FREQUENCY LIMITED MODEL ORDER REDUCTION TECHNIQUE

By

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A thesis submitted to the faculty of Electrical Engineering Department, Military College of Signals, National University of Sciences and Technology, Islamabad, Pakistan, in partial fulfillment of the requirements for the degree of MS in Electrical Engineering

September 2017

ABSTRACT

Derivation of a mathematical system is a critical parameter for analysis, design and simulation of a dynamic system. While deriving from physical systems large higher order complex models are obtained. These models are represented by partial differential equations, odinary differential equations. For simplification and ease in solution of these models, reduced order models are required that approximates with the original system as closely as possible. Considerable amount of research has been done on different features of model order reduction. Existing techniques have the drawbacks of lacking properties like stability, passivity, large approximation with error and lack of apriori error bounds etc. This thesis includes frequency limited Gramians based model order reduction techniques for standard continous and discrete time systems . The proposed techniques produce easily computable error bounds and comparable approximation error. Numerical problems are also illustrated to exhibit the compatibility and effictievness of the proposed techniques to the existing ones. Some of practical applications of MOR are

- Fabrication industries
- Missiles analysis and launching
- Industrial real time applications

DEDICATION

This thesis is committed to

MY FAMILY, FRIENDS AND TEACHERS

for their adoration, unending backing and support.

ACKNOWLEDGEMENTS

I am appreciative to Allah Almighty who has gave me with the quality and the energy to fulfill this postulation and I am grateful to Him for His leniency and kindheartedness. Without his assent I couldn't have entertained myself with this undertaking.

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ACRONYMS

Balanced Truncation	BT
Model Order Reduction	MOR
Reduced Order Models	ROMs
Frequency Limited Model Reduction	FLMOR
Continuous Time Systems	CTS
Gawronski and Juang's Technique	GJ
Ghafoor ad Sreeram's Technique	GS
Imran and Ghafoor's Technique	IG

Chapter 1

Introduction

1.1 Introduction

1.1.1 Overveiw

Each physical framework can be spoken to as numerical model. Demonstrating of physical systems into scientific model produce complex higher order models. These higher order models produce complex differential condition that are hard to analyze, design , mimic and store, and take much memory for capacity. To address these issues, a strategy is required to bring down the computational cost by reducing the order of the system that contains the basic parameters like input, output parameters stability and lower approximation error of original systems. The required outcome can be acheived by a procedure called Model order Reduction (MOR). MOR contributes an essential part in control theory analysis and design.

Balanced truncation (BT)

The most as often as possible utilized MOR method is Balanced truncation (BT) [1] that holds stability in reduce order models (ROMs). In BT [1], controlability and observibility Gramians are altered into an system that is internally balanced. The slightest controllable and minimum obsersvable states are truncated to get ROMs. This makes the estimation error smaller in utilizing BT [1] system, which is viewed as a decent execution of ROMs. Other than BT, other such plans, for example, Hankel ideal estimate [2], Pade apprximation [3] , Krylov technique [4] and so on are valuable for limiting MOR disadvantages. For higher frequencies BT is a decent option as it creates good results; At lower frequencies adjusted particular annoyance guess (BPSA) is utilized for better execution. The ROMs acquired through the BSPA [5] are stable and balanced.

Frequency Weighted model Order Reduction

In MOR the approximation error between original and ROM is required to be small for all frequencies, in a few situations this error is more essential over a specific interval of frequency instead of whole frequencies. The case for this situation when ROMs are utilized are the input control plan. This gives utilizing frequency weights in MOR strategy, otherwise called weighted model reduction (FWMR) issue. Give a stable continous system $G(s) = C(sI - A)^{-1}B + D$, $V_i(s) = C_{vi}(sI - A_{vi})^{-1}B_{vi} + D_{vi}$ is the steady state system, and the steady yield weighting system $W_o(s) = C_{wo}(sI - A_wo)^{-1}B_{wo} + D_{wo}$ where A, B, C, D, with output, input and weightings. $A_{vi}, B_{vi}, C_{vi}, D_{vi}$, $A_{wo}, B_{wo}, C_{wo}, D_{wo}$ are n^{th}, p^{th}, q^{th} arrange separately that is least acknowledgment , the fundamental point is to locate a steady ROM $G_{rr}(s) = C_{rr}(sI - A_{rr})^{-1}D_{rr} + D_{rr}$ where $A_{rr}, B_{rr}, C_{rr}, D_{rr}$ becomes a rr^{th} arrange $(rr < n)$ negligible acknowledgment, in that case $\|W_o(s)(G(s) - G_{rr}(s))V_i(s)\|_\infty$ should be as little as would be prudent. This is known as two sided FWMR issue. On the off chance that the weighting is one sided, the issue is known as uneven FWMR, where the point is to discover a ROM $G_{rr}(s)$, therefore $\|(G(s) - G_{rr}(s))V_i(s)\|\infty$ (in case of input weighting) and $\|W_o(s)(G(s) - G_{rr}(s))\|_\infty$ (in case of output weighting) should be as little as could be expected under the circumstances. Enns [6] hypothesize this issue by acquainting frequency weights with the BT [1] to present frequency weights. These weights change the are helpful for the recurrence of the MOR error, input weights, output weights or both sided weights might be utilized as a part of Enns system [6]. Be that as it may, for uneven weights, stability of ROMs is ensured yet for two sided sided weighting case, stability is not guanranteed. To conquer this shakiness issue of two sided weighting, a few modifications to enns system [6] have been proposed [7]- [11].

To conquer Enns method [6] disadvantages, Lin and Chiu [8] has proposed an alternate procedure that guarantees stability when two sided weightings are available. In any case, their strategy has a confinement that can work just when entirely appropriate weighting capacity is utilized as a part of and no pole, zero cancelation happens while shaping the augumented system. These shortcomings of Lin and Chui [8] method were later adjusted by Sreeram et al [12] and Varga and Anderson [10], where [12] summed up [8] to incorporate weights, while [10] holds the dependability of the system notwithstanding when shaft zero cancelation happen. Varga and Anderson [10] produces an indistinguishable outcomes from Enns [6] particularly in controller decrease applications. So far controller reduction issue, if Enns system [6] produces unstable ROMs, so does by Varga and Anderson [10] method.

Wang et al's method [13] has likewise solve the stability issue of Enns [6], which not just give stable ROMs within the sight of two sided weightings additionally inferred error bound. The approximation error of Wang et al method [13] was later enhanced by Varga and Anderson [10] as pointed out by Sreeram [9]. This system and its adjustment by Varga and Anderson [10] are acknowledgment free. This implies for a similar unique system , diverse models can be gotten from various acknowledge

Frequency limited model reduction Problem

Gawronski and Juang (GJ) [14] presented interval based model reduction technique (FLMR) in this case there are no unequivocally predefined frequency weightings, rather categorical estimation frequency intervals is considered $[\omega_1, \omega_2]$ without struction of information and yield weightings by recurrence space portrayal of Gramians. in this system, Gramians are characterized for a fancied recurrence interims. Notwithstanding it can likewise yield insecure ROMs for stable unique system (like Enns strategy [6]). In addition , there are no error bounds . Spurred by [13] . Gugercin and Antoulas [15] has changed Gawronski and Juang system to give ROMs. Roused from Varga and Anderson [10] change to wang et al's procedure [13]), Ghafoor and Sreeram [16] exhibited another alteration to Gawronski and Juang [14] system to give stable ROMs. Both methods [16], [18] carries frequency response error bound to satisfy rank conditions. However, like [10], [13] these methods are additionally realization dependant.

The FWMR with given weightings and FLMR without predefined weightings gets to be distinctly identical as demonstrated [18].

Gawronski and Juang [14] has additionally presented an idea of time restricted Gramians based model reduction (TLMR). Tragically, TLMR likewise does not have the security of ROMs and does not have frequency response error bound. Summed up descriptor systems are valuable and discover their nearness in various applications which incorporate semidiscretization of fractional differential conditions, multibody elements with requirements, electrical circuit recreation and small scale electro-mechanical framework.

1.2 Problem Summary

Existing FWMR, FLMR systems may yeild unsteady ROMs, and yield more estimation error.

1.3 Summary of contributions

Different FWMR, FLMR and TLMR procedures [19]- [21] for standard and summed up state space representatin for both continous and discrete time are proposed which dependably yield stable ROMs, have effortlessly calculatable approximation error and generally yield less error bound.

1.4 Contributions

The summary of the thesis are condensed as,

- The systems are proposed which guarantee the stability of ROM.
- Proposed techniques deliver less approximation error when contrasted with existing stability preserving techniques.

1.5 Thesis Outline

This thesis is separated into four parts:

- Chapter 1: In this chapter, outline of existing MOR procedures in writing is portrayed.
- Chapter 2: This chapter incorporates all the current FLMOR methods for continous time systems and afterward given examinations existing procedure are given.
- Chapter 3: This chapter incorporates all the current FLMOR strategies for discrete time systems.
- Chapter 4: Future works and Conclusion are presented in this section

Chapter 2

Frequency Limited Gramians based Model Reduction Technique with Error Bounds for Continous Time Systems

In this chapter, a new FLMOR technique is proposed. The technique addresses stability problem of GJ [14] and yeild better approximation error and carries error bound also. Moreover, it does not cause a similar effect on all eigen values. IG [17] tackled the issue by having the same effect on all eigenvalues by subtracting the smallest value from all the eigenvalues.

2.1 Preliminaries

Let a n^{th} order stable system $G(s) = C(sI - A)^{-1}B + D$ since $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times n}$, $C \in \mathcal{R}^{p\times n}$, $D \in \mathcal{R}^{p\times m}$ where inputs and outputs are defined as m and p respectively. A MOR problem is to find

$$G_{rr}(s) = C_1(sI - A_{11})^{-1}B_1 + D_1 \tag{2.1}$$

which proximates the original system (in the frequency range $[\omega_1, \omega_1]$, $0 \leq \omega_1 \leq \omega_2$), in that case $A_{11} \in \mathcal{R}^{r\times r}$, $B_1 \in \mathcal{R}^{r\times m}$, $C_1 \in \mathcal{R}^{p\times r}$, $D_1 \in \mathcal{R}^{p\times m}$, $r < n$. Let P_i and Q_o are controlability and observibility Gramians respectively, satisfy following Lyapunov equations:

$$\begin{aligned} P_i &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(j\omega I - A)^{-1}BB^T(-j\omega I - A^T)^{-1}D\omega \\ Q_o &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(-j\omega I - A^T)^{-1}C^TC(j\omega I - A)^{-1}D\omega \end{aligned}$$

Let P_i and Q_o are controlability and observibility Gramians separately, fulfill taking after continous time Lyapunov conditions

$$\begin{aligned} AP_i + P_iA^T + BB^T &= 0 \\ A^TQ_o + Q_oA + C^TC &= 0 \end{aligned}$$

Gawronski and Juang's technique (GJ)

GJ introduced the frequency limited controllability $P_{i_{GJ}} = P_i(w_2) - P_i(w_1)$ and observability $Q_{o_{GJ}} = Q_o(w_2) - Q_o(w_1)$ Gramians satisfying :

$$\begin{aligned} AP_{i_{GJ}} + A^T P_{i_{GJ}} + X_{GJ} &= 0 \\ A^T Q_{o_{GJ}} + Q_{o_{GJ}} A Q_o + Y_{GJ} &= 0 \end{aligned}$$

where

$$\begin{aligned} X_{GJ} &= (E(w_2) - E(w_1))BB^T + BB^T(E^*(w_2) - E^*(w_1)) \\ Y_{GJ} &= (E(w_2) - E(w_1))C^T C +^T C(E^*(w_2) - E^*(w_1)) \\ E(w) &= \frac{j}{2\pi} \ln((j\omega I + A)(-j\omega I + A)^{-1}) \\ X_{GJ} &= U \begin{bmatrix} S_1 & 0 \\ 0 & S_2 \end{bmatrix} U^T, Y_{GJ} = V \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix} V^T \end{aligned}$$

$S_1 = diag(si_1, si_2, \cdots, si_l), S_2 = diag(si_{l+1}, si_{l+2}, \cdots, si_n)$,
$R_1 = diag(ri_1, ri_2, \cdots, ri_k), R_2 = diag(ri_{k+1}, ri_{k+2}, \cdots, ri_n)$. $l \leq n$ and $k \leq n$ are the number of positive eigenvalues of X_{GJ} and Y_{GJ} respectively.

Remark 1 *For approximation, multiple frequency intervals can be considered. For example, for two intervals $[\omega_1, \omega_2]$ and $[\omega_3, \omega_4]$, $\omega_1 < \omega_2$, $\omega_3 < \omega_4$, the matrices X_{WZ} and Y_{WZ} may becomes indefinite and stability of ROM is not gauranteed. Therefore the ROM got by GJ are not gauranteed stable.*

Gugercin and Antoulas's technique (GA) [15]

The stability issue of GJ [14] was highlighted by GA [15]. GA introduced the frequency limited controllability $P_{i_{GA}} = P_i(w_2) - P_i(w_1)$ and observability $Q_{o_{GA}} = Q_o(w_2) - Q_o(w_1)$ Gramians satisfying the following lyaponav equations :

$$\begin{aligned} AP_{i_{GA}} + A^T P_{i_{GA}} + X_{GA} &= 0 \\ A^T Q_{o_{GA}} + Q_{o_{GA}} A + Y_{GA} &= 0 \end{aligned}$$

Let

$$T_c^T Q_{o_{GJ}} T_c = T - c^{-1} P_{i_{GJ}} T_c^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

To transform the original model to ROMs, T_c is a transformation matrix where $\sigma_h \geq \sigma_{h+1},\ h = 1, 2, \ldots, n-1$ and T is a contragredient matrix used to transform the original system realization. Calculation of ROMs are done by segregating the transformed realiation. $B_{GA} = U_{GA}|S_{GA}|^{\frac{1}{2}}$ and $C_{GA} = |R_{GA}|^{\frac{1}{2}} V_{GA}^T$′, respectively . The expressions U_{GA}, S_{GA}, V_{GA} and R_{GA}, where $R_{GA} = diag(ri_1, ri_2, ..., ri_n)$, $R_{GA} = diagri_1, ri_2, ..., ri_n$ $|si_1| \geq |si_2| \geq \cdots |si_n| \geq 0$ and $|si_1| \geq |si_2| \geq \cdots |si_n| \geq 0$. Calculation of ROMs are carried out by segregating the transformed realization.

Remark 2 *In this case $X_{GJ} \leq B_{GA}B_{GA}^T \geq 0, Y_{GJ} \leq C_{GA}^T C_{GA} \geq 0, P_{iGA} > 0 and Q_{oGA} > 0$, the minimality of A, B_{GA}, C_{GA} is guaranteed. Moreover this technique additionally yields frequency response error bounds*

Ghafoor and Sreeram technique (GS)

Ghafoor and Sreeram (GS) [16] likewise addresses the stability issue of Gawronski and Juang [14] method. GS introduced the frequency limited controllability $P_{i_{GS}} = P_i(w_2) - P_i(w_1)$ and observability $Q_{o_{GS}} = Q_o(w_2) - Q_o(w_1)$ Gramians satisfying :

$$\begin{aligned} AP_{i_{GS}} + A^T P_{i_{GS}} + X_{GS} &= 0 \\ AQ_o^T Q_{o_{GS}} + Q_{o_{GS}} A + Y_{GS} &= 0 \end{aligned}$$

Let

$$T_c^T Q_{o_{GS}} T_c = T_c^{-1} P_{i_{GS}} T_c^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

Transformation of original system is carried out by contragredient matrix T_c where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1$. Calculation of ROMs is done by segregating the transformed realization. $B_{GS} = U_{GS_1}|S_{GS_1}|^{\frac{1}{2}}$ and $C_{GS} = |R_{GS_1}|^{\frac{1}{2}} V_{GS_1}^T$, respectively,

$$X_{GJ} = \begin{bmatrix} U_{GS_1} & U_{GS_2} \end{bmatrix} \begin{bmatrix} S_{GS_1} & 0 \\ 0 & S_{GS_2} \end{bmatrix} \begin{bmatrix} U_{GS_1}^T \\ U_{GS_2}^T \end{bmatrix}$$

$$Y_{GJ} = \begin{bmatrix} V_{GS_1} & V_{GS_2} \end{bmatrix} \begin{bmatrix} R_{GS_1} & 0 \\ 0 & R_{GS_2} \end{bmatrix} \begin{bmatrix} V_{GS_1}^T \\ V_{GS_2}^T \end{bmatrix}$$

where
$\begin{bmatrix} S_{GS_1} & 0 \\ 0 & S_{GS_2} \end{bmatrix} = diag\{si_1, si_2, ..., si_n\}, \begin{bmatrix} R_{GS_1} & 0 \\ 0 & R_{GS_2} \end{bmatrix} = diag\{ri_1, ri_2, ri_3, ..., ri_n\},\ si_1 \geq si_2 \geq si_3 \geq ... \geq si_n,\ ri_1 \geq ri_2 \geq r_3 \geq ... \geq r_n,\ S_{GS_1} = diag\{s_1, s_2, s_3, ..., s_e\},\ R_{GS_1} = diag\{ri_1, ri_2, ri_3, ..., ri_e\},\ si_1 \geq si_2 \geq si_3 \geq ... \geq si_e \geq 0,\ ri_1 \geq ri_2 \geq ri_3 \geq ... \geq ri_e \geq 0$. Note that, the realization $\{A, B_{GS_2}, C_{GS_2}, D\}$ is minimal and stable. The reduced system is calculated by transforming and segregating the tranformed system realization. Since the realization $(A, B_{GS_2}, C_{GS_2}, D)$ is negligible, the stability of the reduced system is ensured. The expression for error bound also appears in [23].

2.1.1 Imran and Ghafoor (IG)

In GA technique [15] , the symmteric matrices X_{GJ} and Y_{GJ} are guaranteed positive definite/semipositive definite respectively by taking the square root of absolutes values estimations of the eigenvalues by eigen value decomposition of symmetric X_{GJ} and Y_{GJ}. This occasionally prompts to a substantial change in some eigen values and may not impact other eigen values. Then again , Ghafoor and Sreeram [16] guarantees positive definitness of the matrices X_{GJ} and Y_{GJ} by effecting just positive eigenvalues and by replacing negative eigenvalues with zeros. This system likewise doesnot have comparative impact on all eigenvalues. In IG [17] was proposed where exertion is to similarly affect all eigenvalues of uncertain matrices X_{GJ} and Y_{GJ} . The ROMs got are ensured to be stable . Additionally, it has error bounds and enhanced frequency response error. Take new controlability $P_{i_{IG}}$ and Observability $Q_{o_{IG}}$ Gramians respectively, are calculated by resolving the following Lyaponav equations:

$$\begin{aligned} P_{i_{IG}} A_{IG} + A^T P_{i_{IG}} + X_{IG} &= 0 \\ A^T Q_{o_{IG}} + Q_{o_{IG}} A + Y_{IG} &= 0 \end{aligned}$$

The matrices B_{IG} and C_{IG} are new fictitious input and output matrices respectively defined as :

$$B_{IG} = \begin{cases} U_{IG}(S_{IG} - si_n I)^{1/2} & \text{for } si_n < 0 \\ U_{IG} S_{IG}^{1/2} & \text{for } si_n \geq 0 \end{cases}$$

$$C_{IG} = \begin{cases} (R_{IG} - ri_n I)^{1/2} V_{IG}^T & \text{for } r_n < 0 \\ R_{IG}^{1/2} V_{IG}^T & \text{for } ri_n \geq 0. \end{cases}$$

The terms U_{IG}, S_{IG}, V_{IG}, and R_{IG} are solved as $X_{WZ} = U_{IG} S_{IG} U_{IG}^T$ and $Y_{WZ} = V_{IG} R_{IG} V_{IG}^T$, where $S_{IG} = diag(si_1, si_2, si_3, \cdots, si_n)$, $R_{IG} = diag(ri_1, ri_2, ri_3, \cdots, ri_n)$, $si_1 \geq si_2 \geq \cdots \geq si_n$, and $ri_1 \geq ri_2 \geq \cdots \geq ri_n$.

A consideration is made that to transform a original system a transformation matrix T is obtained as

$$T_c^T Q_{o_I G} T = T_c^{-1} P_{i_I G} T_c^{-T} = diag(\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_n)$$

Calculation of ROMs is carried out by segregating the transformed realization where $\sigma_h \geq \sigma_{h+1},\ h = 1, 2, 3, \ldots, n-1, \sigma_l > \sigma_{l+1}$.

Remark 3 *Since $X_{GJ} \leq B_{IG} B_{IG}^T$, $Y_{GJ} \leq C_{IG}^T C_{IG}$, $B_{IG} B_{IG}^T \geq 0$, $C_{IG}^T C_{IG} \geq 0$, $P_{i_I G} > 0$ and $Q_{o_{IG}} > 0$. Therefore, the realization (A, B_{IG}, C_{IG}) is minimal. In addition to , the ROMs are guaranteed stable.*

Theorem 1 *in IG [17] technique holds the following derivation of error bound provided that the following rank conditions $rank\,[B_{IG}\ \ B] = rank\,[B_{IG}]$ and $rank \begin{bmatrix} C_{IG} \\ C \end{bmatrix} = rank\,[C_{IG}]$ (which follows from [23]) are satisfied*

$$\|G(s)(s)G_{rr}(s)\|_\infty \leq 2\|L_{IG}\|\|K_{IG}\| \sum_{h=l+1}^{n} \sigma_h$$

where

$$L_{IG} = \begin{cases} CV_{IG}(R_{IG} - ri_n I)^{-1/2} & \textit{for } ri_n < 0 \\ CV_{IG} R_{IG}^{-1/2} & \textit{for } ri_n \geq 0 \end{cases}$$

$$K_{IG} = \begin{cases} (S_{IG} - si_n I)^{-1/2} U_{IG}^T b & \textit{for } si_n < 0 \\ S_{IG}^{-1/2} U_{IG}^T b & \textit{for } si_n \geq 0 \end{cases}$$

Proof: Since $rank\,[B_{IG}\ \ B] = rank\,[B_{IG}]$ and $rank \begin{bmatrix} C_{IG} \\ C \end{bmatrix} = rank\,[C_{IG}]$, the relationships $B = B_{IG}K_{IG}$ and $C = L_{IG}C_{IG}$ hold. By partitioning $B_{IG} = \begin{bmatrix} w & B_{IG_1} \\ & B_{IG_2} \end{bmatrix}, C_{IG} = \begin{bmatrix} C_{IG_1} & C_{IG_2} \end{bmatrix}$ and substituting $B_1 = B_{IG_1}K_{IG}$, $C_1 = L_{IG}C_{IG_1}$ respectively produces

$$\begin{aligned}
\|G(s) - G_{rr}(s)\|_\infty &= \|C(sI - A)^{-1}B - C_1(zI - A_{11})^{-1}B_1\|_\infty \\
&= \|L_{IG}C_{IG}(sI - A)^{-1}B_{IG}K_{IG} \\
&\quad - L_{IG}C_{IG_1}(sI - A_{11})^{-1}B_{IG_1}K_{IG}\|_\infty \\
&= \|L_{IG}(C_{IG}(sI - A)^{-1}B_{IG} \\
&\quad - C_{IG_1}(sI - A_{11})^{-1}B_{IG_1})K_{IG}\|_\infty \\
&= \|L_{IG}\|\|(C_{IG}(sI - A)^{-1}B_{IG} \\
&\quad - C_{IG_1}(sI - A_{11})^{-1}B_{IG_1})\|_\infty\|K_{IG}\|
\end{aligned}$$

If $\{A_{11}, B_{IG_1}, C_{IG_1}\}$ is ROM obtained by segregating a balanced realization $\{A, B_{IG}, C_{IG}\}$, we have [2, 11]

$$\|(C_{IG}(sI - A)^{-1}B_{IG} - C_{IG_1}(sI - A_{11})^{-1}B_{IG_1})\|_\infty \leq 2\sum_{h=l+1}^{n}\sigma_h.$$

Therefore,

$$\|G(s) - G_{rr}(s)\|_\infty \leq 2\|L_{IG}\|\|K_{IG}\| \sum_{h=l+1}^{n} \sigma_h$$

Remark 4 *In this case scenario when symmetric matrices $X_{GJ} \geq 0$ and $Y_{GJ} \geq 0$, then $P_{i_{GJ}} = P_{i_{IG}}$ and $Q_{o_{WZ}} = Q_{o_{IG}}$. Otherwise $P_{i_{GJ}} < P_{i_{IG}}$ and $Q_{o_{WZ}} < Q_{o_{IG}}$. In addition , Hankel singular values satisfies: $(\lambda_j[P_{i_{GJ}}Q_{o_{WZ}}])^{1/2} \leq (\lambda_j[P_{i_{IG}}Q_{o_{IG}}])^{1/2}$.*

Remark 5 *When $X_{WZ} \ngeq 0$ and $yY_{WZ} \ngeq 0$, then*

$$\begin{aligned}
&X_{IG} = B_{IG}B_{IG}^T = X_{GJ} - s_nI \\
&Y_{IG} = C_{IG}^TC_{IG} = Y_{GJ} - r_nI \\
&P_{i_{IG}} = P_{i_{GJ}} + P_{i_{ad}} \\
&Q_{o_{IG}} = Q_{o_{GJ}} + Q_{o_{ad}}
\end{aligned}$$

$$A(P_{i_{GJ}}+P_{i_{ad}})A^T-(P_{i_{GJ}}+P_{i_ad})+(X_{GJ}-s_nI)=0, \textit{for } s_n<0$$
$$A^T(Q_{o_{GJ}}+Q_{o_ad})A-(Q_{o_{GJ}}+Q_{o_{ad}})+(Y_{GJ}-r_nI)=0, \textit{for } r_n<0$$
$$AP_{i_{ad}}A^T-P_{i_{ad}}-s_nI=0, \qquad \textit{for } s_n<0$$
$$A^TQ_{o_{ad}}A-Q_{o_ad}-r_nI=0, \qquad \textit{for } r_n<0$$

Remark 6 *When the matrices are symmetric $X_{GJ}\geq 0$ and $Y_{GJ}\geq 0$, therefore $P_{i_{GJ}}=P_{i_{IG}}$ and $Q_{o_{GJ}}=Q_{o_{IG}}$. Otherwise $P_{i_{GJ}}<P_{i_{IG}}$ and $Q_{o_{GJ}}<Q_{o_{IG}}$. In addition, Hankel singular values satisfies: $(\lambda_j[P_{i_{GJ}}Q_{o_{GJ}}])^{1/2}\leq(\lambda_j[P_{i_{IG}}Q_{o_{IG}}])^{1/2}$.*

2.2 Proposed Technique FLBT Continous Case

GA [15] addressed the stability issue by obtaining the square root of the absolute values of eigenvalues of the matrices X_{GJ} and Y_{GJ}. Whereas in GS [16] technique symmetric matrices are made certain to be positive definite by truncating the negative values. IG [17] tackled the issue by having the same effect on all eigenvalues by subtracting the smallest value from all the eigenvalues. The proposed techniques have the target to produce less approximation error in comparison to the existing stability ensuring frequency limited interval based MOR methods. This has been done in the first proposed the technique by subtracting the smallest negative value from S_2 and R_2 respectively. In the second proposed technique the subsequent eigenvalue is subtracted from the previous eigenvalue of respective X_{GJ} and Y_{GJ} matrices. New controlability $P_{i_{J_i}}$ and observability $Q_{o_{J_i}}$ Gramians are:

$$\begin{aligned} AP_{i_{J_i}}+A^TP_{i_{J_i}}+B_{J_i}B_{J_i}^T &= 0 \\ A^TQ_{o_{J_i}}+Q_{o_{J_i}}+C_{J_i}^TC_{J_i} &= 0 \end{aligned}$$

where $B_{J_i} \in \{B_{J_1}; B_{J_2}\}$and $C_{J_i} \in \{C_{J_1}; C_{J_2}\}$

$$B_{J_1} = \begin{cases} U \begin{bmatrix} S_1 & 0 \\ 0 & S_2 - si_n I_{(n-l)*(n-l)} \end{bmatrix}^{1/2} & \text{for } si_n < 0 \\ U(S_1)^{1/2} & \text{for } si_n \geq 0 \end{cases}$$

$$B_{J_2} = \begin{cases} U(\hat{S})^{1/2} & \text{for } si_n < 0 \\ U(S_1)^{1/2} & \text{for } si_n \geq 0 \end{cases}$$

$$C_{J_1} = \begin{cases} \begin{bmatrix} R_1 & 0 \\ 0 & R_2 - ri_n I_{(n-k)*(n-k)} \end{bmatrix}^{1/2} V^T & \text{for } ri_n < 0 \\ (R_1)^{1/2} V^T & \text{for } ri_n \geq 0 \end{cases}$$

$$C_{J_2} = \begin{cases} (\hat{R})^{1/2} V^T & \text{for } ri_n < 0 \\ (R_1)^{1/2} V^T & \text{for } ri_n \geq 0. \end{cases}$$

where $i = 1, 2$; $\hat{s}_1 = s_1$, $\hat{s}_{1+q} = s_{1+q-1} - s_{1+q}$, $\hat{r}_1 = r_1$, $\hat{r}_{1+t} = r_{1+t-1} - r_{1+t}$, $\hat{S} = diag(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$, $\hat{R} = diag(\hat{r}_1, \hat{r}_2, \cdots, \hat{r}_n)$, $q = 1, 2, \cdots, n-1$ and $t = 1, 2, \cdots, n-1$. Let a transformationed T_{J_i} is obtained as

$$T_{J_i}^T Q_{o_{J_i}} T_{J_i} = T_{J_i}^{-1} P_{i_{J_i}} T_{J_i}^{-T} = diag(\sigma_1, \sigma_2, \cdots, \sigma_n)$$

CalcuLation of ROMs is carried out by segregating the transformation of original realization , since $\sigma_h \geq \sigma_{h+1},\ j = 1, 2, 3, \ldots, n-1$.

Remark 7 *In this case $X_{GJ} \leq B_{J_i} B_{J_i}^T$, $Y_{GJ} \leq C_{J_i}^T C_{J_i}$, $B_{J_i} B_{J_i}^T \geq 0$, $C_{J_i}^T C_{J_i} \geq 0$, $P_{i_{J_i}} > 0$ and $Q_{o_{J_i}}] > 0$. Therefore, (A, B_{J_i}, C_{J_i}) is minimal and ROMs are stable.*

Theorem 2 *Let $rank\,[B_{J_i} \ \ B] = rank\,[B_{J_i}]$ and $rank \begin{bmatrix} C_{J_i} \\ C \end{bmatrix} = rank\,[C_{J_i}]$ (which based on results in [16] are satisfied),*

$$\|G(s) - G_{rr}(s)\|_\infty \leq 2\|L_{J_i}\|\|K_{J_i}\| \sum_{j=l+1}^{n} \sigma_j$$

where $L_{J_i} \in \{L_{J_1}; L_{J_2}\}$ *and* $K_{J_i} \in \{K_{J_1}; K_{J_2}\}$

$$L_{J_1} = \begin{cases} CV \begin{bmatrix} R_1 & 0 \\ 0 & R_2 - ri_n I_{(n-k)*(n-k)} \end{bmatrix}^{1/2} & \text{for } ri_n < 0 \\ CV(R_1)^{1/2} & \text{for } ri_n \geq 0 \end{cases}$$

$$L_{J_2} = \begin{cases} CV(\hat{R})^{-1/2} & \text{for } ri_n < 0 \\ CV(R_1)^{-1/2} & \text{for } ri_n \geq 0 \end{cases}$$

$$K_{J_1} = \begin{cases} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 - si_n I_{(n-l)*(n-l)} \end{bmatrix}^{-1/2} U^T b & \text{for } si_n < 0 \\ (S_1)^{1/2} U^T b & \text{for } si_n \geq 0 \end{cases}$$

$$K_{J_2} = \begin{cases} (\hat{S})^{-1/2} U^T b & \text{for } si_n < 0 \\ (S_1)^{-1/2} U^T b & \text{for } si_n \geq 0 \end{cases}$$

Proof: The relationships $B = B_{J_i} K_{J_i}$ and $C = L_{J_i} C_{J_i}$ hold due to rank conditions. By segregating $B_{J_i} = \begin{bmatrix} B_{J_{i1}} \\ B_{J_{i2}} \end{bmatrix}, C_{J_i} = \begin{bmatrix} C_{J_{i1}} & C_{J_{i2}} \\ w & \end{bmatrix}$

and replacing $B_1 = B_{J_{i1}} K_{J_i}$, $C_1 = L_{J_i} C_{J_{i1}}$ respectively produces

$$\begin{aligned} & \|C(sI - A)^{-1} C_1 (sI - A_{11})^{-1} b_1\|_\infty \\ &= \|L_{J_i} C_{J_i} (sI - A)^{-1} B_{J_i} K_{J_i} - L_{J_i} C_{J_{i1}} (sI - A_{11})^{-1} B_{J_{i1}} K_{J_i}\|_\infty \\ &= \|L_{J_i} (C_{J_i} (sI - A)^{-1} B_{J_i} - C_{J_{i1}} (sI - A_{11})^{-1} B_{J_{i1}}) K_{J_i}\|_\infty \\ &= L_{J_i} \| \|(C_{J_i} (sI - A)^{-1} B_{J_i} - C_{J_{i1}} (sI - A_{11})^{-1} B_{J_{i1}})\|_\infty \|K_{J_i}\| \end{aligned}$$

If $\{A_{11}, B_{J_{i1}}, C_{J_{i1}}\}$ is ROM obtained by segregating a balanced realization $\{A, B_{J_i}, B_{J_i}\}$, then

$$\|(C_{J_i} (sI - B)^{-1} B_{J_i} - C_{J_{i1}} (sI - A_{11})^{-1} B_{J_{i1}})\|_\infty \leq 2 \sum_{j=l+1}^{n} \sigma_j.$$

$$\|G(s) - G_{rr}(s)\|_\infty \leq 2 \|L_{J_i}\| \|K_{J_i}\| \sum_{j=l+1}^{n} \sigma_j$$

2.3 *Numerical Examples*

Example 1: Take into consideration a 12^{th} order analogue chebyshev type 1 bandpass filter with passband ripple of 15 dB with the following transfer function representation:

$$G(s) = \frac{\begin{array}{r} -5.329e^{-15}s^{11} + 1.137e^{-13}s^{10} - 4.434e^{-12}s^{9} - \\ 1.368e^{-9}s^{7} + 88.24s^{6} - 2.161e^{-7}s^{5} - 9.537e^{-7}s^{4} - \\ 1.669e^{-5}s^{3} - 0.0003052s^{2} - 0.0005112s - 0.01367 \end{array}}{\begin{array}{r} s^{12} + 0.5788s^{11} + 937.7s^{10} + 450.9s^{9} + \\ 3.605e^{5}s^{8} + 1.379e^{5}s^{7} + 7.269e^{7}s^{6} + 2.068e^{7}s^{5} + \\ 8.11e^{9}s^{4} + 1.522e^{9}s^{3} + 4.747e^{11}s^{2} + 4.395e^{10}s + 1.139e^{1} \end{array}}$$

Fig 2.1 and 2.2 illustrates the ummagnified and magnified veiw respectively of the plot of approximation error of $3rd$ ROM obtained by the techniques G_J [14], G_A [15], G_S [16], I_G [17], Proposed technique I and Proposed technique II , in the aspired frequency range $[w_1, w_2] = [47, 65]rad/s$.

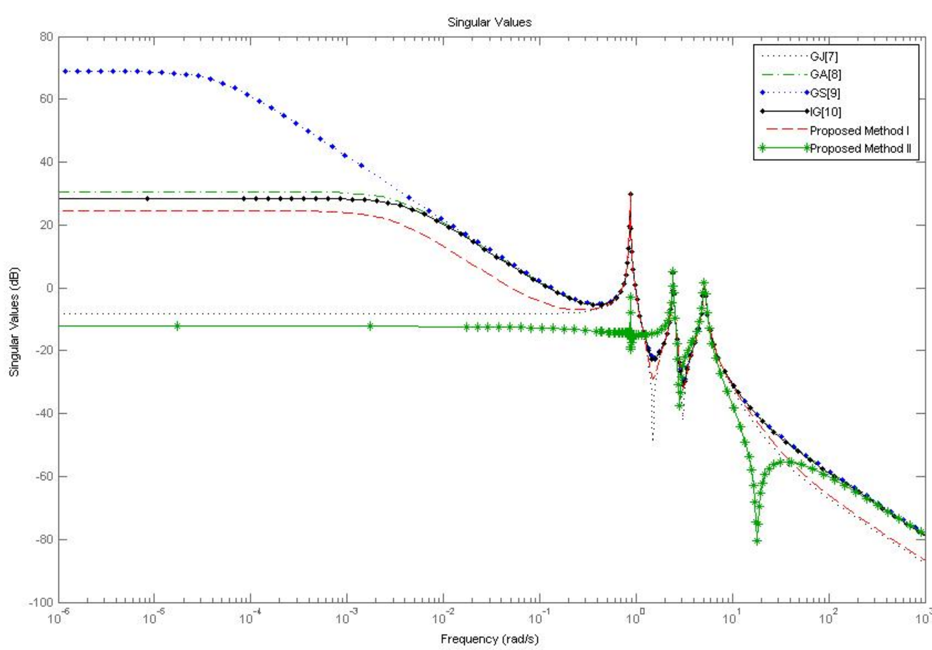

Figure 2.1: $\sigma[G(s) - G_{rr}(s)]$ in the interval $[47, 65]rad/s$

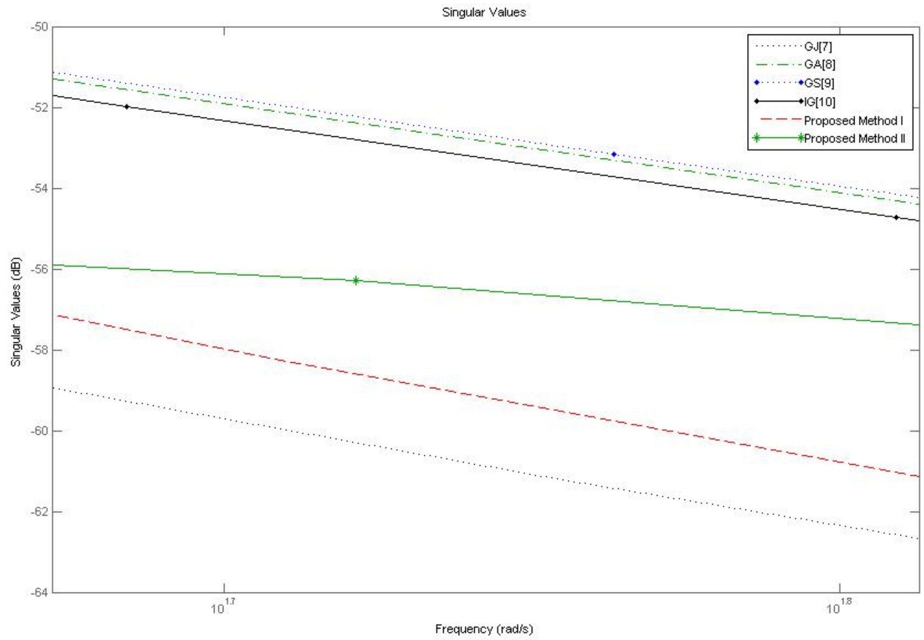

Figure 2.2: $\sigma[G(s) - G_{rr}(s)]$ in the interval $[47, 65]rad/s$ magnified veiw

Example 2: Take into consideration a 8^{th} order stable system elliptic bandpass filter with a lower passband frequency of 9 Hz and a higher passband frequency of 25 Hz, with a passband ripple of 0.2 dB, a stopband attenuation of 75 represented by transfer function:

$$G(s) = \frac{\begin{array}{c} s^8 + 24.5s^7 + 1458s^6 + 2.28e^4s^5 + 5.943e^5s^4 + \\ 5.131e^6s^3 + 7.38e^7s^2 + 2.791e^8s + 2.563e^9 \end{array}}{\begin{array}{c} 0.0001778s^8 + 2.487e^{-14}s^7 + 7.532s^6 + \\ 2.183e^{-11}s^5 + 4.205e^4s^4 - 5.588e^{-9}s^3 + \\ 3.813e^5s^2 - 6.557e^{-7}s + 4.558e^5 \end{array}}$$

Fig 2.3 and 2.4 illustrates the ummagnified and magnified veiw respectively of the plot of approximation error of $3rd$ ROM obtained by the techniques G_J [14], G_A [15], G_S [16], I_G [17], Proposed technique I and Proposed technique II, in the aspired frequency interval $[w_1, w_2] = [40, 52]rad/s$.

Example 3: Take into consideration a stable 6^{th} order system with the following transfer

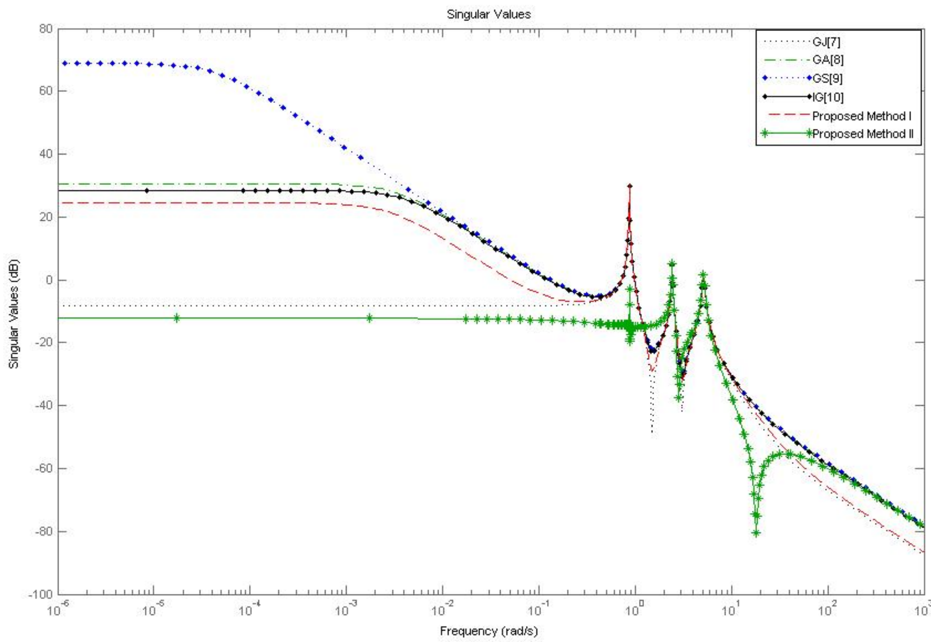

Figure 2.3: $\sigma[G(s) - G_{rr}(s)]$ in the interval $[w_1, w_2] = [40, 52]rad/s$

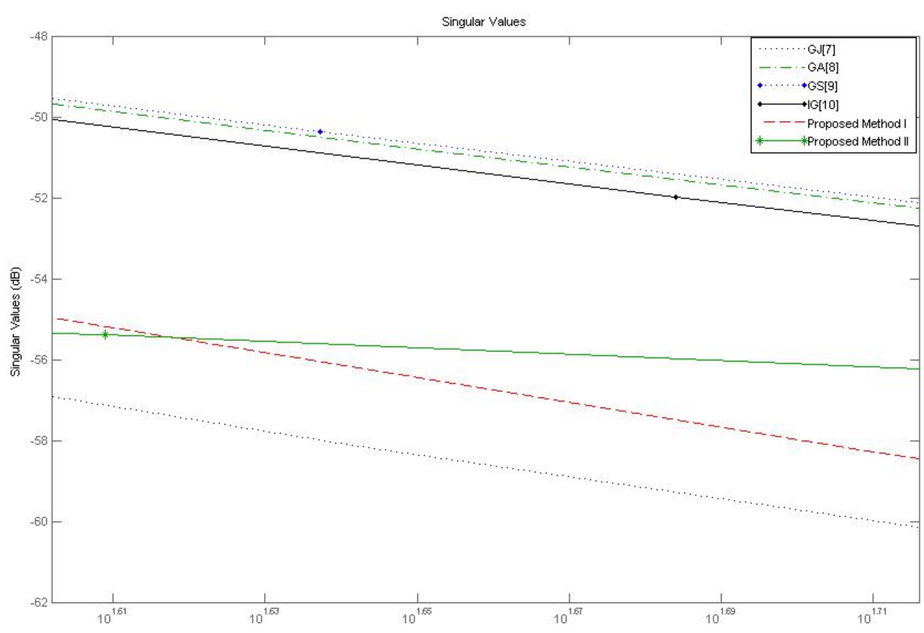

Figure 2.4: $\sigma[G(s) - G_{rr}(s)]$ in the interval $[w_1, w_2] = [40, 52]rad/s$ magnified view

function:

$$G(s) = \frac{-s^6 - 0.3295s^5 - 32.97s^4 - 3.609s^3 - 180.6s^2 - 3.566s - 119.1}{1.665e - 015s^5 + 2.118s^4 + 0.2481s^3 + 24.83s^2 + 0.906s + 45.36}$$

Fig 2.5 and Fig 2.6 illustrates the un magnified and magnified veiw respectively of the error plot of 3^{rd} ROM obtained by the techniques G_J [14], G_A [15], G_S [16], I_G [17], Proposed technique I and Proposed technique II , in the frequency interval $[w_1, w_2] = [23, 28]rad/s$

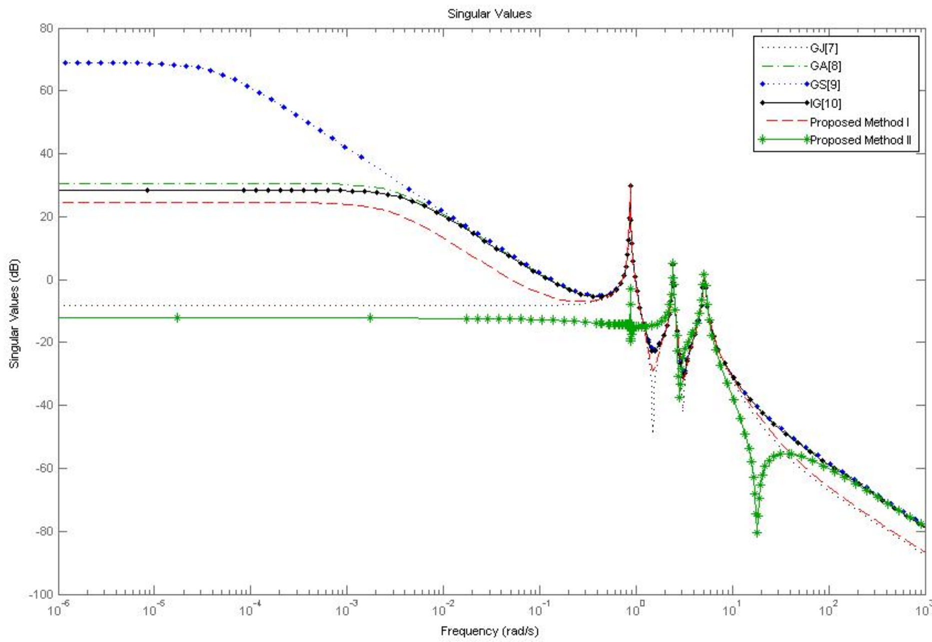

Figure 2.5: $\sigma[G(s) - G_{rr}(s)]$ in the interval $[23, 28]rad/s$

Example 4: Take into consideration a linear time invariant stable 6^{th} order system with the transfer function representation mentioned:

$$G(s) = \frac{-44.1s^3 + 334s^2 + 1034s + 390}{s^6 + 20s^5 + 155s^4 + 586s^3 + 1115s^2 + 1034s + 390}$$

Fig 2.7and Fig 2.8 illustrates the unmagnified and magnified veiw respectively of the error plot of 2^{nd} ROM obtained by the techniques G_J [14], G_A [15], G_S [16], I_G [17], Proposed technique I and II , in the aspired frequency interval $[w_1, w_2] = [17, 37]rad/s$

Example 5: Take into consideration a linear time invariant stable 6^{th} order system with the following transfer function representation

$$G(s) \quad = \quad \frac{s^3 + 2s^2 + s + 1}{s^6 + 3s^5 + 10s^4 + 20s^3 + 15s^2 + 13s + 1}$$

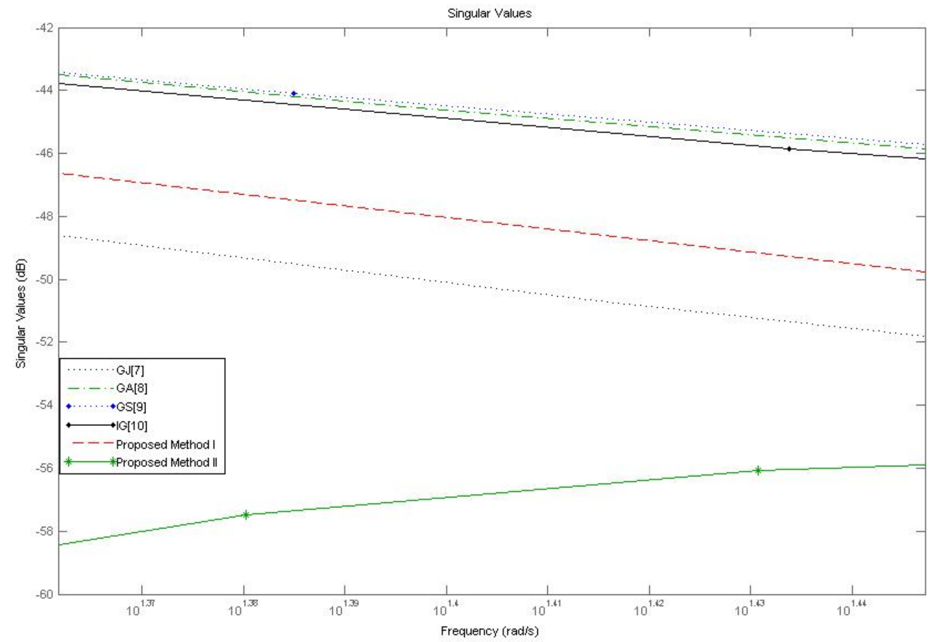

Figure 2.6: $\sigma[G(s) - G_{rr}(s)]$ in the interval $[23, 28]rad/s$
magnified view

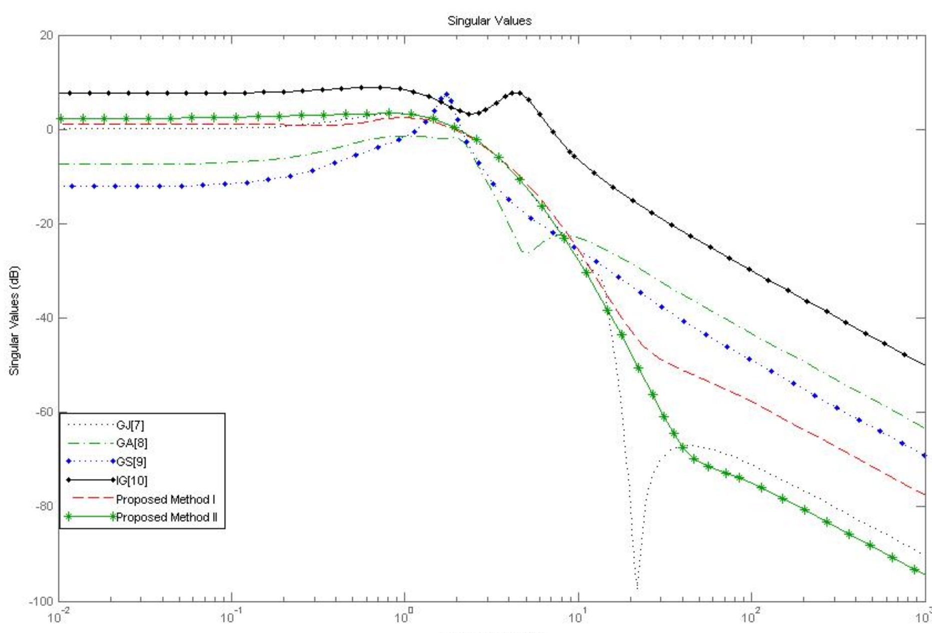

Figure 2.7: $\sigma[G(s) - G_{rr}(s)]$ in the interval $[w_1, w_2] = [17, 37]rad/s$

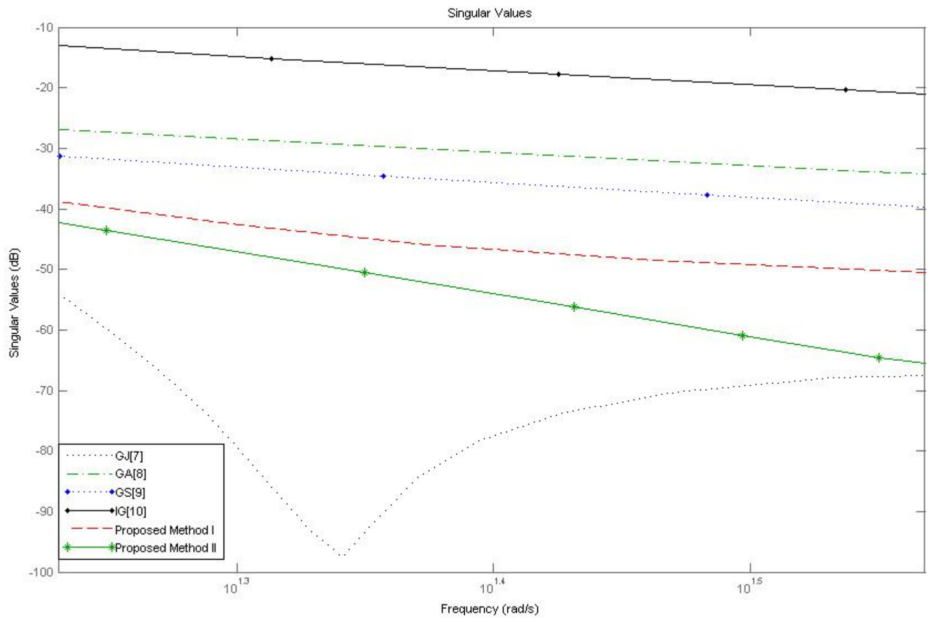

Figure 2.8: $\sigma[G(s) - G_{rr}(s)]$ in the interval $[w_1, w_2] = [17, 37] rad/s$

Fig 2.9 and Fig 2.10 illustrates the unmagnified and magnified veiw respectively of the error plot of 3^{rd} ROM obtained by the techniques G_J [14], G_A [15], G_S [16], I_G [17], Proposed technique I and II, in the aspired frequency interval $[w_1, w_2] = [12, 27] rad/s$

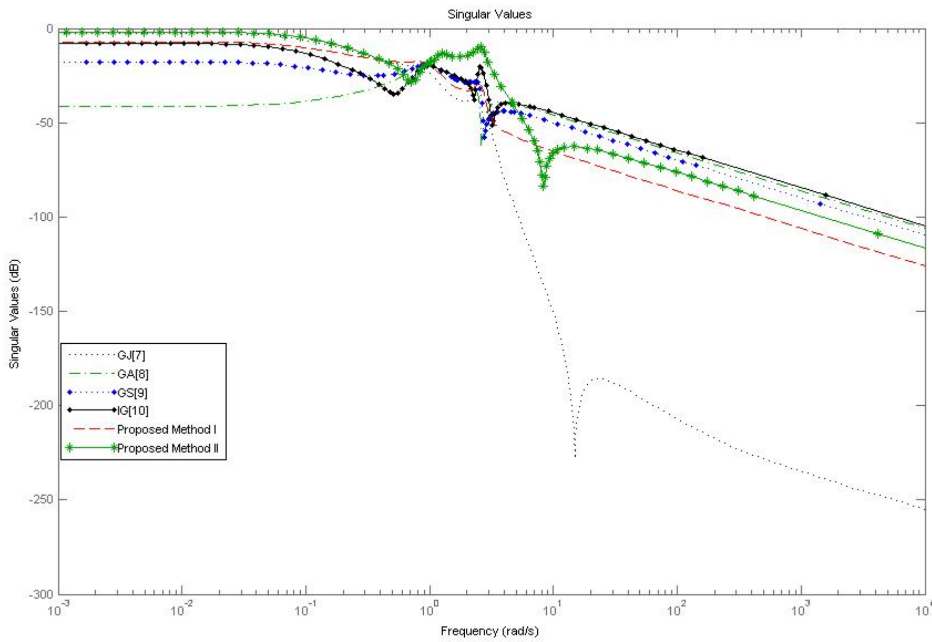

Figure 2.9: $\sigma[G(s) - G_{rr}(s)]$ in the interval $[w_1, w_2] = [12, 27] rad/s$

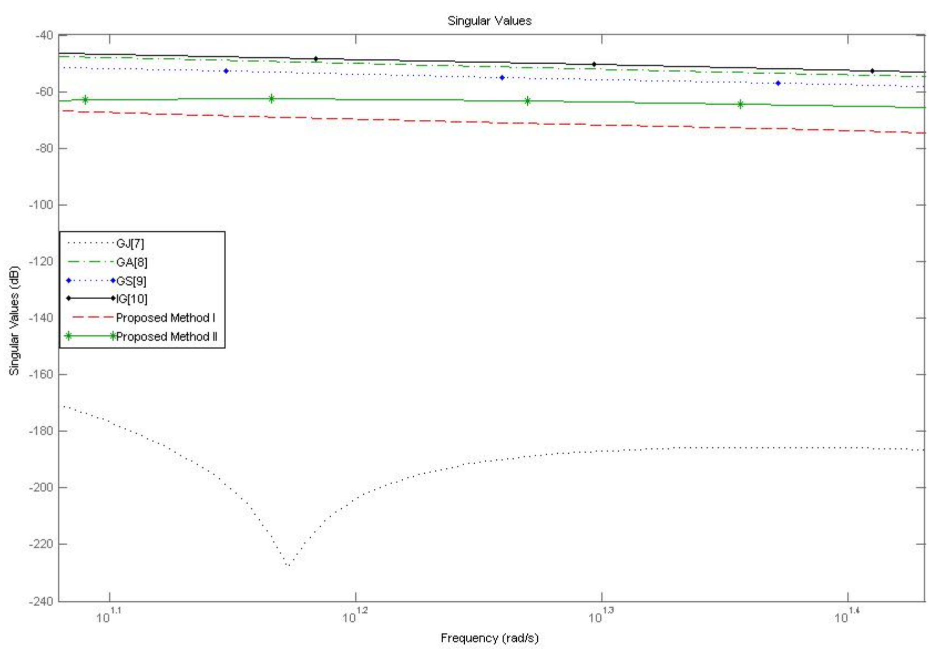

Figure 2.10: $\sigma[G(s) - G_{rr}(s)]$ in the interval $[w_1, w_2] = [12, 27] rad/s$

Frequency Limited Gramians based Model Reduction Technique with Error Bounds for Discrete Time systems

MOR is a method for approximating original system with a reduced order model ROM for ease in simulation, analysis and design of complex systems/filters. Balance truncation [1] is a common and useful scheme to get stable ROM for stable original system. Moreover the scheme also has error bounds.

Enns [6] extended the work of balance truncation technique to introduce frequency weightings. It may use single sided (input/output) and double sided weights and yeilds stable ROM when use only single side weights whereas with double sided weights, ROMs are not guaranteed to be stable. To overcome the problem of Enns [6], many other techniques are given in literature. Wang and Zilouchian [22] proposed a frequency limited technique without explicit weights. It can yield unstable ROM with no error bound. To overcome the problem of Wang and Zilouchians [22], Ghafoor and Sreeram [23] proposed two methods to guarantee the stability of ROM. In first algorithm of [23], synthetic input and output matrices are created by taking the absolute of the eigenvalues of some input and output related matrices. This was to ensure the positive/semipositive definiteness of input and ouput related matrices. For the same purpose, in the second algorithms of [23], the negative eigenvalues of the related matrices was truncated. Imran and Ghafoor [19] proposed a technique to ensure positive/semipositive definiteness of some input and output related matrices by subtracting the largest negative eigenvalue from related input and output matrices . The work in [19] and [23] guarantees stability of ROM and carry error bounds. Two new techniques has been proposed, to improve the approximation error compared with the existing stable techniques and stability condition criteria is ensured and also has frequency response error bounds.

3.1 Preliminaries

Consider an n^{th} order stable discrete system $G_z(z) = C(zI-A)^{-1}B+D$ where $A \in \mathcal{R}^{n\times n}$, $B \in \mathcal{R}^{n\times m}$, $C \in \mathcal{R}^{p\times n}$, $D \in \mathcal{R}^{p\times m}$, the input m and output p respectively. A MOR problem is to find

$$G_{rr}(z) = C_1(sI - A_{11})^{-1}B_1 + D_1 \tag{3.1}$$

which proximates the original system (in the frequency range $[\omega_1, \omega_1]$, $0 \le \omega_1 \le \omega_2$), So $A_{11} \in \mathcal{R}^{r\times r}$, $b_1 \in \mathcal{R}^{r\times m}$, $C_1 \in \mathcal{R}^{p\times r}$, $D_1 \in \mathcal{R}^{p\times m}$, $rr < n$. Let P and Q are controlability and observibility gramians respectively, satisfy following continous Lyapunov equations

$$\begin{aligned} P &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{j\omega}I - A)^{-1}BB^T(e^{-j\omega}I - A^T)^{-1}d\omega \\ Q &= \frac{1}{2\pi}\int_{-\pi}^{\pi}(e^{-j\omega}I - A^T)^{-1}C^TC(e^{j\omega}I - A)^{-1}d\omega \end{aligned}$$

Let P and Q are controlability and observibility Gramians respectively, satisfy following continous time Lyapunov equations

$$\begin{aligned} APA^T + PA^T + BB^T &= 0 \\ A^TQ + QA + C^TC &= 0 \end{aligned}$$

Wang and Zilouchian [22]

Wang and Zilouchain introduced the discrete frequency limited controllability $P_{WZ} = P_C(w_2) - P_C(w_1)$ and observability $Q_{WZ} = Q_O(w_2) - Q_O(w_1)$ Gramians satisfying :

$$\begin{aligned} AP_{WZ}A^T - P_{WZ} + X_{WZ} &= 0 \\ A^TQ_{WZ}A - Q_{WZ} + Y_{WZ} &= 0 \end{aligned}$$

where

$$\begin{aligned} X_{WZ} &= BB^TF^H + FBB^T \\ Y_{WZ} &= C^TCF + F^HC^TC \\ F &= -\frac{\omega2 - \omega1}{4\pi} + \frac{1}{2\pi} \in_{\sigma\omega} (e^{j\omega}I - A)^{-1}d\omega \end{aligned}$$

Let

$$T_{zi}^TQ_{GJ}T_{zi} = T_{zi}^{-1}P_{GJ}T_{zi}^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

Transformaton of orignal system is carried out by a transformation matrix T which is contragredient in nature , in which $\sigma_h \geq \sigma_{h+1},\ j = 1, 2, \ldots, n-1.$ Calculation of ROMs is carried out by segregating the realization which has been transformed.

Remark 8 *Frequency of multiple intervals can be considered for approximation. For example , for two intervals* $[\omega_1, \omega_2]$ *and* $[\omega_3, \omega_4]$ *,* $\omega_1 < \omega_2$*,* $\omega_3 < \omega_4$*, the matrices* X_{WZ} *and* Y_{WZ} *may becomes indefinite sometimes indefinite and stability of ROM is not gauranteed .*

Ghafoor and Sreeram Algorithm I [23]

The instability issue of Wang and zilouchian [22] was solved by Ghafoor and Sreeram algorithm I and II [23]. GSI introduced the frequency limited controllability $P_{GS_I} = P_C(w_2) - P_C(w_1)$ and observability $Q_{GS_I} = Q_o(w_2) - Q_o(w_1)$ Gramians satisfying :

$$\begin{aligned} AP_{GS_I}A^T - P_{GS_I} + X_{GS_I} &= 0 \\ A^T Q_{GS_I} A - Q_{GS_I} A + Y_{GS_I} &= 0 \end{aligned}$$

Let

$$T_{zi}^T Q_{GS_I} T_{zi} = T^{-1} P_{GS_I} T_{zi}^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

Transformaton of original system is carried out by a transformation matrix T which is contragredient in nature, in which $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1.$ Calculation of ROMs is carried out by segregating the realization which has been transformed. $B_{GS_I} = U_{GS_I}|S_{GS_I}|^{\frac{1}{2}}$ and $C_{GS_I} = |R_{GS_I}|^{\frac{1}{2}} V_{GS_I}^T$, respectively . since the expressions U_{GS_I},S_{GS_I}, V_{GS_I} and R_{GS_I}, where $R_{GS_I} = diag(r_1, r_2, ..., r_n)$, $R_{GS_I} = diagri_1, ri_2, ..., ri_n$, $|si_1| \geq |si_2| \geq \cdots |si_n| \geq 0$ and $|si_1| \geq |si_2| \geq \cdots |si_n| \geq 0$. Calculation of ROMs is carried out by segregating the realization which has been transformed.

Remark 9 *Since*$X_{WZ} \leq B_{GS_I} B_{GS_I}^T \geq 0, Y_{WZ} \leq C_{GS_I}^T C_{GS_I} \geq 0, P_{GA} > and Q_{GS_I} > 0$*, the minimality of* A, B_{GS_I}, C_{GS_I} *is guaranteed. This technique also has frequency response error bounds*

Ghafoor and Sreeram technique Algorithm II [23]

Ghafoor and Sreeram II [23] also addresses the instability problem of Wang and Zilouchain [22] technique . GS II introduced the frequency limited controllability $P_{GS_II} = P_c(w_2) -$

$P_c(w_1)$ and observability $Q_{GS_I I} = Q_o(w_2) - Q_o(w_1)$ Gramians satisfying :

$$\begin{aligned} AP_{GS_I I}A^T - P_{GS_I I} + X_{GS_I I} &= 0 \\ A^T Q_{GS_I I} A - Q_{GS_I I} + Y_{GS_I I} &= 0 \end{aligned}$$

Let

$$T_{zi}^T Q_{GS_I I} T_{zi} = T^{-1} P_{GS_I I} T_{zi}^{-T} = diag\{\sigma_1, \sigma_2, \ldots, \sigma_n\}$$

where $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, \ldots, n-1$ and T is a contragredient matrix used to transform the original system realization. The ROMs are derived by segregating the transformed realization.$B_{GS_I I} = U_{GS_I I_1}|S_{GS_I I_1}|^{\frac{1}{2}}$ and $C_{GS_I I} = |R_{GS_I I_1}|^{\frac{1}{2}} V_{GS_I I_1}^T\prime$, respectively .

$$X_{WZ} = \begin{bmatrix} U_{GS_I I_1} & U_{GS_I I_2} \end{bmatrix} \begin{bmatrix} S_{GS_I I_1} & 0 \\ 0 & S_{GS_I I_2} \end{bmatrix} \begin{bmatrix} U_{GS_I I_1}^T \\ U_{GS_I I_2}^T \end{bmatrix}$$
$$Y_{WZ} = \begin{bmatrix} V_{GS_I I_1} & V_{GS_I I_2} \end{bmatrix} \begin{bmatrix} R_{GS_I I_1} & 0 \\ 0 & R_{GS_I I_2} \end{bmatrix} \begin{bmatrix} V_{GS_I I_1}^T \\ V_{GS_I I_2}^T \end{bmatrix}$$

where
$\begin{bmatrix} S_{GS_I I_1} & 0 \\ 0 & S_{GS_I I_2} \end{bmatrix} = diag\{si_1, si_2, ..., si_n\}$,
$\begin{bmatrix} R_{GS_I I_1} & 0 \\ 0 & R_{GS_I I_2} \end{bmatrix} = diag\{ri_1, ri_2, ri_3, ..., ri_n\}$,
$si_1 \geq si_2 \geq si_3 \geq ... \geq si_n,\ ri_1 \geq ri_2 \geq r_3 \geq ... \geq r_n,\ S_{GS_I I_1} = diag\{si_1, si_2, si_3, ..., si_e\},\ R_{GS_I I_1} = diag\{ri_1, ri_2, ri_3, ..., ri_e\},\ s_1 \geq s_2 \geq s_3 \geq ... \geq s_e \geq 0,\ ri_1 \geq ri_2 \geq ri_3 \geq ... \geq ri_e \geq 0$. Note that, the realization $\{A, B_{GS_I I_2}, C_{GS_I I_2}, D\}$ is minimal and stable. The reduced system is calculated by transforming and partitioning the transformed system realization. Since the realization $(A, B_{GS_I I_2}, C_{GS_I I_2}, D)$ is minimal, the ROM is guaranteed to be stable. The error bound expression also appears in [23].

3.1.1 Imran and Ghafoor [19]

In the technique presented by Ghafoor and Sreeram I [23], the symmteric matrices X_{WZ} and Y_{WZ} have certainity of positive/semipositive definite by calculating the square root of mod of the eigenvalues got by Eigen value decomposition (EVD). This sometimes generates to a huge change in only negative eigen entries and doesnot effect other eigen entries. On the contrary , Ghafoor and Sreeram II [23] made certain the positive definitness of the matrices X_{WZ} and Y_{WZ} by taking only positive eigenvalues and replacing negative eigenvalues with zeros. The drawback of this technique technique also have the non-similar effect by only affecting the negative eigenvalues. In Imran and Ghafoor (IM) technique mdifications has been done to create changes on all entries of eigenvalues of matrices X_{WZ} and Y_{WZ} . Stability is made certain in ROMs. Besides, it gets frequency response error bound and better frequency response error. Consider new controlability P_{IG_d} and ObsevabilityQ_{IG_d} Gramians respectively, by solving the following Lyaponav equations:

$$\begin{aligned} AP_{IG_d}A^T + P_{IG_d} + X_{IG_d} &= 0 \\ A^T Q_{IG_d} A + Q_{IG_d} + Y_{IG_d} &= 0 \end{aligned}$$

The matrices B_{IG_d} and C_{IG_d} are new input fictitious input and output fictitious matrices respectively defined as :

$$B_{IG_d} = \begin{cases} U_{IG_d} S_{IG_d} - s_n I)^{1/2} & \text{for } s_n < 0 \\ U_{IG_d} S_{IG_d}^{1/2} & \text{for } s_n \geq 0 \end{cases}$$

$$C_{IG_d} = \begin{cases} (R_{IG_d} - r_n I)^{1/2} V_{IG_d}^T & \text{for } r_n < 0 \\ R_{IG_d}^{1/2} V_{IG_d}^T & \text{for } r_n \geq 0. \end{cases}$$

The terms U_{IG_d}, S_{IG_d}, V_{IG_d}, and R_{IG_d} are calculated as $X_{WZ} = U_{IG_d} S_{IG_d} U_{IG_d}^T$ and $Y_{WZ} = V_{IG_d} R_{IG_d} V_{IG_d}^T$, where $S_{IG_d} = diag(si_1, si_2, si_3, \cdots, s_n)$, $R_{IG_d} = diag(ri_1, ri_2, ri_3, \cdots, r_n)$, $si_1 \geq si_2 \geq \cdots \geq si_n$, and $ri_1 \geq ri_2 \geq \cdots \geq ri_n$. Calculation of ROMs is carried out by segregating the realization which has been transformed. T is a contragredient matrix used to transform the original system realization.

$$T_{zi}^T Q_{IG_d} T_{zi} = T^{-1} P_{IG_d} T_{zi}^{-T} = diag(\sigma_1, \sigma_2, \sigma_3, \cdots, \sigma_n)$$

where $\sigma_h \geq \sigma_{h+1},\ h = 1, 2, 3, \ldots, n-1$, $\sigma_l > \sigma_{l+1}$. Calculation of ROMs is carried out by segregating the realization which has been transformed.

Remark 10 *In this case* $X_{WZ} \leq B_{IG_d}^T$, $Y_{WZ} \leq C_{IG_d}^T C_{IG_d}$, $B_{IG_d} B_{IG_d}^T \geq 0$, $C_{IG_d}^T C_{IG_d} \geq 0$, $P_{IG_d} > 0$ *and* $Q_{IG_d} > 0$. *To make the minimum realization* (A, B_{IG_d}, C_{IG_d}) . *In addition, the stability of ROMs is ensured to be stable.*

Theorem 3 *The error bound for the following proposed technique has the rank conditions* $rank\,[B_{IG_d}\ \ B] = rank\,[B_{IG_d}]$ *and* $rank \begin{bmatrix} C_{IG_d} \\ C \end{bmatrix} = rank\,[C_{IG_d}]$ *(which follows from [23]) are satisfied*

$$\|E(z) - E_l(z)\|_\infty \leq 2\|L_{IG}\|\|K_{IG}\| \sum_{h=l+1}^{n} \sigma_h$$

where

$$L_{IG_d} = \begin{cases} CV_{IG_d}(R_{IG_d} - ri_n I)^{-1/2} & \textit{for } ri_n < 0 \\ CV_{IG)d} R_{IG_d}^{-1/2} & \textit{for } ri_n \geq 0 \end{cases}$$

$$K_{IG_d} = \begin{cases} (S_{IG_d} - si_n I)^{-1/2} U_{IG_d}^T B & \textit{for } si_n < 0 \\ S_{IG_d}^{-1/2} U_{IG_d}^T B & \textit{for } si_n \geq 0 \end{cases}$$

Proof: As $rank\,[B_{IG_d}\ \ B] = rank\,[B_{IG_d}]$ and $rank \begin{bmatrix} C_{IG_d} \\ C \end{bmatrix} = rank\,[C_{IG_d}]$, the relationships $B = B_{IG_d} K_{IG_d}$ and $C = L_{IG_d} C_{IG_d}$ hold. By partitioning $B_{IG_d} = \begin{bmatrix} B_{IG_{d1}} \\ B_{IG_{d2}} \end{bmatrix}$, $C_{IG_d} = \begin{bmatrix} C_{IG_{d1}} & C_{IG_{d2}} \end{bmatrix}$ and substituting $B_1 = B_{IG_{d1}} K_{IG_d}$, $C_1 = L_{IG_d} C_{IG_{d1}}$ respectively yields

$$\begin{aligned} \|G(z) - G_{rr}(z)\|_\infty &= \|C(zI-A)^{-1}B - C_1(zI-A_{11})^{-1}B_1\|_\infty \\ &= \|L_{IG_d} C_{IG_d}(zI-A)^{-1} B_{IG_d} K_{IG_d} \\ &\quad - L_{IG_d} C_{IG_{d1}}(zI - A_{11})^{-1} B_{IG_{d1}} K_{IG_d}\|_\infty \\ &= \|L_{IG_d}(C_{IG_d}(zI-A)^{-1} B_{IG_d} \\ &\quad - C_{IG_{d1}}(zI-A_{11})^{-1} B_{IG_{d1}}) K_{IG_d}\|_\infty \\ &= \|L_{IG_d}\|\|(C_{IG_d}(zI-A)^{-1} B_{IG_{d1}}(zI-A_{11})^{-1} B_{IG_{d1}})\|_\infty \|K_{IG_d}\| \end{aligned}$$

The ROM $\{A_{11}, B_{IG_{d1}}, C_{IG_{d1}}\}$ is calculated by segregating a balanced realization $\{A, B_{IG_d}, C_{IG_d}\}$, we have [11, 2]

$$\|(C_{IG_d}(zI-A)^{-1} B_{IG_d} - C_{IG_{d1}}(zI-A_{11})^{-1} B_{IG_{d1}})\|_\infty \leq 2 \sum_{h=l+1}^{n} \sigma_h.$$

Therefore,

$$\|G(z) - G_{rr}(z)\|_\infty \leq 2\|L_{IG_d}\|\|K_{IG_d}\| \sum_{h=l+1}^{n} \sigma_h$$

Remark 11 *when symmetric matrices $X_{WZ} \geq 0$ and $Y_{WZ} \geq 0$, then $P_{WZ} = P_{IG_d}$ and $Q_{WZ} = Q_{IG_d}$. Otherwise $P_{WZ} < P_{IG}$ and $Q_{WZ} < Q_{IG}$. In addition, Hankel singular values satisfies: $(\lambda_h[P_{GJ}Q_{WZ}])^{1/2} \leq (\lambda_h[P_{IG}Q_{IG}])^{1/2}$.*

Remark 12 *When $X_{WZ} \ngeq 0$ and $Y_{WZ} \ngeq 0$, then*

$$X_{IG_d} = B_{IG_d}B_{IG_d}^T = X_{GJ} - si_nI$$
$$Y_{IG_d} = C_{IG_d}^T C_{IG_d} = Y_{GJ} - ri_nI$$
$$P_{IG_d} = P_{WZ} + P_{ad}$$
$$Q_{IG_d} = Q_{WZ} + Q_{ad}$$

$$A(P_{WZ}+P_{ad})A^T-(P_{WZ}+P_{ad})+(X_{WZ}-si_nI)=0, \textit{for } s_n < 0$$
$$A^T(Q_{GJ}+Q_{ad})A-(Q_{WZ}+Q_{ad})+(Y_{WZ}- !ri_nI)=0, \textit{for } ri_n < 0$$
$$AP_{ad}A^T - P_{ad} - si_nI = 0, \qquad \textit{for } si_n < 0$$
$$A^TQ_{ad}A - Q_{ad} - ri_nI = 0, \qquad \textit{for } ri_n < 0$$

Remark 13 *when symmetric matrices $X_{WZ} \geq 0$ and $Y_{WZ} \geq 0$, then $P_{WZ} = P_{IG_d}$ and $Q_{WZ_d} = Q_{IG_d}$. Otherwise $P_{WZ} < P_{IG_d}$ and $Q_{WZ_d} < Q_{IG_d}$. In addition to, Hankel singular values satisfies: $(\lambda_h[P_{WZ}Q_{WZ}])^{1/2} \leq (\lambda_h[P_{IG_d}Q_{IG_d}])^{1/2}$.*

3.2 Proposed Techniques Discrete Time Case

Ghafoor and Sreeram Algorithm I [23] solved the issue by figuring the square root of the absolute estimations of eigenvalues of the matrices X_{WZ} and Y_{WZ}. Though in Ghafoor and Sreeram Algorithm II [23] procedure symmetric matrices are made sure to be positive definite by truncating the negative entities. IG [19] handled the issue by having a similar impact on all eigenvalues by subtracting the smallest entry from every one eigenvalues. The proposed procedures have the objective to deliver less estimation error contrasted with the prior stablitity guranteed frequency limited MOR methods. This has been done in the initially proposed strategy by subtracting the smallest negative an entry from S_2 and R_2 individually. In the second proposed procedure the ensuing eigenvalue is subtracted from the past eigenvalue of particular X_{WZ} and Y_{WZ} matrices. New controlability P_{f_i} and observability Q_{f_i} Gramians are:

$$\begin{aligned} AP_{f_i} + A^T P_{f_i} + B_{f_i} B_{f_i}^T &= 0 \\ A^T Q_{f_i} + Q_{f_i} + C_{f_i}^T C_{f_i} &= 0 \end{aligned}$$

where $B_{f_i} \in \{B_{f_1}; B_{f_2}\}$and $C_{f_i} \in \{C_{f_1}; C_{f_2}\}$

$$B_{f_1} = \begin{cases} U \begin{bmatrix} S_1 & 0 \\ 0 & S_2 - si_n I_{(n-l)*(n-l)} \end{bmatrix}^{1/2} & \text{for } si_n < 0 \\ U(S_1)^{1/2} & \text{for } si_n \geq 0 \end{cases}$$

$$B_{f_2} = \begin{cases} U(\hat{S})^{1/2} & \text{for } si_n < 0 \\ U(S_1)^{1/2} & \text{for } si_n \geq 0 \end{cases}$$

$$C_{f_1} = \begin{cases} \begin{bmatrix} R_1 & 0 \\ 0 & R_2 - ri_n I_{(n-k)*(n-k)} \end{bmatrix}^{1/2} V^T & \text{for } ri_n < 0 \\ (R_1)^{1/2} V^T & \text{for } ri_n \geq 0 \end{cases}$$

$$C_{f_2} = \begin{cases} (\hat{R})^{1/2} V^T & \text{for } ri_n < 0 \\ (R_1)^{1/2} V^T & \text{for } ri_n \geq 0. \end{cases}$$

where $i = 1, 2$; $\hat{s}_1 = s_1$, $\hat{s}_{1+q} = s_{1+q-1} - s_{1+q}$, $\hat{r}_1 = r_1$, $\hat{r}_{1+t} = r_{1+t-1} - r_{1+t}$, $\hat{S} = diag(\hat{s}_1, \hat{s}_2, \cdots, \hat{s}_n)$, $\hat{R} = diag(\hat{r}_1, \hat{r}_2, \cdots, \hat{r}_n)$, $q = 1, 2, \cdots, n-1$ and $t = 1, 2, \cdots, n-1$. Let a transformationed T_{f_i} is obtained as

$$T_{f_i}^T Q_{f_i} T_{f_i} = T_{f_i}^{-1} P_{f_i} T_{f_i}^{-T} = diag(\sigma_1, \sigma_2, \cdots, \sigma_n)$$

since $\sigma_j \geq \sigma_{j+1},\ j = 1, 2, 3, \ldots, n-1$. The realization that is transformed is segregated to get ROMs as matrices.

Remark 14 *As* $X_{GJ} \leq B_{f_i} B_{f_i}^T$, $Y_{GJ} \leq C_{f_i}^T C_{f_i}$, $B_{f_i} B_{f_i}^T \geq 0$, $C_{f_i}^T C_{f_i} \geq 0$, $P_{f_i} > 0$ *and* $Q_{f_i} > 0$. *Therefore,* (A, B_{f_i}, C_{f_i}) *is minimal and stability of ROMs is preserved.*

Theorem 4 *Consider* $rank\,[B_{f_i}\ \ B] = rank\,[B_{f_i}]$ *and* $rank \begin{bmatrix} C_{f_i} \\ C \end{bmatrix} = rank\,[C_{f_i}]$ *(which* *based on results in [23] are satisfied),*

$$\|G(z) - G_{rr}(z)\|_\infty \le 2\|L_{F_i}\|\|K_{F_i}\|\sum_{j=l+1}^{n}\sigma_j$$

where $L_{f_i} \in \{L_{f_1}; L_{f_2}\}$ *and* $K_{f_i} \in \{K_{f_1}; K_{f_2}\}$

$$L_{f_1} = \begin{cases} CV\begin{bmatrix} R_1 & 0 \\ 0 & R_2 - ri_n I_{(n-k)*(n-k)} \end{bmatrix}^{1/2} & \text{for } ri_n < 0 \\ CV(R_1)^{1/2} & \text{for } ri_n \ge 0 \end{cases}$$

$$L_{f_2} = \begin{cases} CV(\hat{R})^{-1/2} & \text{for } ri_n < 0 \\ CV(R_1)^{-1/2} & \text{for } ri_n \ge 0 \end{cases}$$

$$K_{f_1} = \begin{cases} \begin{bmatrix} S_1 & 0 \\ 0 & S_2 - si_n I_{(n-l)*(n-l)} \end{bmatrix}^{-1/2} U^T B & \text{for } si_n < 0 \\ (S_1)^{1/2} U^T B & \text{for } si_n \ge 0 \end{cases}$$

$$K_{f_2} = \begin{cases} (\hat{S})^{-1/2} U^T B & \text{for } si_n < 0 \\ (S_1)^{-1/2} U^T B & \text{for } si_n \ge 0 \end{cases}$$

Proof: The relationships $B = B_{f_i}K_{f_i}$ and $C = L_{f_i}C_{f_i}$ hold due to rank conditions. By partitioning $B_{f_i} = \begin{bmatrix} B_{f_{i1}} \\ B_{f_{i2}} \end{bmatrix}$, $C_{f_i} = \begin{bmatrix} C_{f_{i1}} & C_{f_{i2}} \end{bmatrix}$

and substituting $B_1 = B_{f_{i1}}K_{f_i}$, $C_1 = L_{f_i}C_{f_{i1}}$ respectively yields

$$\begin{aligned} &\|C(zI - A)^{-1}C_1(zI - A_{11})^{-1}B_1\|_\infty \\ &= \|L_{f_i}C_{f_i}(zI - A)^{-1}B_{f_i}K_{f_i} - L_{f_i}C_{f_{i1}}(zI - A_{11})^{-1}B_{f_{i1}}K_{f_i}\|_\infty \\ &= \|L_{f_i}(C_{f_i}(zI - A)^{-1}B_{f_i} - C_{f_{i1}}(zI - A_{11})^{-1}B_{f_{i1}})K_{f_i}\|_\infty \\ &= \|L_{f_i}\|\|(C_{f_i}(zI - A)^{-1}B_{f_i} - C_{f_{i1}}(zI - A_{11})^{-1}B_{f_{i1}})\|_\infty\|K_{F_i}\| \end{aligned}$$

The ROM $\{A_{11}, B_{f_{i1}}, C_{f_{i1}}\}$ is obtained by segregating a balanced realization $\{A, B_{F_i}, C_{F_i}\}$, then

$$\|(C_{f_i}(zI - A)^{-1}B_{f_i} - C_{f_{i1}}(zI - A_{11})^{-1}B_{f_{i1}})\|_\infty \le 2\sum_{h=l+1}^{n}\sigma_h.$$

$$\|G(z) - G_{rr}(z)\|_\infty \leq 2\|L_{f_i}\|\|K_{f_i}\| \sum_{j=l+1}^{n} \sigma_j$$

Example 1: Let a 4^{th} order stable LTI discrete time system illustrated by

$$A = \begin{bmatrix} 0.2650 & -.6974 & 0.2011 & -0.2819 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 2.1463 & -0.3652 & 0.1734 & -0.2591 \end{bmatrix}, D = 1$$

Figure 3.1 and Figure 3.2 shows the full range and close-up veiw of the approximation error plot of the techniques Wang Zilouchain [22], Ghafoor and Sreeram I and II [23], Imran and Ghafoors [19] and Proposed techniques respectively, in the desired frequency interval $[w_1, w_2] = [0.1, 0.18]rad/s$. The results show that proposed techniques are producing the least approximation error among all the stable techniques.

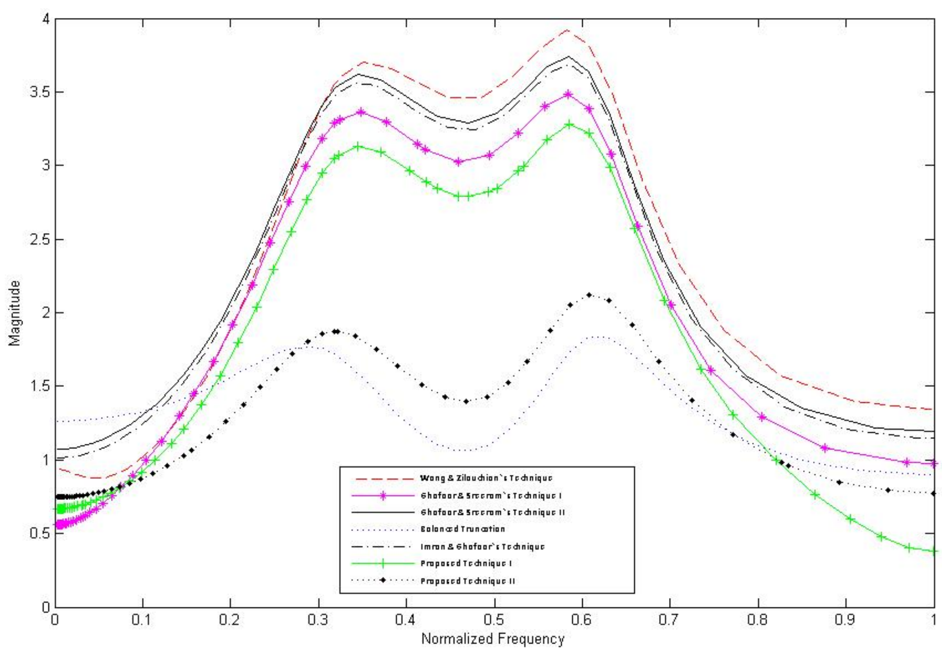

Figure 3.1: $(G(z)G_{rr}(z))$ in the frequency range $[\omega_1, \omega_2] = [.1, .18]rad/s$

Example 2: Let a 6^{th} order Elliptic band-pass $0.4\pi - 0.6\pi$ filter having 30 dB stop band attenuation and 0.1 dB pass band ripple, having following transfer function

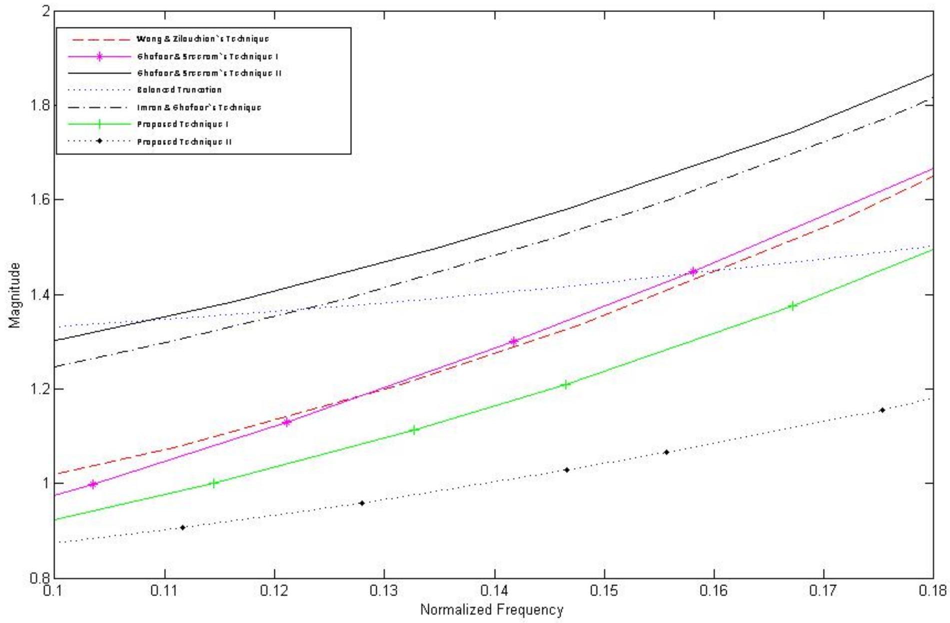

Figure 3.2: $(G(z)G_{rr}(z))$ in the frequency range $[\omega_1, \omega_2] = [.1, .18]rad/s$

$$A = \begin{bmatrix} 0.0000 & -1.6293 & 0.0000 & -1.1809 & 0.0000 & -0.3045 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0000 & -0.1665 & 0.0000 & -0.0250 & 0.0000 & -0.0889 \end{bmatrix}, D = 0.0681$$

Figure 3.3 and Figure 3.4 shows the and full range close-up veiw of the plot of approximation error of the techniques Wang Zilouchain [22] , Ghafoor and Sreeram I and II [23], Imran and Ghafoors [19] and Proposed techniques respectively, in the desired frequency range $[w_1, w_2] = [.75, .86]rad/s$. The results show that proposed techniques are producing the least approxmation error among all the stable techniques.

Example 3: Consider a stable 12^{th} order butter-worth band-pass $0.72\pi - 0.8\pi$ filter, having following transfer function

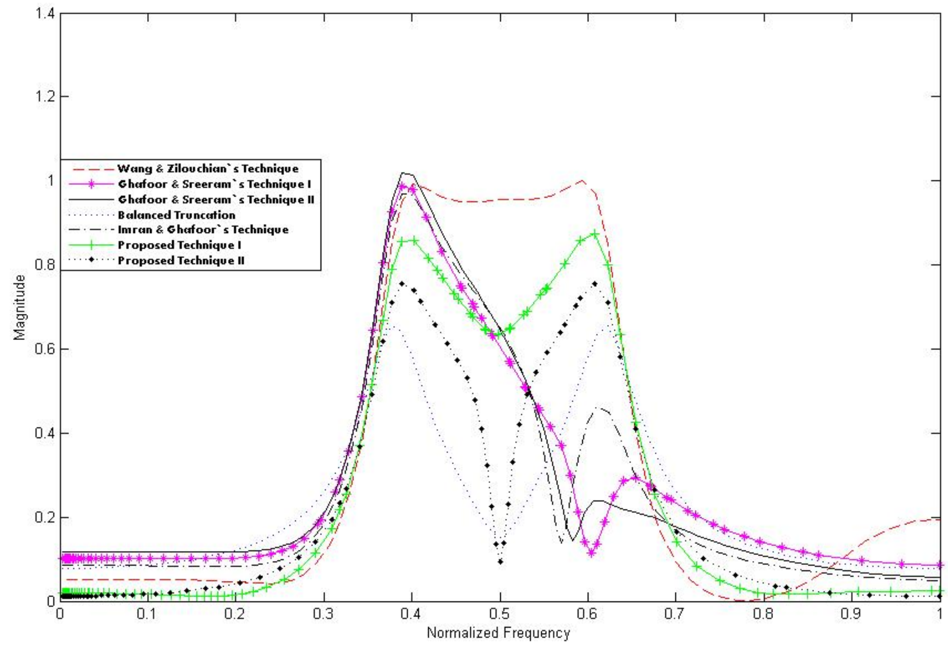

Figure 3.3: $(G(z)G_{rr}(z))$ in the frequency range $[\omega_1, \omega_2] = [.75, .86]rad/s$

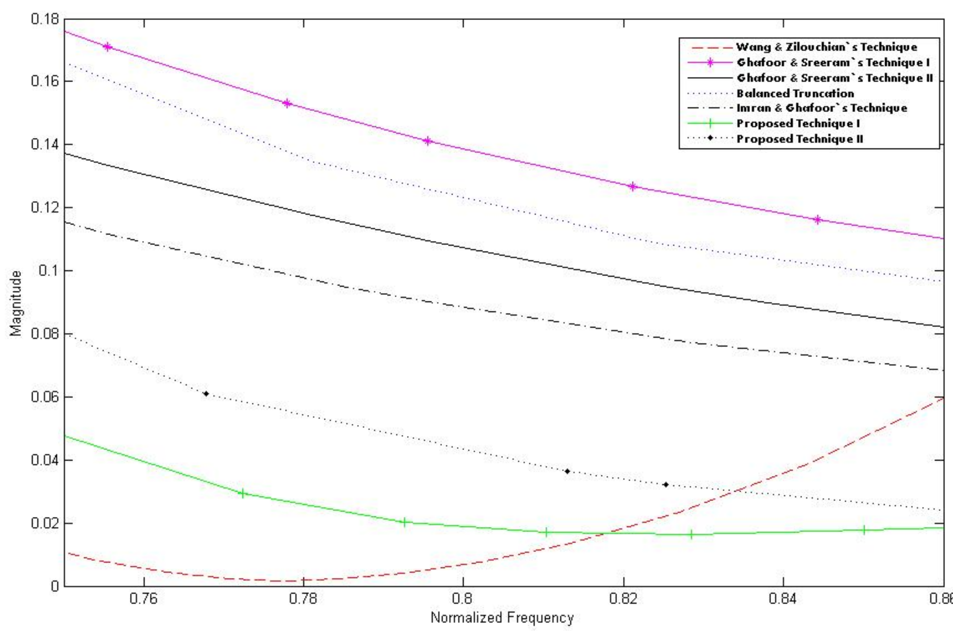

Figure 3.4: $G(z)G_{rr}(z)$ in the frequency range $[_1, _2] = [.75, .86]rad/s$

$$
A = \begin{bmatrix}
-8.10 & -32.43 & -83.53 & -153.31 & -210.40 & -220.98 & -178.85 & -110.79 & -51.31 & -16.93 & -3.59 & -0.37 \\
1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0
\end{bmatrix},
$$

$$
B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}',
$$

$$
C = \begin{bmatrix} -0.0202 & -0.0960 & -0.2086 & -0.3454 & -0.5254 & -0.6018 & -0.4466 & -0.2392 & -0.1281 & -0.0573 & -0.0090 & 0.0016 \end{bmatrix},
$$

$$
D = 2.4972e
$$

Figure 3.5 and Figure 3.6 shows the full range and close-up veiw of the plot of approxi-

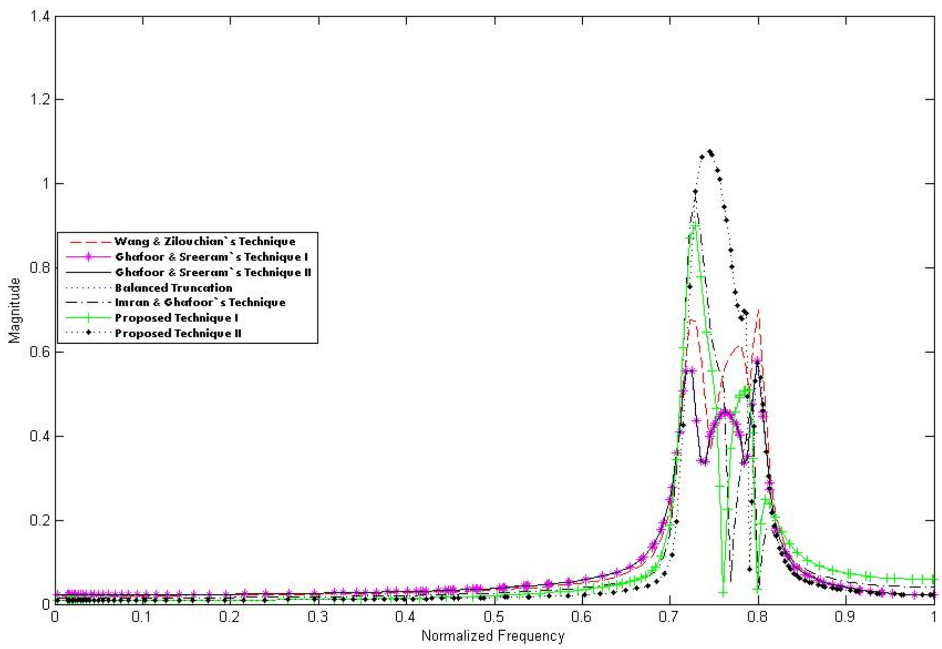

Figure 3.5: $(G(z)G_{rr}(z))$ in the frequency range $[\omega_1, \omega_2] = [.14, .29]rad/s$

mation error of the techniques Wang Zilouchain [22] , Ghafoor and Sreeram I and II [23], Imran and Ghafoors [19] and Proposed techniques respectively, in the desired frequency range $[w1, w2] = [.14, .29]rad/s$. The results show that proposed techniques are producing the least approxmation error among all the stable techniques.

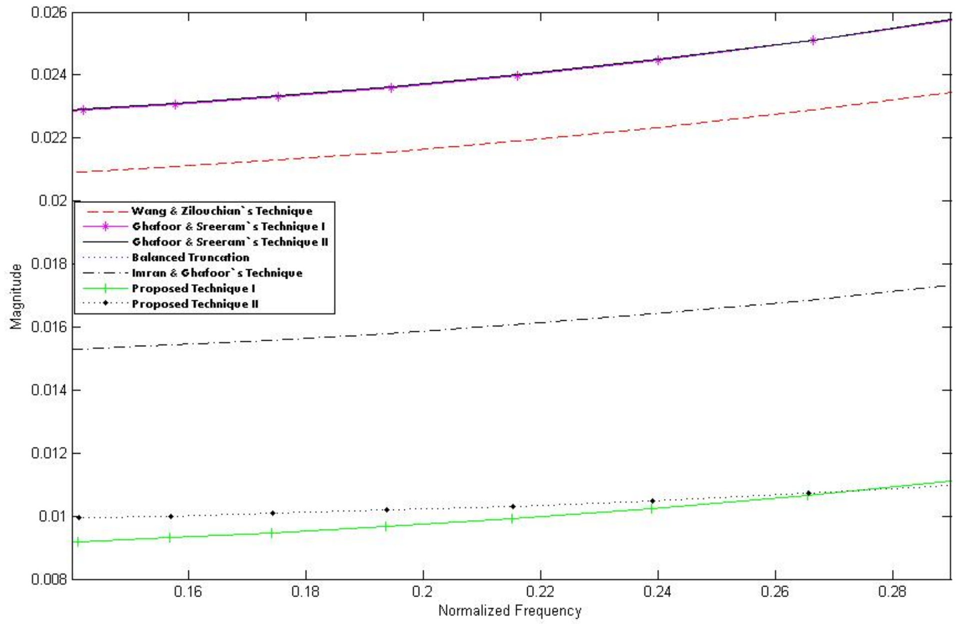

Figure 3.6: $(G(z)G_{rr}(z))$ in the frequency range $[\omega_1, \omega_2] = [.14, .29]rad/s$

Example 4: Let a 6^{th} order stable discrete time system illustrated by

$$A = \begin{bmatrix} 1.4637 & -2.2838 & 2.0587 & -1.4467 & 0.6746 & -0.1825 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.0799 & 0.1351 & 0.2388 & 0.1370 & 0.0776 & -0.0011 \end{bmatrix}, D = 0.0107$$

Figure 3.7 and Figure 3.8 shows the full range andclose-up veiw of the plot of approximation error of the techniques Wang Zilouchain [22], Ghafoor and Sreeram I and II [23], Imran and Ghafoors [19] and Proposed techniques respectively, in the desired frequency range $[w1, w2] = [.58, .69]\frac{r}{a}d/s$. The results show that proposed techniques are producing the least approximation error among all the stable existing techniques.

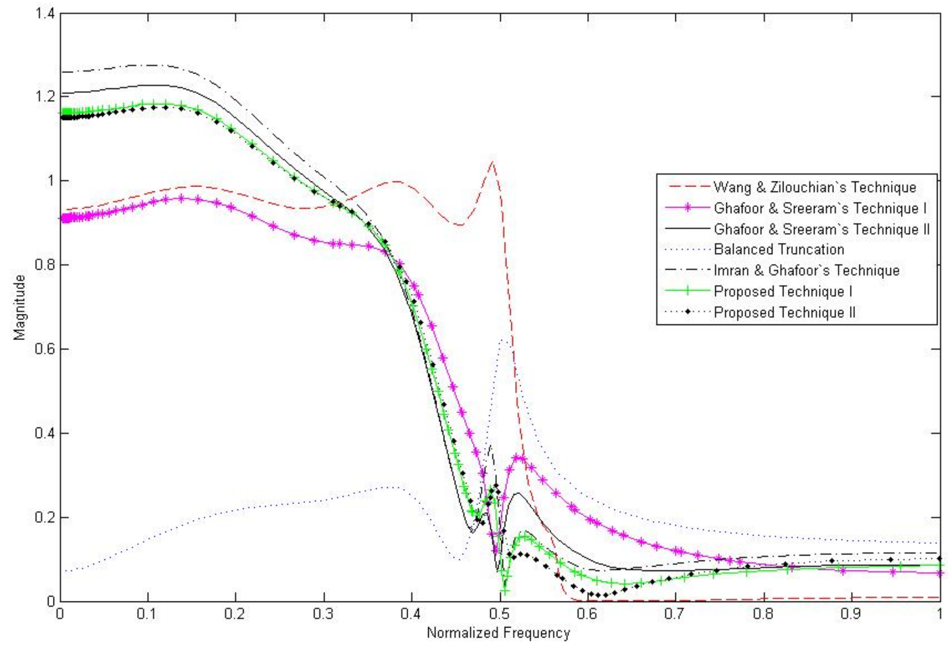

Figure 3.7: $(G(z)G_{rr}(z))$ in the frequency range $[\omega_1, \omega_2] = [.58, .69]rad/s$

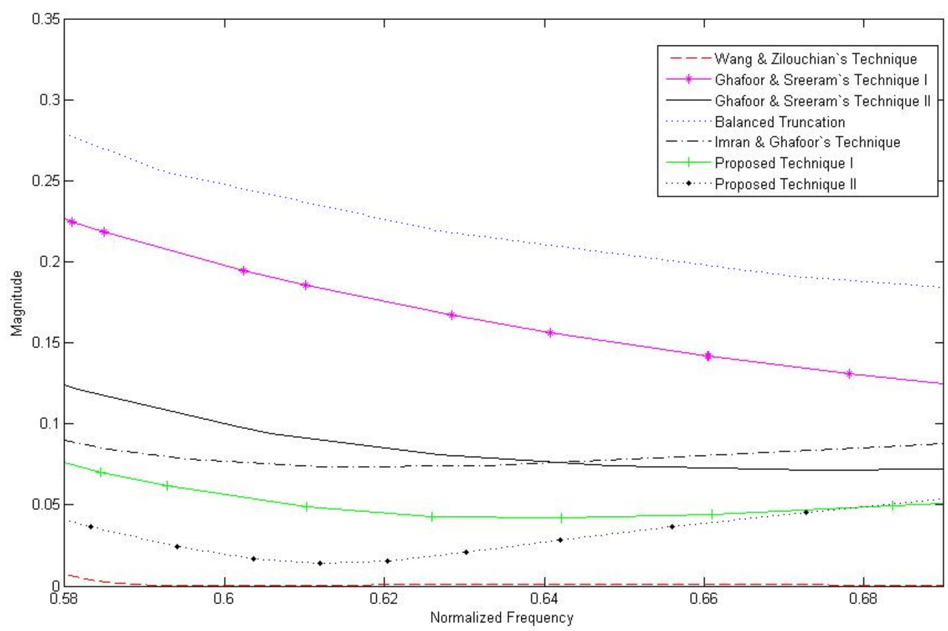

Figure 3.8: $(G(z)G_{rr}(z))$ in the frequency range $[\omega_1, \omega_2] = [.58, .69]rad/s$

Example 5: Take into Consideration a 6^{th} order stable discrete time system illustrated by the mentioned state space functions below:

$$A = \begin{bmatrix} 1.5 & -2.3 & 2.1 & -1.5 & 0.69 & -0.2 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.08 & 0.14 & 0.24 & 0.14 & 0.72 & 0.086 \end{bmatrix}, D = 0.011$$

Figure 3.9 and Figure 3.10 shows the full range and close-up veiw of the plot of approximation error of the techniques Wang Zilouchain [22], Ghafoor and Sreeram I and II [23], Imran and Ghafoors [19] and Proposed techniques respectively, in the desired frequency range $[w1, w2] = [.65, .79]rad/s$. The results show that proposed techniques is producing the least approxmation error among all the stable techniques.

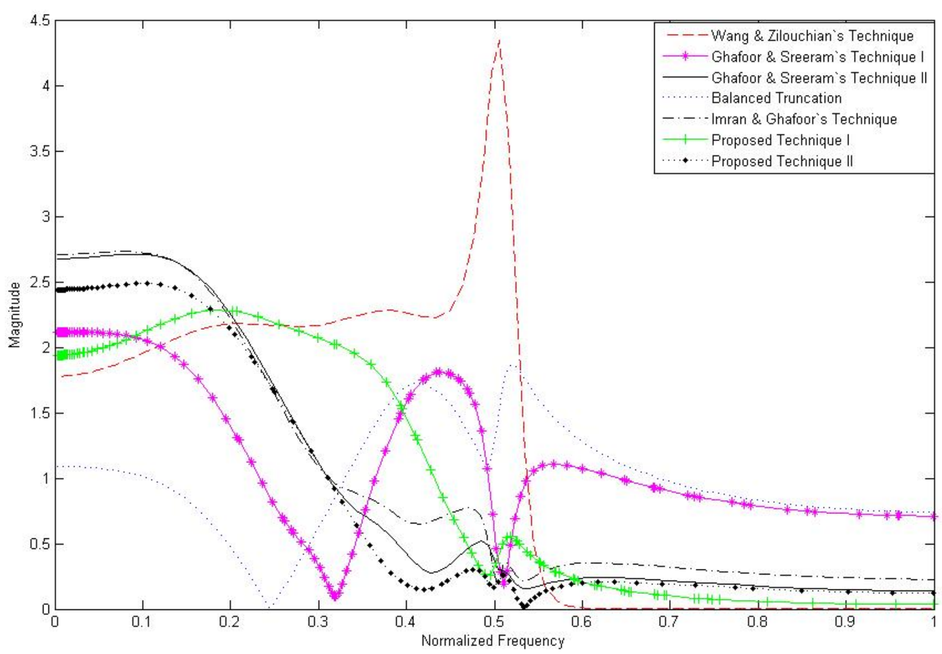

Figure 3.9: $(G(z)G_{rr}(z))$ in the frequency range $[\omega_1, \omega_2] = [.65, .79]rad/s$

Example 6: Consider a 10^{th} order chebychev band-pass $0.4\pi - 0.7\pi$ filter type 1 having 0.3

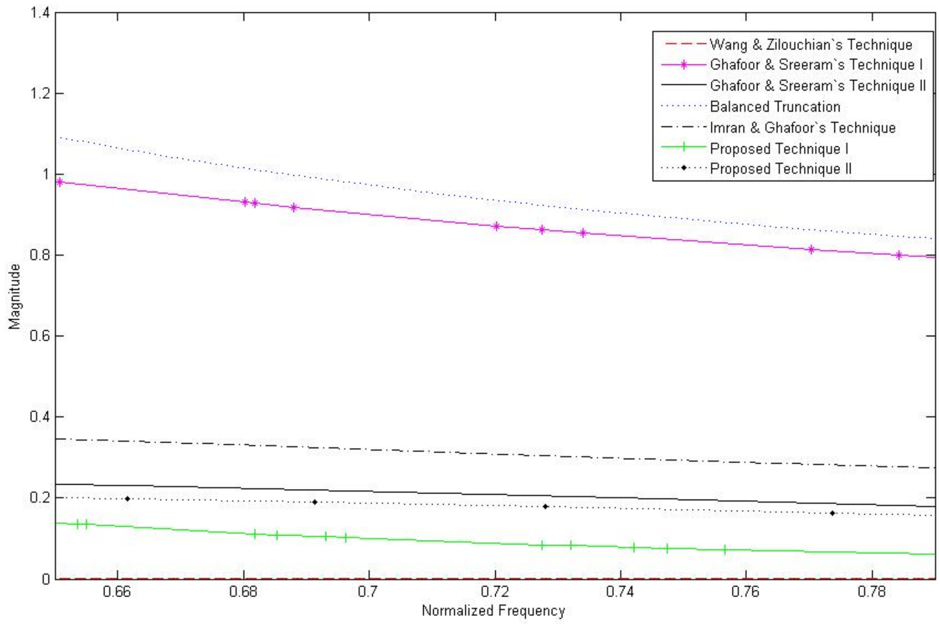

Figure 3.10: $(G(z)G_{rr}(z))$ in the frequency range $[\omega_1, \omega_2] = [.65, .79]rad/s$

dB ripple in the pass band, having following transfer function.

$$A = \begin{bmatrix} -1.3764 & -3.6204 & -3.6144 & -5.5490 & -4.0723 & -4.3847 & -2.2284 & -1.7660 & -0.4954 & -0.2827 \\ 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1.000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1.0000 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.0044 & -0.0273 & -0.0115 & 0.0141 & -0.0129 & -0.0456 & -0.0071 & 0.0103 & -0.0016 & -0.0041 \end{bmatrix}, D = 0.0032$$

Figure 3.11 and Figure 3.12 shows the full range and close-up veiw of the approximation error plot of the techniques Wang Zilouchain [22], Ghafoor and Sreeram I and II [23],Imran and Ghafoors [19] and Proposed techniques respectively, in the desired frequency range $[w_1, w_2] = [.75, .85]rad/s$. The results showing that proposed technique is producing the least approxmation error among all the stable techniques.

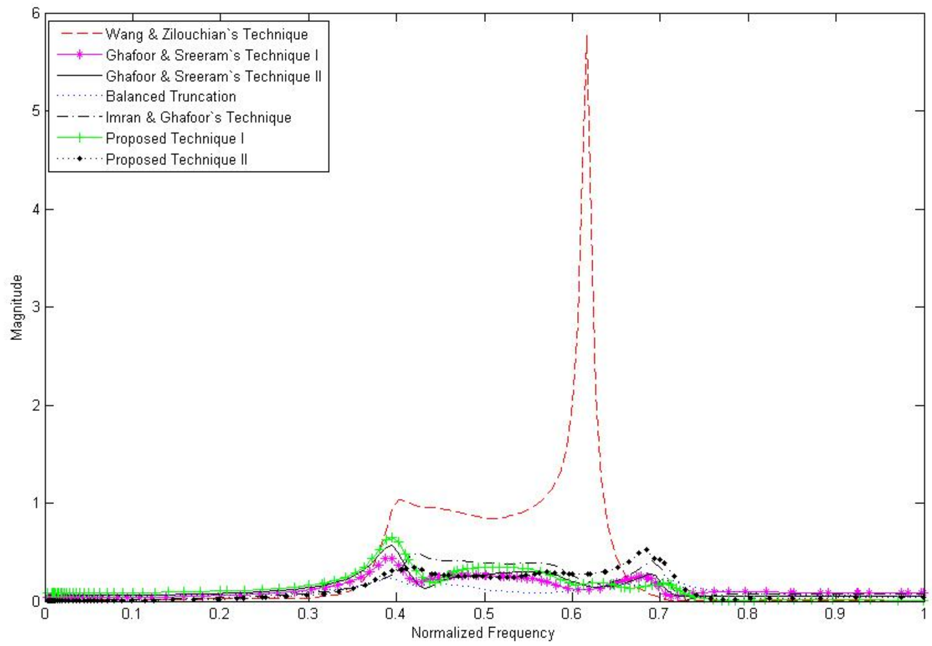

Figure 3.11: $(G(z)G_{rr}(z))$ in the frequency interval $[\omega_1, \omega_2] = [.75, .85]rad/s$

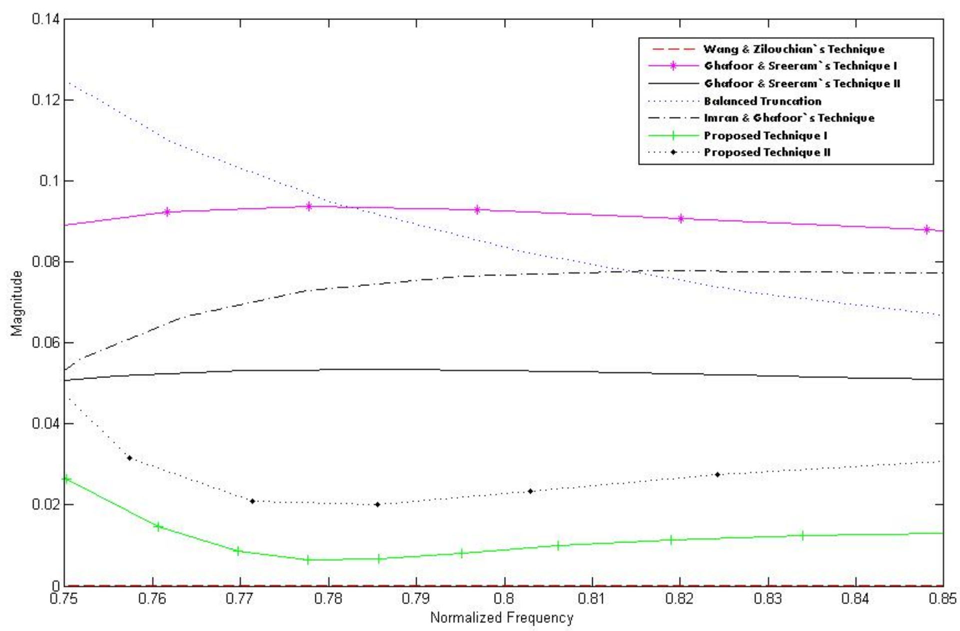

Figure 3.12: $(G(z)G_{rr}(z))$ in the frequency interval $[\omega_1, \omega_2] = [.75, .85]rad/s$

Table 3.1: Poles location of the reduced order systems

Techniques	Example 2	Example 4
Wang and Zilouchian's technique [22]	-0.4361 ,4.7463	1.0716
Proposed technique 1	$0.2971 \pm 0.8249i$	0.1425
Proposed technique 2	$0.2991 \pm 0.8748i$	0.2332

Chapter 4

Conclusions and Suggestions for Future Prospects

4.1 Conclusion

FMLR techniques are proposed in this thesis for linear continous and discrete time systems. In chapter 2, existing techniques of FLMR and Proposed techniques are discussed for linear continous time systems.
In chapter 3, existing techniques of FLMR and Proposed techniques are discussed for linear discrete time systems.
The proposed techniques mostly yield better approximation error in comparison to GA , GS and IG in the aspired frequency interval. ROMs are guranteed to be stable and has error bounds. Numerical examples are also presented.

4.2 Future Directions

As lots of work has been done in this field, some improvements are needed in this area,that are given below:

- Existing techniques like wang et al's, Varga and Anderson, Gugercin and Antoulas, Ghafoor and Sreeram and Imran and Ghafoor and proposed techniques are dependant on realization, where original system realization produces lower approximation error and tight error bounds needs attention.

- It is discussed in this research work in detail that satisfaction of Lyapunov equation stability norm is not a requisite to bear stable ROMs. Therefore, an domain for new FLMR techniques can be developed for futur prospects .

- Error based on norm and error bound derivations are not fit for FLMR techniques, whereas in a interval based frequency limited interval approximation error is important. So, as a effective future direction an error that is approoximated and expressions for error bound equations in interval based frequency limited can be calculated.
- Cost effectiveness of the techniques that are proposed can be improvised by adopting other useful measures and effective expressions.
- FLMR techniques are not relavent for non-linear systems. So, in future FLMR techniques with some suitable improvisations may be applied in non-linear problem.

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