[\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)}{|\mu(x)-\nu(x)|}\int_{\nu(x)}^{\mu(x)}\big(\mathfrak{a}-\nu(x)\big)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a},
\end{aligned}
\tag{3.15}
$$

and

$$
\begin{aligned}
\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)}\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} &\leq \int_{\kappa_1}^{\kappa_2}\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}-\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)\\
&\leq \frac{[\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)}{2}\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)}\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}.
\end{aligned}
\tag{3.16}
$$

A new form of generalized Fejér-type inequality is introduced in inequality (3.15), whereas inequality (3.16) is a straightforward generalization of the classical Fejér inequality for convex functions.

Remark 14. Substituting $x = 0, 1$ in ($\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(0) = \mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(1) = 0$), then we get the following Fejér-type inequality for $\mathfrak{h}$-convex function obtained in equation (3.10):

$$
\begin{aligned}
\frac{1}{2\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{\kappa_1}^{\kappa_2}\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} &\leq \int_{\kappa_1}^{\kappa_2}\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\leq (\kappa_2-\kappa_1)[\mathfrak{F}(\kappa_1)+\mathfrak{F}(\kappa_2)]\int_0^1\mathfrak{h}(\mathfrak{s})\mathfrak{w}(\mathfrak{s}\kappa_1+(1-\mathfrak{s})\kappa_2)\,d\mathfrak{s},
\end{aligned}
\tag{3.17}
$$

similarly, with this assumption in equation (3.14), we get a new $\mathfrak{h}$-convex version of Fejér inequality:

$$\frac{1}{2\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{\kappa_1}^{\kappa_2}\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} \le \int_{\kappa_1}^{\kappa_2}\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}$$
$$\le \frac{\mathfrak{F}(\kappa_1)+\mathfrak{F}(\kappa_2)}{2}\int_{\kappa_1}^{\kappa_2} H\left(\frac{\kappa_2-\mathfrak{a}}{\kappa_2-\kappa_1}\right)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} \quad (3.18)$$
$$= \frac{\mathfrak{F}(\kappa_1)+\mathfrak{F}(\kappa_2)}{2}\int_{\kappa_1}^{\kappa_2} H\left(\frac{\mathfrak{a}-\kappa_1}{\kappa_2-\kappa_1}\right)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}.$$

Now, we discuss different inequalities related to generalized Fejér-type inequality for $\mathfrak{h}$-convex function. For different $\mathfrak{h}(\mathfrak{s})$, $\mathfrak{w}$, and for any $n \in \mathcal{N}$, $n \ge 3$. Inequalities of this nature can be found in [52, 53, 54] along with the references.

Remark 15. Now consider the following for $n \in \mathcal{N}, n \ge 3$,

$$\kappa_1 = \frac{\kappa_1+(n-1)\kappa_1}{n} \le \frac{\kappa_2+(n-1)\kappa_1}{n} \le \frac{\kappa_1+(n-1)\kappa_2}{n} + \frac{n-2}{n}(\kappa_1-\kappa_2)$$
$$\le \frac{\kappa_1+(n-1)\kappa_2}{n} \le \kappa_2,$$

and

$$\mathfrak{w}\left(\frac{\kappa_1+(n-1)\kappa_2}{n} + \frac{\kappa_2+(n-1)\kappa_1}{n} - \mathfrak{a}\right) = \mathfrak{w}(\kappa_1+\kappa_2-\mathfrak{a}) = \mathfrak{w}(\mathfrak{a}).$$

This shows that $\mathfrak{w}$ is symmetric on $[\frac{\kappa_2+(n-1)\kappa_1}{n} + \frac{\kappa_1+(n-1)\kappa_2}{n}]$ with respect to $\frac{\kappa_1+\kappa_2}{2}$. so, in case $x = \frac{1}{n}$ in equation (3.1), and (3.14), we obtain the following inequalities respectively:

$$\frac{1}{2\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{\frac{\kappa_2+(n-1)\kappa_1}{n}}^{\frac{\kappa_1+(n-1)\kappa_2}{n}}\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}$$
$$\le \int_{\frac{\kappa_2+(n-1)\kappa_1}{n}}^{\frac{\kappa_1+(n-1)\kappa_2}{n}}\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} \quad (3.19)$$
$$\le \frac{n-2}{n}(\kappa_2-\kappa_1)\left[\mathfrak{F}\left(\frac{\kappa_2+(n-1)\kappa_1}{n}\right) + \mathfrak{F}\left(\frac{\kappa_1+(n-1)\kappa_2}{n}\right)\right]$$
$$\times \int_0^1 \mathfrak{h}(\mathfrak{s})[\mathfrak{w}\circ\phi_{\mathfrak{s}}]\left(\frac{\kappa_2+(n-1)\kappa_1}{n}, \frac{\kappa_1+(n-1)\kappa_2}{n}\right)d\mathfrak{s},$$

and

$$
\begin{aligned}
&\frac{1}{2\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{\frac{\kappa_2+(n-1)\kappa_1}{n}}^{\frac{\kappa_1+(n-1)\kappa_2}{n}}\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\leq \int_{\frac{\kappa_2+(n-1)\kappa_1}{n}}^{\frac{\kappa_1+(n-1)\kappa_2}{n}}\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\leq \frac{\mathfrak{F}\left(\frac{\kappa_2+(n-1)\kappa_1}{n}\right)+\mathfrak{F}\left(\frac{\kappa_1+(n-1)\kappa_2}{n}\right)}{2}\int_{\frac{\kappa_2+(n-1)\kappa_1}{n}}^{\frac{\kappa_1+(n-1)\kappa_2}{n}} H\left(\left(\frac{n}{n-2}\right)\left(\frac{\kappa_2-\mathfrak{a}}{\kappa_2-\kappa_1}-\frac{1}{n}\right)\right)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}.
\end{aligned}
\tag{3.20}
$$

Inequalities obtained in (3.19) and (3.20) deal with many Hermite-Hadamard and Fejér'-type inequalities of this kind for all $n \geq 3$.

Example 3.2.1. Let $n = 3, 4$ and $\mathfrak{h}(\mathfrak{s}) = \mathfrak{s}$ in (3.20) to get Fejér'-type inequalities:

$$
\begin{aligned}
\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{\frac{\kappa_2+2\kappa_1}{3}}^{\frac{\kappa_1+2\kappa_2}{3}}\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} &\leq \int_{\frac{\kappa_2+2\kappa_1}{3}}^{\frac{\kappa_1+2\kappa_2}{3}}\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\leq \frac{\left[\mathfrak{F}\left(\frac{\kappa_2+2\kappa_1}{3}\right)+\mathfrak{F}\left(\frac{\kappa_1+2\kappa_2}{3}\right)\right]}{2}\int_{\frac{\kappa_2+2\kappa_1}{3}}^{\frac{\kappa_1+2\kappa_2}{3}}\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right)\int_{\frac{\kappa_2+3\kappa_1}{4}}^{\frac{\kappa_1+3\kappa_2}{4}}\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} &\leq \int_{\frac{\kappa_2+3\kappa_1}{4}}^{\frac{\kappa_1+3\kappa_2}{4}}\mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\leq \frac{\mathfrak{F}\left(\frac{\kappa_2+3\kappa_1}{4}\right)+\mathfrak{F}\left(\frac{\kappa_1+3\kappa_2}{4}\right)}{2}\int_{\frac{\kappa_2+3\kappa_1}{4}}^{\frac{\kappa_1+3\kappa_2}{4}}\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}.
\end{aligned}
$$

3.2.2 A New Generalized form of Hermite-Hadamard inequality

Here is the new and improved generalization of the celebrated Hermite-Hadamard inequality related to $\mathfrak{h}$-convex functions and their sub-classes as well (see also [55, 56, 57, 58]).

Now we consider the case $\mathfrak{w} \equiv 1$ in equation (3.1) and (3.14) to get our desired result:

$$\begin{aligned} &\frac{1}{2\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{1}{(\kappa_2-\kappa_1)|1-2x|}\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\,d\mathfrak{a} \\ &\le ([\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x))\int_0^1 \mathfrak{h}(\mathfrak{s})\,d\mathfrak{a} \\ &= \frac{[\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)}{2}\int_0^1 H(\mathfrak{s})\,d\mathfrak{a}. \end{aligned} \tag{3.21}$$

Now with the same assumption in equation (3.17) and(3.18), we obtain the Hermite-Hadamard inequality related to $\mathfrak{h}$-convex functions

$$\frac{1}{2\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{1}{\kappa_2-\kappa_1}\int_{\kappa_1}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\,d\mathfrak{a} \le [\mathfrak{F}(\kappa_1)+\mathfrak{F}(\kappa_2)]\int_0^1 \mathfrak{h}(\mathfrak{s})\,d\mathfrak{s}. \tag{3.22}$$

Remark 16. Thus, we conclude that the equations (3.21) and (3.22) are equivalent, and the only change is in the manner of presentation and consequences. These generalizations of Hermite-Hadamard inequality for $\mathfrak{h}$-convex function provide valuable contributions to the literature.

Further, considering the equations (3.21) and (3.22), taking $\mathfrak{h}(\mathfrak{s}) = \mathfrak{s}^\eta$ $(\eta \in (0,1])$, $\mathfrak{h}(\mathfrak{s}) = \frac{1}{\mathfrak{s}}$ $(\mathfrak{s} \in (0,1))$, $\mathfrak{h}(\mathfrak{s}) = 1$, and $\mathfrak{h}(\mathfrak{s}) = \mathfrak{s}$, we obtain generalized Hermite-Hadamard inequality for $\mathfrak{s}$-convex functions in the second sense, Godunova-Levin functions, $\mathfrak{p}$-convex functions, and convex functions, respectively.

For particular case, if we consider $\mathfrak{h}(\mathfrak{s}) = \mathfrak{s}$, and $x \in [0,1]\backslash\{\frac{1}{2}\}$, in equation (3.21) we found the following inequality:

$$\mathfrak{F}\left(\frac{\kappa_1+\kappa_2}{2}\right) \le \frac{1}{(\kappa_2-\kappa_1)|1-2x|}\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\,d\mathfrak{a} \le \frac{1}{2}([\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu](x)), \tag{3.23}$$

which is dependent on variable x. That is another updated version of Hermite-

Hadamard inequality.

For another particular case, when $x = 0, 1$ we retrieve the classical form of Hermite-Hadamard inequality.

New generalizations of Hermite-Hadamard inequalities for $\mathfrak{h}$-convex function are formed, for different $\mathfrak{h}(\mathfrak{s})$, and any $n \in \mathcal{N}$, $n \geq 3$, we have:

Remark 17. Considering a particular case $\mathfrak{w} \equiv 1$ and $\mathfrak{h}(\mathfrak{s}) = \mathfrak{s}$ in equation (3.19), we obtain

$$\mathfrak{F}\left(\frac{\kappa_1 + \kappa_2}{2}\right) \leq \frac{n}{(\kappa_2 - \kappa_1)(n-2)} \int_{\frac{\kappa_2 + (n-1)\kappa_1}{n}}^{\frac{\kappa_1 + (n-1)\kappa_2}{n}} \mathfrak{F}(\mathfrak{a})\, d\mathfrak{a}$$
$$\leq \frac{1}{2}\left[\mathfrak{F}\left(\frac{\kappa_2 + (n-1)\kappa_1}{n}\right) + \mathfrak{F}\left(\frac{\kappa_1 + (n-1)\kappa_2}{n}\right)\right],$$

which is a particular case for Hermite-Hadamard inequality depending on n.

Remark 18. For fixed $x \in [0, 1]$ and $\mathfrak{a} \in [\kappa_1, \kappa_2]$, the weight function $\mathfrak{w}$ can be written as

$$\mathfrak{w}(\mathfrak{a}) = (D_x(\mu, \nu) - \mathfrak{a})(\mathfrak{a} - C_x(\mu, \nu)),$$

where, $\mathfrak{w}$ is symmetric on $[C_x(\mu, \nu), D_x(\mu, \nu)]$ with respect to $\frac{\kappa_1+\kappa_2}{2}$. By using Lemma 3.2.1, we obtain the following results

$$\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)} \big(D_x(\mu, \nu) - \mathfrak{a}\big)\big(\mathfrak{a} - C_x(\mu, \nu)\big)\big)\, d\mathfrak{a} = \frac{|1 - 2x|^3(\kappa_2 - \kappa_1)^3}{6}$$

and

$$\int_0^1 \mathfrak{h}(\mathfrak{s})[\mathfrak{w} \circ \phi_{\mathfrak{s}}]\big(\mu(x), \nu(x)\big)\, d\mathfrak{s} = |1 - 2x|^2(\kappa_2 - \kappa_1)^2 \int_0^1 \mathfrak{h}(\mathfrak{s})\mathfrak{s}(1 - \mathfrak{s})\, d\mathfrak{s}.$$

By using Equation (3.10), we obtained the following Hermite-Hadamard inequality for $\mathfrak{h}$-convex functions:

$$\begin{aligned}
&\frac{1}{12\mathfrak{h}(\frac{1}{2})}\mathfrak{F}\bigg(\frac{\kappa_1+\kappa_2}{2}\bigg)\\
&\leq \frac{1}{|1-2x|^3(\kappa_2-\kappa_1)^3}\int_{C_x(\mu,\nu)}^{D_x(\mu,\nu)}\mathfrak{F}(\mathfrak{a})\big(D_x(\mu,\nu)-\mathfrak{a}\big)\big(\mathfrak{a}-C_x(\mu,\nu)\big)\,d\mathfrak{a}\\
&\leq \big([\mathfrak{F}\circ\mu](x)+[\mathfrak{F}\circ\nu(x)]\big)\int_0^1 \mathfrak{h}(\mathfrak{s})\mathfrak{s}(\mathfrak{1}-\mathfrak{s})\,d\mathfrak{s}.
\end{aligned}$$

In the particular values for $x=0,1$ and $\mathfrak{h}(\mathfrak{s})=\mathfrak{s}$, we get

$$\mathfrak{F}\bigg(\frac{\kappa_1+\kappa_2}{2}\bigg)\leq\frac{6}{(\kappa_2-\kappa_1)^3}\int_{\kappa_1}^{\kappa_2}\mathfrak{F}(\mathfrak{a})(\kappa_2-\mathfrak{a})(\mathfrak{a}-\kappa_1)\,d\mathfrak{a}\leq\frac{\mathfrak{F}(\kappa_1)+\mathfrak{F}(\kappa_2)}{2},$$

which was obtained from equation (3.17) .

3.3 The mapping $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)$:

Consider two real numbers $\kappa_1<\kappa_2$, consider integrable functions $\mathfrak{F}:[\kappa_1,\kappa_2]=\mathcal{I}\subseteq\mathcal{R}\to\mathcal{R}$ and $\mathfrak{w}:[\kappa_1,\kappa_2]=\mathcal{I}\to\mathcal{R}^+\cup\{0\}$. Define the mapping $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x):[0,1]\to\mathcal{R}$ as

$$\begin{aligned}
\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)=&\int_{\kappa_1}^{C_x(\mu,\nu)}\mathfrak{F}\bigg(x\mathfrak{a}+(1-x)\big(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&+\int_{D_x(\mu,\nu)}^{\kappa_2}\mathfrak{F}\bigg(x\mathfrak{a}+(1-x)\big(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a},
\end{aligned}$$

where

$$C_x(\mu,\nu)=\min\big\{\mu(x),\nu(x)\big\},\qquad D_x(\mu,\nu)=\max\big\{\mu(x),\nu(x)\big\},$$

for $x\in[0,1]$ we define $\mu(x):[0,1]\to[\kappa_1,\kappa_2]$ and $\nu(x):[0,1]\to[\kappa_1,\kappa_2]$ as:

$$\mu(x)=x\kappa_2+(1-x)\kappa_1,\qquad \nu(x)=x\kappa_1+(1-x)\kappa_2.$$

Note that, for $\mathfrak{w} = 1$ in $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ we have

$$\mathbb{H}_{\mathfrak{F}}^{1}(x) = \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}\Big(x\mathfrak{a} + (1-x)\big(\frac{C_x(\mu,\nu) + D_x(\mu,\nu)}{2}\big)\Big)\, d\mathfrak{a}$$
$$+ \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}\big(x\mathfrak{a} + (1-x)\big(\frac{C_x(\mu,\nu) + D_x(\mu,\nu)}{2}\big)\Big)\, d\mathfrak{a}.$$

Remark 19. The Mapping $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ yields similar results to those established for the mapping $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ in Lemma 3.2.1

We obtain some basic properties for the mapping $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ in the following proposition.

Proposition 11. Consider two integrable functions $\mathfrak{F} : [\kappa_1, \kappa_2] \to \mathcal{R}$ and $\mathfrak{w} : [\kappa_1, \kappa_2] \to \mathcal{R}^+ \cup \{0\}$. Then

(i)

$$\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x) = \mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(1-x) \qquad \forall x \in [0,1],$$

which shows that $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ is symmetric on $[\kappa_1, \kappa_2]$ with respect to $\frac{1}{2}$.

(ii) For symmetric $\mathfrak{w}$ on $[\kappa_1, \kappa_2]$ with respect to $\frac{\kappa_1+\kappa_2}{2}$ and $p, q \geq 1$ with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\big|\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)\big| \leq \|\mathfrak{F}\|_p \|\mathfrak{w}\|_q.$$

Also, if $C_x(\mu,\nu) = \mu(x)$, then

$$\big|\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)\big| \leq \left(\frac{1}{2}\right)^{\frac{1}{q}} \big[x(\kappa_2 - \kappa_1)\big]^{\frac{1}{p}} \|\mathfrak{w}\|_q \|\mathfrak{F}\|_\infty,$$

and if $C_x(\mu,\nu) = \nu(x)$, then

$$\big|\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)\big| \leq \left(\frac{1}{2}\right)^{\frac{1}{q}} \big[(1-x)(\kappa_2 - \kappa_1)\big]^{\frac{1}{p}} \|\mathfrak{w}\|_q \|\mathfrak{F}\|_\infty.$$

(iii) Let, the function $(\mathfrak{F}\mathfrak{w})(\mathfrak{a}) = \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})$ is convex on $[\kappa_1, \kappa_2]$. If $C_x(\mu, \nu) = \mu(x)$ for some $x \in (0, 1]$, then the function $\frac{\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)}{x}$ is convex. Also, if $C_x(\mu, \nu) = \nu(x)$ for some $x \in [0, 1)$, then the function $\frac{\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)}{1-x}$ is convex.

(iv) Considering $\mathfrak{F}$ and $\mathfrak{w}$ are two continuous functions on $[\kappa_1, \kappa_2]$. If $\mathfrak{F}$ is non-negative (non-positive) on $[\kappa_1, \kappa_2]$, then the function $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ is increasing (decreasing) on $[0, \frac{1}{2})$ and is decreasing (increasing) on $(\frac{1}{2}, 0]$. Also, $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ has a relative extreme point at $x = \frac{1}{2}$. If $\mathfrak{w} \not\equiv 0$, then corresponding to any $\mathfrak{a} \in [\kappa_1, \kappa_2] \setminus \{\frac{\kappa_1+\kappa_2}{2}\}$ satisfying

$$\mathfrak{F}(\mathfrak{a}) + \mathfrak{F}(\kappa_1 + \kappa_2 - \mathfrak{a}) = 0,$$

there exists a critical point for $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)$.

Proof. The mapping $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ exhibits similar symmetry, convexity, and monotonicity Properties to those of the mapping $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}(x)$ derived in proposition 6. □

Theorem 3.3.1. *Considering an $\mathfrak{h}$-convex function $\mathfrak{F} : [\kappa_1, \kappa_2] \subseteq \mathcal{R} \to \mathcal{R}$ and $\mathfrak{w} : [\kappa_1, \kappa_2] \to \mathcal{R}^+ \cup \{0\}$. The mapping $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}} : [0, 1] \to \mathcal{R}$ defined as above satisfies the following*

(i) The mapping $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}$ is convex on $[0, 1]$.

(ii)

$$\inf_{x\in[0,1]} \mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x) = \mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(0) = \mathfrak{F}\left(\frac{C_x(\mu, \nu) + D_x(\mu, \nu)}{2}\right)\left[\int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{w}(\mathfrak{a})\, d\mathfrak{a}\right]$$

(iii)

$$\sup_{x\in[0,1]} \mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x) = \mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(1) = \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a}.$$

Proof. (i) Let $r, s \geq 0$ with $r+s=1$ and $x_1, x_2 \in [0,1]$. then

$$\begin{aligned}
&\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(rx_1+sx_2)\\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}\bigg((rx_1+sx_2)\mathfrak{a} + \big(1-(rx_1+sx_2)\big)\big(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}\bigg((rx_1+sx_2)\mathfrak{a} + \big(1-(rx_1+sx_2)\big)\big(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}\bigg(r\big(x_1\mathfrak{a} + (1-x_1)\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\quad + \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}\bigg(s\big(x_2\mathfrak{a} + (1-x_2)\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}\bigg(r\big(x_1\mathfrak{a} + (1-x_1)\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}\bigg(s\big(x_2\mathfrak{a} + (1-x_2)\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\leq r\bigg[\int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}\bigg(x_1\mathfrak{a} + (1-x_1)\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}\bigg(x_1\mathfrak{a} + (1-x_1)\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\bigg]\\
&\quad + s\bigg[\int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}\bigg(x_2\mathfrak{a} + (1-x_2)\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}\bigg(x_2\mathfrak{a} + (1-x_2)\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\bigg] = r\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x_1) + s\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x_2).
\end{aligned}$$

(ii) At $x=0$ we have

$$\begin{aligned}
\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(0) &= \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}\bigg((0)\mathfrak{a} + (1-0)\big(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}\bigg((0)\mathfrak{a} + (1-0)\big(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&= \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\\
&= \mathfrak{F}\bigg(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\bigg)\bigg[\int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{w}(\mathfrak{a})\,d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{w}(\mathfrak{a})\,d\mathfrak{a}\bigg] = \inf_{x\in[0,1]} \mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x).
\end{aligned}$$

Now at $x = 1$ we have

$$\begin{aligned}\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(1) &= \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}\bigg((1)\mathfrak{a} + (1-1)\big(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\ &\quad + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}\bigg((1)\mathfrak{a} + (1-1)\big(\frac{C_x(\mu,\nu)+D_x(\mu,\nu)}{2}\big)\bigg)\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} \\ &= \int_{\kappa_1}^{C_x(\mu,\nu)} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} + \int_{D_x(\mu,\nu)}^{\kappa_2} \mathfrak{F}(\mathfrak{a})\mathfrak{w}(\mathfrak{a})\, d\mathfrak{a} = \sup_{x\in[0,1]} \mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}(x).\end{aligned}$$

Hence proved.

□

Chapter 4

CONCLUSIONS AND FUTURE RECOMMENDATION

This thesis has contributed to the understanding and analysis of convex functions particularly focusing on $\mathfrak{p}$-convex functions, $\mathfrak{h}$-convex functions, and strongly reciprocally $(\mathfrak{p}, \mathfrak{h})$-convex functions for higher-order cases. These generalizations are essential tools in various disciplines of pure and applied sciences. This study aims to examine these generalized convex functions in connection with well-known inequalities, as a result, we can deeply understand the behavior, properties, and practical significance.

In the introductory chapter, we gave a detailed review of the convex function that covers the historical background, its applications in both pure and applied sciences, and interpret its various properties, which set a strong base for further study. Moreover, we discussed some well-known inequalities in connection with convexity, as these are pivotal for scientific study. Further in Chapter 2, we focus on strongly reciprocally $(\mathfrak{p}, \mathfrak{h})$-convex functions for higher-order, a new version of convex function. We investigate some basic properties that enable us to explore

Hermite-Hadamard inequality, Fejér inequality, and fractional integral inequalities within this context. An important contribution of this thesis is the introduction of new mappings discussed in Chapter 3. We began with introducing a novel mapping $\mathbb{M}_{\mathfrak{F}}^{\mathfrak{w}}$ for $\mathfrak{h}$-convex functions where its dynamic properties led to the composition of a new generalized Fejér-type inequality and Hermite-Hadamard inequality with a deeper understanding. A second mapping $\mathbb{H}_{\mathfrak{F}}^{\mathfrak{w}}$, along with several associated results offers additional tools for further analysis of $\mathfrak{h}$-convex function. Through the dynamic properties of these mappings, we can extend the current literature to new research avenues, particularly in the study of inequality theory.

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