Formalization of Traffic Flow Theory

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A thesis submitted in partial fulfillment of the requirements for the degree of Master of Science in System Engineering (MS SE-01)

In
Research Center for Modeling and Simulation (RCMS),
National University of Sciences and Technology (NUST),
Islamabad, Pakistan.

(August 2017)

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Dedication

I dedicate this thesis to my parents all closed ones.

Certificate of Originality

I hereby declare that this submission is my own work and to the best of my knowledge it contains no materials previously published or written by another person, nor material which to a substantial extent has been accepted for the award of any degree or diploma at NUST RCMS or at any other educational institute, except where due acknowledgement has been made in the thesis. Any contribution made to the research by others, with whom I have worked at NUST RCMS or elsewhere, is explicitly acknowledged in the thesis.

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Author Name: **Muhammad Umair**
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Acknowledgment

All My thanks goes first to Almighty Allah, Who gives power to a weak and helps the helpless and needy people, The most Gracious and The Most Merciful.

I would like to thank my supervisor, Assistant Professor at School of Electrical Engineering and Computer Sciences, Dr. Osman Hasan, for his moral boosting support, well guiding suggestions and encouragement throughout my research work.
I am also thankful to all the faculty members at RCMS, especially Dr. Jamil Ahmad, Engr. Sikandar Hayat Mirza, Dr. Ammar Mushtaq and Dr. Uzma Habib for their valuable guidance and moral support.

My thanks also go to all of my friends. List goes on but some, Imran Nawaz, Kashif Zaheer, Sohaib Aslam, Hammad Ali Hassan, All SAVe(SEECS) lab members, specially Adnan Rashid, Syed Ali Bukhari and Dr. Waqar Ahmad, for supporting me through thick and thin stages of my thesis.
I also wish to express my appreciation to the Assistant Proffessor at NIT(NUST), Malik Saqib Mahmood for his valuable support for the guidance in some critical phases of my research.

I would like to thank my parents, all closed ones, who encouraged me to work hard with patience and courage for my thesis.

Table of Contents

1 Introduction **1**
- 1.1 Motivation . . . 1
- 1.2 Related Work . . . 2
- 1.3 Problem Statement . . . 3
- 1.4 Proposed Solution . . . 3
- 1.5 Thesis Organization . . . 4

2 Preliminaries **5**
- 2.1 Traffic Flow Theory - Macroscopic Model . . . 5
- 2.2 HOL Light Theorem Prover . . . 11

3 Proposed Framework **13**

4 Formalization of Macroscopic Models **15**
- 4.1 Formalization of Relative Occupancy . . . 15
- 4.2 Formalization of Shockwave . . . 21

5 Case Studies **28**
- 5.1 German Freeway . . . 28
- 5.2 Formal Input-output and Shockwave Analyses and their Consistency . . . 30

6 Conclusions **39**

List of Figures

2.1 Time-space Diagram [21] 6
2.2 Flow Density Diagram 8
2.3 Multiple Shockwaves in Time-Space Diagram [27] 10

3.1 Proposed Framework 14

5.1 German Freeway [32] 29
5.2 Highway Regions 32
5.3 Queuing Dynamics with a Change in Arriving Flow Rate [27] . 32
5.4 Queuing Dynamics with a Change in Discharging Flow Rate [27] 33

List of Tables

2.1 HOL Light Symbols and Functions 12

List of Abbreviations

HOL	Higher Order Logic
Caml	Categorical abstract machine language
I/O	Input-output

Abstract

Traffic flow theory allows us to mathematically describe the behavior of traffic flow and thus deduce interesting properties for transportation engineers. Traditionally, traffic flow problems are analyzed by using paper-and-pencil proof methods, computer-based numerical techniques or computer algebra systems. However all these methods are error-prone and thus the analysis cannot be termed as accurate, which poses a serious threat to the safety-critical domain of transportation systems. To guarantee the correctness of the analysis, we propose to use higher-order-logic theorem proving for analyzing the traffic flow problems and as a first step in this direction, present a logical framework for the formal analysis of macroscopic models of traffic flow. In particular, we present a formalization of some foundational concepts of macroscopic models, namely density, flow rate, mean speed, relative occupancy and shockwave using the higher-order-logic theorem prover HOL Light. To illustrate the practical utilization and effectiveness of the proposed idea, we perform the formal analysis of a German freeway, including its input-output and shockwave analysis by verifying their corresponding properties using our proposed formalization.

Chapter 1

Introduction

1.1 Motivation

Traffic flow theory [21] is widely used to describe the interactions between vehicles, their drivers and the transportation infrastructures, which include the highway and its operational devices, such as highway signals, markings and control devices. All these parameters, contributing towards the dynamics of the transportation systems, are mathematically modeled and analyzed to obtain an optimal and balanced traffic flow with minimal congestion [20].

Traffic flow theory mainly consists of two models, namely *microscopic* and *macroscopic*. Microscopic models [17] capture the dynamic behavior of the underlying transportation system based on the individual behaviors of the vehicles and drivers, and their mutual interaction. On the other hand, the macroscopic model considers the behavior of multiple vehicles simultaneously and it is characterized by its fundamental parameters, such as flow rate, density, mean speed, relative occupancy and shockwave [9]. Thus, in other words, the macroscopic model captures the behavior of all of the vehicles in a certain cross-section as opposed to the microscopic model, which includes the analysis of an individual vehicle. In the macroscopic model, the continuous traffic flow under equilibrium and non-equilibrium conditions is modeled by the continuous-time partial differential equations, known as *conservation equations* or *continuity equations* [1,35] and *random number generations* [29]. These equations can be solved to find a relation between density and flow rate of the traffic. These fundamental parameters are further used to calculate the queue size/number of vehicles. An abrupt change in this queue size, due to some obstruction, i.e., accident, diversion etc., results into the phenomenon of shockwave [15], which is a boundary between two regions having vehicles with different average values of density, flow rate and the speed. As the time progresses, this shockwave moves in the direction of the traffic flow, by creating new shockwaves that replace the earlier earlier showkwaves, depending on the average values of these parameters in the respective regions. The analysis based on these foundations, called shockwave analysis [27], provides the rate of formation or dissipation of the congestion [24] and thus the identification of the congested areas by calculating the queue size/number of vehicles. Traditionally, the above-mentioned analyses are conducted by using paper-and-pencil based proofs and computer simulation methods.

However, the analysis of complex transportation systems using paper-and-pencil based and similar proofs are prone to human errors. Hence more rigorous analyses methods are required in the safety critical domain of transportation systems.

1.2 Related Work

In the paper and pencil based analyses of transportation systems [4, 6, 22], it is customary to not explicitly pen down all the assumptions, which are required for the mathematical analysis on paper. The absence of these underlying assumptions of the analysis adds another dimension of mistrust to the analysis. Numerical algorithm based simulation techniques [3, 31] are also frequently used to solve the conservation equations and thus perform the shockwave and input-output analysis. The simple input-output analysis that describes traffic queues without considering the space dimensions where as the shockwave analysis describes the queue sizes by taking both the dimensions of density and time into consideration [27]. *METANET* (a macroscopic simulation program for motorway) [23] and *FREEFLO* (a macroscopic simulation model of freeway traffic) [26] are some of the widely used simulation tools to perform these analyses. The continuous traffic flow models of a cross-section highway area are discretized in time and space to facilitate their analyses using computer arithmetic and numerical techniques. This kind of discretization compromises the completeness of analysis and thus the accuracy of the results [1]. Just like the case of macroscopic models, the paper-and-pencil proof methods and simulation tools, like VISSIM (a microscopic traffic flow simulator) [3, 7] and MITSIMLab (microscopic traffic simulation laboratory) [2], used for analyzing microscopic models, also suffer from the inaccuracy limitations, described above. Computer algebra systems, such as Mathematica and Maple, have also been used to solve differential equations symbolically and thus overcome the inaccuracies introduced by computer arithmetic based computations and numerical methods. However, the algorithms used by these systems are not rigorously verified and thus can produce error-prone results [5].

1.3 Problem Statement

These above mentioned flaws in the traditional techniques are tremendously undesirable in case of the highly safety-critical domain of transportation, as ignoring some corner cases may lead to dire consequences, such as frequent traffic congestions, road accidents and lost of human lives in worst cases.

1.4 Proposed Solution

Formal methods [14] are computer based system analysis techniques that can overcome the above-mentioned analysis accuracy related limitations. Theorem proving [12] is a widely used formal method that allows the verification of mathematical relations, including continuous variables, by leveraging upon the expressiveness of higher-order logic and thus is quite appropriate for analyzing traffic flow problems. As a first step towards the formal analysis of traffic flow problems, this thesis presents a framework for the formal verification of macroscopic models in traffic flow theory. This choice is primarily motivated by the fact that the macroscopic models play a vital role in planning strategies in allocating resources for implementing optimized and balanced transportation systems [8, 34]. The proposed framework identifies the mathematical foundations of traffic flow theory that are required to conduct such analysis within the sound core of a higher-order-logic theorem prover. Moreover, it describes a step-wise procedure to develop a formal model of the given traffic flow problems in higher-order logic and reason about its corresponding properties using an interactive theorem prover. For this purpose, the thesis presents a higher-order-logic formalization of some of the widely used macroscopic model characteristics, namely relative occupancy [21], density [21], flow rate [16], mean speed [16] and shockwave [27]. Based on this formalization, we formally verify the properties depicting the relationship of relative occupancy and shockwave with the basic parameters of the traffic flow. In order to illustrate the practical effectiveness of our formalization, we present a formal analysis of a German freeway [32] by verifying its traffic flow properties, and input-output [27, 28] and shockwave analysis related expressions [27]. We have used the HOL Light theorem prover [13] for conducting the proposed formal analysis of macroscopic traffic flow models due to its extensive support for formally reasoning about multivariate calculus theories.

1.5 Thesis Organization

The rest of the thesis is organized as follows: We provide a brief overview of traffic flow theory and the HOL Light theorem prover in Chapter 2. Chapter 3 presents the proposed framework for the formalization of the macroscopic traffic flow models and their properties. In Chapter 4, we provide the formalization of the traffic flow theory foundations, which include the density, flow rate, mean speed, relative occupancy and shockwave. Moreover, we utilize our foundational formalization to verify some of the properties depicting the relationship of the relative occupancy and shockwave with the macroscopic model parameter, including flow rate and density. For demonstrating the practical utilization and effectiveness of the proposed formalization, we present a formal input-output and shockwave analysis of a German freeway in Chapter 5. Finally, Chapter 6 concludes the thesis by highlighting some future directions.

Chapter 2

Preliminaries

In this chapter, we provide a brief introduction to the macroscopic model of traffic flow theory and the HOL Light theorem prover to facilitate the understanding of the rest of the thesis.

2.1 Traffic Flow Theory - Macroscopic Model

The macroscopic model of traffic flow theory considers all of the vehicles in a cross-section of a road simultaneously [16, 21]. In order to understand the widely used notions of relative occupancy, flow rate, density and mean speed, consider Fig. 2.1, which depicts two rectangular regions, namely the spatial region and the temporal regioin. The spatial region S_1 corresponds to a measurement over a road section ΔX at a single instant dT, whereas the temporal region S_2 corresponds to the measurement in a fixed location in space dX over a time period ΔT. The area of the spatial region S_1 is $\Delta X dT$, whereas the area of the temporal region S_2 is $\Delta T dX$. Based on the space-time diagram (Fig. 2.1), the characteristics of the macroscopic traffic flow model, i.e., relative occupancy, density, flow rate and mean speed for the single-lane traffic can be defined as follows:

The relative occupancy is the measurement of the fraction of time, for which the measurement location is occupied by the vehicles. In the temporal region S_2, it is given by the following formula [16]:

$$b_{S_2} = \frac{1}{\Delta T} \sum_{i=1}^{n} O_i = \frac{1}{\Delta T} \sum_{i=1}^{n} \frac{L_i}{V_i} \tag{2.1}$$

where ΔT is the length of the temporal region. O_i is the occupancy of the i^{th} vehicle and is equal to the ratio of the length L_i and speed V_i of the vehicle,

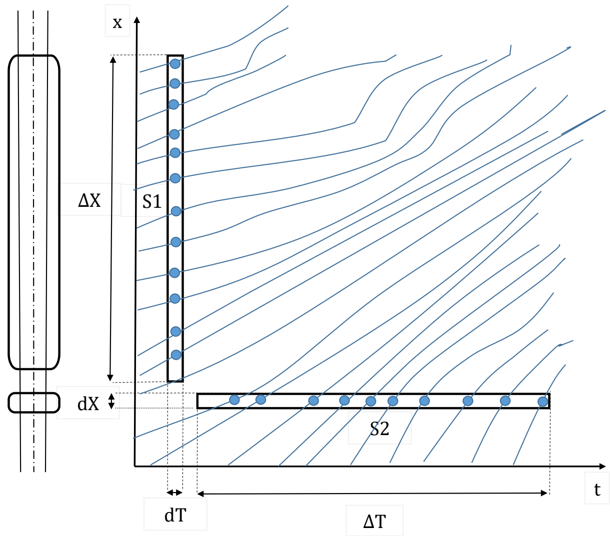

Figure 2.1: Time-space Diagram [21]

whereas n is the total number of vehicles in the temporal region.

Similarly, the occupancy of the vehicles in the spatial region S_1 can be mathematically expressed as [16]:

$$b_{S_1} = \frac{1}{\Delta X} \sum_{i=1}^{m} L_i \tag{2.2}$$

where ΔX represents the length of the spatial region and m is the total number of vehicles in the spatial region.

The flow rate of the traffic can be defined as the number of vehicles in a certain cross-section per unit time or, alternatively, as the ratio of the total distance covered by all vehicles in a region and the area of the region. In the spatial region S_1, it is given by the following formula [16]:

$$q_{S_1} = \frac{1}{\Delta X} \sum_{i=1}^{m} V_i \tag{2.3}$$

where V_i is the velocity of the i^{th} vehicle in the spatial region. Similarly, the flow rate in the temporal region S_2 is given by the following formula [16]:

$$q_{S_2} = \frac{n.dX}{\Delta T.dX} = \frac{n}{\Delta T} \tag{2.4}$$

where dX is the width of the temporal region.

The density of the traffic represents the number of vehicles in a certain cross-section or, alternatively, as the ratio of the total time spent by all vehicles in a region and the area of the region. In the spatial region S_1, it is represented by the following mathematical expression [16]:

$$k_{S_1} = \frac{m.dT}{\Delta X.dT} = \frac{m}{\Delta X} \tag{2.5}$$

where dT is the width of the spatial region. The following formula represents the density in the temporal region S_2 [16]:

$$k_{S_2} = \frac{1}{\Delta T} \sum_{i=1}^{n} \frac{1}{V_i} \tag{2.6}$$

where V_i is the velocity of the i^{th} vehicle in the temporal region.

The mean speed can now be defined as the ratio of the flow rate (q) and the density (k) of the traffic flow in each of the temporal and spatial regions. It is the time mean speed when calculated for the temporal region and space mean speed when calculated for the spatial region.

$$u = \frac{q}{k} \tag{2.7}$$

In general, the time mean speed is the arithmetic mean of speeds observed at some point in a specific time interval and it is generally easier to measure. Whereas, the space mean speed used in the traffic models is calculated as the arithmetic mean of speeds in different time intervals at a spatial region and is generally harder to measure [10] [19].

Now, to understand the phenomenon of shockwave, consider Fig. 2.2, which mainly depicts the flow-density diagram [18]. Consider an area of observation, in which the traffic flows with some density, flow rate and speed where a sudden obstruction of the traffic flow, due to some accident or closed road or some diversion, splits this area in two regions, namely R_1 and R_2. This obstruction results into a phenomenon of shockwave, which basically defines a boundary between the Regions R_1 and R_2 and each of these regions contain vehicles having different values of average density, flow rate and speed, i.e., q_1, k_1 and v_1 in R_1 and q_2, k_2 and v_2 in R_2, respectively at some time instant as depicted in Fig. 2.2. With the passage of time, this shockwave moves along the flow of the traffic with some speed v_w by creating new shockwaves and thus regions and canceling the earlier shockwave and the

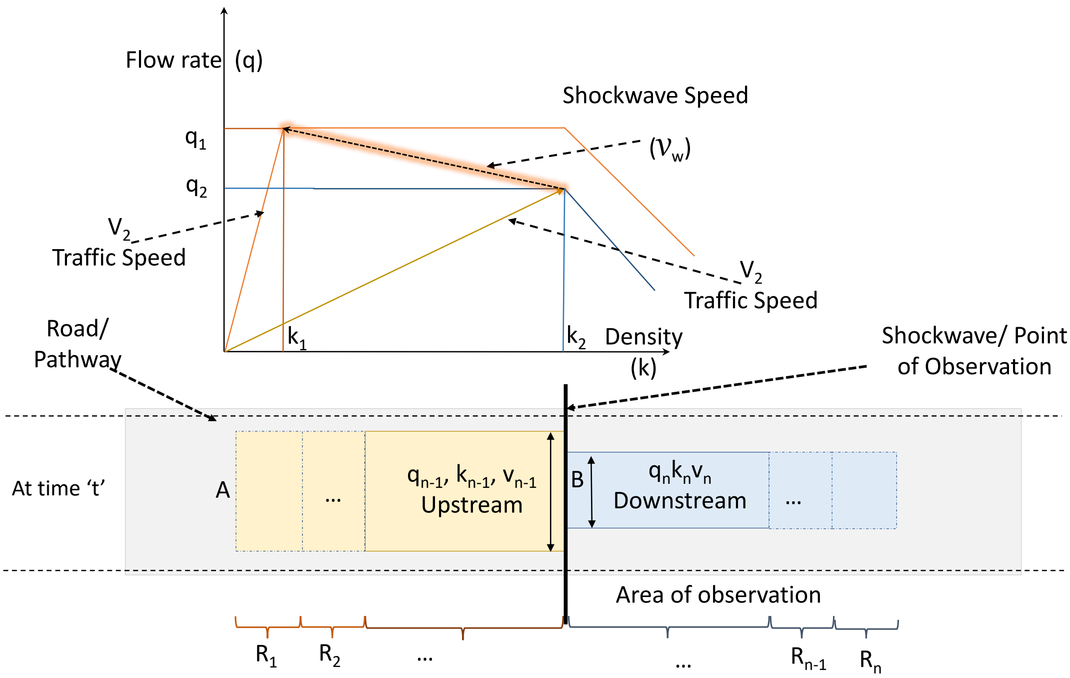

Figure 2.2: Flow Density Diagram

corresponding old regions. The shockwave speed v_w thus plays a vital role in the identification of the congested area by capturing the rate of formation and dissolution of the congestions and finding out the number of vehicles in the respective regions. The shockwave speed, for two adjacent regions R_{n-1} and R_n is mathematically expressed as:

$$v_{w_n} = \frac{q_n - q_{n-1}}{k_n - k_{n-1}} \tag{2.8}$$

where q_n and q_{n-1} are the flow rates in Regions R_n and R_{n-1}, respectively. Similarly, k_n and k_{n-1} are the densities in Regions R_n and R_{n-1}, respectively. These densities and flow rates are related by the following mathematical expressions:

$$q_{n-1} = k_{n-1} v_{n-1} \tag{2.9}$$

and

$$q_n = k_n v_n \tag{2.10}$$

where v_{n-1} and v_n represent the average space mean speeds of the vehicles in Regions R_{n-1} and R_n, respectively.

The relative speed of a vehicle to an observer is defined as the space mean speed relative to the shockwave speed. In Region R_1, it is mathematically represented as:

$$v_{R_1} = v_1 - v_w \tag{2.11}$$

Similarly, the relative speed in Region R_2 is given by the following formula:

$$v_{R_2} = v_2 - v_w \tag{2.12}$$

Fig. 2.3 represents the time-space diagram for the macroscopic model depicting the shockwave speeds in three different regions. The queue size based on a shockwave analysis considering Regions R_1 and R_2 is mathematical expressed as follows [16, 27, 28]:

$$Queue\ Size = -v_{w_1}\Delta t\Delta k = -\frac{q_2 - q_1}{k_2 - k_1}\Delta t\Delta k \tag{2.13}$$

where Δk and Δt are the density range and time length, respectively for the shockwave speed v_{w_1}. Similarly q_2, k_2, q_1 and k_1 are the flow rates and densities in Regions R_2 and R_1 of traffic flow, respectively. Whereas $v_{w_1} = (q_2 - q_1)/(k_2 - k_1)$ in Fig. 2.3. It is important to note that the outgoing flow rate is taken as positive and the ingoing flow rate as negative unlike the input-output model, as the queue size for input-output analysis is mathematically represented as [27, 28]:

$$Queue\ Size = (q_1 - q_2)\Delta t \tag{2.14}$$

The behavior of multiple shockwaves for three regions is depicted in the time space domain [27] in Fig. 2.3. Where two shockwaves v_{w1} and v_{w2} are overlapping in time space graph from t_1 to t_2 [27] and intersect at time point t_2 where their effect disappears and consequently a new shockwave v_{w3} emerges at that point. We use the following generic mathematical expression to analyze the queue size N_{sw} for n regions can be mathematically expressed as:

$$N_{sw} = \sum_{j=1}^{m} -(v_{w_j}\Delta t_j - \sum_{i=0}^{n} v_{w_i}\Delta t_i)\Delta k_j \tag{2.15}$$

where v_{w_j} has the longest existing duration with respect to time as compared to the rest of short ranged shockwaves v_{w_i}, which simultaneously exist in time

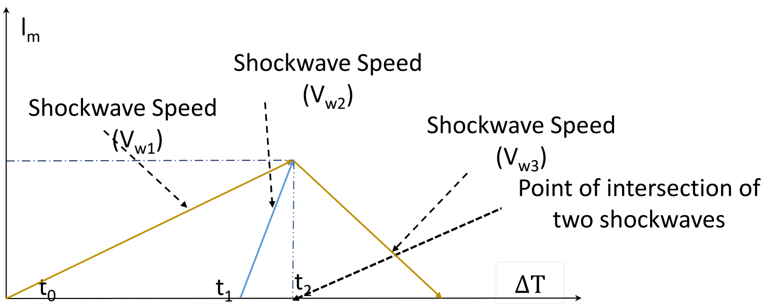

Figure 2.3: Multiple Shockwaves in Time-Space Diagram [27]

domain with v_{w_j}. The negative sign is used in the above equation because in shockwave analysis, the outgoing fow rate is taken as positive and ingoing flow rate as a negative real number

The Input-output analysis models the queue size for n number of regions as follows:

$$N_{io} = \sum_{i=1}^{n} (q_i - q_{i+1})\Delta t_i \tag{2.16}$$

We consider the boundary space between two regions, i.e., between Regions R_1 and R_2, as a separate Region R_w and the average speed of the vehicles in this shockwave region is considered as v_w. This way, the average speed shift between the two regions is $v_1 - v_w$. The density range of this shockwave region is considered as k_1 to accommodate all the incoming vehicles from Region R_1. Similarly consider the time required for the whole queue size of R_1 to exit from the ending point of Region R_1 and to enter the Region R_w as Δt. Hence, Δt and Δk should be the same based on the universal law of conservation. Thus, the number of vehicles crossing the boundry of Region R_1 to R_2 can be mathematically expressed as:

$$N_1 = (v_1 k_1 \Delta t - v_w k_1 \Delta t) = v_{R1} k_1 \Delta t = (v_1 - v_w) k_1 \Delta t = (\frac{q_1}{k_1} - v_w) k_1 \Delta t \tag{2.17}$$

Similarly, the incoming number of vehicles from the rear boundary in Region R_2 is given as:

$$N_2 = v_{R2}k_2\Delta t = (v_2 - v_w)k_2\Delta t = (\frac{q_2}{k_2} - v_w)k_2\Delta t \tag{2.18}$$

2.2 HOL Light Theorem Prover

HOL Light [11] is a higher-order-logic proof assistant that belongs to the HOL family of theorem provers. It provides an interactive theorem proving environment for the construction of the proofs in higher-order logic. In order to ensure secure theorem proving, it uses the Objective CAML (OCaml) language [13], which is a variant of the strongly-typed functional programming language ML [25]. HOL Light users can interactively verify theorems by applying tactics and proof procedures, which can automatically confirm the decidable proof goals. A HOL Light theory consist of types, constants, definitions and theorems. HOL Light theories are build in a hierarchical fashion and new theories can inherit the definitions and theorems of their parent theories. HOL Light consists of a rich set of formalized theories, including sets, natural numbers and the multi variable calculus theories. i.e., real analysis and vector calculus theories. The availability of these theories was the main motivation for choosing HOL Light for the proposed formalization as these foundations are required for reasoning about continuous (real-valued) variables and partial differential equations. Table 2.1 provides the mathematical interpretations of some of the HOL Light symbols and functions used in this thesis.

Table 2.1: HOL Light Symbols and Functions

HOL Symbol	**Standard Symbol**	**Meaning**
/\	and	Logical *and*
\/	or	Logical *or*
$\sim$	not	Logical *negation*
`T`	true	Logical true value
`F`	false	Logical false value
==>	$\longrightarrow$	Implication
<=>	$=$	Equality in Boolean domain
`!x.t`	$\forall x.t$	for all $x : t$
λ`x.t`	$\lambda x.t$	Function that maps x to $t(x)$
`num`	$\{0, 1, 2, \ldots\}$	Positive Integers data type
`real`	All Real numbers	Real data type
`suc n`	$(n + 1)$	Successor of natural number
`HD L`	*head*	Head element of list L
`TL L`	*tail*	Tail of list L
`EL n L`	*element*	n^{th} element of list L
`CONS`	::	Adds a new element to the top of the list
`LENGTH L`	*length*	Length of list L
`FST`	fst (a, b) = a	First component of a pair
`SND`	snd (a, b) = b	Second component of a pair

Chapter 3

Proposed Framework

The proposed framework, shown in Fig. 3.1, outlines the proposed approach for the formal analysis of macroscopic traffic flow models based on higher-order-logic theorem proving, which includes the formalization of their fundamentals, i.e., density, flow rate, means speed, relative occupancy and shockwave. The inputs to the framework are the macroscopic model parameters. For example, to find out the density, flow rate, mean speed and relative occupancy, these input parameters are the lengths and velocities of vehicles and the starting and ending points of the regions S_1 and S_2 (Fig. 2.1). Similarly, to find out the shockwave speed and queue size (number of vehicles), flow rate, density and mean speed are used as the input parameters for our proposed framework.

The first step in conducting formal analysis is the construction of the higher-order-logic based formal model of the given system based on the given macroscopic model parameters. The higher-order-logic formalization, required for developing this model, can be broadly decomposed into two parts, which are depicted by the dotted rectangles in Fig. 3.1. The first part is the core mathematical foundations of macroscopic model of traffic flow theory and the second part is composed of the definitions and theorems of traffic flow theory required for the analysis of macroscopic models. This mathematical foundations include Multivariate calculus theory and the conservation law. The traffic flow theory part builds upon the mathematical foundations and the formalization of the basic concepts of lengths and widths of the rectangular regions, density and flow rates, and the dependencies between them are shown in the Figs. 2.2 and 3.1. We propose to formalize the commonly used macroscopic characteristics i.e., density, flow rate, mean speed, relative occupancy, number of vehicles and shockwave speed for capturing the dynamics of the given transportation system. Furthermore, by using these definitions, we propose to verify the corresponding theorems that capture the character-

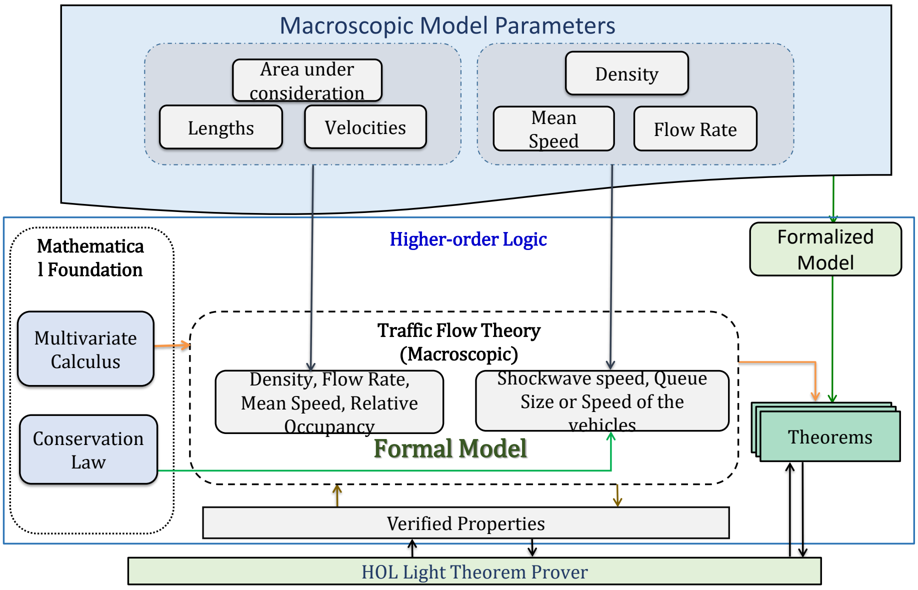

Figure 3.1: Proposed Framework

istics of the macroscopic traffic flow model, e.g., the properties that depict the relationship of the relative occupancy and shockwave velocity with the density, flow rate and number of vehicles of a transportation system.

Once the formal model, corresponding to the given macroscopic model parameters, is constructed then the next step is to verify its properties as higher-order-logic theorems. The proof goals can be expressed in higher-order logic and can be discharged by interacting with the proof assistant of the HOL Light theorem prover. The reasoning process, involved in this interactive verification, would be mainly based on the properties of the above-mentioned formalized notions of macroscopic model of traffic flow theory.

Chapter 4

Formalization of Macroscopic Models

In this chapter, we present a higher-order-logic formalization of the fundamentals of the macroscopic models of the traffic flow theory. This formalization builds upon the formalizations of multivariable calculus of HOL Light. To the best of our knowledge, these mathematical foundations have not been formalized in higher-order logic so far.

4.1 Formalization of Relative Occupancy

In this chapter, we present a higher-order-logic formalization of relative occupancy, which is one of the foremost elements of the macroscopic model of the traffic flow theory, as depicted in Fig. 3.1. This formalization builds upon the formalizations of multi variable calculus and the notions of length and widths of the rectangular regions and velocities, density, flow rate and mean speed from the traffic flow theory part.

A macroscopic model of the traffic flow theory consists of two rectangular regions, namely spatial region S_1 and temporal region S_2 (Fig. 2.1), and the lengths and speeds of the vehicles. We model the length and width of both of the regions in terms of their starting and ending points as a pair of real numbers ($\mathbb{R}$, $\mathbb{R}$), where the first element represents the starting point and the second element represents the ending point of the length and width. For example, taking a measurement of the traffic flow between 2km and 5km on a highway in the spatial region, the starting point of the length of this region is 2 and the ending point is 5 and it can be represented as a pair (2, 5). We formally describe the macroscopic model as the following data type:

Definition 4.1. Time Space Model Datatype for Relative Occupancy

```
new_type_abbrev "ts_macro_traffic_flow",
             :(((real × real) × (real × real) × (real ×
real) × (real × real)) × ((real × real)list × (real ×
real)list ))
```

In the above definition, a time space model is a pair with the first element as a 4-tuple. The first element of the 4-tuple is a pair (`real` × `real`), which represents the starting and ending points of the length of the spatial region (vertical rectangle). The second element of the 4-tuple is also a pair (`real` × `real`), representing the starting and ending points of the width of the spatial region. Similarly, the third and fourth elements of the 4-tuple are also pairs (`real` × `real`) representing the starting and ending points of the length and width of the temporal region (horizontal rectangle). The second element of the macroscopic model pair is itself a pair. The first element (`real` × `real`)`list` of this pair is a list of pairs in which the first element represents the length of a vehicle and the second element is its corresponding speed in the spatial region. Likewise, the second element (`real` × `real`)`list` of this pair is also a list of pairs, where each pair represents the lengths and speed of the vehicles in the temporal region.

In order to obtain the characteristics of the macroscopic model of the traffic flow, i.e., relative occupancy, density, flow rate and mean speed, we need to find out the lengths, widths and the occupancy of the spatial and temporal regions. For this purpose, we use the following function that allows us to find the length and width of a rectangle:

Definition 4.2. Length/Width of the Rectangles

```
⊢ ∀ x_t_rec .  differ x_t_rec = SND x_t_rec - FST x_t_rec
```

The function `differ` accepts the starting and ending point of the length/width of the rectangle in the form of a pair and returns its length/width by taking the arithmetic difference between the elements of the given pair.

The list containing the occupancies of all of the vehicles can be obtained as follows:

Definition 4.3. List Containing the Occupancies of a Collection of Vehicles

```
⊢ occ_list [ ] = [ ] ∧
  occ_list (CONS h t) = CONS (FST h / SND h) (occ_list t)
```

The function `occ_list`:$((real \times real)\ list \rightarrow real\ list)$ accepts a list of pairs, where each pair represents the length and speed of a vehicle, and returns the list of their corresponding occupancies.

In order to obtain the relative occupancy in temporal region S_2, we need the sum of the occupancies of all of the vehicles:

Definition 4.4. Sum of the Occupancies of all the Vehicles

```
⊢ ∀ L . occ_sum L =
  sum (1..LENGTH (occ_list L)) (λi.  EL (i - 1) (occ_list L))
```

The function `occ_sum`:$((real \times real)\ list \rightarrow real)$ accepts a list of pairs, where each pair represents the length and speed of a vehicle, and returns a real number that is the sum of the occupancies of all of the vehicles. This definition uses the HOL Light function `sum` in order to take sum of a function over a range of values.

Now, we can obtain relative occupancy in temporal region S_2 (Equation 2.1) by using Definitions 4.2 and 4.4 as follows:

Definition 4.5. Relative Occupancy in Temporal Region S_2

```
⊢ ∀ xv tv xh lng_spd_v lng_spd_h th.
        rel_occ_s2 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) =
             occ_sum lng_spd_h / differ th
```

The function `rel_occ_s2`:$(ts\_macro\_traffic\_flow \rightarrow real)$ accepts an element of data type $ts\_macro\_traffic\_flow$ and returns the corresponding relative occupancy of the vehicles in the temporal region S_2.

To obtain the relative occupancy in spatial region S_1, we need to find the summation of the lengths of all of the vehicles.

Definition 4.6. Summation of the lengths of all of the vehicles in Spatial Region S_1

```
⊢ ∀ L. sum_l_list L =
        sum (1..LENGTH (l_list L)) (λi.  EL (i - 1) (l_list L))
```

The function `sum_l_list`:$((real \times real)\ list \rightarrow\ real)$ accepts a list of pairs, where each pair represents the length and speed of a vehicle, and returns the sum of the lengths of all of the vehicles in the given list. The function `l_list` used in the above definition, takes the list of pairs containing lengths and speeds of the vehicles and returns a list containing their lengths only.

Now, the relative occupancy in the spatial region S_1 (Equation 2.2) is formalized as follows:

Definition 4.7. Relative Occupancy in Spatial Region S_1

```
⊢ ∀ tv xh th lng_spd_h lng_spd_v xv.
      rel_occ_s1 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) =
           sum_l_list lng_spd_v / differ xv
```

The function `rel_occ_s1` accepts an element of data type ($ts\_macro\_traffic\_flow$) and returns the relative occupancy of the vehicles in the spatial region S_1.

Our next step is to formalize the notion of traffic density in the spatial (Equation 2.5) and temporal regions (Equation 2.6):

Definition 4.8. Density in Spatial Region S_1

```
⊢ ∀ xh th lng_spd_h lng_spd_v tv xv.
      density_s1 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) =
         (&(no_veh lng_spd_v) * differ tv) / (differ xv * differ
tv)
```

The function `density_s1`:(($ts\_macro\_traffic\_flow$) $\rightarrow$ $real$) accepts an element of data type (($ts\_macro\_traffic\_flow$)) and returns the corresponding traffic density in the spatial (S_1) region.

The density in the temporal region (Equation 2.6) can be formalised as follows:

Definition 4.9. Density in Temporal Region S_2

```
⊢ ∀ xv tv lng_spd_v lng_spd_h th xh.
      density_s2 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) =
         sum_v_inv lng_spd_h / (differ th)
```

The function `density_s2` takes an element of data type ($ts\_macro\_traffic\_flow$) and returns the density in temporal region S_2. The function `sum_v_inv` in the above definition accepts a list of pairs, where each pair represents the length and speed of a vehicle, and returns the summation of the inverse of their speeds.

We formally define the traffic flow rate in the spatial region (Equation 2.3) as follows:

Definition 4.10. Flow Rate in Spatial Region S_1

```
⊢ ∀ xh th lng_spd_h lng_spd_v tv xv.
```

```
      flow_rate_s1 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) =
        (sum_v_list lng_spd_v / differ xv)
```

The function `flow_rate_s1` takes a time space macroscopic model and returns the flow rate in the spatial region. In this definition, the function `sum_v_list` accepts a list of pairs, where each pair represents the length and speed of a vehicle, in the region S_1 and returns the sum of their speeds.

Similarly, the flow rate in the temporal region (Equation 2.4) can be defined as follows:

Definition 4.11. Flow Rate in Temporal Region S_2

```
⊢ ∀ xv tv lng_spd_v lng_spd_h th xh.
      flow_rate_s2 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) =
        (&(no_veh lng_spd_h) * differ xh) / (differ th * differ
xh)
```

The function `flow_rate_s2`: $((ts\_macro\_traffic\_flow) \rightarrow \ real)$ accepts an element of data type $(ts\_macro\_traffic\_flow)$ and returns the flow rate in the temporal region. In this function, the function `no_veh` takes a list of pairs containing the lengths and speeds of the vehicles in region S_2 and returns the number of vehicles in the region. This function uses the HOL Light function `LENGTH`, which accepts a list of any data type and returns its length as a positive integer.

We next formalize the mean speed in both of the regions. The mean speed in spatial region S_1 (Equation 2.7) is defined as follows:

Definition 4.12. Mean Speed in Spatial Region S_1

```
⊢ ∀ xv tv xh th lng_spd_v lng_spd_h.
      mean_speed_s1 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) =
        flow_rate_s1 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) /
        density_s1 ((xv,tv,xh,th),lng_spd_v,lng_spd_h)
```

The function `mean_speed_s1` takes an element of data type *ts_macro_traffic_flow* and returns the mean speed in the spatial region.

The mean speed in the temporal region S_2 is given as follows:

Definition 4.13. Mean Speed in Temporal Region S_2

```
⊢ ∀ xv tv xh th lng_spd_v lng_spd_h.
      mean_speed_s2 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) =
```

```
      flow_rate_s2 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) /
      density_s2 ((xv,tv,xh,th),lng_spd_v,lng_spd_h)
```

In order to ensure the correctness and soundness of our definitions, we use them to verify a couple of properties representing some important characteristics of the macroscopic model. The first property deals with the case when length of all of the vehicles is the same then the relative occupancy in the spatial region S_1 is equal to the length times the density of vehicles in the region. The second property captures the same characteristic under the same assumption for the vehicles in the temporal region S_2.

We verify the first property as the following HOL Light theorem:

Theorem 4.1. Relationship of the Relative Occupancy and Density in Spatial Region S_1

```
⊢ ∀ xv tv xh th lng_spd_v lng_spd_h c.
   ~NULL lng_spd_v ∧ &0 < SND xv - FST xv ∧
   &0 < (SND tv - FST tv) ∧ ∀ i.  EL i (l_list lng_spd_v) = c
         ⇒ rel_occ_s1 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) =
           c ∗ density_s1 ((xv,tv,xh,th),lng_spd_v,lng_spd_h)
```

The variable `lng_spd_v` represents the list of pairs having lengths and velocities of the vehicles, whereas, `xv` and `tv` represent the starting and ending points of the length (ΔX) and width (dT) of the spatial region, respectively. The first assumption ensures that the list `lng_spd_v` is not empty. The next two assumptions guarantee that each of the length and width of the region are always positive, as these are the distance and time. The last assumption represents the condition that the lengths of all the vehicles is same. The conclusion of the theorem describes the relationship of the relative occupancy to the density of the vehicle.

The reasoning process of Theorem 1 is primarily based on the definitions of the functions `rel_occ_s1` and `density_s1`, and a lemma that says if all the elements of a list are same, i.e., equal to some constant c, then the summation of this list is equal to c times the length of the list.

Lemma 1. Summation of the List having same element c is equal to c times length of the list

```
⊢ ∀ c L. ∼(NULL L) ∧ (∀ i.  EL i L = c)
        ⇒ sum (1..LENGTH L) (λi.  EL (i - 1) L) = &(LENGTH L)
∗ c
```

Similarly, the second property depicting the relationship of the relative occupancy with the density in temporal region S_2 is given by the following theorem:

Theorem 4.2. Relationship of the Relative Occupancy and Density in Temporal Region S_2

```
⊢ ∀ xv tv xh th lng_spd_v lng_spd_h c.
   ∼(NULL lng_spd_v) ∧ &0 < SND th - FST th ∧
   &0 < SND xh - FST xh ∧ (∀ i.  FST (EL i lng_spd_h) = c)
         ⇒ rel_occ_s2 ((xv,tv,xh,th),lng_spd_v,lng_spd_h) =
            c ∗ density_s2 ((xv,tv,xh,th), lng_spd_v, lng_spd_h)
```

The variable `lng_spd_h` represents the list of pairs having lengths and velocities of the vehicles, whereas, `xh` and `th` represent the starting and ending points of the length (ΔT) and width (dX) of the temporal region, respectively. All the assumptions of this theorem are same as that of Theorem 4.1, but in the context of the temporal region. The conclusion of the Theorem 4.2 describes the relationship of relative occupancy with the density of the vehicles. The verification process of this theorem is similar to the one of Theorem 4.1 and more details can be found in the source code of the formalization [33].

4.2 Formalization of Shockwave

For the shockwave analysis, we have modeled a single region as a pair ((q, k), v), where the first element itself represents a pair i.e., q and k represent the flow rate and density and the second element depicts the shockwave speed, respectively. All these parameters are real-valued, i.e., $q,\ k,\ v \in \mathbb{R}$. For example, traffic flow of 2000 veh/hr, 80 veh/km density and shockwave speed of 1 km/hr on a highway is represented as $((2000, 80), 1)$. Consequently all the regions generating the shockwaves (Fig. 2.2) can be individually modeled using a list of regions. Then the multiple shockwaves, shown in Fig. 2.3, modeled from the pairs of regions, are added for all the regions shown in Fig. 2.2. To simplify the reasoning process about shockwave phenomenon, we encode the above information using the three type abbreviations in HOL Light, namely, `ptrgn`, `sw` and `sw_d` as follows:

Definition 4.14. Macroscopic Model Datatype for Shockwave Analysis

```
new_type_abbrev "ptrgn",   :(real × real) × real
new_type_abbrev "sw", :((ptrgn)list × (real × real))
                             × (num × num)
```

```
new_type_abbrev "sw_d", :((sw)list × (real × real))
```

where the first element of sw is itself a pair, in which the first element represents the list of the regions. The second pair of sw is a pair (real × real) representing *time* points corresponding to start and end of a shockwave. Similarly, the second element of sw is also a pair (num x num), representing the indices of the two adjacent regions as shown in Fig. 2.2.

Similarly a shockwave regional model sw_d is a pair, which models the dynamic behaior of all the shockwaves in an entire area, as shown in Fig. 2.3. The first element of sw_d is a list of sw elements and the second element having data type *real* × *real*, shows the initial and final density points of the area under consideration. Consequently the following data type would be able to model the dynamic behavior for the area of observation on a road or highway as a composition of shockwave's elements, i.e., flow, density, region's index and density range with multiple regions depicted in Fig. 2.2.

Definition 4.15. Point list in the Regions
⊢ ∀ p. pt_list p = FST(FST p)

The function pt_list:($sw \rightarrow (ptrgn)list$) accepts a variable of data type sw and returns the first element of the first pair of variable, i.e., (ptrgn)list. The returned element is itself a list of points and each point in the list represents the flow rate, density and shockwave speed in one region.

Definition 4.16. Time Range
⊢ ∀ p. time p = SND(SND(FST p)) - FST(SND(FST p))

The function time:($sw \rightarrow real$) accepts a variable of data type sw and returns the difference of the second element with the first part of the pair (data type: sw), which represents the considered time length for a shockwave in the area of consideration.

Definition 4.17. First Index of Point from Points/Regions List
⊢ ∀ p. ind_m p = FST(SND p)

The function m:($sw \rightarrow num$) accepts a variable of data type sw and returns the index of the first region for the shockwave considered in the area of consideration.

Definition 4.18. Seond Index of Point from Points/Regions List
```
⊢ ∀ p.  ind_n p = SND(SND p)
```

The function `n`:($sw \rightarrow num$) accepts a variable of data type `sw` and returns the index of the second region for the shockwave considered in the area of consideration.

Definition 4.19. Average Flow Rate of one Point/Region
```
⊢ ∀ t.  flow_rate t = FST(FST t)
```

The function `flow_rate`:($ptrgn \rightarrow real$) accepts an element of data type `ptrgn` and returns the first element of the first pair of a variable, i.e., the average flow rate of the vehicles.

Definition 4.20. Average Density of one Point/Region
```
⊢ ∀ t.  density t = SND(FST t)
```

The function `density`:($ptrgn \rightarrow real$) accepts an element of data type `ptrgn` and returns the second element of the first pair of a variable, i.e., the average density of the vehicles.

Definition 4.21. Shockwave Speed associated with one Point/Region
```
⊢ ∀ t.  shock_wv t = SND t
```

The function `shock_wv`:($ptrgn \rightarrow real$) accepts a variable of data type `ptrgn` and returns the second element of the pair, i.e., the shockwave speed corresponding to underlying region.

Definition 4.22. Number of vehicles crossing line(n) from Region R_{n-1} (during some time period Δt is
```
⊢ ∀ p n.  n_crossing p n =
  (( flow_rate (EL n (pt_list p)) / density (EL n (pt_list p)) -
     shock_wv (EL n (pt_list p))) * density (EL n (pt_list p)))
                                          * time p
```

The function `n_crossing`:(($sw\, num$) $\rightarrow real$) accepts two variables of data types `sw` and `num`, respectively, and returns the number of vehicles crossed to or from the boundary of the n region with index n (according to Equations 2.17 and 2.18). Where `flow_rate/density` represents v and `shock_wv` represents v_w.

Definition 4.23. List Containing the shockwaves of all of the Areas

```
⊢ sw_list [ ] = [ ] ∧
   sw_list (CONS h t) =
  CONS (shock_wv (EL (n h) (pt_list h)) * time h) (sw_list t)
```

The function `sw_list`:$((sw)list \rightarrow real(list))$ accepts an element of data type `(sw)list` and multiplies each element of the list with time (the required time length of the shockwave to be considered for the analysis) and then returns the new list of real values. This represents the space regions in the time space diagram or a shockwave over a time length according to the shockwave analysis [27].

Definition 4.24. Summation of the Shockwaves

```
⊢ ∀ L. sum_sw L =
        sum (1..LENGTH L) (λi.  EL (i - 1) (sw_list L))
```

The function `sum_sw`:$((real)list \rightarrow real)$ accepts a list of real numbers and returns a real number, i.e., the shockwave sums in the required time lengths according to shockwave analysis [27].

Next, we use the above-mentioned formalization for the verification of shockwave equation [6, 28, 30], which elaborates the average speed shift or speed change between two adjacent regions in terms of average flow rates and densities in those regions.

Theorem 4.3. Shockwave Speed Verification in two Regions R_m and R_n

```
 ⊢ ∀ p.
     n_crossing p (ind_n p) = n_crossing p (ind_m p) ∧
    ~(density (EL (ind_n p) (pt_list p)) =
      density (EL (ind_m p) (pt_list p))) ∧
    ~(density (EL (ind_n p) (pt_list p)) = &0) ∧
    ~(density (EL (ind_m p) (pt_list p)) = &0) ∧
     shock_wv (EL (ind_n p) (pt_list p)) =
     shock_wv (EL (ind_m p) (pt_list p)) ∧ &0 < time p
         ⇒ shock_wv (EL (ind_n p) (pt_list p)) =
          (flow_rate (EL (ind_n p) (pt_list p)) -
           flow_rate (EL (ind_m p) (pt_list p))) /
            (density (EL (ind_n p) (pt_list p)) -
             density (EL (ind_m p) (pt_list p)))
```

The variable p is a pair having data type sw, representing a list of points in regions, i.e, indices for the list of points in regions and the time interval, to compute a single shockwave, respectively. The first assumption ensures that the number of vehicles crossing (Defintion 4.22) from one region (Equation 4.17) would be the same as to the other region's incoming number of vehicles crossing (Equation 4.18) from the rear adjacent boundary, according to the universal law of conservation. The next assumption describes that the densities in both regions are not same and thus making them separate regions. The next assumption models the conditions that the densities of vehicles in both of the considered regions are non-zero. The last assumption ensures that the considered time interval is not negative. Finally, the conclusion of this theorem describes the relationship of the shockwave with flow rates and densities in any two regions.

The reasoning process of Theorem 4.3 is primarily based on the definitions of the functions `n_crossing`, `flow_rate`, `density`, `pt_list`, `ind_m` and `ind_n` and a lemma that ensures that the inverse of all non-zero real numbers would also be a non zero quantity and another lemma that describes the cross multiplication of four real numbers. This theorem and the rest of the theorems verified in the thesis can be found at [33].

We formalize the queue size/number of vehicles via an input-output analysis (Equation 2.14) as the following HOL Light defintion:

Definition 4.25. Queue Size (Number of Vehicles) via Input-output Analysis

```
⊢ ∀ p .  n_io p = (flow_rate (EL ind_m (pt_list p)) -
                 flow_rate (EL ind_n (pt_list p))) * time p
```

The function `n_io`:($sw \rightarrow real$) accepts an element of data type sw and returns a real number, which is the number of vehicles in a region.

In order to obtain the number of vehicles in n regions, we write the following HOL Light function:

Definition 4.26. Queue Size for n Regions via Input-output Analysis

```
⊢ io_list [ ] = [ ] ∧
  io_list (CONS h t) = CONS ((n_io h) (io_list t))
```

The function `io_list`:($(sw)list \rightarrow real\ list$) accepts a list of the elements of datatype sw, and a list with each of its element as a real number, and its each element is the number of vehicles in any region.

Definition 4.27. Summation of the number of vehicles via Input-output analysis

```
⊢ ∀ L. sum_io L =
        sum (1..LENGTH L) (λi.  EL (i - 1) (io_list L))
```

The above function sum_io:$((sw)list \rightarrow real)$ accepts sw list and returns the total number of vehicles in all the regions, i.e., the sum of the number of vehicles via input-output analysis according to Equation 2.16.

Next, in order to formalize Equation 2.15, we first model a single term, i.e., for $n = 1$, as the following HOL Light definition:

Definition 4.28. Number of Vehicles in a Single Region

```
⊢ ∀ r.  sw_rgn r =
    --(shock_wv (EL (ind_n (HD(FST r))) (pt_list (HD(FST r))) *
             time (HD(FST r)) - sum_sw (TL(FST r))) *
                         (SND(SND r) - FST(SND r))
```

The function sw_rgn:$(sw\_d \rightarrow real)$ accepts an element of data type sw_d and returns the number of vehicles in a single region via shockwave analysis.

In order to obtain the number of vehicles in n regions, we write the following HOL Light function:

Definition 4.29. Queue Size (Number of Vehicles for n Regions) via Shockwave Analysis

```
⊢ sw_rgn_list [ ] = [ ] ∧
  sw_rgn_list (CONS h t) = CONS (sw_rgn h) (sw_rgn_list t)
```

The function sw_rgn_list:$((sw\_d)list \rightarrow (real)list)$ accepts an (sw_d)list and returns a list containing the number of vehicles in n regions. It uses sw_rgn (Definition 4.29) to obtain the number of vehicles in a single region.

In order to obtain the summation of shockwave in an entire area, we need the to sum up of the individual accumulative shockwaves effect in the individual regions.

Definition 4.30. Sum of the Regional List

```
⊢ ∀ L. sum_sw_rgn L =
          sum (1..LENGTH L) (λ i.  EL (i - 1) (sw_rgn_list L))
```

This function `sum_sw_rgn`:$((sw\_d)list \rightarrow real)$ accepts an `sw_d` list and returns the total number of vehicles in all the regions, i.e., the sum of the number of vehicles via shockwave analysis.

The formalization presented in this chapter took about 500 lines of code and 50 man-hours. All the verified theorems are of generic nature as all the variables are universally quantified and can be specialized to obtain the formal analysis of any transportation system.

Chapter 5

Case Studies

In order to illustrate the utilization and effectiveness of our proposed framework, we formally analyze the German freeway by verifying its foremost property depicting the average vehicle flow in different lanes [32]. We also present the formal shockwave and input-output analysis and consistency between both of these analyses [27].

5.1 German Freeway

We utilize our formalization, presented in Chapter 4.1, to formally model and verify some vital properties of a German freeway macroscopic model [32]. The traffic flow pattern on the German freeway's A8-East from Munich to Salzburg is shown in Fig. 5.1.

There are three traffic lanes on this freeway as shown in Fig. 5.1. In this case study, we consider two, three and four vehicles traveling on lanes 1, 2 and 3 of the freeway, respectively. Based on these parameter the macroscopic traffic flow model for the first lane is given by the following definition:

Definition 5.1. Macroscopic Model of Lane 1

```
⊢ ∀ xv tv xh th L11v V11v L12v V12v L11h V11h L12h V12h.
      german_freeway_lane_1 xv tv xh th L11v V11v L12v V12v
        L11h V11h L12h V12h =
        (xv,tv,xh,th),[L11v,V11v; L12v,V12v],[L11h,V11h;
L12h,V12h]
```

where `xv`, `tv`, `xh` and `th` are the pairs containing the starting and ending points of the lengths and widths of the spatial and temporal regions, respectively. Similarly, `Lijv` and `Lijh` represent the length of the j^{th} vehicle in

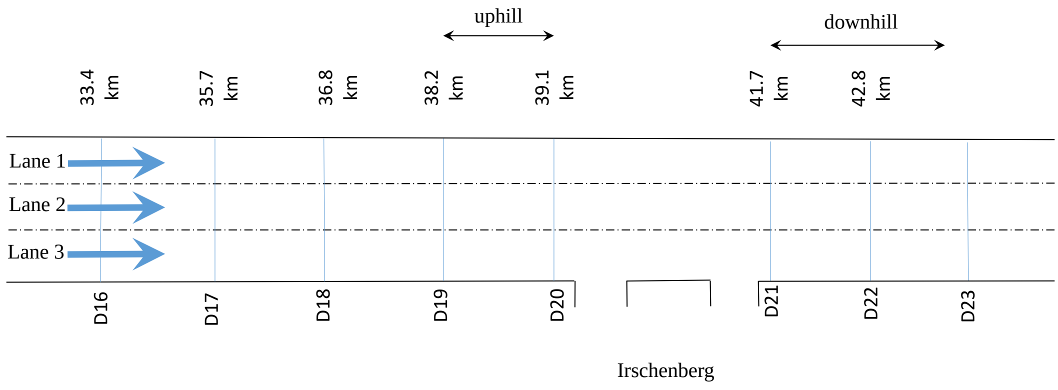

Figure 5.1: German Freeway [32]

the i^{th} lane in the spatial and temporal regions, respectively, whereas `Vijv` and `Vijh` represent the speed of the j^{th} vehicle in the i^{th} lane in the spatial and temporal regions, respectively. The function `german_freeway_lane_1` accepts all of these parameters and returns the time space macroscopic model of the traffic flow on Lane 1 of the German freeway.

Similarly, the following definitions provide the macroscopic models of traffic flow for Lanes 2 and 3.

Definition 5.2. Macroscopic Model of Lane 2

```
⊢ ∀ xv tv xh th L21v V21v L22v V22v L23v V23v L21h V21h
       L22h V22h L23h V23h.
    german_freeway_lane_2 xv tv xh th L21v V21v L22v V22v L23v
V23v
       L21h V21h L22h V22h L23h V23h = (xv,tv,xh,th),
[L21v,V21v;
       L22v,V22v; L23v,V23v],[L21h,V21h; L22h,V22h; L23h,V23h]
```

Definition 5.3. Macroscopic Model of Lane 3

```
⊢ ∀ xv tv xh th L31v V31v L32v V32v L33v V33v L34v V34v
       L31h V31h L32h V32h L33h V33h L34h V34h.
    german_freeway_lane_3 xv tv xh th L31v V31v L32v V32v L33v
```

```
V33v
       L34v V34v L31h V31h L32h V32h L33h V33h L34h V34h =
       (xv,tv,xh,th), [L31v,V31v; L32v,V32v; L33v,V33v;
L34v,V34v],
       [L31h,V31h; L32h,V32h; L33h,V33h; L34h,V34h]
```

Our next step is to formally verify the lane-averaged vehicle flow of the considered German freeway, which is given by the following theorem:

Theorem 5.1. Lane-Averaged Mean Velocity of the German Freeway

```
 ⊢ ∀ xv tv xh th L11v V11v L12v V12v L11h V11h L12h V12h L21v
V21v L22v V22v L23v V23v L21h V21h L22h V22h L23h V23h L31v V31v
L32v V32v L33v V33v L34v V34v L31h V31h L32h V32h L33h V33h L34h
V34h.
   &0 < SND th - FST th ∧ &0 < SND xh - FST xh ∧ L = 3
      ⇒ sum (1..L) (λ i.  EL (i - 1)[
     flow_rate_s2 (german_freeway_lane_1 xv tv xh th
        L11v V11v L12v V12v L11h V11h L12h V12h) / &L;
     flow_rate_s2 (german_freeway_lane_2 xv tv xh th L21v V21v
        L22v V22v L23v V23v L21h V21h L22h V22h L23h V23h) / &L;
     flow_rate_s2 (german_freeway_lane_3 xv tv xh th L31v V31v
        L32v V32v L33v V33v L34v V34v L31h V31h L32h V32h L33h
V33h
        L34h V34h) / &L]) = &9 / (&L * (SND th - FST th))
```

where the first two assumptions ensure that the length and width of the temporal region are positive, as they represent time ΔT and distance dX, respectively. The last assumption represents the number of lanes in the freeway. The conclusion of Theorem 5.1 represents the lane-averaged mean velocity of the freeway. The proof process starts by rewriting with the definitions of the functions `german_freeway_lane_1`, `german_freeway_lane_2`, `german_freeway_lane_3` and `density_s2`. Then the goal is verified using some properties from the `list` theory and the `sum` function and some arithmetic reasoning [33].

5.2 Formal Input-output and Shockwave Analyses and their Consistency

We use our formalization of shockwave, presented in Chapter 4.2, to formally verify the queue size/number of vehicles based on both the input-output and shockwave analysis [27]. The input-output model (also called cumulative arrival and departure model) is commonly used to describe traffic congestions on highways. Conventionally, the queue size at any time can be measured by the difference between the cumulative arrival and the departure curves (shown in the Fig. 5.3). In the same way, shockwave analysis keeps track of the queue propagation, discharging and dissipation. The queue size is measured by the product of the queue length and density at any time via Shockwave analysis (Fig. 5.3). The difference between these two analyses is that the input-output analysis keeps track of queue length and also travel time by considering the time dimensions only unlike the shockwave analysis, which considers both dimensions, i.e., time and density.

The traffic flow patterns for the considered scenario are shown in Figs. 5.3 and 5.4 indicating the changes in arrival and discharging flow rates, where a queue is considered as a lane or sequence of vehicles that are waiting for their turn to be attended. A region where the vehicles experience bottleneck or some obstructions on some highway are called high dense regions. The upstream region is considered along the direction of traffic flow is an area before the point of observation and downstream region is formed after the point of observation. Considered the scenario depicted in Fig. 5.2, having an upstream density and flow rate of q_a and k_a and a downstream flow and density of q_c and k_c at one time instant. After some time, i.e., at time t_1 of Fig. 5.3, congestions is observed. As a result, the density and flow rate in the arrival region gets disturbed due to congestions in the upcoming region (capacity) and thus a new region is formed having flow $q_{a'}$ and density $k_{a'}$.

Then, at the next time instant, when the queue starts dissolving then the discharging (congestion distortion) rate is introduced in the process. Fig. (5.4) shows the comparative sketches for the change of queue dissolving rates for the input-output model and the shockwave analysis. Whereas, at time t_1, another boundary or releasing wave (shockwave), i.e., v_p (due to partial removal of incident) occurs for time $t_1 \rightarrow t_2$. After time instant t_2, when v_p reaches the farthest end of the the queue, another wave v_r is formed (Fig.5.4), due to the introduction of a new Region R_r (Fig. 5.2) in the process, and terminates until the complete removal of incident/congestion till time t_d as shown in Fig. 5.4.

In order to illustrate the effectiveness of our proposed formalization (Section 4.2), we present the formal input-output and shockwave analysis of a highway considering three regions as shown in Fig. 5.2. Moreover, we prove the consistency of both these analyses by verifying that the number of vehicles verified in both cases, i.e., via input-output and shockwave analysis are same for each of the region.

For time interval $t_0 \rightarrow t_1$ (Fig. 5.3), congestion propagates upstream and the queue size can be described by the input-output and shockwave analysis.

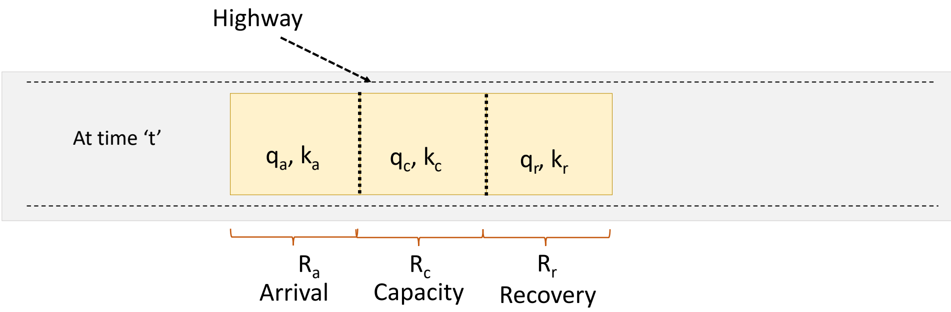

Figure 5.2: Highway Regions

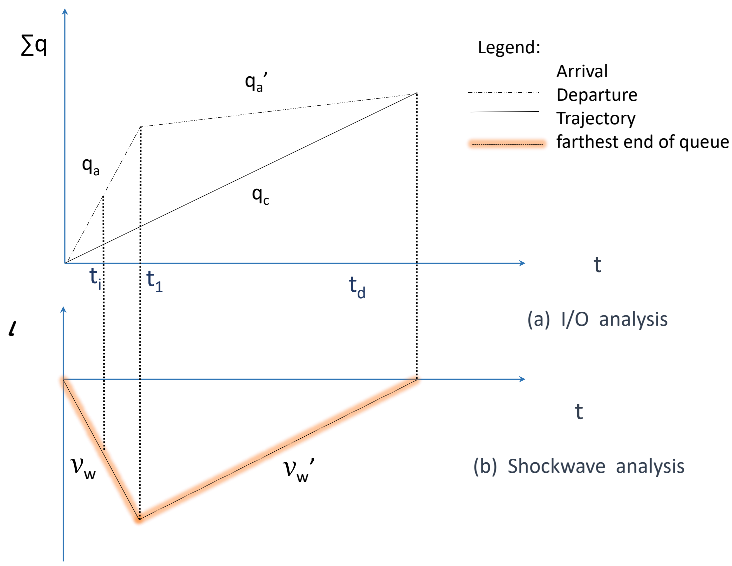

Figure 5.3: Queuing Dynamics with a Change in Arriving Flow Rate [27]

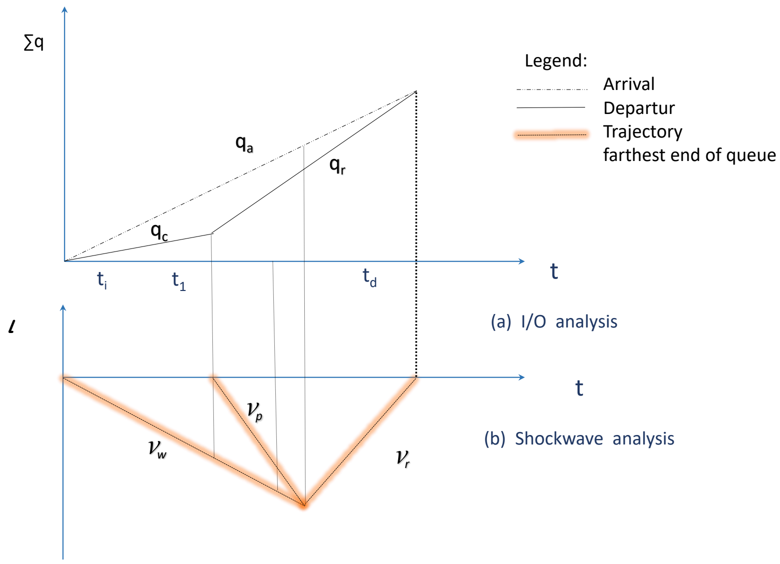

Figure 5.4: Queuing Dynamics with a Change in Discharging Flow Rate [27]

Changes in flow rate in the upstream region is called a change in the arriving demand. Then at t_1, the arriving flow rate is reduced to $q_{a'}$, (or q_{a_2}) due to the accumulation of vehicles in the arrival region. The queue size for the change in arriving demand for two time intervals can be represented, using Equation 2.16, as follows:

$$N_{io} = \sum_{i=1}^{2} (q_{m_i} - q_{n_i}) \Delta t_i \tag{5.1}$$

$$N_{io} = (q_a - q_c)t_1 + (q_{a_2} - q_c)(t - t_1) \tag{5.2}$$

We formally verify the queue size for the case of change in arriving demand via input-output analysis as the following HOL Light theorem:

Theorem 5.2. Queue Size for Change in Arriving Demand via Input-output Analysis

```
⊢ ∀ vw vw2 qa qa2 qc ka ka2 kc t1 t.
    ⇒ sum_io [
      ([(qa ,ka), vw; (qc,kc), vw], &0, t1), 0,1;
      ([(qa2,ka2),vw2; (qc,kc), vw2], t1, t ), 0,1]
           = (qa - qc) * t1 + (qa2 - qc) * (t - t1)
```

where `vw` and `vw2` represent the shockwave speeds before and after the change in arrival flow, respectively. Similarly, `qa, qc, qa2` and `ka, kc, ka2` represent flow rates and densities in approaching/arrival, capacity and changed approaching state, respectively. The reasoning process of Theorem 5.2 is based on rewriting with Definition 4.27, properties of HOL Light function `sum` along with some arithmetic reasoning.

In the same way, Equation 2.15 for the arrival/approaching change, as shown in Fig. 5.3, for shockwave analysis is mathematically expressed as:

$$N_{sw} = \sum_{j=1}^{2} -(v_{w_j}\Delta t_j - \sum_{i=0}^{0} v_{w_i}\Delta t_i)\Delta k_j \tag{5.3}$$

$$N_{sw} = -v_{w_1}(t_1 - t_0) + v_{w_2}(t - t_1) \tag{5.4}$$

Similarly, we verify the queue size as described in the above equation for the case of change in arriving time via shockwave analysis as the following HOL Light theorem:

Theorem 5.3. Queue Size for Change in Arriving Demand via Shockwave Analysis

```
 ⊢ ∀ vw vw2 qc qa qa2 ka kc ka2 t1 t.
    ∼(ka = kc) ∧ ∼(ka2 = kc) ∧
    ∼(ka = &0) ∧ ∼(ka2 = &0) ∧ ∼ (kc = &0 ) ∧
    (∀ p.  n_crossing p (ind_n p) = n_crossing p (ind_m p)) ∧
    (∀ p.  shock_wv (EL (ind_n p) (pt_list p)) =
         shock_wv (EL (ind_m p) (pt_list p))) ∧
     (∀ p.  &0 < time p)
       ⇒ sum_sw_rgn [
         [(([(qa, ka ), vw; (qc, kc), vw], &0, t1), 0, 1], ka
,kc;
         [(([(qa2, ka2), vw2; (qc, kc), vw2], t1, t), 0, 1], ka2,
kc]
            = (qa - qc) * t1 + (qa2 - qc) * (t - t1)
```

where `vw` and `vw2` represent the shockwave speeds before and after the change in arriving flow (Fig. 5.3), respectively. Similarly, `qa, qc, qa2` and `ka, kc, ka2` represent flow rates and densities in arrival, capacity and changed arrival regions, respectively. The first two assumptions of Theorem 5.3 ensure that the densities in both regions are not same. The next three assumptions model the conditions that the densities of vehicles in the considered three regions are non-zero. The next assumption says that the number of vehicles crossing (Definition 4.22) from one region's front boundary (Equation

4.17) would be the same as to the other adjacent region's number of vehicles crossing from the rear adjacent boundary and is thus according to the universal law of conservation. The next assumption models the condition that the shockwave speed in both regions are same. The last assumption models the non-negativity condition for the time interval. Finally, the conclusion of Theorem 5.3 presents the number of vehicles (queue size) for change in arrival demand via shockwave analysis. The reasoning process for Theorem 5.3 is based on Definition 4.30, properties of HOL Light's function `sum` along with some real arithmetic reasoning. More details about its proof can be found at [33].

Then at the next time instant, the arriving flow rate reaches its maximum value and the discharging flow rate in the downstream region starts changing. Thus leads to two cases, i.e., first for the queue propagation when the queue size increases and the second for the queue dissipation when the queue size decreases. This phenomena is shown in the Fig. 5.4.

Changes in queue size, such as the introduction of discharging rate in the process, can be represented as follows:

$$N_{io} = \sum_{i=1}^{2} (q_{m_i} - q_{n_i}) \Delta t_i \tag{5.5}$$

$$N_{io} = (q_a - q_c) t_1 + (q_a - q_r)(t - t_1) \tag{5.6}$$

The queue size (number of vehicles) for the case of queue propagation via input-output analysis is formalized as follows:

Theorem 5.4. Queue Size for the Case of Queue Propagation via Input-output Analysis

```
⊢ ∀ vw vp qa qc ka kc kr t1 t.
     ⇒ sum_io [
       ([(qa, ka), vw; (qc, kc), vw], &0, t1), 0, 1;
       ([(qa, ka), vp; (qr, kr), vp], t1, t), 0, 1]
          = (qa - qc) * t1 + (qa - qr) * (t - t1)
```

where all the input variables are the same as in Theorem 5.2 except `kc` and `kr`, which represent the densities in the capacity and the recovery regions (Fig. 5.4), respectively. Similarly `vp` is the shockwave speed. The verification process of the above theorem is similar to the one for Theorem 5.2, and its details can be found in [33].

The queue size for the case of queue propagation via shockwave analysis, is mathematically expressed as:

$$N_{sw} = \sum_{j=1}^{2} -(v_{w_j}\Delta t_j - \sum_{i=0}^{1} v_{w_i}\Delta t_i)\Delta k_j \tag{5.7}$$

$$N_{sw} = -\{v_w(t - t_0) - v_p(t - t_1)\} + -v_p(t - t_1) \tag{5.8}$$

Similarly, we verify the queue size as described in the above equation for the case of change in propagation via shockwave analysis as the following HOL Light theorem:

Theorem 5.5. Queue Size for the case of Queue Propagation via Shockwave Analysis

```
⊢ ∀ vp vw qa qc qr ka kc kr t1 t.
   ∼(ka = kc) ∧ ∼(kc = kr) ∧
   ∼(ka = &0) ∧ ∼(kc = &0) ∧ ∼(kr = &0) ∧
   (∀ p.  n_crossing p (ind_n p) = n_crossing p (ind_m p)) ∧
   (∀ p.  shock_wv (EL (ind_n p) (pt_list p)) =
        shock_wv (EL (ind_m p) (pt_list p))) ∧
   (∀ p.  &0 < time p)
     ⇒ sum_sw_rgn [[
       ([(qc, kc), vp; (qr, kr), vp], t1, t), 0, 1], ka, kr; [
       ([(qa, ka), vw; (qc, kc), vw], &0, t), 0, 1;
       ([(qc, kc), vp; (qr, kr), vp], t1, t), 0, 1], ka, kc]
              = (qa - qc) * t1 + (qa - qr) * (t - t1)
```

where vw and vp represent the shockwave speeds before the propagation starting in arrival region and after the propagation introduced at time t_1 (Fig. 5.4), respectively. vw is the shockwave between arrival and capacity regions and vp is between capacity and recovery regions (Fig. 5.2). Similarly, qa, qc, qr and ka, kc, kr are representing flow rates and densities in approaching/arrival, capacity and recovery regions, respectively. All the assumptions for Theorem 5.5 are the same as for the the Theorem 5.3 and also describing the same conditions. Similarly, the verification process for the above theorem is the same as that of Theorem 5.3.

The conclusions of both the theorems, i.e., Theorems 5.4 and 5.5, show that the queue size in the case of input-output and shockwave analyses are same, which means both analyses are consistent.

The releasing wave vp reaches at the maximum possible end of the queue at time t_2. This is where the recovery of the normal operation of the highway begins i.e., the congestion of the queue starts dissipating (this phenomena is called queue dissipation) and another wave v_r starts to grow as a result of discharging flow q_r (Fig. 5.4). This emerging wave v_r moves downstream until the complete removal of the queue congestion at t, and the road way section returns to its normal operating condition.

The queue size via Equation 2.16 for the case of queue dissipation via input-output analysis is represented as follows:

$$N_{io} = \sum_{i=1}^{3}(q_{m_i} - q_{n_i})\Delta t_i \tag{5.9}$$

$$N_{io} = (q_a - q_c)t_1 + (q_a - q_r)(t_2 - t_1) + (q_a - q_r)(t - t_2) \tag{5.10}$$

Theorem 5.6. Queue Size for the Case of Queue Dissipation via Input-output Analysis

```
⊢ ∀ vw vp vr qa qc qr ka kc kr t1 t.
   ⇒ sum_io [
     ([(qa, ka), vw; (qc, kc), vw], &0, t1), 0, 1;
     ([(qa, ka), vr; (qr, kr), vr], t1, t2), 0, 1;
     ([(qa, ka), vr; (qr, kr), vr], t2, t ), 0, 1] =
      (qa - qc) * t1 + (qa - qr) * (t2 - t1) + (qa - qr) * (t -
t2)
```

The verification process for the above theorem is very similar to that of Theorems 5.2 and 5.4.

Finally, we verify the queue size for the case of queue dissipation via shockwave analysis as the following modified Equation 2.15:

$$N_{sw} = \sum_{j=1}^{2} -(v_{w_j}\Delta t_j - \sum_{i=0}^{0} v_{w_i}\Delta t_i)\Delta k_j \tag{5.11}$$

$$N_{sw} = [-v_w(t_2 - t_0) + -v_r(t - t_2)](k_r - k_a) \tag{5.12}$$

Theorem 5.7. Queue Size for the Case of Queue Dissipation via Shockwave Analysis

```
⊢ ∀ vp vw vr qa qc qr ka kc kr t1 t2 t.
  ~(ka = kc) ∧ ~(kc = kr) ∧ ~(ka = kr) ∧
  ~(ka = &0) ∧ ~(kc = &0) ∧ ~(kr = &0) ∧
  (∀ p.  n_crossing p (ind_n p) = n_crossing p (ind_m p)) ∧
```

```
   (∀ p.  shock_wv (EL (ind_n p) (pt_list p)) =
        shock_wv (EL (ind_m p) (pt_list p))) ∧
   (∀ p.&0 < time p) ∧
   sw_rgn ([([(qa ,ka), vw; (qc,kc), vw], (&0, t2)), 0, 1], ka,
kr) =
   sum_sw_rgn [
    [([( qc ,kc), vp; (qr, kr), vp], t1, t2), 0, 1], ka, kr;
    [([((qa ,ka), vw); (qc, kc), vw], (&0, t2)), (0, 1);
    ([((qc ,kc), vp); (qr, kr), vp], t1, t2), 0, 1], ka, kc]
      ⇒ sum_sw_rgn [[
        ([(qa ,ka), vw; (qc, kc), vw], (&0, t2)), (0, 1)], ka,
kr;
        [([(qa ,ka), vr; (qr, kr), vr], (t2, t)), (0, 1)], ka,
kr]              = (qa - qc) * t1 + (qa - qr) * (t2 - t1)
             + (qa - qr) * (t - t2)
```

where the first nine assumptions of the above theorem are the same as that of Theorems 5.3 and 5.5. While the last assumption describes that the queue size is already evaluated before time t_2 (in the last time interval) in Theorem 5.5. Finally, the conclusion of the above theorem represents the queue size in the case of queue dissipation.

The verification process of this theorem is similar to Theorems 5.3 and 5.5. More details about whole of the formalization, presented in this chapter, can be found at [33]. Again, the conclusions of both theorems, i.e., Theorems 5.6 and 5.5, show that the queue size in the case of input-output and shockwave analyses are the same, which means both analyses are consistent.

The formal analysis presented, in this chapter, took about 1500 lines of code and 120 man-hours. Moreover, the straightforward proof scripts for the properties, verified in this chapter, clearly indicate the usefulness of our foundational formalization presented in Chapter 4 of this thesis. Our formalization may be utilized to formally reason about may other macroscopic model related properties and the results would be guaranteed to be correct due to the inherent soundness of theorem proving. Moreover, our theorems are generic in nature, i.e., all the variables in these theorems are universally quantified. To the best of our knowledge, no other computer-based analysis technique for traffic flow problems can provide such benefits.

Chapter 6

Conclusions

In this thesis, we propose to use higher-order-logic theorem proving to analyze macroscopic models of traffic flow. Due to the high expressiveness of the underlying logic, we can formally model the continuous components of macroscopic models while capturing their true behavior and the soundness of theorem proving guarantees correctness of results. We formally model the basic parameters of a transportation system, which include density, flow rate, speed, relative occupancy and shockwave and used our formalization to formally analyze a German freeway and a commonly used highway, by performing the input-output and shockwave analyses. The main challenge in the proposed approach is the enormous amount of user intervention required due to the undecidable nature of the logic. We propose to overcome this limitation by formalizing the foundational mathematical theories and core concepts of traffic flow theory so that these available results can be built upon to minimize user interaction. The case studies demonstrated the practicability of this idea.

In future, we plan to develop the formal reasoning support for the microscopic models of traffic flow theory. Modeling of the equations, capturing the dynamics of the microscopic model, would include the formal modeling of many human characteristics as well, which makes the exercise a bit complex [21].

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