ANALYSIS OF SURGE PHENOMENA IN AXIAL FLOW COMPRESSORS

By

MUHAMMAD KAMRAN KHAN TAREEN

2011-NUST-MS PhD-CS&E-33

RESEARCH CENTRE FOR MODELING AND SIMULATION

NATIONAL UNIVERSITY OF SCIENCES AND TECHNOLOGY

ISLAMABAD

2014

ANALYSIS OF SURGE PHENOMENA
IN AXIAL FLOW COMPRESSORS

A Thesis submitted to the Faculty of

Research Centre for Modeling and Simulation,

National University of Sciences and Technology, Islamabad

by

Muhammad Kamran Khan Tareen

In Partial Fulfillment of the Requirements for the Degree of

Master of Science

in

Computational Science and Engineering

August, 2014

STATEMENT OF ORIGINALITY

I hereby certify that the work embodied in this thesis is the result of original research carried out by me, and has not been submitted for a higher degree to any other University or Institution.

8th August, 2014	
Date	Muhammad Kamran Khan Tareen

DEDICATION

To the bittersweet memories of H–12…

ACKNOWLEDGEMENTS

"It is not true that people stop pursuing dreams because they grow old, they grow old because they stop pursuing dreams."

– Gabriel Garcia Marquez

First of all, I am highly grateful to Allah Almighty for enabling me to complete this thesis successfully, and for the uncountable blessings He has conferred upon me in my life. I am thankful for the courage and motivation He poured in my heart, which have led me to this point where I am presenting this humble work before the world.

This work would not have been what it is today without support and encouragement from my thesis supervisor, Dr. Adnan Maqsood. Thank you so much for providing me an opportunity to work under your supervision, and for guiding me so patiently and proficiently throughout the research phase. Your role as a mentor and supervisor has been vital towards the accomplishment of this thesis.

I would also like to thank my G.E.C. members, Dr. Khalid Parvez, Dr. Meraj Mustafa Hashmi and Mr. Ammar Mushtaq; for their encouragement and assistance.

A huge bundle of thanks to *the* friends this life has blessed me with, Mr. Saqib Khan, Mr. Arslan Ahmad and Mr. Syed Sajjad Sattar; thought of whom is a thing that always makes me feel that this life has been worth-living. Time passed with you guys has been so memorable and wonderful that its delight cannot perhaps be put into black and white justly.

I also want to thank all the faculty and staff at RCMS for their responsiveness and for providing a beautiful learning environment.

Special thanks to Mrs. Nusrat Nadeem, Library In-charge RCMS, for arranging every title and reference that I requested her for.

I also feel appreciative for the cooperation and generous attitudes of my colleagues and seniors at Airworthiness Directorate of Pakistan Civil Aviation Authority, Karachi.

I owe a lot of thanks to my family: my parents, Mr. Faizullah Khan Tareen and Mrs. Yasmeen Sultana; my brother, Mr. Ali Nauman Khan Tareen; and sister, Ms. Maryiam Khan Tareen; for their enormous love and affection, and whose support and prayers have brought me here.

In the end, I want to extend my gratitude to all those who have taught me, from kindergarten till date. Where I stand today, I attribute it to my teachers.

"Engineering is not only a learned profession, but it is also a learning profession; one whose practitioners first become and then remain students throughout their active careers."

– William L. Everitt

SUMMARY

Compression systems used in the power generation plants and aerospace propulsion applications are prone to the aerodynamic instabilities of stall and surge. Both of these instabilities result in reduced off-design of the compressors, and raise safety concerns.

This research focuses on the analysis of surge phenomena in axial flow compressors. Mathematical model developed by Moore and Greitzer (1985, 86) [1, 2] has been used for this purpose. The axisymmetric compressor characteristic employed by the Moore-Greitzer Model has been generalized to study the effect of the parameters involved over a wider set of values. The pure surge case of the Moore-Greitzer Model is then subjected to Method of Multiple Time Scales (MTS), and a closed-form solution has been obtained for the problem.

Subsequent application of Bifurcation theory on the closed-form solution reveals that surge manifests itself as a sustained Limit Cycle Oscillation (LCO) subject to fulfillment of a necessary condition dictated by the compressor characteristic. It has been found that the behaviour of surge oscillations is significantly dependent upon the choice of compressor characteristic.

From combination of the various parameters involved in the Moore-Greitzer Model, two new parameters have been obtained, and the condition for the stable limit cycle oscillations has been obtained, which revolves around these parameters.

The analytical solution has been verified against the numerical simulation of the Moore-Greitzer Model, thus implying that Method of Multiple Time Scales has successfully captured the qualitative and quantitative aspects of the surge phenomena.

The results of the analyses presented in this research work can provide useful guidelines to the turbomachinery designers in developing a better understanding towards the surge problem and in improving off-design performance of axial compressors.

TABLE OF CONTENTS

LIST OF FIGURES

"If all the arts aspire to the condition of music, all the sciences aspire to the condition of mathematics."

– George Santayana

Intentionally left blank

Chapter 1

INTRODUCTION

1.1 Background

One of the greatest achievements of mankind is unveiling the principles by which the chemical energy contained in the fossil fuels can be converted into mechanical energy. A heat engine is a machine that simply performs this job. Gas turbines belong to the class of these heat engines that have taken the most significant place in the aerospace propulsion systems. The gas turbine engines intake a mass of air from the surrounding atmosphere, accelerate it, and eject it in the form of a high speed jet. This results in the production of propulsive force, or thrust, that is required for propelling the aircraft.

Anatomy of a gas turbine engine, as shown in Figure 1.1, reveals that there are a few major components that build up any propulsion system. Among these, a fan or a compressor is the first rotating component that the fluid encounters. The basic function of the compressor is to impart kinetic energy to the working fluid, which is air in the case of aerospace vehicles, through stages of rotating blades (rotors) and stationary vanes (stators). The increased kinetic energy of the fluid is converted into an increase in total pressure of the fluid, which is needed in the combustion chamber. It is worth noting that the limits of operation of an engine are often dictated by the compressor [3].

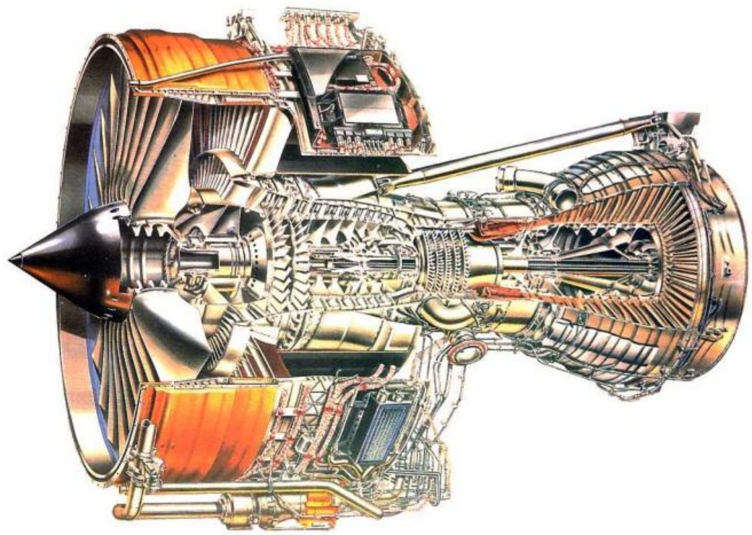

Figure 1.1 Rolls Royce RB211 High Bypass Ratio Turbofan Engine (from [4])

The compressors can be classified into two types, namely axial and centrifugal compressors (Figure 1.2). The axial flow compressors are generally preferred over centrifugal compressors due to higher cross-sectional flow areas per mass flow rate, and lower frontal areas, i.e. diameters. The centrifugal compressors, however, have higher single-stage pressure ratios than axial compressors, and are mostly used in small aircraft applications and helicopters.

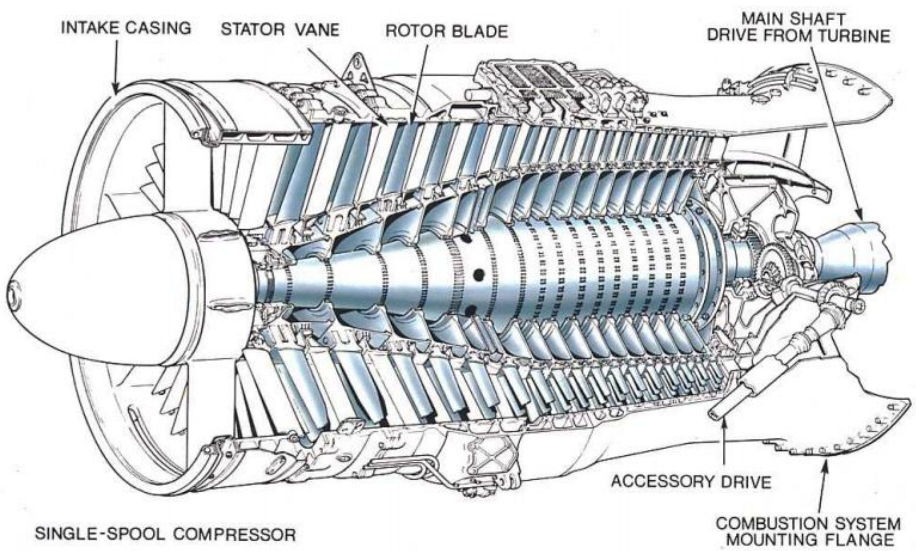

Figure 1.2a Single Spool Axial Flow Compressor (from [4])

Figure 1.2b Typical Impellers for Centrifugal Compressors (from [4])

The dynamics of a compression system is prone to two types of aerodynamic instabilities: Stall and Surge. Both of these are unsteady aerodynamic phenomena, which are difficult to predict in the design of compression systems, and occurrence of these in operation of an engine can lead to serious consequences and hazards to flight safety.

Stall in a compressor may manifest itself as Rotating Stall, Individual Blade Stall, or Stall Flutter. From experimental evidence, the first of these seems the most prevalent [5]. *Rotating Stall* can be thought of as small-amplitude, *multidimensional* flow instability, where regions of non-uniform circumferential flow exist around compressor annulus and which propagate in the downstream stages at some fraction of the rotor speed.

Surge is basically a large-amplitude, *one-dimensional* and axisymmetric flow instability, whereby the *whole engine* exhibits fluctuations of mass flow rate. The frequency of surge oscillations is usually an order of magnitude *less* than those associated with passage of stall cells in rotating stall (Figure 1.3).

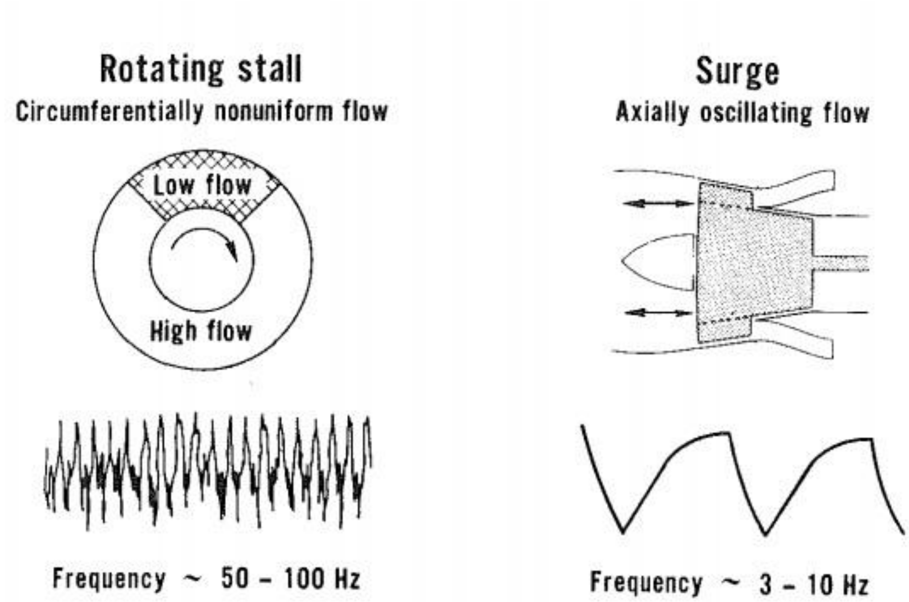

Figure 1.3 Modes of Compression System Instability (from [6])

A compressor may pass in and out of rotating stall during a surge cycle due to unsteady mass flow [7]. Both, rotating stall and surge, induce undesirable vibrating stresses in compressor blades, and can result in reduced off-design performance and structural damage.

In this chapter, the background and the areas of research are presented. These are followed by the objectives and methodology of the research. An outline of the all chapters regarding the organization of this thesis is given at the end.

1.2 Areas of Research

This research is aimed at studying and analyzing the phenomena of surge in axial flow compressors. Axial compressors are an integral subset of turbomachines. The prediction of off-design performance has been the key motivation for researchers in the field for the past several decades. To address the problem at hand, techniques of nonlinear dynamics are studied and applied for understanding the behavior of pumping systems during surge. The areas, encompassing the present research, are briefly described below.

1.2.1 Turbomachinery Off-Design Performance

Turbomachines have gained a vital importance, both in academia and industry because of its broad spectrum applications. Design engineers are interested in predicting the performance of turbomachines within and outside the operational envelopes. Off-design performance prediction is critically important due to safety concerns. The limits of operation of an engine are often dictated by a compressor. Furthermore, the design of an efficient axial flow compressor remains such a complex process that the success or failure of an engine often revolves around the design of a compressor [3]. This research effort has focused on the *Pure Surge* case of the mathematical model developed for the instability of compression systems by Moore and Greitzer (1985, 86) [1, 2], commonly called the *Moore-Greitzer Model*. The deliverables of this study can be helpful in the prediction of off-design performance of axial flow compressors during the initial stages of design, and the methodology may be extended to all other classes of turbomachines for similar treatment.

1.2.2 Nonlinear Dynamics

Most of the real life systems are inherently nonlinear. In order to analyze a system close to its natural response during various scenarios, it is important to take into consideration the intrinsic nonlinearities of the system. Various techniques of Nonlinear Dynamics exist for solving a problem. The present work has focused on the application of Multiple Time Scales

(MTS) method, which is a subset of Perturbation Methods. It is an asymptotic approach to approximate the physical problems that involve perturbations about nominal states specifically in limiting cases. Such solutions have an advantage over usual numerical solutions in that the important parameters and their effects on limit-cycle characteristics, such as amplitude and frequency, can be easily seen in explicit functional relationships. The compressor surge problem, as modeled by the Moore-Greitzer Model, has not yet been handled through this technique. Further, the closed-form solution obtained through MTS has been subjected to Bifurcation analysis, to get a qualitative insight into the governing nonlinearities in the system.

1.3 Research Objectives

This work aims to achieve the following objectives:

- Understanding surge phenomena in axial flow compressors, and the effect of the nonlinearities involved in it on the compression system operation
- Studying Moore-Greitzer Model
- Exploring the potential of Multiple Time Scales technique for the generation of approximate closed-form analytical solution of surge problem
- Analyzing the effect of variations in compressor characteristic curve on the surge oscillations
- Qualitative study of the effect of nonlinearities in the system through Bifurcation analysis, identification of Limit Cycle Oscillations (LCOs) and the conditions necessary for their existence

1.4 Methodology

The methodology adopted to undertake this research has been roughly the same as that described above sequentially in Section 1.3. After sufficient review of the available literature, the problem was defined precisely. The *Pure Surge* case of the famous Moore-Greitzer model is used as the basis of this study. The compressor characteristic used in the Moore-Greitzer model is generalized to study the effect of the parameters involved. Subsequently, Bifurcation theory is used to study the effect of nonlinear dynamics on surge behavior. In the end, numerical simulations of the Moore-Greitzer Model are carried out to verify the closed-form solution obtained from MTS.

The same is elaborated below in a flowchart:

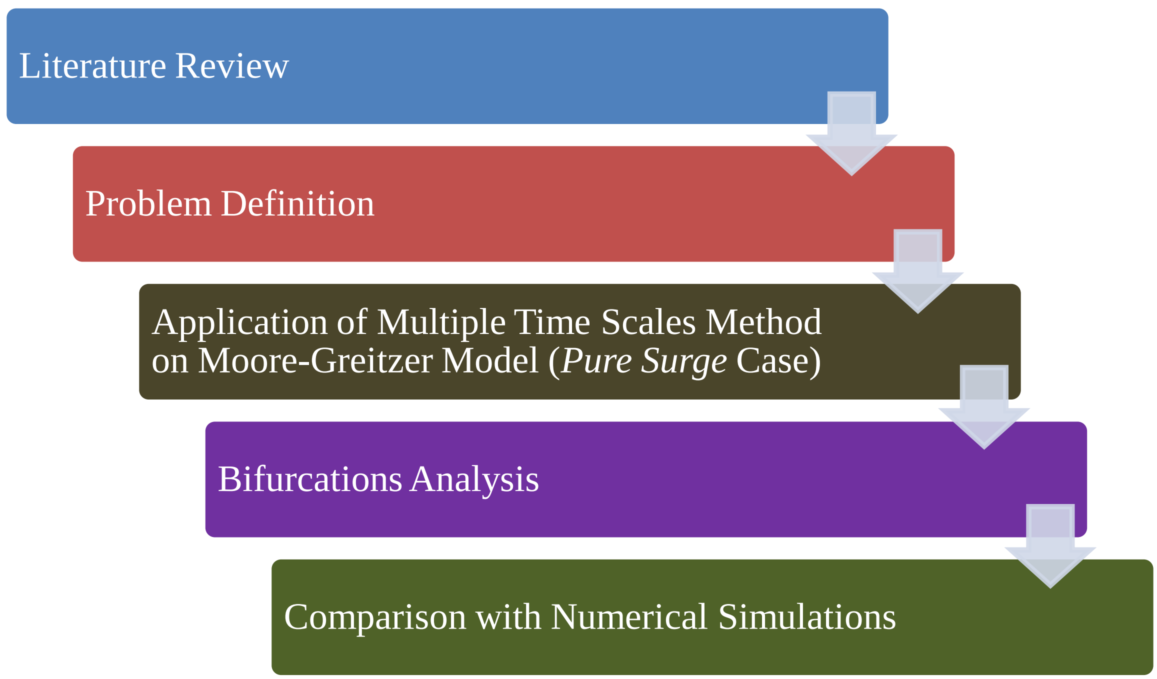

Figure 1.4 Research Methodology

1.5 Contributions

This work extends the research on surge phenomena in axial compressors, and exploration of MTS method. Specifically, it includes the following contributions:

- Analytical solution for the surge problem has been found
- The analytical solution is expressed in terms of amplitude and frequency of surge oscillations, and parametric dependency can be seen explicitly
- Compressor characteristic curve used in the Moore-Greitzer model has been generalized to get a better understanding of the effect of various parameters involved
- Generalized MTS method has been applied on the Moore-Greitzer model for the first time
- Conditions necessary for existence of sustained limit cycle oscillations (surge) have been identified
- This thesis has generated one journal manuscript for *Meccanica* submitted on 05th July, 2014

1.6 Organization of the Thesis

This thesis comprises seven (07) chapters. A brief description of each chapter is given below:

1.6.1 Chapter 1 – Introduction

This is the opening chapter of the thesis. It outlines the background, areas, objectives and methodology of the research and organization of the thesis report.

1.6.2 Chapter 2 – Literature Review

This chapter presents a comprehensive summary of the literature study. Gap in the existing literature has been identified, which has prompted this research.

1.6.3 Chapter 3 – Moore-Greitzer Model

The derivation of the Moore-Greitzer Model has been worked out in detail in this chapter. The assumptions and limitations of the model have also been discussed.

1.6.4 Chapter 4 – Problem Formulation

This chapter discusses the effect of variation of the parameters involved in compressor characteristic curve. The compressor characteristic curve is generalized, and governing equation for surge analysis is developed.

1.6.5 Chapter 5 – Surge Analysis

Method of Multiple Time Scales (MTS) is applied on the generalized governing equation for surge analysis, developed in the previous chapter, to obtain the closed-form solution. Bifurcation analysis is then performed on this solution for identification of Limit Cycle Oscillations (LCOs). Key parameters in the solution are identified, along with the necessary conditions required for the existence of LCOs.

1.6.6 Chapter 6 – Verification Studies

In this chapter, comparisons of numerical simulations of Moore-Greitzer Model with the analytical solution devised in Chapter 5 are shown and discussed.

1.6.7 Chapter 7 – Conclusions and Future Work

This is the final chapter of the thesis. It discusses the conclusions drawn from the research. Recommendations for future work are also laid down.

Chapter 2

LITERATURE REVIEW

2.1 Introduction

This chapter discusses the study of available literature done to reach at a point where the problem could be defined after identifying the gaps in the existing literature. The literature survey was done in two stages. In the first stage, an understanding was developed towards the surge phenomena in axial flow compressors. The second stage focused on the study of nonlinear dynamics in general and Multiple Time Scales technique in particular.

2.2 Stall and Surge in Axial Flow Compressors

The aerodynamic instability in the form of stall and surge has accompanied the operation of compression systems from the beginning. The research on the topic till 1976, was experimental and focused only on qualitative aspects of the phenomena, such as by Pearson (1955) [8] and Huppert (1965) [9]. It was in 1976 when Greitzer [7], for the first time, attempted to study the phenomena analytically and presented a mathematical model for it.

Greitzer (1976) [7] correctly applied the transport equations to the control volume containing compressor and reached upon a set of nonlinear differential equations which models the transient behaviour of compressor instability. The nonlinear model developed by him predicts the transient response of a compression system subsequent to a perturbation from steady state operating conditions. He identified a critical parameter, which was later called the *Greitzer's parameter*, above or below whose value determined the mode of compressor instability (rotating stall or surge). He also performed experimentation to assess his mathematical model [10], and explained the mechanism of surge oscillations, by putting them in analogy to those exhibited by nonlinear mass-spring-damper system.

About a decade later, Moore and Greitzer (1985, 86) [1, 2] reworked the model originally developed by Greitzer [7]. This time, they presented an improved model involving a smooth cubic compressor characteristic, and showed that rotating stall and surge can be modeled separately from each other. Their mathematical model constitutes a set of three nonlinear partial differential equations. These partial differential equations are then converted to ordinary differential equations via a Galerkin procedure. This subsequent set of three nonlinear ordinary differential equations is that what is known as the *Moore-Greitzer Model*.

Since then, a lot of effort has been put in understanding, investigating and improving a compressor's response during stall and surge. Reviews by Greitzer (1980) [6], Greitzer (1981) [11], Longley (1994) [12], Gu et al. (1999) [13] and Paduano et al. (2001) [14] are very informative and are highly recommended for getting an overview of the research carried out in the subject. A seminal development in the field after the work of Moore and Greitzer is the application of classical nonlinear dynamics to rotating stall and surge phenomena (as described by the Moore-Greitzer Model).

This approach was first conceived by McCaughan (1989) [15]. She applied Bifurcation theory on a simplified version of Moore-Greitzer Model, and carried out a parametric study of throttle setting and Greitzer's parameter. Her analysis provides a complete picture of the parametric effects, allowing a good prediction of the compression system response.

The idea was pursued by Abed et al. (1993) [16]. Bifurcations analysis performed by them results in stable oscillations (limit cycle) and unstable oscillations. They termed the former as surge, and the latter as “anti-surge”. Their work gives a convenient and simple explanation of the boundary between surge and rotating stall behaviours.

Liaw and Abed (1996) [17] incorporated the application of Control theory in addition to the bifurcation theory in the Moore-Greitzer Model. They studied the bifurcation behaviour of the model and used it as a basis for nonlinear control design. They found that control, if based solely on the system linearization, cannot ensure stable operation past the operating point owing to uncontrollability of the system eigenvalue, which causes instability as the throttle parameter is varied. Their work was perhaps the first one towards active control of the onset of aerodynamic instabilities in axial compression systems.

Hos et al. (2003) [18] gave special attention to global bifurcations in the Moore-Greitzer Model. They carried out detailed studies for various scenarios; including pure surge, rotating stall and coupled rotating stall and surge. They refined the parameter boundaries of surge and rotating stall, as laid down in the previous works, in terms of global bifurcation theory.

Recent work by Malathi and Kushari (2012) [19] examines the effect of change in geometric parameters on axial compression systems using the Moore-Greitzer Model. They have investigated the effect of each of the major geometric parameters (compressor effective length, annulus area and plenum volume) on the nature of rotating stall and surge for different values of Greitzer’s parameter. The type of instabilities that could be expected for compressors of relatively different sizes has been observed with respect to each of the major geometric parameters. The parametric study conducted by them reveals some interesting results, like among the compressors of equal radii; the longer one will be more prone to rotating stall than surge.

2.3 Method of Multiple Time Scales (MTS)

Moore-Greitzer Model involves coupled nonlinear ordinary differential equations. No generic method to handle all nonlinear differential equations exists till date. Computational Fluid Dynamics (CFD) analyses are impractical because a variety of time and length scales are involved in the internal flow field. However, certain analytical approximations can be made and the equation(s) at hand can be solved in a nice manner to give appropriate results. Perturbation Methods are an example from this class of approximations. As described earlier, the Multiple Time Scales method belongs to the family of perturbation methods. In MTS, the (independent) time variable is resolved into fast and slow(er) components (Figure 2.1); so as to maintain the uniformity of the solution for large times, and an asymptotic approach is developed for obtaining analytical solutions by approximating the problems involving a limiting (perturbation) parameter. By expanding the time variable into its components, an ordinary differential equation is converted into partial differential equations (the number of the PDEs being equal to the number of components of time variable).

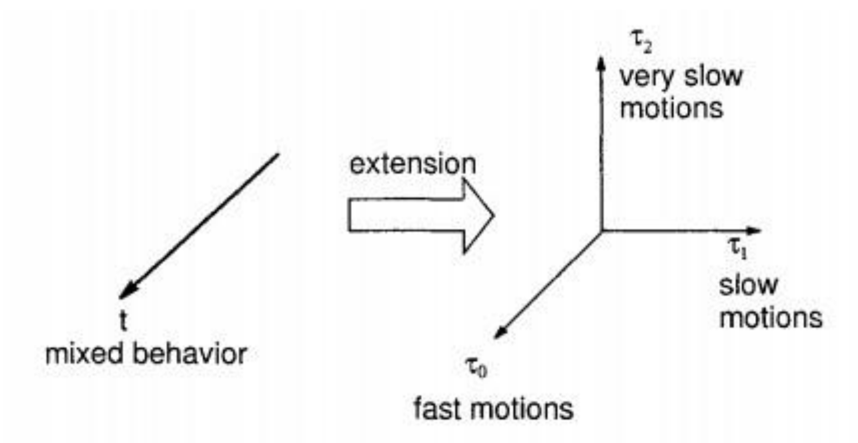

Figure 2.1 Concept of Multiple Time Scales (from [20])

Nayfeh (1981, 85, 95, 2004) [21-24] has given the details of this method and its variants along with various other techniques of nonlinear dynamics in his books. Another good source for learning the basics and usage of the method is Ramnath (2010) [25]. The procedure developed for the application of MTS method in the present study is based on the work of Go & Ramnath (2001) [20] and Maqsood & Go (2012) [26].

Go and Ramnath (2001) [20] applied the MTS method along with the bifurcation theory to study the wing rock dynamics on a rigid aircraft having multiple rotational degrees-of-freedom. They used the derivative expansion version of the MTS method, and obtained closed form solutions in parametric forms. They identified the parts of the solutions governing slow and fast dynamics, and discussed the qualitative aspects of the solutions in the light of bifurcations.

Maqsood and Go (2012) [26] have carried out MTS analysis of longitudinal dynamics of sustained high angle-of-attack flights of an Unmanned Aerial Vehicle (UAV) capable of aerodynamic vectoring. They, too, have used the derivative expansion method in their work. After obtaining data from wind tunnel testing, the values of aerodynamic coefficients are approximated by fitting higher-order polynomials into the experimental data. Subsequently, the governing equations of motion are generated, and are carried to MTS analysis. The closed-form solutions for the governing equations are obtained, and have been then subjected to bifurcation analysis. The analytical solutions have also been shown to be in well agreement with the numerical solutions of the governing differential equations.

2.4 Gaps in Existing Literature and Motivation for the Research

This work is intended to stand at the intersection of Moore-Greitzer Model and Multiple Time Scales method. The compressor surge problem as modeled by the Moore-Greitzer Model has not yet been addressed by MTS method. Although a considerable number of studies on Moore-Greitzer Model have been involving methods of nonlinear dynamics such as bifurcations, the problem has not been subjected to MTS method till the time of writing this thesis.

Identification of the missing link in present literature was the first motivation for this work. The effort to fill in this blank in the realm of Moore-Greitzer Model was stimulated upon realizing the strengths of MTS method.

To mark the scope of this work, following agenda was finalized:

- Method of Multiple Time Scales shall be applied on the *pure surge* case of Moore-Greitzer Model
- The cubic compressor characteristic curve involved in the Moore-Greitzer Model shall be generalized to study the effect on surge oscillations
- The analytical solution obtained from MTS method shall be carried to Bifurcations analysis for studying qualitative behaviour
- Numerical simulations of the pure surge case of Moore-Greitzer Model shall be performed and compared with the MTS solution for verification

The subsequent chapters in this thesis describe how these agenda points were accomplished. Following schematic diagram shows this work as being on crossroads of the two disciplines.

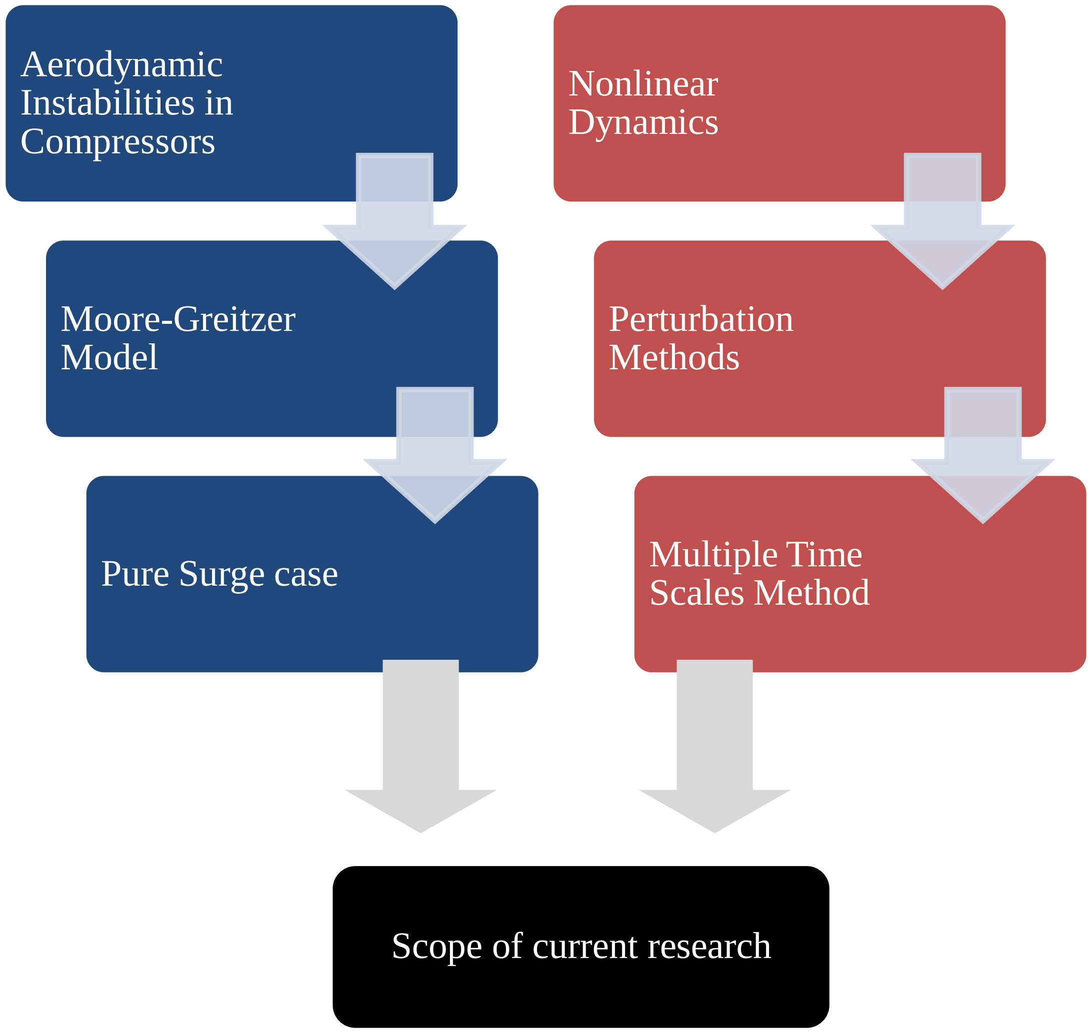

Figure 2.2 Scope of the research

Chapter 3

MOORE-GREITZER MODEL

3.1 Introduction

The Moore-Greitzer model has been proposed and derived in [1] and [2]. It is being reproduced here at length to facilitate the reader. It should, however, be noted that only the Pure Surge case of the Moore-Greitzer will be discussed here, which is pertinent to this work.

3.2 Fluid Dynamic Model

Consider the schematic of the axial compression system in Figure 3.1. It consists of a compressor, a pipe leading the free stream flow into a plenum, a plenum, throttle and its exit duct. The compressor and its ducting are replaced by an actuator disk to account for the pressure-rise C due to the compressor, and a constant area pipe of length L_c, to account for the dynamics of the fluid in the compressor duct.

Similarly, the throttle (which in practice can just be a variable area annular nozzle) is also replaced by this combination of actuator disk (across which the pressure-drop F exists) plus a constant area duct having length L_t.

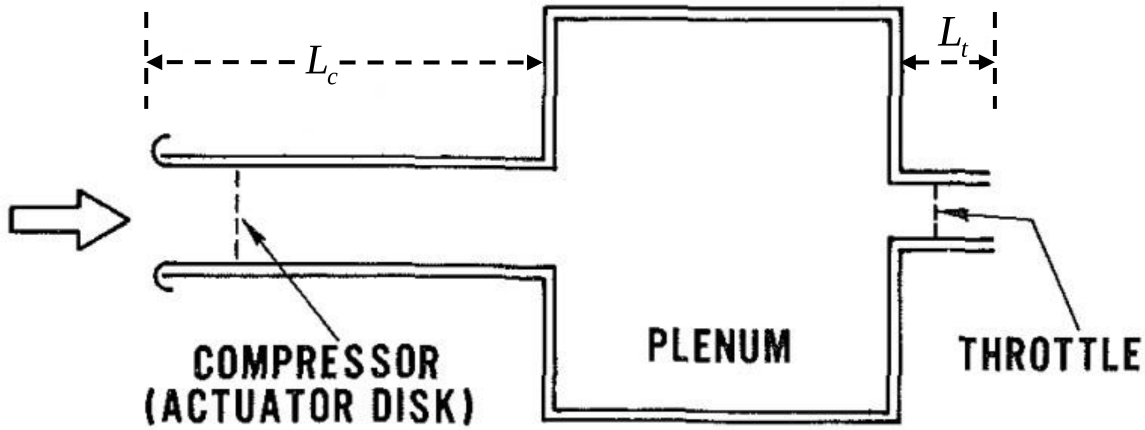

Figure 3.1 The Control Volume for the Compression System (from [7])

3.3 Assumptions

The following assumptions must be kept in mind before proceeding to the derivation of the model. A situation, in which any of these assumptions is no more valid, can bring the results from the model in question.

(i) The oscillations occurring in the system have been modeled in a manner analogous to those of a *Helmholtz resonator*. This means that all the kinetic energy of the oscillations is associated with the motion of the fluid in the compressor and throttle ducts, and all the potential energy is associated with the compression of the gas in the plenum.

(ii) The compression system is subjected to low inlet Mach numbers. Thus, pressure rises are small as compared to the ambient pressure. Since the oscillations associated with surge can generally be regarded as having quite low frequency, the flow in the ducts can be considered to be *incompressible*, with the density taken equal to the ambient value. At any instant, therefore, all the fluid particles in one of these ducts will have the same axial velocity, i.e. *fully developed flow*.

(iii) The plenum dimensions are large as compared to those of the compressor duct, so that the velocities and fluid accelerations in the plenum can be considered to be negligible. Hence, the pressure in the plenum can be assumed to be uniform spatially, but varying in time.

(iv) The flow field is assumed to be *irrotational*.

(v) The effects of viscosity have not been taken into account.

(vi) Body forces acting on the control volume have not been considered.

(vii) The compressor characteristic (plot between two flow properties, typically pressure ratio and mass flow rate, as the fluid passes through the compressor at different compressor speeds) is modeled as a cubic curve, and is representative of a *three-stage low speed* axial flow compressor.

(viii) The inertia in the throttle duct has been neglected. This is generally a good assumption since throttle curves are usually steep and the fluctuations in throttle mass flow rate are substantially smaller than those through the compressor.

(ix) The flow is considered to be *one-dimensional*, and only the axial component of velocity is responsible for the transport processes.

3.4 Equations of Motion

Now the equations of mass and momentum conservation shall be applied on the compressor and throttle ducts. On the plenum, only mass conversation principle will be applied, since we have neglected the fluid velocities and accelerations in the plenum in assumption (iii).

3.4.1 *Compressor Duct*

We start our analysis from the compressor duct. Length of the control volume is L_c (see Figure 3.1). The Momentum equation in the axial or *x*-direction can be written as:

$$\rho\frac{\partial u}{\partial t}+\rho\left(u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial P}{\partial x}+\frac{\partial}{\partial x}\left[\lambda\left(\overline{\nabla}.\overline{V}\right)+2\mu\frac{\partial u}{\partial x}\right]+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]+\rho f_x \tag{3.1}$$

In context of the assumptions made in the previous section, Eq (3.1) reduces to:

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial P}{\partial x} \tag{3.2}$$

Or,

$$\rho\frac{du}{dt}=-\frac{dP}{dx} \tag{3.3}$$

Integrating both sides w.r.t. *x*, we get

$$\rho\frac{du}{dt}\left[x\right]_{\text{length of compressor duct}}=-dP \tag{3.4}$$

$$\Rightarrow\rho\frac{dc_x}{dt}L_c=-\left[\left(P_{plenum}-P_{ambient}\right)-C\right] \tag{3.5}$$

where, $c_x\equiv c_x(t)=\text{Axial velocity}\equiv u$

$C\equiv C(\dot{m}_c)=\text{Pressure-rise across compressor}$

Let $\Delta P=P_{plenum}-P_{ambient}$

Eq (3.5) can then be written as:

$$\rho\frac{dc_x}{dt}L_c=-\Delta P+C \tag{3.6}$$

From Eq (3.6),

$$-\Delta P+C=\frac{L_c}{A_c}\frac{d}{dt}\left(\rho A_c c_x\right) \tag{3.7}$$

where A_c is the frontal area of the compressor duct.

Eq (3.7) can be written as:

$$-\Delta P + C = \frac{L_c}{A_c}\frac{d\dot{m}_c}{dt} \tag{3.8}$$

3.4.2 Plenum

Now consider the plenum to be the control volume having frontal cross-sectional area A_p and volume V_p. The mass conservation principle can be expressed for the plenum as:

$$\dot{m}_{\text{plenum}} = \dot{m}_{\text{what comes in}} - \dot{m}_{\text{what goes out}} \tag{3.9}$$

$$\Rightarrow \dot{m}_p = \dot{m}_c - \dot{m}_t \tag{3.10}$$

Or,

$$\dot{m}_c - \dot{m}_t = \rho_p A_p v_p = \rho_p A_p \frac{d}{dt}\left(x_p\right) \tag{3.11}$$

where ρ_p is the density of the gas in the plenum, and x_p is the length of the plenum.

Eq (3.11) can be written as:

$$\dot{m}_c - \dot{m}_t = \rho_p \frac{d}{dt}\left(A_p x_p\right) \tag{3.12}$$

$$\Rightarrow \dot{m}_c - \dot{m}_t = \dot{\rho}_p V_p \tag{3.13}$$

We need Eq (3.13) to be expressed in terms of change in plenum pressure. If the process in the plenum is polytropic, then the density change is related to the change in plenum pressure by the thermodynamic relation as:

$$\dot{\rho}_p = \frac{\rho_p}{kP_p}\dot{P}_p \tag{3.14}$$

where k is the polytropic constant, and it is usually taken equal to the ratio of specific heats, γ , of the gas. Since the overall pressure and temperature ratios of the compression system under study are near unity (assumption (ii)), therefore, $\frac{\rho_p}{P_p} \approx \frac{\rho}{P}$. Substituting this and Eq (3.14), Eq (3.13) becomes:

$$\dot{m}_c - \dot{m}_t = \left(\frac{\rho}{\gamma P} \dot{P} \right) V_p \tag{3.15}$$

We know that the speed of sound in a medium is given as $a_s = \sqrt{\gamma R T} = \sqrt{\gamma \frac{P}{\rho}}$. Eq (3.15) can therefore be written as:

$$\dot{m}_c - \dot{m}_t = \frac{V_p}{a_s^{\,2}} \dot{P} \tag{3.16}$$

3.4.3 *Throttle Duct*

On the lines of compressor duct, we analyze the flow in the throttle duct. Here, the length of the control volume is L_t.

Reducing the momentum equation (Eq (3.1)) in light of the assumptions, and integrating both sides w.r.t. x, we get:

$$\rho \frac{du}{dt} [x]_{\text{length of throttle duct}} = -dP \tag{3.17}$$

$$\Rightarrow \rho \frac{dc_x}{dt} L_t = -\left[\left(P_{ambient} - P_{plenum} \right) + F \right] \tag{3.18}$$

Where, $F \equiv F(\dot{m}_t) =$ Pressure-drop across throttle

Eq (3.18) can be written as:

$$\Delta P - F = \frac{L_t}{A_t}\frac{d\dot{m}_t}{dt} \tag{3.19}$$

where A_t is the frontal area of the throttle duct.

It can be noted that the continuity equation has been incorporated in momentum equation for both the compressor and throttle ducts while reducing it to Eq (3.4) and Eq (3.17) respectively.

3.5 Non-dimensionalizing the Equations

It is important and helpful to non-dimensionalize the equations developed so far. We non-dimensionalize the mass flow rates by dividing them with $\rho A_c U$ and the pressures by dividing them with ρU^2. Here, U is the reference speed of the rotor blades at mean radius. The time variable is non-dimensionalized by multiplying it with $\frac{U}{R}$, where R is the mean radius of the compressor drum. Then, we have:

$$\text{Eq (3.8)} \Rightarrow l_C \frac{d\Phi}{d\xi} = \psi_C - \Psi \tag{3.20}$$

$$\text{Eq (3.16)} \Rightarrow l_C \frac{d\Psi}{d\xi} = \frac{1}{4B^2}\left(\Phi - \Phi_t\right) \tag{3.21}$$

$$\text{Eq (3.19)} \Rightarrow l_C \frac{d\Phi_t}{d\xi} = \frac{1}{G}\left(\Psi - \psi_t\right) \tag{3.22}$$

Here, $l_C \equiv$ Total length of compressor and ducts, in terms of radius $= \frac{L_c}{R}$

$\Phi \equiv \Phi(\xi) \equiv$ Non-dimensional mass flow rate through compressor $= \frac{\dot{m}_c}{\rho A_c U}$

$\xi \equiv$ Non-dimensional time for wheel to rotate one radian $= \frac{Ut}{R}$

$$B \equiv \text{Greitzer's Parameter} = \frac{U}{2\omega_H L_c} = \frac{U}{2a_s}\sqrt{\frac{V_p}{A_c L_c}}$$

$$\left(\text{where, } \omega_H \equiv \text{Helmholtz frequency (assumption (i))} = a_s\sqrt{\frac{A_c}{V_p L_c}} \right)$$

$$\psi_C \equiv \psi_C(\Phi) = \text{Non-dimensional compressor pressure-rise } = \frac{C}{\rho U^2}$$

$$\Psi \equiv \Psi(\xi) \equiv \text{Pressure coefficient} = \frac{\Delta P}{\rho U^2}$$

$$\Phi_t \equiv \Phi_t(\xi) \equiv \text{Non-dimensional mass flow rate through throttle} = \frac{\dot{m}_t}{\rho A_c U}$$

$$G = \frac{L_t A_c}{L_c A_t}$$

$$\psi_t \equiv \psi_t(\Phi_t) = \text{Non-dimensional throttle pressure-drop } = \frac{F}{\rho U^2}$$

It can be noted that the non-dimensional compressor mass flow rate $\Phi(\xi) = \frac{\dot{m}_c}{\rho A_c U} = \frac{\rho A_c c_x}{\rho A_c U} = \frac{c_x}{U}$ is the axial *flow coefficient* of the compressor.

The non-dimensional parameter B is called the *Greitzer's parameter*. It can be thought of as a measure of the ratio of pressure forces to inertial forces for a given rate of change of mass flow rate. High values of B tend to be associated with surge, while low values tend to lead to rotating stall [1].

By assumption (viii), $\frac{d\Phi_t}{d\xi} = 0$.

$$\therefore \text{Eq (3.22)} \Rightarrow \Psi = \psi_t \qquad (3.23)$$

This approximation will help us in simplifying Eq (3.21), as shown in the next section.

3.6 Compressor and Throttle Characteristics

Up till now, nothing has been said about the nature of compressor and throttle characteristics, except that they are functions of the compressor and throttle mass flow rates, respectively. These characteristics are case-specific, and are dependent on the specifications of the engine or compression system under study. Compressor characteristic, in particular, is decisive in determining the behavior of compressor during instability.

3.6.1 Compressor Characteristic (ψ_C)

Compressor pressure rise is generally a nonlinear function of compressor mass flow rate, i.e. $\psi_C = \psi_C(\Phi)$. It is determined from experimental data for different speeds (rpm) through curve-fitting or regression analysis.

The axisymmetric compressor characteristic used by Moore and Greitzer is obtained from experimental data of a three-stage low speed compressor [1]. It is an S-shaped curve involving a cubic nonlinear term in it:

$$\psi_C \equiv \psi_C(\Phi) = \psi_{C_o} + H\left[1 + \frac{3}{2}\left(\frac{\Phi}{W} - 1\right) - \frac{1}{2}\left(\frac{\Phi}{W} - 1\right)^3\right] \qquad (3.24)$$

The meanings of the various parameters in Eq (3.24) are elaborated in Figure 3.2 below.

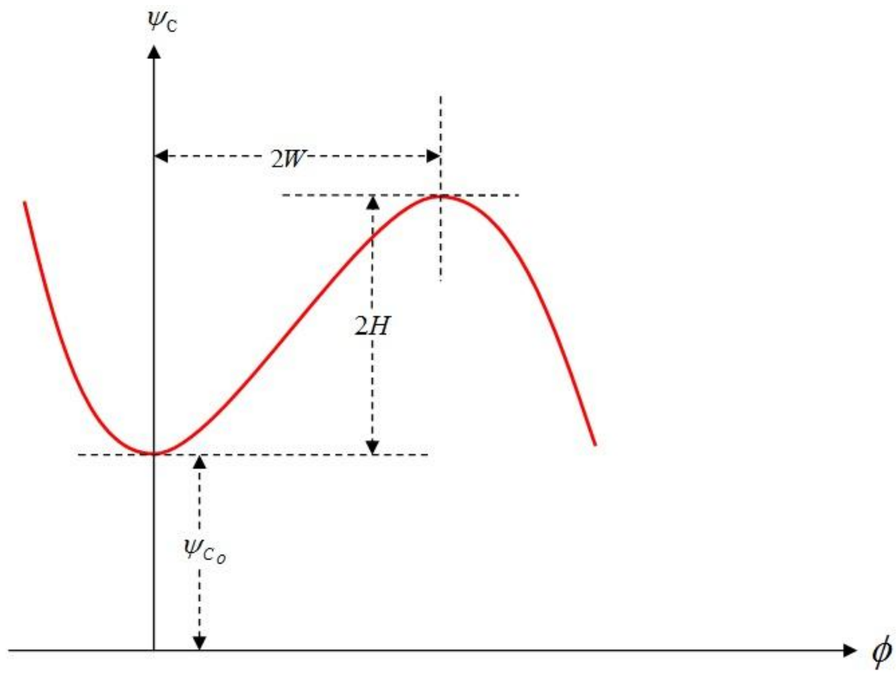

Figure 3.2 Compressor Characteristic

3.6.2 Throttle Characteristic (ψ_t)

Generally, throttle characteristics are parabolic or linear as opposed to the compressor characteristics. Throttle ducts are considerably shorter than compressor ducts, and the throttle flow coefficient Φ_t is also taken as plenum discharge coefficient. This is quite good assumption for realistic throttle ducts.

If we assume the throttle characteristic to be *linear* [1], it is given as:

$$\psi_t \equiv \psi_t(\Phi_t) = \overline{\Psi} + k_T\left(\Phi_t - \overline{\Phi}\right) \tag{3.25}$$

where the bars represent the time-averaged values and k_T is the slope of compressor characteristic.

From Eq (3.23) and Eq (3.25), we get

$$\Phi_t = \overline{\Phi} + \frac{\Psi - \overline{\Psi}}{k_T} \tag{3.26}$$

From Eq (3.21) and Eq (3.26), we have

$$l_C \frac{d\Psi}{d\xi} = \frac{1}{4B^2}\left[\Phi - \overline{\Phi} - \frac{1}{k_T}\left(\Psi - \overline{\Psi}\right)\right] \tag{3.27}$$

Eq (3.20) and Eq (3.27), along with Eq (3.24) for compressor characteristic, are the governing equations for pure surge in Moore-Greitzer Model.

3.7 Governing ODE for Surge

Eq (3.20) and Eq (3.27) can be combined to give the following *second-order nonlinear* ordinary differential equation [1]:

$$\boxed{\frac{d^2\Phi}{d\xi^2} + \frac{1}{l_C}\left(\frac{1}{4B^2 k_T} - \frac{d\psi_C}{d\Phi}\right)\frac{d\Phi}{d\xi} + \frac{1}{4l_C^2 B^2}\left[\left(\Phi - \overline{\Phi}\right) - \frac{\Psi - \overline{\Psi}}{k_T}\right] = 0} \tag{3.28}$$

This equation is the subject of subsequent work and discussion in this thesis. The governing equation for a mass-spring-damper system is given as:

$$\ddot{x} + c\dot{x} + kx = 0 \tag{3.29}$$

Comparing Eq (3.28) with Eq (3.29), it can be seen that the first term on the L.H.S. is the inertial term. The second term is the damping term, and the last term is the restoring term of the system. It is clear that the main part of the *damping* in the system is due to the axisymmetric compressor characteristic ψ_C [1].

Hence, the governing differential equation for surge phenomena is analogous to that describing the behaviour of a second-order mass-spring-damper system with nonlinear damping.

3.8 Concluding Remarks

The Moore-Greitzer model has been derived from the first principles for the pure surge case as Eq (3.28). This is a second-order nonlinear ODE with flow coefficient (non-dimensional mass flow rate) as the dependent variable and (non-dimensional) time as the independent variable. The following analysis in this work is based on this equation.

Chapter 4

PROBLEM FORMULATION

4.1 Introduction

In this chapter, the properties of compressor characteristic for Moore-Greitzer Model have been analyzed, and prepare the governing equation, Eq (3.28) developed in the previous chapter, for application of MTS method. The compressor characteristic is generalized to study the effects for variation in different parameters.

4.2 Simplifications

Consider the Eq (3.28). As stated earlier, the throttle mass excursion and length are significantly smaller than the compressor length, and the throttle slopes are generally steep in nature. Therefore, as a first simplification, we take $k_T \to \infty$ [1]. Now, Eq (3.28) takes the form:

$$\frac{d^2\Phi}{d\xi^2} + \frac{1}{l_C}\left(-\frac{d\psi_C}{d\Phi}\right)\frac{d\Phi}{d\xi} + \frac{1}{4l_C^2B^2}\left(\Phi - \overline{\Phi}\right) = 0 \qquad (4.1)$$

Looking at the above equation, we can see that $\overline{\Phi}$ is a constant, and $\frac{1}{4l_C^2B^2}\overline{\Phi}$ is a non-homogeneous term. It does not affect the properties of the solution qualitatively. It merely shifts the solution on the flow coefficient (Φ) axis by an amount equal to $\frac{\overline{\Phi}}{4l_C^2B^2}$. Therefore, shifting the mean value $\overline{\Phi}$ to the origin of the axes, Eq (4.1) can be expressed as:

$$\frac{d^2\Phi}{d\xi^2}+\frac{\Phi}{4l_C^2B^2}=\frac{1}{l_C}\frac{d\psi_C}{d\Phi}\frac{d\Phi}{d\xi} \qquad (4.2)$$

As described earlier, the R.H.S. of the equation represents the system damping [1], where $\frac{d\psi_C}{d\Phi}$ is the slope of compressor characteristic.

4.3 Generalizing the Compressor Characteristic

The axisymmetric compressor characteristic used in the Moore-Greitzer Model has been explained in Section 3.6.1. We generalize the compressor characteristic, of which Eq (3.24) becomes a special case, as follows:

$$\psi_C(\Phi)=\psi_{C_o}+H\left[1+\alpha\left(\frac{\Phi}{W}-1\right)+\beta\left(\frac{\Phi}{W}-1\right)^3\right] \qquad (4.3)$$

In Moore-Greitzer Model (compare Eq (4.3) with Eq (3.24)), $\alpha=\frac{3}{2}$ and $\beta=-\frac{1}{2}$. It should be noted that the parameters α and β vary for each compressor, and are prone to uncertainty. In Moore-Greitzer Model, α and $\beta=-\frac{1}{2}$ have been obtained experimentally through regression analysis for a three-stage low speed compression system.

In this work, these two parameters are treated in a more generalized way and a closed form expression is derived based on these generalizations. The variation of the characteristic curve subject to parametric variation of α and β is shown in Figure 4.1a & b. it will be shown in the later part of this work that under small variations, the generic shape of the characteristic curve is preserved, however; the onset, amplitude and frequency of the surge oscillations can dramatically change.

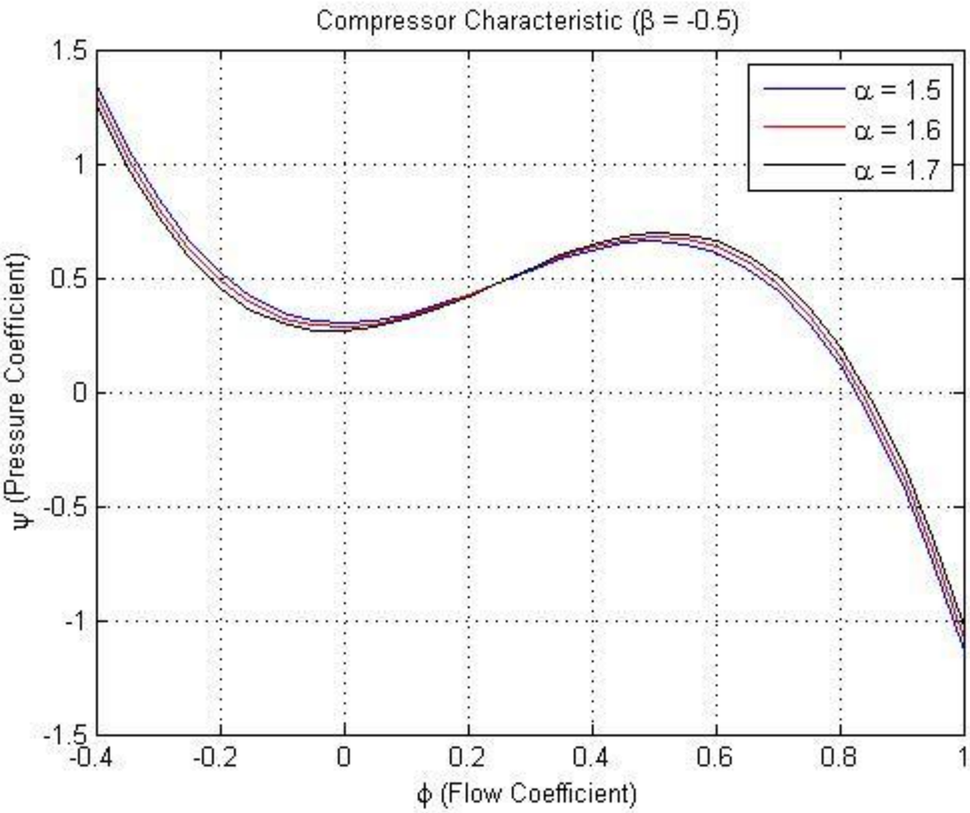

Figure 4.1a Effect of varying α on Compressor Characteristic Curve

It can be seen that the parameter α controls the spread of the curve on the horizontal axis. The value of β is held fixed at $\beta = -\frac{1}{2}$.

The next figure shows the effect of varying β on the compressor characteristic. Here, the value of α has been held constant at α = 3/2.

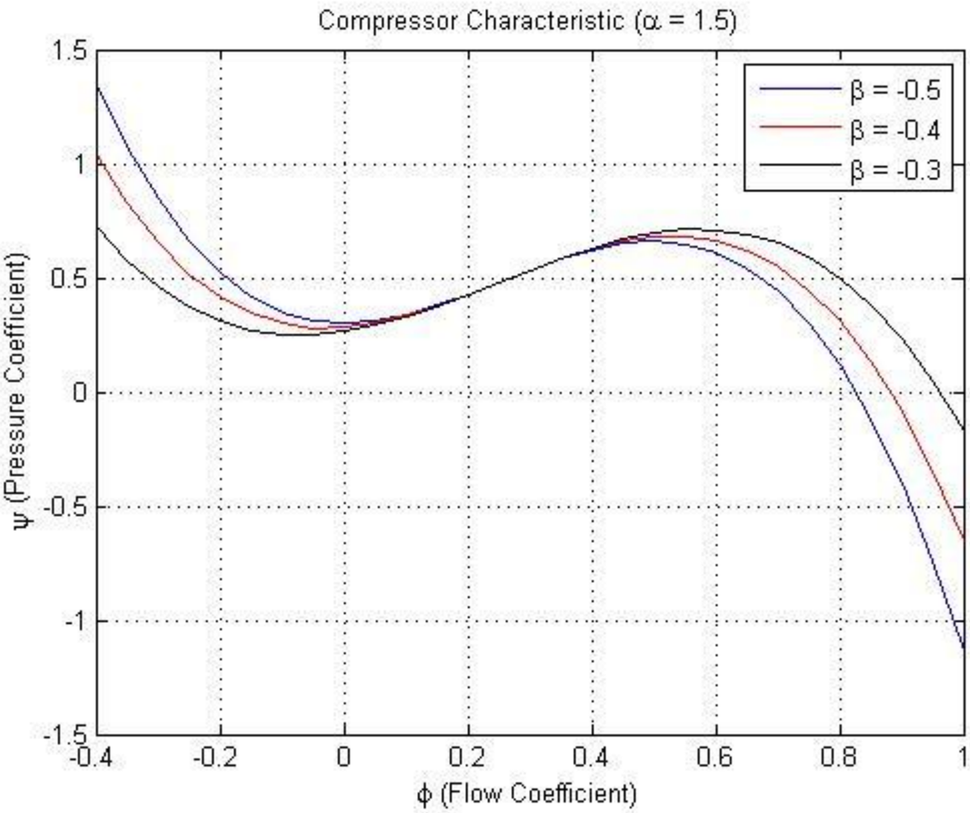

Figure 4.1b Effect of varying β on Compressor Characteristic Curve

It is evident that β has a more pronounced effect on the shape of compressor characteristic as compared to α .

4.4 Development of Governing Equation

We start our analysis from differentiating the compressor characteristic w.r.t. flow coefficient. From Eq (4.3), we get

$$\frac{d\psi_C}{d\Phi} = \left(\frac{3\beta H}{W^3}\right)\Phi^2 + \left(\frac{-6\beta H}{W^2}\right)\Phi + \frac{H}{W}(\alpha + 3\beta) \tag{4.4}$$

It can be noted that the last term goes to zero if $\alpha = \frac{3}{2}$ and $\beta = -\frac{1}{2}$ are substituted (as proposed in the Moore-Greitzer Model). Substituting $\frac{d\psi_C}{d\Phi}$ from Eq (4.4) into Eq (4.2):

$$\frac{d^2\Phi}{d\xi^2} + \frac{1}{4l_C^2 B^2}\Phi = \frac{1}{l_C}\left\{\left(\frac{3\beta H}{W^3}\right)\Phi^2 + \left(\frac{-6\beta H}{W^2}\right)\Phi + \frac{H}{W}\left(\alpha + 3\beta\right)\right\}\frac{d\Phi}{d\xi} \tag{4.5}$$

Eq (4.5) can be expressed in the following convenient and concise form:

$$\boxed{\ddot{\Phi} + \omega^2\Phi = K_1\Phi^2\dot{\Phi} + K_2\Phi\dot{\Phi} + K_3\dot{\Phi}} \tag{4.6}$$

where $\omega = \frac{1}{2l_C B}$, $K_1 = \frac{3\beta H}{l_C W^3}$, $K_2 = \frac{-6\beta H}{l_C W^2}$ and $K_3 = \frac{H}{l_C W}(\alpha + 3\beta)$. It can again be noted that $K_3 = 0$ if $\alpha = \frac{3}{2}$ and $\beta = -\frac{1}{2}$.

Eq (4.6) is the governing equation for the subsequent surge analysis based on the generalized compressor characteristic curve. We shall study the dynamics of surge oscillations modeled by this equation. This is a second-order homogeneous ODE, involving nonlinearities in the dependent variable Φ.

4.5 Concluding Remarks

Eq (4.6) is a simplified form of Eq (3.28), and is suitable for MTS analysis. The next chapter elaborates the application of MTS method on this problem, along with the discussion on qualitative nature of the solution.

Chapter 5

SURGE ANALYSIS

5.1 Introduction

The governing equation for the surge behaviour of axial compression systems has been developed in the previous chapter as Eq (4.6). In this chapter, the said equation is solved by Method of Multiple Time Scales (MTS) to reach at a closed-form solution. The solution is then carried to Bifurcations analysis for qualitative study in later part of the chapter.

5.2 Application of MTS Method

During the surge oscillations, a compression system is operating at off-design conditions, and at the margins of its stability zone. Since at the point of critical stability, the average or effective damping of the system must vanish for the existence of oscillations [9], we parameterize the damping term (R.H.S.) of Eq (4.6) in terms of a small perturbation parameter to carry out the MTS analysis:

$$\ddot{\Phi} + \omega^2 \Phi = \varepsilon \left(K_1 \Phi^2 \dot{\Phi} + K_2 \Phi \dot{\Phi} + K_3 \dot{\Phi} \right) \tag{5.1}$$

where $0 < \varepsilon << 1$. The MTS method is now invoked. Two time scales are selected in this analysis and, therefore, the non-dimensional time ξ is expanded:

$$\xi \rightarrow \{\tau_o, \tau_1\} \text{ where } \tau_o = \xi \text{ and } \tau_1 = \varepsilon \xi \tag{5.2}$$

The dependent variable Φ is now expanded in the following manner:

$$\Phi(\xi) = \Phi_o(\tau_o, \tau_1) + \varepsilon \Phi_1(\tau_o, \tau_1) + O(\varepsilon^2) \tag{5.3}$$

By derivative expansion method [24]:

$$\frac{d}{d\xi} = \frac{\partial}{\partial \tau_o} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + ...,$$

$$\frac{d^2}{d\xi^2} = \frac{\partial}{\partial \tau_o}\left(\frac{\partial}{\partial \tau_o} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + ...\right) + \varepsilon \frac{\partial}{\partial \tau_1}\left(\frac{\partial}{\partial \tau_o} + \varepsilon \frac{\partial}{\partial \tau_1} + \varepsilon^2 \frac{\partial}{\partial \tau_2} + ...\right) + ... \text{ and so on.}$$

Substituting the first and second-order derivatives in Eq (5.1), we get:

$$\left(\frac{\partial^2}{\partial \tau_o{}^2} + \varepsilon \frac{\partial^2}{\partial \tau_o \partial \tau_1} + \varepsilon \frac{\partial^2}{\partial \tau_1 \partial \tau_o}\right)(\Phi_o + \varepsilon \Phi_1) + \omega^2 (\Phi_o + \varepsilon \Phi_1) = \varepsilon K_1 \left(\Phi_o{}^2 + \varepsilon^2 \Phi_1{}^2 + 2\varepsilon \Phi_o \Phi_1\right)$$

$$\left(\frac{\partial}{\partial \tau_o} + \varepsilon \frac{\partial}{\partial \tau_1}\right)(\Phi_o + \varepsilon \Phi_1) + \varepsilon K_2 (\Phi_o + \varepsilon \Phi_1)\left(\frac{\partial}{\partial \tau_o} + \varepsilon \frac{\partial}{\partial \tau_1}\right)(\Phi_o + \varepsilon \Phi_1) + \varepsilon K_3 \left(\frac{\partial}{\partial \tau_o} + \varepsilon \frac{\partial}{\partial \tau_1}\right)(\Phi_o + \varepsilon \Phi_1) \tag{5.4}$$

(Here, the Order symbol $O(\varepsilon^2)$ and dependence of Φ on τ_o and τ_1 has not been explicitely shown for notational simplicity)

Applying the differentiation and equating the terms: (Assuming Φ_o to be a <u>continuous</u> function, $\therefore \frac{\partial^2 \Phi_o}{\partial \tau_o \partial \tau_1} = \frac{\partial^2 \Phi_o}{\partial \tau_1 \partial \tau_o}$)

$$\varepsilon^0 : \quad \frac{\partial^2 \Phi_o}{\partial \tau_o{}^2} + \omega^2 \Phi_o = 0 \tag{5.5}$$

$$\varepsilon^1 : \quad \frac{\partial^2 \Phi_1}{\partial \tau_o{}^2} + \omega^2 \Phi_1 = K_1 \Phi_o{}^2 \frac{\partial \Phi_o}{\partial \tau_o} + K_2 \Phi_o \frac{\partial \Phi_o}{\partial \tau_o} + K_3 \frac{\partial \Phi_o}{\partial \tau_o} - 2\frac{\partial^2 \Phi_o}{\partial \tau_o \partial \tau_1} \tag{5.6}$$

5.2.1 Solving zeroth-order Perturbation

The characteristic equation for Eq (5.5) is:

$$\lambda^2 + \omega^2 = 0 \tag{5.7}$$

$$\Rightarrow \lambda = \pm \omega\, i \quad \text{(complex conjugate roots)} \tag{5.8}$$

Thus, the solution of Eq (5.7) can be expressed as:

$$\Phi_o(\tau_o, \tau_1) = A(\tau_1)\sin\eta \qquad (5.9)$$

where $\eta(\tau_o, \tau_1) = \omega\tau_o + P(\tau_1)$

Here, A is the *amplitude*, η is *phase angle* and P is *phase correction* of the solution. It can be seen that the amplitude and phase of the solution vary with the slow time scale τ_1. Once these variations are known, the zeroth order approximation to the flow coefficient dynamics is complete.

5.2.2 Solving first-order Perturbation

$$\because \Phi_o = A(\tau_1)\sin\eta \qquad \text{where } \eta = \omega\tau_o + P(\tau_1)$$

$$\therefore \frac{\partial\Phi_o}{\partial\tau_o} = \omega A\cos\eta \qquad (5.10)$$

$$\text{and } \frac{\partial^2\Phi_o}{\partial{\tau_o}^2} = \omega\frac{dA}{d\tau_1}\cos\eta - \omega A\frac{dP}{d\tau_1}\sin\eta \qquad (5.11)$$

From Eq (5.9), (5.10) and (5.11): (Again, the dependence of variables on the time scales has not been shown for brevity)

$$\text{Eq (5.6)} \Rightarrow \frac{\partial^2\Phi_1}{\partial{\tau_o}^2} + \omega^2\Phi_1 = K_1\left(A^2\sin^2\eta\right)\left(\omega A\cos\eta\right) + K_2\left(A\sin\eta\right)\left(\omega A\cos\eta\right) + K_3\left(\omega A\cos\eta\right)$$

$$-2\omega\frac{dA}{d\tau_1}\cos\eta + 2\omega A\frac{dP}{d\tau_1}\sin\eta \qquad (5.12)$$

$$= K_1\omega A^3\sin^2\eta\cos\eta + K_2\omega A^2\sin\eta\cos\eta + K_3\omega A\cos\eta - 2\omega\frac{dA}{d\tau_1}\cos\eta + 2\omega A\frac{dP}{d\tau_1}\sin\eta \qquad (5.13)$$

$$= \frac{K_1\omega A^3}{2}\sin\eta\left(2\sin\eta\cos\eta\right) + \frac{K_2\omega A^2}{2}\left(2\sin\eta\cos\eta\right) + K_3\omega A\cos\eta - 2\omega\frac{dA}{d\tau_1}\cos\eta \; + 2\omega A\frac{dP}{d\tau_1}\sin\eta \tag{5.14}$$

$$= \frac{K_1\omega A^3}{2}\sin\eta\sin 2\eta + \frac{K_2\omega A^2}{2}\sin 2\eta + K_3\omega A\cos\eta - 2\omega\frac{dA}{d\tau_1}\cos\eta + 2\omega A\frac{dP}{d\tau_1}\sin\eta \tag{5.15}$$

$$= \frac{K_1\omega A^3}{2}\left(\frac{\cos\eta - \cos 3\eta}{2}\right) + \frac{K_2\omega A^2}{2}\sin 2\eta + K_3\omega A\cos\eta - 2\omega\frac{dA}{d\tau_1}\cos\eta + 2\omega A\frac{dP}{d\tau_1}\sin\eta \tag{5.16}$$

$$\Rightarrow \frac{\partial^2\Phi_1}{\partial\tau_o{}^2} + \omega^2\Phi_1 = \left(\frac{K_1\omega A^3}{4} + K_3\omega A - 2\omega\frac{dA}{d\tau_1}\right)\cos\eta + 2\omega A\frac{dP}{d\tau_1}\sin\eta - \frac{K_1\omega A^3}{4}\cos 3\eta + \frac{K_2\omega A^2}{2}\sin 2\eta \tag{5.17}$$

If the coefficients of first harmonic terms on the right hand side of the Eq (5.17) are non-zero, *secular terms* will appear and collapse the uniformity of the solution. The secular terms are of the form $\tau_o\cos\eta$ and $\tau_o\sin\eta$, and are unbounded as τ_o approaches infinity. Therefore, the terms on the R.H.S. involving $\cos\eta$ and $\sin\eta$ must be put equal to zero in order to maintain the uniformity of the solution. This obtains:

$$\Rightarrow \frac{dA}{d\tau_1} = \frac{K_1A^3}{8} + \frac{K_3A}{2} \tag{5.18}$$

$$\text{and, } \frac{dP}{d\tau_1} = 0 \tag{5.19}$$

We now need to solve these two in order to get the flow coefficient (Eq (5.9)). But first, we see the qualitative information contained in these equations by bifurcation analysis.

5.3 Bifurcation Analysis

Let $\zeta = \frac{K_1}{8}$ and $\sigma = \frac{K_3}{2}$. Then, Eq (5.18) takes the form:

$$\frac{dA}{d\tau_1} = \zeta A^3 + \sigma A \tag{5.20}$$

From Eq (5.19), it can be noted that the phase angle variable is *not* a function of τ_1; implying there is a constant magnitude of phase correction in the phase equation.

The equilibria for the amplitude equation (Eq (5.20)) exist at $A = 0$ and $A = \sqrt{\frac{-\sigma}{\zeta}}$. These equilibria predict the stability properties of the solution. The equilibria consists of the $\sigma-$axis and the parabola $\sigma = -\zeta A^2$. These equilibria are plotted in Figure 5.1 for the values of $\zeta < 0$. It should be noted that $\zeta > 0$ is not possible because $\zeta = (3\beta H)/(8l_C W^3)$ and β cannot have positive value.

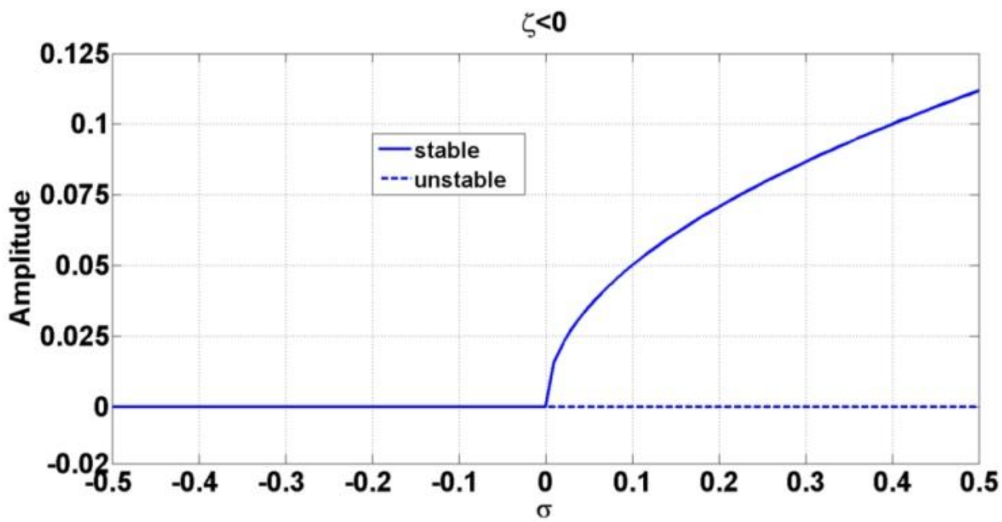

Figure 5.1 Equilibria for Amplitude Equation for $\zeta < 0$

This figure implies that finite amplitude *Limit Cycle Oscillations* (LCOs) appear and disappear in the system as σ is varied across $\sigma = 0$. This phenomenon is termed as *Hopf Bifurcation*. In fact for $\zeta < 0$, it is *supercritical* Hopf bifurcation, since the new branch of equilibria exists only for the values of σ which are larger than that at the onset of bifurcation.

Furthermore, the limit cycle is stable only when $\zeta < 0$. Thus, the amplitude of sustained oscillations is given by:

$$A = \sqrt{\frac{-\sigma}{\zeta}} \tag{5.21}$$

The equations for amplitude and phase correction are now solved to get the complete transient amplitude of the surge oscillations.

5.4 Solving the Amplitude and Phase Equations

Consider the amplitude equation given as Eq (5.20). Separating the variables, we get:

$$d\tau_1 = \frac{dA}{A\left(\zeta A^2 + \sigma\right)} \tag{5.22}$$

By Partial Fraction Expansion:

$$\Rightarrow d\tau_1 = \frac{dA}{\sigma A} - \left[\frac{\zeta A}{\sigma\left(\zeta A^2 + \sigma\right)}\right] dA \tag{5.23}$$

Integrating both sides, we get

$$\Rightarrow \tau_1 = \frac{1}{\sigma}\ln|A| - \frac{1}{2\sigma}\ln\left|\zeta A^2 + \sigma\right| + C_1 \tag{5.24}$$

By laws of logarithms:

$$\Rightarrow 2\sigma\tau_1 = \ln\left|\frac{A^2}{\zeta A^2 + \sigma}\right| + C_2 \tag{5.25}$$

where $C_2 = 2\sigma C_1$. From Eq (5.25):

$$e^{2\sigma\tau_1} = e^{\ln\left|\frac{A^2}{\zeta A^2 + \sigma}\right|} C = \frac{A^2 C}{\zeta A^2 + \sigma} \tag{5.26}$$

where $C = e^{C_2}$

From Eq (5.26):

$$\Rightarrow A^2 = \frac{e^{2\sigma\tau_1}}{C}\left(\zeta A^2 + \sigma\right) \tag{5.27}$$

$$\Rightarrow A^2\left(1 - \frac{\zeta e^{2\sigma\tau_1}}{C}\right) = \frac{\sigma e^{2\sigma\tau_1}}{C} \tag{5.28}$$

$$\Rightarrow A^2\left(\frac{C - \zeta e^{2\sigma\tau_1}}{C}\right) = \frac{\sigma e^{2\sigma\tau_1}}{C} \tag{5.29}$$

$$\Rightarrow A^2 = \frac{\sigma e^{2\sigma\tau_1}}{C}\left(\frac{C}{C - \zeta e^{2\sigma\tau_1}}\right) = \frac{\sigma C e^{2\sigma\tau_1}}{C^2 - \zeta C e^{2\sigma\tau_1}} \tag{5.30}$$

$$\Rightarrow \boxed{A(\tau_1) = \sqrt{\frac{\sigma C e^{2\sigma\tau_1}}{C^2 - \zeta C e^{2\sigma\tau_1}}}} \tag{5.31}$$

where C is a constant that will be determined by the initial conditions.

The solution of phase correction equation can be written as:

$$\boxed{P(\tau_1) = K} \tag{5.32}$$

As stated earlier, phase correction has a constant magnitude in the phase equation.

5.5 Discussion

From the closed form analytical solution for the amplitude of surge oscillations (Eq (5.31)), it can be deduced that ζ and σ are the key parameters that affect the surge dynamics in terms of the onset and the appearance of limit cycles. Once σ is expressed in terms of more intuitive quantities, it becomes:

$$\sigma = \frac{H\left(\alpha + 3\beta\right)}{l_C W} \tag{5.33}$$

And, ζ is expressed as:

$$\zeta = \frac{3\beta H}{8 l_C W^3} \tag{5.34}$$

The scope of this work is to study the behavioral change in surge oscillations as a result of change in compressor characteristic, represented by α and β. It should be noted that since all other parameters in Eq (5.33) and Eq (5.34) have positive sign, therefore, the signs of ζ and σ are explicitly dependent on the nature of compressor characteristic curve. In order to preserve the qualitative shape of the compressor characteristic, the value of β should always be negative as elaborated in Figure 4.1 as well.

5.6 Concluding Remarks

The governing equation for the surge problem has been solved using method of Multiple Time Scales in this chapter. The qualitative nature of the solution has been studied through Bifurcation theory. It has been shown that the sustained limit cycle oscillations exhibit supercritical Hopf bifurcation. Further, the occurrence of limit cycle (surge) is subject to positive sign of σ, and that occurs only when the condition $(\alpha + 3\beta) > 0$ is satisfied.

Chapter 6

VERIFICATION STUDIES

6.1 Introduction

The solution of the surge problem for axial flow compressors has been obtained in the previous chapter in closed-form using MTS method. The present chapter is dedicated to the verification studies carried out through numerical analysis, in order to cross-check the validity and accuracy of the analytical solution obtained through MTS method.

6.2 Numerical Solution of Surge Equation

We need to numerically solve Eq (4.6), in order to approximate the accuracy of analytical results obtained in Eq (5.31) and Eq (5.32). To solve Eq (4.6), a set of two *initial conditions* is required. For flow coefficient Φ, initial conditions are given in [27] as:

$$\Phi(0) = 0.5 \quad (6.1a)$$

$$\dot{\Phi}(0) = 0 \quad (6.1b)$$

For the numerical simulations, MATLAB®'s differential equation solver based on classical fourth-order Runge-Kutta method is used to integrate the governing equation of surge (Eq (4.6)). The values of other parameters also taken from [27] are:

$$H = 0.18 \quad (6.2a)$$

$$W = 0.25 \quad (6.2b)$$

$$l_C = 8 \quad (6.2c)$$

$$B = 1 \quad (6.2d)$$

6.3 Comparison with Numerical Simulations

A simulation of surge phenomena was carried out and was compared with the analytical solution (Eq (5.9), (5.31) and (5.32)) obtained through MTS method. Figure 6.1 shows the results of the simulation.

It is evident that the analytical solution is in well agreement with the numerical solution. It should be noted that there is some discrepancy in the transient behavior but an excellent agreement for the steady state dynamics demonstrates the adequacy of the analytical model derived in this work.

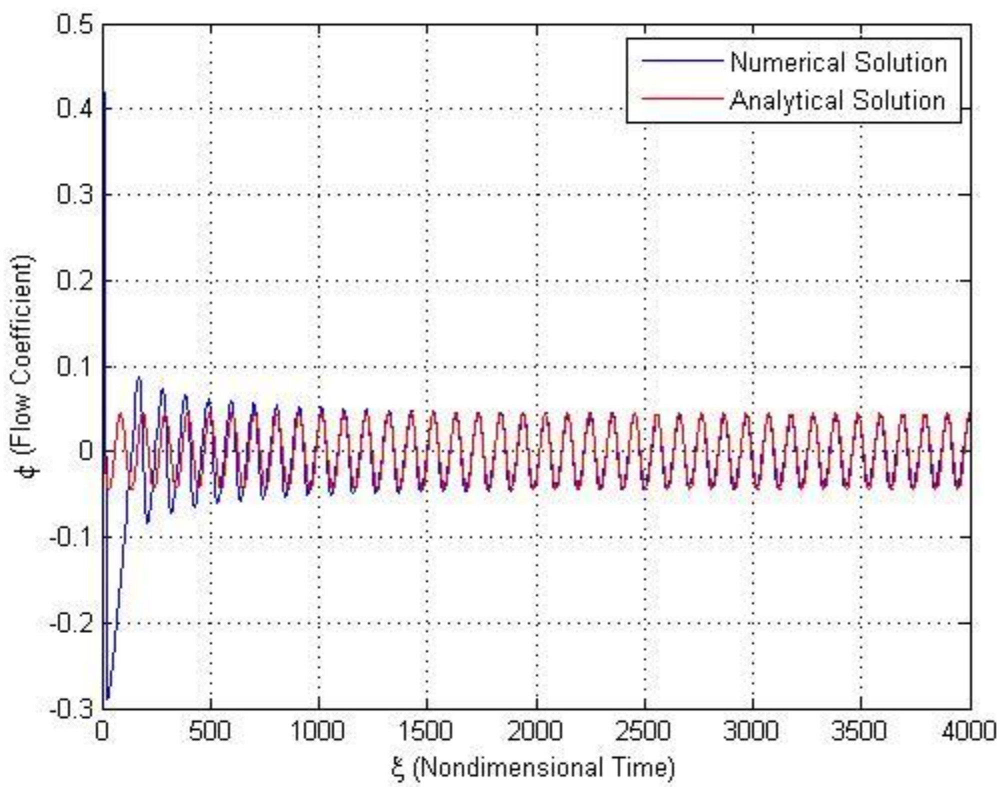

Figure 6.1 Comparison of Analytical and Numerical Solution

The validity of the solution at larger times is the edge that Multiple Time Scales method has over the other perturbation methods, and the same is obvious from the above figure. The phase plot for the governing equation for surge is given below, in which the presence of isolated closed paths in the vicinity of the initial conditions imply the existence of limit cycle oscillations (LCOs).

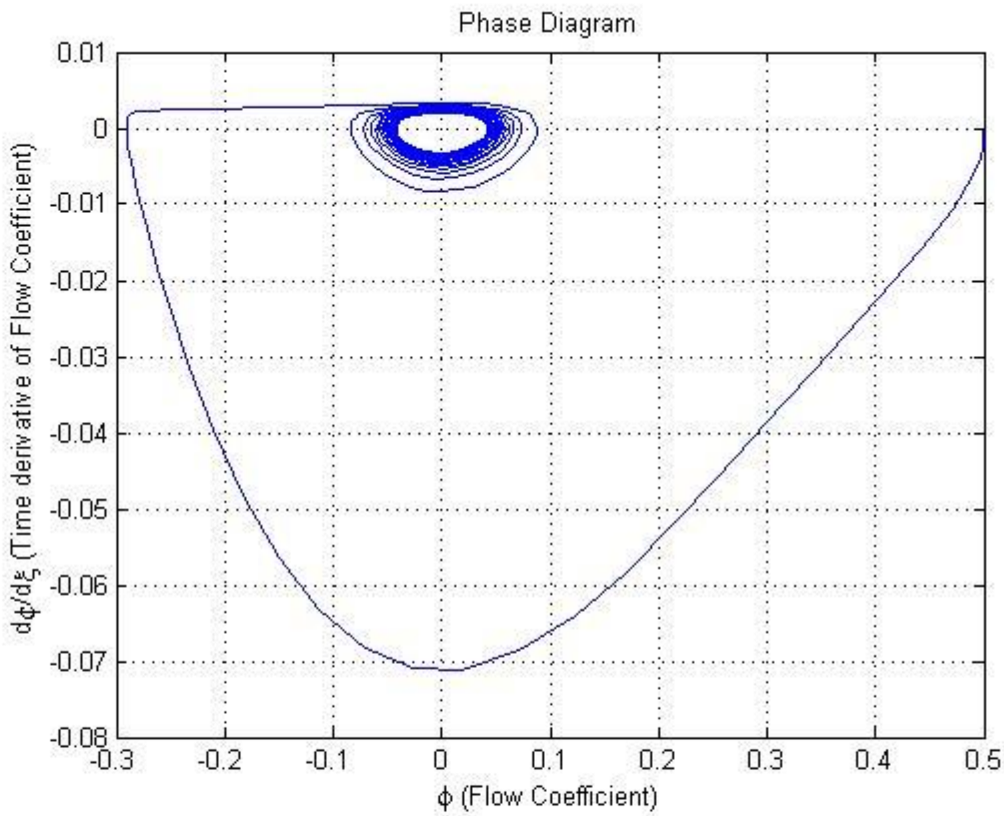

Figure 6.2 Phase Plot for Eq (4.6) with initial conditions Eq (6.1)

6.4 Concluding Remarks

The numerical simulations carried out for ratification of the closed-form solution obtained through MTS method show good agreement between the analytical and numerical solutions. The validity of the analytical solution is thus established, and MTS method has been found to be adequate in handling this class of problems.

Chapter 7

CONCLUSIONS AND FUTURE WORK

7.1 Introduction

This chapter highlights the conclusions drawn from this research. The results are presented, and recommendations for the future work are discussed.

7.2 Conclusions

In this work, the effect of compressor characteristic on surge phenomena in axial flow compressors is analyzed in detail. The generalized effect of compressor characteristic under cubic nonlinearity on the behavior of Limit Cycle Oscillations has been considered. Using the Multiple Time Scales method, approximate solutions are obtained. From these solutions, stability criterion and the necessary condition for sustained limit cycle oscillations is derived. Salient results concluded from this research are listed as under:

(i) The behavior of surge oscillations is significantly dependent on the choice of compressor characteristic. By treating the compressor characteristic used in the Moore-Greitzer Model in a generic manner, it has been found that a small change in α and β effects the dynamics of the system significantly.

(ii) From combination of various parameters involved in the Moore-Greitzer model, two new parameters ζ and σ have been obtained. The analysis shows that these parameters are critical in deciding the nature of surge oscillations. Surge manifests itself as a stable (sustained) Limit Cycle Oscillation when the condition $(\alpha + 3\beta) > 0$ is satisfied.

(iii) The amplitude of surge oscillations exhibit supercritical Hopf bifurcation for the stable limit cycle.

(iv) Method of Multiple Time Scales has successfully captured the qualitative and quantitative aspects of the surge phenomena. The analytical solution obtained via MTS method is well in agreement with the numerical solution.

The crux of this work is realization of the dependence of surge phenomena on the compressor characteristic. In the light of this research, it can be said that a compressor should be designed with such a compressor characteristic for which the condition(s) for stable Limit Cycle Oscillations to exist is (are) stringent, or ruled out if possible.

7.3 Future Work

Following horizons for future research are proposed in the light of this work:

- MTS method may be extended towards the complete Moore-Greitzer Model in contrast with pure surge case only, as worked out in this thesis. Other analytical models for the compression system instability can be explored and made subject to MTS method.
- Future research works may attempt to solve the same problem of pure surge case of Moore-Greitzer Model with other analytical techniques, like Homotopy Analysis Method and Homotopy Perturbation Method. Their results may be compared with the results presented in this thesis. If done, then the present problem of compression system instability may become one of the benchmarks in studying the relative strengths and weaknesses of available analytical methods for solving differential equations.

- The control of surge in compressors can be oriented on the lines as stressed in this work. The control systems may be designed to detect the inception of surge as dictated by the stability condition(s) for the sustained Limit Cycle Oscillations based on the compressor characteristic curve.

It is hoped that this work shall provide some useful guidelines to the turbomachinery designers during conceptual design phase of an axial flow compressor, in improving performance during off-design operations and avoiding surge associated problems.

REFERENCES

1. Moore, F.K. and E.M. Greitzer, *A Theory of Post-Stall Transients in Multistage Axial Compression Systems*. 1985, National Aeronautics and Space Administration.
2. Moore, F.K. and E.M. Greitzer, *A Theory of Post-Stall Transients in Axial Compression Systems: Part I–Development of Equations.* Journal of Engineering for Gas Turbines and Power, January 1986. **108**.
3. Flack, R.D., *Fundamentals of Jet Propulsion with Applications*. Cambridge Aerospace Series, ed. W. Shyy and M.J. Rycroft. 2005: Cambridge University Press.
4. Rolls-Royce, *The Jet Engine*. 5th ed. 1996, The Technical Publications Department, Rolls-Royce plc, Derby, England: Renault Printing Co Ltd, Birmingham, England B44 8BS
5. Graham, R.W. and E.C. Guentert, *Compressor Stall and Blade Vibration*, in *Aerodynamic Design of Axial-Flow Compressors*, I.A. Johnsen and R.O. Bullock, Editors. 1965, National Aeronautics and Space Administration: Washington, D.C. p. 311-330.
6. Greitzer, E.M., *Review–Axial Compressor Stall Phenomena.* Journal of Fluids Engineering, June 1980. **102**.
7. Greitzer, E.M., *Surge and Rotating Stall in Axial Flow Compressors Part I: Theoretical Compression System Model.* Journal of Engineering for Power, April 1976.
8. Pearson, C.E., *Surge Behaviour in a Three-Stage Compressor*. Journal of Aeronautical Science, November 1955: p. 799-802.

9. Huppert, M.C., *Compressor Surge,* in *Aerodynamic Design of Axial-Flow Compressors*, I.A. Johnsen and R.O. Bullock, Editors. 1965, National Aeronautics and Space Administration: Washington, D.C. p. 331–340.

10. Greitzer, E.M., *Surge and Rotating Stall in Axial Flow Compressors Part II: Experimental Results and Comparison.* Journal of Engineering for Power, April 1976.

11. Greitzer, E.M., *The Stability of Pumping Systems—The 1980 Freeman Scholar Lecture.* Journal of Fluids Engineering, June 1981. **103**.

12. Longley, J.P., *A Review of Nonsteady Flow Models for Compressor Stability.* Journal of Turbomachinery, April 1994. **116**.

13. Gu, G., A. Sparks, and S.S. Banda, *An Overview of Rotating Stall and Surge Control for Axial Flow Compressors.* IEEE Transactions on Control Systems Technology, November 1999. **7**(6).

14. Paduano, J.D., E.M. Greitzer, and A.H. Epstein, *Compression System Stability and Active Control.* Annual Review of Fluid Mechanics, 2001. **33**: p. 491–517.

15. McCaughan, F.E., *Application of Bifurcation Theory to Axial Flow Compressor Instability*. Journal of Turbomachinery, October 1989. **111**.

16. Abed, E.H., P.K. Houpt, and W.M. Hosny, *Bifurcation Analysis of Surge and Rotating Stall in Axial Flow Compressors.* Journal of Turbomachinery, October 1993. **115**.

17. Liaw, D.C. and E.H. Abed, *Active Control of Compressor Stall Inception: A Bifurcation-theoretic Approach.* Automatica, 1996. **32**(1).

18. Hos, C., A. Champneys, and L. Kullmann, *Bifurcation analysis of surge and rotating stall in the Moore–Greitzer compression system.* IMA Journal of Applied Mathematics, 2003. **68**: p. 205–228.

19. Malathi, A.S. and A. Kushari, *A Parametric Investigation of Geometric Variation on Fluid Dynamic Instabilities in Axial Compression Systems.* International Journal of Rotating Machinery, 2012. **2012**.

20. Go, T.H. and R.V. Ramnath, *An Analytical Approach to the Aircraft Wing Rock Dynamics*, in *AIAA Atmospheric Flight Mechanics Conference and Exhibit*. August 2001: Montreal,Canada.

21. Nayfeh, A.H., *Introduction to Perturbation Techniques*. 1981, New York: Wiley.

22. Nayfeh, A.H., *Problems in Perturbation*. 1985, New York: Wiley.

23. Nayfeh, A.H., *Applied Nonlinear Dynamics: Analytical, Computational, and Experimental Methods*. 1995, New York: Wiley.

24. Nayfeh, A.H., *Perturbation Methods*. 2004, Weinheim, Germany: Wiley-VCH Verlag GmbH & Co. KGaA.

25. Ramnath, R.V., *Multiple Scales Theory and Aerospace Applications*. AIAA Education Series, ed. J.A. Schetz. 2010: American Institute of Aeronautics and Astronautics, Inc.

26. Maqsood, A. and T.H. Go, *Multiple time scale analysis of aircraft longitudinal dynamics with aerodynamic vectoring.* Nonlinear Dynamics, 2012. **69**: p. 731–742.

27. Greitzer, E.M. and F.K. Moore, *A Theory of Post-Stall Transients in Axial Compression Systems: Part II–Application.* Journal of Engineering for Gas Turbines and Power, April 1986. **108**.